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Authors
Simo, Juan
Ortiz, Miguel

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A UNIFIED APPROACH TO FINITE DEFORMATION PLASTICITY BASED ON THE USE OF HYPERELASTIC CONSTITUTIVE EQUATIONS

by

JUAN C. SIMO

and

MIGUEL ORTIZ

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DEPARTMENT OF CIVIL ENGINEERING
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA

A UNIFIED APPROACH TO FINITE DEFORMATION ELASTOPLASTIC ANALYSIS BASED ON THE USE OF HYPERELASTIC CONSTITUTIVE EQUATIONS

J. C. Simo and M. Ortiz

Department of Civil Engineering, University of California,
Berkeley, CA 94720.
Division of Engineering, Brown University, Providence, RI 02912.

1. Introduction.

In the finite deformation literature it is often found that the elastic response of the material is spatially formulated in rate form, i. e., as an incremental relation between objective rates of stress and spatial deformation. If special care is not exercised, such incremental relations may not be integrable and thus inconsistent with the notion of hyperelasticity, in the sense that a stored energy potential does not exist. This situation may result in aberrant behavior such as hysteretic dissipation inappropriate for an elastic model [1,2]. A familiar example is furnished by the assumption frequently made for computational purposes that the spatial tangent elasticity tensor is constant and isotropic. It has been shown in [3] that this widely employed constitutive model is not only incompatible with the notion of hyperelasticity but even fails to define an elastic (non-dissipative) material in the nonlinear range.

From an algorithmic standpoint, the integration of spatial rate constitutive equations requires elaborate schemes that add significantly to the computational cost of the analyses [4,5,6]. One of the aims of the present paper is to show that this added expense is entirely superfluous. The key fact to be realized is that, even for an inelastic material, the elastic response can be spatially formulated in primitive or non-rate form as a functional relation between stresses and suitable strain measures. This point appears to have passed largely unnoticed in the computational literature. As a result of this formulation, *the need for integration of spatial rate constitutive equations is entirely bypassed, even in the inelastic case.* Furthermore, truly hyperelastic behavior is obtained and the principle of objectivity is trivially satisfied.

To emphasize the applicability of the method to problems involving inelastic behavior, finite deformation elastoplasticity is considered in detail. In this case, stresses are updated in two steps: the spatial elastic stress-strain relations are first evaluated to produce an elastic stress predictor, which is subsequently mapped onto a suitably updated yield surface. Following the pioneering work of Wilkins [7], similar return-mapping notions have been extensively used in the past [8,9,10,11,12] although the scope of such formulations has been by and large restricted to simple plasticity models such as linearly hardening von Mises and to constant elastic moduli. The problem of extending these schemes to the case of nonlinear elasticity with non-constant tangent elasticity tensors is not a trivial one and has not been heretofore considered in the literature. It is shown in Section **3** how the operator splitting methodology can be used to define computationally efficient return mapping algorithms which are applicable to very general materials exhibiting non-associated plasticity, arbitrary yield criteria and hardening laws and variable elastic moduli. This latter aspect is of particular relevance since, as mentioned above, a formulation of the elastic response consistent with the principles of hyperelasticity cannot possibly result in a constant tangent elasticity tensor.

Finally, in Section **4** a numerical examples are presented which demonstrate the excellent performance of the method for very large values of the time step. Further numerical examples are given in [38].

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2. Summary of Constitutive Theory for Finite Deformation Plasticity.

The proper formulation of elastoplastic constitutive laws in the finite deformation range has been the subject of considerable conjecture. Differences of opinion have been voiced concerning elastoplastic kinematics and the formulation of flow rules. Lee [13,14] and others, for instance, have proposed a theory based on a multiplicative decomposition of the deformation gradient

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p, \tag{2.1}$$

where the elastic part of the deformation $\mathbf{F}^e$ is obtained by unloading all infinitesimal neighborhoods of the body. This has the effect of introducing a new configuration into the formulation, commonly termed the intermediate configuration, defined as the collection of all unloaded local neighborhoods. This situation is graphically shown in Fig. 1. A complete account of the geometric concepts underlying the multiplicative decomposition can be found in [15]. For polycrystalline solids, such as metals, multiplicative theories are amenable to an elegant physical interpretation based on dislocation mechanics [16].

The multiplicative relation (2.1) does not exhaust, however, all the possibilities concerning an elastic-plastic decomposition of deformation. Green and Naghdi [17] have advocated an additive decomposition of Lagrangian strain

$$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^p \tag{2.2}$$

in terms of elastic and plastic components $\mathbf{E}^e$ and $\mathbf{E}^p$, respectively. Such an additive decomposition rule has been conclusively shown to enjoy a solid thermodynamic foundation [18,19]. On the other hand, in the computational literature an additive decomposition of the spatial rate of deformation tensor

$$\mathbf{d} = \mathbf{d}^e + \mathbf{d}^p \tag{2.3}$$

has been frequently postulated [e.g.,20,21,22,23,24]

In view of this lack of a standard and generally accepted theoretical framework, this section is devoted to a brief account of a constitutive theory for finite deformation elastoplasticity which will be taken as a basis for subsequent discussions. The multiplicative decomposition (2.1) is adopted as the basic kinematic assumption. However, as recently noted in [15], purely geometric arguments together with the concept of covariance* show that within the multiplicative framework material and spatial formulations can be derived in which strains decompose additively into elastic and plastic parts. The main geometric relations involved are summarized in Box 4. Although in a different context, the correspondence between additive and multiplicative theories has also been investigated by Green and Naghdi [27] and Nemat-Nasser [28], among others.

For simplicity, attention is confined throughout to the isothermal case. Boxes 1, 2 and 3 summarize the relevant relations pertaining to the constitutive framework adopted herein. It should be emphasized, however, that the numerical techniques proposed in this paper are not dependent upon this particular set of constitutive assumptions and can be extended to other models without conceptual changes.

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* The notion of covariance expresses the idea of of form invariance of the basic field equations with respect to arbitrary diffeomorphisms (see [29] and [31, Sec. 2.4]). This notion central to other branches of mechanics such as general relativity, also playss an important role in continuum mechanics (see [25,26]).

Material Formulation.

In the material description, Box 1, the local plastic state of the material is assumed to be characterized by the Lagrangian strain tensor $\mathbf{E}$, the plastic Lagrangian strain $\mathbf{E}^p \equiv \frac{1}{2}(\mathbf{C}^p - \mathbf{I})$ and the plastic internal variables $\mathbf{Q}$. Here, $\mathbf{C}^p$ is the plastic right Cauchy-Green tensor defined as $\mathbf{C}^p = \mathbf{F}^{pT}\overline{\mathbf{G}}\,\mathbf{F}^p$, and $\overline{\mathbf{G}}$ is the metric tensor in the intermediate configuration. Typically, one chooses $\overline{\mathbf{G}} \equiv \mathbf{I}$. In the present context, the elastic Lagrangian strain $\mathbf{E}^e$ is formally defined as the difference $\mathbf{E}^e = \mathbf{E} - \mathbf{E}^p$. The nature of the plastic variables depends on the particular plastic model under consideration. For instance, for isotropic hardening von Mises $\mathbf{Q}$ reduces to the yield stress. For isotropic-kinematic hardening $\mathbf{Q}$ includes both the yield stress and the back-stress tensor defining the location of the elastic domain.

BOX 1: **Material Formulation of Elastoplastic Constitutive Relations.**

- Elastic-plastic decomposition of Lagrangian strain

$$\mathbf{E}^e = \mathbf{E} - \mathbf{E}^p$$

- Stress-strain relations

$$\mathbf{S} = \rho_0 \frac{\partial \Psi}{\partial \mathbf{E}^e}(\mathbf{E}^e, \mathbf{E}^p, \mathbf{Q})$$

- Flow rule

$$\dot{\mathbf{E}}^p = \dot{\gamma}\, \mathbf{R}(\mathbf{S},\mathbf{C},\mathbf{Q})$$

- Hardening laws

$$\dot{\mathbf{Q}} = \dot{\gamma}\, \mathbf{H}(\mathbf{S},\mathbf{C},\mathbf{Q})$$

- Yield criterion

$$\Phi(\mathbf{S},\mathbf{C},\mathbf{Q}) = 0$$

The stress-strain relations may be expressed in terms of a free-energy potential $\Psi(\mathbf{E}^e, \mathbf{E}^p, \mathbf{Q})$. It should be noted that this form of free-energy potential coincides with the one first proposed by Green and Naghdi [17].

General non-associated flow rules and hardening laws governing the evolution of $\mathbf{E}^p$ and $\mathbf{Q}$ can be formulated in terms of a plastic flow direction $\mathbf{R}(\mathbf{S}, \mathbf{C}, \mathbf{Q})$ and plastic moduli $\mathbf{H}(\mathbf{S}, \mathbf{C}, \mathbf{Q})$. Yielding of the material is expressed in terms of a yield function $\Phi(\mathbf{S}, \mathbf{C}, \mathbf{Q})$. For instance, in the particular case of an associated flow rule one has $\mathbf{R} = \partial\Phi/\partial\mathbf{S}$. Note that the dependence on $\mathbf{C}$ of $\mathbf{R}$, $\mathbf{H}$ and Φ need to be included to account for effects such as pressure independence of the plastic response.

Multiplicative Theory.

Relative to the intermediate configuration, Box 2, the local plastic state is characterized by the Lagrangian strain tensors $\overline{\mathbf{E}}^e \equiv \frac{1}{2}(\overline{\mathbf{C}}^e - \overline{\mathbf{G}})$ and $\overline{\mathbf{E}}^p = \frac{1}{2}(\overline{\mathbf{G}} - \overline{\mathbf{b}}^{p-1})$, and some suitable set of plastic internal variables $\overline{\mathbf{Q}}$. Here, $\overline{\mathbf{C}}^e = \mathbf{F}^{eT}\mathbf{g}\,\mathbf{F}^e$ is the elastic right Cauchy-Green relative to the intermediate configuration and $\overline{\mathbf{b}}^{p-1} \equiv \mathbf{F}^{p-T}\mathbf{F}^{p-1}$ is the plastic Finger deformation tensor. Finally, $\mathbf{g}$ denotes the spatial metric tensor. Note that the tensors $\overline{\mathbf{C}}^e$, $\overline{\mathbf{G}}$, $\overline{\mathbf{E}}^p$ and $\overline{\mathbf{E}}^e$ defined on the intermediate configuration, are *covariantly* related through pull-back/ push-forward operations to the material tensors $\mathbf{C}$ $\mathbf{C}^p$, $\mathbf{E}^p$ and $\mathbf{E}^e$, respectively, as indicated in Box 4. They play identical role in different configurations.

The relation between $\overline{\mathbf{E}}^e$ and the second Piola-Kirchhoff stress tensor relative to the intermediate configuration $\overline{\mathbf{S}}$, can be expressed with the aid of a free energy potential $\overline{\Psi}(\overline{\mathbf{E}}^e, \overline{\mathbf{E}}^p, \overline{\mathbf{Q}}, \mathbf{F}^p)$. Note that this choice of arguments is entirely consistent with the the one

BOX 2. **Multiplicative Formulation of Elastoplastic Constitutive Relations.**

- Elastic-plastic decomposition of deformation gradient

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$$

- Stress-strain relations

$$\overline{\mathbf{S}} = \rho_0 \frac{\partial \overline{\Psi}}{\partial \overline{\mathbf{E}}^e}(\overline{\mathbf{E}}^e, \overline{\mathbf{E}}^p, \overline{\mathbf{Q}}, \mathbf{F}^p)$$

- Flow rule

$$L_v^p \, \overline{\mathbf{E}}^p = \overline{\mathbf{D}}^p = \dot{\gamma}\, \overline{\mathbf{R}}(\overline{\mathbf{S}}, \overline{\mathbf{C}}^e, \overline{\mathbf{Q}})$$

- Hardening laws

$$L_v^p \overline{\mathbf{Q}} = \dot{\gamma}\, \overline{\mathbf{H}}(\overline{\mathbf{S}}, \overline{\mathbf{C}}^e, \overline{\mathbf{Q}})$$

- Yield criterion

$$\overline{\Phi}(\overline{\mathbf{S}}, \overline{\mathbf{C}}^e, \overline{\mathbf{Q}}) = 0$$

made in the material formulation, and simply follows from the latter by push-forward with $\mathbf{F}^p$. This accounts for including $\mathbf{F}^p$ in the list of arguments of $\overline{\Psi}$.

The plastic response in the intermediate configuration may be characterized by means of a plastic flow direction $\overline{\mathbf{R}}(\overline{\mathbf{S}}, \overline{\mathbf{C}}^e, \overline{\mathbf{Q}})$, plastic moduli $\overline{\mathbf{H}}(\overline{\mathbf{S}}, \overline{\mathbf{C}}^e, \overline{\mathbf{Q}})$ and a yield function $\overline{\Phi}(\overline{\mathbf{S}}, \overline{\mathbf{C}}^e, \overline{\mathbf{Q}})$. Consistent with the dependence in the material description of the plastic response functions on $\mathbf{C}$, the dependence of $\overline{\mathbf{R}}$, $\overline{\mathbf{H}}$, and $\overline{\Phi}$ on the elastic right Cauchy-Green tensor $\overline{C}^e$ needs to be included to obtain a fully covariant formulation.

It may be noted from Box **2** that rates of tensors defined in the intermediate configuration are taken relative to the plastic flow, which immediately leads to the notion of *plastic Lie derivative.* For example, the plastic Lie derivative of $\overline{\mathbf{C}}^e$ is computed by first pulling $\overline{\mathbf{C}}^e$ back to the material configuration to obtain $\mathbf{C}$, taking the time derivative $\dot{\mathbf{C}}$ and finally pushing it forward into the intermediate configuration according to the expression $L_v^p \overline{\mathbf{C}}^e \equiv \mathbf{F}^{p-T} \dot{\mathbf{C}} \mathbf{F}^{p-1}$. Similar definitions apply to any other tensorial object, in particular to the internal variables $\overline{\mathbf{Q}}$ as recorded in Box **2**.

Spatial Formulation.

By way of background, it may be recalled that in the context of elasticity the stored energy potential in the spatial description is an objective function of the form $\tilde{\psi}(\mathbf{g}, \mathbf{F})$ [25,29,31], where $\mathbf{g}$ is the spatial metric tensor. The tensorial dependence of $\tilde{\psi}$ on $\mathbf{g}$ was first pointed out by Doyle and Ericksen [30] who derived the formula $\boldsymbol{\tau} = 2\rho_0 \partial\tilde{\psi}/\partial\mathbf{g}$, where $\boldsymbol{\tau}$ is the Kirchhoff stress tensor. Since the Almansi strain tensor $\mathbf{e}$ is given by $\mathbf{e} = \frac{1}{2}(\mathbf{g} - \mathbf{b}^{-1})$, with $\mathbf{b}^{-1} = \mathbf{F}^{-T}\mathbf{F}^{-1}$, we have the equivalent form $\psi(\mathbf{e}, \mathbf{F})$ of the stored energy potential, which in turn leads to the alternative expression $\boldsymbol{\tau} = \rho_0 \partial\psi/\partial\mathbf{e}$ for the Doyle-Ericksen formula.

In the present context, similar arguments lead to a spatial elastic potential of the form $\psi(\mathbf{e}^e, \mathbf{e}^p, \mathbf{q}, \mathbf{F})$, where $\mathbf{e}^e = \frac{1}{2}(\mathbf{g} - \mathbf{b}^{e-1})$ is the elastic Almansi strain tensor, and $\mathbf{b}^{e-1} = \mathbf{F}^{e-T} \overline{\mathbf{G}} \mathbf{F}^{e-1}$ is the elastic Finger deformation tensor. As in the material formulation, plastic strains are defined by means of the difference $\mathbf{e}^p \equiv \mathbf{e} - \mathbf{e}^e$. It should be emphasized that, contrary to common practice in the computational literature, the elastic response of the material is thus formulated as a non-rate functional relation between spatial stresses and elastic strains.

By covariance, the flow rule, hardening laws and yield criterion take the form expressed in Box **3**, in terms of $\boldsymbol{\tau}$, the spatial plastic variables $\mathbf{q}$ and $\mathbf{g}$. This latter quantity is the spatial counterpart of $\mathbf{C}$ and is needed in general to obtain scalars from $\boldsymbol{\tau}$ and $\mathbf{q}$ and formulate tensorially meaningful plastic relations.

BOX 3. Spatial Formulation of Elastoplastic Constitutive Relations.

- Elastic-plastic decomposition of Almansi strain

$$\mathbf{e}^p = \mathbf{e} - \mathbf{e}^e$$

- Stress-strain relations

$$\boldsymbol{\tau} = \rho_0 \frac{\partial \psi}{\partial \mathbf{e}^e}(\mathbf{e}^e, \mathbf{e}^p, \mathbf{q}, \mathbf{F})$$

- Flow rule

$$L_{\mathbf{v}}\mathbf{e}^p = \mathbf{d}^p = \dot{\gamma}\, \mathbf{r}(\boldsymbol{\tau},\mathbf{g},\mathbf{q})$$

- Hardening laws

$$L_{\mathbf{v}}\mathbf{q} = \dot{\gamma}\, \mathbf{h}(\boldsymbol{\tau},\mathbf{g},\mathbf{q})$$

- Yield criterion

$$\phi(\boldsymbol{\tau},\mathbf{g},\mathbf{q}) = 0$$

Finally, the Lie derivative $L_{\mathbf{v}}\mathbf{e}^p$ is defined as the push-forward into the current configuration of $\dot{\mathbf{E}}^p$, which takes the form $L_{\mathbf{v}}\mathbf{e}^p = \mathbf{F}^{-T}\dot{\mathbf{E}}^p\mathbf{F}^{-1}$. Similarly, $L_{\mathbf{v}}\mathbf{q}$ is defined as the push-forward of $\dot{\mathbf{Q}}$ into the current configuration. The specific component form of $L_{\mathbf{v}}\mathbf{q}$ depends on the tensorial nature of the plastic variables $\mathbf{q}$ under consideration. The reader unfamiliar with geometric notions such as the push-forward and pull-back operations and the Lie derivative may wish to consult reference [31, Chap.1] excellent reviews.

Remark 2.1. In the context of elasticity, one also have three possible alternative descriptions. The material and spatial descriptions and the *rotated* description [25,26]. The latter is obtained from the spatial description by pull-back with the rotation tensor arising from the polar decomposition of deformation gradient. Equivalently, the rotated and material descriptions are related by pull-back/push-forward with the stretching tensor. In plasticity, the description relative to the intermediate configuration is the counterpart of the rotated description.

Remark 2.2. The multiplicative decomposition (2.1), as it stands, is only defined modulo rigid body motions superimposed on $\mathbf{F}^p$. Attempts to uniquely orient the intermediate configuration have been made by Mandel [e.g.,32,33] and others. In view of the indeterminacy underlying (2.1) some authors have advocated flow rules for the plastic spin in the intermediate configuration [34] or for $\dot{\mathbf{F}}^p$ [16]. It has been recently shown in [15] that there exists a canonical choice of $\mathbf{F}^p$ which preserves the material symmetry group and which renders (2.1) a particular case of the polar decomposition. It would then appear that equations of evolution for the plastic spin are indeed unnecessary and that only flow rules of the type expressed in Box 3 need be formulated.

Model Problem.

Specific examples of simple constitutive models of the form summarized in Box 3 are the following.

Hyperelastic response. An example describing a class of hyperelastic constitutive models considered in [3] which proves convenient in numerical applications. In a spatial description, the model is defined by

$$\begin{aligned} \psi &= \lambda\, U(J^e) + \tfrac{1}{2}\mu\, I^e - \mu\, \log J^e \\ \boldsymbol{\tau} &= \lambda\, J^e \frac{dU(J^e)}{dJ^e}\mathbf{g}^{-1} + \mu\,(\mathbf{b}^e - \mathbf{g}^{-1}) \\ J\,\mathbf{c}^e &= \lambda\, J^e \frac{d}{dJ^e}\Big[J^e \frac{dU(J^e)}{dJ^e}\Big]\mathbf{g}^{-1}\otimes\mathbf{g}^{-1} + \Big[\mu - \lambda\, J^e \frac{dU(J^e)}{dJ^e}\Big]\mathbf{I}_{\mathbf{g}} \end{aligned} \quad , \qquad (2.4)$$

BOX 4. Basic Definition of Kinematical and Stress Variables

	Configurations	
$B_0 \equiv$ material	$\bar{B}_t \equiv$ intermediate	$B_t \equiv$ spatial
	Basic Kinematic Tensors	
$\mathbf{C} = \mathbf{F}^T \mathbf{g}\, \mathbf{F}$; $\mathbf{C}^p = \mathbf{F}^{p^T} \bar{\mathbf{G}}\, \mathbf{F}^p$ $\mathbf{E} = \tfrac{1}{2}(\mathbf{C} - \mathbf{I})$ $\mathbf{E}^p = \tfrac{1}{2}(\mathbf{C}^p - \mathbf{I})$ $\mathbf{E}^e = \tfrac{1}{2}(\mathbf{C} - \mathbf{C}^p)$ $\mathbf{E} = \mathbf{E}^e + \mathbf{E}^p$	$\bar{\mathbf{C}}^e = \mathbf{F}^{e^T} \mathbf{g}\, \mathbf{F}^e$ $\bar{\mathbf{E}} = \tfrac{1}{2}(\bar{\mathbf{C}}^e - \mathbf{F}^{p^{-T}} \mathbf{F}^{p^{-1}})$ $\bar{\mathbf{E}}^p = \tfrac{1}{2}(\bar{\mathbf{G}} - \mathbf{F}^{p^{-T}} \mathbf{F}^{p^{-1}})$ $\bar{\mathbf{E}}^e = \tfrac{1}{2}(\bar{\mathbf{C}}^e - \bar{\mathbf{G}})$ $\bar{\mathbf{E}} = \bar{\mathbf{E}}^e + \bar{\mathbf{E}}^p$	...... $\mathbf{e} = \tfrac{1}{2}(\mathbf{g} - \mathbf{F}^{-T}\mathbf{F}^{-1})$ $\mathbf{e}^p = \tfrac{1}{2}(\mathbf{F}^{e^{-T}} \bar{\mathbf{G}} \mathbf{F}^{e^{-1}} - \mathbf{F}^{-T}\mathbf{F}^{-1})$ $\mathbf{e}^e = \tfrac{1}{2}(\mathbf{g} - \mathbf{F}^{e^{-T}} \bar{\mathbf{G}} \mathbf{F}^{e^{-1}})$ $\mathbf{e} = \mathbf{e}^e + \mathbf{e}^p$
	Rates of Deformation Tensors	
$\dot{\mathbf{E}}$ $\dot{\mathbf{E}}^p$ $\dot{\mathbf{E}}^e$ $\dot{\mathbf{E}} = \dot{\mathbf{E}}^e + \dot{\mathbf{E}}^p$	$\bar{\mathbf{D}} \equiv L_{\mathbf{v}}^p \bar{\mathbf{E}}$ $\bar{\mathbf{D}}^p \equiv L_{\mathbf{v}}^p \bar{\mathbf{E}}^p$ $\bar{\mathbf{D}}^e \equiv L_{\mathbf{v}}^p \bar{\mathbf{E}}^e$ $\bar{\mathbf{D}} = \bar{\mathbf{D}}^e + \bar{\mathbf{D}}^p$	$\mathbf{d} \equiv L_{\mathbf{v}} \mathbf{e}$ $\mathbf{d}^p \equiv L_{\mathbf{v}} \mathbf{e}^p$ $\mathbf{d}^e \equiv L_{\mathbf{v}} \mathbf{e}^e$ $\mathbf{d} = \mathbf{d}^e + \mathbf{d}^p$
	Basic Stress Tensors	
$\mathbf{S}$	$\bar{\mathbf{S}} = \mathbf{F}^p \mathbf{S}\, \mathbf{F}^{p^T}$	$\boldsymbol{\tau} = \mathbf{F}\, \mathbf{S}\, \mathbf{F}^T$

where $J^e = \det(\mathbf{F}^e)$, $J = \det(\mathbf{F})$, I^e is the first invariant of $\mathbf{b}^e$, and $\mathbf{I}_\mathbf{g}$ is the fourth order unit tensor with components $\tfrac{1}{2}[(g^{-1})^{ik}(g^{-1})^{jl} + (g^{-1})^{il}(g^{-1})^{jk}]$. The fourth order tensor $\mathbf{c}^e$ is the *spatial elasticity tensor* of the material which is often defined indirectly by push-forward of the *material* elasticity tensor $\rho_0 \partial \mathbf{S}/\partial \mathbf{E}^e$ (e.g., see [2] p.131, eq (45.2)). The parameters λ and μ are Lame-type material constants.

The form of the stored energy function (2.4a) corresponds to a Neo-Hookean material which is extended into the compressible range by adding the extra function $U(J^e)$. This is in fact a particular case of a Hadamard material. A simple choice of $U(J^e)$ is given by

$$U(J^e) = \tfrac{1}{2}\,(\log J^e)^2, \tag{2.5}$$

which has proven effective in the context of the penalty method applied to incompressible materials such as Mooney-Rivlin [35]. Model (2.4) defines the simplest possible hyperelastic material whose tangent elasticity tensor is isotropic. Further motivation is provided by the fact that the Mooney-Rivlin model is a first approximation to the constitutive behavior of any non-linear incompressible elastic isotropic material.

Plastic response model. An example of a plasticity model widely used in computation is furnished by the von Mises yield criterion with isotropic hardening. To illustrate the role played by the spatial metric tensor $\mathbf{g}$ and the right Cauchy-Green tensor $\mathbf{C}$ in a covariant formulation of plasticity, material and spatial versions of the von Mises model are given next. In a spatial setting, the von Mises yield function reads

$$\phi(\boldsymbol{\tau}, \mathbf{g}, \kappa) \equiv \tfrac{1}{2}\boldsymbol{\tau}' : [\mathbf{I}_{\mathbf{g}} : \boldsymbol{\tau}'] - \kappa^2 \equiv \tfrac{1}{2}\tau'^{ij}\tau'^{kl} g_{ik}\, g_{jl} - \kappa^2 \tag{2.6a}$$

where the components of the deviatoric Kirchhoff stress tensor $\boldsymbol{\tau}'$ are given by

$$\tau'^{ij} = \tau^{ij} - \frac{1}{3}(\tau^{kl} g_{kl})\,(g^{-1})^{ij}, \tag{2.6b}$$

and κ is the shear yield stress. An alternative material formulation involves the yield function

$$\Phi(\mathbf{S}, \mathbf{C}, \kappa) = \tfrac{1}{2} S'^{IJ} S'^{KL} C_{IK}\, C_{JL} - \kappa^2, \tag{2.7a}$$

where the stress deviator consistent with (2.6b) is defined as

$$S'^{IJ} = S^{IJ} - \frac{1}{3}(S^{KL} C_{KL})\,(C^{-1})^{IJ}. \tag{2.7b}$$

Note that the hydrostatic pressure p is given by the equivalent expressions $3\,J\,p = \tau^{ij} g_{ij} \equiv S^{IJ} C_{IJ}$ which are utilized in spatial and material definitions (2.6b) and (2.7b) of the stress deviator. Thus, it is apparent that $\mathbf{g}$ and $\mathbf{C}$ need to be included in the formulation of the plastic response. Note further that an identical role is played by $\bar{\mathbf{C}}^e$ in the intermediate configuration.

3. Numerical Formulation.

In this section, a number of numerical techniques are proposed that allow a systematic treatment of constitutive models of the general type discussed above within the context of finite element analysis. The stress update algorithm herein proposed falls within the category of elastic predictor-return mapping algorithms widely used in computational plasticity. However, our formulation departs from currently employed procedures in that:

i) The need for integration algorithms for elastic rate constitutive relations is entirely bypassed by formulating the elastic relations in non-rate form.

ii) General hyperelastic models are considered and the definition of the return mapping is capable of accommodating non-constant elastic moduli.

iii) The algorithm is applicable to completely general plasticity models including non-associated flow rules and arbitrary yield criteria.

iv) Although the spatial formulation is herein emphasized, the proposed algorithm equally applies in a material setting.

In particular, our formulation of the elastic response is truly hyperelastic and does not involve objective stress rates. Thus, the elastic predictor is reduced to a mere function evaluation. Algorithmic requirements pertaining to integration of spatial rate constitutive equations such as incremental objectivity [4,5] are no longer an issue owing to the inherently objective nature of the hyperelastic constitutive relations. In addition, the elastic predictor is infinitely accurate since no algorithmic approximations are involved.

3.1. Elastic-Plastic Operator Split.

In any numerical scheme employed for the analysis of elastoplastic problems it eventually becomes necessary to update state variables such as stresses, strains and plastic parameters. In the context of finite element analysis using isoparametric elements, stress updates take place at the Gauss points and the incremental deformation is given. The problem to be addressed, therefore, is that of updating the *known* state variables $\mathbf{F}_n$, $\mathbf{F}_n^p$, $\mathbf{q}_n$ and $\boldsymbol{\tau}_n$ associated with a *converged* configuration B_n into their corresponding *updated* values $\mathbf{F}_{n+1}$, $\mathbf{F}_{n+1}^p$, $\mathbf{q}_{n+1}$ and $\boldsymbol{\tau}_{n+1}$ on the *updated* configuration B_{n+1} in a manner consistent with the constitutive assumptions expressed in Box 3. In this process, the incremental displacements $\mathbf{u}$ defining the geometry update $B_n \rightarrow B_{n+1}$ are assumed given.

A number of authors have advocated the use of so-called return mapping algorithms for integration of elastoplastic constitutive relations [7,8,9,10,11,12]. The applicability of such algorithms has been by and large restricted to simple plasticity models such as linearly hardening von Mises with constant elastic moduli. However, many materials of engineering interest such as concrete and soils exhibit nonlinear elastic response, non-associated plasticity and complex yield criteria, flow rules and hardening laws. Furthermore, it has been shown in [3] that an elastic material cannot possibly have constant and isotropic tangent elasticities in the finite deformation range. Therefore, an integration scheme for elastoplastic constitutive relations must be able to accommodate general *non–constant* tangent stiffness compliances to be of value in the context of finite deformation analysis.

We show next how an operator splitting methodology proposed in [36, Chap. 3] in the context of linearized kinematics can be conveniently extended to define efficient return mapping procedures which are capable of dealing with fully nonlinear elastic response and complex plastic models.

Spatial Formulation. For the purpose of this discussion, the deformation gradients are taken to be prescribed functions of time

$$\mathbf{F} = \hat{\mathbf{F}}(t) \tag{3.1}$$

For simplicity, in what follows we shall ignore the possibility of a dependence of the potential ψ on the plastic variables $\mathbf{q}$. Let us start by rephrasing the spatial constitutive relations in Box **3** as a set of *equations of evolution* of the following form

$$\begin{aligned} \mathbf{d} &\equiv \mathbf{d}^e + \mathbf{d}^p = \hat{\mathbf{d}}(t) \\ L_v\boldsymbol{\tau} &= J\,\mathbf{c}^e : \mathbf{d}^e + J\,\mathbf{c}^p : \mathbf{d}^p \\ L_v\mathbf{e}^p &\equiv \mathbf{d}^p = \dot{\gamma}\,\mathbf{r}(\boldsymbol{\tau}, \mathbf{g}, \mathbf{q}) \\ L_v\mathbf{q} &= \dot{\gamma}\,\mathbf{h}(\boldsymbol{\tau}, \mathbf{g}, \mathbf{q}) \end{aligned} \tag{3.2}$$

where $\hat{\mathbf{d}}(t) \equiv (\dot{\hat{\mathbf{F}}}(t)\,\hat{\mathbf{F}}^{-1}(t))^S$ is given. The elastic constitutive equations have been formulated in rate form simply by taking the Lie derivative of stress-strain relations $\boldsymbol{\tau} = \rho_0 \partial\psi/\partial\mathbf{e}^e$. To this end, one defines

$$\mathbf{c}^e \equiv \rho\,\frac{\partial^2\psi}{\partial\mathbf{e}^e\partial\mathbf{e}^e}\,; \quad \mathbf{c}^p \equiv \rho\,\frac{\partial^2\psi}{\partial\mathbf{e}^e\partial\mathbf{e}^p}\,, \tag{3.3}$$

where $\mathbf{c}^e$ is the spatial tangent elasticity tensor. In addition the fact has been used that

$$L_v\mathbf{e}^e = \mathbf{d}^e\,, \qquad L_v\mathbf{e}^p = \mathbf{d}^p\,. \tag{3.4}$$

The plastic rate parameter $\dot{\gamma}$ is determined from the requirement that the yield condition

$$\phi(\boldsymbol{\tau}, \mathbf{g}, \mathbf{q}) = 0 \tag{3.5}$$

be identically satisfied during plastic loading. Note that any other objective stress rate can be used in (3.2b) by suitably adjusting the right hand side. In particular, if an "elastic" Lie derivative relative to the intermediate configuration is employed, the term in (3.2b) involving $\mathbf{d}^p$ no longer appears explicitly.

Elastic-Plastic Operator Split. As first noted in [36], return mapping algorithms are a natural consequence of the fact that constitutive relations (3.2) can be "split" into elastic and plastic parts. The former is *deformation driven* and is given by

$$\begin{aligned} \mathbf{d} &\equiv \mathbf{d}^e + \mathbf{d}^p = \hat{\mathbf{d}}(t) \\ L_v\boldsymbol{\tau} &= J\,\mathbf{c}^e : \mathbf{d} \\ L_v\mathbf{e}^p &\equiv d^p = 0 \\ L_v\mathbf{q} &= 0 \end{aligned} \tag{3.6}$$

On the other hand, the plastic part of the constitutive relations reduces to

$$
\begin{aligned}
&\mathbf{d} \equiv \mathbf{d}^e + \mathbf{d}^p = 0 \\
&\frac{\partial}{\partial t}\boldsymbol{\tau} = -\, J\,(\mathbf{c}^e - \mathbf{c}^p) : \mathbf{d}^p \\
&\frac{\partial}{\partial t}\mathbf{e}^p = \dot{\gamma}\, \mathbf{r}(\boldsymbol{\tau}, \mathbf{g}, \mathbf{q}) \\
&\frac{\partial}{\partial t}\mathbf{q} = \dot{\gamma}\, \mathbf{h}(\boldsymbol{\tau}, \mathbf{g}, \mathbf{q})
\end{aligned} \qquad (3.7)
$$

It should be noted that eqs. (3.6) and (3.7) do indeed add up to the total rate constitutive relations (3.2), consistent with the notion of operator split.

The decoupled eqs. (3.6) and (3.7) are amenable to the following interpretation.

Elastic part of the constitutive equations. In elastic equations (3.6), the plastic response of the material is "frozen", so that plastic strains and plastic variables are merely pushed forward into subsequent configurations. Furthermore, since $\mathbf{d}^p = 0 = \mathbf{F}^{p-T}\dot{\mathbf{C}}^p\mathbf{F}^{p-1}$, it follows that $\dot{\mathbf{C}}^p = 0$ and hence $\dot{\mathbf{F}}^p = 0$. Consequently, the evolution of the state variables introduced by the elastic equations takes place for a *fixed intermediate configuration*, and all of the prescribed deformation $\hat{\mathbf{d}}(t)$ goes into elastically straining the material. Thus, the elastic deformation gradients $\mathbf{F}^e = \hat{\mathbf{F}}(t)\,\mathbf{F}^{p-1} \equiv \tilde{\mathbf{F}}^e(t)$ become themselves given functions of time. Finally, since our formulation is *hyperelastic*, the elastic equations (3.6) are directly *integrable* and stresses are simply given by the elastic relations

$$
\boldsymbol{\tau} = \rho_0 \frac{\partial \psi}{\partial \mathbf{e}^e}\bigg|_{\tilde{\mathbf{F}}^e(t)} \qquad (3.8)
$$

Plastic part of the constitutive equations. In the plastic equations, on the other hand, one has $\mathbf{d} = 0$ and hence the *spatial configuration remains fixed.* Under these conditions, the Lie derivative simply reduces to partial differentiation with respect to time and, in particular, $\partial\mathbf{e}^p/\partial t = \mathbf{d}^p$. Thus, the plastic equations (3.7) may be recast as

$$
\begin{aligned}
&\frac{\partial}{\partial t}\boldsymbol{\tau} = -\,\dot{\gamma}\, J\,(\mathbf{c}^e - \mathbf{c}^p) : \mathbf{r}(\boldsymbol{\tau}, \mathbf{g}, \mathbf{q}) \\
&\frac{\partial}{\partial t}\mathbf{q} = \dot{\gamma}\, \mathbf{h}(\boldsymbol{\tau}, \mathbf{g}, \mathbf{q})
\end{aligned} \qquad (3.9)
$$

or, dividing through by $\dot{\gamma}$

$$
\begin{aligned}
&\frac{d\boldsymbol{\tau}}{d\gamma} = -\, J\,(\mathbf{c}^e - \mathbf{c}^p) : \mathbf{r}(\boldsymbol{\tau}, \mathbf{g}, \mathbf{q}) \\
&\frac{d\mathbf{q}}{d\gamma} = \mathbf{h}(\boldsymbol{\tau}, \mathbf{g}, \mathbf{q})
\end{aligned} \qquad (3.10)
$$

These equations define a *relaxation* of the stresses $\boldsymbol{\tau}$ towards a suitably updated elastic domain. Such a plastic relaxation process is completed as soon as the yield condition (3.5) is satisfied.

Material Formulation. The elastic-plastic splitting methodology defined above can be alternatively formulated in a material setting. In this case, entirely analogous arguments point to the following choice of decoupled equations:

Elastic Equations	*Plastic Equations*	
$\dot{\mathbf{E}} = \dot{\hat{\mathbf{E}}}(t)$	$\dot{\mathbf{E}} = 0$	
$\dot{\mathbf{S}} = \mathbf{A}^e(\mathbf{E}^e, \mathbf{E}^p) : \dot{\mathbf{E}}$	$\dot{\mathbf{S}} = -\,[\mathbf{C}^e(\mathbf{E}^e, \mathbf{E}^p) - \mathbf{A}^p(\mathbf{E}^e, \mathbf{E}^p)] : \dot{\mathbf{E}}^p$	, (3.11)
$\dot{\mathbf{E}}^p = 0$	$\dot{\mathbf{E}}^p = \dot{\gamma}\,\mathbf{R}(\mathbf{S}, \mathbf{C}, \mathbf{Q})$	
$\dot{\mathbf{Q}} = 0$	$\dot{\mathbf{Q}} = \dot{\gamma}\,\mathbf{H}(\mathbf{S}, \mathbf{C}, \mathbf{Q})$	

where $\hat{\mathbf{E}}(t)$ is again a given function of time, $\mathbf{A}^e \equiv \rho_0 \partial^2\Psi/\partial\mathbf{E}^e\partial\mathbf{E}^e$ is the material tangent elasticity tensor, and $\mathbf{A}^p \equiv \rho_0\partial^2\Psi/\partial\mathbf{E}^e\partial\mathbf{E}^p$. As before, the elastic part of the constitutive relations defines a process of elastic straining in which the plastic deformations $\mathbf{E}^p$ remain unchanged while the stresses $\mathbf{S}$ are evaluated through the elastic relations from the known elastic strains $\hat{\mathbf{E}}^e(t) = \hat{\mathbf{E}}(t) - \mathbf{E}^p$. On the other hand, the plastic equations define a relaxation process for stresses and plastic variables which continues until the yield criterion

$$\Phi(\mathbf{S}, \mathbf{C}, \mathbf{Q}) = 0 \tag{3.12}$$

is satisfied.

Remark 3.1. The material tensors $\mathbf{A}^e$ and $\mathbf{A}^p$ defined above are related to the spatial tensors $J\mathbf{c}^e$ and $J\mathbf{c}^p$ given by (3.3), through push-forward with $\mathbf{F}$. As an example, for the hyperelastic constitutive model (2.4), $\mathbf{c}^e$ is given by $(2.4)_3$, whereas $\mathbf{c}^p$ has the expression

$$\mathbf{c}^p = -\lambda\, J^e \frac{dU(J^e)}{dJ^e}\mathbf{I}_{\mathbf{g}}$$

Note that for the choice of $U(J^e)$ given by (2.5) one obtains $J(\mathbf{c}^e - \mathbf{c}^p) \equiv \lambda\,\mathbf{g}\otimes\mathbf{g} + 2\mu\,\mathbf{I}_{\mathbf{g}}$.

3.2. Stress Update: General Return Mapping Algorithm.

Based on the elastic-plastic split (3.6-7), a return algorithm can be conveniently defined by first solving the elastic equations (3.6) to obtain an elastic predictor, which is then taken as an initial condition for the plastic relaxation equations (3.7). The resulting procedure is graphically shown in Fig. **4**. For associated perfect plasticity, for instance, it is seen that the plastic equations define a return path for the stresses which runs along the steepest descents of the yield function, Fig. **2**. It should be noted, however, that the steepest descent direction in stress space is determined based on the tensor $(\mathbf{c}^e - \mathbf{c}^p)$.

For the perfectly plastic von Mises model with infinitesimal isotropic elasticity, the return path for stresses is clearly radial. In general, however, the return path defined by (3.7) is not known in advance nor can it be determined analytically. It becomes therefore necessary to compute the return path for the stresses numerically. An efficient algorithm for this purpose is listed in Box 5, together with the remaining steps in the update procedure. As may be seen, the return mapping is defined iteratively. At every iteration the yield function ϕ is linearized about the current values $\boldsymbol{\tau}_{n+1}^{(i)}$ $\mathbf{q}_{n+1}^{(i)}$ and $\phi_{n+1}^{(i)}$. Such a linearized yield function defines a straight intersection or "cut" on the plane $\phi = 0$ onto which $\boldsymbol{\tau}_{n+1}^{(i)}$ and $\mathbf{q}_{n+1}^{(i)}$ are projected to obtain the next iteration $\boldsymbol{\tau}_{n+1}^{(i+1)}$ and $\mathbf{q}_{n+1}^{(i+1)}$. It should be noted that such projection involves the current elastic moduli $\mathbf{c}_{n+1}^{e(i)}$. The initial conditions for the return procedure $\boldsymbol{\tau}_{n+1}^{(0)}$ and $\mathbf{q}_{n+1}^{(0)}$ are taken to coincide with the elastic predictor.

Geometric Interpretation. A geometric interpretation of the proposed algorithm is given in Fig. **3**. As we may see, the elastic predictor is returned to the yield surface in successive steps. Each one of these steps involves a projection of the stresses onto a straight (linear) approximation to the yield surface or "cut". In the limit, such cuts become tangent to the yield surface and plastic consistency is restored at a *quadratic* convergence rate. For an associated flow rule, the computed return path is indeed an approximation to the steepest descent path as defined by the tensor $(\mathbf{c}^e - \mathbf{c}^p)$.

Stability and Consistency. From general results concerning the operator splitting methodology (e.g., see [36, Chapter 3] for a review) it immediately follows that the proposed algorithm is consistent with the constitutive relations in Box. **3**. Furthermore, unconditional stability follows automatically provided that both the elastic predictor and the plastic corrector are separately unconditionally stable. As for the former, unconditional stability is trivially achieved due to the exact nature of the algorithm. On the other hand, the equations (3.10) defining the return path are clearly dissipative and the corresponding trajectories contractive provided the yield function is convex and the plastic flow direction $\mathbf{r}$ derives from a convex potential or loading function [36, Chap. 3]. Under these conditions, the return algorithm is unconditionally

BOX 5. **Stress Update Algorithm:**

i) Geometric update

$$\boldsymbol{\phi}_{n+1} = \boldsymbol{\phi}_n + \mathbf{u}$$
$$\mathbf{F}_u = \mathbf{I} + \nabla \mathbf{u}$$
$$\mathbf{F}_{n+1} = \mathbf{F}_u \mathbf{F}_n$$

ii) Elastic predictor

$$\mathbf{F}^{p(0)}_{n+1} = \mathbf{F}^p_n$$
$$\mathbf{F}^{e(0)}_{n+1} = \mathbf{F}_{n+1} \mathbf{F}^{p-1}_n$$
$$\boldsymbol{\tau}^{(0)}_{n+1} = \rho_0 \left. \frac{\partial \psi^e}{\partial \mathbf{e}^e} \right|_{\mathbf{F}^{e(0)}_{n+1}}$$
$$\mathbf{q}^{(0)}_{n+1} = \textit{push-forward of } \mathbf{q}_n \textit{ by } \mathbf{F}_{\mathbf{u}}$$

iii) Check for yielding

$$\phi^{(0)}_{n+1} \equiv \phi(\boldsymbol{\tau}^{(0)}_{n+1}, \mathbf{q}^{(0)}_{n+1}) \leqslant 0 \ ?$$

YES $\mathbf{F}^p_{n+1} = \mathbf{F}^{p(0)}_{n+1}$; $\mathbf{q}_{n+1} = \mathbf{q}^{(0)}_{n+1}$; $\boldsymbol{\tau}_{n+1} = \boldsymbol{\tau}^{(0)}_{n+1}$; *EXIT*

NO $i = 0$

iv) Plastic correctors

$$\Delta\gamma = \frac{\phi^{(i)}_{n+1}}{\left(\frac{\partial \phi}{\partial \boldsymbol{\tau}}\right)^{(i)}_{n+1} : J_{n+1}(\mathbf{c}^e - \mathbf{c}^p)^{(i)}_{n+1} : \mathbf{r}^{(i)}_{n+1} - \left(\frac{\partial \phi}{\partial \mathbf{q}}\right)^{(i)}_{n+1} \cdot \mathbf{h}^{(i)}_{n+1}}$$
$$\boldsymbol{\tau}^{(i+1)}_{n+1} = \boldsymbol{\tau}^{(i)}_{n+1} - \Delta\gamma\, J_{n+1} (\mathbf{c}^e - \mathbf{c}^p)^{(i)}_{n+1} : \mathbf{r}^{(i)}_{n+1}$$
$$\mathbf{q}^{(i+1)}_{n+1} = \mathbf{q}^{(i)}_{n+1} + \Delta\gamma\, \mathbf{h}^{(i)}_{n+1}$$

v) Convergence check

$$\phi(\boldsymbol{\tau}^{(i+1)}_{n+1}, \mathbf{q}^{(i+1)}_{n+1}) \leqslant TOL \ ?$$

YES *Compute* $\mathbf{F}^e_{n+1}$ *from* $\boldsymbol{\tau}^{(i+1)}_{n+1}$ (see Remark 3.3. on isotropy)

$\mathbf{F}^p_{n+1} = \mathbf{F}_{n+1} \mathbf{F}^{e-1}_{n+1}$, $\mathbf{q}_{n+1} = \mathbf{q}^{(i+1)}_{n+1}$, $\boldsymbol{\tau}_{n+1} = \boldsymbol{\tau}^{(i+1)}_{n+1}$; *EXIT*

NO $i \leftarrow i+1$; *GO TO* (iv)

stable and so is the overall update procedure.

Remark 3.2. It should be noted that the proposed return algorithm does not require explicit knowledge of the derivatives of the elasticity tensor $\mathbf{c}^e$, the plastic flow direction $\mathbf{r}$ or the plastic hardening moduli $\mathbf{h}$, in spite of which quadratic convergence of the stress iterates towards the elastic domain is achieved.

Remark 3.3. If the elastic response is *isotropic* a standard argument reveals that the *elastic* potential ψ in the spatial description depends on $\mathbf{F}^e$ through the elastic left Cauchy-Green tensor $\mathbf{b}^e$. Thus, knowledge of $\boldsymbol{\tau}$ at a given configuration only determines uniquely $\mathbf{b}^e$ or, equivalently, the left stretch tensor $\mathbf{V}^e = \mathbf{b}^{e^{1/2}}$ in an *unloading* process. However, $\mathbf{F}^e$ and thus $\mathbf{F}^p$ can be uniquely determined from the condition that the multiplicative decomposition $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$ be a *polar* decomposition of $\mathbf{F}$ with metric $\overline{\mathbf{C}}^e$ in the intermediate (*rotated*) configuration. Setting $\mathbf{F} = \mathbf{R}\,\mathbf{U}$, where $\mathbf{R}^T \mathbf{g} \mathbf{R} = \mathbf{I}$, this condition implies that [38]

$$\mathbf{F}^e = \mathbf{V}^e\mathbf{R}, \qquad \mathbf{F}^p = \mathbf{F}\,\mathbf{F}^{e-1}. \tag{3.13}$$

This is the procedure herein adopted in the finite element implementation of the isotropic elastic finite deformation model (2.4). Further computational details may be found in [38].

Remark 3.4. The stress update algorithm proposed above is amenable to an entirely equivalent formulation in the material setting, based on the elastic-plastic split (3.11). Some features of such material implementation are noteworthy. For instance, the plastic strains $\mathbf{E}^p$ are seen to remain constant during the elastic predictor and thus play the role of fixed parameters in the computation of the elastic tangent moduli $\mathbf{C}^e(\mathbf{E}^e, \mathbf{E}^p)$. On the other hand, the plastic relaxation takes places under constant deformations $\mathbf{C}$ which thus behave as fixed parameters in the definition of the plastic flow direction $\mathbf{R}(\mathbf{S}, \mathbf{C}, \mathbf{Q})$, plastic moduli $\mathbf{H}(\mathbf{S}, \mathbf{C}, \mathbf{Q})$ and yield function $\Phi(\mathbf{S}, \mathbf{C}, \mathbf{Q})$.

3.3. Extension to Viscoplasticity.

The formulation heretofore presented readily extends to the case of viscoplasticity. Assuming for simplicity linear viscosity, the flow rule and hardening laws take the same form as in Box 3 with λ being replaced by ϕ/η, where η is the viscosity coefficient. Accordingly, an analogous operator split now applies in which eq. (3.6) remains unaltered and the plastic relaxation equations (3.9) are replaced by

$$\begin{aligned} \frac{\partial}{\partial t}\boldsymbol{\tau} &= -\frac{\phi}{\eta} J\,(\mathbf{c}^e - \mathbf{c}^p) : \mathbf{r}(\boldsymbol{\tau}, \mathbf{g}, \mathbf{q}) \\ \frac{\partial}{\partial t}\mathbf{q} &= \frac{\phi}{\eta}\mathbf{h}(\boldsymbol{\tau}, \mathbf{g}, \mathbf{q}) \end{aligned} \tag{3.14}$$

where plastic loading ($\phi > 0$) is assumed. The rate of change of ϕ is governed by the following equation

$$\frac{\partial}{\partial t}\phi = \frac{\partial\phi}{\partial\boldsymbol{\tau}} : \frac{\partial\boldsymbol{\tau}}{\partial t} + \frac{\partial\phi}{\partial\mathbf{q}} \cdot \frac{\partial\mathbf{q}}{\partial t} = -\frac{\phi}{\eta}\left[\frac{\partial\phi}{\partial\boldsymbol{\tau}} : J\,(\mathbf{c}^e - \mathbf{c}^p) : \mathbf{r} - \frac{\partial\phi}{\partial\mathbf{q}} \cdot \mathbf{h}\right], \tag{3.15}$$

which is the viscous counterpart of the consistency condition of inviscid plasticity. Defining an *instantaneous relaxation time* by the expression

$$\bar{t} \equiv \frac{\eta}{\dfrac{\partial\phi}{\partial\boldsymbol{\tau}} : J\,(\mathbf{c}^e - \mathbf{c}^p) : \mathbf{r} - \dfrac{\partial\phi}{\partial\mathbf{q}} \cdot \mathbf{h}}, \tag{3.16}$$

eq. (3.15) may rephrased as

$$\frac{\partial\phi}{\partial t} = -\frac{\phi}{\bar{t}}. \tag{3.17}$$

In view of equations (3.14) and (3.17) the following conclusions may be drawn:

(i) The return path defined by (3.14) coincides with the return path corresponding to inviscid plasticity.

(ii) As the plastic relaxation proceeds, the location of the stress point in the return path defined by (3.14) is governed by eq. (3.17).

These results enable one to formulate an algorithm for the numerical integration of the elasto-viscoplastic constitutive relations based upon the following notions. The elastic predictor and return path are computed as in the inviscid case. In particular the return path comprises a sequence of straight segments in stress space directed towards the yield surface. The difference, however, is that only partial relaxation now takes place; thus the stress point does not reach the yield surface. To compute the final location of the stress point in the return path, one may proceed as follows.

(a) Within a generic straight segment (i) in the return path, the relaxation time is taken to be constant and equal to a value $\bar{t}^{(i)}$ computed according to (3.16) from the initial conditions for the segment $\boldsymbol{\tau}_{n+1}^{(i)}$, and $\mathbf{q}_{n+1}^{(i)}$.

(b) Thus, within a typical straight segment the variation of the yield function value is given by the exponential relation

$$\phi \approx \phi_{n+1}^{(i)} \exp(-\Delta t / \bar{t}^{(i)}) \tag{3.18}$$

where Δt is the time elapsed since the entrance of the stress point into segment (i).

(c) The total time $\Delta t^{(i)}$ spent by the stress point in segment (i) is therefore given by

$$\Delta t^{(i)} = \bar{t}^{(i)} \log \frac{\phi_{n+1}^{(i)}}{\phi_{n+1}^{(i+1)}} \tag{3.19}$$

(d) The end of the relaxation process is characterized by the condition that $\sum_i \Delta t^{(i)} = h$, where h is the time step size and the sum extends to all traversed segments.

The resulting algorithm is summarized in Box **6**. The simplicity and generality of the algorithm should be noted. The same general results mentioned aboved regarding *consistency* and *stability* of the algorithm apply in the present context. We finally note that as $\eta \to 0$ the integration scheme for *inviscid* plasticity is recovered. This is consistent with well-known results concerning the inviscid limit of viscoplasticity (e.g, see [36, Chap. 3] for a review).

3.4. Boundary Value Problem: Consistent Tangent Operator.

In order to formulate a well-defined problem, in addition to the constitutive relations we need to consider the momentum balance equation and suitable boundary and initial conditions. A weak formulation of the resulting boundary value problem can be given as follows. Let $\mathbf{b}(\mathbf{x})$ be the body force, $\mathbf{a}(\mathbf{x})$ the spatial acceleration field and $\bar{\mathbf{t}}(\mathbf{x})$ the traction vector specified on the Neumann boundary $\partial_\sigma B_n$. Furthermore, let the deformation mapping $\boldsymbol{\phi}$ be prescribed on the Dirichlet boundary $\partial_u B_0$ as $\boldsymbol{\phi}|_{\partial_u B_0} = \bar{\boldsymbol{\phi}}$. As usual we require that $\partial_u B_n \cap \partial_\sigma B_n = \varnothing$ and $\partial_u B_n \cup \partial_\sigma B_n = \partial B_n$, where one has $\partial_u B_n = \boldsymbol{\phi}(\partial_u B_0)$. Then, the weak form of the momentum balance equation at time t_n reads

$$G(\boldsymbol{\phi},\boldsymbol{\eta}) \equiv \int_{B_n} \rho\, \mathbf{a}\cdot\boldsymbol{\eta}\, dB + \int_{B_n} \boldsymbol{\sigma} : \nabla\boldsymbol{\eta}\, dB - \int_{B_n} \mathbf{b}\cdot\boldsymbol{\eta}\, dB - \int_{\partial_\sigma B_n} \bar{\mathbf{t}}\cdot\boldsymbol{\eta}\, dS = 0, \tag{3.20}$$

for any admissible variation $\boldsymbol{\eta}$ such that $\boldsymbol{\eta}|_{\partial_u B_n} = 0$. Since the treatment of the transient dynamic problem plays no role in the present discussion, we shall ignore inertia effects and confine our attention to the static case.

Within the context of finite element analysis, the solution of problem (3.20) is accomplished by an iterative scheme such as Newton's method. Typically, one solves a sequence of linearized problems defined as

$$DG(\boldsymbol{\phi}_{n+1}^{(i)},\boldsymbol{\eta})\cdot\mathbf{u}_{n+1}^{(i)} \equiv \int_{B_n} [\frac{1}{J}\, tr\,(\nabla\boldsymbol{\eta}\,\boldsymbol{\tau}\,\nabla\mathbf{u}) + \nabla\boldsymbol{\eta} : (\mathbf{c}_{n+1}^{(i)} : \nabla\mathbf{u}_{n+1}^{(i)})]\, dB = -\, G(\boldsymbol{\phi}_{n+1}^{(i)},\boldsymbol{\eta}) \tag{3.21}$$

until the residual $G(\boldsymbol{\phi}_{n+1}^{(i)},\boldsymbol{\eta})$ vanishes to within a prescribed tolerance.

The convergence rate of iterative scheme (3.4) is by and large governed by the choice of tangent moduli $\mathbf{c}_{n+1}^{(i)}$ which, as recently noted in [37], depends in turn on the iteration scheme adopted. In a typical iteration i+1 within a time step $[t_n, t_{n+1}]$, the variables $(\bullet)_{n+1}^{(i+1)}$ may be obtained by means of the algorithm in Box 5 and from either

(a) the values $(\bullet)_{n+1}^{(i)}$ corresponding to the previous *non–converged* iteration, or

(b) the *converged* values $(\bullet)_n$ from the previous time step.

BOX 6. **Stress Update Algorithm: Viscoplasticity**

i) Geometric update

ii) Elastic predictor

iii) Check for yielding

$$\phi_{n+1}^{(0)} \equiv \phi(\boldsymbol{\tau}_{n+1}^{(0)}, \mathbf{q}_{n+1}^{(0)}) \leqslant 0 \ ?$$

YES $\quad \mathbf{F}_{n+1}^{p} = \mathbf{F}_{n+1}^{p(0)} \ ; \ \mathbf{q}_{n+1} = \mathbf{q}_{n+1}^{(0)} \ ; \ \boldsymbol{\tau}_{n+1} = \boldsymbol{\tau}_{n+1}^{(0)} \ ;$ *EXIT*

NO $\quad i = 0, \quad t^{(0)} = 0$

iv) Plastic correctors

$$t^{(i)} = \frac{\eta}{\left(\dfrac{\partial \phi}{\partial \boldsymbol{\tau}}\right)_{n+1}^{(i)} : J_{n+1}(\mathbf{c}^e - \mathbf{c}^p)_{n+1}^{(i)} : \mathbf{r}_{n+1}^{(i)} - \left(\dfrac{\partial \phi}{\partial \mathbf{q}}\right)_{n+1}^{(i)} \cdot \mathbf{h}_{n+1}^{(i)}}$$

$$\Delta\gamma = \frac{\phi_{n+1}^{(i)} \, \overline{t}^{(i)}}{\eta}$$

$$\boldsymbol{\tau}_{n+1}^{(i+1)} = \boldsymbol{\tau}_{n+1}^{(i)} - \Delta\gamma \, J_{n+1}(\mathbf{c}^e - \mathbf{c}^p)_{n+1}^{(i)} : \mathbf{r}_{n+1}^{(i)}$$

$$\mathbf{q}_{n+1}^{(i+1)} = \mathbf{q}_{n+1}^{(i)} + \Delta\gamma \, \mathbf{h}_{n+1}^{(i)}$$

$$\Delta t^{(i)} = \overline{t}^{(i)} \log(\phi_{n+1}^{(i)} / \phi_{n+1}^{(i+1)})$$

$$t^{(i+1)} = t^{(i)} + \Delta t^{(i)}$$

v) Check for end of relaxation process

$$t^{(i+1)} \geqslant h$$

YES $\quad \Delta t^{(i)} = h - t^{(i)}$

$$\Delta\gamma = \frac{\phi_{n+1}^{(i)} \, \overline{t}^{(i)}}{\eta} [1 - \exp(-\Delta t^{(i)} / \overline{t}^{(i)})]$$

$$\boldsymbol{\tau}_{n+1} = \boldsymbol{\tau}_{n+1}^{(i)} - \Delta\gamma \, J_{n+1}(\mathbf{c}^e - \mathbf{c}^p)_{n+1}^{(i)} : \mathbf{r}_{n+1}^{(i)}$$

$$\mathbf{q}_{n+1} = \mathbf{q}_{n+1}^{(i)} + \Delta\gamma \, \mathbf{h}_{n+1}^{(i)}$$

Compute $\mathbf{F}_{n+1}^{e} \quad \boldsymbol{\tau}_{n+1}^{(i+1)}$ (see Remark **3.3.** on isotropy)

$\mathbf{F}_{n+1}^{p} = \mathbf{F}_{n+1} \mathbf{F}_{n+1}^{e-1}, \quad$ *EXIT*

NO $\quad i \leftarrow i+1$; *GO TO (iv)*

Both procedures define algorithms which are *consistent* with the field equations. However, scheme (a) introduces a "history dependence" of the converged values on intermediate *non-converged* iterates. This may pose difficulties due to the strong *path dependence* of plasticity models. Spurious unloadings at some Gauss points may also occur as a result of this procedure. By contrast, history dependence on intermediate *non-converged* values is eliminated with the use of scheme (b), and "fictitious" numerical unloading is therefore prevented.

By virtue of an stress update algorithm such as the one proposed in Section **3.1**, the updated stresses $\boldsymbol{\tau}_{n+1}^{(i+1)}$ become a function of the incremental displacements $\nabla \mathbf{u}$, as well as of suitable initial conditions consistent with the update strategy (a) or (b). The essential point emphasized in [37] is that the tangent moduli appearing in (3.4) must be derived by consistent linearization of the update procedure, in order to achieve the asymptotic rate of quadratic convergence characteristic of Newton's method. If scheme (a) is adopted, the consistent tangent moduli coincide with the classical elastoplastic ones. For the spatial constitutive equations in Box **3** such elastoplastic moduli take the form

$$\mathbf{c}^{ep} \equiv \mathbf{c}^{e} - \frac{[(\mathbf{c}^{e}-\mathbf{c}^{p}):\mathbf{r}]\otimes[\mathbf{c}^{e}:\dfrac{\partial\phi}{\partial\boldsymbol{\tau}}]}{\dfrac{\partial\phi}{\partial\boldsymbol{\tau}}:(\mathbf{c}^{e}-\mathbf{c}^{p}):\mathbf{r}-\dfrac{1}{J}\dfrac{\partial\phi}{\partial\mathbf{q}}\cdot\mathbf{h}} \tag{3.22}$$

If, on the other hand the update scheme (b) is adopted, the consistent tangent moduli are no longer given by expression (3.22). In fact, use of the elastoplastic moduli (3.22) may result in a dramatic loss of the quadratic rate of asymptotic convergence, as shown in [37].

In the context of return mapping algorithms and for linearized kinematics, explicit expressions for the tangent moduli consistent with the update strategy (b) and associated with several widely used plastic models are given in [37]. For simple yield conditions such as von Mises, and nonlinear isotropic and kinematic hardening rules, the results in [37] may be extended to the present setting. We refer to [38] for further details. In general, however, the task of evaluating the consistent tangent moduli in closed form may prove exceedingly laborious. It would appear, therefore, that a general purpose implementation of the physically more compelling algorithm based on the updating procedure (b), may require the use of quasi-Newton or secant-Newton methods [39] for the solution of resulting nonlinear algebraic problem.

4. A Numerical Example.

There are a number of important issues involved in the finite element implementation of the formulations and corresponding stress update algorithms discussed in the previous sections. Among them should be mentioned:

(i) The treatment of constraints in the finite deformation range such as incompressibility of the plastic flow. In the context of the linear theory, the importance of a proper treatment of this constraint was first recognized by Nagtegaal, Parks and Rice [40].

(ii) The use of consistent linearization procedures, as discussed in [31, Chap. **4**], to obtain elastoplastic tangent moduli consistent with the stress update algorithm for specific models.

These and related computational issues are treated in [38] together with the discussion of several numerical experiments.

Our intention here is simply to illustrate the formulation heretofore discussed by means of a classical example: the thick wall cylinder under internal pressure. For this purpose we consider perfect plastic behavior with a von Mises yield condition, and elastic response governed by the hyperelastic constitutive equations (2.4-5).

The results of the numerical experiment are shown in Fig. **5** and Fig. **6**, together with the exact solution for the case of a rigid plastic material. In this figure, a designates the inner radius in the current configuration. The following features pertaining to these results are noteworthy:

(i) The final configuration of the cylinder, with inner radius $a = 85.10$ and an outer radius of 88.8, is attained in 15 time steps. Note that the initial values of the inner and outer radius are 10 and 20, respectively.

(ii) Within each time step, convergence with the classical Newton's method, in the energy norm and to a tolerance of $TOL = 10^{-18}$, is achieved in 4-5 iterations.

(iii) For the purpose of comparison with the rigid plastic solution the material properties are chosen as to obtain *infinitesimal* elastic deformations. In spite of the extremely large loading step employed, excellent agreement between the asymptotically exact analytic solution and the computed results is found.

The reason for the excellent rate of convergence exhibited by the solution process may be found in the use of tangent moduli consistent with the stress update algorithm. These tangent moduli *are not* given by (3.19), but obtained from the stress update algorithm by a consistent linearization procedure.

5. Summary and Conclusions.

A unified approach to finite deformation elastoplasticity which embodies both additive and multiplicative theories has been presented. Taking the multiplicative decomposition of the deformation gradient as a point of departure, an additive decomposition of Lagrangian and Almansi strains follows. Such an additive decomposition carries over to the corresponding material and spatial rates of deformation.

From a numerical standpoint, the proposed formulation has several far-reaching consequences. First, the use of hyperelastic constitutive models entirely avoids the need for integration of rate constitutive relations. In particular, so called incrementally objective integration algorithms are no longer needed, even in the context of rate-dependent viscoplastic models. Second, an operator-splitting methodology can be used to exploit the additive decomposition of the deformation rates. On this basis, a general class of return-mapping algorithms capable of accommodating arbitrary yield criteria, flow rules, hardening laws and variable tangent elastic compliances has been derived. It should be emphasized that the elastic predictor in the stress update procedure reduces to a mere function evaluation. The return mapping takes an iterative form whereby stresses converge towards a suitably updated yield surface at a quadratic rate. Accuracy and unconditional stability are guaranteed by general results pertaining to the operator-splitting methodology. The proposed numerical schemes apply to both rate-dependent and rate-independent plastic models.

Owing to the covariant character of the theoretical framework and of the proposed algorithms adopted, selection of the material, intermediate or spatial descriptions as a basis for numerical computations is a simple matter of choice. In fact, the results obtained from any one of the descriptions are identical.

The high accuracy of the method, even for very large time steps has been illustrated by means of numerical experiment. It is also noted that use of consistent tangent moduli results in excellent rates of convergence.

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FIGURE LEGENDS

Figure 1. Schematic representation of material, intermediate and spatial configurations.

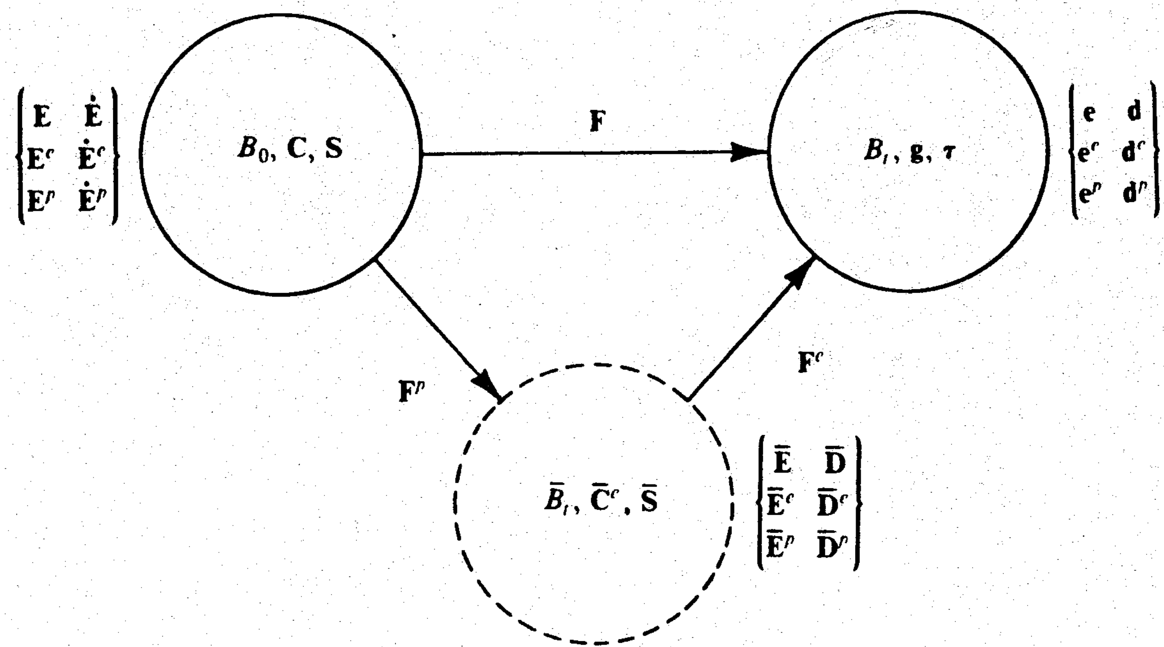
Figure 2. Geometric interpretation for the case of perfect plasticity of a general return mapping algorithm based on an elastic-plastic split of the constitutive equations. The elastic relations define an elastic predictor $\tilde{\boldsymbol{\sigma}}_{n+1}$ which is subsequently returned to the yield surface along the steepest descent path of the yield function ϕ. The steepest direction is determined in terms of the local metric defined by the elastic tangent moduli.

Figure 3. Numerical implementation of the return mapping algorithm shown in Fig. 2. The elastic predictor $\tilde{\boldsymbol{\sigma}}_{n+1}$ is returned to the yield surface in successive steps. At every step, the updated stresses $\boldsymbol{\sigma}_{n+1}^{(i+1)}$ are computed by projecting the previous iteration $\boldsymbol{\sigma}_{n+1}^{(i)}$ onto the trace on the plane $\phi = 0$ of a linear approximation to the yield function at $\boldsymbol{\sigma}_{n+1}^{(i)}$ or "cut". In the limit, such cuts become tangent to the yield surface and plastic consistency is recovered at a quadratic convergence rate.

Figure 4. Geometric aspects of the elastic-plastic splitting methodology. (a) The elastic predictor takes place under constant intermediate configuration and the incremental deformations $\mathbf{F}_u$ strain the body elastically. (b) The plastic corrector leaves the updated spatial configuration invariant while the intermediate configuration relaxes plastically.

Figure 5. Thick wall cylinder subjected to internal pressure. Stress component σ_{rr} versus inner radius a in current configuration.

Figure 6. Thick wall cylinder. Distribution of the radial stress component σ_{rr} over thickness in current configuration.



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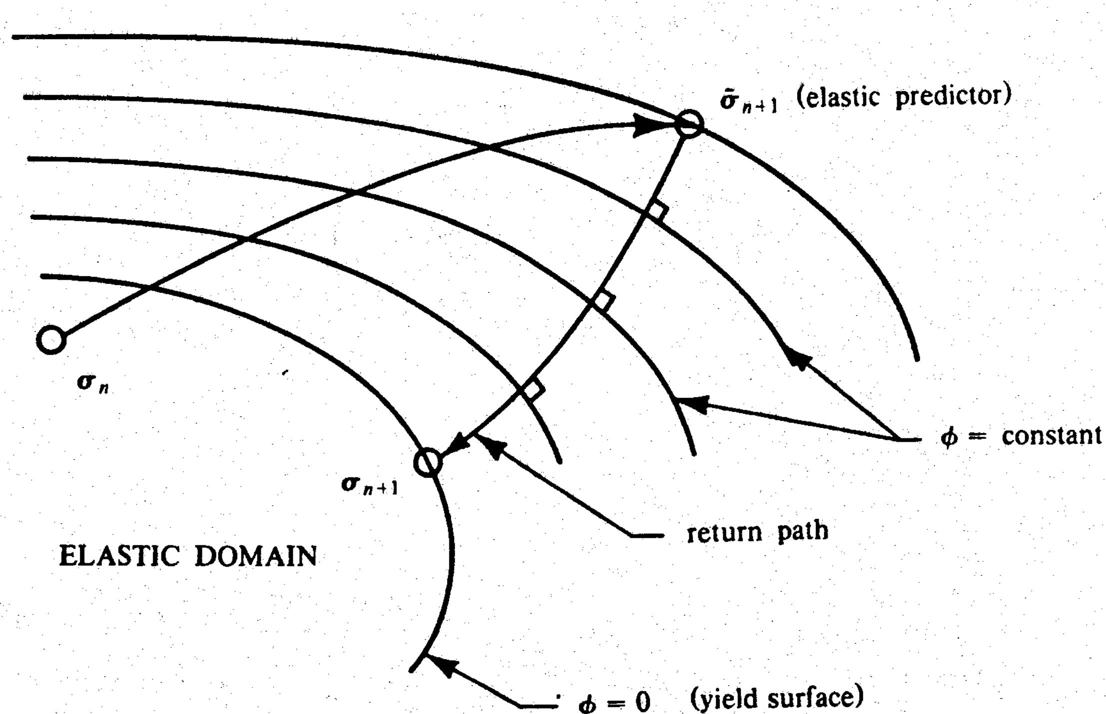

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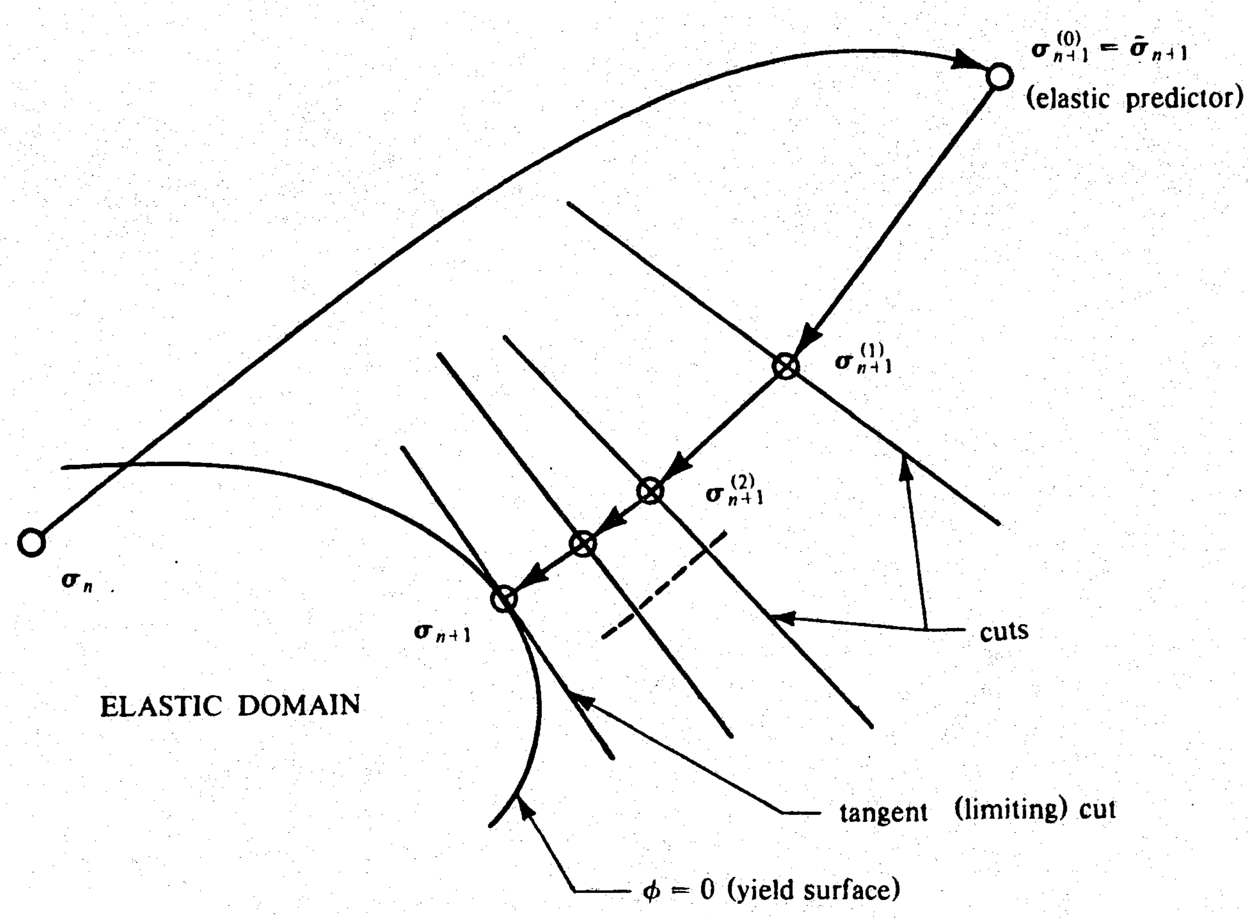
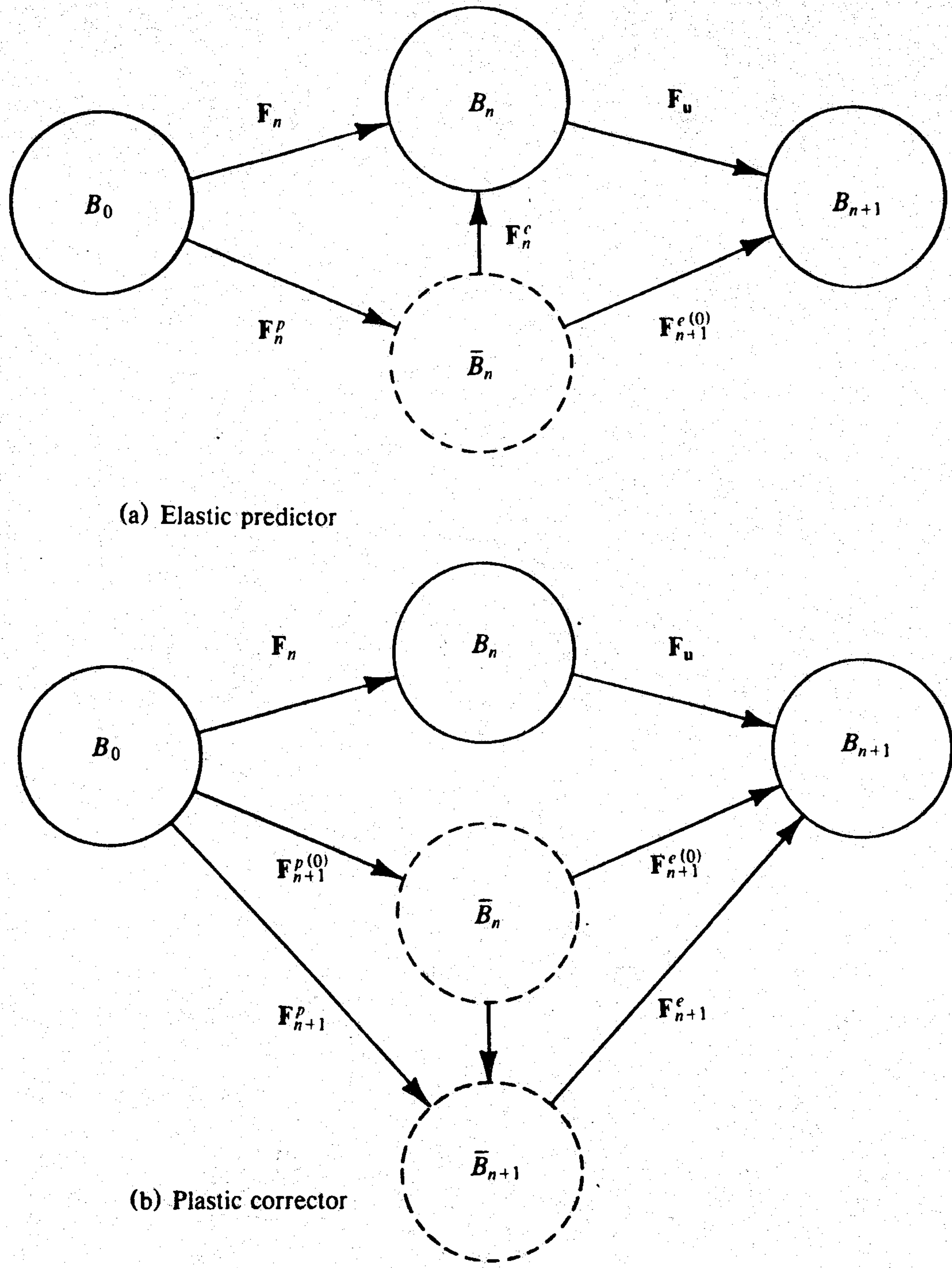


Figure 3. Numerical implementation of the return mapping algorithm shown in Fig. 2. The elastic predictor $\bar{\sigma}_{n+1}$ is returned to the yield surface in successive steps. At every step, the updated stresses $\sigma^{(i+1)}_{n+1}$ are computed by projecting the previous iteration $\sigma^{(i)}_{n+1}$ onto the trace on the plane $\phi = 0$ of a linear approximation to the yield function at $\sigma^{(i)}_{n+1}$ or "cut". In the limit, such cuts become tangent to the yield surface and plastic consistency is recovered at a quadratic convergence rate.



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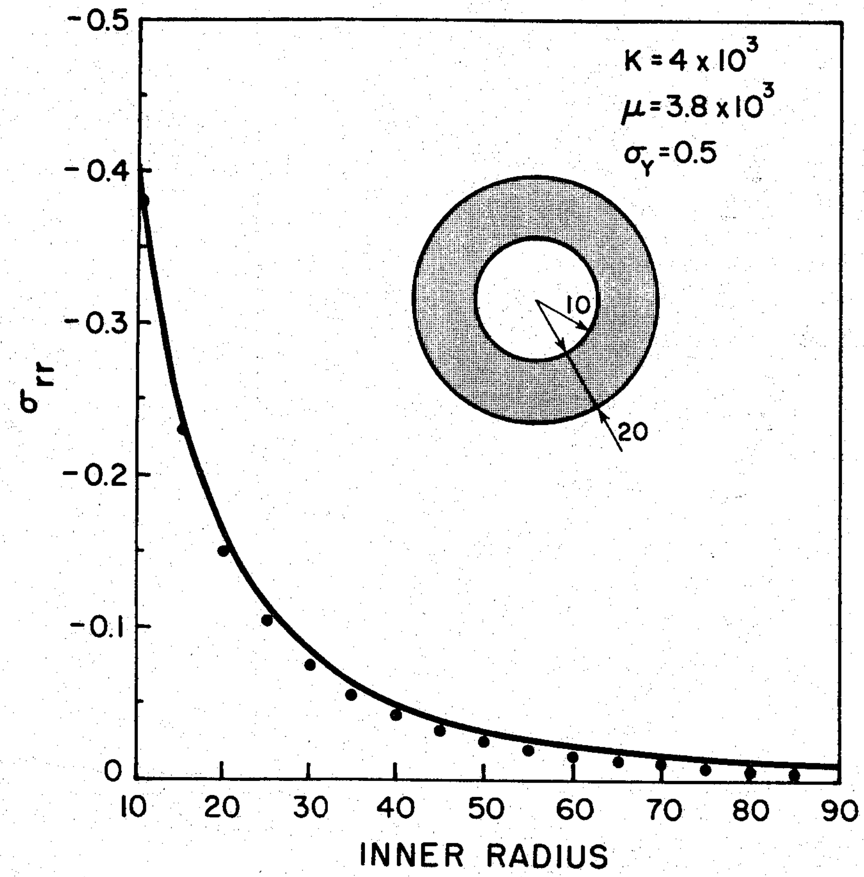

FIG. 5 THICK WALL CYLINDER UNDER INTERNAL PRESSURE.

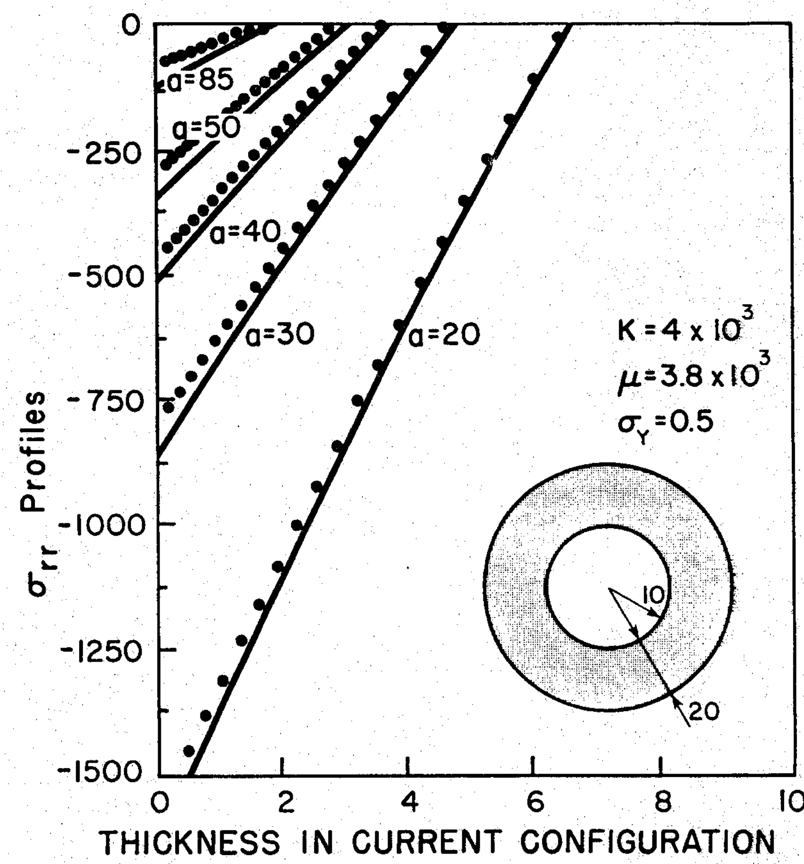

FIG. 6 THICK WALL CYLINDER UNDER INTERNAL PRESSURE.