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On the Existence and Regularity Theory of Yang-Mills Fields

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Casey Lynn Kelleher

Dissertation Committee:
Richard Melvin Schoen, Chair
Professor Vladimir Baranovsky
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2017

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ACKNOWLEDGMENTS

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Streets)
Transactions of the American Mathematical Society. DOI: 1090/tran/6949.

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The Journal of Geometric Analysis. DOI: 10.1007/s12220-015-9641-3.

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Miller, T. Osborn, A. Weston)
Journal of Mathematical Analysis and Applications. DOI: 10.1016/j.jmaa.2014.01.063.

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born, A. Weston)
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OTHER COMPLETED WORKS

Limits of Yang-Mills α-connections   **May 2017**
arXiv:1705.06104 [math.DG]

Singularity formation of the Yang-Mills flow (with J. Streets)   **February 2016**
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ABSTRACT OF THE DISSERTATION

On the Existence and Regularity Theory of Yang-Mills Fields

By

Casey Lynn Kelleher

Doctor of Philosophy in Mathematics

University of California, Irvine, 2017

Richard Melvin Schoen, Chair

This work investigates two regularization techniques designed for identifying critical points of the Yang-Mills energy.

In the first half of the dissertation, we define a family of higher order functionals generalizing the Yang-Mills functional. We study the corresponding gradient flows and prove long-time existence and convergence results for subcritical dimensions as well as a bubbling criterion for critical dimensions. Consequently, we generalize the results of the convergence of Yang-Mills flow in dimensions 2 and 3 given by Råde [Råd92] and the bubbling criterion in dimension 4 of Struwe [Str94] in the case where the initial flow data is smooth. This encompasses the contents of the author's paper [Kel14].

In the second half of the dissertation we study an alternate type of regularization. In the spirit of recent work of Lamm, Malchiodi and Micallef in the setting of harmonic maps [LMM15], we identify Yang-Mills connections obtained by approximations with respect to the Yang-Mills α-energy. More specifically, we show that for the SU(2) Hopf fibration over $\mathbb{S}^4$, for sufficiently small α values the SO(4) invariant ADHM instanton is the unique α-critical point which has Yang-Mills α-energy lower than a specific threshold. This is an overview of the author's paper [Kel16].

Chapter 1

Overview

1.1 Introduction

We first begin with a brief introduction to Yang-Mills theory, stating corresponding energies, gradient flows and past major results. We then conclude with a statement of the main results which will be the focus of this dissertation.

Yang-Mills theory originated from classical field theory in particle physics, but has since revealed deep applications to pure mathematics. In differential geometric terms, Yang-Mills theory seeks to understand the relationship between connections on vector bundles, curvature and topology. The identification and investigation of minimizers of the energies is key to extending and applying the theories. The methods which have been successfully used to construct such minimizers have been quite analogous for the two theories. In our work we examine and utilize a variety of tools drawn from analysis, geometry, and partial differential equations such as geometric flows and regularization methods.

1.1.1 Preliminaries of Yang-Mills field theory

Let $(E, h) \to (M, g)$ be a smooth vector bundle over a closed Riemannian manifold. For a connection $\triangledown \in W^{1,2}(\mathcal{A}_E(M))$ the *Yang-Mills energy of* $\triangledown$ is given by

$$\mathcal{YM}(\triangledown) := \tfrac{1}{2} \int_M |F_\triangledown|^2_{g,h} \ dV_g, \tag{1.1}$$

where $F_\triangledown$ denotes the curvature tensor of $\triangledown$. In a local trivialization of E, we can consider the decomposition $\triangledown = \partial + \Gamma$, where $\Gamma \in \Lambda^1 (\operatorname{Ad} E)$ is the connection matrix. A connection $\triangledown$ is a *Yang-Mills connection* if it is a critical point of the Yang-Mills energy, i.e.,

$$D^*_\triangledown F_\triangledown = 0.$$

Inspired by the seminal work of Eells-Sampson [ES64] in the setting of harmonic maps, in [AB82] Atiyah and Bott proposed using the *Yang-Mills flow* generated from the negative gradient of the energy functional to establish the existence of Yang-Mills connections. The flow is given by

$$\frac{\partial F_{\triangledown_t}}{\partial t} = -D^*_{\triangledown_t} F_{\triangledown_t}. \tag{1.2}$$

The Yang-Mills energy and corresponding gradient flow admits natural scaling laws dependent on dimensions of the manifold M which reflect into the behavior of the corresponding flow. In the subcritical dimensions $(\dim M = 2, 3)$ long time existence and convergence was verified by Råde in [Råd92]. In supercritical dimensions, examples of finite time singularities were constructed by Gastel [Gas02], Grotowski [Gro00], and Naito [Nai94]. In the critical dimension setting $(\dim M = 4)$, the related question has yet to be answered. However, various types of symmetries have been shown to exhibit long time existence (m-equivariance [HT04], $SO(m)$-invariance etc [SSTZ98]). Interestingly, it is expected by many members of the community that the Yang-Mills flow in fact *exists for long time*, which would be a striking difference with the harmonic map theory analogue.

1.1.2 Regularized functionals

Higher order approximations

We introduce a family of higher order Yang-Mills energy modifications which will be the focus of the first portion of the dissertation, and sketch out motivational background.

> For every $k \in \mathbb{N} \cup \{0\}$ and $\nabla \in W^{2+k,2}\left(\mathcal{A}_E\left(M\right)\right)$, the **Yang-Mills k-energy** is given by
>
> $$\mathcal{YM}_k\left(\nabla\right) := \tfrac{1}{2} \int_M \left| \nabla^{(k)} F_\nabla \right|_{g,h}^2 \, dV_g. \tag{1.3}$$

Critical points of (1.3) satisfy the *Yang-Mills k-equation*,

$$0 = (-1)^k D_\nabla^* \triangle^{(k)} F_\nabla + P_1^{(2k+1)}\left[F_\nabla\right] + P_2^{(2k-1)}\left[F_\nabla\right],$$

where $\triangle^{(k)}$ means k iterations of the rough Laplacian, the P notation is defined in (1.7), and D_∇^* defined in (1.4). In preparation for future applications we construct a generalization of the flow, called *generalized Yang-Mills k-flow* given by, for a one-parameter family ∇_t of connections,

$$\tfrac{\partial \nabla_t}{\partial t} = (-1)^{k+1} D_{\nabla_t}^* \triangle_t^{(k)} F_{\nabla_t} + \mho_k(\nabla_t), \tag{gYMkf}$$

where $\mho_k(\nabla)$ is a lower order tensor featuring terms of the background manifold M to be more precisely defined in Chapter 2.

In Yang-Mills theory, the question of long time existence and convergence of Yang-Mills gradient flow over four dimensional manifolds has yet to be determined and is an area of particular interest. For many flows, the critical dimension offers interesting results but requires nonstandard approaches to study. One of the advantages to our proposed study of the functionals above is that the corresponding family of flows have increasing critical dimension, so with appropriate strategies one may be able to provide some insight on Yang-Mills flow and the space of connections in higher dimensions. This can be accomplished by considering the regularization inspired by work performed by Hong, Tian and Yin [HTY15] in their study of the Yang-Mills

α-flow. Consider the *Yang-Mills (ρ, k)-energy* given by, for $\rho \in [0, \infty)$,

$$\mathcal{YM}_k^\rho(\nabla) := \rho \mathcal{YM}_k(\nabla) + \mathcal{YM}(\nabla) \cdot \tag{YMρkE}$$

By studying the corresponding negative gradient flow and sending $\rho \searrow 0$ one expects to, as in the case of [HTY15], identify solutions to the Yang-Mills flow. The advantage of approaching with this quantity is that one is not restricted to any particular dimension; while [HTY15] focuses on dim $M = 4$, in our case by choosing appropriate choices of k one can regularize in any dimension.

In reference to the Yang-Mills k-energy, in the case $k = 1$ the properties of the Yang-Mills 1-energy compare to those of the *bi-Yang-Mills energy*. This functional, studied by demonstrated by the analysis of Ichiyama, Inoguchi and Urakawa in [IIU09], is given by

$$\mathcal{BYM}(\nabla) := \tfrac{1}{2} \int_M |D_\nabla^* F_\nabla|^2 \, dV_g. \tag{BYME}$$

We will reflect on these two energies and their relationship in §2.3.1. Another consequence of our analysis in this paper is the last key result, a statement on the properties in subcritical dimensions of *bi-Yang-Mills flow*, given by

$$\tfrac{\partial \nabla_t}{\partial t} = \triangle_t D_{\nabla_t}^* F + \left[D_{\nabla_t}^* F_{\nabla_t}, F_{\nabla_t} \right]^\# .$$

where here the 'pound bracket' featured on the right term is defined in (1.6). Recently, two families of functionals whose flows are included within this generalized family (gYMkf) were studied in detail by Gastel and Scheven in [GS15]. Roughly speaking, they essentially perform an elliptic analogue of our parabolic analysis demonstrated in Chapter 2. To discuss their results, we introduce the operator

$$D_\nabla^{* \wedge q} := \begin{cases} (D_\nabla D_\nabla^*)^{q/2} & q \in 2\mathbb{N}, \\ D_\nabla^* (D_\nabla D_\nabla^*)^{(q-1)/2} & q \in (2\mathbb{N} - 1). \end{cases} \tag{1.4}$$

We also define

$$\mathcal{Y}_k(\nabla) := \int_M \left(\left| D_\nabla^{* \wedge (k-2)} F_\nabla \right|_{g,h}^2 + |F_\nabla|_{g,h}^k \right) dV_g, \tag{YkE}$$

$$\mathcal{Z}_k(\nabla) := \int_M \left(\left| \nabla^{(k-2)} F_\nabla \right|_{g,h}^2 + |F_\nabla|_{g,h}^2 \right) dV_g. \tag{ZkE}$$

Each energy is chosen to appeal to different qualities. The first, (YkE), is gauge invariant and scaling invariant for $\dim M = 2k$, as well as nondegenerate and coercive with respect to the Uhlenbeck's gauge (cf. Theorem 3.8.12). The latter, (ZkE), is the case $\rho = 1$ of the (ρ, k)-flow we will address. Though the gauge invariance is preserved, this flow is degenerate, does not satisfy the scaling law perfectly, but is a perturbation of the Yang-Mills energy and thus is a strong candidate for study. Through (1.4) the authors perform an elliptic analysis on (YkE) and (ZkE), within which the $|F_\nabla|$ quantities featured in the integrand is a necessary addition for their results.

α-energy

Another approach to existence results within Yang-Mills theory is by studying a regularization of a functional in the static setting. This method seems perhaps better suited to finding unstable critical points of the Yang-Mills energy which cannot easily be found via flow methods. A difficulty of studying the Yang-Mills energy in their corresponding critical dimensions is that Sobolev embeddings which yield sufficient regularity fail. For this reason, it is natural to perturb the functional to break conformal invariance and obtain properties such as the Palais Smale condition. For this purpose, inspired by the harmonic map counterpart, Hong, Tian and Yin [HTY15] introduced the following perturbation of the Yang-Mills energy:

> For every $\alpha > 1$ and $\nabla \in W^{1,2\alpha}\left(\mathcal{A}_E\left(M\right)\right)$, the **Yang-Mills α-energy** is given by
>
> $$\mathcal{YM}_\alpha\left(\nabla\right) := \tfrac{1}{2} \int_M \left(1 + |F_\nabla|^2_{g,h}\right)^\alpha dV_g.$$

The aforementioned harmonic map counterpart was explored in the work of Sacks and Uhlenbeck in [SU81], who developed an existence theory for minimal 2-spheres in compact Riemannian manifolds by studying the corresponding *harmonic map α-energy*. The purpose of this construction can be illustrated by focusing on the setting of over $\mathbb{S}^2$. A difficulty that arises when identifying critical points is that the set of critical maps of the harmonic map energy is noncompact, since it is comprised of conformal transformations of $\mathbb{S}^2$. Thus, the α-energy was chosen to perturb in a way which 'breaks' conformal symmetry in the sense of dilations (the energy is still invariant under rotations). For $\alpha > 1$, the α-energy satisfies Ljusternik Schnirelman theory (yielding lower bounds for the number of critical points) and a Morse theory in addition to the Palais Smale condition. The authors obtained the following result which we sketch.

Theorem of [SU81]. Let $u \in C^\infty\left(M^2, N\right)$, with M and N closed Riemannian manifolds. The energy $\mathcal{E}_\alpha$

given by $\mathcal{E}_\alpha(u) = \frac{1}{2} \int_M \left(1 + |du|^2\right)^\alpha dV$ satisfies the Palais-Smale condition. If $\{u^\alpha\} : M \to N$ is a sequence of critical points with $\alpha \searrow 1$, then if $\{\mathcal{E}_\alpha(u^\alpha)\}$ is uniformly bounded a subsequence converges smoothly to a harmonic map away from at most finitely many points. Furthermore, a sophisticated blow-up phenomenon occurs around such singularities.

Following the initial work of [SU81], Hong, Tian and Yin developed a parallel theory for the Yang-Mills setting in [HTY15], and additionally explored the corresponding negative gradient flow (the *Yang-Mills α-flow*). We emphasize that unlike the Yang-Mills energy, this quantity is not conformally invariant for $\dim M = 4$. Critical points of this energy are smooth up to gauge due to work of Isobe [Iso08] (stated explicitly in Hong and Schabrun's work, §4 of [HS13]) and satisfy

$$D^*_{\nabla_t} F_{\nabla_t} - (\alpha - 1) \frac{\star \left(d|F_{\nabla_t}|^2_{g,h} \wedge \left(\star F_{\nabla_t}\right)\right)}{1 + |F_{\nabla_t}|^2_{g,h}} = 0, \tag{1.5}$$

where $\star$ denotes the Hodge star operator. Note by setting $\alpha = 1$ on obtains the equation satisfied precisely by a *Yang-Mills connection* emerges. In [HTY15], the authors studied the corresponding negative gradient flow of the α-energy. Hong and Schabrun [HS13] continued exploring the α-energy by verifying the Palais Smale condition and applied this to Yang-Mills flow to acquire an energy identity (cf. [HS13] Theorem 1).

1.2   Main results

We outline the main results which will be investigated throughout this dissertation. We first state our results regarding the higher order energies introduced in [Kel14], followed by those regarding the α-energy featured in [Kel16].

Theorem A. Let $(E, h) \to (M, g)$ be a vector bundle over a closed Riemannian manifold. Let ∇ be a smooth metric compatible connection on E and F_∇ its curvature tensor.

(S) (Subcritical) If $\dim M \in [2, 2(k+2)) \cap \mathbb{N}$, there is a unique solution ∇_t to Yang-Mills k-flow with $\nabla_0 = \nabla$ existing for $t \in \mathbb{R}_{\geq 0}$.

(C) (Critical) If $\dim M = 2(k+2)$, there is a unique solution ∇_t to Yang-Mills k-flow, with $\nabla_0 = \nabla$, existing on $[0, T)$ for some maximal $T \in \mathbb{R}_{>0} \cup \{\infty\}$. If $T < \infty$, then there exists a sequence $\{(x_i, t_i)\} \subset M \times [0, T)$ where $(x_i, t_i) \to (X, T)$ and for all $r > 0$,

$$\limsup_{i \to \infty} \left|\left| F_{\nabla_{t_i}} \right|\right|^{k+2}_{L^{k+2}(B_r(X))} \geq \epsilon,$$

where $B_r(X)$ denotes the geodesic ball of radius r centered about X.

Remark 1.2.1. Although in our setting we assume smoothness of the initial connection ∇, if one assumes instead that ∇ lies in certain Sobolev spaces depending on k we can conclude a wider generalization of results. In particular for $k = 0$ we may confirm Theorem 1 of Råde [Råd92] and a slightly weaker version of Theorem 2.3 of Struwe [Str94] in the case of Yang-Mills flow.

The second key result is a useful consequence of the analysis done to prove Theorem A.

Theorem B. Let $(E, h) \to (M, g)$ be a vector bundle over a closed Riemannian manifold. Let ∇ be a smooth metric compatible connection on E and F_∇ its curvature tensor. For all $\rho > 0$, if $\dim M \in [2, 2(k+2)) \cap \mathbb{N}$ there exists a unique solution ∇_t to Yang-Mills (ρ, k)-flow with $\nabla_0 = \nabla$ existing for $t \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $\boldsymbol{\nabla} := \lim_{t \to \infty} \nabla_t$ is a solution to the *Yang-Mills (ρ, k)-equation*, given by

$$0 = \rho \left((-1)^k D^*_\nabla \triangle^{(k)} F_\nabla + P_1^{(2k+1)} [F_\nabla] + P_2^{(2k-1)} [F_\nabla] \right) + D^*_\nabla F_\nabla.$$

Remark 1.2.2. For Theorem B the uniqueness of the solution to the flow at $t = \infty$ can be demonstrated by proving complete convergence rather than sequential. The proof hinges on verifying that the Yang-Mills k-energy satisfies the Łojasiewicz-Simon gradient inequality, as in the proof of Råde in [Råd92]. However, this verification is nontrivially technical and geometrically uninformative, so we exclude it from this particular paper. We refer the reader to §7 of [Fee14] for more information regarding the inequality including a discussion of its use by Råde.

Theorem C. Let $(E, h) \to (M, g)$ be a vector bundle over a smooth compact finite-dimensional boundariless Riemannian manifold. Let ∇ be a smooth metric compactible connection on E and F_∇ its curvature tensor. If $\dim M \in [2, 6) \cap \mathbb{N}$ there exists is a solution ∇_t to bi-Yang-Mills flow with $\nabla_0 = \nabla$ existing for $t \in \mathbb{R}_{\geq 0}$.

It is important to note that the behavior of bi-Yang-Mills flow in the critical dimension (6) remains mysterious. Let's turn our attention to instead the Yang-Mills α-energy. Motivated by the computations featured in (3.4) of Part 3, we have the following.

Theorem D. Let $E \to \left(\mathbb{S}^4, \mathring{g}\right)$ be the adjoint bundle associated to the SU(2) Hopf fibration. There exists $\epsilon > 0$ and $\alpha_0 > 1$ such that for any $\alpha \leq \alpha_0$ the only critical point ∇^α of the Yang-Mills α-energy which satisfies $\mathcal{YM}_\alpha\left(\nabla^\alpha\right) \leq 6^\alpha \frac{4}{3}\pi^2 + \epsilon$ is the basic ADHM connection $\widetilde{\nabla}$.

A natural corollary follows, which adds to the work of [HTY15] regarding the corresponding flow.

Theorem E. Let $\{\nabla_t^\alpha\}$ be a family of solutions to Yang-Mills α-flow satisfying $\mathcal{YM}_\alpha\left(\nabla_0^\alpha\right) \leq 6^\alpha \frac{4}{3}\pi^2 + \epsilon$ and $\alpha \leq \alpha_0$ as in the assumptions of Theorem D. Then $\{\nabla_t^\alpha\}$ converges smoothly in α and t to $\widetilde{\nabla}$.

1.3 Reflections and future directions

Yang-Mills k-energy

By constructing a generalization of Yang-Mills k-flow our analysis was extended to a broader range of flows. One wonders if, for each family of generalized Yang-Mills k-flows, there is a canonical representative for each k, and a corresponding canonical functional. One trait which could distinguish this canonical member is conformal invariance. In the case of Yang-Mills flow, the conformal invariance in dimension 4 is crucial to the work of Taubes in [Tau87] regarding the process of constructing Yang-Mills instantons via gluing procedures.

Yang-Mills α-energy

Naturally, one asks how this type of α-limiting process behaves in other settings. More precisely, given a sequence of $\{\nabla^\alpha\}$ be a sequence of α-harmonic connections over a charge ℓ bundle $E \to (M, g)$, which critical points of the Yang-Mills energy could $\{\nabla^\alpha\}$ possibly converge to? Some cases to consider are SU(2) type bundles over the following spaces, which admit ADHM constructions (as stated in [DK90] pp. 127–129): $\overline{\mathbb{CP}}^2$ (charge 1), $\mathbb{CP}^2$ (charge 2), and $\mathbb{S}^2 \times \mathbb{S}^2$ (charge 2). It would be interesting to know which canonical connections the Yang-Mills α-energy identifies in these various settings, especially cases which do *not* have as obvious structural symmetry as the setting above. Based on our initial work as well as that of [LMM15], we conjecture that in general settings the limits coming from α-approximations lie in a compact subset of the moduli space.

1.4   Notation and conventions

Let $(E, h) \to (M, g)$ be a vector bundle over a closed Riemannian manifold. Let $S(E)$ denote the smooth sections of E. For each point $x \in M$ choose a local orthonormal basis of $T_x M$ given by $\{\partial_i\}$ with dual basis $\{e^i\}$ and a local basis for E given by $\{\mu_\alpha\}$ with dual basis $\{(\mu^*)^\alpha\}$ for the dual E^*. Let $\Lambda^p(M)$ denote the set of smooth p-forms over M and set $\Lambda^p(E) := \Lambda^p(M) \otimes S(E)$. Next set $\mathrm{End}\, E := E \otimes E^*$, where E^* denotes the dual space of E and take

$$\Lambda^p(\mathrm{Ad}\, E) := \{\omega \in \Lambda^p(\mathrm{End}\, E) \mid h_{\mu\gamma}\omega_\beta^\gamma = -h_{\beta\gamma}\omega_\mu^\gamma\}.$$

The affine space of all bundle metric compatible connections on E will be denoted by $\mathcal{A}_E(M)$. Given a chart about $x \in M$ the action of a connection ∇ on E is captured by the *local coefficient matrices* $\Gamma = (\Gamma_{i\zeta}^\beta e^i \otimes \mu_\beta \otimes (\mu^*)^\zeta)$, where

$$\nabla\mu_\beta = \Gamma_{i\beta}^\zeta e^i \otimes \mu_\zeta.$$

The inner products on the bundle indices of E induce a product on the tensor product; in the case of $\mathrm{Ad}\, E$ this inner product extends to the negative of the Killing form. Pairing these connections with the Levi-Civita connection on M also allows us to define multiple iterations of the connection on any element from combinations of E, TM and their respective duals.

Let D_∇ be the *exterior derivative*, or skew symmetrization of ∇ over the tensor products of TM where for each $p \in \mathbb{N} \cup \{0\}$,

$$D_{\nabla,p} : \Lambda^p(E) \to \Lambda^{p+1}(E)$$

$$: \omega \mapsto \mathrm{Alt}(\nabla\omega),$$

where Alt is the unscaled alternating map on tensors. The p subscript will be typically supressed. The *curvature tensor on E* and its coordinate formulation are given by

$$F_\nabla := D_{\nabla,1} \circ D_{\nabla,0} : \Lambda^0(E) \to \Lambda^2(E)$$

$$: F_{ij\alpha}^\beta = \partial_i \Gamma_{j\alpha}^\beta - \partial_j \Gamma_{i\alpha}^\beta - \Gamma_{j\delta}^\beta \Gamma_{i\alpha}^\delta + \Gamma_{i\delta}^\beta \Gamma_{j\alpha}^\delta.$$

The pointwise and global (L^2) inner products are given respectively by

$$\langle \cdot, \cdot \rangle, (\cdot, \cdot) : \ \Lambda^p(\mathrm{Ad}\, E) \times \Lambda^p(\mathrm{Ad}\, E) \to \mathbb{R},$$

$$\langle \omega, \zeta \rangle := -\left(\prod_{v=1}^p g^{i_v j_v} \right) \omega_{i_1 \dots i_p \alpha}^\beta \zeta_{j_1 \dots j_p \beta}^\alpha, \quad |\zeta| := \sqrt{\langle \zeta, \zeta \rangle},$$

$$(\omega, \zeta) := \int_M \langle \omega, \zeta \rangle \, dV_g, \quad \|\zeta\|_{L^2(M)} := \sqrt{(\zeta, \zeta)},$$

where dV_g denotes the canonical volume form with respect to the metric g. The definition of $\langle \cdot, \cdot \rangle$, and thus $(\cdot, \cdot)$, can be extended when necessary as follows. Let $p, q \in \mathbb{N}$ with $p < q$. Let $K = (k_i)_{i=1}^p$ and $L = (l_i)_{i=1}^q$ multiindices, and let $\omega \in \Lambda^p(\mathrm{Ad}\, E)$ and $\xi \in \Lambda^q(\mathrm{Ad}\, E)$ respectively. Then

$$\langle \omega, \xi \rangle := -\left(\prod_{i=1}^p g^{k_i l_i} \right) \omega_{K\delta}^\beta \xi_{L\beta}^\delta.$$

Henceforth, we will suppress g notation and match indices. Considering the nonextended inner product, we set ∇^* to be the formal L^2 adjoint of ∇. For computational purposes, D_∇^* will denote a rescaled version of the formal L^2 adjoint of D_∇, satisfying for $\omega \in \Lambda^p(\mathrm{Ad}\, E)$ and $\psi \in \Lambda^{p-1}(\mathrm{Ad}\, E)$,

$$\int_M \langle D_\nabla^* \omega, \psi \rangle \, dV_g = \frac{1}{p} \int_M \langle \omega, D_\nabla \psi \rangle \, dV_g.$$

In coordinates, this is given simply by $(D_\nabla^* \omega)_{i_1 \dots i_p \mu}^\beta = -g^{j i_1} \nabla_j \omega_{i_1 \dots i_p \mu}^\beta$. The *rough* and *Hodge Laplacian* are

given respectively by

$$\triangle : \Lambda^p(E) \to \Lambda^p(E)$$

$$: \omega \mapsto -\nabla^*\nabla\omega,$$

$$\triangle_{D_\nabla} : \Lambda^p(E) \to \Lambda^p(E)$$

$$: \omega \mapsto (D_\nabla^* D_\nabla + D_\nabla D_\nabla^*)\omega.$$

Note that with this convention, the Bochner formula yields that the two Laplacian operators differ by a sign and lower order terms (cf. Proposition 3.7.1).

Operations.

Let $\omega, \zeta \in \Lambda^p(E)$. Let $\omega * \zeta$ express any normal-valued multilinear form depending on ω and ζ in a universal bilinear way. Furthermore by the Cauchy-Schwartz inequality there exists some $C > 0$ such that

$$|\omega * \zeta| \leq C \, |\omega| \, |\zeta| \, .$$

Let $J := (j_i)_{i=1}^{|J|}$ and $K := (k_i)_{i=1}^{|K|}$ be multiindices and let j, k be a distinct indices from those in J and K respectively. The operation *pound* is given by

$$\# : (T^*M)^{\otimes|J|+1} \otimes (\operatorname{End} E) \times (T^*M)^{\otimes|K|+1} \otimes (\operatorname{End} E) \to (T^*M)^{|J|+|K|} \otimes (\operatorname{End} E),$$

$$(\omega\#\zeta)\left(\partial_{j_1}, ..., \partial_{j_{|J|}}, \partial_{k_1}, ..., \partial_{k_{|K|}}\right) = \sum_{i=1}^{n} \omega\left(\partial_i, \partial_{j_1}, ..., \partial_{j_{|J|}}\right) \zeta\left(\partial_i, \partial_{k_1}, ..., \partial_{k_{|K|}}\right).$$

In coordinates this is written in the form $(\omega\#\zeta)_{JK\alpha}^{\beta} = g^{jk}\omega_{jJ\delta}^{\beta}\zeta_{kK\alpha}^{\delta}$. Roughly speaking, $\#$ is matrix multiplication with respect to the bundle combined with contraction of the first two base manifold indices. The corresponding *pound bracket* is

$$[\omega, \zeta]^\#\left(\partial_{j_1}, ..., \partial_{j_{|J|}}, \partial_{k_1}, ..., \partial_{k_{|K|}}\right) := (\omega\#\zeta)\left(\partial_{j_1}, ..., \partial_{j_{|J|}}, \partial_{k_1}, ..., \partial_{k_{|K|}}\right)$$
$$- (\zeta\#\omega)\left(\partial_{j_1}, ..., \partial_{j_{|J|}}, \partial_{k_1}, ..., \partial_{k_{|K|}}\right). \tag{1.6}$$

Derivatives.

When not performing coordinate computations, for $\nabla \in \mathcal{A}_E$, we reserve upper indices without parentheses for indexing sequences and with parentheses for iterations of differentiation. That is,

$$\{\nabla^i\}_{i\in\mathbb{N}} = \{\nabla^1, \nabla^2, \cdots\} \text{ and } \nabla^{(i)} = \underbrace{\nabla \cdots \nabla}_{i \text{ times}}.$$

We utilize the P notation introduced in [KS86]. Let $s, v \in \mathbb{N}$ and let R represent a generic background tensor dependent only on g, so that

$$P_v^{(s)}[\omega] := \sum_{w_1+\cdots+w_v=s} \left(\nabla^{(w_1)}\omega\right) * \cdots * \left(\nabla^{(w_v)}\omega\right) * R. \tag{1.7}$$

Notation.

Many quantities are one-parameter families, and we will often call this parameter the 'temporal' parameter. This parametrization will be denoted with t parameter, however when understood, particularly in computations, the t subscript will be dropped. Differentiation with respect to t will sometimes be indicated with '$\cdot$' for notational convenience. A geodesic ball centered at a point $x \in M$ with radius r will be denoted by $B_r(x)$.

There is a simple but interesting fact regarding multiplication over $\mathrm{Ad}\, E$. In particular, if we denote the product

$$(A_1)_{\rho_2}^{\rho_1} \cdots (A_{n-1})_{\rho_n}^{\rho_{n-1}} (A_n)_{\rho_1}^{\rho_n},$$

then we have two simple identities. First, the *cyclic property*:

$$(A_1)_{\rho_1}^{\rho_2} \cdots (A_n)_{\rho_1}^{\rho_n} = (A_i)_{\rho_{i+1}}^{\rho_i} \cdots (A_{n-1})_{\rho_n}^{\rho_{n-1}} (A_n)_{\rho_1}^{\rho_n} (A_1)_{\rho_2}^{\rho_1} \cdots (A_{i-1})_{\rho_i}^{\rho_{i-1}},$$

and the *flipping property*, specifically using the fact we are working in $\mathrm{Ad}\, E$,

$$(A_1)_{\rho_2}^{\rho_1} \cdots (A_{n-1})_{\rho_n}^{\rho_{n-1}} (A_n)_{\rho_1}^{\rho_n} = (-1)^n (A_1)_{\rho_1}^{\rho_2} \cdots (A_{n-1})_{\rho_{n-1}}^{\rho_n} (A_n)_{\rho_n}^{\rho_1}$$

$$= (-1)^n (A_n)_{\rho_n}^{\rho_1} (A_{n-1})_{\rho_{n-1}}^{\rho_n} \cdots (A_1)_{\rho_1}^{\rho_2}.$$

Note that these manipulations are completely independent of the relationship of each A_i to the base manifold (roughly speaking, the 'base manifold indices get dragged along'). To reduce notation, we often write

$$\text{tr}_h\left[A_1\cdots A_n\right] = (A_1)^{\rho_1}_{\rho_2}\cdots (A_n)^{\rho_n}_{\rho_1}.$$

Connection Sobolev spaces

The space of connections is an affine space of the form

$$\mathcal{A}_E\left(M\right) = \left\{\nabla = \nabla_{\text{ref}} + A : A \in \Lambda^1(\text{Ad}\, E)\right\}.$$

Definition 1.4.1. Fix ∇_{ref} a background connection on E. The space $W^{l,p}(\Lambda^i(\text{Ad}\, E))$ is the completion of the space of smooth sections of $\Lambda^i(\text{Ad}\, E)$ with respect to the norm

$$\|A\|_{W^{l,p}} := \left(\sum_{k=0}^{l}\left\|\nabla^{(k)}_{\text{ref}} A\right\|^p_{L^p}\right)^{1/p} < \infty, \quad A \in \Lambda^i(\text{Ad}\, E).$$

We will say A is of *Sobolev class* $W^{l,p}$. This space is preserved (up to equivalent norms) regardless of choice of reference connections. In Part 3, we will be using the SO(4)-invariant ADHM connection as our reference connection.

Chapter 2

Limits of Yang-Mills k-connections

2.1 Introduction to Chapter 2

In this portion of the dissertation we investigate the higher order Yang-Mills k-energy and its corresponding flow with and eye towards proving Theorem A, B and C.

2.1.1 General variations

The variations of one-parameter families will be computed in preparation for the work of §2.2. For the remainder of this paper set $\mathcal{I}$ to denote a simply connected parametrization subset of $\mathbb{R}$.

Lemma 2.1.1. *Suppose ∇_t and ω_t are smooth one-parameter families of connections and elements of $\Lambda^p(\mathrm{End}\,E)$ respectively. Then for $\ell \in \mathbb{N}$, if $\dot{\Gamma} := \frac{\partial \nabla}{\partial t}$,*

$$\frac{\partial}{\partial t}\left[\nabla_t^{(\ell)}\omega_t\right] = \sum_{i=0}^{\ell-1}\left(\nabla_t^{(i)}\dot{\Gamma}_t\right) * \left(\nabla_t^{(\ell-i-1)}\omega_t\right) + \left(\nabla_t^{(\ell)}\dot{\omega}_t\right). \tag{2.1}$$

Proof. The proof follows by induction on $\ell \in \mathbb{N}$ satisfying (2.1). Let $J := (j_w)_{w=1}^{|J|}$ be a multiindex and set

$$J(w,s) := \begin{cases} j_r & \text{if } r \neq w, \\ s & \text{if } r = w. \end{cases}$$

Roughly speaking $J(w, s)$ substitutes the wth element of the J multiindex with an s. For $\ell = 1$ applying normal coordinates yields

$$\frac{\partial}{\partial t}\left[\nabla_i \omega_{J\alpha}^\beta\right] = \partial_i \dot{\omega}_{L\alpha}^\beta - \sum_{w=1}^{|P|}\left(\dot{G}_{ij_w}^s \omega_{J(w,s)\alpha}^\beta + G_{ij_w}^s \dot{\omega}_{J(w,s)\alpha}^\beta\right) - \omega_{J\delta}^\beta \dot{\omega}_{i\alpha}^\delta - \dot{\omega}_{J\delta}^\beta \Gamma_{i\alpha}^\delta + \dot{\Gamma}_{i\delta}^\beta \omega_{J\alpha}^\delta + \Gamma_{i\delta}^\beta \dot{\omega}_{J\alpha}^\delta.$$

(2.2)

Hence the base case holds, giving

$$\frac{\partial}{\partial t}\left[\nabla\omega\right] = \nabla\dot{\omega} + \dot{\Gamma} * \omega.$$

(2.3)

Now assume the hypothesis (2.1) is satisfied for $\ell \in \mathbb{N}$ and let L be a multiindex with $|L| = \ell$. We compute

$$\begin{aligned}
\left(\frac{\partial}{\partial t}\left[\nabla^{(\ell+1)}\omega\right]\right)_{jLP\alpha}^\beta &= \partial_j\left[\frac{\partial}{\partial t}\nabla^{(\ell)}\omega\right]_{LP\alpha}^\beta + \frac{\partial}{\partial t}\left[\Gamma_{p\delta}^\beta(\nabla^{(\ell)}\omega_{LP\alpha}^\delta) - (\nabla^{(\ell)}\omega_{LP\delta}^\beta)\Gamma_{j\alpha}^\delta\right] \\
&= \partial_j\left[\nabla^{(i)}\dot{\Gamma} * \nabla^{(\ell-i-1)}\omega + \nabla^{(\ell)}\dot{\omega}\right]_{LP\alpha}^\beta \\
&\quad - \dot{\Gamma}_{j\alpha}^\delta(\nabla^{(\ell)}\omega_{LP\delta}^\beta) + \dot{\Gamma}_{j\delta}^\beta(\nabla^{(\ell)}\omega_{LP\alpha}^\delta).
\end{aligned}$$

Or, written coordinate invariantly,

$$\begin{aligned}
\frac{\partial}{\partial t}\left[\nabla^{(\ell+1)}\omega\right] &= \nabla^{(i+1)}\dot{\Gamma} * \nabla^{(\ell-i-1)}\omega + \nabla^{(i)}\dot{\Gamma} * \nabla^{(\ell-i)}\omega + \nabla^{(\ell+1)}\dot{\omega} + \dot{\Gamma} * \nabla^{(\ell)}\omega \\
&= \nabla^{(i)}\dot{\Gamma} * \nabla^{(\ell-i-1)}\omega + \nabla^{(\ell+1)}\dot{\omega}.
\end{aligned}$$

Hence $\ell + 1$ satisfies the induction hypothesis, so the result follows. $\square$

Corollary 2.1.2. *Suppose ∇_t is a smooth one-parameter family of connections. Then for $\ell \in \mathbb{N}$*

$$\frac{\partial}{\partial t}\left[\nabla_t^{(\ell)} F_{\nabla_t}\right] = \sum_{i=0}^{\ell-1} \nabla_t^{(i)}\dot{\Gamma}_t * \nabla_t^{(\ell-i-1)} F_{\nabla_t} + \nabla_t^{(\ell)} D_{\nabla_t}\dot{\Gamma}_t.$$

(2.4)

Furthermore, $\frac{\partial F_{\nabla_t}}{\partial t} = D_{\nabla_t}^* \dot{\Gamma}_t$.

2.1.2 Flow specific variations

We first compute the Euler Lagrange equation of the Yang-Mills k-energy to determine the corresponding Yang-Mills k-flow, and introduce a generalized version of the flow to expand the scope of our results. We

then demonstrate that this flow is a weakly parabolic system.

Proposition 2.1.3. *The variation of the Yang-Mills k-energy is given by*

$$\frac{d}{dt}\left[\mathcal{YM}_k(\nabla_t)\right] = \int_M \left\langle \mathrm{Grad}\,\mathcal{YM}_k(\nabla_t), \frac{\partial \nabla_t}{\partial t}\right\rangle dV_g,$$

where

$$\mathrm{Grad}\,\mathcal{YM}_k(\nabla) := (-1)^k D_\nabla^* \triangle^{(k)} F_\nabla + \sum_{v=0}^{2k-1} P_1^{(v)}[F_\nabla] + P_2^{(2k-1)}[F_\nabla].$$

Proof. Differentiating the Yang-Mills k-energy with respect to t yields

$$\frac{d}{dt}\left[\frac{1}{2}\int_M |\nabla^{(k)} F|^2 \, dV_g\right] = \int_M \left\langle \frac{\partial}{\partial t}\left[\nabla^{(k)} F\right], \nabla^{(k)} F\right\rangle dV_g.$$

Appealing to Corollary 2.1.2 for the variation of $\nabla^{(k)} F_\nabla$ yields

$$\frac{d}{dt}\left[\frac{1}{2}\int_M |\nabla^{(k)} F|^2 \, dV_g\right] = \int_M \left\langle \left(\sum_{i=0}^{k-1} \nabla^{(i)}\dot{\Gamma} * \nabla^{(k-i-1)} F + \nabla^{(k)} D\dot{\Gamma}\right), \nabla^{(k)} F\right\rangle dV_g$$

$$= \int_M \left\langle \left(\sum_{i=0}^{k-1} \nabla^{(i)}\dot{\Gamma} * \nabla^{(k-i-1)} F\right), \nabla^{(k)} F\right\rangle dV_g + \int_M \left\langle \nabla^{(k)} D\dot{\Gamma}, \nabla^{(k)} F\right\rangle dV_g.$$

$$(2.5)$$

For the first integral of (2.5) integration by parts yields

$$\int_M \left\langle \left(\sum_{i=0}^{k-1} \nabla^{(i)}\dot{\Gamma} * \nabla^{(k-i-1)} F\right), \nabla^{(k)} F\right\rangle dV_g = \int_M \left\langle \dot{\Gamma}, P_2^{(2k-1)}[F]\right\rangle dV_g.$$

The second integral of (2.5) is addressed with Lemma 3.7.4 to recursively integrate by parts,

$$\int_M \left\langle \nabla^{(k)} D\dot{\Gamma}, \nabla^{(k)} F\right\rangle dV_g = \int_M \left\langle D\dot{\Gamma}, \sum_{v=1}^{2k-2}\sum_{w=0}^{v} \nabla^{(w)}\mathrm{Rm} * \nabla^{(2k-2-w)} F + P_2^{(2k-2)}[F]\right\rangle dV_g$$

$$+ \int_M \left\langle D\dot{\Gamma}, (-1)^k \triangle^{(k)} F\right\rangle dV_g$$

$$= \int_M \left\langle D\dot{\Gamma}, \sum_{v=0}^{2k-2} P_1^{(v)}[F] + P_2^{(2k-2)}[F]\right\rangle dV_g + \int_M \left\langle D\dot{\Gamma}, (-1)^k \triangle^{(k)} F\right\rangle dV_g$$

$$= \int_M \left\langle \dot{\Gamma}, \sum_{v=0}^{2k-1} P_1^{(v)}[F] + P_2^{(2k-1)}[F]\right\rangle dV_g + 2\int_M \left\langle \dot{\Gamma}, (-1)^k D^* \triangle^{(k)} F\right\rangle dV_g.$$

Combining the integrands we conclude the result. □

Given ∇_t a smooth one-parameter family of connections define *Yang-Mills k-flow* by

$$\frac{\partial \nabla_t}{\partial t} = -\operatorname{Grad} \mathcal{YM}_k(\nabla_t) = (-1)^{k+1} D_{\nabla_t}^* \triangle_t^{(k)} F_{\nabla_t} + \sum_{v=0}^{2k-1} P_1^{(v)} [F_{\nabla_t}] + P_2^{(2k-1)} [F_{\nabla_t}]. \tag{YMkf}$$

Setting $k = 0$ in (YMkf) and omitting the lower order terms immediately yields Yang-Mills flow,

$$\frac{\partial \nabla_t}{\partial t} = -D_{\nabla_t}^* F_{\nabla_t}.$$

For future work it is advantageous to perform a more general analysis of these flows. For $k \in \mathbb{N}$ set,

$$\mho_k(\nabla) := \sum_{i=1}^{k} \sum_{j=2}^{i+1} P_j^{(2i+3-2j)} [F_\nabla]. \tag{2.6}$$

We additionally set $\mho_0 \equiv 0$. Define *generalized Yang-Mills k-flow* by

$$\frac{\partial \nabla_t}{\partial t} = (-1)^{k+1} D_{\nabla_t}^* \triangle_t^{(k)} F_{\nabla_t} + \mho_k(\nabla_t). \tag{gYMkf}$$

Next we demonstrate the weak ellipticity of generalized Yang-Mills k-flow, which is a result of the gauge invariance of the Yang-Mills k-energy (cf. Corollary 3.8.8) from which the flow is constructed.

Proposition 2.1.4. *Set*

$$\Phi_k : \mathcal{A}_E \to \Lambda^1(\operatorname{End} E)$$

$$: \nabla \mapsto (-1)^{k+1} D_\nabla^* \triangle^{(k)} F_\nabla.$$

Then Φ_k is a weakly elliptic operator.

Remark 2.1.5. Note that Φ_k is the highest order term of generalized Yang-Mills k-flow and thus the symbol of the flow is completely determined by this.

Proof. Let $I = (i_v)_{v=1}^k$ be some multiindex. Given some one-parameter family $\nabla_t \in \mathcal{A}_E \times \mathcal{I}$ and appealing to Corollary 2.1.2 for the variation of $\nabla_t^{(2k+1)} F_{\nabla_t}$ yields,

$$\frac{\partial}{\partial t} \left[(\Phi_k(\nabla_t))_{r\alpha}^\beta \right] = (-1)^k \frac{\partial}{\partial t} \left[\nabla_p \nabla_{i_1 i_1 \cdots i_k i_k} F_{pr\alpha}^\beta \right]$$

$$= (-1)^k \left(\nabla_p \nabla_{i_1 i_1 \cdots i_k i_k} D_p \dot{\Gamma}_{r\alpha}^\beta + \left(\sum_{i=0}^{k-1} \nabla^{(i)} \dot{\Gamma} * \nabla^{(k-i-1)} F \right)_{pi_1 i_1 \cdots i_k i_k pr\alpha}^\beta \right).$$

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The left hand term is of dominating order and thus determines the symbol, which is given by, for $B \in \Lambda^1(\text{End } E)$,

$$\left(\sigma\left[\Phi_k\right](B)\right)_{r\alpha}^{\beta} = (-1)^k \partial_p \partial_{i_1 i_1 \cdots i_k i_k} \left(\partial_p B_{r\alpha}^{\beta} - \partial_r B_{p\alpha}^{\beta}\right),$$

and thus

$$\left(L_{\Phi_k}^{\xi}(B)\right)_{r\alpha}^{\beta} := (-1)^k \xi_p \left(\prod_{s=1}^{k} \xi_{i_s} \xi_{i_s}\right) \left(\xi_p B_{r\alpha}^{\beta} - \xi_r B_{p\alpha}^{\beta}\right)$$
$$= (-1)^k |\xi|^{2k} \left(|\xi|^2 B_{r\alpha}^{\beta} - \xi_r \langle B, \xi \rangle_{\alpha}^{\beta}\right). \tag{2.7}$$

Therefore it follows that

$$\left\langle L_{\Phi_k}^{\xi}(B), B \right\rangle = (-1)^{k+1} |\xi|^{2k} \left(|\xi|^2 |B|^2 - |\langle B, \xi \rangle|^2\right).$$

This term is nonnegative by the Cauchy Schwartz inequality. This completes the proof. Let us proceed and identify the kernel of $L_{\Phi_k}^{\xi}$. After changing bases (via rotation and dilation of the space of ξ) one can take $\xi := (\delta_1^{\ell})_{\ell=1}^n$. Evaluating at such ξ gives that

$$\left\langle L_{\Phi_k}^{\xi}(B), B \right\rangle = (-1)^{k+1} \left(|B|^2 - B_{1\alpha}^{\delta} B_{1\delta}^{\alpha}\right).$$

Therefore the kernel of the operator

$$\mathfrak{L}_{\Phi_k}^{\xi} : \Lambda^1(\text{End } E) \to \mathbb{R}$$
$$: B \mapsto \langle L_{\Phi_k}^{\xi}(B), B \rangle.$$

is sections $B \in S(T^*M \otimes \text{End } E)$ of the form $B = (\delta_1^k B_{k\alpha}^{\beta})$. Thus $\dim(\ker \mathfrak{L}_{\xi}) = \dim(\text{End } E)$. $\square$

2.2 Existence and regularity results

2.2.1 Short time existence

We next demonstrate the short time existence of generalized Yang-Mills k-flow. Despite the weak ellipticity of the operator Φ_k demonstrated in Proposition 2.1.4, one may construct a 'moving gauge' which actively

shifts the flow to an equivalent parabolic system, thus ensuring short time existence.

The active shift of gauge transformations within the gauge group $\mathcal{G}_E$ is achieved by solving the one-parameter family of gauge transformations which satisfy the following ordinary differential equation. A one-parameter family of connections ∇_t and a one-parameter family ς_t of gauge transformations satisfy this flow if

$$\frac{\partial \varsigma_t}{\partial t} = (-1)^{k+1} \left(\triangle_t^{(k)} D_{\nabla_t}^* (\nabla_t - \nabla_0) \right) \varsigma_t, \tag{2.8}$$

with the initial condition $\varsigma_0 := \mathrm{Id}$. The corresponding parabolic system will be written with respect to the following operator:

$$\Psi_k(\nabla, \overline{\nabla}) : \mathcal{A}_E \times \mathcal{A}_E \to \Lambda^1(\mathrm{End}\, E) \tag{ΨkF}$$
$$: (\nabla, \overline{\nabla}) \mapsto (-1)^{k+1} D_\nabla^* \triangle^{(k)} F_\nabla + \mho_k(\nabla) + (-1)^k D_\nabla \triangle^{(k)} D_\nabla^* (\nabla - \overline{\nabla}).$$

Definition 2.2.1 ((Ψ, k)-flow)**.** A smooth one-parameter family ∇_t with initial condition ∇_0 satisfies (Ψ, k)-*flow if*

$$\frac{\partial \nabla_t}{\partial t} = \Psi_k(\nabla_t, \nabla_0).$$

Lemma 2.2.2. *For a fixed connection $\nabla_0 \in \mathcal{A}_E$, $\Psi_k(\cdot, \nabla_0)$ is an elliptic operator.*

Proof. The symbol of Ψ_k will be computed as follows: since the variation of the first term, which is Φ_k, was computed in Proposition 2.1.4, it is sufficient to first consider the variation of the latter quantities. Set

$$(\Theta_k(\nabla, \overline{\nabla}))_{p\alpha}^\beta := (-1)^k \left(D_\nabla \triangle^{(k)} D_\nabla^* (\Gamma - \overline{\Gamma}) \right)_{p\alpha}^\beta$$
$$= (-1)^{k+1} \left(\nabla_p \nabla_{i_1 i_1 \cdots i_k i_k} \nabla_q (\Gamma - \overline{\Gamma})_{q\alpha}^\beta \right).$$

Then set $\overline{\nabla} = \nabla_0$, and consider a one-parameter family ∇_t. We differentiate temporally and appeal to Lemma 2.1.1 to observe that there is only one term of highest order (specifically order $2k + 3$).

$$\frac{\partial}{\partial t} \left[(\Theta_k(\nabla_t, \nabla_0))_{p\alpha}^\beta \right] = (-1)^{k+1} \left(\nabla_p \nabla_{i_1 i_1 \cdots i_k i_k} \nabla_q \dot{\Gamma}_{q\alpha}^\beta \right).$$

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Therefore for $B \in \Lambda^1(\operatorname{End} E)$,

$$(\sigma[\Theta_k](B))_{p\alpha}^{\beta} = (-1)^k \partial_p (\partial_{i_1} \partial_{i_1} \cdots \partial_{i_k} \partial_{i_k} \partial_q B_{q\alpha}^{\beta}),$$

and so

$$\left(L_{\Theta_k}^{\xi}(B)\right)_{p\alpha}^{\beta} = (-1)^k \xi_p |\xi|^{2k} \langle \xi, B \rangle_{\alpha}^{\beta}. \tag{2.9}$$

Then by combining (2.7) and (2.9) and noting they have the same orders we conclude that

$$\left(L_{\Psi_k}^{\xi}(B)\right)_{p\alpha}^{\beta} = \left(L_{\Phi_k}^{\xi}(B) + L_{\Theta_k}^{\xi}(B)\right)_{p\alpha}^{\beta} = (-1)^{k+1} |\xi|^{2k+2} B_{p\alpha}^{\beta}.$$

So

$$\left\langle L_{\Psi_k}^{\xi}(B), B \right\rangle = (-1)^k |\xi|^{2k+2} |B|^2.$$

Thus $\langle L_{\Psi_k}^{\xi}(\cdot), \cdot \rangle$ is either strictly positive definite or negative definite depending on the parity of k. We conclude that Ψ_k is an elliptic operator and the result follows. $\qquad \square$

We now develop some necessary identities regarding the action of gauge transformations on various quantities. The majority are included within the appendix in the gauge transformations section (§3.8), though the most relevant will be stated here.

Remark 2.2.3. For $\varsigma \in S(\mathcal{G}_E)$ and $\nabla \in \mathcal{A}_E$, set

$$\triangle_{\varsigma} := (\varsigma[\nabla])_i (\varsigma[\nabla])_i.$$

Let $\omega \in \Lambda^p(\operatorname{End} E)$. With this notation, the action of ς on $\triangle$ is given by

$$\varsigma[\triangle](\omega) := \varsigma^{-1} \triangle (\varsigma[\omega]) = \triangle_{\varsigma}(\varsigma[\omega]).$$

Additionally an analogous statement of Lemma 3.8.7 applies where $\triangle$ replaces ∇,

$$\varsigma[\triangle(\omega)] = (\varsigma[\triangle])(\varsigma[\omega]) = \triangle_{\varsigma}(\varsigma[\omega]).$$

To understand the intuition behind the proof of short time existence of the flow, we recall its primary inspiration and most basic case (Yang-Mills flow, $k = 0$) given in ([DK90], pp. 233-235), though we have simplified the strategy. The short time existence is not immediately clear since Yang-Mills flow itself fails to be parabolic due to the infinite-dimensional gauge symmetry group. One correctly expects, given a solution ∇_t, that its the geometric content should be preserved in its projection $[\nabla_t]$ within $\mathcal{A}_E/\mathcal{G}_E$, the space of connections modulo gauge transformation. One chooses another family within $[\nabla_t]$ which is not gauge invariant but moves smoothly and transversely to the action of the gauge group. This ensures that the degeneracy is removed and thus the family is parabolic and so exists for short time. This new flow can be represented uniquely by a family of gauge transformations ς_t applied to the initial flow.

Before continuing we define a notational convention which will condense more complicated terms produced from the lower order terms in differentation.

Definition 2.2.4 (Partition strings). Let $L := (l_i)_{i=1}^{|L|}$ denote some multiindex, $\mathcal{P}_r(L)$ denote the *partition strings of L*, that is, the collection of multiindices of length $r \leq |L|$ which contains entries ordered with respect to L,

$$\mathcal{P}_r(L) := \{(l_{s_v})_{v=1}^r : s_v \in [1, m], s_v < s_{v+1}\}.$$

For example,

$$\mathcal{P}_1(L) = \{(l_s) : s \in [1, m] \cap \mathbb{N}\} \text{ and } \mathcal{P}_2(L) = \{(l_{s_1}, l_{s_2}) : s_1, s_2 \in [1, m] \cap \mathbb{N}, s_1 < s_2\}.$$

Given $\mathsf{P} \in \mathcal{P}_r(L)$ of the form $\mathsf{P} := (l_{s_v})_{v=1}^r$ we let P^c denote the complimentary string, where $\mathsf{P}^c \in \mathcal{P}_{m-r}(L)$ and, roughly speaking as sets, $(\mathsf{P} \cup \mathsf{P}^c) = L$.

We next define the following operator which is formulated for notational convenience. Its construction is motivated by the following Lemma 2.2.5, and is purely a technical quantity in terms of the lower order

objects. This is utilized in in demonstrating uniqueness of the flow.

$$\mathfrak{a}_k : S(\mathcal{G}_E) \times \mathcal{A}_E^{\times 2} \to S(\operatorname{End} E),$$

$$: (\varsigma, \nabla, \overline{\nabla}) \mapsto \varsigma_\tau^\beta \nabla^{(k)} \left(D_\nabla^* \left(\Gamma - \overline{\Gamma} \right) \right)_\alpha^\tau + g^{ij} \varsigma_\tau^\beta \triangle^{(k)} \nabla_j \left((\varsigma^{-1})_\zeta^\tau \right) (D_\nabla \varsigma)_{i\alpha}^\zeta$$

$$+ g^{ij} \varsigma_\tau^\beta \triangle^{(k)} \left(\varsigma^{-1} \right)_\zeta^\tau \triangle (\varsigma)_\alpha^\zeta \tag{2.10}$$

$$+ g^{ij} \varsigma_\tau^\beta \left(\prod_{\ell=0}^k g^{i_\ell j_\ell} \right) \sum_{r=1}^{k-1} \sum_{\mathsf{P} \in \mathcal{P}_r(L)} \left(\nabla_\mathsf{P} (\varsigma^{-1})_\zeta^\tau \right) \left(\nabla_{\mathsf{P}^c} (\triangle) (\varsigma)_\alpha^\zeta \right).$$

Lemma 2.2.5. *Let* $\nabla, \widetilde{\nabla} \in \mathcal{A}_E$ *and* $\varsigma \in S(\mathcal{G}_E)$. *The following equality holds*

$$\left(\triangle^{(k)} \left(D_\nabla^* \left(\Gamma - \overline{\Gamma} \right) \right) \right) \varsigma = - \triangle_\varsigma^{(k+1)} \varsigma + \mathfrak{a}_k \left(\varsigma, \varsigma \left[\nabla \right], \overline{\nabla} \right). \tag{2.11}$$

Remark 2.2.6. The equality given in (2.11) is the key to establishing uniqueness of the generalized Yang-Mills k-flow by establishing the correspondence between this flow and the (Ψ, k)-flow. This relationship lies in this miraculous 'dictionary' equality between the flows. On the left side of (2.11) is an algebraic interaction of tensors on the gauge transformation. On the right is an act of differentiation of the gauge transformation plus lower order terms.

Proof. First an expression for the difference of the connection coefficient matrices Γ and $\overline{\Gamma}$ with respect to $\Gamma_{\varsigma[\nabla]}$ will be attained through first forming 'ς-conjugations' of Γ and the coordinate expansion of $\Gamma_{\varsigma[\nabla]}$ (Lemma 3.8.2).

$$\Gamma_{i\alpha}^\beta - \overline{\Gamma}_{i\alpha}^\beta = \varsigma_\delta^\beta (\varsigma^{-1})_\rho^\delta \Gamma_{i\tau}^\rho \varsigma_\theta^\tau (\varsigma^{-1})_\alpha^\theta - \overline{\Gamma}_{i\alpha}^\beta$$

$$= \varsigma_\delta^\beta (\Gamma_{\varsigma[\nabla]})_\gamma^\delta (\varsigma^{-1})_\alpha^\gamma - \varsigma_\delta^\beta (\varsigma^{-1})_\theta^\delta (\partial_i \varsigma_\gamma^\theta)(\varsigma^{-1})_\alpha^\gamma - \overline{\Gamma}_{i\alpha}^\beta$$

$$= \varsigma_\delta^\beta \left((\Gamma_{\varsigma[\nabla]})_\gamma^\delta - (\varsigma^{-1})_\theta^\delta (\partial_i \varsigma_\gamma^\theta) \right) (\varsigma^{-1})_\alpha^\gamma - \overline{\Gamma}_{i\alpha}^\beta.$$

Using one more identity,

$$(D_{\varsigma[\nabla]}\varsigma)_{i\alpha}^\beta = \partial_i \varsigma_\alpha^\theta + (\Gamma_{\varsigma[\nabla]})_{i\delta}^\beta \varsigma_\alpha^\delta - (\Gamma_{\varsigma[\nabla]})_{i\alpha}^\delta \varsigma_\delta^\beta,$$

one sees that

$$\Gamma_{i\alpha}^{\beta} - \overline{\Gamma}_{i\alpha}^{\beta} = \varsigma_{\delta}^{\beta}\left((\varsigma\,[\nabla])_{i\gamma}^{\delta} - (\varsigma^{-1})_{\theta}^{\delta}\left((D_{\varsigma[\nabla]}\varsigma)_{i\gamma}^{\theta} - (\Gamma_{\varsigma[\nabla]})_{i\tau}^{\theta}\varsigma_{\gamma}^{\tau} + \varsigma_{\tau}^{\theta}(\Gamma_{\varsigma[\nabla]})_{i\gamma}^{\tau}\right)\right)(\varsigma^{-1})_{\alpha}^{\gamma} - \overline{\Gamma}_{i\alpha}^{\beta}$$

$$= \varsigma_{\delta}^{\beta}\left(-(\varsigma^{-1})_{\theta}^{\delta}\left((D_{\varsigma[\nabla]}\varsigma)_{i\gamma}^{\theta} - (\Gamma_{\varsigma[\nabla]})_{i\tau}^{\theta}\varsigma_{\gamma}^{\tau}\right) - (\varsigma^{-1})_{\zeta}^{\delta}\overline{\Gamma}_{i\theta}^{\zeta}\varsigma_{\gamma}^{\theta}\right)(\varsigma^{-1})_{\alpha}^{\gamma}$$

$$= -(D_{\varsigma[\nabla]}\varsigma)_{i\gamma}^{\beta}(\varsigma^{-1})_{\alpha}^{\gamma} + (\Gamma_{\varsigma[\nabla]})_{i\alpha}^{\beta} - \overline{\Gamma}_{i\alpha}^{\beta}.$$

We apply this to the following computation.

$$\triangle^{(k)}D_{\nabla}^{*}(\Gamma - \overline{\Gamma})_{\delta}^{\beta}\varsigma_{\alpha}^{\delta} = (\varsigma^{-1}\varsigma)\left[\triangle^{(k)}D_{\nabla}^{*}(\Gamma - \overline{\Gamma})_{\delta}^{\beta}\right]\varsigma_{\alpha}^{\delta}$$

$$= -(\varsigma^{-1}\varsigma)\left[\triangle^{(k)}D_{\nabla}^{*}\left((D_{\varsigma[\nabla]}\varsigma)_{i\gamma}^{\beta}(\varsigma^{-1})_{\delta}^{\gamma} - (\Gamma_{\varsigma[\nabla]})_{i\delta}^{\beta} + \overline{\Gamma}_{i\delta}^{\beta}\right)\right]\varsigma_{\alpha}^{\delta}$$

$$= -(\varsigma^{-1})\left[(\triangle_{\varsigma})^{(k)}D_{\varsigma[\nabla]}^{*}\left((\varsigma^{-1})_{\zeta}^{\beta}(D_{\varsigma[\nabla]}\varsigma)_{i\delta}^{\zeta}\right)\right]\varsigma_{\alpha}^{\delta}$$

$$\quad - (\varsigma^{-1})\left[(\triangle_{\varsigma})^{(k)}D_{\varsigma[\nabla]}^{*}\left(-(\varsigma^{-1})_{\zeta}^{\beta}(\Gamma_{\varsigma[\nabla]})_{i\rho}^{\zeta}\varsigma_{\delta}^{\rho} + (\varsigma^{-1})_{\zeta}^{\beta}\overline{\Gamma}_{i\rho}^{\zeta}\varsigma_{\delta}^{\rho}\right)\right]\varsigma_{\alpha}^{\delta}$$

$$= -\varsigma_{\tau}^{\beta}(\triangle_{\varsigma})^{(k)}D_{\varsigma[\nabla]}^{*}\left((\varsigma^{-1})_{\zeta}^{\tau}(D_{\varsigma[\nabla]}\varsigma)_{\alpha}^{\zeta} - \varsigma\left[(\Gamma_{\varsigma[\nabla]})_{i\alpha}^{\tau} - \overline{\Gamma}_{i\alpha}^{\tau}\right]\right)$$

$$= -\varsigma_{\tau}^{\beta}(\triangle_{\varsigma})^{(k)}D_{\varsigma[\nabla]}^{*}\left((\varsigma^{-1})_{\zeta}^{\tau}(D_{\varsigma[\nabla]}\varsigma)_{\alpha}^{\zeta}\right) + \varsigma_{\tau}^{\beta}\left[(\triangle_{\varsigma})^{(k)}D_{\varsigma[\nabla]}^{*}\left(\varsigma\left[(\Gamma_{\varsigma[\nabla]})_{\alpha}^{\tau} - \overline{\Gamma}_{\alpha}^{\tau}\right]\right)\right]_{T_1}.$$

Expanding the left side term

$$(\triangle_{\varsigma})^{(k)}\left[D_{\varsigma[\nabla]}^{*}\left((\varsigma^{-1})_{\zeta}^{\tau}(D_{\varsigma[\nabla]}\varsigma)\right)\right]_{\alpha}^{\zeta}$$

$$= -g^{ji}(\triangle_{\varsigma})^{(k)}(\varsigma\,[\nabla])_{j}\left[(\varsigma^{-1})_{\zeta}^{\tau}(D_{\varsigma[\nabla]}\varsigma)_{i\alpha}^{\zeta}\right]$$

$$= -g^{ij}(\triangle_{\varsigma})^{(k)}\left[(\varsigma\,[\nabla])_{j}\left[(\varsigma^{-1})_{\zeta}^{\tau}\right](D_{\varsigma[\nabla]}\varsigma)_{i\alpha}^{\zeta} + (\varsigma^{-1})_{\zeta}^{\tau}(\varsigma\,[\nabla])_{j}\,[(\varsigma\,[\nabla])(\varsigma)]_{i\alpha}^{\zeta}\right]$$

$$= -\left[g^{ij}(\triangle_{\varsigma})^{(k)}\left[(\varsigma\,[\nabla])_{j}\left[(\varsigma^{-1})_{\zeta}^{\tau}\right](D_{\varsigma[\nabla]}\varsigma)_{i\alpha}^{\zeta}\right]\right]_{T_2} - g^{ij}(\triangle_{\varsigma})^{(k)}(\varsigma^{-1})_{\zeta}^{\tau}(\varsigma\,[\nabla])_{j}\,[(\varsigma\,[\nabla])(\varsigma)]_{i\alpha}^{\zeta}.$$

We further expand the term on the right, with the intent of drawing out the ς^{-1} term. Using Lemma 3.8.9 applied to $\varsigma^{-1} \in S(\mathcal{G}_E)$ we have

$$(\triangle_{\varsigma})^{(k)}\left((\varsigma^{-1})_{\zeta}^{\beta}(\triangle_{\varsigma}\varsigma)_{\alpha}^{\zeta}\right) = (\varsigma^{-1})_{\zeta}^{\beta}(\triangle_{\varsigma}^{(k+1)}\varsigma)_{\alpha}^{\zeta} + \left[(\triangle_{\varsigma})^{(k)}[\varsigma^{-1}]_{\zeta}^{\beta}(\triangle_{\varsigma})[\varsigma]_{\alpha}^{\zeta}\right]_{T_3}$$

$$+ \left[\left(\prod_{\ell=0}^{k}g^{i_{\ell}j_{\ell}}\right)\sum_{r=1}^{k-1}\sum_{\mathsf{P}\in\mathcal{P}_r(L)}\left((\varsigma^{*}\nabla)_{\mathsf{P}}(\varsigma^{-1})_{\zeta}^{\beta}\right)\left((\varsigma^{*}\nabla)_{\mathsf{P}^c}(\triangle_{\varsigma})\,[\varsigma]_{\alpha}^{\zeta}\right)\right]_{T_4}.$$

Therefore

$$\triangle_{\varsigma}^{(k)}\left(D_{\nabla}^{*}(\Gamma - \overline{\Gamma})_{\delta}^{\beta}\right)\varsigma_{\alpha}^{\delta} = -(\triangle_{\varsigma}^{(k+1)}\varsigma)_{\alpha}^{\beta} + (\mathfrak{a}_{k}(\varsigma, \varsigma\,[\nabla], \overline{\nabla}))_{\alpha}^{\beta},$$

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where $\mathfrak{a}_k \left(\varsigma, \varsigma \left[\nabla \right], \overline{\nabla} \right)_\alpha^\beta = \varsigma_\tau^\beta \left(T_1 + T_2 + T_3 + T_4 \right)_\alpha^\tau$. Note that $\mathfrak{a}_k \left(\varsigma, \varsigma \left[\nabla \right], \overline{\nabla} \right)$ is lower order than $(\triangle_\varsigma)^{(k+1)} \varsigma$. The result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

We now demonstrate the short time existence and uniqueness of the flow.

Proposition 2.2.7. *Let $(E, h) \to (M^n, g)$ be a vector bundle over a compact manifold. Given some metric compatible connection ∇_0 on E, there exists some $\varepsilon > 0$ such that the generalized Yang-Mills k-flow with initial condition ∇_0 has a unique solution ∇_t for $t \in [0, \varepsilon)$.*

Proof. We first prove existence and then uniqueness. Let $\nabla_0 \in \mathcal{A}_E$ and consider the following two flows. First, a one-parameter family of connections ∇_t over $\mathcal{I}$ given by

$$
\begin{cases}
\frac{\partial \nabla_t}{\partial t} & = \Psi_k(\nabla_t, \nabla_0) \\[2mm]
\nabla_t|_{t=0} & = \nabla_0.
\end{cases}
\tag{2.12}
$$

Next, a flow of gauge transformations ς_t over $\mathcal{I}$ satisfying

$$
\begin{cases}
\frac{\partial \varsigma_t}{\partial t} & = (-1)^{k+1} \left(\triangle^{(k)} D_t^* (\Gamma_t - \Gamma_0) \right) \varsigma_t \\[2mm]
\varsigma_0 & = \mathrm{Id}.
\end{cases}
\tag{2.13}
$$

Existence. Consider the flow ∇_t with initial condition ∇_0. Since Ψ_k is an elliptic operator by Lemma 2.2.2, a solution ∇_t to the parabolic system (2.12), the (Ψ, k)-flow, exists on some $t \in [0, \epsilon)$ for $\epsilon > 0$. Choose the unique solution ς_t to the system (2.13) and consider $\varsigma_t^* \nabla_t$. This is a solution to the generalized Yang-Mills k-system with initial condition ∇_0 as seen through the following computation which utilizes (3.83),

$$
\begin{aligned}
\left(\frac{\partial \varsigma [\nabla_t]}{\partial t} \right)_{r\alpha}^\beta &= (D_{\varsigma[\nabla]})_r (\varsigma^{-1} \dot{\varsigma})_{r\alpha}^\beta + (\varsigma^{-1})_\delta^\beta (\dot{\Gamma}_{r\theta}^\delta) \varsigma_\alpha^\theta \\
&= (-1)^{k+1} (D_{\varsigma[\nabla]})_r \left((\varsigma^{-1})_\delta^\beta (\triangle_\varsigma)^{(k)} \left(D_\nabla^* (\Gamma - \Gamma_0)_\zeta^\delta \right) \varsigma_\alpha^\zeta \right) + (-1)^{k+1} (\varsigma^{-1})_\delta^\beta \left((D_\nabla^* (\triangle_\varsigma)^{(k)} F_\nabla)_{r\zeta}^\delta \right) \\
&\quad + (-1)^{k+1} (\mho_k(\nabla))_{r\zeta}^\delta - (D_\nabla)_r (\triangle_\varsigma)^{(k)} \left(D_\nabla^* (\Gamma - \Gamma_0)_\zeta^\delta \right) \varsigma_\alpha^\zeta \\
&= (-1)^{k+1} \left(D_{\varsigma[\nabla]}^* (\triangle_\varsigma)^{(k)} F_{\varsigma[\nabla]} \right)_{r\alpha}^\beta + (\mho_k(\varsigma [\nabla]))_{r\alpha}^\beta.
\end{aligned}
$$

Therefore $\varsigma_t [\nabla_t]$ is a solution to the generalized Yang-Mills k-system. The first result follows.

Uniqueness. Suppose that ∇_0 is some connection with two solutions ∇_t and $\overline{\nabla}_t$ to the generalized Yang-Mills

k-flow. Let ϱ_t be the solution to the following system of gauge transformations over $\mathcal{I}$:

$$\begin{cases} \frac{\partial \varrho_t}{\partial t} &= (-1)^k (\triangle_{\varrho_t})^{(k+1)} \varrho_t + (-1)^k \mathfrak{a}_k(\varrho, \nabla_t, \nabla_0) \\ \varrho_0 &= \mathrm{Id}. \end{cases}$$

Similarly let $\overline{\varrho}_t$ be the solution to the following flow over $\mathcal{I}$:

$$\begin{cases} \frac{\partial \overline{\varrho}_t}{\partial t} &= (-1)^k (\triangle_{\overline{\varrho}_t})^{(k+1)} \overline{\varrho}_t + (-1)^k \mathfrak{a}_k \left(\overline{\varrho}, \overline{\nabla}_t, \overline{\nabla}_0 \right) \\ \overline{\varrho}_0 &= \mathrm{Id}. \end{cases}$$

These are strictly parabolic and lower order hence the solutions exists for all time. The next task is to verify that with the initial condition ∇_0, the one-parameter family $(\varrho_t^{-1})[\nabla_t]$ is a solution to (2.12). Observe that by the equivalence demonstrated by Lemma 2.2.5,

$$\left(\frac{\partial \varrho}{\partial t} \right)^{\beta}_{\alpha} = (-1)^{k+1} (\triangle_{\varrho^{-1}})^{(k)} \left(D^*_{(\varrho^{-1})[\nabla]} \left((\varrho^{-1}) \left[\Gamma - \Gamma_0 \right] \right) \right)^{\beta}_{\delta} \varrho^{\delta}_{\alpha}.$$

With this in mind and utilizing the expression for the derivative of a gauge acting on a connection (3.83),

$$\begin{aligned} \left(\frac{\partial \left((\varrho^{-1})[\nabla] \right)}{\partial t} \right)^{\beta}_{i\alpha} &= (D_{(\varrho^{-1})[\nabla]})_i \left(\varrho^{\beta}_{\delta} \partial_t (\varrho^{-1})^{\delta}_{\alpha} \right) + \varrho^{\beta}_{\delta} \dot{\Gamma}^{\delta}_{i\zeta} (\varrho^{-1})^{\zeta}_{\alpha} \\ &= -(D_{(\varrho^{-1})[\nabla]})_i \left(\dot{\varrho}^{\beta}_{\delta} (\varrho^{-1})^{\delta}_{\alpha} \right) + \varrho^{\beta}_{\delta} \left((-1)^{k+1} D^*_{\nabla} (\triangle)^{(k)} F_{\nabla} + \mho_k(\nabla) \right)^{\delta}_{i\zeta} (\varrho^{-1})^{\zeta}_{\alpha} \\ &= (-1)^k (D_{(\varrho^{-1})[\nabla]})_i \left((\triangle_{\varrho^{-1}})^{(k)} \left(D^*_{(\varrho^{-1})*\nabla} \left((\varrho^{-1}) \left[\Gamma - \Gamma_0 \right] \right) \right)^{\beta}_{\alpha} \right) \\ &\quad + (-1)^{k+1} D^*_{(\varrho^{-1})[\nabla]} (\triangle_{\varrho^{-1}})^{(k)} \left(F_{(\varrho^{-1})[\nabla]} \right)^{\beta}_{i\alpha} + \left(\mho_k ((\varrho^{-1})[\nabla]) \right)^{\beta}_{i\alpha}. \end{aligned}$$

This is precisely $\Psi_k((\varrho_t^{-1})[\nabla_t], \nabla_0)$. The computation could be done identically with $(\overline{\varrho}_t^{-1})[\overline{\nabla}_t]$ instead, giving that $(\overline{\varrho}_t^{-1})[\overline{\nabla}_t]$ and $(\varrho_t^{-1})[\nabla_t]$ are both solutions to the flow (2.12) with the same initial condition. Since solutions to the flow (2.12) are unique, $(\varrho_t^{-1})[\nabla_t] = (\overline{\varrho}_t^{-1})[\overline{\nabla}_t]$. Hence ϱ_t and $\overline{\varrho}_t$ must satisfy the flow (2.13), but since this is a linear ordinary differential equation (2.13) on a compact manifold with no boundary, $\varrho_t \equiv \overline{\varrho}_t$, which implies that $\nabla_t \equiv \overline{\nabla}_t$. Therefore uniqueness follows, and the proof is complete. $\square$

2.2.2 Smoothing estimates

In this section our goal is to compute, assuming a supremal bound on $|F_{\nabla_t}|$, the associated local bounds on the L^2 norms of covariant derivatives of F_{∇_t}. To accomplish this we first compute variational identities.

Lemma 2.2.8. *Suppose ∇_t is solution to generalized Yang-Mills k-flow. For $\ell \in \mathbb{N}$ the following holds,*

$$
\frac{\partial}{\partial t}\left[\nabla_t^{(\ell)} F_{\nabla_t}\right] = (-1)^k \triangle_t^{(k+1)} \left[\nabla_t^{(\ell)} F_{\nabla_t}\right] + P_2^{(\ell+2k)}[F_{\nabla_t}] + \sum_{s=0}^{\ell+2k} P_1^{(s)}[F_{\nabla_t}]
$$
$$
+ \sum_{i=1}^{k}\sum_{j=2}^{i+1} P_j^{(\ell+2i+4-2j)}[F_{\nabla_t}] + \sum_{i=1}^{k}\sum_{j=2}^{i+1} P_{j+1}^{(\ell+2i+2-2j)}[F_{\nabla_t}],
$$

(2.14)

and for $\ell = 0$,

$$
\frac{\partial F_{\nabla_t}}{\partial t} = (-1)^{k+1} \triangle_t^{(k+1)} F_{\nabla_t} + P_1^{(2k)}[F_{\nabla_t}] + P_2^{(2k)}[F_{\nabla_t}] + \sum_{i=1}^{k}\sum_{j=2}^{i+1} P_j^{(2i+4-2j)}[F_{\nabla_t}].
$$

(2.15)

Proof. To vary F_{∇_t}, we differentiate and then apply the Bochner formula (Proposition 3.7.1) to obtain

$$
\frac{\partial F}{\partial t} = D\dot{\Gamma}
$$
$$
= D((-1)^{k+1}D^* \triangle^{(k)} F + \mho_k(\nabla))
$$
$$
= (-1)^{k+1} \triangle_D \triangle^{(k)} F + D(\mho_k(\nabla))
$$
$$
= (-1)^{k+1} \triangle^{(k+1)} F + (-1)^{k+1}(\mathrm{Rm} + F) * (\triangle^{(k)} F) + D(\mho_k(\nabla))
$$
$$
= (-1)^{k+1} \triangle^{(k+1)} F + P_1^{(2k)}[F] + P_2^{(2k)}[F] + \left(\sum_{i=1}^{k}\sum_{j=2}^{i+1} P_j^{2i+4-2j}[F]\right).
$$

To vary $\nabla_t^{(\ell)} F_{\nabla_t}$ we apply (2.1.2) and then insert the equation of generalized Yang-Mills k-flow,

$$
\frac{\partial \nabla^{(\ell)} F}{\partial t} = \nabla^{(\ell)} D\dot{\Gamma} + \sum_{i=0}^{\ell-1}\left(\nabla^{(i)}\dot{\Gamma} * \nabla^{(\ell-i-1)} F\right)
$$
$$
= \left[\nabla^{(\ell)} D\left((-1)^{k+1} D^* \triangle^{(k)} F + \mho_k(\nabla)\right)\right]_{T_1}
$$
$$
+ \left[\sum_{i=0}^{\ell-1}\left(\nabla^{(i)}\left[(-1)^{k+1} D^* \triangle^{(k)} F + \mho_k(\nabla)\right] * \nabla^{(\ell-i-1)} F\right)\right]_{T_2}.
$$

We manipulate T_1 first. Using the Bochner formula (Proposition 3.7.1) to decompose the first quantity,

$$T_1 = (-1)^{k+1} \nabla^{(\ell)} \triangle_D \triangle^{(k)} F + \nabla^{(\ell)} D \mho_k(\nabla)$$

$$= \nabla^{(\ell)} \left[\triangle^{(k+1)} F \right] + \nabla^{(\ell)} \left[(\mathrm{Rm} + F) * \nabla^{(2k)} F \right] + \nabla^\ell D \left[\mho_k(\nabla) \right]$$

$$= \nabla^{(\ell)} \left[\triangle^{(k+1)} F \right] + \sum_{q=0}^{\ell} P_1^{(2k+q)} [F] + P_2^{(2k+\ell)} [F] + \sum_{i=1}^{k} \sum_{j=2}^{i+1} P_j^{(\ell+2i+4-2j)} [F].$$

Using Corollary 3.7.6 yields

$$\nabla^{(\ell)} \left[\triangle^{(k+1)} F \right] = (-1)^k \triangle^{(k+1)} \nabla^{(\ell)} F + \sum_{v=0}^{\ell-1} \sum_{j=0}^{2k+1} \left(\nabla^{(v+j)} [\mathrm{Rm} + F] * (\nabla^{(\ell-v+2k-j)} F) \right)$$

$$= (-1)^k \triangle^{(k+1)} F \left[\nabla^{(\ell)} F \right] + \sum_{s=0}^{\ell+2k} P_1^{(s)} [F] + P_2^{(\ell+2k)} [F].$$

Which gives that

$$T_1 = (-1)^k \triangle^{(k+1)} \left[\nabla^{(\ell)} F \right] + \sum_{s=0}^{\ell+2k} P_1^{(s)} [F] + P_2^{(\ell+2k)} [F] + \sum_{q=0}^{\ell} P_1^{(2k+q)} [F]$$

$$+ \sum_{i=1}^{k} \sum_{j=2}^{i+1} P_j^{(\ell+2i+4-2j)} [F].$$

Next we manipulate T_2,

$$T_2 = \sum_{i=0}^{\ell-1} \left(\nabla^{(i)} \left((-1)^{k+1} D^* \triangle^{(k)} F + \mho_k(\nabla) \right) * \nabla^{(\ell-i-1)} F \right)$$

$$= \sum_{i=0}^{\ell-1} \left((\nabla^{(i+2k+1)} F) * (\nabla^{(\ell-i-1)} F) + \sum_{v=1}^{k} \sum_{j=2}^{v+1} P_j^{(2v+3-2j+i)} [F] * (\nabla^{(\ell-i-1)} F) \right)$$

$$= P_2^{(2k+\ell)} [F] + \sum_{v=1}^{k} \sum_{j=2}^{v+1} \left(P_{j+1}^{(\ell+2v+2-2j)} [F] \right).$$

Combining T_1 and T_2 yields the result. $\qquad\qquad\square$

We next begin the discussion of our local smoothing estimates. While the inclusion of a bump function forces the computations to be significantly more involved, they are highly necessary. During the blowup analysis in Proposition 2.2.23, while working within a coordinate chart, we will require these local estimates to address that the domains of the connections under consideration are restricted to open subsets of $\mathbb{R}^n$.

Definition 2.2.9. Let $\mathcal{B} := \{ \eta \in C_c^\infty(M) : 0 \leq \eta \leq 1 \}$, that is, the family of 'bump' functions. Let $\ell \in \mathbb{N}$

and set, for a given $\nabla \in \mathcal{A}_E$,

$$\jmath_\eta^{(\ell)} := \sum_{q=0}^{\ell} \left\| \nabla^{(q)} \eta \right\|_{L^\infty(M)}.$$

We first prove the following lemma, which will be essential in the manipulations of Lemma 2.2.11. This is a technical result demonstrating how to shift derivatives within the integrands terms which will be commonly featured. This relies primarily on integration by parts while taking the interaction of the bump function into account.

Lemma 2.2.10. *Let* $p, q, r, s \in \mathbb{N}$, $\nabla \in \mathcal{A}_E$ *and* $\eta \in \mathcal{B}$. *Then if* $s \in \mathbb{N} \backslash \{1\}$,

$$\int_M \left(P_1^{(p)} [F_\nabla] * P_1^{(q+r)} [F_\nabla] \right) \eta^s dV_g \leq \sum_{j=0}^{r-1} \jmath_\eta^{(1)} \int_M \left(P_1^{(p+j)} [F] * P_1^{(q+(r-1-j))} [F] \right) \eta^{s-1} dV_g \tag{2.16}$$
$$+ \int_M \left(P_1^{(p+r)} [F] * P_1^{(q)} [F] \right) \eta^s dV_g.$$

Proof. The proof follows by induction on r. For the base case, observe that by integration by parts,

$$\int_M \left(P_1^{(p)} [F] * P_1^{(q+1)} [F] \right) \eta^s dV_g \leq \jmath_\eta^{(1)} \int_M \left(P_1^{(p)} [F] * P_1^{(q)} [F] \right) \eta^{s-1} dV_g$$
$$+ \int_M \left(P_1^{(p+1)} [F] * P_1^{(q)} [F] \right) \eta^s dV_g.$$

The base case follows, now we assume the induction hypothesis (2.16) holds for r. Then by instead applying the identity with p replaced by $p+1$ and q,

$$\int_M \left(P_1^{(p)} [F] * P_1^{(q+r+1)} [F] \right) \eta^s dV_g \leq \int_M \left(P_1^{(p+1)} [F] * P_1^{(q+r)} [F] \right) \eta^s dV_g$$
$$+ \jmath_\eta^{(1)} \int_M \left(P_1^{(p)} [F] * P_1^{(q+r)} [F] \right) \eta^{s-1} dV_g$$
$$= \int_M \left(P_1^{(p+(r+1))} [F] * P_1^{(q)} [F] \right) \eta^s dV_g$$
$$+ \sum_{j=0}^{r} \jmath_\eta^{(1)} \int_M \left(P_1^{(p+j)} [F] * P_1^{(q+(r-j))} [F] \right) \eta^{s-1} dV_g.$$

The result follows. $\square$

Lemma 2.2.11. *Let* $\ell \in \mathbb{N}$, $\eta \in \mathcal{B}$, *and suppose* ∇_t *a solution to generalized Yang-Mills k-flow over $\mathcal{I}$ with* $\sup_{M \times \mathcal{I}} |F_{\nabla_t}| < \infty$. *Set* $\varphi_t := \nabla_t^{(\ell)} F_{\nabla_t}$ *and choose* $K > \max \left\{ \sup_{M \times \mathcal{I}} |F_{\nabla_t}|, 1 \right\}$. *Then for* $s \geq 2(k + \ell + 1)$

there exists $C := C\left(\dim M, \operatorname{Rank} E, k, s, \ell, g, \jmath_\eta^{(s)}\right)$ *such that*

$$\frac{d}{dt}||\eta^{s/2}\varphi_t||^2_{L^2(M)} \leq -||\eta^{s/2}(\nabla_t^{(k+1)}\varphi_t)||^2_{L^2(M)} + CK^{2k+2}||F_{\nabla_t}||^2_{L^2(M),\eta>0}.$$

Proof. We differentiate $||\eta^{s/2}\varphi_t||^2_{L^2}$ using the variation computation in Lemma 2.2.8,

$$
\begin{aligned}
\frac{d}{dt}\left[\int_M \eta^s |\varphi|^2 dV_g\right] &= \int_M 2\left\langle \tfrac{\partial}{\partial t}[\varphi], \eta^s\varphi\right\rangle dV_g \\
&= \left[(-1)^k 2\int_M \left\langle \triangle^{(k+1)}\varphi, \eta^s\varphi\right\rangle dV_g\right]_{T_1} + \left[\int_M \left\langle P_2^{(\ell+2k)}[F], \eta^s\varphi\right\rangle dV_g\right]_{T_2} \\
&\quad + \left[\sum_{q=0}^{\ell+2k}\int_M \langle P_1^{(q)}[F], \eta^s\varphi\rangle dV_g\right]_{T_3} \\
&\quad + \left[\sum_{i=1}^{k}\sum_{j=2}^{i+1}\int_M \left\langle P_j^{(\ell+2i+4-2j)}[F], \eta^s\varphi\right\rangle dV_g\right]_{T_4} \\
&\quad + \left[\sum_{i=1}^{k}\sum_{j=2}^{i+1}\left\langle \int_M P_{j+1}^{(\ell+2i+2-2j)}[F], \eta^s\varphi\right\rangle dV_g\right]_{T_5}.
\end{aligned}
$$

We address each labelled term separately. Note that the analysis of the constraint on s contributed by each term requires two main considerations. Let $\alpha, \beta, \zeta, r \in \mathbb{N}$ and $(i_j)_{j=1}^r$ be some multiindex. First, an application of Corollary 3.6.5 requires that, if applied to $\left|\left|\eta^{\alpha/2}\nabla^{(\beta)}F_\nabla\right|\right|$, we must have $\alpha \geq 2\beta$. The application of Lemma 3.7.4 requires that to estimate $\int_M \eta^\alpha \nabla^{(i_1)}F_\nabla \cdots \nabla^{(i_r)}F_\nabla dV_g$ with $\sum_{j=1}^r i_j = 2\zeta$, then $\alpha \geq 2\zeta$. The constant $C \in \mathbb{R}_{>0}$ to appear in the following manipulations will be updated, increasing through computations.

$\boxed{T_1 \text{ estimate.}}$ We manipulate T_1 using Lemma 3.7.4 to shift ∇ across the inner product.

$$
\begin{aligned}
T_1 &= (-1)^k 2\int_M \left\langle \triangle^{(k+1)}\varphi, \eta^s\varphi\right\rangle dV_g \\
&= -2\int_M \left\langle \nabla^{(k+1)}\varphi, \nabla^{(k+1)}[\eta^s\varphi]\right\rangle dV_g + \left\langle \sum_{q=1}^{2k-2}\sum_{w=0}^{q}\left(\nabla^{(w)}[\operatorname{Rm}+F]*\nabla^{(2k-2-w)}\varphi\right), \varphi\right\rangle \\
&= \left[-2\int_M \left\langle \nabla^{(k+1)}\varphi, \nabla^{(k+1)}[\eta^s\varphi]\right\rangle dV_g\right]_{T_{11}} + \left[\int_M \left(P_3^{(2k-2+2\ell)}[F]\right)\eta^s dV_g\right]_{T_{12}} \\
&\quad + \left[\sum_{q=0}^{2\ell-2}\int_M \left(P_2^{(2k+q)}[F]\right)\eta^s dV_g\right]_{T_{13}}.
\end{aligned}
$$

We address each term above separately. For T_{11}, we differentiate, resulting in a summation, draw out one

term to 'absorb' the others and address the rest of the indices.

$$T_{11} = -2 \sum_{j=0}^{k+1} \int_M \nabla^{(j)} [\eta^s] * \left\langle \nabla^{(k+1-j)} \varphi, \nabla^{(k+1)} \varphi \right\rangle dV_g$$

$$\leq -2\|\eta^{s/2} \nabla^{(k+1)} \varphi\|_{L^2}^2 + \sum_{j=1}^{k+1} \left(J_\eta^{(k+1)} \int_M \eta^{s-j} \left\langle \nabla^{(k+1-j)} \varphi, \nabla^{(k+1)} \varphi \right\rangle dV_g \right).$$

We manipulate each term in the summation on the right, first by a weighted Hölder's inequality and then applying Corollary 3.6.5,

$$J_\eta^{(k+1)} \int_M \eta^{s-j} \left\langle \nabla^{(k+1-j)} \varphi, \nabla^{(k+1)} \varphi \right\rangle dV_g$$

$$\leq \epsilon \left\| \eta^{s/2} \nabla^{(k+1)} \varphi \right\|_{L^2}^2 + C \left\| \eta^{(s-2j)/2} \nabla^{(k+1-j)} \varphi \right\|_{L^2}^2$$

$$= \epsilon \left\| \eta^{s/2} \nabla^{(k+1)} \varphi \right\|_{L^2}^2 + C \left\| \eta^{(s-2j)/2} \nabla^{(k+1+\ell-j)} F \right\|_{L^2}^2$$

$$\leq \epsilon \left\| \eta^{s/2} \nabla^{(k+1)} \varphi \right\|_{L^2}^2 + \epsilon \left\| \eta^{(s-2j+2j)/2} \nabla^{(k+1+\ell)} F \right\|_{L^2}^2 + C \left\| F \right\|_{L^2, \eta>0}^2$$

$$\leq 2\epsilon \left\| \eta^{s/2} \nabla^{(k+1)} \varphi \right\|_{L^2}^2 + C \left\| F \right\|_{L^2, \eta>0}^2.$$

Therefore we conclude, by summing over all terms,

$$T_{11} \leq -2 \left\| \eta^{s/2} \nabla^{(k+1)} \varphi \right\|_{L^2}^2 + (k+1) \left(2\epsilon \left\| \eta^{s/2} \nabla^{(k+1)} \varphi \right\| + C \left\| F \right\|_{L^2, \eta>0}^2 \right)$$

$$\leq (-2 + 2\epsilon(k+1)) \left\| \eta^{s/2} \nabla^{(k+1)} \varphi \right\|_{L^2}^2 + C \left\| F \right\|_{L^2, \eta>0}^2.$$

T_{11} bump function constraints. Now we analyze the maximum power of the bump function in this setting. Corollary 3.6.5 requires that $s - 2j > 2(k+1+\ell-j)$, namely $s > 2(k+1+\ell)$. There are no other constraints on the bump function.

Next we estimate T_{12} by applying Lemma 3.6.6 and then Corollary 3.6.5.

$$T_{12} \leq \int_M \left(P_3^{(2k-2+2\ell)} [F] \right) \eta^s dV_g$$

$$\leq Q_{(3,k-1+\ell)} K \left(\|\eta^{s/2} \nabla^{(k-1)} \varphi\|_{L^2}^2 + \|F\|_{L^2, \eta>0}^2 \right)$$

$$= CK \left(\|\eta^{s/2} \nabla^{(k+\ell+1-2)} F\|_{L^2}^2 + \|F\|_{L^2, \eta>0}^2 \right)$$

$$\leq \epsilon \|\eta^{(s+2)/2} \nabla^{(k+1)} \varphi\|_{L^2}^2 + CK^2 \|F\|_{L^2, \eta>0}^2$$

$$\leq \epsilon \|\eta^{s/2} \nabla^{(k+1)} \varphi\|_{L^2}^2 + CK^2 \|F\|_{L^2, \eta>0}^2.$$

T₁₂ bump function constraints. The application of Lemma 3.6.6 and Corollary 3.6.5 require that $s \geq 2(k - 1 + \ell)$, giving the restraint here.

Next we estimate T_{13}. We divide up the summation into cases when the index q is either odd or even and apply Lemma 2.2.10 to 'balance out' the order of the connection application across terms,

$$
T_{13} = \sum_{q:q\in 2\mathbb{N}\cup\{0\}}^{2\ell-2} \int_M \left(P_2^{(2k+q)}[F]\right)\eta^s dV_g + \sum_{q:q\in 2\mathbb{N}-1}^{2\ell-3} \int_M \left(P_2^{(2k+q)}[F]\right)\eta^s dV_g
$$

$$
= \left[\sum_{q:q\in 2\mathbb{N}\cup\{0\}}^{2\ell-2} \int_M \left(P_2^{(2k+q)}[F]\right) \jmath_\eta^{(1)}\eta^{s-1} dV_g\right]_{T_{13,E}}
$$

$$
+ \left[\sum_{q:q\in 2\mathbb{N}-1}^{2\ell-3} \int_M \left(P_1^{\lceil \frac{2k+q}{2}\rceil}[F] * P_1^{\lfloor \frac{2k+q}{2}\rfloor}[F]\right)\eta^s dV_g\right]_{T_{13,O}}.
$$

For each index q of $T_{13,E}$ we apply Lemma 3.6.6 and then Corollary 3.6.5, noting that we maximize q at $2\ell - 2$ to obtain the final line,

$$
\int_M \left(P_2^{(2k+q)}[F]\right)\jmath_\eta^{(1)}\eta^{s-1} dV_g \leq Q_{(2,k+\frac{q}{2})}\left(\|\eta^{(s-1)/2}\nabla^{(k+\frac{q}{2})}F\|_{L^2}^2 + \|F\|_{L^2,\eta>0}^2\right)
$$

$$
= C\left(\|\eta^{(s-1)/2}\nabla^{(k+\ell+1-(\ell-\frac{q}{2}))}\varphi\|_{L^2}^2 + \|F\|_{L^2,\eta>0}^2\right)
$$

$$
\leq \epsilon\left\|\eta^{\frac{s-1+2\ell-q}{2}}\nabla^{(k+1)}\varphi\right\|_{L^2}^2 + C\|F\|_{L^2,\eta>0}^2
$$

$$
\leq \epsilon\left\|\eta^{\frac{s-1+2\ell-(2\ell-2)}{2}}\nabla^{(k+1)}\varphi\right\|_{L^2}^2 + C\|F\|_{L^2,\eta>0}^2
$$

$$
= \epsilon\left\|\eta^{\frac{s+1}{2}}\nabla^{(k+1)}\varphi\right\|_{L^2}^2 + C\|F\|_{L^2,\eta>0}^2
$$

$$
\leq \epsilon\left\|\eta^{\frac{s}{2}}\nabla^{(k+1)}\varphi\right\|_{L^2}^2 + C\|F\|_{L^2,\eta>0}^2.
$$

Therefore we conclude that

$$
T_{13,E} \leq \ell\epsilon\left(\left\|\eta^{s/2}\nabla^{(k+1)}\varphi\right\|_{L^2}^2\right) + C\|F\|_{L^2,\eta>0}^2.
$$

T₁₃,ₑ bump function constraints. The application of Lemma 3.6.6 and Corollary 3.6.5 require that $s - 1 \geq 2k + q$, which is at worst when q is maximized ($q = 2\ell - 2$). Thus $s \geq 2(k + \ell) - 1$.

Next we address $T_{13,O}$. For each term in the summation we apply Hölder's inequality followed by an

application of Lemma 3.6.6 to each term, then lastly and application of Corollary 3.6.5.

$$\int_M \left(P_1^{\lceil \frac{2k+q}{2} \rceil}[F] * P_1^{\lfloor \frac{2k+q}{2} \rfloor}[F] \right) \eta^s dV_g$$

$$\leq \frac{1}{2} \int_M P_2^{\left(2\lceil \frac{2k+q}{2} \rceil \right)}[F]\, \eta^s dV_g + \frac{1}{2} \int_M P_2^{\left(2\lfloor \frac{2k+q}{2} \rfloor \right)}[F]\, \eta^s dV_g$$

$$\leq \frac{1}{2} Q_{\left(2, \lfloor \frac{2k+q}{2} \rfloor \right)} \left(\left\| \eta^{s/2} \nabla^{\left(\lfloor \frac{2k+q}{2} \rfloor \right)} F \right\|_{L^2}^2 + \|F\|_{L^2,\eta>0}^2 \right)$$

$$+ \frac{1}{2} Q_{\left(2, \lceil \frac{2k+q}{2} \rceil \right)} \left(\left\| \eta^{s/2} \nabla^{\left(\lceil \frac{2k+q}{2} \rceil \right)} F \right\|_{L^2}^2 + \|F\|_{L^2,\eta>0}^2 \right)$$

$$= C \left(\left\| \eta^{s/2} \nabla^{\left(k+1+\ell-\left(\ell-\lfloor \frac{q}{2} \rfloor \right) \right)} F \right\|_{L^2}^2 + \left\| \eta^{s/2} \nabla^{\left(k+1+\ell-\left(\ell-\lceil \frac{q}{2} \rceil \right) \right)} F \right\|_{L^2}^2 \right) + C\, \|F\|_{L^2,\eta>0}^2$$

$$\leq \epsilon \left\| \eta^{\frac{s+\left(\ell-\lfloor \frac{q}{2} \rfloor \right)}{2}} \nabla^{(k+1)} \varphi \right\|_{L^2}^2 + \epsilon \left\| \eta^{\frac{s+\left(\ell-\lceil \frac{q}{2} \rceil \right)}{2}} \nabla^{(k+1)} \varphi \right\|_{L^2}^2 + C\, \|F\|_{L^2,\eta>0}^2$$

$$\leq \epsilon \left\| \eta^{\frac{s+\ell-\left(\ell-\lceil \frac{3}{2} \rceil \right)}{2}} \nabla^{(k+1)} \varphi \right\|_{L^2}^2 + \epsilon \left\| \eta^{\frac{s+\ell-\left(\ell-\lfloor \frac{3}{2} \rfloor \right)}{2}} \nabla^{(k+1)} \varphi \right\|_{L^2}^2 + C\, \|F\|_{L^2,\eta>0}^2$$

$$\leq 2\epsilon \left\| \eta^{s/2} \nabla^{(k+1)} F \right\|_{L^2}^2 + C\, \|F\|_{L^2,\eta>0}^2.$$

$T_{13,O}$ bump function constraints. The applications of both Lemma 3.6.6 and Corollary 3.6.5 require that $s \geq 2 \left(k + \lceil \frac{q}{2} \rceil \right)$. We note that $q \leq 2\ell - 3$, so we conclude that $s \geq 2 \left(k + \ell - \lfloor \frac{3}{2} \rfloor \right) = 2 \left(k + \ell - 1 \right)$. Therefore we conclude that

$$T_1 \leq \left(-2 + \epsilon \left(5 + \ell \right) \right) \left\| \eta^{s/2} \nabla^{(k+1)} \varphi \right\|_{L^2}^2 + CK^2 \, \|F\|_{L^2,\eta>0}^2.$$

T_1 bump function constraints. Based on the above computations we conclude the cumulative constraint across all subterms T_{1i} that $s \geq 2 \left(k + \ell + 1 \right)$.

$\boxed{\mathbf{T_2}\ \textbf{estimate.}}$ For T_2 we apply Lemma 3.6.6 and then Corollary 3.6.5,

$$
\begin{aligned}
T_2 &= \int_M \left\langle P_2^{(\ell+2k)}\left[F\right], \eta^s \varphi \right\rangle dV_g \\
&= \int_M \left(P_3^{(2\ell+2k)}\left[F\right] \right) \eta^s dV_g \\
&\leq Q_{(3,\ell+k)} K \left(\left\| \eta^{s/2} \left(\nabla^{(k+\ell)} F \right) \right\|_{L^2}^2 + \|F\|_{L^2,\eta>0}^2 \right) \\
&= CK \left(\left\| \eta^{s/2} \left(\nabla^{(k+1+\ell-1)} F \right) \right\|_{L^2}^2 + \|F\|_{L^2,\eta>0}^2 \right) \\
&\leq \epsilon \left\| \eta^{(s+1)/2} \nabla^{(k+1)} \varphi \right\|_{L^2}^2 + CK^2 \|F\|_{L^2,\eta>0}^2 \\
&\leq \epsilon \left\| \eta^{s/2} \nabla^{(k+1)} \varphi \right\|_{L^2}^2 + CK^2 \|F\|_{L^2,\eta>0}^2.
\end{aligned}
$$

T_2 bump function constraints. For the application of Lemma 3.6.6 and Corollary 3.6.5 we required $s \geq 2(\ell + k)$.

$\boxed{\mathbf{T_3}\ \textbf{estimate.}}$ For T_3 we divide up terms between an even and odd number of derivatives and have, via integration by parts and collecting up derivatives of η accordingly, noting that $s > k + \ell \geq \left\lfloor \frac{q+\ell}{2} \right\rfloor$,

$$
\begin{aligned}
T_3 &= \sum_{q=0}^{\ell+2k} \int_M \left\langle P_1^{(q)}[F], \eta^s \varphi \right\rangle dV_g \\
&= \sum_{q=0}^{\ell+2k} \int_M \left(P_2^{(q+\ell)}[F] \right) \eta^s dV_g \\
&= \sum_{q:q+\ell\in 2\mathbb{N}}^{2k+\ell} \int_M \left(P_2^{(q+\ell)}[F] \right) \eta^s dV_g + \sum_{q:q+\ell\in 2\mathbb{N}-1}^{2k+\ell-1} \int_M \left(P_2^{(q+\ell)}[F] \right) \eta^s dV_g \\
&\leq \left[\sum_{q:q+\ell\in 2\mathbb{N}}^{2k+\ell} J_\eta^{(1)} \int_M \left(P_2^{(q+\ell)}[F] \right) \eta^{s-1} dV_g \right]_{T_{3,E}} + \left[\sum_{q:q+\ell\in 2\mathbb{N}-1}^{2k+\ell-1} \int_M \left(P_1^{\left(\lceil \frac{q+\ell}{2} \rceil\right)}[F] * P_1^{\left(\lfloor \frac{q+\ell}{2} \rfloor\right)}[F] \right) \eta^s dV_g \right]_{T_{3,O}}.
\end{aligned}
$$

For $T_{3,E}$ we have that by Lemma 3.6.6 and then Corollary 3.6.5, noting that since $q \leq 2k+\ell$, then we have that $k + \frac{\ell-q}{2} \geq 0$, each term of the summation becomes

$$
\begin{aligned}
J_\eta^{(1)} \int_M \left(P_2^{(q+\ell)}[F] \right) \eta^{s-1} dV_g &\leq Q_{\left(2, \frac{q+\ell}{2}\right)} \left(||\eta^{(s-1)/2} \nabla^{\left(\frac{q+\ell}{2}\right)} F||_{L^2}^2 + ||F||_{L^2,\eta>0} \right) \\
&= C \left(||\eta^{(s-1)/2} \nabla^{\left(k+\ell+1-\left(k+1+\frac{\ell-q}{2}\right)\right)} F||_{L^2}^2 + ||F||_{L^2,\eta>0}^2 \right) \\
&\leq \epsilon ||\eta^{(s+k+\frac{\ell-q}{2})/2} \nabla^{(k+1)} \varphi||_{L^2}^2 + C||F||_{L^2,\eta>0}^2 \\
&\leq \epsilon ||\eta^{s/2} \nabla^{(k+1)} \varphi||_{L^2}^2 + C||F||_{L^2,\eta>0}^2.
\end{aligned}
$$

$T_{3,E}$ bump function constraints. The applications of Lemma 3.6.6 and Corollary 3.6.5 required $s \geq q + \ell + 1$. Maximizing the right side of the inequality with respect to q we conclude that $s \geq 2(k + \ell) + 1$. For the second term we manipulate with Hölder's inequality, apply Lemma 3.6.6 and then Corollary 3.6.5, noting that since $q \leq 2k + \ell - 1$, then we have that $\lceil \frac{q+\ell}{2} \rceil \leq k + \ell$, so

$$
\begin{aligned}
\int_M & \left(P_1^{\left(\lceil \frac{q+\ell}{2} \rceil\right)}[F] * P_1^{\left(\lfloor \frac{q+\ell}{2} \rfloor\right)}[F] \right) \eta^s dV_g \\
&\leq \int_M \left(P_2^{\left(2\lceil \frac{q+\ell}{2} \rceil\right)}[F] \right) \eta^s dV_g + \int_M \left(P_2^{\left(2\lfloor \frac{q+\ell}{2} \rfloor\right)}[F] \right) \eta^s dV_g \\
&\leq Q_{(2, \lfloor \frac{q+\ell}{2} \rfloor)} \left(||\eta^{s/2} \nabla^{(\lfloor \frac{q+\ell}{2} \rfloor)} F||_{L^2}^2 + ||F||_{L^2, \eta>0}^2 \right) + Q_{(2, \lceil \frac{q+\ell}{2} \rceil)} \left(||\eta^{s/2} \nabla^{(\lceil \frac{q+\ell}{2} \rceil)} F||_{L^2}^2 + ||F||_{L^2, \eta>0}^2 \right) \\
&= C ||\eta^{s/2} \nabla^{(k+\ell+1-(k+\ell+1-\lfloor \frac{q+\ell}{2} \rfloor))} F||_{L^2}^2 + C ||F||_{L^2, \eta>0}^2 + C ||\eta^{s/2} \nabla^{(k+\ell+1-(k+\ell+1-\lceil \frac{q+\ell}{2} \rceil))} F||_{L^2}^2 \\
&\leq \epsilon ||\eta^{s+(k+\ell+1-\lceil \frac{q+\ell}{2} \rceil)/2} \nabla^{(k+1)} \varphi||_{L^2}^2 + C ||F||_{L^2, \eta>0}^2 + \epsilon ||\eta^{s+(k+\ell+1-\lfloor \frac{q+\ell}{2} \rfloor)/2} \nabla^{(k+1)} \varphi||_{L^2}^2 \\
&\leq 2\epsilon ||\eta^{s/2} \nabla^{(k+1)} \varphi||_{L^2}^2 + C ||F||_{L^2, \eta>0}^2.
\end{aligned}
$$

We explain the manipulation of the bump function power from the second to last line. Note here that since the maximum value of q is $2k + \ell - 1$, then we have $k + \ell + 1 - \lceil \frac{q+\ell}{2} \rceil \geq 1$, which implies that the powers of η are always larger than s.

$T_{3,O}$ bump function constraint. The applications of Lemma 3.6.6 and then Corollary 3.6.5 require that $s \geq \lceil \frac{q+\ell}{2} \rceil$. Maximizing q at the value $2k + \ell - 1$, we have $s \geq 2(k + \ell)$.

We thus conclude that

$$
T_3 \leq 3\epsilon ||\eta^{s/2} \nabla^{(k+1)} \varphi||_{L^2}^2 + C ||F||_{L^2, \eta>0}^2.
$$

T_3 bump function constraint. We combine the cumulative lower bounds of $T_{3,E}$ and $T_{3,O}$ with the constraint that $s \geq 2(k + \ell)$.

$\boxed{\textbf{T}_4 \textbf{ estimate.}}$ For T_4 we apply Lemma 3.6.6 and then Corollary 3.6.5.

$$T_4 = \sum_{i=1}^{k}\sum_{j=2}^{i+1} \int_M \left(P_{j+1}^{(2\ell+2i+2-2j)}[F] \right) \eta^s dV_g$$

$$\leq \sum_{i=1}^{k}\sum_{j=2}^{i+1} Q_{(j+1,\ell+1-j)} K^{j-1} \left(||\eta^{s/2}\nabla^{(\ell+i+1-j)}F||_{L^2}^2 + ||F||_{L^2,\eta>0}^2 \right)$$

$$= \sum_{i=1}^{k}\sum_{j=2}^{i+1} C K^{j-1} \left(||\eta^{s/2}\nabla^{(k+\ell+1-(k+j-i))}F||_{L^2}^2 + ||F||_{L^2,\eta>0}^2 \right)$$

$$\leq \sum_{i=1}^{k}\sum_{j=2}^{i+1} \epsilon ||\eta^{(s+(k+j-i))/2}\nabla^{(k+1)}\varphi||_{L^2}^2 + C K^{2(j-1)}||F||_{L^2,\eta>0}^2$$

$$\leq \tfrac{k(k+1)}{2}\epsilon ||\eta^{s/2}\nabla^{(k+1)}\varphi||_{L^2}^2 + C K^{2k}||F||_{L^2,\eta>0}^2.$$

Note that the second to last line results from the fact that we must consider the minimizing bump function power by implementing the bounds on i and j: with $k-i+j \geq k-k+2 = 2$, and thus the lowest power of η is $s+2$, strictly greater than s.

T_4 bump function constraint. For the application of Lemma 3.6.6 and Corollary 3.6.5 we require $s \geq 2\,(\ell+i+1-j)$. The right hand side of this inequality is maximized when $i = k$ and $j = 2$, that is when $s \geq 2\,(k+\ell-1)$.

$\boxed{\textbf{T}_5 \textbf{ estimate.}}$ For T_5 we apply Lemma 3.6.6 and then Corollary 3.6.5,

$$T_5 \leq \sum_{i=1}^{k}\sum_{j=2}^{i+1} \int_M \left(P_{j+2}^{(2\ell+2i+2-2j)}[F] \right) \eta^s dV_g$$

$$\leq \sum_{i=1}^{k}\sum_{j=2}^{i+1} Q_{(j+2,\ell+i+1-j)} K^{j} \left(||\eta^{s/2}\nabla^{(\ell+i+1-j)}F||_{L^2}^2 + ||F||_{L^2,\eta>0}^2 \right)$$

$$= \sum_{i=1}^{k}\sum_{j=2}^{i+1} C K^{j} \left(||\eta^{s/2}\nabla^{(k+\ell+1-(k-i+j))}F||_{L^2}^2 + ||F||_{L^2,\eta>0}^2 \right)$$

$$\leq \sum_{i=1}^{k}\sum_{j=2}^{i+1} \left(\epsilon ||\eta^{(s+(k-i+j))/2}\nabla^{(k+1)}\varphi||_{L^2}^2 + C K^{2j}||F||_{L^2,\eta>0}^2 \right)$$

$$\leq \sum_{i=1}^{k}\sum_{j=2}^{i+1} \left(\epsilon ||\eta^{s/2}\nabla^{(k+1)}\varphi||_{L^2}^2 + C K^{2j}||F||_{L^2,\eta>0}^2 \right)$$

$$\leq \tfrac{k(k+1)}{2}\epsilon ||\eta^{s/2}\nabla^{(k+1)}\varphi||_{L^2}^2 + C K^{2k+2}||F||_{L^2,\eta>0}^2.$$

Note that the second to last line results from the fact that we must consider the minimizing bump function power by implementing the bounds on i and j, as done for T_4 previously.

T₅ bump function constraint. The application of Lemma 3.6.6 and Corollary 3.6.5 require $s \geq 2\left(\ell + i + 1 - j\right)$. Using the bounds on the i and j the right side is maximized when $i = k$ and $j = 2$, that is, when $s \geq 2\left(k + \ell - 1\right)$.

$\boxed{\text{Final estimate.}}$ Now we combine everything. Summing up the T_i estimates and combining the constraints on s we conclude that

$$\frac{d}{dt}\left[\int_M |\varphi|^2 \, \eta^{s/2} dV_g\right] \leq \sum_{i=1}^{5} T_i$$
$$\leq -2\left|\left|\eta^{s/2}\nabla^{(k+1)}\varphi\right|\right|_{L^2}^2 + (k(k+1) + \ell + 9)\,\epsilon\left|\left|\eta^{s/2}\nabla^{(k+1)}\varphi\right|\right|_{L^2}^2 + CK^{2k+2}\left|\left|F\right|\right|_{L^2,\eta>0}^2,$$

where $s \geq 2(k + \ell + 1)$. Taking $\epsilon = (k(k+1) + \ell + 9)^{-1}$ yields the desired result. $\qquad\square$

Remark 2.2.12. The bounds on $|\nabla^{(i)}\operatorname{Rm}|$ for $i \in [0, i_{k,\ell}] \cap \mathbb{N}$ (where $i_{k,\ell}$ is an index dependent on its subscripts) contribute to the coefficient C in the above estimates, though these are undisplayed and incorporated into the $P_v^{(w)}$ notation. In the later blowup analysis (Proposition 2.2.23), although the manifold M is changing, the bounds on Rm are actually decreasing since the base manifold is tending toward $\mathbb{R}^n$ in the blowup sequence.

Theorem 2.2.13. *Let $q \in \mathbb{N}$, $\eta \in \mathcal{B}$, and suppose ∇_t is a solution to generalized Yang-Mills k-flow on $\mathcal{I}$ with $\sup_{M \times \mathcal{I}} |F_{\nabla_t}| < \infty$. Choose $s > (k + q + 1)$ and $K > \max\{\sup_{M \times \mathcal{I}} |F_{\nabla_t}|, 1\}$. Then for $t \in [0, T) \subset \mathcal{I}$ with $T < K^{-2(k+1)}$ there exists $C_q := C_q\left(\dim M, \operatorname{Rank} E, k, s, q, g, \jmath_\eta^{(s)}\right) \in \mathbb{R}_{>0}$, such that the following estimates hold.*

$$\left|\left|\eta^s\nabla_t^{(q)}F_{\nabla_t}\right|\right|_{L^2(M)}^2 \leq C_q t^{-\frac{q}{k+1}}\left(\sup_{[0,T)} ||F_{\nabla_t}||_{L^2(M),\eta>0}^2\right). \tag{2.17}$$

Proof. Set $\alpha_q := 1$, and let $\{\alpha_\ell\}_{\ell=0}^{q-1} \subset \mathbb{R}$ be coefficients to be determined. Then set

$$\Phi(t) := \sum_{\ell=0}^{q} \alpha_\ell t^\ell \left|\left|\eta^s\nabla_t^{((k+1)\ell)}F_{\nabla_t}\right|\right|_{L^2}^2.$$

Then differentiating, reindexing and applying Lemma 2.2.11 yields

$$\frac{d\Phi}{dt} = \sum_{\ell=1}^{q} \ell \alpha_\ell t^{\ell-1} ||\eta^s \nabla^{((k+1)\ell)} F_{\nabla_t}||_{L^2}^2 + \sum_{\ell=0}^{q} \alpha_\ell t^\ell \frac{d}{dt}\left[||\eta^s \nabla^{((k+1)\ell)} F_{\nabla_t}||_{L^2}^2\right]$$

$$\leq \sum_{\ell=0}^{q-1} \left((\ell+1)\alpha_{\ell+1} t^\ell ||\eta^s \nabla^{((k+1)(\ell+1))} F_{\nabla_t}||_{L^2}^2\right)$$

$$+ \sum_{\ell=0}^{q} \left(\alpha_\ell t^\ell \left(-\left\|\eta^s\left(\nabla^{(k+1)(\ell+1)} F_{\nabla_t}\right)\right\|_{L^2}^2 + CK^{2(k+1)} \|F_{\nabla_t}\|_{L^2,\eta>0}^2\right)\right)$$

$$= -t^q \left\|\eta^s\left(\nabla^{((k+1)(q+1))} F_{\nabla_t}\right)\right\|_{L^2}^2 + \sum_{\ell=0}^{q-1} \left(\alpha_{\ell+1}(\ell+1) - \alpha_\ell\right) t^\ell \left\|\eta^s\left(\nabla^{((k+1)(\ell+1))} F_{\nabla_t}\right)\right\|_{L^2}^2$$

$$+ CK^{2(k+1)} \sum_{\ell=0}^{q} \alpha_\ell t^\ell ||F_{\nabla_t}||_{L^2,\eta>0}^2.$$

Using the initial condition $\alpha_q = 1$, we choose constants satisfying the recursion relation

$$\alpha_{\ell+1}(\ell+1) - \alpha_\ell \leq 0,$$

so in particular, we choose constants which satisfy $\alpha_\ell \geq \frac{q!}{\ell!}$. Then incorporating the fact that t is bounded above by $K^{-2(k+1)}$ and choosing $C_{(k+1)q} \geq C(\sum_{\ell=0}^{q} \alpha_\ell)$,

$$\frac{d\Phi}{dt} = CK^{2(k+1)} \sum_{\ell=0}^{q} \alpha_\ell t^\ell ||F_{\nabla_t}||_{L^2,\eta>0}^2 \leq C_{(k+1)q} K^{2(k+1)} ||F_{\nabla_t}||_{L^2,\eta>0}^2$$

integrating both sides with respect to the temporal variable yields that

$$\Phi(t) - \Phi(0) \leq C_{(k+1)q} K^{2(k+1)} \int_0^t ||F_{\nabla_\tau}||_{L^2,\eta>0}^2 \, d\tau,$$

therefore

$$t^q ||\eta^s \nabla^{((k+1)q)} F_{\nabla_t}||_{L^2}^2 \leq C_{(k+1)q} t K^{-2(k+1)} \left(\sup_{t\in[0,T]} ||F_{\nabla_t}||_{L^2,\eta>0}^2\right) + \Phi(0)$$

$$\leq C_{(k+1)q} \left(\sup_{t\in[0,T]} ||F_{\nabla_t}||_{L^2,\eta>0}^2\right) + q! \, ||F_{\nabla_0}||_{L^2,\eta>0}^2.$$

We conclude

$$||\eta^s \nabla^{((k+1)q)} F_{\nabla_t}||_{L^2}^2 \leq C_{(k+1)q} t^{-q} \left(\sup_{t\in[0,T]} ||F_{\nabla_t}||_{L^2,\eta>0}^2\right).$$

To complete the proof we consider remaining derivative types (that is, $||\eta^s \nabla^{((k+1)\ell+w)} F_{\nabla_t}||_{L^2}^2$ where $m \in$

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$\mathbb{N} \cup \{0\}$ and $w \in [1, k] \cap \mathbb{N}$). We observe that by Corollary 3.6.5 combined with the fact that $T < 1$, then

$$\left\|\nabla^{((k+1)\ell+w)}F_{\nabla_t}\right\|_{L^2}^2 \leq \epsilon \left\|\nabla^{((k+1)(\ell+1))}F_{\nabla_t}\right\|_{L^2}^2 + C_\epsilon \left\|F_{\nabla_t}\right\|_{L^2,\eta>0}^2$$

$$\leq C_{(k+1)(\ell+1)}t^{-\ell-1}\left(\sup_{[0,T)}\|F_{\nabla_t}\|_{L^2,\eta>0}^2\right) + C_\epsilon t^{-\ell-1}\|F_{\nabla_t}\|_{L^2,\eta>0}^2$$

$$\leq C\sup_{[0,T)}\|F_{\nabla_t}\|_{L^2,\eta>0}^2 t^{-\frac{(k+1)(\ell+1)+w}{(k+1)}}.$$

We have established the inequality for all $q \in \mathbb{N}$, and the result follows. $\qquad\square$

Corollary 2.2.14. *Suppose ∇_t is a solution to generalized Yang-Mills k-flow on $[0, \tau]$ and $\eta \in \mathcal{B}$. Set $\bar{\tau} := \min\{\tau, 1\}$ and $K \geq \sup_{M \times [0,\bar{\tau}]} |F_{\nabla_t}|$. Then for $s, \ell \in \mathbb{N}$ with $s > (k + \ell + 1)$ there exists $Q_\ell := Q_\ell\left(K, s, \ell, \jmath_\eta^{(s)}, \operatorname{Rank} E, g, \dim M, \bar{\tau}\right) \in \mathbb{R}_{>0}$ such that*

$$\sup_M \left|\eta^s \nabla^{(\ell)} F_{\nabla_{\bar{\tau}}}\right|^2 \leq Q_\ell \left(\sup_{M \times [0,\bar{\tau})} \|F_{\nabla_t}\|_{L^2(M),\eta>0}^2\right). \tag{2.18}$$

Remark 2.2.15. Note that the corollary results have no dependency on the initial connection ∇_0.

Proof. By the smoothing estimates of Theorem 2.2.13 we have that

$$\sup_M \|\eta^s \nabla_{\bar{\tau}}^{(\ell)} F_{\nabla_{\bar{\tau}}}\|_{L^2}^2 \leq Q_\ell \sup_{M \times [0,\bar{\tau})} \|F_{\nabla_t}\|_{L^2,\eta>0}^2.$$

Since $|\eta^s \nabla^{(\ell)} F|^2$ is a real valued function on $M \times [0, \infty)$, and any $j > \frac{n}{2}$, we use Kato's Inequality combined with the second Sobolev Imbedding Theorem, which gives that $H_j^2 \subset C_B^0$, yielding the Sobolev constant S_ℓ so that

$$\sup_M \left|\eta^s \nabla^{(\ell)} F_{\bar{\tau}}\right| \leq \mathsf{S}_\ell \sum_{w=0}^j \|\nabla^{(w)}\left(|\eta^s \nabla_{\bar{\tau}}^{(\ell)} F_{\nabla_{\bar{\tau}}}|\right)\|_{L^2} \leq \mathsf{S}_\ell \sum_{w=0}^j \|\eta^s \nabla_{\bar{\tau}}^{(k+w)} F_{\nabla_{\bar{\tau}}}\|_{L^2}.$$

Therefore, combining the two above inequalities,

$$\sup_M \left|\eta^s \nabla_{\bar{\tau}}^{(\ell)} F_{\nabla_{\bar{\tau}}}\right|^2 \leq \mathsf{S}_\ell \sum_{w=0}^j \left(C_{\ell+w} \sup_{[0,\bar{\tau})} \|F_{\nabla_t}\|_{L^2,\eta>0}^2\right) \leq \mathsf{S}_\ell \left(\max_{w \in [0,j] \cap \mathbb{N}} C_{\ell+w}\right)\left(\sup_{M \times [0,\bar{\tau})} \|F_{\nabla_t}\|_{L^2,\eta>0}^2\right).$$

Setting $Q_\ell := \mathsf{S}_\ell \left(\max_{w \in [0,j] \cap \mathbb{N}} C_{\ell+w}\right)$, the result follows. $\qquad\square$

Corollary 2.2.16. *Suppose ∇_t is a solution to generalized Yang-Mills k-flow on $[0, T)$ for $T \in [0, \infty)$ and*

that $\eta \in \mathcal{B}$. Furthermore suppose that

$$\max \left\{ \sup_{[0,T)} ||F_{\nabla_t}||_{L^2(M)} , \sup_{[0,T)} ||F_{\nabla_t}||_{L^\infty(M)} \right\} \leq K \in [1, \infty).$$

Then for $t \in [0,T)$, $s, \ell \in \mathbb{N}$ with $s > 2(k + \ell + 1)$ there, exists some

$$Q_\ell := Q_\ell \left(\nabla_0, T, K, \ell, s, g, \mathrm{Rank}(E), \dim(M), \jmath_\eta^{(s)} \right) \in \mathbb{R}_{>0}$$

such that

$$\sup_{M \times [0,T)} \left| \eta^s \nabla_t^{(\ell)} F_{\nabla_t} \right|^2 \leq Q_\ell.$$

Proof. Since ∇_t exists and is smooth on the compact interval $\left[0, \frac{T}{2}\right]$ over M, which is also compact, then for each $\ell \in \mathbb{N}$ there exists a constant $B_\ell > 0$ dependent on ∇_0 such that

$$\sup_{\left[0, \frac{T}{2}\right]} \left|\left| \eta^s \nabla_t^{(\ell)} F_{\nabla_t} \right|\right|_{L^2}^2 \leq B_\ell.$$

Let $\tau := \min \left\{ \frac{1}{K^{2(\ell+k)}}, \frac{T}{2} \right\}$. Then if we consider any time $t > \frac{T}{2}$ we can consider the estimate given on the interval $[t - \tau, t]$ by applying Corollary 2.2.14 to obtain a local pointwise bound. However, since this bound is independent of each t (only relying on the time $\frac{T}{2}$), then we in fact have a uniform bound over $\left[\frac{T}{2}, T\right)$. Taking the maximum of this and B_ℓ we achieve the desired result. $\square$

2.2.3 Long time existence obstruction

We first prove two general lemmas which are in fact independent of the flow. The first is a completely general manipulation, and the second only relies on the bounds on $|\nabla_t^{(\ell)} F_{\nabla_t}|$ for $\ell \in \mathbb{N}$ for a given solution ∇_t over $\mathcal{I}$.

Lemma 2.2.17. *Let $\nabla, \overline{\nabla} \in \mathcal{A}_E$ and set $\Upsilon := \overline{\nabla} - \nabla$. Then for all ζ in some tensor product of TM, E, and their corresponding duals,*

$$\widetilde{\nabla}^{(\ell)} [\zeta] = \nabla^{(\ell)} [\zeta] + \sum_{j=0}^{\ell-1} \sum_{i=0}^{j} \left(P_{\ell-1-i}^{(i)} [\Upsilon] * P_1^{(j-i)} [\zeta] \right)$$

$$= \nabla^{(\ell)} [\zeta] + \sum_{j=0}^{\ell-1} \sum_{i=0}^{j} \left(\overline{P}_{\ell-1-i}^{(i)} [\Upsilon] * \overline{P}_1^{(j-i)} [\zeta] \right).$$

Proof. The first summation follows simply from direct computation, so we address the second. Note that any quantity in $P_w^{(v)}[\Upsilon]$ has the form

$$P_w^{(v)}[\Upsilon] = \sum_{r_1+\cdots+r_w=v} \nabla^{(r_1)}[\Upsilon] * \cdots * \nabla^{(r_w)}[\Upsilon].$$

Then replacing $\nabla = \overline{\nabla} + \Upsilon$ yields

$$
\begin{aligned}
P_w^{(v)} &= \sum_{r_1+\cdots+r_w=v} \nabla^{(r_1)}[\Upsilon] * \cdots * \nabla^{(r_w)}[\Upsilon] \\
&= \sum_{r_1+\cdots+r_w=v} (\overline{\nabla} + \Upsilon*)^{(r_1)}[\Upsilon] * \cdots * (\overline{\nabla} + \Upsilon*)^{(r_w)}[\Upsilon] \\
&= \sum_{r_1+\cdots+r_w=v} \left(\sum_{q=0}^{r_1}\sum_{p=0}^{q} \left(\overline{P}_{r_1+1-p}^{(q)}[\Upsilon]\right) \right) * \cdots * \left(\sum_{q=0}^{r_w}\sum_{p=0}^{q} \left(\overline{P}_{r_w+1-p}^{(q)}[\Upsilon]\right) \right) \\
&= \sum_{q=0}^{v}\sum_{p=0}^{q} \left(\overline{P}_{v+w-p}^{(q)}[\Upsilon]\right).
\end{aligned}
$$

Thus we have that

$$\left(P_{\ell-1-i}^{(i)}[\Upsilon] * P_1^{(j-i)}[\zeta] \right) = \left(\sum_{q=0}^{i}\sum_{p=0}^{q} \left(\overline{P}_{\ell-1-p}^{(q)}[\Upsilon]\right) \right) * P_1^{(j-i)}[\zeta],$$

and thus

$$\sum_{j=0}^{\ell-1}\sum_{i=0}^{j} \left(\sum_{q=0}^{i}\sum_{p=0}^{q} \left(\overline{P}_{\ell-1-p}^{(q)}[\Upsilon]\right) \right) * P_1^{(j-i)}[\zeta] = \sum_{j=0}^{\ell-1}\sum_{i=0}^{j} \left(\overline{P}_{\ell-1-i}^{(i)}[\Upsilon] * \overline{P}_1^{(j-i)}[\zeta] \right).$$

The result follows. $\qquad\square$

For the following proof, given ∇_t a one-parameter family over $[0,T)$ for some $T < \infty$, and set

$$\Upsilon_s := \int_0^s \tfrac{\partial \nabla_t}{\partial t} dt.$$

Note that for all $s < T$, we have $\Upsilon_s = \nabla_s - \nabla_0$.

Proposition 2.2.18. *Let ∇_t be a solution to generalized Yang-Mills k-flow over $[0,T)$ for some $T < \infty$. Suppose further that for all $\ell \in \mathbb{N}$ there exists $C_\ell \in \mathbb{R}_{>0}$ such that*

$$\sup_{M \times [0,T)} \left| \nabla_t^{(\ell)}\left[\tfrac{\partial \nabla}{\partial t} \right] \right| \leq C_\ell.$$

Then $\lim_{t \to T} \nabla_t =: \nabla_T$ *exists and is smooth.*

Proof. We first demonstrate that ∇_T as defined above exists in $C^0(M)$. For all $s \le T$ we have

$$|\Upsilon_s| = \left| \int_0^s \tfrac{\partial \nabla_t}{\partial t} dt \right| \le TC_0. \tag{2.19}$$

This implies, since ∇_0 is continuous, that ∇_T is continuous.

Next we demonstrate smoothness of ∇_T. The proof proceeds by induction on the $\ell \in \mathbb{N}$ satisfying $\left| \nabla_0^{(\ell)} [\Upsilon_T] \right| < \infty$. Let $s < T$. For the base case,

$$\begin{aligned}
|\nabla_0 [\Upsilon_s]| &= \left| \int_0^s \nabla_0 \left[\tfrac{\partial \nabla_t}{\partial t} \right] dt \right| \\
&= \left| \int_0^s \left(\nabla_t \left[\tfrac{\partial \nabla_t}{\partial t} \right] - \Upsilon_t * \left(\tfrac{\partial \nabla_t}{\partial t} \right) \right) dt \right| \\
&\le \int_0^s \left(\left| \nabla_t \left[\tfrac{\partial \nabla_t}{\partial t} \right] \right| + C|\Upsilon_t| \left| \tfrac{\partial \nabla_t}{\partial t} \right| \right) dt,
\end{aligned}$$

where $C = C(\dim M, \operatorname{Rank} E) \in \mathbb{R}_{\ge 0}$. Applying this to the above computations,

$$|\nabla_0 [\Upsilon_s]| \le \int_0^s \left(\left| \nabla_t \left[\tfrac{\partial \nabla_t}{\partial t} \right] \right| + C|\Upsilon_t| \left| \tfrac{\partial \nabla_t}{\partial t} \right| \right) dt \le TC_1 + CTC_0^2 < \infty.$$

Since Υ_s is continuous and the bound above is uniform across $s \in [0, T)$ then $|\nabla_0 [\Upsilon_s]| < \infty$, completing the proof of the base case.

Now let $\ell \in \mathbb{N}$ and suppose the induction hypothesis is satisfied for $\{1, \dots, \ell - 1\}$. Expanding $\nabla_0^{(\ell)} [\Upsilon_s]$ and applying Lemma 2.2.17,

$$\begin{aligned}
\nabla_0^{(\ell)} [\Upsilon_s] &= \int_0^s \nabla_0^{(\ell)} \left[\tfrac{\partial \nabla_t}{\partial t} \right] dt \\
&= \int_0^s \left(\nabla_t^{(\ell)} \left[\tfrac{\partial \nabla_t}{\partial t} \right] + \sum_{j=0}^{\ell-1} \sum_{i=0}^{j} \left(P_{\ell-1-i}^{(i)} [\Upsilon_t] * P_1^{(j-i)} \left[\tfrac{\partial \nabla_t}{\partial t} \right] \right) \right) dt.
\end{aligned}$$

Where here the derivatives within P are taken with respect to ∇_t. Taking the norm of the quantities gives, for $C > 0$,

$$\left| \nabla_0^{(\ell)} [\Upsilon_s] \right| = \int_0^s \left(\left| \nabla_t^{(\ell)} \left[\tfrac{\partial \nabla_t}{\partial t} \right] \right| + \sum_{j=0}^{\ell-1} \sum_{i=0}^{j} \left(P_{\ell-1-i}^{(i)} [\Upsilon_t] * P_1^{(j-i)} \left[\tfrac{\partial \nabla_t}{\partial t} \right] \right) \right) dt.$$

Each term is bounded by assumption, and in particular, all terms on the right are bounded by the induction hypothesis, $\left| \nabla_0^{(\ell)} [\Upsilon_s] \right| < \infty$. Since our choice of bounds are uniform for all $t \in [0, T)$ and Υ_s is continuous, it follows that $\left| \nabla_0^{(\ell)} [\Upsilon_T] \right| < \infty$, so the induction hypothesis is satisfied by ℓ. Thus Υ_T is smooth, so since

$$\Upsilon_T = \lim_{t \to T} \Upsilon_t = \lim_{t \to T} (\Gamma_0 - \Gamma_t) = \Gamma_0 - \lim_{t \to T} \Gamma_t,$$

then ∇_t may be extended to $\nabla_T := \lim_{t \to T} \nabla_t$, which is smooth. The result follows. $\qquad \square$

Using the previous results we demonstrate that the only obstruction to long time existence of the flow is a lack of supremal bound on the curvature tensor.

Theorem 2.2.19. *Suppose ∇_t is a solution to generalized Yang-Mills k-flow for some maximal $T < \infty$. Then $\sup_{M \times [0,T)} |F_{\nabla_t}| = \infty$.*

Proof. Suppose to the contrary that $\sup_{M \times [0,T)} |F_{\nabla_t}| \leq K < \infty$. Then by Corollary 2.2.16 for all $t \in [0, T)$ and $\ell \in \mathbb{N}$, we have $\sup_M |\nabla^{(\ell)} F_{\nabla_t}|$ is uniformly bounded and so by Proposition 2.2.18, $\nabla_T := \lim_{t \to T} \nabla_t$ exists and is a smooth solution to generalized Yang-Mills k-flow for such t. However, by Proposition 2.2.7, there exists $\epsilon > 0$ such that ∇_t exists over the extended temporal domain $[0, T + \epsilon)$, which contradicts the assumption that T was maximal. Thus $\sup_{M \times [0,T)} |F_{\nabla_t}| = \infty$, and the result follows. $\qquad \square$

2.2.4   Blowup analysis

We now address the possibility of Yang-Mills k-flow singularities given no bound on the spatial supremum of curvature. To do so we require the *Coulomb gauge theorem of Uhlenbeck* [Uhl82a] (located in the Appendix, cf. Theorem 3.8.12). For our setting we will need to extend this choice of gauges over a large region- this will be accomplished via the *gauge patching theorem* (located in the Appendix, Theorem 3.8.13). Now we establish some preliminary scaling laws of Yang-Mills k-flow, and discuss the effect that this scaling has on the generalized flow, which will be key in the proceeding blowup analysis argument.

Lemma 2.2.20. *Suppose ∇_t is a solution to Yang-Mills k-flow with local coefficient matrices Γ_t. Define the one-parameter family ∇_t^λ with coefficient matrices given by*

$$\Gamma_t^\lambda(x) := \lambda \Gamma_{\lambda^{2(k+1)}t}(\lambda x). \tag{2.20}$$

Then ∇_t^λ is also a solution to Yang-Mills k-flow.

Proof. Using the scaling as in (2.20) we set $F_t^\lambda := F_{\nabla_t^\lambda}$ and $D_t^\lambda := D_{\nabla_t^\lambda}$. Then we insert ∇_t^λ into the Yang-Mills k-flow equation. First, we take the temporal derivative.

$$\frac{\partial \nabla_t^\lambda}{\partial t} = \frac{\partial \Gamma_t^\lambda}{\partial t} = \lambda^{2k+3} \frac{\partial \Gamma_t}{\partial t} = \lambda^{2k+3} \frac{\partial \nabla_t}{\partial t}.$$

Thus the desired scaling law holds through the Yang-Mills k-flow equation. $\square$

Remark 2.2.21. Now we compare this to the scaling of $\operatorname{Grad} \mathcal{YM}_1$. If we revisit the proof of Proposition 2.1.3, rather than commute connections and perform integration by parts (in order to get clean Laplacian pairings), if we instead just directly integrate by parts, one confirms that

$$\operatorname{Grad} \mathcal{YM}_1(\nabla_t) = P_{1,-R}^{(2k+1)}[F_{\nabla_t}] + P_{2,-R}^{(2k-1)}[F_{\nabla_t}],$$

where here $P_{v,-R}^{(u)}$ notation is that of Definition 1.4 with the added constraint that the quantities are written entirely in terms of derivatives of F_∇. Thus,

$$\operatorname{Grad} \mathcal{YM}_1(\nabla_t^\lambda) = \lambda^{2k+3} \operatorname{Grad} \mathcal{YM}_1(\nabla_t).$$

Remark 2.2.22. While the proof of Lemma 2.2.20 only applies in the case of Yang-Mills k-flow, we may utilize the result on the generalized flow in the subsequent blow up analysis. To see this we must verify that the order of $\mho_k$ works favorably with respect to the scaling law (that is, if we send $\lambda \to 0$). Rescaling with $\Gamma_t^\lambda(x) := \lambda \Gamma_{\lambda^{2(k+1)}t}(\lambda x)$ we have,

$$\mho_k(\nabla_t^\lambda) = \sum_{i=1}^{k} \sum_{j=2}^{i+1} (P_\lambda)_j^{(2i+3-2j)}\left[F_{\nabla_t^\lambda}\right] = \sum_{i=1}^{k} \sum_{j=2}^{i+1} \lambda^{2i+3} P_j^{(2i+3-2j)}[F_{\nabla_t}],$$

where here P_λ means P with respect to ∇_t^λ derivatives. We have that $5 \le 2i + 3 \le 2k + 3$. As $\lambda \to 0$ we have that all terms are dominated except those of the form

$$\sum_{j=2}^{k+1} \lambda^{2k+3} P_j^{(2k+3-2j)}[F_\nabla].$$

These are the dominant quantities of $\mho_k$ in the context of rescaling for small λ and agree with the scaling law of Yang-Mills k-flow, thus ultimately preserving the behavior of the blowup limit.

We now demonstrate the construction of a generalized Yang-Mills k-flow blowup limit in the following proposition.

Proposition 2.2.23. *Suppose there exists a maximal $T \in [0, \infty]$ such that ∇_t is a solution to generalized Yang-Mills k-flow over $[0, T)$ and $\lim_{t \to T} ||F_{\nabla_t}||_{L^\infty(M)} = \infty$. Then a blowup sequence $\{\nabla_t^i\}$ exists and converges pointwise to a smooth solution $\boldsymbol{\nabla}_t$ to generalized Yang-Mills k-flow with domain $\mathbb{R}^n \times \mathbb{R}_{\leq 0}$. Additionally for each $q \in \mathbb{N}$ there exists $C_q = C_q(K, q, g, \mathrm{Rank}\, E \dim M)$ such that*

$$\sup_{(x,t) \in \mathbb{R}^n \times \mathbb{R}_{\leq 0}} \left| \boldsymbol{\nabla}_t^{(q)} F_{\boldsymbol{\nabla}_t} \right| \leq C_q,$$

and furthermore $|F_{\boldsymbol{\nabla}_0}(0)| = 1$.

Proof. Choose a sequence $\tau_i \nearrow T$ within $[0, T)$. For each τ_i, there exists a point $(x_i, t_i) \in M \times [0, T)$ so that

$$\left| F_{\nabla_{t_i}}(x_i) \right| = \sup_{(x,t) \in M \times [0, \tau_i]} |F_{\nabla_t}(x)|.$$

Choose a subsequence so that $\{x_i\}$ converges to some $x_\infty \in M$. There exists some chart about the blowup center x_∞ so that the tail of the sequence $\{x_i\}$ is contained within the single chart mapping into B_1. We will only consider the sequence tail, so it is therefore sufficient to utilize the coordinate chart and assume that the sequence is contained in $\mathbb{R}^n$. Thus we may identify connections with their coefficient matrices in this argument.

Let $\{\lambda_i\} \subset \mathbb{R}_{>0}$ be constants to be determined, and set

$$\Gamma_t^i(x) := \lambda_i^{1/(2k+2)} \Gamma_{\lambda_i t + t_i}(\lambda_i^{1/(2k+2)} x + x_i).$$

Note the corresponding curvatures $F_t := F_{\nabla_t}$ is scaled in the following manner,

$$F_t^i(x) = \lambda_i^{1/(k+1)} F_{\lambda_i t + t_i}(\lambda_i^{1/(2k+2)} x + x_i).$$

By Lemma 2.2.20 all corresponding ∇_t^i are also solutions to the generalized Yang-Mills k-flow (though with different initial conditions and scaled lower order terms). The domain for each ∇_t^i is $B_{\left((\lambda_i)^{-1/(2k+2)}\right)} \times [\frac{-t_i}{\lambda_i}, \frac{T - t_i}{\lambda_i}]$. We will choose λ_i to mitigate the 'blowing up' of the sequence curvatures F_t^i by observing the

following:

$$\sup_{t\in\left[-\frac{t_i}{\lambda_i},\frac{T-t_i}{\lambda_i}\right]}\left|F_t^i(x)\right| = \left|\lambda_i^{1/(k+1)}\right|\sup_{t\in\left[-\frac{t_i}{\lambda_i},\frac{T-t_i}{\lambda_i}\right]}\left|F_{\lambda_i t+t_i}(\lambda_i^{1/(2k+2)}x+x_i)\right|$$

$$= \left|\lambda_i^{1/(k+1)}\right|\sup_{t\in[0,t_i]}\left|F_t(x)\right|$$

$$= \left|\lambda_i^{1/(k+1)}\right|\left|F_{t_i}(x_i)\right|.$$

Setting $\lambda_i = |F_{t_i}(x_i)|^{-(k+1)}$ gives that

$$1 = \left|F_0^i(0)\right| = \lambda_i^{1/(k+1)}|F_{0+t_i}(0+x_i)| = \sup_{t\in\left[-\frac{t_i}{\lambda_i},0\right]}|F_t^i(x)|. \tag{2.21}$$

Note that we have no information on the behavior of the F_t^i for $t > 0$, so this bound (2.21) induced by the choices of λ_i is only guaranteed on $\mathbb{R}_{\leq 0}$.

Next we construct smoothing estimates for the sequence $\{\nabla_t^i\}$. Let $y \in \mathbb{R}^n$, $\tau \in \mathbb{R}_{\leq 0}$ and henceforth consider only sequence indices $i \in \mathbb{N}$ such that the domain of ∇_t^i contains $B_1(y) \times [\tau-1, \tau]$. Let $\eta_y \in C_c^\infty(M)$ be chosen so that $0 \leq \eta_y \leq 1$, $\text{supp}\,\eta_y = B_1(y)$, and on $B_{\frac{1}{2}}(y)$, $\eta_y \equiv 1$. For any $s \in \mathbb{N}$ note that $\sup_{[\tau-1,\tau]}|\eta_y^s F_t^i| \leq 1$. Since each $\eta_y^s F_t^i$ is smooth on $[\tau-1, \tau]$ then by Corollary 2.2.14 for all $q \in \mathbb{N}$ one may choose $s > 2(k+q+1)$ so that there exists $Q_q \in \mathbb{R}_{>0}$ such that

$$\sup_{\{\tau\}\times B_{\frac{1}{2}}(y)}\left|\left(\nabla_t^i\right)^{(q)}F_t^i\right| \leq \sup_{\{\tau\}\times B_1(y)}\left|\eta_y^s\left(\nabla_t^i\right)^{(q)}F_t^i\right| \leq Q_q.$$

Utilizing the same bump function and recentering it at each point, we establish uniform smoothing estimates across all of $\mathbb{R}^n$. Therefore, we obtain uniform pointwise bounds for $|(\nabla_t^i)^{(q)}F_t^i|$ for all $q, i \in \mathbb{N}$.

Let $\epsilon, R \in \mathbb{R}_{>0}$ and $\tau \in \mathbb{R}_{\leq 0}$. For any $m \in \mathbb{N}$ we consider the compact time interval $[\tau - m, -\frac{1}{m}]$ and all $i \in \mathbb{N}$ such that the domain of ∇_t^i contains $[\tau - m, -\frac{1}{m}] \times B_{R+m+\epsilon}$. Since the F_t^i are all pointwise uniformly bounded by 1, then there exists some $\delta > 0$ so that for any $y \in \mathbb{R}^n$, we have $||F_t^i||_{L^{n/2}B_\delta(y)} \leq \kappa_n$, where κ_n is as defined in the Coulomb Gauge Theorem (Theorem 3.8.12). We rescale coordinates of the sequence of connections ∇_t^i restricted to $B_\delta(y)$ (so that $B_\delta(y) \mapsto B_1$), again using the scalings laws to preserve that ∇_t^i is a solution to Yang-Mills k-flow up to highest order terms, by setting

$$\overline{\Gamma}_t^i(x) := \frac{1}{\delta^2}\Gamma_{t\delta^{-2}}^i\left(\frac{x-y}{\delta}\right).$$

We apply the Coulomb Gauge Theorem (Theorem 3.8.12) for $t = \delta^2(\tau - m)$ to obtain a sequence of connections $\overline{\Upsilon}_t^i$ which are gauge equivalent to $\overline{\Gamma}_t^i$ on B_1 and some $c_n > 0$ satisfying, for all $i \in \mathbb{N}$,

$$\left\| \overline{\Upsilon}_t^i(\delta^2(\tau - m)) \right\|_{C^{\rho,1}} \leq c_n.$$

Note that for $t = \delta^2(\tau - m)$ the curvatures corresponding to $\overline{\Upsilon}_t^i$ coincide with the curvatures corresponding to $\overline{\Gamma}_t^i$. We will denote these by $\overline{F}_t^i$. As a result of Lemma 2.2.20 the curvatures scale with the coefficient matrices and so the derivatives of curvatures are also uniformly bounded, that is, for all $q \in \mathbb{N}$ there exists $C_q \in \mathbb{R}_{>0}$ such that

$$\sup_{B_1 \times [\delta^2(\tau - m), \frac{-\delta^2}{m}]} \left| (\overline{\nabla}^i)_t^{(q)} \overline{F}_t^i \right| \leq C_q.$$

Consequently, as demonstrated in Proposition 2.2.18, given the short time existence of $\widetilde{\nabla}_t^i$ by Proposition 2.2.7, it follows that it exists for $t \in [\delta^2(\tau - m), \frac{-\delta^2}{m}]$ such that for some $C_{(\tau - m)} \in \mathbb{R}_{>0}$,

$$\sup_{B_1 \times [\delta^2(\tau - m), \frac{-\delta^2}{m}]} \| \overline{\Upsilon}_t^i \|_{C^{\rho,1}} \leq \delta^2 C_{(\tau - m)}.$$

We redilate and shift the coordinates to recover the domain $B_\delta(y)$ and thus obtain a new sequence of connections Υ_i defined on $B_\delta(y)$ by

$$\Upsilon_t^i(x) := \delta^2 \overline{\Upsilon}_{\delta^2 t}^i(\delta x + y),$$

which satisfies

$$\sup_{B_\delta(y) \times [\tau - m, \frac{-1}{m}]} \left\| \Upsilon_t^i \right\|_{C^{\rho,1}} \leq C_{(\tau - m)}.$$

Again, we emphasize that in fact each Υ_t^i is a solution to generalized Yang-Mills k-flow by virtue of the steps used to construct it, and furthermore Υ_t^i is gauge equivalent to Γ_t^i. Taking a countable covering of these balls for $y \in B_{R+m}$, we apply the Gauge Patching Theorem (Theorem 3.8.13) and obtain sequence of connection matrices $\{\Upsilon_t^i\}$ defined over all of B_{R+m}.

Recursively consider $\rho \in \mathbb{N}$ starting with $m = 1$ and choose $\alpha, \alpha' \in (0, 1)$ with $\alpha' < \alpha$. Given $\{\Upsilon_t^i\}$, with bounds of this sequence in $C^{\rho,\alpha}$, by the Arzela-Ascoli Theorem there exists some subsequence $\{\Upsilon_t^{i_j}\}$ which converges with respect to $C^{\rho,\alpha'}$ to some $(\Upsilon_\rho)_t^\infty$. Note that for any $\rho_1, \rho_2 \in \mathbb{N}$ with $\rho_1 < \rho_2$, $(\Upsilon_{\rho_1})_t^\infty = (\Upsilon_{\rho_2})_t^\infty$

since $C^{\rho_2, \alpha'}$ is a topological subspace of $C^{\rho_1, \alpha'}$ so convergence coincides. Relabel the subsequence terms $\Upsilon_t^{i_j} \mapsto \Upsilon_t^i$ and repeat this refinement through all $\rho \in \mathbb{N}$. We observe that as $\rho \to \infty$, the resulting sequence, $\{\Upsilon_t^i\}$ converges with respect to C^∞ to some limit term Υ_t^∞ on $B_{R+m}(y)$. We perform the above construction for all $m \in \mathbb{N}$ and take a diagonal subsequence of coefficient matrices which converge on any compact subset of $\mathbb{R}_{<0} \times \mathbb{R}^n$ to a connection coefficient matrix $\boldsymbol{\Gamma}_t$. The connection $\boldsymbol{\nabla}_t$ with $\boldsymbol{\Gamma}_t$ as its coefficient matrix is a solution to generalized Yang-Mills k-flow, in particular, given by

$$\frac{\partial \boldsymbol{\nabla}_t}{\partial t} = (-1)^{k+1} D_{\boldsymbol{\nabla}_t}^* \triangle_t^{(k)} F_{\boldsymbol{\nabla}_t} + P_2^{2k-2} \left[F_{\boldsymbol{\nabla}_t} \right].$$

Furthermore, note that $\boldsymbol{\nabla}_t$ inherits all supremal bounds on the derivatives, as desired. $\qquad \square$

Remark 2.2.24. Throughout the blowup procedure each connection satisfied a *different* generalized Yang-Mills k-flow since the lower order terms of $\mho_k$ scaled differently than the highest order term (cf. Remark 2.2.22).

2.3   Long time existence results

We now hone in our attention to specifically the Yang-Mills k-flow rather than its generalization. The explicit form allows for properties necessary to prove the main two parts of Theorem A. We now demonstrate that for finite times the Yang-Mills k-flow yields control over the Yang-Mills energy of the curvature.

Lemma 2.3.1. *Suppose that ∇_t is a solution to Yang-Mills k-flow over $[0, \bar{T})$. Then for all $T \leq \bar{T}$ with $T \in \mathcal{I}$ we have that $\sup_{[0,T)} ||F_{\nabla_t}||_{L^2(M)} < \infty$.*

Proof. Let ∇_t be a solution to the Yang-Mills k-flow. Then differentiating the Yang-Mills k-energy with such ∇_t as the argument yields (referring to the proof of Proposition 2.1.3)

$$\frac{d}{dt} \left[\mathcal{YM}_k(\nabla_t) \right] = -2 \int_M \left\langle \operatorname{Grad} \mathcal{YM}_k(\nabla_t), \frac{\partial \nabla_t}{\partial t} \right\rangle dV_g = -2 || \operatorname{Grad} \mathcal{YM}_k(\nabla_t) ||_{L^2}^2.$$

This, unsurprisingly, indicates that the flow monotonically decreases the Yang-Mills k-energy. With the above computation in mind we will estimate $||F_t||_{L^2}^2$ on $[0, T]$. Differentiating with respect to the temporal

parameter gives

$$\frac{d}{dt}\left[\mathcal{YM}(\nabla_t)\right] = \int_M \left\langle \frac{\partial F_t}{\partial t}, F_t \right\rangle dV_g$$

$$= \int_M \left\langle D_t \dot{\Gamma}_t, F_t \right\rangle dV_g$$

$$= 2 \int_M \left\langle \dot{\Gamma}_t, D_t^* F_t \right\rangle dV_g$$

$$= -2 \int_M \left\langle \operatorname{Grad}\mathcal{YM}_k(\nabla_t), D_t^* F_t \right\rangle dV_g.$$

Integrating both sides with respect to t and manipulating with Young's Inequality followed by the weighted interpolation identity of Corollary 3.6.5, for any $\epsilon > 0$, we obtain

$$\mathcal{YM}(\nabla_T) - \mathcal{YM}(\nabla_0) = -2 \int_0^T \left(\int_M \left\langle \operatorname{Grad}\mathcal{YM}_k(\nabla_t), D_{\nabla_t}^* F_{\nabla_t} \right\rangle dV_g \right) dt$$

$$\leq C \int_0^T \left(\|\operatorname{Grad}\mathcal{YM}_k(\nabla_t)\|_{L^2}^2 + \|D_{\nabla_t}^* F_{\nabla_t}\|_{L^2}^2 \right) dt$$

$$\leq -C \int_0^T \frac{d}{dt}\left[\mathcal{YM}_k(\nabla_t)\right] dt + C \int_0^T \|\nabla_t F_{\nabla_t}\|_{L^2}^2 \, dt$$

$$\leq C\left(\mathcal{YM}_k(\nabla_0) - \mathcal{YM}_k(\nabla_T)\right) + \int_0^T \left(C_\epsilon \left\|\nabla_t^{(k)} F_{\nabla_t}\right\|_{L^2}^2 + \epsilon \|F_{\nabla_t}\|_{L^2}^2 \right) dt$$

$$\leq CT\left(\mathcal{YM}_k(\nabla_0)\right) + \epsilon T \sup_{t \in [0,T]} \mathcal{YM}(\nabla_t).$$

We thus have that

$$\sup_{t \in [0,T]} \left(\mathcal{YM}\left(\nabla_t\right)\right) \leq \frac{CT}{1-\epsilon}\left(\mathcal{YM}_k(\nabla_0) + \mathcal{YM}(\nabla_0)\right).$$

Thus, choosing ϵ sufficiently small yields the desired result. $\qquad\square$

Theorem 2.3.2. *Suppose* $\dim M < 2p$ *and* ∇_t *a solution to generalized Yang-Mills k-flow over* $[0,T)$ *and* $\sup_{[0,T)} \|F_{\nabla_t}\|_{L^p(M)} < \infty$. *Then* $\sup_{[0,T)} \|F_{\nabla_t}\|_{L^\infty(M)} < \infty$.

Proof. Set $\dim M = n < 2p$. We suppose to the contrary $\lim_{t\to T} \|F_t\|_{L^\infty} = \infty$ and construct a blowup limit $\{\nabla_t^i\}$ with limit $\boldsymbol{\nabla}_t$ as described in Proposition 2.2.23. Since $\|F_t^i(x)\|_{L^\infty} = 1 < \infty$, by Fatou's Lemma and natural scaling law,

$$\|F_{\boldsymbol{\nabla}_t}\|_{L^p([0,1]\times^n)}^p \leq \liminf_{i\to\infty} \|F_{\nabla_t}^i\|_{L^p([0,1]\times^n)}^p \leq \lim_{i\to\infty} \lambda_i^{\frac{2p-n}{2k+2}} \|F_{\nabla_t}\|_{L^p(\mathbb{R}^n)}^p.$$

Since $\lim_{i\to\infty} \lambda_i = 0$ then whenever $2p > n$ the right hand side of the inequality converges to zero, which is

a contradiction since the blowup limit is constructed for nontrivial curvatures. The result follows. □

Utilizing this we may prove the complete long time existence of the Yang-Mills k-flow for subcritical dimensions.

Proof of Theorem A (S). Set $\dim M = n$. By the Sobolev embedding theorem, we solve for p such that $H_2^k \subset H_0^p$, namely one satisfying the formula $\frac{1}{p} = \frac{1}{2} - \frac{(k-0)}{n}$. We additionally impose that $p > \frac{n}{2}$ to utilize Theorem 2.3.2 and solve to obtain that $2(k+2) > n$. In this case, then we have that, using the interpolation identities of Corollary 3.6.5, where $\mathsf{S}_{k,p}$ is the Sobolev constant and C is the constant induced by interpolation of these derivatives via Corollary 3.6.5,

$$||F_{\nabla_t}||_{L^p(M)} \leq \mathsf{S}_{k,p} \sum_{j=0}^{k} \left|\left|\nabla^{(j)} F_{\nabla_t}\right|\right|_{L^2(M)} = C\mathsf{S}_{k,p} \left(\sqrt{\mathcal{YM}_k(\nabla_t)} + \sqrt{\mathcal{YM}(\nabla_t)}\right). \tag{2.22}$$

Referring to Lemma 2.3.1, since $\mathcal{YM}_k(\nabla_t)$ is decreasing along Yang-Mills k-flow and we have control over $\mathcal{YM}(\nabla_t)$ for any finite time, we conclude that the flow exhibits smooth long time existence. □

Remark 2.3.3. Note that this proof does not conclude that the flow exists at $t = \infty$, so it may be the case that the Yang-Mills k-flow admits singularities at infinite time.

We now state a theorem which generalizes the characterization of energy concentration of Yang-Mills flow in dimension 4 introduced by Struwe [Str94]. First we characterize bubbling of L^p norms in relation to the base manifold dimension. With this proposition we then may conclude the bubbling in critical dimensions.

Proposition 2.3.4. *Let $p \in \mathbb{N}$ and suppose $\dim M = 2p$ and ∇_t is a solution to generalized Yang-Mills k-flow for $t \in [0, T)$ with T maximal. Then there exists some $\epsilon > 0$ such that if $\{(x_i, t_i)\} \subset M \times [0, T)$ with $(x_i, t_i) \to (X, T)$ has the property that $\lim_{i \to \infty} \left|F_{\nabla_{t_i}}(x_i)\right| = \infty$, then for all $r > 0$, $\lim_{i \to \infty} ||F_{\nabla_{t_i}}||_{L^p(B_r(X))} \geq \epsilon$.*

Proof. Choose a corresponding blowup sequence $\{\nabla_t^i\}$ as described in Proposition 2.2.23 with limit $\boldsymbol{\nabla}_t$. Then by construction $|F_{\boldsymbol{\nabla}_0}(0)| = 1$. By the derivative bounds on $\boldsymbol{\nabla}_t$ of Proposition 2.2.23, since $|\boldsymbol{\nabla}_t F_{\boldsymbol{\nabla}_t}|$ is bounded, combined with the smoothness of $\boldsymbol{\nabla}_t$ over time, one has that that for $(y, t) \in B_\delta \times (-\delta, 0]$ we have

$|F_{\boldsymbol{\nabla}_t}(y)| \geq \frac{1}{2}$. Observing this we have

$$\limsup_{t\nearrow 0} ||F_{\boldsymbol{\nabla}_t}||^p_{L^p(B_\delta)} = \limsup_{t\nearrow 0} \int_{B_\delta} |F_{\boldsymbol{\nabla}_t}|^p \, dV_g \geq \frac{\text{Vol}[B_\delta]}{2^p}.$$

Conversely, using the computations in Theorem 2.3.2 yields,

$$\begin{aligned}
||F_{\boldsymbol{\nabla}_t}||^p_{L^p(B_\delta)} &= \int_{B_\delta} |F_{\boldsymbol{\nabla}_t}|^p dV_g \\
&= \int_{B_\delta} \lim_{i\to\infty} |F_{\nabla_t^i}|^p dV_g \\
&= \lim_{i\to\infty} \lambda_i^{\frac{2p-n}{2k+2}} ||F_{\nabla_t}||^p_{L^p\left(B_{\delta\lambda_i^{1/(2k+2)}}(x_i)\right)} \\
&= \lim_{i\to\infty} ||F_{\nabla_t}||^p_{L^p\left(B_{\delta\lambda_i^{1/(2k+2)}}(x_i)\right)}.
\end{aligned}$$

Since $\lim_{i\to\infty} \lambda_i^{1/(2k+2)} = 0$ then for any $r > 0$ and i large enough so that $\max\{|T - t_i|\} < \delta$,

$$\frac{\text{Vol}[B_\delta]}{2^p} \leq \limsup_{i\to\infty} \left|\left|F_{\nabla_{t_i}}\right|\right|^p_{L^p\left(B_{\delta\lambda_i^{1/(2k+2}(x_i)}\right)} \leq \limsup_{i\to\infty} \left|\left|F_{\nabla_{t_i}}\right|\right|^p_{L^p(B_r(X))}.$$

Taking $\epsilon = \frac{\text{Vol}[B_\delta]}{2^p}$ yields the result. $\qquad\square$

Proof of Theorem A (C). Note that the lower bound on the amount of energy at a singularity location given in Proposition 2.3.4 is ϵ is independent of the point about which the blowup procedure occurred, and yields the conclusion of the theorem. $\qquad\square$

2.3.1   Extensions

Here we state the proof of our second main result, Theorem B, and reflect on possible extensions of the flow.

Regularized flow

As stated in the introduction, we study the Yang-Mills (ρ, k)-energy and corresponding gradient flow (cf. Definition YMρkE). Utilizing the work of the previous sections combined with the presence of the Yang-Mills energy, we demonstrate subcritical long time existence and convergence.

Proof of Theorem B. The corresponding flow of this particular functional is given by the weighted sums of the negative gradient flows of the two participating functionals.

$$\frac{\partial \nabla_t}{\partial t} = \rho \left((-1)^{k+1} D^*_{\nabla_t} \triangle^{(k)}_t F_{\nabla_t} + P^{(2k-1)}_2 [F_{\nabla_t}] \right) - D^*_{\nabla_t} F_{\nabla_t}$$
$$= - \left(\rho \operatorname{Grad} \mathcal{YM}_k(\nabla_t) + \mathcal{YM}(\nabla_t) \right).$$

(2.23)

As in the work of [HTY15], one would hope to apply a regularization argument on the Yang-Mills (ρ, k)-flow by sending $\rho \searrow 0$ to identify Yang-Mills connections. The advantage to using this flow over that of the Yang-Mills α-flow is that the Yang-Mills (ρ, k)-flow has long time existence and convergence in dimensions less than $2(k+2)$. This follows from simply temporally rescaling the gradient flow of (2.23) to shift the dependence of ρ on the highest order term to the others. Since these lower order terms satisfy the requirements of possible quantities represented by $\mho_k(\nabla)$ (cf. (2.6)), then we can simply apply the arguments of §2.2 to obtain short time existence and uniqueness, necessary smoothing estimates, and construct blowup limits as desired. However, since this is the negative gradient flow of the weighted sum of the Yang-Mills k-energy and the Yang-Mills energy, we have that each individual energy is bounded over time above by a scaled multiple of $\rho \mathcal{YM}_k(\nabla_0) + \mathcal{YM}(\nabla_0)$ for all time. Therefore we obtain a subsequential limit at $t = \infty$ since a singularity cannot occur, and we conclude the result. $\square$

As in the case of the Yang-Mills α-flow, one could pursue further results as in the works of [HTY15] and [HS13], such as verifying the Yang-Mills k-energy satisfies the Palais-Smale condition, which will guarantee the existence of minimizers, or proving an energy identity as in [HS13].

Yang-Mills 1-flow versus bi-Yang-Mills

In particular we now turn to the study of the Yang-Mills 1-energy, given by

$$\mathcal{YM}_1(\nabla) := \tfrac{1}{2} \int_M |\nabla F_\nabla|^2 \, dV_g.$$

(YM1E)

The Yang-Mills 1-energy is closely tied to the bi-Yang-Mills energy (cf. (BYME)) studied in [IIU09]. While the bi-Yang-Mills energy is arguably more 'natural' to study (with reference to the gradient flow of Yang-Mills), so far only we are only able to make statements about the existence of the Yang-Mills 1-flow in its critical dimension (cf. Remark 2.3.7). Roughly speaking, this is due to the fact that the Yang-Mills 1-energy measures 'all' of ∇F_∇, while the bi-Yang-Mills energy only measures a portion (since D^* is a scaled trace

over ∇). The relationship between the two energies is displayed in the following lemma. We omit the proof, as it is simply reshuffling of terms.

Lemma 2.3.5. *For* $\nabla \in \mathcal{A}_E$,

$$\mathcal{BYM}(\nabla) = \tfrac{1}{2} \mathcal{YM}_1(\nabla) - \tfrac{1}{4} \int_M \left(\mathrm{Rc}_i^p F_{pk\alpha}^\beta - \mathrm{Rc}_k^p F_{pi\alpha}^\beta - \mathrm{Rm}_{i\ell q}^p F_{qp\alpha}^\beta \right) F_{i\ell\beta}^\alpha \, dV_g - \int_M F_{pi\gamma}^\beta F_{p\ell\alpha}^\gamma F_{i\ell\beta}^\alpha \, dV_g.$$

With this in mind, it is natural that the actual bi-Yang-Mills flow will only differ by lower order terms. One may compute the variation of the bi-Yang-Mills energy as in ([IIU09] Theorem 26) and conclude that *bi-Yang-Mills flow* is given by

$$\frac{\partial \nabla_t}{\partial t} = - \operatorname{Grad} \mathcal{BYM}(\nabla_t) = \triangle_t D_{\nabla_t}^* F_{\nabla_t} + [D_\nabla^* F_{\nabla_t}, F_{\nabla_t}]^\#. \tag{BYMf}$$

Lemma 2.3.6. *Suppose that* ∇_t *is a solution to bi-Yang-Mills flow on* $\mathcal{I}$. *Then for all* $T < \infty$ *with* $T \in \mathcal{I}$ *we have that* $\sup_{[0,T)} ||F_{\nabla_t}||_{L^2(M)} < \infty$.

Proof. In a similar vein of thought to Lemma 2.3.1, we have that for bi-Yang-Mills flow,

$$\begin{aligned}
\mathcal{YM}(\nabla_T) - \mathcal{YM}(\nabla_0) &= -2 \int_0^T \left(\int_M \langle \operatorname{Grad} \mathcal{BYM}(\nabla_t), D_t^* F_t \rangle \, dV_g \right) dt \\
&\le C \int_0^T \left(||\operatorname{Grad} \mathcal{BYM}(\nabla_t)||_{L^2}^2 + \mathcal{BYM}(\nabla_t) \right) dt \\
&\le C \int_T^0 \tfrac{d}{dt} \left[\mathcal{BYM}(\nabla_t) \right] dt + CT \mathcal{BYM}(\nabla_0) \\
&\le CT \mathcal{BYM}(\nabla_0).
\end{aligned}$$

Thus the Yang-Mills energy is bounded in $L^2(M)$ for all finite times along bi-Yang-Mills flow. $\qquad \square$

With this result and the identity of Lemma 2.3.5 we may at last prove Theorem C.

Proof of Theorem C. By the manipulations of the proof of Theorem A (S), is enough to show that we may bound $||\nabla_t F_{\nabla_t}||_{L^2}^2$ for $t < \infty$ to apply the Sobolev embedding theorem. Here we have that, by the

computations of Lemma 2.3.5,

$$\mathcal{YM}_1(\nabla_t) \leq 2\mathcal{BYM}(\nabla_t) + \int_M \left(P_3^{(0)}\left[F_{\nabla_t}\right] + P_2^{(0)}\left[F_{\nabla_t}\right] \right) \leq 2\mathcal{BYM}(\nabla_t) + C\left(||F_{\nabla_t}||^3_{L^3} + ||F_{\nabla_t}||^2_{L^2} \right),$$

(2.24)

where here C is some constant depending on the manifold. For $n \in [2,5]$ we have that $H_1^q \subset H_0^3$ for q satisfying $q = \frac{3n}{n+3}$. If we differentiate with respect to n one obtains

$$\frac{d}{dn}\left[\frac{3n}{n+3} \right] = \frac{9}{(n+3)^2} > 0,$$

so q monotonically increases with respect to n, implying that

$$\left.\frac{3n}{n+3}\right|_{n=2} = \frac{6}{5} \leq q \leq \frac{15}{8} = \left.\frac{3n}{n+3}\right|_{n=5} < 2.$$

Thus for all $n \in [2,5] \cap \mathbb{N}$, this implies that, where S denotes the necessary Sobolev constant,

$$||F_{\nabla_t}||^3_{L^3} \leq \mathsf{S}\left(||F_{\nabla_t}||^q_{L^q} + ||\nabla_t F_{\nabla_t}||^q_{L^q} \right).$$

Since M is compact and we have control over $\mathcal{YM}(\nabla)$ for all finite times, then by the containment $L^2(M) \subset L^q(M)$ for $q < 2$ we know that the term $||F_\nabla||^q_{L^q}$ is bounded. For the second term we apply Young's inequality (weighting accordingly),

$$|\nabla_t F_{\nabla_t}|^q = \left(\epsilon^{q/2} \left(\tfrac{2}{q} |F_{\nabla_t}|^2 \right)^{q/2} \right) \left(\tfrac{2\epsilon}{q} \right)^{-q/2}$$

$$\leq \epsilon |\nabla_t F_{\nabla_t}|^2 + \frac{\left(\tfrac{2\epsilon}{q} \right)^{-(2/q)^* q/2}}{(2/q)^*},$$

where here $(a)^*$ is the number satisfying $\frac{1}{a} + \frac{1}{(a)^*} = 1$. Integrating these over the manifold yields

$$||\nabla_t F_{\nabla_t}||^q_{L^q} \leq \epsilon ||\nabla_t F_{\nabla_t}||^2_{L^2} + \frac{C \operatorname{Vol} M}{\epsilon}.$$

(2.25)

Therefore we conclude that, combining the results into (2.24) and compiling the lingering constant term of (2.25) in with the $C ||F_{\nabla_t}||^2_{L^2}$ of (2.24),

$$\mathcal{YM}_1(\nabla_t) \leq \left(1 - \tfrac{\mathsf{S}}{\epsilon} \right)^{-1} \left(\mathsf{S} C \mathcal{YM}(\nabla_t) + 2\mathcal{BYM}(\nabla_t) \right).$$

Thus, for finite time, the Yang-Mills 1-energy is controlled, so we may apply the Sobolev Embedding Theorem to conclude that for finite time, we have smooth long time existence. □

Remark 2.3.7. Note that for $\dim M = 6$ we cannot apply the necessary weighted Young's inequality manipulation to redistribute weight on $||\nabla_t F_{\nabla_t}||^2_{L^2}$. This is why, given the work above, a statement about the critical dimension of the bi-Yang-Mills functional cannot be made. This represents the delicate distinction between (BYME) and (YM1E).

Bi-Yang-Mills flow admits an isolation phenomena displayed in [IIU09], which was in turn inspired by the work of [BL81]. One would hope that an analogous isolation phenomena can be demonstrated for Yang-Mills 1-flow. However, it appears that this trade off for properties discussed above by measuring the full energy calls for more thought in demonstrating (if possible) an isolation phenomena in the case of Yang-Mills 1-flow.

Chapter 3

Limits of Yang-Mills α-connections

3.1 Introduction to Chapter 3

We now investigate the Yang-Mills α-energy with the goal of proving Theorem D and Theorem E. For this, we restrict to a very canonical setting of Yang-Mills theory.

> Henceforth E denotes the adjoint bundle associated to the Hopf fibration $\mathrm{SU}(2) \hookrightarrow \mathbb{S}^7 \to \mathbb{S}^4$. Here the second Chern class (or charge) of E is 1, and only antiself dual connections exist.

We begin reminding the reader of the notion of Chern classes and how they play in to four dimensional Yang-Mills theory.

Definition 3.1.1 (Chern classes)**.** The *Chern classes* of a bundle E are defined as

$$c_j [E] := \left[P^j \left(\tfrac{\sqrt{-1}}{2\pi} F_\nabla \right) \right] \in H^{2j} (M),$$

where P^j is the jth elementary symmetric polynomial and F_∇ is the curvature of an arbitrary connection $\nabla \in \mathcal{A}_E (M)$. We define the bundle's jth Chern number $C_j [E] \in \mathbb{N}$ by

$$C_j [E] := \int_M c_j [E] \, dV_g.$$

It is known for $E \to M$, with E an SU-type bundle (see [Jï1]) that since the curvature is traceless,

$$c_1[E] = 0, \quad c_2[E] - \frac{m-1}{2m} c_1[E] \wedge c_1[E] = \frac{1}{8\pi^2} \mathrm{tr}_h (F_\nabla \wedge F_\nabla).$$

Consequently we have that

$$C_2[E \to M] := \frac{1}{8\pi^2} \int_M \mathrm{tr}_h (F_\nabla \wedge F_\nabla), \text{ invariant for all } \nabla \in \mathcal{A}_E(M). \tag{3.1}$$

Remark 3.1.2. For $\nabla \in \mathcal{A}_E(M^4)$ by decomposing the curvature tensor $F_\nabla = F_\nabla^+ + F_\nabla^-$, where $F_\nabla^\mp$ denotes the (anti)self dual pieces of F_∇, we obtain the equality

$$\mathrm{tr}_h (F_\nabla \wedge F_\nabla) = \left(-\left| F_\nabla^+ \right|_{g,h}^2 + \left| F_\nabla^- \right|_{g,h}^2 \right) dV_g.$$

The first fundamental observation is a universal lower bound on the Yang-Mills α-energy.

Proposition 3.1.3. *For all $\nabla \in W^{1,2\alpha} \left(\mathcal{A}_E \left(\mathbb{S}^4 \right) \right)$,*

$$\mathcal{YM}_\alpha (\nabla) \geq 6^\alpha \tfrac{4}{3} \pi^2. \tag{3.2}$$

Proof. Recalling the general formula for the second Chern class combined with the decomposition of the curvature tensor $F_\nabla = F_\nabla^+ + F_\nabla^-$ (where $F_\nabla^\mp$ denotes the (anti)self dual pieces of F_∇) we have the following

$$C_2 \left[E \to \mathbb{S}^4 \right] := \frac{1}{8\pi^2} \int_{\mathbb{S}^4} \mathrm{tr}_h (F_\nabla \wedge F_\nabla) = \frac{1}{8\pi^2} \int_{\mathbb{S}^4} \left(-\left| F_\nabla^+ \right|_{\mathring{g}}^2 + \left| F_\nabla^- \right|_{\mathring{g}}^2 \right) dV_g. \tag{3.3}$$

From this we derive the following inequality

$$
\begin{aligned}
\tfrac{16}{3}\pi^2 &= \operatorname{Vol}(\mathbb{S}^4) + \tfrac{8}{3}\pi^2 C_2[E \to \mathbb{S}^4] \\
&= \int_{\mathbb{S}^4} dV_{\mathring{g}} + \tfrac{1}{3}\int_{\mathbb{S}^4} \operatorname{tr}_h(F_\nabla \wedge F_\nabla) \\
&= \int_{\mathbb{S}^4} \left(1 + \tfrac{1}{3}\left(|F_\nabla^-|_{\mathring{g}}^2 - |F_\nabla^+|_{\mathring{g}}^2\right)\right) dV_{\mathring{g}} \\
&\leq \int_{\mathbb{S}^4} \left(1 + \tfrac{1}{3}|F_\nabla|_{\mathring{g}}^2\right) dV_{\mathring{g}} \qquad\qquad (\star_1) \\
&= \left\|1 + \tfrac{1}{3}|F_\nabla|_{\mathring{g}}^2\right\|_{L^1} \\
&\leq \left\|1 + \tfrac{1}{3}|F_\nabla|_{\mathring{g}}^2\right\|_{L^\alpha} \|1\|_{L^{\alpha^*}} \qquad (\star_2) \\
&= \tfrac{2^{1/\alpha}}{3}\left(\tfrac{1}{2}\int_{\mathbb{S}^4}\left(3 + |F_\nabla|_{\mathring{g}}^2\right)^\alpha dV_{\mathring{g}}\right)^{1/\alpha}\left(\tfrac{2^3\pi^2}{3}\right)^{(\alpha-1)/\alpha} \\
&= \tfrac{2^{1/\alpha}}{3}\left(\mathcal{YM}_\alpha(\nabla)\right)^{1/\alpha}\left(\tfrac{2^3\pi^2}{3}\right)^{(\alpha-1)/\alpha},
\end{aligned}
\tag{3.4}
$$

where α^* appearing in $(\star_2)$ denotes the dual of α in the sense of Hölder's inequality. Raising both sides of the resultant inequality of (3.4) to the α power and rearranging accordingly yields the result. $\qquad\square$

We introduce a special class of connections, the ADHM instantons. However, we withhold discussion of quaternionic coordinate system over which these are defined (cf. [Nab10], §6.3 pp.353-361).

Definition 3.1.4 (ADHM instanton). Let $\xi \in \mathbb{H}^1$ and $\lambda \in \mathbb{R}$. Then the (ξ, λ)-*ADHM instanton* is the connection $\nabla^{\xi,\lambda} := \partial + \Gamma^{\xi,\lambda}$ where $\Gamma^{\xi,\lambda}$ is the $\operatorname{Im}\mathbb{H}^1$ valued 1-form given by, with curvature

$$
\Gamma^{\xi,\lambda}(\zeta) := \Im\left[\tfrac{\bar{\zeta}-\bar{\xi}}{|\zeta-\xi|^2+\lambda^2}\, d\zeta\right], \quad F_\nabla^{\xi,\lambda}(\zeta) = \tfrac{\lambda^2}{\left(|\zeta-\xi|^2+\lambda^2\right)}\, d\bar{\zeta}\wedge d\zeta.
$$

For $\lambda \equiv 1$ define the *basic connection* $\widetilde{\nabla} := \nabla^{\xi,1}$ (invariant under choice of ξ). Regarded as a connection on the sphere via Uhlenbeck's Removeable Singularities Theorem of [Uhl82b], the curvature has constant pointwise norm ($|F_{\widetilde{\nabla}}|_{\mathring{g}}^2 \equiv 3$). Recall the following result of Atiyah, Drinfeld, Hitchin and Manin.

Theorem 3.1.5 ([ADHM78]). *Every Yang-Mills energy minimizer is an ADHM instanton.*

In contrast to the Yang-Mills energy, the Yang-Mills α-energy differentiates the basic connection from other members of the ADHM family. This provides the guiding intuition for our main result, Theorem D.

Proposition 3.1.6. $\widetilde{\nabla}$ *is the only Yang-Mills α-energy minimizer.*

Proof. We compute the α-energy of $\widetilde{\nabla}$ by recalling the formulas introduced above

$$
\begin{aligned}
\mathcal{YM}_\alpha\left(\widetilde{\nabla}\right) &= \tfrac{1}{2}\int_{\mathbb{S}^4}\left(3+|F_{\widetilde{\nabla}}|_{\hat{g}}^2\right)^\alpha dV_{\hat{g}}\\
&= \tfrac{1}{2}\int_{\mathbb{H}^1}\left(3+\tfrac{1}{16}\left(|\zeta|^2+1\right)^4|F_{\widetilde{\nabla}}|^2\right)^\alpha 16\left(|\zeta|^2+1\right)^{-4}dV\\
&= \tfrac{1}{2}\int_{\mathbb{H}^1}\left(3+\tfrac{1}{16}\left(|\zeta|^2+1\right)^4\tfrac{48}{(|\zeta|^2+1)^4}\right)^\alpha 16\left(|\zeta|^2+1\right)^{-4}dV\\
&= 6^\alpha 2^3\int_{\mathbb{H}^1}\left(|\zeta|^2+1\right)^{-4}dV\\
&= 6^\alpha 2^3\int_{\mathbb{S}^4}\tfrac{1}{16}\,dV_{\hat{g}}\\
&= 6^\alpha \tfrac{4}{3}\pi^2.
\end{aligned}
$$

Comparing against (3.2), above, we see that only in the case of antiself dual connections ($\star_1$) with pointwise curvature norm precisely 3 ($\star_2$) the inequalities are in fact equalities, that is, when $\nabla \equiv \widetilde{\nabla}$. $\qquad\square$

Quaternionic coordinates

The most convenient way to describe and work with our setup is to identify four dimensional Euclidean space with the linear space of quaternions, $\mathbb{H}^1 := \{x^\nu\kappa_\nu : x^\nu \in \mathbb{R}\}$, where the elements $\{\kappa_i\}_{i\in\{1,2,3,4\}}$ span the *algebra of quaternions*:

$$\kappa_1 = \mathbf{1},\ \kappa_2 = \mathbf{i},\ \kappa_3 = \mathbf{j},\ \kappa_4 = \mathbf{k}, \quad \text{where } \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

The space $\mathbb{H}^1$ a four dimensional real vector space on which is defined a multiplication $(x,y) \mapsto xy$ which satisfies the following laws for $x,y,z \in \mathbb{H}^1$ and $a \in \mathbb{R}$,

$$(xy)z = x(yz), \quad x(y+z) = xy + xz \quad (x+y)z = xz + yz, \quad a(xy) = (ax)y = x(ay),$$

and in which there exists a distinguished basis satisfying the quaternionic relations

$$\mathbf{i}^2 = \mathbf{j}^2\mathbf{k}^2 = -\mathbf{1}, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

A representation for the spanning set of the quaternions is given by

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{i} = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}. \tag{3.5}$$

The distinguished basis $\{\kappa_i\}$ has the group structure of the algebraic group of quaternions and forms the standard set of generators. We can summarize the multiplication above via

$$\kappa_\mu \kappa_\nu = \delta_{1\mu} \kappa_\nu + \delta_{1\nu} \kappa_\mu - \delta_{\mu\nu} \kappa_1 + \varepsilon_{1\mu\nu}^\lambda \kappa_\lambda \quad 1 \leq \mu, \nu \leq 4, \tag{3.6}$$

where here ε is the *generalized Kronecker delta* given

$$\varepsilon_{\rho\mu\nu}^\lambda := \begin{cases} 1 & \text{if } (\rho\,\mu\,\nu\,\lambda) \text{ is an even 4-cycle,} \\ -1 & \text{if } (\rho\,\mu\,\nu\,\lambda) \text{ is an odd 4-cycle,} \\ 0 & \text{otherwise.} \end{cases}$$

For $\zeta = \zeta^\mu \kappa_\mu \in \mathbb{H}^4$, complex conjugation is given by

$$\overline{\zeta} := \zeta^1 \kappa_1 - \sum_{\mu=2}^4 \zeta^\mu \kappa_\mu.$$

The unit sphere in $\mathbb{H}^4$ is given by $\mathbb{U} := \{\zeta \in \mathbb{H}^4 : \overline{\zeta}\zeta = 1\}$. Under the quaternionic product of (3.6), we have that SU(2) is isomorphic to $\mathbb{S}^3$. Since $\mathbb{H}^1$ is isomorphic to $\mathbb{C}^2$, we may apply stereographic projection in terms of the latter setting. Via Uhlenbeck's Removable Singularity theorem in [Uhl82b], a connection $\nabla \in \mathcal{A}_E \left(\mathbb{S}^4, \mathring{g}\right)$ may be identified as a connection on the compactified complex 2-plane, $\left(\widehat{\mathbb{C}^2}, g\right)$ with standard metric. We will convert the Yang-Mills α-energy to the latter context using stereographic projection

$$\Pi_{\text{ster}} : \mathbb{S}^4 \to \mathbb{C}^2$$

$$: \left(\sigma^1, \ldots, \sigma^n, \varsigma\right) \mapsto \left(\zeta^1, \ldots, \zeta^n\right), \quad \zeta^j := \frac{\sigma^j}{1-\varsigma}.$$

With this in mind we have that the metric transforms as follows

$$\left(\Pi_{\text{ster}}^{-1}\right)^* \mathring{g} = \left(4\,|x|^2 + 1\right)^{-2} g,$$

where g is the Euclidean metric and $\mathring{g}$ is the spherical metric. Then the volume form changes in the following way:

$$
\begin{aligned}
\left(\Pi_{\text{ster}}^{-1}\right)^* (dV_{\mathring{g}}) &= \sqrt{\det((\sigma^{-1})^* \, \mathring{g})} \, dV \\
&= \sqrt{\det\left[4 \left(|x|^2 + 1\right)^{-2} g\right]} \, dV \\
&= 16 \left(|x|^2 + 1\right)^{-4} dV.
\end{aligned}
$$

Now we must compute how the curvature is effected by this transformation. Note that

$$
(\Pi_{\text{ster}}^{-1})^* \left[|F_\nabla|_{\mathring{g}}^2\right] = \tfrac{1}{16} \left(|x|^2 + 1\right)^4 |F_\nabla|^2 \, .
$$

It is an elementary computation to show the Yang-Mills α-energy translates to

$$
\int_{\mathbb{R}^4} \left(3 + \tfrac{1}{16} \left(|x|^2 + 1\right)^4 |F_\nabla|^2\right)^\alpha 16 \left(|x|^2 + 1\right)^{-4} dV,
$$

or if we convert to quaternionic coordinates (identifying $\mathbb{H}^1$ with $\mathbb{C}^2$),

$$
\int_{\mathbb{H}^1} \left(3 + \tfrac{1}{16} \left(|\zeta|^2 + 1\right)^4 |F_\nabla|^2\right)^\alpha 16 \left(|\zeta|^2 + 1\right)^{-4} dV.
$$

3.1.1   Gauge group and enlargement

Many of the main complications and interesting properties of Yang-Mills theory stem from the interactions of the gauge group with the connections. For four dimensional manifolds, the gauge group can be naturally extended to a larger class of objects which will be key for our upcoming analysis. We discuss the interactions of connections, conformal automorphisms and gauge transformations as well as define key quantities which identify their action on the α-energy.

Gauge group

We refer the reader to §3.8 in the appendix for an overview of basic aspects of gauge theory.

Conformal automorphisms enlarging the group

Given a connection, we are interested in how various pointwise and L^2-norms as well as the α-energy depends on actions by conformal automorphisms of the four sphere. The special indefinite orthogonal group $\mathrm{SO}(5,1)$ is isomorphic to the space of conformal automorphisms of the four sphere (a detailed summary of this isomorphism is outlined in pages pp.49-52 in [Slo92]).

For any $\varphi \in \mathrm{SO}(5,1)$, considered the centered dilation of magnitude $1/\lambda \in \mathbb{R} \backslash \{0\}$, denoted by λ^{-1}, chosen specifically so that $\lambda^{-1} : \varphi(0) \mapsto 0$. This composition produces an action is the same as a pure rotation centered at the origin, we will denote this R_0, giving that

$$\left(\left(\lambda^{-1} \circ \varphi \right) = R_0 \right) \Rightarrow \left(\varphi = \lambda \circ R_0 \right).$$

In particular, we have that $|\varphi| = |m_\lambda| \, |R_0| = \lambda$, and call λ as the *dilation factor of φ*.

The gauge group $\mathcal{G}_E$ is the group of automorphisms in each fiber, and thus projects onto the base manifold as the identity map on M. Since for four dimensional base manifolds, the Yang-Mills energy depends only on the conformal class of the base metric, we can enlarge the gauge group by considering automorphisms of E which project onto M as conformal automorphisms (cf. [BL81] pp.198). We call the *enlarged gauge group $\mathcal{G}_E$*. For $\varphi : \mathbb{S}^4 \to \mathbb{S}^4$ a conformal automorphism, we may lift to a uniquely determined corresponding element $\widehat{\varphi} \in S(\widehat{\mathcal{G}}_E)$ by lifting with respect to $\widetilde{\nabla}$ which satisfies the following:

$$\widehat{\varphi} : \mathcal{A}_E \left(\mathbb{S}^4 \right) \to \mathcal{A}_E \left(\mathbb{S}^4 \right)$$
$$: (\widehat{\varphi}^* \nabla)_V \left(\widehat{\varphi}^* \mu \right) = \widehat{\varphi}^* \left(\nabla_{\varphi_*(V)} \mu \right) \text{ for } V \in S(T\mathbb{S}^4), \mu \in S(E).$$

that is, given any $\varphi \in \mathrm{SO}(5,1)$ and $\nabla \in \mathcal{A}_E \left(\mathbb{S}^4 \right)$ with local expression $\nabla = \partial + \Gamma$,

$$(\widehat{\varphi}^* \nabla) := \partial + \widehat{\varphi}^* \Gamma, \text{ where } (\widehat{\varphi}^* \Gamma)^\beta_{i\mu} := (\partial_i \varphi^j) \Gamma^\beta_{j\mu}. \tag{3.7}$$

The curvature of $\widehat{\varphi}^* \nabla$ is

$$F_{\widehat{\varphi}^* \nabla} = (\partial_j \varphi^k) \left(\partial_i \Gamma^\beta_{k\mu} \right) - (\partial_i \varphi^k) \left(\partial_j \Gamma^\beta_{k\mu} \right) + (\partial_i \varphi^k)(\partial_j \varphi^l) \Gamma^\beta_{k\zeta} \Gamma^\zeta_{l\mu} - (\partial_j \varphi^k)(\partial_i \varphi^l) \Gamma^\beta_{k\zeta} \Gamma^\zeta_{l\mu}. \tag{3.8}$$

A Coulomb-type projection

We will be considering a specific type of Coulomb gauge that is 'centered' with respect to $\widetilde{\nabla}$. This particular transformation is key to our argument and naturally interacts with the enlarged gauge group.

Definition 3.1.7. A gauge transformation $\varsigma \in S\left(\mathcal{G}_E\right)$ puts $\nabla \in \mathcal{A}_E\left(\mathbb{S}^4\right)$ in *(global)* $\widetilde{\nabla}$-*Coulomb gauge* if

$$D_{\widetilde{\nabla}}^*\left[\varsigma\left[\nabla\right] - \widetilde{\nabla}\right] \equiv 0. \tag{3.9}$$

Call ς satisfying (3.9) the $\widetilde{\nabla}$-*Coulomb gauge transformation (associated to ∇)*. Define the $\widetilde{\nabla}$-*Coulomb gauge projection* by

$$\widetilde{\Pi} : \mathcal{A}_E\left(\mathbb{S}^4\right) \to \mathcal{A}_E\left(\mathbb{S}^4\right)$$

$$: \nabla \mapsto \varsigma\left[\nabla\right].$$

The action of these $\widetilde{\nabla}$-Coulomb gauges commutes with that of conformal automorphisms.

Proposition 3.1.8. *For any $\varphi \in \mathrm{SO}(5,1)$, one has $\widetilde{\Pi}\left[\widehat{\varphi}^*\nabla\right] \equiv \widehat{\varphi}^*\widetilde{\Pi}\left[\nabla\right]$.*

Proof. This result can be confirmed by considering dilations and rotations individually since every element of $\mathrm{SO}(5,1)$ is the composition of these two types. The first follows immediately from natural commutation of dilations with gauge transformations (cf. (3.7)), and the second from the rotational invariance of $\widetilde{\nabla}$, which we describe now. By the definition of our $\widetilde{\nabla}$-Coulomb projection, and application of appropriate pullback gauge transformations, given a rotation $\varrho : \mathbb{S}^4 \to \mathbb{S}^4$,

$$0 = D_{\widetilde{\nabla}}^*\left[\widehat{\varrho}^*\nabla - \widetilde{\nabla}\right] = \varrho_* D_{\left(\widehat{\varrho^{-1}}\right)^*\widetilde{\nabla}}^*\left[\nabla - \left(\widehat{\varrho^{-1}}\right)^*\widetilde{\nabla}\right] = \varrho_* D_{\widetilde{\nabla}}^*\left[\nabla - \widetilde{\nabla}\right].$$

Thus, the $\widetilde{\nabla}$-Coulomb gauge for $\widehat{\varrho}^*\nabla$ coincides with the $\widetilde{\nabla}$-Coulomb gauge for ∇. $\qquad\square$

Behavior of energies

We explore the behavior of energy quantities undergoing conformal dilation and propose a modified energy which distinguishes the dilations' effects. When considering the dilation $\zeta \mapsto \lambda\zeta$ *we abuse notation by*

referring to the dilation map as λ. Using (3.8),

$$\left|F_{\widehat{\lambda}^*\nabla}(\zeta)\right|^2_{\mathring{g}} = \lambda^4 \tfrac{1}{16} \left|F_\nabla(\zeta)\right|^2 \left(|\zeta|^2+1\right)^4,$$

$$|F_\nabla(\lambda\zeta)|^2_{\mathring{g}} = \tfrac{1}{16}\left(|\lambda\zeta|^2+1\right)^4 |F_\nabla(\lambda\zeta)|^2.$$

Combining these yields

$$\left|F_{\widehat{\lambda}^*\nabla}(\zeta)\right|^2_{\mathring{g}} = \lambda^4 \left(\tfrac{|\zeta|^2+1}{1+|\lambda\zeta|^2}\right)^4 |F_\nabla(\lambda\zeta)|^2_{\mathring{g}}.$$

Then setting

$$\chi_\lambda(\zeta) := \tfrac{1}{\lambda^4}\left(\tfrac{1+|\lambda\zeta|^2}{|\zeta|^2+1}\right)^4, \tag{3.10}$$

it follows that for $\lambda > 0$,

$$\begin{aligned}
\mathcal{YM}_\alpha(\nabla) &= \tfrac{1}{2}\int_{\mathbb{S}^4}\left(3+|F_\nabla(\zeta)|^2_{\mathring{g}}\right)^\alpha dV_{\mathring{g}}(\zeta)\\
&= \tfrac{1}{2}\int_{\mathbb{S}^4}\left(3+|F_\nabla(\lambda\zeta)|^2_{\mathring{g}}\right)^\alpha dV_{\mathring{g}}(\lambda\zeta)\\
&= \tfrac{3^\alpha}{2}\int_{\mathbb{H}^1}\left(1+\tfrac{1}{3}\chi_\lambda(\zeta)\left|F_{\widehat{\lambda}^*\nabla}(\zeta)\right|^2_{\mathring{g}}\right)^\alpha \tfrac{16\lambda^4}{\left(1+|\lambda\zeta|^2\right)^4}\,dV\\
&= \tfrac{3^\alpha}{2}\int_{\mathbb{H}^1}\left(1+\tfrac{1}{3}\chi_\lambda(\zeta)\left|F_{\widehat{\lambda}^*\nabla}(\zeta)\right|^2_{\mathring{g}}\right)^\alpha \tfrac{16}{\left(1+|\zeta|^2\right)^4\chi_\lambda(\zeta)}\,dV.
\end{aligned}$$

Set

$$\mathcal{YM}_{\alpha,\lambda}(\nabla) := \tfrac{1}{2}\int_{\mathbb{S}^4}\left(3+\chi_\lambda|F_\nabla|^2_{\mathring{g}}\right)^\alpha \tfrac{1}{\chi_\lambda}\,dV_{\mathring{g}}, \tag{3.11}$$

resulting in a natural relationship and symmetry, where for $\varphi \in \mathrm{SO}(5,1)$ with $|\varphi| = \lambda$,

$$\mathcal{YM}_\alpha(\nabla) = \mathcal{YM}_{\alpha,\lambda}\left(\widehat{\varphi}^*\nabla\right) = \mathcal{YM}_{\alpha,\lambda^{-1}}\left(\left(\widehat{\varphi^{-1}}\right)^*\nabla\right). \tag{3.12}$$

Thus ∇ is a critical point of $\mathcal{YM}_\alpha$ if and only if $\widehat{\lambda}^*\nabla$ is a critical point of $\mathcal{YM}_{\alpha,\lambda}$. By utilizing this symmetry of $\mathcal{YM}_{\alpha,\lambda}$ with respect to λ about 1, it suffices to consider $\lambda \geq 1$.

Remark 3.1.9. Henceforth we refer to $\blacktriangledown$ when discussing critical points of $\mathcal{YM}_{\alpha,\lambda}$ to help notify the reader that these connections are λ-dilations of α-critical points (rather than α-critical points).

Proposition 3.1.10. *With χ_λ is as in (3.10), the gradient equation of $\mathcal{YM}_{\alpha,\lambda}$ is*

$$(\mathrm{Grad}_{\blacktriangledown}\,\mathcal{YM}_{\alpha,\lambda}) = D^*_{\blacktriangledown}F_{\blacktriangledown} + \Theta_1(\blacktriangledown) + \Theta_2(\blacktriangledown),$$

where

$$
\begin{cases}
(\Theta_1(\blacktriangledown))^{\beta}_{i\mu} & := -\frac{2\chi_\lambda(\alpha-1)}{(3+\chi_\lambda|F_{\blacktriangledown}|^2_{\mathring{g}})}\left(\blacktriangledown_j F^{\delta}_{pq\zeta}\right)(F_{\blacktriangledown})^{\zeta}_{pq\delta}(F_{\blacktriangledown})^{\beta}_{ij\mu}, \\[2mm]
(\Theta_2(\blacktriangledown))^{\beta}_{i\mu} & := \frac{\chi_\lambda(\alpha-1)}{(3+\chi_\lambda|F_{\blacktriangledown}|^2_{\mathring{g}})}\left(\blacktriangledown_j\log\chi_\lambda\right)|F_{\blacktriangledown}|^2_{\mathring{g}}(F_{\blacktriangledown})^{\beta}_{ij\mu}.
\end{cases}
\tag{3.13}
$$

Proof. Let $\blacktriangledown_t$ be a one parameter family of smooth connections, and consider

$$
\begin{aligned}
\frac{\partial}{\partial t}\left[\mathcal{YM}_{\alpha,\lambda}(\blacktriangledown_t)\right] &= \frac{1}{2}\int_{\mathbb{S}^4}\frac{\partial}{\partial t}\left[\left(3+\chi_\lambda|F_{\blacktriangledown_t}|^2_{\mathring{g}}\right)^{\alpha}\right]\frac{1}{\chi_\lambda}dV_{\mathring{g}} \\
&= \frac{1}{2}\int_{\mathbb{S}^4}\alpha\left(3+\chi_\lambda|F_{\blacktriangledown_t}|^2_{\mathring{g}}\right)^{\alpha-1}\frac{\partial}{\partial t}\left[|F_{\blacktriangledown_t}|^2_{\mathring{g}}\right]dV_{\mathring{g}} \\
&= \int_{\mathbb{S}^4}\alpha\left(3+\chi_\lambda|F_{\blacktriangledown_t}|^2_{\mathring{g}}\right)^{\alpha-1}\left\langle D_{\blacktriangledown_t}\left[\tfrac{\partial\blacktriangledown_t}{\partial t}\right],F_{\blacktriangledown_t}\right\rangle_{\mathring{g}}dV_{\mathring{g}} \\
&= 2\int_{\mathbb{S}^4}\alpha\left(3+\chi_\lambda|F_{\blacktriangledown_t}|^2_{\mathring{g}}\right)^{\alpha-1}\left\langle\blacktriangledown_t\left[\tfrac{\partial\blacktriangledown_t}{\partial t}\right],F_{\blacktriangledown_t}\right\rangle_{\mathring{g}}dV_{\mathring{g}} \\
&\quad + 2\int_{\mathbb{S}^4}\alpha(\alpha-1)\left(3+\chi_\lambda|F_{\blacktriangledown_t}|^2_{\mathring{g}}\right)^{\alpha-2}\blacktriangledown_i\left[\chi_\lambda|F_{\blacktriangledown_t}|^2_{\mathring{g}}\right]\mathrm{tr}_h\left[(F_{\blacktriangledown_t})_{ij}(\tfrac{\partial\blacktriangledown_t}{\partial t})_j\right]dV_{\mathring{g}}.
\end{aligned}
$$

The result follows. $\qquad\square$

3.2 General closeness to $\mathrm{SO}(5,1)$ pullbacks

Initially we will prove a general result on connections with sufficiently controlled $\mathcal{YM}_\alpha$ energy (not necessarily α-critical) which asserts existence of and characterizes the magnitude of a conformal automorphism required to 'pull' the connection sufficiently close to $\widetilde{\triangledown}$.

Proposition 3.2.1. *There exists some $\delta_0 > 0$ so that for any $\delta \in (0,\delta_0)$, there exists some $\epsilon > 0$ so that if $\alpha \in [1,2]$ and $\mathcal{YM}_\alpha(\triangledown) \leq 6^\alpha\frac{4}{3}\pi^2 + \epsilon$, then there exists $\varphi \in \mathrm{SO}(5,1)$ such that*

$$\left\|\widetilde{\Pi}\left[\widehat{\varphi}^*\triangledown\right] - \widetilde{\triangledown}\right\|_{W^{1,2}} + \left\|F_{\widetilde{\Pi}[\widehat{\varphi}^*\triangledown]} - F_{\widetilde{\triangledown}}\right\|_{L^2} \leq \delta, \tag{3.14}$$

furthermore there is some fixed constant $C > 0$ such that if $\lambda \geq 1$ is the dilation factor of φ, then

$$(\alpha - 1)(\log \lambda)\min\{\log \lambda, 1\} \leq C\delta. \tag{3.15}$$

Remark 3.2.2. The proof of this result relies on the following

- Lemma 3.2.3. The smallness in terms of curvature difference in (3.14) is shown.

- Theorem 3.2.6. A Poincaré inequality (Proposition 3.2.5) is used to establish a property of $\widetilde{\nabla}$-Coulomb gauges with an eye toward concluding the complete smallness of (3.14).

- Lemma 3.2.7. The Yang-Mills α-energy of an arbitrary connection is compared to that of $\widetilde{\nabla}$.

- Lemma 3.2.8. The gap between $\mathcal{YM}_{\alpha,\lambda}$ energy and $\mathcal{YM}_\alpha$ is characterized on $\widetilde{\nabla}$ as a function of α and λ.

The proof of Proposition 3.2.1 tying together these results will be concluded at end of this section.

This first lemma proves a result similar to Proposition 3.2.1 above, though here we will fix one value of δ rather than a range of values. It requires a concentration compactness result for α-connections (Theorem 3.9.4) which follows quickly from results of [HTY15]. We state this result the appendix and supply a sketch of the proof (cf. Theorem 3.9.4).

Lemma 3.2.3. *Given $\delta > 0$ there exists $\epsilon > 0$ sufficiently small with the following property: for all $\alpha \geq 1$, if $\nabla \in W^{1,2\alpha}(\mathcal{A}_E(\mathbb{S}^4))$ and $\mathcal{YM}_\alpha \leq 6^\alpha \frac{4}{3}\pi^2 + \epsilon$, then there exists $\varphi \in \mathrm{SO}(5,1)$ so that*

$$\left\|F_{\widetilde{\Pi}[\widehat{\varphi}^*\nabla]} - F_{\widetilde{\nabla}}\right\|_{L^2} \leq \delta. \tag{3.16}$$

Proof. Suppose $\mathcal{YM}_\alpha(\nabla) \leq 6^\alpha \frac{4}{3}\pi^2 + \epsilon$. As a consequence of (3.4) we have that

$$\begin{aligned}
\mathcal{YM}_1(\nabla) &= \tfrac{1}{2}\int_{\mathbb{S}^4}\left(3 + |F_\nabla|_{\mathring{g}}^2\right)dV_{\mathring{g}} \\
&\leq \left(\mathcal{YM}_\alpha(\nabla)\left(\tfrac{4\pi^2}{3}\right)^{(\alpha-1)}\right)^{1/\alpha} \\
&\leq \left(\left(1 + \tfrac{\epsilon}{6^\alpha \frac{4}{3}\pi^2}\right)6^\alpha \tfrac{4}{3}\pi^2 \left(\tfrac{4\pi^2}{3}\right)^{(\alpha-1)}\right)^{1/\alpha} \\
&= \left(\left(1 + \tfrac{\epsilon}{6^\alpha \frac{4}{3}\pi^2}\right)6^\alpha \left(\tfrac{4\pi^2}{3}\right)^\alpha\right)^{1/\alpha} \\
&\leq 8\pi^2 + \epsilon.
\end{aligned}$$

Suppose to the contrary that the statement were not true. This would imply that the Coulomb gauge may not actually exist (as in Theorem 3.2.6) and thus no gauge transformation could get the curvature 'close' to that of $\widetilde{\nabla}$. More precisely, if the contrary statement holds, then there is a sequence $\epsilon_n \searrow 0$, and a sequence $\{\nabla^n\} \subset W^{1,2}(\mathcal{A}_E(\mathbb{S}^4))$ with $\mathcal{YM}_1(\nabla^n) \leq 6^\alpha \frac{4}{3}\pi^2 + \epsilon_n$, some $\delta > 0$ so that

$$\left\|F_{\sigma_n[\widehat{\varphi}_n^* \nabla^n]} - F_{\widetilde{\nabla}}\right\|_{L^2} > \delta \text{ for all } \{\varphi_n\} \subset \mathrm{SO}(5,1), \{\sigma_n\} \subset S(\mathcal{G}_E). \tag{3.17}$$

For each $\{\nabla^n\}$, let $\xi \in \mathbb{S}^4$ be a point such that $|F_{\nabla^n}(\xi)| = \sup_{\zeta \in \mathbb{S}^4} |F_{\nabla^n}|$. For each n there exists a conformal automorphism $\widehat{\varphi}_n$ which consists of a translation of ξ to the north pole $N \in \mathbb{S}^4$, combined with a dilation so that $\left|F_{\widehat{\varphi}_n^* \nabla^n}(N)\right| = 3$. Via the Removable Singularities Theorem of [Uhl82b], we can argue via a standard gauge patching argument that there exists a sequence of gauge transformations $\{\sigma_n\}$ such that the sequences $\{\sigma_n[\widehat{\varphi}_n^* \nabla^n]\}$ converges to some connection ∇, and furthermore due to the assumed energy bounds of these quantities we know that ∇ is antiself dual (to see this in more detail, see our argument in Theorem 3.9.4). This implies, in particular that $\nabla = \widehat{\psi}^* \widetilde{\nabla}$ for $\psi \in \mathrm{SO}(5,1)$. Thus

$$\left(\widehat{\psi}^{-1}\right)^* \sigma_n [\widehat{\varphi}_n^* \nabla^n] \to \widetilde{\nabla}.$$

We note in particular the enlarged gauge group $\widehat{\mathcal{G}}_E$ acts naturally on $\mathcal{G}_E$ by conjugation, giving

$$\left(\widehat{\psi^{-1}}\right)^* \sigma_n [\widehat{\varphi}^* \nabla^n] = \left(\widehat{\psi}^{-1} \sigma_n \widehat{\psi}\right) \left[\left(\widehat{\psi^{-1}\varphi}\right)^* \nabla^n\right].$$

Indeed, this is of the form of a gauge transformation applied to a pullback. Therefore there exists $\{\nu_n\} \subset S(\mathcal{G}_E)$ and $\{\phi_n\} \subset \mathrm{SO}(5,1)$ so that

$$\left\|F_{\nu_n[\widehat{\phi}_n^*[\nabla^n]]} - F_{\widetilde{\nabla}}\right\|_{L^2} \to 0.$$

In turn, as a result of Theorem 3.2.6,

$$\left\|F_{\widetilde{\Pi}[\widehat{\phi}_n^* \nabla^n]} - F_{\widetilde{\nabla}}\right\|_{L^2} \to 0,$$

which contradictions (3.17), and so the result follows. □

Next we will show the implications of this smallness of L^2 in terms of curvature difference directly on the norm of connection difference itself. This is a proof in the spirit of Theorem 1.3 of [Uhl82b], where the

fundamental difference is that it is given on a *global scale* and bounds the difference of connections in terms of the difference of curvatures. As of now, there is no such proof present in the literature. This will be highly necessary in the following section. To do so we first establish a fundamental Poincaré inequality. The proof, which is stated in the appendix (§3.9.3), relies on the following result (Lemma 3.2.4), which gives crucial control over commutator type terms which is used regularly in our arguments. The result was inspired by Lemma 2.30 of [BL81], and the proof is included in the appendix (cf. §3.9.3). Combining these estimates with the geometry of the setup at play is a crucial technique used multiple times through our remaining arguments within this paper.

Lemma 3.2.4. *Let* $A \in \Lambda^1 (\operatorname{Ad} E)$ *and* $B \in \left(\Lambda^1 (\operatorname{Ad} E) \right)^{\otimes 2}$. *Then*

$$\left\langle \widetilde{F}_{ij}, [A_i, A_j] \right\rangle_{\mathring{g}} \leq |A|_{\mathring{g}}^2, \;\; and \;\; \left\langle \widetilde{F}_{ij}, [B_{ki}, B_{kj}] \right\rangle_{\mathring{g}} \leq 4 \, |B|_{\mathring{g}}^2 . \tag{3.18}$$

One of the many consequences of this result is the following

Proposition 3.2.5 (Global $\widetilde{\nabla}$-Poincaré inequalities). *Given* $A \in \Lambda^1 (\operatorname{Ad} E)$ *there exists* $C_P > 0$ *such that*

$$||A||_{L^2}^2 \leq C_P \left|\left| \widetilde{\nabla} A \right|\right|_{L^2} \;\; and \;\; \left|\left| \widetilde{\nabla} A \right|\right|_{L^2}^2 \leq C_P \left|\left| \widetilde{\nabla}^{(2)} A \right|\right|_{L^2} . \tag{3.19}$$

A proof of a localized version of these Poincaré inequalities is given in the appendix, with a simple gluing argument one can construct this global argument over $\mathbb{S}^4$ (a rough constant will do for our purposes). With this we characterize a norm-controlling behavior of our $\widetilde{\nabla}$-Coulomb gauge.

Theorem 3.2.6. *If* $\delta \in (0, 1)$ *is sufficiently small, then every connection* ∇ *such that there exists a* $\varsigma \in \mathcal{G}_E$ *so that*

$$\left|\left| F_{\varsigma[\nabla]} - F_{\widetilde{\nabla}} \right|\right|_{L^2} \leq \delta \tag{3.20}$$

in fact admits a $\widetilde{\nabla}$-*Coulomb projection* $\widetilde{\Pi} [\nabla]$ *which obeys the bounds*

$$\left|\left| \widetilde{\Pi} [\nabla] - \widetilde{\nabla} \right|\right|_{L^2} \leq C \left|\left| F_{\widetilde{\Pi}[\nabla]} - F_{\widetilde{\nabla}} \right|\right|_{L^2}, \;\;\; \left|\left| F_{\widetilde{\Pi}[\nabla]} - F_{\widetilde{\nabla}} \right|\right|_{L^2} \leq \delta . \tag{3.21}$$

Through the next two lemmata we will put a bound on the dilation factor λ of φ in Lemma 3.2.1. To obtain it we will use the fact that due to the closeness characterized in (3.16) we expect that $\mathcal{YM}_{\alpha,\lambda}(\varphi^* \nabla)$ should

be close to $\mathcal{YM}_{\alpha,\lambda}(\widetilde{\nabla})$. We will compute explicitly how $\mathcal{YM}_{\alpha,\lambda}(\widetilde{\nabla})$ grows with λ (that is, how the α-energy of pullbacks of the basic connection grow).

Lemma 3.2.7. *If $\lambda \geq 1$ and $\alpha \in [1,2]$*

$$\mathcal{YM}_{\alpha,\lambda}(\blacktriangledown) - \mathcal{YM}_{\alpha,\lambda}(\widetilde{\nabla}) \geq -\alpha \left(\frac{3^{\alpha-1}}{2} \right) \left(1 + \lambda^4 \right)^{\alpha-1} \left\| |F_{\blacktriangledown}|_{\mathring{g}}^2 - 3 \right\|_{L^1}. \tag{3.22}$$

Proof. As a result of the mean value theorem, there exists some positive function $f : \mathbb{S}^4 \to [0, \infty)$ whose value at $\xi \in \mathbb{S}^4$ lies between $|F_{\nabla}|_{\mathring{g}}^2$ and $3 \equiv |F_{\widetilde{\nabla}}|_{\mathring{g}}^2$ and satisfies

$$\mathcal{YM}_{\alpha,\lambda}(\blacktriangledown) - \mathcal{YM}_{\alpha,\lambda}(\widetilde{\nabla}) = \frac{\alpha}{2} \int_{\mathbb{S}^4} (3 + \chi_\lambda f)^{\alpha-1} \left(|F_{\blacktriangledown}|_{\mathring{g}}^2 - 3 \right) dV_{\mathring{g}}. \tag{3.23}$$

Now, set

$$A_+ := \left\{ \zeta \in \mathbb{S}^4 : |F_{\blacktriangledown}(\zeta)|_{\mathring{g}}^2 \geq 3 \right\}, \quad A_- := \left\{ \zeta \in \mathbb{S}^4 : |F_{\blacktriangledown}(\zeta)|_{\mathring{g}}^2 < 3 \right\}.$$

Then $f \geq 3$ on A_+, and $f \leq 3$ on A_-. Considering integration on these individual sets we have

$$\int_{A_\pm} (3 + \chi_\lambda f)^{\alpha-1} \left(|F_{\blacktriangledown}|_{\mathring{g}}^2 - 3 \right) dV_{\mathring{g}} \geq 3^{\alpha-1} \int_{A_\pm} (1 + \chi_\lambda)^{\alpha-1} \left(|F_{\blacktriangledown}|_{\mathring{g}}^2 - 3 \right) dV_{\mathring{g}}.$$

It follows that over the entire region

$$\int_{\mathbb{S}^4} (3 + \chi_\lambda f)^{\alpha-1} \left(|F_{\blacktriangledown}|_{\mathring{g}}^2 - 3 \right) dV_{\mathring{g}} \geq 3^{\alpha-1} \int_{\mathbb{S}^4} (1 + \chi_\lambda)^{\alpha-1} \left(|F_{\blacktriangledown}|_{\mathring{g}}^2 - 3 \right) dV_{\mathring{g}}. \tag{3.24}$$

Now since $\sup_{\mathbb{S}^4} \chi_\lambda = \lambda^4$,

$$\left| \int_{\mathbb{S}^4} (1 + \chi_\lambda)^{\alpha-1} (\alpha - 1) \left(|F_{\blacktriangledown}|_{\mathring{g}}^2 - 3 \right) dV_{\mathring{g}} \right| \leq \left(1 + \lambda^4 \right)^{\alpha-1} \left\| |F_{\blacktriangledown}|_{\mathring{g}}^2 - 3 \right\|_{L^1}. \tag{3.25}$$

Combining the estimates, (3.23), (3.24), (3.25) we obtain the result (3.22). $\qquad \square$

Lemma 3.2.8. *One has*

$$\mathcal{YM}_{\alpha,\lambda}\left(\widetilde{\nabla} \right) = \mathcal{YM}_\alpha \left(\left(\widehat{\lambda^{-1}} \right)^* \widetilde{\nabla} \right) = \mathcal{YM}_\alpha \left(\widehat{\lambda}^* \widetilde{\nabla} \right). \tag{3.26}$$

By setting

$$\mho(\alpha, \lambda) := \mathcal{YM}_\alpha\left(\widehat{\lambda}^*\widetilde{\nabla}\right) - 6^\alpha \tfrac{4}{3}\pi^2,$$

there exists a fixed constant $C > 0$ such that for $\alpha \in (1, 2]$,

$$\mho(\alpha, \lambda) \geq \begin{cases} C\lambda^{4\alpha-4} & \text{if } (\alpha-1)\log\lambda \geq 5 \\ C(\alpha-1)\log\lambda & \text{if } (\alpha-1) \leq (\alpha-1)\log\lambda \leq 5 \\ C(\alpha-1)(\log\lambda)^2 & \text{if } 0 \leq \log\lambda \leq 1. \end{cases} \tag{3.27}$$

Furthermore $\mathcal{YM}_\alpha\left(\widehat{\lambda}^*\nabla\right)$ is increasing in λ and for $0 \leq (\alpha-1)\log\lambda \leq 2$,

$$\frac{\partial}{\partial\log\lambda}\left[\mathcal{YM}_\alpha(\widehat{\lambda}^*\widetilde{\nabla})\right] = \frac{\partial}{\partial\log\lambda}\left[\mathcal{YM}_{\alpha,\lambda}(\widetilde{\nabla})\right] \geq C(\alpha-1)\frac{|\log\lambda|_{\mathring{g}}}{1+|\log\lambda|_{\mathring{g}}}. \tag{3.28}$$

Proof. First observe that

$$\mathcal{YM}_{\alpha,\lambda}(\widetilde{\nabla}) := \frac{3^\alpha}{2}\int_{\mathbb{S}^4}(1+\chi_\lambda)^\alpha\frac{1}{\chi_\lambda}\,dV_{\mathring{g}}. \tag{3.29}$$

We will apply change of variables to the integrand of $\mathcal{YM}_\alpha(\widehat{\lambda}^*\widetilde{\nabla})$. First we convert (3.29) to a radial integral; let $r := |\zeta|$ and then, akin to the computations in §3.1.1,

$$\left|F_{\widehat{\lambda}^*\widetilde{\nabla}}\right|_{\mathring{g}}^2 = \frac{\lambda^4}{16}\left|F_{\widetilde{\nabla}}(\lambda\zeta)\right|^2\left(|\zeta|^2+1\right)^4 = \frac{3\lambda^4\left(|\zeta|^2+1\right)^4}{\left(1+|\lambda\zeta|^2\right)^4} = \frac{3}{\chi_\lambda(\zeta)},$$

and thus

$$\mathcal{YM}_\alpha\left(\widehat{\lambda}^*\widetilde{\nabla}\right) = \frac{3^\alpha}{2}\frac{8\pi^2}{3}\int_0^\infty\left(1 + \frac{\lambda^4\left(r^2+1\right)^4}{(1+\lambda^2r^2)^4}\right)^\alpha\frac{r^3}{(1+r^2)^4}\,dr. \tag{3.30}$$

We will next change variables, gathering up '$(\chi_\lambda)^{1/4}$' type quantities:

$$\begin{cases} w & := \lambda\frac{1+r^2}{1+\lambda^2r^2} \\ dw & := 2r\lambda\left(\frac{1-\lambda^2}{(1+\lambda^2r^2)^2}\right)dr. \end{cases}$$

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Observing that $r^2 = \frac{(\lambda - w)}{\lambda(\lambda w - 1)}$, we update (3.30) and obtain

$$\frac{2\pi^2 3^{\alpha-1}}{(\lambda - \lambda^{-1})^3} \int_{1/\lambda}^{\lambda} (1 + w^4)^\alpha \frac{(\lambda - w)(w - \lambda^{-1})}{w^4} \, dw. \tag{3.31}$$

It is now clear that the symmetry between λ and $\frac{1}{\lambda}$ is preserved since

$$\mathcal{YM}_\alpha\left(\widehat{(\lambda^{-1})}^* \widetilde{\nabla}\right) = \mathcal{YM}_\alpha\left(\widehat{\lambda}^* \widetilde{\nabla}\right).$$

Combining this with (3.12) yields (3.26). We apply the change of variables $w = e^t$ and $\lambda = e^\tau$ to (3.31),

$$\frac{2\pi^2 3^{\alpha-1}}{(e^\tau - e^{-\tau})^3} \int_{-\tau}^{\tau} (1 + e^{4t})^\alpha \frac{e^t (e^\tau - e^t)(e^t - e^{-\tau})}{e^{4t}} \, dt$$

$$= \frac{\pi^2 3^{\alpha-1}}{4(\sinh\tau)^3} \int_{-\tau}^{\tau} (e^{-2t} + e^{2t})^\alpha \frac{e^{2\alpha t}(e^\tau - e^t)(e^t - e^{-\tau})}{e^{3t}} \, dt$$

$$= \frac{\pi^2 2^\alpha 3^{\alpha-1}}{4(\sinh\tau)^3} \int_{-\tau}^{\tau} (\cosh 2t)^\alpha e^{(\alpha-1)2t} \frac{(e^\tau - e^t)(e^t - e^{-\tau})}{e^t} \, dt$$

$$= \frac{\pi^2 2^\alpha 3^{\alpha-1}}{4(\sinh\tau)^3} \int_{-\tau}^{\tau} (\cosh 2t)^\alpha e^{(\alpha-1)2t} \underbrace{\frac{(e^\tau - e^t)(e^t - e^{-\tau})}{e^t}}_{} \, dt.$$

We compute out the underbraced term

$$\frac{(e^\tau - e^t)(e^t - e^{-\tau})}{e^t} = e^{-t}\left(e^{t+\tau} - 1 - e^{2t} + e^{t-\tau}\right) = 2\left(\cosh\tau - \cosh t\right),$$

and observe the following symmetry identity

$$\int_0^\tau (\cosh 2t)^\alpha \, e^{(\alpha-1)2t} \left(\cosh\tau - \cosh t\right) \, dt = \int_{-\tau}^0 (\cosh 2(-t))^\alpha \, e^{(\alpha-1)2(-t)} \left(\cosh\tau - \cosh(-t)\right) \, dt.$$

Applying this symmetry yields

$$\mathcal{YM}_\alpha(\widehat{\lambda}^* \widetilde{\nabla}) = \frac{\pi^2 2^\alpha 3^{\alpha-1}}{(\sinh\tau)^3} \int_0^\tau (\cosh 2t)^\alpha \cosh[(\alpha-1)2t] \left(\cosh\tau - \cosh t\right) \, dt.$$

The idea of the next portion of the proof is to provide lower bounds depending on λ and α for the gap function $\mho$ measuring the difference of the α-energy of $\lambda^* \widetilde{\nabla}$ from the α-energy of $\widetilde{\nabla}$. We set

$$\mathcal{YM}_\alpha\left(\left(\widehat{e^\tau}\right)^* \widetilde{\nabla}\right) := 6^\alpha \tfrac{4}{3} \pi^2 G(\sigma). \tag{3.32}$$

We will solve for G, and apply the change of variables

$$\begin{cases} \beta & := (\alpha - 1), \\ s & := \beta t, \\ \sigma & := \beta \tau = (\alpha - 1) \log \lambda. \end{cases}$$

Now, applying the change of variables (denoted 'c.o.v' below),

$$G(\sigma) = \tfrac{1}{4(\sinh \tau)^3} \int_0^\tau (\cosh 2t)^\alpha \cosh\left[2(\alpha - 1)t\right] (\cosh \tau - \cosh t)\, dt$$

$$\overset{c.o.v}{\mapsto} \tfrac{1}{4\beta\left(\sinh \frac{\sigma}{\beta}\right)^3} \int_0^\sigma \left(\cosh \tfrac{2s}{\beta}\right)^{\beta+1} (\cosh 2s)\left(\cosh \tfrac{\sigma}{\beta} - \cosh \tfrac{s}{\beta}\right)\, ds. \tag{3.33}$$

With this new formulation we address the various cases of (3.27).

$\boxed{(\alpha - 1)\log \lambda \geq 5}$ Noting that

$$\tfrac{\partial}{\partial s}\left[\left(\sinh \tfrac{s}{\beta}\right)^3\right] = \tfrac{3}{\beta}\left(\sinh \tfrac{s}{\beta}\right)^2 \cosh \tfrac{s}{\beta},$$

we rewrite (3.33) (and decrease the region of integration) as follows

$$G(\sigma) \geq \tfrac{1}{12\left(\sinh \frac{\sigma}{\beta}\right)^3} \int_{\sigma-2}^{\sigma-1} \tfrac{\partial}{\partial s}\left[\left(\sinh \tfrac{s}{\beta}\right)^3\right] \left(\cosh \tfrac{2s}{\beta}\right)^{\beta+1} \underbrace{\cosh 2s \tfrac{\left(\cosh \frac{\sigma}{\beta} - \cosh \frac{s}{\beta}\right)}{\cosh \frac{s}{\beta}\left(\sinh \frac{s}{\beta}\right)^2}}\, ds.$$

We estimate the underbraced part, which we call $Q(s)$ on the interval $[\sigma - 1, \sigma]$,

$$Q(s) = \cosh 2s \left(\sinh \tfrac{s}{\beta}\right)^{-2} \left(\cosh \tfrac{2s}{\beta}\right)^{\beta+1} \left(\tfrac{\cosh \frac{\sigma}{\beta}}{\cosh \frac{s}{\beta}} - 1\right)$$

$$= [\cosh 2s]_{T_1} \left[\cosh \tfrac{2s}{\beta}\left(\sinh \tfrac{s}{\beta}\right)^{-2}\right]_{T_2} \left[\left(\cosh \tfrac{2s}{\beta}\right)^\beta\right]_{T_3} \left[\left(\tfrac{\cosh \frac{\sigma}{\beta}}{\cosh \frac{s}{\beta}} - 1\right)\right]_{T_4}.$$

For the T_1, we estimate that

$$T_1 \geq \left[\tfrac{e^{2s}}{2}\right]_{\sigma-2}^{\sigma-1} \geq \tfrac{e^{2(\sigma-2)}}{2} = \tfrac{e^{2\sigma}}{2e^4}.$$

For the T_2, using the hyperbolic sine additive angle identity,

$$T_2 \geq \left(\sinh \tfrac{2s}{\beta}\right)\left(\sinh \tfrac{s}{\beta}\right)^{-2} = 2.$$

For T_3 we have that

$$T_3 \geq \left(\frac{e^{\frac{2s}{\beta}}}{2}\right)^\beta = \left[\frac{e^{2s}}{2^\beta}\right]\Big|_{\sigma-2}^{\sigma-1} = \frac{e^{2(\sigma-2)+2\frac{(\sigma-2)}{\beta}}}{2^\beta} \geq \frac{e^{2(\sigma-2)}}{2^\beta} = \frac{e^{2\sigma}}{2^\beta e^4}.$$

For T_4, we estimate

$$\frac{\cosh\frac{\sigma}{\beta}}{\cosh\frac{s}{\beta}} \geq \frac{e^{\frac{\sigma}{\beta}}+e^{-\frac{\sigma}{\beta}}}{e^{\frac{\sigma-1}{\beta}}+e^{-\frac{\sigma-1}{\beta}}} \geq \frac{e^{\frac{\sigma}{\beta}}+e^{-\frac{\sigma}{\beta}}}{2e^{\frac{\sigma-1}{\beta}}} \geq \frac{1}{2}\left(e^{\frac{1}{\beta}}+e^{\frac{-2\beta+1}{\beta}}\right) \geq \frac{1}{2}e^{\frac{1}{\beta}} \geq \frac{e}{2},$$

and so we conclude that

$$Q(s)\big|_{\sigma-2}^{\sigma-1} = \Pi_{i=1}^4 T_i\big|_{\sigma-2}^{\sigma-1} = \frac{e^{4\sigma}}{2^\alpha e^8}\left(e-2\right).$$

Now, we compute, noting that for $x > y > 0$, we have $x^3 - y^3 \geq (x-y)^3$, giving

$$\frac{1}{\left(\sinh\frac{\sigma}{\beta}\right)^3} \int_{\sigma-2}^{\sigma-1} \frac{\partial}{\partial s}\left[\left(\sinh\frac{s}{\beta}\right)^3\right] ds = \frac{1}{\left(\sinh\frac{\sigma}{\beta}\right)^3}\left(\left(\sinh\frac{(\sigma-1)}{\beta}\right)^3 - \left(\sinh\frac{(\sigma-2)}{\beta}\right)^3\right)$$

$$\geq \frac{1}{\left(\sinh\frac{\sigma}{\beta}\right)^3}\left(\sinh\frac{(\sigma-1)}{\beta} - \sinh\frac{(\sigma-2)}{\beta}\right)^3$$

$$= \left(\frac{e^{\frac{\sigma-1}{\beta}}-e^{-\frac{\sigma-1}{\beta}}-e^{\frac{\sigma-2}{\beta}}+e^{-\frac{\sigma-2}{\beta}}}{e^{\frac{\sigma}{\beta}}-e^{-\frac{\sigma}{\beta}}}\right)^3$$

$$= \left(e^{\frac{-1}{\beta}}-e^{\frac{-2}{\beta}}\right)^3$$

$$\geq \frac{(e-1)^3}{e^6}.$$

Note that the last line follows from the monotonicity of $f(x) = \frac{x-1}{x^2}$. Now we combine everything together,

$$G(\sigma) \geq e^{4\sigma}\left(\frac{(e-2)(e-1)^3}{2^{\beta+3}3e^{14}}\right) \geq e^{4\sigma}\left(\frac{(e-2)(e-1)^3}{2^4 3e^{14}}\right).$$

Taking $\sigma \geq 5$, we conclude that

$$G(\sigma) - 1 \geq e^{4\sigma}\left(\frac{(e-2)(e-1)^3}{2^4 3e^{14}} - \frac{1}{e^{4\sigma}}\right) \geq 0,$$

which concludes the first estimate.

Now we make some necessary preparations for the remaining two cases. We start with

$$G(\sigma) := \frac{1}{4\beta\left(\sinh\frac{\sigma}{\beta}\right)^3} \int_0^\sigma \left(\cosh\frac{2s}{\beta}\right)^{\beta+1} (\cosh 2s) \left(\cosh\frac{\sigma}{\beta} - \cosh\frac{s}{\beta}\right) ds.$$

We differentiate, and obtain

$$G'(\sigma) = \frac{1}{4\beta^2\left(\sinh\frac{\sigma}{\beta}\right)^4} \int_0^\sigma \left(\cosh\frac{2s}{\beta}\right)^{\beta+1} (\cosh 2s) \left(3\left(\cosh\frac{\sigma}{\beta}\cosh\frac{s}{\beta} - \cosh^2\frac{\sigma}{\beta}\right) + \sinh^2\frac{\sigma}{\beta}\right) ds.$$

Now we set $g(s,\sigma) = \left(\cosh\frac{2s}{\beta}\right)^{\beta} (\cosh 2s)$ and observe that

$$\frac{\partial g}{\partial s}(s,\sigma) = 2\left(\cosh\frac{2s}{\beta}\right)^{\beta-1} \left(\sinh\frac{2s\alpha}{\beta}\right).$$

Next, set $h(s)$ to be the function

$$h(s) = -\beta\sinh\frac{s}{\beta}\left(\cosh\frac{\sigma}{\beta} - \cosh\frac{s}{\beta}\right)\left(2\cosh\frac{\sigma}{\beta}\cosh\frac{s}{\beta} - 1\right),$$

which is negative and satisfies $h(0) = h(\sigma) = 0$. Differentiating this, we obtain the familiar quantity

$$\frac{\partial h}{\partial s}(s) := \left(\cosh\frac{2s}{\beta}\right)\left(3\left(\cosh\frac{\sigma}{\beta}\cosh\frac{s}{\beta} - \cosh^2\frac{\sigma}{\beta}\right) + \sinh^2\frac{\sigma}{\beta}\right).$$

From this we can observe the following identity and apply an appropriate integration by parts,

$$\begin{aligned}
G'(\sigma) &= \frac{1}{4\beta^2\left(\sinh\frac{\sigma}{\beta}\right)^4} \int_0^\sigma g(s,\sigma)\left(\frac{\partial h(s)}{\partial s}\right) ds \\
&= -\frac{1}{4\beta^2\left(\sinh\frac{\sigma}{\beta}\right)^4} \int_0^\sigma \left(\frac{\partial g(s,\sigma)}{\partial s}\right) h(s)\, ds \\
&= \frac{1}{2\beta\left(\sinh\frac{\sigma}{\beta}\right)^4} \int_0^\sigma \left(\cosh\frac{2s}{\beta}\right)^{\beta-1}\left(\sinh\frac{2s\alpha}{\beta}\right)\sinh\frac{s}{\beta}\left(\cosh\frac{\sigma}{\beta} - \cosh\frac{s}{\beta}\right)\left(2\cosh\frac{\sigma}{\beta}\cosh\frac{s}{\beta} - 1\right) ds.
\end{aligned}$$

$$(3.34)$$

We estimate G' from below in the two different cases.

$\boxed{(\alpha - 1) \le (\alpha - 1)\log\lambda \le 5}$ This is equivalent to considering $0 < \beta \le \sigma \le 5$. For this, we show that G' is bounded below by a positive constant which is independent of β. To do so we apply the following:

$$\frac{\cosh\frac{\sigma}{\beta}}{\sinh\frac{\sigma}{\beta}} > 1, \quad \frac{\sinh\frac{\alpha s}{\beta}}{\cosh\frac{s}{\beta}} \ge \tanh\frac{\alpha s}{\beta}.$$

Then for $\theta \in (0,1)$ and $\beta \le \sigma$, utilizing further the fact that $\tanh \alpha \theta \ge \tanh \theta$ and $\cosh \frac{s}{\beta} \ge 1$,

$$G'(\sigma) = \frac{1}{2\beta \left(\sinh \frac{\sigma}{\beta}\right)^4} \int_0^\sigma \left(\cosh \frac{2s}{\beta}\right)^\beta \left(\frac{\sinh \frac{2s\alpha}{\beta}}{\cosh \frac{2s}{\beta}}\right) \left(\beta \sinh \frac{s}{\beta}\right) \left(\cosh \frac{\sigma}{\beta} - \cosh \frac{s}{\beta}\right) \left(2 \cosh \frac{\sigma}{\beta} \cosh \frac{s}{\beta} - 1\right) ds$$

$$\ge \frac{1}{2\beta \left(\sinh \frac{\sigma}{\beta}\right)^4} \int_{\beta\theta}^\sigma \left(\cosh \frac{2s}{\beta}\right)^\beta \left(\tanh \frac{2\alpha s}{\beta}\right) \left(\beta \sinh \frac{s}{\beta}\right) \left(\cosh \frac{\sigma}{\beta} - \cosh \frac{s}{\beta}\right) \left(\cosh \frac{\sigma}{\beta} \cosh \frac{s}{\beta}\right) ds$$

$$\ge \frac{\tanh 2\alpha\theta}{2\left(\sinh \frac{\sigma}{\beta}\right)^3} \int_{\beta\theta}^\sigma \left(\frac{1}{\beta} \sinh \frac{s}{\beta}\right) \left(\cosh \frac{\sigma}{\beta} - \cosh \frac{s}{\beta}\right) \cosh \frac{s}{\beta} \, ds$$

$$\ge \frac{\tanh 2\alpha\theta}{2\left(\sinh \frac{\sigma}{\beta}\right)^3} \left(\cosh \frac{\sigma}{\beta} \int_{\beta\theta}^\sigma \frac{1}{\beta} \sinh \frac{s}{\beta} \cosh \frac{s}{\beta} \, ds - \int_{\beta\theta}^\sigma \frac{1}{\beta} \sinh \frac{s}{\beta} \cosh^2 \frac{s}{\beta} \, ds\right)$$

$$= \frac{\tanh 2\alpha\theta}{2\left(\sinh \frac{\sigma}{\beta}\right)^3} \left(\frac{1}{6} \cosh^3 \frac{\sigma}{\beta} - \frac{1}{2} \cosh \frac{\sigma}{\beta} \cosh^2 \theta + \frac{1}{3} \cosh^3 \theta\right)$$

$$\ge \frac{\tanh 2\theta}{2\left(\sinh \frac{\sigma}{\beta}\right)^3} \left(\frac{1}{6} \cosh \frac{\sigma}{\beta} \left(\cosh^2 \frac{\sigma}{\beta} - 3 \cosh^2 \theta\right)\right).$$

Choose θ so that $\cosh \theta \le \frac{1}{\sqrt{6}} \cosh 1$, then we can conclude that there is some constant $C > 0$ independent of anything such that if $\alpha > 1$ and $\lambda \ge e$, that is, $\tau \ge 1$ and $0 < \beta \le \sigma$ then

$$G'(\sigma) \ge \frac{\tanh 2\theta}{24\left(\sinh \frac{\sigma}{\beta}\right)^3} \left(\cosh \frac{\sigma}{\beta}\right)^3 \ge C > 0.$$

Therefore, we have that for $0 < \beta \le \sigma$ we have

$$G(\sigma) \ge G(\beta) + C(\sigma - \beta). \tag{3.35}$$

$\boxed{0 < (\alpha - 1) \log \lambda \le (\alpha - 1) \le 1}$ Next let's consider the lower bound for G' in this setting, which is equivalent to $0 < \sigma \le \beta \le 1$. Beginning again from the inequality (3.34), we apply the identities

$$\left(\cosh \frac{s}{\beta}\right)^\beta \le 1, \quad \frac{\sin \frac{2s\alpha}{\beta}}{\cosh \frac{2s}{\beta}} \ge \tanh \frac{2s}{\beta}, \quad \left(2 \cosh \frac{\sigma}{\beta} \cosh \frac{s}{\beta} - 1\right) \ge \cosh \frac{\sigma}{\beta} \cosh \frac{s}{\beta}.$$

Applying them in, we have

$$G'(\sigma) \ge \frac{1}{2\beta \left(\sinh \frac{\sigma}{\beta}\right)^4} \int_0^\sigma \left(\tanh \frac{2s\alpha}{\beta}\right) \sinh \frac{s}{\beta} \left(\cosh \frac{\sigma}{\beta} - \cosh \frac{s}{\beta}\right) \cosh \frac{\sigma}{\beta} \cosh \frac{s}{\beta} \, ds. \tag{3.36}$$

Then we apply three more identities:

$$\sinh x \ge x, \quad \cosh x \ge 1 + \frac{x^2}{2}, \quad \tanh x \ge \frac{x}{(\cosh 2)^2} \text{ for } x \in [0,2], \quad \sinh x \le x \cosh x,$$

and update (3.36) to obtain

$$G'(\sigma) \geq \frac{\beta^4}{2\beta\sigma^4\left(\cosh\frac{\sigma}{\beta}\right)^4} \int_0^\sigma \frac{1}{(\cosh 2)^2} \frac{2s}{\beta} \frac{s}{\beta} \left(1 + \frac{\sigma^2}{2\beta^2}\right) \left(\cosh\frac{\sigma}{\beta} - \cosh\frac{s}{\beta}\right) ds$$

$$\geq \frac{\beta}{2\sigma^4(\cosh 1)^4(\cosh 2)^2} \int_0^\sigma \frac{1}{(\cosh 2)^2} s^2 \left(1 + \frac{\sigma^2}{2\beta^2}\right) \left(\cosh\frac{\sigma}{\beta} - \cosh\frac{s}{\beta}\right) ds. \tag{3.37}$$

Lastly, we implement an identity from standard ODE theory that

$$(\cosh x - \cosh y) \geq \tfrac{1}{2}(x - y)^2.$$

Applying this to (3.37) we can finally conclude that

$$G'(\sigma) \geq \frac{1}{2\beta\sigma^4(\cosh 1)^4(\cosh 2)^2} \left(1 + \frac{\sigma^2}{2\beta^2}\right) \int_0^\sigma s^2(\sigma - s)^2 \, ds$$

$$\geq \frac{\sigma}{60(\cosh 1)^4(\cosh 2)^2\beta} \left(1 + \frac{\sigma^2}{2\beta^2}\right) \tag{3.38}$$

$$\geq \frac{\sigma}{60(\cosh 1)^4(\cosh 2)^2\beta},$$

so for $0 \leq \sigma \leq \beta$,

$$G(\sigma) - G(0) \geq \frac{\sigma^2}{60(\cosh 1)^4(\cosh 2)^2\beta} \geq \frac{(\alpha-1)(\log\lambda)^2}{60(\cosh 1)^4(\cosh 2)^2}. \tag{3.39}$$

Now we can establish the last two estimates. In the event that $(\alpha - 1) \leq (\alpha - 1)\log\lambda \leq 5$, then by rescaling (3.35) and applying the identities of β and σ we have

$$\mho(\alpha, \lambda) \geq 6^\alpha \tfrac{4}{3}\pi^2 \left((G(\alpha - 1) - 1) + C(\alpha - 1)(\log\lambda - 1)\right) \geq C(\alpha - 1)\log\lambda.$$

If $\log\lambda \leq 1$ (the latter case) we again obtain from rescaling from (3.39) that

$$\mho(\alpha, \lambda) \geq 6^\alpha \tfrac{2^5}{15}\pi^2 (\alpha - 1)(\log\lambda)^2.$$

Now we prove the last two claims. We know that $\mathcal{YM}_{\alpha,\lambda}$ is monotonically increasing with respect to λ since G' is positive (cf. (3.34)). To show the final expression, (3.28), we note that from (3.32) that

$$\frac{\partial}{\partial\log\lambda}\left[\mathcal{YM}_{\alpha,\lambda}\left(\widetilde{\nabla}\right)\right] \geq C(\alpha - 1) \geq C(\alpha - 1)\frac{|\log\lambda|}{1 + |\log\lambda|}.$$

For $0 < \log \lambda \leq 1$ we use (3.38) to conclude

$$\frac{\partial}{\partial \log \lambda} \left[\mathcal{YM}_{\alpha,\lambda} \left(\widetilde{\nabla} \right) \right] \geq C \left(\alpha - 1 \right) \log \lambda \geq C \left(\alpha - 1 \right) \frac{|\log \lambda|}{1 + |\log \lambda|}.$$

This concludes the argument. □

Proof of Proposition 3.2.1. Since we have Lemma 3.2.3 and thus appropriate control over connection and curvature differences, it remains to prove (3.15), the resultant control over λ (though dependent on α). To do so we will apply Lemma 3.2.7 to $\varphi^*\nabla$, where φ a conformal automorphism which enforces curvature difference smallness in the sense of (3.16), and $\lambda \geq 1$ its corresponding dilation factor. Obtaining the inequality

$$\left\| \left| F_{\widehat{\varphi}^*\nabla} \right|_{\mathring{g}}^2 - 3 \right\|_{L^1} \leq \| F_{\widehat{\varphi}^*\nabla} - F_{\widetilde{\nabla}} \|_{L^2} \| F_{\widehat{\varphi}^*\nabla} + F_{\widetilde{\nabla}} \|_{L^2} \leq \delta \sqrt{\left(\tfrac{16}{3}\pi^2 + \epsilon \right)\left(\tfrac{16}{3}\pi^2 \right)} \leq \delta \left(\tfrac{16}{3} \right)\pi^2,$$

we apply this to (3.22) to obtain

$$6^\alpha \tfrac{4}{3}\pi^2 + \epsilon \geq \mathcal{YM}_\alpha(\nabla) = \mathcal{YM}_{\alpha,\lambda}(\widehat{\varphi}^*\nabla) \geq \mathcal{YM}_{\alpha,\lambda}\left(\widetilde{\nabla} \right) - \alpha 6^\alpha \tfrac{4}{3}\pi^2 \lambda^{4(\alpha-1)}\delta. \tag{3.40}$$

Additionally, from Lemma 3.2.8,

$$\mathcal{YM}_{\alpha,\lambda}\left(\widetilde{\nabla} \right) = \mathcal{YM}_\alpha(\widehat{\lambda}^*\widetilde{\nabla}) = \tfrac{16}{3}\pi^2 + \mho(\alpha,\lambda), \tag{3.41}$$

and note that ϵ in Lemma 3.2.3 is always bounded above by δ. Then we rewrite (3.40) as,

$$\delta \left(1 + C'\lambda^{4\alpha-4} \right) \geq \mho(\alpha,\lambda) \text{ for some } C' \in \mathbb{R}. \tag{3.42}$$

Now we need to consider the regions mentioned in Lemma 3.2.8. In the case that $(\alpha - 1)\log \lambda \geq 5$, that is (multiplying both sides by 5 and exponentiating) $\lambda^{5(\alpha-1)} \geq e^{10}$ (i.e. $\lambda^{4(\alpha-1)} \geq e^8$). Then by (3.27) we have deduced the lower bound $\mho(\alpha,\lambda) \geq C\lambda^{4(\alpha-1)}$, which implies that (3.42) is false when $0 \leq \delta < \delta_0 := \min\left\{ \frac{C}{2C'}, \frac{C}{2}e^8 \right\}$. Therefore $\lambda^{4(\alpha-1)} < e^8$ and thus, combining the latter two cases of (3.27) and (3.42), we have

$$\delta \left(1 + C'e^8 \right) \geq C \left(\alpha - 1 \right) \left(\log \lambda \right) \min\left\{ \log \lambda, 1 \right\},$$

yielding the result. □

3.3 Closeness of α-connections in the $W^{2,p}$-norm

Now we prove a refinement of Proposition 3.14 which demonstrates closeness between $\varphi^*\nabla$ and $\widetilde{\nabla}$ in $W^{2,p}$ for $p \in (2, \frac{12}{5}]$, with the further restriction that ∇ is a Yang-Mills α-connection. The determination of this range will be clarified in Proposition 3.4.3 (the fundamental reason being to apply necessary Sobolev embeddings). At this point in the proof we reach a key fundamental difference between the arguments of [LMM15] and ours. Since we are working in the four rather than two dimensional setting, our Sobolev embeddings are not as favorable in that we require more degrees of differentiability. To address this we introduce the notion of Morrey space of maps.

Definition 3.3.1. Take $\Omega \subset \mathbb{S}^4$ and set $\Omega(\zeta_0, \rho) := \Omega \cap B_\rho(\zeta_0)$ and for every $p \in [1, +\infty]$, $\lambda \geq 0$ set,

$$
\mathcal{M}_\lambda^p(\Omega) := \left\{ u \in L^p(\Omega) : \sup_{\substack{\zeta_0 \in \Omega \\ \rho > 0}} \rho^{-\lambda} \int_{\Omega(\zeta_0,\rho)} |u|^p \, dV_{\hat{g}} < \infty \right\},
$$

with associated norm defined by

$$
||u||_{\mathcal{M}_\lambda^p} := \left(\sup_{\substack{\zeta_0 \in \Omega \\ \rho > 0}} \rho^{-\lambda} \int_{\Omega(\zeta_0,\rho)} |u|^p \, dV_{\hat{g}} \right)^{1/p} < \infty.
$$

Using this space we intend to use the following result of Morrey.

Theorem 1.1 of [Gia83]. Assume $p \geq n$ and let $u \in W_{\text{loc}}^{1,p}$, $Du \in \mathcal{M}_{\text{loc}}^{p,n-p+\varepsilon}$ for some $\varepsilon > 0$. Then $u \in C_{\text{loc}}^{0, \frac{\varepsilon}{p}}$.

Remark 3.3.2 (Notational conventions)**.** For the proofs of this following proposition as well as that of Lemma 3.9.10, we will indicate applications of either Sobolev embeddings or Poincaré inequality with a subscript S or P on constants. This is nonstandard but will help the reader follow manipulations.

Proposition 3.3.3. *There exists* $\alpha_0 > 1$, $\delta_0, C = C(\alpha_0, \delta_0) > 0$ *depending on only* α_0 *and* δ_0 *such that for every* $\alpha \in (1, \alpha_0]$, *every* $\delta \in (0, \delta_0]$ *and for every critical point* $\blacktriangledown \in W^{1,2\alpha}(\mathcal{A}_E(\mathbb{S}^4))$ *of* $\mathcal{YM}_{\alpha,\lambda}$ *satisfying* (3.14) *and* (3.15) *then for* $p \in \left(2, \frac{12}{5}\right]$ *sufficiently small,*

$$
\left|\left|\widetilde{\Pi}[\blacktriangledown] - \widetilde{\nabla}\right|\right|_{L^\infty} + \left|\left|F_{\widetilde{\Pi}[\blacktriangledown]} - F_{\widetilde{\nabla}}\right|\right|_{W^{1,p}} \leq C(\delta + (\alpha - 1)). \tag{3.43}
$$

Proof. We summarize the proof. Via Proposition 3.2.1 we have that the difference of curvatures is small in L^2, i.e. (3.20). For notational convenience and without loss of generality we may set $\blacktriangledown \equiv \widetilde{\Pi}[\blacktriangledown]$. Initially, we

will demonstrate

$$\left|\left|\blacktriangledown - \widetilde{\nabla}\right|\right|_{W^{2,2}} + \left|\left|F_{\blacktriangledown} - F_{\widetilde{\nabla}}\right|\right|_{W^{1,2}} < C\left(\delta + (\alpha - 1)\right). \tag{3.44}$$

Using this, we will apply a hole-filling argument to obtain the necessary Morrey type estimates, and apply appropriate embeddings to obtain the main result. We will first conclude (3.44) by obtaining bounds on the first derivative of $F_{\blacktriangledown} - F_{\widetilde{\nabla}}$, and second derivative of $\blacktriangledown - \widetilde{\nabla}$ (in fact, the curvature bounds follow from the polarization of curvature). Now, recall that the Yang-Mills α-energy is also preserved under gauge transformation, so using Proposition 3.1.10, since $\blacktriangledown$ is a critical point of $\mathcal{YM}_{\alpha,\lambda}$ and $\widetilde{\nabla}$ is a critical point of $\mathcal{YM}$ we take the difference of the corresponding equations,

$$0 = \left[D_{\blacktriangledown}^{*} F_{\blacktriangledown} - D_{\widetilde{\nabla}}^{*} F_{\widetilde{\nabla}}\right] + \Theta_1(\blacktriangledown) + \Theta_2(\blacktriangledown). \tag{3.45}$$

Our goal is to first estimate $\Upsilon := \blacktriangledown - \widetilde{\nabla}$ in the $W^{2,2}$ sense. To do so we first identify key pointwise estimates on Θ_i for $i \in \{1, 2\}$.

$$\begin{aligned}
|\Theta_1(\blacktriangledown)|_{\mathring{g}} &\leq 4\chi_\lambda(\alpha - 1)\frac{|\blacktriangledown F_{\blacktriangledown}|_{\mathring{g}}|F_{\blacktriangledown}|_{\mathring{g}}^2}{\left|3 + \chi_\lambda|F_{\blacktriangledown}|_{\mathring{g}}^2\right|} \\
&\leq 4\chi_\lambda(\alpha - 1)\frac{|\blacktriangledown F_{\blacktriangledown}|_{\mathring{g}}|F_{\blacktriangledown}|_{\mathring{g}}^2}{\left|\chi_\lambda|F_{\blacktriangledown}|_{\mathring{g}}^2\right|} \\
&\leq C(\alpha - 1)|\blacktriangledown F_{\blacktriangledown}|_{\mathring{g}} \\
&\leq C(\alpha - 1)\left(\left|\widetilde{\nabla} F_{\blacktriangledown}\right|_{\mathring{g}} + \left|[\Upsilon, F_{\blacktriangledown}]\right|_{\mathring{g}}\right) \\
&\leq C(\alpha - 1)\left(\left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}} + \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}|\Upsilon|_{\mathring{g}} + |\Upsilon|_{\mathring{g}}^3\right).
\end{aligned} \tag{3.46}$$

$$\begin{aligned}
|\Theta_2(\blacktriangledown)|_{\mathring{g}} &\leq \frac{2\chi_\lambda(\alpha - 1)}{\left|3 + \chi_\lambda|F_{\blacktriangledown}|_{\mathring{g}}^2\right|}|\blacktriangledown \log \chi_\lambda|_{\mathring{g}}|F_{\blacktriangledown}|_{\mathring{g}}|F_{\blacktriangledown}|_{\mathring{g}}^2 \\
&\leq \frac{2\chi_\lambda(\alpha - 1)}{\left|\chi_\lambda|F_{\blacktriangledown}|_{\mathring{g}}^2\right|}|\blacktriangledown \log \chi_\lambda|_{\mathring{g}}|F_{\blacktriangledown}|_{\mathring{g}}^3 \\
&\leq C(\alpha - 1)|\blacktriangledown \log \chi_\lambda|_{\mathring{g}}|F_{\blacktriangledown}|_{\mathring{g}} \\
&\leq C(\alpha - 1)|\nabla \log \chi_\lambda|_{\mathring{g}}\left(1 + \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}} + |\Upsilon|_{\mathring{g}}^2\right).
\end{aligned} \tag{3.47}$$

Via integration by parts combined with (3.18) and applying the formula of the curvature tensor,

$$
\begin{aligned}
\left\|\widetilde{\nabla}^{(2)}\Upsilon\right\|_{L^2}^2 &= -\int_{\mathbb{S}^4}\left\langle\widetilde{\nabla}\Upsilon,\widetilde{\triangle}\widetilde{\nabla}\Upsilon\right\rangle_{\mathring{g}}dV_{\mathring{g}} \\
&= \left\|\widetilde{\triangle}\Upsilon\right\|_{L^2}^2 - \int_{\mathbb{S}^4}\left\langle\widetilde{\nabla}_l\Upsilon_i,\widetilde{\nabla}_k\left[\widetilde{\nabla}_k,\widetilde{\nabla}_l\right]\Upsilon_i\right\rangle_{\mathring{g}}dV_{\mathring{g}} - \int_{\mathbb{S}^4}\left\langle\widetilde{\nabla}_l\Upsilon_i,\left[\widetilde{\nabla}_k,\widetilde{\nabla}_l\right]\widetilde{\nabla}_k\Upsilon_i\right\rangle_{\mathring{g}}dV_{\mathring{g}} \\
&= \left\|\widetilde{\triangle}\Upsilon\right\|_{L^2}^2 - \int_{\mathbb{S}^4}\left\langle\widetilde{\nabla}_l\Upsilon_i,\widetilde{\nabla}_k\left[\mathrm{Rm}_{klis}^{\mathbb{S}^4}\Upsilon_s\right]+\mathrm{Rm}_{klks}^{\mathbb{S}^4}\widetilde{\nabla}_s\Upsilon_i+\mathrm{Rm}_{klis}^{\mathbb{S}^4}\widetilde{\nabla}_k\Upsilon_s\right\rangle_{\mathring{g}}dV_{\mathring{g}} \\
&\quad - 2\int_{\mathbb{S}^4}\left\langle\widetilde{\nabla}_l\Upsilon_i,\left[\widetilde{F}_{kl},\widetilde{\nabla}_k\Upsilon_i\right]\right\rangle_{\mathring{g}}dV_{\mathring{g}} \\
&\leq \left\|\widetilde{\triangle}\Upsilon\right\|_{L^2}^2 + 11\left\|\widetilde{\nabla}\Upsilon\right\|_{L^2}^2 - 2\int_{\mathbb{S}^4}\left\langle\left[\widetilde{\nabla}_i,\widetilde{\nabla}_l\right]\Upsilon_i,\Upsilon_l\right\rangle_{\mathring{g}}dV_{\mathring{g}} \\
&= \left\|\widetilde{\triangle}\Upsilon\right\|_{L^2}^2 + 11\left\|\widetilde{\nabla}\Upsilon\right\|_{L^2}^2 - 2\int_{\mathbb{S}^4}\left\langle\mathrm{Rm}_{ilis}^{\mathbb{S}^4}\Upsilon_s+\left[\widetilde{F}_{il},\Upsilon_s\right],\Upsilon_l\right\rangle_{\mathring{g}}dV_{\mathring{g}} \\
&\leq \left\|\widetilde{\triangle}\Upsilon\right\|_{L^2}^2 + 11\left\|\widetilde{\nabla}\Upsilon\right\|_{L^2}^2 + 8\left\|\Upsilon\right\|_{L^2}^2 \\
&< C\delta + \left\|\widetilde{\triangle}\Upsilon\right\|_{L^2}^2.
\end{aligned}
\tag{3.48}
$$

We will estimate the latter term. Recall from Proposition 3.9.2 in the appendix combined with (3.45),

$$
\begin{aligned}
\widetilde{\triangle}\Upsilon_{i\theta}^\beta &= -3\Upsilon_{i\theta}^\beta - 2\Upsilon_{k\mu}^\beta\widetilde{F}_{ki\theta}^\mu + 2\widetilde{F}_{ki\mu}^\beta\Upsilon_{k\theta}^\mu \\
&\quad - \Upsilon_{i\zeta}^\beta\Upsilon_{k\mu}^\zeta\Upsilon_{k\theta}^\mu + 2\Upsilon_{k\mu}^\beta\Upsilon_{i\zeta}^\mu\Upsilon_{k\theta}^\zeta - \Upsilon_{k\mu}^\beta\Upsilon_{k\zeta}^\mu\Upsilon_{i\theta}^\zeta + (\Theta_1+\Theta_2)_{i\alpha}^\beta \\
&\quad - \left(\widetilde{\nabla}_i\Upsilon_{k\mu}^\beta\right)\Upsilon_{k\theta}^\mu + \Upsilon_{k\mu}^\beta\left(\widetilde{\nabla}_i\Upsilon_{k\theta}^\mu\right) - 2\Upsilon_{k\mu}^\beta\left(\widetilde{\nabla}_k\Upsilon_{i\theta}^\mu\right) + 2\left(\widetilde{\nabla}_k\Upsilon_{i\mu}^\beta\right)\Upsilon_{k\theta}^\mu.
\end{aligned}
\tag{3.49}
$$

With this in mind we compute each term of the following

$$
\begin{aligned}
\left\|\widetilde{\triangle}\Upsilon\right\|_{L^2}^2 &= \int_{\mathbb{S}^4}\left(-3\left\langle\Upsilon,\widetilde{\triangle}\Upsilon\right\rangle_{\mathring{g}}+2\left\langle\left[\widetilde{F}_{ki},\Upsilon_k\right],\widetilde{\triangle}\Upsilon_i\right\rangle_{\mathring{g}}\right)dV_{\mathring{g}} + \int_{\mathbb{S}^4}\left\langle\widetilde{\triangle}\Upsilon,\Upsilon^{*_{\mathring{g}}3}\right\rangle_{\mathring{g}}dV_{\mathring{g}} \\
&\quad + \int_{\mathbb{S}^4}\left(\widetilde{\nabla}\Upsilon*_{\mathring{g}}\Upsilon*_{\mathring{g}}\widetilde{\triangle}\Upsilon\right)dV_{\mathring{g}} + \int_{\mathring{S}^4}\left\langle(\Theta_1+\Theta_2),\widetilde{\triangle}\Upsilon\right\rangle_{\mathring{g}}dV_{\mathring{g}}.
\end{aligned}
$$

For the first term we apply integration by parts combined with (3.18), noting $F_{\widetilde{\nabla}}$ is $\widetilde{\nabla}$-parallel

$$
\int_{\mathbb{S}^4}\left(-3\left\langle\Upsilon,\widetilde{\triangle}\Upsilon\right\rangle_{\mathring{g}}+2\left\langle\left[\widetilde{F}_{ki},\Upsilon_k\right],\widetilde{\triangle}\Upsilon_i\right\rangle_{\mathring{g}}\right)dV_{\mathring{g}} = \int_{\mathbb{S}^4}3\left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2+2\left\langle\left[\widetilde{\nabla}_j\Upsilon_i,\widetilde{F}_{ki}\right],\widetilde{\nabla}_j\Upsilon_i\right\rangle_{\mathring{g}}dV_{\mathring{g}}
$$

$$
\leq C\left((\alpha-1)+\delta\right),
$$

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For the second term, via integration by parts, Hölder's inequality and Sobolev embedding

$$
\begin{aligned}
\int_{\mathbb{S}^4} \left\langle \widetilde{\triangle}\Upsilon, \Upsilon^{*3} \right\rangle_{\mathring{g}} dV_{\mathring{g}} &\leq C \int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2 |\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g}} \\
&\leq C \left(\int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^4 dV_{\mathring{g}} \right)^{1/2} \left(\int_{\mathbb{S}^4} |\Upsilon|_{\mathring{g}}^4 \, dV_{\mathring{g}} \right)^{1/2} \\
&\leq C \left((\alpha - 1) + \delta\right) \left(C\left((\alpha - 1) + \delta\right) + \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} \right) \\
&\leq C \left((\alpha - 1) + \delta\right) \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} + C\left((\alpha - 1) + \delta\right).
\end{aligned}
\tag{3.50}
$$

Likewise we have that, using a weighted Hölder's inequality and then applying the same equation as above in (3.50),

$$
\begin{aligned}
\int_{\mathbb{S}^4} \left(\widetilde{\nabla}\Upsilon *_{\mathring{g}} \Upsilon *_{\mathring{g}} \widetilde{\triangle}\Upsilon \right) dV_{\mathring{g}} &\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\triangle}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} + \frac{C}{\nu} \int_{\mathbb{S}^4} |\Upsilon|_{\mathring{g}}^2 \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} \\
&\leq C\left(\nu + (\alpha - 1) + \delta\right) \int_{\mathbb{S}^4} \left|\widetilde{\triangle}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} + C\left((\alpha - 1) + \delta\right).
\end{aligned}
\tag{3.51}
$$

Decomposing

$$
\begin{aligned}
\int_{\mathbb{S}^4} \left\langle \widetilde{\triangle}\Upsilon, \Theta_1 \right\rangle_{\mathring{g}} dV_{\mathring{g}} &\leq C\left(\alpha - 1\right) \left[\int_{\mathbb{S}^4} \left|\widetilde{\nabla}^2\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} + \int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}} |\Upsilon|_{\mathring{g}} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}} dV_{\mathring{g}} + \int_{\mathbb{S}^4} |\Upsilon|_{\mathring{g}} \left|\widetilde{\triangle}\Upsilon\right|_{\mathring{g}} dV_{\mathring{g}} \right] \\
&\quad + C(\alpha - 1) \int_{\mathbb{S}^4} \left|\widetilde{\triangle}\Upsilon\right|_{\mathring{g}} |\Upsilon|_{\mathring{g}}^3 dV_{\mathring{g}} \\
&= (\alpha - 1) \sum_{i=1}^{4} T_i.
\end{aligned}
$$

For the second term,

$$
\begin{aligned}
T_2 &\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} + \frac{C}{\nu} \int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2 |\Upsilon|_{\mathring{g}}^2 dV_{\mathring{g}} \\
&\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} + \left(\int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^4 dV_{\mathring{g}} \right)^{1/2} \left(\int_{\mathbb{S}^4} |\Upsilon|_{\mathring{g}}^4 dV_{\mathring{g}} \right)^{1/2} \\
&\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} + C\left(\delta + (\alpha - 1)\right) \left(C\left(\delta + (\alpha - 1)\right) + \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} \right) \\
&\leq C\left(\delta + (\alpha - 1) + \nu\right) \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} + C\left(\delta + (\alpha - 1)\right).
\end{aligned}
$$

For the third term,

$$T_3 \leq \int_{\mathbb{S}^4} |\Upsilon|_{\mathring{g}} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}} dV_{\mathring{g}}$$

$$\leq C \left(\int_{\mathbb{S}^4} |\widetilde{\nabla}\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g}}\right)^{1/2} \left(\int_{\mathbb{S}^4} |\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g}}\right)^{1/2}$$

$$\leq C_P \int_{\mathbb{S}^4} |\widetilde{\nabla}\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g}}.$$

T_4 follows exactly as in (3.50) and (3.51) above.

$$\int_{\mathbb{S}^4} \left\langle \widetilde{\triangle}\Upsilon, \Theta_2 \right\rangle_{\mathring{g}} dV_{\mathring{g}} \leq (\alpha - 1) \left[\int_{\mathbb{S}^4} |\widetilde{\nabla}\log\chi_\lambda|_{\mathring{g}} |\widetilde{\nabla}\Upsilon|_{\mathring{g}} \left|\widetilde{\triangle}\Upsilon\right|_{\mathring{g}} dV_{\mathring{g}} + \int_{\mathbb{S}^4} |\widetilde{\nabla}\log\chi_\lambda|_{\mathring{g}} |\widetilde{\nabla}\Upsilon|_{\mathring{g}} |\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g}}\right]$$

$$= \sum_{i=1}^{2} Q_i.$$

We estimate these two terms. First we estimate

$$Q_1 \leq C \int_{\mathbb{S}^4} \left(|\widetilde{\nabla}\log\chi_\lambda|_{\mathring{g}}^2 |\widetilde{\nabla}\Upsilon|_{\mathring{g}}^2 + \left|\widetilde{\triangle}\Upsilon\right|_{\mathring{g}}^2 \right) dV_{\mathring{g}}$$

$$\leq C \left(\int_{\mathbb{S}^4} |\widetilde{\nabla}\log\chi_\lambda|_{\mathring{g}}^4\right)^{1/2} \left(\int_{\mathbb{S}^4} |\widetilde{\nabla}\Upsilon|_{\mathring{g}}^4\right)^{1/2} + C \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g}}$$

$$\leq C \log\lambda \left(C(\alpha - 1 + \delta) + \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g}}\right) + C \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g}}.$$

Next we estimate

$$Q_2 \leq \int_{\mathbb{S}^4} |\Upsilon|_{\mathring{g}}^4 \, dV_{\mathring{g}} + \int_{\mathbb{S}^4} |\widetilde{\nabla}\log\chi_\lambda|_{\mathring{g}}^2 |\widetilde{\nabla}\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g}}$$

$$\leq C(\alpha - 1 + \delta) + C \left(\int_{\mathbb{S}^4} |\widetilde{\nabla}\log\chi_\lambda|_{\mathring{g}}^4 \, dV_{\mathring{g}}\right)^{1/2} \left(\int_{\mathbb{S}^4} |\widetilde{\nabla}\Upsilon|_{\mathring{g}}^4 \, dV_{\mathring{g}}\right)^{1/2}$$

$$\leq C(\alpha - 1 + \delta) + C \log\lambda \left(C(\alpha - 1 + \delta) + \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g}}\right).$$

Therefore we have that, combining everything

$$\left\|\widetilde{\nabla}^{(2)}\Upsilon\right\|_{L^2}^2 \leq C(\alpha - 1 + \delta) \left\|\widetilde{\nabla}^{(2)}\Upsilon\right\|_{L^2}^2 + C(\alpha - 1 + \delta).$$

Thus provided δ_0, $(\alpha_0 - 1)$ are sufficiently small, we may absorb the $\widetilde{\nabla}^{(2)}\Upsilon$ type terms into the left side of (3.48), allowing us to conclude the control

$$\left\|\widetilde{\nabla}^{(2)}\Upsilon\right\|_{L^2} \leq C(\delta + (\alpha - 1)). \tag{3.52}$$

It is not difficult to believe that derivatives of curvature differences are intrinsically related to derivatives of Υ. Consider the case for the L^2 control of the $\widetilde{\nabla}$ derivative of $(F_\nabla - F_{\widetilde{\nabla}})$. Applying Proposition 3.85,

$$\left| \widetilde{\nabla} \left[F_{\blacktriangledown} - F_{\widetilde{\nabla}} \right] \right|_{\mathring{g}} \leq C \left(\left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}} + \left| \Upsilon * \widetilde{\nabla} \Upsilon \right|_{\mathring{g}} \right).$$

Consequently the estimates in L^2 on this term follow directly from those above.

Our next goal is to demonstrate that $\widetilde{\nabla} \Upsilon \in \mathcal{M}_\beta^4$ for some $\beta > 0$. To do so we use the Sobolev embedding $\mathcal{M}_\beta^{1,2} \hookrightarrow \mathcal{M}_\beta^4$ and focus our attention on demonstrating containment in $\mathcal{M}_\beta^{1,2}$ (for the sake of brevity we include this step in the appendix, Lemma 3.9.10). Provided the estimate $||\Upsilon||_{\mathcal{M}_\beta^4} \leq C(\alpha - 1 + \delta)$ we will translate our setting to a functional perspective to apply Theorem 1.1 of [Gia83]. In a $\widetilde{\nabla}$-adapted frame we note the control $\left| \widetilde{\Gamma} \right| \leq CR$. Therefore using the coordinate decomposition $\widetilde{\nabla}_i \Upsilon_{j\alpha}^\beta = \partial_i \Upsilon_{j\alpha}^\beta + \left[\widetilde{\Gamma}_i, \Upsilon_j \right]_\alpha^\beta$,

$$R^{-\beta} \int_{B_R} \left| \partial_i \Upsilon_{j\alpha}^\beta \right|^4 dV_{\mathring{g}} \leq R^{-\beta} \int_{B_R} \left(\left| \widetilde{\nabla}_i \Upsilon_{j\alpha}^\beta \right|^4 + CR^4 \left| \Upsilon_{j\alpha}^\beta \right|^4 \right) dV_{\mathring{g},\eta}$$
$$\leq R^{-\beta} \int_{B_R} \left| \widetilde{\nabla} \Upsilon \right|_{\mathring{g}}^4 dV_{\mathring{g},\eta} + R^{4-\beta} \int_{B_R} \left| \Upsilon \right|_{\mathring{g}}^4 dV_{\mathring{g},\eta}$$
$$\leq C \left((\alpha - 1) + \delta \right).$$

Applying Theorem 1.1 of [Gia83] to each coefficient function of Υ, with $p = n = 4$ we thus have desired Hölder continuity for all coefficients of Υ, and so

$$||\Upsilon||_{C^{0,\frac{\beta}{4}}} \leq C \left((\alpha - 1) + \delta \right).$$

This is a particularly strong and immediately implies that $||\Upsilon||_{L^\infty} \leq C(\alpha - 1 + \delta)$. Furthermore, as discussed on [GM12] pp.76-78 (combining Proposition 5.4 and Theorem 5.5 of the text) we have $C^{0,\frac{\beta}{4}}$ is equivalent to $\mathcal{M}_\nu^2$ in the sense of functions, where $\nu := \frac{\beta}{2} + 4$. Applying these results to the coefficient functions of $\widetilde{\nabla}$ on a local level as above in a $\widetilde{\nabla}$-adapted frame, we obtain that

$$||\Upsilon||_{\mathcal{M}_\nu^2} \leq C \left((\alpha - 1) + \delta \right).$$

Via polarization of $F_{\blacktriangledown} - F_{\widetilde{\nabla}}$ in terms of Υ it follows that

$$||F_{\blacktriangledown} - F_{\widetilde{\nabla}}||_{\mathcal{M}_\mu^{1,2}} \leq C \left((\alpha - 1) + \delta \right), \quad \text{where } \mu := \min\{\nu, \beta\}.$$

Lastly, we use the fact that for ε sufficiently small (in particular $\varepsilon \leq \frac{2\mu}{4-\mu}$),

$$||F_{\blacktriangledown} - F_{\widetilde{\nabla}}||_{\mathcal{M}_\gamma^{1,2+\varepsilon}} \leq C \, ||F_{\blacktriangledown} - F_{\widetilde{\nabla}}||_{\mathcal{M}_\mu^{1,2}}, \quad \text{where } \gamma \leq \mu\left(1 + \tfrac{\varepsilon}{2}\right) - 2\varepsilon.$$

Provided α_0 is chosen sufficiently small so that the range of p values lies in $2 + \varepsilon$, in particular, $\alpha_0 < \frac{2\varepsilon+2}{(2-\varepsilon)}$ (cf. (3.57) for the reasoning for such choice) then we finally attain

$$||F_{\blacktriangledown} - F_{\widetilde{\nabla}}||_{W^{1,p}} \leq C\left((\alpha - 1) + \delta\right),$$

and thus concluding (3.43). □

3.4   Bound on λ

We next demonstrate how the estimates (3.15) and (3.43) imply small growth of $\frac{\partial}{\partial \log \lambda}\left[\mathcal{YM}_{\alpha,\lambda}(\widetilde{\nabla})\right]$ which, when coupled with (3.28), yields a bound on λ (which is independent of how close α is to 1). We compute $\frac{\partial}{\partial \log \lambda}\left[\mathcal{YM}_{\alpha,\lambda}(\widetilde{\nabla})\right]$ directly from (3.10) and (3.11).

Lemma 3.4.1. *We have the following equalities:*

$$\tfrac{\partial \chi_\lambda}{\partial \log \lambda}(\zeta) = \chi_\lambda(\zeta)\left(\tfrac{4\left(\lambda^2|\zeta|^2 - 1\right)}{\lambda^2|\zeta|^2 + 1}\right).$$

Proof. Recalling that $\chi_\lambda(\zeta) = \frac{1}{\lambda^4}\left(\frac{1+|\lambda\zeta|^2}{|\zeta|^2+1}\right)^4$, we have

$$(\log \chi_\lambda)(\zeta) = 4\log\left(1 + \lambda^2\,|\zeta|^2\right) - 4\log\lambda - 4\log\left(|\zeta|^2 + 1\right)$$
$$\tfrac{\partial[\log[\chi_\lambda]]}{\partial \lambda}(\zeta) = \tfrac{4\left(\lambda^2|\zeta|^2 - 1\right)}{\lambda^2|\zeta|^2 + 1}.$$

Remanipulating this accordingly, one obtains $\frac{\partial \chi_\lambda}{\partial \log \lambda} = \chi_\lambda \frac{\partial[\log \chi_\lambda]}{\partial \log \lambda}$, giving the result. □

Proposition 3.4.2. *The following inequality holds.*

$$\frac{\partial}{\partial \log \lambda} \left[\mathcal{YM}_{\alpha,\lambda} \left(\widetilde{\triangledown} \right) \right] - \frac{\partial}{\partial \log \lambda} \left[\mathcal{YM}_{\alpha,\lambda} \left(\blacktriangledown \right) \right]$$

$$\leq C \left(\alpha - 1 \right) \left(1 + \lambda^{4(\alpha - 1)} \right) \left\| F_{\widetilde{\triangledown}} - F_{\widetilde{\Pi}[\blacktriangledown]} \right\|_{L^2} \left(\| F_{\widetilde{\triangledown}} \|_{L^2} + \left\| F_{\widetilde{\Pi}[\blacktriangledown]} \right\|_{L^2} \right)$$

$$+ C \left(\alpha - 1 \right)^2 \left(1 + \lambda^{4(\alpha - 1)} \right) \left(\| F_{\widetilde{\triangledown}} \|_{L^{2\alpha+2}} + \left\| F_{\widetilde{\Pi}[\blacktriangledown]} \right\|_{L^{2\alpha+2}} \right) \left\| F_{\widetilde{\triangledown}} - F_{\widetilde{\Pi}[\blacktriangledown]} \right\|_{L^{2\alpha+2}} \left\| F_{\widetilde{\Pi}[\blacktriangledown]} \right\|_{L^{2\alpha+2}}^{2\alpha}.$$

$$(3.53)$$

Proof. Again, via gauge invariance of $\mathcal{YM}_{\alpha,\lambda}$ we can assume that $\blacktriangledown = \widetilde{\Pi}[\blacktriangledown]$. We differentiate and obtain

$$\frac{\partial}{\partial \log \lambda} \left[\mathcal{YM}_{\alpha,\lambda}(\blacktriangledown) \right] = \frac{1}{2} \frac{\partial}{\partial \log \lambda} \left[\int_{\mathbb{S}^4} \left(3 + \chi_\lambda \, |F_\blacktriangledown|_{\mathring{g}}^2 \right)^\alpha \frac{1}{\chi_\lambda} \, dV_{\mathring{g}} \right]$$

$$= \frac{1}{2} \int_{\mathbb{S}^4} \left(3 + \chi_\lambda \, |F_\blacktriangledown|_{\mathring{g}}^2 \right)^{\alpha - 1} \left((\alpha - 1) \, |F_\blacktriangledown|_{\mathring{g}}^2 - \frac{2}{\chi_\lambda} \right) \left(\frac{\partial \chi_\lambda}{\partial \log \lambda} \right) \frac{1}{\chi_\lambda} \, dV_{\mathring{g}}$$

$$= 2 \int_{\mathbb{S}^4} \left(3 + \chi_\lambda \, |F_\blacktriangledown|_{\mathring{g}}^2 \right)^{\alpha - 1} \left((\alpha - 1) \, |F_\blacktriangledown|_{\mathring{g}}^2 - \frac{2}{\chi_\lambda} \right) \underline{\left(\frac{\lambda^2 |\zeta|^2 - 1}{\lambda^2 |\zeta|^2 + 1} \right)} \, dV_{\mathring{g}}.$$

For computational ease, set $\mu(\zeta) := \left(\frac{|\zeta|^2 - 1}{|\zeta|^2 + 1} \right) \in [-1, 1)$ so that the underlined quantity is $\mu(\lambda \zeta)$. Then

$$\frac{\partial}{\partial \log \lambda} \left[\mathcal{YM}_{\alpha,\lambda}(\widetilde{\triangledown}) \right] - \frac{\partial}{\partial \log \lambda} \left[\mathcal{YM}_{\alpha,\lambda}(\blacktriangledown) \right]$$

$$= - \int_{\mathbb{S}^4} \left((3 + 3\chi_\lambda)^{\alpha - 1} - \left(3 + \chi_\lambda \, |F_\blacktriangledown|_{\mathring{g}}^2 \right)^{\alpha - 1} \right) \frac{2\mu(\lambda\zeta)}{\chi_\lambda} \, dV_{\mathring{g}} \qquad (3.54)$$

$$+ (\alpha - 1) \int_{\mathbb{S}^4} \left(3 \, (3 + 3\chi_\lambda)^{\alpha - 1} - |F_\blacktriangledown|_{\mathring{g}}^2 \left(3 + \chi_\lambda \, |F_\blacktriangledown|_{\mathring{g}}^2 \right)^{\alpha - 1} \right) \mu(\lambda\zeta) \, dV_{\mathring{g}}.$$

Similar to the proof of Lemma 3.2.7, there exists some $f : \mathbb{S}^4 \to [0, \infty)$ whose value at $\zeta \in \mathbb{S}^4$ lies between $|F_\blacktriangledown(\zeta)|_{\mathring{g}}^2$ and $3 = |F_{\widetilde{\triangledown}}(\zeta)|_{\mathring{g}}^2$ such that

$$\left(3^\alpha \, (1 + \chi_\lambda)^{\alpha - 1} - \left(3 + \chi_\lambda \, |F_\blacktriangledown|_{\mathring{g}}^2 \right)^{\alpha - 1} \right) = (\alpha - 1) \, (3 + f\chi_\lambda)^{\alpha - 2} \, \chi_\lambda \left(3 - |F_\blacktriangledown|_{\mathring{g}}^2 \right).$$

Then multiplying through by $|F_\triangledown|_{\mathring{g}}^2$ and remanipulating,

$$3 \, (3 + 3\chi_\lambda)^{\alpha - 1} - |F_\blacktriangledown|_{\mathring{g}}^2 \left(3 + \chi_\lambda \, |F_\blacktriangledown|_{\mathring{g}}^2 \right)^{\alpha - 1}$$

$$= (3 + 3\chi_\lambda)^{\alpha - 1} \left(3 - |F_\blacktriangledown|_{\mathring{g}}^2 \right) + (\alpha - 1) \, (3 + f\chi_\lambda)^{\alpha - 2} \, \chi_\lambda \left(3 - |F_\blacktriangledown|_{\mathring{g}}^2 \right) |F_\blacktriangledown|_{\mathring{g}}^2.$$

Note for $\alpha \leq 2$ one has $(3 + f\chi_\lambda)^{\alpha-2} \leq 1$. Furthermore

$$\frac{\chi_\lambda |F_\triangledown|_{\mathring{g}}^2}{3 + f\chi_\lambda} \leq \begin{cases} \frac{1}{3} |F_\blacktriangledown|_{\mathring{g}}^2 & \text{if } |F_\blacktriangledown|_{\mathring{g}}^2 \geq 3 \\ 1 & \text{if } |F_\blacktriangledown|_{\mathring{g}}^2 \leq 3 \end{cases}$$

$$\leq 1 + |F_\blacktriangledown|_{\mathring{g}}^2.$$

and furthermore,

$$(3 + 3\chi_\lambda)^{\alpha-1} = 3^{\alpha-1} \left(1 + \tfrac{\chi_\lambda}{3}\right)^{\alpha-1} \leq 6^{\alpha-1} \lambda^{4(\alpha-1)}, \quad (3 + f\chi_\lambda)^{\alpha-1} \leq 6^{\alpha-1} \lambda^{4(\alpha-1)} \left(1 + |F_\blacktriangledown|_{\mathring{g}}^{2\alpha-2}\right).$$

Then using the fact that $|\mu| \leq 1$,

$$\left| \left((3 + 3\chi_\lambda)^{\alpha-1} - \left(3 + \chi_\lambda |F_\blacktriangledown|_{\mathring{g}}^2\right)^{\alpha-1} \right) \frac{2\mu(\lambda\zeta)}{\chi_\lambda} \right| \leq 2(\alpha-1) \left| 3 - |F_\blacktriangledown|_{\mathring{g}}^2 \right|,$$

and

$$\left| 3(3 + 3\chi_\lambda)^{\alpha-1} - |F_\blacktriangledown|_{\mathring{g}}^2 \left(3 + \chi_\lambda |F_\blacktriangledown|_{\mathring{g}}^2\right)^{\alpha-1} \mu(\lambda\zeta) \right| \leq C\lambda^{4(\alpha-1)} \left| 3 - |F_\blacktriangledown|_{\mathring{g}}^2 \right| \left(1 + (\alpha-1)|F_\blacktriangledown|_{\mathring{g}}^{2\alpha}\right). \quad (3.55)$$

Therefor, applying (**??**), (3.55) into (3.54)

$$\frac{\partial}{\partial \log \lambda} \left[\mathcal{YM}_{\alpha,\lambda}\left(\widetilde{\triangledown}\right) \right] - \frac{\partial}{\partial \log \lambda} \left[\mathcal{YM}_{\alpha,\lambda}\left(\blacktriangledown\right) \right]$$

$$\leq C(\alpha-1)\left(1 + \lambda^{4(\alpha-1)}\right) \int_{\mathbb{S}^4} \left| 3 - |F_\blacktriangledown|_{\mathring{g}}^2 \right| \left(1 + (\alpha-1)|F_\blacktriangledown|_{\mathring{g}}^{2\alpha}\right) dV_{\mathring{g}}$$

$$\leq \left[C(\alpha-1)\left(1 + \lambda^{4(\alpha-1)}\right) \int_{\mathbb{S}^4} \left| 3 - |F_\blacktriangledown|_{\mathring{g}}^2 \right| dV_{\mathring{g}} \right]_{T_1}$$

$$+ \left[C(\alpha-1)^2 \left(1 + \lambda^{4(\alpha-1)}\right) \int_{\mathbb{S}^4} \left| 3 - |F_\blacktriangledown|_{\mathring{g}}^2 \right| |F_\blacktriangledown|_{\mathring{g}}^{2\alpha} dV_{\mathring{g}} \right]_{T_2}.$$

For T_1 we compute out, first applying Hölder's inequality followed by triangle inequality

$$\int_{\mathbb{S}^4} \left| |F_{\widetilde{\triangledown}}|_{\mathring{g}}^2 - |F_\blacktriangledown|_{\mathring{g}}^2 \right| dV_{\mathring{g}} \leq \int_{\mathbb{S}^4} \left(|F_{\widetilde{\triangledown}}|_{\mathring{g}} - |F_\blacktriangledown|_{\mathring{g}} \right) \left(|F_{\widetilde{\triangledown}}|_{\mathring{g}} + |F_\blacktriangledown|_{\mathring{g}} \right) dV_{\mathring{g}}$$

$$\leq \int_{\mathbb{S}^4} |F_{\widetilde{\triangledown}} - F_\blacktriangledown|_{\mathring{g}} |F_{\widetilde{\triangledown}} + F_\blacktriangledown|_{\mathring{g}} dV_{\mathring{g}}$$

$$\leq \|F_{\widetilde{\triangledown}} - F_\blacktriangledown\|_{L^2} \|F_{\widetilde{\triangledown}} + F_\blacktriangledown\|_{L^2}$$

$$\leq \|F_{\widetilde{\triangledown}} - F_\blacktriangledown\|_{L^2} \left(\|F_{\widetilde{\triangledown}}\|_{L^2} + \|F_\blacktriangledown\|_{L^2} \right).$$

For T_2 this follows in a similar fashion but rather than apply the standard Hölder's inequality we apply the triple version, namely, for $f, g, h \in C^1(\mathbb{S}^4)$,

$$\int fgh \, dV = ||f||_{L^p} \, ||g||_{L^q} \, ||h||_{L^r}, \quad \tfrac{1}{p} + \tfrac{1}{q} + \tfrac{1}{r} = 1.$$

where in our case we will take $p = q = 2\alpha + 2$ and $r = \frac{\alpha}{2\alpha+2}$. Now we compute

$$\int_{\mathbb{S}^4} \left| |F_{\widetilde{\nabla}}|_{\mathring{g}}^2 - |F_{\blacktriangledown}|_{\mathring{g}}^2 \right| |F_{\blacktriangledown}|_{\mathring{g}}^{2\alpha} \, dV_{\mathring{g}} \leq \int_{\mathbb{S}^4} |F_{\widetilde{\nabla}} + F_{\blacktriangledown}|_{\mathring{g}} \, |F_{\widetilde{\nabla}} - F_{\blacktriangledown}|_{\mathring{g}} \, |F_{\blacktriangledown}|_{\mathring{g}}^{2\alpha} \, dV_{\mathring{g}}$$

$$\leq ||F_{\widetilde{\nabla}} + F_{\blacktriangledown}||_{L^{2\alpha+2}} \, ||F_{\widetilde{\nabla}} - F_{\blacktriangledown}||_{L^{2\alpha+2}} \left| \left| |F_{\blacktriangledown}|_{\mathring{g}}^{2\alpha} \right| \right|_{L^{\frac{\alpha+1}{\alpha}}}$$

$$\leq \left(||F_{\widetilde{\nabla}}||_{L^{2\alpha+2}} + ||F_{\blacktriangledown}||_{L^{2\alpha+2}} \right) ||F_{\widetilde{\nabla}} - F_{\blacktriangledown}||_{L^{2\alpha+2}} \, ||F_{\blacktriangledown}||_{L^{2\alpha+2}}^{2\alpha}.$$

Note that the manipulation of the last quantity follows from the fact that

$$\left| \left| |F_{\blacktriangledown}|_{\mathring{g}}^{2\alpha} \right| \right|_{L^{\frac{\alpha+1}{\alpha}}} = \left(\int_{\mathbb{S}^4} \left(|F_{\blacktriangledown}|_{\mathring{g}}^{2\alpha} \right)^{\frac{\alpha+1}{\alpha}} dV_{\mathring{g}} \right)^{\frac{\alpha}{\alpha+1}} = \left(\left(\int |F_{\blacktriangledown}|_{\mathring{g}}^{2\alpha+2} \, dV_{\mathring{g}} \right)^{\frac{1}{2\alpha+2}} \right)^{2\alpha} = ||F_{\blacktriangledown}||_{L^{2\alpha+2}}^{2\alpha}.$$

Combining these estimates we conclude (3.53). $\qquad\qquad\qquad\square$

Proposition 3.4.3. *There exist $\alpha_0 > 1$, $\delta_0 > 0$ possibly smaller than those in Proposition 3.3.3 such that if $\blacktriangledown \in W^{1,2\alpha}\left(\mathcal{A}_E(\mathbb{S}^4)\right)$ is a critical point of $\mathcal{YM}_{\alpha,\lambda}$ satisfying (3.14) and (3.15), with $\alpha \in (1, \alpha_0]$ and $\delta \in (0, \delta_0]$, then*

$$\log \lambda \leq C \left(\delta + \alpha - 1 \right). \tag{3.56}$$

Proof. As in Proposition 3.4.3 we will be considering the relationship between the connections ∇ and $\widetilde{\nabla}$. Then we can apply a Sobolev embedding to obtain that

$$\left| \left| F_{\widetilde{\Pi}[\blacktriangledown]} - F_{\widetilde{\nabla}} \right| \right|_{L^{2\alpha+2}} \leq C_{S,\alpha} \left| \left| F_{\widetilde{\Pi}[\blacktriangledown]} - F_{\widetilde{\nabla}} \right| \right|_{W^{1,p}}, \quad \text{where } p := \tfrac{4(\alpha+1)}{\alpha+3}.$$

This choice is explained as follows: we have that $W^{1,p} \hookrightarrow L^{p^*}$, so if we desire $p^* = 2\alpha + 2$ we compute

$$\left(\tfrac{1}{p^*} = \tfrac{1}{p} - \tfrac{1}{4} \right) \rightarrow \left(p = \tfrac{4(\alpha+1)}{\alpha+3} \right). \tag{3.57}$$

Since we can assume that $\alpha_0 \leq 2$, we have that $p \in (2, \frac{12}{5}]$, as in Proposition 3.3.3. Then $C_{S,\alpha}$ can in fact then be chosen independent of α, so by taking α_0 and δ_0 as in Proposition 3.3.3, we obtain that, from (3.43),

$$\left\| F_{\widetilde{\Pi}[\blacktriangledown]} - F_{\widetilde{\triangledown}} \right\|_{L^{2\alpha+2}} \leq C_S \left(\delta + \alpha - 1 \right). \tag{3.58}$$

Consequently, we have that,

$$\| F_{\blacktriangledown} \|_{L^{2\alpha+2}} = \left\| F_{\widetilde{\Pi}[\blacktriangledown]} \right\|_{L^{2\alpha+2}} \leq \left\| F_{\widetilde{\Pi}[\blacktriangledown]} - F_{\widetilde{\triangledown}} \right\|_{L^{2\alpha+2}} + \| F_{\widetilde{\triangledown}} \|_{L^{2\alpha+2}} \leq C. \tag{3.59}$$

Furthermore by (3.15)

$$\lambda^{4(\alpha-1)} < \max \left\{ e^{4C\delta}, e^{4\alpha_0 - 4} \right\}. \tag{3.60}$$

Since $\triangledown$ is a critical point of $\mathcal{YM}_{\alpha,\lambda}$ we claim that $\frac{\partial}{\partial \log \tau} \left[\mathcal{YM}_{\alpha,\tau}(\triangledown) \right]_{\tau=\lambda} = 0$. To see this, first note that, via (3.12),

$$\mathcal{YM}_{\alpha,\tau}(\blacktriangledown) = \mathcal{YM}_\alpha((\widehat{\tau^{-1}})^* \blacktriangledown) = \mathcal{YM}_{\alpha,\lambda}(\widehat{\lambda}^* (\widehat{\tau}^{-1})^* \blacktriangledown),$$

and thus

$$\frac{\partial}{\partial \log \tau} \left[\mathcal{YM}_{\alpha,\tau}(\blacktriangledown)) \right]_{\tau=\lambda} = \left(\tau \frac{\partial}{\partial \tau} \left[\mathcal{YM}_{\alpha,\tau}(\blacktriangledown) \right] \right)\Big|_{\tau=\lambda} = \mathcal{YM}'_{\alpha,\lambda}(\blacktriangledown)(\Xi), \tag{3.61}$$

where $\Xi \in \Lambda^1(\mathrm{Ad}\, E)$ is given by

$$\Xi := \left(\tau \frac{\partial}{\partial \tau} \left[\left(\widehat{\lambda \tau^{-1}} \right)^* \blacktriangledown \right] \right)\Big|_{\tau=\lambda}.$$

But, $\blacktriangledown$ is a critical point of $\mathcal{YM}_{\alpha,\lambda}$ and thus $\mathcal{YM}'_{\alpha,\lambda}(\blacktriangledown) = 0$, which forces (3.61) to vanish. Consequently it follows from combining (3.28) (for a lower bound), (3.53), (3.58), (3.59) and (3.60) (for the upper bound) that

$$\frac{(\alpha-1)}{C'} \frac{\log \lambda}{1 + \log \lambda} \leq \frac{\partial}{\partial \log \lambda} \left[\mathcal{YM}_{\alpha,\lambda}(\widetilde{\triangledown}) \right] \leq C (\alpha - 1)(\delta + \alpha - 1). \tag{3.62}$$

We obtain the estimate (3.56) by taking α_0 and δ sufficiently close to 1 and 0 respectively. $\qquad\square$

3.5 Proof of Main Result

Ultimately our goal is to prove the dilation factor of the conformal automorphism is precisely 1, though at this point in our argument, we cannot to better than (3.56); a bound dependent on the closeness in curvature and α-value. Proposition 3.2.1 hints to choose a $\varphi \in \mathrm{SO}(5,1)$ minimizing

$$\left\| F_{\widetilde{\Pi}[\widehat{\varphi}^* \nabla]} - F_{\widetilde{\nabla}} \right\|^2_{L^2,(\mathbb{S}^4, \mathring{g})} = \left\| F_{\widehat{(\varphi^{-1})}^* \widehat{\varphi}^* \nabla} - F_{(\widehat{\varphi}^{-1})^* \widetilde{\nabla}} \right\|^2_{L^2,(\mathbb{S}^4, (\widehat{\varphi}^{-1})^* \mathring{g})} = \left\| F_\nabla - F_{(\widehat{\varphi}^{-1})^* \widetilde{\nabla}} \right\|^2_{L^2,(\mathbb{S}^4, \mathring{g})}. \quad (3.63)$$

For our purposes, however, we will be even more selective in our choice of minimizer to help ourselves in later computations. Noting the relationship of Theorem 3.2.6 between connection and curvature difference, for a fixed $\nabla \in \mathcal{A}_E \left(\mathbb{S}^4 \right)$ we may instead choose to minimize

$$\mathcal{Z}^\nabla (\varphi) := \left\| F_{\widetilde{\Pi}[\widehat{\varphi}^* \nabla]} - F_{\widetilde{\nabla}} \right\|^2_{L^2,(\mathbb{S}^4, \mathring{g})} + \left\| \widetilde{\Pi} [\widehat{\varphi}^* \nabla] - \widetilde{\nabla} \right\|^2_{L^2,(\mathbb{S}^4, \mathring{g})}. \quad (3.64)$$

To justify that this minimization is possible within $\mathrm{SO}(5,1)$, we prove the following.

Lemma 3.5.1. *It is sufficient to minimize $\mathcal{Z}^\nabla (\varphi)$ over a compact subset of $\mathrm{SO}(5,1)$.*

Proof. To do this, we first note that minimizing $\mathcal{Z}^\nabla (\varphi)$ is equivalent to minimizing (3.63). We make the key observation that (up to rotations), φ goes to infinity only if it approaches a dilation from the south pole towards the north pole by large $\lambda > 0$, so that the energy of the dilation portion of φ, given by λ, is concentrating on a small disk $\overline{B}$ centered at the south pole. Since we will be working in various regions of integration we will denote these in our subscripts.

Consider $\overline{B}$ so small that $||F_\nabla||^2_{L^2,(\overline{B})} < \epsilon$. We separate out and divide up the equality, applying the conformal and gauge invariance of the norm,

$$\left\| F_{\widetilde{\Pi}[\nabla]} - F_{\widehat{(\varphi^{-1})}^* \widetilde{\nabla}} \right\|^2_{L^2} = \left\| F_{\widetilde{\Pi}[\nabla]} \right\|^2_{L^2} + 2 \left\langle F_{\widetilde{\Pi}[\nabla]}, F_{\widehat{(\varphi^{-1})}^* \widetilde{\nabla}} \right\rangle_{L^2} + \left\| F_{\widehat{(\varphi^{-1})}^* \widetilde{\nabla}} \right\|^2_{L^2}$$

$$= ||F_\nabla||^2_{L^2} + 2 \left\langle F_{\widetilde{\Pi}[\nabla]}, F_{\widehat{(\varphi^{-1})}^* \widetilde{\nabla}} \right\rangle_{L^2} + ||F_{\widetilde{\nabla}}||^2_{L^2}. \quad (3.65)$$

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Furthermore, we can decompose the middle term into their corresponding parts on or outside of $\overline{B}$:

$$\left\langle F_{\widetilde{\Pi}[\nabla]}, F_{(\varphi^{-1})^*\widetilde{\nabla}} \right\rangle_{L^2} \leq \|F_\nabla\|^2_{L^2,(\overline{B})} \|F_{\widetilde{\nabla}}\|^2_{L^2,(\overline{B})} + \left| \left\langle F_{\widetilde{\Pi}[\nabla]}, F_{\widehat{(\varphi^{-1})}^*\widetilde{\nabla}} \right\rangle_{L^2,(\mathbb{S}^4 - \overline{B})} \right|$$

$$\leq 8\pi^2\epsilon + (\lambda^{-1})^2 \left| \left\langle F_{\widetilde{\Pi}[\nabla]}, F_{\lambda^*\widehat{(\varphi^{-1})}^*\widetilde{\nabla}} \right\rangle_{L^2,(\mathbb{S}^4 - \overline{B})} \right|$$

$$\leq C\epsilon.$$

This implies that $\left\langle F_{\widetilde{\Pi}[\nabla]}, F_{\widehat{(\varphi^{-1})}^*\widetilde{\nabla}} \right\rangle_{L^2}$ is small, and so in particular we can update (3.65),

$$\left\| F_{\widetilde{\Pi}[\nabla]} - F_{\widehat{(\varphi^{-1})}^*\widetilde{\nabla}} \right\|^2_{L^2} \geq 16\pi^2 - C\epsilon \ggg 0,$$

concluding that the minimization of (3.64) occurs on a compact subset of $\mathrm{SO}(5,1)$, away from conformal infinity, as desired. $\qquad\square$

Recall the Yang-Mills energy Jacobi operator given by

$$\left(\mathcal{J}^\nabla(\Xi) \right)^\beta_{i\mu} := -(D_\nabla^* D_\nabla \Xi)^\beta_{i\mu} - (F_\nabla)^\beta_{ki\delta} \Xi^\delta_{k\mu} + \Xi^\beta_{k\delta} (F_\nabla)^\delta_{ki\mu}. \tag{3.66}$$

This can be derived as follows. Varying the Euler-Lagrange equation for Yang-Mills energy,

$$\frac{\partial}{\partial s} \left[D^*_{\nabla_s} F_{\nabla_s} \right]_i \Big|_{\nabla_s = \nabla} = - \frac{\partial}{\partial s} \left[\nabla_k F_{ki} \right] \Big|_{\nabla_s = \nabla} = (D_\nabla^* D_\nabla \dot\Gamma)_i + \left[F_{ki}, \dot\Gamma_k \right].$$

Using the the Bochner formula, which we substitute into (3.66),

$$\triangle_D \Xi_i = - \triangle \Xi_i + 3\Xi_i + [F_{ki}, \Xi_k],$$

we can write the Jacobi operator in the form

$$\left(\mathcal{J}^\nabla(\Xi) \right)_i = \triangle \Xi_i + (DD^*\Xi)_i - 3\Xi_i - 2[F_{ki}, \Xi_k].$$

We will consider $\mathcal{J}^{\widetilde{\nabla}}$, the Jacobi operator based at the identity. The kernel this operator is precisely the tangent space to the moduli space of instantons (this follows from Proposition 4.2.23 of [DK90]), which in turn is precisely the tangent space of the orbit of $\widetilde{\nabla}$ under the action of $\mathrm{SO}(5,1)$. Let $\overline{A}$ denote the orthogonal projection of the 1-form A onto the kernel of $\mathcal{J}^{\widetilde{\nabla}}$ (the Jacobi operator based at $\widetilde{\nabla}$) with respect to the L^2

inner product. Then from ellipticity of the system, we have

$$||A||_{W^{2,p}} \leq C \left(\left|\left| \mathcal{J}^{\widetilde{\nabla}} A \right|\right|_{L^p} + ||\overline{A}||_{L^p} \right). \tag{3.67}$$

Now, assume that φ minimizes $\mathcal{Z}^{\widetilde{\nabla}}(\varphi)$, and again set $\blacktriangledown \equiv \widetilde{\Pi}\left[\widehat{\varphi}^* \nabla\right]$ with $|\varphi| = \lambda$. We now estimate the two terms on the right side of the inequality in the case $A \equiv \Upsilon (= \blacktriangledown - \widetilde{\nabla})$, keeping in mind two main identities:

$$D_{\widetilde{\nabla}}^* \Upsilon = 0 \text{ (due to } \widetilde{\nabla}\text{-Coulomb gauge)}, \qquad \mathcal{J}^{\widetilde{\nabla}}\left(\overline{\Upsilon}\right) = 0 \text{ (due to } \Upsilon \text{ in Ker } \mathcal{J}^{\widetilde{\nabla}}).$$

Proposition 3.5.2. *Assuming the results above*

$$\left|\left| \widetilde{\Pi}[\blacktriangledown] - \widetilde{\nabla} \right|\right|_{W^{2,p}} \leq C(\alpha - 1)\log\lambda. \tag{3.68}$$

Proof. We first estimate the L^p norm of $\overline{\Upsilon}$. From the minimizing property of $\mathcal{Z}^{\nabla}(\varphi)$ we have that,

$$\begin{aligned}
0 &= \int_{\mathbb{S}^4} \left\langle F_\blacktriangledown - F_{\widetilde{\nabla}}, \nabla\overline{\Upsilon} \right\rangle_{\mathring{g}} dV_{\mathring{g}} + \int_{\mathbb{S}^4} \left\langle \Upsilon, \overline{\Upsilon} \right\rangle_{\mathring{g}} dV_{\mathring{g}} \\
&= \int_{\mathbb{S}^4} \left\langle F_\blacktriangledown - F_{\widetilde{\nabla}}, \widetilde{\nabla}\overline{\Upsilon} + [\Upsilon, \overline{\Upsilon}] \right\rangle_{\mathring{g}} dV_{\mathring{g}} + \int_{\mathbb{S}^4} \left\langle \Upsilon, \overline{\Upsilon} \right\rangle_{\mathring{g}} dV_{\mathring{g}} \\
&= \left[\int_{\mathbb{S}^4} \left\langle F_\blacktriangledown - F_{\widetilde{\nabla}}, \widetilde{\nabla}\overline{\Upsilon} \right\rangle_{\mathring{g}} dV_{\mathring{g}} \right]_{T_1} + \left[\int_{\mathbb{S}^4} \left\langle F_\blacktriangledown - F_{\widetilde{\nabla}}, [\Upsilon, \overline{\Upsilon}] \right\rangle_{\mathring{g}} dV_{\mathring{g}} \right]_{T_2} + \int_{\mathbb{S}^4} \left\langle \Upsilon, \overline{\Upsilon} \right\rangle_{\mathring{g}} dV_{\mathring{g}}.
\end{aligned}$$

Using the polarization identity of the curvature, Proposition 3.9.1, we break the terms T_1 and T_2 apart into four labelled integrals:

$$\begin{aligned}
||\overline{\Upsilon}||_{L^2}^2 &= \left[\int_{\mathbb{S}^4} \left(\widetilde{D}_i \Upsilon_{j\zeta}^\beta\right) \left(\widetilde{\nabla}_i \overline{\Upsilon}_{j\beta}^\zeta\right) dV_{\mathring{g}} \right]_{T_{11}} + \left[\int_{\mathbb{S}^4} \left(\widetilde{D}_i \Upsilon_{j\zeta}^\beta\right) [\Upsilon_i, \overline{\Upsilon}_j]_\beta^\zeta \, dV_{\mathring{g}} \right]_{T_{21}} \\
&\quad + \left[\int_{\mathbb{S}^4} [\Upsilon_i, \Upsilon_j]_\zeta^\beta \left(\widetilde{\nabla}_i \overline{\Upsilon}_{j\beta}^\zeta\right) dV_{\mathring{g}} \right]_{T_{12}} + \left[\int_{\mathbb{S}^4} [\Upsilon_i, \Upsilon_j]_\zeta^\beta [\Upsilon_i, \overline{\Upsilon}_j]_\beta^\zeta \, dV_{\mathring{g}} \right]_{T_{22}}.
\end{aligned}$$

Let's manipulate these each separately. We have that

$$T_{11} = \int_{\mathbb{S}^4} \left(\widetilde{D}_i \Upsilon_{j\zeta}^\beta\right) \left(\widetilde{\nabla}_i \overline{\Upsilon}_{j\beta}^\zeta\right) dV_{\mathring{g}} = \int_{\mathbb{S}^4} \left(\widetilde{\nabla}_i \Upsilon_{j\zeta}^\beta\right) \left(\widetilde{D}_i \overline{\Upsilon}_{j\beta}^\zeta\right) dV_{\mathring{g}} = \int_{\mathbb{S}^4} \Upsilon_{j\zeta}^\beta \left(\widetilde{D}^* \widetilde{D}\overline{\Upsilon}\right)_{j\beta}^\zeta dV_{\mathring{g}}.$$

Next we approach T_{21}. Here we have

$$T_{21} = \int_{\mathbb{S}^4} \left(\widetilde{D}_i \Upsilon_{j\zeta}^\beta\right) [\Upsilon_i, \overline{\Upsilon}_j]_\beta^\zeta \, dV_{\mathring{g}} = 2 \int_{\mathbb{S}^4} \left(\widetilde{\nabla}_i \Upsilon_{j\zeta}^\beta\right) [\Upsilon_i, \overline{\Upsilon}_j]_\beta^\zeta \, dV_{\mathring{g}} = 2 \int_{\mathbb{S}^4} \Upsilon_{i\zeta}^\beta \left[\Upsilon_j, \left(\widetilde{\nabla}_i \overline{\Upsilon}_j\right)\right]_\zeta^\beta dV_{\mathring{g}}.$$

If we apply the fact that $D^*\Upsilon \equiv 0$ we quickly obtain that $T_{21} = -\frac{1}{2}T_{12}$. For T_{22}, we use the skew symmetry

with respect to the bundle indices,

$$T_{22} = \int_{\mathbb{S}^4} [\Upsilon_i, \Upsilon_j]_\zeta^\beta [\Upsilon_i, \overline{\Upsilon}_j]_\beta^\zeta \, dV_{\hat{g}} = 2 \int_{\mathbb{S}^4} \Upsilon_{i\zeta}^\beta \Upsilon_{j\delta}^\zeta [\Upsilon_i, \overline{\Upsilon}_j]_\beta^\delta \, dV_{\hat{g}}.$$

Combining these together, we have that

$$||\overline{\Upsilon}||_{L^2}^2 = \sum_{i,j=1}^2 T_{ij} = \int_{\mathbb{S}^4} \Upsilon_{j\mu}^\beta \left(\widetilde{D}^* \widetilde{D} \overline{\Upsilon} \right)_{j\beta}^\mu + \Upsilon_{i\mu}^\beta \left[\Upsilon_j, \left(\widetilde{\nabla}_i \overline{\Upsilon}_j \right) \right]_\mu^\beta + 2 \Upsilon_{i\zeta}^\beta \Upsilon_{j\mu}^\zeta [\Upsilon_i, \overline{\Upsilon}_j]_\beta^\mu$$

Applying (3.66) noting that $\mathcal{J}^{\widetilde{\nabla}} \left(\Upsilon \right) \equiv 0$ to the first term on the right, we have that

$$\begin{aligned}
||\overline{\Upsilon}||_{L^2}^2 &= \int_{\mathbb{S}^4} -\Upsilon_{j\mu}^\beta \left[\widetilde{F}_{kj}, \overline{\Upsilon}_k \right]_\beta^\mu + \Upsilon_{i\mu}^\beta \left[\Upsilon_j, \left(\widetilde{\nabla}_i \overline{\Upsilon}_j \right) \right]_\mu^\beta + 2 \Upsilon_{i\zeta}^\beta \Upsilon_{j\mu}^\zeta [\Upsilon_i, \overline{\Upsilon}_j]_\beta^\mu \\
&\leq C \left\| \Upsilon \right\|_{L^2} \left\| \overline{\Upsilon} \right\|_{L^2} + \left\| \Upsilon \right\|_{L^\infty}^2 \left\| \widetilde{\nabla} \overline{\Upsilon} \right\|_{L^2} + \left\| \overline{\Upsilon} \right\|_{L^2} \left\| \Upsilon^{*3} \right\|_{L^2} \\
&\leq C\delta \left(\left\| \overline{\Upsilon} \right\|_{L^2} + \left\| \widetilde{\nabla} \overline{\Upsilon} \right\|_{L^2} \right).
\end{aligned} \tag{3.69}$$

Now, we claim furthermore that $\left\| \widetilde{\nabla} \overline{\Upsilon} \right\|_{L^2} \leq C \left\| \overline{\Upsilon} \right\|_{L^2}$. To see this, consider the ratio $\left\| \widetilde{\nabla} \overline{\Upsilon} \right\|_{L^2} \left\| \overline{\Upsilon} \right\|_{L^2}^{-1}$. This is scale invariant in $\overline{\Upsilon}$ and when restricted to $\{ A \in \Lambda^1 \left(\mathrm{Ad}\, E \right) : |A|_{\mathring{g}} \equiv 1 \}$ has a maximum and minimum, and is thus bounded. So we can update (3.69) by applying this estimate and dividing through by $\left\| \overline{\Upsilon} \right\|_{L^2}$ to conclude that $\left\| \overline{\Upsilon} \right\|_{W^{1,2}} \leq C\delta$, and using the Sobolev embedding $W^{1,2} \hookrightarrow L^p$, we conclude $\left\| \overline{\Upsilon} \right\|_{L^p}^2 \leq C_S \delta$.

Now, we estimate the Jacobi operator term, by observing that, by subtracting Θ_1 and Θ_2 from both sides of (3.45) and inserting in (3.49), and then finally observing the presence of the terms of the $\mathcal{J}^{\widetilde{\nabla}}$ (this is the first line of the right hand side of (3.49)), with rearrangement we obtain

$$\begin{aligned}
\left(\mathcal{J}^{\widetilde{\nabla}} \left(\Upsilon \right) \right)_{i\theta}^\beta &= \left(\widetilde{\nabla}_i \Upsilon_{k\mu}^\beta \right) \Upsilon_{k\theta}^\mu - \Upsilon_{k\mu}^\beta \left(\widetilde{\nabla}_i \Upsilon_{k\theta}^\mu \right) + 2 \left(\widetilde{\nabla}_k \Upsilon_{i\mu}^\beta \right) \Upsilon_{k\theta}^\mu \\
&\quad - \Upsilon_{i\zeta}^\beta \Upsilon_{k\mu}^\zeta \Upsilon_{k\theta}^\mu + \Upsilon_{k\mu}^\beta \Upsilon_{k\zeta}^\mu \Upsilon_{i\theta}^\zeta + \left(\Theta_1 \left(\nabla \right) \right)_{i\theta}^\beta + \left(\Theta_2 \left(\nabla \right) \right)_{i\theta}^\beta.
\end{aligned}$$

In fact, we actually obtained all of these term types from the proof of Proposition 3.3.3. Therefore

$$\left\| \mathcal{J}^{\widetilde{\nabla}} \left(\Upsilon \right) \right\|_{W^{1,2}} \leq C \left(\delta + (\alpha - 1) \right) \left(1 + \log \lambda \right).$$

Applying the Sobolev embedding, $L^p \hookrightarrow W^{1,2}$, by taking δ_0 and $(\alpha_0 - 1)$ both sufficiently small, we may conclude that by inserting our estimates into (3.67) and absorbing proper terms across the inequality, we obtain (3.68). $\qquad \square$

Proof of Theorem D. We refer back to the proof of Proposition 3.4.3 using the improved estimate in (3.68) obtain

$$\left\| F_{\widetilde{\Pi}[\blacktriangledown]} - F_{\widetilde{\nabla}} \right\|_{L^{2\alpha+2}} \leq C\left(\alpha - 1\right) \log \lambda.$$

Then the inequalities in (3.62) transform to

$$\frac{(\alpha-1)}{C'} \frac{\log \lambda}{1+\log \lambda} \leq \frac{\partial}{\partial \log \lambda} \left[\mathcal{YM}_{\alpha,\lambda}\left(\widetilde{\nabla}\right) \right] \leq C\left(\alpha - 1\right)^2 \log \lambda.$$

By taking α sufficiently close to (but not equal to 1) and knowing that λ is greater than 1 and bounded, we conclude that $\lambda \equiv 1$. However, using (3.68) we conclude that Υ vanishes, that is, $\nabla \equiv \widetilde{\nabla}$, as desired. $\qquad \square$

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Appendix

3.6 Analytic background

We begin by recalling key analytical theorems used throughout our work.

Theorem 3.6.1 (Arzela-Ascoli Theorem). *Let $M = \mathbb{R}^n$. Let $\alpha \in (0,1]$ and $R \in \mathbb{R}_{>0}$. Given some $K > 0$, $\alpha' < \alpha$ and a sequence $\{f_i\}$ such that*

$$||f_i||_{C^{\varrho,\alpha}(B_R)} \leq K,$$

there exists a subsequence $\{f_{i'}\}$ and some $f_\infty \in C^{\varrho,\alpha'}(B_R)$ such that $f_{i'} \to f_\infty$ with respect to the $C^{\varrho,\alpha'}(B_R)$ norm.

Theorem 3.6.2 (Sobolev Imbedding Theorems, pp.35 of [Aub98]). *Set $M = \mathbb{R}^n$. Let $i, j \in \mathbb{N} \cup \{0\}$ with $i < j$, and $p, q \in [1, \infty)$ with $1 \leq q < p$ such that $\frac{1}{p} = \frac{1}{q} - \frac{(j-i)}{n}$. Then $H_j^q \subset H_i^p$ and the identity operator is continuous, and the following holds.*

1. *(First Sobolev Imbedding Theorem) If there exists $\varrho \in \mathbb{N} \cup \{0\}$ such that $\frac{(j-\varrho)}{n} > \frac{1}{q}$ then $H_j^q \subset C^{\varrho,0}$ and the identity operator is continuous.*

2. *(Second Sobolev Imbedding Theorem) If there exists $\varrho \in \mathbb{N} \cup \{0\}$ such that $\frac{(j-\varrho-\alpha)}{n} \leq \frac{1}{q}$, then $H_j^q \subset C^{\varrho,\alpha}$.*

Lemma 3.6.3 (Kato's inequality). *Suppose L is some multiindex and $\omega \in (TM)^{\otimes|L|}$. Then if $|\omega| \neq 0$,*

$$|\nabla|\omega_L|| \leq |\nabla \omega_L|.$$

Convexity estimates

We next extend two convexity estimates (Corollary 5.3 and 5.5) of [KS86] to be applied to L^q norms of elements of $\Lambda^p(\text{End}\, E)$ for $\ell \in \mathbb{N}$ (rather than elements of $C^\infty(M)$). The resulting corollary (cf. Corollary 3.6.5) of the first result mentioned combined with the second result (cf. Lemma 3.6.6) are key in the smoothing estimates of §2.2.2. For a given $\eta \in \mathcal{B}$, recall the definition of $\jmath_\eta^v$ which is a constant bounding the $L^\infty(M)$ norms of η (cf. Definition 2.2.9). We now state an analogue of Corollary 5.3 of [KS86]

Lemma 3.6.4. *Let* $\nabla \in \mathcal{A}_E$ *and* $\eta \in \mathcal{B}$. *For* $2 \le p < \infty$, $\ell \in \mathbb{N}$, $s \ge \ell p$, *there exists*

$$C_\epsilon = C_\epsilon\left(\dim M, \text{Rank}\, E, p, q, s, \ell, g, \jmath_\eta^{(1)}\right) \in \mathbb{R}_{>0},$$

such that for $\phi \in C^\infty(M)$ *we have,*

$$\left(\int_M \left|\nabla^{(\ell)}\phi\right|^p \eta^s dV_g\right)^{1/p} \le \epsilon \left(\int_M \left|\nabla^{(\ell+1)}\phi\right|^p \eta^{s+p} dV_g\right)^{1/p} + C_\epsilon \left(\int_{\text{supp}\,\eta} |\phi|^p \eta^{s-\ell p} dV_g\right)^{1/p}.$$

Proof. Note that this is simply Corollary 5.3 of [KS86] applied to $\left|\nabla^{(\ell)}\phi\right|$. $\qquad\square$

An immediate consequence of this lemma is an interpolation identity obtained via iterating the inequality is the following corollary.

Corollary 3.6.5. *Let* $\nabla \in \mathcal{A}_E$ *and* $\eta \in \mathcal{B}$. *For* $2 \le p < \infty$, $\ell \in \mathbb{N}$, $s \ge \ell p$, *there exists some* $C_\epsilon = C_\epsilon\left(\dim M, \text{Rank}\, E, p, q, s, \ell, g, \jmath_\eta^{(1)}\right) \in \mathbb{R}_{>0}$ *such that for* $\phi \in C^\infty(M)$,

$$\left\|\eta^{\frac{s}{p}}\nabla^{(\ell)}\phi\right\|_{L^p} \le \epsilon \left\|\eta^{\frac{s+jp}{p}}\nabla^{(\ell+j)}\phi\right\|_{L^p} + C_\epsilon \left\|\phi\right\|_{L^p, \eta>0}. \tag{3.70}$$

In particular for $p = 2$ *and some constant* $K \ge 1$,

$$K\left\|\eta^{\frac{s}{2}}\nabla^{(\ell)}\phi\right\|_{L^2}^2 \le \epsilon \left\|\eta^{\frac{s+2j}{2}}\nabla^{(\ell+j)}\phi\right\|_{L^2}^2 + C_\epsilon K^2 \left\|\phi\right\|_{L^2, \eta>0}^2. \tag{3.71}$$

Proof. This simply follows by induction. The base case is given by Lemma 3.6.4. Now assume that for $j \in \mathbb{N}$ that (3.70) holds. Without loss of generality, we consider the equality with ϵ replaced by $\sqrt{\epsilon}$. Manipulating the first term on the right we have, applying Lemma 3.6.4,

$$\left\|\eta^{\frac{s+jp}{p}}\nabla^{(\ell+j)}\phi\right\|_{L^p} \le \sqrt{\epsilon}\left\|\eta^{\frac{s+p(j+1)}{p}}\nabla^{(\ell+j+1)}\phi\right\|_{L^p} + C_{\sqrt{\epsilon}}\left\|\phi\right\|_{L^p, \eta>0}.$$

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Inserting this into (3.70) with ϵ replaced with $\sqrt{\epsilon}$, we conclude the result. Consequently we have the case for $j + 1$, so inductively the result holds for all $\mathbb{N}$.

The second identity (3.71) is essentially representing the direct application of this lemma in §2.2.2, which is strictly in the setting where $p = 2$ and the computations feature quantities with their L^2 norm *squared*. Therefore we note one more manipulation where we square both sides of the inequality and apply Hölder's inequality. Given $a, b, c \in \mathbb{R}_{\geq 0}$, if $a \leq b + c$, then

$$a^2 \leq (b + c)^2 = b^2 + 2bc + c^2 \leq b^2 + 2\left(\frac{b^2}{2} + \frac{c^2}{2}\right) + c^2 \leq 2\left(b^2 + c^2\right).$$

Therefore if we are using the weighted versions of the inequality then they hold for the squared norms too. This is a minor manipulation but should be noted. Additionally, note that the shift of the 'weight' K featured in (3.71) is merely a consequence of weighted Hölder's inequality being featured through each of these iterated computations. $\square$

Lemma 3.6.6 (Analogue of Corollary 5.5 of [KS86]). *Let $\nabla \in \mathcal{A}_E$ and $\eta \in \mathcal{B}$. Let $r, w, s \in \mathbb{N}$ and $\phi \in \Lambda^p(\mathrm{End}\,E)$ with $s \geq 2w$ and $0 \leq i_1, \ldots, i_r \leq w$, so that $\sum_{j=1}^{r} i_j = 2w$. Then there exists some $Q_{(w,r)} := Q\left(\dim M, \mathrm{Rank}\,E, p, s, g, j_\eta^{(1)}, w, r\right) \in \mathbb{R}_{>0}$ such that*

$$\int_M \eta^s \left(\nabla^{(i_1)}\phi * \ldots * \nabla^{(i_r)}\phi\right) dV_g \leq Q_{(w,r)} ||\phi||_{L^\infty}^{r-2} \left(||\eta^{s/2}\nabla^{(w)}\phi||_{L^2(M)}^2 + ||\phi||_{L^2(M),\eta>0}^2\right).$$

3.7 Connection identities

We next provide some key identities regarding manipulations applied to the connections throughout various identities. We first state standard elementary manipulations and key formulas such as Bochner formula (cf. Proposition 3.7.1) in the preliminary identities section (§3.7), then state some basic scaling laws of connections and curvatures (§3.7) and then introduce manipulations to address the higher order differential operators which appear within the various studied flows.

Preliminary identities

We first begin with a statement of the main Bochner formulas which are utilized through our various arguments.

Proposition 3.7.1 (Bochner formula). *Let $\nabla \in \mathcal{A}_E$ and $\omega \in \Lambda^p(\operatorname{End} E)$. Then the following equality holds.*

$$\triangle_D \omega = -\triangle \omega + \operatorname{Rm} * \omega + F_\nabla * \omega. \tag{3.72}$$

In particular, for $p = 1$,

$$(\triangle_D \omega)_{i\alpha}^\beta = -\nabla^k(\nabla_k \omega_{i\alpha}^\beta) + \operatorname{Rc}_i^p \omega_{p\alpha}^\beta + g^{jk}\left((F_\nabla)_{ij\alpha}^\delta \omega_{k\delta}^\beta - (F_\nabla)_{ij\delta}^\beta \omega_{k\alpha}^\delta\right), \tag{3.73}$$

given invariantly by $\triangle_D \omega = -\triangle \omega + \operatorname{Rc}(\omega^\sharp, \cdot) + [F_\nabla, \omega]^\#$. For $p = 2$,

$$(\triangle_D \omega)_{i\ell\alpha}^\beta = -\nabla^k \nabla_k \omega_{i\ell\alpha}^\beta + \left(-\operatorname{Rc}_\ell^p \omega_{pi\alpha}^\beta + \operatorname{Rc}_i^p \omega_{p\ell\alpha}^\beta - g^{qk}\operatorname{Rm}_{i\ell q}^p \omega_{kp\alpha}^\beta\right) + \left([\omega, F_\nabla]^\#\right)_{i\ell\alpha}^\beta - \left([\omega, F_\nabla]^\#\right)_{\ell i\alpha}^\beta. \tag{3.74}$$

The Bochner formula can be seen as a consequence of the following technical lemma which states the terms that appear when commuting connections. We state this result in terms of explicit coordinates and demonstrate the proof for future use. We then follow with a generalized Bochner formula statement.

Lemma 3.7.2. *For $K = (k_s)_{s=1}^{|K|}$ be a multiindex and $\omega \in S\left((T^*M)^{\otimes|K|} \otimes \operatorname{End} E\right)$,*

$$[\nabla_i, \nabla_j]\omega_{K\alpha}^\beta = \operatorname{Rm}_{ijk_\ell}^p(\omega_{K(\ell,p)\alpha}^\beta) - (F_\nabla)_{ij\alpha}^\delta \omega_{K\delta}^\beta + (F_\nabla)_{ij\delta}^\beta \omega_{K\alpha}^\delta. \tag{3.75}$$

Where '$K(s,p)$' means replacing k_s with p in the multiindex K.

Proof. Use of normal coordinates in the computation of the commutator of connections yields

$$\begin{aligned}
[\nabla_i, \nabla_j]\omega_{K\alpha}^\beta &= \nabla_i(\nabla_j \omega_{K\alpha}^\beta) - \nabla_j(\nabla_i \omega_{K\alpha}^\beta) \\
&= \partial_i\left(\partial_j \omega_{K\alpha}^\beta - G_{jk_\ell}^p \omega_{K(\ell,p)\alpha}^\beta - \Gamma_{j\alpha}^\delta \omega_{K\delta}^\beta + \Gamma_{j\delta}^\beta \omega_{K\alpha}^\delta\right) \\
&\quad - \partial_j\left(\partial_i \omega_{K\alpha}^\beta - G_{ik_\ell}^p \omega_{K(\ell,p)\alpha}^\beta - \Gamma_{i\alpha}^\delta \omega_{K\delta}^\beta + \Gamma_{i\delta}^\beta \omega_{K\alpha}^\delta\right) \\
&= \left(\partial_j G_{ik_\ell}^p - \partial_i G_{jk_\ell}^p\right)\omega_{K(\ell,p)\alpha}^\beta + \left(\partial_j \Gamma_{i\alpha}^\delta - \partial_i \Gamma_{j\alpha}^\delta\right)\omega_{K\delta}^\beta \\
&\quad + \left(-\partial_j \Gamma_{i\delta}^\beta + \partial_i \Gamma_{j\delta}^\beta\right)\omega_{K\alpha}^\delta \\
&= \operatorname{Rm}_{ijk_\ell}^p(\omega_{K(\ell,p)\alpha}^\beta) - F_{ij\alpha}^\delta \omega_{K\delta}^\beta + F_{ij\delta}^\beta \omega_{K\alpha}^\delta.
\end{aligned}$$

The result follows. $\qquad\square$

Higher order identities

The following identities are key in manipulating higher order terms which appear in generalized Yang-Mills k-flow and are primarily results of recursive integration by parts and commuting of connections.

This next technical lemma prepares an expression to draw out iterations of the Laplacian after integrating by parts, which is performed in the following result (cf. Lemma 3.7.4).

Lemma 3.7.3. *Suppose* $\nabla \in \mathcal{A}_E$ *and* ξ *in the domain of* ∇. *Let* $I = (i_v)_{v=1}^{|I|}$ *and* $J = (j_v)_{v=1}^{|J|}$ *be two multiindices with* $|I| = |J| = k$. *Let* S *and* Q *be two multiindices consisting of entries corresponding to both bundle and base manifold. The following identity holds.*

$$\left(\nabla_{i_k}\cdots\nabla_{i_1}\nabla_{j_1}\cdots\nabla_{j_k}\xi_Q^S\right) = \left(\nabla_{i_k}\nabla_{j_k}\nabla_{i_{k-1}}\cdots\nabla_{i_1}\nabla_{j_1}\xi_Q^S\right) + \sum_{\ell=1}^{2k-2}\sum_{w=0}^{\ell}\left(\nabla^{(w)}(\mathrm{Rm}+F_\nabla)*\nabla^{(k-2-w)}\xi_Q^S\right).$$

$$(3.76)$$

Proof. For notational simplicity, given a multiindex $I = (i_v)_{v=1}^{|I|}$, set $\nabla_{i_1\cdots i_{|I|}} := \nabla_{i_1}\cdots\nabla_{i_{|I|}}$. Iterating Lemma 3.7.2 as covariant derivatives are interchanged one obtains

$$
\begin{aligned}
\left(\nabla_{i_\ell}\cdots\nabla_{i_1}\nabla_{j_1}\cdots\nabla_{j_\ell}\xi_Q^S\right) &= \left(\nabla_{i_\ell}\nabla_{j_\ell}\nabla_{i_{\ell-1}\cdots i_1}\nabla_{j_1\cdots j_{\ell-1}}\xi_Q^S\right) \\
&\quad + \sum_{v=1}^{\ell-1}\left(\nabla_{i_\ell\cdots i_{v-1}}[\nabla_{i_v},\nabla_{j_\ell}]\nabla_{i_{v+1}\cdots i_\ell}\nabla_{j_1\cdots j_{\ell-1}}\right)\xi_Q^S \\
&\quad + \sum_{v=1}^{\ell-1}\left(\nabla_{i_1\cdots i_\ell}\nabla_{j_1\cdots j_{v-1}}[\nabla_{j_v},\nabla_{j_\ell}]\nabla_{j_{v+1}\cdots j_{\ell-1}}\right)\xi_Q^S \\
&= \left(\nabla_{i_\ell}\nabla_{j_\ell}\nabla_{i_{\ell-1}\cdots i_1}\nabla_{j_1\cdots j_{\ell-1}}\xi_Q^S\right) \\
&\quad + \sum_{v=1}^{2\ell-2}\left(\nabla^{(v)}\left((\mathrm{Rm}+F)*\nabla^{(2\ell-2-\ell)}\xi\right)\right)_{i_1\cdots i_\ell j_1\cdots j_\ell Q}^{S} \\
&= \left(\nabla_{i_\ell}\nabla_{j_\ell}\nabla_{i_{\ell-1}}\cdots\nabla_{i_1}\nabla_{j_1}\xi_Q^S\right) \\
&\quad + \sum_{v=1}^{2\ell-2}\sum_{w=0}^{v}\left(\nabla^{(w)}(\mathrm{Rm}+F)*\nabla^{(2\ell-2-w)}\xi\right)_{i_1\cdots i_\ell j_1\cdots j_\ell Q}^{S}.
\end{aligned}
$$

The result follows. $\qquad\square$

Lemma 3.7.4. *Let $\triangledown \in \mathcal{A}_E$, and ζ, ξ in the domain of $\triangledown$. Then for $\ell \in \mathbb{N}$,*

$$\int_M \left\langle \triangledown^{(\ell)}\zeta, \triangledown^{(\ell)}\xi \right\rangle dV_g = \int_M \langle \zeta, (-1)^k \triangle^{(\ell)} \xi \rangle dV_g + \left\langle \zeta, \sum_{v=1}^{2\ell-2} \sum_{w=0}^{v} \left(\triangledown^{(w)} [\mathrm{Rm} + F_{\triangledown}] * \triangledown^{(2\ell-2-w)} \xi \right) \right\rangle.$$

(3.77)

Proof. Let P, Q, S represent multiindices consisting of base manifold and bundle indices (roman and greek letters respectively), and let $\mathfrak{g}$ denote products of the bundle metric h and the manifold metric g corresponding to such multiindices. Integrating by parts yields

$$\int_M \left\langle \triangledown^{(\ell)}\zeta, \triangledown^{(\ell)}\xi \right\rangle dV_g = \int_M \left(\prod_{v=0}^{\ell} g^{i_v j_v} \right) \mathfrak{g}^{PQ} \mathfrak{g}_{QS} \left(\triangledown_{i_1 \cdots i_\ell} \zeta_P^R \right) \left(\triangledown_{j_1 \cdots j_\ell} \xi_Q^S \right) dV_g$$

$$= (-1)^\ell \int_M \left(\prod_{v=0}^{\ell} g^{i_v j_v} \right) \mathfrak{g}^{PQ} \mathfrak{g}_{RS} \zeta_P^R \left(\triangledown_{i_\ell \cdots i_1} \triangledown_{j_1 \cdots j_\ell} \xi_Q^S \right) dV_g.$$

Using Lemma 3.7.3 to manipulate the quantity yields

$$\int_M \langle \triangledown^{(\ell)}\zeta, \triangledown^{(\ell)}\xi \rangle dV_g = \int_M \langle \zeta, (-1)^\ell \triangle^{(\ell)} \xi \rangle dV_g + \left\langle \zeta, \sum_{v=1}^{2\ell-2} \sum_{w=0}^{v} \left(\triangledown^{(w)} [\mathrm{Rm} + F_{\triangledown}] * \triangledown^{(2\ell-2-w)} \xi \right) \right\rangle.$$

The result follows. $\square$

The next two lemmas are formal manipulations of commuting connections and Laplacians in order to keep track of the lower order terms which appear when these are performed.

Lemma 3.7.5. *For $\triangledown \in \mathcal{A}_E$, $\ell \in \mathbb{N}$ and $\omega \in \Lambda^p(\mathrm{End}\, E)$,*

$$\triangledown \triangle^{(\ell)} \omega = \triangle^{(\ell)} \triangledown\omega + \sum_{j=0}^{2\ell-1} \triangledown^{(j)} [\mathrm{Rm} + F_{\triangledown}] * \triangledown^{(2\ell-1-j)}\omega.$$

(3.78)

Proof. The proof follows by induction over $\ell \in \mathbb{N}$ satisfying (3.78). For the following computations we will

excise the indices from ω for computational ease. For the base case, computing $\nabla\triangle$ using Lemma 3.7.2 gives

$$
\begin{aligned}
\nabla_i \triangle \omega &= g^{jk}\nabla_i\nabla_j\nabla_k\omega \\
&= g^{jk}\left(\nabla_j\nabla_i\nabla_k\omega + [\nabla_i,\nabla_j]\nabla_k\omega\right) \\
&= g^{jk}\left(\nabla_j\nabla_k\nabla_i\omega + \nabla_j\left([\nabla_i,\nabla_k]\omega\right) + [\nabla_i,\nabla_j]\nabla_k\omega\right) \\
&= \triangle\nabla_i\omega + \left(\nabla\left[(\mathrm{Rm}+F)*\omega\right]\right)_i + \left((\mathrm{Rm}+F)*\nabla\omega\right)_i \\
&= \triangle\nabla_i\omega + \left(\sum_{j=0}^{1}\nabla^{(j)}\left[\mathrm{Rm}+F\right]*\nabla^{(1-j)}\omega\right)_i.
\end{aligned}
$$

The base case follows. Now let $\ell \in \mathbb{N}$ and suppose (3.78) is satisfied by $\ell - 1$. Applying this to the ℓ case yields

$$
\begin{aligned}
\nabla\triangle^{(\ell)}\omega &= \nabla\triangle^{(\ell-1)}\left(\triangle\omega\right) \\
&= \triangle^{(\ell-1)}\nabla\triangle\omega + \sum_{j=0}^{2\ell-3}\nabla^{(j)}\left[\mathrm{Rm}+F\right]*\nabla^{(2\ell-3-j)}\triangle\omega.
\end{aligned}
\tag{3.79}
$$

Expanding and manipulating the left term gives

$$
\begin{aligned}
\triangle^{(\ell-1)}\nabla\triangle\omega &= \triangle^{(\ell-1)}\left[\triangle\nabla\omega + \sum_{j=0}^{1}\nabla^{(j)}\left[\mathrm{Rm}+F\right]*\nabla^{(1-j)}\omega\right] \\
&= \triangle^{(\ell)}\nabla\omega + \triangle^{(\ell-1)}\left[\sum_{j=0}^{1}\nabla^{(j)}\left[\mathrm{Rm}+F\right]*\nabla^{(1-j)}\omega\right] \\
&= \triangle^{(\ell)}\nabla\omega + \left(\sum_{j=0}^{1}\nabla^{(2\ell-2)}\left(\nabla^{(j)}\left[\mathrm{Rm}+F\right]*\nabla^{(1-j)}\omega\right)\right) \\
&= \triangle^{(\ell)}\nabla\omega + \left(\sum_{j=0}^{1}\sum_{(p+q=2\ell-2)}\left(\nabla^{(p+j)}\left[\mathrm{Rm}+F\right]*\nabla^{(1-j+q)}\omega\right)\right).
\end{aligned}
$$

Updating (3.79) we obtain

$$
\nabla\triangle^{(\ell)}\omega = \triangle^{(\ell)}\nabla\omega + \sum_{j=0}^{2\ell-1}\nabla^{(j)}\left[\mathrm{Rm}+F_\nabla\right]*\nabla^{(2\ell-1-j)}\omega.
$$

The result follows. $\qquad\square$

Corollary 3.7.6. *Let* $\triangledown \in \mathcal{A}_E$, *take* $v, w \in \mathbb{N}$ *and* $\omega \in \Lambda^p(\mathrm{End}\, E)$. *Then*

$$\triangledown^{(v)} \triangle^{(w)} \omega = \triangle^{(w)} \triangledown^{(v)} \omega + \sum_{b=0}^{v-1} \sum_{j=0}^{2w-1} \left(\triangledown^{(b+j)} \left[\mathrm{Rm} + F_{\triangledown} \right] * \triangledown^{(\ell - b + 2w - 2 - j)} \omega \right).$$

Scaling laws

We introduce key scaling properties of connections and corresponding quantities. This determines the critical dimension of Yang-Mills k-flow, and is applied primarily in the blowup analysis (§2.2.4) and flow long time existence results (§2.3). We first show how iterations of a scaled connection act on a similarly scaled 1-form.

Lemma 3.7.7. *Suppose* $\triangledown$ *is a connection and let* $x \in M$ *such that in a coordinate chart containing* x *the coefficient matrix of* $\triangledown$ *is* Γ. *Let* $\omega \in S(E)$ *and set*

$$\Gamma^{\lambda}(x) := \lambda \Gamma(\lambda x) \ and \ \omega_{\lambda}(x) := \omega(\lambda x).$$

Let $\triangledown^{\lambda}$ *denote the connection with coefficient matrix* Γ^{λ}. *Then for all* $\ell \in \mathbb{N}$,

$$\triangledown^{(\ell)}_{\lambda} \omega_{\lambda} = \lambda^{\ell} \triangledown^{(\ell)} \omega.$$

Proof. We observe that in the case $\ell = 1$,

$$(\triangledown^{\lambda})_i (\omega_{\lambda})^{\alpha} = \partial_i (\omega_{\lambda})^{\alpha} + (\Gamma^{\lambda})^{\alpha}_{i\delta} (\omega_{\lambda})^{\delta} = \lambda \partial_i \omega^{\alpha} + \lambda \Gamma^{\alpha}_{i\delta} \omega^{\delta}.$$

Iterating this operation of $\triangledown$ yields the desired result. $\qquad\square$

Remark 3.7.8. Since $F_{\triangledown} = D_{\triangledown} \circ D_{\triangledown}$, then it follows that for $\triangledown^{\lambda}$ as defined above over the appropriate vector bundles, $F_{\triangledown^{\lambda}} = \lambda^2 F_{\triangledown}$.

We now demonstrate how the above scaling effects the L^p norm of the curvature. This is key in determining the critical dimension of the Yang-Mills k-energies as well as performing their blowup analyses. In preparation for this, we consider the scaling of the L^p norm of rescaled curvature.

Proposition 3.7.9. *For* $\lambda \in \mathbb{R}$, *set* $F^{\lambda}(x) := \lambda^q F(\lambda^r x)$. *Then it follows that*

$$||F^{\lambda}||^p_{L^p(B_1)} = \lambda^{qp - nr} ||F||^p_{L^p(B_{\lambda^r})}.$$

Proof. Expressing the L^p norm of F_λ in terms of F, with condensed notation $dx^{1\cdots n} = dx^1 \wedge \cdots \wedge dx^n$ yields

$$||F^\lambda||_{L^p}^p = \int_{B_1} |F^\lambda|^p dx^{1\cdots n} = \int_{B_1} \lambda^{qp} |F(\lambda^r x)|^p dx^{1\cdots n}.$$

We change variables by to $y^i = \lambda^r x^i$, giving $dy^{1\cdots n} = \lambda^{nr} dx^{1\cdots n}$. Applying this to the above equality,

$$||F^\lambda||_{L^p}^p = \lambda^{qp} \int_{B_{\lambda^r}} |F(\lambda^r)|^p \tfrac{1}{\lambda^{nr}} dy^{1\cdots n} = \lambda^{qp-nr} ||F||_{L^p(B_{\lambda^r})}^p.$$

The result follows. □

3.8   Connections and gauge transformations

The main complications and interesting properties of the Yang-Mills flow stem from the interactions with the gauge group with the connections. We provide multiple identities which characterize these interactions and their consequences.

Definition 3.8.1 (Gauge transformation). A gauge transformation ς is a section of $\operatorname{Aut} E$. The group of gauge transformations is called the *gauge group* of the metric bundle E, denoted by $\mathcal{G}_E$. The action of a gauge transformation ς on a connection ∇ is denoted by $\varsigma[\nabla]$ and given by

$$\varsigma : \mathcal{A}_E \to \mathcal{A}_E$$
$$: \nabla \mapsto \varsigma[\nabla] := \varsigma^{-1} \circ \nabla \circ \varsigma,$$

that is, for some $\mu \in S(E)$ we have $\varsigma[\nabla] = \varsigma^{-1}\nabla(\varsigma\mu)$. Furthermore, for $\phi \in S(\operatorname{End} E)$, define the action of ς on ϕ by

$$(\varsigma[\phi])_\alpha^\beta := (\varsigma^{-1})_\zeta^\beta \phi_\tau^\zeta \varsigma_\alpha^\tau.$$

That is, a gauge transformation simply conjugates an endomorphism. Similarly for any $\omega \in \Lambda^p(\operatorname{End} E)$ for $p \in \mathbb{N}$ set

$$\varsigma\left[\omega_{i\alpha}^\beta\right] := (\varsigma^{-1})_\zeta^\beta \omega_\tau^{i\zeta} \varsigma_\alpha^\tau.$$

The following results demonstrate the action of gauges on various objects besides those mentioned above. To obtain these, we first perform some explicit coordinate computations regarding these actions. The following lemma demonstrates how the connection coefficient matrix transforms under the action of gauge transformation.

Lemma 3.8.2. *Let* $\varsigma \in S(\mathrm{Aut}\,E)$ *and* $\nabla \in \mathcal{A}_E$, *and set* $\nu = \nu^\beta \mu_\beta \in S(E)$. *The coordinate expression of* $\varsigma\,[\nabla]$ *is*

$$(\varsigma\,[\nabla]\,\nu)_i^\beta = \partial_i \nu^\beta + (\Gamma_{\varsigma[\nabla]})_{i\theta}^\beta \nu^\theta,$$

where

$$(\Gamma_{\varsigma[\nabla]})_{i\theta}^\beta := (\varsigma^{-1})_\delta^\beta (\partial_i \varsigma_\theta^\delta) + (\varsigma^{-1})_\delta^\beta \Gamma_{i\gamma}^\delta (\varsigma_\theta^\gamma).$$

Remark 3.8.3. We emphasize that $\Gamma_{\varsigma^*\nabla}$ is *not* equivalent to $\varsigma\,[\Gamma] = \varsigma^{-1}\Gamma\varsigma$.

Proof. Simply computing yields

$$
\begin{aligned}
(\varsigma\,[\nabla])_i\,\nu^\beta &:= (\varsigma^{-1})_\delta^\beta \left(\nabla_i(\varsigma_\theta^\delta \nu^\theta)\right) \\
&= (\varsigma^{-1})_\delta^\beta \left[\partial_i \varsigma_\theta^\delta \nu^\theta + \varsigma_\theta^\delta (\partial_i \nu^\theta) + \Gamma_{i\gamma}^\delta (\varsigma^\gamma \nu^\theta)\right] \\
&= \partial_i \nu^\beta + (\varsigma^{-1})_\delta^\beta (\partial_i \varsigma_\theta^\delta) \nu^\theta + (\varsigma^{-1})_\delta^\beta \Gamma_{i\gamma}^\delta (\varsigma_\theta^\gamma \nu^\theta).
\end{aligned}
$$

The result follows. $\square$

We next compute the curvature of a gauge transformed connection in coordinates, and demonstrate that the curvature of a gauge transformed connection is equal to the conjugation of the connection's curvature by gauge transformation. Note that this agrees with the declaration of the action of the gauge transformation on a 1-form.

Lemma 3.8.4. *Suppose that* $\nabla \in \mathcal{A}_E$ *and* $\varsigma \in S\,(\mathrm{Aut}\,E)$. *Then*

$$(F_{\varsigma[\nabla]})_{ij\alpha}^\beta = (\varsigma^{-1})_\delta^\beta (F_\nabla)_{ij\theta}^\delta \varsigma_\alpha^\theta.$$

Proof. Observe that, by carefully matching terms yields

$$
\begin{aligned}
(F_{\varsigma[\nabla]})^{\beta}_{ij\alpha} &= \partial_i(\Gamma_{\varsigma[\nabla]})^{\beta}_{j\alpha} - \partial_j(\Gamma_{\varsigma[\nabla]})^{\beta}_{i\alpha} + (\Gamma_{\varsigma[\nabla]})^{\beta}_{i\delta}(\Gamma_{\varsigma[\nabla]})^{\delta}_{j\alpha} - (\Gamma_{\varsigma[\nabla]})^{\beta}_{j\delta}(\Gamma_{\varsigma[\nabla]})^{\delta}_{i\alpha} \\
&= \partial_i\left((\varsigma^{-1})^{\beta}_{\delta}(\partial_j\varsigma^{\delta}_{\alpha}) + (\varsigma^{-1})^{\beta}_{\delta}\Gamma^{\delta}_{j\gamma}\varsigma^{\gamma}_{\alpha}\right) - \partial_j\left((\varsigma^{-1})^{\beta}_{\delta}(\partial_i\varsigma^{\delta}_{\alpha}) + (\varsigma^{-1})^{\beta}_{\delta}\Gamma^{\delta}_{i\gamma}\varsigma^{\gamma}_{\alpha}\right) \\
&\quad + \left((\varsigma^{-1})^{\beta}_{\theta}(\partial_i\varsigma^{\theta}_{\delta}) + (\varsigma^{-1})^{\beta}_{\theta}\Gamma^{\theta}_{i\gamma}\varsigma^{\gamma}_{\delta}\right)\left((\varsigma^{-1})^{\delta}_{\zeta}(\partial_j\varsigma^{\zeta}_{\alpha}) + (\varsigma^{-1})^{\delta}_{\zeta}\Gamma^{\zeta}_{j\eta}\varsigma^{\eta}_{\alpha}\right) \\
&\quad - \left((\varsigma^{-1})^{\beta}_{\theta}(\partial_j\varsigma^{\theta}_{\delta}) + (\varsigma^{-1})^{\beta}_{\theta}\Gamma^{\theta}_{j\gamma}\varsigma^{\gamma}_{\delta}\right)\left((\varsigma^{-1})^{\delta}_{\zeta}(\partial_i\varsigma^{\zeta}_{\alpha}) + (\varsigma^{-1})^{\delta}_{\zeta}\Gamma^{\zeta}_{i\eta}\varsigma^{\eta}_{\alpha}\right) \\
&= (\varsigma^{-1})^{\beta}_{\theta}\left(\partial_i\Gamma^{\theta}_{j\mu} - \partial_j\Gamma^{\theta}_{i\mu} + \Gamma^{\beta}_{i\delta}\Gamma^{\delta}_{j\mu} - \Gamma^{\theta}_{j\delta}\Gamma^{\delta}_{i\mu}\right)\varsigma^{\mu}_{\alpha} \\
&= (\varsigma^{-1})^{\beta}_{\theta}(F_{\nabla})^{\theta}_{ij\delta}\varsigma^{\delta}_{\alpha}.
\end{aligned}
$$

The result follows. $\qquad\square$

The following lemma demonstrates the action of a gauge on a commutation bracket.

Lemma 3.8.5. *For $\omega, \psi \in \Lambda^p(E)$ and $\varsigma \in S(\operatorname{Aut} E)$,*

$$
\varsigma\left[[\omega, \psi]^{\#}\right] = [\varsigma[\omega], \varsigma[\psi]]^{\#}. \tag{3.80}
$$

Proof. Let K and L be multiindices of length p and q respectively, with $K = (k_i)_{i=1}^{|K|}$ and $L = (l_i)_{i=1}^{|L|}$. Then computing yields

$$
\begin{aligned}
\varsigma\left[[\omega, \psi]\right]^{\beta}_{KL\alpha} &= \varsigma\left[\omega^{\beta}_{K\delta}\psi^{\delta}_{L\alpha} - \psi^{\beta}_{L\delta}\omega^{\delta}_{K\alpha}\right] \\
&= \left((\varsigma^{-1})^{\beta}_{\zeta}\omega^{\zeta}_{K\delta}\psi^{\delta}_{L\rho}\varsigma^{\rho}_{\alpha} - (\varsigma^{-1})^{\beta}_{\zeta}\psi^{\zeta}_{L\delta}\omega^{\delta}_{K\rho}\varsigma^{\rho}_{\alpha}\right) \\
&= \left((\varsigma^{-1})^{\beta}_{\zeta}\omega^{\zeta}_{K\delta}\varsigma^{\delta}_{\theta}(\varsigma^{-1})^{\theta}_{\tau}\psi^{\tau}_{L\rho}\varsigma^{\rho}_{\alpha} - (\varsigma^{-1})^{\beta}_{\zeta}\psi^{\zeta}_{L\delta}\varsigma^{\delta}_{\theta}(\varsigma^{-1})^{\theta}_{\tau}\omega^{\tau}_{K\rho}\varsigma^{\rho}_{\alpha}\right) \\
&= ((\varsigma^*\omega)^{\beta}_{K\delta}(\varsigma[\psi])^{\delta}_{L\alpha} - (\varsigma^*\psi)^{\beta}_{L\delta}(\varsigma[\omega])^{\delta}_{K\alpha}) \\
&= ([\varsigma[\omega], \varsigma[\psi]])^{\beta}_{KL\alpha}.
\end{aligned}
$$

The result follows. $\qquad\square$

Remark 3.8.6. Note that since contraction occurs across base indices and the gauge transformation acts on the bundle, a consequence of this computation is that the gauge transformation also respects the pound bracket (cf. Definition (1.6)), that is,

$$
\varsigma\left[[\omega, \psi]^{\#}\right] = [\varsigma[\omega], \varsigma[\psi]]^{\#}.
$$

The next lemma demonstrates the action of gauge on a connection applied to endomorphism, and how the action distributes between the two objects.

Lemma 3.8.7. *Let* $\triangledown \in \mathcal{A}_E$, $\phi \in S(\operatorname{End} E)$ *and* $\varsigma \in S(\operatorname{Aut} E)$. *Then*

$$\varsigma\left[\triangledown \phi\right] = (\varsigma\left[\triangledown\right])(\varsigma\left[\phi\right]).$$

Proof. Expanding $\varsigma\left[\triangledown \phi\right]$ yields

$$\varsigma\left[(\triangledown)_i \phi^\beta_\alpha\right] = \left[(\varsigma^{-1})^\beta_\theta(\partial_i\phi^\theta_\gamma)\varsigma^\gamma_\alpha\right]_{T_1} - \left[(\varsigma^{-1})^\beta_\theta\phi^\theta_\delta\Gamma^\delta_{i\gamma}\varsigma^\gamma_\alpha\right]_{T_2} + \left[(\varsigma^{-1})^\beta_\theta\Gamma^\theta_{i\delta}\phi^\delta_\gamma\varsigma^\gamma_\alpha\right]_{T_3} = T_1 + T_2 + T_3.$$

Now observe that

$$
\begin{aligned}
(\varsigma\left[\triangledown\right])(\varsigma^{-1}\phi\varsigma) &= \partial_i(\varsigma^{-1}\phi^\theta_\gamma\varsigma^\gamma_\alpha) + ((\varsigma^{-1})^\beta_\theta(\partial_i\varsigma^\theta_\delta) + (\varsigma^{-1})^\beta_\theta\Gamma^\theta_{i\rho}\varsigma^\rho_\delta)(\varsigma^{-1}\phi\varsigma)^\delta_\alpha \\
&\quad - (\varsigma^{-1}\phi\varsigma)^\beta_\delta((\varsigma^{-1})^\delta_\rho\partial_i\varsigma^\rho_\alpha + (\varsigma^{-1})^\delta_\rho\Gamma^\rho_{i\theta}\varsigma^\theta_\alpha) \\
&= (\partial_i\varsigma^{-1})^\beta_\theta\phi^\theta_\gamma\varsigma^\gamma_\alpha + (\varsigma^{-1})^\beta_\theta(\partial_i\phi^\theta_\gamma)\varsigma^\gamma_\alpha + (\varsigma^{-1})^\beta_\theta\phi^\theta_\gamma(\partial_i\varsigma^\gamma_\alpha) \\
&\quad + (\varsigma^{-1})^\beta_\theta(\partial_i\varsigma^\theta_\delta)(\varsigma^{-1})^\delta_\rho\phi^\rho_\zeta s^\zeta_\alpha + (\varsigma^{-1})^\beta_\theta\Gamma^\theta_{i\rho}\phi^\rho_\zeta\varsigma^\zeta_\alpha \\
&\quad - (\varsigma^{-1})^\beta_\gamma\phi^\gamma_\delta\partial_i\varsigma^\delta_\alpha - (\varsigma^{-1})^\beta_\gamma\phi^\gamma_\delta\Gamma^\delta_{i\theta}\varsigma^\theta_\alpha. \\
&= -\left[(\varsigma^{-1})^\beta_\theta(\partial_i\varsigma^\theta_\tau)(\varsigma^{-1})^\tau_\zeta\phi^\zeta_\gamma\varsigma^\gamma_\alpha\right]_{T_4} + \left[(\varsigma^{-1})^\beta_\theta(\partial_i\phi^\theta_\gamma)\varsigma^\gamma_\alpha\right]_{T_1} \\
&\quad + \left[(\varsigma^{-1})^\beta_\theta\phi^\theta_\gamma(\partial_i\varsigma^\gamma_\alpha)\right]_{T_5} + \left[(\varsigma^{-1})^\beta_\theta(\partial_i\varsigma^\theta_\delta)(\varsigma^{-1})^\delta_\rho\phi^\rho_\zeta\varsigma^\zeta_\alpha\right]_{T_4} \\
&\quad + \left[(\varsigma^{-1})^\beta_\theta\Gamma^\theta_{i\rho}\phi^\rho_\zeta\varsigma^\zeta_\alpha\right]_{T_3} - \left[(\varsigma^{-1})^\beta_\gamma\phi^\gamma_\delta\partial_i\varsigma^\delta_\alpha\right]_{T_5} - \left[(\varsigma^{-1})^\beta_\gamma\phi^\gamma_\delta\Gamma^\delta_{i\theta}\varsigma^\theta_\alpha\right]_{T_2} \\
&= T_1 + T_2 + T_3.
\end{aligned}
$$

The equality holds and the result follows. $\qquad\square$

A key property of gauges with respect to the Yang-Mills k-energy, which determines the nonellipticity of the flow (cf. Proposition 2.1.4) is demonstrated in the following lemma. Namely, that the Yang-Mills k-energy is invariant under gauge transformation.

Corollary 3.8.8. *For* $\varsigma \in S(\operatorname{Aut} E)$ *and* $\triangledown \in \mathcal{A}_E$,

$$\mathcal{YM}_k(\triangledown) = \mathcal{YM}_k(\varsigma\left[\triangledown\right]).$$

Proof. This is a consequence of Lemma 3.8.4 and the definition of the action of gauge on a connection and

on a 2-form (cf. Definition 3.8.1). $\square$

Lemma 3.8.9. *Let $L := (i_1, j_1, \cdots i_k, j_k)$, $\varsigma \in S(\operatorname{Aut} E)$, and ζ some element of a tensor product of T^*M and E and their corresponding duals. Then*

$$\triangle^{(k)} \left[\varsigma_\alpha^\beta \zeta_R^Q \right] = \triangle^{(k)} [\varsigma]_\alpha^\beta \zeta_R^Q + \left(\prod_{v=0}^{k} g^{i_v j_v} \right) \left(\sum_{r=1}^{k-1} \sum_{\mathsf{P} \in \mathcal{P}_r(L)} \left(\nabla_{\mathsf{P}} \varsigma_\alpha^\beta \right) \left(\nabla_{\mathsf{P}^c} \zeta_R^Q \right) \right) + \varsigma_\alpha^\beta \triangle^{(k)} [\zeta]_R^Q, \tag{3.81}$$

where the quantity $\mathcal{P}_r(L)$ is defined in Definition 2.2.4

Proof. This is simply an application of the Leibniz rule and being aware of the distribution of connection pairings (coming from each Laplacian). $\square$

In the following lemma we investigate the action of this particular connection with a one-parameter family of gauge transformation and is essential in the following result.

Lemma 3.8.10. *Let $\nabla \in \mathcal{A}_E$ and $\varsigma_t \in S(\operatorname{Aut} E) \times \mathcal{I}$. Then*

$$\left(\varsigma_t \left[\nabla \right] \left[\varsigma_t^{-1} \dot{\varsigma}_t \right] \right)_\alpha^\beta = (\varsigma_t^{-1})_\delta^\beta (\partial_i \dot{\varsigma}_t)_\alpha^\delta + (\varsigma_t^{-1})_\theta^\beta (\Gamma)_{i\gamma}^\theta (\dot{\varsigma}_t)_\alpha^\gamma - (\Gamma_{\varsigma_t [\nabla]})_{i\alpha}^\delta \left(\varsigma_t^{-1} \dot{\varsigma}_t \right)_\delta^\beta. \tag{3.82}$$

Proof. Note that despite the assumed time dependence of ς_t the notational dependency will be omitted. Simply computing yields

$$(\varsigma [\nabla])(\varsigma^{-1} \dot{\varsigma}) = \partial_i \left((\varsigma^{-1})_\delta^\beta (\dot{\varsigma})_\alpha^\delta \right) + (\Gamma_{\varsigma [\nabla]})_{i\delta}^\beta \left(\varsigma^{-1} \dot{\varsigma} \right)_\alpha^\delta - (\Gamma_{\varsigma [\nabla]})_{i\alpha}^\delta \left(\varsigma^{-1} \dot{\varsigma} \right)_\delta^\beta$$

$$= -(\varsigma^{-1})_\rho^\beta (\partial_i \varsigma)_\theta^\rho (\varsigma^{-1})_\delta^\theta (\dot{\varsigma})_\alpha^\delta + (\varsigma^{-1})_\delta^\beta (\partial_i \dot{\varsigma})_\alpha^\delta + (\Gamma_{\varsigma [\nabla]})_{i\delta}^\beta (\varsigma^{-1} \dot{\varsigma})_\alpha^\delta$$

$$\quad - (\Gamma_{\varsigma [\nabla]})_{i\alpha}^\delta \left(\varsigma^{-1} \dot{\varsigma} \right)_\delta^\beta.$$

$$= -(\varsigma^{-1})_\rho^\beta (\partial_i \varsigma)_\theta^\rho (\varsigma^{-1})_\delta^\theta (\dot{\varsigma})_\alpha^\delta + (\varsigma^{-1})_\delta^\beta (\partial_i \dot{\varsigma})_\alpha^\delta + (\varsigma^{-1})_\rho^\beta (\partial_i \varsigma)_\theta^\rho (\varsigma^{-1})_\delta^\theta (\dot{\varsigma})_\alpha^\delta$$

$$\quad + (\varsigma^{-1})_\theta^\beta (\Gamma)_{i\gamma}^\theta (\dot{\varsigma})_\alpha^\gamma - (\varsigma [\Gamma])_{i\alpha}^\delta \left(\varsigma^{-1} \dot{\varsigma} \right)_\delta^\beta.$$

$$= (\varsigma^{-1})_\delta^\beta (\partial_i \dot{\varsigma})_\alpha^\delta + (\varsigma^{-1})_\theta^\beta \Gamma_{i\gamma}^\theta (\dot{\varsigma})_\alpha^\gamma - (\Gamma_{\varsigma [\nabla]})_{i\alpha}^\delta \left(\varsigma^{-1} \dot{\varsigma} \right)_\delta^\beta.$$

The result follows. $\square$

In the following lemma both the connection and the gauge transformation are one-parameter families, though the gauge transformation does not necessarily determine how the family of connections varies, as was the case in the prior lemma.

Lemma 3.8.11. *Let ς_t and ∇_t be one-parameter families of gauge transformations and connections respectively. The linearized gauge action of ς_t on ∇_t is given in coordinates by*

$$\left(\tfrac{\partial}{\partial t}\left[\varsigma_t\left[\nabla_t\right]\right]\right)^{\beta}_{k\alpha} = (\varsigma_t\left[\nabla_t\right])(\varsigma_t^{-1}\dot\varsigma_t)^{\beta}_{k\alpha} + (\varsigma_t^{-1})^{\beta}_{\delta}\left(\dot\Gamma_t\right)^{\delta}_{k\theta}(\varsigma_t)^{\theta}_{\alpha}. \tag{3.83}$$

Proof. As in the previous lemma, note that despite the assumed time dependence of ς_t and ∇_t the notational dependency will be omitted. Differentiating $\varsigma\left[\Gamma\right]$ with respect to t gives

$$\left(\tfrac{\partial}{\partial t}\left[\varsigma\left[\nabla\right]\right]\right)^{\beta}_{k\alpha} = \partial_t((\varsigma^{-1})^{\beta}_{\delta}\Gamma^{\delta}_{k\theta}\varsigma^{\theta}_{\alpha}) + \partial_t((\varsigma^{-1})^{\beta}_{\delta}\partial_i s^{\delta}_{\alpha})$$

$$= \partial_t(\varsigma^{-1})^{\beta}_{\delta}\Gamma^{\delta}_{k\theta}\varsigma^{\theta}_{\alpha} + (\varsigma^{-1})^{\beta}_{\delta}(\dot\Gamma^{\delta}_{k\theta})\varsigma^{\theta}_{\alpha} + (\varsigma^{-1})^{\beta}_{\delta}\Gamma^{\delta}_{k\theta}(\dot\varsigma^{\theta}_{\alpha})$$

$$\quad + \partial_t(\varsigma^{-1})^{\beta}_{\delta}(\partial_i\varsigma^{\delta}_{\alpha}) + (\varsigma^{-1})^{\beta}_{\delta}(\partial_i\dot\varsigma^{\delta}_{\alpha})$$

$$= -\left((\varsigma^{-1})^{\beta}_{\gamma}(\dot\varsigma^{\gamma}_{\zeta})(\varsigma^{-1})^{\zeta}_{\delta}\right)\Gamma^{\delta}_{k\theta}\varsigma^{\theta}_{\alpha} + (\varsigma^{-1})^{\beta}_{\delta}\dot\Gamma^{\delta}_{k\theta}\varsigma^{\theta}_{\alpha} + (\varsigma^{-1})^{\beta}_{\delta}\Gamma^{\delta}_{k\theta}\dot\varsigma^{\theta}_{\alpha}$$

$$\quad - \left((\varsigma^{-1})^{\beta}_{\gamma}(\dot\varsigma^{\gamma}_{\zeta})(\varsigma^{-1})^{\zeta}_{\delta}\right)(\partial_i\varsigma^{\delta}_{\alpha}) + (\varsigma^{-1})^{\beta}_{\delta}(\partial_i\dot\varsigma^{\delta}_{\alpha}).$$

Applying the computations of Lemma 3.8.10 and then Lemma 3.8.2,

$$\left(\tfrac{\partial}{\partial t}\left[\varsigma\left[\nabla\right]\right]\right)^{\beta}_{k\alpha} = -\left((\varsigma^{-1})^{\beta}_{\gamma}(\dot\varsigma^{\gamma}_{\zeta})(\varsigma^{-1})^{\zeta}_{\delta}\right)\Gamma^{\delta}_{k\theta}\varsigma^{\theta}_{\alpha} + (\varsigma^{-1})^{\beta}_{\delta}\dot\Gamma^{\delta}_{k\theta}\varsigma^{\theta}_{\alpha} + (\varsigma^{-1})^{\beta}_{\delta}\Gamma^{\delta}_{k\theta}\dot\varsigma^{\theta}_{\alpha}$$

$$\quad - \left((\varsigma^{-1})^{\beta}_{\gamma}(\dot\varsigma^{\gamma}_{\zeta})(\varsigma^{-1})^{\zeta}_{\delta}\right)(\partial_i\varsigma^{\delta}_{\alpha}) + (\varsigma^{-1})^{\beta}_{\delta}(\partial_i\dot\varsigma^{\delta}_{\alpha})$$

$$= \left[(\varsigma^{-1})^{\beta}_{\delta}(\partial_k\dot\varsigma^{\delta}_{\alpha}) + (\varsigma^{-1})^{\beta}_{\delta}\Gamma^{\delta}_{k\theta}\dot\varsigma^{\theta}_{\alpha} - (\varsigma^*\Gamma)^{\delta}_{k\alpha}(\varsigma^{-1}\dot\varsigma)^{\beta}_{\delta}\right]$$

$$\quad + \left[-\left((\varsigma^{-1})^{\beta}_{\gamma}(\dot\varsigma^{\gamma}_{\zeta})(\varsigma^{-1})^{\zeta}_{\delta}\right)\Gamma^{\delta}_{k\theta}\varsigma^{\theta}_{\alpha} - \left((\varsigma^{-1})^{\beta}_{\gamma}(\dot\varsigma^{\gamma}_{\zeta})(\varsigma^{-1})^{\zeta}_{\delta}\right)(\partial_k\varsigma^{\delta}_{\alpha})\right]$$

$$\quad + (\varsigma\left[\Gamma\right])^{\delta}_{k\alpha}(\varsigma^{-1}\dot\varsigma)^{\beta}_{\delta} + (\varsigma^{-1})^{\beta}_{\delta}(\dot\Gamma^{\delta}_{k\theta})\varsigma^{\theta}_{\alpha}$$

$$= (\varsigma\left[\nabla\right])(\varsigma^{-1}\dot\varsigma)^{\beta}_{k\alpha} + (\varsigma^{-1})^{\beta}_{\delta}(\dot\Gamma^{\delta}_{k\theta})\varsigma^{\theta}_{\alpha}.$$

The result follows. $\square$

Key results

Theorem 3.8.12 ([Uhl82a] Theorem 1.3 'Coulomb gauge'). *Let $E \to M = B_1 \subset \mathbb{R}^n$, $2p \geq n$, and $\nabla = \partial + \Gamma$ for $\Gamma \in L^p_1\left(S(\mathrm{Ad}\,E) \otimes \Lambda^1(B_1)\right)$. Then there exists some $\kappa_n, c_n > 0$ so that if $\|F_\nabla\|_{L^{n/2}_{n/2}(B_1)} \leq \kappa_n$, then ∇ is*

gauge equivalent via some gauge in $L_2^p \left(S(\mathcal{G}_E) \right)$ to a connection $\nabla = \partial + \mathbf{\Gamma}$ satisfying

$$\begin{cases} d^* \mathbf{\Gamma} & = 0, \\ \|\mathbf{\Gamma}\|_{L_1^p(B_1)} & \leq c_n \|F_\nabla\|_{L_0^p}. \end{cases} \tag{3.84}$$

Theorem 3.8.13 (Gauge patching theorem, Corollary 4.4.8, pp.159 of [DK90]). *Suppose $\{\nabla^i\}$ is a sequence of connections on E over M with the following property: for each $x \in M$ there is a neighborhood U_x and a subsequence $\{\nabla^{i_j}\}$ with corresponding sequence of gauge transformations s_{i_j} defined over M such that $s_{i_j}\left[\nabla^{i_j}\right]$ converges over U_x. Then there is a single subsubsequence $\{\nabla^{i_{j_k}}\}$ defined over M such that $s_{i_{j_k}}\left[\nabla^{i_{j_k}}\right]$ converges over all of M.*

Theorem 3.8.14 ([DK90] Lemma 2.3.11, pp.61). *For all ∇ and $\ell \in \mathbb{N}$, set*

$$Q_\ell(\nabla) := \|F_\nabla\|_{L^\infty} + \sum_{i=1}^{\ell} \left\| \nabla^{(i)} F_\nabla \right\|_{L^2}.$$

There is a constant $\eta > 0$ such that if the connection ∇ on the trivial bundle over $\mathbb{S}^4$ in Coulomb gauge relative to the product connection (i.e. with $D_\nabla^* \Gamma = 0$, as described in Theorem 3.8.12) satisfies $\|\Gamma\|_{L^2} < \eta$, then for each $\ell \in \mathbb{N}$ we have some universal continuous function f_ℓ, independent of Γ, such that

$$\|\Gamma\|_{L_{\ell+1}^2} \leq f_\ell \left(Q_\ell(\nabla) \right).$$

3.9   Chapter 3 supportive material

We supplement Chapter 3 by stating a variety of key polarization identities, which require elementary computations to show. For the following let $\nabla, \blacktriangledown \in C^2 \left(\mathcal{A}_E(M) \right)$, and $\Upsilon := \nabla - \blacktriangledown$. We record Proposition 3.9.1 and 3.9.2 in terms of more general formulas; not assuming $M = \mathbb{S}^4$.

Proposition 3.9.1 (Curvature polarization). *We have*

$$(F_\nabla - F_\blacktriangledown)_{ij\theta}^\beta = \left(\blacktriangledown_i \Upsilon_{j\theta}^\beta \right) - \left(\blacktriangledown_j \Upsilon_{i\theta}^\beta \right) + \Upsilon_{i\mu}^\beta \Upsilon_{j\theta}^\mu - \Upsilon_{j\mu}^\beta \Upsilon_{i\theta}^\mu. \tag{3.85}$$

Proposition 3.9.2 ($D_\nabla^* F_\nabla$ polarization)**.** *We have that*

$$
-(D_\nabla^* F_\nabla - D_\blacktriangledown^* F_\blacktriangledown)^\beta_{i\theta} = \left(\blacktriangledown_k \blacktriangledown_k \Upsilon^\beta_{i\theta}\right) - \left(\blacktriangledown_i \blacktriangledown_k \Upsilon^\beta_{k\theta}\right) - \mathrm{Rm}^p_{kik}\,\Upsilon_{p\theta} + 2\Upsilon^\beta_{k\mu}\,(F_\blacktriangledown)^\mu_{ki\theta} - 2\,(F_\blacktriangledown)^\beta_{ki\mu}\,\Upsilon^\mu_{k\theta}
$$

$$
+ \Upsilon^\beta_{i\zeta}\Upsilon^\zeta_{k\mu}\Upsilon^\mu_{k\theta} - 2\Upsilon^\beta_{k\mu}\Upsilon^\mu_{i\zeta}\Upsilon^\zeta_{k\theta} + \Upsilon^\beta_{k\mu}\Upsilon^\mu_{k\zeta}\Upsilon^\zeta_{i\theta}
$$

$$
+ \left(\blacktriangledown_i \Upsilon^\beta_{k\mu}\right)\Upsilon^\mu_{k\theta} - \Upsilon^\beta_{k\mu}\left(\blacktriangledown_i \Upsilon^\mu_{k\theta}\right) + 2\Upsilon^\beta_{k\mu}\left(\blacktriangledown_k \Upsilon^\mu_{i\theta}\right) - 2\left(\blacktriangledown_k \Upsilon^\beta_{i\mu}\right)\Upsilon^\mu_{k\theta}
$$

$$
+ \left(\blacktriangledown_k \Upsilon^\beta_{k\mu}\right)\Upsilon^\mu_{i\theta} - \Upsilon^\beta_{i\mu}\left(\blacktriangledown_k \Upsilon^\mu_{k\theta}\right).
$$

3.9.1 Various technical computations

Lemma 3.9.3 (Estimate for χ_λ)**.** *There is a constant $C > 0$ which is independent of $\lambda \geq 1$ so that*

$$
\|\nabla \log \chi_\lambda\|_{L^2} + \left\|\nabla^{(2)} \log \chi_\lambda\right\|_{L^2} \leq
\begin{cases}
C \log \lambda & \text{when } \lambda \in [1, e] \\[2mm]
C \left(\log \lambda\right)^{1/2} & \text{when } \lambda \in [e, \infty)
\end{cases}.
\tag{3.86}
$$

Proof. To apply spherical coordinates we use the appropriate change of variables

$$
\zeta_1 = r \sin \vartheta_1 \sin \vartheta_2 \cos \vartheta_3, \quad \zeta_2 = r \sin \vartheta_1 \sin \vartheta_2 \sin \vartheta_3, \quad \zeta_3 = r \sin \vartheta_1 \cos \vartheta_2, \quad \zeta_4 = r \cos \vartheta_4
$$

$$
g_{rr} = 1, \quad g_{11} := g_{\vartheta_1 \vartheta_1} = r^2, \quad g_{22} := g_{\vartheta_2 \vartheta_2} = r^2 \sin^2 \vartheta_1, \quad g_{33} := g_{\vartheta_3 \vartheta_3} = r^2 \sin^2 \vartheta_1 \sin^2 \vartheta_2.
$$

Inserting these into the formula for the Levi Civita connection, we have

$$
\Gamma^r_{ij} = 0 \,(i \neq j), \quad \Gamma^r_{11} = r, \quad \Gamma^r_{22} = r \sin^2 \vartheta_1, \quad \Gamma^r_{33} = r \sin^2 \vartheta_1 \sin^2 \vartheta_2,
\tag{3.87}
$$

Recalling the formula for $\chi_\lambda(\zeta) = \frac{1}{\lambda^4}\left(\frac{1+|\lambda\zeta|^2}{|\zeta|^2+1}\right)^4$ via (3.10), take $r = |\zeta|$. Observe that

$$
\nabla \log \chi_\lambda = \frac{\partial}{\partial r}\left[\log \chi_\lambda\right] \quad \text{and} \quad \nabla^{(2)} \log \chi_\lambda = \frac{\partial^2}{\partial r^2}\log \chi_\lambda - \left(\sum_{i=1}^3 \Gamma^r_{ii}\right)\frac{\partial}{\partial r}\left[\log \chi_\lambda\right].
\tag{3.88}
$$

First we compute

$$
\frac{\partial}{\partial r}\left[\log \chi_\lambda\right] = \frac{8r(\lambda^2-1)}{(1+\lambda^2 r^2)(1+r^2)}, \quad \frac{\partial^2}{\partial r^2}\left[\log \chi_\lambda\right] = -\frac{8\left(\lambda^2-1\right)\left(3\lambda^2 r^4+\left(\lambda^2+1\right)r^2-1\right)}{(r^2+1)^2(\lambda^2 r^2+1)^2}.
$$

Thus we have that, for the first derivative of $\log \chi_\lambda$,

$$
\begin{aligned}
||\nabla \log \chi_\lambda||_{L^2}^2 &= \left(\int_0^{2\pi} \int_0^\pi \int_0^\pi d\vartheta_1 \, d\vartheta_2 \, d\vartheta_3 \right) \left(\int_0^\infty \tfrac{r^3}{(1+r^2)^2} \left(\tfrac{\partial}{\partial r} [\log \chi_\lambda] \right)^2 dr \right) \\
&= 2\pi^3 \left(\int_0^{1/\lambda} + \int_{1/\lambda}^1 + \int_1^\infty \right) \tfrac{r^3}{(1+r^2)^2} \left(\tfrac{\partial}{\partial r} [\log \chi_\lambda] \right)^2 dr \\
&= 2^7 \pi^3 \left(\lambda^2 - 1 \right)^1 \left(\int_0^{1/\lambda} r^5 \, dr + \int_{1/\lambda}^1 \tfrac{r}{\lambda^4} \, dr + \int_0^{1/\lambda} \tfrac{1}{\lambda^4 r^7} \, dr \right) \\
&= 2^7 \pi^3 \left(\lambda^2 - 1 \right)^2 \left(\tfrac{1}{6\lambda^6} + \tfrac{1}{2\lambda^4} - \tfrac{1}{2\lambda^6} + \tfrac{1}{6\lambda^4} \right) \\
&= 2^7 \pi^3 \left(\tfrac{\lambda-1}{\lambda} \cdot \tfrac{\lambda+1}{\lambda} \right)^2 \left(\tfrac{2}{3} - \tfrac{1}{3\lambda^2} \right) \\
&= \tfrac{2^7}{3} \pi^3 \left(\tfrac{\lambda-1}{\lambda} \right)^2 \left(\tfrac{\lambda+1}{\lambda} \right)^2 .
\end{aligned}
$$

Now we approach the second derivative of $\log \chi_\lambda$. Utilizing (3.88) above, we note that

$$
\left|\left| \nabla^{(2)} \log \chi_\lambda \right|\right|_{L^2}^2 \leq \left|\left| \tfrac{\partial^2}{\partial r^2} [\log \chi_\lambda] \right|\right|_{L^2}^2 + ||\Gamma^r \partial_r \log \chi_\lambda||_{L^2}^2 . \tag{3.89}
$$

In this case we have that

$$
\begin{aligned}
&\left|\left| \tfrac{\partial^2}{\partial r^2} [\log \chi_\lambda] \right|\right|_{L^2}^2 \\
&= \left(\int_0^{2\pi} \int_0^\pi \int_0^\pi d\vartheta_1 \, d\vartheta_2 \, d\vartheta_3 \right) \left(\int_0^\infty \tfrac{r^3}{(1+r^2)^4} \left(\tfrac{\partial^2}{\partial r^2} [\log \chi_\lambda] \right)^2 dr \right) \\
&= 2\pi^3 \left(\int_0^{1/\lambda} + \int_{1/\lambda}^1 + \int_1^\infty \right) \tfrac{r^3}{(1+r^2)^4} \left(\tfrac{\partial^2}{\partial r^2} [\log \chi_\lambda] \right)^2 dr \\
&\leq 2^7 \pi^3 \left(\lambda^2 - 1 \right)^2 \int_0^{1/\lambda} \left(r^{11} 9\lambda^4 + r^9 \left(6\lambda^4 + 6\lambda^2 \right) + r^7 \left(\lambda^4 - 4\lambda^2 + 1 \right) \right) dr \\
&\quad + 2^7 \pi^3 \left(\lambda^2 - 1 \right)^2 \int_0^{1/\lambda} \left(r^5 \left(-2\lambda^2 - 2 \right) + r^3 \right) dr \\
&\quad + 2^7 \pi^3 \left(\lambda^2 - 1 \right)^2 \int_{1/\lambda}^1 \tfrac{1}{r^8 \lambda^8} \left(r^{11} 9\lambda^4 + r^9 \left(6\lambda^4 + 6\lambda^2 \right) + r^7 \left(\lambda^4 - 4\lambda^2 + 1 \right) \right) dr \\
&\quad + 2^7 \pi^3 \left(\lambda^2 - 1 \right)^2 \int_{1/\lambda}^1 \tfrac{1}{r^8 \lambda^8} \left(r^5 \left(-2\lambda^2 - 2 \right) + r^3 \right) dr \\
&\quad + 2^7 \pi^3 \left(\lambda^2 - 1 \right)^2 \int_1^\infty \tfrac{1}{r^{24} \lambda^8} \left(r^{11} 9\lambda^4 + r^9 \left(6\lambda^4 + 6\lambda^2 \right) + r^7 \left(\lambda^4 - 4\lambda^2 + 1 \right) \right) dr \\
&\quad + 2^7 \pi^3 \left(\lambda^2 - 1 \right)^2 \int_1^\infty \tfrac{1}{r^{24} \lambda^8} \left(r^5 \left(-2\lambda^2 - 2 \right) + r^3 \right) dr \\
&= 2^7 \pi^3 \left(\tfrac{\lambda-1}{\lambda} \right)^2 \left(\tfrac{\lambda+1}{\lambda} \right)^2 \left(-\tfrac{2177}{720} \tfrac{1}{\lambda^4} - \tfrac{209}{1260} \tfrac{1}{\lambda^2} + \tfrac{1943}{336} \right) \\
&= 2^7 \tfrac{1307}{504} \pi^3 \left(\tfrac{\lambda-1}{\lambda} \right)^2 \left(\tfrac{\lambda+1}{\lambda} \right)^2 .
\end{aligned}
$$

For the second component of (3.89), we have

$$\int_0^{2\pi} \int_0^\pi \int_0^\pi \int_0^\infty \frac{r^3}{(1+r^2)^4} \left(\Gamma^r \frac{\partial}{\partial r} [\log \chi_\lambda]\right)^2 dr \, d\vartheta_1 \, d\vartheta_2 \, d\vartheta_3$$

$$= \left(\int_0^{2\pi} \int_0^\pi \int_0^\pi \left(1 + \sin^2 \vartheta_1 + \sin^2 \vartheta_1 \sin^2 \vartheta_2\right)^2 d\vartheta_1 \, d\vartheta_2 \, d\vartheta_3\right) \left(\int_0^\infty \frac{64\left(\lambda^2-1\right)^2 r^9}{(r^2+1)^6(\lambda^2 r^2+1)^2} dr\right)$$

$$= \frac{217\pi^2}{32} \left(\int_0^\infty \frac{64\left(\lambda^2-1\right)^2 r^9}{(r^2+1)^6(\lambda^2 r^2+1)^2} dr\right)$$

$$\leq 2 \cdot 217\pi^2 \left(\lambda^2 - 1\right)^2 \left(\int_0^{1/\lambda} r^9 \, dr + \int_{1/\lambda}^1 \frac{r^5}{\lambda^4} dr + \int_1^\infty \frac{1}{\lambda^4 r^7} dr\right)$$

$$= 2 \cdot 217\pi^2 \left(\frac{\lambda-1}{\lambda}\right)^2 \left(\frac{\lambda+1}{\lambda}\right)^2 \left(\frac{1}{3} - \frac{1}{15\lambda^6}\right)$$

$$= \frac{2^3}{15} \cdot 217\pi^2 \left(\frac{\lambda-1}{\lambda}\right)^2 \left(\frac{\lambda+1}{\lambda}\right)^2.$$

Using (3.87) above we have that

$$\left|\Gamma^r \partial_r \chi_\lambda\right|^2 = r^2 \left(1 + \sin^2 \vartheta_1 + \sin^2 \vartheta_1 \sin^2 \vartheta_2\right)^2 \left|\frac{\partial}{\partial r} \chi_\lambda\right|^2.$$

With this we compute

$$\left\|\Gamma^r \partial_r \chi_\lambda\right\|_{L^2}^2 = 2\pi \left(\int_0^\pi \int_0^\pi \left(1 + \sin^2 \vartheta_1 + \sin^2 \vartheta_1 \sin^2 \vartheta_2\right)^2 d\vartheta_1 \, d\vartheta_2\right) \left(\int_0^\infty r^5 \left|\chi_\lambda\right|^2 dr\right)$$

$$= \frac{\pi}{64} \int_0^\pi \left(-68\cosh\left(2\vartheta_2\right) + 3\cosh(4\vartheta_2) + 217\right) d\vartheta_2 \left(\int_0^\infty r^5 \left|\chi_\lambda\right|^2 dr\right)$$

$$= \frac{217\pi^2}{64} \int_0^\infty r^5 \left|\chi_\lambda\right|^2 dr$$

$$= 217\pi^2(\lambda^2 - 1)^2 \left(\int_0^\infty \frac{r^7}{(1+\lambda^2 r^2)^2(1+r^2)^6} dr\right)$$

$$= 217\pi^2(\lambda^2 - 1)^2 \left(\int_0^{1/\lambda} r^7 \, dr + \frac{1}{\lambda^4} \int_{1/\lambda}^1 r^3 \, dr + \frac{1}{\lambda^4} \int_1^\infty r^{-9} \, dr\right)$$

$$= \frac{217}{8}\pi^2 \left(\frac{\lambda-1}{\lambda}\right)^2 \left(\frac{\lambda+1}{\lambda}\right)^2 \left(\frac{3}{8} - \frac{1}{8\lambda^4}\right)$$

$$\leq \frac{217}{2^7}\pi^2 \left(\frac{\lambda-1}{\lambda}\right)^2 \left(\frac{\lambda+1}{\lambda}\right)^2.$$

To all of these estimates, we note that $\left(\frac{\lambda+1}{\lambda}\right) \leq 1 + \frac{1}{\lambda} \leq 2$. It is a standard fact that

$$\frac{\lambda - 1}{\lambda} \leq \begin{cases} (\log \lambda) & \text{when } \lambda \in [1, \infty), \\[2mm] (\log \lambda)^{1/2} & \text{when } \lambda \in [e, \infty). \end{cases}$$

Applying these, we conclude (3.88). The result follows. $\square$

3.9.2 α-connection concentration compactness result

Theorem 3.9.4. *Let $\{\alpha_i\} \subset [1, 2)$ with $\lim_{i \to \infty} \alpha_i = 1$. Given corresponding Yang-Mills α_i-energy min-imizing connections $\{\nabla^i\}$, there exists sequences $\{\varphi_i\} \subset \mathrm{SO}(5, 1)$ and $\{\sigma_i\} \subset S(\mathcal{G}_E)$ such that there is a subsequence $\{\sigma_{i_j} [\widehat{\varphi_{i_j}}^* \nabla^{i_j}]\}$ which converges strongly in C^∞ to a antiself dual connection ∇^∞.*

Proof. First, assume that the pointwise curvature norms $|F_{\nabla^i}|_{\mathring{g}}$ do not concentrate as $i \to \infty$. If so, then the derivatives of curvature are also controlled via the ϵ-regularity and derivative estimates results of [HTY15] (Lemmata 3.5 and 3.6) (note that these results are for the Yang-Mills α-flow, but for our purposes we can apply them assuming the stationary setting). Thus up to gauge transformation, the sequence $\{\sigma_{i_j} [\widehat{\varphi_{i_j}}^* \nabla^{i_j}]\}$ converges to a minimal energy critical value of the Yang-Mills energy, which implies antiself duality.

If the pointwise curvature norms concentrate as $i \to \infty$, then we do a maximal blowup along the sequence, to identify a sequence of points $\{\zeta_{i'}\}$ with $\zeta_{i'}$ admitting supremal pointwise curvature norm for all $k \le i'$. There exists a further subsequence $\{\zeta_{i''}\}$ converging to $\zeta_\infty \in \mathbb{S}^4$, so that

$$\lim_{i'' \to \infty} |F_{\nabla^{i''}} (\zeta_{i''})|_{\mathring{g}} = \infty.$$

Stereographically projecting $\mathbb{S}^4$ onto $\mathbb{H}^1$, with ζ_∞ as the center point, we see that on $\mathbb{H}^1$, dilation centered at the origin is equivalent to performing a conformal automorphism on $\mathbb{S}^4$. We normalize the curvature via dilations in the blowup so that one has (identifying back to the corresponding setting on $\mathbb{S}^4$),

$$\lim_{i'' \to \infty} \left| F_{\widehat{\varphi_{i''}}^* \nabla^{i''}} (\zeta_{i''}) \right|_{\mathring{g}} = 1.$$

This modified sequence satisfies the initial case, above, namely that the pointwise norms do not concentrate, which concludes the result. $\qquad\square$

3.9.3 Poincaré Inequalities

In this section we will compute global and localized $\widetilde{\nabla}$-Poincaré inequalities which rely heavily on the strong structure of the $\widetilde{\nabla}$ and its corresponding curvature. To do so, we first need to establish Lemma 3.2.4, namely, pointwise bounds on commutator type terms.

Proof of Lemma 3.2.4. Using the formula for $\widetilde{F}_{ij}$ written out in coordinates, identifying $\operatorname{Ad} E$ with $\Im \mathbb{H}^1$ (as discussed in [Nab10]), we see that on $\mathbb{H}^1$,

$$F_{\widetilde{\nabla}} = \frac{2}{\left(1+|\zeta|^2\right)^2} \left(\left(d\zeta^{12} - d\zeta^{34}\right) \mathbf{i} + \left(d\zeta^{13} + d\zeta^{24}\right) \mathbf{j} + \left(d\zeta^{14} - d\zeta^{23}\right) \mathbf{k} \right),$$

where here $d\zeta^{ij} := d\zeta^i \wedge d\zeta^j$. We aim to compute

$$\left\langle \widetilde{F}_{ij}, [A_i, A_j] \right\rangle = \widetilde{F}_{ij}^{\mathbf{i}} [A_i, A_j]^{\mathbf{i}} + \widetilde{F}_{ij}^{\mathbf{j}} [A_i, A_j]^{\mathbf{j}} + \widetilde{F}_{ij}^{\mathbf{k}} [A_i, A_j]^{\mathbf{k}}.$$

Computing strictly in coordinates of $\mathbb{H}^1$, we compute the following:

$$[A_i, A_j]^{\mathbf{i}} = 2 \left(A_i^{\mathbf{j}} A_j^{\mathbf{k}} - A_i^{\mathbf{k}} A_j^{\mathbf{j}} \right).$$

Then we have that

$$\widetilde{F}_{ij}^{\mathbf{i}} [A_i, A_j]^{\mathbf{i}} = \frac{4}{\left(1+|\zeta|^2\right)^2} \left(\left(A_1^{\mathbf{j}} A_2^{\mathbf{j}} - A_2^{\mathbf{j}} A_1^{\mathbf{j}} \right) - \left(A_3^{\mathbf{j}} A_4^{\mathbf{j}} - A_4^{\mathbf{j}} A_3^{\mathbf{j}} \right) \right).$$

We have that for $\mathbf{l}, \mathbf{m} \in \Im \mathbb{H}$ then $2 \left| A_i^{\mathbf{l}} A_j^{\mathbf{m}} \right| \leq \left| A_i^{\mathbf{l}} \right|^2 + \left| A_j^{\mathbf{m}} \right|^2$, so we have

$$\left| \widetilde{F}_{ij}^{\mathbf{i}} [A_i, A_j]^{\mathbf{i}} \right| \leq \frac{2}{\left(1+|\zeta|^2\right)^2} \left(\left| A_1^{\mathbf{j}} \right|^2 + \left| A_1^{\mathbf{k}} \right|^2 + \left| A_2^{\mathbf{j}} \right|^2 + \left| A_2^{\mathbf{k}} \right|^2 + \left| A_3^{\mathbf{j}} \right|^2 + \left| A_3^{\mathbf{k}} \right|^2 + \left| A_4^{\mathbf{j}} \right|^2 + \left| A_4^{\mathbf{k}} \right|^2 \right).$$

Consequently it follows that (applying the round metric)

$$\left| \left\langle \widetilde{F}_{ij}, [A_i, A_j] \right\rangle_{\mathring{g}} \right| \leq |A|_{\mathring{g}}^2.$$

The second inequality of (3.18) follows similarly (noting the contraction within the commutator adds an extra dimensional factor of 4). $\qquad\square$

Proposition 3.9.5 (Localized $\widetilde{\nabla}$-Poincaré inequalities). *For $R > 0$, $\ell \in \mathbb{N}$, and $A \in \left(\Lambda^1 (B_R) \otimes \operatorname{Ad} E \right)$ where $B_R \subset \mathbb{S}^4$ there exists $C_P > 0$ such that*

$$\int_{B_R} |A|_{\mathring{g}}^\ell \, dV_{\mathring{g}} \leq C_P R^\ell \int_{B_R} \left| \widetilde{\nabla} A \right|_{\mathring{g}}^\ell \, dV_{\mathring{g}}, \text{ and } \int_{B_R} \left| \widetilde{\nabla} A \right|_{\mathring{g}}^\ell \, dV_{\mathring{g}} \leq C_P R^\ell \int_{B_R} \left| \widetilde{\nabla}^{(2)} A \right|_{\mathring{g}}^\ell \, dV_{\mathring{g}}. \tag{3.90}$$

Proof. We provide a proof by contradiction. If the inequality above were false, we can find a normalized

sequence $\{A^i\}$ satisfying

$$\int_{B_R} \left|\widetilde{\nabla} A^i\right|^\ell_{\mathring{g}} dV_{\hat{g}} \to 0, \quad \int_{B_R} \left|A^i\right|^\ell_{\mathring{g}} dV_{\hat{g}} = 1.$$

Via theorems of Rellich and Banach-Alaoglu, we choose a normalized subsequence $\{A^{i'}\}$ satisfying

$$A^{i'} \xrightarrow{L^p} A, \quad A^{i'} \overset{W^{1,p}}{\rightharpoonup} A \text{ and so } \widetilde{\nabla} A \equiv 0, \text{ and } \|A\|_{L^\ell(\Lambda^1(B_R)\otimes \operatorname{Ad} E)} \equiv 1.$$

It follows from the Ambrose-Singer Theorem ([AS53] Theorem 2) that this cannot hold. Implicitly the theorem relates the curvature of the connection to its holonomy. Thus, if one finds a local parallel section of $\Lambda^1(B_R) \otimes \operatorname{Ad} E$ in a neighborhood of some point, the holonomy is reduced, which is a contradiction since $\widetilde{\nabla}$ has full holonomy. This concludes the first inequality of (3.90).

For the second inequality of (3.90), we again perform a proof by contradiction and construct a normalizing sequence $\{A_j\}$, this time satisfying

$$\int_{B_R} \left|\widetilde{\nabla}^{(2)} A_j\right|^2_{\mathring{g}} dV_{\hat{g}} \to 0, \quad \int_{B_R} \left|\widetilde{\nabla} A\right|^2_{\mathring{g}} dV_{\hat{g}} = 1.$$

Again by Rellich's and Banach-Alaoglu's theorems there is a further subsequence $\{A_{j'}\}$ such that

$$A_{j'} \overset{W^{1,2}}{\to} A, \quad A_{j'} \rightharpoonup A \text{ and so } \widetilde{\nabla}^{(2)} A \equiv 0, \text{ and } \left\|\widetilde{\nabla} A\right\|_{L^\ell\left((\Lambda^1(B_R))^{\otimes 2}\otimes \operatorname{Ad} E\right)} \equiv 1.$$

In particular $\widetilde{\nabla}^{(2)} A \equiv 0$. Since this is true on the coordinate level, we also have

$$0 = \widetilde{\nabla}_i \widetilde{\nabla}_j A_k - \widetilde{\nabla}_j \widetilde{\nabla}_i A_k = \left[\widetilde{\nabla}_i, \widetilde{\nabla}_j\right] A_k.$$

In particular, this implies that (using (3.2.4))

$$0 = \left\langle \left[\widetilde{\nabla}_i, \widetilde{\nabla}_j\right] A_i, A_j\right\rangle_{\hat{g}} = -3\left|A\right|^2_{\mathring{g}} + \left\langle \left[\widetilde{F}_{ij}, A_i\right], A_j\right\rangle_{\hat{g}} < 0,$$

an obvious contradiction. □

Note that Proposition 3.2.5 follows naturally from a simple covering argument over $\mathbb{S}^4$.

3.9.4   Properties of $\widetilde{\nabla}$-Coulomb gauge

Here we include a proof of Theorem 3.2.6, an global adaptation of Tao and Tian's local result ([TT04] Theorem 4.6) which in turn was inspired by Theorem 1.3 of [Uhl82b]. Set $K > 1$ to be an absolute constant we to be chosen later, and define two sets

$$\mathcal{U}_\epsilon := \left\{ \nabla \in \mathcal{A}_E \left(\mathbb{S}^4 \right) : \inf_{\varsigma \in \mathcal{G}_E} \left| \left| F_{\varsigma[\nabla]} - F_{\widetilde{\nabla}} \right| \right|_{L^2} \le \epsilon \right\}$$

$$\mathcal{U}_\epsilon^* := \left\{ \nabla \in \mathcal{U}_\epsilon \left(\mathbb{S}^4 \right) : \left| \left| \widetilde{\Pi} \left[\nabla \right] - \widetilde{\nabla} \right| \right|_{W^{1,2}} \le K\epsilon \right\}.$$

Our goal is to show $\mathcal{U}_\epsilon^* \equiv \mathcal{U}_\epsilon$, thus establishing Theorem 3.2.6. The preliminary step needed in the proof is a bootstrap estimate. Throughout the proof, we warn the reader to be cautious of the meaning of Υ, as it changes periodically throughout the argument (it will either be $\nabla - \widetilde{\nabla}$ *or* $\widetilde{\Pi} \left[\nabla \right] - \widetilde{\nabla}$).

Lemma 3.9.6 (Bootstrap estimate). *For any $\nabla \in \mathcal{U}_\epsilon^*$, the following estimate may be bootstrapped*

$$\left| \left| \widetilde{\Pi} \left[\nabla \right] - \widetilde{\nabla} \right| \right|_{L^2} \le K\epsilon,$$

to instead obtain

$$\left| \left| \widetilde{\Pi} \left[\nabla \right] - \widetilde{\nabla} \right| \right|_{L^2} \le K\frac{\epsilon}{2},$$

additionally, (3.21) holds.

Proof. Set $\Upsilon := \widetilde{\Pi} \left[\nabla \right] - \widetilde{\nabla}$. Via Proposition 3.2.5 and Proposition 3.9.1,

$$
\begin{aligned}
||\Upsilon||_{L^2} &\le C \, ||D_{\widetilde{\nabla}} \Upsilon||_{L^2} \\
&= C \left| \left| F_{\widetilde{\Pi}[\nabla]} - F_{\widetilde{\nabla}} + [\Upsilon, \Upsilon] \right| \right|_{L^2} \\
&\le C \left| \left| F_{\widetilde{\Pi}[\nabla]} - F_{\widetilde{\nabla}} \right| \right|_{L^2} + C \, ||\Upsilon||_{L^2}^2 \\
&\le C\epsilon + C \, ||\Upsilon||_{L^2}^2 \, .
\end{aligned}
$$

Applying the estimates yields the desired results. $\qquad\qquad\square$

Now take $\nabla \in \mathcal{U}_\epsilon$ and consider the one-parameter family of connections for $s \in [0,1]$ by

$$\nabla_s(x) := \widetilde{\nabla}(x) + s\left(\nabla - \widetilde{\nabla}\right)(sx).$$

One can see that for $s = 0$, $\nabla_0 \equiv \widetilde{\nabla}$ and $\nabla_1 \equiv \nabla$. We next verify this entire family lies inside of $\mathcal{U}_\epsilon$.

Proposition 3.9.7. *Using the notation above* $\nabla_s \in \mathcal{U}_\epsilon$.

Proof. Take $\Upsilon_s := \nabla_s - \widetilde{\nabla}$ and observe that

$$\begin{aligned}
\left.[F_{\nabla_s} - F_{\widetilde{\nabla}}]\right|_x &= \left.[D_{\widetilde{\nabla}}(\Upsilon_s) + [\Upsilon_s, \Upsilon_s]]\right|_x \\
&= \left.s\,[D_{\widetilde{\nabla}}(\Upsilon) + [\Upsilon, \Upsilon]]\right|_{sx} \\
&= \left.s\,[F_\nabla - F_{\widetilde{\nabla}}]\right|_{sx}.
\end{aligned}$$

Consequently $\|F_{\nabla_s} - F_{\widetilde{\nabla}}\|_{L^2} = s\,\|F_\nabla - F_{\widetilde{\nabla}}\|_{L^2} \leq \frac{s\epsilon}{2}$ as desired. $\qquad\square$

Proposition 3.9.8 (Continuity of the Coulomb gauge construction in smooth norms). *Let* $X \in (0, \infty)$, $p \in (2, 4)$ *and let* $\nabla \in \mathcal{U}_\epsilon$ *be such that*

$$\left\|\nabla - \widetilde{\nabla}\right\|_{W^{1,p}} \leq X. \tag{3.91}$$

Then there exists a quantity $\delta_X > 0$ *depending only on* $X, \mathcal{G}_E, p, \epsilon$, *such that*

$$\{\nabla + A \in \mathcal{U}_\epsilon : \|A\|_{W^{1,p}} \leq \delta_X\} \subset \mathcal{U}_\epsilon^*.$$

Proof. Fix p (all constants are allowed to depend on p), and let $C_X \geq 0$ denote quantities dependent on X, which can be updated as necessary. As in [TT04], the argument consists of three steps.

Step 1. Estimation of the $\widetilde{\nabla}$-Coulomb gauge in smooth norms. Note that via Proposition 3.9.1 combined with Hölder's inequality, there exists a constant $C_X \geq 0$ such that

$$\|F_\nabla - F_{\widetilde{\nabla}}\|_{L^p} \leq C_X.$$

This constant C_X can be updated as necessary to bound above

$$\left\lVert F_{\widetilde{\Pi}[\nabla]} - F_{\widetilde{\nabla}}\right\rVert_{L^p} \leq C_X.$$

Manipulating as in the lemma above,

$$\left\lVert \widetilde{\Pi}\left[\nabla\right] - \widetilde{\nabla}\right\rVert_{W^{1,p}} \leq \left\lVert F_{\widetilde{\Pi}[\nabla]} - F_{\widetilde{\nabla}}\right\rVert_{L^p} + C\left\lVert \left(\widetilde{\Pi}\left[\nabla\right] - \widetilde{\nabla}\right)\wedge\left(\widetilde{\Pi}\left[\nabla\right] - \widetilde{\nabla}\right)\right\rVert_{W^{1,p}} \leq C_X.$$

Let ς denote the gauge transformation such that $\varsigma\left[\nabla\right] = \widetilde{\Pi}\left[\nabla\right] = \partial + \Sigma$. Then

$$\Sigma_{i\theta}^{\beta} := \left(\varsigma^{-1}\right)_{\delta}^{\beta}\left(\partial_i \varsigma_{\theta}^{\delta}\right) + \left(\varsigma^{-1}\right)_{\delta}^{\beta}\Gamma_{i\gamma}^{\delta}\varsigma_{\theta}^{\gamma}.$$

Remanipulating, and setting $\Upsilon := \varsigma\left[\nabla\right] - \widetilde{\nabla}$ and $\mathrm{Y} := \nabla - \widetilde{\nabla}$ yields

$$\left(\partial_i \varsigma_{\theta}^{\rho}\right) = \varsigma_{\beta}^{\rho}\Sigma_{i\theta}^{\beta} - \Gamma_{i\gamma}^{\rho}\varsigma_{\theta}^{\gamma}$$

$$\left(\widetilde{\nabla}_i \varsigma_{\theta}^{\rho}\right) = \varsigma_{\beta}^{\rho}\Sigma_{i\theta}^{\beta} - \Gamma_{i\gamma}^{\rho}\varsigma_{\theta}^{\gamma} - \varsigma_{\mu}^{\rho}\widetilde{\Gamma}_{i\theta}^{\mu} + \widetilde{\Gamma}_{i\mu}^{\rho}\varsigma_{\theta}^{\mu} = \varsigma_{\beta}^{\rho}\Upsilon_{i\theta}^{\beta} - \mathrm{Y}_{i\mu}^{\rho}\varsigma_{\theta}^{\mu}.$$

Now we note that

$$\widetilde{\nabla}_j \widetilde{\nabla}_i \varsigma_{\theta}^{\rho} = \left(\widetilde{\nabla}_j \varsigma_{\beta}^{\rho}\right)\Upsilon_{i\theta}^{\beta} + \varsigma_{\beta}^{\rho}\left(\widetilde{\nabla}_j \Upsilon_{i\theta}^{\beta}\right) - \left(\widetilde{\nabla}_j \mathrm{Y}_{i\mu}^{\rho}\right)\varsigma_{\theta}^{\mu} - \mathrm{Y}_{i\mu}^{\rho}\left(\widetilde{\nabla}_j \varsigma_{\theta}^{\mu}\right)$$

$$= \left(\varsigma_{\gamma}^{\rho}\Upsilon_{j\beta}^{\gamma} - \mathrm{Y}_{j\gamma}^{\rho}\varsigma_{\beta}^{\gamma}\right)\Upsilon_{i\theta}^{\beta} + \varsigma_{\beta}^{\rho}\left(\widetilde{\nabla}_j \Upsilon_{i\theta}^{\beta}\right) - \left(\widetilde{\nabla}_j \mathrm{Y}_{i\mu}^{\rho}\right)\varsigma_{\theta}^{\mu} - \mathrm{Y}_{i\mu}^{\rho}\left(\varsigma_{\zeta}^{\mu}\Upsilon_{j\theta}^{\zeta} - \mathrm{Y}_{j\zeta}^{\mu}\varsigma_{\theta}^{\zeta}\right).$$

We thus have that

$$\left\lvert \widetilde{\nabla}^{(2)}\varsigma\right\rvert_{\mathring{g}} \leq \lvert\varsigma\rvert_{\mathring{g}}\left(\lvert\Upsilon\rvert_{\mathring{g}} + \lvert\mathrm{Y}\rvert_{\mathring{g}}\lvert\Upsilon\rvert_{\mathring{g}}\right) + \lvert\varsigma\rvert_{\mathring{g}}\left(\lvert\widetilde{\nabla}\Upsilon\rvert_{\mathring{g}} + \lvert\widetilde{\nabla}\mathrm{Y}\rvert_{\mathring{g}}\right)$$

$$\leq C\lvert\varsigma\rvert_{\mathring{g}}\left(\lvert\Upsilon\rvert_{\mathring{g}} + \lvert\Upsilon\rvert_{\mathring{g}}^2 + \lvert\widetilde{\nabla}\Upsilon\rvert_{\mathring{g}} + \lvert\mathrm{Y}\rvert_{\mathring{g}}^2\lvert\widetilde{\nabla}\mathrm{Y}\rvert_{\mathring{g}}\right).$$

Consequently we have

$$\lvert\varsigma\rvert_{\mathring{g}} + \lvert\widetilde{\nabla}\varsigma\rvert_{\mathring{g}} + \left\lvert\widetilde{\nabla}^{(2)}\varsigma\right\rvert_{\mathring{g}} \leq C\lvert\varsigma\rvert_{\mathring{g}}\left(\lvert\Upsilon\rvert_{\mathring{g}} + \lvert\Upsilon\rvert_{\mathring{g}}^2 + \lvert\widetilde{\nabla}\Upsilon\rvert_{\mathring{g}} + \lvert\mathrm{Y}\rvert_{\mathring{g}} + \lvert\mathrm{Y}\rvert_{\mathring{g}}^2 + \lvert\widetilde{\nabla}\mathrm{Y}\rvert_{\mathring{g}}\right).$$

Combining these all together we conclude that $\lVert\varsigma\rVert_{W^{2,p}} \leq C_X$, concluding the first step.

Step 2. Pass to the $\widetilde{\nabla}$-Coulomb gauge. As a consequence of **Step 1**, it follows that the action of

a $\widetilde{\nabla}$-Coulomb gauge transformation is uniformly continuous in the sense of the $W^{2,p}$-topology in a small neighborhood of ∇. Furthermore, since both $\mathcal{U}_\epsilon$ and $\mathcal{U}_\epsilon^*$ are in fact invariant under gauge transformation (the $\widetilde{\nabla}$-projection always overrides any gauge action) we can prove Proposition 3.9.8 specifically in the setting $\nabla = \widetilde{\Pi}\,[\nabla]$.

Step 3. Apply perturbation theory to the Coulomb gauge. Fix the perturbation parameter A as in Proposition 3.9.8. To show $\nabla + A \in \mathcal{U}_\epsilon^*$, we must construct a gauge transformation ς satisfying

$$D_{\widetilde{\nabla}}^* \left(\varsigma\,[\nabla + A] - \widetilde{\nabla} \right) = 0. \tag{3.92}$$

To do so, we give a perturbative argument. Set $\varsigma := e^\sigma$, recall the formula of a gauge action on a connection. We have that $\varsigma\,[\nabla + A] - \widetilde{\nabla}$ is given by

$$
\begin{aligned}
\left(\varsigma\,[\nabla + A] - \widetilde{\nabla} \right)_{i\theta}^\beta &= \left(\varsigma^{-1}\right)_\delta^\beta \left(\partial_i \varsigma_\theta^\delta \right) + \left(\varsigma^{-1}\right)_\delta^\beta \left[\Gamma - A \right]_{i\gamma}^\delta \varsigma_\theta^\gamma - \widetilde{\Gamma}_{i\theta}^\beta \\
&= \left(\varsigma^{-1}\right)_\delta^\beta \left(\widetilde{\nabla}_i \varsigma_\theta^\delta \right) - \left(\varsigma^{-1}\right)_\delta^\beta \widetilde{\Gamma}_{i\tau}^\delta \varsigma_\theta^\tau + \left(\varsigma^{-1}\right)_\delta^\beta \left[\Gamma + A \right]_{i\gamma}^\delta \varsigma_\theta^\gamma \\
&= \left(\varsigma^{-1}\right)_\delta^\beta \left(\widetilde{\nabla}_i \varsigma_\theta^\delta \right) + \left(\varsigma^{-1}\right)_\delta^\beta \left(\Upsilon + A \right)_{i\gamma}^\delta \varsigma_\theta^\gamma.
\end{aligned}
$$

Converting this to be in terms of σ (and simultaneously defining our term $\mathcal{W}$) we have

$$
\begin{aligned}
\mathcal{W}\left(\sigma, \nabla + A\right) &= \left(\varsigma\,[\nabla + A] - \widetilde{\nabla} \right)_{i\theta}^\beta - \widetilde{\nabla}_i \sigma_\theta^\beta \\
&= \left(e^{-\sigma} \right)_\delta^\beta \left(\Upsilon + A \right)_{i\gamma}^\delta \left(e^\sigma \right)_\theta^\gamma \\
&= \left(\mathrm{Id}_\delta^\beta - \sigma_\delta^\beta + \cdots \right) \left(\Upsilon + A \right)_{i\gamma}^\delta \left(\mathrm{Id}_\delta^\beta + \sigma_\delta^\beta + \cdots \right) \\
&= \left(\Upsilon + A \right)_{i\theta}^\beta - \sigma_\delta^\beta \left(\Upsilon + A \right)_{i\theta}^\delta + \left(\Upsilon + A \right)_{i\delta}^\beta \sigma_\theta^\delta + \sigma_\zeta^\beta \left(\Upsilon + A \right)_{i\delta}^\zeta \sigma_\theta^\delta + \cdots.
\end{aligned} \tag{3.93}
$$

Expanding out $0 = D_{\widetilde{\nabla}}^* \left(\mathcal{W}\left(\sigma, \nabla + A\right) + \nabla\sigma \right)$ gives

$$
\begin{aligned}
0 &= -\widetilde{\nabla}_i \left[\left(\varsigma^{-1}\right)_\gamma^\beta \left(\partial_i \varsigma_\theta^\gamma \right) + \left(\varsigma^{-1}\right)_\delta^\beta \left(\Gamma + A \right)_{i\gamma}^\delta \left(\varsigma_\theta^\gamma \right) \right] \\
&= -\widetilde{\nabla}_i \left[\left(\varsigma^{-1}\right)_\gamma^\beta \left(\widetilde{\nabla}_i \varsigma_\theta^\gamma - \widetilde{\Gamma}_{i\varsigma}^\gamma \varsigma_\theta^\varsigma + \varsigma_\zeta^\gamma \widetilde{\Gamma}_{i\theta}^\zeta \right) + \left(\varsigma^{-1}\right)_\delta^\beta \left(\Gamma + A \right)_{i\gamma}^\delta \left(\varsigma_\theta^\gamma \right) \right] \\
&= -\left(\varsigma^{-1}\right)_\gamma^\beta \widetilde{\triangle} \varsigma_\theta^\gamma + \left(\varsigma^{-1}\right)_\omega^\beta \left(\widetilde{\nabla}_i \varsigma_\delta^\omega \right) \left(\varsigma^{-1}\right)_\gamma^\delta \left(\widetilde{\nabla}_i \varsigma_\theta^\gamma \right) + \left(\varsigma^{-1}\right)_\gamma^\beta \widetilde{\Gamma}_{i\zeta}^\gamma \left(\widetilde{\nabla}_i \varsigma_\theta^\zeta \right) - \left(\varsigma^{-1}\right)_\gamma^\beta \left(\widetilde{\nabla}_i \varsigma_\zeta^\gamma \right) \widetilde{\Gamma}_{i\theta}^\zeta \\
&\quad + \left(\varsigma^{-1}\right)_\mu^\beta \left(\widetilde{\nabla}_i \varsigma_\nu^\mu \right) \left(\varsigma^{-1}\right)_\delta^\nu \left(\Gamma + A \right)_{i\gamma}^\delta \varsigma_\theta^\gamma - \left(\varsigma^{-1}\right)_\delta^\beta \widetilde{\nabla}_i \left(\Gamma + A \right)_{i\gamma}^\delta \varsigma_\theta^\gamma - \left(\varsigma^{-1}\right)_\delta^\beta \left(\Gamma + A \right)_{i\gamma}^\delta \left(\widetilde{\nabla}_i \varsigma_\theta^\gamma \right).
\end{aligned}
$$

Inserting the fact that $\varsigma \equiv e^\sigma$, we have that

$$
\begin{aligned}
0 = & -\widetilde{\triangle}\sigma_\theta^\beta + \left(\widetilde{\nabla}_i\sigma_\delta^\beta\right)\left(\widetilde{\nabla}_i\sigma_\theta^\delta\right) + (\exp(-\sigma))_\gamma^\beta \widetilde{\Gamma}_{i\zeta}^\gamma (e^\sigma)_\omega^\zeta \left(\widetilde{\nabla}_i\sigma_\theta^\omega\right) - \left(\widetilde{\nabla}_i\sigma_\zeta^\beta\right)\widetilde{\Gamma}_{i\theta}^\zeta \\
& + \left(\widetilde{\nabla}_i\sigma_\beta^\mu\right)\left(e^{-\sigma}\right)_\delta^\nu (\Gamma + A)_{i\gamma}^\delta (e^\sigma)_\theta^\gamma - \left(e^{-\sigma}\right)_\delta^\beta \widetilde{\nabla}_i (\Gamma + A)_{i\gamma}^\delta (e^\sigma)_\theta^\gamma \\
& - \left(e^{-\sigma}\right)_\delta^\beta (\Gamma + A)_{i\gamma}^\delta (e^\sigma)_\zeta^\gamma \left(\widetilde{\nabla}_i\sigma_\theta^\zeta\right) \\
= & -\widetilde{\triangle}\sigma_\theta^\beta + \left(\widetilde{\nabla}_i\sigma_\delta^\beta\right)\left(\widetilde{\nabla}_i\sigma_\theta^\delta\right) + \left(\widetilde{\nabla}_i\sigma_\beta^\mu\right)\left[\left(e^{-\sigma}\right)_\delta^\nu (\Gamma + A)_{i\gamma}^\delta (e^\sigma)_\theta^\gamma - \widetilde{\Gamma}_{i\theta}^\mu\right] \\
& - \left(e^{-\sigma}\right)_\delta^\beta \widetilde{\nabla}_i (\Upsilon + A)_{i\gamma}^\delta (e^\sigma)_\theta^\gamma - \left(e^{-\sigma}\right)_\delta^\beta A_{i\gamma}^\delta (e^\sigma)_\zeta^\gamma \left(\widetilde{\nabla}_i\sigma_\theta^\zeta\right).
\end{aligned}
$$

Therefore we have that,

$$
\begin{aligned}
\widetilde{\triangle}\sigma_\theta^\beta = & \left(\widetilde{\nabla}_i\sigma_\delta^\beta\right)\left(\widetilde{\nabla}_i\sigma_\theta^\delta\right) + \left(\widetilde{\nabla}_i\sigma_\nu^\beta\right)\left[\left(e^{-\sigma}\right)_\delta^\nu (\Upsilon + A)_{i\gamma}^\delta (e^\sigma)_\theta^\gamma + \left(e^{-\sigma}\right)_\delta^\nu \widetilde{\Gamma}_{i\gamma}^\delta (e^\sigma)_\theta^\gamma - \widetilde{\Gamma}_{i\theta}^\nu\right] \\
& - \left(e^{-\sigma}\right)_\delta^\beta \widetilde{\nabla}_i A_{i\gamma}^\delta (e^\sigma)_\theta^\gamma - \left(e^{-\sigma}\right)_\delta^\beta (\Upsilon + A)_{i\gamma}^\delta (e^\sigma)_\zeta^\gamma \left(\widetilde{\nabla}_i\sigma_\theta^\zeta\right).
\end{aligned}
\tag{3.94}
$$

By combining (3.93) and (3.92), and noting that here, $D_{\widetilde{\nabla}}^* \left[\mathcal{W}\left(\sigma^{(j)}, \nabla + A\right)\right]$ is precisely the right hand side of (3.94), we conclude

$$
\widetilde{\triangle}\sigma = D_{\widetilde{\nabla}}^* \left[\mathcal{W}\left(\sigma, \nabla + A\right)\right].
$$

Consider the following iteration scheme, with initial condition $\sigma^{(0)} \equiv 0$,

$$
\widetilde{\triangle}\sigma^{(\ell+1)} := D_{\widetilde{\nabla}}^* \left[\mathcal{W}\left(\sigma^{(\ell)}, \nabla + A\right)\right].
$$

Note $\sigma^{(\ell+1)}$ is uniquely defined by standard elliptic regularity. We will derive bounds on $\sigma^{(\ell+1)}$. First, based off of the system above, we have

$$
\left\|\sigma^{(\ell+1)}\right\|_{W^{2,p}} \leq C \left\|D_{\widetilde{\nabla}}^* \left[\mathcal{W}\left(\sigma^{(\ell)}, \nabla + A\right)\right]\right\|_{W^{2,p}} + C \left\|\mathcal{W}\left(\sigma^{(\ell)}, \widetilde{\nabla} + A\right)\right\|_{W^{1,p}}
$$

We estimate each term on the right hand side. For both terms, we apply the exponential power series expansion for some $C > 0$ depending on $\widetilde{\nabla}$. First by (3.93),

$$
\left|\mathcal{W}\left(\sigma^{(\ell)}, \widetilde{\nabla} + A\right)\right|_{\hat{g}} \leq C |A|_{\hat{g}} \left(1 + \left|\sigma^{(\ell)}\right|_{\hat{g}}\right).
$$

From this we can deduce, using Hölder's inequality, that

$$\left\|\mathcal{W}\left(\sigma^{(\ell)},\widetilde{\nabla}+A\right)\right\|_{W^{1,p}} \leq C\left(\left\|\sigma^{(\ell)}\right\|_{W^{2,p}}^{2} + \|A\|_{W^{1,p}}\right).$$

Using (3.94) above, we see that

$$\left|D_{\widetilde{\nabla}}^{*}\left[\mathcal{W}\left(\sigma^{(\ell)},\nabla+A\right)\right]\right|_{\mathring{g}} \leq C\left(\left|\widetilde{\nabla}\sigma^{(\ell)}\right|_{\mathring{g}}^{2} + \left(1+\left|\sigma^{(\ell)}\right|_{\mathring{g}}\right)\left(\left|\widetilde{\nabla}\sigma^{(\ell)}\right|_{\mathring{g}}\left(|\Upsilon+A|_{\mathring{g}}+1\right) + |\widetilde{\nabla}A|_{\mathring{g}}\right)\right).$$

Therefore we can expand out with Holder's inequality, noting that $\nabla \in \mathcal{U}_{\epsilon}^{*}$ and conclude that

$$\left\|D_{\widetilde{\nabla}}^{*}\mathcal{W}\left(\sigma^{(\ell)},\nabla+A\right)\right\|_{W^{1,p}} \leq C\left\|\sigma^{(\ell)}\right\|_{W^{2,p}}^{2} + K\epsilon\left\|\sigma^{(\ell)}\right\|_{W^{2,p}} + \|A\|_{W^{2,p}}.$$

Thus, as long as δ_X is sufficiently small, we can obtain inductively that

$$\left\|\sigma^{(\ell)}\right\|_{W^{2,p}} \leq C\delta_X.$$

We adapt this iteration scheme to conclude that $\sigma^{(\ell)}$ converges in $W^{2,p}$ to a solution σ satisfying

$$\|\sigma\|_{W^{2,p}} \leq C\delta_X.$$

As a result of standard Sobolev embeddings, σ has some Hölder regularity, which, when elliptic regularity is applied, can be bootstrapped to conclude that σ is in fact smooth. If we exponentiate σ and apply Hölder's inequality we obtain a smooth $\widetilde{\nabla}$-Coulomb gauge $\varsigma\left[\nabla+A\right]$ satisfying

$$\|\varsigma - \mathrm{Id}\|_{W^{2,p}}, \left\|\varsigma^{-1} - \mathrm{Id}\right\|_{W^{2,p}} \leq C\delta_X.$$

As a consequence of the gauge transformation action and (3.91) of the assumptions on A,

$$\|\varsigma\left[\nabla+A\right] - \nabla\|_{W^{1,p}} \leq C_X\delta_X.$$

therefore since $p \in (2,4)$ we have

$$\|\varsigma\left[\nabla+A\right] - \nabla\|_{W^{1,2}} \leq C_X\delta_X.$$

If δ_X is sufficiently small, as a consequence of our bootstrapping estimate of Lemma 3.9.6 we have

$$\left|\left|\varsigma\left[\nabla + A\right] - \widetilde{\nabla}\right|\right|_{W^{1,2}} \leq K\epsilon.$$

which is precisely the desired result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.9.5   Morrey-type Inequalities

Let $R > 0$ and $\eta \in C^\infty$ be a nonnegative function, where

$$\eta(x) = \begin{cases} 1 & \text{when } x \in B_{R/2}, \\[2mm] 0 & \text{when } x \notin B_R. \end{cases}$$

Remark 3.9.9 (More notational conventions)**.** We will be using an unusual convention when working with cutoff functions in this argument. Our notation simply notifies the reader that *some power* of the cutoff is present. Ultimately this makes the proof easier to read; the choice of power of the cutoff is not necessary to the argument, but it is clear it is finite. Take

$$dV_{\mathring{g},\eta} := \eta^K dV_{\mathring{g}}, \ \text{ where } K \in \mathbb{N} \text{ is sufficiently large.}$$

Refer to Remark 3.3.2 regarding our notation for scaling coefficients.

Lemma 3.9.10. *Given the assumptions of Proposition 3.3.3, (3.44), and (3.52) there exists $\beta > 0$ such that*

$$\left|\left|\widetilde{\nabla}\Upsilon\right|\right|_{\mathcal{M}_\beta^{1,2}} \leq C\left((\alpha - 1) + \delta\right).$$

Proof. We will compute the Morrey inequalities for $\widetilde{\nabla}\Upsilon$ and $\widetilde{\nabla}^{(2)}\Upsilon$ separately. We also point out that due to the estimate (3.15) of Proposition 3.2.1 combined with Lemma 3.9.3 give that

$$(\alpha - 1)\left|\left|\nabla \log \chi_\lambda\right|\right|_{L^2}^\mu + (\alpha - 1)\left|\left|\nabla^{(2)} \log \chi_\lambda\right|\right|_{L^2}^\mu \leq C\delta \quad \mu \in \{1, 2\}.$$

We show below that $\widetilde{\nabla}\Upsilon \in \mathcal{M}_2^2$ and $\widetilde{\nabla}^{(2)}\Upsilon \in \mathcal{M}_\beta^2$ to conclude $\widetilde{\nabla}\Upsilon \in \mathcal{M}_\beta^{1,2}$ with necessary bounds.

$\boxed{\widetilde{\nabla}\Upsilon \in \mathcal{M}_2^2}$ For this, we simply have the following by applying Hölder's inequality followed by applying

our global estimates on $\widetilde{\nabla}^{(k)}\Upsilon$ for $k \in \{0, 1, 2\}$,

$$
\int_{B_R} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} \leq \left(\int_{B_R} dV_{\mathring{g},\eta}\right)^{1/2} \left(\int_{B_R} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^4 dV_{\mathring{g},\eta}\right)^{1/2}
$$

$$
\leq \left(\int_{B_R} dV_{\mathring{g},\eta}\right)^{1/2} \left(\int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^4 dV_{\mathring{g}}\right)^{1/2}
$$

$$
\leq C_S \left(\delta + (\alpha - 1)\right) R^2.
$$

Therefore $\widetilde{\nabla}\Upsilon \in \mathcal{M}_2^2$.

$\boxed{\widetilde{\nabla}^{(2)}\Upsilon \in \mathcal{M}_\beta^2}$ We perform a hole-filling argument. To begin,

$$
\int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} \leq \left[C \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}} \left|\nabla\eta\right|_{\mathring{g}} dV_{\mathring{g},\eta}\right]_{T_1} + \left[-\int_{\mathbb{S}^4} \left\langle \widetilde{\nabla}\Upsilon, \widetilde{\triangle}\widetilde{\nabla}\Upsilon\right\rangle_{\mathring{g}} dV_{\mathring{g},\eta}\right]_{T_2}. \tag{3.95}
$$

For the first term we have that, using a weighted Young's inequality for $\nu > 0$ to be chosen and applying the local Poincaré inequality (Proposition 3.9.5)

$$
T_1 \leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + \frac{C}{\nu R^2} \int_{B_R \setminus B_{R/2}} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}}
$$

$$
\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + C_P \int_{B_R \setminus B_{R/2}} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}}.
$$

We next manipulate T_2, commuting derivatives and applying (3.49),

$$
T_2 = -\int_{\mathbb{S}^4} \left\langle \widetilde{\nabla}\Upsilon, \widetilde{\triangle}\widetilde{\nabla}\Upsilon\right\rangle_{\mathring{g}} dV_{\mathring{g},\eta}
$$

$$
= \left[-\int_{\mathbb{S}^4} \left\langle \widetilde{\nabla}_i \Upsilon_j, \widetilde{\nabla}_k \left[\widetilde{\nabla}_k, \widetilde{\nabla}_i\right] \Upsilon_j\right\rangle_{\mathring{g}} dV_{\mathring{g},\eta}\right]_{T_{21}}
$$

$$
+ \left[-\int_{\mathbb{S}^4} \left\langle \widetilde{\nabla}_i \Upsilon_j, \left[\widetilde{\nabla}_k, \widetilde{\nabla}_i\right] \widetilde{\nabla}_k \Upsilon_j\right\rangle_{\mathring{g}} dV_{\mathring{g},\eta}\right]_{T_{22}} + \left[-\int_{\mathbb{S}^4} \left\langle \widetilde{\nabla}\Upsilon, \widetilde{\nabla}\widetilde{\triangle}\Upsilon\right\rangle_{\mathring{g}} dV_{\mathring{g},\eta}\right]_{T_{23}}.
$$

Note that the estimates of T_{21} and T_{22} follow in suit with the manipulations of (3.48), and thus

$$
T_{21} + T_{22} \leq 11 \int_{B_R} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + 8 \int_{B_R} \left|\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} \leq C_P \left(R^4 \int_{B_R} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} + R^2 \left((\alpha - 1) + \delta\right)\right)
$$

$$
\leq C \left((\alpha - 1) + \delta\right) R^2.
$$

Now we approach T_{23}. Applying (3.49) coming from the α-critical equation,

$$T_{23} = \left[-\int_{\mathbb{S}^4} \left\langle \widetilde{\nabla}\Upsilon, 3\widetilde{\nabla}\Upsilon \right\rangle_{\mathring{g}} dV_{\mathring{g},\eta} - \int_{\mathbb{S}^4} \left\langle \widetilde{\nabla}\Upsilon, \left[\widetilde{F}, \widetilde{\nabla}\Upsilon \right] \right\rangle_{\mathring{g}} dV_{\mathring{g},\eta} \right]_{T_{231}}$$

$$+ \left[\int_{\mathbb{S}^4} \left(\widetilde{\nabla}\Upsilon \right)^{*3} dV_{\mathring{g},\eta} \right]_{T_{232}} + \left[\int_{\mathbb{S}^4} \widetilde{\nabla}^{(2)}\Upsilon * \widetilde{\nabla}\Upsilon * \Upsilon \, dV_{\mathring{g},\eta} \right]_{T_{233}}$$

$$+ \left[-\int_{\mathbb{S}^4} \left\langle \widetilde{\nabla}\Upsilon, \widetilde{\nabla}\Theta_1 \right\rangle_{\mathring{g}} dV_{\mathring{g},\eta} \right]_{T_{234}} + \left[-\int_{\mathbb{S}^4} \left\langle \widetilde{\nabla}\Upsilon, \widetilde{\nabla}\Theta_2 \right\rangle_{\mathring{g}} dV_{\mathring{g},\eta} \right]_{T_{235}}.$$

For the first term, note that using (3.18) once more,

$$T_{231} \leq \int_{\mathbb{S}^4} \left| \widetilde{\nabla}\Upsilon \right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} \leq C \left((\alpha - 1) + \delta \right) R^2.$$

Next we have that, applying Hölder's inequality, then global Sobolev embedding $W^{1,2} \hookrightarrow L^4$, and localized Poincaré inequality and finally incorporating the global L^2-estimate for $\widetilde{\nabla}^{(2)}\Upsilon$.

$$T_{232} \leq \int_{\mathbb{S}^4} \left| \widetilde{\nabla}\Upsilon \right|_{\mathring{g}}^3 dV_{\mathring{g},\eta}$$

$$\leq \left(\int_{\mathbb{S}^4} \left| \widetilde{\nabla}\Upsilon \right|_{\mathring{g}}^4 dV_{\mathring{g},\eta} \right)^{1/2} \left(\int_{\mathbb{S}^4} \left| \widetilde{\nabla}\Upsilon \right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} \right)^{1/2}$$

$$\leq C_S \delta \left(\int_{\mathbb{S}^4} \left| \widetilde{\nabla}\Upsilon \right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + \int_{\mathbb{S}^4} \left| \widetilde{\nabla}\left[\eta\widetilde{\nabla}\Upsilon \right] \right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} \right)$$

$$\leq C\delta \left(\int_{B_R} \left| \widetilde{\nabla}\Upsilon \right|_{\mathring{g}}^2 dV_{\mathring{g}} + \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)}\Upsilon \right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + \int_{\mathbb{S}^4} \left| \widetilde{\nabla}\eta \right|_{\mathring{g}}^2 \left| \widetilde{\nabla}\Upsilon \right|^2 dV_{\mathring{g},\eta} \right)$$

$$\leq C\delta \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)}\Upsilon \right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + C\delta \left(\int_{B_R} \left| \widetilde{\nabla}\Upsilon \right|_{\mathring{g}}^2 dV_{\mathring{g}} + \frac{1}{R^2} \int_{B_R \setminus B_{R/2}} \left| \widetilde{\nabla}\Upsilon \right|_{\mathring{g}}^2 dV_{\mathring{g}} \right)$$

$$\leq C\delta \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)}\Upsilon \right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + C_P \delta \left(R^2 \int_{B_R} \left| \widetilde{\nabla}^{(2)}\Upsilon \right|_{\mathring{g}}^2 dV_{\mathring{g}} + \int_{B_R \setminus B_{R/2}} \left| \widetilde{\nabla}^{(2)}\Upsilon \right|_{\mathring{g}}^2 dV_{\mathring{g}} \right)$$

$$\leq C\delta \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)}\Upsilon \right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + C\delta \int_{B_R \setminus B_{R/2}} \left| \widetilde{\nabla}^{(2)}\Upsilon \right|_{\mathring{g}}^2 dV_{\mathring{g}} + C \left((\alpha - 1) + \delta \right) R^2.$$

Next, applying a weighted Young's inequality, Hölder's inequality, applying the global L^4-bound on Υ, and

Sobolev embedding $W^{1,2} \hookrightarrow L^4$ and then applying the localized Poincaré inequality,

$$
\begin{aligned}
T_{233} &\leq C \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}} \left| \widetilde{\nabla} \Upsilon \right|_{\mathring{g}} \left| \Upsilon \right|_{\mathring{g}} \, dV_{\mathring{g},\eta} \\
&\leq \nu \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + \frac{C}{\nu} \int_{\mathbb{S}^4} \left| \widetilde{\nabla} \Upsilon \right|_{\mathring{g}}^2 \left| \Upsilon \right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} \\
&\leq \nu \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + C \left(\int_{\mathbb{S}^4} \left| \widetilde{\nabla} \Upsilon \right|_{\mathring{g}}^4 \, dV_{\mathring{g},\eta} \right)^{1/2} \left(\int_{\mathbb{S}^4} \left| \Upsilon \right|_{\mathring{g}}^4 \, dV_{\mathring{g},\eta} \right)^{1/2} \\
&\leq \nu \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + C_S \delta \left(\int_{\mathbb{S}^4} \left| \widetilde{\nabla} \Upsilon \right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + \int_{\mathbb{S}^4} \left| \widetilde{\nabla} \left[\eta \widetilde{\nabla} \Upsilon \right] \right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} \right) \\
&\leq \nu \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} \\
&\qquad + C \delta \left(\int_{\mathbb{S}^4} \left| \widetilde{\nabla} \Upsilon \right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + \frac{1}{R^2} \int_{B_R \setminus B/2} \left| \widetilde{\nabla} \Upsilon \right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} \right) \\
&\leq \nu \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} \\
&\qquad + C_S \delta \left(R^2 \int_{B_R} \left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}}^2 \, dV_{\mathring{g}} + \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + \int_{B_R \setminus B/2} \left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}}^2 \, dV_{\mathring{g}} \right) \\
&\leq \nu \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} \\
&\qquad + C_S \left(\delta + (\alpha - 1) \right) R^2 + C \delta \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + C \int_{B_R \setminus B/2} \left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}}^2 \, dV_{\mathring{g}}.
\end{aligned}
$$

We now approach the first term with α dependence,

$$
T_{234} = \left[\int_{\mathbb{S}^4} \left\langle \widetilde{\triangle} \Upsilon, \Theta_1 \right\rangle_{\mathring{g}} \, dV_{\mathring{g},\eta} \right]_{T_{2341}} + \left[\int_{\mathbb{S}^4} \widetilde{\nabla} \Upsilon * \Theta_1 * \nabla \eta \, dV_{\mathring{g},\eta} \right]_{T_{2342}}.
$$

We compute these separately. We have that

$$
\begin{aligned}
T_{2341} &\leq \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)} \Upsilon \right| \left| \Theta_1 \right| \, dV_{\mathring{g},\eta} \\
&\leq C (\alpha - 1) \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + C (\alpha - 1) \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}} \left| \widetilde{\nabla} \Upsilon \right|_{\mathring{g}} \left| \Upsilon \right|_{\mathring{g}} \, dV_{\mathring{g},\eta} \qquad (3.96) \\
&\quad + C (\alpha - 1) \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}} \left| \Upsilon \right|_{\mathring{g}} \, dV_{\mathring{g},\eta} + C (\alpha - 1) \int_{\mathbb{S}^4} \left| \widetilde{\nabla}^{(2)} \Upsilon \right|_{\mathring{g}} \left| \Upsilon \right|_{\mathring{g}}^3 \, dV_{\mathring{g},\eta}.
\end{aligned}
$$

The first term can be absorbed, and the second is precisely T_{233}. For the third term of (3.96) we apply a

weighted Young's inequality, localized Poincaré inequality twice, and the global estimate to $\widetilde{\nabla}^{(2)}\Upsilon$.

$$
\int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}} |\Upsilon| \, dV_{\mathring{g},\eta} \leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + \frac{C}{\nu} \int_{\mathbb{S}^4} |\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta}
$$

$$
\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + C \int_{B_R} |\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g}}
$$

$$
\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + C_P \left(R^4 \int_{B_R} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} + R^2 \delta \right)
$$

$$
\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + CR^2 \left((\alpha - 1) + \delta \right).
$$

For the last integral in (3.96) we have

$$
\int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}} |\Upsilon|_{\mathring{g}}^3 \, dV_{\mathring{g},\eta}
$$

$$
\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + \frac{C}{\nu} \left(\int_{\mathbb{S}^4} |\Upsilon|_{\mathring{g}}^6 \, dV_{\mathring{g},\eta} \right)^{1/3} \left(\int_{\mathbb{S}^4} |\Upsilon|_{\mathring{g}}^6 \, dV_{\mathring{g},\eta} \right)^{2/3}
$$

$$
\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + C \left((\alpha - 1) + \delta \right) \left(\int_{\mathbb{S}^4} |\Upsilon|_{\mathring{g}}^6 \, dV_{\mathring{g},\eta} \right)^{1/3}
$$

$$
\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta}
$$

$$
+ C_S \left((\alpha - 1) + \delta \right) \left(\int_{\mathbb{S}^4} |\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + \int_{\mathbb{S}^4} \left|\widetilde{\nabla}\left[\eta \Upsilon\right]\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\left[\eta \Upsilon\right]\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} \right)
$$

$$
\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + C \left((\alpha - 1) + \delta \right) \left(\int_{\mathbb{S}^4} |\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + \int_{\mathbb{S}^4} |\widetilde{\nabla}\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + \int_{\mathbb{S}^4} |\Upsilon|_{\mathring{g}}^2 |\nabla \eta|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} \right)
$$

$$
+ C \left((\alpha - 1) + \delta \right) \left(\int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + \int_{\mathbb{S}^4} |\widetilde{\nabla}\Upsilon|_{\mathring{g}}^2 |\nabla \eta|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + \int_{\mathbb{S}^4} |\Upsilon|_{\mathring{g}}^2 \left|\nabla^{(2)}\eta\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} \right)
$$

$$
\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + C \left((\alpha - 1) + \delta \right) \left(\int_{B_R} |\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g}} + \int_{B_R} |\widetilde{\nabla}\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g}} + \frac{1}{R^2} \int_{B_R \setminus B_{R/2}} |\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g}} \right)
$$

$$
+ C \left((\alpha - 1) + \delta \right) \left(\int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + \frac{1}{R^2} \int_{B_R \setminus B_{R/2}} |\widetilde{\nabla}\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g}} + \frac{1}{R^4} \int_{B_R \setminus B_{R/2}} |\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g}} \right)
$$

$$
\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + C_P \left((\alpha - 1) + \delta \right) \left(\int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + \int_{B_R \setminus B_{R/2}} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} \right).
$$

We now address the next term

$$
T_{2342} = \int_{\mathbb{S}^4} \widetilde{\nabla}\Upsilon * \Theta_1 * \nabla \eta \, dV_{\mathring{g},\eta}
$$

$$
\leq C (\alpha - 1) \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}} |\widetilde{\nabla}\Upsilon|_{\mathring{g}} |\nabla \eta|_{\mathring{g}} \, dV_{\mathring{g},\eta} + C (\alpha - 1) \int_{\mathbb{S}^4} |\widetilde{\nabla}\Upsilon|_{\mathring{g}}^2 |\Upsilon|_{\mathring{g}} |\nabla \eta|_{\mathring{g}} \, dV_{\mathring{g},\eta} \tag{3.97}
$$

$$
+ C (\alpha - 1) \int_{\mathbb{S}^4} |\widetilde{\nabla}\Upsilon|_{\mathring{g}} |\Upsilon|_{\mathring{g}} |\nabla \eta|_{\mathring{g}} \, dV_{\mathring{g},\eta} + C (\alpha - 1) \int_{\mathbb{S}^4} |\widetilde{\nabla}\Upsilon|_{\mathring{g}} |\Upsilon|_{\mathring{g}}^3 |\nabla \eta|_{\mathring{g}} \, dV_{\mathring{g},\eta}.
$$

The first term is exactly T_1, so we can apply the same estimate. Now we approach the second term of (3.97),

applying Young's inequality, Hölder's inequality, a global Sobolev embedding of $W^{1,2} \hookrightarrow L^4$, and localized Poincaré inequalities, and global L^2 control of $\widetilde{\nabla}^{(2)}\Upsilon$,

$$
\int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2 |\Upsilon|_{\mathring{g}} |\nabla\eta|_{\mathring{g}} \, dV_{\mathring{g},\eta}
$$

$$
\leq C \int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^4 \, dV_{\mathring{g},\eta} + \frac{C}{R^2} \int_{B_R\setminus B_{R/2}} |\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g}}
$$

$$
\leq C\left(\delta + (\alpha-1)\right) \left(\int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^4 \, dV_{\mathring{g},\eta}\right)^{1/2} + R^2 C_P \int_{B_R\setminus B_{R/2}} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g}}
$$

$$
\leq C_S\left((\alpha-1)+\delta\right) \left(\int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + \int_{\mathbb{S}^4} \left|\widetilde{\nabla}\left[\eta\widetilde{\nabla}\Upsilon\right]\right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta}\right) + CR^2\left((\alpha-1)+\delta\right)
$$

$$
\leq C \int_{B_R\setminus B_{R/2}} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g}} + C_P\left((\alpha-1)+\delta\right) \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + CR^2\left((\alpha-1)+\delta\right).
$$

For the third term, applying weighted Young's Inequality in preparation for an application of Poincaré inequality, then a Holder's inequality followed by applying global L^4 control of Υ,

$$
\int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}} |\Upsilon|_{\mathring{g}} |\nabla\eta|_{\mathring{g}} \, dV_{\mathring{g},\eta} \leq \frac{\nu}{C_P R^2} \int_{B_R\setminus B_{R/2}} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g}} + \frac{CC_P}{\nu} \int_{B_R\setminus B_{R/2}} |\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g}}
$$

$$
\leq \nu \int_{B_R\setminus B_{R/2}} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g}} + C\left(\int_{\mathbb{S}^4} dV_{\mathring{g}}\right)^{1/2} \left(\int_{B_R\setminus B_{R/2}} |\Upsilon|_{\mathring{g}}^4 \, dV_{\mathring{g},\eta}\right)^{1/2}
$$

$$
\leq \nu \int_{B_R\setminus B_{R/2}} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g}} + C\delta R^2.
$$

For the fourth integral we apply weighted Young's inequality followed by localized Poincaré inequality and Hölder's inequality, using the global Sobolev embedding $W^{2,2} \hookrightarrow L^8$ and applying localized Poincaré,

$$
\int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}} |\Upsilon|_{\mathring{g}}^3 |\nabla\eta|_{\mathring{g}} \, dV_{\mathring{g},\eta}
$$

$$
\leq \frac{1}{R^2} \int_{B_R\setminus B_{R/2}} |\Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g}} + \int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2 |\Upsilon|_{\mathring{g}}^4 \, dV_{\mathring{g},\eta}
$$

$$
\leq C_P R^2 \int_{B_R\setminus B_{R/2}} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g}}
$$

$$
+ C\left(\int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^4 \, dV_{\mathring{g},\eta}\right)^{1/2} \left(\int_{\mathbb{S}^4} |\Upsilon|_{\mathring{g}}^8 \, dV_{\mathring{g},\eta}\right)^{1/2}
$$

$$
\leq C \int_{B_R\setminus B_{R/2}} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g}} + C_S\delta \left(\int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + \int_{\mathbb{S}^4} \left|\widetilde{\nabla}\left[\eta\widetilde{\nabla}\Upsilon\right]\right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta}\right)
$$

$$
\leq C_P\delta \left(\int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + \int_{B_R\setminus B_{R/2}} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g}}\right) + CR^2\left((\alpha-1)+\delta\right).
$$

Next we have that

$$T_{235} = \left[\int_{\mathbb{S}^4} \langle \triangle \Upsilon, \Theta_2 \rangle_{\mathring{g}} \, dV_{\mathring{g},\eta}\right]_{T_{2351}} + \left[\int_{\mathbb{S}^4} \nabla \Upsilon * \Theta_2 * \nabla \eta \, dV_{\mathring{g},\eta}\right]_{T_{2352}}.$$

Then we expand out

$$T_{2351} \leq C(\alpha - 1) \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)} \Upsilon\right|_{\mathring{g}} |\Theta_2|_{\mathring{g}} \, dV_{\mathring{g},\eta}$$

$$\leq C(\alpha - 1) \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)} \Upsilon\right|_{\mathring{g}} \left|\widetilde{\nabla} \Upsilon\right|_{\mathring{g}} |\nabla \log \chi_\lambda|_{\mathring{g}} \, dV_{\mathring{g},\eta} + C(\alpha - 1) \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)} \Upsilon\right|_{\mathring{g}} |\nabla \log \chi_\lambda|_{\mathring{g}} \, dV_{\mathring{g},\eta} \quad (3.98)$$

$$+ C(\alpha - 1) \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)} \Upsilon\right|_{\mathring{g}} |\Upsilon|^2 |\nabla \log \chi_\lambda|_{\mathring{g}} \, dV_{\mathring{g},\eta}.$$

For the first term of (3.98), applying a weighted Young's inequality and Hölder's inequality, followed by global Sobolev embedding $W^{1,2} \hookrightarrow L^4$,

$$C(\alpha - 1) \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)} \Upsilon\right|_{\mathring{g}} |\widetilde{\nabla} \Upsilon| |\nabla \log \chi_\lambda|_{\mathring{g}} \, dV_{\mathring{g},\eta}$$

$$\leq (\alpha - 1) \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)} \Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + \frac{C(\alpha-1)}{\nu} \int_{\mathbb{S}^4} |\nabla \log \chi_\lambda|_{\mathring{g}}^2 \left|\widetilde{\nabla} \Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta}$$

$$\leq (\alpha - 1) \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)} \Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta}$$

$$+ C(\alpha - 1) \left(\int_{\mathbb{S}^4} |\nabla \log \chi_\lambda|_{\mathring{g}}^4 \, dV_{\mathring{g},\eta}\right)^{1/2} \left(\int_{\mathbb{S}^4} \left|\widetilde{\nabla} \Upsilon\right|_{\mathring{g}}^4 \, dV_{\mathring{g},\eta}\right)^{1/2}$$

$$\leq (\alpha - 1) \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)} \Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + C_S \delta \left(\int_{\mathbb{S}^4} |\widetilde{\nabla} \Upsilon|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + \int_{\mathbb{S}^4} \left|\widetilde{\nabla} \left[\eta \widetilde{\nabla} \Upsilon\right]\right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta}\right)$$

$$\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)} \Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta}$$

$$+ C_P \delta \left(\int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)} \Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + \int_{B_R \setminus B_{R/2}} \left|\widetilde{\nabla}^{(2)} \Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g}} + R^2 (\alpha - 1 + \delta)\right).$$

Next we have that, for the second term of (3.98), using weighted Young's inequality followed by Holder's inequality and applying the global bounds to $\log \chi_\lambda$ type terms,

$$C(\alpha - 1) \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)} \Upsilon\right|_{\mathring{g}} |\nabla \log \chi_\lambda|_{\mathring{g}} \, dV_{\mathring{g},\eta}$$

$$\leq (\alpha - 1) \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)} \Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + \frac{C}{\nu} (\alpha - 1) \int_{\mathbb{S}^4} |\nabla \log \chi_\lambda|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta}$$

$$\leq (\alpha - 1) \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)} \Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + C(\alpha - 1) \left(\int_{\mathbb{S}^4} |\nabla \log \chi_\lambda|_{\mathring{g}}^4 \, dV_{\mathring{g},\eta}\right)^{1/2} \left(\int_{\mathbb{S}^4} dV_{\mathring{g},\eta}\right)^{1/2}$$

$$\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)} \Upsilon\right|_{\mathring{g}}^2 \, dV_{\mathring{g},\eta} + CR^2 \delta.$$

Lastly for the third term of (3.98), we apply a weighted Young's inequality, Hölder's inequality, a global

Sobolev embedding and then localized Poincaré to the remaining pieces,

$$C\left(\alpha-1\right)\int_{\mathbb{S}^4}\left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}|\Upsilon|^2\left|\nabla\log\chi_\lambda\right|_{\mathring{g}}dV_{\mathring{g},\eta}$$

$$\leq\left(\alpha-1\right)\nu\int_{\mathbb{S}^4}\left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2dV_{\mathring{g},\eta}+\left(\alpha-1\right)\frac{C}{\nu}\int_{\mathbb{S}^4}|\Upsilon|_{\mathring{g}}^4\left|\nabla\log\chi_\lambda\right|_{\mathring{g}}^2dV_{\mathring{g},\eta}$$

$$\leq\left(\alpha-1\right)\nu\int_{\mathbb{S}^4}\left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2dV_{\mathring{g},\eta}$$

$$+\left(\alpha-1\right)C\left(\int_{\mathbb{S}^4}|\Upsilon|_{\mathring{g}}^8dV_{\mathring{g},\eta}\right)^{1/2}\left(\int_{\mathbb{S}^4}\left|\nabla\log\chi_\lambda\right|_{\mathring{g}}^4dV_{\mathring{g},\eta}\right)^{1/2}$$

$$\leq\left(\alpha-1\right)\nu\int_{\mathbb{S}^4}\left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2dV_{\mathring{g},\eta}$$

$$+C_S\delta\left(\int_{\mathbb{S}^4}|\Upsilon|_{\mathring{g}}^2dV_{\mathring{g},\eta}+\int_{\mathbb{S}^4}\left|\widetilde{\nabla}\left[\eta\Upsilon\right]\right|_{\mathring{g}}^2dV_{\mathring{g},\eta}+\int_{\mathbb{S}^4}\left|\widetilde{\nabla}^{(2)}\left[\eta\Upsilon\right]\right|_{\mathring{g}}^2dV_{\mathring{g},\eta}\right)^2$$

$$\leq\nu\int_{\mathbb{S}^4}\left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2dV_{\mathring{g},\eta}$$

$$+C_P\delta^2\left(\int_{B_R\setminus B_{R/2}}\left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2dV_{\mathring{g}}+\int_{\mathbb{S}^4}\left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2dV_{\mathring{g},\eta}+R^2\left(\alpha-1+\delta\right)\right).$$

We next expand out

$$T_{2352}\leq C\int_{\mathbb{S}^4}\left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}|\Theta_2|_{\mathring{g}}|\nabla\eta|_{\mathring{g}}dV_{\mathring{g},\eta}$$

$$\leq C\left(\alpha-1\right)\int_{\mathbb{S}^4}\left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}\left|\nabla\log\chi_\lambda\right|_{\mathring{g}}|\nabla\eta|_{\mathring{g}}dV_{\mathring{g},\eta}+C\left(\alpha-1\right)\int_{\mathbb{S}^4}\left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2\left|\nabla\log\chi_\lambda\right|_{\mathring{g}}|\nabla\eta|_{\mathring{g}}dV_{\mathring{g},\eta}\quad(3.99)$$

$$+C\left(\alpha-1\right)\int_{\mathbb{S}^4}\left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}|\Upsilon|_{\mathring{g}}^2\left|\nabla\log\chi_\lambda\right|_{\mathring{g}}|\nabla\eta|_{\mathring{g}}dV_{\mathring{g},\eta}.$$

For the first term of (3.99), applying a weighted Young's inequality in preparation for a Poincaré inequality, then Hölder's inequality and applying the global bounds of $\log\chi_\lambda$ type terms,

$$C\left(\alpha-1\right)\int_{\mathbb{S}^4}\left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}\left|\nabla\log\chi_\lambda\right|_{\mathring{g}}|\nabla\eta|_{\mathring{g}}dV_{\mathring{g},\eta}$$

$$\leq\frac{\nu}{C_P R^2}\int_{B_R\setminus B_{R/2}}\left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2dV_{\mathring{g}}+\left(\alpha-1\right)\frac{CC_P}{\nu}\int_{\mathbb{S}^4}\left|\nabla\log\chi_\lambda\right|_{\mathring{g}}^2dV_{\mathring{g}}$$

$$\leq\nu\int_{B_R\setminus B_{R/2}}\left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2dV_{\mathring{g}}+C\left(\alpha-1\right)\left(\int_{\mathbb{S}^4}dV_{\mathring{g},\eta}\right)^{1/2}\left(\int_{\mathbb{S}^4}\left|\nabla\log\chi_\lambda\right|_{\mathring{g}}^4dV_{\mathring{g},\eta}\right)^{1/2}$$

$$\leq\nu\int_{B_R\setminus B_{R/2}}\left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2dV_{\mathring{g}}+C_S\delta R^2.$$

For the second term of (3.99) we apply Hölder's inequality twice followed by global L^4 bounds of $\nabla\log\chi_\lambda$,

a global Sobolev embedding $W^{1,2} \hookrightarrow L^4$ and the localized Poincaré inequality

$$C\left(\alpha-1\right) \int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2 \left|\nabla \log \chi_\lambda\right|_{\mathring{g}} \left|\nabla \eta\right|_{\mathring{g}} dV_{\mathring{g},\eta}$$

$$\leq \frac{C}{R}\left(\alpha-1\right) \left(\int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^4 dV_{\mathring{g},\eta}\right)^{1/2} \left(\int_{\mathbb{S}^4} \left|\nabla \log \chi_\lambda\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta}\right)^{1/2}$$

$$\leq \frac{C}{R}\left(\alpha-1\right) \left(\int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^4 dV_{\mathring{g},\eta}\right)^{1/2} \left(\int_{B_R} dV_{\mathring{g}}\right)^{1/4} \left(\int_{\mathbb{S}^4} \left|\nabla \log \chi_\lambda\right|_{\mathring{g}}^4 dV_{\mathring{g},\eta}\right)^{1/4}$$

$$\leq C_S \delta \left(\int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + \int_{\mathbb{S}^4} \left|\widetilde{\nabla}\left[\eta\Upsilon\right]\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta}\right)$$

$$\leq C_P \left(\delta + \left(\alpha-1\right)\right) R^2 + C_P \delta \int_{B_R \backslash B_{R/2}} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}}.$$

For the third term of (3.99), applying weighted Young's inequality ('preparing' for the application of the Poincaré inequality with our choice of weight), then Poincaré inequality and Hölder's inequality, then applying Sobolev embedding $W^{2,2} \hookrightarrow L^8$ and Poincaré inequalities once more

$$C\left(\alpha-1\right) \int_{\mathbb{S}^4} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}} \left|\Upsilon\right|_{\mathring{g}}^2 \left|\nabla \log \chi_\lambda\right|_{\mathring{g}} \left|\nabla \eta\right|_{\mathring{g}} dV_{\mathring{g},\eta}$$

$$\leq \frac{\nu}{C_P R^2} \int_{B_R \backslash B_{R/2}} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + \frac{C(\alpha-1)C_P}{\nu} \int_{\mathbb{S}^4} \left|\Upsilon\right|_{\mathring{g}}^4 \left|\nabla \log \chi_\lambda\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta}$$

$$\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + C\left(\alpha-1\right) \left(\int_{\mathbb{S}^4} \left|\Upsilon\right|_{\mathring{g}}^8 dV_{\mathring{g},\eta}\right)^{1/2} \left(\int_{\mathbb{S}^4} \left|\nabla \log \chi_\lambda\right|_{\mathring{g}}^4 dV_{\mathring{g},\eta}\right)^{1/2}$$

$$\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta}$$

$$+ C_S \delta \left(\int_{\mathbb{S}^4} \left|\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + \int_{\mathbb{S}^4} \left|\widetilde{\nabla}\left[\eta\Upsilon\right]\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\left[\eta\Upsilon\right]\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta}\right)^2$$

$$\leq \nu \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta}$$

$$+ C_P \delta \left(\int_{B_R \backslash B_{R/2}} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} + \int_{\mathbb{S}^4} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g},\eta} + R^2 \left(\left(\alpha-1\right)+\delta\right)\right).$$

Take $\nu \leq \frac{1}{10}$ so that the terms scaled by ν may be absorbed into the left hand side of (3.95). Furthermore, choose δ, α sufficiently small so that remaining terms may be absorbed over,

$$\int_{B_{\frac{R}{2}}} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} \leq C \left(\int_{B_R \backslash B_{\frac{R}{2}}} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} + R^2 \left(\left(\alpha-1\right)+\delta\right)\right). \tag{3.100}$$

Define

$$f(R) := \int_{B_R \backslash B_{\frac{R}{2}}} \left|\widetilde{\nabla}^{(2)}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} + R^2 \left(\left(\alpha-1\right)+\delta\right).$$

Remanipulating (3.100) above, we have that $f\left(\frac{R}{2}\right) \leq \frac{C}{C+1} f(R)$, and therefore for all $k \geq 1$,

$$f\left(\frac{R}{2^k}\right) \leq \left(\frac{C}{C+1}\right)^k \left(\int_{B_R} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} + R^2\left((\alpha-1)+\delta\right)\right) \leq \left(\frac{C}{C+1}\right)^k C\left((\alpha-1)+\delta\right).$$

This implies that there exists some $\beta > 0$ such that

$$\int_{B_R} \left|\widetilde{\nabla}\Upsilon\right|_{\mathring{g}}^2 dV_{\mathring{g}} \leq C\left(\alpha-1+\delta\right) R^{2\beta},$$

which yields the desired Morrey bound. $\qquad\square$