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Title
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Permalink
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Journal
Canadian Journal of Mathematics, 45(1)

ISSN
0008-414X

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Publication Date
1993-02-01

DOI
10.4153/cjm-1993-004-0

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Canad. J. Math. Vol. **45 (1)** 1993, pp. 33–87.

CLASSIFICATION OF ATOMIC FACIALLY SYMMETRIC SPACES

YAAKOV FRIEDMAN AND BERNARD RUSSO

ABSTRACT A Banach space satisfying some physically significant geometric properties is shown to be the predual of a JBW*-triple If one considers the unit ball of this Banach space as the state space of a physical system, the result shows that the set of observables is equipped with a natural ternary algebraic structure This provides a spectral theory and other tools for studying the quantum mechanical measuring process

Facially symmetric spaces were introduced in [12] and studied in [15]. In this paper we obtain the complete structure of *atomic* facially symmetric spaces. We have thus solved the problem posed in [12] by showing that atomic neutral strongly facially symmetric spaces are preduals of atomic JBW*-triples. We show, more precisely, that an irreducible neutral strongly facially symmetric space is linearly isometric to the predual of one of the Cartan factors of types 1 to 6, provided that it satisfies some natural and physically significant axioms, four in number, which are known to hold in the preduals of all JBW*-triples. As was done in [4] to study the state spaces of Jordan algebras, we shall refer to our axioms as the pure state properties, abbreviated PSP. Since we can regard the entire unit ball of a facially symmetric space as the "state space" of a physical system, *cf.* [12, Introduction], we have given a geometric characterization of such state spaces.

The project of classifying facially symmetric spaces was started in [16], where, using two of the PSP's, denoted by STP and FE, geometric characterizations of complex Hilbert spaces and complex spin factors were given. The former is precisely a rank 1 JBW*-triple and a special case of a Cartan factor of type 1, and the latter is the Cartan factor of type 4 and a special case of a JBW*-triple of rank 2. For a description of all of the Cartan factors, see [8, pp. 292-3].

In §1 we recall some basic facts about facially symmetric spaces and prove a few general results which are needed later. In order to find a geometric characterization of the Cartan factor of type 3, it was necessary to add the third PSP, denoted by ERP. With only this additional axiom, we are able to characterize the Cartan factor of type 3 among *atomic* facially symmetric spaces. This property is introduced and studied in §1 and used in §2 to develop some important properties of the geometric Peirce 1-space of a minimal geometric tripotent. Of special importance is a dichotomy property called the "two case lemma". We give a detailed description of each of the cases occuring in the dichotomy property and use them to prove some hereditary properties of the 1-space.

Supported in part by NSF Grant No DMS8805256 and BSF Grant No 8800066
Received by the editors August 15, 1991
AMS subject classification Primary 46, secondary 17, 52
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https://doi.org/10.4153/CJM-1993-004-0 Published online by Cambridge University Press

After proving the characterization theorem for the Cartan factor of type 3 in §3, in §4 we provide some tools for obtaining geometric characterizations of the remaining Cartan factors, *i.e.*, those of types 1, 2, 5, and 6. We introduce the fourth axiom JP and use it to obtain further properties of the geometric Peirce 1-space of a minimal geometric tripotent. The classification scheme developed in §4 is very similar in spirit to the one in [8], being based in part on a reduction to previous cases by using the properties of the geometric Peirce 1-space prepared in §2. Indeed, the main technical result of §4 is the fact that the pure state properties STP and ERP are inherited by the geometric Peirce 1-space of a minimal geometric tripotent (Corollary 4.12). The difficulty in the proof of this fact arises from the non-neutrality of the geometric Peirce 1-projection. The fact that this space is atomic and satisfies FE is less troublesome and is proved in §2.

Also in this section, we define the basic building blocks for facially symmetric spaces, namely the *geometric quadrangles*, which are used in the characterizations of the Cartan factors of types 1, 2, 5, 6. A helpful feature of the classification from this point on is that we deal only with so-called ortho-colinear (geometric) grids. Thus the technically difficult case of a governing geometric tripotent which occurs in §3 is not present in the rest of the classification. We introduce an invariant of a facially symmetric space which we call the *spin degree* and show that this invariant is either infinite, or one of the integers 3, 4, 6, 8, 10.

In §5 (resp.§6) we show that if Z is a neutral SFS space of spin degree 4 (resp. spin degree 6) which satisfies the four PSP's, then Z has an L-summand which is linearly isometric to the predual of a Cartan factor of type 1 (resp. type 2). In §7 we show that if Z is a neutral SFS space of spin degree 8 (resp. spin degree 10), which satisfies the above axioms, then Z has an L-summand which is linearly isometric to the predual of a Cartan factor of type 5 (resp. type 6). Finally, in §8 our main result is stated and we indicate how its proof follows simply from results in the earlier sections. We also include a discussion of which of the PSP's we believe are essential and we discuss some areas for further investigation.

The complex and real fields are denoted by $\mathbf{C}$ and $\mathbf{R}$ respectively. The unit circle is denoted by $\mathbf{T}$. We use $\Re z$ for the real part of the complex number z.

1. **Preliminaries: extreme rays property.** In this section we shall recall some basic facts and notation about facially symmetric spaces from [15] and prove one new result which supplements those in [15]. We then introduce the axiom ERP and use it to establish an important property concerning the least upper bound of two minimal geometric tripotents.

Let Z be a complex normed space. Elements $f, g \in Z$ are *orthogonal*, notation $f \diamond g$, if $\|f+g\| = \|f-g\| = \|f\|+\|g\|$. A *norm exposed face* of the unit ball Z_1 of Z is a non-empty set (necessarily $\neq Z_1$) of the form $F_x = \{f \in Z_1 : f(x) = 1\}$, where $x \in Z^*$, $\|x\| = 1$. Recall that a *face* G of a convex set K is a non-empty convex subset of K such that if $g \in G$ and $h, k \in K$ satisfy $g = \lambda h + (1 - \lambda)k$ for some $\lambda \in (0, 1)$, then $h, k \in G$. An element $u \in Z^*$ is called a *projective unit* if $\|u\| = 1$ and $\langle u, F_u^{\diamond}\rangle = 0$. Here, for any subset

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S, $S^\diamond$ denotes the set of all elements orthogonal to each element of S. $\mathcal{F}$ and $\mathcal{U}$ denote the collections of norm exposed faces of Z_1 and projective units in Z^*, respectively.

Motivated by measuring processes in quantum mechanics, we define a *symmetric face* to be a norm exposed face F in Z_1 with the following property: there is a linear isometry S_F of Z onto Z, with $S_F^2 = I$ (we call such maps *symmetries*), such that the fixed point set of S_F is $(\overline{\text{sp}}F) \oplus F^\diamond$ (topological direct sum). A complex normed space Z is said to be *weakly facially symmetric* (WFS) if every norm exposed face in Z_1 is symmetric. For each symmetric face F we define contractive projections $P_k(F)$, $k = 0, 1, 2$ on Z as follows. First $P_1(F) = (I - S_F)/2$ is the projection on the -1 eigenspace of S_F. Next we define $P_2(F)$ and $P_0(F)$ as the projections of Z onto $\overline{\text{sp}}F$ and $F^\diamond$ respectively, so that $P_2(F) + P_0(F) = (I + S_F)/2$. These projections were called *generalized Peirce projections* in [12] and [15]. A *generalized tripotent* is a projective unit $u \in \mathcal{U}$ with the property that $F := F_u$ is a symmetric face and $S_F^* u = u$ for some choice of symmetry S_F corresponding to F. In [16] and this paper we call such u *geometric tripotents*, and the projections $P_k(F_u)$ will be called *geometric Peirce projections*.

$\mathcal{GT}$ and $\mathcal{SF}$ denote the collections of geometric tripotents and symmetric faces respectively, and the map $\mathcal{GT} \ni u \longmapsto F_u \in \mathcal{SF}$ is a bijection [15, Proposition 1.6]. For each geometric tripotent u in the dual of a WFS space Z, we shall denote the geometric Peirce projections by $P_k(u) = P_k(F_u), k = 0, 1, 2$. Also we let $U := Z^*, Z_k(u) = Z_k(F_u) := P_k(u)Z$ and $U_k(u) = U_k(F_u) := P_k(u)^*(U)$, so that we have the *geometric Peirce decompositions* $Z = Z_2(u) + Z_1(u) + Z_0(u)$ and $U = U_2(u) + U_1(u) + U_0(u)$. A symmetry corresponding to the symmetric face F_u will sometimes be denoted by S_u. Elements a and b of U are *orthogonal* if one of them belongs to $U_2(u)$ and the other to $U_0(u)$ for some geometric tripotent u. Two geometric tripotents u and v are said to be *compatible* if their associated geometric Peirce projections commute, *i.e.*, $[P_k(u), P_j(v)] = 0$ for $k,\ j \in \{0, 1, 2\}$. For each $G \in \mathcal{F}$, v_G denotes the unique geometric tripotent with $F_{v_G} = G$.

A contractive projection Q on a normed space X is said to be *neutral* if for each $\xi \in X$, $\|Q\xi\| = \|\xi\|$ implies $Q\xi = \xi$. A normed space Z is *neutral* if for every symmetric face F, the projection $P_2(F)$ corresponding to some choice of symmetry S_F, is neutral.

The following proposition is a consequence of neutrality and the uniqueness of Hahn-Banach extensions from certain geometric Peirce subspaces.

PROPOSITION 1.1. *If T is a linear isometry of a neutral* WFS *space Z onto Z, then for each projective unit u,*

$$TS_uT^{-1} = S_{T^{*-1}u}.$$

Also $T\big(Z_k(u)\big) = Z_k(T^{*-1}u)$ for $k = 0, 1, 2$ and

$$(T^{-1})^* S_u^* T^* = S_{T^{*-1}u}^*. \tag{1}$$

PROOF. Let $u' := T^{*-1}u$. From [12, Lemma 2.4], u' is a projective unit and $TF_u = F_{u'}$. Therefore $T\big(Z_2(u)\big) = T(\overline{\text{sp}}F_u) = \overline{\text{sp}}T(F_u) = Z_2(u')$. Obviously, $F_u^\diamond \supset (\overline{\text{sp}}F_u)^\diamond$. By [15, Proposition 1.1], $F_u^\diamond \subset (\overline{\text{sp}}F_u)^\diamond$, and thus $Z_0(u) = Z_2(u)^\diamond$. Hence $T\big(Z_0(u)\big) = \big(TZ_2(u)\big)^\diamond =$

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$\big(Z_2(u')\big)^{\diamond} = Z_0(u')$, and thus for $k = 0$ and 2, $TP_k(u)T^{-1}$ is a contractive projection with range $Z_k(u')$.

For $k = 0$ or 2, the projections $R := TP_k(u)T^{-1}$ and $Q := P_k(u')$ have the same range and are both neutral. In order to apply [15, Lemma 2.2] to conclude $R = Q$, we must show that $R^*(U) = Q^*(U)$. Since $T^*v_G = v_{T^{-1}G}$, and $T\big(Z_k(u)\big) = Z_k(u')$, we have, by [15, Theorem 2.3], $Q^*(U) = \overline{\mathrm{sp}}^{w^*}\{v_H : H \subset Z_k(u')\} = \overline{\mathrm{sp}}^{w^*}\{v_G : T^{-1}G \subset Z_k(u)\}$. Thus $T^*Q^*(U) = \overline{\mathrm{sp}}^{w^*}\{T^*v_G : T^{-1}G \subset Z_k(u)\} = \overline{\mathrm{sp}}^{w^*}\{v_H : H \subset Z_k(u)\} = U_k(u) = P_k(u)^*(U)$. Hence $Q^*(U) = T^{*-1}P_k(U)^*(U) = T^{*-1}P_k(u)^*T^*(U) = R^*(U)$.

We now have, from [15, Lemma 2.2], that $TP_k(u)T^{-1} = P_k(u')$ for $k = 0$ and 2, and since $P_2(u') + P_1(u') + P_0(u') = I$, we have $TP_1(u)T^{-1} = P_1(u')$. Finally, $S_{u'} = P_2(u') - P_1(u') + P_0(u') = TS_uT^{-1}$, which implies (1). ∎

If Y is a closed subspace of a normed space Z, the collections of norm exposed faces and symmetric faces in Y_1, will be denoted by $\mathcal{F}_Y$ and $\mathcal{SF}_Y$ respectively. Similarly for $\mathcal{U}_Y$, $\mathcal{GT}_Y$. Recall that if Y is a closed subspace of a normed space Z, and if K is a norm exposed face of the unit ball of Y, then $K = F_x \cap Y$ for some $x \in Z^*$. Moreover, by [15, Theorem 3.6], if Z is neutral and WFS and if K is a symmetric face, then the geometric Peirce projections associated with K are the restrictions of the geometric Peirce projections associated with F_x (*cf.* [15, Theorem 3.6]). We also have $\mathcal{GT}_{Z_k(u)} = \mathcal{GT} \cap U_k(u)$ (*cf.* [16, Proposition 2.1]).

A WFS space Z is *strongly facially symmetric* (SFS) if for every norm exposed face F in Z_1 and every $y \in Z^*$ with $\|y\| = 1$ and $F \subset F_y$, we have $S_F^* y = y$, where S_F denotes a symmetry associated with F. Let Z be a strongly facially symmetric space and suppose that $u, v \in \mathcal{GT}_Z$. If $F_u \subset F_v$, we write $u \le v$ (See [15, Definition 2, Lemma 4.2]).

The principal examples of neutral strongly facially symmetric spaces are preduals of JBW*-triples, in particular, the preduals of von Neumann algebras, *cf.* [14]. In these cases, as shown in [14], geometric tripotents correspond to tripotents in a JBW*-triple and partial isometries in a von Neumann algebra.

In a neutral strongly facially symmetric space Z, every non-zero element has a *polar decomposition* [15, Theorem 4.3]: for $0 \ne f \in Z$ there exists a unique geometric tripotent $v = v(f)$ with $f(v) = \|f\|$ and $\langle v, \{f\}^{\diamond} \rangle = 0$.

The following lemma will be needed in §4 and in §8. It holds more generally, with the same proof, for the intersection of the geometric Peirce spaces of any family of mutually compatible geometric tripotents (*cf.* Proposition 4.11 below).

LEMMA 1.2. *Let* $\{v_\alpha\}$ *be a family of geometric tripotents in a neutral* SFS *space* Z *satisfying* $v_\alpha \in U_1(v_\beta)$ *for all* $\alpha \ne \beta$. *Then* $\bigcap_\alpha Z_1(v_\alpha)$ *is a neutral* SFS *space.*

PROOF. Since the infinite product of commuting contractive projections is a contractive projection with range the intersection of the ranges of the family, the subspace $Y := \bigcap_\alpha Z_1(v_\alpha)$ is the range of a contractive projection Q.

Let K be a norm exposed face in the unit ball Y_1 of Y, and write $K = F_x \cap Y$ for some $x \in \bigcap_\alpha U_1(v_\alpha)$ $\big(\cong Q^*(U)\big)$ with $\|x\| = 1$. By [15, Theorem 2.5], the geometric tripotent w with $F_w = F_x$ belongs to $\bigcap_\alpha U_1(v_\alpha)$ and therefore, by [15, Theorem 3.3], as in the

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proof of [15, Theorem 3.6 and Proposition 4.1], Q commutes with the symmetry S_{F_x}. It follows that Y is a neutral SFS space. ∎

The following property is needed in the characterization of Cartan factors of type 3, which will be proved in §3. We use the notation $\operatorname{ext} K$ for the set of extreme points of the convex set K.

DEFINITION 1.3. A neutral SFS space Z is said to satisfy the *"extreme rays property"* (ERP) if for every $u \in \mathcal{GT}$ and every $f \in \operatorname{ext} Z_1$, it follows that $P_2(u)f$ is a scalar multiple of some element in $\operatorname{ext} Z_1$. We also say that $P_2(u)$ preserves extreme rays.

REMARK 1.4. It follows exactly as in [15, Lemma 2.1] that if $P_2(u)$ preserves extreme rays for every geometric tripotent u, then so does $P_0(u)$. Indeed, if $f \in \operatorname{ext} Z_1$, and $g := P_0(u)f$, then $\nu(g) \in U_0(u)$, and by [15, Corollary 3.4(a)], $P_0(u)f = P_2\big(\nu(g)\big)P_0(u)f = P_2\big(\nu(g)\big)f$ is a multiple of an extreme point.

If Z is the predual of a JBW*-triple, then it is known ([11, Proposition 7]) that Z satisfies ERP. Moreover, for each tripotent u, and $k = 0$ or 2, $P_k(u)^*$ maps a minimal tripotent to a multiple of a minimal tripotent [11, Proposition 6]. The following proposition establishes this property in atomic facially symmetric spaces, and will be used in Proposition 3.7.

Recall that $\mathcal{M}$ denotes the collection of all minimal geometric tripotents in U, *i.e.*, $\mathcal{M} = \{v \in \mathcal{GT} : Z_2(v) \text{ is one dimensional}\}$, and that the conjugage linear mapping π and the symmetric sesquilinear form $\langle \cdot \mid \cdot \rangle$ are defined in [16, Proposition 2.9]. Also, for definitions of the PSP's STP and PE and the property "atomic", see [16, Definitions 2.5, 2.6, 2.8].

PROPOSITION 1.5. *In an atomic neutral* SFS *space Z satisfying* PE, STP *and* ERP, *for each geometric tripotent $u \in \mathcal{GT}$, and for $k = 0$ or 2, $P_k(u)^*$ maps a minimal geometric tripotent to a multiple of a minimal geometric tripotent.*

PROOF. We show first that for every pair of extreme points f, g of Z_1,

$$\langle P_k(u)f, \nu(g)\rangle = \overline{\langle P_k(u)g, \nu(f)\rangle}. \tag{2}$$

To prove (2), we first show that

$$\pi P_k(u) = P_k(u)^* \pi P_k(u). \tag{3}$$

By atomicity, it suffices to check (3) on an extreme point f of Z_1. By ERP, $P_k(u)f = \lambda h$ for some $\lambda \in \mathbf{C}$ and extreme point h. Since $\pi(h) = \nu(h) \in U_k(u)$, we have $\pi P_k(u)f = \lambda \nu(h) = P_k(u)^*(\lambda h) = P_k(u)^* \pi P_k(u)f$, proving (3).

We can now prove (2):

$$\begin{aligned}\langle P_k(u)f, \nu(g)\rangle &= \langle P_k(u)f \mid g\rangle = \overline{\langle g, \pi P_k(u)f\rangle} = \overline{\langle P_k(u)g, \pi P_k(u)f\rangle}\\ &= \langle P_k(u)f, \pi P_k(u)g\rangle = \langle f, \pi P_k(u)g\rangle = \overline{\langle P_k(u)g, \pi(f)\rangle}.\end{aligned}$$

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We shall show that (2) implies that for every extreme point g and every $v \in \mathcal{M}$

$$\langle P_k(u)^* v, g\rangle = \langle \pi P_k(u)\pi^{-1}(v), g\rangle. \tag{4}$$

Then, since Z is atomic, we will have $P_k(u)^* v = \pi P_k(u)\pi^{-1}(v)$ and by ERP and [16, Proposition 2.9], $P_k(u)^* v$ will be a scalar multiple of an element of $\mathcal{M}$.

To prove (4), note first that its left side is equal to $\langle v, P_k(u)g\rangle$, whereas its right side, with $P_k(u)\pi^{-1}(v) = \lambda k$ ($\lambda \in \mathbf{C}, k \in \operatorname{ext} Z_1$) equals (by STP and (2))

$$\overline{\lambda}\langle \pi(k), g\rangle = \overline{\langle \lambda k, v(g)\rangle} = \overline{\langle P_k(u)\pi^{-1}(v), v(g)\rangle} = \langle P_k(u)g, v\rangle.$$

■

REMARK 1.6. The preceding proposition implies that the geometric Peirce projections $P_k(u)$ ($k = 0, 1, 2$) are self-adjoint operators with respect to the symmetric sesquilinear form defined in [16, Proposition 2.9].

Recall ([15, Proposition 4.5]) that for a fixed geometric tripotent w, the collection of all geometric tripotents u which satisfy $u \leq w$, *i.e.*, $F_u \subset F_w$, is a complete orthomodular lattice.

PROPOSITION 1.7. *Let Z be a neutral* SFS *space satisfying* PE *and* ERP. *Let f and g be distinct elements of* $\operatorname{ext} Z_1$ *which belong to the same norm exposed face* (*so that $v(f) \vee v(g)$ exists*). *With $v = v(f)$, we have*

$$v \vee v(g) = v + \tilde{v}$$

for some minimal geometric tripotent $\tilde{v}$ orthogonal to v. In other words, the smallest norm exposed face containing f and g is of rank 2, i.e., of the form $F_{v+\tilde{v}}$, with $v, \tilde{v}$ orthogonal minimal geometric tripotents (cf. [16, Definition 3.1]).

This should be compared with [1, Theorem 12.1 and Property $\mathcal{R}$].

PROOF. If $f \diamond g$, then we may take $\tilde{v} = v(g)$. Now assume that f and g belong to the norm exposed face F_w and are not orthogonal.

We first show that $P_0(v)g \neq 0$. Otherwise, if $P_0(v)g = 0$, then since $w - v \in U_0(v)$,

$$1 = g(w) = g(v) + g(w - v) = g(v),$$

which implies that $g \in F_v$, *i.e.* $f = g$, a contradiction. Thus $P_0(v)g \neq 0$.

Set $\tilde{f} := \|P_0(v)g\|^{-1}P_0(v)g$, so that $\tilde{v} := v(\tilde{f}) \in U_0(v)$ and $\tilde{f}$ is an extreme point of Z_1 by Remark 1.4.

By [15, Theorem 4.3(d)], the contractiveness of $P_2 + P_0 = (I + S_v)/2$, and the formula $P_2(w - v) = P_2(w - v)P_0(v)$ ([15, Corollary 3.4]), we have

$$\begin{aligned} 1 = g(w) &= g\big(v + (w - v)\big) = g(v) + g(w - v) = \|P_2(v)g\| + \|P_2(w - v)g\| \\ &\leq \|P_2(v)g\| + \|P_0(v)g\| = \big\|\big(P_2(v) + P_0(v)\big)g\big\| \leq 1. \end{aligned}$$

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Since $1 = \langle \tilde{f}, \tilde{v} \rangle = \|P_0(v)g\|^{-1}\langle P_0(v)g, \tilde{v}\rangle$, we have $\langle g, \tilde{v}\rangle = \|P_0(v)g\|$ and hence $g(v + \tilde{v}) = \|P_2(v)g\| + \|P_0(v)g\| = 1$ so that $g \in F_{v+\tilde{v}}$.

Now let $u := v(g)$. We shall show that $u \vee v = v + \tilde{v}$. Since $g \in F_{v+\tilde{v}}$, $u \leq v + \tilde{v}$ and therefore $u \vee v \leq v + \tilde{v}$. Suppose now that $u \leq s$ and $v \leq s$ for some geometric tripotent $s \leq w$. We must show that $v + \tilde{v} \leq s$. Since $g \in F_s$ and $F_v \subset F_s$, by strong facial symmetry, $S_v^* s = s$ and $\langle S_v g, s\rangle = \langle g, s\rangle = 1$, *i.e.*, $S_v g \in F_s$. Now

$$\|P_2(v)g\| \frac{P_2(v)g}{\|P_2(v)g\|} + \|P_0(v)g\| \frac{P_0(v)g}{\|P_0(v)g\|} = \frac{I + S_v}{2} g \in F_s.$$

By definition of face, $\tilde{f} = \|P_0(v)g\|^{-1}P_0(v)g \in F_s$ so that $\tilde{v} \leq s$. Since $f \in F_s$, $v \leq s$ and therefore $v + \tilde{v} \leq s$. ∎

2. **Properties of the geometric Peirce 1-space of a minimal geometric tripotent.** In this section we introduce the fundamental geometric relations (other than orthogonality) between geometric tripotents, namely colinearity and governing, and show that any geometric tripotent in the geometric Peirce 1-space of a *minimal* geometric tripotent must satisfy one of these relations. These relations among tripotents in Jordan triple systems are standard tools in the algebraic theory, *cf.* [20], [19], [8]. Additional information about the facial structure is obtained in each case, and for the geometric Peirce 1-space in general.

DEFINITION 2.1. Let Z be a WFS space, and let u and v be geometric tripotents. We say that u *governs* v, if $u \in U_1(v)$ and $v \in U_2(u)$, notation $u \vdash v$. We say that u and v are *colinear* if $u \in U_1(v)$ and $v \in U_1(u)$, notation $u \top v$.

PROPOSITION 2.2 (TWO CASE LEMMA). *Let Z be a neutral* WFS *space, let v be a minimal geometric tripotent and let u be a geometric tripotent which lies in $U_1(v)$. Then either $v \in U_2(u)$ or $v \in U_1(u)$, i.e. either u governs v, or u and v are colinear.*

PROOF. By [15, Theorem 3.3], u and v are compatible. Hence, for $j \in \{0, 1, 2\}$,

$$P_j(u)^* v = P_j(u)^* P_2(v)^* v = P_2(v)^* P_j(u)^* v = \lambda_j v$$

with $\lambda_j = 0$ or 1 since $P_j(u)$ is a projection. If λ_0 were 1, we would have $u \diamond v$. Hence $\lambda_0 = 0$. Since $0 = P_2(u)^* P_1(u)^* v = \lambda_1 \lambda_2 v$, exactly one of λ_1 or λ_2 is zero. If $\lambda_1 = 0$, then $v \in U_2(u)$. If $\lambda_2 = 0$, then $v \in U_1(u)$. ∎

COROLLARY 2.3. *Let Z be a neutral* WFS *space and let v be a minimal geometric tripotent. If $u, \tilde{u}$ are orthogonal geometric tripotents in $U_1(v)$, then u and $\tilde{u}$ are each colinear with v.*

PROOF. By the proposition, either u governs or is colinear with v. In the first case, by [15, Corollary 3.4], $U_1(v) \cap U_0(u) = \{0\}$. But $\tilde{u} \in U_1(v) \cap U_0(u)$, a contradiction so that $u \top v$. By symmetry, $\tilde{u}$ is also colinear with v. ∎

We consider now the colinear case of Proposition 2.2. The conclusion $F_u \subset Z_1(v)$ of the following proposition implies that $S_v = -I$ on F_u. Note that, for arbitrary geometric

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tripotents u and v, by [15, Theorem 2.5], $u \in U_1(v)$ implies only that $S_v(F_u) = -F_u$ *setwise*. For the definition of FE, see [16, Definition 3.3].

PROPOSITION 2.4. *Let Z be an atomic neutral* SFS *space which satisfies the properties* FE *and* ERP. *Let u and v be geometric tripotents with $u \in U_1(v)$, and suppose that $Z_2(u) \cap Z_2(v) = \{0\}$. Then $F_u \subset Z_1(v)$, and hence $P_2(u)P_1(v) = P_2(u)$. The same conclusion holds (with the same proof) if $Z_2(u) \cap Z_0(v) = \{0\}$. In particular, if v is minimal in U, if $u \top v$, and if u is minimal in $U_1(v)$, then u is minimal in U.*

PROOF. Assume that $F_u \not\subset Z_1(v)$. Since F_u is the closed convex hull of its extreme points ([16, Proposition 3.4]), there is an extreme point ρ of F_u such that $\rho \notin Z_1(v)$. Then, since by [15, Theorem 2.5], $S_vF_u = -F_u$, $\sigma := -S_v\rho \in F_u$ and since $\rho \notin Z_1(v)$, $\sigma \neq \rho$. By Proposition 1.7, $F_{v(\rho)\vee v(\sigma)}$ is a rank 2 face. Moreover, with $u' := v(\rho) \vee v(\sigma)$,

$$P_2(v)\rho + P_0(v)\rho = (I + S_v)\rho/2 = (\rho - \sigma)/2 \in Z_2(u').$$

By compatibility of u and v, $P_2(v)\rho \in Z_2(u)$ so that $P_2(v)\rho = 0$ and $P_0(v)\rho$ is a non-zero member of $Z_2(u)$. By Remark 1.4, $P_0(v)\rho = \mu\tilde{f}$, for some $\mu > 0$ and some extreme point $\tilde{f}$ of Z_1. Note that $\tilde{f} \in \mathrm{ext}(\mathrm{sp}_{\mathbf{R}} F_{u'})_1$ and so by the Jordan decomposition for rank 2 faces ([16, Remark 3.13]), $\tilde{f}(u') = \pm 1$, and $\langle P_0(v)\rho, u'\rangle = \mu\tilde{f}(u') = \pm\mu = \big(\rho(u') - \sigma(u')\big)/2 = 0$, contradicting $\mu > 0$.

With $F_u \subset Z_1(v)$ proved, we have $Z_2(u) \subset Z_1(v)$ and $P_2(u)P_1(v) = P_1(v)P_2(u) = P_2(u)$.

Finally, if v is minimal in U, and $u \top v$, then $v \notin U_2(u)$ so $U_2(v) \cap U_2(u) = \mathrm{sp}_{\mathbf{C}}\{v\} \cap U_2(u) = \{0\}$. By the first statement, $P_2(u)P_1(v) = P_2(u)$ which implies that $U_2(u) = U_2(u) \cap U_1(v)$ is one dimensional since it equals $U_2\big(u, Z_1(v)\big)$, the geometric Peirce 2-space of u considered as a geometric tripotent of $Z_1(v)$. ∎

We next consider the governing case in Proposition 2.2. Unlike Proposition 2.4, we must assume that v is minimal.

PROPOSITION 2.5. *Let Z be an atomic neutral* SFS *space which satisfies* FE *and* ERP. *Let v be a minimal geometric tripotent and let u be a geometric tripotent which governs v. Then F_u is a rank 2 face.*

PROOF. Choose an extreme point $\varphi \in F_u$ such that $\langle \varphi, v\rangle \neq 0$. The element φ exists by [16, Proposition 3.4]), since $v \in U_2(u)$ and $U_2(u)$ is the Banach space dual of $Z_2(u) = \overline{\mathrm{sp}}_{\mathbf{C}} F_u$. With S_v the symmetry determined by v, since $u \in U_1(v)$,

$$\langle S_v(\varphi), u\rangle = \langle \varphi, S_v^* u\rangle = -1$$

so that $\psi := -S_v(\varphi) \in \mathrm{ext}\, F_u$. Let $u' := v(\varphi) \vee v(\psi)$ so that $u' \leq u$ and $\varphi, \psi \in Z_2(u')$. We shall show that $u' = u$, which completes the proof by Proposition 1.7.

Since v is minimal, with $v = v(f)$ for some extreme point f,

$$\langle \varphi, v\rangle f + P_0(f)\varphi = \big(P_2(f) + P_0(f)\big)\varphi = (I + S_v)\varphi/2 = (\varphi - \psi)/2 \in Z_2(u'),$$

so by [15, Remark 3.2], $f \in Z_2(u')$. Therefore $v \diamond (u - u')$, so that S_v^* fixes $u - u'$. From $u = u' + (u - u')$ we have $P_2(u - u')u = u - u'$. On the other hand, $-u = S_v^* u = S_v^* u' + S_v^*(u - u')$, $S_v^* u' \in U_0(u - u')$ and $S_v^*(u - u') = u - u'$. Therefore, $P_2(u - u')(-u) = u - u'$ and hence $u = u'$. ∎

The following definition is made for convenience of exposition. Let $M_2(\mathbf{C})$ (resp. $S_2(\mathbf{C})$) denote the JBW*-triple of all 2 by 2 complex matrices (resp. symmetric complex matrices).

DEFINITION 2.6. We say that $\mathrm{sp}_{\mathbf{C}}\{v, u, \tilde{v}\}$ is *canonically isomorphic* to $S_2(\mathbf{C})$ if

(i) v and $\tilde{v}$ are orthogonal minimal geometric tripotents;

(ii) u is a geometric tripotent governing each of v and $\tilde{v}$;

(iii) the correspondence $v \longmapsto E_{11},\ u \longmapsto E_{12} + E_{21},\ \tilde{v} \longmapsto E_{22}$, where

$$E_{11} := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ E_{12} := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ E_{21} := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ E_{22} := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

extends linearly to an isometry of $\mathrm{sp}_{\mathbf{C}}\{v, u, \tilde{v}\}$ onto $S_2(\mathbf{C})$;

(iv) $(v + u + \tilde{v})/2$ is a minimal geometric tripotent of U.

Similarly, we say that $\mathrm{sp}_{\mathbf{C}}\{u_1, \tilde{u}_1, u_2, \tilde{u}_2\}$ is *canonically isomorphic* to $M_2(\mathbf{C})$ if the correspondence

$$u_1 \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ u_2 \longmapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \tilde{u}_1 \longmapsto \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ \tilde{u}_2 \longmapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

extends linearly to an isometry of $\mathrm{sp}_{\mathbf{C}}\{u_1, u_2, \tilde{u}_1, \tilde{u}_2\}$ onto $M_2(\mathbf{C})$ in such a way that $(u_1 + u_2 + \tilde{u}_1 + \tilde{u}_2)/2$ is a minimal geometric tripotent of U.

If we apply the main result of [16] to a complex space $Z_2(u)$, with u as in Proposition 2.5, we obtain the following corollary, whose proof uses some facts from [8] concerning spin factors. The last assertion in Corollary 2.7 will be needed in the proof of Lemma 4.15.

COROLLARY 2.7. *Let Z be a complex atomic neutral* SFS *space satisfying* FE, STP, *and* ERP. *Let v be a minimal geometric tripotent and let u be a geometric tripotent which governs v. Then $U_2(u)$ is isometric to a complex spin factor and there exists a minimal geometric tripotent $\tilde{v} \in U_2(u) \cap U_0(v)$ governed by u such that $U_2(u) = U_2(v + \tilde{v})$. Moreover, $\tilde{v}$ is uniquely determined by the property that* $\mathrm{sp}_{\mathbf{C}}\{v, u, \tilde{v}\}$ *is canonically isomorphic to $S_2(\mathbf{C})$. Furthermore, if $u = u_1 + \tilde{u}_1$ is the sum of two orthogonal minimal geometric tripotents of U lying in $U_1(v)$, then $\tilde{v}$ can be chosen in such a way that* $\mathrm{sp}_{\mathbf{C}}\{v, u_1, \tilde{v}, \tilde{u}_1\}$ *is canonically isomorphic to $M_2(\mathbf{C})$.*

PROOF. By neutrality of $P_2(u)$, the properties atomic, FE, and STP are valid for $Z_2(u)$, which is of type I_2 by the proposition and a neutral SFS space (by [15, Theorem 3.6 and Proposition 4.1]). Thus by [16, Theorem 4.16], $U_2(u)$ is isometric to a complex spin factor.

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In a spin factor, every minimal tripotent has a one dimensional Peirce 0-space of rank 1, so the existence of a minimal v' orthogonal to v with $U_2(u) \subset U_2(v+v')$ follows. On the other hand, since $v+v' \in U_2(u)$, we have by [15, Theorem 2.3] that $U_2(v+v') \subset U_2(u)$. Identifying $U_2(u)$ with a spin factor, by the joint Peirce decomposition in a JBW*-triple *cf.* [8, Lemma 2.4], we have $u \in U_2(v+v') = U_2(v) + U_2(v') + U_1(v) \cap U_1(v')$ which implies, since $u \in U_1(v)$, that $u \in U_1(v')$, *i.e.*, $u \vdash v'$.

Consider the Peirce 1-space $V := U_1\big(v, U_2(u)\big) = U_1(v) \cap U_2(u)$ of v in the spin factor $U_2(u)$. From the properties of spin factors and spin grids developed in [8] (*cf.* Proposition 2.1 and Classification Scheme, Cases 2 and 3, page 305), we have the following. The (geometric) tripotent u is minimal in V if and only if $U_2(u) = \mathrm{sp}\{v, u, v'\} \cong S_2(\mathbf{C})$. Otherwise, there exist orthogonal geometric tripotents $u_1, \tilde{u}_1$ in V such that either $U_2(u) = \mathrm{sp}\{v, u_1, \tilde{u}_1, v'\} \cong M_2(\mathbf{C})$ or V is a spin factor of dimension at least 3. In the first case choose $\tilde{v}$ to be $\{uvu\}$. Then $w := (u+v+\tilde{v})/2$, corresponding to $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ in $S_2(\mathbf{C})$, is minimal in $U_2(u)$, and by neutrality, w is minimal in U. In the second case, we define, as in the proof of [8, Proposition, p. 312], $v' := -2\{u_1 v \tilde{u}_1\}$, to obtain an odd quadrangle $(v, u_1, v', \tilde{u}_1)$. Then, by the proof of [8, Proposition, p. 312], there is a spin grid for $U_2(u)$ containing $\{v, u_1, v', \tilde{u}_1\}$ (*cf.* [8, Corollary, p. 313]). With respect to this spin grid, the element

$$w = \frac{v - v' + u}{2} = \frac{v - v' + u_1 + \tilde{u}_1}{2}$$

has determinant zero, so is minimal in $U_2(u)$ ([8, Proposition 3.3]). By neutrality, w is minimal in U. Thus, $\tilde{v} := -v'$ does the job in this case. The existence of $\tilde{v}$ is now proved in all cases.

If v'' satisfies the same properties as does $\tilde{v}$, then $v'' = \lambda\tilde{v}$ for some $\lambda \in \mathbf{C}$ since $U_0(v) \cap U_2(u)$ is one-dimensional. It follows that $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & \lambda/2 \end{bmatrix}$ has determinant 0 in $S_2(\mathbf{C})$. This proves uniqueness.

The last statement of the corollary follows from the fact that the members of any quadrangle in a spin factor span a four-dimensional space which is canonically isomorphic to $M_2(\mathbf{C})$. ■

If Z is the predual of a JBW*-triple, then we have already remarked that Z is a neutral SFS space which satisfies ERP. It also satisfies STP ([11, Lemma 2.2]) and FE ([10, Theorem 4.4]). Moreover, if u is a tripotent, then the Peirce spaces $Z_k(u)$ $(k = 0, 1, 2)$ are preduals of JBW*-triples and hence also satisfy these properties. If Z is the predual of an atomic JBW*-triple, then so are $Z_k(u)$ $(k = 0, 1, 2)$.

We next consider this situation in the setting of facially symmetric spaces. If Z is a neutral SFS space, and u is a geometric tripotent, then the geometric Peirce spaces $Z_k(u)$ $(k = 0, 1, 2)$ are neutral SFS spaces by [15, Theorem 3.6 and Proposition 4.1]. Moreover, because of neutrality of $P_2(u)$ and $P_0(u)$, the spaces $Z_2(u)$ and $Z_0(u)$ inherit separately each of the properties "atomic", FE, STP, ERP from Z. We now show that, in case v is minimal, the space $Z_1(v)$ inherits the first two of these properties. Later, in

https://doi.org/10.4153/CJM-1993-004-0 Published online by Cambridge University Press

Corollary 4.12 we will show that, with the fourth PSP JP in force, STP and ERP also hold in $Z_1(v)$.

LEMMA 2.8. *Let v be a minimal geometric tripotent in an atomic complex neutral* SFS *space Z satisfying* FE, STP *and* ERP. *Then $Z_1(v)$ is atomic and satisfies* FE.

PROOF. We show first that FE holds in $Z_1(v)$. Let K be a proper norm closed face in the unit ball of $Z_1(v)$. Since K is convex and contained in the unit sphere of $Z_1(v)$, by a standard separation theorem there is an element $x \in U_1(v)$ with $\|x\| = 1$ and $K \subset F_x$. Since $Z_1(v)$ is WFS, by [15, Proposition 1.6], there is $u \in \mathcal{GT}_{Z_1(v)}$ such that $F_x \cap Z_1(v) = F_u \cap Z_1(v)$. By [16, Proposition 2.1], $u \in \mathcal{GT} \cap U_1(v)$, and therefore by Proposition 2.2, either $u \vdash v$ or $u \top v$.

If $u \vdash v$, then by Corollary 2.7, $U_2(u)$ is isometric with a spin factor and $K \subset F_u \cap Z_1(v) \subset Z_2(u) \cap Z_1(v) = Z_1\big(v, Z_2(u)\big)$ (= the geometric Peirce 1-space of v considered as a geometric tripotent of $Z_2(u)$). Under this isometry, K corresponds to a norm closed face in the Peirce 1-space of a tripotent. Since FE holds in the predual of any JBW*-triple, K is a norm exposed face in $Z_1\big(v, Z_2(u)\big)$, *i.e.* $K = F_w \cap Z_1(v) \cap Z_2(u)$ for some geometric tripotent $w \in U_2(u) \cap U_1(v)$. However, since, by [15, Theorem 2.3] $F_w \subset Z_2(u)$, we have $K = F_w \cap Z_1(v)$, proving that K is a norm exposed face in $Z_1(v)$.

If $u \top v$, by Proposition 2.4, $F_u \subset Z_1(v)$. In this case K is already a subset of F_u, and hence K is a closed face of Z_1. By the validity of FE in Z, K is a norm exposed face of Z_1, and hence of the unit ball of $Z_1(v)$. Thus FE holds in $Z_1(v)$.

To prove that $Z_1(v)$ is atomic, let K be any norm exposed face in the unit ball of $Z_1(v)$. Then $K = F_u \cap Z_1(v)$ for some geometric tripotent $u \in U_1(v)$. Again we have the two cases $u \vdash v$ or $u \top v$.

In the first case, $Z_2(u)$ is isometric to the predual of a spin factor and K is a norm exposed face in the unit ball of $Z_1\big(v, Z_2(u)\big)$, which corresponds to the Peirce 1-space of a tripotent in a complex spin factor. Since this latter space is atomic, K has an extreme point.

In the second case, as above, we have $F_u \subset Z_1(v)$. But in this case, $K = F_u$ so it has an extreme point. Thus $Z_1(v)$ is atomic. ∎

In [16, Proposition 2.9], it was shown that if Z is atomic and satisfies PE, then each indecomposable geometric tripotent is minimal. Indecomposability (the impossibility of being written as the sum of two orthogonal geometric tripotents) is easier to establish than minimality (one-dimensionality of the geometric Peirce 2-space). Thus the following corollary will be of great use in the rest of this paper.

COROLLARY 2.9. *Let v be a minimal geometric tripotent in an atomic complex neutral* SFS *space Z satisfying* FE, STP *and* ERP. *Then every indecomposable geometric tripotent of $Z_1(v)$ is a minimal geometric tripotent of $Z_1(v)$.*

3. **The type 3 case.** In this section, we give a geometric characterization of the Cartan factor of type 3.

https://doi.org/10.4153/CJM-1993-004-0 Published online by Cambridge University Press

In a Hilbert space $\mathcal{H}$, considered as (the dual of) a facially symmetric space, every unit vector is a minimal geometric tripotent, and (since $P_0(u) = 0$) the symmetry S_u, corresponding to the unit vector u is given by $S_u^* x = \big(2P_2(u)^* - I\big)x = 2(x|u)u - x$, for $x \in \mathcal{H}$, where $(\cdot|\cdot)$ is the inner product in $\mathcal{H}$.

REMARK 3.1. If ξ and η are unit vectors in a Hilbert space $\mathcal{H}$ with $\xi \neq -\eta$, then the isometry S_t corresponding to the unit vector $t = (\xi + \eta)/\|\xi + \eta\|$ interchanges ξ and η if and only if the inner product of ξ and η is real. Also, $S_t|_{\{t\}^\perp} = -I$.

PROOF. With $\alpha := \|\xi + \eta\|$, we have $\alpha^2 = \|\xi\|^2 + \|\eta\|^2 + (\xi|\eta) + \overline{(\xi|\eta)} = 2(1 + \Re\langle \xi \mid \eta\rangle)$, and

$$S_t\xi = 2(\xi|t)t - \xi = 2\Big(\xi\Big|\frac{\xi+\eta}{\alpha}\Big)\frac{\xi+\eta}{\alpha} - \xi = \frac{2}{\alpha^2}[1 + (\xi|\eta)]\xi - \xi + \frac{2}{\alpha^2}[1 + (\xi|\eta)]\eta.$$

Therefore $S_t\xi = \eta \Leftrightarrow \frac{2}{\alpha^2}[\|\xi\|^2 + (\xi|\eta)] = 1 \Leftrightarrow (\xi|\eta) = \overline{(\xi|\eta)}$.

The last statement follows since in a Hilbert space, colinearity is equivalent to orthogonality with respect to the inner product in the Hilbert space. ∎

In the following lemma we cannot use [16, Corollary 2.11] since it is not yet proved that $Z_1(v)$ satisfies STP.

LEMMA 3.2. *Suppose that Z is an atomic neutral* SFS *space satisfying* FE, STP *and* ERP, *and that v is a minimal geometric tripotent with $Z_1(v)$ of rank 1 ([16, Definition 2.10]). Then $Z_1(v)$ is isometric to a Hilbert space. If in addition, there is a geometric tripotent $u \vdash v$, then any $w \in U_1(v)$ with $\|w\| = 1$ is a geometric tripotent satisfying $w \vdash v$.*

PROOF. Let $f \in Z_1(v)$ be normalized. Then $u := v(f) \in U_1(v)$ and since $Z_1(v)$ is of rank 1, by Corollary 2.9, u is a minimal geometric tripotent of $Z_1(v)$. By Proposition 2.2, either $u \top v$ in which case by Proposition 2.4 u is minimal in U and f is an extreme point of Z_1; or $u \vdash v$ in which case $U_2(u) = \mathrm{sp}_{\mathbf{C}}\{v, u, \tilde{v}\}$ by Corollary 2.7 so that f is the sum of two orthogonal extreme points of Z_1. Letting $\|\cdot\|_2$ denote the norm coming from the sesquilinear form $\langle\cdot \mid \cdot\rangle$ we have $\|f\|_2 = 1$ if f is an extreme point and $\|f\|_2 = 1/\sqrt{2}$ in the second case. Thus the continuous function $\partial\big(Z_1(v)\big)_1 \ni f \longmapsto \|f\|_2 \in \{0, 1/\sqrt{2}\}$ is a constant. It follows that $\|\cdot\|_Z$, restricted to $Z_1(v)$, is a multiple of the semi-norm $\|\cdot\|_2$, so $Z_1(v)$ is a Hilbert space.

Now suppose that u exists with $u \vdash v$. Then $\|\cdot\|_Z = \sqrt{2}\|\cdot\|_2$ on $Z_1(v)$ and thus any norm 1 element w of $U_1(v)$ is a geometric tripotent of rank 2 and minimal in $U_1(v)$. By Proposition 2.4, w is not colinear with v, so by Proposition 2.2 $w \vdash v$. ∎

In the remainder of this section we shall make use of the following assumption.

ASSUMPTION 3.3. *Z is an atomic neutral* SFS *space satisfying* FE, STP, *and* ERP, *v is a minimal geometric tripotent, $Z_1(v)$ is of rank 1 and there exists a geometric tripotent u with $u \vdash v$ (Note that u is automatically minimal in $U_1(v)$).*

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We now introduce a canonical construction of a family of geometric tripotents, to be called a *canonical basis*, which will be shown in Proposition 3.12 to be a weak* basis for an M-summand of U under suitable assumptions.

CONSTRUCTION 3.4. Let Z satisfy Assumption 3.3. Define $u_{11} = v$, and by Lemma 3.2, let $\{u_{1j} : j \in I\}$ be any orthonormal basis of the Hilbert space $U_1(v)$, where I is an index set not containing 1. Thus, $\{u_{1j} : j \in I\}$ is a maximal family of mutually colinear geometric tripotents each of which governs u_{11}. Apply Corollary 2.7 to the data u_{11}, u_{1j} to obtain unique minimal geometric tripotents $\{u_{jj} : j \in I\}$ such that for all $j \in I$,

$$u_{jj} \diamond u_{11}, \quad u_{1j} \vdash u_{jj}, \tag{5}$$

and $\mathrm{sp}_{\mathbf{C}}\{u_{11}, u_{1j}, u_{jj}\}$ is canonically isomorphic to $S_2(\mathbf{C})$, *i.e.*, the map in which u_{11}, u_{1j}, u_{jj} correspond to the natural basis $E_{11}, E_{12}+E_{21}, E_{22}$ in $S_2(\mathbf{C})$ extends linearly to an isometry (See Definition 2.6). Furthermore,

$$w_{1j} := (u_{11} + u_{jj} + u_{1j})/2 \text{ and } \tilde{w}_{1j} := (u_{11} + u_{jj} - u_{1j})/2.$$

are minimal geometric tripotents of U.

For $i,\ j \in I, i \neq j$, define

$$u_{ij} := -S^*_{w_{1i}} u_{1j}. \tag{6}$$

In Proposition 3.7 below, we will show that $u_{ij} = u_{ji}$. For completeness we can define $u_{j1} = u_{1j}$ for $j \in I$.

Note that w_{1j} is minimal if and only if $\tilde{w}_{1j}$ is, since they are interchanged by the symmetry $S^*_{u_{11}}$.

The reason for the minus sign in (6) is apparent by comparison with the case that Z is the predual of the JBW*-triple of n by n symmetric complex matrices, where n denotes an arbitrary cardinal. For example, if $n = 3$ then with

$$U_{13} = E_{13} + E_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } W_{12} = \frac{1}{2}(E_{11} + E_{12} + E_{21} + E_{22}) = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we have $S^*_{W_{12}} U_{13} = -U_{23}$, where we have used the formulas $S_{W_{12}} = 2[P_2(W_{12}) + P_0(W_{12})] - I$, and for any matrix A,

$$P_2(W_{12})A = W_{12}AW_{12} \text{ and } P_0(W_{12})A = (1 - W_{12})A(1 - W_{12}).$$

Similarly, by using the correspondence of u_{11}, u_{1i}, u_{ii} with the natural basis in $S_2(\mathbf{C})$, matrix calculations show that

$$S^*_{w_{1i}} u_{11} = u_{ii} \text{ and } S^*_{w_{1i}} u_{1i} = u_{1i}. \tag{7}$$

Moreover, by Proposition 1.1, $S_{w_{1i}} S_{u_{11}} S_{w_{1i}} = S_{u_{ii}}$, and therefore

$$P_1(u_{ii}) = (I - S_{u_{ii}})/2 = (I - S_{w_{1i}} S_{u_{11}} S_{w_{1i}})/2 = S_{w_{1i}}(I - S_{u_{11}}) S_{w_{1i}}/2 = S_{w_{1i}} P_1(u_{11}) S_{w_{1i}}.$$

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Thus $S^*_{w_{1\iota}}U_1(u_{11}) = U_1(u_{\iota\iota})$ and since $U_1(u_{11}) = \overline{\mathrm{sp}}_{\mathbf{C}}\{u_{1\jmath}\}_{\jmath\in I}$, we have by (6) and (7)

$$U_1(u_{\iota\iota}) = \overline{\mathrm{sp}}_{\mathbf{C}}\{u_{\iota\jmath}\}_{\jmath\in(I\cup\{1\})\setminus\{\iota\}}. \tag{8}$$

DEFINITION 3.5. The family $\{u_{\iota\jmath}\}_{\iota,\jmath\in I\cup\{1\}}$ given in Construction 3.4 is called the *canonical basis* determined by u_{11} and the orthonormal basis $\{u_{1\jmath}\}_{\jmath\in I}$ of $U_1(u_{11})$.

By a straightforward application of Proposition 1.1, we have the following lemma.

LEMMA 3.6. *Let Z satisfy Assumption 3.3, and let T be a linear isometry of Z onto Z. Then T^* maps any canonical basis onto a canonical basis. More precisely, if $\{u_{\iota\jmath}\}$ is the canonical basis determined by u_{11} and the orthonormal basis $\{u_{1\jmath}\}_{\jmath\in I}$ of $U_1(u_{11})$, then $\{T^*u_{\iota\jmath}\}$ is the canonical basis determined by T^*u_{11} and the orthonormal basis $\{T^*u_{1\jmath}\}$ of $U_1(T^*u_{11})$.*

PROOF. We only need to show that $T^*u_{\iota\jmath} = -S^*_{w'_{1\iota}}T^*u_{1\jmath}$, where $w'_{1\iota} = (T^*u_{11} + T^*u_{1\iota} + T^*u_{\iota\iota})/2 = T^*w_{1\iota}$. This reduces to $T^*S^*_{w_{1\iota}}T^{*-1} = S^*_{T^*w_{1\iota}}$, which is (1). ∎

The following proposition shows that the family $\{u_{\iota\jmath}\}_{\iota,\jmath\in\{1\}\cup I}$ behaves locally in the same way as the standard canonical basis $\{U_{\iota\jmath}\}_{\iota,\jmath\in\{1\}\cup I}$ (where $U_{\iota\iota} = E_{\iota\iota}$ and for $i \neq j$, $U_{\iota\jmath} = E_{\iota\jmath} + E_{\jmath\iota}$) in the symmetric matrices.

PROPOSITION 3.7. *Let Z satisfy Assumption 3.3 and let $\{u_{\iota\jmath}\}$ be a canonical basis.*

1. *The family $\{u_{\jmath\jmath}\}_{\jmath\in\{1\}\cup I}$ is a set of pairwise orthogonal minimal geometric tripotents;*
2. *$u_{\iota\jmath} \in U_2(u_{\iota\iota} + u_{\jmath\jmath})$ for $i \neq j$, $i \in \{1\}\cup I$, $j \in I$;*
3. *$w_{\iota\jmath} := (u_{\iota\iota}+u_{\jmath\jmath}+u_{\iota\jmath})/2$ and $\tilde{w}_{\iota\jmath} := (u_{\iota\iota}+u_{\jmath\jmath}-u_{\iota\jmath})/2$ are minimal geometric tripotents of U for $i \neq j$, $i \in \{1\}\cup I$, $j \in I$;*
4. *$U_2(u_{\iota\jmath}) = U_2(u_{\iota\iota} + u_{\jmath\jmath})$ for $i \neq j$, $i \in \{1\}\cup I$, $j \in I$;*
5. *$u_{\iota\jmath} = u_{\jmath\iota}$ for $i,\ j \in \{1\}\cup I$.*

PROOF. Let $j, k \in \{1\}\cup I$ with $j \neq k$. If $j = 1$ or $k = 1$, the first statement is true by construction. Since $u_{1\jmath} \vdash u_{11}$ and $u_{1\jmath}\top u_{1k}$, we have $S^*_{u_{1\jmath}}u_{11} = u_{11}$ and $S^*_{u_{1\jmath}}u_{1k} = -u_{1k}$. Thus $S^*_{u_{1\jmath}}w_{1k} = (u_{11} - u_{1k} + S^*_{u_{1\jmath}}u_{kk})/2$, and $S^*_{u_{1\jmath}}w_{1k}$ is minimal. Since $S^*_{u_{1\jmath}}u_{kk} \diamond u_{11}$ and $u_{1k} \vdash S^*_{u_{1\jmath}}u_{kk}$, this implies, by the uniqueness in Corollary 2.7, that $S^*_{u_{1\jmath}}u_{kk} = u_{kk}$. Thus u_{kk} belongs to the fixed point set of $S^*_{u_{1\jmath}}$, *i.e.*, $u_{kk} \in U_2(u_{1\jmath}) + U_0(u_{1\jmath})$. Writing $u_{kk} = a + b$, by Proposition 1.5, $a = P_2(u_{1\jmath})^*u_{kk} = \lambda_1 w_1$ and $b = P_0(u_{1\jmath})^*u_{kk} = \lambda_2 w_2$ are multiples of orthogonal minimal geometric tripotents. Since u_{kk} is indecomposable, we must have, for example, $|\lambda_2| < 1$, in which case, $F_{u_{kk}} = F_{\lambda_1 w_1}$, implying $\lambda_2 = 0$. Thus, u_{kk} belongs to one of the spaces $U_2(u_{1\jmath})$ or $U_0(u_{1\jmath})$. To show that it belongs to the latter, which will complete the proof of 1. since $u_{\jmath\jmath} \in U_2(u_{1\jmath})$, assume to the contrary that $u_{kk} \in U_2(u_{1\jmath})$. Then $u_{kk} \in U_2(u_{1\jmath}) \cap U_0(u_{11}) = \mathbf{C}u_{\jmath\jmath}$, and with $u_{kk} = \lambda u_{\jmath\jmath}$ for some $\lambda \in \mathbf{T}$, we have $U_2(u_{1\jmath}) = U_2(u_{11} + u_{\jmath\jmath}) = U_2(u_{11} + \overline{\lambda}u_{kk}) = U_2(u_{1k})$ which contradicts $u_{1\jmath}\top u_{1k}$. Hence $u_{kk} \in U_0(u_{1\jmath})$, completing the proof of the first statement.

Statement 2. is obvious if $i = 1$. Since $w_{1\iota} \in U_2(u_{1\iota}) = U_2(u_{11} + u_{\iota\iota})$ and since by 1., $u_{\jmath\jmath} \in U_0(u_{11}) \cap U_0(u_{\iota\iota}) = U_0(u_{11} + u_{\iota\iota})$, we have $u_{\jmath\jmath} \diamond w_{1\iota}$ and therefore $u_{\jmath\jmath}$ is fixed by $S^*_{w_{1\iota}}$.

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Thus we have the following for the action of $S^*_{w_{1i}}$ on a part of the canonical basis:

$$S^*_{w_{1i}}:\ \begin{array}{ccc} u_{11} & u_{1i} & u_{1j}\\ & u_{ii} & u_{ij}\\ & & u_{jj}\end{array} \longrightarrow \begin{array}{ccc} u_{ii} & u_{1i} & -u_{ij}\\ & u_{11} & -u_{1j}\\ & & u_{jj}\end{array} \tag{9}$$

In particular,

$$S^*_{w_{1i}}:\ \begin{array}{cc} u_{11} & u_{1j}\\ & u_{jj}\end{array} \longrightarrow \begin{array}{cc} u_{ii} & -u_{ij}\\ & u_{jj}\end{array} \tag{10}$$

which implies statements 2,3, and 4. Moreover, if we interchange $i,\ j$ in (10), and let u_{ji} correspond to $\begin{bmatrix} a & b\\ b & c\end{bmatrix}$ under the canonical isomorphism $\mathrm{sp}_{\mathbf{C}}\{u_{ii}, -u_{ij}, u_{jj}\} \cong S_2(\mathbf{C})$, then since u_{ji} governs u_{ii}, $a = c = 0$, and since $\tilde{w}_{ij}$ is minimal, $b^2 = 1$. Therefore, as u_{ij} corresponds to $\begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}$, $u_{ij} = \pm u_{ji}$. We shall show that the plus sign prevails, proving 5., and that (12) below holds.

In the proof of 5. we may obviously assume that $i,\ j \in I$ and $i \neq j$. Let $T = S_{w_{1i}}$. Then by Proposition 1.1,

$$S^*_{\tilde{w}_{ij}} = T^* S^*_{w_{1j}} T^* = S^*_{w_{1i}} S^*_{w_{1j}} S^*_{w_{1i}}. \tag{11}$$

If $u_{ij} = u_{ji}$, then by (9),(11) and (7),

$$S^*_{\tilde{w}_{ij}}:\ \begin{array}{ccc} u_{11} & u_{1i} & u_{1j}\\ & u_{ii} & u_{ij}\\ & & u_{jj}\end{array} \longrightarrow \begin{array}{ccc} u_{11} & u_{1j} & u_{1i}\\ & u_{jj} & u_{ij}\\ & & u_{ii}\end{array} \tag{12}$$

Suppose instead that $u_{ij} = -u_{ji}$. Then in place of (12) we would have

$$S^*_{\tilde{w}_{ij}}:\ \begin{array}{ccc} u_{11} & u_{1i} & u_{1j}\\ & u_{ii} & u_{ij}\\ & & u_{jj}\end{array} \longrightarrow \begin{array}{ccc} u_{11} & -u_{1j} & -u_{1i}\\ & u_{jj} & u_{ij}\\ & & u_{ii}\end{array}$$

and thus

$$S^*_{u_{11}} S^*_{\tilde{w}_{ij}}:\ \begin{array}{ccc} u_{11} & u_{1i} & u_{1j}\\ & u_{ii} & u_{ij}\\ & & u_{jj}\end{array} \longrightarrow \begin{array}{ccc} u_{11} & u_{1j} & u_{1i}\\ & u_{jj} & u_{ij}\\ & & u_{ii}\end{array}$$

Hence, by Lemma 3.6,

$$\begin{array}{ccc} u_{11} & u_{1j} & u_{1i}\\ & u_{jj} & u_{ij}\\ & & u_{ii}\end{array} \tag{13}$$

is the part of the canonical basis determined by $\{u_{1k}\}_{k\in\{1\}\cup I}$ corresponding to $\{u_{11}, u_{1j}, u_{1i}\}$. On the other hand, by construction,

$$\begin{array}{ccc} u_{11} & u_{1j} & u_{1i}\\ & u_{jj} & u_{ji}\\ & & u_{ii}\end{array} \tag{14}$$

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is also the part of the canonical basis determined by $\{u_{1k}\}_{k\in\{1\}\cup I}$ which corresponds to $\{u_{11}, u_{1j}, u_{1i}\}$. From (13) and (14), $u_{ij} = u_{ji}$, a contradiction. So 5. is proved and (12) holds. ■

It follows from Proposition 3.7 that $\langle u_{ij}, \hat{u}_{kl}\rangle = 0$ if $(i, j) \neq (k, l)$. Thus a canonical basis is a linearly independent set. From this we obtain $U_1(u_{11}) \cap U_1(u_{ii}) = \mathbf{C}u_{1i}$, and by [16, Lemma 2.3],

$$U_2(u_{11} + u_{ii}) = \mathrm{sp}_{\mathbf{C}}\{u_{11}, u_{1i}, u_{ii}\}. \tag{15}$$

In the proof of the main theorem of this section, we shall use the following two remarks concerning the construction of a canonical basis.

First, if we use $T = S_{w_{1i}}$ again in Lemma 3.6, then, since $w_{1i} \in U_2(u_{11} + u_{ii})$ and $u_{kl} \in U_2(u_{kk} + u_{ll})$, so that T permutes the basis, we have that for each i, the canonical basis determined by u_{11} and the orthonormal basis $\{u_{1j}\}_{j\in I}$ of $U_1(u_{11})$ is the same as the canonical basis determined by u_{ii} and the orthonormal basis $\{u_{ij}\}_{j\in(I\cup\{1\})\setminus\{i\}}$ of $U_1(u_{ii})$.

Second, if we delete a "diagonal" basis vector u_{ii} $(i \neq 1)$, then $\{u_{1j}\}_{j\neq i,1}$ is an orthonormal basis in the Hilbert space $U_1\big(u_{11}, U_0(u_{ii})\big) = U_1(u_{11}) \cap U_0(u_{ii})$ so that by the construction, $\{u_{kj}\}_{k\neq i,\, j\neq i}$ is a canonical basis for $U_0(u_{ii})$. By the same argument, $\{u_{ij}\}_{2\leq i\leq j\leq n}$ is the canonical basis for $U_0(u_{11})$ determined by u_{22} and the orthonormal basis $\{u_{2j}\}_{2<j\leq n}$

The previous proposition describes the behavior of rank 2 facially symmetric subspaces. Its proof, *e.g.* (12) reveals some information about the structure of rank 3 subspaces. These will be studied further in the following proposition. We first need a couple of definitions and a lemma.

Let Z satisfy Assumption 3.3 and let $\{u_{ij}\}$ be a canonical basis. Since u_{ii} is a minimal geometric tripotent, $U_2(u_{ii}) = \mathbf{C}u_{ii}$. Also since, for $i \neq j$, $\mathrm{sp}_{\mathbf{C}}\{u_{ii}, u_{ij}, u_{jj}\} \cong S_2(\mathbf{C})$ canonically, u_{ij}, which is the sum of the minimal geometric tripotents w_{ij} and $-\tilde{w}_{ij}$, is a minimal geometric tripotent of $U_1(u_{ii})$ which governs u_{ii}. Thus $U_1(u_{ii}) \cap U_2(u_{ij}) = \mathbf{C}u_{ij}$.

For a geometric tripotent u, introduce the notation $\hat{u}$ for a norm 1 functional in Z with $u = v(\hat{u})$. Thus, if u is a minimal geometric tripotent and Z is atomic, then $\hat{u} = \pi^{-1}(u)$ while if u is the sum of two orthogonal minimal geometric tripotents, then $\hat{u} = \frac{1}{2}\pi^{-1}(u)$.

DEFINITION 3.8. Let Z satisfy Assumption 3.3 and let $\{u_{ij}\}$ be a canonical basis. The *coordinates* $\{x_{ij}\} \subset \mathbf{C}$ of an element $a \in U$ are defined by

$$x_{ii}u_{ii} = P_2(u_{ii})^*a, \quad i \in I \cup \{1\} \text{ and}$$

$$x_{ij}u_{ij} = P_2(u_{ij})^*P_1(u_{ii})^*a, \quad i, j \in I \cup \{1\}, i \neq j.$$

Similarly, the *coordinates* $\{x_{ij}\} \subset \mathbf{C}$ of an element $f \in Z$ are defined by

$$x_{ii}\hat{u}_{ii} = P_2(u_{ii})f, \quad i \in I \cup \{1\} \text{ and}$$

$$x_{ij}\hat{u}_{ij} = P_2(u_{ij})P_1(u_{ii})f, \quad i, j \in I \cup \{1\}, i \neq j.$$

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Then the coordinates of $a \in U$ satisfy $x_{ij} = \langle a, \hat{u}_{ij}\rangle$ and the coordinates of $f \in Z$ satisfy $x_{ij} = \langle f, u_{ij}\rangle$.

LEMMA 3.9. *Let Z satisfy Assumption 3.3 and let $\{u_{ij}\}$ be a canonical basis. Then, for $i, j \in I$ with $i \neq j$, we have a canonical isometry*

$$\mathrm{sp}_{\mathbf{C}}\left\{u_{11}, \frac{u_{1j}+u_{1i}}{\sqrt{2}}, w_{ij}\right\} \cong S_2(\mathbf{C}). \tag{16}$$

In particular,

$$\frac{u_{11} + \frac{u_{1j}+u_{1i}}{\sqrt{2}} + w_{ij}}{2}$$

is a minimal geometric tripotent of U.

PROOF. By (12), the symmetry $S^*_{\tilde{w}_{ij}}$ fixes $u_{1j}+u_{1i}$. Thus, $u_{1j}+u_{1i} \in U_2(\tilde{w}_{ij}) \oplus U_0(\tilde{w}_{ij})$. We will show that $u_{1j}+u_{1i} \in U_0(\tilde{w}_{ij})$. By 4. of Proposition 3.7, the minimal geometric tripotent $\tilde{w}_{ij}$ belongs to $U_2(u_{ii}+u_{jj})$, so is orthogonal to u_{11}. Thus $P_2(\tilde{w}_{ij}) = P_2(\tilde{w}_{ij})P_0(u_{11})$. Hence $P_2(\tilde{w}_{ij})^*(u_{1j}+u_{1i}) = P_2(\tilde{w}_{ij})^*P_0(u_{11})^*(u_{1j}+u_{1i}) = 0$, so that $u_{1j}+u_{1i} \in U_0(\tilde{w}_{ij})$, *i.e.*, $(u_{1j}+u_{1i}) \diamond \tilde{w}_{ij}$.

Since $u_{1j} \in U_2(u_{11}+u_{jj}) \subset U_2(u_{11}+u_{jj}+u_{ii})$, it follows that $u_{1j}+u_{1i} \in U_2(u_{11}+u_{ii}+u_{jj})$. Also $u_{11}+w_{ij}+\tilde{w}_{ij} = u_{11}+u_{ii}+u_{jj}$ and thus $u_{1j}+u_{1i} \in U_2(u_{11}+w_{ij}+\tilde{w}_{ij}) \cap U_0(\tilde{w}_{ij}) = U_2(w_{ij}+u_{11})$. It follows that $\mathrm{sp}\{u_{11}, (u_{1j}+u_{1i})/\sqrt{2}, \lambda w_{ij}\}$ is (canonically) isometric with $S_2(\mathbf{C})$ for some $\lambda \in \mathbf{T}$, in such a way that

$$\frac{\hat{u}_{11} + \frac{\hat{u}_{1j}+\hat{u}_{1i}}{\sqrt{2}} + \overline{\lambda}\hat{w}_{ij}}{2} \tag{17}$$

is an extreme point of Z_1. Apply $P_2(u_{11}+u_{ii})$ to (17) and use ERP to conclude that $\hat{u}_{11}+\frac{1}{\sqrt{2}}\hat{u}_{1i}+\frac{\overline{\lambda}}{2}\hat{u}_{ii}$ is a multiple of an extreme point of Z_1. Keeping in mind that canonically, $\mathrm{sp}_{\mathbf{C}}\{u_{11}, u_{1i}, u_{ii}\} \cong S_2(\mathbf{C})$, this implies that $\det\begin{bmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & \lambda/2 \end{bmatrix} = 0$, *i.e.*, $\lambda = 1$ and therefore (16) holds. ∎

Let $[x_{ij}]$ be the formal matrix obtained from the coordinates of an element $f \in Z$ with respect to a canonical basis. The following proposition shows that the determinants of a large number of 2 by 2 submatrices of this matrix vanish if f is a multiple of an extreme point.

PROPOSITION 3.10. *Let Z satisfy Assumption 3.3 and let $\{u_{ij}\}_{i,j\in I\cup\{1\}}$ be a canonical basis. Let f be an extreme point of Z_1 with coordinates x_{ij} with respect to this canonical basis. Then for all $i, j \in I \cup \{1\}$,*

$$x_{11}x_{ij} = x_{1i}x_{1j} \tag{18}$$

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PROOF. By ERP, $P_2(u_{11}+u_{ii})f$ is a multiple of an extreme point of Z_1. Since, by (15), $P_2(u_{11}+u_{ii})f = x_{11}\hat{u}_{11} + x_{1i}\hat{u}_{1i} + x_{ii}\hat{u}_{ii}$, $\det\begin{bmatrix} x_{11} & x_{1i} \\ x_{1i} & x_{ii} \end{bmatrix} = 0$, and thus (18) holds for $i = j$. This implies that (18) holds for all i and j if $x_{11} = 0$.

Similarly, $P_2(u_{jj}+u_{ii})f$ is a multiple of an extreme point of Z_1, so $x_{jj}x_{ii} = x_{ij}^2$, and thus $x_{11}^2x_{ij}^2 = (x_{11}x_{ii})(x_{11}x_{jj}) = x_{1i}^2x_{1j}^2$, *i.e.*,

$$x_{11}x_{ij} = \pm x_{1j}x_{1i}. \tag{19}$$

We shall show that the plus sign prevails, which will complete the proof. We may assume, without loss of generality, that $x_{11} \neq 0$. By Lemma 3.9 as in (15), we have $U_2(u_{11}+w_{ij}) = \mathrm{sp}_{\mathbf{C}}\{u_{11}, \frac{u_{1j}+u_{1i}}{\sqrt{2}}, w_{ij}\}$ so that $P_2(u_{11}+w_{ij})f$ belongs to the complex span of $\{\hat{u}_{11}, (\hat{u}_{1i}+\hat{u}_{1j})/\sqrt{2}, \hat{w}_{ij}\}$. We shall show

$$P_2(u_{11}+w_{ij})f = x_{11}\hat{u}_{11} + \frac{x_{1i}+x_{1j}}{\sqrt{2}}(\hat{u}_{1i}+\hat{u}_{1j})/\sqrt{2} + \frac{(x_{1i}\pm x_{1j})^2}{2x_{11}}\hat{w}_{ij}. \tag{20}$$

Since, by ERP, $P_2(u_{11}+w_{ij})f$ is a multiple of an extreme point of Z_1, the plus sign holds in the last term, and as we shall see below, this implies (18).

To prove (20), note first that the coefficient of $\hat{u}_{11}$ is given by $\langle P_2(u_{11}+w_{ij})f, \hat{u}_{11}\rangle = \langle f, \hat{u}_{11}\rangle = x_{11}$. Since $P_2(u_{11}+w_{ij}) = P_2\big((u_{1j}+u_{1i})/\sqrt{2}\big)$, the coefficient of $(\hat{u}_{1j}+\hat{u}_{1i})/\sqrt{2}$ is obtained by applying $P_1(u_{11})$ to (20) and is given by the inner product $\big(P_1(u_{11})f \,|\, \frac{\hat{u}_{1i}+\hat{u}_{1j}}{\sqrt{2}}\big)$ in the Hilbert space $Z_1(u_{11})$. It therefore equals $\frac{x_{1i}+x_{1j}}{\sqrt{2}}$. Finally, the coefficient of $\hat{w}_{ij}$ is obtained by applying $P_2(w_{ij})$ to (20) and is therefore given by

$$\langle f, \hat{w}_{ij}\rangle = \frac{1}{2}\langle f, \hat{u}_{ii}+\hat{u}_{jj}+2\hat{u}_{ij}\rangle = \frac{1}{2}(x_{ii}+x_{jj}+2x_{ij}) = \frac{1}{2}\Big(\frac{x_{1i}^2}{x_{11}} + \frac{x_{1j}^2}{x_{11}} \pm \frac{2x_{1i}x_{1j}}{x_{11}}\Big).$$

As mentioned above, the plus sign must prevail and (18) follows. ∎

REMARK 3.11. Proposition 3.10 says that for an extreme point f, if, with respect to a canonical basis, the coordinate $x_{11} \neq 0$, then the coordinates of $P_0(u_{11})f$ are determined uniquely and universally by the coordinates of $P_2(u_{11})f + P_1(u_{11})f$. More precisely, if f and g are extreme points in spaces Z and Y respectively, with dual spaces U and V containing canonical bases $\{u_{ij}\}$ and $\{v_{ij}\}$ with the same index set, if $x_{11} \neq 0$ and $y_{11} \neq 0$ and if $x_{1j} = y_{1j}$ for every j, then $x_{ij} = y_{ij}$ for every i and j.

PROPOSITION 3.12. *Let Z satisfy Assumption 3.3 and let $\{u_{ij}\}$ be a canonical basis. Then with $\mathcal{S} = \text{weak}^* - \mathrm{sp}_{\mathbf{C}}\{u_{ij}\}$, we have*

$$U = \mathcal{S} \oplus^{\ell^\infty} \mathcal{S}^\diamond.$$

The family $\{\hat{u}_{ij}\}$ is norm total in $\mathcal{S}_*$, the predual of $\mathcal{S}$, and

$$Z = \mathcal{S}_* \oplus^{\ell^1} (\mathcal{S}_*)^\diamond. \tag{21}$$

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PROOF. For each i, $Z = Z_2(u_{ii}) + Z_1(u_{ii}) + Z_0(u_{ii})$ and by (8), $Z = X + Z_0(u_{ii})$ where $X = \overline{\mathrm{sp}}^{\mathrm{norm}}\{\hat{u}_{ij}\}$. Continuing, for each finite set $A \subset I \cup \{1\}$, we have $Z = X + \bigcap_{i \in A} Z_0(u_{ii})$.

Let $Q_A = \prod_{i \in A} P_0(u_{ii})$. The net Q_A converges strongly to a contractive projection Q with range $Y = \bigcap_{i \in I \cup \{1\}} Z_0(u_{ii})$. Thus $Z = X + Y$, and these last two summands are orthogonal. To prove the orthogonality it suffices to notice that for each $g \in Y$, we have $\hat{u}_{ij} \in Z_2(u_{ii} + u_{jj})$ and $g \in Z_0(u_{ii}) \cap Z_0(u_{jj}) = Z_0(u_{ii} + u_{jj})$. Also, since $\{u_{ii}, u_{jj}, u_{ij}\}^\diamond = \{u_{ii}, u_{jj}\}^\diamond$, we have $Y = X^\diamond$ (Note also that $Y = \left\{\hat{u}_{1j} : j \in I \cup \{1\}\right\}^\diamond$ (*cf.* Lemmas 5.5, 6.6, and 7.3). This proves (21) and hence the first statement.

Let $x \in \mathcal{S}$ vanish on all $\hat{u}_{ij}$, so that all coordinates x_{ij} of x are zero. We know that x is the weak*-limit of a net of finite sums of the form $\sum \alpha_{ij} u_{ij}$. Applying the functional $\hat{u}_{kl}$ to this limit shows that $\alpha_{ij} = x_{ij}$ and therefore $x = 0$. Thus, $\{\hat{u}_{ij}\}$ is norm total in $\mathcal{S}_*$. ∎

DEFINITION 3.13. A WFS space Z is said to be *irreducible* if it is not the direct sum of two non-zero orthogonal subspaces.

We are not requiring that these subspaces be WFS, or closed. However, they are automatically closed because they form an ℓ^1-sum, *i.e.*, they are *L-summands*. Such a subspace therefore inherits from Z any of the four PSP's as well as atomicity, neutrality and SFS. Recall that subspaces which form ℓ^∞-sums are called *M-summands*.

For example, the predual of any JBW*-triple factor is irreducible. To see this, note that if Z is the predual of a JBW*-triple M, and Z is the direct sum of two orthogonal subspaces, then it follows that there is a weak*-continuous bicontractive projection on M with range the dual of one of these subspaces. Since bicontractive projections have JB^*-subtriples for their ranges ([13, Proposition 3.1]), U is the orthogonal sum of JBW*-subtriples. By orthogonality, these summands are ideals, so the projection is either the identity or 0 since U is a factor.

The following is the main result of this section.

THEOREM 3.14. *Let Z be an atomic neutral* SFS *space, and assume the pure state properties* FE, ERP, *and* STP. *Assume that there exists a minimal geometric tripotent v with $U_1(v)$ of rank 1 and a geometric tripotent u with $u \vdash v$. Then U has an M-summand which is linearly isometric with the complex* JBW*-*triple of all symmetric "matrices" on a complex Hilbert space (Cartan factor of type 3). In particular, if Z is irreducible, then Z^* is isometric to a Cartan factor of type 3.*

PROOF. Let $\mathcal{S}$ be as in Proposition 3.12. Then $\mathcal{S}_*$ is an atomic neutral strongly facially symmetric space satisfying the properties FE, STP, and ERP (since it is an L-summand in Z). We shall show that $\mathcal{S}$ is linearly isometric to a Cartan factor of type 3. For simplicity of notation, and without loss of generality, we shall assume that $\mathcal{S}_* = Z$.

Since a Cartan factor of type 3 is the dual of a neutral strongly facially symmetric space satisfying the same assumptions as Z, it suffices to prove the following proposition.

PROPOSITION 3.15. *If Y is a space satisfying the same assumptions as Z in Theorem 3.14, and if $\{u_{ij}\}_{i,j \in \{1\} \cup I}$ and $\{v_{ij}\}_{i,j \in \{1\} \cup I}$ are canonical bases in U and $V := Y^*$ respectively, then the map $\kappa \colon \mathrm{sp}_{\mathbf{C}}\{\hat{u}_{ij}\} \longrightarrow \mathrm{sp}_{\mathbf{C}}\{\hat{v}_{ij}\}$ taking $\hat{u}_{ij}$ onto $\hat{v}_{ij}$ extends linearly to an isometry of Z onto Y.*

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PROOF. We shall prove the proposition first in the case that Z is finite dimensional. In this case, κ is a linear bijection of Z onto Y which we shall show is contractive. By symmetry, this will show that it is isometric, completing the proof.

To show that κ is contractive, we first prove the following lemma.

LEMMA 3.16. *If f is an extreme point of Z_1, then $\kappa(f)$ is an extreme point of Y_1.*

Assuming this lemma, since Z is atomic, it follows that κ maps the unit ball Z_1 into Y_1, completing the proof of Proposition 3.15 in the finite dimensional case.

It remains to prove Lemma 3.16 (in the finite dimensional case) and to remove the restriction of finite dimensionality in Proposition 3.15. We shall prove Lemma 3.16 by induction on the cardinality n of the set indexing the canonical basis. The case $n = 1$ is trivial and the case $n = 2$ holds because then U and V are spin factors. Let x_{ij} be the coordinates of the extreme point f. For $n > 2$, we consider first the case that $x_{11} = 0$. Then, since $x_{1j}^2 = x_{11}x_{jj}$, we have $x_{1j} = 0$ for all $j \in I$ and thus $P_1(u_{11})f = 0$ so that $f \in U_0(u_{11})$. Since $\{u_{ij}\}_{2\leq i\leq j\leq n}$ is a canonical basis for $U_0(u_{11})$, it follows by induction that $\kappa(f)$ is extreme in $Y_0(v_{11})$, and hence by neutrality extreme in Y.

We may now assume in our proof that $x_{11} \neq 0$. Next, suppose that $P_1(u_{11})f = 0$. Then $f = x_{11}\hat{u}_{11} + P_0(u_{11})f$, and since f is extreme, we must have $f = x_{11}\hat{u}_{11}$ so that $\kappa(f) = x_{11}\hat{v}_{11}$ is extreme in Y. We may therefore assume that $P_1(u_{11})f \neq 0$ as well.

By multiplying f by a scalar, we may assume that x_{12} is real. Let $g := P_1(u_{11})f/\gamma$, where $\gamma := \|P_1(u_{11})f\|$. Since κ is an isometry of the Hilbert space $Z_1(u_{11})$ onto the Hilbert space $Y_1(v_{11})$, $\gamma = \|P_1(v_{11})\kappa(f)\|$ and $h := \kappa(g) = P_1(v_{11})\kappa(f)/\gamma$. Moreover, the inner products $(g|\hat{u}_{12})$ and $(h|\hat{v}_{12})$ are real. Let R be the symmetry (defined in Remark 3.1) on Z which interchanges g and $\hat{u}_{12}$, and similarly, let T be the symmetry on Y which interchanges h and $\hat{v}_{12}$.

The element $R(f)$ belongs to $\mathrm{sp}_{\mathbf{C}}\{\hat{u}_{11}, \hat{u}_{12}, \hat{u}_{22}\}$; indeed if we write $f = x_{11}\hat{u}_{11} + P_1(\hat{u}_{11})f{+}{+}P_0(\hat{u}_{11})f$, then, using the fact that $P_1(u_{11})f = \gamma g$, we have $R(f) = x_{11}\hat{u}_{11}+\gamma\hat{u}_{12}+ R\big(P_0(u_{11})f\big)$ and $R\big(P_0(u_{11})f\big) \in \mathbf{C}\hat{u}_{22}$ by the extremality of $R(f)$ and Proposition 3.10. Thus

$$R(f) = x_{11}\hat{u}_{11} + \tilde{x}_{12}\hat{u}_{12} + \tilde{x}_{22}\hat{u}_{22} \tag{22}$$

with $x_{11}\tilde{x}_{22} = \tilde{x}_{12}^2$ and $|x_{11}|^2 + 2|\tilde{x}_{12}|^2 + |\tilde{x}_{22}|^2 = 1$ and also

$$\kappa\big(R(f)\big) = x_{11}\hat{v}_{11} + \tilde{x}_{12}\hat{v}_{12} + \tilde{x}_{22}\hat{v}_{22} \tag{23}$$

is extreme in $Y_2(v_{12})$, having determinant 0 and norm 1 in $S_2(\mathbf{C})$. Thus by neutrality, $\kappa R(f)$ and hence $T\kappa R(f)$ is extreme in Y. If we show that $T\kappa R(f) = \kappa(f)$, or equivalently, that the coordinates of $T\kappa R(f)$ with respect to v_{ij} are the same as the coordinates of f with respect to u_{ij}, it will follow that $\kappa(f)$ is extreme in Y, completing the proof of Lemma 3.16.

Let us write

$$R\hat{u}_{ij} = \sum_{k\leq l=1}^{n} \alpha_{ijkl}\hat{u}_{kl}, \qquad T\hat{v}_{ij} = \sum_{k\leq l=1}^{n} \beta_{ijkl}\hat{v}_{kl}.$$

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We must show

$$\langle T\kappa R(f), v_{ij}\rangle = \langle f, u_{ij}\rangle (= x_{ij}). \tag{24}$$

From (23), $T\kappa R(f) = \sum_{k,l}(x_{11}\beta_{11kl} + \tilde{x}_{12}\beta_{12kl} + \tilde{x}_{22}\beta_{22kl})\hat{v}_{kl}$ so that $\langle T\kappa R(f), v_{ij}\rangle = x_{11}\beta_{11ij} + \tilde{x}_{12}\beta_{12ij} + \tilde{x}_{22}\beta_{22ij}$. From (22), $f = R^2 f = \sum_{k,l}(x_{11}\alpha_{11kl} + \tilde{x}_{12}\alpha_{12kl} + \tilde{x}_{22}\alpha_{22kl})\hat{u}_{kl}$ so that $\langle f, u_{ij}\rangle = x_{11}\alpha_{11ij} + \tilde{x}_{12}\alpha_{12ij} + \tilde{x}_{22}\alpha_{22ij}$.

By the construction of R and T, for all indices j, k, l,

$$\alpha_{1jkl} = \beta_{1jkl}. \tag{25}$$

Therefore it suffices to prove that for all indices k, l,

$$\alpha_{22kl} = \beta_{22kl}. \tag{26}$$

We shall use Proposition 3.10 to show that (26) follows from (25).

We consider the extreme point of Z_1

$$\rho = R(\hat{w}_{12}) = \frac{1}{2}(R\hat{u}_{11} + R\hat{u}_{12} + R\hat{u}_{22}) = \frac{1}{2}\sum_{k,l}(\alpha_{11kl} + \alpha_{12kl} + \alpha_{22kl})\hat{u}_{kl}.$$

Since $R\hat{u}_{11} = \hat{u}_{11}$, we have $\alpha_{11kl} = 0$ unless $k = l = 1$. Since $RZ_1(u_{11}) = Z_1(u_{11})$, we have $\alpha_{12kl} = 0$ unless $k = 1$ and $l \neq 1$. Since $u_{22} \diamond u_{11}$, we have $R\hat{u}_{22} \diamond \hat{u}_{11}$ so that $\alpha_{22kl} = 0$ unless $k \neq 1$. Thus, for $p \neq 1$ and $q \neq 1$, if y_{pq} denotes the coordinates of ρ with respect to $\{u_{ij}\}$,

$$y_{pq} = (\alpha_{11pq} + \alpha_{12pq} + \alpha_{22pq})/2 = \alpha_{22pq}/2, \quad y_{1p} = (\alpha_{111p} + \alpha_{121p} + \alpha_{221p})/2 = \alpha_{121p}/2$$

and by Proposition 3.10,

$$\alpha_{22pq}/2 = y_{pq} = y_{1p}y_{1q}/y_{11} = (\alpha_{121p}/2)(\alpha_{121q}/2)/(1/2),$$

i.e.

$$\alpha_{22pq} = \alpha_{121p}\alpha_{121q}. \tag{27}$$

Since a similar relation holds for the β_{ijkl}, equations (25) and (27) imply that (26) holds and Lemma 3.16 is proved. ∎

We now remove the restriction that Z be finite dimensional, thereby completing the proof of Proposition 3.15 and hence of Theorem 3.14. For any canonical basis $\{u_{ij}\}_{i,j\in I\cup\{1\}}$ and for any finite subset $\{1, 2, \ldots, n\}$ of I, by the neutrality of $P_2(u_{11} + \cdots + u_{nn})$, the PSP's are satisfied in $Z_2(u_{11} + \cdots + u_{nn})$. Therefore, the restriction of κ to the facially symmetric space $Z_2(u_{11} + \cdots + u_{nn}) = \mathrm{sp}\{\hat{u}_{ij}\}_{1\leq i,j\leq n}$ is an isometry of $\mathrm{sp}\{\hat{u}_{ij}\}_{1\leq i,j\leq n}$ onto $\mathrm{sp}\{\hat{v}_{ij}\}_{1\leq i,j\leq n}$, and hence κ is an isometry of $\mathrm{sp}\{\hat{u}_{ij}\}_{i,j\in I\cup\{1\}}$ onto $\mathrm{sp}\{\hat{v}_{ij}\}_{i,j\in I\cup\{1\}}$. By Proposition 3.12, κ extends to an isometry of Z onto Y. ∎

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REMARK 3.17. A trivial modification of the proof of Lemma 3.16 shows that it is valid in arbitrary dimensions. To state the infinite dimensional version, let $\mathcal{H}_Z$ denote the Hilbert space which is the completion of Z with respect to the norm given by the symmetric sesquilinear form $\langle \cdot \mid \cdot \rangle$ defined in [16, Proposition 2.9]. Because of the existence of a canonical basis, this form is positive definite, and so $\mathcal{H}_Z$ exists. The map κ extends to a linear map $\tilde{\kappa}$ of the Hilbert space $\mathcal{H}_Z$ onto $\mathcal{H}_Y$ and the proof of Lemma 3.16 given above shows that $\tilde{\kappa}(\operatorname{ext} Z_1) \subset \operatorname{ext} Y_1$. In this case, α_{ijkl} are the coordinates of $R\hat{u}_{ij}$ with respect to u_{kl} and β_{ijkl} are the coordinates of $T\hat{v}_{ij}$ with respect to v_{kl}

REMARK 3.18. A canonical basis is an example of a hermitian grid (*cf.* [8, Definition and Proposition, p. 308]). See also Definitions 5.6 and 6.4 below for other types of grids.

4. **Classification scheme: geometric quadrangles and spin degree.** In this section we prepare some tools for the classification of atomic facially symmetric spaces. To do so we need some additional properties of the relations between minimal geometric tripotents discussed in §2. These properties do not seem to follow from the properties developed heretofore. Thus we introduce an additional property which we call JP.

DEFINITION 4.1. A WFS space Z satisfies JP if for any pair u, v of orthogonal minimal geometric tripotents, we have

$$S_u S_v = S_{u+v}, \tag{28}$$

where for any geometric tripotent w, S_w is the symmetry associated with the symmetric face F_w.

REMARK 4.2. In a neutral WFS space Z satisfying JP, we have for orthogonal minimal geometric tripotents u and v,

$$Z_2(u+v) = Z_2(u) + Z_2(v) + Z_1(u) \cap Z_1(v) \tag{29}$$

$$Z_1(u+v) = Z_1(u) \cap Z_0(v) + Z_1(v) \cap Z_0(u) \tag{30}$$

$$Z_0(u+v) = Z_0(u) \cap Z_0(v). \tag{31}$$

PROOF. For arbitrary orthogonal geometric tripotents, equation (31) was established in [15, Lemma 1.8]. For arbitrary orthogonal geometric tripotents, inclusion in one direction in (29) was shown in [16, Lemma 2.3], implying $P_2(u+v) = P_2(u+v)[P_2(u) + P_2(v) + P_1(u)P_1(v)]$. Using these facts and [15, Corollary 3.4(a)] in

$$\big(P_2(u) - P_1(u) + P_0(u)\big)\big(P_2(v) - P_1(v) + P_0(v)\big) = P_2(u+v) - P_1(u+v) + P_0(u+v)$$

yields (30). It remains to show

$$Z_1(v) \cap Z_1(u) \subset Z_2(u+v). \tag{32}$$

If $f \in Z_1(v) \cap Z_1(u)$, then $P_1(u+v)f + P_0(u+v)f = P_1(u)P_0(v)f + P_1(v)P_0(u)f + P_0(u)P_0(v)f = 0$, *i.e.* $f \in Z_2(u+v)$. ∎

https://doi.org/10.4153/CJM-1993-004-0 Published online by Cambridge University Press

In a JB*-triple, if u and v are orthogonal tripotents, then equations (29)–(31) hold for the Peirce projections and constitute the *joint Peirce decomposition* of the JB*-triple. Hence the notation JP.

In the Hilbert space model for quantum mechanics, property JP is trivially satisfied as follows. Choose $\xi \otimes \xi$ to be the state exposed by v and $\eta \otimes \eta$ to be the state exposed by u, and complete $\{\xi, \eta\}$ to an orthonormal basis. For any state vector ζ expressed in this basis, the symmetry S_v (resp. S_u) changes the sign of the coefficient of ξ (resp. η) and S_{u+v} changes the sign of both coefficients.

By using JP we shall now obtain further properties of the relations between minimal geometric tripotents.

In the remainder of this section, we will make use of the following assumption.

ASSUMPTION 4.3. *Z is an atomic neutral* SFS *space over* **C** *satisfying* FE, STP, ERP, *and* JP.

PROPOSITION 4.4. *Let Z satisfy Assumption 4.3. If u_1 and u_2 are orthogonal geometric tripotents in $U_1(v)$ for some minimal geometric tripotent v, then u_1 and u_2 are minimal geometric tripotents and $(u_1 + u_2) \vdash v$.*

PROOF. First of all, $u_j \top v$ by Corollary 2.3. Suppose u_1 is not a minimal geometric tripotent of $Z_1(v)$. Then, by Corollary 2.9, there are orthogonal geometric tripotents w_1 and w_2 of $Z_1(v)$ such that $u_1 = w_1 + w_2$. Again by Corollary 2.3, $w_j \top v$. By JP, we have $v \in U_1(w_1) \cap U_1(w_2) \subset U_2(w_1 + w_2)$ which says that $(w_1 + w_2) \vdash v$. Then by [15, Corollary 3.4], $U_0(w_1 + w_2) \cap U_1(v) = \{0\}$, a contradiction, since u_1 is there. Thus u_1 is minimal in $U_1(v)$ and by Proposition 2.4, u_1 is minimal in U. As above, $(u_1 + u_2) \vdash v$. ∎

COROLLARY 4.5. *If v is a minimal geometric tripotent in U and $u \top v$, then u is a minimal geometric tripotent in U.*

PROOF. If u were not minimal in $U_1(v)$, we would have $u = u_1 + u_2$ with $u_j \in U_1(v)$, and $u \vdash v$ by the proposition, contradiction. Thus u must be minimal in $U_1(v)$ and so by Proposition 2.4 it is minimal in U. ∎

REMARK 4.6. Another consequence of Proposition 4.4 is that the property JP is satisfied in the geometric Peirce 1-space of any minimal geometric tripotent.

The following important fact about the rank of the geometric Peirce 1-space of a minimal geometric tripotent plays a key role in the classification scheme. Note that the following definition is consistent with the definition of rank 1 given in [16, Definition 2.10].

DEFINITION 4.7. A facially symmetric space Z is of *rank* n ($n = 1, 2, \ldots$) if every orthogonal family of geometric tripotents has cardinality at most n, and if there is at least one orthogonal family of geometric tripotents containing exactly n elements.

PROPOSITION 4.8. *Let Z satisfy Assumption 4.3. If v is a minimal geometric tripotent, then the rank of $U_1(v)$ is at most 2.*

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PROOF. If u_1, u_2, u_3 are orthogonal geometric tripotents in $U_1(v)$, then $(u_1 + u_2) \vdash v$ by Proposition 4.4, contradicting $U_0(u_1 + u_2) \cap U_1(v) = \{0\}$. ■

LEMMA 4.9. *Let Z satisfy Assumption 4.3. For a minimal geometric tripotent v, the norm of $Z_1(v)$ is equivalent to a Hilbert space norm. Hence $Z_1(v)$ is a reflexive Banach space.*

PROOF. For any $f \in Z$, $\|f\|_2 = \langle f, \pi(f)\rangle^{1/2} \leq \|f\|$ always holds. If $0 \neq f \in Z_1(v)$, then $v(f) \in U_1(v)$ so $v(f)$ is either minimal in U by Corollary 4.5, or the sum of two orthogonal minimal geometric tripotents by Proposition 2.5. In the first case $\|f\|_2 = \|f\|$ and in the second case, by [16, Lemma 3.6], $\|f\|^{-1}f = \lambda\rho + (1-\lambda)\tilde{\rho}$ for some pair of orthogonal extreme points $\rho, \tilde{\rho}$ in $F_{v(f)}$ and $\lambda \in (0,1)$. Thus $\left\|(f/\|f\|)\right\|_2^2 = \lambda^2 + (1-\lambda)^2 \geq 1/2$, so the norms $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent on $Z_1(v)$. ■

A special case of the following lemma was proved in Lemma 3.2 without the assumption of JP.

LEMMA 4.10. *Let Z satisfy Assumption 4.3. Let v be a minimal geometric tripotent and let Y be a neutral strongly facially symmetric subspace of $Z_1(v)$ of rank 1. Then Y is isometric to a Hilbert space.*

PROOF. By the spectral theorem in reflexive strongly facially symmetric spaces ([12, Theorem 1]), every element of $V := Y^* \subset U$ is a multiple of a geometric tripotent of Y. As in the proof of [16, Proposition 2.1], each geometric tripotent of Y is also a geometric tripotent of Z. Thus, for each $x \in V$ of norm 1, x is either a minimal geometric tripotent of U or, by Proposition 4.8, a sum of two orthogonal minimal geometric tripotents of U. By considering the continuous function $V_1 \ni x \longmapsto \|\pi^{-1}(x)\|_2$, it follows that either *each* x of norm 1 in V is a minimal geometric tripotent of U, or *each* x in V of norm 1 is the sum of two orthogonal minimal geometric tripotents of U. In the first case, Y is a Hilbert space by [16, Corollary 2.11]. In the second case, an inner product on V inducing a multiple of the norm of V is given by $(x|y) := \langle x, \pi^{-1}(y)\rangle$, since if $x/\|x\| = w_1 + w_2$, then $(x|x) = \|x\|^2\langle w_1 + w_2, \hat{w}_1 + \hat{w}_2\rangle = 2\|x\|^2$. ■

PROPOSITION 4.11. *Let Z satisfy Assumption 4.3. Let v be a minimal geometric tripotent and let u be a minimal geometric tripotent of $U_1(v)$ such that $u \vdash v$. Then $Z_1(v)$ is isometric to a Hilbert space.*

PROOF. Let $\{u_\alpha\}$ be a maximal family of mutually colinear geometric tripotents such that, for each α, $u_\alpha \vdash v$ and u_α is a minimal geometric tripotent of the neutral strongly facially symmetric space $Z_1(v) \cap \bigcap_{\beta\neq\alpha} Z_1(u_\beta)$ (*cf.* Lemma 1.2). We shall show first that

$$U_1(v) = \overline{\operatorname{span}}^{\text{weak}^*}\{u_\alpha\}. \tag{33}$$

In the first place, by [15, Corollary 3.4],

$$P_0(u_\alpha)P_1(v) = 0, \tag{34}$$

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and by minimality

$$P_2(u_\alpha)^* P_1(v)^* \prod_{\beta\neq\alpha} P_1(u_\beta)^*(U) = \mathbf{C}u_\alpha. \tag{35}$$

Moreover, by mutual colinearity

$$P_2(u_\alpha)P_2(u_\beta)P_1(v) = 0 \text{ for } \alpha \neq \beta. \tag{36}$$

To prove (36), it is enough to check that, with $u_\alpha = u'_\alpha + u''_\alpha$ (u'_α, u''_α orthogonal minimal geometric tripotents of U lying in $U_1(v)$) we have, for $\alpha \neq \beta$,

$$P_2(u'_\alpha)P_2(u'_\beta)P_1(v) = 0; \tag{37}$$

$$P_2(u'_\alpha)P_1(u'_\beta)P_1(u''_\beta)P_1(v) = 0; \tag{38}$$

$$P_1(u'_\alpha)P_1(u'_\beta)P_1(u''_\alpha)P_1(u''_\beta)P_1(v) = 0. \tag{39}$$

If (37) is false, then u'_α is a multiple $\lambda u'_\beta$ of u'_β and by (30),

$$\lambda u'_\beta = u'_\alpha = P_2(u'_\alpha)u_\alpha = P_2(u'_\alpha)P_1(u_\beta)u_\alpha = P_2(u'_\alpha)[P_1(u'_\beta)P_0(u''_\beta) + P_1(u''_\beta)P_0(u'_\beta)]u_\alpha.$$

Hence $u'_\alpha = P_2(u'_\beta)u'_\alpha = 0$, a contradiction. If (38) is false, then $u'_\alpha \in U_1(u'_\beta) \cap U_1(u''_\beta)$, so that $u'_\alpha \top u'_\beta$, $u'_\alpha \top u''_\beta$ and $u_\beta \in U_1(u'_\alpha)$, implying $u_\beta \vdash u'_\alpha$. Again by (30)

$$u_\beta = P_1(u'_\alpha)P_1(u_\alpha)u_\beta = P_1(u'_\alpha)[P_1(u'_\alpha)P_0(u''_\alpha) + P_1(u''_\alpha)P_0(u'_\alpha)]u_\beta = P_1(u'_\alpha)P_0(u''_\alpha)u_\beta.$$

Therefore $u''_\alpha \in U_0(u_\beta) \cap U_1(v) = \{0\}$, a contradiction. Finally, if (39) is false, choose a geometric tripotent w there. Then either $w \top v$ or $w \vdash v$. In the first case, w is minimal in U and colinear to both u'_α and u''_α so that $u_\alpha \in U_1(w)$ implying the minimality of u_α in U, contradiction. In the second case, w governs each of $u'_\alpha, u'_\beta, u''_\alpha, u''_\beta$ so that u_α, u_β both belong to the spin factor $U_2(w)$, implying $U_2(u_\alpha) = U_2(w)$, another contradiction. This proves (36).

Since $U_1(v) = P_1(v)^* \prod_\alpha \big(P_2(u_\alpha) + P_1(u_\alpha) + P_0(u_\alpha)\big)^*(U)$, (34)–(36) imply

$$U_1(v) = \overline{\text{span}}^{\text{weak}^*}\{u_\alpha\} + \Big[U_1(v) \cap \bigcap_\alpha U_1(u_\alpha)\Big]. \tag{40}$$

We shall show that (33) holds by showing that the second summand in (40) is the zero space, or equivalently, that its predual $X := Z_1(v) \cap \bigcap_\alpha Z_1(u_\alpha)$ is zero. Suppose that X is not zero. Since it is a closed subspace of the reflexive space $Z_1(v)$, by the Lindenstrauss-Troyanski theorem ([5, page 60]), the unit ball of X has an exposed point, and since X is a neutral SFS space, by [16, Proposition 2.4], there is a minimal geometric tripotent $w \in X^*$ of X. As before, w is a geometric tripotent of U. To arrive at the desired contradiction, we now show that $w \top u_\alpha$ for all α and that $w \vdash v$.

First of all, if $w \top v$, then w is minimal in U and both of the possibilities $w \vdash u_\alpha$ and $w \top u_\alpha$ lead to the conclusion that u_α is minimal in U, which is a contradiction. Thus $w \vdash v$.

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To prove that $w\top u_\alpha$, assume to the contrary that $w \vdash u_\alpha$. Then u_α belongs to the spin factor $U_2(w)$ and $u_\alpha \vdash v$, so that $U_2(u_\alpha) = U_2(w)$, contradicting the fact the $w \in U_1(u_\alpha)$.

Having proved (33) we next show that for each $x = \sum_{\iota=1}^n \lambda_\iota u_\iota \in \operatorname{span}\{u_\alpha\}$, we have,

$$\|x\|^2 = \sum_{\iota=1}^n |\lambda_\iota|^2. \tag{41}$$

This, together with (33) will complete the proof as follows. For arbitrary $x \in U$, let $\lambda_\alpha = \langle x, \hat{u}_\alpha\rangle$. With A a finite set and $Q = \prod_{\beta\notin A} P_1(u_\beta)$, we have

$$\sum_{\alpha\in A} |\lambda_\alpha|^2 = \|Qx\|^2 \leq \|x\|^2$$

and so $\sum_\alpha |\lambda_\alpha|^2 < \infty$. Hence for $A \subset B$,

$$\Big\|\sum_A \lambda_\alpha u_\alpha - \sum_B \lambda_\alpha u_\alpha\Big\|^2 = \Big\|\sum_{B\setminus A} \lambda_\alpha u_\alpha\Big\|^2 = \sum_{B\setminus A} |\lambda_\alpha|^2 \longrightarrow 0.$$

Hence $\sum_\alpha \lambda_\alpha u_\alpha$ converges in norm. But this sum converges in the weak*-topology to x if $x \in U_1(v)$, as is easily seen by applying the functionals $\hat{u}_\alpha$. Thus the family $\{u_\alpha\}$ is norm total in $U_1(v)$, showing that $U_1(v)$ is a Hilbert space.

We return to the proof of (41). Let $Y = \operatorname{span}_{\mathbf{C}}\{\hat{u}_1, \hat{u}_2\} = Z_1(v) \cap \bigcap_{\alpha\neq 1,2} Z_1(u_\alpha)$, which is a neutral strongly facially symmetric space by Lemma 1.2. The space Y is of rank 1, since if there were two orthogonal unit vectors φ, ψ of Y, then the extreme points of Y_1 would consist of scalar multiples of φ and ψ and could not contain $\hat{u}_1$ for example. By Lemma 4.10, Y is isometric to a Hilbert space. This proves the case $n = 2$ of the following assertion: *For $n = 2, 3, \ldots,$ and $u_1, \ldots, u_n$ from $\{u_\alpha\}$,* $\operatorname{span}\{u_1, \ldots, u_n\}$ *is a Hilbert space with orthonormal basis $u_1, \ldots, u_n$.*

For $n \geq 2$ let $x = \sum_{k=1}^{n+1} \lambda_k u_k = x' + \lambda_{n+1} u_{n+1}$ and let T be the symmetry S_t, where $t = (u_1 + x'/\|x'\|)/\big\|(u_1 + x'/\|x'\|)\big\|$. Then, since $t\top u_{n+1}$, we have $Tx = \|x'\| u_1 - \lambda_{n+1} u_{n+1}$ and $\|x\|^2 = \|Tx\|^2 = \|x'\|^2 + |\lambda_{n+1}|^2$, which completes the induction. ∎

COROLLARY 4.12. *If v is a minimal geometric tripotent, then $Z_1(v)$ satisfies* ERP *and* STP. *Hence, if Z satisfies Assumption 4.3, then so does $Z_1(v)$.*

PROOF. We first prove ERP. Suppose that f is an extreme point of the unit ball of $Z_1(v)$, and that w is any geometric tripotent of $U_1(v)$. It suffices to show that $P_2(w)f$ is a multiple of an extreme point of the ball of $Z_1(v)$. If $v(f)\top v$, the result follows by ERP in Z. If on the other hand $v(f) \vdash v$, then since $v(f)$ is minimal in $U_1(v)$, $Z_1(v)$ is a Hilbert space, hence satisfies ERP.

Now suppose that f, g are two extreme points of the unit ball of $Z_1(v)$. If both of $v(f)$ and $v(g)$ are colinear to v, then the result follows by STP in Z. If one of them governs v, then $Z_1(v)$ is a Hilbert space and hence satisfies STP. ∎

In the following we show that, using property JP, the hypothesis of rank 1 in Theorem 3.14 can be weakened.

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COROLLARY 4.13. *Let Z be an atomic neutral* SFS *space over* $\mathbf{C}$, *and assume the pure state properties* FE, STP, ERP, *and* JP. *Assume that there exists a minimal geometric tripotent v and a geometric tripotent u with $u \vdash v$ which is minimal in $U_1(v)$. Then U has an M-summand which is linearly isometric with the complex* JBW*-*triple of all symmetric matrices on a complex Hilbert space (Cartan factor of type 3).*

PROOF. By the proposition, $Z_1(v)$ is of rank 1, so the result follows from Theorem 3.14. ∎

The basic building blocks for atomic facially symmetric spaces will be shown to be the (geometric) quadrangles (and odd quadrangles), as is the case for quadrangles in atomic JBW*-triples.

DEFINITION 4.14. A quadruple (u_1, u_2, u_3, u_4) of minimal geometric tripotents is called a *geometric quadrangle* if $u_1 \diamond u_3$, $u_2 \diamond u_4$, $u_1 \top u_2$, $u_2 \top u_3$, $u_3 \top u_4$, $u_4 \top u_1$, and $\frac{1}{2} \sum_j u_j$ is a minimal geometric tripotent. A triple (u_1, u_2, u_3) of minimal geometric tripotents is called a *geometric prequadrangle* if $u_1 \diamond u_3$, $u_1 \top u_2$, $u_2 \top u_3$. Schematically, we represent a geometric quadrangle by

$$\begin{matrix} u_1 & u_2 \\ u_4 & u_3. \end{matrix}$$

LEMMA 4.15. *Let Z satisfy Assumption 4.3. For each geometric prequadrangle (u_1, u_2, u_3), there is a unique minimal geometric tripotent u_4 such that (u_1, u_2, u_3, u_4) is a geometric quadrangle. The span of any geometric quadrangle is canonically isomorphic to $M_2(\mathbf{C})$.*

PROOF. By Proposition 4.4, $(u_1 + u_3) \vdash u_2$, so we may apply Corollary 2.7 to obtain a minimal geometric tripotent u_4 orthogonal to u_2 such that $(u_1 + u_3) \vdash u_4$ and $U_2(u_1 + u_3) = U_2(u_2 + u_4)$. The proof of Corollary 2.7 shows that (u_1, u_2, u_3, u_4) is a geometric quadrangle. ∎

In $M_2(\mathbf{C})$, let

$$W = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}; \; L = WW^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \; R = W^*W = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then for $X \in M_2(\mathbf{C})$, $S_W^*(X) = 4LXR + X - 2LX - 2XR$ implies $S_W^*(E_{11}) = E_{12}$ and $S_W^*(E_{22}) = -E_{21}$. Thus we have:

COROLLARY 4.16. *For a geometric quadrangle (u_1, u_2, u_3, u_4), the symmetry $S = S_{\frac{u_1+u_2}{\sqrt{2}}}$ satisfies*

$$S^* \colon \begin{matrix} u_1 & u_2 \\ u_4 & u_3 \end{matrix} \longrightarrow \begin{matrix} u_2 & u_1 \\ -u_3 & -u_4 \end{matrix} \tag{42}$$

A slightly more detailed analysis yields the following Corollary, which for $x_1 = 0$ reduces to the previous one.

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COROLLARY 4.17. *For a geometric quadrangle* (u_1, u_2, u_3, u_4), *let* $f = \sum_{j=1}^4 x_j \hat{u}_j$ *with* x_1 *real,* $x_2 \neq 0$, *and* $x_1 x_3 - x_2 x_4 = 0$. *Then there is a symmetry* T *depending only on* x_1, x_2 *and* u_1, u_2 *such that* $Tf = c_1 \hat{u}_1 + c_4 \hat{u}_4$ *for suitable constants* c_1, c_4. *Moreover, if also* $x_1 \neq 0$, *then there is a complex number* λ, *depending only on* x_1, x_2 *such that* $c_1 = \lambda x_1$ *and* $c_4 = -\lambda x_4$.

PROOF. We know that $\mathrm{sp}_{\mathbf{C}}\{u_1, u_2, u_3, u_4\}$ is canonically isomorphic to $M_2(\mathbf{C})$. Let T be the symmetry on $\mathrm{sp}_{\mathbf{C}}\{\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4\}$ corresponding to the symmetry R on $M_2(\mathbf{C})_*$ which interchanges the unit vectors

$$(x_1^2 + |x_2|^2)^{-1/2} \begin{bmatrix} x_1 & x_2 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that with $\alpha = (x_1^2 + |x_2|^2)^{1/2}$, $T = S_w$ where

$$w = \frac{\frac{x_1 u_1 + x_2 u_2}{\alpha} + u_1}{\left\|\frac{x_1 u_1 + x_2 u_2}{\alpha} + u_1\right\|}.$$

Both statements are consequences of the following simple matrix calculations. With α as above, let $\beta = [2(1 + \frac{x_1}{\alpha})]^{1/2}$ and (by an obvious abuse of notation) let

$$f = \begin{bmatrix} x_1 & x_2 \\ x_4 & x_3 \end{bmatrix}; \; w = \frac{1}{\beta} \begin{bmatrix} \frac{x_1}{\alpha} + 1 & \frac{x_2}{\alpha} \\ 0 & 0 \end{bmatrix}.$$

Then

$$\ell = ww^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \; r = w^* w = \frac{1}{\beta^2} \begin{bmatrix} (\frac{x_1}{\alpha} + 1)^2 & (\frac{x_1}{\alpha} + 1)\frac{x_2}{\alpha} \\ (\frac{x_1}{\alpha} + 1)\frac{\bar{x}_2}{\alpha} & \frac{|x_2|^2}{\alpha^2} \end{bmatrix},$$

and a tedious calculation yields

$$S_w f = 4\ell f r + f - 2\ell f - 2fr = \begin{bmatrix} \lambda x_1 & 0 \\ -\lambda x_4 & 0 \end{bmatrix},$$

where $\lambda = \alpha / x_1$. ∎

The following lemma will be useful for verifying that certain quadruples occurring in the construction of a natural basis are geometric quadrangles. The process can be called "side by side glueing".

LEMMA 4.18. *Let* Z *satisfy Assumption 4.3. If* (u_1, u_2, u_3, u_4) *and* (u_1, u_2, u_5, u_6) *are geometric quadrangles with* $u_4 \top u_6$, *then* (u_3, u_4, u_6, u_5) *is a geometric quadrangle. Schematically,*

$$\begin{matrix} u_4 & u_1 & u_6 \\ u_3 & u_2 & u_5 \end{matrix}$$

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PROOF. Let $S = S_{\frac{u_1+u_4}{\sqrt{2}}}$. By Corollary 4.16, $S^*u_2 = -u_3$. On the other hand, $\frac{u_1+u_4}{\sqrt{2}} \in U_1(u_6)$ and is minimal (since $(u_1+u_4)/\sqrt{2}$ has spin determinant 0 in $U_2(u_1+u_3)$), hence colinear with u_6 so that $S^*u_6 = -u_6$.

From $u_2 \diamond u_6$ we have $u_3 \diamond u_6$. Therefore, letting $T = S_{\frac{u_1+u_6}{\sqrt{2}}}$ we have

$$T^*: \begin{matrix} u_1 & u_2 \\ u_4 & u_3 \end{matrix} \longrightarrow \begin{matrix} u_6 & -u_5 \\ -u_4 & u_3 \end{matrix} \tag{43}$$

which implies (by applying $S^*_{u_6}$), that (u_6, u_5, u_3, u_4) is a geometric quadrangle. ∎

To proceed with the classification scheme, we need the notion of *spin degree*. Recall that by [16, Theorem 3.8], a rank 2 face in the unit ball Z_1 of an atomic neutral SFS space which satisfies FE and STP is affinely isomorphic to the unit ball of a real Hilbert space. By the dimension of the face, we mean the dimension of this Hilbert space. This is also equal to one less than the complex dimension of $Z_2(u)$, where F_u is the rank 2 face (*cf.* [16, Remark 4.12]).

DEFINITION 4.19. The *spin degree* of a neutral strongly facially symmetric space Z satisfying FE and STP is one greater than the (finite or infinite) dimension of some rank 2 face in Z_1, or 0, if no such face exists.

Thus the spin degree is the dimension of a spin factor sitting in U as the geometric Peirce 2-space of some rank 2 face. *A priori*, spin degree can vary with the face. In the case of spin factors (Cartan factor of type 4), the spin degree is the complex dimension of the factor. The types of the remaining Cartan factors are determined by the spin degree. For example, the Cartan factor of type 3 (symmetric matrices) has spin degree 3, and the Cartan factor of type 1 (full rectangular matrices) has spin degree 4.

The following proposition about the spin degree plays an important role in the classification scheme by allowing a reduction to a previous case.

PROPOSITION 4.20. *Let Z be an atomic neutral* SFS *space over* $\mathbf{C}$ *satisfying* FE, STP, ERP, *and* JP. *Suppose that Z is of finite spin degree $n \geq 5$ and has no L-summand of type I_2. Then there is a minimal geometric tripotent v such that $Z_1(v)$ has spin degree $n-2$ and has no L-summand of type I_2. Moreover, the spin degree n of such Z, when finite, is even.*

PROOF. Let $v, \tilde{v}$ be orthogonal minimal geometric tripotents such that the dimension of $U_2(v+\tilde{v})$ is $n \geq 5$. Since Z has no L-summand of type I_2, $U_1(v+\tilde{v}) \neq \{0\}$. By (30) and (32), $P_1(v+\tilde{v}) = P_1(v+\tilde{v})^2 = P_1(v+\tilde{v})(P_1(v)P_0(\tilde{v}) + P_0(v)P_1(\tilde{v}) = P_1(v+\tilde{v})P_1(v)[I - P_1(\tilde{v}) - P_0(\tilde{v})] + P_1(v+\tilde{v})P_1(\tilde{v})[I - P_1(v) - P_0(v)] = P_1(v+\tilde{v})P_1(v) + P_1(v+\tilde{v})P_1(\tilde{v})$. Therefore,

$$U_1(v+\tilde{v}) = U_1(v+\tilde{v}) \cap U_1(v) + U_1(v+\tilde{v}) \cap U_1(\tilde{v}).$$

Using the fact that there is a symmetry of Z exchanging v and $\tilde{v}$, we have that both terms in this decomposition are non-zero.

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Now choose $u_1, \tilde{u}_1$ such that $(v, u_1, \tilde{v}, \tilde{u}_1)$ is an odd quadrangle in the spin factor $U_2(v + \tilde{v})$. Since $U_0(u_1 + \tilde{u}_1) \cap U_1(v) = \{0\}$ and $u_1 + \tilde{u}_1 \in U_1(v)$, the joint Peirce decomposition of $u_1 + \tilde{u}_1$ in $U_1(v)$ is

$$U_1(v) = U_2\big(u_1 + \tilde{u}_1, Z_1(v)\big) + U_1\big(u_1 + \tilde{u}_1, Z_1(v)\big).$$

The first term is a spin factor (since it is the Peirce 1-space of the minimal tripotent v in the spin factor $U_2(u_1 + \tilde{u}_1)$) and is of dimension $n - 2$ by (29). The second term equals $U_1(v + \tilde{v}) \cap U_1(v) \neq \{0\}$ so that $U_1(v)$ is not of type I_2. Since $Z_1(v)$ has rank at most 2, it has no L-summand of type I_2, proving the first statement.

If the spin degree was odd for some SFS space with no L-summand of type I_2, then by repeated use of the first statement, there would be an SFS space satisfying Assumption 4.3 of spin degree 3 with no L-summand of type I_2. By Corollary 4.13 and Proposition 4.8, this space is isomorphic to $S_2(\mathbf{C})$, and is hence of type I_2, a contradiction. ∎

The following Lemma will be needed in Section 7.

LEMMA 4.21. *Let Z satisfy Assumption 4.3. Let $u_1, \ldots, u_n$ be mutually colinear minimal geometric tripotents, each pair of which lies in some geometric quadrangle. Then each geometric tripotent w in the linear span of $u_1, \ldots, u_n$ is a minimal geometric tripotent.*

PROOF. We shall show by induction that if $x = \sum_{j=1}^{k} x_j u_j$ $(1 \leq k \leq n)$, then there exist an isometry T_k and $\lambda_k \in \mathbf{C}$ such that $T_k x = \lambda_k u_1$ and $T_k u_j = \pm u_j$ for $j > k$. For $k = 1$, there is nothing to prove, so let $x = \sum_{j=1}^{k+1} x_j u_j = x' + x_{k+1} u_{k+1}$ and choose T_k and λ_k such that $T_k x' = \lambda_k u_1$ and $T_k u_j = \pm u_j$ for $j > k$. Then $T_k x = \lambda_k u_1 \pm x_{k+1} u_{k+1}$ and since u_1 and u_{k+1} belong to a geometric quadrangle, there exists a symmetry S_w such that $S_w^* T_k x = \lambda_{k+1} u_1$ for some $\lambda_{k+1} \in \mathbf{C}$ and $S_w^* u_j = -u_j$ for $j = 2, \ldots, n$. Let $T_{k+1} := S_w T_k$. Then $T_{k+1} x = \lambda_{k+1} u_1$ and for $j > k + 1$, $T_{k+1} u_j = \pm S_w^* u_j = \pm u_j$. ∎

5. The type 1 case.

In this section we consider facially symmetric spaces of spin degree 4. We will show that, under suitable hypotheses, such a space is isometric to a neutral strongly facially symmetric space containing the predual of a Cartan factor of type 1 as an L-summand.

For a minimal geometric tripotent v, we shall show first that $U_1(v)$ is the orthogonal direct sum of two Hilbert spaces. By choosing orthonormal bases in these Hilbert spaces and completing a family of geometric prequadrangles, we obtain a family of geometric tripotents which we show is a (geometric) rectangular grid. Finally, we show that the natural map from the span of the dual of this grid to the span of the dual of a concrete rectangular grid is isometric so extends to an isometry of the norm closed span onto the predual of a Cartan factor of type 1.

ASSUMPTION 5.1. *Z is an atomic neutral strongly facially symmetric space of spin degree 4 which satisfies* FE, STP, ERP, *and* JP. *Thus, there is a geometric quadrangle $(v, u_1, \tilde{v}, \tilde{u}_1)$ canonically isomorphic to $M_2(\mathbf{C})$ such that $U_2(v + \tilde{v}) = \mathrm{sp}_{\mathbf{C}}\{v, u_1, \tilde{v}, \tilde{u}_1\}$.*

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LEMMA 5.2. *Let Z satisfy Assumption 5.1. Then $U_1(v)$ is the direct sum of two orthogonal Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. More precisely, $U_1(v) = [U_0(\tilde{u}_1) \cap U_1(v)] + [U_0(u_1) \cap U_1(v)]$, and the summands are orthogonal facially symmetric spaces of rank 1 which are isometric to Hilbert spaces.*

PROOF. Since $U_2(v+\tilde{v}) = U_2(u_1+\tilde{u}_1) = U_2(u_1) + U_2(\tilde{u}_1) + [U_1(u_1) \cap U_1(\tilde{u}_1)]$ has dimension 4 and

$$\begin{aligned} U_1(v) \cap U_1(\tilde{v}) &= U_2(v+\tilde{v}) \cap U_1(v) \\ &= [U_2(u_1) + U_2(\tilde{u}_1) + U_1(u_1) \cap U_1(\tilde{u}_1)] \cap U_1(v) \\ &= [U_2(u_1) \cap U_1(v)] + [U_2(\tilde{u}_1) \cap U_1(v)] + [U_1(u_1) \cap U_1(\tilde{u}_1) \cap U_1(v)], \end{aligned}$$

we have $U_1(u_1) \cap U_1(\tilde{u}_1) \cap U_1(v) = \{0\}$.

Since $U_0(v+\tilde{v}) \cap U_1(v) = [U_0(v) \cap U_0(\tilde{v})] \cap U_1(v) = \{0\}$, we have by (29) and (30)

$$\begin{aligned} U_1(v) &= [U_2(u_1+\tilde{u}_1) \cap U_1(v)] + [U_1(u_1+\tilde{u}_1) \cap U_1(v)] + \{0\} \\ &= [U_2(u_1) \cap U_1(v)] + [U_2(\tilde{u}_1) \cap U_1(v)] + \{0\} + \\ &\qquad [U_1(u_1) \cap U_0(\tilde{u}_1) \cap U_1(v)] + [U_1(\tilde{u}_1) \cap U_0(u_1) \cap U_1(v)] \\ &= \mathbf{C}u_1 + [U_1(u_1) \cap U_0(\tilde{u}_1) \cap U_1(v)] + \mathbf{C}\tilde{u}_1 + [U_1(\tilde{u}_1) \cap U_0(u_1) \cap U_1(v)] \\ &= U_0(\tilde{u}_1) \cap U_1(v) + U_0(u_1) \cap U_1(v) := \mathcal{H}_1 + \mathcal{H}_2, \end{aligned}$$

where, for example,

(44)
$$\mathcal{H}_1 = U_0(\tilde{u}_1) \cap U_1(v) = \mathbf{C}u_1 + [U_1(u_1) \cap U_0(\tilde{u}_1) \cap U_1(v)] = [U_2(u_1) + U_1(u_1)] \cap U_1(v).$$

Since $u_1 \diamond \tilde{u}_1$, in order to show that the summands are orthogonal, it suffices to show $e \diamond e'$ where $e \in U_1(u_1) \cap U_0(\tilde{u}_1) \cap U_1(v)$ and $e' \in U_1(\tilde{u}_1) \cap U_0(u_1) \cap U_1(v)$ are geometric tripotents. Since $e \vdash v$ would imply $U_0(e) \cap U_1(v) = \{0\}$, we must have $e \top v$ so e and similarly e' are minimal geometric tripotents. Thus $P_2(e)e'$ is a multiple of e and since e, u_1 are compatible,

$$P_2(e)e' = P_1(u_1)P_2(e)e' = P_2(e)P_1(u_1)e' = P_2(e)P_1(u_1)P_0(u_1)e' = 0. \tag{45}$$

On the other hand, let S be the symmetry $S_{\frac{u_1+e}{\sqrt{2}}}$, which exists by completion (by Lemma 4.15) of the geometric prequadrangle $(e, v, \tilde{u}_1)$ to the geometric quadrangle $(e, v, \tilde{u}_1, v')$ and "side by side" glueing (by Lemma 4.18) of $(u_1, v, \tilde{u}_1, \tilde{v})$ and $(e, v, \tilde{u}_1, v')$. By (44), we have $\mathcal{H}_1 = S^*(\mathcal{H}_1) = [U_2(e) + U_1(e)] \cap U_1(v)$. Therefore, since e is compatible with v, $P_1(e)e' \in U_1(v) \cap U_1(e) \subset \mathcal{H}_1 \subset U_2(u_1) + U_1(u_1)$, so

$$P_1(e)e' = \big(P_2(u_1) + P_1(u_1)\big)P_1(e)e' = P_1(e)[P_2(u_1) + P_1(u_1)]P_0(u_1)e' = 0. \tag{46}$$

From (45) and (46), we have $e \diamond e'$. Thus $\mathcal{H}_1 \diamond \mathcal{H}_2$ and each is of rank 1, hence a Hilbert space by [16, Corollary 2.11]. ∎

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CONSTRUCTION 5.3. Let Z satisfy Assumption 5.1. Define $u_{11} = v$ and by Lemma 5.2, let $\{u_{1j} : j \in J\}$ be any orthonormal basis, including u_1, of the Hilbert space $U_0(\tilde{u}_1) \cap U_1(v)$, where J is an index set not containing 1. Similarly, let $\{u_{i1} : i \in I\}$ be any orthonormal basis, including $\tilde{u}_1$, of the Hilbert space $U_0(u_1) \cap U_1(v)$, where I is an index set not containing 1. Note that $u_{1j} \top u_{11}$ for $j \neq 1$ since $u_1 \top v$ and u_1, u_{1j} can be exchanged by a symmetry sending v to $-v$, namely $S_{\frac{u_1+v}{\sqrt{2}}}$. Thus for $(i,j) \in I \times J$, by Lemma 4.15, there exist u_{ij} such that

$$(u_{11}, u_{1j}, u_{ij}, u_{i1}) \tag{47}$$

is a geometric quadrangle.

REMARK 5.4. Let Z satisfy Assumption 5.1 and let $\{u_{ij}\}$ be given by Construction 5.3. For fixed $p \in I \cup \{1\}$, $\{u_{pj}\}_{j \in J \cup \{1\}}$ is a maximal family of mutually colinear geometric tripotents. Similarly, for fixed $q \in J \cup \{1\}$, $\{u_{iq}\}_{i \in I \cup \{1\}}$ is a maximal family of mutually colinear geometric tripotents.

PROOF. Let $S = S_{\frac{u_{11}+u_{p1}}{\sqrt{2}}}$. By Corollary 4.16, for $j \neq 1$, $S^* u_{1j} = -u_{pj}$. Since isometries preserve colinearity, the first statement follows. Similar proof holds for the second statement. ∎

LEMMA 5.5. *Let Z satisfy Assumption 5.1 and let $\{u_{ij}\}$ be given by Construction 5.3. Then with $\mathcal{S} = \overline{\mathrm{sp}}_{\mathbf{C}}^{\mathrm{weak}^*} \{u_{ij}\}_{i \in I \cup \{1\}, j \in J \cup \{1\}}$,*

$$U = \mathcal{S} \oplus^{\ell^\infty} \mathcal{S}^\diamond.$$

The family $\{\hat{u}_{ij}\}$ is norm total in $\mathcal{S}_*$, the predual of $\mathcal{S}$, and $Z = \mathcal{S}_* \oplus^{\ell^1} (\mathcal{S}_*)^\diamond$.

PROOF. By the proof of Remark 5.4, for $(i,j) \in I \times J$, $U_1(u_{1j}) \subset \mathcal{S}$ and $U_1(u_{i1}) \subset \mathcal{S}$. Moreover,

$$\begin{aligned} U_0(u_{11}) &= U_2(u_{1j}) \cap U_0(u_{11}) + U_1(u_{1j}) \cap U_0(u_{11}) + U_0(u_{1j}) \cap U_0(u_{11}) \\ &\subset \mathcal{S} + U_0(u_{1j}) \cap U_0(u_{11}) \\ &= \mathcal{S} + U_0(u_{1j}) \cap U_0(u_{11}) \cap U_1(u_{i1}) + U_0(u_{1j}) \cap U_0(u_{11}) \cap U_0(u_{i1}) \\ &\subset \mathcal{S} + U_0(u_{1j}) \cap U_0(u_{11}) \cap U_0(u_{i1}). \end{aligned}$$

Thus, for finite sets $A \subset I, B \subset J$

$$U = \mathcal{S} + U_0(u_{11}) \cap \bigcap_{i \in A, j \in B} [U_0(u_{1j}) \cap U_0(u_{i1})].$$

Let $Q_{A,B} = P_0(u_{11})^* \prod_{j \in B} P_0(u_{1j})^* \prod_{i \in A} P_0(u_{i1})^*$. The net $Q_{A,B}$ converges strongly to a contractive projection Q with range $V := U_0(u_{11}) \cap [\bigcap_{i \in J} U_0(u_{1j})] \cap [\bigcap_{i \in I} U_0(u_{i1})]$. Thus $U = \mathcal{S} + V$, and these last two summands are orthogonal. To prove the orthogonality it suffices to show that $z \diamond u_{pq}$ for each $(p,q) \in I \times J$ and each z in the intersection. For such z, we have $z \diamond u_{p1}$ and $z \diamond u_{1q}$ and since $(u_{p1} + u_{1q}) \vdash u_{pq}$, we have $u_{pq} \in U_2(u_{p1} + u_{1q})$ and $z \in$

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$U_0(u_{p1}) \cap U_0(u_{1q}) = U_0(u_{p1} + u_{1q})$, *i.e.*, $z \diamond u_{pq}$. Since $\{u_{i1}, u_{11}, u_{1j}, u_{ij}\}^\diamond = \{u_{i1}, u_{11}, u_{1j}\}^\diamond$, we have $V = \mathcal{S}^\diamond$.

The proof of the second statement is exactly like the proof of the corresponding statement in Proposition 3.12. ∎

DEFINITION 5.6. A *geometric rectangular grid* is a family of minimal geometric tripotents $\{u_{ps}\}_{(p,s)\in P\times S}$ such that

$$(u_{ps}, u_{qs}, u_{qr}, u_{pr}) \text{ is a geometric quadrangle}$$

whenever $p, q \in P$, $s, r \in S$ satisfy $p \neq q$ and $r \neq s$.

PROPOSITION 5.7. *Let Z satisfy Assumption 5.1 and let $\{u_{ij}\}$ be given by Construction 5.3. Then $\{u_{ij}\}$ is a geometric rectangular grid.*

PROOF. First apply Lemma 4.18 to the geometric quadrangles

$$(u_{11}, u_{1l}, u_{il}, u_{i1}) \text{ and } (u_{11}, u_{1l}, u_{jl}, u_{j1})$$

to obtain the geometric quadrangle $(u_{il}, u_{i1}, u_{j1}, u_{jl})$. Schematically,

$$\begin{matrix} u_{i1} & u_{il} \\ u_{11} & u_{1l} \\ u_{j1} & u_{jl} \end{matrix} \Longrightarrow \begin{matrix} u_{i1} & u_{il} \\ u_{j1} & u_{jl} \end{matrix}.$$

Next apply Lemma 4.18 to the geometric quadrangles

$$(u_{11}, u_{1k}, u_{ik}, u_{i1}) \text{ and } (u_{11}, u_{1k}, u_{jk}, u_{j1})$$

to obtain the geometric quadrangle $(u_{ik}, u_{i1}, u_{j1}, u_{jk})$. Schematically,

$$\begin{matrix} u_{i1} & u_{ik} \\ u_{11} & u_{1k} \\ u_{j1} & u_{jk} \end{matrix} \Longrightarrow \begin{matrix} u_{i1} & u_{ik} \\ u_{j1} & u_{jk} \end{matrix}.$$

Then apply Lemma 4.18 to the geometric quadrangles

$$(u_{i1}, u_{j1}, u_{jl}, u_{il}) \text{ and } (u_{i1}, u_{j1}, u_{jk}, u_{ik})$$

to obtain the geometric quadrangle $(u_{jl}, u_{il}, u_{ik}, u_{jk})$. Schematically,

$$\begin{matrix} u_{i1} & u_{il} & u_{ik} \\ u_{j1} & u_{jl} & u_{jk} \end{matrix} \Longrightarrow \begin{matrix} u_{il} & u_{ik} \\ u_{jl} & u_{jk} \end{matrix}.$$

In the last application of Lemma 4.18 we used the fact that, by Remark 5.4, $u_{il} \top u_{ik}$. ∎

DEFINITION 5.8. Let Z satisfy Assumption 5.1 and let $\{u_{ij}\}$ be a geometric rectangular grid. The *coordinates* $\{x_{ij}\} \subset \mathbf{C}$ of an element $a \in U$ are defined by

$$x_{ij} = \langle a, \hat{u}_{ij} \rangle,$$

and the *coordinates* $\{x_{ij}\} \subset \mathbf{C}$ of an element $f \in Z$ are defined by

$$x_{ij} = \langle f, u_{ij} \rangle.$$

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Note that if f is an extreme point of the unit ball of Z, with coordinates x_{ij}, then $v(f)$ has coordinates $\bar{x}_{ij}$, and if $v \in \mathcal{M}$ has coordinates x_{ij}, then $\hat{v}$ has coordinates $\bar{x}_{ij}$. Also, for arbitrary $a \in \mathcal{S}$, $a = \text{weak}^*\text{-}\lim \sum x_{ij} u_{ij}$, and for arbitrary $f \in Z$, $f = \text{norm-}\lim \sum x_{ij} \hat{u}_{ij}$. Note that, as in Definition 3.8, we are using the same notation, namely x_{ij} for the coordinates of an $a \in U$ and for an $f \in Z$.

LEMMA 5.9. *Let Z satisfy Assumption 5.1 and let $\{u_{ij}\}$ be the geometric rectangular grid given by Construction 5.3. Then for each extreme point f of the unit ball of $\mathcal{S}_*$, with coordinates x_{ij}, we have*

(48)
$$\sum_{ij} |x_{ij}|^2 = 1 \textit{ and } \det \begin{bmatrix} x_{ij} & x_{ik} \\ x_{lj} & x_{lk} \end{bmatrix} = 0 \textit{ for } i,l \in I \cup \{1\},\ i \neq l;\ j,k \in J \cup \{1\},\ j \neq k.$$

Conversely, if $\mathcal{S}$ is of finite rank, i.e., if one or both of the index sets I, J are finite, and if an element $f \in \mathcal{S}_*$ with $\|f\| = 1$ satisfies (48), then f is an extreme point of the unit ball of $\mathcal{S}_*$.

PROOF. If $i \neq l$ and $j \neq k$, then $u_{ij} \diamond u_{lk}$ so $u_{ij} + u_{lk}$ is a geometric tripotent. By ERP, $P_2(u_{ij} + u_{lk})f$ is a multiple of an extreme point of the unit ball of $Z_2(u_{ij} + u_{lk})$. Since $M_2(\mathbf{C}) \cong U_2(u_{ij} + u_{lk}) = \mathrm{sp}\{u_{ij}, u_{ik}, u_{lk}, u_{lj}\}$ canonically, the determinant is 0. Also,

$$1 = \langle f, \pi(f) \rangle = \left\langle \sum x_{ij} \hat{u}_{ij} \sum \bar{x}_{ij} u_{ij} \right\rangle = \sum |x_{ij}|^2.$$

Conversely, suppose that $\mathcal{S}$ is of finite rank, say $J = \{1, \ldots n\}$, let f be an element of $(\mathcal{S}_*)_1$ of norm 1 satisfying (48) and let x_{ij} be the coordinates of f with respect to $\{u_{ij}\}$. Symbolically, we shall write

$$f = \begin{matrix} x_{11} & x_{12} & \cdots & x_{1j} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2j} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ x_{i1} & x_{i2} & \cdots & x_{ij} & \cdots & x_{in} \\ \vdots & \vdots & & \vdots & & \vdots \end{matrix}.$$

There exist p, q such that $x_{pq} \neq 0$. Let $R_1 = S_{\frac{u_{11}+u_{p1}}{\sqrt{2}}}$. The action of R_1 on the (dual) geometric rectangular grid $\{\hat{u}_{ij}\}$ is given by

$$R_1\colon \begin{matrix} \hat{u}_{11} & \hat{u}_{12} & \cdots & \hat{u}_{1q} & \cdots & \hat{u}_{1n} \\ \hat{u}_{21} & \hat{u}_{22} & \cdots & \hat{u}_{2q} & \cdots & \hat{u}_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \hat{u}_{p1} & \hat{u}_{p2} & \cdots & \hat{u}_{pq} & \cdots & \hat{u}_{pn} \\ \vdots & \vdots & & \vdots & & \vdots \end{matrix} \longrightarrow \begin{matrix} \hat{u}_{p1} & -\hat{u}_{p2} & \cdots & -\hat{u}_{pq} & \cdots & -\hat{u}_{pn} \\ -\hat{u}_{21} & \hat{u}_{22} & \cdots & \hat{u}_{2q} & \cdots & \hat{u}_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \hat{u}_{11} & -\hat{u}_{12} & \cdots & -\hat{u}_{1q} & \cdots & -\hat{u}_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix}.$$

https://doi.org/10.4153/CJM-1993-004-0 Published online by Cambridge University Press

Let $f_1 = R_1 f$ have coordinates x^1_{ij}, so that $x^1_{1q} \neq 0$. With $R_2 = S_{\frac{u_{11}+u_{1q}}{\sqrt{2}}}$ it is clear that $f_2 := R_2 f_1$ has coordinates x^2_{ij} with $x^2_{11} \neq 0$ and f_2 satisfies (48). We now apply Corollary 4.17 to the geometric quadrangle $(u_{11}, u_{1n}, u_{in}, u_{i1})$ and the element $g_i = x^2_{11}\hat{u}_{11} + x^2_{1n}\hat{u}_{1n} + x^2_{in}\hat{u}_{in} + x^2_{i1}\hat{u}_{i1}$. We obtain a symmetry R_3, independent of i such that

$$R_3 f_2 = \begin{matrix} \lambda x^2_{11} & -x^2_{12} & \cdots & -x^2_{1,n-1} & 0 \\ -\lambda x^2_{21} & x^2_{22} & \cdots & x^2_{2,n-1} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -\lambda x^2_{j1} & x^2_{j2} & \cdots & x^2_{j,n-1} & 0 \\ \vdots & \vdots & & \vdots & \vdots \end{matrix}.$$

We note that $R_3 f_2$ also satisfies (48), and we repeat this process a finite number of times to obtain an isometry T such that

$$Tf = \begin{matrix} \tilde{x}_{11} & 0 & \cdots & 0 \\ \tilde{x}_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \tilde{x}_{i1} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \end{matrix}.$$

This is an extreme point of the unit ball of Z, since it is a unit vector in the Hilbert space $\mathcal{H}_2$. It follows that $f = T^{-1}(Tf)$ is an extreme point. ∎

THEOREM 5.10. *Let Z be an atomic neutral* SFS *space of spin degree 4 which satisfies* FE, STP, ERP, *and* JP. *Then Z has an L-summand which is linearly isometric to the predual of a Cartan factor of type 1. In particular, if Z is irreducible, then Z^* is isometric to a Cartan factor of type 1.*

PROOF. Let $\{u_{ij}\}$ be the geometric rectangular grid given by Construction 5.3 from an arbitrary geometric quadrangle. Let $\{v_{ij}\}$ be the rectangular grid over the same index sets which is known to exist in a Cartan factor V of type 1. Then $Y := V_*$ satisfies Assumption 5.1 and $\{v_{ij}\}$ can be chosen to be the geometric rectangular grid obtained by Construction 5.3 from the quadrangle $(v_{11}, v_{12}, v_{22}, v_{21})$. The correspondence $\hat{u}_{ij} \longmapsto \hat{v}_{ij}$ extends to a linear map κ of a norm dense subset of $\mathcal{S}$ onto a norm dense subset of Y.

Suppose first that the space $\mathcal{S}$ is of finite rank. Then, by Lemma 5.9, extreme points are mapped to extreme points by κ, so κ is contractive. By symmetry so is κ^{-1}. Therefore κ is isometric, proving the theorem in this case.

Suppose now that $\mathcal{S}$ is of arbitrary dimension and infinite rank. In this case, any element φ in the span of the dual rectangular grid $\{\hat{u}_{ij}\}$ belongs to a finite dimensional subspace of the form $\mathrm{sp}\{\hat{u}_{ij}\}_{i\in A, j\in B}$ for suitable finite subsets $A \subset I$, $B \subset J$ of the same finite cardinality. Since this subspace is of the form $Z_2(u_{11} + \cdots u_{nn})$ and satisfies the four PSP's, it follows from the finite dimensional case just proved that $\|\kappa(\varphi)\| = \|\varphi\|$. By the density of this span, the proof is completed. ∎

https://doi.org/10.4153/CJM-1993-004-0 Published online by Cambridge University Press

6. The type 2 case. In this section we consider facially symmetric spaces of spin degree 6. We will show that, under appropriate hypotheses, such a space is isometric to a neutral strongly facially symmetric space containing the predual of a Cartan factor of type 2 as an L-summand.

For a suitable minimal geometric tripotent v, we shall show that $U_1(v)$ is a Cartan factor of type 1 and rank 2, and using a rectangular grid in $U_1(v)$, we construct a family of geometric tripotents which we show to be a geometric symplectic grid. Finally, we show that the natural map from the span of the dual of this grid to the span of the dual of a concrete symplectic grid is isometric so extends linearly to an isometry onto the predual of a Cartan factor of type 2.

ASSUMPTION 6.1. *Z is an atomic neutral strongly facially symmetric space of spin degree 6 which satisfies* FE, STP, ERP, *and* JP, *and $v, \tilde{v}$ are a pair of orthogonal minimal geometric tripotents for which $U_2(v+\tilde{v})$ is of dimension 6 and $U_1(v+\tilde{v}) \neq \{0\}$.*

LEMMA 6.2. *Let Z satisfy Assumption 6.1. Then $U_1(v)$ is isometric to a Cartan factor of type 1 and rank 2.*

PROOF. By Proposition 4.8, $U_1(v)$ has rank ≤ 2, and by Proposition 4.20, v can be chosen such that $Z_1(v)$ has spin degree 4. Let $v_2, \tilde{v}_2, v_3, \tilde{v}_3$ be chosen such that $v, \tilde{v}, v_2, \tilde{v}_2, v_3, \tilde{v}_3$ is a spin grid for $U_2(v+\tilde{v})$. Since $v_2, \tilde{v}_2, v_3, \tilde{v}_3$ belong to $U_1(v)$ and their span is isometric to $M_2(\mathbf{C})$, the space $U_1(v)$ is of rank 2. By Corollary 4.12, Remark 4.6, and Lemma 2.8, $Z_1(v)$ satisfies the hypotheses of Theorem 5.10. Thus $U_1(v)$ has an M-summand isometric to a Cartan factor of type 1. This Cartan factor is of rank at least 2 since it contains a geometric quadrangle. Therefore the above mentioned M-summand coincides with $U_1(v)$ and the lemma follows. ■

CONSTRUCTION 6.3. Let Z satisfy Assumption 6.1 and define $u_{12} = v$. Using Lemma 6.2, let $\{u_{1j}, u_{2j} : j \in \tilde{J}\}$ be a (geometric) rectangular grid for $U_1(v)$, where $\tilde{J}$ is an index set not containing 1 or 2. Let $u_{j1} := -u_{1j}$ and $u_{j2} := -u_{2j}$ for $j \in \tilde{J}$. For $i, j \in \tilde{J}, i \neq j$, apply Lemma 4.15 to u_{i2}, u_{12}, u_{1j} to obtain u_{ij} such that

$$(u_{i2}, u_{12}, u_{1j}, u_{ij}) \tag{49}$$

is a geometric quadrangle.

Schematically,

$$\begin{array}{ll} u_{12} & u_{1j} \\ u_{i2}\ (= -u_{2i}) & u_{ij}. \end{array}$$

Note that, since $(-u_{2i}, u_{12}, u_{1j}, u_{ij})$ is a (geometric) quadrangle, $(u_{2i}, u_{12}, u_{1j}, u_{ij})$ is an odd (geometric) quadrangle. The odd quadrangles (which are obtained from quadrangles by a single sign change) are the appropriate building blocks for spin factors (*cf.* [8, Corollary, p. 313]). Note also that $\{u_{1j}\}_{j\in\tilde{J}}$ and $\{u_{2j}\}_{j\in\tilde{J}}$ are each an orthonormal basis for a Hilbert space, by well-known properties of Cartan factors of type 1 and rank 2.

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DEFINITION 6.4. A *geometric symplectic grid* is a family of minimal geometric tripotents $\{u_{ij}\}_{i,j\in J, i\neq j}$, where $J = \{1,2\} \cup \tilde{J}$ and $\tilde{J}$ does not contain 1 or 2, such that $u_{ij} = -u_{ji}$ for all $i, j \in J$ and

$$(u_{jl}, u_{il}, u_{ik}, u_{jk}) \text{ is a geometric quadrangle}$$

whenever i, j, k, l are all distinct.

Note that if some pair of indices are equal, then either one of u_{pq} is not defined, being of the form u_{pp}, or the elements coincide in pairs.

In the following proposition we shall use the fact that for a spin grid $u_1, \tilde{u}_1; u_2, \tilde{u}_2; \ldots,$ $(u_i, u_j, \tilde{u}_i, -\tilde{u}_j)$ is a quadrangle for $i \neq j$ ([8, Corollary, p. 313]).

PROPOSITION 6.5. *Let Z satisfy Assumption 6.1 and let $\{u_{ij}\}$ be given by Construction 6.3. Then $\{u_{ij}\}$ is a geometric symplectic grid.*

PROOF. We first show that whenever i, j, k, l are all distinct,

$$(u_{il}, u_{ik}, u_{jk}, u_{jl}) \text{ is a geometric quadrangle.} \tag{50}$$

First apply Lemma 4.18 to the geometric quadrangles

$$(u_{12}, u_{1l}, u_{il}, u_{i2}) \text{ and } (u_{12}, u_{1l}, u_{jl}, u_{j2})$$

to obtain the geometric quadrangle $(u_{il}, u_{i2}, u_{j2}, u_{jl})$. Schematically,

$$\begin{matrix} u_{12} & u_{1l} \\ u_{i2} & u_{il} \\ u_{j2} & u_{jl} \end{matrix} \Longrightarrow \begin{matrix} u_{i2} & u_{il} \\ u_{j2} & u_{jl} \end{matrix}$$

Next apply Lemma 4.18 to the geometric quadrangles

$$(u_{12}, u_{1k}, u_{ik}, u_{i2}) \text{ and } (u_{12}, u_{1k}, u_{jk}, u_{j2})$$

to obtain the geometric quadrangle $(u_{ik}, u_{i2}, u_{j2}, u_{jk})$. Schematically,

$$\begin{matrix} u_{12} & u_{1k} \\ u_{i2} & u_{ik} \\ u_{j2} & u_{jk} \end{matrix} \Longrightarrow \begin{matrix} u_{i2} & u_{ik} \\ u_{j2} & u_{jk} \end{matrix}$$

Note that $u_{il} \top u_{ik}$. To see this, let $S := \frac{u_{12}+u_{1l}}{\sqrt{2}}$. Since $u_{ik} \top u_{i2}$ we have $S u_{ik} \top (-u_{il})$. But $u_{ik} \diamond u_{12}$ and $u_{ik} \diamond u_{1l}$, so $S u_{ik} = u_{ik}$. Hence we may again apply Lemma 4.18 to the geometric quadrangles

$$(u_{il}, u_{i2}, u_{j2}, u_{jl}) \text{ and } (u_{ik}, u_{i2}, u_{j2}, u_{jk})$$

to obtain the geometric quadrangle $(u_{jl}, u_{il}, u_{ik}, u_{jk})$. This proves (50). Schematically,

$$\begin{matrix} u_{i2} & u_{il} & u_{ik} \\ u_{j2} & u_{jl} & u_{jk} \end{matrix} \Longrightarrow \begin{matrix} u_{il} & u_{ik} \\ u_{jl} & u_{jk} \end{matrix}$$

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We next show that $u_{ij} = -u_{ji}$ for $i, j \in \tilde{J}$. By Corollary 2.7, $U_2(u_{1j}+u_{i2}) = U_2(u_{12}+u_{ij})$ and thus

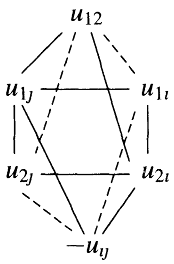
$$\begin{aligned} U_2(u_{1j} + u_{i2}) &= U_2(u_{12} + u_{ij}) = \mathbf{C}u_{12} + \mathbf{C}u_{ij} + U_1(u_{12}) \cap U_1(u_{ij}) \\ &\subset \mathbf{C}u_{12} + \mathbf{C}u_{ij} + U_1(u_{12}) \cap U_2(u_{1j} + u_{i2}) \subset U_2(u_{1j} + u_{i2}), \end{aligned}$$

so equality holds.

By [11, Remark 1.1] there is an isometry of $U_1(u_{12})$ which is a finite product of symmetries and scalar multiplications, taking u_{13} into u_{1j} and u_{24} into u_{2i} respectively (for details of this, see step 2 of the proof of Lemma 6.9). Therefore, $U_2(u_{1j} + u_{2i})$ has dimension 6 and hence $U_1(u_{12}) \cap U_2(u_{1j} + u_{i2})$ is four dimensional. Since $U_1(u_{12})$ is spanned by the rectangular grid $\{u_{1k}, u_{2k} : k \in \tilde{J}\}$, $U_1(u_{12}) \cap U_2(u_{1j} + u_{i2})$ is spanned by the elements of the geometric quadrangle $(u_{2j}, u_{1j}, u_{1i}, u_{2i})$. Indeed, it obviously contains u_{1j} and u_{2i}, and by JP, $U_2(u_{1j} + u_{2i}) \supset U_1(u_{1j}) \cap U_1(u_{i2})$ so it also contains u_{1i} and u_{2j}. Since $U_2(u_{1j} + u_{i2})$ is isometric to a spin factor, with quadrangles corresponding to geometric quadrangles, we can apply [8, Lemma 1.8] to the pair of *concrete* quadrangles (see the diagram below)

$$(u_{2j}, u_{1j}, u_{1i}, u_{2i}) \text{ and } (u_{12}, u_{1j}, -u_{ij}, -u_{i2}) = (u_{12}, u_{1j}, -u_{ij}, u_{2i}),$$

in which $u_{2j} \top u_{12}$, to obtain a geometric quadrangle $\big(u_{2j}, u_{12}, u_{1i}, -(-u_{ij})\big)$.



On the other hand, by definition, u_{ji} is the completion of the geometric prequadrangle (u_{j2}, u_{12}, u_{1i}). Thus $(u_{2j}, u_{12}, u_{1i}, -u_{ji})$ is a geometric quadrangle. Thus $u_{ij} = -u_{ji}$. ∎

LEMMA 6.6. *Let Z satisfy Assumption 6.1 and let $\{u_{ij}\}$ be the geometric symplectic grid given by Construction 6.3. Then with $\mathcal{S} = \overline{\mathrm{sp}}_{\mathbf{C}}^{\mathrm{weak}^*}\{u_{ij}\}_{i,j \in J, i \neq j}$, we have*

$$U = \mathcal{S} \oplus^{\ell^\infty} \mathcal{S}^\diamond.$$

The family $\{\hat{u}_{ij}\}$ is norm total in $\mathcal{S}_*$, the predual of $\mathcal{S}$, and $Z = \mathcal{S}_* \oplus^{\ell^1} (\mathcal{S}_*)^\diamond$.

PROOF. We first show that for $p \neq 2$, $U_1(u_{1p}) \subset \mathcal{S}$. Indeed, with $S_1 := S_{\frac{u_{12}+u_{1p}}{\sqrt{2}}}$, $S_1^*(u_{12}) = u_{1p}$ and $S_1^*(u_{1j}) = -u_{1j}$ for $j \neq 2, j \neq p$. Moreover, $S_1^*(u_{2k}) = -u_{kp}$ for $k \neq p$ by Corollary 4.16, and $S_1^*(u_{2p}) = -u_{2p}$ since $u_{2p} \top u_{1p}$ and $u_{2p} \top u_{12}$.

Thus

$$U_1(u_{1p}) = S_1^* U_1(u_{12}) = S_1^*(\overline{\mathrm{sp}}\{u_{1k}, u_{2k}\}_{k \in \tilde{J}}) = \overline{\mathrm{sp}}(\{u_{1k}\}_{k \in \{2\} \cup \tilde{J} \setminus \{p\}} \cup \{u_{kp}\}_{k \in \tilde{J}}).$$

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Similarly, we use $S_2 := S_{\frac{u_{12}+u_{2q}}{\sqrt{2}}}$ to show that $U_1(u_{2q}) \subset \mathcal{S}$. As in the proof of Lemma 5.5, we have, for any finite set $B \subset J$

$$U = \mathcal{S} + U_0(u_{12}) \cap \bigcap_{j \in B} U_0(u_{1j}),$$

and for any finite subset $A \subset J$

$$U = \mathcal{S} + U_0(u_{12}) \cap \Big(\bigcap_{j \in B} U_0(u_{1j})\Big) \cap \Big(\bigcap_{i \in A} U_0(u_{2i})\Big).$$

Let $Q_{A,B} = P_0(u_{12})^* \prod_{j \in B} P_0(u_{1j})^* \prod_{i \in A} P_0(u_{2i})^*$. The net $Q_{A,B}$ converges strongly to a contractive projection Q with range $V := U_0(u_{12}) \cap [\bigcap_{j \in \tilde{J}} U_0(u_{1j})] \cap [\bigcap_{i \in \tilde{J}} U_0(u_{2i})]$. Thus $U = \mathcal{S} + V$, and these last two summands are orthogonal. To prove this orthogonality it suffices to show that $z \diamond u_{pq}$ for each $p, q \in \tilde{J}$ and each $z \in V$. For such z, we have $z \diamond u_{1p}$ and $z \diamond u_{2q}$ and since $(u_{1p} + u_{q2}) \vdash u_{pq}$, we have $z \diamond u_{pq}$. Since $\{u_{2i}, u_{12}, u_{1j}, u_{ij}\}^\diamond = \{u_{2i}, u_{12}, u_{1j}\}^\diamond$, we have $V = \mathcal{S}^\diamond$.

The proof of the second statement is exactly like the proof of the corresponding statement in Proposition 3.12. ∎

LEMMA 6.7. *Let Z satisfy Assumption 6.1 and let $\{u_{ij}\}$ be the geometric symplectic grid given by Construction 6.3. Then for each extreme point f of the unit ball of $\mathcal{S}_*$, with coordinates $x_{ij} = \langle f, u_{ij} \rangle$, we have $\sum_{i,j} |x_{ij}|^2 = 1$ and for distinct $\{i, j, k, l\}$,*

$$x_{ij}x_{lk} + x_{kj}x_{il} + x_{ik}x_{jl} = 0. \tag{51}$$

PROOF. By ERP, $P_2(u_{ij} + u_{lk})f$ is a multiple of an extreme point of Z_1. Since $U_2(u_{ij} + u_{lk}) = \mathrm{sp}\{u_{ij}, u_{ik}, u_{lk}, u_{lj}, u_{jk}, u_{il}\}$ is isometric to a spin factor, the determinant in that spin factor is zero (*cf.* [8, Proposition 3.3]). Since $u_{ij}, u_{lk}; u_{ik}, -u_{lj}; u_{il}, -u_{jk}$ is a spin grid for that spin factor, (51) holds. Also,

$$1 = \langle f, \pi(f) \rangle = \Big\langle \sum x_{ij}\hat{u}_{ij} \sum \bar{x}_{ij}u_{ij} \Big\rangle = \sum |x_{ij}|^2.$$

∎

In the proof of the following theorem, $\mathcal{H}_Z$ denotes the Hilbert space which is the completion of $\mathcal{S}_*$ with respect to the norm given by the symmetric sesquilinear form $\langle \cdot \mid \cdot \rangle$ defined in [16, Proposition 2.9]. By Lemma 6.6, this form is positive definite, and so $\mathcal{H}_Z$ exists.

THEOREM 6.8. *Let Z be an atomic neutral* SFS *space of spin degree 6 which satisfies* FE, STP, ERP, *and* JP. *Then Z has an L-summand which is linearly isometric to the predual of a Cartan factor of type 2. In particular, if Z is irreducible, then Z^* is isometric to a Cartan factor of type 2.*

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PROOF. Let $\{u_{ij}\}$ be the geometric symplectic grid given by Construction 6.3. Let $\{v_{ij}\}$ be the symplectic grid over the same index sets which is known to exist in a Cartan factor V of type 2. Then $Y := V_*$ also satisfies Assumption 6.1 and $\{v_{ij}\}$ can be chosen to be the geometric symplectic grid given by Construction 6.3. The correspondence $\hat{u}_{ij} \longmapsto \hat{v}_{ij}$ extends to a linear map κ of a dense subset of $\mathcal{S}_*$ onto a dense subset of Y and to a unitary map $\tilde{\kappa}$ of the Hilbert space $\mathcal{H}_Z$ onto $\mathcal{H}_Y$.

We shall use the following lemma to complete the proof of the theorem.

LEMMA 6.9. *If f is an extreme point of the unit ball of $\mathcal{S}_*$, then $\tilde{\kappa}(f)$, which is* a priori *an element of $\mathcal{H}_Y$, actually belongs to Y and is an extreme point of Y_1.*

PROOF. Let f be an extreme point of $(\mathcal{S}_*)_1$ and let x_{ij} be the coordinates of f with respect to $\{u_{ij}\}$.

STEP 1. If $x_{12} = 0$, then there exist p, q such that $x_{pq} \neq 0$. Let $R_1 = S_{\frac{u_{12}+u_{p2}}{\sqrt{2}}}$. As in the proof of Lemma 6.6 for $S_{\frac{u_{12}+u_{1p}}{\sqrt{2}}}$, the action of R_1 on the (dual) geometric symplectic grid $\{\hat{u}_{ij}\}$ is the following:

$$\begin{array}{lll} R_1\hat{u}_{12} = -\hat{u}_{2p} & R_1\hat{u}_{1j} = -\hat{u}_{pj} \quad (j \neq 2) & R_1\hat{u}_{2k} = -\hat{u}_{2k} \quad (k \neq p) \\ R_1\hat{u}_{ij} = \hat{u}_{ij} \quad (i \neq p, j \neq p) & R_1\hat{u}_{pj} = -\hat{u}_{1j} & R_1\hat{u}_{ip} = \hat{u}_{1i}. \end{array}$$

Since this amounts to a permutation with some changes of sign, it is clear that if we set $T_1 = S_{\frac{v_{12}+v_{p2}}{\sqrt{2}}}$, then

$$T_1 \circ \tilde{\kappa} \circ R_1 = \tilde{\kappa} \text{ on } \mathcal{H}_Z. \tag{52}$$

If $f_1 := R_1 f$ has coordinates x^1_{ij}, then $x^1_{1q} \neq 0$. With $R_2 = S_{\frac{u_{12}+u_{1q}}{\sqrt{2}}}$, and $T_2 = S_{\frac{v_{12}+v_{1q}}{\sqrt{2}}}$, it is also clear that

$$T_2 \circ \tilde{\kappa} \circ R_2 = \tilde{\kappa} \text{ on } \mathcal{H}_Z, \tag{53}$$

and that $f_2 := R_2 f_1$ has coordinates x^2_{ij} with $x^2_{12} \neq 0$.

If $x_{12} \neq 0$, define $f_2 := f$ and $x^2_{ij} := x_{ij}$.

STEP 2. Let

$$P_1(u_{12})f_2 = s_1\tilde{u}_{13} + s_2\tilde{u}_{24}$$

be the spectral decomposition of $P_1(u_{12})f_2$ in the rank 2 Cartan factor $U_1(u_{12})$ of type 1. By [11, Remark 1.1], there is a geometric tripotent w_1 and a complex number λ_1 such that, with $R_3 := \lambda_1 S_{w_1}$, we have $R_3^* u_{13} = \tilde{u}_{13}$. Let $u'_{24} = R_3^* u_{24}$. Again choose a geometric tripotent w_2 and complex number λ_2 such that, with $R_4 := \lambda_2 S_{w_2}$, we have $R_4^* u'_{24} = \tilde{u}_{24}$. Note that $R_4^*\tilde{u}_{13} = \lambda_2\tilde{u}_{13}$. Let $f_4 := R_4 R_3 f_2$ and denote the coordinates of f_4 by x^4_{ij}. Then $x^4_{1j} = 0$ for $j \geq 4$, $x^4_{23} = 0$, and $x^4_{2j} = 0$ for $j \geq 5$.

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STEP 3. Since f_4 is an extreme point, by Lemma 6.7, for all $2 < i < j$, $j \geq 5$, we have

$$x^4_{12}x^4_{ij} + x^4_{j2}x^4_{1i} + x^4_{1j}x^4_{2i} = 0,$$

which implies $x^4_{ij} = 0$ in this case. Thus

$$f_4 = x^4_{12}u_{12} + x^4_{13}u_{13} + x^4_{24}u_{24} + x^4_{34}u_{34}.$$

Let T_3 and T_4 be the symmetries on Y which are constructed in the same way as R_3 and R_4. We shall show below that

$$T_3 \circ \tilde{\kappa} \circ R_3(f_2) = \tilde{\kappa}(f_2), \tag{54}$$

and

$$T_4 \circ \tilde{\kappa} \circ R_4(f_3) = \tilde{\kappa}(f_3). \tag{55}$$

We can now complete the proof of the lemma assuming (54) and (55). The element $\kappa(f_4)$ belongs to $Y_2(v_{12} + v_{34})$, which is isometric to the predual of a (six dimensional) spin factor. Since its (spin) determinant is 0, $\kappa(f_4)$ is extremal in $Y_2(v_{12} + v_{34})$, and by neutrality, it is extremal in the unit ball of Y. By (52)–(55), $\tilde{\kappa}(f) = T_1T_2T_3T_4\tilde{\kappa}(f_4)$ is extremal, completing the proof of the lemma under the assumptions (54) and (55).

STEP 4. We return to the proofs of (54) and (55). We shall prove (54), the proof of (55) being similar. For this, consider a symmetry $R = S_t$, with t a minimal geometric tripotent of $U_1(u_{12})$. For example, $R = R_3$ or $R = R_4$ above. The symmetry R has the following properties, for suitable scalars α_{ijkl}:

- $R\hat{u}_{12} = -\hat{u}_{12}$ (since $u_{12} \top t$)
- For $j \neq 2$, $R\hat{u}_{1j} = \sum_{l\neq 2, k=1,2} \alpha_{1jkl}\hat{u}_{kl}$ (since $R(Z_1(u_{12}) = Z_1(u_{12}))$
- For $j \neq 1$, $R\hat{u}_{2j} = \sum_{l\neq 2, k=1,2} \alpha_{2jkl}\hat{u}_{kl}$ (since $R(Z_1(u_{12}) = Z_1(u_{12}))$
- For $i \geq 3$, $j \geq 4$, $R\hat{u}_{ij} = \sum_{k\neq 3, l\geq 4} \alpha_{ijkl}\hat{u}_{kl}$ (since $u_{ij} \diamond u_{12}$ and $u_{ij} \diamond \mathcal{S}^{\diamond}$).

For $i \neq j, i \neq 2, j \neq 2, \rho := \frac{1}{2}(\hat{u}_{12}+\hat{u}_{1j}+\hat{u}_{i2}+\hat{u}_{ij})$ is an extreme point of Z_1. Symbolically, if we write

$$f = \begin{array}{cccccc} x_{12} & x_{13} & \cdots & x_{1j} & \cdots \\ & x_{23} & \cdots & x_{2j} & \cdots \\ \vdots & \vdots & & \vdots & \\ & & \cdots & x_{ij} & \cdots \\ \vdots & \vdots & & \vdots & \end{array},$$

then

$$2R(\rho) = \begin{array}{ccccc} -1 & \alpha_{1j13} - \alpha_{2i13} & \cdots & \alpha_{1j1l} - \alpha_{2i1l} & \cdots \\ & \alpha_{1j23} - \alpha_{2i23} & \cdots & \alpha_{1j2l} - \alpha_{2i2l} & \cdots \\ & & \alpha_{ij34} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ & & \cdots & \alpha_{ijkl} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}.$$

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From Lemma 6.7, for $i \geq 3$, $j \geq 4$, we have

$$(-1)\alpha_{ijkl} + (\alpha_{1j2k} - \alpha_{2i2k})(\alpha_{1j1l} - \alpha_{2i1l}) + (\alpha_{1j1k} - \alpha_{2i1k})(\alpha_{1j2l} - \alpha_{2i2l}) = 0,$$

i.e.,

$$\alpha_{ijkl} = (\alpha_{1j2k} - \alpha_{2i2k})(\alpha_{1j1l} - \alpha_{2i1l}) + (\alpha_{1j1k} - \alpha_{2i1k})(\alpha_{1j2l} - \alpha_{2i2l}). \tag{56}$$

The element s in $V_1(v_{12})$ with the same coordinates (obvious definition) with respect to v_{ij} as t has with respect to u_{ij} determines a symmetry T (for example $T = T_3$ or $T = T_4$ above) on Y and numbers β_{ijkl} which satisfy

$$\beta_{ijkl} = (\beta_{1j2k} - \beta_{2i2k})(\beta_{1j1l} - \beta_{2i1l}) + (\beta_{1j1k} - \beta_{2i1k})(\beta_{1j2l} - \beta_{2i2l}). \tag{57}$$

Since $\tilde{\kappa}$ is an isometry of the Cartan factor $U_1(u_{12})$ of type 1 onto the Cartan factor $V_1(v_{12})$ of type 1,

$$\alpha_{pqrs} = \beta_{pqrs} \text{ whenever } r \in \{1,2\}. \tag{58}$$

From (58), (56) and (57) it follows that

$$\alpha_{ijkl} = \beta_{ijkl} \text{ for all } i, j, k, l. \tag{59}$$

We now prove (54), which is to say

$$\langle T_3\tilde{\kappa}R_3(f_2), v_{ij}\rangle = \langle f_2, u_{ij}\rangle. \tag{60}$$

We have, with $R = R_3$ and $T = T_3$,

$$\tilde{\kappa}R_3(f_2) = \sum_{k,l}\Big(\sum_{i,j} x_{ij}^2\alpha_{ijkl}\Big)\hat{v}_{kl} \tag{61}$$

and

$$T_3\tilde{\kappa}(f_2) = T_3\Big(\sum_{i,j} x_{ij}^2\hat{v}_{ij}\Big) = \sum_{k,l}\Big(\sum_{i,j} x_{ij}^2\beta_{ijkl}\Big)\hat{v}_{kl}. \tag{62}$$

By (59), (61) and (62), we have (60), and hence (54). The proof of (55) uses $R = R_4$, $T = T_4$. This completes the proof of the lemma. ∎

We now complete the proof of the theorem. Suppose first that the space $\mathcal{S}$ is finite dimensional. Then since extreme points are mapped to extreme points by κ, by finite dimensionality, κ is contractive. By symmetry so is κ^{-1}. Therefore κ is isometric, proving the theorem.

Suppose now that $\mathcal{S}$ is of arbitrary dimension. Since any element φ in the span of $\hat{u}_{ij}$ belongs to a neutral SFS subspace of the form $U_2(u_{12} + \cdots + u_{2n-1,2n})$, and since by the neutrality of $P_2(u_{12} + \cdots + u_{2n-1,2n})$, the PSP's are satisfied in $Z_2(u_{12} + \cdots + u_{2n-1,2n})$, by the finite dimensional case just proved $\|\kappa(\varphi)\| = \|\varphi\|$. By the density of this span, the proof is completed. ∎

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7. The type 5 and 6 cases. In this section we consider facially symmetric spaces of spin degree 8 (resp. 10). We will show that, under the usual hypotheses, the dual of such a space has an M-summand which is isometric to a Cartan factor of type 5 (resp. 6). In §8 we shall see that there are no neutral strongly facially symmetric spaces of spin degree greater than 10 which have no L-summands of type I_2, thereby enabling the completion of the classification of atomic facially symmetric spaces.

7.1. *The type 5 case.* In this subsection, we shall prove the following theorem.

THEOREM 7.1. *Let Z be an atomic neutral* SFS *space which satisfies* FE, STP, ERP, *and* JP, *and let $v, \tilde{v}$ be orthogonal minimal geometric tripotents in $U := Z^*$ such that the dimension of $U_2(v+\tilde{v})$ is 8 and $U_1(v+\tilde{v}) \neq \{0\}$. Then there is an L-summand of Z which is isometric to the predual of a Cartan factor of type 5, i.e., the 16 dimensional* JBW**-triple of 1 by 2 matrices over the Octonions. In particular, if Z is irreducible, then Z^* is isometric to the Cartan factor of type 5.*

PROOF. By Proposition 4.20, v can be chosen in such a way that $U_1(v)$ is of spin degree 6 and is not of type I_2. Thus, by Theorem 6.8, $U_1(v)$ has an M-summand $\mathcal{S}$ which is isometric to a Cartan factor of type 2. Because $U_1(v)$ has rank at most 2 and $\mathcal{S}$ contains a geometric quadrangle, $U_1(v)$ is a Cartan factor of type 2 and rank 2. Since $U_1(v)$ is not of type I_2, it must be isometric to $A_5(\mathbf{C})$, the JBW*-triple of all 5 by 5 anti-symmetric complex matrices. Let $\Phi = \{1,2,3,4,5\}$ and $\{u_{ij}\}_{i\neq j, i,j\in\Phi}$ be a symplectic grid for $U_1(v)$. For each $m \in \Phi$, there is, by Lemma 4.15, a unique minimal geometric tripotent v_m such that, with $\Phi = \{m,i,j,k,l\}$,

$$v, v_m;\ u_{ij}, u_{lk};\ u_{jk}, u_{il};\ u_{ik}, u_{jl}, \tag{63}$$

is, up to signs, a spin grid for $U_2(u_{ij}+u_{kl})$.

PROPOSITION 7.2. *For each $m \in \Phi$, the following is, up to signs, a spin grid:*

$$v_i, u_{im};\ v_j, u_{jm};\ v_k, u_{km};\ v_l, u_{lm}. \tag{64}$$

PROOF. This follows from an application of Lemma 4.18 to the geometric quadrangles $(v, u_{lk}, -v_j, u_{im})$ and (v, u_{lk}, v_i, u_{jm}). Schematically,

$$\begin{matrix} u_{im} & v & u_{jm} \\ -v_j & u_{lk} & v_i \end{matrix}$$

∎

We remark that the linear span of each of the spin grids (63) and (64) is an 8-dimensional complex spin factor and is therefore isometric with the Cayley algebra, or Octonions, in the spectral norm (*cf.* [17]).

Note that since the four digits occurring in a geometric quadrangle are distinct, each quadrangle from the symplectic grid $\{u_{ij}\}$ is determined uniquely, up to equivalence (*i.e.*, having the same span) by an index $m \in \Phi$.

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LEMMA 7.3. *$U = \mathcal{S} \oplus^{\ell^\infty} \mathcal{S}^\diamond$, where $\mathcal{S}$ is the linear span of $\{u_{ij}\}_{i,j\in\Phi, i\neq j} \cup \{v, v_1, v_2, v_3, v_4, v_5\}$.*

Schematically, we can depict this basis as

$$\begin{array}{c|ccccc} & u_{12} & u_{13} & u_{14} & u_{15} & \\ & & u_{23} & u_{24} & u_{25} & \\ v & & & u_{34} & u_{35} & \\ & & & & u_{45} & \\ \hline & v_1 & v_2 & v_3 & v_4 & v_5 \end{array}$$

PROOF. Suppose we have shown

(65) $$U_1(u_{kl}) \subset \mathcal{S} \text{ for all } k, l \in \Phi.$$

By the mutual compatibility of the family $\{u_{ij}\}$,

$$U = \mathcal{S} + U_0(v) \cap \Big[\bigcap_{k,l\in\Phi, k\neq l} U_0(u_{kl}) \Big]$$

and these two summands are orthogonal.

To show the orthogonality let $z \in \mathcal{S}$ and $y \in U_0(v) \cap [\bigcap_{k,l} U_0(u_{kl})]$. If $z = u_{ij}$ or v, then $z \diamond y$ is trivially true. If $z = v_m$, then $z \in U_2(u_{ij}+u_{kl})$ and $y \in U_0(u_{ij}) \cap U_0(u_{kl}) = U_0(u_{ij}+u_{kl})$, proving the orthogonality.

To prove (65), note first that since v and u_{kl} belong to a spin grid, the symmetry $S := S_{\frac{v+u_{kl}}{\sqrt{2}}}$ exists. Since $S^*v = u_{kl}$, it suffices to prove that $S^*U_1(v) \subset \mathcal{S}$. For this it is enough to show $S^*u_{ij} \in \mathcal{S}$.

CASE 1. $\{i, j\} \cap \{k, l\} = \emptyset$; then, by Corollary 4.16, $S^*u_{ij} = -v_m \in \mathcal{S}$.

CASE 2. $\{i, j\} \cap \{k, l\} = \{k\}$, say $i = k$; then $u_{kj} \top u_{kl}$ and $u_{kj} \top v$ so that $(\frac{v+u_{kl}}{\sqrt{2}}) \top u_{kj}$ and thus $S^*u_{kj} = -u_{kj} \in \mathcal{S}$.

CASE 3. $\{i, j\} \cap \{k, l\} = \{k, l\}$; then $S^*u_{kl} = v \in \mathcal{S}$.

Finally, since $\{v, u_{ij}, u_{kl}, v_m\}^\diamond = \{v, u_{ij}, u_{kl}\}^\diamond$, we have $U = \mathcal{S} \oplus^{\ell^\infty} \mathcal{S}^\diamond$. ∎

LEMMA 7.4. *For each extreme point f of the unit ball of $\mathcal{S}_*$, with coordinates*

$$x_0, \quad x_m \ (1 \leq m \leq 5), \quad x_{ij} \ (i, j \in \Phi, i \neq j),$$

i.e.,

$$f = \sum_{i,j} x_{ij}\hat{u}_{ij} + x_0\hat{v} + \sum_{j=1}^{5} x_j\hat{v}_j,$$

we have

(66) $$\sum_{j=0}^{5} |x_j|^2 + \sum_{i,j} |x_{ij}|^2 = 1$$

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and for each $m \in \Phi$, writing $\Phi = \{m, i, j, k, l\}$,

$$x_0 x_m + \epsilon_{ijlk} x_{ij} x_{lk} + \epsilon_{jkil} x_{jk} x_{il} + \epsilon_{ikjl} x_{ik} x_{jl} = 0 \tag{67}$$

and

$$x_i x_{im} + \epsilon_{jm} x_j x_{jm} + \epsilon_{km} x_k x_{km} + \epsilon_{lm} x_l x_{lm} = 0, \tag{68}$$

where the ϵ_{pq} and ϵ_{pqrs} belong to $\{-1, 1\}$.

PROOF. By ERP, $P_2(u_{ij} + u_{kl})f$ is a multiple of an extreme point of Z_1. Since $U_2(u_{ij} + u_{kl}) = \mathrm{sp}\{v, v_m, u_{ij}, u_{ik}, u_{kl}, u_{jl}, u_{jk}, u_{il}\}$ is isometric to a spin factor, the (spin) determinant of $P_2(u_{ij} + u_{kl})f$ in the dual of that spin factor is zero, so (67) holds. Similarly, using $P_2(v_i + u_{im})$ and Proposition 7.2 leads to (68). The formula (66) follows from $1 = \langle f, \pi(f)\rangle$. ∎

Since a Cartan factor of type 5 is the dual of a neutral strongly facially symmetric space satisfying the same assumptions as S_*, to prove the theorem, it suffices to prove the following proposition.

PROPOSITION 7.5. *If Z' is another space satisfying the same assumptions as S_* and if $\{u_{ij}\}_{i,j\in\Phi,i\neq j} \cup \{v, v_1, v_2, v_3, v_4, v_5\}$ and $\{u'_{ij}\}_{i,j\in\Phi,i\neq j} \cup \{v', v'_1, v'_2, v'_3, v'_4, v'_5\}$ are the corresponding generating sets for S and U' (constructed in the first paragraph of the proof of Theorem 7.1), then the map $\kappa\colon S_* \longrightarrow Z'$ taking $\hat{u}_{ij}$ onto $\hat{u}'_{ij}$ ($i, j \in \Phi, i \neq j$), $\hat{v}$ onto $\hat{v}'$, and $\hat{v}_m$ onto $\hat{v}'_m$ ($1 \leq m \leq 5$) extends to a linear isometry of S_* onto Z'.*

Not surprisingly, this will be accomplished with the help of the following lemma.

LEMMA 7.6. *If f is an extreme point of $(S_*)_1$, then $\kappa(f)$ is an extreme point of Y_1.*

PROOF. Let x_0, x_m, x_{ij} be the coordinates of f. Symbolically, we shall write

$$f = \begin{array}{c|cccc} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_0 & & x_{23} & x_{24} & x_{25} \\ & & & x_{34} & x_{35} \\ & & & & x_{45} \\ \hline & x_1 & x_2 & x_3 & x_4 \;\; x_5 \end{array} \tag{69}$$

Suppose first that $P_1(v)f = 0$. Then either $f = x_0\hat{v}$ or $f = \sum_{j=1}^{5} x_j \hat{v}_j$. In the first case, $\kappa(f) = x_0\hat{v}'$ is extreme. In the second case, since $\hat{v}_1, \ldots, \hat{v}_5$ are mutually colinear (for example, apply Lemma 4.18 to the geometric quadrangles (v, u_{45}, v_1, u_{23}) and (v, u_{45}, v_2, u_{13}) to obtain $(v_1, u_{23}, u_{13}, v_2)$ in which $v_1 \top v_2$) and span the facially symmetric space $Z_0(v)$, this space is a Hilbert space and $\kappa(f)$ is an extreme point of the Hilbert space $Z'_0(v')$ spanned by the colinear family $\hat{v}'_1, \ldots, \hat{v}'_5$. By neutrality, $\kappa(f)$ is extreme in Z'.

In what follows we may thus assume that $P_1(v)f \neq 0$. Let

$$P_1(v)f = s_1\tilde{u}_{12} + s_2\tilde{u}_{34}$$

be the spectral decomposition of $P_1(v)f$ in the predual of the rank 2 Cartan factor $U_1(v) \cong A_5(\mathbf{C})$. By [11, Remark 1.1], there is a geometric tripotent w_1 and a complex

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number λ_1 such that, with $R_1 := \lambda_1 S_{w_1}$, we have $R_1^* u_{12} = \tilde{u}_{12}$, and $R_1^* u_{34} = u'_{34}$ say. Again choose a geometric tripotent w_2 and a complex number λ_2 such that, with $R_2 := \lambda_2 S_{w_2}$, we have $R_2^* u'_{34} = \tilde{u}_{34}$. Note that $R_2^* \tilde{u}_{12} = \lambda_2 \tilde{u}_{12}$. We then have

$$f_1 := R_2 R_1 f = \begin{array}{c|ccccc} & & \tilde{x}_{12} & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ \tilde{x}_0 & & & & \tilde{x}_{34} & 0 \\ & & & & & 0 \\ \hline & \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \tilde{x}_4 & \tilde{x}_5 \end{array}$$

We shall use (68), and we may assume that $\tilde{x}_{12} \neq 0$ (since $s_1 \geq s_2 \geq 0$). First, with $m = 2$, we get $\tilde{x}_1 \tilde{x}_{12} = 0$, so $\tilde{x}_1 = 0$. Then, with $m = 1$, we get $\tilde{x}_2 \tilde{x}_{21} = 0$, so $\tilde{x}_2 = 0$. Similarly, with $m = 3$ and 4, we get $\tilde{x}_4 = 0 = \tilde{x}_3$.

We now have that

(70) $$f_1 = \begin{array}{c|ccccc} & & \tilde{x}_{12} & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ \tilde{x}_0 & & & & \tilde{x}_{34} & 0 \\ & & & & & 0 \\ \hline & 0 & 0 & 0 & 0 & \tilde{x}_5 \end{array}$$

belongs to the spin factor $Z_2(v + v_5)$, in fact to the span of a quadrangle (corresponding to the original (v, u_{12}, v_5, u_{34})).

Let T_1, T_2 be the symmetries acting on Z' corresponding to R_1, R_2. We shall now show that

(71) $$T_k \circ \kappa \circ R_k = \kappa \text{ for } k = 1, 2.$$

Indeed, let $R = S_t$ with $t \in U_1(v)$. Then $R\hat{v} = -\hat{v}$ and since $v_m \diamond v$, we have $R\hat{v}_m \in \mathrm{sp}\{\hat{v}_1, \ldots, \hat{v}_5\}$, say $R\hat{v}_m = \sum_{j=1}^{5} \alpha_{mj} \hat{v}_j$. Obviously, $RZ_1(v) = Z_1(v)$, so we have $R\hat{u}_{ij} = \sum_{i,j} \alpha_{ijkl} \hat{u}_{kl}$. Since (v, u_{ij}, v_m, u_{kl}) is a geometric quadrangle, $\rho := R(\hat{v} + \hat{u}_{ij} + \hat{v}_m + \hat{u}_{kl})$ is a scalar multiple of an extreme point. Now

$$\rho = \begin{array}{c|ccccc} & & \alpha_{ij12} + \alpha_{kl12} & \alpha_{ij13} + \alpha_{kl13} & \alpha_{ij14} + \alpha_{kl14} & \alpha_{ij15} + \alpha_{kl15} \\ -1 & & & \alpha_{ij23} + \alpha_{kl23} & \alpha_{ij24} + \alpha_{kl24} & \alpha_{ij25} + \alpha_{kl25} \\ & & & & \alpha_{ij34} + \alpha_{kl34} & \alpha_{ij35} + \alpha_{kl35} \\ & & & & & \alpha_{ij45} + \alpha_{kl45} \\ \hline & \alpha_{m1} & \alpha_{m2} & \alpha_{m3} & \alpha_{m4} & \alpha_{m5} \end{array}$$

By (67) the numbers α_{mj} are expressed uniquely and universally in terms of the numbers α_{ijkl}, *i.e.*, if T denotes the corresponding symmetry acting on Z' and giving rise to numbers β_{ijkl} and β_{mj}, then $\alpha_{ijkl} = \beta_{ijkl}$ and $\alpha_{mj} = \beta_{mj}$. It follows as in earlier sections, that $T \circ \kappa \circ R = \kappa$.

We can now complete the proof of the lemma. The element $\kappa(f_1)$ belongs to $Z'_2(v' + v'_5)$ which is isometric to a spin factor. Since its (spin) determinant is 0, it is extremal in $Z'_2(v' + v'_5)$, and by neutrality, it is extremal in Z'_1. By (71), $\kappa(f) = T_1 T_2 \kappa(f_1)$ is extremal in Z'_1.

This completes the proof of Lemma 7.6. The fact that f_1 lies in a geometric quadrangle (*cf.* (70)) leads to the following Corollary.

COROLLARY 7.7. *Any two minimal tripotents in the Cartan factor of type 5 can be exchanged by at most three symmetries (in the sense of [11, Remark 1.1]).*

We now complete the proof of Proposition 7.5 and hence of Theorem 7.1. Since extreme points are mapped to extreme points by κ, by finite dimensionality, κ is contractive. By symmetry so is κ^{-1}. Therefore κ is isometric. ∎

7.2. *The type 6 case.* In this subsection, we shall prove the following theorem.

THEOREM 7.8. *Let Z be an atomic neutral* SFS *space of spin degree 10 which satisfies* FE, STP, ERP, *and* JP, *and has no L-summand of type I_2. Then Z contains an L-summand which is isometric to the predual of a Cartan factor of type 6, i.e., the 27 dimensional* JBW**-triple of all 3 by 3 hermitian matrices over the Octonions. In particular, if Z is irreducible, then Z^* is isometric to the Cartan factor of type 6.*

PROOF. Let v and $\tilde{v}$ be orthogonal minimal geometric tripotents such that the dimension of $U_2(v+\tilde{v})$ is 10 and $U_1(v+\tilde{v}) \neq \{0\}$. By Proposition 4.20, v can be chosen in such a way that $U_1(v)$ is of spin degree 8 and has no L-summand of type I_2. Thus, by Theorem 7.1, $U_1(v)$ has an M-summand $\mathcal{S}$ which is isometric to a Cartan factor of type 5. As in the proof of Theorem 7.1, $U_1(v) = \mathcal{S}$. Let us denote v by $-u_0^+$ and let $\{u_0^-, u_{ij}, u_m^+\}$ for $i, j, m \in \Phi := \{1, 2, 3, 4, 5\}$ be a canonical grid for $U_1(v)$ (*cf.* Lemma 7.3).

Recall that by (63) and (64), for each $m \in \Phi$, we have the following two spin grids:

$$u_0^-, u_m^+;\ u_{ij}, u_{kl};\ u_{jk}, u_{il};\ u_{ik}, u_{jl}, \tag{72}$$

and

$$u_i^+, u_{im};\ u_j^+, u_{jm};\ u_k^+, u_{km};\ u_l^+, u_{lm}. \tag{73}$$

All the elements of these spin grids are colinear to $-u_0^+$. Thus, for each of these spin grids, there is a unique element which together with $-u_0^+$ and the spin grid forms a spin grid with 10 elements. Denote these two elements by u_m^- and u_{0m}. Schematically, we can depict this basis as

$$\begin{array}{c|c}
 & \begin{array}{c|ccccc}
 & u_{12} & u_{13} & u_{14} & u_{15} & \\
 u_0^- & & u_{23} & u_{24} & u_{25} & \\
 & & & u_{34} & u_{35} & \\
 & & & & u_{45} & \\ \hline
 & u_1^+ & u_2^+ & u_3^+ & u_4^+ & u_5^+
 \end{array} \\
 -u_0^+ & \\ \hline
 & \begin{array}{cccccccccc}
 u_1^- & & u_2^- & & u_3^- & & u_4^- & & u_5^- & \\
 & u_{01} & & u_{02} & & u_{03} & & u_{04} & & u_{05}
 \end{array}
\end{array}$$

For later use we list here some relations between various pairs of elements in the above array.

Applying "side by side" glueing to the geometric quadrangles

$$\begin{matrix} -u_0^+ & u_j^+ \\ u_{mj} & u_{0m} \end{matrix} \quad \text{and} \quad \begin{matrix} -u_0^+ & u_{il} \\ u_{jm} & u_k^- \end{matrix}$$

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yields the geometric quadrangle

$$\begin{matrix} u_j^+ & u_{il} \\ u_{0m} & u_k^- \end{matrix}.$$

Therefore

(74) $$u_{0m} \top u_k^-,\ u_{0m} \top u_j^+,\ u_{0m} \diamond u_{il},\ u_j^+ \diamond u_k^-,\ u_j^+ \top u_{il}.$$

Applying "side by side" glueing to the geometric quadrangles

$$\begin{matrix} u_j^+ & u_{0m} \\ u_{il} & u_k^- \end{matrix} \text{ and } \begin{matrix} u_j^+ & u_{0k} \\ u_{il} & u_m^- \end{matrix}$$

yields the geometric quadrangle

$$\begin{matrix} u_{0m} & u_{0k} \\ u_k^- & u_m^- \end{matrix}.$$

Thus $\{u_{0j}, u_j^- : j \in \Phi\}$ is a spin grid of dimension 10. Let $\mathcal{S}$ be the span of the family

(75) $$\{u_k^+, u_k^-, u_{ij} : 0 \leq k \leq 5,\ 0 \leq i < j \leq 5\}.$$

We show next that $\mathcal{S}$ is an M-summand in U. Since obviously, $\mathcal{S} \subset U_2(u_0^+ + u_1^- + u_{01})$, it suffices to prove that

(76) $$U_1(u_0^+ + u_1^- + u_{01}) = \{0\}.$$

The symmetry S_w^* with $w = (-u_0^+ + u_{23} + u_1^- + u_{45})/2$ exchanges $-u_0^+$ with u_1^-, leaves u_{01} fixed, and maps the family (75) into multiples of elements of this family. From this it follows that $U_1(u_0^+)$ is mapped to $U_1(u_1^-)$ and the geometric Peirce decomposition with respect to $u_0^+ + u_1^- + u_{01}$ is left invariant by S_w^*. Since

(77) $$U_1(u_0^+) \subset \mathcal{S} \text{ and } U_1(u_0^+) \cap U_1(u_0^+ + u_1^- + u_{01}) = \{0\},$$

we obtain

(78) $$U_1(u_1^-) \subset \mathcal{S} \subset U_2(u_0^+ + u_1^- + u_{01}) \text{ and } U_1(u_1^-) \cap U_1(u_0^+ + u_1^- + u_{01}) = \{0\}.$$

Similarly, we obtain

(79) $$U_1(u_{01}) \subset \mathcal{S} \subset U_2(u_0^+ + u_1^- + u_{01}) \text{ and } U_1(u_{01}) \cap U_1(u_0^+ + u_1^- + u_{01}) = \{0\}.$$

Thus (76) holds and it now follows that $U = \mathcal{S} + U_0(u_0^+ + u_1^- + u_{01})$, *i.e.*, $\mathcal{S}$ is an M-summand of U.

DEFINITION 7.9. The *coordinates*

$$x_m^+\ (0 \leq m \leq 5), \quad x_j^-\ (0 \leq j \leq 5), \quad x_{ij}\ (0 \leq i < j \leq 5),$$

of $f \in \mathcal{S}_*$ are defined by

$$f = \sum_{m=0}^{5} x_m^+ \hat{u}_m^+ + \sum_{j=0}^{5} x_j^- \hat{u}_j^- + \sum_{i,j} x_{ij} \hat{u}_{ij}.$$

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Since a Cartan factor of type 6 is the dual of a neutral strongly facially symmetric space satisfying the same assumptions as $\mathcal{S}_*$, to prove the theorem, it suffices to prove the following proposition.

PROPOSITION 7.10. *If Y is another space satisfying the same assumptions as Z and if*

$$\{u_k^{\pm}\}_{0\le k\le 5} \cup \{u_{ij}\}_{0\le i<j\le 5},$$

and

$$\{v_k^{\pm}\}_{0\le k\le 5} \cup \{v_{ij}\}_{0\le i<j\le 5},$$

are the corresponding generating sets for $\mathcal{S}$ and $V(=Y^*)$ (constructed in the first paragraph of the proof of Theorem 7.8), then the map $\kappa\colon \mathcal{S}_* \longrightarrow Y$ taking $u_k^{\pm}$ onto $v_k^{\pm}$ $(0\le k\le 5)$ and $\hat{u}_{ij}$ onto $\hat{v}_{ij}$ $(0\le i<j\le 5)$ extends to a linear isometry of $\mathcal{S}_*$ onto Y.

This will be accomplished with the help of the following lemma.

LEMMA 7.11. *If f is an extreme point of $(\mathcal{S}_*)_1$, then $\kappa(f)$ is an extreme point of Y_1.*

PROOF. Let $x_m^{\pm}$, x_{ij} be the coordinates of f. Symbolically, we shall write

$$f = \begin{array}{c|c}
x_0^+ & \begin{array}{c|c}
x_0^- & \begin{array}{cccc} x_{12} & x_{13} & x_{14} & x_{15} \\ & x_{23} & x_{24} & x_{25} \\ & & x_{34} & x_{35} \\ & & & x_{45} \end{array} \\ \hline
 & x_1^+\ x_2^+\ x_3^+\ x_4^+\ x_5^+
\end{array} \\ \hline
 & \begin{array}{cccccccccc} x_1^- & & x_2^- & & x_3^- & & x_4^- & & x_5^- & \\ & x_{01} & & x_{02} & & x_{03} & & x_{04} & & x_{05} \end{array}
\end{array}$$

Suppose first that $P_1(u_0^+)f = 0$. Then by extremality of f, it is either a multiple of $\hat{u}_0^+$ or in $Z_0(u_0^+)$, which as shown above is a 10 dimensional spin factor. Therefore $\kappa(f)$ is either a multiple of the extreme point $\hat{v}_0^+$, or an extreme point of the 10 dimensional spin factor $Y_0(v_0^+)$, and by neutrality, an extreme point of Y_1.

In what follows, we may therefore assume that $P_1(u_0^+)f \neq 0$. Let

$$P_1(u_0^+)f = s_1\tilde{u}_0^- + s_2\tilde{u}_1^+$$

be the spectral decomposition of $P_1(u_0^+)f$ in the predual of the rank 2 Cartan factor $U_1(u_0^+)$ of type 5. By Corollary 7.7, there is a product R_1 of at most three symmetries such that $R_1^*u_0^- = \tilde{u}_0^-$, and $R_1^*u_1^+ = u_1'^+$ say. Again choose a product R_2 of at most three symmetries such that $R_2^*u_1'^+ = \tilde{u}_1^+$. Note that $R_2^*\tilde{u}_0^- = \tilde{u}_0^-$. We then have

$$f_1 := R_2R_1f = \begin{array}{c|c}
y_0^+ & \begin{array}{c|c}
y_0^- & \begin{array}{cccc} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{array} \\ \hline
 & y_1^+\ 0\ 0\ 0\ 0
\end{array} \\ \hline
 & \begin{array}{cccccccccc} y_1^- & & y_2^- & & y_3^- & & y_4^- & & y_5^- & \\ & y_{01} & & y_{02} & & y_{03} & & y_{04} & & y_{05} \end{array}
\end{array}$$

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The spin factor $U_2(u_0^- + u_{0i})$ contains $\{u_j^-, u_{ij} : 0 \leq i < j \leq 5\}$. By ERP, the spin determinant of $P_2(u_0^- + u_{0i})f_1$ is zero. But this determinant is $y_0^- y_{0i}$, and since $P_1(u_0^+)f \neq 0$, by the spectral decomposition $y_0^- = s_1 \neq 0$. Thus $y_{0i} = 0$ for each i.

We next assume that $y_0^+ \neq 0$. For $m \neq 1$, the spin determinant of $P_2(u_0^+ + u_m^-)f_1$ in $U_2(u_0^+ + u_m^-)$ is zero, implying, since $y_m^+ = 0$ that $y_m^- = 0$. We now have

$$f_1 = \begin{array}{c|c|c} & & 0\ 0\ 0\ 0 \\ & & 0\ 0\ 0 \\ y_0^+ & y_0^- & 0\ 0 \\ & & 0 \\ \cline{2-3} & & y_1^+\ 0\ 0\ 0\ 0 \\ \hline & y_1^-\ \ 0\ \ 0\ \ 0\ \ 0 & \\ & 0\ \ 0\ \ 0\ \ 0\ \ 0 & \end{array}$$

which lies in a geometric quadrangle in a spin factor and the proof of the lemma is completed as in earlier cases.

We now consider the case $y_0^+ = 0$. We then have

$$f_1 = \begin{array}{c|c|c} & & 0\ 0\ 0\ 0 \\ & & 0\ 0\ 0 \\ 0 & y_0^- & 0\ 0 \\ & & 0 \\ \cline{2-3} & & y_1^+\ 0\ 0\ 0\ 0 \\ \hline & y_1^-\ \ y_2^-\ \ y_3^-\ \ y_4^-\ \ y_5^- & \\ & 0\ \ 0\ \ 0\ \ 0\ \ 0 & \end{array}.$$

If also $y_1^+ = 0$, then since $u_0^- \top u_m^-$ for $m = 1, \ldots 5$, $\pi(f_1)$ lies in the linear span of a finite number of mutually colinear minimal geometric tripotents, each pair of which lies in some geometric quadrangle. Under these conditions, by Lemma 4.21, $\pi(f_1)$ is a minimal geometric tripotent so that f_1 is an extreme point.

So we may assume that $y_1^+ \neq 0$. In this case, since the spin determinant of $P_2(u_1^+ + u_m^-)f_1$ is zero, we have $y_m^- = 0$ for $m = 2, \ldots, 5$. This completes the proof of Lemma 7.11.

We now complete the proof of Proposition 7.10 and hence of Theorem 7.8. Since extreme points are mapped to extreme points by κ, by finite dimensionality κ is contractive. By symmetry so is κ^{-1}. Therefore κ is isometric. ∎

8. **Structure of atomic facially symmetric spaces.** In this section we use the results of earlier sections to prove the main result of the paper, embodied in Theorems 8.2 and 8.3.

PROPOSITION 8.1. *Let Z satisfy Assumption 4.3. Let v be a minimal geometric tripotent such that the rank of $Z_1(v)$ is one, and there is a geometric tripotent $u \in U_1(v)$ such that $u \top v$. Then $J(v) := Z_2(v) + Z_1(v)$ is an L-summand of Z containing $\hat{v}$ which is isometric to a Hilbert space.*

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PROOF. By Lemma 4.10, $Z_1(v)$ is isometric to a Hilbert space. Let $\{u_\iota\}$ be an orthonormal basis of $U_1(v)$ containing u. Since u is a minimal geometric tripotent by Corollary 4.5, all u_ι are minimal geometric tripotents by the same corollary, and $u_\iota \top v$ for all i.

To show that $J(v)$ is a summand, assume first that $U_0(v) \cap U_1(u_\iota) \neq \{0\}$ for some i. If w is a geometric tripotent in $U_0(v) \cap U_1(u_\iota)$, then by Proposition 4.4, $(v + w) \vdash u_\iota$, and by Corollary 2.7, there is a geometric tripotent $\tilde{u} \in U_0(u_\iota) \cap U_1(v)$, contradicting the assumption rank $Z_1(v) = 1$. Thus $Z_0(v) = Z_0(v) \cap Z_0(u_\iota)$ for all i and this shows that $J(v)$ is a summand.

To show that $J(v)$ is isometric to a Hilbert space, it is enough to prove that $\|f + \lambda g\| = \sqrt{1 + |\lambda|^2}$ for $v(f) = v$ and any $g \in Z_1(v)$, $\|g\| = 1$. Let $\{g\} \cup \{g_\alpha\}_{\alpha\in\Lambda}$ be an orthonormal basis for the Hilbert space $Z_1(v)$. Then $Y := \mathrm{sp}\{f, g\} = \bigcap_{\alpha\in\Lambda} Z_1(g_\alpha)$ is a neutral SFS space by Lemma 1.2. This two dimensional facially symmetric space is of rank 1 since if it contained two orthogonal norm one elements φ, ψ, we would have ext $Y_1 = \{\pm\varphi, \pm\psi\}$ which does not contain f. As in Lemma 4.10, this implies that Y is a Hilbert space. ∎

THEOREM 8.2. *Let Z be an atomic neutral strongly facially symmetric space satisfying* FE, STP, ERP, *and* JP. *For any minimal geometric tripotent v in U, there is an L-summand $J(v)$ of Z isometric to the predual of a Cartan factor of one of the types 1-6 such that $\hat{v} \in J(v)$.*

PROOF. By Proposition 4.8, the rank of $U_1(v)$ is at most 2. If $U_1(v) = \{0\}$, then $J(v) = \mathbf{C}v$ is an L-summand of Z. If the rank of $U_1(v)$ is one, choose any geometric tripotent u in $U_1(v)$. By Proposition 2.2, either $u \top v$ in which case the theorem follows by Proposition 8.1; or $u \vdash v$, in which case the theorem follows from Corollary 4.13. Thus we may assume that the rank of $U_1(v)$ is 2.

Choose two orthogonal geometric tripotents u_1, u_2 in $U_1(v)$. By Proposition 4.4, u_1, u_2 are minimal and by Lemma 4.15 there is a $\tilde{v}$ completing (u_1, v, u_2) to a geometric quadrangle. If $Z_1(v + \tilde{v}) = \{0\}$, then Z has an L-summand $Z_2(v + \tilde{v})$ isometric to a Cartan factor of type 4 (by [16, Theorem 4.16]). Now, suppose that $Z_1(v + \tilde{v}) \neq \{0\}$. Let m be the spin degree of Z and suppose first that m is finite. By Proposition 4.20, m is even. If $m \leq 10$, then the theorem follows from Theorems 5.10, 6.8, 7.1 and 7.8. If m is finite and greater than 10, then by several applications of Proposition 4.20 there would be an atomic neutral strongly facially symmetric space satisfying the PSP's and having spin degree 10 and rank 2. By Theorem 7.8 this space is isometric to a Cartan factor of type 6 and therefore has rank 3, a contradiction.

It remains to consider the case $m = \infty$. Write $v = u_1, \tilde{v} = \tilde{u}_1$ and assume that $U_2(v + \tilde{v})$ is infinite dimensional with a spin grid of the form $v, \tilde{v}; u_2, \tilde{u}_2; \ldots; u_6, \tilde{u}_6; u_j, \tilde{u}_j, \ldots,$ $j \in J$, where J is an index set not containing any of the symbols $1, 2, 3, 4, 5, 6$. For each $j \in \{2, 3, 4, 5, 6\} \cup J$,

$$\begin{aligned} \tilde{U} := U_1(v + \tilde{v}) &= [U_1(u_j) \cap U_0(\tilde{u}_j)] + [U_1(\tilde{u}_j) \cap U_0(u_j)] \\ &= P_0(u_j^1)\tilde{U} + P_0(u_j^0)\tilde{U} = P_1(u_j^0)\tilde{U} + P_1(u_j^1)\tilde{U}, \end{aligned}$$

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where we are now using the notation $u_J^0 := u_J$ and $u_J^1 := \tilde{u}_J$. Then

$$\tilde{U} = \bigoplus_{\epsilon\ J \to \{0,1\}} \Big[\prod_{J\in J} P_0(u_J^{\epsilon(J)})\Big] \tilde{U}$$

and

$$\tilde{U} = \bigoplus_{\epsilon\ J \to \{0,1\}} \Big[\prod_{J\in J} P_1(u_J^{1-\epsilon(J)})\Big] \tilde{U}.$$

Pick ϵ_0 such that $\prod_{J\in J} P_1(u_J^{\epsilon_0(J)})\tilde{U} \neq \{0\}$ and set $W := \prod_{J\in J} P_1(u_J^{\epsilon_0(J)})U$, which is an atomic neutral SFS space satisfying the four PSP's by the neutrality of $P_0(u_J^{\epsilon})$. Then $W_2(v+\tilde{v})$ is a twelve dimensional spin factor generated by $v, \tilde{v}, u_2, \tilde{u}_2, \ldots, u_6, \tilde{u}_6$, and $W_1(v+\tilde{v}) \neq \{0\}$. Thus implies that $W_1(v)$ has spin degree 10 leading to a contradiction as in the first part of the proof. ∎

THEOREM 8.3. *Let Z be an atomic neutral strongly facially symmetric space satisfying* FE, STP, ERP, *and* JP. *Then $Z = \oplus_\alpha^{\ell^1} J_\alpha$ where each J_α is isometric to the predual of a Cartan factor of one of the types 1-6. Thus Z^* is isometric to an atomic* JBW*-*triple. If Z is irreducible, then Z^* is isometric to a Cartan factor.*

PROOF. For each minimal geometric tripotent v we have $Z = J_v \oplus^{\ell^1} J_v^\diamond$ where J_v is isometric to the predual of a Cartan factor. For v_1 and v_2 minimal geometric tripotents, let J_ι denote J_{v_ι}. Then either $J_1 = J_2$ or $J_1 \diamond J_2$. To see this, suppose that J_1 is the weak*-span of a grid $\{u_\gamma\}$. We may assume that each u_γ is a minimal (geometric) tripotent. With $u_\alpha = u'_\alpha + u''_\alpha \in J_2 + J_2^\diamond$ we have that for each α, either $u_\alpha \in J_2$ or $u_\alpha \in J_2^\diamond$. If some $u_\alpha \in J_2$ then $J_1 = J_2$. Otherwise all $u_\alpha \in J_2^\diamond$ and therefore $J_1 \subset J_2^\diamond$.

Let $A = \oplus_{\iota\in I} J_\iota$ and $N = \bigcap_{\iota\in I} J_\iota^\diamond$, where $\{J_\iota\}$ are the distinct J_v arising from all minimal geometric tripotents. Then $Z \supset A \oplus^{\ell^1} N$. Let P_ι be the projection on Z with range J_ι. Then $\sum_\iota P_\iota$ converges strongly and for $j \in I$ and $f \in Z$, $P_j(f - \sum_\iota P_\iota f) = P_j f - \sum_\iota P_j P_\iota f = 0$ so that $f - \sum_\iota P_\iota f \in N$ and $Z = A \oplus^{\ell^1} N$. However, since Z is atomic, $N = \{0\}$. ∎

The following basic result in the theory of JBW*-triples was first proved in [11]: *Let U be a* JBW*-*triple. Then there exist orthogonal* weak*-*closed ideals A and N such that $U = A \oplus^{\ell^\infty} N$, A is an atomic* JBW*-*triple and N is a* JBW*-*triple with no minimal tripotents.*

Later, proofs were given in [9],[20], and [8].

This suggests the following problem, which in view of Theorem 8.3 reduces the complete structure of facially symmetric spaces to the "continuous" case.

PROBLEM 1 (ATOMIC DECOMPOSITION). Let Z be a neutral strongly facially symmetric space satisfying FE, STP, ERP, and JP. Then $Z = A \oplus^{\ell^1} N$ where A^* is isometric to an atomic JBW*-triple, N is a neutral SFS space whose unit ball has no extreme points.

In the following problem, for Z = the predual of a JBW*-triple, the result was proved in [11, Proposition 5]. Although this problem would follow from the solution to the previous problem, a direct proof would provide a useful tool.

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PROBLEM 2 (HILBERT BALL PROPERTY). Let Z be a neutral strongly facially symmetric space satisfying FE, STP, ERP, and JP. Then Z satisfies the *Hilbert Ball Property*, *i.e.*, for any two extreme points f, g of the unit ball of Z, the smallest facially symmetric space containing f, g has dimension at most 4 and is isometric to the predual of one of the following spaces

$$\mathbf{C},\ M_{1,2}(\mathbf{C}),\ \mathbf{C}\oplus\mathbf{C},\ S_2(\mathbf{C}),\ M_2(\mathbf{C}).$$

The following is the non-ordered analog of the main theorem of Alfsen-Shultz [4]. It is phrased in such a way that the existence of sufficiently many extreme points is insured. Since the bidual of a JB*-triple is a JBW*-triple, the converse is known to be true, so the solution to Problem 3 will provide a geometric characterization of the duals of JB*-triples.

PROBLEM 3. Let Z be a Banach space with a predual. Then there exists a JB^*-triple B such that $B^* \cong Z$ if and only if Z is a neutral strongly facially symmetric space satisfying FE, STP, ERP, and JP.

PROBLEM 4. Let Z be a Banach space. Find necessary and sufficient conditions in order that there exist a JBW*-triple U such that $Z^* \cong U$.

This last problem is the analog of Iochum-Shultz [18]. Its solution will require some new ideas because of the possible lack of any extreme points.

We now give a review and a discussion of the appropriateness of the PSP's and other possible axioms.

The most general condition on a linear space needed to obtain an algebraic structure based on geometric and physically significant axioms is *facial linear complementation*. Recall ([15, p. 109]) that a normed space Z is said to be *facially linearly complemented*, notation FLC, if the orthogonal complement $F^\diamond$ of every norm exposed face F of the unit ball is a linear subspace of Z. From this axiom alone, the existence of a projective unit in Z^* supporting a given element of Z can be proved ([15, Proposition 1.4]), thereby suggesting the existence of a polar decomposition in this setting.

Now let F_x be a norm exposed face in a FLC space and let $Y := Y_F = \operatorname{span} F_x \oplus^{\ell^1} F_x^\diamond$. Ideas from [3, Proposition 1.2] suggest the following decomposition of $U := Z^*$ which can be called the geometric Peirce decomposition: $U = U_2(F) + U_1(F) + U_0(F)$, where

- $U_1(F) = Y^\perp$
- $U_2(F) = \overline{\operatorname{span}}^{\text{weak}^*}\{w \in \mathcal{U} : F_w \subset \overline{\operatorname{span}}^{\text{norm}} F_x\}$
- $U_0(F) = \overline{\operatorname{span}}^{\text{weak}^*}\{w \in \mathcal{U} : F_w \subset F_x^\diamond\}$.

The property WFS was introduced in [15, p. 111]. It was used, together with neutrality, to establish three properties: the correspondence between geometric tripotents and symmetric faces ([15, Proposition 1.6]), uniqueness of the symmetry corresponding to a symmetric face ([15, Theorem 2.4]), and compatibility of geometric tripotents, one of

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which lies in a geometric Peirce space of the other ([15, Theorem 3.3]). We believe it should be possible to derive WFS from FLC and the geometric condition (akin to neutrality) asserting that the subspace span(F_x) has unique Hahn-Banach extensions. Moreover, we believe that the faithfulness, uniqueness and minimality of the polar decomposition should be derivable from neutrality and WFS.

We now discuss which axioms are needed at various steps of the classification, and which ones we believe to be redundant. For atomic spaces, the axioms STP and PE are needed in order to define the generalized Riesz map π, and thereby obtain a Hilbert space in the rank 1 case. To obtain the geometric characterization of spin factors, we found it necessary to introduce the axiom FE, which we believe should follow from PE in atomic spaces. We used FE mainly to show the existence of sufficiently many extreme points. Now FE is known to be true in the predual of a JBW*-triple ([10]), and in that category, atomic is equivalent to both the Krein-Milman and Radon-Nikodym properties ([7],[6]). This is the basis for our belief in the redundancy of FE.

In order to obtain the characterization of the Cartan factor of type 3, it was necessary to introduce the axiom ERP, and in order to complete the classification, the axiom JP was needed. Although it is possible that ERP can be deduced from the other axioms, as suggested by [1], we are unsure of this, although we do believe it should be possible to remove the axiom JP.

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