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Los Angeles

Essays in Economic Theory

A dissertation submitted in partial satisfaction of the

requirements for the degree Doctor of Philosophy

in Economics

by

Omer Ali

2017

ABSTRACT OF THE DISSERTATION

Essays in Economic Theory

by

Omer Ali

Doctor of Philosophy in Economics

University of California, Los Angeles, 2017

Professor Ichiro Obara, Co-Chair

Professor Marek G. Pycia, Co-Chair

My dissertation is composed of three chapters. In the first, I study the incentive role of information – how the strategic release of information can induce an agent to exert more effort on a project. More specifically, I focus on how feedback can be provided to a worker who is uninformed about the progress they make on a long term project. I show that delaying feedback about their performance can induce the worker to continue working on the project longer than they would were they to learn about their performance without delay. Negative feedback, due to the absence of good news, received in the early stages of the project can cause them to quit prematurely. In the second chapter, I study a model of matching between individuals and institutions. Matching models allow researchers to identify optimal allocations of individuals to school seats, medical residency programs and other positions over which individuals have preferences

and for which they may differ in suitability. While we know that in models in which individuals only care about the institution they match with, stable matchings always exist, I show that when individuals also care about the the number of matches made by the institution they join, stable matchings no longer exist in general. I show that stable matchings can only be found under a set of conditions I identify. Relaxing any of these conditions leads to examples of markets with no stable matchings. In the third chapter, I set out to understand why elected politicians choose to toe the party line instead of voting on issues according to their own preferences. I find that despite the short term benefits of voting for their preferred policies, there are long-term benefits from coordinating their voting behavior among like-minded legislators. These findings provide a rationale for why political parties form among politicians with similar policy positions.

The dissertation of Omer Ali is approved.

Moshe Buchinsky

Maciej Kotowski

Romain Wacziarg

Ichiro Obara, Committee Co-Chair

Marek G. Pycia, Committee Co-Chair

University of California, Los Angeles

2017

To Aala, and my parents.

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I am grateful to my co-chairs, Ichiro Obara and Marek Pycia, for their guidance throughout my PhD. I am also thankful for the advice of my committee members, Moshe Buchinsky, Maciej Kotowski and Romain Wacziarg. Remaining errors are my sole responsibility.

Education

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Chapter 1

Disclosure in Multistage Projects

1.1 Introduction

When and how individuals receive information about their progress on a task has significant effects on the effort they provide. While the incentive role of monetary payments is well understood, economists have only recently begun exploring the incentive role of information. In this paper, I study how information supplied by a sender affects a receiver's incentives to keep working on a project.

There are many applications of this framework: an entrepreneur who must convince her investors that the project is going well has an incentive to share good news as soon as it arrives. However, if she commits to this disclosure policy, the absence of good news makes investors more pessimistic about the prospects of the venture, and they may withdraw their funding. A physician who would like to keep her patient on an experimental treatment for as long as possible and promises to disclose positive results of the treatment risks making her patient more pessimistic when there is no good news to share. Finally, a firm that benefits from R&D may want researchers to keep working on a new technology even when there is uncertainty about the prospects of successfully marketing the innovation. When the firm receives information indicating that the prospects are good, it would like to make this known to its researchers. Committing to do so, however, means that as long as no good news is received, researchers conclude that things are not going well.

In all of the above examples, the sender must balance between the benefits of sharing good news and the discouragement effect of its absence. I study this trade-off in a model

in which the sender learns about the underlying quality of a project through the arrival of successes over time and must choose whether (and how) to share her information with the receiver. The project is completed when two successes arrive, which captures the idea that it has two distinct stages. While the first success is only observable to the sender, the second success is public and ends the game. Only good projects yield successes, which arrive stochastically in every period in which the receiver chooses to work. At the beginning of the relationship, neither player knows the type of the project and they share a common prior belief, p_0, that it is, in fact, a good project. The sender does not incur any cost herself, but also benefits from the completion of the project, an event I will refer to as a 'breakthrough'. When the sender and receiver choose their disclosure policy and quitting strategy (respectively) simultaneously, there is a class of equilibria characterized by the length of time the receiver is willing to work from the beginning of the game without receiving any information, and the threshold beliefs they use thereafter to decide whether or not to quit. We study two focal disclosure mechanisms: promise policies keep the receiver working until a pre-specified time at which point the sender releases some (or all) information; the indifference policy involves releasing just enough information in each period to make the receiver indifferent between working and quitting. While neither policy is the optimal mechanism in all cases, a number of partial results indicate that promise strategies are superior for the sender earlier in the game, while the indifference policy is better in its later stages.

One of the main trade-offs faced by the sender in the model is that information can have both a positive as well as a negative effect on incentives. Good news (the arrival of

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the first success) motivates the receiver to keep working, but its absence is discouraging. This trade-off is present in static models of persuasion such as Kamenica and Gentzkow (2011).[1] An additional trade-off that the sender takes into account is introduced by the dynamics of the problem and relates to the timing of information revelation. While the sender always prefers to delay revealing information, the receiver prefers to learn about the outcome history as early as possible. Suppose that there is a certain amount of information that the sender can reveal to ensure that the receiver stays in the game whenever they are asked to do so. Can the sender decrease this amount in return for giving the receiver more information later in the game? The answer to this question determines whether promise policies improve upon the indifference policy described above. When it is possible to delay information revelation with a promise of a future reward, the sender considers whether the reward is affordable and, if so, changes their disclosure policy accordingly. A novel feature of the framework I study is that this trade-off changes over time because of the non-stationarity of the problem. While in the early stages of the game, the receiver's beliefs are in a region of the belief space where the necessary rewards are affordable, over time, they move into a region where the sender prefers to stick to the indifference policy.

A growing literature studies how the provision of information can induce a receiver to take certain actions, even when the incentives of the receiver and the information provider are not aligned. Kamenica and Gentzkow (2011) study a static model in which a sender knows the state and commits to a disclosure policy that partially reveals

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[1]Similar frameworks are also studied in Calzolari and Pavan (2006) and Ivanov (2010).

it to the receiver. Whereas the sender's most preferred action is state-independent, the receiver conditions their choice on their beliefs about the state. The sender must calibrate their disclosure policy in a way that maintains the credibility of their messages. While the sender would always prefer to inform the receiver that the state is such that the sender's most preferred action is the appropriate one, the receiver would, in response, disregard this information and instead choose according to their prior. This principle is also at play in the setting I study: the sender would like the receiver to continue working for as long as possible, but sending them good news regardless of the outcome history observed renders the messages unpersuasive.

A dynamic version of this problem is studied in Ely (2017): a sender observes the true state as it evolves according to a Markov process. The receiver knows the transition rule, but does not observe the state. Messages from the sender can be conditioned on their information in such a way that sometimes induces the receiver to take the sender's preferred action. In this setting, the myopic optimal disclosure policy is also optimal when the receiver is patient and strategic. This is in stark contrast with the results I find in this paper: the sender can do strictly better when they promise to reveal information in the future, thereby taking advantage of the receiver's patience.[2]

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[2]Smolin (2015) studies the optimal feedback policy when the sender receives private information about the receiver's performance over time. This is the closest framework in spirit to the one studied in this paper. Payoffs accrue to the players in every period and the game is played for an infinite number of periods with no end date. These features ensure that, unlike the case here, values are stationary in beliefs and independent of calendar time. When the receiver prefers to quit earlier than the sender would like, an optimal policy involves a coarsening of the information received by the sender: they only communicate to the receiver whether or not their beliefs are above a certain threshold. In the model I study, however, the sender can sometimes increase their payoff by randomly disclosing bad news and keeping the receiver in the game with strictly positive probability in every period. In this way, the receiver continues working even when the sender's beliefs drift arbitrarily close to zero.

While the framework studied in this paper is novel, it combines features present in existing work. The multistage nature of the project is similar to the models in Bimpikis et al. (2016) and Green and Taylor (2016). Both of these papers, however, assume that the first success is the receiver's private information, not the sender's. Dynamic information disclosure is present in Ely (2017) and Orlov et al. (2016). Learning about the underlying productivity of the project and choosing how to share this information with the receiver is present in Orlov (2015) and Smolin (2015). In Pei (2016), an intermediary shares their information about the quality of the project/agent with the market. Che and Horner (2015) study the optimal recommendation strategy when a platform maximizes the welfare of consumers who sample a product over time.

Below, we describe some examples that motivate the structure of the game described above. The rest of the paper proceeds as follows: section 2.2 describes the framework and the evolution of beliefs; section 1.3 describes the mechanism design problem facing the sender; section 1.4 describes the simultaneous move game between sender and receiver when the latter also has commitment power, and section 2.5 concludes. Proofs are relegated to section 1.6.

1.1.1 Examples

Venture capital

Consider a venture capitalist involved in funding a project run by an entrepreneur. Suppose the project requires building a prototype before the final product is produced.

Funding occurs in discrete stages (periods) and allows the entrepreneur to continue working. If the project is good, a prototype is successfully built with some probability in each stage; a bad project never yields a functioning prototype. The entrepreneur observes whether or not the prototype built in each period is functioning and can choose whether (and how) to share this information with the investor. While the investor never observes the state of prototypes, the ultimate success of the project (say, the development of the marketable product) is public information. This occurs with some probability in every period but only after a prototype has been successfully developed. The prototype, therefore, represents an intermediate stage of the production process that reveals the quality of the project being undertaken by the entrepreneur, as well as the news that development of the product is halfway complete. The entrepreneur would like to work on the project for as long as possible, but the investor would only like to invest if the rewards are sufficiently high. How should the entrepreneur reveal information about the state of the prototype to the investor?

Experimental treatment

Consider a physician treating a patient with a condition that has no known cure. There is an experimental drug that can either be effective ($\omega = 1$) or not. Since the drug is unproven, neither party knows its quality for sure and they both begin with a shared common prior. The drug is taken once a week in pill form and is guaranteed to generate painful side effects that only the patient suffers ($c > 0$). During the week, the patient undergoes some tests that only the physician has access to. The weekly tests perfectly

reveal whether the week's treatment was successful, but the patient doesn't feel any better. The condition is cured when two weeks of treatment succeed, only at which point does the patient know that the drug must have worked. The physician can choose whether and how to share information about the weekly tests with the patient. The patient would like to continue with the treatment but only if there is a reasonably high chance of success. The physician, on the other hand, would prefer to keep the patient on the drug for as long as possible. How should the physician communicate the results of the weekly tests to the patient?

Research and development

Consider a firm with an active research department. The firm ultimately monetizes the results of successful research projects and incentivizes its researchers by rewarding them with a financial stake in the project. They receive a payoff only if/when their research is successfully developed into a marketable product. Researchers can choose whether or not to contribute to a project, but they can only keep their financial stake alive if they continue working. Let's examine a researcher's decision whether or not to start and continue working on a particular project. If the project is good, it can be successfully sold by the marketing department, but in each time period this happens only with some probability. If the project is bad, it can never be marketed successfully and no matter how long the researcher remains involved, their stake will never bear fruit. The firm understands the researchers' incentives, but would like them to continue exploring an idea even when marketing efforts prove unsuccessful, because their planning horizon

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is longer than the researcher's, who may not necessarily spend their entire career at the firm. The marketing department receives information about the success of their efforts over time. Suppose there are two stages of a successful marketing campaign: (i) developing a strong core group of enthusiastic customers, and (ii) mass market success. The second stage can never be achieved before the first, and neither stage is possible with a bad project. When a project is successfully mass marketed, the researchers receive their payoffs. However, they do not learn about whether the marketing department has successfully mobilized a core group of enthusiasts. The firm's management can commit to sharing information about the first stage with researchers to keep them involved in the project for as long as possible. How should the firm share information about the outcome of its marketing efforts with its researchers?

1.2 Model

1.2.1 Environment

A sender and receiver collaborate on a joint project over discrete time. The project, which can either be good quality or bad, has the following characteristics: when the project is good, there is a positive probability of a success in every period in which the receiver chooses to work. When the project is bad, however, successes never arrive. The receiver must decide whether to work or quit in each period, and once they choose to quit, they can never work on the project again. Completion of the project - which we refer to as a "breakthrough" - requires the arrival of two successes, only at which

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point do the players receive a payoff. It is helpful to view the first success as a project milestone, such as a functioning prototype, and the second success as the end of the project. Examples of such projects are common in venture financing, medicine, and R&D in firms (see subsection 1.1.1). While working makes the realization of a payoff more likely, it is costly for the receiver. As a result, they only choose to work when the likelihood of a breakthrough is sufficiently high.

Only the sender observes the arrival of the first success and decides whether (and how) to share this information with the receiver. We assume that the sender commits to a disclosure policy at the beginning of the game and cannot deviate from their chosen policy. After observing the outcome in each period, they can send a message to the receiver, which may depend on the history of outcomes realized and messages sent thus far. The receiver interprets this message in light of the sender's commitment to send certain messages only when certain histories arise. Messages inform the receiver about likely outcome histories and about the quality of the project. A message that the sender commits to sending more often when a success has arrived makes the receiver more optimistic about the quality of the project. The absence of such a message, however, has the opposite effect. The sender's decision problem reflects the trade-off faced, for example, by an entrepreneur who commits to sharing information about her progress on a project of uncertain viability with her investors. While a breakthrough (in this case, the ultimate success of the venture) is observable to both the entrepreneur as well as her investors, progress is only known to the entrepreneur and she must consider the effect of regularly sharing news of this progress (or lack thereof) on the investors'

enthusiasm for her project, and the likelihood of securing their continued support. The sequence of actions for the sender and receiver within a given period are described in figure 1.1.
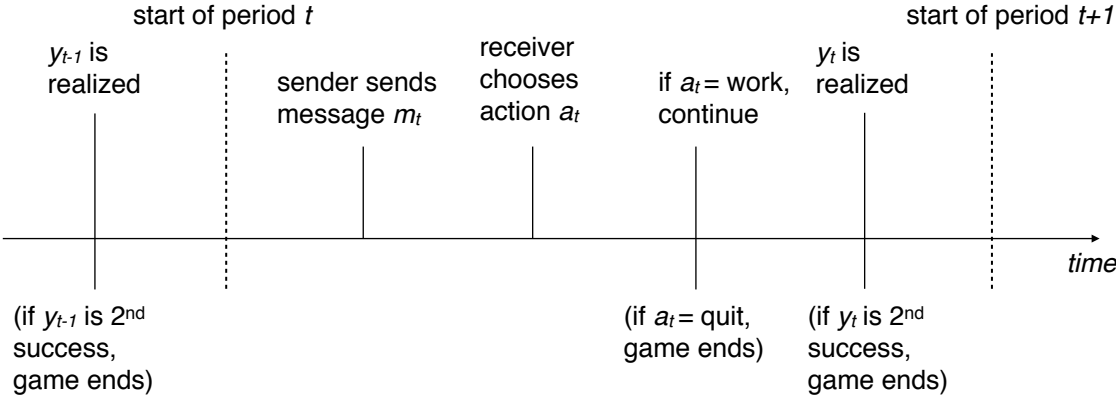
Figure 1.1: The sequence of actions within a period.

Let $\omega \in \Omega = \{0, 1\}$ denote the quality of the project. Neither player knows this parameter for sure at the beginning of the game.[3] The project is good ($\omega = 1$) with common prior probability p_0, in which case successes arrive with probability $\theta \in (0, 1)$ in every period in which the receiver chooses to work. The receiver incurs a per-period cost of $c > 0$ whenever they choose to work. If the receiver chooses to quit, the game ends and each player receives a payoff of zero. While the receiver never observes the arrival of an intermediate success, a breakthrough is observable to both players and ends the game. When a breakthrough occurs, each player receives a lump sum payoff of $B > 0$. The sender observes the entire history of outcomes (successes and failures in

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[3]This can alternatively be interpreted as the receiver's type, which is unknown to both players.

every period). They choose when and how to share this information with the receiver. As such, both players hold beliefs about the quality of the project that evolve over time. Notice that the benefit of a breakthrough is the same for both players, yet a misalignment of incentives exists because only the receiver incurs the cost of working.
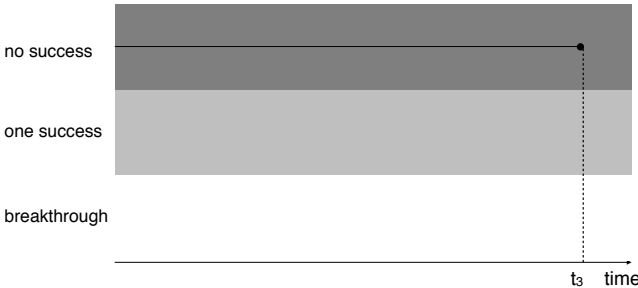
(a) An outcome history with no successes.
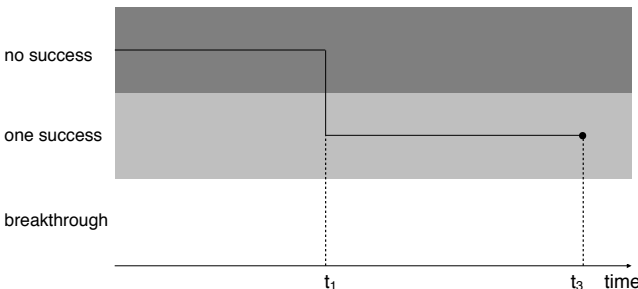
(b) An outcome history with one success.
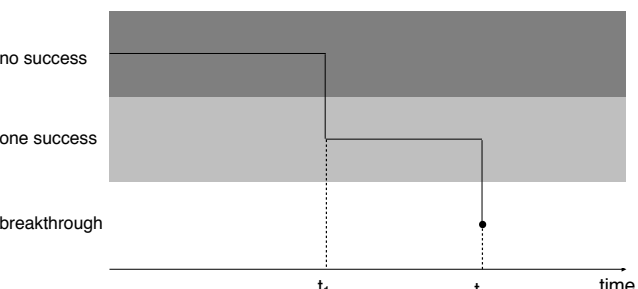
(c) An outcome history with a breakthrough.

Figure 1.2: Possible outcome histories when the receiver quits at time t_3.
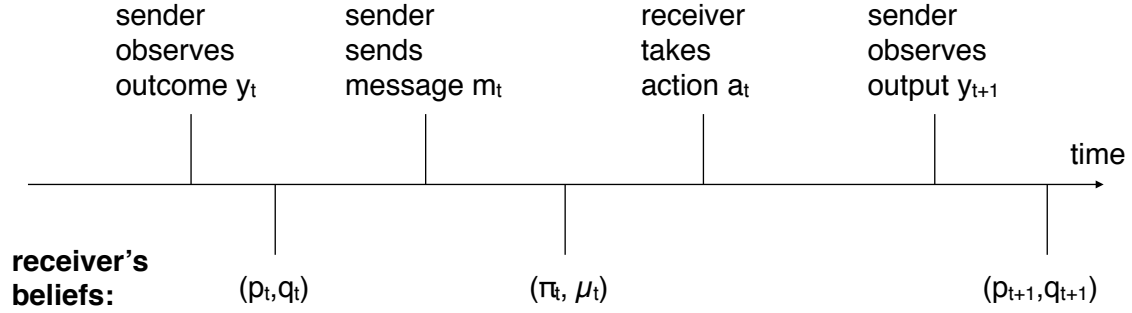
Figure 1.3: Within-period actions and beliefs

1.2.2 Outcomes, payoffs and preferences

Recall that the sender and receiver only receive a payoff of B when a breakthrough occurs. Along the way, the receiver incurs a cost of c in every period, but the sender does not. Let $y_t \in \{0,1\}$ denote the <u>outcome</u> in period t when the receiver chooses to work, with 1 representing a success and 0 a failure. The <u>outcome history</u>, $y^t = (y_1, \ldots, y_t)$, is the sequence of outcomes up to time t. Let the set of all histories up to time t be Y^t. Define the function $S : Y^t \to \{0,1,2\}$ in the following way:

$$
S\left(y^t\right) =
\begin{cases}
2 & if \ \sum_{s=1}^{t} y_s = 2 \ and \ y_t = 1 \\[2ex]
1 & if \ \sum_{s=1}^{t} y_s = 1 \\[2ex]
0 & if \ \sum_{s=1}^{t} y_s = 0
\end{cases}
$$

The function S counts the number of successes in a history. Since the project ends when two successes arrive, histories have at most two successes, and the second must arrive in the final period. Y^t is the set of all t-period histories satisfying this restriction:

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$$Y^t = \left\{ y^t = (y_1, \ldots, y_t) \in \{0,1\}^t : S\left(y^t\right) \leq 2,\ S\left(y^t\right) = 2 \iff y_t = 1 \right\}$$

Notice that both players have preferences over outcome histories, not outcomes per se, since intermediate successes are not valued independently of the history in which they occur. A 10-period history with a success in the 5th period and no successes thereafter does not yield a payoff, whereas a 10-period history with successes in the 5th and 10th periods does. The 5th period success, therefore, is not independently valuable.

In addition to the players' utilities being defined over the space of outcome histories, they are also time-dependent. Consider the 10-period outcome history with a breakthrough in the final period. A receiver evaluating this history at the beginning of the game can look forward to 10 periods in which they incur the per period cost c and a reward at the end. Their utility from this outcome history at the beginning of the game is, therefore, $\delta^9 B - \sum_{t=0}^{9} \delta^t c$, where the reward B is discounted, and there is a cost of c in every period leading up to the breakthrough. Now consider the same outcome history evaluated at the beginning of the 6th period. The costs incurred in the first five periods are now sunk and the reward is closer in time; the receiver's utility is now $\delta^4 B - \sum_{t=0}^{4} \delta^t c$; the reward is closer in time, and there are only 5 more periods in which to incur the cost of working. In general, denote the receiver's payoff from outcome history y^t at the beginning of period $\tau \leq t$ by $u\left(y^t | \tau\right)$:

$$u\left(y^t\mid\tau\right)=\begin{cases}\delta^{t-\tau}B-\left(\sum_{s=0}^{t-\tau}\delta^s\right)c & if\ S\left(y^t\right)=2\\[2em] -\left(\sum_{s=0}^{t-\tau}\delta^s\right)c & otherwise\end{cases}$$

When the receiver evaluates an outcome history, they take into account (i) how long they work, and (ii) whether or not a breakthrough occurs. On the other hand, when the sender evaluates an outcome history, they only care about whether or not a breakthrough occurs, since they do not incur the per-period cost of working, c. Denote the sender's payoff from outcome history y^t at time $\tau \leq t$ by $v\left(y^t\mid\tau\right)$:

$$v\left(y^t\mid\tau\right)=\begin{cases}\delta^{t-\tau}B & if\ S\left(y^t\right)=2\\[2em] 0 & otherwise\end{cases}$$

This payoff structure is a departure from similar models in the literature, which generally feature flow payoffs that accrue to players over time. In Smolin (2015), the principal and agent receive flow payoffs generated by an underlying state and the agent's actions in each period. In Ely (2017), the players receive flow payoffs determined by an evolving state and the agent's actions in each period. When flow payoffs depend on the state, or a player's beliefs about the state, the problem can be expressed in a recursive manner with these beliefs as state variables. In this paper, both the receiver's as well as the sender's beliefs are relevant state variables. Moreover, the the non-linear evolution of beliefs precipitated by the information released at the start of each period implies that tools used in linear environments cannot be applied here.

1.2.3 Strategies

It is clear that there is a misalignment of incentives between the sender and receiver. The former would like the receiver to work on the project forever, since even arbitrarily long sequences of failures never convince the sender that the project is bad for sure (as long as $p_0 > 0$). The receiver, on the other hand, would only like to work when the probability that the project is good is sufficiently high, since working is costly. As such, the sender would like to maintain the receiver's beliefs about the quality of the project by sending good news. The receiver interprets the messages they receive with this in mind. In the model we study, the sender must commit to an information disclosure policy at the beginning of the game.[4] This policy specifies the distribution of messages sent to the receiver as a function of the outcome histories and message histories up to that point in the game. Recall that Y^t is the space of outcome histories up to time t. Let M_t denote the message space containing all messages sent by the sender in period t. $M^{t-1} = M_1 \times \ldots \times M_{t-1}$ denotes the space of message histories up to time $t - 1$. Then $H^{t-1} = Y^{t-1} \times M^{t-1}$ is the space of histories containing past outcomes as well as past messages. Formally, the sender's strategy, $\sigma = \{\sigma_t\}_{t=1}^{\infty}$, is a sequence of functions mapping histories to probability distributions over messages:

$$\sigma_t : H^{t-1} \to \Delta\left(M_t\right),$$

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[4]This is similar to the sender choosing a signal in Kamenica and Gentzkow (2011), the principal committing to a feedback policy in Smolin (2015), or the principal committing to an information policy in Ely (2017).

where $\Delta(M_t)$ is the space of probability distributions over an arbitrary message space, M_t. With a slight abuse of notation, we will denote the probability distribution over M_t induced by history h^{t-1} by $\sigma_t(.|h^{t-1})$. Knowing σ and observing $m_t \in M_t$ at time t informs the receiver about the likely outcome histories that arose. A perfectly informative disclosure policy, for example, would specify distributions with disjoint support whenever histories are distinct: $\sigma_t(m|h^{t-1}) > 0$, $\sigma_t\left(m|\hat{h}^{t-1}\right) > 0 \iff h^{t-1} = \hat{h}^{t-1}$. Receiving a message would perfectly reveal the outcome history to the receiver. Denote such a policy by σ^{full}. A disclosure policy that is completely uninformative would specify the same distribution over messages for every history. Denote a policy that provides no information to the receiver by σ^{null}, and let Σ be the space of all disclosure policies.

The receiver's strategy specifies whether or not they choose to work in each period after observing the sender's history of messages. Formally, their strategy, $\beta = \{\beta_t\}_{t=1}^{\infty}$, is a sequence of functions mapping each message history to a probability distribution over their action space:

$$\beta_t : M^t \to \Delta\{work, quit\},$$

Let $\mathcal{B}$ denote the space of all possible strategies, and a_t denote the chosen action in period t.

1.3 Analysis

1.3.1 Beliefs

Messages inform the receiver (who does not directly observe outcomes) about outcome histories that are likely to have arisen thus far in the game. Given a disclosure policy, σ, a message $m_t \in M_t$ induces a probability distribution, $d_t\left(.|\,m_t;\sigma\right) \in \Delta\left(Y^{t-1}\right)$, over possible outcome histories via Bayes' rule. This distribution, which represents the receiver's beliefs, depends on σ, the sender's disclosure policy, and the message, m_t, received in period t. The former is the framework through which the receiver interprets messages, and the latter is the new information released as the game progresses. This, in turn, induces beliefs about the quality of the project, and whether an intermediate success has arrived - these derived beliefs turn out to be convenient parameters in the analysis of the game. We denote the receiver's beliefs at the beginning of period t about outcome histories of length $t-1$ <u>before</u> they receive a message by $d_t \in \Delta\left(Y^{t-1}\right)$. This is their belief about outcome histories given that the game has not ended in period t, and given all their prior information.

We begin by defining the receiver's beliefs over outcome histories, $d_t\left(.|\,m_t;\sigma\right)$:

$$d_t\left(y^{t-1}\big|\,m_t;\sigma\right) = \frac{\mathbb{P}\left(y^{t-1}, m_t;\sigma\right)}{\mathbb{P}\left(m_t;\sigma\right)},$$

where $\mathbb{P}\left(m_t;\sigma\right)$ is the probability that message m_t is sent in period t given that the sender is using disclosure policy σ, and $\mathbb{P}\left(y^{t-1}, m_t;\sigma\right)$ is the probability that message

m_t is sent and outcome history y^{t-1} is observed, given that the sender is using disclosure policy σ. The former is a function of the receiver's beliefs about outcome histories at the beginning of period t $(d_t \in \Delta(Y^{t-1}))$. The expression for $\mathbb{P}(m_t; \sigma)$ is given below.

$$\mathbb{P}(m_t; \sigma) = \sum_{y^{t-1} \in Y^{t-1}} \sigma_t\left(m_t|\, y^{t-1}\right) d_t\left(y^{t-1}\right)$$

The numerator of the expression for $d_t\left(y^{t-1}|\, m_t; \sigma\right)$ can be further expanded into the following:

$$\mathbb{P}\left(y^{t-1}, m_t; \sigma\right) = \mathbb{P}\left(m_t|\, y^{t-1}; \sigma\right) d_t\left(y^{t-1}\right),$$

where $\mathbb{P}\left(m_t|\, y^{t-1}; \sigma\right)$ is the conditional probability of observing message m_t following outcome history y^{t-1}. We can now give the expression for $d_t\left(y^{t-1}|\, m_t; \sigma\right)$ in terms of the receiver's beliefs.

$$d_t\left(y^{t-1}\big|\, m_t; \sigma\right) = \frac{\sigma_t\left(m_t|\, y^{t-1}\right) d_t\left(y^{t-1}\right)}{\sum\limits_{y^{t-1} \in Y^{t-1}} \sigma_t\left(m_t|\, y^{t-1}\right) d_t\left(y^{t-1}\right)}$$

The receiver is interested in the project's quality, and whether a success has already occurred, conditional on the project being good. For any outcome history, y^{t-1}, there is a corresponding belief about the likelihood that this history was generated by a project of quality ω. It is clear to see that whenever $S\left(y^{t-1}\right) = 1$, it must be the case that $\omega = 1$.[5] However, when $S\left(y^{t-1}\right) = 0$, it may well be that $\omega = 1$, but a success has not

---

[5]We don't need to deal with the case in which $S\left(y^{t-1}\right) = 2$, since those histories are revealed to all players when they occur.

yet arrived; it could also be the case that $\omega = 0$. Let the function $\phi : Y^t \to [0,1]$ be the mapping from outcome histories (of any length) to beliefs about ω, so that $\phi(y^t)$ is the receiver's beliefs about ω when they know for sure that y^t occurred. For example, when $S(y^t) = 1$, $\phi(y^t) = 1$; on the other hand, when $S(y^t) = 0$, $\phi(y^t) = \frac{p_0(1-\theta)^t}{p_0(1-\theta)^t + 1 - p_0}$, which is the probability that the project is good and t failures have occurred. Similarly, let the function $\psi : Y^t \to [0,1]$ map outcome histories to beliefs about the arrival of the first success. $S(y^t) = 1$ means that $\psi(y^t) = 1$, and the receiver knows for sure that a success has arrived; $S(y^t) = 0$ implies that $\psi(y^t) = 0$, and the receiver knows for sure that no success has arrived. Interior values for this belief arise when the receiver holds non-degenerate beliefs about outcome histories. Suppose, for example, that the receiver believes that y_0^t arose with probability $1 - \rho$ and y_1^t arose with probability ρ, where $S(y_0^t) = 0$, and $S(y_1^t) = 1$. Their beliefs about the number of successes conditional on $\omega = 1$ is then $\rho \times 1 + (1 - \rho) \times 0 = \rho$.

Over the course of the game, the receiver's beliefs about outcome histories evolve in response to information about the continuation of the game as well as messages from the sender. In the first period, the fact that the game has not yet ended is uninformative because two successes are required to end the game and only one success can arrive per period. In the absence of any message from the sender, their beliefs about ω remain unchanged from p_0, and their belief that a success has occurred conditional on $\omega = 1$ is simply θ. Denote these beliefs by (p_0, θ) in the space $\Delta(\Omega) \times \Delta(X)$. When the sender plays the no-information strategy, σ^{null}, these beliefs persist until the beginning of the second period. Suppose instead that the sender plays the fully informative strategy,

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σ^{full}. When $y_1 = 1$, the receiver's beliefs before they observe the sender's message are (p_0, θ) and evolve to $(1, 1)$ after they observe it. When $y_1 = 0$, the receiver's beliefs evolve from (p_0, θ) to $\left(\frac{p_0(1-\theta)}{1-p_0\theta}, 0\right)$. Notice that the receiver's beliefs about ω obey the martingale property, since $p_0 = p_0\theta \times 1 + (1 - p_0\theta) \times \frac{p_0(1-\theta)}{1-p_0\theta}$. Their beliefs about the arrival of the first success, conditional on $\omega = 1$ also obey the martingale property, so long as we condition all probabilities on the event $\omega = 1$: $\theta = \theta \times 1 + (1 - \theta) \times 0$.

We can now define the mapping from beliefs about outcome histories, $d_t \in \Delta(Y^{t-1})$, to beliefs about the project's type, and whether a success has arrived, conditional on the project being good: $\Delta(\Omega) \times \Delta(X)$. We will use p_t and π_t to denote the receiver's belief that $\omega = 1$, and q_t and μ_t to denote their beliefs that a success has occurred, conditional on $\omega = 1$. The pair (p_t, q_t) will represent their beliefs after they learn that the game has not yet ended in period t, but before they observe the sender's message. Meanwhile, the pair (π_t, μ_t) will represent their beliefs after they observe the sender's period-t message. Given any distribution over outcome histories in Y^{t-1}, the pair of functions (ϕ, ψ) derive a pair of beliefs in $\Delta(\Omega) \times \Delta(X)$ induced by this distribution. With some abuse of notation, let $\phi(d_t)$ denote the beliefs about ω induced by the probability distribution over outcome histories, $d_t \in \Delta(Y^{t-1})$:

$$\phi(d_t) = \sum_{y^{t-1} \in Y^{t-1}} \phi\left(y^{t-1}\right) d_t\left(y^{t-1}\right)$$

Similarly, let $\psi(d_t)$ denote the beliefs about the number of successes achieved, conditional on $\omega = 1$ when the receiver's beliefs about outcome histories is d_t:

$$\psi\left(d_t\right) = \sum_{y^{t-1} \in Y^{t-1}} \psi\left(y^{t-1}\right) d_t\left(y^{t-1}\right)$$

We summarize these observations below. A pair of disclosure policy and message, (σ, m_t), induce a belief distribution over outcome histories, which, in turn, induces beliefs about the quality of the project and whether a success has arrived.[6]

$$(\sigma, m_t) \in \Sigma \times M_t$$

$$\Big\downarrow d$$

$$d_t\left(.\,|\,m_t; \sigma\right) \in \Delta\left(Y^{t-1}\right)$$

$$\phi \Big| \psi$$

$$(\pi_t, \mu_t) \in \Delta\left(\Omega\right) \times \Delta\left(X\right)$$

The belief pairs (p_t, q_t) and (π_t, μ_t) are key parameters in the model and will feature prominently in the analysis that follows. We refer to (p_t, q_t) as the receiver's <u>period-t prior beliefs</u> and (π_t, μ_t) as their <u>period-t posterior beliefs</u>. Figure 1.4 depicts the evolution of the receiver's beliefs within a given period.

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[6]The message need not be informative. If $m_t = \emptyset$, for example, then $d_t\left(.\,|\,m_t; \sigma\right) = d_t$, which are the receiver's beliefs after learning that the game has not yet ended, but before receiving a message from the sender.
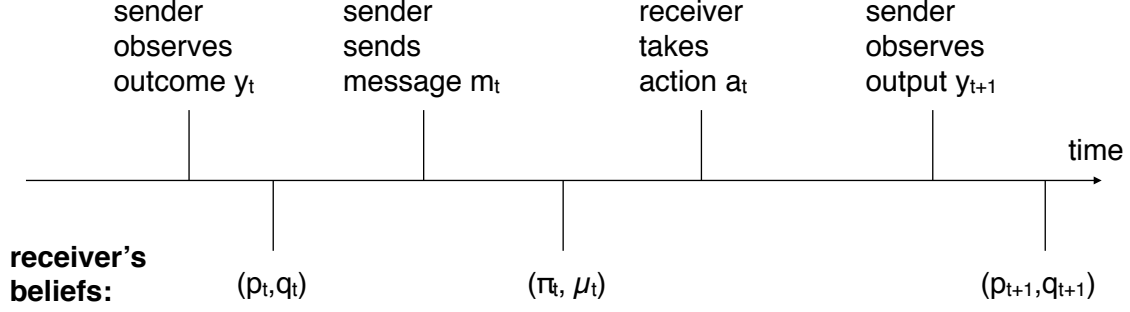
Figure 1.4: Within-period actions and beliefs

There are two benchmark disclosure policies briefly mentioned earlier: σ^{null}, the no-information disclosure policy, and σ^{full}, the full information disclosure policy. These two policies represent the worst and best payoffs, respectively, for the receiver. Whereas the full information policy yields the lowest expected payoff to the sender, the no-information policy is not their most preferred. In the following subsection, we describe how the receiver's beliefs evolve over time under the two policies.

1.3.2   Expected payoffs

A pair of strategies, $(\sigma, \beta) \in \Sigma \times \mathcal{B}$, along with a player's beliefs about the project induce a probability distribution over future outcome histories. The collection of possible outcome histories from the beginning of period τ onwards is $Y_\tau = \bigcup_{t=\tau}^{\infty} Y^t$. The receiver's expected payoff at time τ, given the strategy profile (σ, β), can be expressed as follows:

$$U\left(\sigma, \beta \,|\, \tau\right) = \mathbb{E}_{\sigma, \beta}\left[u\left(\tilde{y}\,|\,\tau\right)\big|\left\{\sigma_t\right\}_{t=1}^{\tau-1}\right],$$

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where $\tilde{y} \in Y_\tau$ is random, and the expectation is taken with respect to the distribution over Y_τ determined by the receiver's beliefs and the strategy pair (σ, β). The receiver's beliefs at time τ are determined by the sender's disclosure policy up to time τ. Similarly, the sender's expected payoff at time τ is $V(\sigma, \beta|\tau)$:

$$V(\sigma, \beta|\tau) = \mathbb{E}_{\sigma, \beta}\left[v(\tilde{y}|\tau)| \{y_t\}_{t=1}^{\tau-1}\right],$$

where the sender's beliefs are determined by the actual outcome realizations. Players' expected payoffs at the beginning of the game will be expressed by $U(\sigma, \beta|0)$ and $V(\sigma, \beta|0)$.

Although the sender can use an unrestricted message space, they ultimately care about whether or not the receiver chooses to work. Moreover, since the sender's strategy is known to the receiver, every message sent induces a unique choice of action by the receiver.[7]

As such, the message space may as well be the receiver's action space. This insight was leveraged by Kamenica and Gentzkow (2011) and Smolin (2015) to define and consider underlined straightforward signals and underlined recommendation policies, respectively. We define recommendation policies in our context as those policies with codomain equal to probability distributions over the receiver's action space.

Definition 1. A disclosure policy, $\sigma = \{\sigma_t\}_{t=1}^{\infty}$, is a **recommendation policy** if:

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[7]We assume that ties are broken in favor of working, a technical assumption that does not drive any of the results.

$$\sigma_t : H^{t-1} \rightarrow \Delta\left(\{work, \ quit\}\right)$$

Considering these policies is without loss of generality, as the next result will show. By considering only those policies that recommend an action for the receiver to take, we can reduce the space of disclosure policies over which the sender maximizes, simplifying their decision problem.

Proposition 1. *Consider the pair of disclosure policy and strategy (σ, β). There exists a recommendation policy, R^σ, such that:*

$$V\left(\sigma, \beta \,|\, t\right) = V\left(R^\sigma, \beta \,|\, t\right) \ \ \forall t \in \mathbb{N}$$

$$U\left(\sigma, \beta \,|\, t\right) = U\left(R^\sigma, \beta \,|\, t\right) \ \ \forall t \in \mathbb{N}$$

Proof. See subsection 1.6.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

In addition to the simplification afforded by proposition 1, we can also show that recommendation policies need only depend on the outcome history. As the receiver learns about outcome histories through the sender's messages over time, they update their beliefs about the state. Often, models in which a player learns about a parameter associated with the productivity, quality, type of a project or bandit arm reduce to optimal stopping problems. The optimal strategy involves tracking some parameter, usually the player's beliefs, which triggers the player to stop investing, experimenting,

or working when it falls below some threshold.[8] Since our framework bears many similarities with these models, optimal strategies in our setting also reduce to cut-off strategies, but only in the later stage of the game. We define a generalization of cut-off strategies, which we call eventually cut-off strategies. The relevant belief in our case is the receiver's belief that a breakthrough will occur in the next period. In the beginning of the game, the receiver knows that not enough time has elapsed for a breakthrough to occur, even if they are fairly confident that the project is good. In the extreme, they are sure that a breakthrough will never occur in the first period because two successes are required, and a maximum of one can arrive in one period. In the second period, their beliefs that a breakthrough will arrive can be positive but small. In the third period, beliefs will be positive and larger still etc. Once a sufficient amount of time has elapsed, their beliefs about the quality of the project begin to deteriorate, and their belief that a breakthrough will occur in the next period decreases. Under some disclosure policies, beliefs decline monotonically for the remainder of the game. In this phase of the game, they find it optimal to employ a cut-off strategy (see Proposition 2).

Definition 2. A strategy, $\beta \in \mathcal{B}$, is **eventually a cut-off strategy** if there exists some $T \in \mathbb{N}$ and $b \in [0, 1]$ such that:

_____

[8]See, for example, Smolin (2015) among many others.

$$\beta_t\left(m_t\right) = \begin{cases} work & \forall t < T \\ \\ work & if\ t \geq T,\ \pi_t\mu_t\theta \geq b \\ \\ quit & if\ t \geq T,\ \pi_t\mu_t\theta < b \end{cases}$$

The receiver works in every period $t < T$, and for all $t \geq T$, they choose to work if and only if they believe that a breakthrough will occur in the next period with probability greater than b.

Recall from subsection 1.3.4 that the receiver's subjective belief that a breakthrough will occur in the current period is $\pi_t\mu_t\theta$. This belief decreases monotonically as long as $\pi_t\mu_t \geq \pi_{t+1}\mu_{t+1}$ for all remaining t. Notice that it is always the case that $\mu_{t+1} > \mu_t$, since the longer the receiver stays in the game, the more likely it is that a success has arrived, conditional on the project being good. However, μ_t is bounded above by 1, and over time, in the absence of good news from the sender, π_t tends towards 0. This moves the receiver's belief about the probability of a breakthrough downwards.

Proposition 2. *Suppose the sender chooses a recommendation policy R^σ. If $\left\{\left(\pi_t, \mu_t\right)\right\}_{t=1}^\infty$ is the sequence of the receiver's posterior beliefs after they receive the message "work" and $\pi_t\mu_t \geq \pi_{t+1}\mu_{t+1}\ \forall t \geq T$ for some $T \in \mathbb{N}$, then a strategy that is eventually a cutoff strategy is optimal for the receiver.*

Proof. See subsection 1.6.2. □

Every disclosure policy chosen by the sender induces a sequence of posterior beliefs for the receiver. The receiver evaluates whether (and for how long) to work by using

these beliefs to calculate their expected payoff. For a pair of posterior beliefs (π, μ) such that π is the receiver's belief about ω and μ is their belief about whether a success has already occurred, we refer to the quantity $\pi\mu\theta B - c$ as the <u>receiver's flow payoff</u>. This is the expected payoff the receiver anticipates less the cost of working for an additional period. The first term is the probability of a breakthrough arriving in the next period $(\pi\mu\theta)$ multiplied by the benefit (B). When the flow payoff is positive, it is myopically optimal for the receiver to work for an additional period. When the flow payoff is negative, a myopic receiver would quit, since working for one period costs strictly more than they expect to gain. However, a forward looking receiver may stay if the sum of future flow payoffs is positive. At the beginning of the game, this consideration determines whether the receiver begins working on the project or not. Let $\{(\pi_t, \mu_t)\}$ be the sequence of posterior beliefs induced by the sender's disclosure policy. If there is some T such that:

$$-c + (\pi_1\mu_1\theta B - c) + \sum_{t=2}^{T} \prod_{s=1}^{t-1} (1 - \pi_s\mu_s\theta)(\pi_t\mu_t\theta B - c) \geq 0,$$

the receiver chooses to work until period T.

Before we present the main result of this section, let $\bar{T}$ be the time at which the receiver chooses to quit when they receive no information from the sender. Recall that under $\bar{\sigma}$, the receiver's beliefs about the project deteriorate monotonically over time (see figure 3.7). Their beliefs about the arrival of a breakthrough in the next period $(\pi_t\mu_t\theta)$ also decrease monotonically beyond some point in time. In this phase of the

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game, they employ a cut-off strategy and quit whenever the benefit they can expect
to gain in the next period $(\pi_t \mu_t \theta B)$ no longer exceeds the cost of working. Since the
probability of a breakthrough is decreasing, if this inequality fails in any period, it also
fails in all future periods. As a result, the receiver quits in the first period in which
this inequality fails. We call this period, $\bar{T}$ and define it formally below. When the
receiver does not learn any new information, their posterior beliefs are equivalent to
their priors: $\pi_t = p_t$ and $\mu_t = q_t$.

$$\bar{T} = \min_t \left\{ t : p_s q_s < \frac{c}{\theta B} \ \forall s \geq t; \ \sigma = \sigma^{null} \right\}$$

To ensure that the receiver begins working on the project even under a no informa-
tion policy, we make the following assumption:

Assumption 1. *The receiver always chooses to start working on the project:*

$$-c + p_1 q_1 \theta B - c + \sum_{t=2}^{\bar{T}-1} \prod_{s=1}^{t-1} (1 - p_s q_s \theta) (p_t q_t \theta B - c) \geq 0$$

1.3.3 Evolution of beliefs under σ^{null} and σ^{full}

Under σ^{null}, the only source of information about ω and the number of successes
achieved comes from the fact that a breakthrough has not yet occurred. In the first
period, there is no new information, since a breakthrough never occurs after only one
success. The receiver's beliefs about ω remain at their initial level, p_0. However, in the
second period, if the game has not yet ended, the receiver can begin to draw inferences

about the project's quality. Let the number of successes achieved through time t be X_t. The probability that the project is good, conditional on having less than two successes by the second period is:

$$p_2 = \mathbb{P}\left(\omega = 1 \mid X_t < 2\right) = \frac{\mathbb{P}\left(\omega = 1, X_t < 2\right)}{\mathbb{P}\left(X_t < 2\right)} = \frac{\left(1 - \theta^2\right) p_0}{\left(1 - \theta^2\right) p_0 + 1 - p_0},$$

where $\left(1 - \theta^2\right)$ is the probability that two successes have not yet occurred, given that the project is good. When the project is good, the number of successes up to time t follows a Binomial distribution with t draws and success probability θ: $X_t \mid \omega = 1 \sim B\left(t, \theta\right)$. Let p_t denote the receiver's beliefs that $\omega = 1$ at time t, when the game has not yet ended:

$$p_t = \mathbb{P}\left(\omega = 1 \mid X_t < 2\right) = \frac{\mathbb{P}\left(\omega = 1, X_t < 2\right)}{\mathbb{P}\left(X_t < 2\right)}.$$

Since $\mathbb{P}\left(X_t < 2\right) \to 0$ as $t \to \infty$, $p_t \to 0$, and the receiver becomes increasingly pessimistic about the quality of the project the longer they stay in the game.

Now consider the case in which the sender reveals all information to the receiver $\left(\sigma^{full}\right)$. This differs from the preceding case in that now the receiver knows when the first success arrives, not just when the second success ends the game. As such, beliefs evolve differently: in the absence of good news, the receiver becomes more pessimistic about the quality of the project. When they do see the first success, the receiver's beliefs jump to 1. Denote the receiver's period-t beliefs when they observe no successes by $\hat{p}_t$.

$$\hat{p}_t = \mathbb{P}\left(\omega = 1 \middle| X_t < 1\right) = \frac{\mathbb{P}\left(X_t = 0\right) p_0}{\mathbb{P}\left(X_t = 0\right) p_0 + 1 - p_0}$$

It can be shown that $\hat{p}_t \leq p_t$ for all t for histories in which no successes arrive. However, any history in which a success arrives induces beliefs $\hat{p}_t$ to jump to one, while beliefs p_t continue to deteriorate. Below, we depict the evolution of beliefs under no information and full information for two outcome histories to illustrate this point.

(a) No successes.   (b) Successes in the 11th and 15th periods.
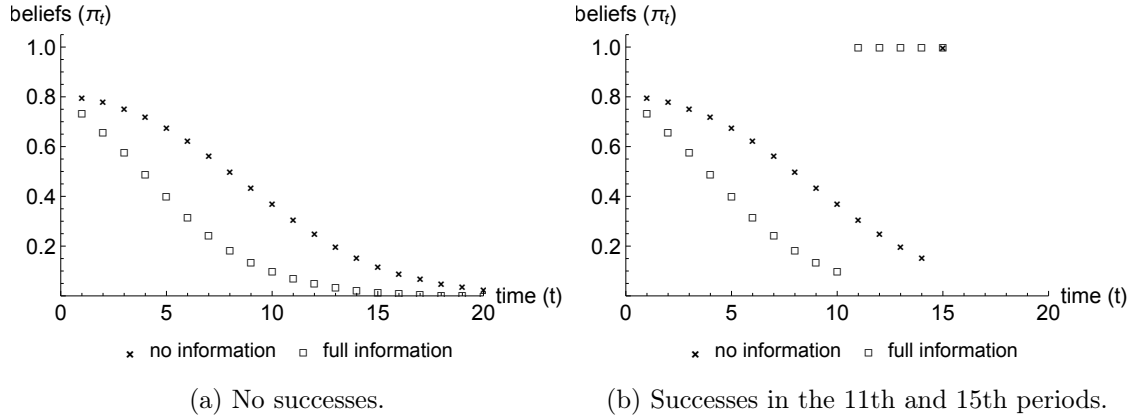
Figure 1.5: The receiver's beliefs for two outcome histories under no information (σ^{null}) and full information (σ^{full}) when the prior belief that $\omega = 1$ is $p_0 = 0.8$, and the probability of success in each period when the receiver works on a good project is $\theta = 0.3$.

1.3.4   Probability of a breakthrough

The sender's period-t message induces period-t posterior beliefs (π_t, μ_t) about $\omega \in \Omega$ (the quality of the project) and $x \in X$ (the number of successes conditional on the project being good). Given these beliefs, the receiver can calculate the probability of achieving a breakthrough in the current period if they choose to work. The probability of a breakthrough in period t depends on their beliefs (π_t, μ_t), since a breakthrough only

occurs when (i) the project is good, and (ii) one success has already occurred in the past. Indeed, it is simply the probability that the project is good, π_t, multiplied by the probability that a success has already occurred, conditional on the project being good, μ_t, multiplied by the probability that another success arrives this period, conditional on the project being good, θ: $\pi_t \times \mu_t \times \theta$. When the receiver does not observe an informative message in period t, their prior beliefs are unchanged, and hence their belief that a breakthrough will occur in the current period is $p_t q_t \theta$ instead.

Consider the mechanism design problem in which the sender commits to a disclosure policy and the receiver best responds. One class of disclosure policies, which we refer to as promise policies, consists of the following two stages: (i) a phase of no information disclosure, then (ii) a promise by the sender to reveal some information after a certain amount of time has elapsed. When the sender reveals all of their information at the promised time, we call the policy a perfect promise policy; otherwise, it is a partial promise policy.

Another focal policy we analyze is the indifference policy: the sender conceals all information until the receiver is at the cusp of quitting, at which point they begin releasing information randomly with just enough probability to keep the receiver indifferent between staying and quitting. These two policies are described below.

1.3.5 Indifference policy

Under the indifference policy, the sender conceals all information (by always recommending work) until time $\bar{T}$, after which point they (i) always recommend work after

outcome histories with one success, and (ii) sometimes recommend <u>work</u> after outcome histories with no successes. Let $1 - r_t$ be the probability with which the sender recommends work following a $t - 1$ period outcome history with no success. The probability r_t is chosen such that the receiver is just indifference between working and quitting when they receive the message <u>work</u>.

Payoffs from the indifference policy

Notice that since the sender always recommends to the receiver that they should work following histories with one success, their continuation value is B following any such history. An argument for this statement is provided below.

Proposition 3. *Under the indifference policy, the sender's value is B following any history with a success.*

Proof. See subsection 1.6.3. □

Let R^* be the indifference policy and β^* the receiver's best response. There is always some positive probability that the receiver keeps working. This probability rapidly approaches 0 as time progresses. Nevertheless, this means that the sender's value is composed of an infinite sum. We describe it in full below, then provide upper and lower bounds that become arbitrarily tight the more terms we include. The sender's value at time $\bar{T}$ from the indifference policy is described below:

$$V\left(R^*, \beta^* | \bar{T}\right) = (1 - r_{\bar{T}})\left(B\theta b_{\bar{T}} + (1 - \theta b_{\bar{T}}) V\left(R^*, \beta^* | \bar{T} + 1\right)\right)$$

The sender's value at the beginning of the game is below:

$$V\left(R^*, \beta^* \mid 0\right) = p_0 \mathbb{P}\left(X_{\bar{T}-1} > 0\right) B + \left[p_0 \mathbb{P}\left(X_{\bar{T}-1} = 0\right) + 1 - p_0\right] V\left(R^*, \beta^* \mid \bar{T}\right),$$

where $X_{\bar{T}-1}$ follows a binomial distribution with $\bar{T} - 1$ trials and success probability θ. The expression above captures the idea that, since the receiver stays for sure until $\bar{T}$, the sender can look forward to $\bar{T} - 1$ trials that may yield a success. Beyond period $\bar{T}$, the sender's value is described above and depends on the sequence of revelation probabilities, $\{r_t\}_{t=\bar{T}}^{\infty}$. Since all policies considered by the sender involve an initial $\bar{T}$ period phase of no-information, this does not play a role in our comparison between indifference and promise policies below.

The receiver's value from the indifference policy depends on the revelation probabilities chosen by the sender, $\{r_t\}_{t=\bar{T}}^{\infty}$. Let $U\left(R^*, \beta^* \mid \bar{T}\right)$ be their continuation value at the beginning of period $\bar{T}$. Since the sender chooses these probabilities to ensure that the receiver is just indifferent between staying and quitting, his continuation value is equal to zero:

$$U\left(R^*, \beta^* \mid \bar{T}\right) = \left(1 - r_{\bar{T}}\left(1 - p_{\bar{T}} q_{\bar{T}}\right)\right)\left[\pi_{\bar{T}} \mu_{\bar{T}} \theta B - c + \left(1 - \pi_{\bar{T}} \mu_{\bar{T}} \theta\right) U\left(R^*, \beta^* \mid \bar{T}+1\right)\right] = 0$$

In fact, this is true in every period beyond $\bar{T}$. The next result shows that when the

receiver plays an eventually cutoff strategy with $T = \bar{T}$, it is optimal for the sender to choose the indifference policy.

Theorem 1. *Suppose assumption 1 holds, and that the receiver chooses an eventually cutoff strategy with $T = \bar{T}$. The recommendation policy, R^* is optimal for the sender:*

$$
R_t^* \left(y^{t-1} \right) = \begin{cases} work & if \ t < \bar{T} \\\\ work & if \ t \geq \bar{T} \ and \ S \left(y^{t-1} \right) = 1 \\\\ quit \ with \ probability \ r_t = \frac{c - p_t q_t \theta B}{c(1 - p_t q_t)} & if \ t \geq \bar{T} \ and \ S \left(y^{t-1} \right) = 0 \\\\ work \ with \ probability \ (1 - r_t) & if \ t \geq \bar{T} \ and \ S \left(y^{t-1} \right) = 0 \end{cases}
$$

$$
\beta_t^* \left(m_t \right) = \begin{cases} work & if \ m_t = work \\\\ quit & if \ m_t = quit \end{cases}
$$

Proof. See subsection 1.6.5. $\qquad\square$

When this restriction on the receiver's strategies is relaxed, the indifference policy may no longer be optimal. For example, if the receiver can choose a voluntary initial period larger than $\bar{T}$, the sender can benefit by delaying information. To study this, we introduce promise policies in the next section.

1.3.6 Promise policies

Payoffs from a promise policy

Consider a promise policy that the sender proposes to the receiver at the beginning of period t, in which the sender promises to reveal the entire history of outcomes at time $T > t$. Denote the sender's beliefs about ω by b_t, and the receiver's period-t prior beliefs by (p_t, q_t). If the receiver chooses to stay until period T, the values of the players are described as follows, where P^T is the perfect promise strategy that reveals all information at time T, and β^T is the receiver's best response:

$$U\left(P^T, \beta^T \big| t\right) = (p_t q_t \theta B - c) + (1 - p_t q_t \theta)(p_{t+1} q_{t+1} \theta B - c) + \ldots + \prod_{s=t}^{T-1}(1 - p_s q_s \theta)\, p_T q_T \left(B - \frac{c}{\theta}\right)$$

$$V\left(P^T, \beta^T \big| t\right) = b_t \theta B + (1 - b_t \theta) b_{t+1} \theta B + \ldots + \prod_{s=t}^{T-2}(1 - b_s \theta)\, b_{T-1} \theta B$$

Incentive compatibility

At the time the sender offers a promise policy to the receiver, the receiver must decide whether or not to accept. This decision involves not only deciding whether or not to work in the period of the offer, it also involves deciding whether or not to work in every subsequent period before receiving the information from the sender. The sender's recommendation that the receiver should work in every period before the one in which they reveal the history of outcomes must be incentive compatible if the receiver is

expected to follow through. The next result shows that if the receiver prefers to follow
the recommendations of the promise strategy at time t, then they also prefer to do so
at every subsequent time before the final period.

Proposition 4. *Let P^T be a promise policy proposed by the sender to the receiver at
time $t > \bar{T}$. If $U_s\left(P^T\right) \geq 0$, then $U_{s+1}\left(P^T\right) \geq 0$ for all $t \leq s < T$.*

Proof. See subsection 1.6.4. □

1.3.7 Conditions under which information delay is optimal

Sender

In general, the sender prefers to delay revealing information to the receiver, since a
negative message delivered earlier in the game hastens the receiver's departure. We ask
the following question: for an ϵ decrease in revelation probability today, what is the
maximum δ increase in revelation probability tomorrow that the sender is willing to
give, assuming the receiver continues obeying the recommendations? Let the sender's
value at time t from a status-quo revelation policy (with revelation probabilities r_t and
r_{t+1}) be V_t, and let their value from the new $\epsilon - \delta$ perturbed policy be $\hat{V}_t$.

$$
\begin{aligned}
V_t \;=\; & \left(1 - r_t\right) \theta b_t B + \ldots \\
& \left(1 - r_t\right)\left(1 - r_{t+1}\right)\left(1 - \theta b_t\right) \theta b_{t+1} B + \ldots \\
& \left(1 - r_t\right)\left(1 - r_{t+1}\right)\left(1 - \theta b_t\right)\left(1 - \theta b_{t+1}\right) V_{t+2}
\end{aligned}
$$

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$$\hat{V}_t = (1 - r_t + \epsilon)\,\theta b_t B + \dots$$

$$(1 - r_t + \epsilon)\,(1 - r_{t+1} - \delta)\,(1 - \theta b_t)\,\theta b_{t+1} B + \dots$$

$$(1 - r_t + \epsilon)\,(1 - r_{t+1} - \delta)\,(1 - \theta b_t)\,(1 - \theta b_{t+1})\,\hat{V}_{t+2}$$

Since we assume that the only differences between the two disclosure policies occur in periods t and $t + 1$, it follows that the continuation value at time $t + 2$ is the same. A sufficient condition for the perturbed policy to be preferred by the sender is for the following condition to hold: $(1 - r_t + \epsilon)\,(1 - r_{t+1} - \delta) \geq (1 - r_t)\,(1 - r_{t+1})$. This inequality can be rearranged into the expression in condition 1 for δ in terms of ϵ, r_t and r_{t+1}.

Condition 1. Sufficient condition for the sender to prefer decreasing the revelation probability in period t by ϵ and increasing it by δ in period $t + 1$:

$$\delta \leq \hat{\hat{\delta}}_{max} = \frac{\epsilon\,(1 - r_{t+1})}{1 - r_t + \epsilon}$$

In general, the sender's period-$t + 2$ values may differ, but it will always be the case that $\hat{V}_{t+2} \geq V_{t+2}$. To see why, recall that the sender must compensate the receiver for lower flow payoffs in period t by increasing the revelation probability in period $t + 1$ and, therefore, increasing the receiver's period-$t + 1$ posterior beliefs beyond their value under the status quo.

While the condition on δ is sufficient, it is not necessary. Decreasing revelation probability at time t by ϵ strictly increases the sender's flow payoff in period t. This strict increase can allow for the condition we require to be violated and for the sender to still prefer the perturbed policy to the status quo.

Receiver

Now consider the receiver's willingness to accept a decrease in revelation probability by ϵ in period t in exchange for an increase in period $t+1$ revelation probability by δ. A decrease in r_t decreases the receiver's posterior beliefs about the prospect of a breakthrough in period t, while an increase in r_{t+1} increases these beliefs. As such, the receiver's flow payoffs are lower in period t and higher in period $t+1$. Let their period-t prior beliefs be (p, q) and their value at time t from the status quo policy and the perturbed policy be U_t and $\hat{U}_t$, respectively.

$$U_t\left(work\,|\,m_t = work; R, \beta\right) = \left(\pi\mu\theta B - c\right) + \ldots$$

$$\ldots + \left(1 - \pi\mu\theta\right)\left(1 - r_{t+1}\left(1 - p'q'\right)\right)\left(\pi'\mu'\theta B - c + \left(1 - \pi'\mu'\theta\right)U_{t+2}\right)$$

where (π, μ) are the receiver's period-t posterior beliefs when their prior beliefs are (p, q) and the revelation probability is r_t; (p', q') are the receiver's period-$t+1$ prior beliefs, and (π', μ') are their period-$t+1$ posterior beliefs.

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$$\hat{U}_t\left(work|\, m_t = work; \hat{R}, \hat{\beta}\right) = (\hat{\pi}\hat{\mu}\theta B - c) + \dots$$

$$\dots + (1 - \hat{\pi}\hat{\mu}\theta)\left(1 - (r_{t+1} + \delta)\left(1 - \hat{p}'\hat{q}'\right)\right)\left(\hat{\pi}'\hat{\mu}'\theta B - c + (1 - \hat{\pi}'\hat{\mu}'\theta)\,\hat{U}_{t+2}\right)$$

where terms with hats reflect the fact that under the perturbed policy, the evolution of the receiver's beliefs differs from their evolution under the status quo. Since the receiver's value at time $t+2$ is determined by their period-$t+2$ prior beliefs, and these, in turn, are determined by the revelation probabilities in time t and $t+1$, $\hat{U}_{t+2}$ may not be equal to U_{t+2}. Unlike the sender's, the receiver's beliefs (and, hence, their values) are affected by the revelation probabilities. For any $\epsilon > 0$, the corresponding δ that keeps the receiver's period t value from working weakly greater than the status quo value must raise their period-$t + 1$ posterior beliefs about the prospects of a breakthrough above the status quo level. This is necessary to compensate the receiver for the lower flow payoff in period t under the perturbed strategy. In addition to generating higher flow payoffs in period $t+1$, higher period-$t+1$ posterior beliefs also induce high period -$t+2$ prior beliefs, which increase the receiver's value. As such, $\hat{U}_{t+2} \geq U_{t+2}$. In general, when the value of δ is chosen such that $U_t = \hat{U}_t$, say, it is possible to choose a value that increases $\hat{U}_t$ through a mixture of (i) higher period-$t+1$ flow payoffs, as well as (ii) a higher value of $\hat{U}_{t+2}$. It is sufficient to require the increase in period-$t + 1$ revelation probability, δ, to guarantee that $\hat{U}_t \geq U_t$ through the flow payoffs in periods t and $t + 1$ alone, because the receiver's period-$t + 2$ continuation value will, by the previous

argument, automatically be higher. For any decrease in period-t revelation probability, an increase in period-$t+1$ revelation at least as large as δ guarantees that the receiver's value at time t is at least as high as their value under the status quo policy.

Condition 2. Sufficient condition for the receiver to accept ϵ lower revelation probability in period t and δ higher probability in period $t+1$:

$$\delta \geq \hat{\delta}_{min} = \frac{\epsilon pq}{1 - r_t + \epsilon} \left[\frac{B\theta \left[2 \left(1 - pq \left(1 - \theta \right) \right) - \theta \left(1 + p \right) \right] + c \left[\left(1 - pq \right) \theta - \left(1 - p \left(q \left(1 - \theta \right) + \theta \right) \right) r_{t+1} \right]}{c \left(1 - \left(1 - pq \right) r_t \right) \left(1 - p \left(q \left(1 - \theta \right) + \theta \right) \right)} \right]$$

When the δ required to keep the receiver's payoff sufficiently high is lower than the maximal δ the sender is willing to offer, the sender can improve their payoff by decreasing the revelation probability. This is guaranteed to hold whenever $\hat{\delta}_{min} \leq \hat{\hat{\delta}}_{max}$.

1.3.8   Examples

A priori, it is not clear whether the sender earns a higher payoff from the indifference policy or from the maximal promise policy - the one in which they keep the receiver in the game for the longest time possible. The examples below show that for some parameter values, the sender can do strictly better by adopting the indifference strategy, and for others, they can do better by adopting the maximal promise strategy. Recall that any policy used by the sender involves an initial $\bar{T}$ period with no information sharing. The indifference policy differs from promise policies beyond this initial phase. Therefore, we consider the sender's value beyond this initial point in the comparison

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below. The sender's payoff from the indifference policy involves an infinite number of terms, since the receiver remains in the game with positive probability in every period. Below, we derive bounds on their value beyond period $\bar{T}$:

$$(1 - r_t) [\theta b_t B + (1 - \theta b_t) (1 - r_{t+1}) \theta b_{t+1} B] \leq V(R^*, \beta^* | t) \leq \frac{(1 - r_t) \theta b_t B}{1 - (1 - r_t) (1 - \theta b_t)}$$

These inequalities follow from the fact that $b_t \geq b_{t+1}$, and $r_t \leq r_{t+1}$ for all t. The sender's value from promise policies involve multiple stages of computation: first, we find the maximal time the receiver is willing to stay before receiving information. Given this time, we calculate the sender's expected payoff from the receiver remaining in the game until that time. When $B = 4$, $\theta = 0.4$, $c = 0.4$ and $p_0 = 0.8$, the sender conceals all information for the first 8 periods of the game. After this point, they can use a maximal promise policy to keep the receiver in the game for an additional period after which they fully reveal the outcome history. The receiver is only willing to stay in the game for one more period beyond the voluntary 8 during which they stay without receiving any information. However, the sender can do strictly better by using the indifference policy. When, on the hand, $\theta = 0.35$, the receiver still stays in the game for 8 periods without receiving any information, but the sender can now use a maximal promise policy to keep them in the game for an additional 3 periods beyond the voluntary phase. They now prefer the maximal promise policy to the indifference policy. An increase in θ has two effects: the probability of a success in a given period is now higher, conditional

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on the project being good. However, in the absence of good news, beliefs deteriorate faster. Lower values of θ make players more lenient on outcome histories. It may well be the case that the project is good but a success is yet to arrive.

1.4 Commitment

In this section, we consider the game in which the receiver has commitment power, and both players choose their strategies at the beginning of play. The sender chooses a disclosure policy, and the receiver chooses a mapping from disclosure policies to strategies in $\mathcal{B}$, simultaneously. To distinguish between these strategies and those described above, we refer to them as commitment strategies. A commitment strategy, C, is a mapping $C : \Sigma \to \mathcal{B}$, and the space of all such strategies is $\mathcal{C}$.

The strategies defined in subsection 1.2.3 map the sender's messages to distributions over actions. When the players commit to their strategies at the beginning of the game, messages can have different meanings, depending on the disclosure policy in use by the sender (recall that beliefs are induced by the pair of disclosure policy and message). The receiver's decisions are determined by their beliefs. When the disclosure policy is known in advance (in the mechanism design problem analyzed earlier), messages alone determine the receiver's beliefs. When the disclosure policy is not known, however, the pair of disclosure policy and message together determine their beliefs. Both strategies as well as commitment strategies are equivalent to mappings from beliefs to distributions over actions.

A pair, (σ^*, C^*), is an equilibrium if $(\sigma^*, C(\sigma^*))$ is a Nash equilibrium of the simultaneous move game, and the receiver's strategy, $\beta^* = C^*(\sigma^*)$, is sequentially rational on the equilibrium path. Namely, it must be the case that, given that the sender is committed to σ^*, the receiver would not want to deviate from β^* at any point during the game. Conversely, given that the receiver is committed to β^*, the sender would not want to deviate from σ^* at the beginning of the game. Whereas the receiver makes decisions along the path of play, the sender makes all their choices at the beginning of the game, since their disclosure policy determines how messages are interpreted. Changes in messages over time would render this commitment meaningless, and the receiver would not be able to correctly interpret messages.

Definition 3. An **equilibrium** is a pair (σ^*, C^*) such that:

$$U(\sigma^*, C^*(\sigma^*)|0) \geq U(\sigma^*, C(\sigma^*)|0) \ \forall \ C \in \mathcal{C},$$

$$V(\sigma^*, C^*(\sigma^*)|0) \geq V(\sigma, C(\sigma)|0) \ \forall \ \sigma \in \Sigma,$$

We restrict attention to commitment strategies that induce the same mappings from beliefs to actions for any $\sigma \in \Sigma$. For example, suppose that for some $\sigma_1 \neq \sigma_2 \in \Sigma$, and $C \in \mathcal{C}$, $C(\sigma_1) = \beta_1$ and $C(\sigma_2) = \beta_2$. Let (π_t, μ_t) be the receiver's period-t posterior beliefs under (σ_1, β_1) as well as under (σ_2, β_2) induced by messages m_{1t} and m_{2t}, respectively. If $\beta_1(m_{1t}) \neq \beta_2(m_{2t})$, then C does not induce the same mapping from beliefs to actions for σ_1 and σ_2. Essentially, we restrict attention to commitment

strategies that map a receiver's beliefs to action distributions. Call this restricted space $\tilde{\mathcal{C}} \subseteq \mathcal{C}$.

The main result of this section shows that a range of outcomes, including outcomes obtained through promise policies, can be sustained in equilibrium. In particular, the first best (for the receiver) full information outcome can be sustained. Before we state the result, let $\hat{T}$ denote the time at which information is released in the maximal promise policy.

Theorem 2. *Suppose assumption 1 holds, and suppose that the receiver chooses commitment strategies from $\tilde{\mathcal{C}}$. For every $T \in \left[\bar{T}, \hat{T} \right]$, there exists an equilibrium pair $\left(\beta^T, C^T \right)$ such that the sender conceals all information until time T, and the receiver chooses to stay until time T, at which point they quit if a success has not arrived.*

1.5 Conclusion

In this paper, I study the role of disclosure policies on the incentives of a receiver who decides when to quit a project. Although the payoffs that accrue to the players from the success of the project are the same, the sender does not incur the cost of working. As a result, she would like the project to continue indefinitely. The receiver, on the other hand, does incur a cost, and only prefers to stay on the project when he is sufficiently optimistic about the project's chances of success. There are two key forces at play: (i) the balance between promising to give the receiver good news, and the resulting bad news generated when this good news does not arrive; and (ii) the trade-off

between releasing some information today, and delaying revelation at the same time as increasing the amount of information released to the agent in order to compensate them for waiting. The first trade-off appears in many settings in which an agent commits to an information revelation policy used to persuade another agent to take some actions that may not be in line with their interests. The second trade-off appears in dynamic settings, but often does not play a crucial role. In Ely (2017), for example, the myopic optimal disclosure policy is also an optimal policy when the receiver is patient and strategic. In the above setting, I provide conditions under which, and examples in which, this is not the case. The sender can take advantage of the receiver's patience by promising them more information if they choose to stay in the project longer. Why does the sender benefit from the receiver's patience in this framework? There are two important differences between the model I study here and the one studied in Ely (2017): (i) the sender does not receive flow payoffs in my setting, and (ii) she is uncertain about the quality of the project. These two differences imply that the value to the sender from the receiver staying for one more period is large: she may receive a perfect good news signal about the project in this additional period, which would guarantee her receipt of the lump sum payoff, B. Promising to reveal information in the future allows the sender the opportunity to learn that the project is good, and relay that information to the receiver. These considerations do not arise when the sender knows the state and can choose when to communicate it to the receiver. Other than the flow payoff that accrues, there is no informational value from waiting an additional period.

I study two natural disclosure policies: the indifference policy and the maximal

promise policy. I derive conditions under which the sender would prefer to deviate from the indifference policy by delaying revelation and compensating the receiver with more information later on. Through examples, I demonstrate that neither the indifference policy nor the maximal promise policy is optimal for all parameter values. This is surprising because the indifference policy keeps the receiver's continuation value at 0, making his individual rationality constraint bind. On the other hand, the maximal promise policy generates strictly positive continuation payoffs for the receiver: when they choose to stay until some period, T, their continuation value grows as they approach T and they anticipate receiving the promised information.

Many open questions remain. Are there sufficient conditions for the optimality of the indifference policy? This questions is more difficult to answer than whether or not there are conditions for the optimality of the maximal promise policy. This is because continuation payoffs after a deviation in which the sender delays releasing information are strictly positive, hence higher than those under the indifference policy. This gives the sender more room to delay information further, giving the promise policy an advantage over the indifference policy. A sufficient condition must, therefore, take this possibility into account. In general, expressions for the receiver's continuation values are less tractable than those in existing models since their relevant beliefs are two dimensional with non-linear transition rules. This makes studying deviations from the indifference policy more cumbersome. Nevertheless, the findings above point towards an important role that promise policies can play in dynamic persuasion games.

1.6 Proofs

1.6.1 Proof of Proposition 1

Fix a disclosure policy, $\sigma = \{\sigma_t\}$, such that $\sigma_t\left(.\,|\,y^{t-1}\right) \in \Delta\left(M_t\right)$. We can partition M_t into Q_t and $M_t \backslash Q_t$ such that the receiver quits whenever they observe a message $m_t \in Q_t$ and chooses to work otherwise. Let $r_t = \int_{Q_t} \sigma_t\left(m_t|\,y^{t-1}\right) dm_t$, and define R_t^σ as follows:

$$
R_t^\sigma\left(m|\,y^{t-1}\right) = \begin{cases} r_t & if\ m = quit \\[2mm] 1 - r_t & if\ m = work \end{cases}
$$

Since the receiver chooses to quit whenever they receive a message $m_t \in Q_t$ under σ, they choose to quit whenever they receive the message "quit" under R^σ. Similarly, they choose to work whenever they receive the message "work" under R^σ, since working is preferred to quitting whenever they receive a message $m_t \in M_t \backslash Q_t$ under σ. To see why the players' values are the same under (σ, β) as they are under (R^σ, β), notice that the receiver stays in the game after each history with exactly the same probability under both pairs.

$\square$

1.6.2 Proof of Proposition 2

Let T be the first period such that $\pi_t \mu_t \geq \pi_{t+1} \mu_{t+1}$ for all $t \geq T$. The receiver's expected payoff from T onwards when they choose to work whenever they receive the message ɪ*work*ɪ is the following:

$$U\left(R^\sigma, \beta \mid T-1\right) = \pi_T \mu_T \theta B - c + (1 - \pi_T \mu_T \theta) U\left(R^\sigma, \beta \mid T\right)$$

$$= \pi_T \mu_T \theta B - c + (1 - \pi_T \mu_T \theta) \left[\pi_{T+1} \mu_{T+1} \theta B - c + (1 - \pi_{T+1} \mu_{T+1} \theta) U\left(R^\sigma, \beta \mid T+1\right)\right]$$

$$= \sum_{t=T}^{\infty} (\pi_t \mu_t \theta B - c) \prod_{s=1}^{t-T} (1 - \pi_{T+s-1} \mu_{T+s-1} \theta)$$

Since $\pi_t \mu_t \geq \pi_{t+1} \mu_{t+1}$, it is clear that once $(\pi_t \mu_t \theta B - c) < 0$ (or $\pi_t \mu_t < \frac{c}{\theta B}$) for some t, the expression is negative for all subsequent periods, and quitting in period t maximizes the receiver's payoff conditional on reaching period T. For any choice rule prior to period T, the receiver maximizes their expected payoff from period T onwards with a cutoff strategy. Hence, their optimal strategy is eventually a cutoff strategy.

□

1.6.3 Proof of proposition 3

Suppose that a success has occurred. The sender recommends that the receiver works in every period thereafter. Since the policy is incentive compatible, the receiver complies with the recommendation. Only good projects yield successes, which means that the

project must be good. As such, in every period, there is a probability of θ that a breakthrough arrives, with a payoff of B. With probability $(1 - \theta)$ there is a next period with the same prospects. The sender's value is below:

$$
\begin{aligned}
V &= \theta B + (1 - \theta) \theta B + (1 - \theta)^2 \theta B + (1 - \theta)^3 \theta B + \ldots \\
&= \theta B \left(1 + (1 - \theta) + (1 - \theta)^2 + (1 - \theta)^3 + \ldots\right) \\
&= \theta B \left(\frac{1}{1 - (1 - \theta)}\right) = B
\end{aligned}
$$

$\square$

1.6.4   Proof of proposition 4

Suppose that $U_s\left(P^T\right) \geq 0$ for some $t \leq s < T$. We can write this condition, while expanding $U_s\left(P^T\right)$, as follows:

$$
U_s\left(P^T\right) = (p_s q_s \theta B - c) + (1 - p_s q_s \theta)(p_{s+1} q_{s+1} \theta B - c) + \ldots + \prod_{\tau=s}^{T-1}(1 - p_\tau q_\tau \theta) p_T q_T \left(B - \frac{c}{\theta}\right)
$$

Since $t > \bar{T}$, every term of the form $(p_s q_s \theta B - c)$ in the above expression is negative. The expression for $U_{s+1}\left(P^T\right)$ can be written in terms of $U_s\left(P^T\right)$ in the following way:

$$
U_{s+1}\left(P^T\right) = \frac{U_s\left(P^T\right) - (p_s q_s \theta B - c)}{(1 - p_s q_s \theta)} \geq U_s\left(p^T\right),
$$

where the inequality follows from the fact that $(p_s q_s \theta B - c) \leq 0$ and $(1 - p_s q_s \theta) < 1$.

$\square$

1.6.5 Proof of Theorem 1

We will show that (1) given R^*, β^* is (i) sequentially rational on the equilibrium path, and (ii) optimal. (2) Given β^*, R^* is optimal for the sender.

Let b_t be the sender's period t beliefs that the project is good, where $t \geq \bar{T}$. Let p_t be the receiver's belief that $\omega = 1$, and q_t be their belief that there has been one success, conditional on $\omega = 1$, all at the beginning of period t (their period t prior beliefs - see subsection 1.3.1 for details). We consider the class of disclosure policies in which the sender always recommends to the receiver that they stay when $S(y^t) = 1$, and recommends that they quit with some probability r_t when $S(y^t) = 0$. The choice of r_t affects the receiver's period-t posterior beliefs, which are a function of their prior beliefs and the probability r_t in the following way (where we have suppressed time subscripts):

$$\pi(p, q, r) = \frac{p(1 - r(1 - q))}{1 - r(1 - pq)}$$

$$\mu(q, r) = \frac{q}{1 - r(1 - q)}$$

It can be shown that both π and μ are increasing in r. The reason is that the probability r represents the credibility of the sender's messages, or the extent to which

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their disclosure policy is in line with the receiver's interests. For example, consider a disclosure policy with $r_t = 1$: this means that whenever the sender observes a history with no successes, they tell the receiver to quit, which is exactly what the receiver would like to do. When $r_t = 1$, and the sender recommends to the receiver that they should stay, the receiver knows for sure that the sender must have observed a success. The receiver's period-t posterior beliefs, therefore, jump to 1. In the other extreme, when $r_t = 0$, the sender always tells the receiver to stay. As a result, the message does not convey any credible information, and the receiver's period-t posterior beliefs are unchanged - they quit even when the sender sends the message "stay". The value of r_t, therefore, simultaneously determines (i) how often the sender recommends that the receiver quit when they observe no successes, and (ii) how likely it is that a success has occurred given that the sender recommends "stay". The sender would like to minimize how often they recommend that the receiver quit (by minimizing r_t), as well as ensuring that their ıstay] message is credible. This trade-off is captured by the sender's value function from the optimal strategy after histories in which no success has been observed. The value function, $V(b, p, q)$, takes their beliefs, as well as the prior beliefs of the receiver into account, and can be described as follows:

$$V(b, p, q) = \max_{r \in [0,1]} \{(1 - r)[b\theta V(1, \pi(p, q, r), \mu(q, r)) + (1 - b\theta) V(b', p'(\pi, \mu), q'(\mu))]\}$$

$$subject\ to \quad \pi(p, q, r) \mu(q, r) \theta B \geq c$$

The constraint ensures that the sender's "stay" message is credible - the receiver's period-t posterior beliefs must be such that when they do receive the stay message, they actually prefer to stay. $V(1, \pi, \mu)$ is the sender's value when a success arrives and their beliefs that $\omega = 1$ jumps to 1. When a success does not arrive, their beliefs about the project deteriorate in the next period to b':

$$b' = \frac{(1 - \theta) b}{1 - \theta b}$$

Recall that the transition rule for receiver's beliefs is the following:

$$p' = \frac{\pi (1 - \mu) + \pi \mu (1 - \theta)}{1 - \pi \mu \theta}$$

$$q' = \frac{(1 - \mu) \theta + \mu (1 - \theta)}{1 - \mu \theta}$$

Namely, whenever the game does not end, the receiver's prior beliefs in the next period are given by (p', q') given that their previous period's posterior beliefs were (π, μ). Suppose the sender plays the strategy σ^* in which they reveal the state to the receiver with probability r_t in period t such that $\pi_t \mu_t \theta B = c$. Namely, the sender reveals just enough information to bring the receiver's beliefs up to the level at which they are indifferent between staying and quitting (we assumed that the receiver stays when they are indifferent). In period t, given a history with no success, the sender's value function is described below:

$$V_t^* = (1 - r_t) \left[\theta b_t V\left(1, \pi_t, \mu_t\right) + (1 - \theta b_t)\left(1 - r_{t+1}\right)\left[\theta b_{t+1} V\left(1, \pi_{t+1}, \mu_{t+1}\right) + (1 - \theta b_{t+1}) V_{t+2}\right]\right]$$

Consider an alternative strategy, $\hat{\sigma}$, in which, whenever a success has not yet occurred, the sender reveals the state with higher probability in period t $(\hat{r}_t > r_t)$, pushing the receiver's posterior beliefs above the threshold level required for them to stay. In period $t+1$, the sender reveals the state with smaller probability $(\hat{r}_{t+1} < r_{t+1})$, since the receiver's period-$t + 1$ prior beliefs are now higher than they would have been had the sender sent the message "quit" with probability r_t instead of $\hat{r}_t$. This is beneficial for the sender, since they now send the message "stay" with higher probability. The trade-off is the following: the sender must send the message "stay" with lower probability in period t in order to send it with higher probability in period $t + 1$. The sender's value from this strategy, $\hat{V}$, is described below.

$$\hat{V}_t = (1 - \hat{r}_t)\left[\theta q_t V\left(1, \hat{\pi}_t, \hat{\mu}_t\right) + (1 - \theta q_t)\left(1 - \hat{r}_{t+1}\right)\left[\theta q_{t+1} V\left(1, \hat{\pi}_{t+1}, \hat{\mu}_{t+1}\right) + (1 - \theta q_{t+1}) V_{t+2}\right]\right]$$

First, notice that $\pi_{t+1} = \hat{\pi}_{t+1}$ and $\mu_{t+1} = \hat{\mu}_{t+1}$, since the sender only needs to push the receiver's period-$t + 1$ posterior beliefs to the threshold required for them to stay. To determine whether such a deviation is profitable, we calculate the difference $V_t^* - \hat{V}_t$:

$$V_t^* - \hat{V}_t = \theta q_t \left[(1 - r_t) V (1, \pi_t, \mu_t) - (1 - \hat{r}_t) V (1, \hat{\pi}_t, \hat{\mu}_t) \right] + \ldots$$

$$\ldots + (1 - \theta q_t) \theta q_{t+1} V (1, \pi_{t+1}, \mu_{t+1}) \left[(1 - r_t) (1 - r_{t+1}) - (1 - \hat{r}_t) (1 - \hat{r}_{t+1}) \right] + \ldots$$

$$\ldots + (1 - \theta q_t) (1 - \theta q_{t+1}) V_{t+2} \left[(1 - r_t) (1 - r_{t+1}) - (1 - \hat{r}_t) (1 - \hat{r}_{t+1}) \right]$$

$$= \theta q_t \left[(1 - r_t) V (1, \pi_t, \mu_t) - (1 - \hat{r}_t) V (1, \hat{\pi}_t, \hat{\mu}_t) \right] + \ldots$$

$$\ldots + (1 - \theta q_t) \left[(1 - r_t) (1 - r_{t+1}) - (1 - \hat{r}_t) (1 - \hat{r}_{t+1}) \right] \times \ldots$$

$$\ldots \times \left[\theta q_{t+1} V (1, \pi_{t+1}, \mu_{t+1}) + (1 - \theta q_{t+1}) V_{t+2} \right]$$

First, note that the first bracketed expression is positive since $(1 - r_t) > (1 - \hat{r}_t)$ and $V (1, \pi_t, \mu_t) = V (1, \hat{\pi}_t, \hat{\mu}_t)$. Indeed, it can be shown that $V (1, \pi_t, \mu_t) = V (1, \hat{\pi}_t, \hat{\mu}_t) = V (1, \pi_{t+1}, \mu_{t+1}) = B$. To see this, notice that the sender always recommends to the receiver that they should stay after every period with one success. The receiver obeys this recommendation and stays forever, eventually achieving a breakthrough. Since there is no discounting, and a breakthrough arrives with probability 1 whenever the receiver stays forever, the sender's payoff is simply B (see proposition 3 for a proof).

The expression above can be simplified as follows:

$$V_t^* - \hat{V}_t = \theta q_t \left[\hat{r}_t - r_t \right] B + (1 - \theta q_t) \left[(1 - r_t) (1 - r_{t+1}) - (1 - \hat{r}_t) (1 - \hat{r}_{t+1}) \right] \left[\theta q_{t+1} B + (1 - \theta q_{t+1}) V_{t+2} \right]$$

Note that r_t is determined by the condition $\pi_t \mu_t \theta B = c$, which is a function of p_t,

q_t, θ, B and c. Solving this identity for r_t yields the following expression:

$$r_t = \frac{c - p_t q_t \theta B}{c(1 - p_t q_t)},$$

which is positive since $c > p_t q_t \theta B$ for all $t \geq \bar{T}$. Consider any $\hat{r}_t = r_t + \epsilon$ for some $\epsilon > 0$, and notice that the resulting period-t posterior beliefs are as follows:

$$\hat{\pi}_t (p_t, q_t, \hat{r}_t) = \frac{p_t (1 - \hat{r}_t (1 - q_t))}{p_t (1 - \hat{r}_t (1 - q_t)) + (1 - \hat{r}_t) (1 - p_t)}$$

$$\hat{\mu}_t (q_t, \hat{r}_t) = \frac{q_t}{1 - \hat{r}_t (1 - q_t)}$$

Applying the transition rules for b', p' and q' from above yields the following expressions, where hats denote quantities under the alternative strategy, $\hat{\sigma}$:

$$b_{t+1} = \frac{(1 - \theta) b_t}{(1 - \theta) b_t + 1 - b_t}$$

$$\hat{p}_{t+1} = \frac{\hat{\pi}_t (1 - \hat{\mu}_t) + \hat{\pi}_t \hat{\mu}_t (1 - \theta)}{\hat{\pi}_t (1 - \hat{\mu}_t) + \hat{\pi}_t \hat{\mu}_t (1 - \theta) + 1 - \hat{\pi}_t}$$

$$\hat{q}_{t+1} = \frac{(1 - \hat{\mu}_t) \theta + \hat{\mu}_t (1 - \theta)}{1 - \hat{\mu}_t \theta}$$

Next, $\hat{r}_{t+1}$ chosen such that $\hat{\pi}_{t+1} \hat{\mu}_{t+1} \theta B = c$. Recall that $\hat{\pi}_{t+1}$ is a function of period-$t + 1$ prior beliefs, $(\hat{p}_{t+1}, \hat{q}_{t+1})$, as well as $\hat{r}_{t+1}$:

$$\hat{\pi}_{t+1} = \frac{\hat{p}_{t+1} \left(1 - \hat{r}_{t+1} \left(1 - \hat{q}_{t+1}\right)\right)}{\hat{p}_{t+1} \left(1 - \hat{r}_{t+1} \left(1 - \hat{q}_{t+1}\right)\right) + \left(1 - \hat{r}_{t+1}\right)\left(1 - \hat{p}_{t+1}\right)}$$

$$\hat{\mu}_{t+1} = \frac{\hat{q}_{t+1}}{1 - \hat{r}_{t+1} \left(1 - \hat{q}_{t+1}\right)}$$

We now have formulae for all the terms in the expression for $V_t^* - \hat{V}_t$ expressed in terms of p_t, q_t, θ, B, c and ϵ. It can be shown that the expression in brackets, which is the only one that may turn out to be negative, simplifies to the following fraction, which is always positive:

$$\left(1 - r_t\right)\left(1 - r_{t+1}\right) - \left(1 - \hat{r}_t\right)\left(1 - \hat{r}_{t+1}\right) = \frac{\epsilon p_t \theta \left(B\theta - c\right)\left(1 - q_t\right)}{c \left(1 - p_t \left(q_t + \left(1 - q_t\right)\theta\right)\right)} > 0$$

This shows that no "one-shot" two period, deviation by the sender improves upon their payoff from policy σ^*. Now consider multiple period deviations in which the sender keeps the receiver's posterior beliefs after sending message "stay" above the threshold belief required to keep them in the game when they stick to strategy β^*. This is the only other type of admissible deviation, because the receiver quits whenever their beliefs fall below this threshold.

Consider an alternative strategy, $\tilde{\sigma}$, in which the sender deviates in periods t, $t + 1$, and $t + 2$, such that $\tilde{r}_t = \hat{r}_t > r_t$, $\tilde{r}_{t+1} > \hat{r}_{t+1}$ and $\tilde{r}_{t+2} < \hat{r}_{t+2} = r_{t+2}$. Let $\tilde{V}_t$ denote the sender's value in period t from the strategy $\tilde{\sigma}$, $\hat{V}_t$ their value from $\hat{\sigma}$, and V_t^* their value from σ^*. Since $\hat{r}_t = \tilde{r}_t$, it follows that the receiver's period-$t + 1$ prior beliefs are

the same under the two strategies: $(\hat{p}_{t+1}, \hat{q}_{t+1}) = (\tilde{p}_{t+1}, \tilde{q}_{t+1})$. Starting at period $t+1$, the strategies $\tilde{\sigma}$ and $\hat{\sigma}$ now correspond to a pair of strategies one of which restores the receiver's beliefs to the threshold level and the other inflates their beliefs for one period before restoring it in the next, respectively. But these are precisely what the strategies σ^* and $\hat{\sigma}$ do at time t. The result above shows that $V_t^* > \hat{V}_t$ for any $\hat{\sigma}$ with $\hat{r}_t > r_t$ and $\hat{r}_{t+1}$ such that $\hat{\pi}_{t+1} = \pi_{t+1}$. This implies that $\hat{V}_{t+1} > \tilde{V}_{t+1}$. Using this fact, and the expression for $\tilde{V}_t$, we show that $V_t^* > \tilde{V}_t$.

$$\tilde{V}_t = (1 - \tilde{r}_t) \left[\theta b_t B + (1 - \theta b_t) \tilde{V}_{t+1} \right] < (1 - \tilde{r}_t) \left[\theta b_t B + (1 - \theta b_t) \hat{V}_{t+1} \right] = \hat{V}_t < V_t^*$$

Now consider any strategy, σ^r, that deviates from σ^* for r periods - the sender changes the recommendation probabilities r_t for $r - 1$ periods, potentially changing the receiver's posterior beliefs, then restores the receiver's posterior beliefs in the rth period to the threshold level and returns to the strategy σ^* thereafter. We have shown that the sender does not benefit from such a deviation when $r = 2$ or 3.[9] We will prove that such a strategy is never beneficial for any $r \in \mathbb{N}$ by induction. We begin by assuming that it is not beneficial for $r = k$. Denote the sender's value at time s by V_s^k when the sender deviates deviates for k periods, and by V_s^{k+1} when they deviate for $k + 1$ periods. Let the first period of deviation be t and assume that $V_t^* \geq V_t^k$; we will show

[9]It is immediate that the sender does not benefit when $r = 1$, since inflating the receiver's beliefs for one period then returning to σ^* is costly for that one period (since $\hat{r}_t$ is higher than r_t), and there is no benefit in subsequent periods.

that $V_t^* \geq V_t^{k+1}$. Notice that V_t^k can be expressed as follows:

$$V_t^k = \theta B \left[\sum_{s=0}^{k-2} q_{t+s} \prod_{i\leq s} (1 - \hat{r}_{t+i}) \prod_{j\leq s-1} (1 - \theta b_{t+j}) \right] + \prod_{i\leq k-2} (1 - \hat{r}_{t+i}) (1 - \theta b_{t+i}) V_{t+k-1}^k,$$

where $t + k - 1$ is the last period in which the sender deviates from σ^* under σ^k -

the period in which the receiver's posterior beliefs are restored to the threshold level.

Let the receiver's prior beliefs at the start of this period be $(\hat{p}_{t+k-1}, \hat{q}_{t+k-1})$ and the

sender's be b_{t+k-1},; the sender chooses $\hat{r}_{t+k-1}$ such that the receiver's posterior beliefs

are at the threshold. Indeed, V_{t+k-1}^k can be expressed as follows:

$$V_{t+k-1}^k = (1 - \hat{r}_{t+k-1}) \left[B\theta b_{t+k-1} + (1 - \theta b_{t+k-1}) V_{t+k}^* \right] = V_{t+k-1}^* (b_{t+k-1}, \hat{p}_{t+k-1}, \hat{q}_{t+k-1})$$

Now consider V_t^{k+1} and suppose that the first $k - 1$ deviations are exactly those

employed in σ^k (such a strategy is always possible to construct). The sender's value in

period t, V_t^{k+1}, can be expressed as follows:

$$V_t^{k+1} = \theta B \left[\sum_{s=0}^{k-2} b_{t+s} \prod_{i\leq s} (1 - \hat{r}_{t+i}) \prod_{j\leq s-1} (1 - \theta b_{t+j}) \right] + \prod_{i\leq k-2} (1 - \hat{r}_{t+i}) (1 - \theta b_{t+i}) V_{t+k-1}^{k+1},$$

Notice that V_{t+k-1}^{k+1} can be further unpacked:

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$$V_{t+k-1}^{k+1} = (1 - \tilde{r}_{t+k-1}) \left[B\theta q_{t+k-1} + (1 - \theta q_{t+k-1})(1 - \tilde{r}_{t+k}) \left[B\theta q_{t+k} + (1 - \theta q_{t+k}) V_{t+k+1}^* \right] \right],$$

where $\tilde{r}_{t+k-1} \geq \hat{r}_{t+k-1}$ and $\tilde{r}_{t+k} \leq \hat{r}_{t+k}$. But this is the same as the sender's value at time $t + k - 1$ with priors $(b_{t+k-1}, \hat{p}_{t+k-1}, \hat{q}_{t+k-1})$ of a 2-period deviation from σ^*, which we have already shown is never beneficial. This implies the following:

$$V_{t+k-1}^* (b_{t+k-1}, \hat{p}_{t+k-1}, \hat{q}_{t+k-1}) \geq V_{t+k-1}^{k+1}$$

Combining the inequalities established above, we can conclude that σ^* is preferred by the sender to any $k + 1$ period deviation:

$$V_t^* \geq V_t^k \geq V_t^{k+1}$$

This completes the proof.

$\square$

Chapter 2

Stability in a many-to-one Matching Model with Externalities among Colleagues

2.1 Introduction

Externalities can arise among agents in many two-sided settings including school choice and the labor market. A student's decision to attend a school, for example, may depend on the characteristics of the school itself as well as on the other students attending that school. A worker's decision to join a particular firm may depend on the firm itself as well as on their prospective colleagues. In this paper, I study a two-sided matching model in which agents on one side of the market exert externalities on one another, and investigate the conditions under which stable matchings exist.

I complement the existing literature on matching with externalities by analyzing a many-to-one setting in which firms match with many workers who prefer to join larger firms. I find that stable matchings exist but only under certain conditions: there can only be two firms in the market, and these firms must have responsive preferences.

The results in this paper are related to those in Dutta and Masso (1997), who also study a many-to-one matching model with externalities among workers. They derive restrictions on the preferences of agents that precipitate a non-empty core.[1] They find, for example, that core matchings exist when firms have substitutable preferences and workers care about firms first, then consider the workers they employ in a lexicographic manner. In contrast, I focus on stable matchings, and allow workers to have preferences that are not necessarily lexicographic. The main take-away from their analysis is that allowing for more general preferences may result in an empty core and, hence, no stable

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[1]All matchings in the core are stable.

matching.

The results below are also related to those in Fisher and Hafalir (2016), who study a one-to-one matching model with externalities among all agents. In their set-up, both workers and firms prefer assignments with more matches, externalities affect all agents in a symmetric way, and agents do not take the effect of their actions on the level of externalities into account. They show that under these conditions, a stable matching exists. In contrast to this approach, I assume that externalities only affect workers, and allow them to take the effect of their actions on the level of externalities into account when they make their matching decisions. I use similar fixed point methods to show that a stable matching exists under these assumptions.

Many-to-one matching models with externalities are also analyzed by Bando (2012) and Salgado-Torres (2013) using an approach pioneered by Sasaki and Toda (1996) and refined by Hafalir (2008). In these papers, agents are endowed with 'estimation functions' that describe their beliefs about the outcomes that arise when they deviate from existing assignments. Pycia and Yenmez (2015) study a much more general many-to-many matching model with contracts in which agents on both sides of the market are affected by externalities. They derive restrictions on preferences that guarantee the existence of stable outcomes. A similar setting to the present model (but with transfers) has been studied by Lee (2014). The study investigates the existence of market tipping (in which only one platform prevails) or market splitting (in which two platforms coexist) equilibria in a model with firms that contract with two competing platforms.

The existence of stable matchings in the presence of externalities turns out to be uncommon. Whereas agents in markets without externalities have preferences over agents on the other side of the market, in the presence of externalities, each agent may have preferences over all possible matching outcomes that can arise. As such, models with externalities often do not admit stable matchings for arbitrary preference profiles.[2] Motivated by this observation, and eschewing restrictions on preferences, Echenique and Yenmez (2007) propose an algorithm that finds core matchings (when they exist) in a model with externalities among colleagues. The approach I take in this paper is different and in line with studies that look for sufficient conditions for existence. Despite the simplicity of the setting I study in this paper, stable matchings only exist when there are two firms in the market. This result highlights the tradeoff between allowing for more complex interdependence between agents on the one hand, and finding stable matchings on the other.

The main result of the paper comes at the end of section 2.3. The proof uses Tarski's fixed point theorem to show that a fixed point, which corresponds to the externalities generated by a myopic-stable matching, exists. Roughly speaking, existence of myopic-stable matchings is driven by a monotonic relationship between the number of workers a firm starts out with, and the number of workers it ends up with at the end of a 'round' of matching. Assuming that firms have responsive preferences ensures that a

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[2]A stable matching always exists in the classical one-to-one matching model (Gale and Shapley (1962)). In the many-to-one matching model (see chapters 5 and 6 in Roth and Sotomayor (1990)) and the many-to-many matching model (See Echenique and Oviedo (2006)), a stable matching exists when preferences satisfy a condition called 'substitutability' .

firm always ends up with more workers than it would have had it begun the round with fewer workers. To reach a stable matching from this point requires verifying that workers move around in a way that does not result in any cycles.

Section 2.2 describes the model with two firms, and Section 2.3 outlines the existence proof. In Section 2.4, I show examples of cases in which either no stable matchings exist, or the existence proof fails to go through when the sufficient conditions I introduce in Section 2.3 are violated. I also describe an example in which even when these conditions hold, stable matchings fail to exist in general with more than two firms. Section 2.5 concludes and discusses directions for future research.

2.2 Model

Consider a model in which firms are matched with sets of workers. Let the set of firms be $\mathcal{F} = \{f_1, f_2\}$, the set of workers be $\mathcal{W} = \{w_1, \ldots, w_n\}$, and the set of all agents be $\mathcal{N} = \mathcal{F} \cup \mathcal{W}$. Each firm f_i has a quota, q_i, which determines the number of workers they are matched with (empty slots are filled by multiple copies of the firm itself).[3] Workers have preferences over the firm they match with and the number of workers employed by that firm. Firms have preferences over sets of workers. A matching game, G, is a quintuple, $\{\mathcal{F}, \mathcal{W}, \mu, u_w, u_f\}$, of firms, workers, a matching (defined below), a utility function for workers, and a utility function for firms. Both firms and workers have strict preferences.

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[3]Throughout the remainder of the paper, I assume that $q_i = n$ for both firms and show in Section 2.4 an example of a game with no stable matchings when this assumption is relaxed.

Definition 4. A mapping $\mu : \mathcal{N} \to \mathcal{N}$ is a **matching** if it satisfies the following conditions:

- $\mu(w) \in \mathcal{F} \cup \{w\}$ for each $w \in \mathcal{W}$

- $\mu(f_i) \subset \mathcal{W} \cup f_i : \ |\mu(f_i)| = q_i$ for each $f_i \in \mathcal{F}$

- For each w and f, $\mu(w) = f \iff w \in \mu(f)$

Let $\mathcal{M}$ be the set of all matchings. We define what we mean by an externality correspondence below.[4] When a firm employs a smaller number of workers than its capacity allows, the remaining slots are filled by multiple copies of itself, hence $\mu(f_i) \subset \mathcal{W} \cup f_i$.

Definition 5. An **externality correspondence** is a mapping $e : \mathcal{M} \to \mathbb{R}^2$ that associates a two-dimensional, real-valued vector, $e(\mu)$, to each matching, μ. Elements of this vector are indexed by the firm, so that $e_{f_i}(\mu)$ is the externality associated with firm f_i in matching μ.

The externality $e_{f_i}(\mu)$ is the number of workers matched with firm f_i in matching μ. Next, we define the payoffs of firms and workers at a given matching, μ. **Firm f's payoff** at matching μ is the utility they obtain from matching with the set of workers $\mu(f)$:

$$u_f(\mu) = u_f(\mu(f))$$

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[4]The terminology we use is borrowed from Fisher and Hafalir (2016).

Worker w's payoff derives from the firm they are matched with as well as the level of externality associated with that firm in the going matching:

$$u_w(\mu) = u_w\left(\mu(w), e_{\mu(w)}(\mu)\right)$$

If worker w is single (i.e. $\mu(w) = w$), then $e_{\mu(w)} = 0$. The next set of definitions describe what we mean by stability in the model. The first definition describes individual rationality - one of the requirements that a stable matching must satisfy. A matching is individually rational if no worker prefers to be single than remain with their current match, and no firm prefers to relinquish any one of its workers. Since I assume that firms have responsive preferences, whenever they are willing to fire a set of workers, there must exist one worker they would prefer to exchange for an empty slot.

Definition 6. A matching is **individually rational** if $u_w(\mu) \geq u_w(w, 0)$ for every $w \in \mathcal{W}$, and $u_f(\mu(f)) \geq u_f\left(\{\mu(f) \backslash w\} \cup f\right)$ for every $f \in \mathcal{F}$ and any $w \in \mu(f)$.

Another way in which a matching may be unstable is when there is a worker who prefers to join another firm over the firm they are currently matched with, and that firm prefers to employ the worker.[5] The deviating firm and worker are said to block the going matching.

Definition 7. A matching, μ, is **blocked** by worker w and firm f if $u_w(f, e_f(\mu) + 1) > u_w(\mu(w), e_{\mu(w)}(\mu))$ and $u_f(\mu(f) \cup w) > u_f(\mu(f))$.

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[5]If the firm has already filled its quota of workers, then it must prefer to employ the deviating worker instead of one of its existing employees.

When the firm has already filled its quota, the blocking conditions for the worker and firm become $u_w(f, e_f(\mu)) > u_w(\mu(w), e_{\mu(w)}(\mu))$ and $u_f(\{\mu(f)\backslash w'\} \cup w) > u_f(\mu(f))$ for some $w' \in \mu(f)$, respectively.

Definition 8. A matching is **stable** if it is individually rational, and not blocked by any worker-firm pair.

This is the standard definition of stability, and the only difference between our setting and the classical notion is that workers now take the number of other workers employed at their target firm into account when contemplating whether or not to deviate.[6] Note that although the present setting is very similar to that in Fisher and Hafalir (2016), I allow workers to take their own deviation into account when blocking a matching. This endows workers with some rationality as they anticipate the effect their own deviation has on the size of a firm's workforce. The next section outlines the existence proof.

2.3 Existence of stable matchings

Let $E = \{e(\mu) : \mu \in \mathcal{M}\} \subset \mathbb{R}^{\mathcal{F}}$ be the set of possible vectors of externalities. For each $x \in E$, the auxiliary matching game, $G(x)$, is $\{\mathcal{F}, \mathcal{W}, \mu, u_w^x, u_f\}$, where the only

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[6]A similar notion of stability is used in both Fisher and Hafalir (2016) and Pycia and Yenmez (2015) in which agents take other agents' matching decisions as fixed. This is an alternative approach to the one that models what agents believe will occur after they deviate through estimation functions (Sasaki and Toda (1996); Hafalir (2008); Salgado-Torres (2013)). Bando (2012) takes an intermediate approach and requires that group-wise deviations by a single firm and a set of workers only occur when they are credible. That is, when there are no deviations by a subset of the deviating workers and those already at the firm that make workers in this subset as well as the firm better off.

change made to the ingredients of the matching game G is that workers' utilities are now indexed by the externality vector x and defined below.

Definition 9. For a given matching μ, and vector of externalities x, **worker w's payoff in the auxiliary game** $G(x)$ derives from the firm they are matched with as well as the level of externality associated with this firm according to the externality vector x:

$$u_w^x(\mu) = u_w\left(\mu(w), x_{\mu(w)}\right)$$

Notice that each auxiliary matching game is now a potentially different matching game without externalities. Workers take the externalities associated with a firm as a fixed feature of that firm and, therefore, have a strict ranking of firms that is independent of the matching decisions of other agents. For a given auxiliary game, $G(x)$, we define what it means for a matching to be stable in the usual sense (i.e. in settings without externalities) in $G(x)$ and refer to this concept as auxiliary-stability to draw a distinction between stability in the game with externalities and stability in auxiliary games. As usual, stability requires individual rationality and no-blocking. The corresponding requirements for auxiliary games are defined below.

Definition 10. A matching μ is **auxiliary-rational in** $G(x)$ if $u_f(\mu(f)) \geq u_f\left(\{\mu(f)\backslash w\} \cup f\right)$ for every $f \in \mathcal{F}$ and any $w \in \mu(f)$, and $u_w(\mu(w), x_{\mu(w)}) = u_w^x(\mu(w)) \geq u_w^x(w) = u_w(w, 0)$ for all $w \in \mathcal{W}$.

Auxiliary-rationality is the same as individual rationality in the absence of exter-

nalities. In a matching that satisfies this condition, no firm prefers to relinquish any of its workers, and no worker prefers to be unmatched over remaining with the firm they are matched with.

Definition 11. A matching μ is **auxiliary-blocked in** $G(x)$ by worker w and firm f if $u_w(f, x_f) > u_w^x(\mu(w)) = u_w\left(\mu(w), x_{\mu(w)}\right)$ and $u_f\left(\{\mu(f)\backslash w'\} \cup w\right) > u_f\left(\mu(f)\right)$ for some $w' \in \mu(f)$.

A matching is auxiliary-blocked when, for a fixed vector of externalities, there is some firm-worker pair that would prefer to jointly deviate over remaining with their current matches. When no such firm-worker pair exists, and a matching is auxiliary-rational, it is auxiliary-stable.

Definition 12. A matching μ is **auxiliary-stable in** $G(x)$ if it is auxiliary-rational and not auxiliary-blocked in $G(x)$.

Each auxiliary game, $G(x)$, is a many-to-one matching game without externalities. As a result, an auxiliary-stable matching exists if firms' preferences are responsive (defined below). Responsiveness is a common restriction on preferences in the matching literature (see, for example, Roth and Sotomayor (1990)). Let $S(x)$ denote the set of stable matchings in the auxiliary game $G(x)$: $S(x) = \{\mu : \mu \text{ is auxiliary stable in } G(x)\}$. Below is the definition of responsive preferences, and a result guaranteeing that auxiliary games always admit an auxiliary-stable matching.

Definition 13. (Definition 5.2 in Roth and Sotomayor (1990)) A firm's preference relation, $\succ_f$, over sets of workers is **responsive** (to preference relation $\tilde{\succ}$ over

individual workers) if for any set of workers S, and any $w \notin S$ and $w' \in S$, $S \succ \{S \backslash w'\} \cup \{w\} \iff w \tilde{\succ} w'$.

For each vector of externalities $x \in E$, if firms' preferences are responsive, then $S(x)$, the set of stable matchings of auxiliary game $G(x)$, is non-empty.

Proof. See the appendix. $\square$

Definition 14. A worker's preferences, $\succ_w$, are **increasing in the number of colleagues** if for any two auxiliary games, $G(e_1)$ and $G(e_2)$, such that $e_1 \geq e_2$,

$$f_1 \succ_w f \ in \ G(e_2) \implies f_1 \succ_w f \ in \ G(e_1) \ for \ f \in \{f_2, \emptyset\}$$

$$f_2 \succ_w f \ in \ G(e_1) \implies f_2 \succ_w f \ in \ G(e_2) \ for \ f \in \{f_1, \emptyset\}$$

Preferences are increasing in the number of colleagues if, whenever a worker prefers some firm, say firm 1, over either firm 2 or remaining unmatched when firm one is matched with some number of workers, then they also prefer firm 1 to either firm 2 or being unmatched when firm 1 is matched with more workers. Before stating the next result, we introduce the concept of myopic-stability. This form of stability requires that matchings are auxiliary-stable with respect to the vector of externalities they generate. Formally, let $x_\mu = e(\mu)$ be the set of externalities generated by matching μ. Matching μ is **myopic-stable** if it is auxiliary stable in $G(x_\mu)$.

While we will be able to relax the assumption of myopia, we use it in an intermediate

step in the existence proof. The next step in showing existence of stable matchings in G is to define a correspondence, $T : E \rightarrow E$, in the following way:

$$T(x) = \{e(\mu) : \mu \in S(x)\}$$

In words, $T(x)$ is the set of possible externalities associated with auxiliary-stable matchings of the game $G(x)$. The fixed points of T correspond to myopic-stable matchings, as the next result shows.

Proposition 5. *There is a myopic-stable matching μ^* of the matching game G iff there is an x^* such that $x^* \in T(x^*)$, where $x^* = e(\mu^*)$.*

Proof. See the appendix. $\square$

Having shown that fixed points of T correspond to externalities generated by myopic-stable matchings, we now show that fixed points of T do exist. Our strategy is to apply Tarski's fixed point theorem (Tarski (1955)), which means that we need to guarantee that there is an increasing selection of T (since T is a correspondence) and that E is a partially ordered complete lattice.[7] A selection of the correspondence T is a function $f : E \rightarrow E$ such that for any $x \in E$, $f(x) \in T(x)$. E endowed with a partial order is a complete lattice if any subset of E has a greatest lower bound and least upper bound that exist in E.

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[7]Tarski's fixed point theorem states the following: let $F : X \rightarrow X$ be a map from a complete lattice, $(X, \leq)$, onto itself. If F is increasing ($x \geq y \implies F(x) \geq F(y)$), the set of fixed points of F, $\{x \in X : x = F(x)\}$, is a non-empty complete lattice.

Proposition 6. *If there exists an increasing selection f of T, then a myopic-stable matching, μ^*, exists.*

Proof. By Tarski's fixed point theorem, an increasing selection of T has a fixed point, which is also a fixed point of T. By Proposition 5, this fixed point corresponds to the externalities generated by a myopic-stable matching, μ^*. □

We are now ready to show that myopic-stable matchings exist, as long as three conditions hold: (i) firms have responsive preferences, (ii) workers have preferences that are increasing in the number of colleagues, and (iii) firms do not have capacity constraints. First, we define an appropriate partial order on the set of externalities, E, and show that the resulting partially ordered set is a complete lattice. Then we check that T is increasing with respect to the partial order defined. Once these two conditions are satisfied, Tarski's fixed point theorem applies. The partial order is described in lemma 2.6 in the appendix, which also shows that E equipped with this partial order is a complete lattice. We obtain the existence of myopic-stable matchings in the next proposition.

Proposition 7. *For workers with preferences that are increasing in the number of colleagues, and two firms with responsive preferences and $q_i = n$, a myopic-stable matching exists.*

Proof. See the appendix. □

Showing that stable matchings exist once we have a myopic stable matching is

relatively straightforward, and we present the argument in the penultimate result in this section.

Proposition 8. *For firms with responsive preferences, workers with preferences that are increasing in the number of colleagues and $q_i = n$, if a myopic-stable matching exists, then so does a stable matching.*

Proof. See the appendix. □

Putting these results together yields the main result of the paper: a stable matching exists whenever the preferences of firms and workers satisfy the restrictions above.

Theorem 3. *For two firms with responsive preferences, workers with preferences that are increasing in the number of colleagues and $q_i = n$, a stable matching exists.*

Proof. This follows from propositions 1-4. □

The assumptions we imposed on preferences to ensure that a stable matching exists allow for the application of Tarski's fixed point theorem in Proposition 7. The three restrictions we require, (1) that there are only two firms, (2) that firms have responsive preferences, and (3) that firms have no capacity constraints, are all necessary, as will be shown in the next section.

2.4   Examples

This section contains examples of games without stable matchings that arise when the assumptions on firms' capacities, their preferences and their number are relaxed. In

the first example, I show that when there are capacity constraints, it is possible to find a preference profile in which firms' preferences are responsive, workers' preferences are increasing in the number of colleagues, and yet no stable matching exists.

Responsive preferences are a special case of preferences that satisfy 'substitutability'.[8] In the standard many-to-one matching model without externalities, a stable matching exists whenever firms have substitutable preferences. However, example 2.4 shows that substitutable preferences are not sufficient to guarantee that stable matchings exist - it describes a game with two firms, workers whose preferences are increasing in the number of colleagues and no stable matching.

Finally, example 2.4 shows that when there are three firms (with responsive preferences) and three workers with preferences that are increasing in the number of colleagues, it is possible to construct preferences for workers that preclude the existence of stable matchings.

An example of a game with firms that have capacity constraints in which no stable matching exists:[9] Consider a game with two firms and two workers. Let both firms have responsive preferences with the following underlying ranking of workers: $w_1 \succ w_2 \succ \emptyset$. Let worker i's preferences ($\succ_i$) be increasing in the number of colleagues and order firm-employee pairs in the following way:

$$(f_1, 2) \succ_1 (f_2, 1) \succ_1 (f_1, 1) \succ_1 \emptyset$$

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[8]A preference relation $\succ$ satisfies substitutability if for any $w, w' \in W$, if $w \in S \subseteq W$ and $S \succ S'$, $\forall S' \subseteq W$, then $w \in T \subseteq W \backslash w'$ where $T \succ T'$, $\forall T' \subseteq W \backslash w'$

[9]I would like to thank Ichiro Obara for suggesting this example.

$$(f_2, 1) \succ_2 (f_1, 2) \succ_2 (f_1, 1) \succ_2 \emptyset$$

Namely, worker 1 is preferred by both firms to worker 2, and worker 1 prefers firm 1 to firm 2 only when firm 1 employs one other worker. Meanwhile, worker 2 always prefers firm 2 to firm 1. Finally, suppose that firm 2's capacity is 1: it can only employ one worker, while firm 1 can employ 2 workers. To see that there is no stable matching, consider the following cycle, beginning at matching $\mu_1 = \{\{f_1, w_2\}, \{f_2, w_1\}\}$:

$\mu_1 = \{\{f_1, w_2\}, \{f_2, w_1\}\}$ is blocked by (f_1, w_1), leading to μ_2.

$\mu_2 = \{\{f_1, w_1, w_2\}, \{f_2, \emptyset\}\}$ is blocked by (f_2, w_2), leading to μ_3.

$\mu_3 = \{\{f_1, w_1\}, \{f_2, w_2\}\}$ is blocked by (f_2, w_1), leading to μ_4.

$\mu_4 = \{\{f_1, \emptyset\}, \{f_2, w_1\}, \{\emptyset, w_2\}\}$ is blocked by (f_1, w_2), leading back to μ_1. Notice that starting at any matching of this game leads to some point in this cycle through either an individual or a pair-wise block. Since all agents prefer to be matched, it is clear that no matching in which there is an unmatched agent is stable, so we only need to consider matchings in which all agents are matched. μ_1, μ_2 and μ_3 are, therefore, the only possible candidates for stable matchings. But these matchings are part of the cycle above, hence no stable matchings exist.

An example of a game with firms that have substitutable preferences in which no stable matching exists: Let $F = \{f_1, f_2\}$, $W = \{w_1, w_2\}$, and suppose that firms' preferences are such that $\{w_1\} \succ_f \{w_1, w_2\} \succ_f \{w_2\} \succ_f \emptyset$. First, notice that although these preferences are substitutable, they are not responsive. If they were

responsive to some underlying preference relation, $\succ$, then $w_2 \succ \emptyset$ would imply that $\{w_1, w_2\} \succ_f \{w_1, \emptyset\} = w_1$. Let workers' preferences be such that $(f_1, 2) \succ_w (f_2, 2) \succ_w (f_1, 1) \succ_w (f_2, 1)$, and notice that there are no stable matchings in which any worker is unmatched. This is because any unmatched worker would prefer to match with some firm. Moreover, if one worker is unmatched, then there must be a vacant firm, which would prefer to employ them than remain unmatched. However, any matching in which all workers are matched is not stable either:

$\{\{f_1, w_1, w_2\}, f_2\}$ is blocked by f_1

$\{\{f_1, w_1\}, \{f_2, w_2\}\}$ is blocked by (f_2, w_1)

$\{\{f_1\}, \{f_2, w_1, w_2\}\}$ is blocked by f_2

$\{\{f_1, w_2\}, \{f_2, w_1\}\}$ is blocked by (f_1, w_1) If there is no stable matching in which some workers are unmatched, and none in which all are matched, then stable matchings don't exist in this example.

In the final example, I show that whenever there are 3 firms and 3 workers, it is possible to construct preferences for workers such that no stable matching exists even when firms have responsive preferences and no capacity constraints. **An example of a game with 3 firms in which no stable matching exists:** suppose there are 3 firms and 3 workers.[10] Firms have responsive preferences and all workers are acceptable. The following describes the preferences of workers.

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[10]When there are more than 3 firms and 3 workers, a preference profile can be constructed so that the resulting game has no stable matching by assuming that firms only find 3 workers acceptable, and these 3 workers only find three firms acceptable.

$$w_1 : (f_3, 2) \succ_1 (f_2, 2) \succ_1 (f_2, 1) \succ_1 (f_1, 3) \succ_1 (f_3, 1) \succ_1 \emptyset$$

$$w_2 : (f_2, 2) \succ_2 (f_1, 2) \succ_2 (f_1, 1) \succ_2 (f_2, 1) \succ_2 \emptyset$$

$$w_3 : (f_1, 2) \succ_3 (f_3, 2) \succ_3 (f_3, 1) \succ_3 (f_1, 1) \succ_3 \emptyset$$

Workers' preferences are increasing in the number of colleagues, hence satisfy the condition in Definition 14. However, in every matching between workers and firms, there exists a series of deviations that lead to the cycle described below. The starting matching is $\mu_1 = \{\{f_1, w_2, w_3\} \{f_2, w_1\}, f_3\}$.

μ_1 is blocked by (f_2, w_2), leading to $\mu_2 = \{\{f_1, w_3\} \{f_2, w_1, w_2\} f_3\}$.

μ_2 is blocked by (f_3, w_3), leading to $\mu_3 = \{f_1, \{f_2, w_1, w_2\}, \{f_3, w_3\}\}$.

μ_3 is blocked by (f_3, w_1), leading to $\mu_4 = \{f_1, \{f_2, w_2\}, \{f_3, w_1, w_3\}\}$.

μ_4 is blocked by (f_1, w_2), leading to $\mu_5 = \{\{f_1, w_2\}, f_2, \{f_3, w_1, w_3\}\}$.

μ_5 is blocked by (f_1, w_3), leading to $\mu_6 = \{\{f_1, w_2, w_3\}, f_2, \{f_3, w_1\}\}$.

μ_6 is blocked by (f_2, w_1), leading back to μ_1.

The existence of a cycle alone does not prove that there is no stable matching. It can be shown that starting at any potentially stable matching leads, through a sequence of blocks, to the cycle above. The remainder of this example is relegated to the appendix.

2.5 Conclusion

In this paper, I described a matching model in which workers exert externalities on their colleagues. I showed that a stable matching exists when workers' preferences are increasing in the number of colleagues, and two firms have responsive preferences with no capacity constraints. The results contribute to the literature on matching models with externalities among colleagues by providing sufficient conditions for the existence of stable matchings and examples in which stable matchings no longer exist when these conditions are relaxed. The framework is reminiscent of an oligopolistic labor market with two firms in which workers prefer to join larger firms, or a school choice problem with two schools and students who prefer larger cohorts.

There are some open questions that remain. While the examples in section 2.4 show that stable matchings may not exist, it may be the case that stronger assumptions on the preferences of firms and workers that rule out the type of cycles presented above would allow for stable matchings to exist. One natural question to ask is whether stable matchings emerge when preferences of workers are such that they only care about the relative difference in the number of workers between firms. This would shrink the number of deviations workers make, and could reasonably result in more stable matchings.

Finally, agents see ahead up to the effect of their own deviation. This is a departure from Fisher and Hafalir (2016), in which agents are myopic. However, it would be reasonable to assume that agents can also anticipate the responses of others. In which

case, a suitable notion of stability (for example, farsighted stability as in Mauleon et al. (2011) and Ray and Vohra (2015)) may yield a different conclusion.

2.6 Appendix

Proof. (of lemma 2.3)

Responsive preferences satisfy substitutability. Any many-to-one matching game without externalities in which firms' preferences satisfy substitutability admits a stable matching by theorem 6.5 in Roth and Sotomayor (1990). □

Proof. (of proposition 5)

First, we will show that if μ^* is stable, then $e(\mu^*) \in T(x^*)$.

Suppose, to the contrary, that $e(\mu^*) \notin T(x^*)$. Notice that this implies that $\mu^* \notin S(x^*)$, i.e. that μ^* is not auxiliary stable in $G(x^*)$. This means that either (i) μ^* is not auxiliary-rational, or (ii) it is auxiliary-blocked in $G(x^*)$.

If (i) is true, then either there exists some $w \in \mathcal{W}$ such that $u_w\left(\mu^*(w), x^*_{\mu^*(w)}\right) < u_w(w,0)$, or there exists some $f \in \mathcal{F}$ with $u_f(\mu^*) < u_f(f)$. If the latter is true, then μ^* cannot be stable (a contradiction). Moreover, μ^* cannot be stable if the former is true, since $u_w\left(\mu^*(w), x^*_{\mu^*(w)}\right) = u_w\left(\mu^*(w), e(\mu^*)_{\mu^*(w)}\right) = u_w(\mu^*)$ and $u_w(\mu^*) < u_w(w,0)$ implies that μ^* is not individually rational.

If (ii) is the case, then there exists $w \in \mathcal{W}$ and f$\in \mathcal{F}$ such that $u_w(f, x^*_f) > u_w\left(\mu^*(w), x^*_{\mu^*(w)}\right)$ and $u_f\left(\mu^*(f) \cup w\right) > u_f\left(\mu^*(f)\right)$. Since $x^* = e(\mu^*)$, this violates the stability of μ^*. It must, therefore be the case that a stable μ^* is a fixed point of T.

Next, we show that a fixed point of T must be generated by a stable matching. Let $x^* \in T(x^*)$ such that $x^* = e(\mu^*)$ - such a matching exists by non-emptiness of T. Now suppose that μ^* is not stable. Either (i) μ^* is not individually rational, or else (ii) it is blocked by some $w \in \mathcal{W}$ and $f \in \mathcal{F}$.

(i) in the first case, the existence of some $f \in \mathcal{F}$ such that $u_f(\mu^*) < u_f(f)$ violates the auxiliary stability of μ^* in $G(x^*)$, which is guaranteed by the definition of μ^* as a matching such that $e(\mu^*) \in T(x^*)$. If, instead, individual rationality is violated by the existence of some $w \in \mathcal{W}$ with $u_w(\mu^*) < u_w(w, 0)$, then since $u_w(\mu^*) = u_w(\mu^*(w), e(\mu^*)_{\mu^*(w)}) = u_w(\mu^*(w), x^*_{\mu^*(w)})$, this violates the auxiliary stability of μ^* in $G(x^*)$.

(ii) in the second case, the existence of some $w \in \mathcal{W}$ and $f \in \mathcal{F}$ such that $u_f(\mu^*(f) \cup w) > u_f(\mu^*(f))$ and $u_w(f, e(\mu^*)_{\mu^*(w)})) > u_w(\mu^*(f))$ violates auxiliary stability of μ^* in $G(x^*)$, since $e(\mu^*) = x^*$ by definition. $\qquad \square$

$E = \{(x, y, n - x - y) \in \mathbb{R}^3 : x, y \geq 0, x + y \leq n\}$ endowed with the partial order $(x_1, y_1, n - x_1 - y_1) \geq (x_2, y_2, n - x_2 - y_2) \iff x_1 \geq x_2$ and $y_1 \leq y_2$ is a complete lattice.

Proof. We want to show that for any $A \subseteq E$, there exists a supremum and infimum in E. Let $\bar{x} = \max_{(x,y,n-x-y) \in A} \{x\}$, and $\underline{y} = \min_{(x,y,n-x-y) \in A} \{y\}$ and define $s = (\bar{x}, \underline{y}, n - \bar{x} - \underline{y})$. Then s is the supremum of A, and is in E. To see that s is the supremum of A, notice that $\bar{x} \geq x$ for any $(x, y, n - x - y) \in A$ and $\underline{y} \leq y$ for any $(x, y, n - x - y) \in A$, by definition. To see that $s \in E$, notice that since $\bar{x} = x$ for some

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$(x, \tilde{y}, n - x - \tilde{y}) \in A \subseteq E$, it must be the case that $\bar{x} + \tilde{y} \leq n$. Moreover, since $\underline{y} \leq \tilde{y}$,

it follows that $\bar{x} + \underline{y} \leq \bar{x} + \tilde{y} \leq n$.

Similarly, let $\underline{x} = \min_{(x,y,n-x-y) \in A} \{x\}$, $\bar{y} = \max_{(x,y,n-x-y) \in A} \{y\}$, and $n = (\underline{x}, \bar{y}, n -$

$\underline{x} - \bar{y})$. Then n is an infimum of A since $\underline{x} \leq x$ and $\bar{y} \geq y$ for any $(x, y, n - x - y) \in A$.

To see that $n \in E$, notice that $\bar{y} = y$ for some $(\tilde{x}, y, n - \tilde{x} - y) \in A$, and so $\tilde{x} + \bar{y} \leq n$.

Since $\underline{x} \leq \tilde{x}$, it must be the case that $\underline{x} + \bar{y} \leq \tilde{x} + \bar{y} \leq n$. $\qquad \square$

Proof. (of proposition 7) We will show that there exists an increasing (with respect to

the partial order defined in lemma 2.6) selection of T. Let $(x_1, y_1, n - x_1 - y_1), (x_2, y_2, n -$

$x_2 - y_2) \in E$ be such that $(x_1, y_1, n - x_1 - y_1) \geq (x_2, y_2, n - x_2 - y_2)$. We will show that

for any $\mu_1 \in T(x_1, y_1, n - x_1 - y_1)$ and $\mu_2 \in T(x_2, y_2, n - x_2 - y_2)$, $e(\mu_1) \geq e(\mu_2)$. This

implies that any selection of T is increasing.

Let $G(1) := G(x_1, y_1, n - x_1 - y_1)$ and $G(2) := G(x_2, y_2, n - x_2 - y_2)$, and notice that

if $W_1(1) = \{w \in W : f_1 \succ_w f_2 \text{ in } G(1)\}$ and $W_1(2) = \{w \in W : f_1 \succ_w f_2 \text{ in } G(2)\}$,

then $W_1(2) \subseteq W_1(1)$.

To proceed, we make use of the following chain of implications: $e(\mu_1) \geq e(\mu_2) \iff$

$|\mu_1(f_1)| \geq |\mu_2(f_1)|$ *and* $|\mu_1(f_2)| \leq |\mu_2(f_1)|$ $\impliedby$ $\mu_1(f_1) \supseteq \mu_2(f_1)$ *and* $\mu_1(f_2) \subseteq \mu_2(f_2)$.

$\mu_1(f_1) \supseteq \mu_2(f_1)$

Proof. Suppose that $\mu_1(f_1) \not\supseteq \mu_2(f_1)$. Then there exists some $w \in \mu_2(f_1) \backslash \mu_1(f_1)$.

$w \in \mu_2(f_1)$ means that $w \succ_{f_1} \emptyset$ and that $f_1 \succ_w \emptyset$ in $G(2)$. In addition it means that

either $f_1 \succ_w f_2$, or that $f_2 \succ_w f_1$ and $\emptyset \succ_{f_2} w$ in $G(2)$. First, notice that $f_1 \succ_w \emptyset$

in $G(2)$ implies that $f_1 \succ_w \emptyset$ in $G(1)$. If it is the case that $f_1 \succ_w f_2$ in $G(2)$, then it

follows that $f_1 \succ_w f_2$ in $G(1)$, and we arrive at a contradiction to the statement that $w \notin \mu_1(f_1)$. Suppose, instead, that $f_2 \succ_w f_1$ and $\emptyset \succ_{f_2} w$ in $G(2)$. Since preferences of firms are unchanged across games, this means that $\emptyset \succ_{f_2} w$ in $G(1)$ as well, hence $w \notin \mu_1(f_2)$. But since $f_1 \succ_w \emptyset$ in $G(1)$ and $w \succ_{f_1} \emptyset$, it must be the case that $w \in \mu_1(f_1)$ - a contradiction. $\square$

$\mu_1(f_2) \subseteq \mu_2(f_2)$

Proof. Suppose that $\mu_1(f_2) \not\subseteq \mu_2(f_2)$. Then there exists some $w \in \mu_1(f_2)\backslash\mu_2(f_2)$. $w \in \mu_1(f_2) \implies w \succ_{f_2} \emptyset$ and $f_2 \succ_w \emptyset$ in $G(1)$. In addition, $w \in \mu_1(f_2) \implies$ either (i) $f_2 \succ_w f_1$ in $G(1)$, or (ii) $f_1 \succ_w f_2$ and $\emptyset \succ_{f_1} w$ in $G(1)$.

Case (i): $f_2 \succ_w \emptyset$ in $G(1) \implies f_2 \succ_w \emptyset$ in $G(2)$. $f_2 \succ_w f_1$ in $G(1) \implies f_2 \succ_w f_1$ in $G(2)$. Since $w \succ_{f_2} \emptyset$ and $\{f_2 \succ_w f_1, f_2 \succ_w \emptyset\}$ in $G(2)$, w and f_2 are a blocking pair in $G(2)$ - a contradiction. $\square$

We have shown that $e(\mu_1) \geq e(\mu_2)$ for any $\mu_1 \in T(x_1, y_1, n - x_1 - y_1)$ and $\mu_2 \in T(x_2, y_2, n - x_2 - y_2)$. Pick any selection f of T - by the above, it must be an increasing selection. $\square$

Proof. (of proposition 8)

Start at some myopic-stable matching μ with some workers employed at the two firms while some are unemployed. Define the following sets of workers:

$$W^0(\mu) := \{w \in \mu(f_1) \cup \mu(f_2) : \emptyset \succ_w (\mu(w), |\mu(\mu(w))|)\}$$

$$W^1(\mu) := \{w \in \mu(f_2) : (f_1, |\mu(f_1)| + 1) \succ_w (f_2, |\mu(f_2)|) \ and \ w \succ_{f_1} \emptyset\}$$

$$W^2(\mu) := \{w \in \mu(f_1) : (f_2, |\mu(f_2)| + 1) \succ_w (f_1, |\mu(f_1)|) \ and \ w \succ_{f_2} \emptyset\}$$

Notice that since μ is myopic-stable, $W^0(\mu) = \emptyset$. There are 4 possible cases we need to consider:

(1) $W^1(\mu) = W^2(\mu) = \emptyset$

(2) $W^1(\mu) = \emptyset, \ W^2(\mu) \neq \emptyset$

(3) $W^1(\mu) \neq \emptyset, \ W^2(\mu) = \emptyset$

(4) $W^1(\mu) \neq \emptyset, \ W^2(\mu) \neq \emptyset$

Case 1:

Suppose that case (1) is true. Workers in the unemployment pool may prefer to join one of the two firms when they take the effect of their deviation on the number of the firm's employees into account. Define the following sets of workers:

$$U^1(\mu) := \{w : \mu(w) = \emptyset, \ (f_1, |\mu(f_1)| + 1) \succ_w \emptyset, \ and \ w \succ_{f_1} \emptyset\}$$

$$U^2(\mu) := \{w : \mu(w) = \emptyset, \ (f_2, |\mu(f_2)| + 1) \succ_w \emptyset, \ and \ w \succ_{f_2} \emptyset\}$$

There are 4 possible cases:

(i) $U^1(\mu) = U^2(\mu) = \emptyset$

(ii) $U^1(\mu) = \emptyset$, $U^2(\mu) \neq \emptyset$

(iii) $U^1(\mu) \neq \emptyset$, $U^2(\mu) = \emptyset$

(iv) $U^1(\mu) \neq \emptyset$, $U^2(\mu) \neq \emptyset$

If (i) is true, then μ is stable.

Cases 1-ii (and, by symmetry, case 1-iii):

Suppose (ii) is true. Let all workers in $U^2(\mu)$ move to f_2. This may cause other workers to want to move from the unemployment pool to f_2. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of unemployed workers is finite. Call the new matching $\tilde{\mu}$. Since f_2 now has more workers, $W^2(\tilde{\mu})$ may be non-empty, while $W^1(\tilde{\mu})$ must be empty because $W^1(\mu)$ is empty, and f_2 now has more workers relative to f_1. Notice that for any matching μ^*, if $W^1(\mu^*) = W^2(\mu^*) = U^1(\mu^*) = U^2(\mu^*) = \emptyset$, then μ^* is stable.

- Step 1:

    - Allow all workers in $W^0(\tilde{\mu}) \subseteq \tilde{\mu}(f_1)$ (if any) to move to the unemployment pool. This may cause other workers to want to move from f_1 to the unemployment pool. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of workers matched with f_1 is finite; call the resulting new matching μ^1.

    - If $W^2(\mu^1)$ is empty, consider the sets $U^1(\mu^1)$ and $U^2(\mu^1)$. Since $U^1(\mu)$ is empty, so is $U^1(\mu^1)$. It may be the case, however, that $U^2(\mu^1)$ is non-empty. If so, allow workers to move to f_2. This may cause other workers to want to

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move from the unemployment pool to f_2. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of unemployed workers is finite. Call the new matching $\tilde{\mu}^1$.

- If $W^2(\mu^1)$ is non-empty, allow workers to move to f_2. This may cause other workers to want to move from f_1 to f_2. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of workers employed by f_1 is finite, and call the new matching $\tilde{\mu}^1$. Consider the sets $U^1(\tilde{\mu}^1)$ and $U^2(\tilde{\mu}^1)$. Since $U^1(\mu)$ is empty, so is $U^1(\tilde{\mu}^1)$. It may be the case, however, that $U^2(\tilde{\mu}^1)$ is non-empty. If so, allow workers to move to f_2. This may cause other workers to want to move from the unemployment pool to f_2. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of unemployed workers is finite. Call the new matching $\tilde{\mu}^1$.

- Step k:

  - Allow all workers in $W^0(\tilde{\mu}^{k-1}) \subseteq \tilde{\mu}^{k-1}(f_1)$ (if any) to move to the unemployment pool. This may cause other workers to want to move from f_1 to the unemployment pool. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of workers employed by f_1 is finite. Call the new matching μ^{k-1}.

  - If $W^2(\mu^{k-1})$ is empty, consider the sets $U^1(\mu^{k-1})$ and $U^2(\mu^{k-1})$. Since $U^1(\mu)$ is empty, so is $U^1(\mu^{k-1})$. If $U^2(\mu^{k-1})$ is non-empty, allow workers to move to

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f_2. This may cause other workers to want to move from f_1 to f_2. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of workers employed by f_1 is finite, and call the new matching $\tilde{\mu}^k$.

- If $W^2(\mu^{k-1})$ is non-empty, allow workers to move to f_2. This may cause other workers to want to move from f_1 to f_2. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of workers employed by f_1 is finite. Call the new matching $\tilde{\mu}^{k-1}$. Consider the sets $U^1(\tilde{\mu}^{k-1})$ and $U^2(\tilde{\mu}^{k-1})$. Since $U^1(\mu)$ is empty, so is $U^1(\tilde{\mu}^{k-1})$. It may be the case, however, that $U^2(\tilde{\mu}^{k-1})$ is non-empty. If so, allow workers to move to f_2. This may cause other workers to want to move from the unemployment pool to f_2. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of unemployed workers is finite. Call the new matching $\tilde{\mu}^k$.

This process must stop at some finite k^*, since the number of agents who are either matched with firm 1 or unemployed is finite: it must be the case that $\tilde{\mu}^{k^*}$ is stable. Notice that the same process can be implemented for case (iii) when $U^1(\mu) \neq \emptyset$.

Case 1-iv:

Without loss of generality, let workers in $U^1(\mu)$ move to firm 1 and call the resulting matching $\tilde{\mu}$. $W^2(\tilde{\mu})$ is empty since $W^2(\mu)$ is empty and there are more workers in firm 1 relative to firm 2 in $\tilde{\mu}$ than in μ.

- Step 1:

  - Allow all workers in $W^0(\tilde{\mu}) \subseteq \tilde{\mu}(f_2)$ (if any) to move to the unemployment pool. This may cause other workers to want to move from f_2 to the unemployment pool. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of workers matched with f_2 is finite; call the resulting new matching μ^1.

  - If both $W^1(\mu^1)$ and $U^1(\mu^1)$ are empty, follow the steps for case **1-ii**.

  - If $W^1(\mu^1)$ is empty, and $U^1(\mu^1)$ is non-empty, allow workers to move to firm 1. This may cause other workers to want to move from the unemployment pool to f_1. If so, allow them and subsequent sets of similar workers to move. The process ends at some point since the number of workers in the unemployment pool is finite; call the new matching $\tilde{\mu}^1$.

  - If $W^1(\mu^1)$ is non-empty, allow workers to move to firm 1. This may cause other workers to want to move from f_2 to f_1. If so, allow them and subsequent sets of similar workers to move. The process ends at some point since the number of workers matched with f_2 is finite; call the new matching $\hat{\mu}^1$.

    * If $U^1(\hat{\mu}^1)$ is non-empty, allow workers to move to firm 1. This may cause other workers to want to move from the unemployment pool to f_1. If so, allow them and subsequent sets of similar workers to move. The process ends at some point since the number of workers in the unemployment pool is finite; and call the new matching $\tilde{\mu}^1$.

    ∗ If $U^1(\hat{\mu}^1)$ is empty, follow the steps for case **1-ii**.

- Step k:

  - Allow all workers in $W^0(\tilde{\mu}^{k-1}) \subseteq \tilde{\mu}^{k-1}(f_2)$ (if any) to move to the unemployment pool. This may cause other workers to want to move from f_2 to the unemployment pool. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of workers matched with f_2 is finite; call the resulting new matching μ^k.

  - If both $W^1(\mu^k)$ and $U^1(\mu^k)$ are empty, follow the steps for case **1-ii**.

  - If $W^1(\mu^k)$ is empty, and $U^1(\mu^k)$ is non-empty, allow workers to move to firm 1. This may cause other workers to want to move from the unemployment pool to f_1. If so, allow them and subsequent sets of similar workers to move. The process ends at some point since the number of workers in the unemployment pool is finite; call the new matching $\tilde{\mu}^k$.

  - If $W^1(\mu^k)$ is non-empty, allow workers to move to firm 1. This may cause other workers to want to move from f_2 to f_1. If so, allow them and subsequent sets of similar workers to move. The process ends at some point since the number of workers matched with f_2 is finite; call the new matching $\hat{\mu}^k$.

    ∗ If $U^1(\hat{\mu}^k)$ is non-empty, allow workers to move to firm 1. This may cause other workers to want to move from the unemployment pool to f_1. If so, allow them and subsequent sets of similar workers to move. The process

ends at some point since the number of workers in the unemployment

pool is finite; and call the new matching $\tilde{\mu}^k$.

* If $U^1(\hat{\mu}^k)$ is empty, follow the steps for case **1-ii**.

The process leads to case **1-ii** at some k^* since the number of workers that are either

matched with firm 2 or unemployed is finite.

Note that if some workers moved from firm 2 to firm 1 during the process, then

$U^2(\mu^{k^*})$ is empty, since μ is myopic-stable. Namely, for all workers with $\mu(w) = \emptyset$, it

must be the case that one of the following statements is true:

- $\emptyset \succ_w (f_1, |\mu(f_1)|)$ and $\emptyset \succ_w (f_2, |\mu(f_2)|)$

- $\emptyset \succ_w (f_1, |\mu(f_1)|)$, $(f_2, |\mu(f_2)|) \succ_w \emptyset$, and $\emptyset \succ_{f_2} w$

- $\emptyset \succ_w (f_2, |\mu(f_2)|)$, $(f_1, |\mu(f_1)|) \succ_w \emptyset$, and $\emptyset \succ_{f_1} w$

- $(f_1, |\mu(f_1)|) \succ_w \emptyset$, $\emptyset \succ_{f_1} w$, $(f_2, |\mu(f_2)|) \succ_w \emptyset$, and $\emptyset \succ_{f_2} w$

If at least one worker moves from firm 2 to firm 1, $|\mu^{k^*}(f_2)| < |\mu(f_2)|-1$, and workers

in $U^2(\mu)$ now prefer to remain unemployed even after taking their own deviations into

account.

Had we allowed workers to move from $U^2(\mu)$ instead of $U^1(\mu)$, we would return to

case **1-iii** instead of **1-ii**.

Case 2:

Suppose $W^1(\mu) = \emptyset$ and $W^2(\mu)$ is non-empty. Allow workers in $W^2(\mu)$ to move to

firm 2 and call the new matching μ^1. Since $W^1(\mu)$ is empty, so is $W^1(\mu^1)$. If $W^2(\mu^1)$ is

non-empty, allow workers to move to firm 2 and call the resulting matching μ^2. Continue the process until $W^2(\mu^{k^*})$ is empty - such a k^* must exist since the number of workers matched with firm 1 is finite. Since $W^2(\mu)$ was non-empty and μ is a myopic-stable matching, $W^1(\mu^{k^*}) = \emptyset$, and we can follow the steps in **case 1**.

Case 3:

This is the same as **case 2** but with workers moving to firm 1 instead of firm 2.

Case 4:

When both $W^1(\mu)$ and $W^2(\mu)$ are non-empty, allowing any one of the sets of workers to move to its preferred firm (and allowing subsequent sets of similar workers to move in the same direction) takes us back to **case 1**, since μ is myopic-stable. To see this, suppose we let workers in $W^1(\mu)$ move to firm 1 and call the new matching μ^1. Since μ is myopic-stable, $W^2(\mu^1) = \emptyset$.

- Step 1:

  - Suppose $W^1(\mu^1) \neq \emptyset$. Allow workers in $W^1(\mu^1)$ to move to firm 1 and call the new matching μ^2.

- Step $k - 1$:

  - Suppose $W^1(\mu^{k-1}) \neq \emptyset$. Allow workers in $W^1(\mu^{k-1})$ to move to firm 1, and call the new matching μ^k.

The process must end at some point, and there must exist a k^*: $W^1(\mu^{k^*}) = \emptyset$,

since the number of workers matched with firm 2 is finite. Moreover, $W^2(\mu^{k^*}) = \emptyset$ by myopic-stability of μ. This takes us back to **case 1**.

$\square$

Example 2.4 (continued) Since all the workers prefer to be matched with some firm than remain unemployed, and firms find all workers acceptable, there will be no unemployed workers at any stable matching. It suffices, therefore, to consider matchings in which all workers are employed. There are 27 such matchings (including the 6 shown in the main text). In 6 of these 27 matchings, one worker is matched with each firm. For example, $\mu = \{\{f_1, w_1\}, \{f_2, w_2\}, \{f_3, w_3\}\}$ is such a matching. None of these matchings can be stable, since workers always prefer to join another firm with an existing employee than remain at a firm that only employs one worker. Next, consider matchings in which two workers are matched with one firm and the remaining worker is matched with another firm. There are 18 such matchings, none of which are stable. To see this, notice that in 6 of these matchings, worker 1 is the only employee in the firm they are matched with. If that firm is either firm 1 or 3, they would prefer to move to firm 2 (even when it doesn't employ the other two workers). When that firm is firm 2, and the other two workers are employed by firm 3, then worker 1 prefers to deviate and join them. Finally, when worker 1 is matched with firm 2 and the other two workers are matched with firm 1, we are back to μ_1, which we know is not stable.

In 6 of the matchings in which two workers are matched with one firm and one worker is matched with another, worker 2 is the only worker employed by some firm.

If they are employed by either firm 2 or 3, they block the going matching and move to firm 1. If they are employed by firm 1 and the remaining workers are employed by firm 2, they block the matching by moving to firm 2. Finally, if they are employed by firm 1 and the rest are employed by firm 3, we are back to μ_5, which is not stable.

In the remaining 6 matchings, worker 3 is the sole employee of some firm. This firm cannot be firm 1 in a stable matching, since worker 3 strictly prefers to join firm 3 with any number of employees than remain at firm 1. The worker can also never be employed by firm 2 since it individually blocks such a matching. If the worker is employed by firm 3 and the remaining workers are employed by firm 1, then it prefers to deviate and join them. If they are employed by firm 2, then worker 1 deviates and joins firm 3.

Finally, there are 3 matchings in which all 3 workers are matched with the same firm. Call them μ^1, μ^2, and μ^3, where μ^j is the matching in which all workers are matched with firm j. μ^2 is not stable because worker 3 prefers to be single. μ^1 is not stable because worker 1 prefers to join firm 2. μ^3 is not stable because worker 2 prefers to be single.

This shows that the game presented in example 2.4 has no stable matching.

Chapter 3

A Simple Model of Party Formation

3.1 Introduction

Baron and Ferejohn (1989) was the first study to model the process of legislative bargaining. They modeled the bargaining process as a dynamic game between agents bargaining over the split of a finite resource. A proposer is chosen with some exogenous probability at the beginning of each period, and they propose a split of the resources that must garner majority support in order to be implemented. The equilibrium strategies that emerge are ones in which the proposer courts a minimal winning coalition that approves the proposal, while the other legislators oppose it. There is no role for political parties in this model. As a result, every proposal is passed with a minimal winning coalition and legislators vote independently of any strategic considerations.

There is, as yet, no consensus on what the most important functions of political parties are Dhillon (2004). Baron (1993) focuses on parties' wish to influence the ultimate policy that the government implements, as well as their incentives to garner wide electoral support. In his model, parties consider the eventual government formation process as well as the process of bargaining between parties into account when choosing their policies. They also take into account the potential electoral support that their policies are likely to attract. The model has two main results: in a 3-party set-up with a two dimensional policy space, either all parties choose the median policy, or else, for some parameter values, they each choose different positions that are equidistant and

symmetric from the median position. Although the title of the paper[1] alludes to the process through which parties form, the main contribution of the paper is its attempt to consider multiple goals that parties have simultaneously.

Jackson and Moselle (2002) extend the basic bargaining model in Baron and Ferejohn (1989) by adding a policy that the proposer can choose. This choice is characterized as a location in the interval $[0, 1]$. In this setting, parties are exogenous constraints on individual legislators ensuring that they propose a particular vector of policy and allocations whenever they are selected. In a 3 legislator example (left, right and media legislators), the authors show that *stable parties*[2] emerge only between players that are ideologically adjacent - a party between the two extreme legislators, for example, is not stable.

Morelli (2004) studies a three party model, where party formation is modeled as two of the three parties agreeing to a coalition. In this setting, the parties serve two functions. They are a commitment device for politicians who want to run on platforms that are not their own. The second function they serve is as a coordination device for voters. Eguia (2012), on the other hand, models parties as coalitions that vote according to the wishes of the majority of their members. A peculiar assumption in the model is that legislators face uncertainty over their own preferences. At the beginning of each stage,

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[1]Government formation and endogenous parties.

[2]A political party is defined to be stable if no agent wishes to leave in anticipation of higher utility elsewhere.

legislators form parties prior to the resolution of uncertainty, and credibly commit to vote according to the party line once their preferences are revealed. In a model with repeated stages of voting, Eguia (2012) shows that a stable coalition of legislators (a party) can consistently shift equilibrium policy away from the status quo median - this conclusion contradicts one of the results in Baron (1993), which argues that under some circumstances, parties will simply offer the median policy.

We follow the approach of Jackson and Moselle (2002) and adopt their framework. In their study, they show that an equilibrium in stationary strategies exists. In this paper, we show that a symmetric equilibrium exists in a legislative game with 3 players, and that the equilibrium takes a particular form. Moreover, we show that an equilibrium exists in some neighborhood of this symmetric equilibrium. Finally, we demonstrate the properties of equilibrium quantities with numerical simulations. Section 2 describes the model, section 3 discusses existence and properties of equilibrium, section 4 explores the stability of political parties, and section 5 concludes.

3.2 The model

3.2.1 Legislative game

Consider the legislative bargaining model in Jackson and Moselle (2002) with three players. Let the ideal point of legislator $i \in \{0, m, 1\}$ be $v_i \in [0, 1]$, and let $v_0 = 0$ and $v_1 = 1$ and m be the median legislator. A legislator is picked proposer with probability p_i (this

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will be assumed to be 1/3 for the remainder of the paper) at the beginning of each period to propose a *decision* composed of a policy y^i and a split of the finite resources available, X: $\{x_i\}_i$ such that $\sum_i x_i = 1$. A decision is then voted on by all players who can specify if they vote "for" or "against" the decision. It is adopted if it receives majority support. There are an infinite number of periods denoted by $t \in \{1, 2, \ldots\}$. The game ends when a decision is adopted. A strategy for each player is a decision that they propose whenever they are chosen proposer, and a voting rule whenever they are not proposing that can be characterized by a function assigning a "yes" or "no" to each possible decision . A strategy for a player is *stationary* if their continuation strategy at the beginning of every subgame is the same regardless of history. An equilibrium is *stationary* if it is a subgame perfect equilibrium and each legislator's strategy is stationary.

A *simple equilibrium* is a stationary equilibrium in which (i) each legislator randomizes over at most $M < \infty$ proposals, (ii) each such proposal can be identified with a distinct coalition C such that $i \notin C$, and $\#C = \frac{n-1}{2}$. In our version of the model with only three legislators, each C is simply another player.

3.2.2 Preferences

Let legislator i evaluate a decision $d = (y, x) = (y, x_0, x_m, x_1)$ with utility function u_i, defined as follows:

$$u_i(d) = x_i X + 1 - (v_i - y)^2$$

This function is separable in x_i and y, is concave in the decision, d, and is single peaked for every x for legislator i in y at their ideal point, v_i. Therefore, it satisfies the assumptions in Jackson and Moselle (2002). It is commonly assumed that legislators are office-seeking: they only care about getting re-elected. This form of utility function can be interpreted in that light if we consider the ideal point v_i to be the ideal point of the legislator's district, and the resources $x_i X$ as the resources that the legislator then spends on their district (or redistributes among voters). In either case, we can view the legislator as having internalized the preferences of their district. Since pursuing the best interest of the district (or the district's median voter) is the best way to ensure reelection, this formulation is in line with the assumption that legislators are office-seeking. Alternatively, if we assume that legislators are policy-motivated, then v_i can be interpreted as their own ideal point, and they maximize their own utility.

3.3 Equilibrium

In this section, we show the existence of a novel kind of equilibrium in the three player game. One in which the median player randomizes between proposing to the other two players, and where the extreme players always propose to the median, when either one is chosen proposer. This equilibrium can be seen as exhibiting some form of party structure since legislators only propose to adjacent legislators - any proposal that passes does so with a contiguous coalition. In the next section, we explore an explicit party structure using the notion of the stability of coalitions.

Proposition 2[3] in Jackson and Moselle (2002) shows that a simple equilibrium exists in the general legislative game with unspecified concave utility functions and n legislators. Here, we will show that a particular kind of simple equilibrium exists. First, we show that a symmetric simple equilibrium exists whenever the median legislator is equidistant from the extreme legislators. By Proposition 3 in Jackson and Moselle (2002), any approved decision in any stationary equilibrium distributes resources among an exact majority. This means that each proposer targets another player, and offers that player a share of the resources that would induce them to vote for the proposal. In an arbitrary symmetric equilibrium, it may be the case that legislators randomize between proposing to one of the other two players. We show, however, that whenever a symmetric equilibrium exists, it is one where only the median player randomizes (with probability $1/2$) between proposing to 0 and 1, and where the other two players always propose to the median player.

A symmetric equilibrium exists whenever $v_m = 1/2$ in which:

- m proposes $(1 - x^m, 1/4)$ to 0 with probability $1/2$ and $(1 - x^m, 3/4)$ to 1 with probability $1/2$

- 0 proposes $(1 - x^0, 1/4)$ to m whenever they are chosen proposer

- 1 proposes $(1 - x^0, 3/4)$ to m whenever they are chosen proposer

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[3]If u_i is concave for each i, then there exists a simple equilibrium. Moreover, if each u_i is strictly concave then all stationary equilibria are simple (Jackson and Moselle (2002)).

- players that are proposed to always accept

Having shown the existence of a symmetric equilibrium, we will now show that it must be one where both extreme players always propose to the median player, and the median player randomizes with probability 1/2 between proposing to either one. The proof will consider two cases: first, we will consider the case of a symmetric equilibrium in which the two extreme players propose to one another with certainty and show that, for both players, proposing to the median player instead is a profitable deviation. Second, we will consider a symmetric equilibrium in which the two extreme players randomize between proposing to the other extreme player and proposing to the median. We will then show that it is profitable for both players to propose to the median player every time, and so randomizing is not equilibrium behavior.

It is important to point out that a crucial assumption throughout this paper is that the amount of resources, X, is sufficiently large to allow for an interior solution (where the proportion of resources shared with the legislator being proposed to is strictly between 0 and 1). This allows the proposed policy to always be at the midpoint between the two legislators' (the proposer and the legislator accepting the proposal) ideal points.

The first two results we prove show that the equilibrium in which the median player randomizes between proposing to 0 and 1, and where both 0 and 1 propose to the median with certainty is the unique symmetric equilibrium in the three player game

specified above.

Let $v_m = 1/2$. For some values of δ and large enough values of X, a symmetric equilibrium in which 0 and 1 always propose to m, and m randomizes with probability $1/2$ between proposing to either extreme player exists. Moreover, it is the unique symmetric equilibrium.

Next, we show that, in addition to a symmetric equilibrium, whenever $v_m = 1/2$, there exist equilibria when the position of the median legislator is near v_m. These equilibria will not be symmetric and some of them are described in the numerical analysis we conduct in the appendix.

An equilibrium in which 0 and 1 always propose to m, and m randomizes between proposing to either player exists for v_m in the neighborhood of $1/2$.

The appendix contains numerical simulations of equilibrium quantities and demonstrates how they change with parameters δ and X. We can see that the range of δ in which an equilibrium exists is increasing in X: the larger X is, the larger the range of δ at which an equilibrium exists (appendix sections 5.1, 6.1, 6.2). Moreover, we can see that at any value of X, the proportion kept by the median player whenever they propose to the extreme players is increasing in δ (appendix section 6.4).

3.4 Political Parties

Jackson and Moselle (2002) define a *party* as a subset of legislators $P \subset N$ who propose the same policy whenever any member is chosen proposer. Moreover, they each vote

for this proposal whenever it is tabled. Such a party is *stable*[4] if the following condition holds:

$$u_i(P) \geq u_i(P') \ \forall i \in P, \ \forall P' \ni i$$

In their examples, Jackson and Moselle (2002) find that the identity of stable parties depends on the specific utility parameters. Although our framework specifies the utility function, it is parameter-free[5]. To this extent, there is an obvious question to ask: what is the identity of stable parties in our framework? How do they differ with the location of the median legislator? Is it possible to have a non-contiguous stable party (one made up of legislators 0 and 1)?

Notice that a party generates some joint utility for its members:

$$u^P = u_i(P) + u_j(P) = X + u(v_i - v^P) + u(v_j - v^P),$$

where v^P is the policy proposed by party members. For $u_i(P)$ to be well defined, the split of this total utility among the members of the party must be specified. Since there are always two players in a party in our example, and following Jackson and Moselle

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[4]Although the notion of stability defined here applies more generally than the three player model we describe, the utility that an individual player obtains from being a member of a party is not well defined for more than three players. We use the Nash bargaining solution to determine the policy that a party chooses and split the revenue between two legislators in a party. However, in a party with more than 3 players, this approach is no longer applicable. The definition of stability above, therefore, applies in those cases, provided utility is defined in some way.

[5]As opposed to Jackson and Moselle (2002), who must specify a parameter that determines the relative preference for policy vis-a-vis resources for each legislator, we have a utility function that is common for all legislators. One possible variation to explore may be the concavity of the utility function, specified by the power to which we raise the distance between policy and ideal point.

(2002), we let the split of utility be determined by the Nash bargaining solution[6].

Let $P(i,j)$ be the party formed by players i and j. We begin with the equilibrium described above (where both extreme players propose to m, and m randomizes) and investigate which parties are stable. First, consider the case of $P(0,m)$ and let v_{0m}^p be the chosen party policy and x_{0m}^0 be the proportion of the resources kept by player 0 in the split. We're looking for the Nash bargaining solution, and so (v_{0m}^p, x_{0m}^0) solve the following maximization:

$$(v_{0m}^p, x_{0m}^0) = \arg\max_{v,x} \left\{ (xX + 1 - v^2 - \bar{u}^0)((1-x)X + 1 - (v_m - v)^2 - \bar{u}^m) \right\},$$

where $\bar{u}^i$ is expected utility of player i in the equilibrium described above. Since we're considering the symmetric case, $v_m = 1/2$. The first order conditions are then simply:

$$FOC(x): \ x = \frac{1}{2X} \left(X - (v_m - v)^2 - \bar{u}^m + v^2 + \bar{u}^0 \right)$$

$$FOC(v): (v_m - v)\left(xX + 1 - v^2 - \bar{u}^0\right) - v\left((1-x)X + 1 - (v_m - v)^2 - \bar{u}^m\right) = 0$$

Solving these two simultaneous equations, we get $v_{0m}^P = v_m/2$ and $x_{0m}^0 = \frac{X + \bar{u}^0 - \bar{u}^m}{2X}$. We can repeat this exercise for the party $P(0,1)$ and find $v_{01}^P = 1/2$ and $x_{01}^0 = \frac{X + \bar{u}^0 - \bar{u}^1}{2X}$.

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[6]The agreement $x^* \in X$, the set of possible consequences, is a Nash solution of the bargaining problem $\{X, (d_1, d_2), \succsim_1, \succsim_2\}$ if and only if:

$$(u_1(x^*) - d_1)(u_2(x^*) - d_2) \geq (u_1(x) - d_1)(u_2(x) - d_2) \forall x \in X,$$

where (d_1, d_2) is the disagreement point, $\succsim_i$ is the preference relation of i over lotteries over X (page 302, Osborne & Rubinstein (1994)).

Finally, for the party $P(m, 1)$, the quantities are $v_{m1}^P = (1 + v_m)/2$ and $x_{m1}^m = \frac{X + \bar{u}^m - \bar{u}^1}{2X}$.

These results are summarized in the following proposition.

Proposition 9. *Let X be sufficiently large, and let $P(i, j)$ be a party composed of players i and j. If $\bar{u}^k$ is player k's expected utility from the independent voting game and the players determine the policy of the party, v_{ij}^P, and the share of resources, x_{ij}^k, allocated to player k through the Nash bargaining solution, then these quantities take the following form:*

$$v_{ij}^P = \frac{v_i + v_j}{2} \qquad x_{ij}^i = \frac{X + \bar{u}^i - \bar{u}^j}{2X}$$

Notice that the party's policy is the midpoint between the two legislators' ideal points, and the share of resources allocated to a legislator is proportional to their expected utility in the independent voting game and decreasing in the other player's expected utility in the independent voting game. This makes intuitive sense since more influential players (those with higher expected utility in the independent voting game) command a larger proportion of the party's resources.

According to the above definition of stability, a party is stable if neither member wishes to be part of another party. Let's consider the party $P(0, m)$. 0 does not want to leave the party whenever $U^0(P(0, m)) \geq U^0(P(0, 1))$, while m doesn't want to leave whenever $U^m(P(0, m)) \geq U^m(P(m, 1))$. Below, we derive the general conditions for stability of $P(0, m)$ and proceed to show that these hold in the symmetric equilibrium we discuss above.

Notice that since the threat points are the expected utilities from the independent voting game, it must be the case that $\bar{u}^0 = \bar{u}^1$, otherwise m would not randomize between proposing to the extreme players. This observation makes it clear to see that $U^m(P(0,m)) \geq U^m(P(m,1)) \implies v_m \leq 1/2$, the first condition for stability of $P(0,m)$.

The second condition for stability comes from the condition for player 0 preferring not to leave the party and it simplifies to[7]:

$$\frac{X + \bar{u}^0 - \bar{u}^m - 1}{2} + \frac{1 - v_m^2}{4} \geq 0$$

Consider the symmetric case ($v_m = 1/2$, and m randomizes with probability $1/2$ between the two players) and notice that the first condition is weakly satisfied. To check the second condition, we need to recall the expressions for $\bar{u}^0$ and $\bar{u}^m$. We reproduce these below and, since we are looking at the symmetric case, simplify the expressions[8] so that $x_1^m = x_0^m = x^m$ and $x_m^0 = x_m^1 = x^0$.

$$\bar{u}^0 = \frac{1}{3}\left[x^0 X + 1 - (1/4)^2 + 1 - (3/4)^2 + (1/2)\left((1 - x^m)X + 1 - (1/4)^2\right) + (1/2)(1 - (1/4)^2)\right]$$

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[7]It's straight forward to derive this inequality from the party policies and proportion of party resources allocated to each member in parties $P(0,m)$ and $P(0,1)$ and the following definition:

$$u^i(P(i,j)) = x_{ij}^i X + 1 - (v_i - v_{ij}^P)$$

[8]Namely, the proportions offered by m to both 0 and 1 are the same and the proportions offered by both 0 and 1 to m are the same. We chose to denote these with a 0 superscript for notational convenience, and a 1 subscript could have also been used.

$$\bar{u}^m = \frac{1}{3}\left[(1/2)(x^m X + 1 - (1/4)^2) + (1/2)(x^m X + 1 - (1/4)^2) + \left((1 - x^0)X + 1 - (1/4)^2\right)\right.$$

$$\left. + \left((1 - x^0)X + 1 - (1/4)^2\right)\right]$$

A straight forward substitution and simplification shows that the second condition for stability of $P(0, m)$ can be expressed as follows:

$$\frac{1}{2}\left(X + \frac{1}{3}\left(X(3x^0 - (3/2)x^m - (3/2)) + (1/4)^2 - (3/4)^2\right) - 1\right) + \frac{1 - (1/2)^2}{4} \geq 0$$

We will show that this inequality holds for large values of X. To do this, notice that we only need to show that the coefficient on X is positive, in which case, as X increases the left hand side increases and eventually becomes positive, since the negative terms are constant. The coefficient on X can be expressed as follows:

$$\frac{1}{2} \times \frac{1}{3}\left(3 + 3x^0 - \frac{3}{2}x^m - \frac{3}{2}\right) = \frac{1}{2}\left(1 + x^0 - \frac{1 + x^m}{2}\right) > 0$$

The inequality follows from the fact that $x^m \leq 1$. This result is summarized in the following proposition.

Proposition 10. *In a symmetric game with a sufficiently large X, parties $P(0, m)$ and $P(m, 1)$ are stable, while $P(0, 1)$ is not.*

Proof. We've already shown the stability of $P(0, m)$ above. The stability of $P(m, 1)$ follows from the symmetry of the set up. $P(0, 1)$ is unstable since both 0 and 1 prefer to be in a party with m. □

3.5   Conclusion

In this paper, we showed that a particular kind of symmetric equilibrium exists in a 3-legislator version of Jackson and Moselle's legislative bargaining model with quasi-linear utilities. This is a first step to studying political party formation in this framework. Although Jackson and Moselle set out to study political party formation, they only study this under restrictive assumptions on the utility function[9]. We also characterize the set of stable parties.

There are some outstanding questions. The first is whether we can show the existence result in claim 2 analytically. We conjecture that this is indeed possible. Second, it remains to be seen whether claim 3 can be shown more completely (as it stands, the final step of the proof is assumed to hold).

The results shown here apply to a restricted model: there are only three legislators with a particular utility function. The first question we might ask is whether these results hold for any quasi-linear utility function. This is quite possibly true. Whether or not the results hold for more than three players, however, is a less straightforward question. Most importantly, the utility of a legislator from belonging to a party is only well defined for parties consisting of two legislators since the internal split of the resources is decided by the Nash bargaining solution. With parties that have more than

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[9]They look at linear utility functions with a coefficient that determines how much a legislator values the proximity of a policy to their ideal point relative to consumption of resources.

just two members, this is no longer possible. How would the internal split of resources be decided in this case? This is an open question.

Furthermore, we only considered the symmetric case. However, it would be interesting to explore how the identity of stable parties, for example, changes a the position of the median legislator changes. In a model with more than three players, one questions that would be interesting to explore is whether parties are always composed of contiguous coalitions of legislators.

It may be the case that the current framework would not lend itself well to exploring these questions, particularly those that extend the model to more than 3 legislators. As we can see, the model quickly becomes unwieldy and to obtain any properties of the model usually requires solving some set of simultaneous equations. With more legislators, this set would increase, making us resort even more frequently to numerical analysis, instead of analytical results.

3.6 Appendix

3.6.1 Proofs

Proof of Claim 1

Proof. A symmetric equilibrium is a vector

$$(q_m^0, x_m^0, q_1^0, p, x_0^m, x_1^m, q_m^1, x_m^1, x_0^1)$$

such that $q = q_m^0, = q_m^1$, $x_m = x_m^0 = x_m^1$, $x = x_1^0 = x_0^1$, $x_0^m = x_1^m = x^m$, and $p = 1/2$. Namely, the probabilities with which 0 and 1 randomize between proposing to one another and proposing to m are the same, and the amount of resources they offer is the same. In addition, the amount of resources offered by m to either 0 or 1 is the same, and m randomizes between offering to 0 and 1 with equal probabilities. A symmetric equilibrium, therefore, can be characterized by 4 quantities:

$$(q, x_m, x, x^m)$$

Define $A_i^s(l) := \{d \in D : u_i(d) \geq \delta v_i(l)\}$, where $l \in L^s$, the space of symmetric strategy profiles. We want to show that $A_i^s(l)$ is (i) non-empty, (ii) compact, and (iii) continuous.

(i) Non-empty: Since $A_i^s(l)$ is non-empty for $l \in L$ (Jackson and Moselle (2002)), and $L^s \subset L$, it follows that $A_i^s(l)$ is non-empty.

(ii) Compact: $A_i^s(l)$ is closed, and since $A_i^s(l) \subset [0,1]^6$, which is compact, it follows that $A_i^s(l)$ is compact.

(iii) Continuous: We want to show that $A_i^s(l)$ is both uhc and lhc:

uhc: A correspondence $f : X \to Y$ is uhc if $\forall$ compact $B \subset X$, the set $f(B) = \{y \in Y : y \in f(x) \; for \; some \; x \in B\}$ is bounded.

Since D^s is compact, $A^s(B) \subset D^s$ is compact for any compact B, so $A_i^s(.)$ is uhc.

lhc: A correspondence $f : X \to Y$ is lhc if $\forall x^n \to x \in X : x^n \in X \forall n$, $\forall y \in f(x)$, $\exists y^n \to y$ and $N \in \mathbb{N} : y^n \in f(x^n) \forall n > N$

Let $l^n \to l \in L^s$, $l^n \in L^s \; \forall n$. Pick $d \in A_i^s(l) = \{d \in D : u_i(d) \geq \delta v_i(l)\}$. We want $d^n \to d$ and N such that $d^n \in A_i^s(l^n) \; \forall n > N$. Let d_i be the best decision for player i - they get all the resources $(x_i = 1)$, and the proposed policy is their ideal point $(y = v_i)$. It's clear that the following inequality holds:

$$u_i(d_i) > \delta v_i(l^n)$$

Let $d^k = \alpha_k d_i + (1 - \alpha_k)d$, where $\alpha_k \to 0$ (i.e. $d^k \to d$). For each l^n, we can find a k_n such that $u_i(d^{k_n})) > \delta v_i(l^n)$. Then $d^{k_n} \to d$ and $d^{k_n} \in A_i^s(l^n) \; \forall n$, hence A_i^s is lhc.

Define $A_{-i}^s(l) = \cup_{j \neq i} A_j^s(l)$ - the set of decisions that would be approved if proposed by player i. We want to show that $A_{-i}^s(l)$ is (i) nonempty, (ii) compact, and (iii)

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continuous.

(i) Non-empty: Since each $A_j^s(l)$ is non-empty, so is $A_{-i}^s(l)$.

(ii) Compact: Since a finite union of compact sets is compact, $A_{-i}^s(l)$ is compact.

(iii) Continuous: The union of uhc (lhc) correspondences is uhc (lhc). The footnote contains a

direct proof [10].

Define $A_i^*(l) = \arg\max_{d \in A_{-i}}\{u_i(d)\}$. By the maximum theorem, $A_i^*(l)$ is non-empty, compact-valued, and uhc.

Define $H^s(l) = \{\tilde{l} \in L^s : \tilde{d}_{iC} \in A_i^*(l)\ \forall i\}$

We want to show that $H^s(l)$ is (i) non-empty, (ii) compact, (iii) uhc, and (iv) convex.

(i) Non-empty: Each $A_i^*(l)$ is non-empty. To show that $\tilde{l}$ is symmetric, notice that if $d_{01} \in A_0^*(l)$,

---

[10] We want to show that $A_i^s(l)$ is both uhc and lhc:

uhc: Let $B \subset L^s$ be a compact set. Since $A_{-i}^s(B) = \cup_{j \neq i} A_j^s(B)$ and each $A_j^s(B)$ is bounded, so is $A_{-i}^s(B)$.

lhc: Suppose we can express l as $(\ldots, d_{iC}, \pi_{iC}, \ldots)$, where π_{iC} is the probability with which player i proposes d_{iC} to coalition C. Define $E[l]$ in the following way:

$$E[l] = \sum_i p_i \left(\sum_C \pi_{iC} d_{iC} \right)$$

Since individuals are risk averse,
$$u_i(E[l]) > \delta v_i(l)\ \forall i$$

To see this, notice that $v_i(l) = \sum_j p_j \left(\sum_C \pi_{jC} u_i(d_{jC}) \right)$.

Let $d^k = \alpha_k E[l] + (1 - \alpha_k)d : \alpha_k \to 0$. For each $j \neq i$,

For each $j \neq i$, $\exists K_j : u_j(d^k) > \delta v_i(l^n)\ \forall k \geq K_j$

Let $N_n = \max\{K_j, K_k\}$. Then $u_j(d^k) > \delta v_j(l)\ \forall k \geq N_n$, Hence, $d^{N_n} \in A_{-i}^s(l)$. Since $d^{N_n} \to d$ and $d^{N_n} \in A_{-i}^s(l^n)$, $A_i^s(l)$ is lhc.

then, by symmetry, it must be that $d'_{01} \in A^*_1(l)$ (where d'_{01} is the proposal in which 1 offers to 0 what 0 offers to 1 in d_{01}).

(ii) Compact: Since $H^s(l) \subset L^s$, a compact space, it suffices to show that $H^s(l)$ is closed. Let $l^n \to l$, where $l^n \in H^s(l') \; \forall n$. Then it follows that $d^n_{iC} \in A^*_i(l') \; \forall i$. Let $l^n = (l^n_0, l^n_m, l^n_1)$, where l^n_i is the continuation strategy of player i - $l^n_i = (\ldots, (d^n_{iC}, \pi^n_{iC}), \ldots)$. If $l^n \to l$, it must be the case that $l^n_i \to l_i \; \forall i$ and hence $(d^n_{iC}, \pi^n_{iC}) \to (d_{iC}, \pi_{iC})$.

$l^n \in H^s(l') \implies (\pi^n_{iC} > 0 \Rightarrow d^n_{iC} \in A^*_i(l')) \; \forall i$. This means that $u_i(d^n_{iC}) = \bar{u}_i \geq u_i(d) \; \forall d \in A^s_{-i}(l') \; \forall n$. Since $u_i(.)$ is continuous, $u_i(d_{iC}) = \bar{u}_i \implies d_{iC} \in A^*_i(l')$. Now, to show that $l \in H^s(l')$, we simply need to show that $\pi_{iC} > 0 \implies d_{iC} \in A^*_i(l')$. Suppose $\pi_{iC} > 0$, then it must have been the case that $\pi^n_{iC} > 0$ for $n \geq N$, for some large N. But if this is true, then we have shown that it must be the case that $d_{iC} \in A^*_i(l')$.

(iii) Uhc: Since L^s is bounded, it follows that $H^s(B) \subset L^s$ is bounded for any compact B, therefore $H^s(.)$ is uhc.

(iv) Convex: Suppose $l, l' \in H^s(l'')$ and let $\lambda \in (0, 1)$, then if $l_\lambda = \lambda l + (1 - \lambda)l'$, $l_\lambda \in H^s(l'')$. To see this, notice that we simply need to show that l_λ is a symmetric strategy profile. Recall that the entries of a symmetric continuation strategy are decisions and probabilities:

$$((d_{01}, 1 - q), (d_{0m}, q), (d_{m0}, 1/2), (d_{m1}, 1/2), (d_{1m}, q), (d_{10}, 1 - q)),$$

where each proposal d_{iC} is identified with a distinct C (one of the other players). The difference between l and l' are the probability, q, and the details of d_{iC}, by which we mean the particular proportions being offered to C. Since each $A_j^s(l'')$ is convex, if both d_{iC}' and d_{iC} maximize u_i over $A_j^s(l'')$, then, by the convexity of $A_j^s(l'')$ and concavity of $u_i(.)$, it must be that d_{iC}^λ also maximizes $u_i(.)$. This means that $d_{iC}^\lambda = \lambda d_{iC} + (1 - \lambda)d_{iC}' \in A_i^*(l'')$. If both l and l' are symmetric (and hence have the form above), then l^λ is also symmetric. Then if l^λ is symmetric and each d_{iC}^λ maximizes u_i over $A_i^*(l'')$, it follows that $l^\lambda \in H^s(l'')$.

By Kakutani's fixed point theorem, $H^s : L^s \to L^s$ has a fixed point. This fixed point is a symmetric equilibrium of the legislative game[11]. $\qquad\square$

Proof of Claim 2

Proof. First, notice that when X is large enough, the optimal policy y_j^i for player i to propose to player j is $y_j^i = \frac{v_i + v_j}{2}$, which is the midpoint between the ideal points of i and j. To see why this is the case, let x_j^i be the proportion that i keeps when they propose to j and notice that whenever player i chooses what policy to propose, they solve the following maximization problem:

$$\max_{y_j^i, x_j^i} \left\{ U_i(y_j^i, x_j^i) : U_j(y_j^i, x_j^i) \geq \delta v_j \right\},$$

where v_j is j's continuation payoff - a function of all players' strategies.

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[11]The final claim follows from the final step of the existence proof in Jackson and Moselle (2002).

Since $U_i(y, x) = u_i(y) + x_j^i X$, and $v_j = \frac{\delta}{3} \left(U_j(P_j) + u_j(y_j^i) + x_j^i + U_j(P_k) \right)$, the Lagrangian of the above maximization problem is the following:

$$\mathcal{L} = u_i(y) + x_j^i X + \lambda \left(u_j(y) + (1 - x_j^i)X - \frac{\delta}{3} \left(U_j(P_j) + u_j(y) + (1 - x_j^i)X + U_j(P_k) \right) \right)$$

The first order conditions of the above maximization problem with respect to y and x_j^i are the following condition:

$$FOC(y) : \frac{\partial u_i(y)}{\partial y} + \lambda \left(\frac{\partial u_j(y)}{\partial y} - \frac{\delta}{3} \frac{\partial u_j(y)}{\partial y} \right) = \frac{\partial u_i(y)}{\partial y} + \lambda \frac{\partial u_j(y)}{\partial y} \frac{3 - \delta}{3} = 0$$

$$FOC(x_j^i) : X + \lambda \left(-X + \frac{\delta}{3}X \right) = X \left(1 - \lambda \frac{3 - \delta}{3} \right) = 0$$

The first order condition with respect to x_j^i yields $\lambda = \frac{3}{3-\delta}$. Substituting this into the expression for the first order condition with respect to y yields:

$$\frac{\partial u_i(y)}{\partial y} + \frac{\partial u_j(y)}{\partial y} = 0$$

Since $\partial u_i(y)/\partial y = 2(v_i - y)$, it follows that

$$\frac{\partial u_i(y)}{\partial y} + \frac{\partial u_j(y)}{\partial y} = 2(v_i - y) + 2(v_j - y) = 2(v_i + v_j - 2y) = 0 \iff y = (v_i + v_j)/2$$

Case 1: Suppose the symmetric equilibrium is one where the two extreme players propose to one another, i.e. 1 proposes to 0 and 0 proposes to 1. We will show that at the beginning of any subgame at which one of the extreme players is to propose to the

Figure 3.1: Existence of a profitable deviation for 0.

If m's continuation payoff, u_m lies between $\bar{u}$ and u_1, then there exists a profitable deviation for 0 where they propose to m instead of 1, moving to the outer utility frontier and increasing their own continuation value from u_0.

other according to the equilibrium strategy profile, it would be profitable to propose

to m instead. More precisely, there exists a proposal that offers some share of the

resources to m that player 0, say, would prefer to the one that offers a share to 1, and

that this proposal would be accepted by m.

Let the total utility that can be "shared" between legislators i and j whenever i

proposes to j be denoted by $U(i,j) = U_i(P_j^i) + U_j(P_j^i)$, where P_j^i is i's proposal to j.

We will show that such a deviation exists for player 0, but the symmetry of the setup

implies that the argument is exactly the same for 1. Notice that whenever the proposals

are interior, the proposed policy is located at the midpoint between the two legislators'

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ideal points. It's clear to see that since the ideal point of m is closer to 0 than is 1, $U(0, m) > U(0, 1)$:

$$U(0, m) = U_0(P_m^0) + U_m(P_m^0) = 1 - \left(0 - \frac{1}{4}\right)^2 + x_m^0 X + 1 - \left(\frac{1}{2} - \frac{1}{4}\right)^2 + \left(1 - x_m^0\right) X = \frac{15}{8} + X$$

$$U(0, 1) = U_0(P_1^0) + U_1(P_1^0) = 1 - \left(0 - \frac{1}{2}\right)^2 + x_1^0 X + 1 - \left(1 - \frac{1}{2}\right)^2 + \left(1 - x_1^0\right) X = \frac{3}{2} + X$$

The argument can be most clearly seen by looking at figure 1. The continuation payoffs of m and 1 are on the y-axis, and 0's continuation payoff is on the x-axis. The utility frontier when 0 proposes to m is everywhere above that for the case when 0 proposes to 1 since $U(0, m) > U(0, 1)$. If a strategy profile defines continuation payoffs u_0 and u_1 as depicted, then if u_m lies strictly below $\bar{u}$, a profitable deviation exists where 0 can propose some split of the resources and the policy $y_m^0 = 1/4$ that yield immediate utility for m greater than u_m. This means that m accepts this offer from 0. Moreover, by the definition of $\bar{u}$, this proposal would yield utility for 0 greater than u_0, their utility from proposing to 1. This follows by the observation that a horizontal line from any point on the y-axis below $\bar{u}$ intersects the outer utility frontier to the right of u_0.

To show that a profitable deviation for 0 exists, therefore, it suffices to show that $u_m < \bar{u}$. This can be shown by solving for u_m, u_1 and $\bar{u}$ from the five simultaneous equations (in five unknowns x^m, u_0, u_m, u_1, and x^0) below that derive from the binding incentive constraints associated with the strategy profile we specified. Notice that x^0 is

proportion of resources kept by 0 when they propose to 1, and, by symmetry, it is also

the proportion kept by 1 when they propose to 0, and $u(x) = 1 - x^2$.

1) 0 accepts m's offer:

$$(1 - x^m)X + u(1/4) = u_0$$

2) 1 accepts m's offer:

$$(1 - x^m)X + u(1/4) = u_1$$

3) 1 accepts 0's offer:

$$(1 - x^0)X + u(1/2) = u_1$$

4) m's continuation payoff:

$$u_m = \frac{\delta}{3}\left(0 + u(1/4) + x^m X + u(1/4) + 0 + u(1/4)\right)$$

5) 1's continuation payoff:

$$u_1 = \frac{\delta}{3}\left[(1 - x^0)X + u(1/2) + \frac{1}{2}(0 + u(3/4)) + \frac{1}{2}((1 - x^m)X + u(1/4)) + x^0 X + u(1/2)\right]$$

Once we solve for u_0, we can solve for $\bar{u}$ by the equality $U(0, m) = u_0 + \bar{u}$[12]. Since

---

[12]This is clear to see from figure 1.

this exercise involves solving five simultaneous equations in five unknowns, we find the solutions numerically for some X and δ and verify that $u_m < \bar{u}$ holds.

Case 2: Now suppose the symmetric equilibrium is one in which the extreme players, 0 and 1, randomize between proposing to m and one another. Namely, they each propose to m with probability q, and with probability $(1-q)$ they propose to the other extreme player. An equilibrium, therefore, is characterized by a vector

$$(x^m, q, x_m^0, x_1^0),$$

where m proposes $(1 - x^m)$ of the resources to each extreme player with probability $1/2$, each extreme player proposes x_m^0 of the resources to m with probability q and x_1^0 to the other extreme player with probability $(1-q)$. We want to show that an extreme player can profitably deviate in the subgame in which they should propose to the other extreme player by proposing, instead, to m. In addition, we want to show that this proposal will be accepted by m. Similarly to the above case, we solve for u_m and u_1 from the equations characterizing the equilibrium. We have 6 unknowns: the 4 equilibrium quantities and 2 continuation payoffs (0 and 1's continuation payoffs are the same, so we only consider u_0 in the system below).

1) 0 accepts m's offer:

$$(1 - x^m)X + u(1/4) = u_0$$

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2) 0 accepts 1's offer:

$$(1 - x_1^0)X + u(1/2) = u_1$$

3) m accepts 0 & 1's offers:

$$(1 - x_m^0)X + u(1/4) = u_m$$

4) m's continuation payoff:

$$u_m = \frac{\delta}{3} \left(q(1 - x_m^0)X + u(1/4) + x^m X + u(1/4) + q(1 - x_m^0)X + u(1/4) \right)$$

5) 1's continuation payoff:

$$u_1 = \frac{\delta}{3} \Big[(1 - q)[(1 - x_1^0)X + u(1/2)] + q[(1 - x_m^0)X + u(1/4)] \dots$$
$$\dots + \frac{1}{2}(0 + u(3/4)) + \frac{1}{2}((1 - x^m)X + u(1/4)) + x^0 X + u(1/2) \Big]$$

6) m's continuation payoff equals both extreme players' continuation payoffs, since only then can any one player be indifferent (and hence randomize) between propos-

ing to m and proposing to the other extreme player.

$$u_m = u_0$$

The above system of equations has a solutions since it has 6 equations in 6 unknowns. From the value of u_0, we can find $\bar{u}$, and check to see if $\bar{u} < u_m$. Since this exercise involves solving 6 equations in 6 unknowns, we check it numerically. □

Proof of Claim 3

Proof. An equilibrium of this form is a vector

$$(q_m^0, x_m^0, q_1^0, p, x_0^m, x_1^m, q_m^1, x_m^1, x_0^1),$$

such that $q^0 = q^1 = 1$. An equilibrium, therefore can be characterized by 5 quantities:

$$x = (x_1, x_2, x_3, x_4, x_5) = (x_m^0, p, x_0^m, x_1^m, x_m^1)$$

At an equilibrium, it must be that the following list of conditions hold. Although these conditions are characterized by inequalities, in equilibrium, they must hold with equality since otherwise the proposer can increase the proportion they keep while still ensuring his proposal is accepted. This gives us 5 equation in 5 unknowns:

1) 0 accepts m's offer

$$f_1(x, v_m) = (1 - x_0^m)X + u\left(\frac{v_m}{2}\right) - u_0 = 0$$

2) 1 accepts m's offer

$$f_2(x, v_m) = (1 - x_1^m)X + u\left(\frac{1 - v_m}{2}\right) - u_1 = 0$$

3) m accepts 0's offer

$$f_3(x, v_m) = (1 - x_m^0)X + u\left(\frac{v_m}{2}\right) - u_m = 0$$

4) m accepts 1's offer

$$f_4(x, v_m) = (1 - x_m^1)X + u\left(\frac{1 - v_m}{2}\right) - u_m = 0$$

5) m is indifferent between proposing to 0 and 1

$$f_5(x, v_m) = x_0^m X + u\left(\frac{v_m}{2}\right) - x_1^m X - u\left(\frac{1 - v_m}{2}\right) = 0$$

u_i is i's continuation payoff:

$$u_0 = \frac{\delta}{3}\left[x_m^0 X + u\left(\frac{v_m}{2}\right) + p\left((1 - x_0^m)X + u\left(\frac{v_m}{2}\right)\right) + (1 - p)\left(0 + u\left(\frac{1 + v_m}{2}\right)\right) + 0 + u\left(\frac{1 + v_m}{2}\right)\right]$$

$$u_m = \frac{\delta}{3}\left[(1-x_m^0)X + u\left(\frac{v_m}{2}\right) + p\left(x_0^m X + u\left(\frac{v_m}{2}\right)\right) + (1-p)\left(x_1^m X + u\left(\frac{1-v_m}{2}\right)\right) + (1-x_m^1)X + u\left(\frac{1-v_m}{2}\right)\right]$$

$$u_1 = \frac{\delta}{3}\left[0 + u\left(\frac{2-v_m}{2}\right) + p\left(0 + u\left(\frac{2-v_m}{2}\right)\right) + (1-p)\left((1-x_1^m)X + u\left(\frac{1-v_m}{2}\right)\right) + x_m^1 X + u\left(\frac{1-v_m}{2}\right)\right]$$

Let $x = (x_m^0, p, x_0^m, x_1^m, x_m^1)$ be the unknowns of this system of equations, and v_m be the parameter. We know that a symmetric equilibrium exists, and so the system of equations can be solved at $(\bar{x}, 1/2)$. To show that there exists another v_m close to $1/2$ at which an equilibrium exists, we need to show that we can locally solve these equations at $(\bar{x}, 1/2)$. By the implicit function theorem, this is equivalent to showing that the Jacobian of this system with respect to x evaluated at $\bar{x}$ is non-singular:

$$Det(J) = \begin{vmatrix} \frac{\partial f_1(\bar{x},1/2)}{\partial x_1} & \cdots & \frac{\partial f_1(\bar{x},1/2}{\partial x_5} \\ \vdots & & \vdots \\ \frac{\partial f_5(\bar{x},1/2)}{\partial x_1} & \cdots & \frac{\partial f_5(\bar{x},1/2}{\partial x_5} \end{vmatrix} \neq 0$$

$$Det(J) = \begin{vmatrix} -\frac{\delta}{3}X & -\frac{\delta}{3}\left((1-x_0^m)X + u\left(\frac{v_m}{2}\right)\right) - u\left(\frac{1+v_m}{2}\right) & (p-1)X & 0 & 0 \\ 0 & \frac{-\delta}{3}\left(u\left(\frac{2-v_m}{2}\right) - (1-x_1^m)X - u\left(\frac{1-v_m}{2}\right)\right) & 0 & \left(\frac{\delta}{3}(1-p)-1\right)X & \frac{-\delta}{3}X \\ X\left(\frac{\delta}{3}-1\right) & -\frac{\delta}{3}\left[x_0^m X + u\left(\frac{v_m}{2}\right) - x_1^m X - u\left(\frac{1-v_m}{2}\right)\right] & -\frac{\delta}{3}pX & -\frac{\delta}{3}(1-p)X & \frac{\delta}{3}X \\ \frac{\delta}{3}X & -\frac{\delta}{3}\left[x_0^m X + u\left(\frac{v_m}{2}\right) - x_1^m X - u\left(\frac{1-v_m}{2}\right)\right] & -\frac{\delta}{3}pX & -\frac{\delta}{3}(1-p)X & X\left(\frac{\delta}{3}-1\right) \\ 0 & 0 & X & -X & 0 \end{vmatrix}$$

We want to show that $Det(J) \neq 0$. Notice that J is a matrix of the following form:

$$
\begin{array}{ccccc}
a & b & c & 0 & 0 \\
0 & d & 0 & e & f \\
g & h & i & j & k \\
l & m & n & o & p \\
0 & 0 & q & r & 0
\end{array}
$$

Moreover, for any matrices A $(n \times n)$, B $(n \times m)$ and C $(m \times m)$, the following equality holds:

$$
Det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = Det(A)Det(D)
$$

We want to transform J into this form. To do so for the general matrix above, we need to do the following operations that leave the determinant unchanged:

- multiply the first row by $-l/a$ and add it to the 4th

- multiply the second row by $\frac{1}{d}\left(\frac{l}{a}b - m\right)$ and add it to the 4th

These two operations yield a matrix of the form:

$$
\begin{array}{ccccc}
a & b & c & 0 & 0 \\
0 & d & 0 & e & f \\
g & h & i & j & k \\
0 & 0 & n - \frac{lc}{a} & o + \frac{e}{d}\left(\frac{l}{a}b - m\right) & p + \frac{f}{d}\left(\frac{l}{a}b - m\right) \\
0 & 0 & q & r & 0
\end{array}
$$

The determinant of this matrix is then simply the following:

$$
Det(J) = Det \begin{pmatrix} a & b & c \\ 0 & d & 0 \\ g & h & i \end{pmatrix} \times Det \begin{pmatrix} o + \frac{e}{d}\frac{l}{a}b - m & p + \frac{f}{d}(\frac{l}{a}b - m) \\ r & 0 \end{pmatrix}
$$

$$
= (adi + 0 + 0 - cdg - 0 - 0) \times -r\left[p + \frac{f}{d}\left(\frac{l}{a}b - m\right)\right]
$$

$$
= (adi - cdg) \times r\left[\frac{f}{d}\left(m - \frac{l}{a}b\right) - p\right]
$$

Whenever $\delta^2 p/9 \neq (p-1)(\delta - 3)/3$, the first term in the multiplication is not zero. Whether the second term is equal to zero or not is tedious to verify and, for the time being, we conjecture that it's not. The numerical exercises we conduct, the results of some of which are included in the appendix support this conjecture: that there exist equilibria of this form for $v_m \neq 1/2$. □

3.6.2 Numerical exercises
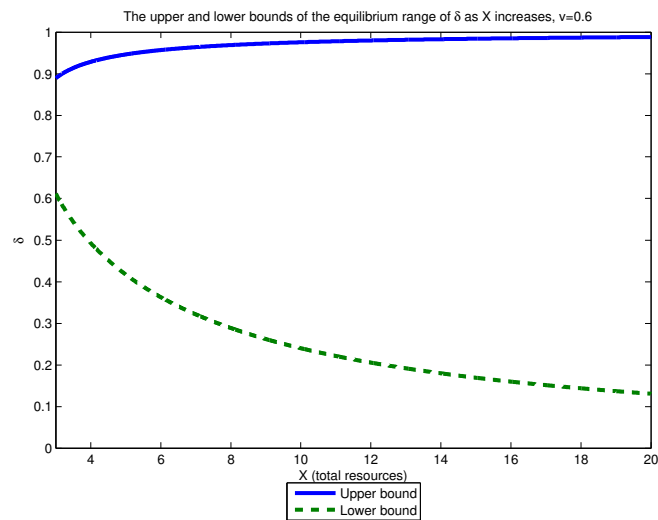
Figure 3.2: The range of δ for which an equilibrium exists (all x^i's $\in [0,1]$) as X increases, $v_m = 0.6$
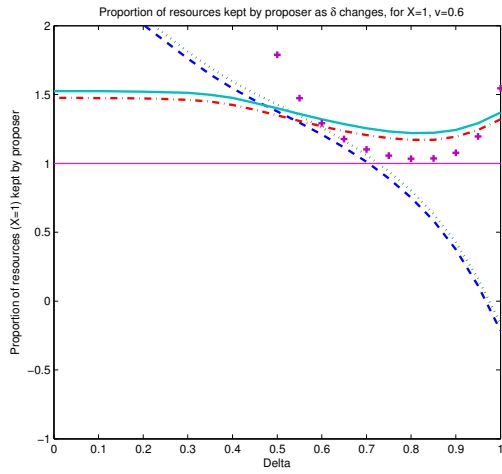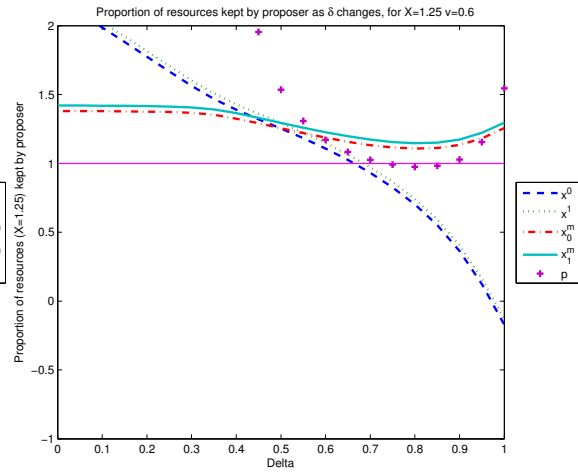
(a) Caption 1   (b) Caption 2
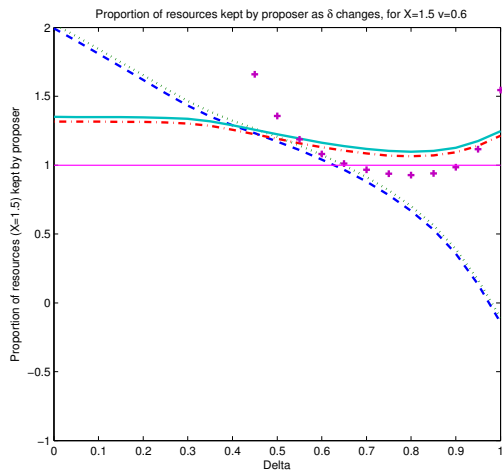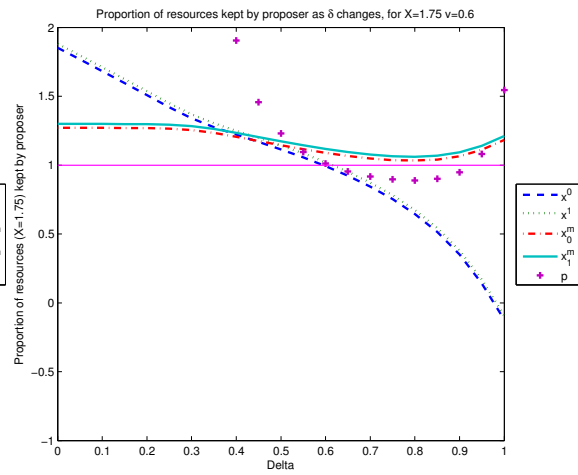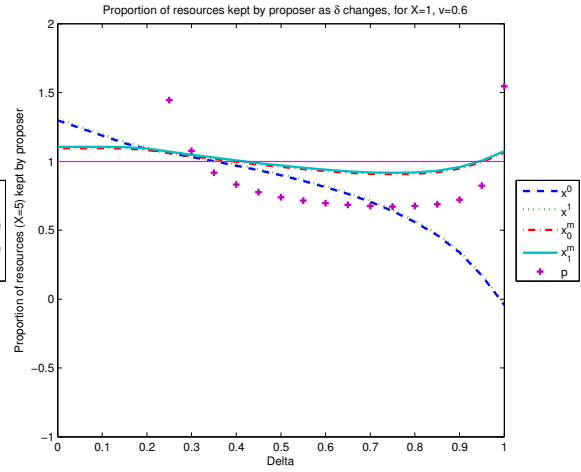
(c) Caption 3   (d) Caption 4
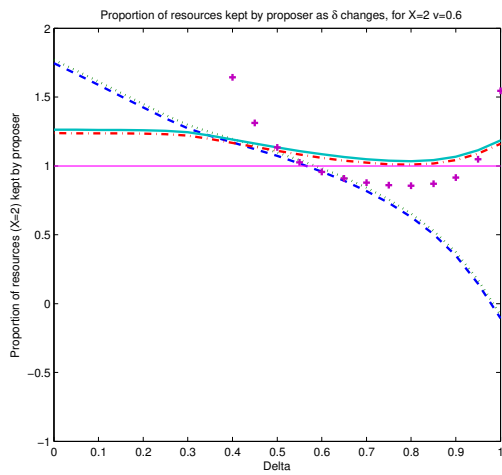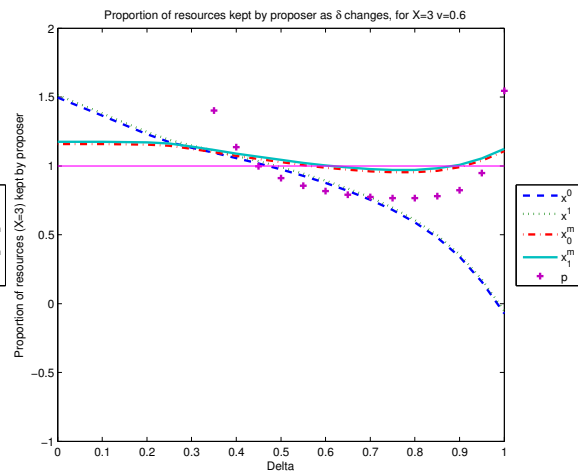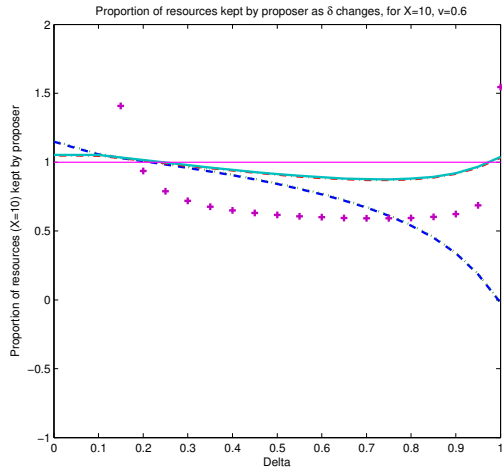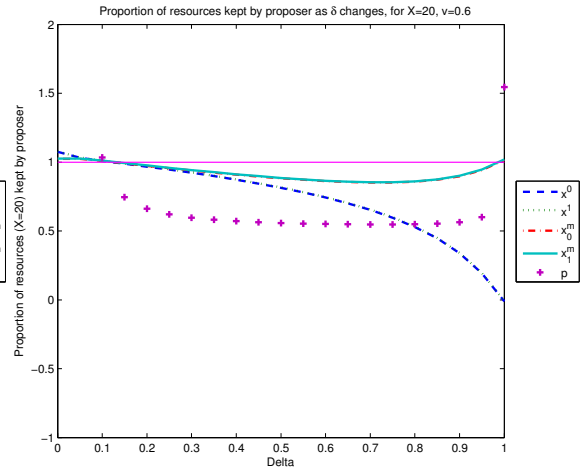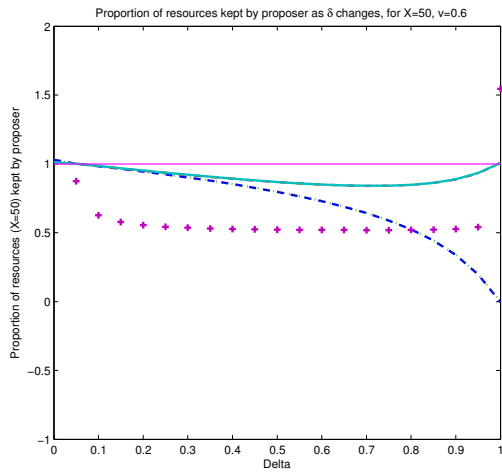
(e) Caption 5   (f) Caption 6

Figure 3.3: Equilibrium strategies for a fixed amount of resources, as δ changes with $v_m = 0.6$ (part 1)
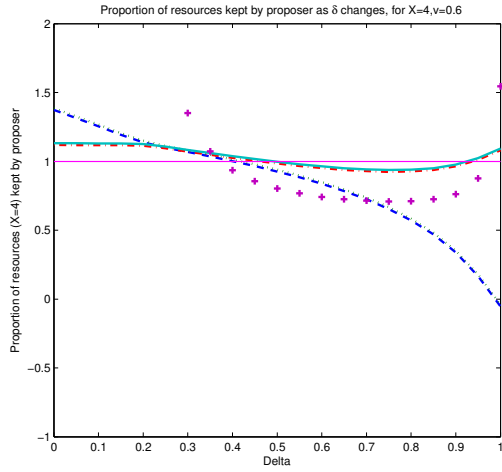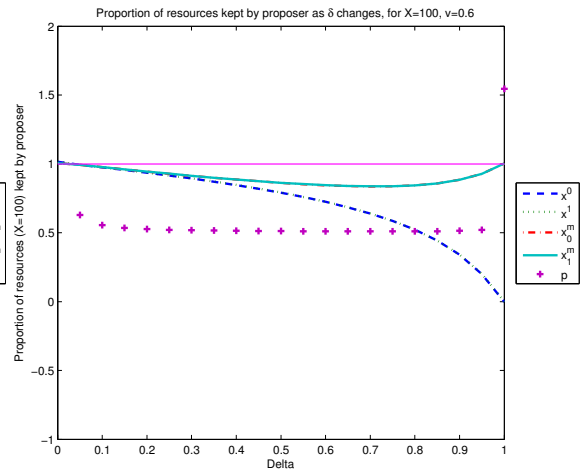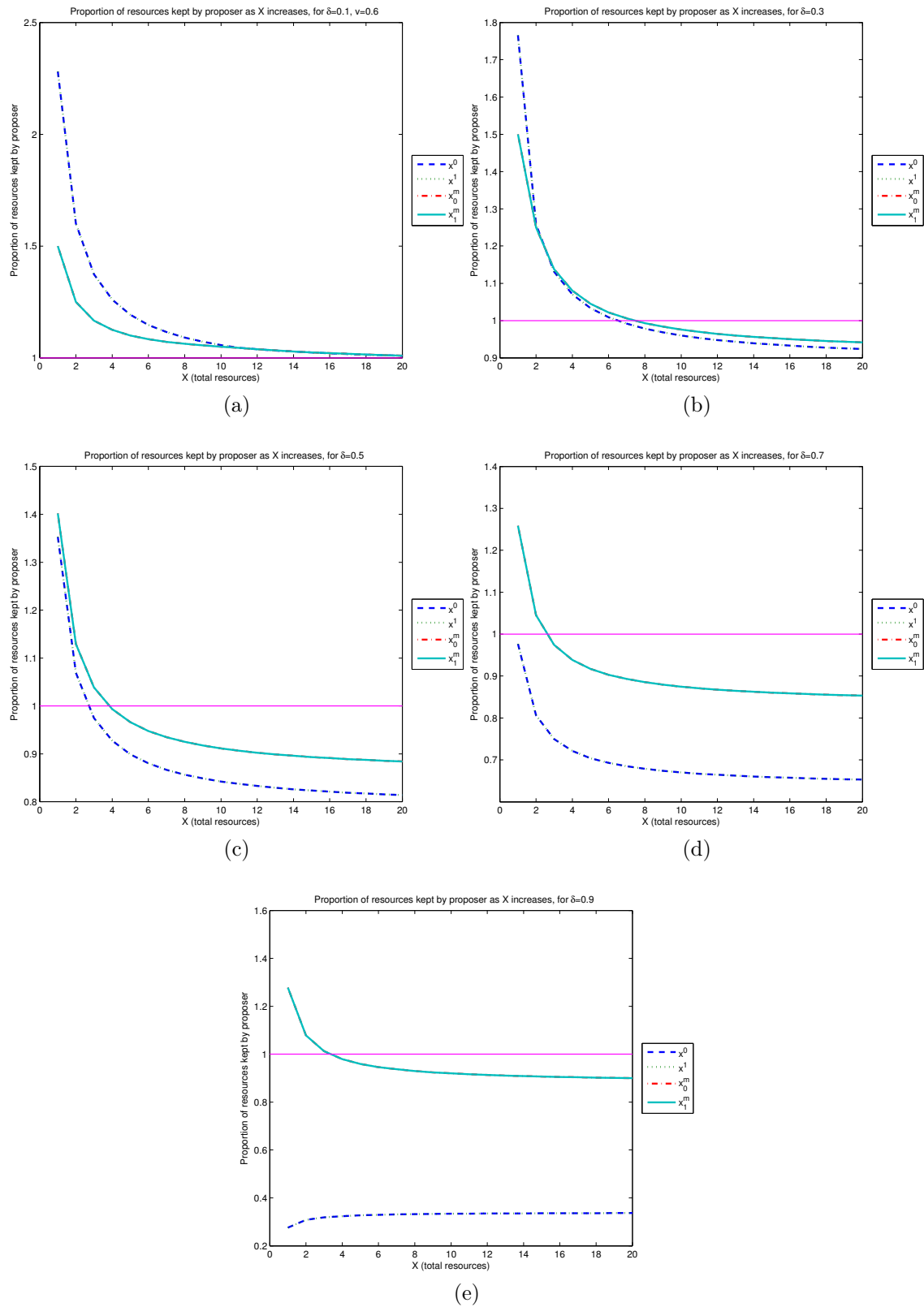
(a) Caption 7

(b) Caption 8

(c) Caption 9

(d) Caption 10

(e) Caption 11

(f) Caption 12

Figure 3.4: Equilibrium strategies for a fixed amount of resources, as δ changes with $v_m = 0.6$ (part 2)

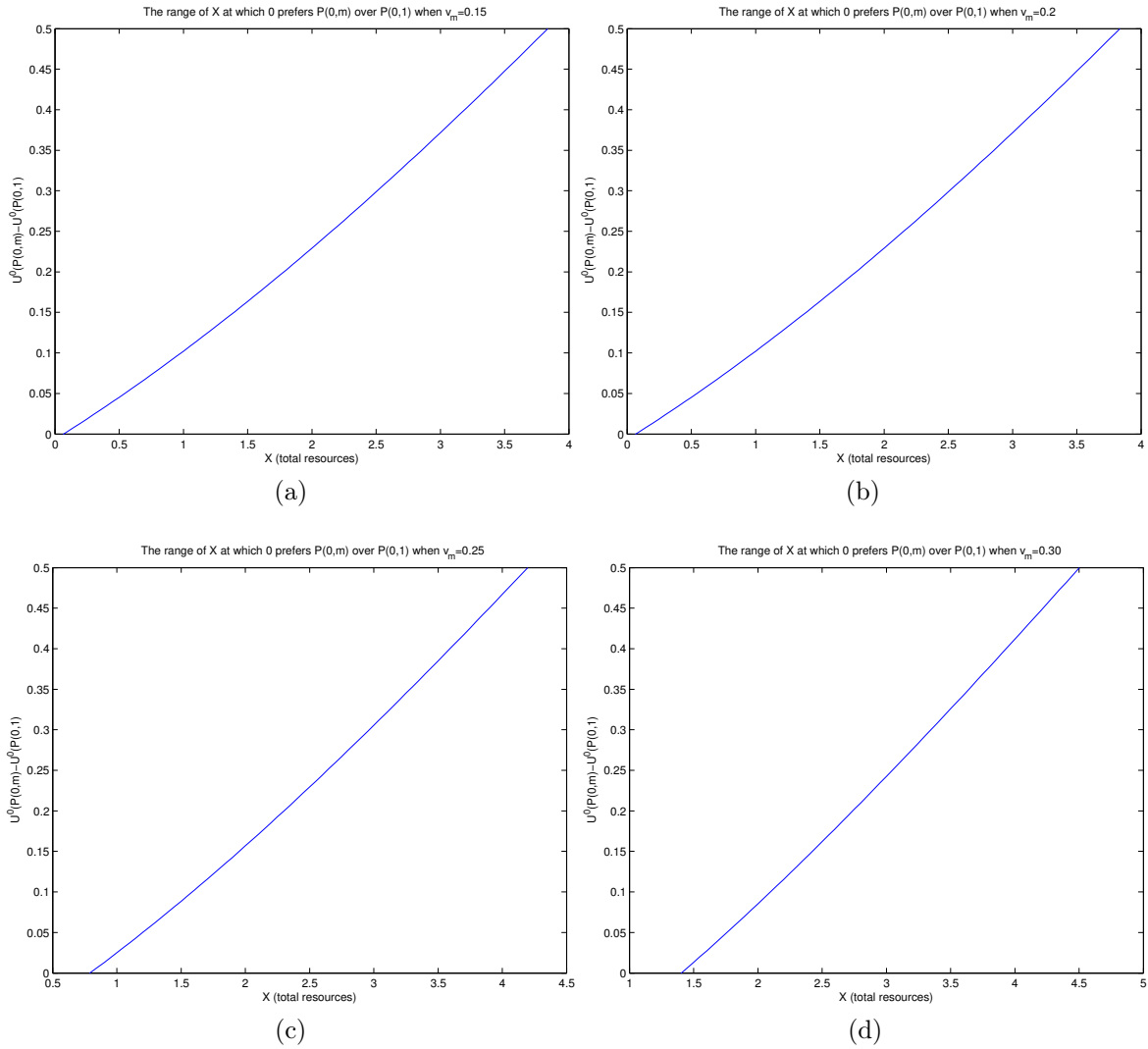Figure 3.5: Equilibrium strategies for a fixed δ as X changes

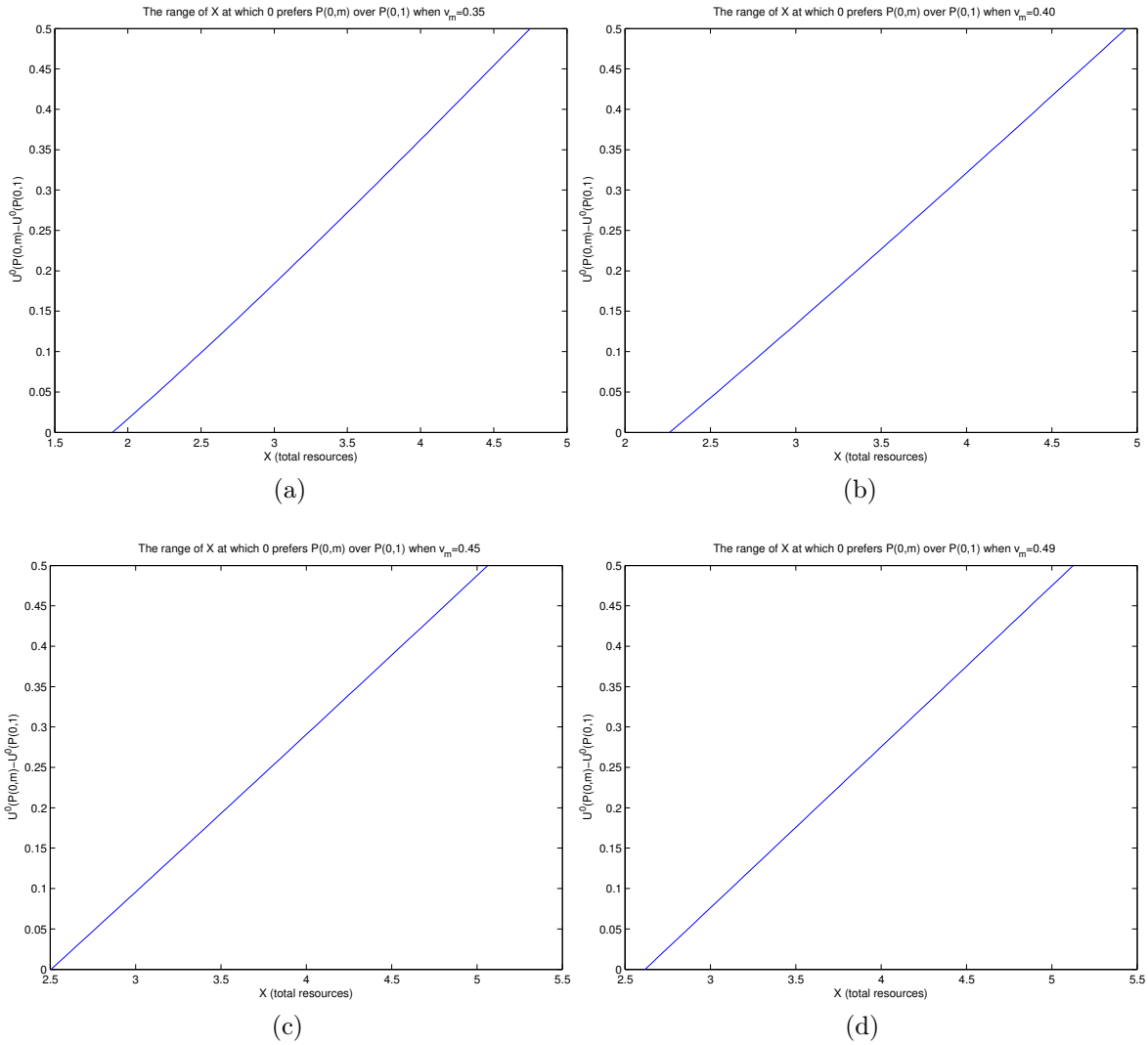Figure 3.6: The range of X for which 0 prefers P(0,m) over P(0,1) (part 1)

Figure 3.7: The range of X for which 0 prefers P(0,m) over P(0,1) (part 2)

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