Title
Competitive pricing and the core: With reference to matching

Journal
Games and Economic Behavior, 108(C)

ISSN
0899-8256

Author
Ostroy, Joseph M

Publication Date
2018-03-01

DOI
10.1016/j.geb.2018.01.011

Peer reviewed

Games and Economic Behavior ••• (••••) •••–•••

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

Competitive pricing and the core: With reference to matching ☆

Joseph M. Ostroy

Dept of Economics, UCLA, United States

ARTICLE INFO

Article history:
Received 1 September 2016
Available online xxxx

JEL classification:
C71
C78
D51

Keywords:
Subdifferential
Core
Market game
Assignment model
Walrasian equilibrium
Perfect substitutability

ABSTRACT

Lloyd Shapley's contributions with respect to the core are interpreted as *subdifferentiability* characterizations of the pricing of individuals that is similar to the pricing of commodities in economic models of exchange with transferable utility. *Differentiability* of the core is interpreted as perfect substitutability with respect to the pricing of individuals. Differentiability implies, but is not implied by, equivalence of the core and Walrasian equilibria. Differentiability eliminates opportunities for strategic misrepresentation of utilities. The assignment model with transferable utility is framed in the setting of exchange economies and its individual and commodity pricing is extended to non-transferable utility.

1. Introduction

The first application of the Gillies–Shapley concept of the core was to the assignment model (Shapley, 1955), more fully elaborated in the influential paper by Shapley and Shubik (1972). Necessary and sufficient conditions for a non-empty core of a transferable utility game were given by Bondareva (1963) and Shapley (1967) in terms of balanced sets. Shapley and Shubik (1969) exhibited the intimate connections between the core and Walrasian for market games with transferable, also known as quasilinear, utility.

Shapley (1969) also pioneered extensions of results from transferable to non-transferable utility through the introduction of the λ-transfer scheme. Gale and Shapley (1962) formulated the non-transferable version of the assignment model, called two-sided matching by Roth and Sotomayor (1990), along with an algorithm for achieving a stable matching. With some exceptions (e.g., Roth et al., 1993), the non-transferable utility version of the assignment model has been analyzed without regard to duality properties that figure prominently in the transferable utility version. The purpose of this paper is to use convex analysis to extend Shapley's contributions of the pricing of individuals, including to the assignment model without transfers.

The emphasis, below, is on the derivation of prices as replacements for constraints. Just as prices of commodities are derived from the value of perturbing aggregate commodity resource constraints, so the prices of individuals can be derived

☆ I benefited from the editor's guidance and numerous comments by Joon Song.
E-mail address: ostroy@ucla.edu.

https://doi.org/10.1016/j.geb.2018.01.011

from the value of perturbing population constraints. This will require that individuals be regarded as elements in a linear space along with, but different from, the linear space of commodities.

In Section 2, a transferable utility game in characteristic function form is compared to a polyhedral concave positively homogeneous function and its subdifferentiability is related to the core. Properties of discrete differences and directional derivatives of concave functions and their connections to differentiability are reviewed.

In Section 3, market games, i.e., models of exchange economies, are formulated as concave positively homogeneous functions that are not typically polyhedral; and, their subdifferentiability properties are characterized as Walrasian equilibria. Differences between the cores of market games and their Walrasian equilibria are attributed to differences in the generators of the positively homogeneous functions defining them. The absence of a difference distinguishes Shapley and Shubik's (1969) *direct markets*, where the linear spaces of individuals and commodities coincide, from market games. Extensions of these constructions to market games with non-transferable utility are introduced as preparation for demonstrating existence of Walrasian equilibria in assignment models without transfers.

In Section 4, the assignment model is regarded as a transferable utility exchange economy, but with indivisible commodities. Along with direct markets, the assignment model is also defined by a finite number of generators and is such that Walrasian equilibria coincide with the core. This feature allows an elementary proof of the existence of Walrasian equilibria without transfers for the assignment model, i.e., for two-sided matching.

In Section 5, an individual in a transferable utility exchange economy is defined as *perfectly substitutable* if there is price for that individual (derived from perturbing the population constraint) achieving the upper bound of an inequality associated with the subdifferential. The utility received by an individual who is perfectly substitutable is shown to be such that efforts to gain by misrepresentation would be counterproductive. All individuals are perfectly substitutable in an economy with a finite number of individuals if and only if the prices of individuals exhibit *discrete differentiability*. Differentiability is an asymptotic property defined by limiting ratios, ensuring that if infinitesimal individuals are perfect substitutes, there are finite approximations that are nearly so.

The prices corresponding to differentiability with respect to individuals, discrete or infinitesimal, are consistent with the rewards individuals receive in Walrasian equilibrium, and with the more stringent requirement that the core and Walrasian equilibrium coincide, but not conversely: neither the existence of Walrasian equilibrium, nor its equivalence with the core, implies differentiability. In the exchange economies with large numbers (where individuals are infinitesimal) considered, below, "typically" — but not universally — all individuals are perfectly substitutable.

Typically, no individual is perfectly substitutable in an exchange economy with finite numbers. A distinctive feature of the assignment model *with* transfers is that, even with finite numbers, there are prices at which some individuals, if not all, can be perfectly substitutable. Again, large numbers typically implies differentiability, but not because replication shrinks the size of the core. In the finite assignment model *without* transfers, the qualification to guarantee existence of Walrasian equilibria (equivalent to the existence of stable matches) requires strict preference. Strict preference precludes the possibility of indifference required for perfect substitutability. Section 5.4 outlines how perfect substitutability can be approximated in the assignment model with large numbers without transfers. Section 6 contains a concluding comment.

2. Preliminaries

2.1. Subdifferentiability and the core

A transferable utility game in characteristic function form is defined by a function $\mathbf{v}$ on the non-empty subsets $S \subseteq I = \{1, 2, \ldots, n\}$, where $\mathbf{v}(S) \geq 0$, and $\mathbf{v}(\emptyset) = 0$. To exploit its connections to convex analysis, $\mathbf{v}$ is restated as a function on $\mathbb{R}^I$. Throughout the following, $S \subseteq I$ *will refer to non-empty sets* and the empty set will be replaced by $\mathbf{0}$, the zero element of $\mathbb{R}^I$.

Defining $\mathbb{R}^S = \{r = \langle r_i \rangle : r_i = 0, \forall i \notin S \subseteq I\}$, let $\mathbf{e}_S \in \mathbb{R}^S$ be the indicator of S, i.e., $r_i = 1$, $i \in S$. Rewrite the set function $\mathbf{v}$ as $v : \mathbb{R}^I \to \mathbb{R}_+ \cup \{-\infty\}$, where

$$\operatorname{dom} v := \{r : v(r) > -\infty\} = \{\mathbf{e}_S : S \subseteq I\} \cup \{\mathbf{0}\}$$

$$v(\mathbf{e}_S) = \mathbf{v}(S), \; S \subseteq I, \quad v(\mathbf{0}) = 0.$$

Denoting $\sum_{\{S:S\subseteq I\}}$ by $\sum_S$ throughout, two extensions of v are:

- the balanced cover v^B of v has $\operatorname{dom} v^B = \operatorname{dom} \; v$ and

$$v^B(\mathbf{e}_T) = \sup\left\{\sum_S \alpha_S v(\mathbf{e}_S) : \sum_S \alpha_S \mathbf{e}_S = \mathbf{e}_T, \alpha_S \geq 0\right\}, \; T \subseteq I;$$

- the smallest positively homogeneous concave function $v^\infty \geq v$:

$$v^\infty(r) = \sup\left\{\sum_S \alpha_S v(\mathbf{e}_S) : \sum_S \alpha_S \mathbf{e}_S = r, \alpha_S \geq 0\right\}.$$

Please cite this article in press as: Ostroy, J.M. Competitive pricing and the core: With reference to matching. Games Econ. Behav. (2018), https://doi.org/10.1016/j.geb.2018.01.011

The first is due to Bondareva (1963) and Shapley (1967) and the second is taken from convex analysis (Rockafellar, 1970). By construction, dom $v(= \text{dom } v^B) \subset \text{dom } v^\infty = \mathbb{R}^I_+$; and

$$v^\infty \geq v^B \geq v \quad \text{and} \quad v^\infty = v^B \text{ on dom } v.$$

The inequality $v^B(\mathbf{e}_T) \geq v(\mathbf{e}_T)$, $T \subseteq I$, means that $v^B(\mathbf{e}_T) = v(\mathbf{e}_T)$ is a statement of *integral optimality*: there are no gains to making fractional assignments of $\mathbf{e}_T$. The construction of v^∞ says that it is *polyhedral*: $\{\mathbf{e}_S, S \subseteq I\}$ are the *generators*, i.e., the basis for positive linear combinations defining v^∞.

Denoting the inner product of $q, r \in \mathbb{R}^I$ by $q \cdot r$, the *concave version of the subdifferential of* v at $\bar{r} \in \text{dom } v$, also known as the *superdifferential*, is

$$\partial v(\bar{r}) := \{q : q \cdot r - v(r) \geq q \cdot \bar{r} - v(\bar{r}), \forall r\}. \tag{1}$$

A similar definition applies to $\partial v^\infty(\bar{r})$. However, because v^∞ is positively homogeneous, it is readily established that it can be written as

$$\partial v^\infty(\bar{r}) = \{q : q \cdot r - v^\infty(r) \geq q \cdot \bar{r} - v^\infty(\bar{r}) = 0, \forall r\}. \tag{2}$$

Another well-known feature is that ∂v^∞ is homogeneous of degree zero:

$$\partial v^\infty(\alpha \bar{r}) = \partial v(\bar{r}), \quad \forall \alpha > 0. \tag{3}$$

Without further qualification $\partial v(\mathbf{e}_S)$, $S \subseteq I$, may be empty, but $\partial v^\infty(r)$, $r \in \mathbb{R}^I_+$, is not.

Throughout the following, when $S = I$, *abbreviate* $\mathbf{e}_I$ *as* $\mathbf{e}$. The *core* of the game v is the set of q, if any, such that

$$q \cdot \mathbf{e}_S - v(\mathbf{e}_S) \geq 0, \forall S \text{ and } q \cdot \mathbf{e} - v(\mathbf{e}) = 0. \tag{4}$$

The core of v combines the subdifferential inequalities for $\partial v(\mathbf{e})$ with the added requirement $q \cdot \mathbf{e} - v(\mathbf{e}) = 0$. Hence, (4) can be rewritten as

$$\partial_C v(\mathbf{e}) := \partial v(\mathbf{e}) \cap \{q : q \cdot \mathbf{e} = v(\mathbf{e})\}.$$

The Bondareva–Shapley characterization for a non-empty core is that v should exhibit integral optimality at $\mathbf{e}$; i.e.,

$$\partial_C v(\mathbf{e}) \neq \emptyset \iff v(\mathbf{e}) = v^B(\mathbf{e}).$$

By construction, $v^B(\mathbf{e}) = v^\infty(\mathbf{e})$. Another characterization of the core is:

Proposition 1. $\partial_C v(\mathbf{e}) \neq \emptyset \iff \partial_C v(\mathbf{e}) = \partial v^\infty(\mathbf{e})$.

Proof. Replacing $\bar{r}$ with $\mathbf{e}$ in (2), it follows that $\partial v^\infty(\mathbf{e}) \subseteq \partial_C v(\mathbf{e})$, since the former must satisfy a superset of the restrictions defining the latter, i.e., in (4).

For the converse, if $q \in \partial_C v(\mathbf{e})$, then $q \cdot \mathbf{e}_S \geq v(\mathbf{e}_S)$, $\forall S \subset I$ and $q \cdot \mathbf{e} = v(\mathbf{e})$; hence $\alpha_S q \cdot \mathbf{e}_S \geq \alpha_S v(\mathbf{e}_S)$ for all $\alpha_S \geq 0$. Therefore, if $r = \sum_S \alpha_S \mathbf{e}_S$, then $\sum_S \alpha_S q \cdot \mathbf{e}_S \geq \sum_S \alpha_S v(\mathbf{e}_S)$; so, $q \cdot r \geq v^\infty(r)$. Hence, $q \in \partial_C v^B(\mathbf{e})$ implies $q \in \partial v^\infty(\mathbf{e})$. $\square$

The definition of balance at $\mathbf{e}$ can be extended to $\mathbf{e}_S$ as $v(\mathbf{e}_S) = v^B(\mathbf{e}_S)$. A similar restriction can be applied to

$$\partial_C v(\mathbf{e}_S) := \left\{q : q \cdot \mathbf{e}_T - v(\mathbf{e}_T) \geq q \cdot \mathbf{e}_S - v(\mathbf{e}_S) = 0, \forall T \subset S\right\},$$

as the core of the (sub-)game on subsets of S. Then, as defined by Shapley, v is *totally balanced* if $v^B(\mathbf{e}_S) = v(\mathbf{e}_S)$, $S \subseteq I$. Extending Proposition 1, if v is totally balanced,

$$\partial_C v(\mathbf{e}_S) = \partial v^\infty(\mathbf{e}_S).$$

2.2. Local properties and differentiability

To introduce further properties of functions appealed to, below, let $g : \mathbb{R}^m \to \mathbb{R} \cup \{-\infty\}$ be a concave and Lipschitz continuous function on dom g that is not necessarily polyhedral. Assuming $x, x + d \in \text{dom } g$, discrete differences are

$$\Delta g(x; k^{-1}d) := g(x + k^{-1}d) - g(x), \ k = 1, 2, \ldots.$$

Facts (Rockafellar, 1970, Section 23):

(i) $k\Delta g(x; k^{-1}d)$ is non-decreasing in k.
(ii) $Dg(x; d) := \lim_{k\to\infty} k\Delta g(x; k^{-1}d)$ is the directional derivative of g at x in the direction d.
(iii) $x^* \in \partial g(x)$ implies $-k\Delta g(x; -k^{-1}d) \geq x^* \cdot d \geq k\Delta g(x; k^{-1}d)$.

Summary Comparison: For $x^* \in \partial g(x)$, Facts (i)–(iii) imply

$$-\Delta g(x; -d) \geq -Dg(x : -d) \geq x^* \cdot d \geq Dg(x : d) \geq \Delta g(x; d). \tag{5}$$

Infinitesimal changes are captured by the subdifferential in the sense that

(iv) $-Dg(x; -d) = \max\left\{x^* \cdot d : x^* \in \partial g(x)\right\}$ and $Dg(x : d) = \min\left\{x^* \cdot d : x^* \in \partial g(x)\right\}$.

If g is also positively homogeneous,

(v) $\lim_{k\to\infty} \Delta g(kx; d) = \lim_{k\to\infty} k[g(x; k^{-1}d) - g(x)] = Dg(x; d)$.

Fact (v) says that with positive homogeneity, the consequences of the (infinitesimal) directional derivative d at x can be reproduced by holding (the non-infinitesimal) d fixed while scaling up x. This will be interpreted, below, as: a discrete change in a finite population can be translated into a directional derivative in a population with large numbers.

Denote by $\mathbf{e}_h \in \mathbb{R}^m_+$, $h = 1, \ldots, m$, the hth coordinate vector and assume $x = \langle x_h \rangle \in \mathbb{R}^m_{++}$ belongs to the interior of dom g. Existence of a partial derivative of g at x is $\frac{\partial g(x)}{\partial x_h} := -Dg(x; -\mathbf{e}_h) = Dg(x; \mathbf{e}_h)$: g is *differentiable* when there is equality for all h.

(vi) A concave function g is differentiable almost everywhere in the interior of dom g.

If g is also positively homogeneous, it suffices for g to be *differentiable* at x that there exist $x^* = \langle x^*_h \rangle \in \partial g(x)$ such that

(vii) $x^*_h = -Dg(x; -\mathbf{e}_h)$, $\forall h \iff x^* \cdot x = \sum_h -Dg(x; -\mathbf{e}_h)x_h = g(x)$.

It is *discretely differentiable* at x if there exist $x^* = \langle x^*_h \rangle \in \partial g(x)$ such that

(viii) $x^*_h = -\Delta g(x; -\mathbf{e}_h)$, $\forall h \iff x^* \cdot x = \sum_h -\Delta g(x; -\mathbf{e}_h)x_h = g(x)$.

3. Market games: prices of individuals from prices of commodities

3.1. Transferable utility

A market game in Shapley and Shubik (1969) is formulated as a family of models that vary in their populations of a fixed finite set of types of individuals and their aggregate resource constraints. Perturbations within this family are used to derive the connections between the pricing of individuals and the pricing commodities in general equilibrium models with transfers, i.e., with quasilinear utility, that highlights the role of positive homogeneity.

The set I is as above. The characteristics of $i \in I$ are now defined by the utility function for (non-money) commodities, $\nu_i : \mathbb{R}^\ell \to \mathbb{R} \cup \{-\infty\}$. The zero element of $\mathbb{R}^\ell$ will also be denoted by $\mathbf{0}$. The characteristics include the set of trades i can feasibly make, defined by

$$\text{dom}\, \nu_i := \{z_i : \nu_i(z_i) > -\infty\} \subset \mathbb{R}^\ell.$$

Assume throughout this Section that:

- $\mathbf{0} \in \text{dom}\, \nu_i$, $\nu_i(\mathbf{0}) = 0$, and dom ν_i is compact
- ν_i is concave and Lipschitz continuous on dom ν_i.

These properties guarantee that for every $z_i \in \text{dom}\, \nu_i$, there exists $p \in \mathbb{R}^\ell$ such that

$$\begin{aligned} \nu_i^*(p) = p \cdot z_i - \nu_i(z_i) &= \inf_{y_i} \left\{p \cdot y_i - \nu_i(y_i)\right\} \\ &= -\sup_{y_i} \{\nu_i(y_i) - p \cdot y_i\} \\ &= -\sup_{(y_i, m_i)} \{\nu_i(y_i) + m_i : p \cdot y_i + m_i = 0\}. \end{aligned} \tag{6}$$

Please cite this article in press as: Ostroy, J.M. Competitive pricing and the core: With reference to matching. Games Econ. Behav. (2018), https://doi.org/10.1016/j.geb.2018.01.011

Thus, $-v_i^*(p) = \sup_{y_i}\{v_i(y_i) - p\cdot y_i\}$ is the indirect utility function (the negative of the concave conjugate function of v_i) associated with maximization of $v_i(y_i) + m_i$ subject to the budget constraint $p\cdot y_i + m_i = 0$. Since the money commodity, with a normalized price of unity, enters each individual's utility in the same way, the data of quasilinear exchange economy can be summarized as $\mathcal{E} = \langle v_i\rangle := (v_1, v_2, \ldots, v_n)$.

Based on $\mathcal{E}$, when the population of individuals is $\bar r = \langle \bar r_i\rangle \in \mathbb{R}^I_+$ and the aggregate resource constraint is $\bar z \in \mathbb{R}^\ell$, the maximum gains in the quasilinear model is

$$\Phi_{\mathcal{E}}(\bar r, \bar z) = \sup\Big\{\sum_i r_i v_i(y_i) : \sum_i r_i \mathbf{e}_i = \bar r, \sum_i r_i y_i = \bar z, r_i \geq 0\Big\},$$

a positively homogeneous concave function on $\operatorname{dom}\Phi_{\mathcal{E}}$. Pricing of individuals and commodities at $(\bar r, \bar z) \in \operatorname{dom}\Phi_{\mathcal{E}}$ is determined by $(q, p) \in \partial\Phi_{\mathcal{E}}(r, \bar z)$ as

$$q\cdot r + p\cdot z - \Phi_{\mathcal{E}}(r, z) \geq q\cdot \bar r + p\cdot \bar z - \Phi_{\mathcal{E}}(\bar r, \bar z) = 0, \quad \forall (r, z) \in \mathbb{R}^I_+ \times \mathbb{R}^\ell.$$

When $q_i = -v_i^*(p) = v_i(z_i) - p\cdot z_i$, $\sum_i r_i \mathbf{e}_i = \bar r$ and $\sum_i r_i z_i = \bar z$,

$$-\sum_i r_i v_i^*(p) + p\cdot \bar z = \sum_i r_i[v_i(z_i) - p\cdot z_i] + p\cdot \bar z = \Phi_{\mathcal{E}}(\bar r, \bar z).$$

Fixing $\bar z = \mathbf{0}$, the gains from varying the population is the positively homogeneous concave function

$$w^\infty_{\mathcal{E}}(r) = \Phi_{\mathcal{E}}(r, \mathbf{0}) = \max\Big\{\sum_i r_i v_i(y_i) : \sum_i r_i \mathbf{e}_i = r, \sum_i r_i y_i = \mathbf{0}, r_i \geq 0\Big\}. \tag{7}$$

[The superscript $^\infty$ emphasizes that $w^\infty_{\mathcal{E}}$ considers any population $r \in \mathbb{R}^I_+$ along with its aggregate resource constraint, $\sum_i r_i y_i = \mathbf{0}$.] Letting $\mathbb{Z}^I_+$ denote the integer-valued vectors in $\mathbb{R}^I_+$, the concavity of v_i implies integral optimality: if $r \in \mathbb{Z}^I_+$, the maximum in (7) can be achieved when each r_i is an integer.

Fixing r, the gains from varying the aggregate resource constraint is the concave function

$$f_{\mathcal{E}}(\bar z\,|\,r) := \Phi_{\mathcal{E}}(r, \bar z); \tag{8}$$

so

$$w^\infty_{\mathcal{E}}(r) = \Phi_{\mathcal{E}}(r, \mathbf{0}) = f_{\mathcal{E}}(\mathbf{0}\,|\,r)$$

measures the total gains either in terms of the commodity resource constraint or the population constraint. Pricing of individuals at r is defined by

$$\partial w^\infty_{\mathcal{E}}(r) = \{q : q\cdot r' - w^\infty_{\mathcal{E}}(r') \geq q\cdot r - w^\infty_{\mathcal{E}}(r) = 0, \forall r'\}.$$

When the commodity resource constraint is $\mathbf{0}$, pricing of commodities at r is

$$\partial f_{\mathcal{E}}(\mathbf{0}\,|\,r) = \{p : p\cdot z \geq f_{\mathcal{E}}(z\,|\,r) - f_{\mathcal{E}}(\mathbf{0}\,|\,r), \forall z\}.$$

[Note: Positive homogeneity of $w^\infty_{\mathcal{E}}$ implies $\partial w^\infty_{\mathcal{E}}(\alpha r) = \partial w^\infty_{\mathcal{E}}(r)$, $\alpha > 0$. Similarly, $\alpha f_{\mathcal{E}}(\mathbf{0}\,|\,r) = f_{\mathcal{E}}(\mathbf{0}\,|\,\alpha r)$ implies $\partial f_{\mathcal{E}}(\mathbf{0}\,|\,\alpha r) = \partial f_{\mathcal{E}}(\mathbf{0}\,|\,r)$.]

Definition 1. A *Walrasian equilibrium for the quasilinear exchange economy* $\mathcal{E}$ when the population is r is a pair $(p, \langle z_i\rangle)$ satisfying $\sum_{i\in I} r_i z_i = \mathbf{0}$ [market clearance] and $-v_i^*(p) = v_i(z_i) - p\cdot z_i$, $\forall i$ [utility maximization].

Proposition 2 *(Characterization of Walrasian Equilibrium). $(p, \langle z_i\rangle)$ is a Walrasian equilibrium for $\mathcal{E}$ at r if and only if $\sum_i r_i z_i = \mathbf{0}$, $\sum_i v_i(z_i) = w^\infty_{\mathcal{E}}(\mathbf{e}) = -\sum_i v_i^*(p)$, where $p \in \partial f_{\mathcal{E}}(\mathbf{0}; r)$, and*

$$\partial w^\infty_{\mathcal{E}}(r) = \{q = \langle q_i = -v_i^*(p)\rangle : p \in \partial f_{\mathcal{E}}(\mathbf{0}\,|\,r)\}. \tag{9}$$

The equality in (9) says that the pricing of individuals at r, i.e., $q \in \partial w^\infty_{\mathcal{E}}(r)$, is derived from the pricing of commodities, $p \in \partial f_{\mathcal{E}}(\mathbf{0}\,|\,r)$. The main population points of reference will be $r = k\mathbf{e}$, $k = 1, 2, \ldots$, with $r \neq k\mathbf{e}$ as points of comparison.

$\mathcal{E}$ defines a *market game* in characteristic function form: for $S \subseteq I$,

$$v_{\mathcal{E}}(\mathbf{e}_S) := \max\Big\{\sum_{i\in S} v_i(y_i^S) : \sum_{i\in S} y_i^S = \mathbf{0}\Big\}.$$

The function $v_{\mathcal{E}}$ on $\{\mathbf{e}_S : S \subseteq I\}$ is extended to the positively homogeneous and superadditive, therefore concave, function on $\mathbb{R}^I_+$ exactly as v is extended to v^∞ in Section 2.1:

$$v_{\mathcal{E}}^{\infty}(r) := \max\left\{\sum_S \alpha_S v_{\mathcal{E}}(\mathbf{e}_S) : \sum_S \alpha_S \mathbf{e}_S = r, \alpha_S \geq 0\right\}.$$

When v_i is concave, Shapley and Shubik show that $v_{\mathcal{E}}$ is totally balanced. Consequently,

$$w_{\mathcal{E}}^{\infty}(\mathbf{e}_S) = v_{\mathcal{E}}^{\infty}(\mathbf{e}_S) = v_{\mathcal{E}}^{B}(\mathbf{e}_S) = v_{\mathcal{E}}(\mathbf{e}_S).$$

And, by Proposition 1,

$$\partial_C v_{\mathcal{E}}(\mathbf{e}) = \partial v_{\mathcal{E}}^{\infty}(\mathbf{e}).$$

A well-known difference between $w_{\mathcal{E}}^{\infty}$ and $v_{\mathcal{E}}^{\infty}$ is

$$\partial w_{\mathcal{E}}^{\infty}(\mathbf{e}) \subseteq \partial_C v_{\mathcal{E}}(\mathbf{e}). \tag{10}$$

I.e., the prices of individuals defined by (9) is contained in the prices defined by the core, and the containment is typically strict.

The difference in (10) arises from the fact that while $v_{\mathcal{E}}^{\infty}$ is defined by the finite generators $\{\mathbf{e}_S : S \subseteq I\}$, the generators of $w_{\mathcal{E}}^{\infty}$ are a superset that need not be finite. To demonstrate, define the game $v_{\mathcal{E}}(\cdot\,|\,k\mathbf{e})$ based on the integer-valued coalitions

$$\operatorname{dom} v_{\mathcal{E}}(\cdot\,|\,k\mathbf{e}) := \{r \in \mathbb{Z}_+^I : r \leq k\mathbf{e}\}, \quad k = 1, 2, \ldots$$

The value for each coalition is the maximum gains it can achieve on its own,

$$v_{\mathcal{E}}(r\,|\,k\mathbf{e}) := w_{\mathcal{E}}^{\infty}(r), \quad r \in \operatorname{dom} v_{\mathcal{E}}(\cdot\,|\,k\mathbf{e}).$$

Therefore, while

$$v_{\mathcal{E}}(k\mathbf{e}\,|\,k\mathbf{e}) = w_{\mathcal{E}}^{\infty}(k\mathbf{e}) = v_{\mathcal{E}}^{\infty}(k\mathbf{e}), \quad k = 1, 2, \ldots,$$

the gains in $v_{\mathcal{E}}(\cdot\,|\,k\mathbf{e})$ are bounded below by $v_{\mathcal{E}}^{\infty}$ (defined by the generators $\{\mathbf{e}_S : S \subseteq I\}$),

$$v_{\mathcal{E}}(r\,|\,k\mathbf{e}) \geq v_{\mathcal{E}}^{\infty}(r), \quad \forall r \in \operatorname{dom} v_{\mathcal{E}}(\cdot\,|\,k\mathbf{e});$$

and bounded above by $w_{\mathcal{E}}^{\infty}$,

$$w_{\mathcal{E}}^{\infty}(r) \geq v_{\mathcal{E}}(r\,|\,k\mathbf{e}), \quad \forall r.$$

Definition 2. The core of $v_{\mathcal{E}}(\cdot\,|\,k\mathbf{e})$ is

$$\partial_C v_{\mathcal{E}}(k\mathbf{e}\,|\,k\mathbf{e}) := \left\{q : q \cdot r - v_{\mathcal{E}}(r\,|\,k\mathbf{e}) \geq q \cdot k\mathbf{e} - v_{\mathcal{E}}(k\mathbf{e}\,|\,k\mathbf{e}) = 0, \forall r\right\}.$$

Letting $v_{\mathcal{E}}^{\infty}(\cdot\,|\,k\mathbf{e})$ be the smallest concave positively homogeneous function such that $v_{\mathcal{E}}^{\infty}(\cdot\,|\,k\mathbf{e}) \geq v_{\mathcal{E}}(\cdot\,|\,k\mathbf{e})$: its generators are $\operatorname{dom} v_{\mathcal{E}}(\cdot\,|\,k\mathbf{e})$. A direct extension of Proposition 1 implies

$$\partial_C v_{\mathcal{E}}(k\mathbf{e}\,|\,k\mathbf{e}) = \partial v_{\mathcal{E}}^{\infty}(k\mathbf{e}\,|\,k\mathbf{e}).$$

As k increases the generators of $v_{\mathcal{E}}^{\infty}(\cdot\,|\,k\mathbf{e})$ expand to fill out the generators of $w_{\mathcal{E}}^{\infty}$.

Convergence of the core to Walrasian equilibrium can be expressed as

$$\lim_{k\to\infty} \partial_C v_{\mathcal{E}}(k\mathbf{e}\,|\,k\mathbf{e}) = \partial w_{\mathcal{E}}^{\infty}(\mathbf{e}). \tag{11}$$

If the equality in (11) is achieved for $\bar{k}$, it is readily seen that $\partial_C v_{\mathcal{E}}(k\mathbf{e}\,|\,k\mathbf{e}) = \partial w_{\mathcal{E}}^{\infty}(\mathbf{e})$, $k > \bar{k}$. Also, (11) may occur whether or not it satisfies a differentiability condition, below. When that differentiability condition is satisfied, Proposition 5 in Section 5 describes a more parsimonious way to achieve (11) than the conditions defining the core.

3.1.1. Direct markets: a market game version of Proposition 1

Shapley and Shubik (1969) define a *direct market* in which the individuals are, in effect, also the commodities. In a direct market, (11) is achieved when $k = 1$.

The commodity space is $\mathbb{R}^\ell = \mathbb{R}^I$. Denote by $U : \mathbb{R}_+^\ell \to \mathbb{R}_+$ a positively homogeneous concave function, with the interpretation that U is a utility function on commodities. A *direct market* $\mathcal{E}_D = \langle v_i^D \rangle$ is an exchange economy where

$$\operatorname{dom} v_i^D = \{y_i : y_i + \mathbf{e}_i \in \mathbb{R}_+^I\},$$

and

$$\nu_i^D(y_i) = U(y_i + \mathbf{e}_i).$$

In $\mathcal{E}_D$ individuals differ only with respect to their endowments, $\mathbf{e}_i$. [Unlike the convention $\nu_i(\mathbf{0}) = 0$ in $\mathcal{E}$, $\nu_i^D(\mathbf{0}) = U(\mathbf{e}_i) \geq 0$.]

The analog for $\mathcal{E}_D$ of the indirect utility function (6) for $\mathcal{E}$ is

$$\begin{aligned} -(\nu_i^D)^*(p) &= \sup\{\nu_i^D(y_i) - p \cdot y_i\} \\ &= \sup\{U(y_i + \mathbf{e}_i) - p \cdot y_i\}. \end{aligned}$$

A Walrasian equilibrium for $\mathcal{E}_D$ at $\mathbf{e}$ is a $(p, \langle z_i \rangle)$ such that $\sum_i z_i = \mathbf{0}$ and

$$-(\nu_i^D)^*(p) = U(z_i + \mathbf{e}_i) - p \cdot z_i, \quad \forall i.$$

As in $\mathcal{E}$, the maximum total gains for $\mathcal{E}_D$ at $(\mathbf{e}, \mathbf{0})$ is

$$w^\infty_{\mathcal{E}_D}(\mathbf{e}) = \Phi_{\mathcal{E}_D}(\mathbf{e}, \mathbf{0}) = f_{\mathcal{E}_D}(\mathbf{0}),$$

where

$$\begin{aligned} \Phi_{\mathcal{E}_D}(\mathbf{e}, \mathbf{0}) &= \sup\left\{\sum_i r_i \nu_i^D(y_i) : \sum_i r_i \mathbf{e}_i = \mathbf{e}, \sum_i r_i y_i = \mathbf{0}, r_i \geq 0\right\} \\ &= \sup\left\{\sum_i r_i U(y_i + \mathbf{e}_i) : \sum_i r_i \mathbf{e}_i = \mathbf{e}, \sum_i r_i y_i = \mathbf{0}, r_i \geq 0\right\} \\ &= U(\mathbf{e}). \end{aligned}$$

I.e., the population $\mathbf{e}$ having total resources $\mathbf{e}$ achieves a maximum utility when $z_i + \mathbf{e}_i = \lambda_i \mathbf{e}$, $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$, and $\sum_i \lambda_i U(z_i + \mathbf{e}_i) = U(\sum_i \lambda_i [z_i + \mathbf{e}_i]) = U(\mathbf{e})$.

As a choice for U, let $U = v^\infty$, the positively homogeneous concave functions defined in Section 2.1 by the generators $\{\mathbf{e}_S : S \subseteq I\}$. Therefore,

$$w^\infty_{\mathcal{E}_D} = U = v^\infty_{\mathcal{E}_D}. \tag{12}$$

The core of $\mathcal{E}_D$ at $\mathbf{e}$ is

$$\partial U(\mathbf{e}) = \partial v^\infty_{\mathcal{E}_D}(\mathbf{e}) = \{q : q \cdot r - v^\infty_{\mathcal{E}_D}(r) \geq q \cdot \mathbf{e} - v^\infty_{\mathcal{E}_D}(\mathbf{e}) = 0, \forall r\}.$$

Consequently, the core coincides with Walrasian equilibria

$$\partial w^\infty_{\mathcal{E}_D}(\mathbf{e}) = \partial v^\infty_{\mathcal{E}_D}(\mathbf{e}).$$

In $\mathcal{E}_D$, the prices of individuals, $q \in \partial w^\infty_{\mathcal{E}_D}(\mathbf{e})$, are also the prices of commodities, $p \in \partial U(\mathbf{e}) = \partial f_{\mathcal{E}_D}(\mathbf{0})$.

3.2. Non-transferable utility

The characteristic function of a game with non-transferable utility is described by sets of vector-valued utilities $V : \mathbb{R}^I \to 2^{\mathbb{R}^I}$, where

$$\operatorname{dom} V = \{r : V(r) \neq \emptyset\} = \{\mathbf{e}_S : S \subseteq I\} \cup \{\mathbf{0}\},$$

where $V(\mathbf{e}_S)$ is a closed set, bounded above, and

$$\mathbb{R}^S_- \subset V(\mathbf{e}_S) \subset \mathbb{R}^S \text{ and } V(\mathbf{0}) = 0.$$

A $q \in V(\mathbf{e})$ is in the *core of the non-transferable utility game* V if

$$V(\mathbf{e}_S) - \{q_S\} \cap \mathbb{R}^S_{++} = \emptyset, \quad \forall S \subseteq I,$$

i.e., there is no $S \subseteq I$ and $u_S \in V(\mathbf{e}_S)$ such that $u_S \gg q_S$.

To construct a non-transferable utility V from the data of $\mathcal{E}$, let

$$V_{\mathcal{E}}(\mathbf{e}_S) = \left\{u_S = \langle \nu_i(z_i) \rangle : \sum_{i \in S} z_i = \mathbf{0}\right\} + \mathbb{R}^S_-, \quad S \subseteq I.$$

(Concavity of ν_i implies $V_{\mathcal{E}}(\mathbf{e}_S)$ is convex, but not necessarily polyhedral.) As $v_{\mathcal{E}}$ is the basis for the positively homogeneous extensions, $v^\infty_{\mathcal{E}}$ and $w^\infty_{\mathcal{E}}$, $V_{\mathcal{E}}$ can also be extended in two ways as:

Please cite this article in press as: Ostroy, J.M. Competitive pricing and the core: With reference to matching. Games Econ. Behav. (2018), https://doi.org/10.1016/j.geb.2018.01.011

$$V^{\infty}_{\mathcal{E}}(r) = \Big\{ \sum_S \alpha_S \langle \nu_i(z^S_i)\rangle : \sum_S \alpha_S \mathbf{e}_S = r, \ \sum_{i \in S} z^S_i = \mathbf{0}, \alpha_S \geq 0 \Big\},$$

$$W^{\infty}_{\mathcal{E}}(r) = \Big\{ \sum_i r_i \langle \nu_i(z_i)\rangle : \ \sum_i r_i \mathbf{e}_i = r, \ \sum_i r_i z_i = \mathbf{0}, r_i \geq 0 \Big\},$$

with the vector-valued comparison

$$V^{\infty}_{\mathcal{E}}(r) \subseteq W^{\infty}_{\mathcal{E}}(r), \tag{13}$$

similar to the scalar-valued comparison in (10).

For $\lambda = \langle \lambda_i \rangle \gg \mathbf{0}$, let $\mathcal{E}[\lambda] = \langle \lambda_i \nu_i \rangle$. From the non-quasilinear, i.e., non-transferable utility, perspective $\lambda_i \nu_i$ is equivalent to ν_i. However, when $\mathcal{E}[\lambda]$ is regarded as a model with transferable utility, different (relative) weights lead to different optimal allocations and prices because $\lambda_i \nu_i(z_i)$ changes the tradeoff between non-money commodities and the money commodity. A transferable utility representation suitable for defining a Walrasian equilibrium for $\mathcal{E}[\lambda]$ is

$$\partial w^{\infty}_{\mathcal{E}[\lambda]}(r) := \sup\{\lambda \cdot W^{\infty}_{\mathcal{E}}(r)\}.$$

Extending (9) from $\mathcal{E}$ to $\mathcal{E}[\lambda]$,

$$q = \{\langle q_i = -\lambda_i \nu^*_i(p[\lambda])\rangle\} \in \partial w^{\infty}_{\mathcal{E}[\lambda]}(\mathbf{e}), \tag{14}$$

where $(p[\lambda], \langle z_i[\lambda]\rangle)$ is a Walrasian equilibrium for $\mathcal{E}[\lambda]$.

To demonstrate existence of Walrasian equilibrium without transfers for the non-quasilinear $\mathcal{E}$, it would suffice to show the existence of a $\lambda \gg 0$ satisfying (14) that also has the property

$$q = \langle q_i = \lambda_i \nu_i(z_i)\rangle \in \partial w^{\infty}_{\mathcal{E}[\lambda]}(\mathbf{e}), \tag{15}$$

because in that case $p[\lambda] \cdot z_i[\lambda] = 0$, for all i. More formally stated,

Proposition 3 *(Characterization of Walrasian Equilibrium without Transfers). If there exists $\lambda \gg \mathbf{0}$ such that*

$$q = \langle q_i = \lambda_i v_i(z_i[\lambda])\rangle \in \partial w^{\infty}_{\mathcal{E}[\lambda]}(\mathbf{e}),$$

where $\sum_i z_i[\lambda] = \mathbf{0}$ and $\sum_i \lambda_i \nu_i(z_i) = w^{\infty}_{\mathcal{E}[\lambda]}(\mathbf{e})$, then there exists $p[\lambda]$ such that

$$q = \langle q_i = -\lambda_i v^*_i(p[\lambda] = \lambda_i v_i(z_i[\lambda])\rangle \in \partial w^{\infty}_{\mathcal{E}[\lambda]}(\mathbf{e}), \ \textit{hence}, \ p[\lambda] \cdot z_i[\lambda] = 0, \quad \forall i,$$

i.e., $(p[\lambda], \langle z_i[\lambda]\rangle)$ is a Walrasian without transfers for the non-quasilinear version of $\mathcal{E}[\lambda]$.

With the qualification that the underlying convex preferences can be represented by concave utilities, this construction shows how the more elementary problem of demonstrating existence of Walrasian equilibrium for a family of quasilinear models $\mathcal{E}[\lambda]$ is related to the existence of its non-quasilinear counterpart. However, the decisive step of showing the existence of a $\lambda \gg 0$ for which transfers are zero requires a fixed-point argument. In the assignment model, a duality demonstration suffices.

4. The assignment model

4.1. Transferable utility

In the assignment model individuals exchange with each other, one-on-one. $I = A \cup B$ is divided into two disjoint groups, individual members of A to be matched with individual members of B. The data of the model is defined by $\mathcal{V} : A \times B \to \mathbb{R}_+$, where $\mathcal{V}(a, b)$ is the value of matching $a \in A$ with $b \in B$.

Denote by $\mathcal{E}_{\mathcal{A}} = \langle \nu_i \rangle$ an exchange economy description of an assignment model. A commodity space compatible with transferable and non-transferable utility versions of the assignment model is $\mathbb{R}^{\ell} := \mathbb{R}^{A \times B}$. For $(a, b) \in A \times B$, e_{ab} is the unit vector in $\mathbb{R}^{A \times B}$. The restrictions on $\operatorname{dom} \nu_i$ are:

$$\operatorname{dom} \nu_i = \begin{cases} \{-e_{ab} : b \in B\} \cup \{\mathbf{0}\} & i = a \in A, \\ \{e_{ab} : a \in A\} \cup \{\mathbf{0}\} & i = b \in B. \end{cases} \tag{16}$$

Individuals in A supply themselves to some individual in B, or do not trade; and individuals in B buy themselves from someone in A, or do not trade.

The same conditions on $z = \langle z_i \rangle$, i.e., $\sum_i \nu_i(z_i) > -\infty$ and $\sum_i z_i = \mathbf{0}$ defining a feasible allocation in $\mathcal{E}$, i.e., when the population is $\mathbf{e}$ and the aggregate resource constraint is $\mathbf{0}$, also apply to $\mathcal{E}_{\mathcal{A}}$. However, in $\mathcal{E}_{\mathcal{A}}$ aggregate balance is superseded by bilateral balance: if $z_a \neq \mathbf{0}$, there exists a b such that

$$z_a[= -e_{ab}] + z_b[= e_{ab}] = \mathbf{0}.$$

Therefore, $\max\{v_a(z_a) + v_b(z_b), 0\} = \mathcal{V}(a,b)$ reproduces the data of a transferable utility/quasilinear version of the assignment model.

Prices have the same dimension as the commodity space, i.e., $p \in \mathbb{R}^{A\times B}$. The definition of Walrasian equilibrium from $\mathcal{E}$ and its characterization in Proposition 2 also applies to $\mathcal{E}_{\mathcal{A}}$. If $(p, \langle z_i \rangle)$ is a Walrasian equilibrium at $\mathbf{e}$, let $q_i = -v_i^*(p) = v_i(z_i) - p\cdot z_i$. If a and b are paired in $\mathcal{E}_{\mathcal{A}}$, then $z_a = -z_b$. Hence, $p\cdot(z_a + z_b) = \mathbf{0}$ and

$$\begin{aligned} q_a + q_b &= -[v_a^*(p) + v_a^*(p)] = [v_a(z_a) - p\cdot z_a] + [v_b(z_b) - p\cdot z_b] \\ &= v_a(z_a) + v_b(z_b) = \mathcal{V}(a,b). \end{aligned}$$

Existence of money transfers allows the division of the joint gains $\mathcal{V}(a,b)$ according to $-p\cdot z_a = p\cdot z_b$; e.g., if $-p\cdot z_a = -p\cdot[-e_{ab}] > 0$, then in addition to $v_a(-e_{ab})$, individual a receives $m_i = -p\cdot z_a$ from b. [Note: Proposition 3 also applies to $\mathcal{E}_{\mathcal{A}}[\lambda]$, $\lambda \gg \mathbf{0}$. If $(p[\lambda], \langle z_i[\lambda]\rangle)$ is a Walrasian *without transfers* for the non-quasilinear version of $\mathcal{E}_{\mathcal{A}}[\lambda]$, and a and b are paired, $-p[\lambda]\cdot z_a = -p[\lambda]\cdot z_b = 0$.]

A key difference between $\mathcal{E}$ and $\mathcal{E}_{\mathcal{A}}$ is the indivisibility of commodities in (16). The absence of concavity means that maximization of total gains may require that individuals of same type i are not necessarily given the same allocation, z_i. This changes the description of $\Phi_{\mathcal{E}}$ to

$$\Phi_{\mathcal{E}_{\mathcal{A}}}(r,\bar{z}) = \sup\Big\{\sum_i\sum_h r_i^h v_i(z_i^h) : \sum_i\sum_h r_i^h \mathbf{e}_i = r, \sum_i\sum_h r_i^h z_i^h = \bar{z}, r_i^h \geq 0\Big\},$$

where z_i^h denotes the allocation to an individual with utility v_i and $z_i^h \neq z_i^{h'}$, $h \neq h'$. However, the well-known integer optimality properties of the assignment model implies the non-concavity of v_i has especially benign consequences.

Recall the definition of $v_{\mathcal{E}}(\cdot\,|\,k\mathbf{e})$ in Section 3.1 and its extension to $v_{\mathcal{E}}^{\infty}(\cdot\,|\,k\mathbf{e})$. The source of the difference between $w_{\mathcal{E}}^{\infty}$ and $v_{\mathcal{E}}^{\infty}$ — and therefore Walrasian equilibria $(\partial w_{\mathcal{E}}^{\infty})$ and the core $(\partial_C v_{\mathcal{E}}^{\infty})$ — is the fact that the generators of $v_{\mathcal{E}}^{\infty}(\cdot\,|\,k\mathbf{e})$ are increasing with k. The generators of the analogous constructions $v_{\mathcal{E}_{\mathcal{A}}}^{\infty}(\cdot\,|\,k\mathbf{e})$ do not vary. They are limited to those $r = \mathbf{e}_S$ that are singletons, i.e., $\mathbf{e}_i$, $i \in I$, and the pairs $\mathbf{e}_a + \mathbf{e}_b$, $a \in A$, $b \in B$. Since the generators do not vary,

$$\begin{aligned} v_{\mathcal{E}_{\mathcal{A}}}^{\infty}(r) &= \max\Big\{\sum_S \alpha_S\Big(\sum_{i\in S} v_i(z_i^S)\Big) : \sum_S \alpha_S \mathbf{e}_S = r, \sum_{i\in S} z_i^S = \mathbf{0}, \alpha_S \geq 0\Big\} \\ &= \max\Big\{\sum_i\sum_h r_i^h v_i(z_i^h) : \sum_i\sum_h r_i^h \mathbf{e}_i = r, \sum_i r_i^h z_i^h = \mathbf{0}, r_i^h \geq 0\Big\} \\ &= w_{\mathcal{E}_{\mathcal{A}}}^{\infty}(r). \end{aligned} \tag{17}$$

The equality in (17) is similar to the equality in (12) for a direct market, $\mathcal{E}_D$. But the generators of $v_{\mathcal{E}_{\mathcal{A}}}^{\infty} = w_{\mathcal{E}_{\mathcal{A}}}^{\infty}$ are a restricted subset of $\{\mathbf{e}_S : S \subseteq I\}$, the generators of the direct market yielding $v_{\mathcal{E}_D}^{\infty} = w_{\mathcal{E}_D}^{\infty}$. Consequences of this added restriction are examined in Section 5.2.

4.2. Non-transferable utility

Whether quasilinear or not, Walrasian equilibrium requires that individuals with the same tastes and trading opportunities are such that the utility received by one will not be different from the utility received by the other. The following example exploits this property to illustrate that without quasilinearity pricing equilibrium need not exist.

Example 1 *(Non-existence in the Assignment Model without Transfers).* Assume $A = \{a\}$ and $B = \{b_1, b_2\}$: a prefers trade to no trade, but is indifferent between b_1 and b_2; and both b's prefer to trade with a. I.e.,

$$v_a(-e_{ab_1}) = v_a(-e_{ab_2}) > v_a(\mathbf{0}) = 0, \quad v_{b_k}(e_{ab_k}) > v_{b_k}(\mathbf{0}) = 0,\ k = 1, 2.$$

In terms of preferences, b_1 and b_2 have the same tastes and are regarded by a as perfect substitutes. If a were to trade with b_1 and money transfers were required to cancel, then $p_{ab_1} = 0$ and

$$\begin{aligned} \lambda_a v_a^*(p) &= \lambda_a v_a(-e_{ab_1}) + p_{ab_1} = \lambda_a v_a(-e_{ab_1}) \\ \lambda_{b_1} v_{b_1}^*(p) &= \lambda_{b_1} v_{b_1}(e_{ab_1}) - p_{ab_1} = \lambda_{b_1} v_{b_1}(e_{ab_1}). \end{aligned}$$

For prices to discourage trade by b_2, $p_{ab_2} \geq \lambda_{b_2} v_{b_2}(e_{ab_2}) > 0$. But at a positive p_{ab_2}, a would prefer to trade with b_2. A similar contradiction would occur if a were to trade with b_2.

Please cite this article in press as: Ostroy, J.M. Competitive pricing and the core: With reference to matching. Games Econ. Behav. (2018), https://doi.org/10.1016/j.geb.2018.01.011

Example 2 *(Existence with strict preference).* Eliminate indifference by assuming a prefers trade with b_1 compared to b_2; specifically, $\nu_a(-e_{ab_1}) = 1$ and $\nu_a(-e_{ab_2}) = 1 - \epsilon$, $0 < \epsilon < 1$. Now, each of the three individuals is a different type. With prices such that a wants to supply b_1, and b_1 wants to buy a without transfers, i.e., $p_{ab_1} = 0$,

$$\begin{aligned}\lambda_a \nu_a^*(p) &= \lambda_a \nu_a(-e_{ab_1}) + p_{ab_1} = \lambda_a \nu_a(-e_{ab_1}) = \lambda_a \\ &\geq \lambda_a \nu_a(-e_{ab_2}) + p_{ab_2} = \lambda_a(1-\epsilon) + p_{ab_2}.\end{aligned}$$

Again, b_2 does not gain from trade when $p_{ab_2} \geq \lambda_{b_2} \nu_{b_2}(e_{ab_2}) > 0$. For any $\epsilon > 0$, the inequalities are satisfied if

$$\frac{\epsilon \lambda_a}{\lambda_{b_2}} \geq \frac{p_{ab_2}}{\lambda_{b_2}} \geq \nu_{b_2}(e_{ab_2}) > 0.$$

However, as $\epsilon \to 0$, $\lambda_a \to \infty$, i.e., the relative weight of λ_{b_2} compared to λ_a (and λ_{b_1}) must be going to 0, confirming that as ν_a approaches indifference, there are, equivalently, no prices and no strictly positive weights to sustain an equilibrium without transfers. The limiting failure of prices to exist without transfers when there is indifference in $\mathcal{E}_\mathcal{A}$ is similar to what may occur in $\mathcal{E}$ when ν_i does not satisfy a Lipschitz condition, leading to a failure of subdifferentiability.

4.2.1. *Stable matchings*

Definition 3. $z = \langle z_i \rangle$ is a *stable matching* for $\mathcal{E}_\mathcal{A}$ if $\sum_i \nu_i(z_i) > -\infty$, $\sum_i z_i = \mathbf{0}$ and there does not exist $y_a + y_b = \mathbf{0}$ such that

$$(\nu_a(y_a), \nu_b(y_b)) - (\nu_a(z_a), \nu_b(z_a)) \in \mathbb{R}^2_{++}.$$

Gale and Shapley (1962) formulated the problem and demonstrated, via a simple algorithm, the existence of stable matchings assuming:

Definition 4. Preferences are *strict* in $\mathcal{E}_\mathcal{A}$ if for each i and $z_i, y_i \in \operatorname{dom} \nu_i$, $\nu_i(z_i) = \nu_i(y_i)$ only if $z_i = y_i$.

With strict preferences — the default hypothesis in two-sided matching (Roth and Sotomayor, 1990), the pricing properties of the transferable utility version of $\mathcal{E}_\mathcal{A}$ can be translated, via Shapley's λ-transfer method, to show that a Walrasian equilibrium without transfers exists for any stable matching.

Proposition 4 *(Walrasian Equilibrium for $\mathcal{E}_\mathcal{A}$ without Transfers). If preferences are strict in $\mathcal{E}_\mathcal{A}$, then for any stable matching $z = \langle z_i \rangle$ there exists a $\lambda \gg 0$ and $p[\lambda]$ such that $(p[\lambda], \langle z_i[\lambda] = z_i \rangle)$ is a Walrasian equilibrium and $p[\lambda] \cdot z_i[\lambda] = 0, \forall i$.*

Proof. Vector-valued utilities in the non-transferable version of the assignment model are

$$V_{\mathcal{E}_\mathcal{A}}(\mathbf{e}_S) = \{u_S = \langle \nu_i(z_i^S) \rangle : \sum_{i \in S} z_i^S = \mathbf{0}\} + \mathbb{R}^S_-.$$

As a feasible allocation, the stability of z implies that setting

$$q^* = \langle q_i^* = \nu_i(z_i) \rangle,$$

q^* is in the core of $V_{\mathcal{E}_\mathcal{A}}$, i.e., $V_{\mathcal{E}_\mathcal{A}}(\mathbf{e}_S) - \{q_S^*\} \cap \mathbb{R}^I_{++} = \emptyset$, $S \subseteq I$. Moreover, as illustrated in Example 2, with strict preferences,

$$V_{\mathcal{E}_\mathcal{A}}(\mathbf{e}_S) - \{q_S^*\} \cap \mathbb{R}^I_+ \backslash \{\mathbf{0}\} = \emptyset, \quad S \subseteq I.$$

To demonstrate, suppose the contrary that there exist $S \subseteq I$ such that $u_S \in V_{\mathcal{E}_\mathcal{A}}(\mathbf{e}_S)$, $u_S - q_S^* \in \mathbb{R}^S_+ \backslash \{\mathbf{0}\}$. Assume a gainer is an $a \in A \cap S$. Then a must trade with some $b \in B \cap S$, i.e., b must leave his current trading partner in the stable matching (where he is possibly not trading) and go with a. But strict preference implies that, by trading, b changed his utility. And if it does not go down, it must go up, which contradicts the hypothesis that z is a stable matching. Consequently,

$$\alpha_S \big(V_{\mathcal{E}_\mathcal{A}}(\mathbf{e}_S) - \{q_S^*\}\big) \cap \mathbb{R}^I_+ \backslash \{\mathbf{0}\} = \emptyset, \quad \forall \alpha_S \geq 0, \ S \subseteq I. \tag{†}$$

Also, if z is a stable matching, and $S, T \subseteq I$, there is the following additivity condition:

$$V(\mathbf{e}_S) - \{q_S^*\} + V(e_T) - \{q_T^*\} \cap \mathbb{R}^I_+ \backslash \{\mathbf{0}\} = \emptyset. \tag{‡}$$

Again, supposing the contrary, since gains can only be achieved by pairs, there must exist $a, b \in S \cup T$ and $y_a + y_b = \mathbf{0}$ such that

$$\big(v_a(y_a), v_b(y_b)\big) - \big(q_a^* = v_a(z_a), q_b^* = v_b(z_b)\big) \in \mathbb{R}_+^I \backslash \{\mathbf{0}\}.$$

But, as above, this would also contradict the hypothesis that the utilities q^* represent a stable matching.

Properties (†) (scalar multiplicity) and (‡) (additivity) imply that the convex cone

$$K(V_{\mathcal{E}_{\mathcal{A}}}, q^*) = \Big\{ \sum_S \alpha_S \big[V_{\mathcal{E}_{\mathcal{A}}}(\mathbf{e}_S) - \{q_S^*\}\big] : \alpha_S \geq 0, S \subseteq I \Big\} \cap \mathbb{R}_+^I \backslash \{\mathbf{0}\} = \emptyset,$$

with polar $K^0(V_{\mathcal{E}_{\mathcal{A}}}, q^*) = \big\{\lambda : \lambda \cdot K(V_{\mathcal{E}_{\mathcal{A}}}, q^*) \leq 0\big\}$ has the property that

$$K^0(V_{\mathcal{E}_{\mathcal{A}}}, q^*) \cap \mathbb{R}_{++}^I \neq \emptyset.$$

From Proposition 2,

$$q = \langle \lambda_i q_i^* \rangle = \langle \lambda_i v_i(z_i) \rangle \in \partial v^\infty_{\mathcal{E}_{\mathcal{A}[\lambda]}}(\mathbf{e})$$

for $\lambda \gg 0 \in K^0(V_{\mathcal{E}_{\mathcal{A}}}, q^*)$.

From (25), $v^\infty_{\mathcal{E}_{\mathcal{A}[\lambda]}}(\mathbf{e}) = w^\infty_{\mathcal{E}_{\mathcal{A}[\lambda]}}(\mathbf{e})$. Hence, there exists $p[\lambda] \in \mathbb{R}^{A \times B}$ such that

$$q = \langle q_i = -\lambda_i v^*(p[\lambda]) \rangle \in \partial w^\infty_{\mathcal{E}_{\mathcal{A}[\lambda]}}(\mathbf{e}).$$

Therefore,

$$q_i = -\lambda_i v^*(p[\lambda]) = \lambda_i v_i(z_i) - p[\lambda] \cdot z_i = \lambda_i v_i(z_i),$$

or $p[\lambda] \cdot z_i = 0$, $\forall i$. This fulfills the conditions of Proposition 3 characterizing Walrasian equilibrium without transfers. □

5. Differentiability with respect to individuals

5.1. Differentiability as perfect substitutability

Pricing has, so far, been described as subdifferentiability. Subdifferentiability is defined for all perturbations which, assuming the effective domain of the function is convex, includes its infinitesimal counterparts. The emphasis, below, is on differentiability, i.e., on infinitesimal perturbations. Hence, differentiability with respect to individuals is more immediately interpreted in the infinite case when each individual is infinitesimal. To emphasize the meaning of differentiability with respect to individuals, a discrete version can be defined that is applicable to models with finite numbers. The relation between discrete differences and directional derivatives describes the sense in which increasing the number of individuals may, or may not, lead to differentiability.

Restricting attention to $r = k\mathbf{e}$, $k = 1, 2, \ldots$, concavity and positive homogeneity of $w^\infty_{\mathcal{E}}$ and Fact (v) imply

$$w^\infty_{\mathcal{E}}(k\mathbf{e}) - w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i) \equiv -\Delta w^\infty_{\mathcal{E}}(k\mathbf{e}; -\mathbf{e}_i) \geq \max\big\{q \cdot \mathbf{e}_i : q = \partial w^\infty_{\mathcal{E}}(k\mathbf{e}) = \partial w^\infty_{\mathcal{E}}(\mathbf{e})\big\}. \tag{18}$$

The inequality says that for any $q \in \partial w^\infty_{\mathcal{E}}(k\mathbf{e})$ the maximum value of $q \cdot \mathbf{e}_i = q_i$ cannot exceed the extra gains that i's participation adds to the total gains. When the inequality is strict, prices are such that i is always contributing more to $k\mathbf{e} - \mathbf{e}_i$ than they can achieve on their own.

Definition: Individual i is *perfectly substitutable* at $k\mathbf{e}$ if

$$-\Delta w^\infty_{\mathcal{E}}(k\mathbf{e}; -\mathbf{e}_i) = \max\{q \cdot \mathbf{e}_i : q = \partial w^\infty_{\mathcal{E}}(k\mathbf{e})\}.$$

Equivalently, i is perfectly substitutable when

$$\partial w^\infty_{\mathcal{E}}(k\mathbf{e}) \cap \partial w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i) \neq \emptyset. \tag{19}$$

The non-empty intersection in (19) says there is a $q \in \partial w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i)$ such that there is no need to change q_i to accommodate an additional i. Geometrically, (19) implies that $w^\infty_{\mathcal{E}}$ is "flat" along $[k\mathbf{e} - \mathbf{e}_i, k\mathbf{e}]$. When $k = 1$, there are no individuals of type i in $\mathbf{e} - \mathbf{e}_i$. Then $q \in \partial w^\infty_{\mathcal{E}}(\mathbf{e} - \mathbf{e}_i)$ means that $q \cdot \mathbf{e}_i = q_i$ is the *reservation price* for i, i.e., the price such that individuals in $\mathbf{e} - \mathbf{e}_i$ are *indifferent* between trading or not trading with i. The same statement also applies for perfect substitutability when $k > 1$. [As the assignment model will illustrate when $k = 1$, (19) can hold for a subset of individuals.]

From (9) in Proposition 2, $\partial w^\infty_{\mathcal{E}}(k\mathbf{e})$ is derived from $\partial f_{\mathcal{E}}(\mathbf{0} \,|\, k\mathbf{e})$ via $q_i = -v_i^*(p)$. To elaborate on that dependence, the value $w^\infty_{\mathcal{E}}(k\mathbf{e})$ can be obtained by maximizing the incremental gains to $\Phi_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i, \cdot) = f_{\mathcal{E}}(\cdot \,|\, k\mathbf{e} - \mathbf{e}_i)$ from trading with i as

$$w^\infty_{\mathcal{E}}(k\mathbf{e}) = \sup\{v_i(y_i) + f_{\mathcal{E}}(-y_i \,|\, k\mathbf{e} - \mathbf{e}_i)\}.$$

Recalling that $f_{\mathcal{E}}(\mathbf{0} \,|\, k\mathbf{e} - \mathbf{e}_i) = w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i)$, a restatement of (18) is

Please cite this article in press as: Ostroy, J.M. Competitive pricing and the core: With reference to matching. Games Econ. Behav. (2018), https://doi.org/10.1016/j.geb.2018.01.011

$$w^{\infty}_{\mathcal{E}}(k\mathbf{e}) - w^{\infty}_{\mathcal{E}}(k\mathbf{e}-\mathbf{e}_i) = \sup\left\{v_i(y_i) + [f_{\mathcal{E}}(-y_i\,|\,k\mathbf{e}-\mathbf{e}_i) - f_{\mathcal{E}}(\mathbf{0}\,|\,k\mathbf{e}-\mathbf{e}_i)]\right\}$$
$$\geq \sup\left\{-v^*_i(p) : p \in \partial f_{\mathcal{E}}(\mathbf{0}\,|\,k\mathbf{e})\right\} = \max\left\{q\cdot\mathbf{e}_i : q \in \partial w^{\infty}_{\mathcal{E}}(k\mathbf{e})\right\}.$$

Proposition 2 says that elements of $\partial f_{\mathcal{E}}(\mathbf{0}\,|\,k\mathbf{e}-\mathbf{e}_i)$ are Walrasian equilibrium prices for $k\mathbf{e}-\mathbf{e}_i$ and elements of $\partial f_{\mathcal{E}}(\mathbf{0}\,|\,k\mathbf{e})$ are Walrasian equilibrium prices for $k\mathbf{e}$. Therefore, if

$$\partial f_{\mathcal{E}}(\mathbf{0}\,|\,k\mathbf{e}) \cap \partial f_{\mathcal{E}}(\mathbf{0}\,|\,k\mathbf{e}-\mathbf{e}_i) \neq \emptyset, \tag{20}$$

then p belonging to the intersection means there is no need for a change in commodity prices at $k\mathbf{e}-\mathbf{e}_i$ to accommodate a utility maximizing trade at p when $\mathbf{e}_i$ is added. I.e., $f_{\mathcal{E}}(\cdot\,|\,k\mathbf{e}-\mathbf{e}_i)$ is "flat" along $[\mathbf{0}, -z_i]$ for $-v^*_i(p) = v_i(z_i) - p\cdot z_i$. Evidently, (20) implies (19) since there exists p such that

$$\sup\{v_i(y_i) + [f_{\mathcal{E}}(-y_i\,|\,k\mathbf{e}-\mathbf{e}_i)\} - f_{\mathcal{E}}(\mathbf{0}\,|\,k\mathbf{e}-\mathbf{e}_i)] = v_i(z_i) + p\cdot z_i.$$

[The converse can also be shown.]

When all individuals are perfectly substitutable at $k\mathbf{e}$, the function $w^{\infty}_{\mathcal{E}}$ is *discretely differentiable* at $k\mathbf{e}$, i.e.,

$$\partial w^{\infty}_{\mathcal{E}}(k\mathbf{e}) \bigcap_{i\in I} \partial w^{\infty}_{\mathcal{E}}(k\mathbf{e}-\mathbf{e}_i) \neq \emptyset. \tag{21}$$

(21) says that the gains from $w^{\infty}_{\mathcal{E}}(k\mathbf{e})$ can be distributed so that each i receives the amount its participation adds to the total: there is a $q \in \partial w^{\infty}_{\mathcal{E}}(k\mathbf{e})$ such that

$$k\sum_i q\cdot\mathbf{e}_i = k\sum_i -\Delta w^{\infty}_{\mathcal{E}}(k\mathbf{e}; -\mathbf{e}_i) = w^{\infty}_{\mathcal{E}}(k\mathbf{e}).$$

The commodity pricing analog of (21) is

$$\partial f_{\mathcal{E}}(\mathbf{0}\,|\,k\mathbf{e}) \bigcap_i \partial f_{\mathcal{E}}(\mathbf{0}\,|\,k\mathbf{e}-\mathbf{e}_i) \neq \emptyset. \tag{22}$$

I.e., there are Walrasian equilibrium commodity prices p for $k\mathbf{e}$ that are also Walrasian equilibrium prices for all $k\mathbf{e}-\mathbf{e}_i$.

The discrete connections, above, are readily extended to their (more standard) infinitesimal counterparts. The discrete difference $-\Delta w^{\infty}_{\mathcal{E}}(k\mathbf{e}; -\mathbf{e}_i)$ is non-increasing in k (Fact (i) Section 2). Moreover, the definition of a directional derivative (Fact (ii)) and its relation to the subdifferential (Fact (v)) implies that for any i, there is always a q such that any one i is *asymptotically* perfectly substitutable, i.e.,

$$\lim_{k\to\infty} -\Delta w^{\infty}_{\mathcal{E}}(k\mathbf{e}; -\mathbf{e}_i) = -Dw^{\infty}_{\mathcal{E}}(\mathbf{e}; -\mathbf{e}_i) = \max\{q\cdot\mathbf{e}_i : q = \partial w^{\infty}_{\mathcal{E}}(k\mathbf{e}) = \partial w^{\infty}_{\mathcal{E}}(\mathbf{e})\}.$$

The asymptotic extension of (21) is the *differentiability* of $w^{\infty}_{\mathcal{E}}$ at $\mathbf{e}$: the existence of a q such that

$$q_i = \lim_{k\to\infty} -\Delta w^{\infty}_{\mathcal{E}}(k\mathbf{e}; -\mathbf{e}_i) = -Dw^{\infty}_{\mathcal{E}}(\mathbf{e}; -\mathbf{e}_i), \quad \forall i. \tag{23}$$

Then, $\{q\} = \partial w^{\infty}_{\mathcal{E}}(k\mathbf{e})$, $k = 1, 2, \ldots$.

As in the discrete case, differentiability with respect to the pricing of individuals is based on differentiability with respect to the pricing of commodities. From positive homogeneity, $f_{\mathcal{E}}(z\,|\,k\mathbf{e}) - f_{\mathcal{E}}(\mathbf{0}\,|\,k\mathbf{e}) = k[f_{\mathcal{E}}(k^{-1}z\,|\,\mathbf{e}) - f_{\mathcal{E}}(\mathbf{0}\,|\,\mathbf{e})]$. Therefore,

$$\lim_{k\to\infty}[f_{\mathcal{E}}(z\,|\,k\mathbf{e}) - f_{\mathcal{E}}(z\,|\,k\mathbf{e})] = \lim_{k\to\infty} k[f_{\mathcal{E}}(k^{-1}z\,|\,\mathbf{e}) - f_{\mathcal{E}}(z\,|\,\mathbf{e})]$$
$$= Df_{\mathcal{E}}(\mathbf{0}; z\,|\,\mathbf{e}) = \max\{p\cdot z : p \in \partial f_{\mathcal{E}}(\mathbf{0}\,|\,\mathbf{e})\}.$$

Hence, there are commodity prices at which a trade by an infinitesimal individual can be regarded as perfectly substitutable. When, in addition, $f_{\mathcal{E}}(\cdot\,|\,\mathbf{e})$ is differentiable at $\mathbf{0}$, then $\partial f_{\mathcal{E}}(\mathbf{0}\,|\,\mathbf{e}) = \{p\}$ and the continuity of the derivative at $\mathbf{0}$ implies the asymptotic analog of (22),

$$\lim_{k\to\infty} \partial f_{\mathcal{E}}(\mathbf{0}\,|\,k\mathbf{e}-\mathbf{e}_i) = \lim_{k\to\infty} \partial f_{\mathcal{E}}(\mathbf{0}\,|\,\mathbf{e}-k^{-1}\mathbf{e}_i) = p, \quad \forall i. \tag{24}$$

Hence, all utility maximizing trades at p are perfectly substitutable.

Discrete differentiability is defined by *equalities* with respect to "marginal" conditions on individuals compared to the marginal and infra-marginal inequalities with respect to coalitions defining the core. Differentiability is similarly defined by asymptotic marginal equalities on individuals. Nevertheless, these marginal conditions defining perfect substitutability suffice to conclude:

Please cite this article in press as: Ostroy, J.M. Competitive pricing and the core: With reference to matching. Games Econ. Behav. (2018), https://doi.org/10.1016/j.geb.2018.01.011

Proposition 5 *(Differentiable Core). (I): If $w^\infty_{\mathcal{E}}$ is discretely differentiable at $k\mathbf{e}$ (all individuals in $k\mathbf{e}$ are perfectly substitutable), then $q \in \partial_C v_{\mathcal{E}}(k\mathbf{e}\,|\,k\mathbf{e})$ if and only if*

$$q_i = v_{\mathcal{E}}(k\mathbf{e}\,|\,k\mathbf{e}) - v_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i\,|\,k\mathbf{e}) = w^\infty_{\mathcal{E}}(k\mathbf{e}) - w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i), \quad \forall i.$$

(II): If $w^\infty_{\mathcal{E}}$ is differentiable at $\mathbf{e}$ (all infinitesimal individuals in $\mathbf{e}$ are perfectly substitutable), then $q \in \lim_{k\to\infty} \partial_C v_{\mathcal{E}}(k\mathbf{e}\,|\,k\mathbf{e}) = \partial w^\infty_{\mathcal{E}}(\mathbf{e})$ if and only if

$$q_i = \lim_{k\to\infty} w^\infty_{\mathcal{E}}(k\mathbf{e}) - w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i) = \lim_{k\to\infty} -\Delta w^\infty_{\mathcal{E}}(k\mathbf{e}; -\mathbf{e}_i), \quad \forall i.$$

When the subdifferential exhibits differentiability it has the following desirable property. Applied to $w^\infty_{\mathcal{E}}$, if $r \in \mathbb{R}^I_{++}$ and $\partial w^\infty_{\mathcal{E}}(r) = \{q\}$, then for any $\epsilon > 0$, there is a $\bar{k}$ such that if $k \geq \bar{k}$ and $q' \in \partial w^\infty_{\mathcal{E}}(kr - \mathbf{e}_i)$, then $\|q' - q\| < \epsilon$. I.e., differentiability implies that the larger the number of individuals, the closer an individual is to being perfectly substitutable. Without differentiability, continuity fails: If $q \in \partial w^\infty_{\mathcal{E}}(r)$, but $\partial w^\infty_{\mathcal{E}}(r)$ is not a singleton, there is a q^k and i^k such that $q^k \in \partial w^\infty_{\mathcal{E}}(kr - \mathbf{e}_{i^k})$ and $\|q - q^k\| > \epsilon$, $k = 1, 2, \ldots$. Without differentiability, an individual's ability to change prices does not vanish as the number of individuals increases.

The definition of perfect substitutability for models with transferable utility was originally formulated for models with ordinal preferences, called *no-surplus* (Ostroy, 1980; Makowski, 1980). An ordinal characterization of differentiability as asymptotically no-surplus is given in Ostroy (1981) for a sequence of exchange economies where both the number of individuals and the number of commodities in increasing.

5.2. Perfect substitutability precludes favorable manipulation

The subdifferentials of the optimal value functions associated with $\mathcal{E} = \langle v_i \rangle$ measure how those function vary as the commodity constraints, $f_{\mathcal{E}}(z\,|\,r) = \Phi_{\mathcal{E}}(r, z)$, or population constraints, $w^\infty_{\mathcal{E}}(r) = \Phi_{\mathcal{E}}(r, \mathbf{0})$, vary. A change in utility is another kind of perturbation that changes the way the constraints are valued. This Section shows the implications of perfect substitutability for individual perturbations of utilities.

Let v_0 be a concave function satisfying the same restrictions as the utility functions, above, and having the same trading possibilities as v_i, i.e., $\operatorname{dom} v_0 = \operatorname{dom} v_i$. Denote by $(k\mathbf{e} - \mathbf{e}_i + \mathbf{e}_0)$ the perturbation of $k\mathbf{e}$ in which one individual with utility v_i is replaced by another with utility v_0; i.e., a model having k individuals with utilities v_j, $j \neq i$, $(k-1)$ individuals with utilities v_i, and *one individual* with utility v_0. The linear space of populations is now $\mathbb{R}^{I+1}$, where the added dimension accommodates the introduction of v_0.

The maximum gains for that model is denoted $w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i + \mathbf{e}_0)$. An element of $\partial w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i + \mathbf{e}_0)$ is a $\tilde{q} = (\langle \tilde{q}_j \rangle_{j \neq i}, \tilde{q}_i, \tilde{q}_0)$. (As above, when $k = 1$ there are no individuals with utility v_i; hence, $\tilde{q}_i$ is a reservation price, a price at which it would not be profitable to introduce anyone with utility v_i.)

By construction, the trading opportunities underlying $w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i + \mathbf{e}_0)$ and $w^\infty_{\mathcal{E}}(k\mathbf{e})$ are identical. Let $w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i + \mathbf{e}_0\,|\,v_i)$ represent the utility gains of the allocation maximizing $w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i + \mathbf{e}_0)$ when the allocation to v_0 is evaluated using v_i. Evidently,

$$w^\infty_{\mathcal{E}}(k\mathbf{e}) \geq w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i + \mathbf{e}_0\,|\,v_i).$$

If i (having v_i) is perfectly substitutable at q in $w^\infty_{\mathcal{E}}(k\mathbf{e})$,

$$q_i = -\Delta w^\infty_{\mathcal{E}}(k\mathbf{e}; -\mathbf{e}_i) = w^\infty_{\mathcal{E}}(k\mathbf{e}) - w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i).$$

For any v_0,

$$\tilde{q} \cdot (k\mathbf{e} - \mathbf{e}_i + \mathbf{e}_0 - \mathbf{e}_0) = \tilde{q} \cdot (k\mathbf{e} - \mathbf{e}_i) \geq w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i + \mathbf{e}_0 - \mathbf{e}_0) = w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i).$$

Since q_i allows i to obtain $w^\infty_{\mathcal{E}}(k\mathbf{e}) - w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i)$ while $\tilde{q}_i$ allows i at most $w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i + \mathbf{e}_0\,|\,v_i) - w^\infty_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i)$,

Proposition 6. *If there is a q such that i is perfectly substitutable at $k\mathbf{e}$, i cannot improve its payoff by claiming to be v_0.*

Proposition 6 is a sufficient condition for non-manipulability of utilities. When the set of possible misrepresentations v_0 for each i is sufficiently large, perfect substitutability for all individuals is also necessary for efficient incentive compatible mechanism design for both transferable and non-transferable utility (Makowski et al., 1999). Hence, the robust possibility of perfect substitutability requires large numbers where differentiability with respect to individuals can be a robust conclusion.

Please cite this article in press as: Ostroy, J.M. Competitive pricing and the core: With reference to matching. Games Econ. Behav. (2018), https://doi.org/10.1016/j.geb.2018.01.011

5.3. Perfect substitutability in the transferable utility assignment model

The equality $w^{\infty}_{\mathcal{E}_{\mathcal{A}}} = v^{\infty}_{\mathcal{E}_{\mathcal{A}}}$ in (17) means $\partial w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(k\mathbf{e}) = \partial v^{\infty}_{\mathcal{E}_{\mathcal{A}}}(k\mathbf{e})$. Shapley and Shubik (1972) point out that replication does not change the size of the core in the assignment model. This can be expressed as

$$\partial w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(k\mathbf{e}) = \partial_C v_{\mathcal{E}_{\mathcal{A}}}(\mathbf{e}), k = 1, 2, \ldots.$$

Core equivalence was essential to the demonstration of Proposition 4, above.

The special properties of $\mathcal{E}_{\mathcal{A}}$ also admit perfect substitutability. To highlight that feature, its failure in $\mathcal{E}$ arises from the inequality in (18),

$$w^{\infty}_{\mathcal{E}}(k\mathbf{e}) - w^{\infty}_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i) \geq \max\{q \cdot e_i : q \in \partial w^{\infty}_{\mathcal{E}}(k\mathbf{e}) = \partial w^{\infty}_{\mathcal{E}}(\mathbf{e})\},$$

that is non-increasing in k. Hence, the failure of perfect substitutability, i.e., when the inequality is strict, occurs when $-\Delta w^{\infty}_{\mathcal{E}}(k\mathbf{e}; -\mathbf{e}_i) = w^{\infty}_{\mathcal{E}}(k\mathbf{e}) - w^{\infty}_{\mathcal{E}}(k\mathbf{e} - \mathbf{e}_i)$ is decreasing in k, a consequence of the strict superadditivity

$$w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(k\mathbf{e} - \mathbf{e}_i) > w^{\infty}_{\mathcal{E}_{\mathcal{A}}}([k-1]\mathbf{e}) + w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(\mathbf{e} - \mathbf{e}_i).$$

In $\mathcal{E}_{\mathcal{A}}$, where the generators consist of single individuals and matched pairs, the indivisibility of the unit amounts an individual can contribute implies that

$$w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(k\mathbf{e} - \mathbf{e}_i) = w^{\infty}_{\mathcal{E}_{\mathcal{A}}}([k-1]\mathbf{e}) + w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(\mathbf{e} - \mathbf{e}_i), \quad \forall k.$$

Therefore,

$$\begin{aligned} w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(k\mathbf{e}) - w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(k\mathbf{e} - \mathbf{e}_i) &= w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(\mathbf{e}) - w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(\mathbf{e} - \mathbf{e}_i) \\ &= \max\{q \cdot \mathbf{e}_i : q \in \partial w^{\infty}_{\mathcal{E}}(k\mathbf{e}) = \partial w^{\infty}_{\mathcal{E}}(\mathbf{e})\}. \end{aligned} \tag{25}$$

From the characterization of Walrasian equilibrium in (9) (Proposition 2) applied to $\mathcal{E}_{\mathcal{A}}$, for any $q \in \partial w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(\mathbf{e})$ there exists p such that for all i, $q_i = -\nu^*_i(p) = \nu_i(z_i) - p \cdot z_i$. To elaborate on its implications, let $\langle z_i \rangle$ be *any* optimal assignment for $\mathbf{e}$. Then there are prices such that $(p, \langle z_i \rangle)$ is a Walrasian equilibrium. If a and b are matched in $\langle z_i \rangle$,

$$p \cdot z_a = -p \cdot e_{ab} = -p_{ab} = p \cdot (-e_{ab}) = p \cdot z_b.$$

Since

$$q_a = \nu_a(z_a) - [-p_{ab}],$$

the greater is p_{ab}, the greater is the gain to a. Holding $z = \langle z_i \rangle$ fixed and assuming $p_{ab} > 0$, raise p_{ab} to the maximum value, and $p_{ba}(= -p_{ab})$ to the minimum value such a would be perfectly substitutable, i.e., b could find partners in $\mathbf{e} - \mathbf{e}_a$ such that together they could achieve the total gains available in $w_{\mathcal{E}_{\mathcal{A}}}(\mathbf{e} - \mathbf{e}_a) = v_{\mathcal{E}_{\mathcal{A}}}(\mathbf{e} - \mathbf{e}_a)$.

The equalities in (25) imply that there exist prices $\overline{p}^a \in \mathbb{R}^{A \times B}$ and $\overline{q}^a = \langle -\nu^*_i(\overline{p}^a) \rangle \in \partial w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(\mathbf{e})$ such that

$$\overline{q}^a \cdot \mathbf{e}_a = \overline{q}^a_a = -D w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(\mathbf{e}; -\mathbf{e}_a) = -\Delta w^{\infty}_{\mathcal{E}}(\mathbf{e}; -\mathbf{e}_a).$$

The gains a receives at $\overline{p}^a$ is the highest p_{ab} consistent with an optimal assignment. It achieves the upper bound on what a can hope to obtain in the core since it makes the individuals in $\mathbf{e} - \mathbf{e}_i$ just indifferent between trading and not trading with a: a achieves the highest value in a match with b by reducing the value to b to the lowest such that b is just indifferent between pairing with b or matching with $a' \neq a$ (or not being matched). This does not interfere with $a' \neq a$ also achieving the highest value in its optimally assigned match with $b' \neq b$. I.e., there exists Walrasian prices $\overline{p}^A$ such that all members of A receive their upper bounds. Therefore, there exists $\overline{q}^A = \langle -\nu^*_i(\overline{p}^A) \rangle \in \partial w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(\mathbf{e})$,

$$\overline{q}^A_a = -D w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(\mathbf{e}; -\mathbf{e}_a) = -\Delta w^{\infty}_{\mathcal{E}}(\mathbf{e}; -\mathbf{e}_a), \; \forall a \in A.$$

Analogous conditions hold for the existence of $\overline{p}^B$ and $\overline{q}^B_b$. This feature of the assignment model is equivalent to Shapley and Shubik's (1972) characterization of pricing in $\mathcal{E}_{\mathcal{A}}$ as exhibiting a lattice property.

Evidently, $\overline{q}^A_a \geq \overline{q}^B_a$, $\forall a$ and $\overline{q}^B_b \geq \overline{q}^A_b$, $\forall b$. The condition that duplicates in $\mathcal{E}_{\mathcal{A}}$ what replication typically achieves for $\mathcal{E}$ is that *all* individuals simultaneously receive their highest payoffs, i.e., $\overline{q}^A = q = \overline{q}^B$; hence, the highest payoff for each individual is also the lowest. In that case, $\{q\} = \partial w^{\infty}_{\mathcal{E}_{\mathcal{A}}}$, and

$$q_i = -\Delta w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(\mathbf{e}; -\mathbf{e}_i) = -D w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(\mathbf{e}; -\mathbf{e}_i), \quad \forall i. \tag{26}$$

[Note: Example 1 in Section 4.2 illustrated non-existence without transfers. With transfers, it illustrates (26). To elaborate, assume b_1 and b_2 have the same utilities, $\nu_{b_1}(e_{ab_1}) = \nu_{b_2}(e_{ab_2}) = \alpha > 0$ (the utility of no trade is normalized to 0).

Please cite this article in press as: Ostroy, J.M. Competitive pricing and the core: With reference to matching. Games Econ. Behav. (2018), https://doi.org/10.1016/j.geb.2018.01.011

Equilibrium prices are $p_{ab_1} = p_{ab_2} = \alpha$, leaving b_1 and b_2 indifferent between buying or not buying, although one b trades and the other does not. The same prices would also describe an equilibrium if any two of the three were present. (Without individual a, $p_{ab_1} = p_{ab_2} = \alpha$ are reservation prices for the commodities available to $\mathbf{e} - \mathbf{e}_a$.)

As an assignment model with transfers, the uniqueness of prices in Example 1 is not typical; and populations $k\mathbf{e}$, $k = 1, 2, \ldots$, in $\mathcal{E}_{\mathcal{A}}$ with the same number of each type are typically incompatible with perfect substitutability for all i. However, the (polyhedral) concavity of $w^{\infty}_{\mathcal{E}_{\mathcal{A}}}$ means that instances of non-differentiability are exceptional, and since differentiability implies discrete differentiability in the assignment model, most large (finite) populations with unequal numbers of types will satisfy (26). Hence, unequal numbers of types in $\mathcal{E}_{\mathcal{A}}$ serves as a substitute for replicating $\mathbf{e}$ in $\mathcal{E}$.]

5.4. Approximating perfect substitutability in the non-transferable utility assignment model

The best outcome for any $i \in I$ in the quasilinear version of $\mathcal{E}_{\mathcal{A}}$ can be obtained from any optimal matching by varying prices, i.e., money payments. Such adjustments are precluded in the non-quasilinear version. Substitution possibilities are, therefore, limited to variations in the "dual," i.e., stable matchings in the assignment model without transfers can incorporate the role of prices when there are transfers. For example, if $\partial w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(\mathbf{e}) = \{q\}$ in the transferable utility model, no individual has an incentive to manipulate his utility. If there is a unique stable matching in the non-transferable version, the same conclusion holds (Gale and Sotomayor, 1985). Relatedly, the lattice property of $\partial w^{\infty}_{\mathcal{E}_{\mathcal{A}}}(\mathbf{e})$ with transferable utility, allowing each of the individuals of one side of the market to be perfectly substitutable, is analogous to the lattice property for stable matches without transfers, at which there is also no gain from preference manipulation (see Roth and Sotomayor, 1990).

The parallels are incomplete. Replication, which plays a central role with transfers, is inconsistent with strict preferences: if $k\mathbf{e}$, $k > 1$, an $a \in A$ would be indifferent between being matched with two individuals of the same type in B. Nevertheless, if differences among individuals were small, there would be a meaningful sense in which substitution possibilities could be almost perfect, with conclusions that are approximately similar to the ideal limiting case. To this end, a sequence $\mathcal{E}_{\mathcal{A}}(\mathbf{e}^k)$, where $\mathbf{e}^k$ refers to I_k, below, representing assignment models in which preferences are strict while the number of distinct types is increasing, allows individuals to become increasingly closer substitutes.

Represent the set $I_k = A_k \cup B_k$, where A_k and B_k are k equally spaced points in separate unit intervals, $[0, 1]$, $1/k, 2/k, \ldots, k/k$. For each $a \in A_k$, $v^k_a(-e_{ab})$ can be abbreviated as $v^k_a(b)$ to designate the unique $b \in B_k \cup \{\mathbf{0}\}$ with whom a could be matched. Preferences are strict if $v^k_a : B_k \cup \{\mathbf{0}\} \to \mathbb{R}$ is invertible. Similar statements apply to $v^k_b : A_k \cup \{\mathbf{0}\} \to \mathbb{R}$. For example, the condition for assortative matching is: for each $a \in A_k$ and $b, b' \in B_k$, $v^k_a(b) > v^k_a(b')$ if $b > b'$; and for each $b \in B_k$ and $a, a \in A_k$, $v^k_b(a) > v^k_b(a')$ if $a > a'$ ($\mathbf{0}$ is the least preferred choice). If the uniformly kth ranked b by the a's were paired with the uniformly kth ranked a by the b's, that would constitute a stable matching.

From Proposition 4, a stable matching for $\mathcal{E}_{\mathcal{A}}(\mathbf{e}^k)$ can be represented as a pricing equilibrium (p^k, z^k) without transfers for some $(\lambda^k_i v^k_i)$, $\lambda^k_i > 0$. A condition that "nearby" individuals, in the sense of their assigned locations in $[0, 1]$, are becoming closer substitutes is the following equi-continuity condition on the preferences of individuals in the sequence $\mathcal{E}_{\mathcal{A}}(\mathbf{e}^k)$: For any $\delta > 0$, there exists $\bar{k}$ such that if $k > \bar{k}$, $b, b' \in B_k$ and $|b - b'| < \bar{k}^{-1}$, then

$$\max_{a \in A_k} |v^k_a(b) - v^k_a(b')| < \delta;$$

and similarly for $a, a' \in A_k$ and $\max_{b \in B_k} |v^k_b(a) - v^k_b(a')|$. If $\lambda^k : A_k \cup B_k \to \mathbb{R}_{++}$ is similarly equicontinuous, the property of pricing equilibrium without transfers that

$$(\lambda^k_i v^k_i)^*(p_k) = \max\{\lambda^k_i v^k_i(y^k_i) - p^k \cdot y^k_i\} = \lambda^k_i v^k_i(z^k_i),$$

implies that prices of nearby commodities are nearly equal. In the quasilinear version of the assignment model with equi-continuity, the genericity of perfect competition (and the meaning of 'genericity') is established in Gretsky et al. (1999).

6. Concluding remark

Differentiability of Walrasian equilibrium *with respect to individuals* yields a description of the competitive pricing of individuals as exhibiting perfect substitutability that subsumes competitive pricing of commodities. This description builds on Shapley's contributions: the definition of differentiability requires that the general equilibrium model of economic interdependence be viewed from the perspective of a game in characteristic function form, i.e., as a *market game*; and on the subdifferentiability of the prices of individuals that characterizes the *core*. In those settings with large numbers of individuals where differentiability is likely to hold, differentiability is a generic feature of subdifferentiability.

References

Bondareva, Olga, 1963. Some applications of linear programming to the theory of cooperative games. Probl. Kibern. 14, 9–25 (in Russian).
Gale, David, Shapley, Lloyd, 1962. College admissions and the stability of marriage. Amer. Math. Monthly 69, 9–15.
Gale, David, Sotomayor, Marilda, 1985. Ms. Machiavelli and the stable matching problem. Amer. Math. Monthly 92 (4), 261–268.
Gretsky, Neil, Ostroy, Joseph, Zame, William, 1999. Perfect competition in the continuous assignment model. J. Econ. Theory 88, 60–118.

Please cite this article in press as: Ostroy, J.M. Competitive pricing and the core: With reference to matching. Games Econ. Behav. (2018), https://doi.org/10.1016/j.geb.2018.01.011

Makowski, Louis, 1980. A characterization of perfectly competitive economies with production. J. Econ. Theory 22, 208–221.
Makowski, Louis, Ostroy, Joseph, Segal, Uzi, 1999. Efficient incentive compatible economies are perfectly competitive. J. Econ. Theory 85, 169–225.
Ostroy, Joseph, 1980. The no-surplus condition as a characterization of perfectly competitive equilibrium. J. Econ. Theory 22, 183–207.
Ostroy, Joseph, 1981. Differentiability as convergence to perfectly competitive equilibrium. J. Math. Econ. 8, 59–81.
Rockafellar, R.T., 1970. Convex Analysis. Princeton University Press.
Roth, Alvin, Rothblum, Uriel, Vende Vate, John, 1993. Stable matchings, optimal assignments and linear programming. Math. Oper. Res. 18, 803–828.
Roth, Alvin, Sotomayor, Marilda, 1990. Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis. Cambridge University Press.
Shapley, Lloyd, 1955. Markets as Cooperative Games. Technical Report P-B29. Rand Corporation, Santa Monica, CA. Unpublished lecture notes from 1953–54 Princeton seminar.
Shapley, Lloyd, 1967. On balanced sets and the core. Nav. Res. Logist. Q. 14, 453–460.
Shapley, Lloyd, 1969. Utility comparisons and the theory of games. In: La Decision, vol. 1, pp. 251–263.
Shapley, Lloyd, Shubik, Martin, 1969. On market games. J. Econ. Theory 1, 9–29.
Shapley, Lloyd, Shubik, Martin, 1972. The assignment game I: The core. Int. J. Game Theory 1, 111–130.

Please cite this article in press as: Ostroy, J.M. Competitive pricing and the core: With reference to matching. Games Econ. Behav. (2018), https://doi.org/10.1016/j.geb.2018.01.011