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Author
Young, Jonathan D.

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Jonathan D. Young

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by

Jonathan D. Young

Lawrence Berkeley Laboratory
University of California
Berkeley, California

ABSTRACT

We present a method for approximating a harmonic function on and within a simple polygonal domain when values of the function are specified at seven or more points on the boundary. Function values must be specified at all the vertices of the polygon. Further, for any simple domain, the boundary may be approximated by a simple polygon and the above process applied. Thus, we are able to approximate the solution to Laplace's equation with Dirichlet conditions on rather general simple domains. We are also able to approximate the harmonic conjugate of the original function to within an arbitrary constant. This constant may be determined if the value of the harmonic conjugate is specified at one point.

The original function and its harmonic conjugate are treated as, respectively, the real and imaginary parts of a complex function. The approximation of this latter function is by means of a complex cubic spline (see Reference 1).

SECTION 1. INTRODUCTION

We consider a real function, u, of real variables, x and y. With

$$z = x + i\,y$$

we have u(z), a real function of the complex variable z.

We shall be concerned with K, ($K \geq 7$), points z_n, distinct and not collinear. The points, z_n, can then be ordered so that they define a simple closed counterclockwise path

$$P:\ z_1 \text{ --- } z_2 \text{ --- } z_3 \text{ --- } \cdots\ z_k \text{ --- } z_1 .$$

We shall assume this ordering for the z_n. For convenience in indicing, in view of the cyclic ordering, we define

$$n + 1 \equiv 1 \quad \text{for} \quad n = K$$

$$n - 1 \equiv K \quad \text{for} \quad n = 1 .$$

We shall require that u satisfies Laplace's equation,

$$u_{xx} + u_{yy} = 0$$

on and within P. Thus, u may be regarded as the real part of an analytic complex function, w(z), on the domain of P and its interior. Then, there exists a real function, v(z) such that

$$w = u + i\,v$$

with the Cauchy conditions

$$v_y = u_x$$

$$v_x = u_y$$

satisfied. Such a function, v, is called a harmonic conjugate of u.

Next we note that any other harmonic conjugate of u is obtainable from

$$v + C, \quad C, \text{ an arbitrary real constant.}$$

This means that if v is unknown, we can arbitrarily assign it a value at one point, or conversely if v is specified at one point it is uniquely determined by u.

We shall, initially, treat v as unknown and assign it the value

$$v_1 = 0 \quad \text{at } z_1.$$

We now wish to approximate w as a complex cubic spline in z. From 1, we know that a unique cubic spline is determined only if both u and v are known at all the points, z_k. Even with v_1 specified as zero, values for v_n for n equal 2 to K are arbitrary insofar as constructing a complex cubic spline is concerned. We, therefore, consider finding values for v_n, n equal 2 to K in some optimal manner.

Since, in general, complex cubic splines have discontinuities in the third derivative at the points, z_k, we shall consider choosing the v_n for n equal 2 K in such a way that this discontinuity is minimized (see Section 3). Now we note that for K = 7, the specification of u_n for n equal 1, to K and of v_1 determines a unique complex cubic, that is a function with no third derivative discontinuities. However, if we are permitted $K < 7$, there would be an infinitude of such cubics satisfying the specifications, hence we require $K \geq 7$. With $K > 7$, (except in extremely fortuitous circumstances) there would be third derivative discontinuity. We consider now the space of all complex cubic splines which meet the specifications.

SECTION 2. LINEAR SPACE OF COMPLEX CUBIC SPLINES

As a basis for a linear space, S, of complex cubic splines, let us consider

$$S_j \text{ with } u_{jn} = u_n \text{(specified) and } v_1 = 0 \text{(specified)}$$
$$\text{and } v_{jm} = S_{jm} \text{ for m equal 2 to K.}$$

We note first that the S_j are linearly independent and that any linear combination

$$S = \sum_{j=1}^{K} a_j S_j \quad \text{with all } a_j \text{ real}$$

is a complex cubic spline. Now we note that although any one of the basic splines by itself satisfies the specifications, the linear combination, S, will satisfy them if and only if

$$\sum_{j=1}^{K} a_j = 1.$$

From the above, we must have

$$a_1 = 1 - \sum_{j=2}^{K} a_j$$

and

$$S = [1 - \sum_{j=2}^{K} a_j]S_1 + \sum_{j=2}^{K} a_j S_j .$$

Hence

$$S = S_1 + \sum_{j=2}^{K} a_j(S_j - S_1).$$

Any such S will satisfy the specifications and we need only to determine the a_j for j equal 2 to K in some optimal manner. We consider an optimization process in the next section.

SECTION 3. OPTIMIZATION - MINIMIZATION OF THIRD DERIVATIVE DISCONTINUITY

For each basic spline, S_j, the discontinuity of S_j''' at any point, z_n, is the complex quantity

$$d_{jn} = S_{jn}''' - S_{j,n-1}'''.$$

Let

$$d_{jn} = r_{jn} + i\, t_{jn}$$

where r and t are respectively the real and imaginary parts of d, then

$$r_{jn} = \text{Real}\ (S_{jn}''') - \text{Real}\ (S_{j,n-1}''')$$

$$t_{jn} = \text{Imaginary}\ (S_{jn}''') - \text{Imaginary}\ (S_{n,n-1}''').$$

For the linear combination spline, S, last defined in the previous section, we have

$$S''' = S_1''' + \sum_{j=2}^{K} a_j\ (S_j''' - S_1''').$$

For the discontinuity d_n of S''' at z_n, we have

$$d_n = S_{1n}''' - S_{1,n-1}''' + \sum_{j=2}^{K} a_j \left[(S_{j,n}''' - S_{j,n-1}''') - (S_{1,n}''' - S_{1,n-1}''')\right]$$

$$= \left[r_{1,n} + \sum_{j=2}^{K} a_j (r_{j,n} - r_{1,n})\right] + i \left[t_{1,n} + \sum_{j=2}^{K} a_j (t_{j,n} - t_{1,n})\right].$$

Let

$$\left.\begin{aligned} p_{j,n} &\equiv r_{j,n} - r_{1,n} \\ q_{j,n} &= t_{j,n} - t_{1,n} \end{aligned}\right\} \quad \text{for } j \text{ equal 2 to } K$$

we have

$$|d_n|^2 = \left[r_{1,n} + \sum_{j=2}^{K} a_j\, p_{j,n}\right]^2 + \left[t_{1,n} + \sum_{j=2}^{K} a_j\, q_{j,n}\right]^2$$

We shall "minimize" the third derivative discontinuities by defining

$$\sigma = \sum_{n=1}^{K} |d_n|^2$$

and finding those a_j for j equal 2 to K for which σ is a minimum.

$$\sigma = \sum_{n=1}^{K}\left\{\left[r_{1,n} + \sum_{j=2}^{K} a_j\, p_{j,n}\right]^2 + \left[t_{1,n} + \sum_{j=2}^{K} a_j\, q_{j,n}\right]^2\right.$$

Note that σ is a real number greater than or equal to zero, hence does have a minimum. This minimum can only occur for the choice of the a's such that

$$\frac{\partial\sigma}{\partial a_i} = 0 \qquad \text{for i equal 2 to K}$$

hence

$$\sum_{n=1}^{K}\left\{2\left[r_{1,n} + \sum_{j=2}^{K} a_j\, p_{j,n}\right]p_{i,n} + 2\left[t_{1,n} + \sum_{j=2}^{K} a_j\, q_{j,n}\right]q_{i,n}\right\} = 0$$

for i equal 2 to K.

Consequently,

$$\sum_{j=2}^{K}\ \sum_{n=1}^{K}(p_{i,n}\, p_{j,n} + q_{i,n}\, q_{j,n})a_j = -\sum_{n=1}^{K}(r_{1,n}\, p_{i,n} + t_{1,n}\, q_{i,n})$$

giving us K - 1 equations in the K - 1 unknowns, a_j, for j equal 2 to K and provided this linear system is determinate we can find the a_j for j equal 2 to K which minimize σ.

We now note that at any point, z_n, for n equal 2 to K

$$v_n = \sum_{j=2}^{K} a_j \text{ Imaginary } (S_j(z_n)) \qquad \text{for n equal 2 to K}$$

but for $j \neq n$, $S_j(z_n) = 0$ and for $j = n$, $S_j(z_n) = 1$, hence

$$v_n = a_n \qquad \text{for n equal 2 to K.}$$

Thus, we have found optimum values for v_n for n equal to to K with $v_1 = 0$.

Now suppose that some point, z_ℓ, the value of $v(z_\ell)$ is specified, then with the v_ℓ obtained above we find

$$C = v(z_\ell) - v_\ell$$

and set

$$v_1 = C$$

and add C to all the v_n for n equal 2 to K obtained above. We then have values for v pertinent to this specification.

SECTION 4. INTERPOLATION

Now that values of u_n and v_n are known (whether by the arbitrary assumption $v_1 = 0$ or by specification of $v(z_\ell)$), we have

$$w_n = u_n + i\,v_n \qquad \text{for n equal 1 to K}$$

and can compute derivatives, w_n', w_n'', and w_n''' as described in 1.

Then for any z on P, we have some n such that

$$z \in [z_n, z_{n+1}]$$

then with

$$h = z - z_n$$

we have

$$w(z) = w_n + hw_n' + h^2 w_n''/2 + h^3\, w_n'''/6$$

and

$$u(z) - \text{Real } (w(z))$$

$$v(z) = \text{Imaginary } (w(z)).$$

Now for any ξ within P, we can approximate w by

$$w(\xi) = \frac{1}{2\pi i} \int_P \frac{w(z)}{z-\xi}\, dz$$

or

$$w(\xi) = \frac{1}{2\ T} \sum_{n=1}^{K} \int_{z_n}^{z_{n+1}} \frac{w(z)}{z-\xi}\, dz\ .$$

The integration on each segment z_n — $z_{n\mp 1}$ can be performed functionally or numerically as shown in 1. Then we have

$$u(\xi) = \text{Real } (w(\xi))$$

$$v(\xi) = \text{Imaginary } (w(\xi))$$

and we have an approximation for u and v at any point interior to P.

SECTION 5. COMPUTER CODE

A computer code, HYRMYC, has been written in FORTRAN for the CDC 7600 to perform all the above described computation. The integration for interior interpolation is performed numerically. A flow chart for this code follows:

100 | READ HD,K,L,NB,NIN
PRINT HD,K,L,NB,NIN |

↓ IF (K < 7) STOP

110 | READ x_n, y_n, u_n
120 | for n = 1, K |

↓

130 | Subroutine CMPQ
Constructs Basic Splines
Computes and stores
$r_{1,n}$, $t_{1,n}$, $p_{j,n}$, $q_{j,n}$
for n = 1, K j = 2, K. |

↓

140 | Subroutine CMPV
Computes optimum
v_j for j = 2, K
with $v_1 = \sigma$. |

↓

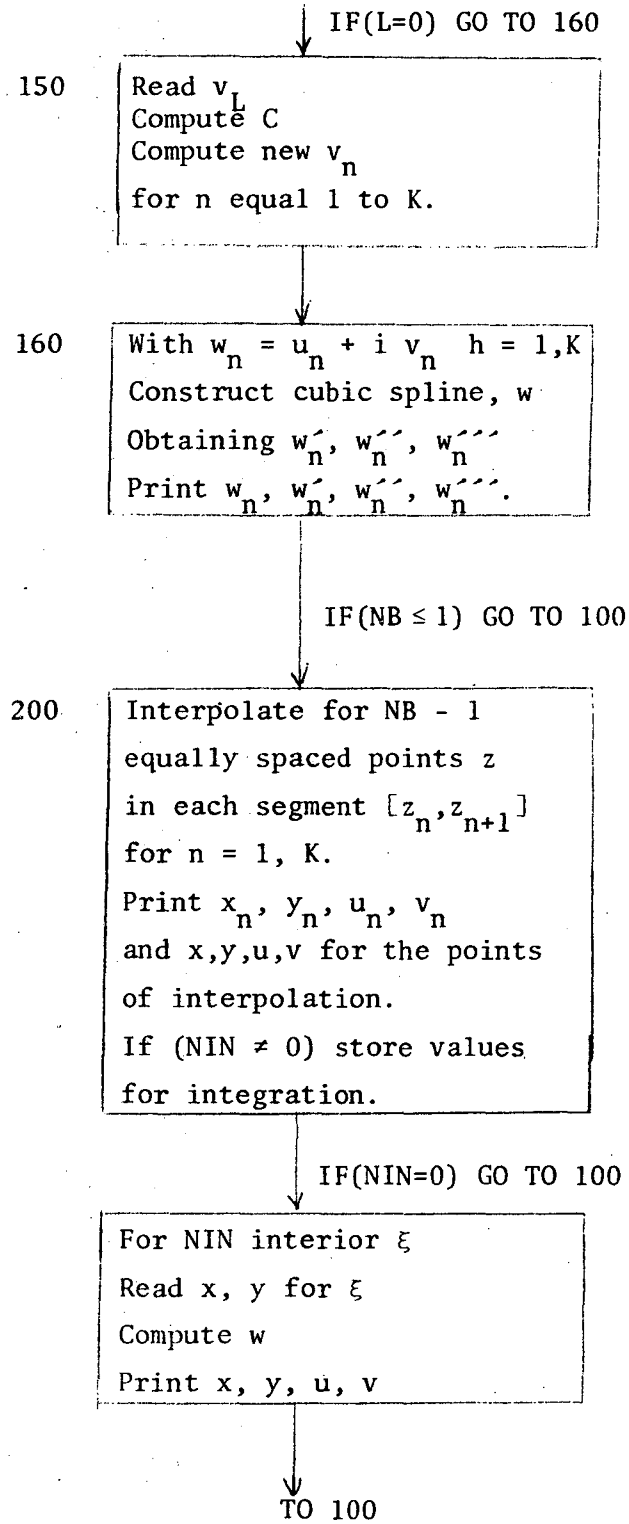
HD is a header of 30 characters

K is the number of points (≥ 7)

L is the point at which v is specified
if L = 0 no v is specified.

NB is number of subintervals in each $[z_n, z_{n+1}]$ for boundary interpolation.

NIN is number of interior points for interpolation


IF(L=0) GO TO 160
150
Read v_L
Compute C
Compute new v_n
for n equal 1 to K.
160
With $w_n = u_n + i\, v_n$ h = 1,K
Construct cubic spline, w
Obtaining w'_n, w''_n, w'''_n
Print w_n, w'_n, w''_n, w'''_n.
IF(NB ≤ 1) GO TO 100
200
Interpolate for NB - 1
equally spaced points z
in each segment $[z_n, z_{n+1}]$
for n = 1, K.
Print x_n, y_n, u_n, v_n
and x,y,u,v for the points
of interpolation.
If (NIN ≠ 0) store values
for integration.
IF(NIN=0) GO TO 100
For NIN interior ξ
Read x, y for ξ
Compute w
Print x, y, u, v
TO 100

SECTION 6. NUMERICAL EXAMPLES.

In all the following examples, the boundary values specified for u were those for

$$u = \ell^x \cos y$$

and the value specified for v at z_1 was for

$$v = \ell^x \sin y$$

It will be noted that this function u is a potential function and the function v is a harmonic conjugate of u. By using values pertinent to these known functions we were able to compare computed values with actual values of the functions u and v.

Example 1. Circle, Nine Equally Spaced Points (K = 9).
Comparison of v function with v computed.

n	x	y	u = $\ell^x \cos y$	v = $\ell^x \sin y$	v computed	Error v
1	0.00000	0.00000	1.00000	0.00000	0.00000	-
2	0.32139	0.11698	1.36962	0.16095*	0.16028**	-.00067
3	0.49240	0.41318	1.49856	0.65699*	0.65691**	-.00008
4	0.43301	0.75000	1.12819	1.05101*	1.05057**	-.00044
5	0.17101	0.96985	0.67088	0.97863*	0.97830**	-.00043
6	-.17101	0.96985	0.47655	0.69515*	0.69492**	-.00023
7	-.43301	0.75000	0.47454	0.44208*	0.44161**	-.00047
8	-.49240	0.41318	0.55973	0.24539*	0.24528**	-.00011
9	-.32139	0.11698	0.72018	0.08463*	0.08403**	-.00060

* Functional values of v not specified

** Computed values of v

Comparison of first derivatives functional and computed

n	$u_x = \ell^x \cos y$	u_x Computed	Error u_x	$u_y = -\ell^x \sin y$	u_y Computed	Error u_y
1	1.00000	0.99721	-.00279	0.00000	0.00071	0.00071
2	1.36962	1.37255	0.00293	-.16095	-.15983	0.00112
3	1.36962	1.49692	-.00164	-.65699	-.65907	-.000208
4	1.12819	1.12865	0.00046	-1.05101	-1.04935	0.00166
5	0.67088	0.67074	-.00014	-.97863	-.97957	-.00094
6	0.47655	0.47682	0.00017	-.69515	-.69447	0.00068
7	0.47454	0.47421	-.00033	-.44208	-.44305	-.00097
8	0.55973	0.55956	-.00018	-.24539	-.24383	0.00156
9	0.72018	0.72164	0.00146	-.08463	-.08635	-.00162

Comparison of midpoint functional and computed values

n/ n+1	$n = \ell^x \cos y$	u computed	Error u	v $\ell^x \sin y$	v Computed	Error v
1 2	1.17232	1.17208	-.00024	0.06865	0.06820	-.00045
2 3	1.44968	1.44992	0.00024	0.39354	0.39332	-.00032
3 4	1.32722	1.32707	-.00017	0.87257	0.87215	-.00032
4 5	0.88255	0.88262	0.00007	1.02497	1.02473	-.00024
5 6	0.56542	0.56542	0.00000	0.82480	0.82442	-.00038
6 7	0.48241	0.48237	-.00004	0.56026	0.55996	-.00030
7 8	0.52607	0.52616	0.00009	0.84586	0.34556	-.00030
8 9	0.64246	0.64232	-.00014	0.17441	0.17402	-.00039
9 1	0.85010	0.85029	0.00019	0.04978	0.04954	-.00024

Comparison of some interior points (NB=6)

(With trapezoidal integration over 6 subintervals in each interval)

X	Y	$\ell^x \cos y$	u Computed	Error u	$\ell^x \sin y$	v Computed	Error v
0.00000	0.25000	0.96891	0.96693	-.00198	0.24740	0.24659	-.00081
-.25000	0.50000	0.6846	0.68208	-.00138	0.37338	0.37234	-.00104
0.00000	0.50000	0.87758	0.87583	-.00175	0.47943	0.47814	-.00129
0.25000	0.50000	1.12684	1.12461	-.00223	0.61559	0.61400	-.00159
0.00000	0.75000	0.73169	0.73026	-.00143	0.68164	0.67998	-.00166

Example 2 Rectangle, Twelve points (K=12)

Comparison of v functional with v computed

n	x	y	u = ℓ^x cos y	v = ℓ^x sin y	v computed	Error v
1	0.0	0.0	1.00000	0.00000	0.00000	--
2	0.5	0.0	1.64872	0.00000	-.01084	-.01084
3	1.0	0.0	2.71829	0.00000	-.01708	-.01708
4	1.0	0.5	2.38552	1.30321	1.27477	-.02523
5	1.0	1.0	1.46869	2.28736	2.26000	-.02736
6	0.5	1.0	0.89081	1.38735	1.34720	-.04015
7	0.0	1.0	0.54030	0.84147	0.80640	-.03507
8	-.5	1.0	0.32771	0.51038	0.49059	-.01979
9	-1.0	1.0	0.19877	0.30956	0.29844	-.01112
10	-1.0	0.5	0.32284	0.17637	0.16899	-.00738
11	-1.0	0.0	0.36788	0.00000	-.00272	-.00272
12	-.5	0.0	0.60653	0.00000	0.00008	0.00008

Comparison of first derivatives functional and computed

n	u_x = ℓ^x cos y	u_x Computed	Error u_x	u_y = $-\ell^x$ sin y	u_y Completed	Error u_y
1	1.00000	0.99962	-.00038	0.00000	0.01129	0.01129
2	1.64872	1.64781	-.00091	0.00000	0.02744	0.02744
3	2.71829	2.71882	0.00053	0.00000	-.01853	-.01853
4	2.38552	2.36058	-.01494	-1.30321	-1.29144	0.01177
5	1.46869	1.50136	0.03267	-2.28736	- ..31323	-.02587
6	0.89081	0.88155	-.00926	-1.38735	-1.39840	-.01105
7	0.54030	0.54277	0.00247	-.84147	-.81479	0.02668
8	0.32771	0.32594	-.00177	-.51038	-.48210	0.02828
9	0.19877	0.20269	0.00392	-.30956	-.30458	0.00498
10	0.32284	0.30963	-.01321	-.17637	-.17765	-.00128
11	0.36788	0.36574	-.00204	0.00000	0.00052	0.00051
12	0.60653	0.60684	0.00031	0.00000	-.00703	-.00703

Comparison of midpoint values

$n/_{n+1}$	$u = \ell^x \cos y$	u Computed	Error u	$v = \ell^x \sin y$	v Computed	Error v
1 2	1.28403	1.28385	-.00018	0.00000	-.00441	-.00441
2 3	2.11700	2.11656	-.00044	0.00000	-.01684	-.01684
3 4	2.63378	2.63146	-.00232	0.67251	0.65123	-.02128
4 5	1.98894	1.99097	0.00203	1.85289	1.82109	-.03180
5 6	1.14382	1.14101	-.00281	1.78139	1.74642	-.03497
6 7	0.69376	0.69438	0.00062	1.08047	1.04032	-.04015
7 8	0.42079	0.42045	-.00034	0.65534	0.62770	-.02764
8 9	0.25522	0.25553	0.00031	0.39748	0.38342	-.01406
9 10	0.26917	0.26874	-.00043	0.25076	0.24040	-.01036
10 11	0.35644	0.35650	0.00006	0.09101	0.08664	-.00437
11 12	0.47237	0.47214	-.00023	0.00000	-.00179	-.00179
12 1	0.77880	0.77872	-.00008	0.00000	0.00119	0.00119

Comparison at some interior points.

(Integration as in Example 1)

x	y	$\ell^x \cos y$	u Computed	Error u	$\ell^x \sin y$	v Computed	Error v
0.00	0.25	0.96891	0.96697	-.00194	0.24740	0.23254	-.01486
-.25	0.50	0.68346	0.68603	0.00257	0.37338	0.35794	-.01544
0.00	0.50	0.87758	0.87774	0.00016	0.47943	0.46176	-.01767
0.25	0.50	1.12684	1.12409	-.00275	0.61559	0.59562	-.01997
0.00	0.75	0.73169	0.73447	0.00278	0.68146	0.66128	-.02036

Example 3. Keyhole 20 Points

Comparison of v-function with v-computed

n	x	y	$u = \ell^x \cos y$	$v = \ell^x \sin y$	v Computed	Error v
1	1.00000	0.00000	2.71828	0.00000	0.00000	0.00000
2	1.00000	0.33333	2.56866	0.88940	0.88801	-.00139
3	0.75000	0.33333	2.00048	0.69266	0.68993	-.00273
4	0.50000	0.33333	1.55797	0.53945	0.53553	-.00392
5	0.35355	0.68688	1.10117	0.90307	0.90111	-.00196
6	0.00000	0.83333	0.67241	0.74017	0.73933	-.00084
7	-.35355	0.68688	0.54295	0.44528	0.44711	0.00183
8	-.50000	0.33333	0.57315	0.19845	0.20059	0.00214
9	-.75000	0.33333	0.44637	0.15455	0.15568	0.00113
10	-1.00000	0.33333	0.34763	0.12037	0.12100	0.00063
11	-1.00000	0.00000	0.36788	0.00000	0.00000	0.00000
12	-1.00000	-.33333	0.34763	-.12037	-.12100	-.00063
13	-.75000	-.33333	0.44637	-.15455	-.15568	-.00113
14	-.50000	-.33333	0.57315	-.19845	-.20059	-.00214
15	-.35355	-.68688	0.54295	-.44528	-.44711	-.00183
16	0.00000	-.83333	0.67241	-.74017	-.73933	0.00084
17	0.35355	-.68688	1.10117	-.90307	-.90111	0.00196
18	0.50000	-.33333	1.55797	-.53945	-.53553	0.00392
19	0.75000	-.33333	2.00048	-.69266	-.68993	0.00273
20	1.00000	-.33333	2.56866	-.88940	-.88801	0.00139

Comparison of first derivatives, functional and computed

n	$u_x = \ell^x \cos y$	u_x Computed	Error u_x	$u_y = -\ell^x \sin y$	u_y Computed	Error u_y
1	2.71828	2.70986	-.00842	0.00000	0.00000	0.00000
2	2.56866	2.57245	0.00379	-.88940	-/89029	-.00089
3	2.00048	1.99884	-.00164	-.69266	-.69943	-.00677
4	1.55797	1.56046	0.00249	-.53945	-.54167	-.00222
5	1.10117	1.10772	0.00655	-.90307	-.90027	0.00280
6	0.67241	0.66810	-.00431	-.74017	-.73654	0.00363
7	0.54295	0.54270	-.00025	-.44528	-.43928	0.00600
8	0.57315	0.57262	-.00053	-.19845	-.20257	-.00412
9	0.44637	0.44656	0.00019	-.15455	-.15808	-.00353
10	0.34763	0.34732	-.00031	-.12037	-.12021	0.00016
11	0.36788	0.37086	0.00298	0.00000	0.00000	0.00000
12	0.34763	0.34732	-.00031	0.12037	0.12021	-.00016
13	0.44637	0.44656	0.00019	0.15455	0.15808	0.00353
14	0.57315	0.57262	-.00053	0.19845	0.20257	0.00412
15	0.54295	0.54270	-.00025	0.44528	0.43928	-.00600
16	0.67241	0.66810	-.00431	0.74017	0.73654	-.00363
17	1.10117	1.10772	0.00655	0.90307	0.90027	-.00280
18	1.55797	1.56046	0.00249	0.53945	0.54167	0.00222
19	2.00048	1.99884	-.00164	0.69266	0.69943	0.00677
20	2.56866	2.57245	0.00379	0.88940	0.89029	0.00089

Comparison of midpoint values, functional and computed

n/ n+1	u = $\ell^x \cos y$	u Computed	Error u	v = $\ell^x \sin y$	v Computed	Error v
1 2	2,68062	2.68057	-.00005	0.45095	0.44973	-.00122
2 3	2.26684	2.26664	-.00020	0.78489	0.78300	-.00189
3 4	1.76541	1.76552	0.00011	0.61127	0.60780	-.00347
4 5	1.33723	1.22713	-.00010	0.74818	0.74490	-.00328
5 6	0.86491	0.86436	-.00055	0.82222	0.82104	-.00118
6 7	0.60733	0.60758	0.00025	0.57736	0.57779	0.00043
7 8	0.56953	0.56906	-.00047	0.31865	0.32084	0.00219
8 9	0.50580	0.50582	0.00002	0.17513	0.17675	0.00162
9 10	0.39392	0.39390	-.00002	0.13639	0.13716	0.00077
10 11	0.36278	0.36276	-.00002	0.06103	0.06148	0.00045
11 12	0.36278	0.36276	-.00002	-.06103	-.06148	-.00045
12 13	0.39392	0.39390	-.00002	-.13639	-.13716	-.00077
13 14	0.50508	0.50582	0.00002	-.17513	-.17675	-.00162
14 15	0.56953	0.56906	-.00047	-.31865	-.32084	-.00219
15 16	0.60733	0.60758	0.00025	-.57736	-.57779	-.00043
16 17	0.86491	0.86436	-.00055	-.82222	-.82104	0.00118
17 18	1.33723	1.33713	-.00010	-.74818	-.74490	0.00328
18 19	1.76541	1.76552	0.00011	-.61127	-.60780	0.00347
19 20	2.26684	2.26664	-.00020	-.78489	-.78300	0.00189
20 1	2.68062	2.68057	-.00005	-.45095	-.44973	0.00122

Comparison of some interior points

x	y	$\ell^x \cos y$	u Computed	Error u	$\ell^x \sin y$	v Computed	Error v
0.00	0.25	0.96891	0.96835	-.00056	0.24740	0.24714	-.00026
-.25	0.50	0.68346	0.68275	-.00071	0.37338	0.37202	-.00136
0.00	0.50	0.87758	0.87649	-.00109	0.47943	0.47841	-.00102
0.25	0.50	1.12684	1.12379	-.00305	0.61559	0.61496	-.00063
0.00	0.75	0.73169	0.72337	-.00832	0.68164	0.67257	-.00907

Example 4 Droplet 20 points

Comparison of v functional and computed

n	x	y	u: $\ell^x \cos y$	v: $\ell^x \sin y$	v Computed	Error v
1	0.00000	0.00000	1.00000	0.00000	0.00000	0.00000
2	0.15451	0.12447	1.15806	0.14489	0.14490	0.00001
3	0.29389	0.19549	1.31608	0.26061	0.26068	0.00007
4	0.40451	0.30611	1.42890	0.45159	0.45172	0.00013
5	0.47553	0.44549	1.45184	0.69326	0.69341	0.00015
6	0.50000	0.60000	1.36075	0.93094	0.93107	0.00013
7	0.47553	0.75451	1.17223	1.10196	1.10206	0.00010
8	0.40451	0.89389	0.93868	1.16816	1.16821	0.00005
9	0.29389	1.00451	0.71979	1.13221	1.13219	-.00002
10	0.15451	1.07553	0.55468	1.02685	1.02676	-.00009
11	0.00000	1.10000	0.45360	0.89121	0.89103	-.00018
12	-.15451	1.07553	0.40723	0.75388	0.75362	-.00026
13	-.29389	1.00451	0.39989	0.62901	0.62869	-.00032
14	-.40451	0.89389	0.41799	0.52017	0.51980	-.00037
15	-.47553	0.75451	0.45287	0.42572	0.42532	-.00040
16	-.50000	0.60000	0.50059	0.34247	0.34207	-.00040
17	-.47553	0.44549	0.56089	0.26783	0.26746	-.00037
18	-.40451	0.30611	0.63628	0.20109	0.20079	-.00030
19	-.29389	0.19549	0.73116	0.14478	0.14459	-.00019
20	-.15451	0.12447	0.85021	0.10638	0.10630	-.00008

Comparison of first derivative values, functional and computed.

n	u_x $\ell^x \cos y$	u_x Computed	Error u_x	u_y $-\ell^x \sin y$	u_y Computed	Error u_y
1	1.00000	0.99984	-.00016	-.00000	-.00030	-.00030
2	1.15806	1.15815	0.00009	-.14490	-.14519	-.00029
3	1.31608	1.31636	0.00028	-.26061	-.26094	-.00033
4	1.42890	1.42912	0.00022	-.45159	-.45179	-.00020
5	1.45184	1.45181	-.00003	-.69326	-.69329	-.00003
6	1.36075	1.36053	-.00022	-.93094	-.93091	0.00003
7	1.17223	1.17201	-.00022	-1.10196	-1.10200	-.00004
8	0.93868	0.93844	-.00024	-1.16816	-1.16835	-.00019
9	0.71979	0.71953	-.00026	-1.13221	-1.13262	-.00041
10	0.55468	0.55449	-.00019	-1.02685	-.102740	-.00055
11	0.45360	0.45358	-.00002	-.89121	-.89173	-.00052
12	0.40728	0.40735	0.00012	-.75388	-.75429	-.00041
13	0.039989	0.40010	0.00021	-.67901	-.62929	-.00028
14	0.41799	0.41821	0.00022	-.52017	-.52032	-.00015
15	0.45287	0.45298	0.00011	-.42572	-.42576	-.00004
16	0.50059	0.50051	-.00008	-.34247	-.34248	-.00001
17	0.56089	0.56059	-.00030	-.26783	-.26794	-.00011
18	0.63628	0.63582	0.00054	-.20109	-.20143	-.00034
19	0.73116	0.73071	-.00045	-.14478	-.14541	-.00063
20	0.85021	0.84995	-.00026	-.10638	-.10707	-.00069

Comparison of midpoint values

n/n+1	$u = \ell^x \cos y$	u Computed	Error u	$v = \ell^x \sin y$	v Computed	Error v
1 2	1.07823	1.07823	0.00000	0.06719	0.06718	-.00001
2 3	1.23534	1.23534	0.00000	0.19933	0.19937	0.00004
3 4	1.37357	1.37357	0.00000	0.35190	0.35200	0.00010
4 5	1.44438	1.44438	0.00000	0.56988	0.57003	0.00015
5 6	1.41117	1.41116	-.00001	0.81313	0.81327	0.00014
6 7	1.26922	1.26922	0.00000	1.02062	1.02073	0.00011
7 8	1.105453	1.05454	0.00001	1.13972	1.13979	0.00007
8 9	0.82571	0.82571	0.00000	1.15271	1.15273	0.00002
9 10	0.63342	0.63342	0.00000	1.07916	1.07911	-.00005
10 11	0.50177	0.50177	0.00000	0.95672	0.95658	-.00014
11 12	0.42993	0.42994	0.00001	0.81975	0.81953	-.00022
12 13	0.40453	0.40454	0.00001	0.68920	0.68891	-.00029
13 14	0.41069	0.41069	0.00000	0.57333	0.57299	-.00034
14 15	0.43738	0.43739	0.00001	0.47272	0.47233	-.00039
15 16	0.47849	0.47848	-.00001	0.38477	0.38436	-.00041
16 17	0.53200	0.53200	0.00000	0.30654	0.30615	-.00039
17 18	0.59908	0.59908	0.00000	0.23637	0.23603	-.00034
18 19	0.68319	0.68318	-.00001	0.17503	0.17478	-.00025
19 20	0.78895	0.78895	0.00000	0.12730	0.12717	-.00013
20 1	0.92386	0.92388	0.00002	0.05757	0.05756	-.00001

SECTION 7. CONCLUSIONS

No general conclusions are obtained as to the accuracy of the method presented. However, the results obtained in the Examples on a rather varied domains indicate that the complex cubic spline approximation with optimization as described does provide a practical and reasonably accurate numerical solution of Laplace's equation with Dirichlet conditions on a simple domain. In addition, fairly good values for the first derivates and for the harmonic conjugate are obtained.

Examples 1 and 4 indicate that error is less when all the points of specification are vertices as opposed to the case in Examples 2 and 3 when intermediate points on the sides are included. The abrupt "corners" (right angles) in Examples 2 and 3 also have an adverse effect. Other tests run on these examples indicate very little improvement in accuracy if more points are specified.

REFERENCES

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TECHNICAL INFORMATION DIVISION
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720