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Geometry and Conservation Laws for a Class of Second-Order Parabolic Equations

by

Benjamin Blake McMillan

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University of California, Berkeley

Committee in charge:

Professor Robert Bryant, Chair
Professor Ian Agol
Professor Nicolai Reshetikhin
Professor James Casey

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Abstract

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Benjamin Blake McMillan
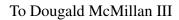
Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Robert Bryant, Chair

I study the geometry and the conservation laws of second-order partial differential equations of parabolic type. The general strategy is to replace the differential equation with an *exterior differential system*—a smooth manifold with extra geometric structure that keeps track of the solutions—and then use geometric methods.

I use Cartan's method of equivalence to determine the essential geometric curvatures of parabolic equations. I then explain the geometric significance of these curvatures, including some normal form results. The study of these curvatures also leads me naturally to a nice class of equations, the parabolic Monge-Ampère equations.

In the second half, I study the relationship between the geometry and the conservation laws of parabolic equations. In particular, I prove that for a specific class of parabolic equations, the *generating function* of any conservation law depends on at most second derivatives of solutions. This is in contrast to examples such as the KdV equation, which have conservation laws of arbitrarily high order.

To Dougald McMillan III

I wish we had had the opportunity to know each other better.
$<8\Omega\sim$

Contents

Acknowledgments

Introduction

The goal of this thesis is to study the geometry and the conservation laws of differential equations of parabolic type. I define *parabolic systems* in arbitrarily many variables and make progress on their equivalence problem. In general, Cartan's method of equivalence is a strategy for determining the local invariants of geometric structures: if one is handed a specific class of geometric structure on smooth manifolds (eg. Riemannian, almost complex, CR, etc.), one can in principal 'turn the crank' and find the local curvatures (eg. Riemannian curvature, the Nijenhuis tensor, the Levi form, respectively), which distinguish non-isomorphic structures .

The reader may not be familiar with what I mean by 'geometry of differential equations.' In short, any differential equation has an associated *exterior differential system*—a manifold M equipped with an ideal $\mathcal{I}$ in the ring of forms—for which certain compatible submanifolds correspond to the solutions of the differential equation. There are a few advantages to taking this perspective. Not least, an exterior differential system is independent of a choice of coordinates, so questions of guage are more easily answered. Another advantage is that one can now ask (and answer) questions about the geometry of differential equations. From the perspective of the method of equivalence, the ideal $\mathcal{I}$ is simply additional geometric structure on M, much the same as (for example) a metric or a contact form. In fact, contact manifolds provide an example of an exterior differential system, albeit with no local geometry. Bryant, Griffiths and Hsu give a good general overview of this philosophy in the monograph *Toward a Geometry of Differential Equations*, [BGH95c].

A large portion of this paper is devoted to the geometry of parabolic systems, which correspond to parabolic, second-order differential equations. In studying the geometry of the EDS $(M, \mathcal{I})$ associated to a second-order differential equation, the first invariant one finds is the *geometric principal symbol*, a symmetric matrix valued function on M. The signature of this symbol controls the behaviour of solutions, which is one explanation for why second-order equations are typically divided into classes: elliptic, hyperbolic, parabolic.

A second-order equation is parabolic if it satisfies the (closed) condition that its symbol is everywhere positive semi-definite, i.e., the symbol has a 1-dimensional kernel at each point. These kernels are closely associated to the *characteristic* directions of the equation. Intuitively, a parabolic system has at every point a direction for which the second derivates involving this direction don't enter the differential equation. For example, the characteristic direction of the heat equation

$$u_t = \Delta u$$

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is the vector $\frac{\partial}{\partial t}$ at each point. More generally, any equation of the form

$$u_t = F\left(x^i, u, \frac{\partial u}{\partial x^i}, \frac{\partial^2 u}{\partial x^i \partial x^j}\right),$$

where F is elliptic, has $\frac{\partial}{\partial t}$ as characteristic direction. Such *evolutionary* equations are what might classically be called parabolic equations.

I also study a more general class of equations, which have geometric symbol of parabolic type yet cannot be written as an evolution equation in any choice of coordinates. For these general parabolic systems, the intuition of a characteristic direction is not precise. Instead, one should consider the *characteristic co-vector*. This co-vector defines hyperplane distributions on (the graphs of) solutions, the *characteristic distributions*. Evolutionary equations are characterized by the property that these distributions be Frobenius on all solutions, which I show in Theorem 2.

Restricted to parabolic systems, the equivalence problem provides 2 new families of invariants: the Monge-Ampère invariants and the $\mathfrak{co}(n)$-valued Goursat invariant. A major element of my thesis is the provision of a geometric interpretation for these.

I introduce the *linear-type Monge-Ampère systems*, which correspond to a particularly simple, and geometrically natural, class of Monge-Ampère equations. Within these, I provide a geometric characterization of *parabolic Monge-Ampère* systems. If all of the Monge-Ampère invariants of a parabolic system vanish identically, then it has a *de-prolongation* to a parabolic Monge-Ampère equation. This should be thought of as a condition on the 'curvatures' of parabolic systems. If the 'first half' of the Monge-Ampère invariants vanish, then there is a partial de-prolongation, which puts the differential equation in a normal form.

Given a parabolic equation in generic coordinates, it may not be obvious whether or not there is a change of variables that puts it into evolutionary form. The Goursat invariant helps to answer this question. The Goursat invariant is a section of a $\mathfrak{co}(n)$-fiber bundle, and naturally splits into its trace component $a\delta_{ij}$ and anti-symmetric component $\mathcal{G}_{ij}$. If the Goursat invariant is a non-zero multiple of the identity matrix everywhere, then Theorem 2 provides a choice of coordinates that puts the equation into evolutionary form. Conversely, if it is not a multiple of the identity matrix somewhere, then the equation is not evolutionary in any choice of coordinates.

Geometrically, the trace component of the Goursat invariant detects the sub-principal symbol, and can be used to determine whether a parabolic system deserves to be called parabolic. Indeed, if it vanishes identically, then there is a choice of coordinates so that the differential equation is of the form

$$F\left(t, x^i, u, \frac{\partial u}{\partial x^i}, \frac{\partial^2 u}{\partial x^i \partial x^j}\right) = 0,$$

with no dependence on the derivatives of t. On the other hand, if the trace component does not vanish, then the parabolic system has a well defined 'positive time' direction. This is useful, for example, for initial value problems, so that one can be sure to look for solutions in the correct direction.

The anti-symmetric component of the Goursat invariant measures the integrability of the characteristic distributions. Indeed, Theorem 2 also states that $\mathcal{G}_{ij}$ vanishes if and only if the characteristic foliation is integrable on any solution. Even better, one can in this case construct a global 'time' foliation, which puts the equation in evolutionary form.

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Both the Monge-Ampère and the Goursat invariants have precedents, of the same name, in work done on low dimensional parabolic systems. The trace component of the Goursat invariant was introduced by Goursat in his study of parabolic equations in 1+1 variables. It also features in work on parabolic system by Bryant & Griffiths [BG95b] and the thesis of Clelland [NC97], where she generalizes it to 2+1 variables. The anti-symmetric component is new. Bryant and Griffiths also introduced the Monge-Ampère invariants for parabolics in 1+1 variables, while Clelland generalized them to 2+1 dimensions. My work here generalizes the previously studied cases in two directions: I study parabolic systems in more than 3 variables and I study non-evolutionary equations.

In the second part of this thesis, I focus on the conservation laws of parabolic systems. The systematic study of conservation laws has a long history, but an important modern development was the introduction by Vinogradov et al. (e.g. [Vin99]) of the variational bi-complex. This allowed the study of conservation laws by homological methods. In the 90's, Bryant and Griffiths generalized the theory in [BG95a], introducing the characteristic cohomology of exterior differential systems. My thesis is heavily indebted to this paper (and its sequel [BG95b], part II)—most of the ideas appear there already. My contribution is applying, and in some cases extending, the tools to the case at hand, parabolic equations. By using the EDS version of the space of conservation laws, I am able to apply the results developed in the first half of this thesis.

Bryant and Griffiths took a similar approach in part II, using their general theory to study parabolic equations in two variables. Their approach cut both ways: Studying conservation laws informed their solution to the equivalence problem, and understanding the local invariants allowed them to classify equations in terms of conservation laws. One important step in this classification was the proof that, for parabolic systems in $1+1$ variables, all conservation laws depend on at most second derivatives of solutions. They also showed that if an equation has a non-trivial conservation law then it must be of a special form, a Monge-Ampère equation. Using this reduction, they were able to determine the invariants obstructing conservation laws and to prove that a non-linearizable parabolic equation has at most 3 conservation laws.

In her thesis ([NC97]), Clelland generalized these results to parabolic equations in $2 + 1$ variables with integrable characteristics. In this case again, all conservation laws depend on at most second derivatives, and the existence of a non trivial conservation law leads to a reduction to a Monge-Ampère equation. With this reduction, she was able to find examples of non-linearizable parabolic equations that nonetheless have infinitely many conservation laws.

Both Clelland's and Bryant and Griffiths' results suggest the conjecture that for parabolic equations the calculation of conservation laws reduces to a finite dimensional problem. I prove this in Theorem 5 for parabolic systems with integrable characteristics. An interesting phenomenon in its own right, this means that the problem of classifying conservation laws is far more tractable than for general PDE. Contrast this situation with that of the KdV equation, which has conservation laws that depend on arbitrarily many derivatives of solutions.

For intuition on conservation laws, consider the EDS $(M, \mathcal{I})$ associated to a *determined* differential equation F for functions in n variable—approximately, F is determined when a compatible $(n-1)$-dimensional manifold in M uniquely specifies the graph of a solution to F, but an $(n-2)$-dimensional manifold does not. On such a system, a conservation law is a differential form Φ on M whose restriction to the graph of any solution of F is closed. A closed form Φ certainly meets

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this criterion, providing an example of a trivial conservation law. Suppose that Φ is not closed on M. By Stokes' Theorem, its integral over the boundary of a solution in M is zero. In particular, Φ provides a necessary condition for an $(n-1)$-dimensional manifold in M to extend to the graph of a solution to F.

To take a simple, but critical, example, the energy of physical systems fall into this discussion. Consider an ordinary differential equation that describes a particle in a conservative force field, and let M be the phase space of the particle. (I omit the description of $\mathcal{I}$ in this case, but there is one.) Solution manifolds are curves, 1-dimensinal submanifolds of M. The energy E is a non-constant function on phase space, but is constant along solutions. This is equivalent to the statement that dE is zero when restricted to a solution.

In order to discuss conservation laws of arbitrary order, I establish some technical results, which extend a given parabolic system to its *infinite prolongation*, which is essentially an infinite phase space for solutions to live in. The *principal structure equations*, which I descibe in Proposition 1, are useful for determining the leading order part of conservation laws, the part that only depends on the principal symbol of an equation. The principal structure equations allow me to filter the space of differential forms by *principal weight*, which is closely related to the number of derivatives of solutions that a form depends on.

The behavior of parabolic equations, and their conservation laws, depends on the sub-principal symbol. Since the principal weight filtration does not detect the sub-principal symbol, it is necessary to find a refinenent. For parabolic systems with integrable characteristics, I calculate finer structure equation in Theorem 3 . These refined structure equations allow me to define a refined filtration of the differential forms, which is critical to the proof of Theorem 5. In effect, this filtration is 'transverse' to the the principal filtration, so by playing them off of each other I can split equations into quite manageable pieces.

The weight filtrations, and the horizontal and vertical derivatives, tools which are crucial to the calculations done here, are introduced in [BG95a]. One contribution that I make is to extend the definition of these derivatives to a more coframe-equivariant setting. This allows me to do calculations without fixing a specific choice of coframing. The philosophy here is in line with the dictum in linear algebra that a gentlemathematician never fixes a basis.

It is not clear yet whether a parabolic system in more than three variables that has a non-trivial conservation law is required to be Monge-Ampère, as happens in fewer variables. However, I demonstrate that at the very least, there are strong restrictions on the Monge-Ampère invariants.

Theorem 5 says that for parabolic systems with integrable characteristics, the generating functions of any conservation laws cannot depend on more than second derivatives of solutions. I remark that this result likely extends to porabolic systems with non-integrable characteristics, which are more generic and so are even more constrained in having conservation laws. The main difficulty in extending the proof of Theorem 5 is calculating the refined structure equations for general parabolic systems.

I hold the following conventions and notations throughout:

- $n + 1$ is the number of independent variables in a parabolic system. One can, in special cases (that is, for *evolutionary equations*), split these into one 'time' variable and n 'space' variables. Parabolic systems with $n = 1$ and $n = 2$ are fairly well understood (except for the non-evolutionary case), and I generally consider parabolic systems with $n \geq 3$.

- Indices i, j, k, l, m always range from 1 to n, while indices a, b, c always range from 0 to n. I typically abuse the Einstein summation notation on repeated indices in two related ways: Repeated spatial indices mean a sum over only the spatial indices and I apply this convention to tensors a_{ab} in $\mathbb{R}^{n+1} \otimes \mathbb{R}^{n+1}$. Precisely, I let

$$a_{ii} = \sum_{i=1}^{n} a_{ii}.$$

  This will be justified by the existence of a partial trace operator, which is invariant under allowed changes of basis. This is not so different from the typical Einstein convention, where the trace of elements in $\mathbb{R}^{n+1} \otimes (\mathbb{R}^{n+1})^\vee$ is justified by the fact that the identity is preserved by the conjugation action of $\mathrm{GL}(\mathbb{R}^{n+1})$.

- Many calculations in this paper can be done by fixing a coframing on a manifold. However, it is often advantageous to instead lift the calculation to the bundle of coframes of the manifold, where the calculations can be performed without making a choice of coframing. I typically underline objects on the base manifold and do not underline their analogues on the coframe bundle.

- There is a collection of $n + 1$ independent 1-forms ω^a, which I define below. I use ω to denote the exterior product of all of them:

$$\omega = \omega^0 \wedge \ldots \wedge \omega^n,$$

  the hat denoting omission. I use the *omitted index* notation, so that

$$\omega_{(a)} = (-1)^a \omega^0 \wedge \omega^1 \wedge \ldots \wedge \hat{\omega}^a \wedge \ldots \wedge \omega^n.$$

  Observe that the sign is chosen so that

$$\omega^a \wedge \omega_{(a)} = \omega.$$

  For pairs of indices, I define

$$\omega_{(ab)} = \pm \prod_{c \neq a,b} \omega^c,$$

  with the sign uniquely specified by the conditions

$$\omega_{(ab)} = -\omega_{(ba)}$$

and

$$\omega^a \wedge \omega_{(ab)} = \omega_{(b)}.$$

These two properties are useful to note when following calculations.

More generally, for an anti-symmetric multi-index I in $\{0, \ldots, n\}$, I define

$$\omega_{(I)} = \pm \prod_{a \notin I} \omega^a,$$

with signs specified recursively by the condition that

$$\omega^a \wedge \omega_{(aI)} = \omega_{(I)}$$

whenever a is not already in the multi-index I, and the condition that

$$\omega_{(\sigma(I))} = \text{sign}(\sigma)\omega_{(I)},$$

where σ is a permutation of the set I.

- I and J are subsets of $\{1, \ldots, n\}$ when discussing Monge-Ampère systems. Otherwise, I is a symmetric multi-index $(i_1 \ldots i_s)$ so that the (possibly repeating) indices $i_1, \ldots, i_s$ are between 1 and n. The index I is of size s, denoted $|I|$. The context will keep these separate uses of I clear.

- F is the principal weight filtration, defined by the principal weight function pwt.

- $\mathcal{F}$ the sub-principal weight filtration, defined by the sub-principal weight function wt.

- Various reductions of structure groups are defined throughout. The notions are defined below, but I collect the most relevant facts here:
  The G_0-structure $\mathcal{B}_0$ is the bundle of 0-adapted coframes of a parabolic system. Matrices in G_0 are first defined in equation (1.8).

  If (most of) the primary Monge-Ampère invariants vanish identically, then $\mathcal{B}_0$ reduces to the G_1-structure $\mathcal{B}_1$, where G_1 consists of matrices in G_0 so that $S^{a0} = 0$.

  If all of the Monge-Ampère invariants vanish, then $\mathcal{B}_0$ reduces to $\mathcal{B}_{MA}$, with structure group G_{MA}, matrices so that $S = 0$.

  If *all* of the primary Monge-Ampère invariants vanish, then there is a reduction to a G_2-structure, on which the primary Goursat invariant is well defined. The non-degeneracy condition of any parabolic system then guarantees a further reduction to $\mathcal{B}_3$, where the trace-Goursat invariant is normalized to $-1/n$.

  If the anti-symmetric component of the Goursat invariant vanishes, then there is a reduction to $\mathcal{B}_4$ so that the component (a_i) of the Goursat invariant vanishes identically.

- $\mathcal{C}^\infty(M)$ is the set of smooth functions on a manifold M.

Chapter 1

Background

1.1 Exterior Differential Systems

Definitions and motivation

Definition 1. On a smooth manifold M, a graded ideal $\mathcal{I}$ in the ring of forms $\Omega^*(M)$ is *differentially closed* if $d\mathcal{I} \subset \mathcal{I}$. Throughout, *ideal* means differentially closed ideal, although I sometimes use *differential ideal* for emphasis and *algebraic ideal* for ideals not assumed to be differentially closed.

An *exterior differential system* $(M, \mathcal{I})$ is a smooth manifold M and a graded, differentially closed ideal $\mathcal{I}$ on M.

A submanifold $\iota\colon N \hookrightarrow M$ is an *integral manifold* of $(M, \mathcal{I})$ if the pullback $\iota^* \mathcal{I}$ is identically zero, or equivalently, if $\phi|_{T_x N} = 0$ for all $\phi \in \mathcal{I}$ and $x \in N$.

An *integral element* at a point $x \in M$ is a subspace $E \subset T_x M$ for which $\phi|_E = 0$ for all $\phi \in \mathcal{I}$.

As I will describe shortly, every sufficiently non-degenerate system of partial differential equations corresponds to an exterior differential system. Under this correspondence, solutions to a PDE are equivalent to integral submanifolds of the associated EDS.

Integral elements can be thought of as potential tangent spaces of integral manifolds. The Cartan-Kähler theory gives a sufficient condition for an integral element to have an integral manifold tangent to it.

The ideal $\mathcal{I}$ determines differential conditions on the tangent planes of integral manifolds. This is one reason why integral manifolds should be thought of as (the graphs of) solutions to a PDE. The following examples provide a more direct explanation.

Example 1 (The empty differential equation)**.** Consider $J^2 = J^2(\mathbb{R}^n, \mathbb{R})$, the bundle over $\mathbb{R}^n$ of 2-jets for functions in n variables. A choice of coordinates (x^i, u) for $J^0(\mathbb{R}^n, \mathbb{R})$—the space of 0-jets—induces coordinates x^i, u, p_i, and p_{ij} on J^2, where the p_i correspond to the first derivatives of u with respect to x^i and p_{ij} to the second derivatives. These coordinates may be used to define

the *contact forms*[1]

$$\hat{\theta}_\varnothing = du - p_i \, dx^i$$
$$\hat{\theta}_i = dp_i - p_{ij} \, dx^j,$$

as well as the canonical *contact ideal*

$$\mathcal{C} = \{\hat{\theta}_\varnothing, \hat{\theta}_i\} = \{\hat{\theta}_\varnothing, \hat{\theta}_i, \, d\hat{\theta}_\varnothing, \, d\hat{\theta}_i\}$$

on J^2. The pair $(J^2, \mathcal{C})$ is an exterior differential system.

I note that the geometric geometric structure of this EDS is independent of a choice of coordinates on J^2, because $\mathcal{C}$ can be defined intrinsically. Let $C_x \subset T_x(J^2)$ be the subspace spanned by the tangent planes of all 2-jet graphs which pass through $x \in J^2$. This defines the $n + n(n+1)/2$ dimmensional *contact distribution* C on J^2, and $\mathcal{C}$ is the differential ideal generated by $C^\perp \subset \Omega^1(M)$.

The integral submanifolds of J^2 that submerse onto $\mathbb{R}^n$ are the graphs of 2-jet lifts. A smooth function $u \colon \mathbb{R}^n \to \mathbb{R}$ naturally induces its 2-*jet lift*, the section of J^2 given by

$$(x^i) \longmapsto \left(x^i, u(x^i), \frac{\partial u}{\partial x^i}(x^i), \frac{\partial^2 u}{\partial x^i \partial x^j}(x^i) \right).$$

On the other hand, a generic section s of J^2 is not the 2-jet lift of any function. Indeed, write s as

$$(x^i) \longmapsto \left(x^i, u(x^i), p_i(x^i), p_{ij}(x^i) \right).$$

Then s is a 2-jet lift if and only if it agrees with the 2-jet lift of the function $u(x^i)$, that is,

$$p_i(x^i) = \frac{\partial u}{\partial x^i}(x^i)$$

and

$$p_{ij}(x^i) = \frac{\partial^2 u}{\partial x^i \partial x^j}(x^i)$$

for all $(x^i) \in \mathbb{R}^n$. The contact ideal provides a geometric test of these conditions, stated in the following lemma.

Lemma 1. *An n-dimensional submanifold N of J^2 is locally the graph of a 2-jet lift if and only if N is an integral manifold of $(J^2, \mathcal{C})$ and $dx^1 \wedge \ldots \wedge dx^n$ is non-vanishing on N.*

Proof. First note that N is locally the graph of a section of J^2 if and only if $dx^1 \wedge \ldots \wedge dx^n$ is non-vanishing on N. In this case, N may be locally parameterized by functions $u(x)$, $p_i(x)$ and $p_{ij}(x)$. It is straightforward to check that the requirement that $\hat{\theta}_\varnothing$ and the $\hat{\theta}_i$ vanish on N is equivalent to the conditions $p_i = \frac{\partial u}{\partial x^i}$ and $p_{ij} = \frac{\partial p_i}{\partial x^j} = \frac{\partial^2 u}{\partial x^i \partial x^j}$ for $1 \leq i, j \leq n$. $\qquad\square$

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[1]These are not contact forms in the sense of contact geometry, in which a contact form defines a totally non-integrable hyperplane distribution. However, the concepts are related. In particular, the form $\hat{\theta}_\varnothing$ can be defined on the space of 1-jets, where it *does* define a maximally non-integrable distribution.

Because of this lemma, $(J^2, \mathcal{C})$ can be reasonably thought of as the exterior differential system corresponding to the system of no 2nd order equations for one function of n variables. Of course, the solutions to such a system are arbitrary functions of n variables.

Example 2 (One second-order equation)**.** Consider now a single 2nd order differential equation for a function of n variables, given in jet coordinates by

$$F\left(x^i, u, p_i, p_{ij}\right) = 0.$$

Let M_0 be the zero locus of F. To avoid degeneracy, I will assume that at each point of M_0,

$$dF \not\equiv 0 \quad \left(\mathrm{mod}\ dx^i,\ du,\ dp_i\right).$$

Apart from ensuring that the equation is truly second order, this guarantees that M_0 is a codimension 1 submanifold in $J^2(\mathbb{R}^n, \mathbb{R})$. Let $\mathcal{I}_0$ denote the restriction of $\mathcal{C}$ to M_0.

We say that $(M_0, \mathcal{I}_0)$ is the exterior differential system associated to the differential equation F. Its generic integral manifolds are locally in correspondence with solutions to F. Indeed, from the lemma, if N is an integral manifold of $(M_0, \mathcal{I}_0)$ that submerses onto $\mathbb{R}^n$ under the natural projection, then near every point of N there is a function $u(x)$ whose 2-jet graph agrees with N. But N lies in M_0, the zero locus of F, so u satisfies the differential equation.

Although I will not need it here, this story generalizes to arbitrary PDE satisfying appropriate non-degeneracy conditions. There is a naturally defined contact ideal on $J^r(\mathbb{R}^n, \mathbb{R}^s)$, and the EDS obtained by restricting this ideal to the zero set of

$$F\colon J^r(\mathbb{R}^n, \mathbb{R}^s) \to \mathbb{R}^l$$

is the EDS associated to a system of l different rth order differential equations for s functions of n variables.

These examples demonstrate a common situation; it is often useful to consider only integral manifolds satisfying a given transversality condition. The integral manifolds described above, with non-zero $dx^1 \wedge \ldots \wedge dx^n$, are *classical solutions*. There are of course integral manifolds that don't satisfy this condition, the *generalized solutions*. These have their uses, but it is clearly important to distinguish the two classes.

The condition that the form $dx^1 \wedge \ldots \wedge dx^n$ restrict to be non-zero is appropriately generalized in the following definition. For clarity, assume that $(M, \mathcal{I})$ is an exterior differential system satisfying the constant rank condition that the degree 1 piece of $\mathcal{I}$,

$$\mathcal{I}^1 = \mathcal{I} \cap \Omega^1(M),$$

has constant rank, say s. This assumption typically holds in examples and does hold for all examples in this paper.

Definition 2. An *independence condition* for $(M, \mathcal{I})$ is a free submodule J in $\Omega^1(M)$ of dimension $n + s$ so that *(i)* $\mathcal{I}^1 \subset J$ and *(ii)* J has everwhere a local basis

$$\underline{\theta}_1, \ldots, \underline{\theta}_s, \underline{\omega}^1, \ldots, \underline{\omega}^n$$

for which
$$\underline{\omega}^1 \wedge \ldots \wedge \underline{\omega}^n \notin \mathcal{I}.$$

An n-dimensional integral manifold Σ of M *satisfies the independence condition* if $J|_\Sigma$ is n-dimensional, or equivalently, if
$$\underline{\omega}^1 \wedge \ldots \wedge \underline{\omega}^n|_\Sigma \neq 0$$
everywhere. Likewise, an n-dimensional integral element E satisfies the independence condition if
$$\underline{\omega}^1 \wedge \ldots \wedge \underline{\omega}^n|_E \neq 0.$$

I will typically call integral manifolds that satisfy a given independence condition *solution manifolds*.

I also remark that both examples belong to a special class of exterior differential system.

Definition 3. An exterior differential system $(M, \mathcal{I})$ is a *Pfaffian system* if it has an independence condition and $\mathcal{I}$ is locally generated by 1-forms and their exterior derivatives.

There is a well-developed theory of Pfaffian systems, see for example [BCG$^+$13], chapter IV. Several of the results there are invaluable in the following.

Morphisms of Exterior Differential Systems

In the category of exterior differential systems, a morphism from $(M, \mathcal{I})$ to $(M', \mathcal{I}')$ is a smooth map $f \colon M \to M'$ that pulls back $\mathcal{I}'$ to a subset of $\mathcal{I}$. From this perspective, an integral manifold N of $(M, \mathcal{I})$ is simply an EDS embedding $\varphi \colon (N, \{0\}) \to (M, \mathcal{I})$. It is occasionally useful to drop the condition that integral manifolds are embeddings. Then morphisms are characterized by the condition that they push forward solutions of $(M, \mathcal{I})$ to solutions of $(M', \mathcal{I}')$.

The most important case is when two exterior differential systems are equivalent. The following definition gives a class of maps which preserve the structure of integral manifolds.

Definition 4. An *equivalence* of exterior differential systems $(M, \mathcal{I})$ and $(M', \mathcal{I}')$ is a diffeomorphism $f \colon M \to M'$ for which $f^* \mathcal{I}' = \mathcal{I}$.

The most classical examples of EDS equivalences are the *point transformations*—equivalences induced by changes of coordinates of a PDE. More precicesly, a change of coordinates transforms an equation F into a new equation F' in the standard manner. On the other hand, the change of coordinates diffeomorphism naturally lifts to a 'prolonged' diffeomorphism of $J^r(\mathbb{R}^n, \mathbb{R}^s)$. It follows from the definitions that this diffeomorphism defines an equivalance from the EDS induced by F to the one induced by F'.

Not all equivalences come from point transformations. For example, consider the map from $J^1(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ to itself, given in coordinates by
$$\varphi(x^i, u, p_i) = (p_i, x^i p_i - u, x^i).$$

It is straightforward to check that φ pulls back the contact ideal $\{du - p_i\, dx^i\}$ to itself. As a consequence, φ induces a morphism from any first order equation to a new one in such a way that

solutions are taken to solutions. For example, the EDS induced by the equation $\frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^i} = x_1$ is taken to the simpler EDS induced by the equation $\frac{\partial u}{\partial x^1} = x^i x^i$. Solutions to the second equation, which is relatively easier to solve, can be pushed forward by φ^{-1} to give solutions to the first. Note that this transformation switches position and derivative variables, so it cannot come from any change of coordinates.

1.2 The method of equivalence for PDEs

Geometry of the empty PDE

I now turn to Cartan's method of equivalence, applied to the geometry of exterior differential systems. I begin with the geometry of the empty PDE, because it will help to understand more interesting equations. The story is analogous to the study of surfaces in Euclidean space, where the flat Riemannian structure, when restricted to a submanifold, gives the first and second fundamental forms. Here the empty equation (J, C) plays the same role as Euclidean space. Upon restricting to a second-order PDE, one recovers the geometric analogue of the symbol, as well deeper invariants.

The first step in any equivalence problem is to restrict attention to coframings that are 'adapted' to the geometry of interest. Before getting to this, I recall the definition for the bundle of all coframes on a manifold.

Definition 5. Given a smooth n-manifold M, a *coframe* at $x \in M$ is an isomorphism

$$u \colon T_x M \longrightarrow \mathbb{R}^n .$$

Denote the set of coframes based at x by $\mathcal{F}_x$. The *coframe bundle* $\mathcal{F}(M)$ is then given by

$$\mathcal{F}(M) = \bigcup_{x \in M} \mathcal{F}_x \subset \mathrm{Hom}(TM, \underline{\mathbb{R}}^n).$$

The bundle projection $\pi \colon \mathcal{F}(M) \to M$ sending $u \in \mathcal{F}_x$ to x makes $\mathcal{F}(M)$ into a principal $\mathrm{GL}(\mathbb{R}^n)$ bundle. For reasons of convention, I consider $\mathcal{F}(M)$ as a *right* principal bundle, where the action on each fiber is given by *inverse post-composition*:

$$g \cdot u = g^{-1} \circ u$$

for any coframe u and element $g \in \mathrm{GL}(\mathbb{R}^n)$.

Up to choice of basis for $\mathbb{R}^n$, a local section $\underline{\eta}$ of $\mathcal{F}(M)$ is the same as a local *coframing* of M. Indeed, the components of $\underline{\eta}$ are a set of n independent 1-forms on M. Conversely, a choice of n independent 1-forms defines a section of $\mathcal{F}(M)$.

By considering subbundles, we can restrict attention to coframings with specific properties. The most important case arises by considering coframings adapted to some extra geometric structure on M. It often happens that the adapted coframes for a geometry form a principal G bundle, where G is a subgroup of $\mathrm{GL}(\mathbb{R}^n)$. In this case, G is the group of 'pointwise' symmetries of the geometry, and the adapted coframings are given by sections of a *G-structure*.

Definition 6. Let $G \subset \mathrm{GL}(\mathbb{R}^n)$ be a matrix Lie group. A *G-structure* on M, with *structure group* G, is a principal G-subbundle of $\mathcal{F}(M)$.

Example 3. Riemannian manifolds are equivalent an $O(n)$-structures. Given a Riemannian manifold (M, g) and a fixed Euclidean structure g_0 on $\mathbb{R}^n$, define $\mathcal{B}$ to be the subbundle of coframes $u \in \mathcal{F}(M)$ for which

$$u^* g_0 = g_{\pi(u)}.$$

Observe that any two coframes in the same fiber differ by an element of $O(n)$, so $\mathcal{B}$ is an $O(n)$-structure.

Conversely, an $O(n)$-structure $\mathcal{B}$ defines a metric on M: at each point $x \in M$, let

$$g_x = u^*(g_0)$$

for *any* u in the fiber of x. The resulting metric is well defined precisely because $\mathcal{B}$ is an $O(n)$-structure. Notice that, up to a choice of orthonormal basis for $(\mathbb{R}^n, g_0)$, an orthonormal coframing of M is equivalent to a section of $\mathcal{B}$. The local geometry (i.e., the Riemannian curvature) of M is wrapped up in the question of how curved $\mathcal{B}$ is as a subbundle of $\mathcal{F}(M)$.

Example 4. In preparation for the definition of parabolic systems, consider the geometry induced by $\mathcal{C}$ on J^2, the empty equation in (now) $n+1$ variables.

At any point x of J^2, the tangent space $T_x J^2$ is isomorphic to

$$\mathbb{R} \oplus \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1} \oplus \mathrm{Sym}^2 \mathbb{R}^{n+1}.$$

For notation, let $W = \mathbb{R}^{n+1}$ and fix a basis

$$e_\varnothing, e_a, f_a, e_{ab} = e_a \circ e_b \tag{1.1}$$

of

$$V = \mathbb{R} \oplus W \oplus W^\vee \oplus \mathrm{Sym}^2 W,$$

as well as dual basis

$$e^\varnothing, e^a, f^a, e^{ab}$$

of $V^\vee$.

I note that the splitting of $T_x J^2$ should be more properly expressed as a partial flag, the one coming from the various submersions of J^2 to lower order jet spaces. In the given basis of V, this flag is

$$\{e_\varnothing\} \subset \{e_\varnothing, e_a\} \subset \{e_\varnothing, e_a, f_a\} \tag{1.2}$$

and, for example,

$$W \cong \{e_\varnothing, e_a\}/\{e_\varnothing\}.$$

It will be useful to also make the identifications

$$W^\vee \cong \{e_\varnothing, e_a, f_a\}/\{e_\varnothing, e_a\}$$

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and
$$\text{Sym}^2 W \cong V/\{e_\varnothing, e_a, f_a\},$$

because they will accord with the geometry and corresponding representation theory.

The vector-valued form

$$\hat{\eta} = \left(\hat{\theta}_\varnothing, \hat{\theta}_a,\ dx^a,\ dp_{ab}\right) \in \Omega^1(J^2, V)$$

gives a coframing of J^2. However, this coframing is not unique, as its definition depended on a choice of coordinates. In particular, it is not invariant under EDS equivalences, which are supposed to be our geometric 'isometries.' Still, this coframing has its uses. For example, the geometry of $\mathcal{C}$ is determined by the structure equations

$$d\hat{\theta}_\varnothing = -\hat{\theta}_a \wedge dx^a$$
$$d\hat{\theta}_a = -dp_{ab} \wedge dx^b,$$

along with the condition

$$\mathcal{C} = \{\hat{\theta}_\varnothing, \hat{\theta}_a\}.$$

Modelled on $\hat{\eta}$, we say that a coframing[2]

$$\underline{\eta} = (\underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\omega}^a, \underline{\pi}_{ab}) \in \Omega^1(J^2, V)$$

is $\mathcal{C}$-adapted if

$$\mathcal{C} = \{\underline{\theta}_\varnothing, \underline{\theta}_a\}$$

and the structure equations

$$d\underline{\theta}_\varnothing \equiv -\underline{\theta}_a \wedge \underline{\omega}^a \quad (\text{mod } \underline{\theta}_\varnothing)$$
$$d\underline{\theta}_a \equiv -\underline{\pi}_{ab} \wedge \underline{\omega}^b \quad (\text{mod } \underline{\theta}_\varnothing, \underline{\theta}_a)$$

hold. These are the properly normalized coframings that are compatible with the following contact invariant ideals and equations, which characterize $\mathcal{C}$ (see for example [Gar67]):

1. The first derived system $\mathcal{C}^{\langle 1 \rangle}$ is equal to $\{\underline{\theta}_\varnothing\}_{alg}$.

2. The degree one piece $\mathcal{C}^1$ is equal to $\{\underline{\theta}_\varnothing, \underline{\theta}_a\}_{alg}$.

3. $d\underline{\theta}_\varnothing \equiv 0 \quad (\text{mod } \underline{\theta}_\varnothing, \underline{\theta}_a)$.

4. The Cartan system of $\mathcal{C}^{\langle 1 \rangle}$ is equal to $\{\underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\omega}^a\}_{alg}$.

5. $d\underline{\theta}_a \equiv 0 \quad (\text{mod } \underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\omega}^a)$.

6. The Cartan system of $\mathcal{C}^1$ is all of $\Omega^*(J^2)$.

---

[2]Since the form $(\underline{\pi}_{ab})$ is valued in $\text{Sym}^2 W$, it will be convenient to consider $\underline{\pi}_{ab}$ and $\underline{\pi}_{ba}$ as the same ($\mathbb{R}$-valued) 1-form. I will do this without comment throughout.

Note in particular that the filtration

$$\{\underline{\theta}_\varnothing\}_{alg} \subset \{\underline{\theta}_\varnothing, \underline{\theta}_a\}_{alg} \subset \{\underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\omega}^a\}_{alg}$$

is a contact invariant of $\mathcal{C}$.

The $\mathcal{C}$-adapted coframings are the sections of a $G_{\mathcal{C}}$-structure, defined as follows. The group $G_{\mathcal{C}}$ is the subgroup of $\mathrm{GL}(V)$ that preserves the flag (1.2), the subspace

$$\{e_\varnothing + e_a \wedge f_a\} \subset V \oplus \Lambda^2 V/(\{e_\varnothing\} \wedge V)$$

and the subspace

$$\{e_a + e_{ab} \wedge f_b \colon a = 0, \ldots n\} \subset V \oplus \Lambda^2 V/(\{e_\varnothing, e_a\} \wedge V).$$

Explicitly, in the splitting

$$V = \mathbb{R} \oplus W \oplus W^\vee \oplus \mathrm{Sym}^2 W,$$

one finds that $G_{\mathcal{C}}$ consists of matrices of the form

$$\begin{pmatrix} k_\varnothing & 0 & 0 & 0 \\ * & B & 0 & 0 \\ * & {}^t B^{-1} S & k_\varnothing \, {}^t B^{-1} & 0 \\ * & * & BT & C_B/k_\varnothing \end{pmatrix}, \tag{1.3}$$

where

$$k_\varnothing \in \mathbb{R}^\times,$$

$$B \in \mathrm{GL}(W),$$

$$S \in \mathrm{Sym}^2 W \subset \mathrm{Hom}(W^\vee, W),$$

the linear map

$$T \colon W^\vee \to \mathrm{Sym}^2 W$$

satisfies the condition that

$$T \in \mathrm{Sym}^3 W \subset \mathrm{Sym}^2 W \otimes W,$$

and finally, the matrix C_B is induced by conjugate transpose action of B on $\mathrm{Sym}^2 W$, so that

$$C_B(e_{ij}) = B_i^k e_{kl} B_j^l.$$

The components labeled with a $*$ are unrestricted.

Define $\mathcal{B}_{\mathcal{C}}$ to be the $G_{\mathcal{C}}$-principal subbundle of $\mathcal{F}(J^2)$ generated by the section $\hat{\eta}$. By construction, a local coframing $\underline{\eta}$ over a neighborhoood $U \subset J^2$ is $\mathcal{C}$-adapted if and only if there is a function $g \colon U \to G_{\mathcal{C}}$ so that

$$\underline{\eta} = g \cdot \hat{\eta}.$$

This in turn holds if and only if the image of $\underline{\eta}$ lies in $\mathcal{B}_{\mathcal{C}}$.

I now recall the tautological 1-form of a G-structure, which can be used to calculate properties of adapted coframings in a uniform way.

Definition 7. On the coframe bundle $\pi\colon \mathcal{F}(M) \to M$ of an n-manifold M, the *tautological form* $\eta \in \Omega^1(\mathcal{F}(M), \mathbb{R}^n)$ is defined by
$$\eta_u(v) = (\pi^* u)(v)$$
for all $u \in \mathcal{F}(M)$ and $v \in T_u\mathcal{F}(M)$.

The tautological form is uniquely characterized by its *reproducing property*, the property that[3]

$$\underline{\eta}^* \eta = \underline{\eta}$$

for any section $\underline{\eta}$ of $\mathcal{F}(M)$. For this reason, the tautological form may be thought of as a 'universal' choice of coframing for M. I will typically denote the restriction of η to a G-structure by η as well. Such a restriction of η is the universal G-coframing.

The components of the tautological form can be used to define Ω^*_{sb}, the *semi-basic forms* on $\mathcal{B}$. Fix a basis of $\mathbb{R}^n$ and let η^i be the corresponding components of η in this basis. Then Ω^*_{sb} is the $\mathcal{C}^\infty(\mathcal{B})$-module generated by the η^i. This agrees with the standard definition of semi-basic forms for a fiber bundle.

The tautological form also provides a direct way to measure the first variational information of a G-structure. *Cartan's first structure equation* states that on a G-structure $\mathcal{B}$, there is a pseudo-connection
$$\varphi \in \Omega^1(\mathcal{B}, \mathfrak{g})$$
(for $\mathfrak{g}$ the Lie algebra of G) and a torsion map
$$T\colon \mathcal{B} \longrightarrow \mathrm{Hom}(\Lambda^2 \mathbb{R}^n, \mathbb{R}^n)$$
so that
$$d\eta = -\varphi \wedge \eta + T(\eta \wedge \eta).$$
Roughly, φ measures the variation of η in the fiber direction and T measures the first order twisting between fibers.

Example 5. On the $G_\mathcal{C}$-structure $\mathcal{B}_\mathcal{C}$, the tautological form takes values in V. Denote the components of η by
$$\theta_\varnothing = e^\varnothing \eta,$$
$$\theta_a = e^a \eta,$$
$$\omega^a = f^a \eta,$$
$$\pi_{ab} = e^{ab} \eta.$$

---

[3]The notation is admittedly awkward in this instance, but note that the underlined $\underline{\eta}$ denotes a specific coframing, a function from M to $\mathcal{F}(M)$, whereas η is a form on $\mathcal{F}(M)$.

(I will without comment adopt similar notation for an arbitrary coframing $\underline{\eta}$.) Then Cartan's structure equation takes the form

$$
d\begin{pmatrix} \theta_\varnothing \\ \theta_a \\ \omega^a \\ \pi_{ab} \end{pmatrix} = -\begin{pmatrix} \kappa_\varnothing & 0 & 0 & 0 \\ * & \beta & 0 & 0 \\ * & \sigma & \kappa_0 - {}^t\beta & 0 \\ * & * & \tau & C_\beta - \kappa_0 \end{pmatrix} \wedge \begin{pmatrix} \theta_\varnothing \\ \theta_a \\ \omega^a \\ \pi_{ab} \end{pmatrix} + \begin{pmatrix} T_{\theta_\varnothing} \\ T_{\theta_a} \\ T_{\omega^a} \\ T_{\pi_{ab}} \end{pmatrix}. \tag{1.4}
$$

Because the psuedo-connection is valued in $\mathfrak{g}_{\mathcal{C}}$, the 1-form β takes values in $\mathfrak{gl}(W)$, while σ takes values in $\mathrm{Sym}^2 W$. By differentiating the action of C_B, one finds that

$$
C_\beta \wedge (\pi_{ab}) = \beta \wedge (\pi_{ab}) - (\pi_{ab}) \wedge {}^t\beta.
$$

The negative sign is counter-intuitive, but it is because the π_{ab} are 1-forms. The sign makes $C_{\beta \wedge}(\pi_{ab})$ into a symmetric matrix valued 2-form.

The general strategy of the method of equivalence is to normalize the torsion terms as much as possible, through absorption into the pseudo-connection, as well as through reductions of the structure group. As this proceeds, the remaining torsion provides the first order 'geometric curvatures,' the local invariants of the geometry. Typically, invariants of a G-structure $\mathcal{B}$ are *relative invariants*, which are G-equivariant functions on $\mathcal{B}$ that determine the torsion. A relative invariant that is constant on each fiber is an *absolute invariant*. In this case, the invariant defines a function on M.

In the next section I will restrict to the submanifold of J^2 cut out by a second order equation, and the resulting invariants will include the geometric symbol, as well as some deeper invariants.

The geometric (principal) symbol of second order PDE

Let F be a single 2nd order differential equation F for functions of $n+1$ variables, as in example 2, and $(M_0, \mathcal{I}_0)$ the corresponding exterior differential system. I now begin to describe the geometry of M_0.

The strategy is to restrict the bundle of $\mathcal{C}$-adapted coframings to M_0 and then focus on coframes in $\mathcal{B}_{\mathcal{C}}|_{M_0}$ that are compatible with M_0. This subbundle will determine a G_0-structure on M_0, which is almost the central object of study in this thesis, and will be once I express it independently of an embedding into J.

Consider a $\mathcal{C}$-coframe $u \in \mathcal{B}_{\mathcal{C}}|_{M_0}$, based at x. There is one relation, $dF_x = 0$, on the restriction of u to $T_x M_0$. To treat this relation uniformly, I pull it up to the coframe bundle, where there is a $\mathrm{Sym}^2 W$-valued function (S_{ab}) on $\mathcal{B}_{\mathcal{C}}|_{M_0}$ so that (omitting the pullback notation)

$$
dF \equiv S_{ab}\pi_{ab} \quad (\text{mod } \theta_\varnothing, \theta_a, \omega^a). \tag{1.5}
$$

Because dF does not vary within each fiber, the function (S_{ab}) is $G_{\mathcal{C}}$-equivariant,

$$
(S_{ab})(g \cdot u) = C_{B^{-1}}(S_{ab})(u)
$$

for any $u \in \mathcal{B}_{\mathcal{C}}$ and g as in (1.3). The function (S_{ab}) is a relative invariant—the *unadapted symbol* of F.

Recall that the signature of any matrix in $\mathrm{Sym}^2 W$ determines its orbit under the conjugate transpose action. Consequently, there is a well defined function on M_0 whose value at a point x is given by the signature of (S_{ab}) at any point of the fiber over x. This function is an absolute invariant of M_0—the *adapted symbol* of F

Now suppose that the signature function takes the constant value (p, q, r) on a neighborhood U in M_0, and let G_s be the subgroup of $G_{\mathcal{C}}$ that stabilizes the signature matrix of (p, q, r). The set of coframes that reduce (S_{ab}) to its signature matrix is a G_s-subbundle of $\mathcal{B}_{\mathcal{C}}|_{M_0}$. It is not quite a G_s-structure on M_0, because it is not a subbundle of $\mathcal{F}(M_0)$. Observe though, that restricted to this subbundle, equation (1.5) simplifies to

$$dF \equiv \sum_{a=1}^{p} \pi_{aa} - \sum_{a=p+1}^{p+q} \pi_{aa} \quad (\mathrm{mod}\ \theta_\varnothing, \theta_a, \omega^a)\,, \tag{1.6}$$

so this coframe adaptation certainly simplifies equations.

It is straightforward to see that the symbol is determined by F: using the coframing $\hat{\eta}$,

$$dF \equiv \frac{\partial F}{\partial p_{ab}}\, dp_{ab} \quad \left(\mathrm{mod}\ \hat{\theta}_\varnothing, \hat{\theta}_a,\ dx^a\right),$$

so $(S_{ab}) = \left(\frac{\partial F}{\partial p_{ab}}\right)$ at one point of each fiber.

For a solution u to F, the classical principal symbol is determined by the linearization (i.e. Frechet derivative) of F around u. Because F is second order, this only depends on the 2-jet of u. If $x \in M_0$ is the 2-jet corresponding to u, then the unadapted symbol

$$\left(\frac{\partial F}{\partial p_{ab}}\right)_x$$

agrees with the classical symbol at u.

At this point the equivalence problem splits into cases, depending on the symbol. From the perspective of their conservation laws, the elliptic and hyperolic cases are well studied. Bryant, Griffiths, and Hsu explored the hyperbolic case in [BGH95a] and [BGH95b]. Bryant and Griffiths explored the parabolic case in dimension $1 + 1$ in [BG95b]. Clelland extended the parabolic case to $2 + 1$ dimensions in her thesis, [NC97]. In this paper I study arbitrary second order PDE's with parabolic symbol. The parabolic case is particularly amenable, because conservation laws tend to be functions in few derivatives of solutions.

1.3 Parabolic Systems

Given a parabolic equation in $n + 1$ variables, the corresponding system $(M_0, \mathcal{I}_0)$ has geometric symbol $(n, 0, 1)$ everywhere. In this case, equation (1.6) is

$$dF \equiv \sum_{i=1}^{n} \pi_{ii} \quad (\mathrm{mod}\ \theta_\varnothing, \theta_a, \omega^a)\,. \tag{1.7}$$

I will establish coframes for M_0 that are adapted to this symbol relation.

Let K be the kernel of the map

$$
\begin{array}{ccc}
\mathrm{Sym}^2 W & \longrightarrow & \mathbb{R} \\
x_{ab} & \longmapsto & \sum_{i=1}^{n} x_{ii}
\end{array}
$$

and define

$$
V_0 = \mathbb{R} \oplus W \oplus W^{\vee} \oplus K
$$

as a subset of V.

A coframe u_0 of M_0 at a point x is 0-*adapted* if there is a $\mathcal{C}$-coframe $u \in (\mathcal{B}_\mathcal{C})_x$ so that the diagram commutes.

$$
\begin{array}{ccc}
T_x J^2 & \xrightarrow{\;\; u \;\;} & V \\
\Big\uparrow & {\scriptstyle \cong} & \Big\uparrow \\
T_x M_0 & \xrightarrow{\;\; u_0 \;\;} & V_0
\end{array}
$$

The set of 0-adapted coframes over M_0 forms a G_0-structure, which I denote by $\mathcal{B}_0$.

To better understand the structure group G_0, consider two 0-adapted coframes u_0 and $\widetilde{u}_0$ over x, and corresponding $\mathcal{C}$-coframes u and $\widetilde{u}$. The latter differ by an element g of $G_\mathcal{C}$ as in (1.3). On the other hand, g restricts to preserve V_0, or equivalently, to preserve the vector space generated by the symbol relation,

$$
V_0^{\perp} \cong \left\{ \sum_{i=1}^{n} e^i \circ e^i \right\} \subset V^{\vee}.
$$

It is not difficult to see that C_B fixes K if and only if B is an element of

$$
\begin{pmatrix} CO(n) & 0 \\ {}^t(\mathbb{R}^n) & \mathbb{R}^{\times} \end{pmatrix} \subset \mathrm{GL}(W).
$$

Furthermore, the linear map T now must have image in K, so may naturally be identified as an element of

$$
K^{(1)} := (W \otimes K) \cap \mathrm{Sym}^3 W.
$$

In the language of [BCG$^+$13], $K^{(1)}$ is the first prolongation of the tableaux K associated to the parabolic symbol.

These calculations demonstrate that G_0 consists of the matrices of the form

$$
\begin{pmatrix}
k_{\varnothing} & 0 & 0 & 0 \\
\vec{k} & B & 0 & 0 \\
* & {}^t B^{-1} S & k_{\varnothing} \, {}^t B^{-1} & 0 \\
* & C & BT & C_B / k_{\varnothing}
\end{pmatrix}
\tag{1.8}
$$

with B and T as just described. The components $\vec{k}$ and C are still unconstrained, but, for notation which will be needed, let

$$
\vec{k} = (k_i) \in W, \quad C = (C_{bc}^a) \in \mathrm{Hom}(W, K),
$$

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as well as

$$S = \begin{pmatrix} S^{ij} & S^{0j} \\ S^{0j} & S^{00} \end{pmatrix} \in \mathrm{Sym}^2 W.$$

and

$$B = \begin{pmatrix} B' & 0 \\ B_0^i & B_0^0 \end{pmatrix}$$

such that

$$B' = B_{\mathrm{tr}}(B_i^j), \quad (B_j^i) \in SO(n), \quad B_{\mathrm{tr}} \in \mathbb{R}^\times.$$

It is from this point on that it will make sense to apply the 'partial' Einstein summation convention. For example, the *partial trace operator* can now be written as

$$e^i \circ e^i = \sum_{i=1}^n e^i \circ e^i.$$

This operator is G_0-invariant by construction.

Another consequence of restricting to G_0 is a natural refinement of the flag (1.2). In the basis (1.1) of V, the flag

$$\begin{aligned} \{e_\varnothing\} &\subset \{e_\varnothing, e_i\} \subset \{e_\varnothing, e_a\} \subset \{e_\varnothing, e_a, f_0\} \subset \{e_\varnothing, e_a, f_a\} \\ &\subset \{e_\varnothing, e_a, f_a, e_{ij}\} \cap V_0 \subset \{e_\varnothing, e_a, f_a, e_{aj}\} \cap V_0 \end{aligned} \tag{1.9}$$

of V_0 is preserved by the action of G_0. In particular, the subspace

$$W' = \{e_\varnothing, e_i\}/\{e_\varnothing\}$$

of W is well defined. Note that the component K of V_0 is isomorphic to

$$\mathrm{Sym}_0^2 W' \oplus W' \oplus \mathbb{R}, \tag{1.10}$$

which is naturally identified with harmonic polynomials in n variables of degree two or less. Unfortunately, there is not a good choice of basis for K, because any choice would privilege a direction that has no geometric significance. On the other hand, we have the isomorphism

$$V_0^\vee \cong V/\{e^i \circ e^i\},$$

and the dual basis to (1.1) provides a natural spanning set for $V_0^\vee$. The difference will be more pronounced after Section 3.1, when the prolongations of K will become central. They will be naturally identified with higher degree harmonic polynomials, which have even less suitable bases. For this reason, I will often refer to 'coframings' which are more properly *extended coframings*, spanning sets of 1-forms.

Example 6. Consider the heat equation

$$p_0 = p_{ii}$$

and the corresponding exterior differential system M_0, given by the set $\{p_0 = p_{ii}\}$. The $\mathcal{C}$-coframing

$$
\begin{aligned}
\underline{\theta}_\varnothing &= du - p_a \underline{\omega}^a & \underline{\pi}_{ij} &= dp_{ij} - \frac{1}{n}\delta_{ij}(dp_{ii} - p_{0b}\underline{\omega}^b) \\
\underline{\theta}_a &= dp_a - p_{ab}\underline{\omega}^b & \underline{\pi}_{0a} &= dp_{0a} \\
\underline{\omega}^a &= dx^a
\end{aligned}
$$

restricts to a 0-adapted extended coframing of M_0. Observe that, when restricted to M_0,

$$
dp_{ii} - p_{0b}\underline{\omega}^b = \underline{\theta}_0.
$$

To summarize, if $(M_0, \mathcal{I}_0)$ has everywhere parabolic symbol, then its embedding into $J^2(\mathbb{R}^{n+1}, \mathbb{R})$ induces a G_0-structure on M_0. These are the motivating examples for the following definition.

Definition 8. A *weakly parabolic system* in $n+1$ variables is a $2n + 2 + (n+1)(n+2)/2$ dimensional[4] exterior differential system $(M_0, \mathcal{I}_0)$ such that any point has a neighborhood equipped with a spanning set of 1-forms

$$
\underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\omega}^a, \underline{\pi}_{ab} \tag{1.11}
$$

that satisfy:

1. The symbol relations $\underline{\pi}_{ab} = \underline{\pi}_{ba}$ and

$$
\underline{\pi}_{ii} \equiv 0 \pmod{\underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\omega}^a}
$$

2. The forms $\underline{\theta}_\varnothing, \underline{\theta}_a$ generate $\mathcal{I}_0$ as a differential ideal.

3. The structure equations

$$
\begin{aligned}
d\underline{\theta}_\varnothing &\equiv -\underline{\theta}_a \wedge \underline{\omega}^a \pmod{\underline{\theta}_\varnothing} \\
d\underline{\theta}_a &\equiv -\underline{\pi}_{ab} \wedge \underline{\omega}^b \pmod{\underline{\theta}_\varnothing, \underline{\theta}_b}.
\end{aligned}
$$

4. The non-degeneracy condition

$$
d(\underline{\theta}_i \wedge \underline{\omega}_{(i)}) \not\equiv 0 \pmod{\underline{\theta}_\varnothing, \underline{\theta}_i}.
$$

(I draw the reader's attention to the fact that this is *not* modulo θ_0! I also note that the notation $\underline{\omega}_{(i)}$ is explained in the introduction)

Any such (extended) coframing of $(M_0, \mathcal{I}_0)$ is called 0-*adapted*. The 0-adapted coframings are sections of a G_0-structure $\mathcal{B}_0$ on M_0.

---

[4]This is 1 less than the dimension of $J^2(\mathbb{R}^{n+1}, \mathbb{R})$.

The non-degeneracy condition is necessary to exclude equations which should not be called parabolic. For example, the 2-dimensional Laplace equation with 1 free parameter,

$$\left(\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} \right) u(x_0, x_1, x_2) = 0, \tag{1.12}$$

behaves like an elliptic equation. However, by modifying the example of the heat equation above, one can construct coframings that satisfy conditions (1) through (3). These coframings do not satisfy condition (4). Once defined, the Monge-Ampère and Goursat invariants (Section 2.1 and 2.3 respectively) will provide another test of non-degeneracy.

It follows from the structure equations that any parabolic system has an EDS embedding into $(J^2(\mathbb{R}^{n+1}, \mathbb{R}), \mathcal{C})$. For this reason, parabolic systems are locally equivalent to parabolic equations. It is still worth talking about abstract parabolic systems, for all of the same reasons that abstract manifolds are useful as compared to embedded ones.

The filtration (1.9) corresponds to the following filtration of ideals, adapted to the geometry of the parabolic system $(M_0, \mathcal{I}_0)$:

$$\begin{aligned} \{\underline{\theta}_\varnothing\} &\subset \{\underline{\theta}_\varnothing, \underline{\theta}_i\} \subset \{\underline{\theta}_\varnothing, \underline{\theta}_a\} \subset \{\underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\omega}^0\} \subset \{\underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\omega}^a\} \\ &\subset \{\underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\omega}^a, \underline{\pi}_{ij}\} \subset \{\underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\omega}^a, \underline{\pi}_{aj}\}. \end{aligned} \tag{1.13}$$

The structure of this filtration will be critical to the analysis throughout. The algebraic ideal

$$\underline{\mathcal{J}} = \{\underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\omega}^a\}_{alg}$$

will be especially important. Note that $\underline{\mathcal{J}}$ defines a natural independence condition on M_0. Observe also that the ideal $\underline{\mathcal{J}}$ is the Cartan system of $\theta_\varnothing$. In particular, $\underline{\mathcal{J}}$ is Frobenius.

I emphasize that the ideals in (1.13) are contact invariant, in that they don't depend on the specific coframing used to define them. It will be convenient to abuse notation and describe invariant ideals by the corresponding ideals on $\mathcal{B}_0$. More generally, any object on $\mathcal{B}_0$ that is invariant under the structure group defines an object on M_0, which I will often call by the same name. This will not cause problems, because any adapted coframing pulls back an invariant object on $\mathcal{B}_0$ to a unique object on M_0. For example, the ideal

$$\mathcal{J} = \{\theta_\varnothing, \theta_a, \omega^a\}_{alg}$$

pulls down by any 0-adapted coframing to $\underline{\mathcal{J}}$. See 3 for another example.

It will be useful to employ the vector notation

$$\theta_\varnothing, \quad \Theta = \begin{pmatrix} \theta_i \\ \theta_0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} \omega^i \\ \omega^0 \end{pmatrix}, \quad \Pi = \begin{pmatrix} \pi_{ij} & \pi_{i0} \\ \pi_{i0} & \pi_{00} \end{pmatrix}$$

for the components of the tautological form on $\mathcal{B}_0$. With analogous notation for a 0-adapted coframing, the structure equations can be written more concisely as

$$\begin{aligned} d\underline{\theta}_\varnothing &\equiv -{}^t\Theta \wedge \Omega \quad (\mathrm{mod}\ \underline{\theta}_\varnothing) \\ d\Theta &\equiv -\Pi \wedge \Omega \quad (\mathrm{mod}\ \underline{\theta}_\varnothing, \theta_a). \end{aligned} \tag{1.14}$$

This form of the structure equations is useful for understanding G_0.

Per Cartan's first structure equation, there is a $\mathfrak{g}_0$-valued psuedo-connection and torsion 2-forms defined on $\mathcal{B}_0$ so that

$$
d\begin{pmatrix} \theta_\varnothing \\ \Theta \\ \Omega \\ \Pi \end{pmatrix} = -\begin{pmatrix} \kappa_\varnothing & 0 & 0 & 0 \\ \vec{\kappa} & \beta & 0 & 0 \\ * & \sigma & \kappa_\varnothing - {}^t\beta & 0 \\ * & \gamma & \tau & C_\beta - \kappa_\varnothing \end{pmatrix} \wedge \begin{pmatrix} \theta_\varnothing \\ \Theta \\ \Omega \\ \Pi \end{pmatrix} + \begin{pmatrix} T_{\theta_\varnothing} \\ T_\Theta \\ T_\Omega \\ T_\Pi \end{pmatrix}. \tag{1.15}
$$

The torsion terms $T_{\theta_\varnothing}, T_\Theta, T_\Omega$ and T_Π are, as usual, semi-basic. The differences from (1.4) are that β has components

$$
\beta = \begin{pmatrix} \beta_i^j & 0 \\ \beta_0^i & \beta_0^0 \end{pmatrix} \in \Omega^1 \left(\mathcal{B}_0, \begin{pmatrix} \mathfrak{co}(n) & 0 \\ {}^t(\mathbb{R}^n) & \mathbb{R} \end{pmatrix} \right)
$$

and

$$
\tau \in \Omega^1 \left(\mathcal{B}_0, K^{(1)} \right).
$$

Note that

$$
\beta_j^i = -\beta_i^j
$$

for $i \neq j$, and define

$$
\beta_{\mathrm{tr}} = \frac{1}{n} \beta_i^i.
$$

I denote the components of σ by

$$
\sigma = \begin{pmatrix} \sigma^{ij} & \sigma^{0j} \\ \sigma^{0j} & \sigma^{00} \end{pmatrix},
$$

the components of $\vec{\kappa}$ by

$$
\vec{\kappa} = (\kappa_i),
$$

and the components of γ by

$$
\gamma = (\gamma_{bc}^a).
$$

The reproducing property of the tautological form immediately determines some of the torsion forms. Because (1.14) holds for any 0-adapted coframing $\underline{\eta}$,

$$
\underline{\eta}^*(T_{\theta_\varnothing}) \equiv \underline{\eta}^* \, d\theta_\varnothing = d\underline{\theta}_\varnothing \equiv -{}^t\underline{\Theta} \wedge \underline{\Omega} \quad (\mathrm{mod} \ \underline{\theta}_\varnothing),
$$

and thus

$$
T_{\theta_\varnothing} = -{}^t\Theta \wedge \Omega + \xi_\varnothing \wedge \theta_\varnothing
$$

for a semi-basic 1-form $\xi_\varnothing$. Adding $\xi_\varnothing$ to κ will not affect Cartan's structure equation, but will absorb the torsion. Doing so simplifies the first component of (1.15) to

$$
d\theta_\varnothing = -\kappa_\varnothing \wedge \theta_\varnothing - {}^t\Theta \wedge \Omega.
$$

It is clear that no other modification of $\kappa_\varnothing$ can be made to absorb the remaining torsion, and that $\kappa_\varnothing$ is uniquely defined up to a multiple of $\theta_\varnothing$.

An analogous calculation shows that there is a matrix of semi-basic 1-forms

$$\xi = \left(\begin{array}{cc} \xi_i^j & \xi_i^0 \\ \xi_0^j & \xi_0^0 \end{array} \right)$$

so that

$$d\Theta = -\vec{\kappa} \wedge \theta_\varnothing - \beta \wedge \Theta - \Pi \wedge \Omega - \xi \wedge \Theta.$$

From this equation it is clear that semi-basic forms may be added to β in such a way that the torsion ξ reduces to

$$\xi = \left(\begin{array}{cc} \xi_i^j & \xi_i^0 \\ 0 & 0 \end{array} \right),$$

where furthermore

$$\xi_j^i = \xi_i^j \text{ and } \xi_i^i = 0.$$

The torsion forms ξ also control the behavior of the ω^a, which is seen as follows. From the structure equations,

$$\begin{aligned} 0 = d^2\theta_\varnothing &\equiv d(-\kappa_\varnothing \wedge \theta_\varnothing - {}^t\Theta \wedge \Omega) \\ &\equiv {}^t\Theta \wedge (-{}^t\xi \wedge \Omega + T_\Omega) \quad (\text{mod } \theta_\varnothing). \end{aligned}$$

An application of the generalized Cartan's lemma shows that

$$T_\Omega \equiv H \wedge \Theta + {}^t\xi \wedge \Omega \quad (\text{mod } \theta_\varnothing),$$

where H is a semi-basic, $\text{Sym}^2 W$-valued 1-form. Thus

$$d\Omega \equiv -(\sigma - H) \wedge \Theta - (\kappa_\varnothing - {}^t\beta) \wedge \Omega + {}^t\xi \wedge \Omega \quad (\text{mod } \theta_\varnothing).$$

By modifying σ accordingly, all of the torsion H may be absorbed.

Finally, I remark without proof that ξ also determines some of the torsion of Π. For example, by considering $d^2\theta_i = 0$, one finds that

$$T_{\pi_{ij}} \equiv -\xi_i^a \wedge \pi_{aj} + \pi_{ia} \wedge \xi_j^a \quad (\text{mod } \mathcal{J}).$$

To summarize the structure equations so far,

$$\begin{aligned} d\theta_\varnothing &= -\kappa_\varnothing \wedge \theta_\varnothing - \theta_a \wedge \omega^a, \\ d\theta_i &= -\kappa_i \wedge \theta_\varnothing - \beta_i^j \wedge \theta_j - \pi_{ia} \wedge \omega^a - \xi_i^0 \wedge \theta_0 - \xi_i^j \wedge \theta_j, \\ d\theta_0 &\equiv -\beta_0^a \wedge \theta_a - \pi_{0a} \wedge \omega^a && (\text{mod } \theta_\varnothing), \\ d\omega^0 &\equiv -\sigma^{0a} \wedge \theta_a - (\kappa_\varnothing - \beta_a^0) \wedge \omega^a + \xi_i^0 \wedge \omega^i && (\text{mod } \theta_\varnothing) \\ d\omega^i &\equiv -\sigma^{ia} \wedge \theta_a - (\delta_j^i \kappa_\varnothing - {}^t\beta_j^i) \wedge \omega^j + \beta_0^i \wedge \omega^0 + \xi_j^i \wedge \omega^j && (\text{mod } \theta_\varnothing) \\ d\pi_{ab} &\equiv -\gamma_{ab}^c \wedge \theta_c - \tau_{abc} \wedge \omega^c + \kappa_\varnothing \wedge \pi_{ab} - \beta_a^c \wedge \pi_{cb} + \pi_{ac} \wedge {}^t\beta_b^c + T_{\pi_{ab}} && (\text{mod } \theta_\varnothing) \end{aligned}$$

Chapter 2

Invariants of parabolic systems

After adapting coframes to the symbol of a parabolic system $(M_0, \mathcal{I}_0)$, the next level of (relative) invariants fall into 2 families: the Monge-Ampère invariants and the Goursat invariants. Within both families there is a further division into primary and secondary invariants. The primary invariants arise from the torsion forms ξ_i^0, while the secondary invariants arise from the forms ξ_i^j. I describe these invariants in this section.

2.1 The Monge-Ampère Invariants

Primary Monge-Ampère Invariants

Because the forms ξ_i^0 are semi-basic, there are functions U_i^{jk}, U_i^j, U_i on $\mathcal{B}_0$ so that

$$\xi_i \equiv U_i^{jk}\pi_{jk} + U_i^j\pi_{j0} + U_i\pi_{00} \pmod{\theta_\varnothing, \theta_a, \omega^a}.$$

Roughly, these functions comprise the primary Monge-Ampère invariants. Instead of treating them all at once, I will filter ξ into simpler pieces using the flag (1.13). In particular, I will show that the functions U_i^{**} split into three levels, and that each level defines relative invariants if the previous levels vanish identically.

To see when the U_i^{**} are relative invariants, I must determine how they vary in each fiber. The method of equivalence often employs a standard trick to determine how such torsion functions vary: consider the exterior derivative of Cartan's structure equation. In particular, $d^2\theta_i = 0$ determines all of the variation of U_i^{**}.

In indices,

$$d\theta_i \equiv -\beta_i^j \wedge \theta_j - \xi_i^j \wedge \theta_j - \xi_i^0 \wedge \theta_0 - \pi_{ij} \wedge \omega^j - \pi_{i0} \wedge \omega^0 \pmod{\theta_\varnothing}.$$

Taking the exterior derivative,

$$0 \equiv \left(\beta_i^j \wedge \xi_j^0 + d(\xi_i^0) + \xi_i \wedge \beta_0^0 + \pi_{ij} \wedge \sigma^{j0} + \pi_{i0} \wedge \sigma^{00}\right) \wedge \theta_0 \pmod{\theta_\varnothing, \theta_i, \omega^a, \Omega_{sb}^3},$$

which in turn proves that

$$0 \equiv d(\xi_i^0) + \beta_i^j \wedge \xi_j^0 + \pi_{ij} \wedge \sigma^{j0} + \pi_{i0} \wedge \sigma^{00} + \xi_i \wedge \beta_0^0 \quad \left(\text{mod } \mathcal{J}, \Omega_{sb}^2\right). \qquad (2.1)$$

This is the equation which I filter using (1.13).

At the 'highest weight,' after plugging in for ξ_i^0, equation (2.1) simplifies to

$$0 \equiv \left(\, dU_i + \beta_i^j U_j - 3\beta_0^0 U_i + \kappa_\varnothing U_i\right) \wedge \pi_{00} \quad \left(\text{mod } \mathcal{J}, \pi_{ij}, \pi_{i0}, \Omega_{sb}^2\right).$$

By an application of Cartan's lemma,

$$dU_i \equiv -\beta_i^j U_j + (3\beta_0^0 - \kappa_\varnothing)U_i \quad \left(\text{mod } \Omega_{sb}^1\right).$$

Integrating this, one finds that the vector-valued function (U_i) on $\mathcal{B}_0$ is G_0-equivariant[1]. Indeed, for a 0-coframe u and a matrix g as in (1.8),

$$(U_i(g \cdot u)) = \frac{(B_0^0)^3}{k_\varnothing}(B')^{-1}(U_i(u)).$$

In other words, (U_i) is a relative invariant. It is the highest weight primary Monge-Ampère invariant.

If the function (U_i) is non-zero, then the function (U_i^j) will not define a relative invariant. However, suppose that (U_i) vanishes identically on $\mathcal{B}_0$. Then (2.1) reduces to

$$0 \equiv (d(U_i^j) + \beta_i^k U_k^j - U_i^k \beta_k^j + U_i^j(\kappa_\varnothing - 2\beta_0^0) - \delta_i^j \sigma^{00}) \wedge \pi_{j0} \quad \left(\text{mod } \mathcal{J}, \pi_{ij}, \Omega_{sb}^2\right).$$

This implies that

$$d(U_i^j) \equiv -\beta_i^k U_k^j + U_i^k \beta_k^j - U_i^j(\kappa_\varnothing - 2\beta_0^0) + \delta_i^j \sigma^{00} \quad \left(\text{mod } \Omega_{sb}^1\right)$$

for all i and j. Integrating, the function (U_i^j) varies by the rule

$$\left(U_i^j(g \cdot u)\right) = \frac{(B_0^0)^2}{k_\varnothing}(B')^{-1}\left(U_i^j(u)\right)B' + S^{00}\delta_i^j.$$

Clearly (U_i^j) is not G_0-equivarant due to the last term. However, this term also means that that there are choices of coframe for which U_i^j is traceless. The subbundle of such coframes has structure group consisting of matrices as in (1.8) for which $S^{00} = 0$. When restricted to this reduced G-structure, the remaining component of (U_i^j) *is* a relative invariant, the next level of the primary Monge-Ampère invariant. Note that the pseudo-connection form σ^{00} becomes semi-basic when restricted to $\mathcal{B}_0$.

---

[1]To be precise, this argument only works for the identity component of G_0. However, one can check the variation of (U_i) for one element in each component of G_0 to see that it really is a relative invariant.

Finally, suppose that (U_i) and (U_i^j) vanish identically and the frame reduction has been carried out. Then (2.1) simplifies to

$$0 \equiv \left(dU_i^{jk} + \beta_i^l U_l^{jk} - \beta_l^j U_i^{lk} + \beta_k^l U_i^{jl} + (\kappa_\varnothing - \beta_0^0)U_i^{jk} - \frac{1}{2}\delta_j^i \sigma^{0k} - \frac{1}{2}\delta_k^i \sigma^{0j} \right) \wedge \pi_{jk}$$

$$\left(\text{mod } \mathcal{J}, \Omega_{sb}^2 \right),$$

so that

$$dU_i^{jk} \equiv -\beta_i^l U_l^{jk} + \beta_l^j U_i^{lk} + U_i^{jl}\beta_l^k + (\beta_0^0 - \kappa_0)U_i^{jk} + \frac{1}{2}\delta_j^i \sigma^{0k} + \frac{1}{2}\delta_i^k \sigma^{0j} \quad \left(\text{mod } \Omega_{sb}^1 \right).$$

Integrating, (U_i^{jk}) transforms as

$$\left(U_i^{jk}(g \cdot u) \right) = \frac{B_0^0}{k_\varnothing} B' \cdot \left(U_i^{jk}(u) \right) - \frac{1}{2}\delta_i^j S^{0k} - \frac{1}{2}\delta_i^k S^{0j}$$

in each fiber, where B' acts by the tensor product representation on $(W')^\vee \otimes \mathrm{Sym}_0^2 W'$. A coframe adaptation may be made to absorb the trace components of this representation, so that

$$U_i^{ik} = 0$$

for each k. Such coframes are called *1-adapted*. After this coframe adaptation, the remaining components of (U_i^{jk}) are relative invariants, the lowest weight piece of the primary Monge-Ampère invariant.

The subbundle of 1-adapted coframes $\mathcal{B}_1$ has structure group G_1, which consists of matrices as in (1.8) so that

$$S^{00} = S^{0i} = 0.$$

The torsion on $\mathcal{B}_1$ is similar to the torsion of $\mathcal{B}_0$, except that the psuedo-connection forms σ^{0a} are semi-basic when restricted to $\mathcal{B}_1$. They now contribute non-absorbable torsion terms.

The most important case for this paper is when all of the primary Monge-Ampère invariants vanish. Note this holds if and only if there are coframings so that

$$\xi_i^0 \equiv 0 \quad (\text{mod } \mathcal{J}).$$

This second condition can be used to quickly show that the exterior derivative of $d\theta_i$ reduces to

$$0 \equiv (\sigma^{0j} \wedge \pi_{ij} + \sigma^{00} \wedge \pi_{i0}) \wedge \theta_0 \quad (\text{mod } \theta_\varnothing, \theta_i, \omega^a).$$

When $n \geq 3$, multiple applications of Cartan's lemma show that

$$\sigma^{0j} \equiv \sigma^{00} \equiv 0 \quad (\text{mod } \mathcal{J}).$$

When $n = 2$, there are functions h_1 and h_2 on $\mathcal{B}_1$ so that

$$\begin{pmatrix} \sigma^{01} \\ \sigma^{02} \end{pmatrix} \equiv \begin{pmatrix} h_1 & h_2 \\ h_2 & -h_1 \end{pmatrix} \begin{pmatrix} \pi_{11} \\ \pi_{12} \end{pmatrix} \quad (\text{mod } \mathcal{J}).$$

Secondary Monge-Ampère Invariants

There are functions $V_i^j, V_i^{jk}, V_i^{jkl}$ on $\mathcal{B}_1$ so that

$$\xi_i^j \equiv V_i^j \pi_{00} + V_i^{jk} \pi_{k0} + V_i^{jkl} \pi_{kl} \quad (\mathrm{mod}\ \theta_\varnothing, \theta_a, \omega^a).$$

If the primary Monge-Ampère invariants vanish identically, the non-absorbable components of these functions are the *secondary Monge-Ampère invariants*. More precisely, if the primary Monge-Ampère invariants vanish, then the functions V_i^{***} filter into three levels, each of which defines relative invariants when the previous ones vanish.

The assumption that the primary Monge-Ampère invariants vanish means that

$$d\theta_i \equiv -\xi_i^j \wedge \theta_j - \beta_i^j \wedge \theta_j - \pi_{ij} \wedge \omega^j - \pi_{i0} \wedge \omega^0 \quad (\mathrm{mod}\ \theta_\varnothing, \Lambda^2 \mathcal{J}),$$

and thus

$$0 \equiv \left(d(\xi_i^j) + \xi_i^k \wedge \beta_k^j + \beta_i^k \wedge \xi_k^j + \pi_{ik} \wedge \sigma^{kj} + d\beta_i^j + \beta_i^k \wedge \beta_k^j \right) \wedge \theta_j \quad (\mathrm{mod}\ \theta_\varnothing, \omega^a, \Lambda^2 \mathcal{J}, \Omega_{sb}^3)$$

By an application of Cartan's lemma,

$$0 \equiv d(\xi_i^j) + \xi_i^k \wedge \beta_k^j + \beta_i^k \wedge \xi_k^j + \pi_{ik} \wedge \sigma^{kj} + d\beta_i^j + \beta_i^k \wedge \beta_k^j \quad (\mathrm{mod}\ \mathcal{J}, \Omega_{sb}^2) \tag{2.2}$$

One would have to prolong the G_1-structure to properly understand the $d\beta' + \beta' \wedge \beta'$ term, but that won't be necessary here. Instead, consider just the component of (2.2) that is symmetric and traceless in i and j,

$$0 \equiv d(\xi_i^j) + \xi_i^k \wedge \beta_k^j + \beta_i^k \wedge \xi_k^j + \pi_{ik} \wedge \sigma^{kj} \quad (\mathrm{mod}\ \mathcal{J}, \Omega_{sb}^2).$$

This equation determines the variation of the secondary Monge-Ampère invariants in each fiber.

Because the argument is very similar to the one used for the primary Monge-Ampère invariants, I simply state the result: Let u be a coframe in $\mathcal{B}_1$ and $g \in G_1$. The highest weight secondary Monge-Ampère invariants vary by the rule

$$\left(V_i^j (g \cdot u) \right) = \frac{(B_0^0)^2}{k_\varnothing} (B')^{-1} \left(V_i^j(u) \right) B'.$$

If (V_i^j) vanishes identically, then at the next level

$$\left(V_i^{jk} (g \cdot u) \right) = \frac{B_0^0}{k_\varnothing} B' \cdot \left(V_i^{jk}(u) \right).$$

Here B' acts by the tensor product representation on $(W')^\vee \otimes W' \otimes W'$. Finally, if (V_i^{jk}) vanishes, then

$$\left(V_i^{jkl} (g \cdot u) \right) = \frac{1}{k_\varnothing} B' \cdot \left(V_i^{jkl}(u) \right) + \frac{1}{2} \delta_i^k S_{lj} + \frac{1}{2} \delta_i^l S_{kj}.$$

Here B' now acts by the tensor product representation on $(W')^\vee \otimes W' \otimes \mathrm{Sym}_0^2 W'$.

There is a reduction of coframes so that

$$V_i^{jik} = 0$$

for all j and k. This results in a G_{MA}-structure $\mathcal{B}_{MA}$, where G_{MA} is the subgroup of matrices as in (1.8) for which

$$S = 0.$$

The pseudo-connection forms σ^{ij} are semi-basic when restricted to $\mathcal{B}_{MA}$.

Now consider a parabolic system M_0, all of whose Monge-Ampère invariants vanish. When restricted to $\mathcal{B}_{MA}$, the torsion form ξ satisfies

$$\xi_i \equiv \xi_i^j \equiv 0 \pmod{\mathcal{J}},$$

and, when $n \geq 3$, the new torsion σ satisfies

$$\sigma^{00} \equiv \sigma^{i0} \equiv \sigma^{ij} \equiv 0 \pmod{\mathcal{J}}.$$

The latter equation holds because

$$d\theta_i \equiv -\beta_i^j \wedge \theta_j - \pi_{ij} \wedge \omega^j - \pi_{i0} \wedge \omega^0 \pmod{\theta_\varnothing, \Lambda^2 \mathcal{J}},$$

and thus

$$0 \equiv \pi_{ij} \wedge \sigma^{jk} \wedge \theta_k \pmod{\theta_\varnothing, \Omega, \Lambda^2 \mathcal{J}}.$$

Several applications of Cartan's lemma show that

$$\sigma^{ij} \equiv 0 \pmod{\mathcal{J}}.$$

2.2 Monge-Ampère Systems

Parabolic systems whose Monge-Ampère systems vanish are closely related to a special class of non-linear differential equations. A second order differential equation for one function of n variables is *Monge-Ampère* if it can be written in the form

$$F(x^i, u, p_i, p_{ij}) = \sum_{|I| = |J|} A_{I,J}(x^i, u, p_i) H_{I,J} = 0, \tag{2.3}$$

where the I, J range over subsets of $\{1, \ldots, n\}$ and $H_{I,J}$ stands for the minor of the hessian matrix

$$H = (p_{ij})$$

with rows I and columns J deleted. Because the Hessian is symmetric, its minors satisfy the relation

$$H_{I,J} = H_{J,I}.$$

For this reason, I will assume without loss of generality that

$$A_{I,J} = A_{J,I}.$$

Monge-Ampère equations arise frequently in differential geometry. One famous example is the equation used by Calabi and Yau to prove the Calabi conjecture,

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}\right) = A_0.$$

In this example the coefficient functions $A_{I,J}$ are functions of the metric $g_{i\bar{j}}$ alone.

Unlike general second order equations, there is a simpler exterior differential system that models equations of Monge-Ampère type.

Definition 9. A *Monge-Ampère system* in n variables is a $2n+1$ dimensional exterior differential system $(M, \mathcal{I})$ such that $\mathcal{I}$ is locally generated by a 1-form $\underline{\theta}_\varnothing$ and an n-form Υ satisfying

1. $\underline{\theta}_\varnothing$ is contact, in the usual sense of maximal non-integrability:

$$\underline{\theta}_\varnothing \wedge (\, d\underline{\theta}_\varnothing)^n \neq 0.$$

2. Υ is not in the (contact) ideal generated by $\underline{\theta}_\varnothing$.

Equivalently, such systems are defined by local coframings

$$\underline{\theta}_\varnothing, \underline{\omega}^i, \underline{\pi}_i$$

so that

$$d\underline{\theta}_\varnothing \equiv \underline{\omega}^i \wedge \underline{\pi}_i \quad (\mathrm{mod}\ \underline{\theta}_\varnothing)$$

and

$$\Upsilon = \sum_{|I|=|J|} A_{I,J}\, \underline{\pi}_I \wedge \underline{\omega}_{(J)} \not\equiv 0 \quad (\mathrm{mod}\ \underline{\theta}_\varnothing,\ d\underline{\theta}_\varnothing) \tag{2.4}$$

for functions $A_{I,J}$ on M.

I remark that the notation, used here for consistency, suggests that there might be a distinction between the $\underline{\omega}^i$'s and the $\underline{\pi}_i$'s. In fact, for general Monge-Ampère systems, neither the ideal $\{\underline{\theta}_0, \underline{\omega}^i\}$ nor the ideal $\{\underline{\theta}_0, \underline{\pi}_i\}$ need be preserved by EDS equivalences. However, parabolic Monge-Ampère systems, defined below, have an invariant independence condition, of the form $\{\underline{\theta}_0, \underline{\omega}^i\}$.

I describe a useful normalization condition. As an element of $\mathcal{I}$, the n-form Υ is only defined up to scaling and addition of multiples of $\underline{\theta}_\varnothing$ and $d\underline{\theta}_\varnothing$. However, by the pointwise Lefschetz decomposition of symplectic linear algebra[2], there is a unique multiple $\gamma \wedge d\underline{\theta}_\varnothing$ so that $\Upsilon + \gamma \wedge d\underline{\theta}_\varnothing$ is *primitive*,

$$(\Upsilon + \gamma \wedge d\underline{\theta}_\varnothing) \wedge d\underline{\theta}_\varnothing \equiv 0 \quad (\mathrm{mod}\ \underline{\theta}_\varnothing).$$

---

[2]See, for example, [BGG03], Proposition 1.1.

Thus I will henceforth assume that

$$\Upsilon \wedge d\underline{\theta}_\varnothing \equiv 0 \quad (\text{mod } \underline{\theta}_\varnothing)$$

without loss of generality. With this assumption, the representative Υ in $\mathcal{I}$ is uniquely defined up to scaling and multiples of $\underline{\theta}_\varnothing$. Furthermore, the primitivity condition guarantees that $A_{I,J} = A_{J,I}$ in equation (2.4).

The Lefshetz decomposition also shows that for $k > n$, the degree k homogenous component of $\mathcal{I}$ contains all k-forms, that is,

$$\mathcal{I}^k = \Omega^k(M), \quad k > n.$$

Monge-Ampère systems model the solutions of Monge-Ampère equations, as follows. Let $M_{-1} = J^1(\mathbb{R}^n, \mathbb{R})$ and fix the coframing

$$\underline{\theta}_\varnothing = du - p_i\, dx^i, \quad \underline{\omega}^i = dx^i, \quad \underline{\pi}_i = dp_i.$$

Corresponding to equation (2.3), define the n-form

$$\Upsilon = \sum_{|I|=|J|} A_{I,J}(x^i, u, p_i)\, \underline{\pi}_I \wedge \underline{\omega}_{(J)}.$$

The Monge-Ampère system $(M_{-1}, \{\underline{\theta}_\varnothing, \Upsilon\})$ has a natural independence condition, defined by the forms $\underline{\omega}^i$. Any solution manifold is locally the 1-jet graph of a function $u(x^i)$, and the condition that Υ vanish forces u to be a solution to (2.3).

Conversely, any Monge-Ampère system $(M_{-1}, \{\underline{\theta}_\varnothing, \Upsilon\})$ is locally modelled by Monge-Ampère differential equations: By the Pfaff Normal Form Theorem, there are local coordinates x^i, u, p_i, and a nonzero function λ on M_{-1} so that

$$\underline{\theta}_\varnothing = \lambda(\,du - p_i\, dx^i).$$

The dx^i's determine an independence condition, albeit not contact invariantly. A solution manifold Σ is in particular a solution to the contact system $\{\underline{\theta}_\varnothing\}$. This and the fact that

$$dx^1 \wedge \ldots \wedge dx^n|_\Sigma \neq 0$$

imply that Σ is locally the graph of functions $u(x^i)$ and $p_i(x^i)$ so that

$$p_i = \frac{\partial u}{\partial x^i}$$

and thus

$$dp_i = \frac{\partial^2 u}{\partial x^i \partial x^j}\, dx^j.$$

Fixing the local coframing

$$\underline{\theta}_\varnothing = du - p_i\, dx^i, \quad \underline{\omega}^i = dx^i, \quad \underline{\pi}_i = dp_i,$$

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let $A_{I,J}$ be functions so that

$$\Upsilon = \sum_{|I|=|J|} A_{I,J}(x^i, u, p_i)\, \underline{\pi}_I \wedge \underline{\omega}_{(J)}.$$

Then the condition that Υ vanishes when restricted to Σ is the same as the condition that $u(x^i)$ solves the equation

$$\sum_{|I|=|J|} A_{I,J} H_{I,J} = 0.$$

The class of Monge-Ampère systems is manifestly preserved by EDS equivalences, so we can now see that the class of Monge-Ampère equations is preserved under changes of variables. (This was certainly not obvious from the initial definition.) By way of comparison, the class of linear differential equations is not preserved by changes of variables. Indeed, a generic change of coordinates takes a linear equation to a non-linear equation.

I have defined two systems that model the solutions of a Monge-Ampère equation: the Monge-Ampère system $(M_{-1}, \mathcal{I}_{-1})$ and the second order system $(M_0, \mathcal{I}_0)$. One should expect that there is a relation between the two, and there is: M_0 is the first prolongation of M_{-1}. In particular, the projection

$$J^2(\mathbb{R}^n, \mathbb{R}) \longrightarrow J^1(\mathbb{R}^n, \mathbb{R})$$

restricts to a submersive EDS map

$$\pi^{(0)} \colon M_0 \longrightarrow M_{-1},$$

and furthermore, solutions to the two systems are identified under $\pi^{(0)}$.

I now want to define the class of parabolic Monge-Ampère systems. First though, note that generic Monge-Ampère equations do not have constant symbol on M_0. For example, the equation

$$F = \sum_{i \neq j} (p_{ii}p_{jj} - p_{ij}^2) = 0$$

is Monge-Ampère, but

$$dF = \sum_{i \neq j} (p_{ii}\, dp_{jj} + p_{jj}\, dp_{ii} - 2p_{ij}\, dp_{ij}).$$

Consequently, F has *any* possible symbol, depending on a suitable choice of $(x^i, u, p_i, p_{ij}) \in M_0$. This means that it does not always make sense to ask what the symbol is at a point x of M_{-1}, because the answer can depend on the choice of a point in the fiber $(M_0)_x$.

I define a class of Monge-Ampère systems that do have a well defined notion of symbol.

Definition 10. Let $(M_{-1}, \mathcal{I}_{-1})$ be a Monge-Ampère system in $n+1$ variables, locally generated by a contact form $\underline{\theta}_\varnothing$ and an n-form Υ. I say that M_{-1} is of *linear type* if it has an independence condition J locally spanned by $\underline{\theta}_\varnothing$ and 1-forms $\underline{\omega}^0, \ldots, \underline{\omega}^n$ so that

1. J is Lagrangian with respect to $d\underline{\theta}_\varnothing$,

$$d\underline{\theta}_\varnothing \equiv 0 \pmod{J}.$$

2. For any 1-form $\underline{\alpha} \in J$,

$$\underline{\alpha} \wedge \Upsilon \equiv 0 \quad \left(\text{mod } \underline{\theta}_\varnothing, \, \underline{\omega}^0 \wedge \ldots \wedge \underline{\omega}^n\right).$$

The ideal J is a *compatible independence condition* for $(M_{-1}, \mathcal{I}_{-1})$.

Note well that linear-type Monge-Ampère systems do not typically arise from linear differential equations. The nomenclature is in line with linear Pfaffian systems, where 'linear' refers to linearity in the complement of the independence condition. Indeed, by condition 2, there are 1-forms $\underline{\eta}_a$ so that

$$\Upsilon \equiv \underline{\eta}_a \wedge \underline{\omega}_{(a)} \quad (\text{mod } \underline{\theta}_\varnothing). \tag{2.5}$$

By condition 1, the forms $\underline{\theta}_\varnothing$ and $\underline{\omega}^a$ can be completed to a coframing of M_{-1} by forms $\underline{\pi}_a$ so that

$$d\underline{\theta}_\varnothing \equiv -\underline{\pi}_a \wedge \underline{\omega}^a \quad (\text{mod } \underline{\theta}_\varnothing).$$

Combined with equation (2.5) and the assumed primitivity of Υ, this shows that

$$d\underline{\theta}_\varnothing \wedge \Upsilon \equiv (\underline{\eta}_a \wedge \underline{\pi}_a) \wedge \underline{\omega}^0 \wedge \ldots \wedge \underline{\omega}^n \equiv 0 \quad (\text{mod } \underline{\theta}_\varnothing).$$

By an application of Cartan's lemma, there is a symmetric-matrix valued function (h_{ab}) so that

$$\underline{\eta}_a \equiv h_{ab}\underline{\pi}_a \quad (\text{mod } \underline{\theta}_\varnothing, \underline{\omega}^a),$$

and thus,

$$\Upsilon \equiv h_{ab}\,\underline{\pi}_a \wedge \underline{\omega}_{(b)} \quad (\text{mod } \underline{\theta}_\varnothing).$$

The function (h_{ab}) is the *unadapted symbol* of the linear-type Monge-Ampère system.

In a story analogous to that described for parabolic systems, one can define a G-structure of coframes adapted to the ideal $\mathcal{I}_{-1}$, where the structure group consists of matrices of the form

$$\begin{pmatrix} k_\varnothing & 0 & 0 \\ * & B & 0 \\ * & {}^tB^{-1}S & {}^tB^{-1} \end{pmatrix}$$

for $k_\varnothing \in \mathbb{R}^\times$, $B \in \mathrm{GL}(W)$, and $S \in \mathrm{Sym}^2 W$. In general, there are compatible choices of coframes that diagonalize h_{ab}, reducing the G-structure.

If the unadapted symbol h_{ab} is positive semi-definite everywhere on a linear-type Monge-Ampère system M_{-1}, then it has coframings

$$\underline{\theta}_\varnothing, \underline{\omega}^a, \underline{\pi}_a \tag{2.6}$$

so that

$$d\underline{\theta}_\varnothing \equiv -\underline{\pi}_a \wedge \underline{\omega}^a \quad (\text{mod } \underline{\theta}_\varnothing)$$

and

$$\Upsilon \equiv \underline{\pi}_i \wedge \underline{\omega}_{(i)} \quad (\text{mod } \underline{\theta}_\varnothing).$$

I call this a 0-adapted coframing of a parabolic Monge-Ampère system, although I show below that sucha coframing pulls back to give Monge-Ampère-adapted coframing of a parabolic system. Observe that, for 0-adapted coframings, $\underline{\omega}^0$ is uniquely specified (up to scaling and addition of multiples of $\underline{\theta}_\varnothing$) by the condition that

$$\underline{\omega}^0 \wedge \Upsilon \equiv 0 \pmod{\underline{\theta}_\varnothing}.$$

In fact, the ideal $\{\underline{\theta}_\varnothing, \underline{\omega}^0\}$ defines the characteristic hyper-plane distribution when restricted to solution manifolds. This provides a useful characterization of parabolic Monge-Ampère systems.

Definition 11. A Monge-Ampère system of linear type is *parabolic* if its compatible independence condition contains a 1-form $\underline{\omega}^0$ so that $\underline{\theta}_\varnothing \wedge \underline{\omega}^0$ is non-vanishing and so that

$$\underline{\omega}^0 \wedge \Upsilon \equiv 0 \pmod{\underline{\theta}_0}.$$

Then $\underline{\omega}^0$ is called a *characteristic* co-vector.

As one might hope, a Monge-Ampère system of parabolic type has a G-structure that reflects much of the G_0-structure of parabolic systems. To see this, fix two adapted coframings of the form (2.6). There is a non-zero function $k_\varnothing$, a function B taking values in

$$\begin{pmatrix} CO(n) & 0 \\ {}^t(\mathbb{R}^n) & \mathbb{R}^\times \end{pmatrix}$$

and a function T taking values in K

$$K \cong \mathrm{Sym}_0^2 W' \oplus W' \oplus \mathbb{R}$$

(this is the same K as defined in (1.10)), so that

$$\begin{aligned}
(\widetilde{\underline{\omega}^a}) &\equiv k_\varnothing\, {}^t B^{-1}(\underline{\omega}^a) \pmod{\underline{\theta}_\varnothing} \\
(\widetilde{\underline{\pi}_a}) &\equiv T(\underline{\omega}^a) + B(\underline{\pi}_a) \pmod{\underline{\theta}_\varnothing}.
\end{aligned} \tag{2.7}$$

I now describe the first prolongation $(M^{(0)}, \mathcal{I}^{(0)})$ of a parabolic system $(M_{-1}, \mathcal{I}_{-1})$. For the sake of explicitness, I only define the open prolongation along the independence condition.

As a manifold, $M^{(0)}$ is the subset of the Grasmannian bundle $\mathrm{Gr}_{n+1}(TM_{-1})$ given by the integral elements that satisfy the independence condition. The projection of the Grassmanian bundle restricts to a submersion

$$\pi^{(0)} \colon M^{(0)} \to M_{-1},$$

which makes $M^{(0)}$ an affine-space fiber bundle with fiber K. The independence condition guarantees that elements E in the fiber $M_x^{(0)}$ are parameterized by numbers p_{ab} so that

$$E = \{\underline{\theta}_\varnothing, \underline{\pi}_a - p_{ab}\underline{\omega}^b\}_x^\perp.$$

Then the fact that E is an integral element of $\mathcal{I}_{-1}$ ensures that

$$p_{ab} = p_{ba}, \quad p_{ii} = 0.$$

As written, it is tempting to consider $M^{(0)}$ as a vector-space bundle over M_{-1}, but this would be incorrect. Indeed, consider a change of coframe (2.7) with $B = \mathbf{1}$. In this change of coframe E will be parameterized by new numbers $\widetilde{p}_{ab}$, related to the old ones by the translation T. In effect, a choice of origin in each fiber would force a reduction of coframing.

Define the forms[3]

$$\underline{\theta}_a \equiv \underline{\pi}_a - p_{ab}\underline{\omega}^b \quad (\mathrm{mod}\ \underline{\theta}_\varnothing)$$

on $M^{(0)}$. The prolonged ideal of $\mathcal{I}_{-1}$ is defined to be the differential ideal

$$\mathcal{I}^{(0)} = \{\underline{\theta}_\varnothing, \underline{\theta}_a\}.$$

It is not difficult to see that $\pi^{(0)}$ is an EDS morphism. Furthermore, because $M^{(0)}$ is an involutive prolongation, its solution manifolds are in bijection with the solution manifolds of M_{-1}. Observe also that the local (extended) coframing

$$\underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\omega}^a,\ dp_{ab} \tag{2.8}$$

exhibits $M^{(0)}$ as a parabolic system.

As promised, given a parabolic Monge-Ampère equation, its corresponding parabolic system M_0 is essentially the same as the prolongation of its Monge-Ampère system. More precisely, there is an EDS equivalence f from M_0 to $M^{(0)}$, defined as follows. At each point $x \in M_0$, consider the plane

$$\widetilde{E}_x = \{\theta_\varnothing, \theta_a\}_x^\perp \subset T_x M_0.$$

It is not difficult to see that its projection to M_{-1}, given by

$$E_x = d\pi^{(0)}(\widetilde{E}_x),$$

is an $n + 1$ dimensional integral element of $\mathcal{I}$ satisfying the independence condition. In particular, E_x is an element of $M^{(0)}$, so we may define f by the rule

$$x \longmapsto E_x.$$

It follows from the structure equations of M_0 both that f is a diffeomorphism and that f is an EDS equivalence.

Given a 2$^{\mathrm{nd}}$ order system, an obvious question to ask is whether it has a Monge-Ampère deprolongation like the one just described. In the parabolic case, the Monge-Ampère invariants answer this question completely:

Theorem 1. *A parabolic system* $(M_0, \mathcal{I}_0)$ *has a parabolic Monge-Ampère deprolongation if and only if its Monge-Ampère invariants vanish identically.*

---

[3]Some readers might object that this depends on a choice of the $\underline{\pi}_a$. Or, they might object that the p_{ab} are only defined up to an affine structure. However, these effects cancel exactly, and the $\underline{\theta}_a$ are well defined independently of any choice of coframing.

Proof. This has been proved for systems in $n + 1 = 2$ and $n + 1 = 3$ variables by Bryant & Griffiths [BG95b] and Clelland [NC97] respectively, so I will assume that $n \geq 3$.

If M_0 has a Monge-Ampère deprolongation, then the coframing (2.8) provides a Monge-Ampère-adapted coframing of M_0, proving the forward implication.

To see the reverse implication, recall that the Cartan system of $\theta_\varnothing$,

$$\mathcal{J} = \{\theta_\varnothing, \theta_a, \omega^a\},$$

is Frobenius. Consider a small enough neighborhood in M_0 so that the space of leaves M_{-1} is a manifold and let

$$\pi^{(0)} \colon M_0 \to M_{-1}$$

denote the submersion of M_0 onto its leaf space. The manifold M_{-1} will be the space of the deprolongation.

Consider the $(n + 1)$-form

$$\Upsilon_0 = \theta_i \wedge \omega_{(i)},$$

which is defined on $\mathcal{B}_0$. The ideal

$$\widetilde{\mathcal{I}} = \{\theta_\varnothing, \ \Upsilon_0, \ \Lambda^{n+2}\mathcal{J}\}$$

will define the ideal $\mathcal{I}_{-1}$ that makes M_{-1} into a parabolic Monge-Ampère system.

If the Monge-Ampère invariants vanish, and the corresponding reductions of coframe have been carried out, the structure equations

$$
\begin{aligned}
d\Theta &\equiv -\beta \wedge \Theta - \Pi \wedge \Omega \quad \left(\mathrm{mod}\ \theta_\varnothing, \Lambda^2 J\right) \\
d\Omega &\equiv -(\kappa_0 - {}^t\beta) \wedge \Omega \quad \left(\mathrm{mod}\ \theta_\varnothing, \Lambda^2 J\right)
\end{aligned}
\tag{2.9}
$$

hold. (This is where the assumption that $n \geq 3$ is used, so that the torsion $\sigma \in \mathcal{J}$.) In this case, with a little work, one finds that

$$d(\theta_i \wedge \omega_{(i)}) \equiv (2\beta_{tr} - n\kappa_0 + \beta_0^0) \wedge \theta_i \wedge \omega_{(i)} \quad \left(\mathrm{mod}\ \theta_\varnothing, \Lambda^{n+2}\mathcal{J}\right). \tag{2.10}$$

It follows from (2.9) and (2.10) that $\widetilde{\mathcal{I}}$ is invariant[4] in each fiber of $\mathcal{B}_{MA}$, so it is the pullback of an ideal on M_0, which I also denote by $\widetilde{\mathcal{I}}$.

Even better, it follows from (2.10) that the Cartan system of $\widetilde{\mathcal{I}}$ is $\mathcal{J}$. By the general theory of Cartan systems, there is a 1-form $\underline{\theta}$ and an $(n + 1)$-form Υ_{-1} on M_{-1} so that the ideal

$$\mathcal{I}_{-1} = \{\underline{\theta}, \Upsilon_{-1}\}$$

pulls back[5] by $\pi^{(0)}$ to $\widetilde{\mathcal{I}}$.

---

[4]The Lie derivative of any element of $\widetilde{\mathcal{I}}$ in any fiber direction is again in $\widetilde{\mathcal{I}}$. This follows from Cartan's formula for the Lie derivative.

[5]To be totally precise, it pulls back to a set which generates $\widetilde{\mathcal{I}}$ as an ideal.

The structure equations of M_0 can be used to prove that the pair $(M_{-1}, \mathcal{I}_{-1})$ is a parabolic type Monge-Ampère system. For example, to see that $\underline{\theta}$ is a contact form, note that there is a non-zero function λ on M_0 so that

$$(\pi^{(0)})^* \underline{\theta} = \lambda \underline{\theta}_{\varnothing}.$$

Thus, if $\underline{\theta}$ were a degenerate contact form, so that $\underline{\theta} \wedge (d\underline{\theta})^{n+1}$ vanished somewhere, then its pullback to M_0 would also vanish. This in turn would contradict the structure equations of $\theta_{\varnothing}$ for a parabolic system.

To see that M_{-1} is of linear parabolic type, observe that (2.9) can also be used to show that the Cartan system of the ideal

$$\widetilde{J} = \{\theta_{\varnothing}, \omega^0, \omega^i\}$$

is $\mathcal{J}$. Consequently, $\widetilde{J}$ pushes down to an ideal J on M_{-1} so that $(\pi^{(0)})^* J = \widetilde{J}$. This ideal is necessarily generated locally by $n+2$ independent 1-forms, including θ, so it defines an independence condition. The fact that it is a compatible independence condition for M_{-1} also follows from the structure equations of M_0. For example, if J were not lagrangian for $d\underline{\theta}$, then $\widetilde{J}$ would not Lagrangian be for $d\underline{\theta}_{\varnothing}$, contradicting the structure equations of M_0.

Finally, the ideal

$$\{\theta_{\varnothing}, \omega^0\}$$

also has Cartan system $\mathcal{J}$, and the corresponding ideal on M_{-1} provides the characteristic form for $(M_{-1}, \mathcal{I}_{-1})$.

The map f defined prior to the proposition gives an EDS equivalence between M_0 and the first prolongation of M_{-1}, so the theorem is proved. $\qquad\square$

2.3   The Goursat Invariant

I now turn to the primary Goursat invariants, which are defined on a parabolic system M_0 whose primary Monge-Ampère invariants vanish. Note that M_0 is not here required to be Monge-Ampère and could well have non-vanishing secondary Monge-Ampère invariants.

Assuming that the primary Monge-Ampère invariants of M_0 vanish, there are functions a_{ij} and a_i on $\mathcal{B}_1$ so that

$$\xi_i^0 \equiv a_{ij}\omega^j + a_i\omega^0 \quad (\text{mod } \theta_{\varnothing}, \theta_a).$$

In this case, the exterior derivative of

$$d\theta_i = -\kappa_i \wedge \theta_{\varnothing} - \beta_i^j \wedge \theta_j - \pi_{ij} \wedge \omega^j - \pi_{i0} \wedge \omega^0 - \xi_i^0 \wedge \theta_0 - \xi_i^j \wedge \theta_j.$$

gives, after an application of Cartan's lemma,

$$d\xi_i^0 \equiv \beta_0^0 \wedge \xi_i^0 - \beta_i^j \wedge \xi_j^0 - \gamma_{ij}^0 \wedge \omega^j \quad \left(\text{mod } \theta_{\varnothing}, \theta_a, \omega^0, \Omega_{sb}^2\right).$$

Plugging in ξ_i^0, one finds that

$$da_{ij} \equiv (\beta_0^0 + \kappa_0)a_{ij} - \beta_i^k a_{kj} - a_{ik}\beta_j^k - \gamma_{ij}^0 \quad \left(\text{mod } \Omega_{sb}^1\right).$$

Integrating, one finds that the functions a_{ij} vary in the fiber of $\mathcal{B}_1$ by the rule

$$(a_{ij}(g \cdot u)) = B_0^0 k_\varnothing (B')^{-1} (a_{ij}(u))({}^t B')^{-1} - C_{ij}^0, \tag{2.11}$$

for all $u \in \mathcal{B}_1$ and $g \in G_1$. Since C_{ij}^0 is symmetric and traceless, there are coframes absorbing the traceless symmetric part of a_{ij}. These adapted coframes form a G_2-structure[6] $\mathcal{B}_2$ on M_0, where G_2 consists of matrices as in (1.8) for which

$$S^{0a} = 0 \quad \text{and} \quad C_{ij}^0 = 0.$$

Restricted to $\mathcal{B}_2$, the remaining non-absorbable components of (a_{ij}) define a relative invariant, which I define to be the *primary Goursat invariant*. It splits into two components,

$$a_{ij} = \mathcal{G}_{ij} + a\delta_{ij},$$

for a real valued function a and anti-symmetric $\mathcal{G}_{ij}$. Geometrically, the trace component a is the higher dimensional analogue of the Goursat invariant studied by Bryant & Griffiths and Clelland. The anti-symmetric component is new. It measures how far the characteristic distribution is from being Frobenius.

Restricted to $\mathcal{B}_2$, the structure equation for θ_i is

$$d\theta_i \equiv -\pi_{ij} \wedge \omega^j - \pi_{i0} \wedge \omega^0 - a_{ij}\omega^j \wedge \theta_0 - a_i \omega^0 \wedge \theta_0 \quad (\text{mod } \theta_\varnothing, \theta_i).$$

In this case, the non-degeneracy condition for parabolic systems simplifies to

$$(d\theta_i) \wedge \omega_{(i)} \equiv na\theta_0 \wedge \omega \not\equiv 0 \quad (\text{mod } \theta_\varnothing, \theta_i).$$

Consequently, the trace component a of the Goursat invariant *never* vanishes. Then it follows from equation (2.11) that coframes may be chosen so that $a = -1/n$ everywhere. (The sign is for consistency with coframing of the heat equation given in example 6.) This reduces $\mathcal{B}_2$ to a G_3-structure $\mathcal{B}_3$, where G_3 is the subgroup of G_1 consisting of matrices in (1.8) for which

$$S^{0a} = 0,$$

$$C_{ij}^0 = 0$$

and

$$B_0^0 = B_{\text{tr}}^2 / k_\varnothing.$$

Now, suppose that $\mathcal{G}_{ij}$ is identically zero. In this case, one can check that (a_i) varies by the rule

$$(a_i(g \cdot u)) = k_\varnothing (B')^{-1}(a_i(u)) + k_i - B_0^i - C_{i0}^0.$$

The reduction of coframes (so that (a_i) is identically zero) is straighforward and the resulting structure group G_4 is given by matrices in G_3 for which

$$C_{i0}^0 = B_0^i - k_i.$$

---

[6]This group is unrelated to the exceptional simple group G_2.

On the G_4-structure $\mathcal{B}_4$, the structure equation for θ_i is

$$d\theta_i \equiv -\pi_{ij} \wedge \omega^j - \pi_{i0} \wedge \omega^0 - \frac{1}{n}\theta_0 \wedge \omega^i \quad (\text{mod } \theta_\varnothing, \theta_i)\,.$$

More generally, equation (2.11) shows that the anti-symmetric matrix $\mathcal{G}_{ij}$ may be diagonalized, even if it is nonzero. However, the corresponding reduction of coframes generally does not result in a G-structure. Indeed, if the eigenvalues are not constant, then the reduced structure group—the stabilizers of the diagonalized $\mathcal{G}_{ij}$—will vary as well. Some kind of constant rank assumption must be made to guarantee a G-structure. Instead, it will be better for now to make no more reductions. Even with suitable constant rank assumptions, the equivalence problem splits into cases depending on $\mathcal{G}_{ij}$.

I remark that there is always a reduction of coframes absorbing the functions a_i, but this also does not generally result in a G-structure.

The Goursat Invariant for general parabolic systems.

Something very similar to the Goursat invariants can be defined for a parabolic system whose primary Monge-Ampère invariants don't all vanish. However, it is only well defined on the first prolongation of M_0.

Using the independence condition $\mathcal{J}$, the process of prolongation results in an EDS $(M^{(1)}, \mathcal{I}^{(1)})$ equipped with an EDS submersion

$$\pi^{(1)} \colon M^{(1)} \to M_0.$$

As usual, the underlying manifold $M^{(1)}$ is defined to be the space of integral elements that satisfy the independence condition $\mathcal{J}$.

The map $\pi^{(1)}$ gives $M^{(1)}$ the structure of a fiber bundle, with fiber the affine space modelled on $K^{(1)}$. Indeed, fixing an adapted coframing of M_0, the integral elements E in each fiber are parameterized by numbers p_{abc} so that

$$E = \{\underline{\theta}_\varnothing,\ \underline{\theta}_i,\ \underline{\theta}_0, \underline{\pi}_{ab} - p_{abc}\underline{\omega}^c\}^\perp \subset T_x M_0.$$

This simply follows from the fact that each E satisfies the independence condition. The condition that E be integral then requires that the numbers p_{abc} be symmetric in all 3 indices and that the 'traces' p_{iia} vanish for each index a.

Now we can define the forms

$$\underline{\theta}_{ab} \equiv \underline{\pi}_{ab} - p_{abc}\underline{\omega}^c \quad (\text{mod } \mathcal{I}_0)$$

at each point of $M^{(1)}$. These define the prolonged ideal of $\mathcal{I}_0$ by the equation

$$\mathcal{I}^{(1)} = \{\underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\theta}_{ab}\},$$

where the obvious pullbacks have been omitted from the notation.

It is now simple to define the Goursat invariant: there are functions $a_{ij}, a_i, a_{ij}^k, a_{i0}^j$ on the adapted coframe bundle of $M^{(1)}$ so that

$$d\theta_i \equiv -\theta_{ia} \wedge \omega^a - (a_{ij}\omega^j + a_i\omega^0) \wedge \theta_0 - (a_{ik}^j\omega^k + a_{i0}^j) \wedge \theta_j \quad \left(\mathrm{mod}\ \theta_\varnothing, \Lambda^2\,\mathcal{I}^{(1)}\right).$$

It is possible to absorb some components of thes functions. However, the story is not as simple as before, and any reductions of coframing are necessarily defined on the coframe bundle over $M^{(1)}$.

I remark that even though the Goursat invariants are functions on the coframe bundle of $M^{(1)}$, they are so in a very controlled way. For example, there are functions on $\mathcal{B}_0$ so that

$$\xi_i^j \equiv \widetilde{a}_{ik}^j\omega^k + V_i^{jlm}\pi_{lm} + V_i^{jl}\pi_{l0} + V_i^j\pi_{00} \quad \left(\mathrm{mod}\ \theta_\varnothing, \theta_a, \omega^0\right),$$

and pulled back to the coframe bundle of $M^{(1)}$ one finds that

$$\xi_i^j \equiv (\widetilde{a}_{ik}^j + V_i^{jlm}p_{klm} + V_i^{jl}p_{kl0} + V_i^j p_{k00})\omega^k \quad \left(\mathrm{mod}\ \theta_\varnothing, \theta_a, \omega^0, \theta_{ab}\right).$$

This shows that a_{ik}^j is linear in the fiber coordinates of $M^{(1)}$, and in particular that

$$da_{ik}^j \equiv V_i^{jlm}\,dp_{klm} + V_i^{jl}\,dp_{kl0} + V_i^j\,dp_{k00} \quad (\mathrm{mod}\ \mathcal{J}, \theta_{ab}) \tag{2.12}$$

Interpreting the Goursat Invariant

The value of the Goursat invariant allows one to draw interesting geometric conclusions about a parabolic system. Intuitively, the trace component detects the sub-principal symbol of a parabolic equation. The anti-symmetric component measures the integrability of the characteristic co-vector. I proceed to make these statements precise.

The general theory of characteristics for linear Pfaffian systems (see [BCG$^+$13], Chapter IV) guarantees that a parabolic system has a well defined characteristic ideal. For a parabolic system M_0, the *characteristic ideal* is given in a coframing by

$$\{\underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\omega}^0\}.$$

The forms $\underline{\theta}_\varnothing$ and $\underline{\theta}_a$ vanish on Σ, so the restriction of this ideal to a solution manifold $\Sigma \subset M_0$ defines a hyperplane distribution on Σ, the *characteristic (hyperplane) distribution*.

A generic parabolic system will have non-integrable characteristic distributions on solution manifolds. Contrast this with the parabolic system M_0 corresponding to an *evolutionary equation*, written in appropriate coordinates as

$$p_0 = F(x^a, u, p_i, p_{ij}). \tag{2.13}$$

In this case, the level sets of the coordinate x_0 define a natural foliation of M_0 into 'time' slices. This restricts to a foliation on any solution manifold. It follows from equation (2.13) and the definition of 0-adapted coframes that the tangent planes of this foliation agree with the characteristic distribution. Thus, by explicit construction, the characteristic distribution of an evolution equations is Frobenius on all solution manifolds.

The following theorem proves the converse of this and makes a connection with the invariants just developed.

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Theorem 2. *For a real analytic parabolic system $(M_0, \mathcal{I}_0)$ in $n + 1 > 3$ variables, the following conditions are equivalent:*

1. *The primary Monge-Ampère invariants and the Goursat invariant $\mathcal{G}_{ij}$ vanish identically.*

2. *M_0 has a 0-adapted coframing so that*
$$d\underline{\omega}^0 \equiv 0 \quad \left(mod\ \theta_\varnothing, \theta_a, \omega^0\right).$$

3. *M_0 is locally equivalent to a parabolic equation in evolutionary form.*

4. *The characteristic distribution is Frobenius on every solution manifold.*

Proof. Conditions 1 and 2 are equivalent by the work of the previous sections. In particular, the coframing given in condition 2 is necessarily 2-adapted. The discussion immediately prior to this theorem shows that 3 implies 4.

For 4 $\Rightarrow$ 2: Fix a local 0-adapted coframing η near $x \in M_0$. For any integral element E, over x and satisfying the independence condition, let p_{abc} be the numbers so that
$$E = \{\underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\pi}_{ab} - p_{abc}\underline{\omega}^c\}^\perp_x.$$

Since $(M_0, \mathcal{I}_0)$ is involutive, the Cartan-Kähler theorem[7] guarantees that there exists an integral manifold
$$\iota \colon \Sigma \to M_0$$
for which $x \in \Sigma$ and
$$T_x\Sigma = E.$$
At x, the pullback of $\underline{\omega}^0$ to Σ will satisfy
$$(\iota^* d\underline{\omega}^0)_x \equiv (\eta\iota)^*(d\omega^0)_x \equiv (\eta\iota)^*(\xi_i \wedge \omega^i)_x \qquad \left(mod\ \iota^*\underline{\omega}^0\right)$$
$$\equiv (\eta\iota)^*((U_i p_{j00} + U_i^k p_{jk0} + U_i^{kl} p_{jkl} + \mathcal{G}_{ji})\omega^j \wedge \omega^i)_x \qquad \left(mod\ \iota^*\underline{\omega}^0\right).$$

On the other hand, the characteristic foliation is integrable on Σ, so
$$(\iota^* d\underline{\omega}^0)_x \equiv 0 \quad \left(mod\ \iota^*\underline{\omega}^0\right)$$

Combining these, the equation
$$(U_i p_{j00} + U_i^k p_{jk0} + U_i^{kl} p_{jkl} + \mathcal{G}_{ij})\omega^j \wedge \omega^i = 0$$
holds at $\underline{\eta(\iota(x))} \in \mathcal{B}_0$. Since the choice of integral element was arbitrary, it is clear that this equation holds for any choice of p_{abc} that is symmetric and trace-free.

Choosing $p_{jk0} = p_{jkl} = 0$ and $p_{j00} \neq 0$ shows that (U_i) vanishes identically. (Recall that as a relative invariant, when (U_i) vanishes anywhere in a fiber, it vanishes in the whole fiber.)

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[7]Here is where the assumption of real analyticity is needed. For reference, see [BCG$^+$13] Chapter III.

By choosing the p_{jk0} arbitrarily, one finds that the matrix U_i^k must be a multiple of the identity. For example, taking $p_{120} = 1$ and all else zero shows that $U_i^1 = U_i^2 = 0$ for $i > 2$ and $U_1^1 = U_2^2$. Thus, I may assume without loss of generality that $\underline{\eta}$ is adapted so that U_j^i vanishes identically.

Similar reasoning shows that the non-absorbable part of U_i^{kl} vanishes, so I may assume that $\underline{\eta}$ is adapted to the Goursat invariant. It is now clear that the $\mathcal{G}_{ij}$ must also vanish.

For 2 $\Rightarrow$ 3: If condition 2 (and thus condition 1) holds, there is a 2-adapted coframing

$$\underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\omega}^a, \underline{\pi}_{ab},$$

of M_0 so that

$$d\underline{\omega}^0 \equiv 0 \quad \left(\mathrm{mod}\ \theta_\varnothing, \theta_a, \omega^0\right).$$

There is a matrix valued function (N_a^b) and an anti-symmetric matrix valued function (M^{ab}) so that

$$d\underline{\omega}^0 \equiv M^{ab}\underline{\theta}_a \wedge \underline{\theta}_b + N_a^b\underline{\theta}_b \wedge \underline{\omega}^a \quad \left(\mathrm{mod}\ \theta_\varnothing, \omega^0\right).$$

Taking the exterior derivative,

$$0 \equiv -N_a^b\underline{\pi}_{bc} \wedge \underline{\omega}^c \wedge \underline{\omega}^a \quad \left(\mathrm{mod}\ \theta_\varnothing, \theta_a, \omega^0\right).$$

In order for this congruence to hold, it must be true that

$$N_a^b = N_{\mathrm{tr}}\delta_a^b,$$

where N_{tr} is a function on M_0. Replacing $\underline{\omega}^0$ with $\underline{\omega}^0 + N_{\mathrm{tr}}\underline{\theta}_\varnothing$ results in a new 2-adapted coframing where furthermore

$$d\underline{\omega}^0 \equiv M^{ab}\underline{\theta}_a \wedge \underline{\theta}_b \quad \left(\mathrm{mod}\ \theta_\varnothing, \omega^0\right).$$

I remark that this is a further adaptation of coframes, and reduces the structure group.

Returning to the exterior derivative of $d\underline{\omega}^0$, now

$$0 \equiv 2M^{ab}\underline{\pi}_{ac} \wedge \underline{\theta}_b \wedge \underline{\omega}^c \quad \left(\mathrm{mod}\ \theta_\varnothing, \omega^0, \Lambda^2\,\mathcal{I}_0\right),$$

which only holds if

$$(M^{ab}) = 0.$$

Finally, there is a 1-form $\underline{\kappa}^0$ so that

$$d\underline{\omega}^0 \equiv -\underline{\kappa}^0 \wedge \underline{\theta}_\varnothing \quad \left(\mathrm{mod}\ \omega^0\right).$$

Taking the exterior derivative of this results in

$$0 \equiv \underline{\kappa}^0 \wedge \underline{\theta}_a \wedge \underline{\omega}^a \quad \left(\mathrm{mod}\ \theta_\varnothing, \omega^0\right),$$

which is enough to guarantee that

$$\underline{\kappa}^0 \equiv 0 \quad \left(\mathrm{mod}\ \theta_\varnothing, \omega^0\right).$$

This shows that there are 2-adapted coframes for which

$$d\underline{\omega}^0 \equiv 0 \quad \left(\mathrm{mod}\ \omega^0\right).$$

Because of this, the Frobenius theorem says that there is locally a function x_0 on M_0 so that $\underline{\omega}^0$ is a multiple of dx_0. Extend this to any jet coordinates so that $\underline{\omega}^0$ is a multiple of dx_0 and fix a 2-adapted coframing $\underline{\eta}$ of M_0. Recall that $\underline{\eta}$, as a 0-adapted coframing of M_0, extends to a $\mathcal{C}$-adapted coframing of $J^2(\mathbb{R}^{n+1}, \mathbb{R})$ along M_0, which I will also denote $\underline{\eta}$. Observe that the $\mathcal{C}$-adapted coframing

$$\hat{\eta} = \left(\hat{\theta}_\varnothing, \hat{\theta}_a,\ dx^a,\ dp_{ab}\right)$$

differs from $\underline{\eta}$ by a function

$$g\colon M_0 \to G_{\mathcal{C}}.$$

By construction, g takes dx_0 to a multiple of $\underline{\omega}^0$, so g takes values in matrices (1.3) so that B preserves the subspace

$$\{e_\varnothing, e_a, f^0\}$$

of V.

But this means that g also preserves the ideal

$$\{\hat{\theta}_\varnothing, \hat{\theta}_a,\ dx^a,\ dp_{ij}\}.$$

As a consequence, this ideal is equal to the ideal

$$\{\underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\omega}^a, \underline{\pi}_{ij}\},$$

which is defined by a 2-adapted coframing. Then, since

$$dF \equiv 0 \quad \left(\mathrm{mod}\ \underline{\theta}_\varnothing, \underline{\theta}_a, \underline{\omega}^a, \underline{\pi}_{ij}\right),$$

one sees that

$$dF \equiv \frac{\partial F}{\partial p_{0a}}\, dp_{0a} \equiv 0 \quad \left(\mathrm{mod}\ \hat{\theta}_\varnothing, \hat{\theta}_a,\ dx^a,\ dp_{ij}\right),$$

and thus

$$F = F(x^a, u, p_a, p_{ij}).$$

Finally, the non-degeneracy condition for parabolic systems guarantees that

$$\frac{\partial F}{\partial p_0} \neq 0,$$

so by the implicit function theorem, F is locally equivalent to an equation in evolutionary form. $\quad\square$

In the course of the proof, I showed that a parabolic system satisfying any of the stated conditions has a coframing so that $\underline{\omega}^0 \wedge d\underline{\omega}^0 = 0$. So I make the following definition.

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Definition 12. A *strongly parabolic* system is a parabolic system that furthermore has an adapted coframing for which

$$d\underline{\omega}^0 \equiv 0 \quad \left(\operatorname{mod} \omega^0\right).$$

Finally, I note an interesting geometric consequence of normalizing the trace component of the Goursat invariant. For coframings $\underline{\eta}$ and $\widetilde{\underline{\eta}}$ which differ by a function valued in G_3, it is not difficult to compute that

$$\widetilde{\underline{\omega}}^0 \equiv \left(\frac{k_\varnothing}{B_{\mathrm{tr}}}\right)^2 \underline{\omega}^0 \quad \left(\operatorname{mod} \theta_\varnothing, \theta_a\right).$$

This means that the characteristic covector ω^0 has a well defined sign on $\mathcal{B}_3$, and thus that the characteristic hyperplanes of an integral manifold are co-oriented. This can be interpreted as a sort of 'arrow of time' result, even for non-evolutionary parabolic systems, which necessarily don't have a definition of 'time'. In the case of integrable characteristics this corresponds to an actual orientation of the characteristic direction x_0.

Chapter 3

Conservation Laws

3.1 Principal Structure Equations

I now turn to the conservation laws of parabolic systems and the connection between conservation laws and the geometric invariants just developed. In order to consider conservation laws that depend on derivatives of arbitrarily high order, it will be necessary to work on the infinite prolongation of M_0. I summarize the necessary results here; the process of prolongation is described in more detail in [BG95a] and [BCG$^+$13]. The reader unfamiliar with prolongations of exterior differential systems can keep in mind the natural submersion of $J^2(\mathbb{R}^n, \mathbb{R})$ onto $J^1(\mathbb{R}^n, \mathbb{R})$ (equipped with the corresponding contact structures) as the canonical example. The prolongations I describe here are the EDS analogue of prolongations in classical PDE theory, where one formally adds higher derivatives as new variables in order to obtain an equivalent system of equations that may be easier to work with.

Consider an involutive exterior differential system $(M^{(0)}, \mathcal{I}^{(0)})$ with an independence condition. Its first prolongation $(M^{(1)}, \mathcal{I}^{(1)})$ is a new EDS, which has the same solutions but 'sees' more of their derivatives. It comes naturally equipped with a submersive EDS morphism

$$\pi^{(1)} \colon M^{(1)} \to M^{(0)}$$

such that any solution in $M^{(0)}$ lifts to a unique solution in $M^{(1)}$. This lifting map gives a local bijection between solution manifolds. (Implicit in this discusion is the fact that the independence condition on $M^{(0)}$ pulls back to one on $M^{(1)}$.) It is a theorem[1] that the involutivity of $M^{(0)}$ guarantees the involutivity of $M^{(1)}$.

The rth prolongation—$(M^{(r)}, \mathcal{I}^{(r)})$—of $M^{(0)}$ is defined inductively to be the first prolongation of $M^{(r-1)}$. This generates a tower of submersive EDS morphisms such that solutions at any level are in bijection with solutions at any other level.

For example, consider a parabolic system M_0 embedded into $J^2(\mathbb{R}^{n+1}, \mathbb{R})$, so that points in M_0 are identified with 2-jets. The rth prolongation of M_0 naturally embeds into $J^{2+r}(\mathbb{R}^{n+1}, \mathbb{R})$, the rth-prolongation of $J^2(\mathbb{R}^{n+1}, \mathbb{R})$. Correspondingly, points of $M_0^{(r)}$ may be considered as $(r+2)$-jets.

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[1] See [BCG$^+$13], Chapter VI.

To avoid restricting the number of derivatives I can discuss, I will work on the infinite prolongation of M_0. This is described in more detail in [BG95a], but I recall some details here.

The infinite prolongation $(M^{(\infty)}, \mathcal{I}^{(\infty)})$ of an exterior differential system $M^{(0)}$ is the inverse limit of its prolongation tower. More precisely, the manifold $M^{(\infty)}$ is given by

$$M^{(\infty)} = \varprojlim M^{(r)},$$

the inverse limit of the underlying tower of manifolds. The ideal that makes $M^{(\infty)}$ into an EDS is by definition

$$\mathcal{I}^{(\infty)} = \bigcup_{r=0}^{\infty} \mathcal{I}^{(r)}.$$

The manifold $M^{(\infty)}$ will generally be infinite dimensional, but will be treated formally, so this will not require any special machinery. In particular, I will study conservation laws of *finite type*, which are certain forms that pull back from a finite prolongation of M_0. The reason for working on the infinite prolongation is to avoid specifying which prolongation in advance.

Accordingly, I will only need to consider finite-type function on $M^{(\infty)}$, those that can be expressed as the pullback of a function on $M^{(r)}$ for some r. The space of finite-type functions is given by

$$\bigcup_{r=0}^{\infty} \mathcal{C}^{\infty}\left(M^{(r)}\right).$$

In general, I will omit the pullback in my notation and simply refer to functions on $M^{(\infty)}$ as if they were functions on some $M^{(r)}$. I will also consider the finite-type differential forms, setting

$$\Omega^*(M^{(\infty)}) = \bigcup_{r=0}^{\infty} \Omega^*(M^{(r)}).$$

From this point on, I will denote the infinite prolongation of a parabolic system $(M_0, \mathcal{I}_0)$ by $(M, \mathcal{I})$. I will also denote the space of finite-type functions on M by $\mathcal{C}^{\infty}$.

The following proposition describes the structure of $(M, \mathcal{I})$. Although the statement is long, it follows immediately from the general theory of exterior differential systems, and in particular from the calculation that the involutive tableaux

$$K = \operatorname{Sym}_0^2 W' \oplus W' \oplus \mathbb{R}$$

(in this context, a tableaux is a linear subspace of $W \otimes W$) has r^{th} prolongation

$$K^{(r)} := (K \otimes W^{\otimes r}) \cap (W \otimes \operatorname{Sym}^{r+1} W) \cong \bigoplus_{s+t=r+2} \operatorname{Sym}_0^s W' \otimes \operatorname{Sym}^t(W/W').$$

Here $\operatorname{Sym}_0^s W'$ is the trace-free symmetric power of W'.

Recall that W comes equipped with a basis e_a so that the subspace W' has basis e_i. Then

$$\operatorname{Sym}^r W = \bigoplus_{s+t=r} \operatorname{Sym}^s W' \otimes \operatorname{Sym}^t\{e_0\}.$$

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Fix the component of $\operatorname{Sym}^r W$ indexed by a choice of s and t. For each multi-index $I = (i_1 \ldots i_s)$ in integers 1 through n, let

$$e_{I,t} = e_{i_1} \circ \ldots \circ e_{i_s} \circ e_0 \circ \ldots \circ e_0,$$

the symmetric product with t copies of e_0. The set of these form a basis for

$$\operatorname{Sym}^s W' \otimes \operatorname{Sym}^t \{e_0\}.$$

On this component, the basis $e_{I,t}$ defines linear coordinate functions $p_{I,t}$. I denote the restriction of $p_{I,t}$ to the trace-free subspace

$$\operatorname{Sym}_0^s W' \otimes \operatorname{Sym}^t \{e_0\}$$

by the same name.

I will use the action of W on $\operatorname{Sym}^* W$, which is essentially just multiplication of polynomials. In the given basis,

$$e_i \cdot e_{I,t} := e_{Ii,t}$$
$$e_0 \cdot e_{I,t} := e_{Ii,t+1}.$$

I will also use the action of $W^\vee$ on $\operatorname{Sym}^* W$, which is essentially a directional derivative. In the given basis,

$$e^i \cdot e_{I,t} := \sum_{i \in I} e_{I \setminus i,t}$$
$$e^0 \cdot e_{I,t} := t e_{I,t-1}.$$

These can be used to define a particularly useful operator, the *trace* on W', by

$$\operatorname{tr} = e^i e^i.$$

These actions induce ones on the linear functions $p_{I,t}$, as well as their differentials. In Proposition 1 I use the functions $p_{I,t}$ to define extended coframings $\theta_{I,t}$ of M, and I extend the action of W to this coframing. In particular, I employ the notation

$$e_i \theta_{I,t} = \theta_{Ii,t}$$
$$e^i \theta_{I,t} = \sum_{i \in I} \theta_{I \setminus i,t}.$$

For consistency with the coframings of M, I denote the pullbacks of $\theta_\varnothing, \theta_a$ to M by

$$\theta_{0,0} = \theta_\varnothing$$
$$\theta_{i,0} = \theta_i$$
$$\theta_{0,1} = \theta_0.$$

Proposition 1 (Principal Structure Equations). *For each $r \geq 0$, the prolongation $M^{(r+1)}$ is an affine-space bundle over $M^{(r)}$, fitting into the sequence*

$$\bigoplus_{s+t=r+3} \operatorname{Sym}_0^s W' \otimes \operatorname{Sym}^t \{e_0\} \longhookrightarrow M^{(r+1)} \xrightarrow{\pi^{(r+1)}} M^{(r)}.$$

Each fiber has affine functions $p_{I,t}$ for all I and t such that $|I| + t = r + 3$. These functions are subject to the trace relations

$$p_{Iii,t} = 0 \qquad \forall\, I, t \text{ s.t. } |I| + t = r + 1.$$

At each point $x \in M^{(r+1)}$, this fiber sequence induces the exact sequence

$$0 \longrightarrow T^\vee_{\pi^{(r+1)}(x)} M^{(r)} \longrightarrow T^\vee_x M^{(r+1)} \longrightarrow \bigoplus_{s+t=r+3} (\operatorname{Sym}_0^s W' \otimes \operatorname{Sym}^t \{e_0\})^\vee \longrightarrow 0.$$

In particular, any choice of 1-forms $\underline{\pi}_{I,t}$ so that

$$\underline{\pi}_{I,t} \equiv dp_{I,t} \quad \left(\operatorname{mod} \mathcal{I}^{(r)}, \omega^a\right)$$

completes a coframing of $M^{(r)}$ to an (extended) coframing of $M^{(r+1)}$, subject to the relations

$$\underline{\pi}_{Iii,t} \equiv 0 \quad \left(\operatorname{mod} \mathcal{I}^{(r)}, \omega^a\right) \quad \forall\, I, t \text{ s.t. } |I| + t = r + 1.$$

The ideal $\mathcal{I}^{(r+1)}$ is generated by $\mathcal{I}^{(r)}$ and the forms

$$\underline{\theta}_{I,t} = \underline{\pi}_{I,t} - p_{Ii,t}\underline{\omega}^i - p_{I,t+1}\underline{\omega}^0, \quad \forall\, I, t \text{ s.t. } |I| + t = r + 2.$$

By induction, one finds on $M^{(r+1)}$ the structure equations

$$d\underline{\theta}_{I,t} \equiv -\underline{\pi}_{Ii,t} \wedge \underline{\omega}^i - \underline{\pi}_{I,t+1} \wedge \underline{\omega}^0 \quad \left(\operatorname{mod} \underline{\theta}_{I',t'} \text{ s.t. } |I'| + t' \leq |I| + t\right)$$

when $|I| + t = r + 2$ and

$$d\underline{\theta}_{I,t} \equiv -\underline{\theta}_{Ii,t} \wedge \underline{\omega}^i - \underline{\theta}_{I,t+1} \wedge \underline{\omega}^0 \quad \left(\operatorname{mod} \underline{\theta}_{I',t'} \text{ s.t. } |I'| + t' \leq |I| + t\right).$$

when $|I| + t < r + 2$.
On M, all the structure equations are of the form

$$d\underline{\theta}_{I,t} \equiv -\underline{\theta}_{Ii,t} \wedge \underline{\omega}^i - \underline{\theta}_{I,t+1} \wedge \underline{\omega}^0 \quad \left(\operatorname{mod} \underline{\theta}_{I',t'} \text{ s.t. } |I'| + t' \leq |I| + t\right).$$

One can see from the structure equations on M that the differential ideal $\mathcal{I}$ is formally Frobenius, that is,

$$\mathcal{I} = \{\underline{\theta}_{I,t}\}_{alg}.$$

This is true for the infinite prolongation of any exterior differential system. Note also that on the infinite prolongation there are no "$\underline{\pi}_{I,t}$" forms:

$$\Omega^*(M) = \{\underline{\omega}^a\} \cup \mathcal{I}.$$

For each r, the G-structure of adapted coframes of $M^{(r)}$ is closely related to $\mathcal{B}_0$. Its structure group is an extension of G_0 by a nilpotent group, which corresponds to the freedom to add multiples of $\theta_{I',t'}$ to $\theta_{I,t}$ (or $\underline{\pi}_{I,t}$) if $|I'|+t' < |I|+t$ without altering the structure equations. This phenomenon can already be seen by comparing the structure group of a Monge-Ampère system to that of its prolongation.

One can define the *full adapted coframe bundle* of M by taking the inverse limit of these bundles, but I will only require a simpler notion. A coframing of M_0 already determines the *principal part* of a coframing on M—the part of the coframing that is independent of the nilpotent translations. To do coframe equivariant calculations it will suffice to work on the *principal adapted coframe bundle*, which is simply the pullback of $\mathcal{B}_0$ to M. Henceforth, I will use $\mathcal{B}_0(M_0)$ to denote the 0-adapted coframe bundle on M_0 and, abusing earlier notation, use $\mathcal{B}_0$ to denote the principal adapted coframe bundle on M. More generally, I will use $\mathcal{B}_0(M^{(r)})$ to denote the pullback of $\mathcal{B}_0(M_0)$ to the rth prolongation of M_0.

Likewise, for a parabolic system with reduction to a G_1-structure (or G_2-structure, etc.), I will denote the pullback of $\mathcal{B}_1$ to $M^{(s)}$ by $\mathcal{B}_1(M^{(s)})$ and the pullback to M by $\mathcal{B}_1$.

3.2 The $\mathcal{C}$-spectral sequence

For an infinitely prolonged parabolic system $(M,\mathcal{I})$, let $\overline{\Omega}^*$ be the chain complex defined by the exact sequence

$$0 \longrightarrow \mathcal{I}^* \longrightarrow \Omega^*(M) \longrightarrow \overline{\Omega}^* \longrightarrow 0.$$

Note that the differential on $\Omega^*(M)$ descends to a well defined differential d_h on $\overline{\Omega}^*$. Following Bryant and Griffiths [BG95a], I make the following definition.

Definition 13. The *characteristic cohomology* $\overline{H}^*$ of an infinitely prolonged parabolic system $(M,\mathcal{I})$ is the cohomology of the complex $\overline{\Omega}^*$,

$$\overline{H}^* = H^*(\overline{\Omega}^*, d_h).$$

These cohomology groups have various interpretations in terms of the geometry of solutions to $(M,\mathcal{I})$. The 'top' cohomology, $\overline{H}^{n+1}$, is the space of functionals on maximal integral manifolds: it consists of the $n + 1$ forms modulo exact forms and forms in $\mathcal{I}$, both classes which integrate to zero on any closed integral manifold.

The next cohomology group can be interpreted as the space of conservation laws, which is the reason for the following definition.

Definition 14. The *space of conservation laws* for a parabolic system is given by the degree n characteristic cohomology,

$$\overline{\mathscr{C}} = \overline{H}^n(M).$$

This is a reasonable definition: an element $\Phi \in \overline{\mathscr{C}}$ is an n-form on M whose exterior derivative is in $\mathcal{I}$. Consequently, the restriction of Φ to any integral manifold is a closed form. By Stokes Theorem, the integral of Φ over the boundary of any integral manifold is zero. For an evolutionary equation this means that the integral of Φ over one time slice agrees with its integral over a later time slice. Or, restricting attention to a region, the integral across its boundary measures the 'divergence' of the conserved quantity that Φ represents. This is exactly the behavior that a local conservation law should have. As with top forms, Φ is defined modulo exact forms and elements of $\mathcal{I}$, both of which are trivial as conservation laws.

For parabolic systems, $\overline{H}^*$ vanishes in the remaining degrees. Any form of degree larger than $n+1$ is automatically in $\mathcal{I}$, so $\overline{\Omega}^{>n+1} = 0$ even before taking homology. For $q < n$ the vanishing of $\overline{H}^q$ follows from results quoted below.

The cohomology of the differential operator d_h is difficult to compute directly. However, the characteristic cohomology fits into a spectral sequence. In turn, the first page of this spectral sequence can be computed using a second spectral sequence. This second spectral sequence is quite amenable to calculations. In fact, its first page is determined by the symbol alone, and the differential linearizes to a function of vector bundles.

Consider the filtration of $\Omega^*(M)$ defined recursively by

$$\mathrm{F}^0 = \Omega^*(M)$$
$$\mathrm{F}^{p+1} = \mathcal{I} \wedge \mathrm{F}^p .$$

Define the *associated graded* spaces to this filtration by

$$Gr^{p,*} := \mathrm{F}^p / \mathrm{F}^{p+1} .$$

Here and in the following, the elements of $Gr^{p,q}$ have degree $(p+q)$ as differential forms. The exterior derivative descends to a well defined operator d_h on each complex, which is used to define the *filtration spectral sequence*, with page 0 given by

$$E_0^{p,*} = Gr^{p,*}.$$

It follows immediately from the definition that

$$E_0^{0,*} = \overline{\Omega}^*,$$

and thus that

$$E_1^{0,*} = \overline{H}^*.$$

Note that

$$E_\infty \Rightarrow H^*(M),$$

so for sufficiently small neighborhoods the spectral sequence converges to zero. Since I am concerned with locally defined conservation laws, I henceforth assume that M is contractible, restricting attention to a neighborhood if necessary.

It follows from the two-line theorem of Vinogradov, [Vin99], or the more general theory of characteristic cohomology, [BG95a], that for parabolic systems, most of the terms in this spectral sequence vanish at the first page. Specifically,

$$E_1^{p,q} = 0 \quad \text{for} \quad q < n,$$

so the E_1 page is

$$0 \longrightarrow \overline{H}^{n+1} \xrightarrow{\ d_v\ } E_1^{1,n+1} \xrightarrow{\ d_v\ } E_1^{2,n+1} \xrightarrow{\ d_v\ } \dots$$

$$0 \longrightarrow \overline{H}^{n} \xrightarrow{\ d_v\ } E_1^{1,n} \xrightarrow{\ d_v\ } E_1^{2,n} \xrightarrow{\ d_v\ } \dots$$

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$$

One immediate consequence is that the bottom row is exact. In particular, $\overline{H}^n$ is isomorphic to the kernel of d_v in $E_1^{1,n}$. This motivates the following definition.

Definition 15. The *space of differentiated conservation laws* is

$$\mathscr{C} = \ker\left(E_1^{1,n} \xrightarrow{\ d_v\ } E_1^{2,n} \right)$$

The operator d_v provides an isomorphism between $\overline{\mathscr{C}}$ and $\mathscr{C}$, which is useful, because the latter space may be computed in two steps: First one computes $E_1^{1,n}$, and then one computes the kernel of d_v.

The reason for calling the zeroth page differential "d_h" and the first page differential "d_v" is explained in section 3.4.

3.3 The weight filtrations

Still following Bryant and Griffiths, I introduce the *principal weight filtration*, defined on the associated gradeds $\mathrm{Gr}^{p,*}$ with $p > 0$. For parabolic systems, the principal weight filtration doesn't see 'lower order' information, such as the Goursat invariant, which means it doesn't detect the sub-principal symbol. So, I also introduce the *sub-principal weight filtration*, which does detect the Goursat invariant. Theorem 5 below shows that the existence of conservation laws for parabolic systems is controlled by the sub-principal symbol, and the proof relies heavily on the sub-principal weight filtration.

Definition 16. A *weight function* is a function $\mathrm{wt}\colon \Omega^*(M) \to \mathbb{Z}$ satisfying the following properties:

1. $\mathrm{wt}(f) = 0$ for $f \in \mathcal{C}^\infty$.

2. $\mathrm{wt}(\alpha \wedge \beta) = \mathrm{wt}(\alpha) + \mathrm{wt}(\beta)$ for $\alpha, \beta \in \Omega^*(M)$.

3. $\mathrm{wt}(\alpha + \beta) = \max(\mathrm{wt}(\alpha), \mathrm{wt}(\beta))$.

Weight functions are used to define the weight filtrations, as in the following examples.

Example 7 (The Principal Weight Filtration). Consider the weight function pwt on $\Omega^*(M)$, uniquely specified by

$$\mathrm{pwt}(\omega^a) = -1$$
$$\mathrm{pwt}(\theta_{I,t}) = |I| + t.$$

For each integer k, define the subspace

$$\mathrm{F}_k^{p,*} = \{\alpha \in \mathrm{F}^p \colon \mathrm{pwt}(\alpha) \leq k\} / \mathrm{F}^{p+1}$$

of $\mathrm{Gr}^{p,*}$, the forms of principal weight less than k. The *principal weight filtration* on $\mathrm{Gr}^{p,*}$ is defined by the sequence

$$\ldots \subset \mathrm{F}_k^{p,*} \subset \mathrm{F}_{k+1}^{p,*} \subset \ldots \subset \mathrm{Gr}^{p,*}.$$

It follows immediately from the principal structure equations that d_h restricts to make each $\mathrm{F}_k^{p,*}$ into a subcomplex.

For each fixed $p > 0$ there is a spectral sequence associated to this filtration, with $\overline{E}_0$ page given by

$$\overline{E}_0^{*,k} = \mathrm{F}_k^{p,*} / \mathrm{F}_{k-1}^{p,*}.$$

This spectral sequence converges to $E_1^{p,*}$.

Any element of $\overline{E}_0^{*,k}$ can be represented by a linear combination of the forms

$$\underline{\theta}_{I_1,t_1} \wedge \ldots \wedge \underline{\theta}_{I_p,t_p} \wedge \underline{\omega}^{a_1} \wedge \ldots \wedge \underline{\omega}^{a_q}$$

of principal weight k. Explicitly, this is the condition that

$$|I_1| + t_1 + \ldots + |I_p| + t_p - q = k.$$

Note that $\overline{E}_0^{*,k}$ is the space of sections of a vector bundle $\overline{\mathcal{E}}_0^{*,k}$ spanned by these forms.

I will abuse notation slightly, also using F_k^p to denote

$$\mathrm{F}_k^p = \{\alpha \in \mathrm{F}^p \colon \mathrm{pwt}(\alpha) \leq k\} \cup \mathrm{F}^{p+1},$$

as well as the notation

$$\mathrm{F}_k^{p,d} = \mathrm{F}_k^p \cap \Omega^d(M)$$

for the forms in F_k^p of degree d. This will allow me to consider an element of $\overline{E}_0^{*,k}$ as any form representing it, modulo F_{k-1}^p. Because the distinct usages occur in different contexts, this should not cause any confusion. Further abusing notation, I will also use F_k^p to denote its pullback to the

principal coframe bundle of M. This will also not be a problem, because F_k^p is invariant under changes of adapted coframing.

The exterior derivative is $\mathcal{C}^\infty$-linear on $\overline{E}_0^{*,k}$, and treats the forms $\underline{\omega}^a$ as constants. Indeed, by $\mathbb{R}$-linearity, it suffices to check that

$$d_h(f\underline{\theta}_{I_1,t_1} \wedge \ldots \wedge \underline{\theta}_{I_p,t_p} \wedge \underline{\omega}^{a_1} \wedge \ldots \wedge \underline{\omega}^{a_q})$$
$$\equiv f\, d_h(\underline{\theta}_{I_1,t_1} \wedge \ldots \wedge \underline{\theta}_{I_p,t_p}) \wedge \underline{\omega}^{a_1} \wedge \ldots \wedge \underline{\omega}^{a_q} \quad (\mathrm{mod}\ \mathrm{F}_{k-1}^{p,*}).$$

But this follows from the observation that d_h decreases weight for functions and for each $\underline{\omega}^a$:

$$d_h f \equiv f_a \underline{\omega}^a \quad (\mathrm{mod}\ \mathrm{F}^1),$$

for functions f_1, so

$$\mathrm{pwt}(d_h f) = -1$$

(unless it happens that $d_h f = 0$) and

$$d_h \underline{\omega}^a \equiv M_{bc}^a \underline{\omega}^b \wedge \underline{\omega}^c \quad (\mathrm{mod}\ \mathrm{F}^1),$$

for functions M_{bc}^a, so

$$\mathrm{pwt}(d_h \underline{\omega}^a) = -2$$

(likewise).

The $\mathcal{C}^\infty$-linearity of d_h means that it is induced from a morphism of vector bundles. That is, each map

$$\overline{E}_0^{*,k} \xrightarrow{\ d_h\ } \overline{E}_0^{*,k}$$

comes from a vector bundle map

$$\overline{\mathcal{E}}_0^{*,k} \longrightarrow \overline{\mathcal{E}}_0^{*,k}.$$

This means that to compute the next page, $\overline{E}_1^{*,k}$, it suffices to determine the vector bundle map.

Example 8 (The Sub-Principal Weight Filtration)**.** The *sub-principal weight function* is the unique weight function such that

$$\mathrm{wt}(\omega^i) = -1$$
$$\mathrm{wt}(\omega^0) = -2$$
$$\mathrm{wt}(\theta_{I,t}) = |I| + 2t.$$

For each integer k, define the subspace

$$\mathcal{F}_k^p = \{\alpha \in \mathrm{F}^p \colon\ \mathrm{wt}(\alpha) \le k\}/\mathrm{F}^{p+1}.$$

of $\mathrm{Gr}^{p,*}$, the forms of sub-principal weight less than k. The *sub-principal weight filtration* on $\mathrm{Gr}^{p,*}$ is defined by the sequence

$$\ldots \subset \mathcal{F}_k^p \subset \mathcal{F}_{k+1}^p \subset \ldots \subset \mathrm{Gr}^{p,*}.$$

In order for d_h to restrict to a differential on the associated graded spaces

$$\mathcal{F}_k^{p,*} / \mathcal{F}_{k-1}^{p,*},$$

it is necessary that each $\mathcal{F}_k^p$ be d_h-closed. Although no longer a trivial fact, this follows from Troposition 3, which shows that d_h does not increase sub-principal weight. Note also that each subspace $\mathcal{F}_k^p$ is coframe equivariant.

The operator d_h restricts to a C^∞-linear operator on $\mathcal{F}_k^{p,*} / \mathcal{F}_{k-1}^{p,*}$. However, if the Goursat invariant is non-zero, then it no longer treats ω^0 as a constant, because

$$d_h\omega^0 \equiv \mathcal{G}_{ij}\omega^i \wedge \omega^j \quad \left(\mathrm{mod}\ \mathcal{F}_{-3}^0\right).$$

As with the principal weight filtration, it will be useful to have the notation

$$\mathcal{F}_k^p = \{\alpha \in \mathrm{F}^p \colon \mathrm{wt}(\alpha) \leq k\} \cup \mathrm{F}^{p+1}$$

and

$$\mathcal{F}_k^{p,d} = \mathcal{F}_k^p \cap \Omega^d(M).$$

3.4   Horizontal and vertical derivatives

The differentials d_h and d_v in the characteristic spectral sequence have natural geometric interpretations, which I describe here. They are the same operators as defined in [BG95a], but modified for the situation at hand.

By definition, the operator d_h is the restriction of d to the associated graded $\mathrm{Gr}^{p,*}$. It has a more "down to earth" interpretation: for $[\alpha] \in \mathrm{Gr}^{p,*}$ represented by $\alpha \in \mathrm{F}^p$, the derivative $d_h[\alpha]$ is represented by $d\alpha$. For φ an element of F^p, I will abuse notation, using $d_h\varphi$ to denote the exterior derivative of φ modulo F^{p+1}, so that

$$d_h\varphi \equiv d\varphi \quad \left(\mathrm{mod}\ \mathrm{F}^{p+1}\right).$$

Used this way, d_h is no different from d, but implies that a calculation fits into the characteristic spectral sequence.

The operator d_h is *horizontal* with respect to any solution manifold Σ, in the sense of the relation

$$(d_h\varphi)|_\Sigma = d(\varphi|_\Sigma)$$

for $\varphi \in \Omega^*(M)$. Note that this fact is trivial for $\varphi \in \mathcal{I}$.

In a given coframing[2]

$$\underline{\omega}^a, \underline{\theta}_\varnothing, \underline{\theta}_{i,0}, \underline{\theta}_{0,1}, \ldots \tag{3.1}$$

of M, the relation

$$d_h\underline{A} \equiv (\underline{D}_a\underline{A})\underline{\omega}^a \quad (\mathrm{mod}\ \mathcal{I})$$

---

[2]More precisely, only a partial coframing $\underline{\omega}^a$ is necessary here.

defines the operators $\underline{D}_a$ for functions $\underline{A} \in \mathcal{C}^\infty$. These are *total directional derivatives* of $\underline{A}$ in the sense that

$$d(A|_\Sigma) = (D_a A \omega^a)|_\Sigma.$$

The operators $\underline{D}_a$ can be expressed coframe equivariantly: letting A denote the pullback of $\underline{A}$ to $\mathcal{B}_0$, the operators D_a are given by

$$d_h A \equiv (D_a A) \omega^a \quad (\text{mod } \mathcal{I}).$$

These operators will be used to define the differential equation that governs conservation laws of parabolic systems.

Now consider the *vertical derivative* d_v. Given a form $\varphi \in \mathrm{F}^p$ so that

$$d_h \varphi \equiv 0 \quad (\text{mod } \mathrm{F}^{p+1}),$$

the element $d_v \varphi$ of F^{p+1} is defined by the equation

$$d_v \varphi \equiv d\varphi \quad (\text{mod } \mathrm{F}^{p+2}).$$

The operator dv on the spectral sequence is only well defined for forms so that $d_h \varphi = 0$, but it is possible to extend d_v to general forms, which I describe.

As a warmup, I define the vertical derivative $\underline{d}_v$ in a fixed coframing. The forms $\underline{\omega}^a$ in this coframing are the *horizontal* forms (for this coframing), because the ideal defined by $\underline{J} = \{\underline{\omega}^a\}$ provides a complement to the ideal $\mathcal{I}$. As a consequence of this splitting of $\Omega^*(M)$, any element $[\varphi] \in \mathrm{Gr}^{p,q}$ is represented by a form

$$\varphi \in (\Lambda^q \underline{J}) \wedge (\Lambda^p \mathcal{I}).$$

Then, even when $d_h \varphi$ is not zero on the nose, it is true that

$$d_h \varphi \in (\Lambda^{q+1} \underline{J}) \wedge (\Lambda^p \mathcal{I}),$$

or in other words,

$$d_h \varphi \equiv 0 \quad (\text{mod } \Lambda^{q+1} \underline{J}, \mathrm{F}^p).$$

For this reason, I define $\underline{d}_v$ by the congruence

$$\underline{d}_v \varphi \equiv d\varphi \quad (\text{mod } \Lambda^{q+1} \underline{J}, \mathrm{F}^{p+2}),$$

which is represented by an element of

$$(\Lambda^q \underline{J}) \wedge (\Lambda^{p+1} \mathcal{I}).$$

This operator clearly agrees with the original definition of $\underline{d}_v$ on d_h-closed forms. In fact, it provides a partial complement to d_h, in the sense that

$$d\varphi \equiv d_h \varphi + \underline{d}_v \varphi \quad (\text{mod } \mathrm{F}^{p+2}).$$

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For $\varphi \in \mathrm{Gr}^{p,q}$, I define the coframe-equivariant vertical derivative d_v by the formula

$$d_v \varphi \equiv d\varphi \quad \left(\mathrm{mod}\ \mathrm{F}^{p+2}, \Lambda^{q+1}\mathcal{J}\right).$$

Here, to maintain coframe equivariance, it is necessary to replace the horizontal ideal $\underline{J}$ with $\mathcal{J}$. As a consequence, the coframe equivariant operator d_v misses some information that $\underline{d}_v$ does not. This may seems disadvantageous, but I argue that it is a benefit. When using the coframe dependent operator $\underline{d}_v$, one has to keep track of what is true in general and what is true for the specific coframing. The equivariant d_v keeps track for free.

The operator d_v does run into trouble if φ is already in $\Lambda^{q+1}\mathcal{J}$. Indeed, $\Lambda^{q+1}\mathcal{J}$ is differentially closed, so in this case

$$d_h \varphi \equiv 0 \quad \left(\mathrm{mod}\ \Lambda^{q+1}\mathcal{J}\right).$$

Still, given any coframe-equivariant ideal $\mathcal{K}$ for which

$$d_h \varphi \equiv 0 \quad \left(\mathrm{mod}\ \mathcal{K}, \mathrm{F}^{p+1}\right)$$

it makes sense to define

$$d_v \varphi \equiv d\varphi \quad \left(\mathrm{mod}\ \mathcal{K}, \mathrm{F}^{p+2}\right).$$

The operator d_v defines *vertical derivatives* for functions $\underline{A} \in \mathcal{C}^\infty$. Indeed, fixing the coframing (3.1), the relation

$$d_v \underline{A} \equiv d\underline{A} \quad (\mathrm{mod}\ \mathcal{J})$$

can be written as

$$d_v \underline{A} \equiv \sum_{I,t} (\underline{\partial}_{I,t}\, \underline{A})\underline{\theta}_{I,t} \quad (\mathrm{mod}\ \mathcal{J}),$$

defining the differential operators $\underline{\partial}_{I,t}$ on $\mathcal{C}^\infty$ for all $|I| + t \geq 2$. Just as with the operators D_a, coframe equivariant operators $\partial_{I,t}$ can be defined by pulling $\underline{A}$ up to $\mathcal{B}_0$.

In jet coordinates, each differential operator $\underline{\partial}_{I,t}$ is given by the partial derivative $\frac{\partial}{\partial p_{I,t}}$ plus partial derivatives in coordinates $p_{I',t'}$ of strictly *higher* weight[3]. To see this, recall from the principal structure equations that, for indices I and t such that $|I| + t > 2$,

$$\underline{\theta}_{I,t} \equiv dp_{I,t} \quad \left(\mathrm{mod}\ \mathcal{J}, \mathrm{F}^1_{|I|+t-1}\right).$$

Now consider a function $\underline{A} \in \mathcal{C}^\infty\left(M^{(r)}\right)$, so that

$$d_v A \equiv \sum_{|I|+t=r+2} \frac{\partial A}{\partial p_{I,t}}\, dp_{I,t} \equiv \sum_{|I|+t=r+2} (\underline{\partial}_{I,t}A)\underline{\theta}_{I,t} \quad \left(\mathrm{mod}\ \mathcal{J}, \mathrm{F}^1_{r+1}\right).$$

Comparing coefficients shows that each $\underline{\partial}_{I,t}$ equals $\frac{\partial}{\partial p_{I,t}}$ for any function that doesn't depend on coordinates $p_{I',t'}$ of weight higher than r.

---

[3]This may be surprising, but it comes down to the nilpotent component of the full adapted structure group, the fact that adding low weight forms in $\mathcal{I}$ to higher weight forms preserves the principal structure equations.

In particular, this shows that the operator d_v gives a test for when a function on $M^{(r)}$ is the pullback of a function on $M^{(r-1)}$. Indeed, for $\underline{A} \in \mathcal{C}^\infty \left(M^{(r)} \right)$, if

$$d_v \underline{A} \equiv 0 \quad \left(\mathrm{mod} \ \mathcal{J}, \mathrm{F}^1_{r+1} \right),$$

then it is constant in each fiber of $M^{(r)}$ over $M^{(r-1)}$.

Finally, while d_v can't be defined for general functions on $\mathcal{B}_0$, in special cases it can. To choose an example that will be relevant, consider the form

$$\underline{\Phi} = \underline{A}\underline{\theta}_{i,0} \wedge \underline{\omega}_{(i)}$$

given in a fixed coframing $\underline{\eta}$. The pullback of $\underline{\Phi}$ to $\mathcal{B}_0$ may be written as

$$\Phi = A\theta_{i,0} \wedge \omega_{(i)},$$

where $A \in \mathcal{C}^\infty$ agrees with $\underline{A}$ along the image of $\underline{\eta}$. By a calculation similar to the one used to show (2.10), one finds that

$$d \left(\theta_{i,0} \wedge \omega_{(i)} \right) \equiv \left(2\beta_{tr} - n\kappa_0 + \beta^0_0 \right) \wedge \theta_{i,0} \wedge \omega_{(i)} \quad \left(\mathrm{mod} \ \theta_{0,0}, \Omega^{d+2}_{sb} \right).$$

Since Φ is basic, the variation of A in each fiber must be given by

$$dA \equiv \left(-2\beta_{tr} + n\kappa_0 - \beta^0_0 \right) A \quad \left(\mathrm{mod} \ \Omega^1_{sb} \right).$$

Observe though, that $\theta_{i,0} \wedge \omega_{(i)}$ is defined on M_0, so the variational form $2\beta_{tr} - n\kappa_0 + \beta^0_0$ is defined on $\mathcal{B}_0(M_0)$. Consequently, for any coframing $\widetilde{\underline{\eta}}$,

$$\widetilde{\underline{\eta}}^*(2\beta_{tr} - n\kappa_0 + \beta^0_0) \equiv 0 \quad \left(\mathrm{mod} \ F^1_2 \right)$$

and thus

$$d\underline{A} \equiv \sum_{I,t}(\underline{\partial}_{I,t}\underline{A})\theta_{I,t} \quad \left(\mathrm{mod} \ \mathcal{J}, F^1_2 \right).$$

I will abuse notation and write

$$d_v A \equiv \sum_{I,t}(\underline{\partial}_{I,t}\underline{A})\theta_{I,t} \quad \left(\mathrm{mod} \ \mathcal{J}, F^1_2 \right)$$

when working on $\mathcal{B}_0$. This level of generality for the definition of d_v is necessary in the proof of Theorem 5.

Recall that a bi-complex with differentials d_1 and d_2 so that $d = d_1 + d_2$ has the useful identity

$$d_1 \, d_2 = - \, d_2 \, d_2.$$

Now, d_h and d_v don't quite split d like this, but on sufficiently nice functions they do split it above a finite weight. Indeed, let A be a function on $\mathcal{B}_0$ so that

$$d_v A \equiv dA \quad \left(\mathrm{mod} \ \mathcal{J}, \mathrm{F}^1_N \right)$$

is well defined. Then

$$dA \equiv d_h A + d_v A \quad \left(\text{mod } \mathrm{F}_N^1\right)$$

and thus

$$0 = d^2 A \equiv d_v(D_a A \wedge \omega^a) + d_h \left(\sum_{|I|+s>N} (\partial_{I,s} A)\theta_{I,s} \right) \quad \left(\text{mod } \mathrm{F}_N^1\right).$$

If $N > 3$, then

$$d_v \omega^a \equiv 0 \quad \left(\text{mod } \mathrm{F}_N^1\right),$$

so the previous equation simplifies to

$$0 \equiv d_v(D_a A) \wedge \omega^a - \sum_{|I|+s>N} (\partial_{I,s} A) e_a \theta_{I,s} \wedge \omega^a \quad \left(\text{mod } \mathrm{F}_N^1\right)$$

and thus

$$d_v(D_a A) \equiv \sum_{|I|+s>N} (\partial_{I,s} A) e_a \theta_{I,s} \quad \left(\text{mod } \mathcal{J}, \mathrm{F}_{N+1}^1\right).$$

Finally, I remark that I will without comment use the same notation d_h and d_v for the respective restrictions to the weighted complexes.

3.5   The Refined Structure Equations

I now turn to the refined structure equations for strongly parabolic systems. This refinement makes it possible to take into account the sub-principal symbol when calculating conservation laws.

Theorem 3. *Let $(M, \mathcal{I})$ be the infinite prolongation of a strongly parabolic system. There are local coframings of M as in Proposition 1 for which furthermore*

$$\begin{aligned}
d_h \theta_{I,t} \equiv &- \theta_{Ii,t} \wedge \omega^i - \theta_{I,t+1} \wedge \omega^0 \\
&- c_{|I|} \left(e^i \theta_{I,t+1} \wedge \omega^i - C_{|I|} e_i e^{jj} \theta_{I,t+1} \wedge \omega^i \right) \quad \left(\text{mod } \mathcal{F}_{|I|+2t-1}^1\right).
\end{aligned} \tag{3.2}$$

Here

$$c_s = \begin{cases} \dfrac{n}{n+2(s-1)} & s > 0 \\ 0 & s = 0 \end{cases}$$

and

$$C_s = \frac{1}{n + 2s - 4}$$

Corollary 1. *The exterior derivative does not increase sub-principal weight. In particular, the subspace $\mathrm{F}_{s+t}^p \cap \mathcal{F}_{s+2t}^p$ of $\mathrm{Gr}^{p,*}$ is differentially closed for any s and t.*

Remark 1. The new terms can be predicted from the representation theory of $CO(n)$. Indeed, one should not expect reductions of the structure group to appear suddenly after a large number of prolongations. This is true a posteriori. The non-existence of such coframe reductions is equivalent to the fact that d_h is G_2-equivariant.

Since d_h is G_2-equivariant, it induces at each point $x \in M$ an equivariant map

$$\operatorname{Sym}_0^s W' \longrightarrow \operatorname{Sym}_0^{s-1} W' \otimes W',$$

where the former space is spanned by vectors $(\theta_{I,t})_x$ with $|I| = s$ and the latter by vectors $(\theta_{I,t+1 \wedge \omega^i})_x$ with $|I| = s - 1$. By Schur's lemma, this map must be a projection to the irreducible component $\operatorname{Sym}_0^s W'$ in $\operatorname{Sym}_0^{s-1} W' \otimes W'$. Explicitly, the map is a multiple of

$$e_I \longmapsto e^j \cdot \left(e_I - \frac{1}{2(n + 2|I| - 4)} e_{ii} e^{kk} e_I \right) \otimes e_j.$$

The second term is necessary to ensure that the result is traceless.

Proof. Suppose the Theorem has been proven for all 1-forms in $\mathcal{I}$ of principal weight less than N. Note that this implies that

$$\mathrm{F}_{N-2}^{1,2} \cap \mathcal{F}_t^{1,2}$$

is differentially closed for each t: Any element is a linear combination of terms $\theta_{I,s \wedge \omega^a}$ such that $\mathrm{pwt}(\theta_{I,s}) \leq N - 1$, so equation (3.2) holds.

Now I proceed by a second induction, in sub-principal weight. Assume that (3.2) holds for 1-forms $\theta_{I,s} \in \mathcal{I}$ such that $\mathrm{pwt}(\theta_{I,s}) = N$ and $\mathrm{wt}(\theta_{I,s}) > 2N - S$. The base case $S = 0$ is

$$d\theta_{0,N} \equiv -\theta_{i,N} \wedge \omega^i - \theta_{0,N+1} \wedge \omega^0 \quad \left(\operatorname{mod} \mathcal{F}_{2N-1}^1 \right),$$

which follows immediately from the principal structure equations, because

$$\mathrm{F}_{N-1}^{1,2} \subset \mathcal{F}_{2N-1}^{1,2}.$$

Observe that the space

$$\mathrm{F}_N^{1,1} \cap \mathcal{F}_{2N-S}^{1,1} / \mathrm{F}_N^{1,1} \cap \mathcal{F}_{2N-S-1}^{1,1}$$

is spanned by the forms $\theta_{I,N-S}$ with $|I| = S$. Recall that the principal structure equations for these forms are

$$d_h \theta_{I,N-S} \equiv -\theta_{Ii,N-S} \wedge \omega^i - \theta_{I,N-S+1} \wedge \omega^0 \quad \left(\operatorname{mod} \mathrm{F}_{N-1}^1 \right).$$

Let (α_I) be a $\operatorname{Sym}_0^S W'$-valued 2-form so that

$$d_h \theta_{I,N-S} \equiv -\theta_{Ii,N-S} \wedge \omega^i - \theta_{I,N-S+1} \wedge \omega^0 - \alpha_I \quad \left(\operatorname{mod} \mathcal{F}_{2N-S-1}^1 \right).$$

By the remark, one expects there to be choices of coframing so that for each I,

$$\alpha_I \equiv c_S \left(e^i \theta_{I,N-S+1} - C_S e_i e^j e^j \theta_{I,N-S+1} \right) \wedge \omega^i$$

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for some function c_S. This is true, but the calculation that proves it is not illuminating, so I omit it. The point is that nothing surprising happens.

With this simplification, it is only left to determine c_S. Fix a multi-index I such that $|I| = S-1$. Then, by the induction hypothesis,

$$d_h\theta_{I,N-S} \equiv -\theta_{Ii,N-S} \wedge \omega^i - \theta_{I,N-S+1} \wedge \omega^0$$
$$- c_{S-1}(e^i\theta_{I,N-S+1} - C_{S-1}e_ie^ke^k\theta_{I,N-S+1}) \wedge \omega^i \pmod{\mathcal{F}^1_{2N-S-2}}.$$

Applying d_h,

$$0 \equiv \alpha_{Ii} \wedge \omega^i + c_{S-1}e_je^i\theta_{I,N-S+1} \wedge \omega^j \wedge \omega^i \pmod{\omega^0, \mathrm{F}^1_{N-3}, \mathcal{F}^1_{2N-S-2}}$$
$$\equiv \left(c_S e^j e_i\theta_{I,N-S+1} - c_S C_S e_j e^k e^k e_i\theta_{I,N-S+1} - c_{S-1}e_ie^j\theta_{I,N-S+1}\right) \wedge \omega^j \wedge \omega^i$$
$$\equiv \left(c_S + 2c_S C_S - c_{S-1}\right)e^je_i\theta_{I,N-S+1} \wedge \omega^j \wedge \omega^i$$

This proves that c_S satisfies the recurrence relation

$$c_S = \frac{n+2S-4}{n+2S-2}c_{S-1}.$$

Because $c_1 = 1$, one finds that

$$c_S = \frac{n}{n+2S-2},$$

as was required. □

3.6 Conservation Laws of Parabolic Systems

The first step in calculating the space of conservation laws for a parabolic system is to compute $E_1^{1,n}$. This is done in the following Theorem, which holds for any parabolic system.

Theorem 4. *For any $\underline{\Phi} \in E_1^{1,n}$, there is a function $\underline{A} \in C^\infty$ and an n-form $\psi_{\underline{A}}$ so that the $\mathcal{I}$-linear piece of $\underline{\Phi}$ is given by*

$$\underline{\Phi}_1 \equiv \underline{A}\Upsilon + \underline{\theta}_{0,0} \wedge \psi_{\underline{A}} \pmod{\mathrm{F}^2},$$

where

$$\Upsilon = \left(\underline{\theta}_{i,0} \wedge \underline{\omega}_{(i)} + \underline{\theta}_{0,0} \wedge (a_{jj}\underline{\omega}_{(0)} + 2a^i_{jj}\underline{\omega}_{(i)})\right).$$

There is a linear first order operator sending $\underline{A}$ to $\psi_{\underline{A}}$, defined by

$$\psi_{\underline{A}} \equiv -(\underline{D}_i\underline{A})\underline{\omega}_{(i)} \pmod{\theta_{0,0}}. \tag{3.3}$$

Furthermore, in any fixed coframing, there is a function $L(\underline{A}, \underline{D}_a\underline{A})$, linear in its arguments and with coefficients determined by the curvatures of M_0, so that the solutions of

$$\mathscr{F}(\underline{A}) = \underline{D}_i\underline{D}_i\underline{A} - L(\underline{A}, \underline{D}_a\underline{A})$$

are in bijection with the elements of $E_1^{1,n}$.

Proof. It suffices to understand the principal weight spectral sequence $\overline{E}_*^{*,k}$ (with $p = 1$), which converges to $E_1^{1,*}$. By the general theory of characteristic spectral sequences, the zeroth page $\overline{E}_0$ is isomorphic to the Spencer complex of the tableaux K. This Spencer complex calculates the minimal free resolution of the symbol module associated to K. See [BG95a], as well as [BCG+13], for details, but the takeaway is that there are no relations between the symbol relations of a parabolic systems, which shows that the following diagram contains the only terms that don't immediately degenerate. Adopting the notation

$$\mathrm{Gr}_k^* = \mathrm{F}_k^{1,*} / \mathrm{F}_{k-1}^{1,*},$$

the relevant part of the $\overline{E}_0$ page is

$$
\begin{array}{ccc}
\mathrm{Gr}_{-n+1}^{n+1} & \mathrm{Gr}_{-n}^{n+1} & \mathrm{Gr}_{-n-1}^{n+1} \\
\bar{d}_h \uparrow & \bar{d}_h \uparrow & \\
\mathrm{Gr}_{-n+1}^{n} & \mathrm{Gr}_{-n}^{n} & 0 \\
\bar{d}_h \uparrow & & \\
\mathrm{Gr}_{-n+1}^{n-1} & 0 & 0
\end{array}
$$

Since $\bar{d}_h$ is a $\mathcal{C}^\infty$-linear vector bundle map between these spaces, it suffices to compute pointwise. Localizing at a point of M results in the following diagram of vector spaces.

$$
\begin{array}{ccc}
\mathbb{R}\{\theta_{ij,0}\wedge\omega, \theta_{i,1}\wedge\omega, \theta_{0,2}\wedge\omega\} & \mathbb{R}\{\theta_{i,0}\wedge\omega, \theta_{0,1}\wedge\omega\} & \mathbb{R}\{\theta_{0,0}\wedge\omega\} \\
\bar{d}_h \uparrow & \bar{d}_h \uparrow & \\
\mathbb{R}\{\theta_{i,0}\wedge\omega_{(b)}, \theta_{0,1}\wedge\omega_{(b)}\} & \mathbb{R}\{\theta_{0,0}\wedge\omega_{(a)}\} & 0 \\
\bar{d}_h \uparrow & & \\
\mathbb{R}\{\theta_{0,0}\wedge\omega_{(ab)}\} & 0 & 0
\end{array}
$$

The $\overline{E}_1$ page is then computed to be

$$
\begin{array}{ccc}
0 & 0 & \mathcal{C}^\infty\{\theta_{0,0}\wedge\omega\} \\
 & & \\
\mathcal{C}^\infty\{\theta_{i,0}\wedge\omega_{(i)}\} & 0 & 0 \\
 & & \\
0 & 0 & 0
\end{array}
$$

Because the differential is no longer guaranteed to be $\mathcal{C}^\infty$-linear, it is not possible to compute

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pointwise. Still, the $\overline{E}_2$ page is easily seen to be

$$
\begin{array}{ccc}
0 & 0 & \mathcal{C}^\infty\{\theta_{0,0}\wedge\omega\} \\
& \nearrow^{\delta} & \\
\mathcal{C}^\infty\{\theta_{i,0}\wedge\omega_{(i)}\} & 0 & 0 \\
& & \\
0 & 0 & 0
\end{array}
$$

After this page the spectral sequence degenerates, so

$$
E_1^{1,n} = \ker(\delta).
$$

The upshot is that any element $\underline{\Phi} \in E_1^{1,d}$ is represented by an element of $\mathcal{C}^\infty\{\theta_{i,0}\wedge\omega_{(i)}, \theta_{0,0}\wedge\omega_{(a)}\}$ that is anihilated the operators $\bar{d}_h$ and δ. Let A and A_a be functions in $\mathcal{C}^\infty$ so that

$$
\underline{\Phi} \equiv \underline{A}\theta_{i,0}\wedge\underline{\omega}_{(i)} + \underline{\theta}_{0,0}\wedge\left(\underline{A}_i\underline{\omega}_{(i)} + \underline{A}_0\underline{\omega}_{(0)}\right) \quad \left(\mathrm{mod}\ \mathrm{F}^2\right).
$$

The condition that $\bar{d}_h$ vanish determines the functions A_a. The calculation is best done on the coframe bundle of M. Denote the pullbacks of $\underline{\Phi}, \underline{A}, \underline{A}_a$ to $\mathcal{B}_0$ by Φ, A, A_a respectively, so that

$$
\Phi_1 \equiv A\theta_{i,0}\wedge\omega_{(i)} + \theta_{0,0}\wedge\left(A_i\omega_{(i)} + A_0\omega_{(0)}\right) \quad \left(\mathrm{mod}\ \mathrm{F}^2\right).
$$

The condition $\bar{d}_h\Phi_1 = 0$ is the same as

$$
0 \equiv (-A_i - \bar{D}_iA + 2a_{jj}^iA)\theta_{i,0}\wedge\omega + (-A_0 + Aa_{ii})\theta_{0,1}\wedge\omega \quad \left(\mathrm{mod}\ \theta_{0,0}, \mathrm{F}^2\right),
$$

which proves (3.3).

The condition that δ vanish on Φ results in a further differential equation on A, which is computed by

$$
d\Phi_1 \equiv d_h(A\left(\theta_{i,0}\wedge\omega_{(i)} + \theta_{0,0}\wedge(a_{jj}\omega_{(0)} + 2a_{jj}^i\omega_{(i)})\right) - (D_iA)\theta_{0,0}\wedge\omega_{(i)}) \equiv 0 \quad \left(\mathrm{mod}\ \mathrm{F}^2\right)
$$

The result depends on the details of the curvatures of M_0. However, it is not difficult to see that the horizontal derivative is of the form

$$
\mathscr{F}(A)\theta_{0,0}\wedge\omega \quad \left(\mathrm{mod}\ \mathrm{F}^2\right).
$$

In particular, observe that the leading order term D_iD_iA comes from the horizontal derivative of

$$
(D_iA)\theta_{0,0}\wedge\omega_{(i)}.
$$

$\square$

Remark 2. Observe that
$$d_h \Upsilon \equiv 0 \pmod{\theta_{0,0}, F^2}.$$

Also, for strongly parabolic system, the trace a_{jj} is normalized to -1 on $\mathcal{B}_4$, so that
$$\Upsilon = \left(\underline{\theta}_{i,0} \wedge \underline{\omega}_{(i)} + \underline{\theta}_{0,0} \wedge (-\underline{\omega}_{(0)} + 2a_{jj}^i \underline{\omega}_{(i)}) \right).$$

Remark 3. For a given parabolic system M_0, one could determine $\mathscr{F}$ explicitly and then try to find solutions to the auxiliary equation. If additionally $d_v(A\Upsilon + \theta_{0,0} \wedge \psi_A) = 0$, then these solutions correspond to conservation laws.

I take a different approach that is simpler for strongly parabolic systems. I use the fact that the sub-principal weight filtration splits this equation into several *algebraic* equations. This idea follows naturally from $\mathcal{C}$-spectral sequence considerations.

Example 9. Recall the coframing of the heat equation in Example 6. This coframing is 4-adapted and has all of the secondary Goursat invariants absorbed. Consequently, any conservation law has $\mathcal{I}$-linear part
$$\Phi_1 \equiv A(\underline{\theta}_i \wedge d\underline{\omega}_{(i)} - \underline{\theta}_\varnothing \wedge d\underline{\omega}_{(0)}) - (D_i A)\underline{\theta}_\varnothing \wedge d\underline{\omega}_{(i)} \pmod{F^2}.$$

Taking the horizontal derivative,
$$0 \equiv (D_0 A + D_i D_i A)\underline{\theta}_\varnothing \wedge \omega \pmod{F^2}.$$

One can see from the theorem that for M to have even a single non-trivial conservation law Φ puts strong constraints on its Monge-Ampère invariants. Indeed, let Φ_1 denote the $\mathcal{I}$-linear piece of Φ, and Φ_2 its $\mathcal{I}$-quadratic piece, so that
$$\Phi_1 = A\Upsilon + \theta_{0,0} \wedge \psi_A$$
for some $A \in \mathcal{C}^\infty$ and
$$\Phi \equiv \Phi_1 + \Phi_2 \pmod{F^3}.$$
If necessary, restrict to a neighborhood where A is not zero.

Since Φ is closed and $d_h \Phi_1 = 0$,
$$0 \equiv d_v \Phi_1 + d_h \Phi_2 \pmod{F^3}$$
$$\equiv (d_v A) \wedge \Upsilon + A d_v \Upsilon + d_h \Phi_2 \pmod{\theta_{0,0}, d\theta_{0,0}, F^3}.$$

Applying d_h and dividing by A,
$$0 \equiv d_h d_v \Upsilon \pmod{\theta_{0,0}, d\theta_{0,0}, \Upsilon, F^3}.$$

On the other hand, by direct calculation, and using the identity
$$d\theta_{0,0} \wedge \omega_{(0i)} \equiv \theta_{0,1} \wedge \omega_{(i)} - \theta_{i,0} \wedge \omega_{(0)} \pmod{\theta_{0,0}},$$

I find that

$$d_v \Upsilon \equiv -2\xi_i^0 \wedge \theta_{i,0} \wedge \omega_{(0)} - 2\xi_i^j \wedge \theta_{j,0} \wedge \omega_{(i)} - \theta_{i,0} \wedge \theta_b \wedge \sigma^{ab} \wedge \omega_{(ai)} \quad \left(\text{mod } \theta_{0,0},\, d\theta_{0,0},\, \Upsilon,\, F^3\right).$$

Taking the horizontal derivative of this shows that

$$0 \equiv d_h(\xi_i^0 \wedge \theta_{i,0}) \wedge \omega_{(0)} + d_h(\xi_i^j \wedge \theta_{j,0}) \wedge \omega_{(i)} \quad \left(\text{mod } \theta_{0,0},\, d\theta_{0,0},\, \Upsilon,\, F^3\right). \tag{3.4}$$

I note that the terms $d_h(\theta_{i,0}\wedge\theta_b\wedge\sigma^{ab}\wedge\omega_{(ai)})$ make no contribution to (3.4). Essentially, this follows from the fact that

$$\Lambda^{n+2}\mathcal{J} \subset \{\theta_{0,0},\, d\theta_{0,0}\},$$

which in turn follows from the Lefschetz decomposition of $\Lambda^*\mathcal{J}$ for the 'symplectic' form $d\theta_{0,0}$. Fixing any coframing $\underline{\eta}$ and functions S_c^{ab} so that

$$\underline{\eta}^*(\sigma^{ab}) \equiv S_c^{ab}\underline{\omega}^c \quad \left(\text{mod } F^1\right),$$

the pullback by $\underline{\eta}$ of $d_h(\theta_{i,0}\wedge\theta_b\wedge\sigma^{ab}\wedge\omega_{(ai)})$ satisfies

$$d_h\left(S_c^{ab}\underline{\theta}_{i,0} \wedge \underline{\theta}_b \wedge \underline{\omega}^c \wedge \underline{\omega}_{(ai)}\right) \equiv 0 \quad \left(\text{mod } \Lambda^{n+2}\mathcal{J},\, F^3\right).$$

Since this holds for any coframing, these terms will never contribute to equation (3.4).

At highest sub-principal weight, equation (3.4) simplifies to

$$0 \equiv d_h(U_i\theta_{0,2} \wedge \theta_{i,0}) \wedge \omega_{(0)} \qquad \left(\text{mod } \theta_{0,0},\, d\theta_{0,0},\, \Upsilon,\, \mathcal{F}^2_{-n+4}\right)$$
$$\equiv U_i\theta_{0,2} \wedge \theta_{i,1} \wedge \omega \qquad \left(\text{mod } \theta_{0,0},\, d\theta_{0,0},\, \Upsilon,\, \mathcal{F}^2_{-n+4}\right).$$

Thus the function (U_i) must vanish identically.

At the next weight, equation (3.4) simplifies to

$$0 \equiv d_h(U_i^j\theta_{j,1} \wedge \theta_{i,0}) \wedge \omega_{(0)} + d_h(V_i^j\theta_{0,2} \wedge \theta_{j,0}) \wedge \omega_{(i)} \qquad \left(\text{mod } \theta_{0,0},\, d\theta_{0,0},\, \Upsilon,\, \mathcal{F}^2_{-n+3}\right)$$
$$\equiv (U_i^j\theta_{j,1} \wedge \theta_{i,1} + V_i^j\theta_{0,2} \wedge \theta_{ij,0}) \wedge \omega \qquad \left(\text{mod } \theta_{0,0},\, d\theta_{0,0},\, \Upsilon,\, \mathcal{F}^2_{-n+3}\right).$$

This shows that the anti-symmetric component of U_i^j vanishes and that the traceless symmetric component of V_i^j vanishes.

Finally, with all of the tools developed here, it is not so hard to prove the following theorem.

Theorem 5. *Let $(M,\mathcal{I})$ be a strongly parabolic system. Any non-trivial conservation law Φ has a representative*

$$\Phi \equiv A\Upsilon + \theta_{0,0} \wedge \psi_A \quad (\text{mod } F^2)$$

so that $A \in C^\infty(M_0)$.

Proof. Let $\underline{\Phi}$ be conservation law of M. By proposition 4, there is a function $\underline{A}$ and corresponding $\psi_{\underline{A}}$ so that the $\mathcal{I}$-linear part of $\underline{\Phi}$ is given by

$$\underline{\Phi}_1 \equiv \underline{A}\Upsilon + \theta_{0,0} \wedge \psi_{\underline{A}} \quad (\text{mod } F^2).$$

Let $\underline{\Phi}_2$ be the $\mathcal{I}$-quadratic part of $\underline{\Phi}$, so that

$$\underline{\Phi} \equiv \underline{A}\Upsilon + \theta_{0,0} \wedge \psi_{\underline{A}} + \underline{\Phi}_2 \pmod{\mathrm{F}^3}.$$

Denote by $\Phi, \Phi_1, \Phi_2, A, Q$ the respective pullbacks to $\mathcal{B}_2$.

Since Φ is closed and $d_h \Phi_1 = 0$,

$$0 \equiv d_v \Phi_1 + d_h \Phi_2 \pmod{\mathrm{F}^3}.$$

Applying d_h shows that

$$d_h d_v \Phi_1 \equiv 0 \pmod{\mathrm{F}^3}.$$

To prove that $A \in \mathcal{C}^\infty(M_0)$ it suffices to show that

$$d_v A \equiv 0 \pmod{\mathcal{J}, \mathrm{F}_2^1}.$$

Using the sub-principal weight filtration, let N be the largest weight so that

$$d_v A \not\equiv 0 \pmod{\mathcal{J}, \mathrm{F}_2^1, \mathcal{F}_{N-1}^1}.$$

Furthermore, let S be the smallest integer so that

$$d_v A \not\equiv 0 \pmod{\mathcal{J}, \mathrm{F}_2^1, \mathrm{F}_{N-S-1}^1, \mathcal{F}_{N-1}^1}.$$

There are functions so that

$$d_v A \equiv \sum_{\substack{S \leq s \leq N/2 \\ |I| = N-2s}} A_{I,s} \theta_{I,s} \pmod{\mathcal{J}, \mathrm{F}_2^1, \mathcal{F}_{N-1}^1}.$$

I will show that each function $A_{I,S}$ vanishes.

Preliminary to this, it is easy to calculate from the structure equations that the following equations hold in any coframing:

$$d_v \underline{\theta}_{0,0} \equiv 0 \pmod{\mathcal{J}, \mathrm{F}_2^2},$$

$$d_v \underline{\theta}_{i,0} \equiv 0 \pmod{\mathcal{J}, \mathrm{F}_3^2},$$

$$d_v \underline{\omega}^a \equiv 0 \pmod{\Lambda^2 \mathcal{J}, \mathrm{F}_1^1}.$$

Note that in each case, taking the horizontal derivative raises the principal weight by at most 2. Furthermore, it immediately follows from equation (2.12) that

$$d_v a_{ij}^j \equiv 0 \pmod{\mathcal{J}, \mathrm{F}_3^1}.$$

Combined, these show that

$$d_v(\Upsilon) \equiv 0 \pmod{\Lambda^{n+2} \mathcal{J}, \mathrm{F}_{-n+3}^2}.$$

Furthermore, since

$$d_v(D_iA) \equiv \sum_{\substack{S \le s \le N/2 \\ |I|=N-2s}} A_{I,s}(\theta_{Ii,s} + c_{|I|}(e^i - C_{|I|}e_ie^je^j)\theta_{I,s+1}) \quad (\mathrm{mod}\ \mathcal{J}, \mathrm{F}_3^1, \mathcal{F}_N^1),$$

one finds that

$$d_v(\theta_{0,0} \wedge \psi_A) \equiv \theta_{0,0} \wedge d_v(D_iA) \wedge \omega_{(i)} \quad (\mathrm{mod}\ \Lambda^{n+2}\mathcal{J}, \mathrm{F}_{-n+3}^2, \mathcal{F}_{N-n-1}^2)$$
$$\equiv \sum_{\substack{S \le s \le N/2 \\ |I|=N-2s}} A_{I,s}\theta_{0,0} \wedge (\theta_{Ii,s} + c_{|I|}(e^i - C_{|I|}e_ie^je^j)\theta_{I,s+1}) \wedge \omega_{(i)}.$$

The upshot of all of this is that

$$d_v\Phi_1 \equiv (d_vA) \wedge (\theta_{i,0} \wedge \omega_{(i)} + \theta_{0,0} \wedge \omega_{(0)}) + \theta_{0,0} \wedge d_v(D_iA) \wedge \omega_{(i)} \quad (\mathrm{mod}\ \mathrm{F}_{-n+3}^2, \mathrm{F}_{N-S-n-3}^2, \mathcal{F}_{N-n-1}^2)$$
$$\equiv \sum_{\substack{S \le s \le S+1 \\ |I|=N-2s}} A_{I,s}\left(\theta_{I,s} \wedge (\theta_{i,0} \wedge \omega_{(i)} + \theta_{0,0} \wedge \omega_{(0)}) + \theta_{0,0} \wedge (\theta_{Ii,s} + c_{|I|}(e^i - C_{|I|}e_ie^je^j)\theta_{I,s+1})\right) \wedge \omega_{(i)}.$$

Taking the horizontal derivative results in

$$0 \equiv \sum_{|I|=N-2S} A_{I,S}\theta_{0,0} \wedge (-1 + c_{|I|+1}(e^i - C_{|I|+1}e_ie^je^j)e_i + c_{|I|}e_i(e^i - C_{|I|}e_ie^je^j))\theta_{I,s+1} \wedge \omega$$

$$(\mathrm{mod}\ \mathrm{F}_{-n+3}^2, \mathrm{F}_{N-S-n-3}^2, \mathcal{F}_{N-n-1}^2)$$

(I remark that the fact that this vanishes modulo $\theta_{0,0}$ and weights is true simply because of the anti-commutativity of d_v and d_h.) The only thing left to check is that the operator

$$(-1 + c_{|I|+1}(e^i - C_{|I|+1}e_ie^je^j)e_i + c_{|I|}e_i(e^i - C_{|I|}e_ie^je^j)),$$

which is a multiple of the identity on $\mathrm{Sym}_0^{N-2S}W'$, is non-zero.

Let $t = N - S$. Then for $\theta_{I,S+1} \in \mathrm{Sym}_0^t W'$,

$$e^ie_i\theta_{I,S+1} = (n+t)\theta_{I,S+1},$$

$$e_ie^i\theta_{I,S+1} = t\theta_{I,S+1},$$

$$e_ie^je^je_i\theta_{I,S+1} = t(n+t)\theta_{I,S+1}.$$

(These calculations may be more familiar if one considers the isomorphism between $\mathrm{Sym}_0^t W'$ and harmonic polynomials of degree t. Then, for example, e_ie^i is the same operator as the divergence vector field $x^i\frac{\partial}{\partial x^i}$.) Taken together, one sees that

$$-1 + c_{|I|+1}(e^i - C_{|I|+1}e_ie^je^j)e_i + c_{|I|}e_i(e^i - C_{|I|}e_ie^je^j)$$
$$= -1 + \frac{n}{n+2t}\left(n+t - \frac{t(n+t)}{n+2t-2}\right) + \frac{nt}{n+2t-2},$$

which is never zero.

$\square$

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