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Vortex Pairs and Dipoles

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Abstract—Point vortices have been extensively studied in vortex dynamics. The generalization to higher singularities, starting with vortex dipoles, is not so well understood. We obtain a family of equations of motion for inviscid vortex dipoles and discuss limitations of the concept. We then investigate viscous vortex dipoles, using two different formulations to obtain their propagation velocity. We also derive an integro-differential for the motion of a viscous vortex dipole parallel to a straight boundary.

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1. INTRODUCTION

Fluid mechanics and more particularly vortex dynamics have been at the origin of a fascinating and well-studied dynamical system: point vortices. Described by one author as a "classical applied mathematical playground" [1], point vortices have been used to examine many different physical systems as well as being an object of interest in the theory of integrable systems. Point vortices are singular solutions to the nonlinear equations of incompressible potential flow. Each vortex corresponds to a delta function in vorticity and moves according to Kirchhoff's laws. The physical justification for these laws is reviewed in [2]; another discussion from a more mathematical perspective is given in [3]. Informally, point vortices move with the fluid.

The natural mathematical step of moving from a delta function singularity to higher singularities goes back to Fridman and Polubarinova [4]. Again, the justification for the resulting evolution equations is reviewed in [2]. Consistent results can be obtained for vortex dipoles (see also [5]), but higher-order singularities are problematic.

Probably the most physically satisfying justification of the Kirchhoff equations is that using Matched Asymptotic Expansions, which goes back, at least conceptually, to J. J. Thomson [6]. Thomson argued that the boundaries of interacting line vortices are deformed, but these deformations are neutral modes and can be ignored if the other vortices are far enough away. Hence vortices move in the far field of the other vortices, which look like point vortices from far enough away. The method of Matched Asymptotic Expansions is used in problems in which there are regions of fast variation, for example close to boundaries where the influence of viscosity cannot be neglected. One solves outer and inner problems and matches them to obtain an asymptotic solution to the full problem. This approach is outlined for the point vortex problem by Ting [7] who obtains the motion of a small Rankine vortex in a large-scale fluid flow (which locally looks like a uniform stream). This calculation leads to the point vortex equations for the motion of the centers of the vortices and neutral modes on the edge of the vortex.

For a dipole, the natural scaling of self-induced velocity leads to arbitrary large velocities of the dipole in an MAE framework. This is problematic and one is left with a question: what is the

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link between the well-posed vortex dipole equations that have been obtained previously and the physical procedure that views the dipole as the limit of two intense nearby vortices?

Other reduced models for dipoles have been obtained. Tchieu *et al.* [8] created finite dipoles by taking pairs of point vortices and fixing the distance between the vortices. Matsumoto *et al.* [9] obtained moment equations for the motion of vortex pairs through an incompressible flow. There also exist exact analytical solutions for vortex pairs with distributed vorticity that propagate steadily. The simplest is the Lamb or Chaplygin vortex. Extensions of these solutions to the beta-plane and other situations have been obtained, e.g. modons.

In this paper, we review existing models of vortex dipoles and pairs, study their propagation, and examine the limitations of different models. We also examine the viscous case.

2. THE VORTEX DIPOLE EQUATION

We start by obtaining the equations for a vortex pair. Consider two point vortices at $z = z_\pm$ with circulation $\pm\Gamma$. The complex velocity is

$$w = f(z) + \frac{\Gamma}{2\pi \mathrm{i}}\frac{1}{z - z_+} - \frac{\Gamma}{2\pi \mathrm{i}}\frac{1}{z - z_-}, \tag{2.1}$$

where $f(z)$ is the non-singular part of the velocity field (this is written $\tilde{w}(z)$ in [2]). We assume there are no other singularities near the vortices. The equations of motion are then

$$\dot{\bar{z}}_+ = f(z_+) - \frac{\Gamma}{2\pi \mathrm{i}}\frac{1}{z_+ - z_-}, \qquad \dot{\bar{z}}_- = f(z_-) + \frac{\Gamma}{2\pi \mathrm{i}}\frac{1}{z_- - z_+}. \tag{2.2}$$

We define the centerpoint of the vortices and their (rotated) separation, both complex numbers, by

$$z_c = \frac{1}{2}(z_+ + z_-), \qquad z_+ - z_- = \mathrm{i}d. \tag{2.3}$$

This gives

$$\dot{\bar{z}}_c = \frac{1}{2}[f(z_+) + f(z_-)] + \frac{\Gamma}{2\pi d}, \qquad \dot{\overline{(\mathrm{i}d)}} = f(z_+) - f(z_-). \tag{2.4}$$

These equations are exact. They are used in [8] as a starting point for dipoles of finite size.

We now take the limit of small d to obtain the equations of a dipole singularity at z_c. We obtain

$$\dot{\bar{z}}_c = f(z_c) + \frac{\Gamma}{2\pi d} + O(d), \qquad \dot{\bar{d}} = -df'(z_c) + O(d^2). \tag{2.5}$$

We pick a time scale T and (outer) length scale L set by the problem as a whole, so that $\dot{z}_c$ and f have the same magnitude $U = LT^{-1}$. We denote the length scale for d by δ which is constant, say the initial value of d. The intrinsic scale of our pair is supposed to be small compared to the outer length scale, so $\delta \ll L$ and we neglect the order terms in (2.5).

The equation for d is linear and indicates the d aligns with the flow if it is allowed to evolve. This is what was found previously in [2] and [5]. The equation for $\dot{z}_c$ in (2.5) is a generalization of what was found before. The first term in this equation is the usual advection by the desingularized velocity field. The second is new. The size of the ratio $\Gamma/\delta U$ determines the importance of the second term. The equations obtained by [2] and others correspond to ignoring the second term, which is the case when $\Gamma \ll \delta U$, i.e. when the exterior flow dominates over the dipole. The case $\Gamma = O(\delta U)$ gives a finite second term (D. J. Hill, personal communication). Finally, for $\Gamma \gg \delta U$, the dipole term dominates formally: this is the singular self-induced velocity mentioned in [2]. The usual dipole limit, say in electrostatics, corresponds to $\Gamma \to \infty$, $d \to 0$ with $\Gamma\delta$ having a finite limit; this is not the relevant limit for the dynamical equations.

There is a problem, however, with this approach. We are really carrying out an expansion in small δ/L. This expansion is not valid for distances of order δ from the vortices. This requirement will become a problem if boundaries or vortices come within a distance $O(\delta)$ of the dipole. We should then solve the original equations since the internal structure of the dipole will be affected by the full dynamics of the problem. For example, if two dipoles were to collide, they might scatter.

This break-up of the dipoles could not be examined using Eqs. (2.5), at least not without positing some exterior interaction force or potential that would have to be obtained from the full problem. This can also be seen from the complex velocity, which in nondimensional form is

$$w = f(z) + \frac{\Gamma d}{2\pi U L^2} \frac{1}{(z - z_c)^2} \tag{2.6}$$

when $|z - z_c| = O(1)$, i.e. on the outer length scale. For $\Gamma \leqslant \delta U$, i.e. the cases when the self-advection term dominates or co-dominates, the second term in the potential is of size $\leqslant (\delta/L)^2 \ll 1$ and is negligible. The dipoles hence have no presence in the far field. This is consistent with the previous comments: for example there are no image vortices until the vortex is within $O(\delta)$ of boundaries, etc. . . Hence it is not clear that calculations of dipoles near boundaries form a useful problem, nor that situations in which dipoles approach close to one another make sense. Another way of looking at this is that the complex potential is a far-field expansion for which there is no correction due to the dipole, whereas the equation of motion comes from a near-field expansion.

The case of a single dipole is degenerate since there are no pre-existing scales T and U to compute the size of terms. In addition, we note that the resulting dipole equations are of fourth order, the same as the original equations for two point vortices. Hence we do not reduce the number of equations, although we have a clearly defined location for the vortex and a single dipole moment.

It is interesting to study (2.5) under conformal mapping. Write $z = z(\zeta)$. The potential is invariant, implying that the dipole strength E in the ζ-plane is related to D by $D\,\mathrm{d}z = E\,\mathrm{d}\zeta$. This is different from the point vortex case in which circulation is the same in any plane. One can now compute the equivalent of the Routh correction, that is the extra term that needs to be added to the velocity when working with the conformal map. We follow the same procedure as above, starting from the equation for a single point vortex at $z(t)$:

$$\dot{z} = \frac{1}{z'} \left(\dot{\bar{\zeta}} + \frac{\mathrm{i}\Gamma}{4\pi} \frac{z''}{z'} \right). \tag{2.7}$$

Then one obtains

$$\dot{\bar{z}}_c = \frac{1}{z'_c} \left(f(\zeta_c) + \frac{\mathrm{i}\Gamma}{4\pi} \frac{z''_c}{z'_c} \right) + \frac{1}{2\pi E}, \tag{2.8}$$

$$\dot{\bar{E}} = -E \left(f'_c - \frac{z''_c}{z'_c} f_c + \frac{\mathrm{i}\Gamma}{4\pi} \left[\frac{z'''_c}{z'_c} - \frac{z''_c}{z'^2_c} \right] \right). \tag{2.9}$$

3. VISCOUS DIPOLES AND PAIRS

Much work on two-dimensional vortex dynamics has considered inviscid flows. This is particularly the case for point vortices, vortex sheets and other singular structures, since viscosity smears out singular structures in vorticity. We now consider the effect of viscosity on vortex dipoles and vortex pairs. Our goal is to obtain analytical or quasi-analytical solutions rather than to carry out full direct numerical simulations.

Two particularly interesting related but different approaches have been proposed by Nagem and co-authors. The first [10] (N2007 hereafter) expands the vorticity field in a series of derivatives of a Gaussian vortex. The second [11] (N2009 hereafter) expands the vorticity field using Hermite functions about different moving centers. It is the basis of a full numerical method for solving the two-dimensional Navier–Stokes equations developed further in [12].

3.1. Motion of a Vortex Pair or Dipole: First Approach

In N2007, the vorticity is expanded as

$$\omega = \Gamma(t)\delta^\lambda + D_j(t)\frac{\partial}{\partial x_j}\delta^\lambda + Q_{jk}(t)\frac{\partial^2}{\partial x_j \partial x_k}\delta^\lambda + \cdots, \tag{3.1}$$

where the function δ^λ is the Oseen or Gaussian vortex, an exact solution of the two-dimensional Navier–Stokes equation given by

$$\delta^\lambda(x_1, x_2, y, t) = \frac{\mathrm{e}^{-r^2/\lambda^2}}{\pi\lambda^2} \tag{3.2}$$

with $r = \sqrt{x_1^2 + x_2^2}$. The distance λ is the width of the vortex and is given explicitly by $\lambda = \sqrt{\lambda_0^2 + 4\nu t}$, where λ_0 is the initial width.

Using the orthogonality property of these expansion functions, one finds

$$\Gamma = \iint \omega\, \mathrm{d}^2\mathbf{x}, \qquad D_j = -\iint \omega x_j\, \mathrm{d}^2\mathbf{x}, \qquad Q_{jk} = \iint \omega \left(\frac{x_j x_k}{2} - \frac{\lambda^2\delta_{jk}}{4}\right) \mathrm{d}^2\mathbf{x}. \tag{3.3}$$

Evolution equations for the coefficients $\Gamma(t)$, $D_j(t)$ and so on come from integrating the vorticity equation over the plane and again using the orthogonality properties of Gaussian functions. The result is

$$\dot{\Gamma} = 0, \qquad \dot{D}_j = 0, \qquad \dot{Q}_{jk} = \cdots, \tag{3.4}$$

where the final expression is an infinite series. One can obtain a closed system by retaining up to quadrupole terms. The resulting equation (21) of N2007 involves Γ, D_j and $Q_{jk}(t)$. An important property of this equation is that Q_{kk} is constant.

We consider symmetric pair solutions with

$$\omega = \gamma\delta^\lambda(x - X, y - Y) - \gamma\delta^\lambda(x - X, y + Y), \tag{3.5}$$

where $X(t)$ and $Y(t)$ are to be found. The form (3.5) is a solution to the truncated equations if, from (3.4),

$$\Gamma = 0, \qquad D_1 = 0, \qquad D_2 = -2\gamma Y, \qquad Q_{11} = Q_{22} = 0, \qquad Q_{12} = Q_{21} = \gamma XY. \tag{3.6}$$

We take γ to be constant in time. The $\dot{\Gamma}$, $\dot{D}_1$, $\dot{Q}_{11}$ and $\dot{Q}_{22}$ equations are then satisfied identically. From the $\dot{D}_2$ equation, $Y = b$ is constant. The $\dot{Q}_{12}$ equation becomes

$$\dot{Q}_{12} = b\gamma\dot{X} = \frac{D_2^2}{16\pi\lambda^2} = \frac{(-2b\gamma)^2}{16\pi(\lambda_0^2 + 4\nu t)}. \tag{3.7}$$

We hence obtain

$$\dot{X} = \frac{b\gamma}{4\pi(\lambda_0^2 + 4\nu t)}. \tag{3.8}$$

The quantities b and γ enter this equation only as the product $b\gamma$, so the resulting motion is the same for all vortex pairs and for vortex dipoles, which correspond to the limit $b \to 0$, $\gamma \to \infty$, $b\gamma \to -\frac{1}{2}D_2$. Keeping dimensional variables allows all the cases to be considered. We integrate (3.8) and obtain

$$X = \frac{b\gamma}{16\pi\nu} \log\left(1 + 4\nu\lambda_0^{-2} t\right) \tag{3.9}$$

with $X(0) = 0$. For small time, the motion is linear in time with $X \sim b\gamma t/(4\pi\lambda_0^2)$; for large time it is logarithmic with $X \sim b\gamma \log t/(16\pi\nu)$.

3.2. Motion of a Vortex Pair: Second Approach

In the approach of N2009, the vorticity is expanded as a sum of vortices

$$\omega(\mathbf{x}, t) = \sum_{j=1}^{N} \omega_j(\mathbf{x} - \mathbf{x}_j(t); t) \tag{3.10}$$

and the vorticity about each center is further expanded in Hermite functions:

$$\omega_j(\mathbf{x},t) = \sum_{k_1,k_2=0}^{\infty} M_j[k_1,k_2;t]\phi_{k_1,k_2}(\mathbf{x},t;\lambda). \tag{3.11}$$

The functions ϕ are Hermite functions obtained from differentiating the Oseen solution, so that

$$\phi_{k_1,k_2}(\mathbf{x},t;\lambda) = \frac{\partial^{k_1}}{\partial x^{k_1}}\frac{\partial^{k_2}}{\partial y^{k_2}}\phi_{0,0}(\mathbf{x},t;\lambda), \tag{3.12}$$

with

$$\phi_{0,0}(\mathbf{x},t;\lambda) = \frac{1}{\pi\lambda^2}\mathrm{e}^{-|\mathbf{x}|^2/\lambda^2} \tag{3.13}$$

with λ evolving as above. For simplicity, we take λ_0 to be the same for all vortices. Taking Hermite moments leads to evolution equations for $\mathbf{x}_j(t)$ and $M_j[k_1,k_2;t]$. Formally the moment expansion is convergent, but we consider low-order truncations to obtain manageable dynamical systems.

The approach of N2009 does not work for a vortex dipole, since each propagating vortex must have non-vanishing circulation, i.e. $M[0,0;t] \neq 0$. We can, however, consider a vortex pair with $M_1 = -M_2 = M$ and $(x_2,y_1) = (x_1,-y_2) = (X,Y)$. One can show that $M_j[0,0;t]$ is constant in time whatever the truncation. If we truncate at $k_1 = k_2 = 0$, we find that $Y = b$ is constant, so that the pair moves in a straight line. We obtain the single equation

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{M}{2\pi}\frac{1-\mathrm{e}^{-4b^2/[\lambda_0^2+4\nu t]}}{2b}. \tag{3.14}$$

(Note that the equations (68) of N2009 are incorrect; the equation (A.42) of [12] give the correct result.)

From (3.14) the velocity for large times is $\dot{X} \sim Mb/(4\pi\nu t)$. The vortex slows down, but never stops. Its position continues to grow logarithmically. We can rewrite this in a form that highlights the parameter dependence of the problem. We nondimensionalize distances using b and velocity using $M/(4\pi b)$. Then, in nondimensional form, (3.14) becomes

$$\frac{\mathrm{d}X}{\mathrm{d}t} = 1 - \mathrm{e}^{-[l_0^2+Rt]^{-1}}, \tag{3.15}$$

where $l_0 \equiv \lambda_0/b$ is the initial vortex width and $R \equiv 16\pi\nu/M$ is the Reynolds number. We see that for $t \leqslant R^{-1}l_0^2$, the velocity is approximate constant and given by $1-\mathrm{e}^{-l_0^{-2}}$. For vortices with small core size, this is approximately 1, which is the velocity of an inviscid vortex pair in this nondimensionalization. For $t \gg R^{-1}l_0^2$, we recover the t^{-1} dependence in velocity.

The differential equation (3.15) can be integrated exactly, yielding

$$X(t) = t - \frac{l_0^2+Rt}{R}\mathrm{e}^{-[l_0^2+Rt]^{-1}} + \frac{l_0^2}{R}\mathrm{e}^{-l_0^{-2}} + \frac{1}{R}E_1\left(\frac{1}{l_0^2+Rt}\right) - \frac{1}{R}E_1(l_0^{-2}) \tag{3.16}$$

with $X(0) = 0$. For large time, we find

$$X(t) = R^{-1}[\log Rt + 1 - l_0^2(1+\mathrm{e}^{-l_0^{-2}}) - E_1(l_0^{-2})] + O(R^{-2}t^{-1}). \tag{3.17}$$

The large-R and large-t limits do not commute.

3.3. Vortex Moving Along Boundary

For an inviscid fluid, the pair solution also gives the motion of a single vortex above a boundary, with the second vortex acting as an image vortex. For a viscous fluid, the no-slip boundary condition is also needed. We use a distribution of elementary time-dependent solutions $\phi_{0,1}$ to satisfy this boundary condition. The strength, or rather distribution, of boundary vortices is determined by the no-slip condition. Note that only $\phi_{0,1}$ is needed along the boundary; this is analogous to the potential flow case.

Working in terms of velocity, we write

$$\mathbf{u} = \mathbf{u}^1(\mathbf{x}-\mathbf{x}^1(t),t) + \mathbf{u}^2(\mathbf{x}-\mathbf{x}^2(t),t) + \int_{-\infty}^{\infty} m(\xi,t)\mathbf{V}_{0,1}(x-\xi,y)\,\mathrm{d}\xi, \tag{3.18}$$

where the travelling vortex at (X,Y) has Hermite moments $M[k_1,k_2;t]$ and the image vortex at $(X,-Y)$ has Hermite moments $(-1)^{k_2}M[k_1,k_2;t]$. The two vortices together satisfy the no normal flow boundary condition. The boundary contribution comes from the velocity field

$$\mathbf{V}_{0,1}(x,0) = \frac{(-1,0)}{2\pi x^2}(1-\mathrm{e}^{-x^2/\lambda(t)^2}). \tag{3.19}$$

On the boundary the vertical velocity component vanishes and the no normal flow condition is automatically satisfied by the integral. The no-slip boundary condition along $y=0$ becomes

$$0 = 2u^1(x-X,-Y,t) - \int_{-\infty}^{\infty} m(\xi,t)\frac{1-\mathrm{e}^{-(x-\xi)^2/\lambda(t)^2}}{2\pi(x-\xi)^2}\,\mathrm{d}\xi = 2u^1(x-X,-Y,t) - \frac{1}{2\pi}[K*m](x). \tag{3.20}$$

The condition (3.20) is a Fredholm integral equation for the unknown function $m(\xi,t)$ which incorporates parametric time dependence. It is in convolution form. The kernel of the integral operator is

$$K(x) = \frac{1-\mathrm{e}^{-x^2/\lambda(t)^2}}{x^2}. \tag{3.21}$$

This has a known Fourier transform

$$\mathcal{K}(k) = \pi\left(k[\operatorname{erf}(k\lambda/2) - \operatorname{sgn} k] + \frac{2}{\sqrt{\pi}}\mathrm{e}^{-k^2\lambda^2/4}\right). \tag{3.22}$$

We hence have a formal solution

$$M(\xi,t) = 2u^1(x-X,-Y,t) * L(x) \tag{3.23}$$

where $L(x)$ is the inverse Fourier transform of $\mathcal{K}(k)^{-1}$. This solution is formal, and some care may be required to obtain appropriately convergent forms.

The equations of motion for the vortex needs to take into account the boundary contribution. The simplest approach here is to think of a collection of discrete vortices along the boundary and then take the continuum limit. Then in the governing equations (A.35)–(A.42) of [12], we make the substitution

$$\sum_{j'\neq j}^{m}\sum_{m_1,m_2} M^{j'}[l_1,l_2;t] \to \int_{-\infty}^{\infty} m(\xi;t)\delta_{l_1,0}\delta_{l_2,1}\,\mathrm{d}\xi. \tag{3.24}$$

Then the motion of the vortex is given by adapting (A.42) of [12]

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}^1}{\mathrm{d}t} &= \frac{1}{M[0,0;t]}\sum_{l_1,l_2}\sum_{m_1,m_2}(-1)^{l_2}M[l_1,l_2;t]M[m_1,m_2;t](-1)^{m_1+m_2}\mathcal{H}^B(m_1+l_1,m_2+l_2)\\ &\quad -\frac{1}{M[0,0;t]}\int_{-\infty}^{\infty} m(\xi;t)\sum_{m_1,m_2}(-1)^{m_1+m_2}\mathcal{H}^B(m_1,m_2+1)\,\mathrm{d}\xi.\end{aligned} \tag{3.25}$$

(As usual self-induced vorticity is zero). The evolution equation for the moments formally remains

$$\frac{\mathrm{d}M[k_1,k_2;t]}{\mathrm{d}t} = A(k_1,k_2) + B(k_1,k_2) + C(k_1,k_2), \tag{3.26}$$

but the various terms are again modified according to (3.24).

One is left with a coupled integro-differential system. As usual, the degree of truncation of the moments can be chosen. The simplest is to take a single moment $M[0,0;t]$. The moment equation becomes trivial since $M[0,0;t]$ is constant in time as has been mentioned previously. The equations of motion become

$$\frac{\mathrm{d}\mathbf{x}^1}{\mathrm{d}t} = M[0,0;t]\mathcal{H}^B(0,0) + \int\limits_{-\infty}^{\infty} m(\xi;t) \sum_{m_1,m_2} M[m_1,m_2;t](-1)^{m_1+m_2}\mathcal{H}^B(0,1)\,\mathrm{d}\xi, \tag{3.27}$$

where the first $\mathcal{H}^B$ function is evaluated at $(0, 2Y)$ and the second at $(X - \xi, Y)$. We do not pursue the calculation any further at this point.

4. CONCLUSION

We have discussed the evolution of vortex dipoles and pairs. The straightforward method of obtaining pair evolution equations from nearby point vortices outlined in § 2 shows explicitly that this is a problem with multiple regions. It yields evolution equations for position and dipole strength, whose exact form depends on the quantity $\gamma/\delta U$. When this quantity is small, one recovers previously obtained equations. When it is order-one, one finds an extra term. In both of these cases, the resulting complex flow velocity has no contribution from the dipole in the far field. and hence is not appropriate in the presence of nearby boundaries for example. Conversely complex velocities that are $O(1)$ in the far field give evolution equations that are dominated by the self-interaction term.

The presence of viscosity overcomes a large part of this difficulty. Using the two different but related methods of N2007 and N2009, evolution equations for a viscous pair were obtained in § 3. The velocities (3.8) and (3.15) and positions (3.9) and (3.17) obtained from N2007 and N2009 are not the same. They have the same logarithmic dependence for large time with the identification $\gamma = 4M$, although the correction terms are different. The positions at small times grow linearly, but the initial velocities are different. In the limit of large Reynolds number, the approach of N2009 gives the appropriate inviscid velocity, while the velocity predicted by N2007 diverges. The approach of N2007 gives the same value for all vortex pairs with the same dipole moment, whereas N2009 cannot be applied in its current form to a vortex dipole.

The motion of a viscous vortex above a boundary can be investigated using the N2009 framework. The result is a coupled integro-differential system. Unavoidably, this is more complicated that the problem of propagation of an inviscid vortex along a straight boundary, for which the boundary can be represented as a single image vortex.

Note that Cantwell and Rott [13] had previously derived a heuristic model for the motion and decay of a viscous vortex pair. Their asymptotic result for large time,

$$U = \frac{D}{32\pi\nu t} \tag{4.1}$$

where $D = 2\Gamma b$ is the hydrodynamic impulse divided by mass, is the same as the long-time limit of (3.9), which was obtained using the consistent and formally convergent expansion procedure of N2007.

A number of authors have found that long-time simulations of a propagating dipole lead to almost self-similar solutions [14, 15]. It would be interesting to see if these observations can be explained using the MMVM and varying the truncation.

Gorshkov *et al.* [16] have derived an expansion procedure to obtain corrections to the motion of vortices in the presence of perturbative effects, for example non-uniform stratification, obtaining evolution equations using a secularity procedure taking into account the discrete and continuous spectrum of the vortices. They examine the motion of pairs of Rankine vortices through a density gradient or past a density interface. These methods could be used to generalize the inviscid results discussed in this paper, although the viscous analogue might well be lengthy.

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