Title
Growth conditions on Hilbert functions of modules

Author
Dellaca, Roger D.

Publication Date
2015

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA,
IRVINE

Growth conditions on Hilbert functions of modules

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Roger D. Dellaca

Dissertation Committee:
Professor Vladimir Baranovsky, Chair
Professor Karl Rubin
Professor Zhiqin Lu

2015

TABLE OF CONTENTS

ACKNOWLEDGMENTS

I would like to thank Vladimir Baranovsky, for his limitless patience, great insight, and good humor. I would also like to thank my other committee members, and the faculty and staff that have made my experience here so rewarding.

CURRICULUM VITAE

Roger D. Dellaca

EDUCATION

Doctor of Philosophy in Mathematics **2015**
University of California *Irvine CA*

Master of Science in Mathematics **2009**
California State University *Long Beach, CA*

Master of Science in Management Science and Information Systems **1998**
California State University *Fullerton, CA*

Bachelor of Science in Business Administration **1995**
California State University *Long Beach, CA*

ABSTRACT OF THE DISSERTATION

Growth conditions on Hilbert functions of modules

By

Roger D. Dellaca

Doctor of Philosophy in Mathematics

University of California, Irvine, 2015

Professor Vladimir Baranovsky, Chair

Gotzmann's Regularity Theorem uses a binomial representation of the Hilbert polynomial of a standard graded algebra to establish a bound on Castelnuovo-Mumford regularity. Using this and his Persistence Theorem, Gotzmann provided an explicit construction of the Hilbert scheme. This author will show that Gotzmann's Regularity Theorem cannot be extended to arbitrary modules. However, under an additional assumption on the generating degrees of a module, Gotzmann's Regularity Theorem will be proven. The modules satisfying the additional assumption will correspond to globally generated coherent sheaves. This will be used to provide an explicit construction of the Quot scheme.

The Gotzmann Regularity bound is known to be strict for standard graded algebras, but not for globally generated coherent sheaves. In order to address this, new representations for the Hilbert function and Hilbert polynomial are given that account for the rank and generating degrees of a module. Generalizations of the theorems of Macaulay, Green, and Gotzmann will be proven using these representations. The generalized Gotzmann number will give a strict upper bound for the regularity of modules generated in degree zero. Additionally, these representations will be used to prove a sharp inequality on the first and second Chern classes of a globally generated coherent sheaf.

Chapter 1

Introduction

Hilbert initiated the study of a graded module through its resolution [21], including Hilbert functions and Hilbert polynomials. Hilbert originally studied these to answer important questions in Invariant Theory, but they have found many applications. For example, one can prove Bèzout's Theorem and Riemann-Roch and define Dimension Theory using Hilbert polynomials, and Hilbert functions on the Clements-Lindstrom ring $\mathbb{C}[x_0, \ldots, x_n]/(x_0^2, \ldots, x_n^2)$ provide numerous results on the combinatorics of simplicial complexes. Hilbert polynomials were also used in the construction of Hilbert schemes and Quot schemes. Hilbert schemes and Quot schemes are important because many moduli spaces of interest to Algebraic Geometers are formed as a quotient of a Hilbert scheme or Quot scheme by a group action.

The resolution of a module can also be used to define Castelnuovo-Mumford regularity, which has proven very useful due to its interpretation of the geometric and computational complexity of a module. In particular, the computational complexity of a Gröbner basis of a module can be formulated in terms of the regularity of the module. The regularity, generating degrees, and projective dimension bound the shape of the Betti table associated to the module. Many recent results and open problems are given in [31] and [33].

A few results on Hilbert functions of standard graded algebras stand out. First, Macaulay characterized precisely what numerical functions can be the Hilbert function of a standard graded algebra, by providing a bound on the growth of the Hilbert function in successive degrees [26]. Second, Gotzmann showed what happens when equality is achieved in Macaulay's growth bound, and found an invariant of a Hilbert polynomial (the Gotzmann number) that bounds the Castelnuovo-Mumford regularity of any standard graded algebra with the given Hilbert polynomial; these are known as Gotzmann's Persistence and Regularity Theorems [14]. Third, Green provided a bound on the Hilbert function of a module to its restriction to a generic hyperplane, thus comparing the same degree and successively smaller dimension [16].

One is interested in extending these results to arbitrary modules. Gasharov extended most of these results to modules; however, Gotzmann's Regularity Theorem cannot be extended to arbitrary modules, as I will show by counterexample. Under an additional assumption on the generating degrees of a module, I will prove that its Hilbert polynomial has a Gotzmann representation, and satisfies Gotzmann Regularity; the modules satisfying the additional assumption will correspond to globally generated coherent sheaves. I will use this to extend Gotzmann's construction of the Hilbert scheme to a construction of the Quot scheme.

The Gotzmann Regularity bound is known to be strict for standard graded algebras, but we will see that it is not strict for globally generated coherent sheaves. In order to address this, I will give representations for the Hilbert function and Hilbert polynomial that account for the rank and generating degrees of a module. I will show that these representations allow for generalizations of Macaulay's Theorem, Green's Theorem, and Gotzmann's Regularity and Persistence Theorems. Furthermore, the rank-and-degree-adjusted representation will give a strict bound for Gotzmann regularity of modules generated in degree zero. I will also use these representations to provide an inequality on the first and second Chern classes of a coherent sheaf, which will be sharp when the rank-and-degree-adjusted representation is used.

The structure of the thesis is as follows. The remainder of this chapter will provide background material on resolutions, Hilbert functions, regularity, Hilbert and Quot schemes, and Chern classes. Chapter 2 will provide the classical results on Hilbert functions of standard graded algebras mentioned earlier, culminating with Gotzmann's construction of the Hilbert scheme. Chapter 3 will give the new results on Gotzmann regularity for globally generated coherent sheaves and the Quot scheme construction. Chapter 4 will give the new results on rank-and-degree adjusted representations of Hilbert functions and Hilbert polynomials.

1.1 Background

Fix the following notation. Let k be an algebraically closed field, $S = k[x_0, \dots, x_n]$ the symmetric algebra on $S_1 = \{x_0, \dots, x_n\}$, with $\mathfrak{m}$ the irrelevant ideal which is generated by S_1. A ring R is *graded* if $R = \bigoplus R_d$ such that each R_d is closed under addition and $R_d R_e \subseteq R_{d+e}$. The ring S will be graded with the standard grading, that is $\deg(x_i) = 1$ for each i. If R is a graded ring, then a *graded R-module* M satisfies $M = \bigoplus M_d$ with each M_d closed under addition and $R_d M_e \subseteq M_{d+e}$. If M is a graded S-module, write M_d for the degree d part of M, and $M(d)$ for the twist of M, that is $M(d)_m = M_{d+m}$ for each m.

Given a graded R-module M, let $\tilde{M}$ be the corresponding quasicoherent sheaf, and given a quasicoherent sheaf $\mathcal{F}$ on $X = \operatorname{Proj}(R)$, let $\Gamma_*(\mathcal{F}) = \oplus_{d \in \mathbb{Z}} (\Gamma(\mathcal{F}(d)))_0$ be the corresponding module. Let $\mathcal{O}_{\mathbb{P}^n}$ or $\mathcal{O}$ denote the structure sheaf, and $\mathcal{F}(d) = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(d)$.

Proposition 1.1. *Let $\mathcal{F}$ be a quasicoherent sheaf on $\mathbb{P}^n$. There is a natural isomorphism $\widetilde{\Gamma_*(\mathcal{F})} \cong \mathcal{F}$.*

Proof. See [18, pg. 119]. □

On the other hand, modules that differ only up to some finite degree correspond to the same sheaf; denote $\mathrm{sat}(M) = \Gamma_*(\widetilde{M})$ for the saturation of M.

Recall that a sheaf $\mathcal{F}$ on X is coherent if it locally has a finite presentation; that is, an open cover $\{U_i\}$ of X such that $\mathcal{F}(U_i)$ is finitely presented for each i. By abuse of notation, a vector bundle on X will be the same as the locally free sheaf of $\mathcal{O}_X$-modules to which it corresponds. A line bundle is a vector bundle of rank 1.

Define the sheaf cohomology functors $H^i(X, \mathcal{F})$ to be the derived functors of the global sections functor $\Gamma(\mathcal{F})$.

If I is a homogeneous ideal of S and M is a graded S-module, define the submodule supported on I

$$H^0_I(M) = \{m \in M | I^d m = 0 \text{ for some } d\}.$$

The local cohomology modules are $H^i_I(M)$, the derived functors of $H^0_I(M)$. Sheaf cohomology can be related to local cohomology as follows.

Theorem 1.1. *Let M be a graded S-module. Then*

1. *There is an exact sequence*

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to \Gamma_*(\tilde{M}) \to H^1_{\mathfrak{m}}(M) \to 0.$$

2. *For $i \geq 2$,*

$$H^i_{\mathfrak{m}}(M) = \oplus_d H^{i-1}(\mathbb{P}^n, \tilde{M}(d)).$$

Proof. See [10, Proposition A1.11]. □

1.2 Resolutions and Hilbert functions

Some of the most important invariants of a graded module are obtained from a minimal graded free resolution; in particular, the Hilbert function, Hilbert polynomial, and graded Betti numbers are described below.

Definition 1.1. *A* graded free resolution *of a graded S-module M is a complex of graded free S-modules*

$$\cdots \to F_2 \to F_1 \to F_0$$

where each F_i appears in homological degree i, the maps are graded (that is, $F_{i+1} \to F_i$ preserves degree), the complex is exact except at F_0, and $\operatorname{coker}(F_1 \to F_0) = M$*.*

The resolution is minimal *if the image of F_{i+1} is contained in $\mathfrak{m}F_i$ for each i.*

When the maps of free modules in the resolution are written as matrices, the last condition is equivalent to having no invertible elements in the matrix. Intuitively, a graded free resolution is minimal if a minimal set of generators is chosen in each step of computing the resolution. The next result shows that the resolutions under consideration exist, are finite, and are essentially unique.

Theorem 1.2. *(Hilbert) Every finitely-generated graded S-module has a minimal free graded resolution of finite length, and each free module appearing in the minimal resolution is finitely generated. Any two minimal free graded resolutions of the same module are isomorphic.*

Proof. See [10, Theorem 1.1, Corollary 1.2, Theorem 1.6]. □

Definition 1.2. *Let M be a finitely generated graded S-module. The* Hilbert function *of M*

is defined to be

$$H(M, d) = \dim_k(M_d),$$

the dimension of the degree-d part of M as a k-vector space.

Note that the Hilbert function of the shifted polynomial ring is

$$H(S(-a), d) = \binom{n+d-a}{n}. \tag{1.1}$$

When the minimal graded free resolution of a module is restricted to a single degree, it becomes an exact sequence of vector spaces; the following proposition is the result, which allows the computation of the Hilbert function from the resolution.

Proposition 1.2. *If the finitely generated graded S-module M has a finite graded free resolution*

$$0 \to F_m \to \cdots \to F_1 \to F_0$$

with $F_i = \oplus_j S(-a_{i,j})$ for each i, then

$$H(M, d) = \sum_{i=0}^{m} (-1)^i \sum_j \binom{n+d-a_{i,j}}{n},$$

with the convention that $\binom{a}{b} = 0$ if $a < b$.

Proof. This follows from induction and the Rank-Nullity theorem for vector spaces and Equation 1.1. □

Once d is large enough so that all the binomial coefficients above are defined, the expression becomes a polynomial. Thus, we have the following corollary.

Corollary 1.1. *There is a polynomial $P_M(d)$ such that $H(M,d) = P_M(d)$ for $d \gg 0$.* $\square$

Definition 1.3. *The* Hilbert polynomial *of M is the polynomial $P_M(d)$ that agrees with the Hilbert function in sufficiently large degree. For a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, the Hilbert function and Hilbert polynomial of $\mathcal{F}$ are $H(\mathcal{F},d) = H(\Gamma_*(\mathcal{F}),d)$ and $P_{\mathcal{F}}(d) = P_{\Gamma_*(\mathcal{F})}(d)$.*

The Hilbert polynomial P of S/I for a homogeneous ideal I gives information about the variety $V(I)$. The degree of P is the dimension of $V(I)$; if $\deg(P) = r$ and the leading coefficient of P is c, then the degree of $V(I)$ is $r!c$.

Example 1.1. *If I is the ideal of a set of m points in the projective plane, then the Hilbert polynomial of S/I is m, showing that $V(I)$ is dimension 0 and degree m.*

As a more concrete example, if I is the ideal of $(1:0:0)$ and $(0:1:0)$, then $I = (xy, z)$, and the Hilbert polynomial of S/I is 2, because the only monomials of degree d that are not in I are x^d and y^d for all $d \geq 1$.

Definition 1.4. *Given a graded S-module M with minimal graded free resolution*

$$0 \to F_m \to \cdots \to F_1 \to F_0$$

for each F_i with a finite set of integers j such that

$$F_i = \bigoplus S(-j)^{\beta_{i,j}},$$

The collection $\beta_{i,j}$ are the graded Betti numbers *of M.*

Note that the graded Betti numbers determine the Hilbert function, and so they also determine the Hilbert polynomial. By Theorem 1.2, they do not depend on the choice of minimal free graded resolution of the module.

Example 1.2. *Set $S = \mathbb{C}[x, y, z, w]$. Consider $I = (x^2 - y^2, xy - z^2)$ and $J = (x^2, xy, y^3)$. The module S/I has a minimal graded free resolution*

$$0 \to S(-4) \to S(-2)^2 \to S,$$

where $S(-2)^2$ is the direct sum of $S(-2)$ with itself. The graded Betti numbers are

$$\beta_{0,0} = 1,\ \beta_{1,2} = 2,\ \beta_{2,4} = 1.$$

The module S/J has a minimal graded free resolution

$$0 \to S(-3) \oplus S(-4) \to S(-2)^2 \oplus S(-3) \to S,$$

with graded Betti numbers

$$\beta_{0,0} = 1,\ \beta_{1,2} = 2,\ \beta_{1,3} = 1,\ \beta_{2,3} = 1,\ \beta_{2,4} = 1.$$

The Hilbert function of S/I is computed as

$$H_{S/I}(d) = \binom{d+3}{d} - 2\binom{d+1}{d-2} + \binom{d}{d-3},$$

remembering that not all of these binomial coefficients are nonzero for small values of d (in particular, $H_{S/I}(0) = 1$, $H_{S/I}(1) = 4$, and $H_{S/I}(2) = 8$). However, each of the binomial coefficients appears in every degree $d \geq 3$, at which point the Hilbert function becomes the Hilbert polynomial $P_{S/I}(d) = 4d$. Note that S/J has precisely the same Hilbert function as S/I. To see this, note that the only difference in the graded Betti numbers is the addition of $\beta_{1,3} = 1$ and $\beta_{2,3} = 1$ for S/J, and these cancel in the formula for the Hilbert function in Theorem 1.2.

1.3 Regularity, Hilbert schemes and Quot schemes

Castelnuovo-Mumford regularity was first defined by Mumford in [27]. I will give this definition first, in terms of sheaf cohomology, which can also be given in terms of local cohomology for modules; another definition in terms of graded Betti numbers will follow. Regularity has great utility, in part due to the equivalence of the two definitions.

Definition 1.5. *A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is called m-*regular *if*

$$H^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0 \text{ for all } i > 0.$$

The Castelnuovo-Mumford regularity *of $\mathcal{F}$, written reg$(\mathcal{F})$, is the smallest m such that $\mathcal{F}$ is m-regular.*

By using the relationship between sheaf cohomology and local cohomology in Theorem 1.1, one can give a similar definition of Castelnuovo-Mumford regularity of a module in terms of local cohomology.

Definition 1.6. *A finitely-generated graded S-module is d regular if*

$$d \geq \max\{e | H^i_{\mathfrak{m}}(M)_e \neq 0\} + i \text{ for all } i \geq 0.$$

The Castelnuovo-Mumford regularity of M is the smallest such d.

Note that a module that is not saturated can have larger regularity than its associated coherent sheaf, but the regularity will agree for a saturated module and its associated sheaf.

An alternative definition is available for the regularity of a module in terms of its graded Betti numbers.

Definition 1.7. *Given a graded S-module M with graded Betti numbers $\beta_{i,j}$, the Castelnuovo-*

Mumford regularity of M is

$$\operatorname{reg}(M) = \max\{j - i | \beta_{i,j} \neq 0\}.$$

Proposition 1.3. *The definitions of Castelnuovo-Mumford regularity of a graded S-module in terms of local cohomology and in terms of graded Betti numbers agree.*

Proof. See [10, Corollary 4.5]. □

In example 1.2, an examination of the minimal free resolutions of S/I and S/J shows that they both have regularity 2.

The following lemma shows how regularity behaves in an exact sequence. It will be used in chapter 4.

Lemma 1.1. *Suppose*

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of finitely-generated graded S-modules.

1. $\operatorname{reg}(M) \leq \max(\operatorname{reg}(M'), \operatorname{reg}(M''))$;
2. $\operatorname{reg}(M') \leq \max(\operatorname{reg}(M), \operatorname{reg}(M'') + 1)$.

Proof. See [31, Corollary 18.7]. □

Mumford used the notion of regularity in his exposition on the Hilbert scheme. I will give the definition in the simplified case of families over $\mathbb{P}^n$; further details can be found in [28]. The *Hilbert functor* $\mathcal{H}ilb_{\mathbb{P}^n}$ from the category of noetherian schemes to sets assigns to each noetherian k-scheme T the family

$$\{X \subset \mathbb{P}^n \times T | X \text{ is flat over } T\}.$$

Grothendieck showed that this functor is in fact representable by the Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^n}$. As a first step in the proof, one shows that the functor naturally decomposes as a coproduct

$$\mathcal{H}ilb_{\mathbb{P}^n} = \coprod \mathcal{H}ilb^P_{\mathbb{P}^n},$$

which gives the family above with the restriction that the scheme X has Hilbert polynomial P in each fiber. In a similar vein, the Quot scheme $\mathrm{Quot}^P_{\mathcal{O}^N}$ represents the Quot functor which assigns to T the family

$$\{\mathcal{F} \text{ a quotient of } \mathcal{O}^N_{\mathbb{P}^n} \otimes \mathcal{O}_T | \mathcal{F} \text{ is flat over } T \text{ with each fiber having Hilbert polynomial } P\}.$$

1.4 Chern classes, Hirzebruch-Riemann-Roch

We will have a need later on for the definition of Chern classes of a coherent sheaf on $\mathbb{P}^n$. First we see a simplified definition of Chern classes of a vector bundle on $\mathbb{P}^n$. For a broader treatment, see for example [11] or [18, Appendix 1]. For a line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$, the *first Chern class* is $c_1(\mathcal{L}) = dH$, where H is the divisor class of the hyperplane section. Where no confusion exists, we may write $c_1 = d$. The Chern polynomial of the line bundle $\mathcal{L}$ is $c(\mathcal{L})(t) = 1 + c_1 t$; in general, the Chern polynomial of a rank r vector bundle will be $c(\mathcal{F}) = 1 + c_1 t + c_2 t^2 + \cdots + c_r t^r$, with Chern classes $c_1, c_2, \cdots, c_r$ that will come from the following splitting principle (see [11, Section 3.2] for more details). If $\mathcal{F}$ is a vector bundle of rank r on a scheme X, there exists a scheme Y and a morphism $f : Y \to X$ such that $\mathcal{F}$

has a filtration by vector bundles

$$0 = \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_r = \mathcal{F}$$

such that the quotients $\mathcal{F}_i/\mathcal{F}_{i+1}$ are line bundles on Y. Each quotient has a Chern polynomial $1 + \alpha_i t$, and $c(\mathcal{F})$ is the product of these; the α_i are called the *Chern roots* of $\mathcal{F}$.

The *Chern character* $ch(\mathcal{F})$ of a vector bundle $\mathcal{F}$ is computed in terms of the Chern roots $\alpha_1, \ldots, \alpha_r$:

$$ch(\mathcal{F}) = e^{\alpha_1} + \cdots + e^{\alpha_r}.$$

Given an exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0,$$

the Chern polynomial of $\mathcal{G}$ is $c_x(\mathcal{G}) = c_x(\mathcal{F}) \cdot c_x(\mathcal{H})$, and the Chern character of $\mathcal{G}$ is $ch(\mathcal{G}) = ch(\mathcal{F}) + ch(\mathcal{H})$.

The *Todd class* of $\mathbb{P}^n$ is the Todd class of the tangent bundle $T_{\mathbb{P}^n}$ (see [18, Appendix 1]):

$$Td(\mathbb{P}^n) = \frac{H^{n+1}}{(1 - \exp(-H))^{n+1}},$$

where H is the class of the hyperplane.

The Hilbert polynomial is related to the Chern classes by the Hirzebruch-Riemann-Roch Theorem, which computes the Euler characteristic of a vector bundle $\mathcal{F}$ on $\mathbb{P}^n$ in terms of the Chern character of $\mathcal{F}$ and the Todd class of $\mathbb{P}^n$, which is the Todd class of the tangent bundle $T_{\mathbb{P}^n}$ [11, Theorem 15.2]:

$$\chi(\mathcal{F}) = \int_{\mathbb{P}^n} ch(\mathcal{F})Td(\mathbb{P}^n) = \deg(ch(F) \cdot Td(\mathbb{P}_n)),$$

where the degree is taken in the dimension-zero part of the Chow ring using the degree map $\deg : A_0(\mathbb{P}^n) \to \mathbb{Z}$. Since the Hilbert polynomial of $\mathcal{F}$ is $\chi(\mathcal{F}(d))$, we can compute the Hirzebruch-Riemann-Roch formula on $\mathcal{F}(d)$ to get the Hilbert polynomial.

In order to use this formula for the Hilbert polynomial of a coherent sheaf that is not a vector bundle, we will observe that the Chern classes and Chern character coefficients of a coherent sheaf $\mathcal{E}$ satisfy the same relationship that they do for vector bundles. Indeed, any coherent sheaf has a resolution by locally free sheaves

$$\mathcal{F}_m \to \cdots \to \mathcal{F}_0,$$

and the computation of Chern classes does not depend on the resolution chosen (see [4], particularly Theorem 2, and [11, section B.8.3]). For each locally free sheaf $\mathcal{F}_i$ in the resolution of $\mathcal{E}$, we can assign Chern roots $\alpha_{i,j}$, thus we can write the Chern polynomial of $\mathcal{E}$ as

$$c(\mathcal{E})(x) = \prod_i c_{\mathcal{F}_i}(x)^{(-1)^i} = \prod_{i,j}(1 + \alpha_{i,j}x)^{(-1)^i}.$$

Consider the logarithmic derivative

$$\begin{aligned}
\frac{c'_{\mathcal{E}}}{c_{\mathcal{E}}} &= \sum_{i,j}(-1)^i \frac{\alpha_{i,j}}{1 + \alpha_{i,j}x} \\
&= \sum_{i,j}(-1)^i \sum_n (-1)^n \alpha_{i,j}^{n+1} x^n \\
&= \sum_n (-1)^n (n+1)! ch_{n+1} x^n.
\end{aligned}$$

Multiplying both sides by $c_{\mathcal{E}}$ and equating coefficients gives the same Newton identities between the coefficients of the Chern polynomial and Chern character for the sheaf $\mathcal{E}$ that we get for vector bundles, namely,

$$jc_j = \sum_{i=1}^{j}(-1)^{i-1}c_{j-i}(i!ch_i).$$

Therefore, the Hirzebruch-Riemann-Roch formula can be applied to a coherent sheaf.

As an example, let us compute the Hilbert polynomial of a rank-2 coherent sheaf on $\mathbb{P}^3$ which will be used in section 3.5.

Example 1.3. *Let $\mathcal{E}$ be a rank-2 coherent sheaf on $\mathbb{P}^3$ with Chern classes c_1H,c_2H^2 and c_3H^3. By [20, Lemma 2.1], the Chern classes of $\mathcal{E}(d)$ are $c_1' = (c_1+2d)H$, $c_2' = (c_2+c_1d+d^2)H^2$, and $c_3' = c_3H^3$ respectively, so the Chern character is*

$$\begin{aligned} ch(\mathcal{E}(d)) &= \mathrm{rank}(\mathcal{E}(d)) + c_1' + \frac{1}{2}((c_1')^2 - 2c_2') + \frac{1}{6}((c_1')^3 - 3c_1'c_2' + 3c_3') \\ &= 2 + (c_1+2d)H + \frac{1}{2}(c_1^2 - 2c_2 + 2c_1d + 2d^2)H^2 \\ &\quad + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3 + 2d^3 + 3c_1d^2 + (3c_1^2 - 6c_2)d)H^3. \end{aligned}$$

The Todd class of $\mathbb{P}^3$ is

$$Td(\mathbb{P}^3) = 1 + 2H + \frac{11}{6}H^2 + H^3.$$

The Hilbert polynomial is the coefficient of H^3 in the intersection:

$$\begin{aligned} P_{\mathcal{E}}(d) &= 1\cdot\frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3 + 2d^3 + 3c_1d^2 + (3c_1^2 - 6c_2)d) \\ &\quad + 2\cdot\frac{1}{2}(c_1^2 - 2c_2 + 2c_1d + 2d^2) + \frac{11}{6}\cdot(c_1+2d) + 1\cdot 2 \\ &= \frac{1}{3}d^3 + (2+\frac{1}{2}c_1)d^2 + (\frac{1}{2}c_1^2 + 2c_1 + \frac{11}{3} - c_2)d \\ &\quad + c_1^3/6 + c_1^2 + 11c_1/6 - c_1c_2/2 - 2c_2 + c_3/2 + 2. \end{aligned}$$

Chapter 2

Characterization of Hilbert functions

This chapter contains previously known results on Hilbert functions and Hilbert polynomials of standard graded algebras, namely, Macaulay's Theorem, Green's Hyperplane Restriction Theorem, and Gotzmann's Persistence and Regularity Theorems.

Macaulay's Theorem was a major early result on Hilbert functions. It characterizes all possible functions that can be a Hilbert function of a standard graded algebra, by establishing an upper bound on the growth of the Hilbert function from degree d to $d+1$. Gotzmann's Persistence Theorem establishes the behavior of the Hilbert function when the Macaulay growth bound is achieved. Green's Hyperplane restriction Theorem provides a different type of bound; in particular, given a scheme X and a generic hyperplane section H, it gives a lower bound on the Hilbert function for the smaller-dimensional $X \cap H$ in degree d in terms of the Hilbert function of X in degree d.

Gotzmann's Regularity Theorem uses the results on Hilbert functions above to establish a bound on the regularity of any ideal with a given Hilbert polynomial. Gotzmann uses this to provide an explicit construction of the Hilbert scheme $\mathrm{Hilb}^P_{\mathbb{P}^n}$ by stating the precise Grassmannian into which the Hilbert scheme embeds, and the condition that characterizes the embedding.

First I will discuss the notions of monomial, lexicographic, and stable modules, which are used later in this chapter and in future chapters. Then I will provide the definitions for Macaulay representations, Macaulay and Green transformations, and the theorems discussed above.

2.1 Monomial, lexicographic and stable modules

The notions of monomial, lex, and stable modules are important because we have operations to deform an arbitrary module successively to these more restrictive classes of modules; furthermore, the behavior of Hilbert functions and Betti numbers under these operations can be controlled.

Let $F = Se_1 + \cdots + Se_r$ with $deg(e_i) = d_i$ for each i.

Definition 2.1. *A* monomial ideal *$I \subseteq S$ is an ideal that is generated by monomials. A* monomial submodule *$M \subseteq F$ is generated by module monomials $m_i e_i$, where each m_i is a monomial of S.*

Note that a monomial submodule must be of the form $I_1 e_1 + \cdots + I_r e_r$ where each I_i is a monomial ideal of S.

Definition 2.2. *Define the* lexicographic *(lex) order on the monomials of S as follows: $x_0^{e_0} \cdots x_n^{e_n} > x_0^{f_0} \cdots x_n^{f_n}$ if the lowest i such that $e_i \neq f_i$ satisfies $e_i > f_i$. A* lex-segment *in S_d is a set of the first $r(d)$ monomials of degree d in the lex order; a* lex-segment ideal *(or* lex *ideal) is an ideal that is a lex-segment in each degree.*

Define the lex order on module monomials by $fe_i > ge_j$ if $i < j$, or if $i = j$ and $f > g$ in the lex order on monomials of S. Lex-segment submodules are defined analogously to lex-segment ideals.

Definition 2.3. *Let $I \subset S$ be a monomial ideal. I is* strongly stable *if whenever $i < j$ and g is a monomial such that $gx_j \in I$, then $gx_i \in I$. For a prime p, we say that $k \leq_p l$ if, when we write $k = \sum k_i p^i$ and $l = \sum l_i p^i$, we have $k_i \leq l_i$ for each i. A monomial ideal $I \subset S$ is* p-stable *if for every monomial $m \in M$, if $x_j^l || m$ and $i < j$ then $(x_i/x_j)^k m \in M$ for every $k \leq_p l$.*

Let $M = I_1 e_1 + \cdots + I_r e_r \subseteq F$ be a monomial submodule. M is strongly stable *(*p-stable*) if each I_i is strongly stable (p-stable) and $\mathfrak{m}^{d_j - d_i} I_j \subseteq I_i$ for each $i < j$.*

When k has characteristic p, strongly stable modules are called standard Borel-fixed modules by Pardue, and p-stable modules are called non-standard Borel-fixed modules by Pardue [29, Definition 7]. Note that strongly stable modules are p-stable for each prime p.

It is clear from the definitions that a lex ideal is strongly stable. Also note that a lex submodule can be written $L = L_1 e_1 + \cdots + L_r e_r$, where each L_i is a lex ideal. Strongly stable ideals have a well-understood minimal free resolution, from which we can deduce the regularity.

Proposition 2.1. *If I is a strongly stable ideal of S, then the regularity of I is equal to the maximal degree in a minimal generating set of I.*

Proof. See [32, Corollary 3.1]. □

In particular, since a lex ideal is strongly stable and a lex submodule is a direct sum of lex ideals, a lex submodule has a *linear* resolution; that is, the maximal generating degree of the minimal resolution increases by one in each homological degree.

Lex modules are known to have maximal Betti numbers, as expressed in the next theorem.

Theorem 2.1. *If M is a graded submodule of F and L is the lex submodule with the same*

Hilbert function as M, then

$$\beta_{i,j}(M) \leq \beta_{i,j}(L) \text{ for all } i, j.$$

Proof. This was proven by Bigatti [3] and Hulett [22] for ideals in characteristic zero, and by Hulett [23] for modules in characteristic zero; Pardue [29] proved the result for modules in arbitrary characteristic. □

A different type of monomial ordering is given in the following definition.

Definition 2.4. *The* weight ordering $<_{\mathbf{w}}$ *with weight vector* $\mathbf{w} = (w_0, \dots w_n, v_1, \dots, v_r)$ *is given by* $\sum x^{\alpha_i} e_i <_{\mathbf{w}} \sum x^{\alpha_j} e^j$ *if* $\sum_i \sum_k \alpha_{i,k} w_k + e_i v_i < \sum_j \sum_k \alpha_{j,k} w_k + e_j v_j$*; that is, the order is given by performing a dot product on the combination of the exponents and basis elements with the weight vector.*

Proposition 2.2. *Assume N is a submodule of F with initial submodule* $\text{in}(N)$ *under a given ordering. Then there exists a weight vector* w *with positive integer entries such that* $\text{in}(N)$ *is the initial module of N under the weight ordering $<_{\mathbf{w}}$.*

Proof. See [2, Proposition 1.8]. □

The following comes from [29]. It is an extension of the polarization used by Hartshorne in the proof of the connectedness of the Hilbert scheme.

Definition 2.5. *Let $P = k[z_{ijk}]$ with $0 \leq i \leq n$, $1 \leq k \leq r$ and $1 \leq j < J$ for some J sufficiently large. Let F' be the free P-module with basis $e'_1, \dots, e'_r$ with the same generating degrees as $e_1, \dots, e_r$. For a monomial ideal I, the* k-polarization $I^{(p_k)}$ *of I is the monomial ideal in P generated by*

$$\{z^{p_k(\mu)} = \prod z_{ijk} | x^{\mu} \text{ is a minimal generator of } I\}.$$

If $N = I_1e_1 + \cdots + I_re_r$ *is a monomial submodule of* F*, then the polarization of* N *is*

$$N^{(p)} = I_1^{(p_1)}e_1' + \cdots + I_r^{(p_r)}e_r'.$$

Let $L = \{h_{ijk}\}$ *be a collection of linear forms in* S*. Define* $\sigma_L : P \to S$ *by* $\sigma_L(z_{ijk}) = h_{ijk}$*, and* $\sigma_L' : F' \to F$ *by* $\sigma_L'(\sum f_ie_i) = \sum \sigma_L(f_i)e_i$*.*

In particular, given a monomial submodule N and a generic set of linear forms L, the operation $\sigma_L'(N^{(p)})$ performs a polarization (that is, adds enough variables so that every term is squarefree), followed by taking generic hyperplane sections. The following proposition shows that the graded Betti numbers are unchanged under k-polarization.

Proposition 2.3. *If* N *is a submodule of* F *and* L *is a generic collection of linear forms, then the graded Betti numbers of* F/N *and* $F/\sigma_L'(N^{(p)})$ *are the same.*

Proof. See [29, Corollary 15]. □

Example 2.1. *Set* $S = \mathbb{C}[x, y, z, w]$*. In example 1.2, we saw the ideals* $I = (x^2 - y^2, xy - z^2)$ *and* $J = (x^2, xy, y^3)$*. Note that* I *is not a monomial ideal, but* J *is; in fact,* $J = \mathrm{in}(I)$ *using the lex ordering. However,* J *is not a lex ideal, since* $y^3 \in J$ *but* $xz^2 \notin J$*. The ideal* $J = (x^2, xy, xz^2, xzw^2, xw^4, y^5, y^4z^2)$ *is a lex ideal; indeed,* I *contains the first 2 monomials in degree 2, the first 8 monomials in degree 3 and so on, under the lex order. In fact, all three of these ideals have the same Hilbert function.*

2.2 Macaulay's and Green's Theorems

Macaulay's and Green's theorems are foundational in the characterization of Hilbert functions. These will depend on the behavior of two transformations on a sort of representation of a positive integer as a sum of binomial coefficients. I will describe these in this section. First we will define the Macaulay representation and its transformations.

Definition 2.6. *Given* $a, d \in \mathbb{N}$*, the* dth Macaulay representation of a *is the unique expression*

$$a = \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \cdots + \binom{k(\delta)}{\delta},$$

with $\delta \in \mathbb{Z}$*, satisfying* $k(d) > \cdots > k(\delta) \geq \delta > 0$*. Given this representation, the* dth Macaulay transformation of a *is*

$$a^{\langle d \rangle} = \binom{k(d)+1}{d+1} + \binom{k(d-1)+1}{d} + \cdots + \binom{k(\delta)+1}{\delta+1},$$

and the Green transformation *is*

$$a_{\langle d \rangle} = \binom{k(d)-1}{d} + \binom{k(d-1)-1}{d-1} + \cdots + \binom{k(\delta)-1}{\delta}.$$

Example 2.2. *The 3rd Macaulay representation of 11 is*

$$\binom{5}{3} + \binom{2}{2},$$

the Macaulay transformation is

$$\begin{aligned} 11^{\langle 3 \rangle} &= \binom{6}{4} + \binom{3}{3} \\ &= 16. \end{aligned}$$

and the Green transformation is

$$\begin{aligned} 11_{\langle 3\rangle} &= \binom{4}{3} + \binom{1}{2} \\ &= 4. \end{aligned}$$

Green's theorem bounds the Hilbert function of a standard graded algebra after slicing by a generic hyperplane. It provides easier proofs of the rest of the main theorems in this chapter.

Theorem 2.2. *(Green's Hyperplane Restriction Theorem) Let $R = S/I$ be a graded k-algebra and $n \geq 1$ and integer. For a general linear form h,*

$$H(R/hR, d) \leq H(R, d)_{\langle d\rangle}.$$

Proof. Following [16, Theorem 1]: Fix the following notation. Let $W = I_d$, let H be the hyperplane given by the vanishing of h, let $c = H(R, d)$ and $c_H = H(R/hR, d)$. We will proceed by induction on dimension and degree; the result is clear when $n = 0$ and when $d = 1$. Write the dth Macaulay representation of c_H as

$$c_H = \binom{k(d)}{d} + \cdots + \binom{k(1)}{1},$$

and set $\delta = \min\{m | k(m) \geq m\}$. From the definition of the Green transformation, it is sufficient to show that

$$c \geq \binom{k(d)+1}{d} + \cdots + \binom{k(\delta)+1}{\delta}.$$

Assume to the contrary. Then

$$c - c_H < \binom{k(d)}{d-1} + \cdots + \binom{k(\delta)}{\delta - 1}.$$

Write $W(-H)$ for the subspace of S_{d-1} consisting of polynomials P such that $PH \in W$. From the exact sequence

$$0 \to W(-H) \to W \to W_H \to 0,$$

it follows that

$$c = c_H + \operatorname{codim}(W(-H)).$$

Assume H^* is another general hyperplane. From the restriction sequence

$$0 \to W_H(-(H \cap H^*)) \to W_H \to W_{H \cap H^*} \to 0$$

it follows that

$$\operatorname{codim}(W_H) = \operatorname{codim}(W_{H \cap H^*}) + \operatorname{codim}(W_H(-(H \cap H^*))).$$

Observe that $W(-H^*)_H \subseteq W_H(-(H \cap H^*))$, so $\operatorname{codim}(W(-H^*)_H) \geq \operatorname{codim}(W_H(-(H \cap H^*)))$; substituting into the last equation, we have

$$\operatorname{codim}(W_H) \leq \operatorname{codim}(W_{H \cap H^*}) + \operatorname{codim}(W(-H^*)_H).$$

Note that $W_{H \cap H^*}$ is the restriction of $W_H \subseteq S/hS$ to a generic hyperplane, and $W(-H^*)_H$ is the restriction of $W(-H) \subseteq S_{d-1}$ to a generic hyperplane, so they satisfy the inductive hypothesis. Thus $\operatorname{codim}(W_{H \cap H^*}) \leq (c_H)_{\langle d \rangle}$ and $\operatorname{codim}(W(-H^*)_H) \leq (c - c_H)_{\langle d-1 \rangle}$. Thus

$$c_H \leq (c_H)_{\langle d \rangle} + (c - c_H)_{\langle d-1 \rangle}.$$

We consider two cases for δ, using the previous Macaulay representation for $c - c_H$. First, if

$\delta = 1$, then $\binom{k(\delta)-1}{\delta-1} > 0$, so

$$(c - c_H)_{\langle d-1 \rangle} \leq \binom{k(d)-1}{d-1} + \cdots + \binom{k(2)-1}{1}$$

and

$$(c_H)_{\langle d \rangle} = \binom{d(d)-1}{d} + \cdots + \binom{k(1)-1}{1}.$$

Adding these two expressions, we have

$$c_H \leq \binom{k(d)}{d} + \cdots + \binom{k(1)-1}{1} < c_H$$

(since $\delta = 1$, so $\binom{k(1)-1}{1} < \binom{k(1)}{1}$), which is a contradiction.

Second, if $\delta > 1$, then using the Macaulay representation for $c - c_H$, we have

$$(c - c_H)_{\langle d-1 \rangle} < \binom{k(d)-1}{d-1} + \cdots + \binom{k(\delta)-1}{\delta-1},$$

where the strict inequality follows since $k(\delta) - 1 > \delta - 1$. Adding this to the expression for $(c_H)_{\langle d \rangle}$ gives

$$c_H < \binom{k(d)}{d} + \cdots + \binom{k(\delta)}{\delta} = c_H,$$

which is also a contradiction. Therefore, $c_H \leq c_{\langle d \rangle}$. $\square$

Macaulay characterized the Hilbert functions of all standard graded algebras. We will first need a lemma.

Lemma 2.1. *Given positive integers a, d, the Macaulay transform satisfies*

$$(a+1)^{\langle d\rangle} = \begin{cases} a^{\langle d\rangle} + k(1) + 1 & \text{if } \delta = 1 \\ a^{\langle d\rangle} + 1 & \text{if } \delta > 1 \end{cases}$$

Proof. See [5, Lemma 4.2.13]. □

Theorem 2.3. *(Macaulay [26]) Assume that $h : \mathbb{Z} \to \mathbb{Z}$ is a numerical function. Then h is the Hilbert function of a standard graded algebra if and only if $h(0) = 1$ and $h(n+1) \le h(n)^{\langle n\rangle}$ for all $n \ge 1$.*

Proof. For the backward implication, see [5, Theorem 4.2.10]. The proof of the forward implication will follow [16, Theorem 2]. Keep the same notation as in the previous theorem. Write $W_1 = I_{d+1}$ and $c_1 = \operatorname{codim}(W_1)$. From the restriction sequence on W_1 and the containment $W \subseteq W_1(-H)$, we have

$$c_1 = \operatorname{codim}(W_1(-H)) + \operatorname{codim}((W_1)_H) \le c + \operatorname{codim}((W_1)_H).$$

Write the $d+1$ Macaulay representation of c_1:

$$c_1 = \binom{k(d+1)}{d+1} + \cdots + \binom{k(1)}{1}.$$

From Green's Theorem, we have

$$\operatorname{codim}((W_1)_H) \le \binom{k(d+1)-1}{d+1} + \cdots + \binom{k(1)-1}{1},$$

so

$$c \ge \binom{k(d+1)-1}{d} + \cdots + \binom{k(1)-1}{0}.$$

As in the last proof, there are two cases for δ. If $\delta = 1$, then $k(d) \geq 1$. The right hand side of the last inequality is not a Macaulay representation, but by applying Lemma 2.1 with

$$a = \binom{k(d+1)-1}{d+1} + \cdots + \binom{k(2)-1}{1},$$

we have

$$c^{\langle d \rangle} = (a+1)^{\langle d \rangle} \geq \binom{k(d+1)}{d+1} + \cdots + \binom{k(2)}{2} + k(2) > c_1,$$

since $k(2) > k(1)$. If $\delta > 1$, then

$$c^{\langle d \rangle} \geq \binom{k(d+1)}{d+1} + \cdots + \binom{k(\delta)}{\delta} = c_1. \quad \square$$

2.3 Gotzmann's Theorems

Gotzmann noted that, considering the Macaulay representation of the Hilbert function in sufficiently high degree, one may build a similar representation for the Hilbert polynomial. This representation will be used in Gotzmann's Regularity Theorem, below.

Definition 2.7. *Given a polynomial $P(d) \in \mathbb{Q}[d]$, a* Gotzmann representation *of P is a binomial expansion*

$$P(d) = \binom{d+a_1}{d} + \binom{d+a_2-1}{d-1} + \cdots + \binom{d+a_s-(s-1)}{d-(s-1)},$$

with $a_1, \cdots, a_s \in \mathbb{Z}$ and $a_1 \geq \cdots \geq a_s \geq 0$.

Proposition 2.4. *The Hilbert polynomial of a subscheme of $\mathbb{P}^n$ has a unique Gotzmann representation.*

Proof. See [34, Corollary B.31]. □

Definition 2.8. *The number of terms in the Gotzmann representation of a scheme's Hilbert polynomial is called the* Gotzmann number *of a scheme.*

Example 2.3. *Assume X is a scheme with Hilbert polynomial $P_X(d) = 3d + 2$. This has Gotzmann representation*

$$3d+2 = \binom{d+1}{d} + \binom{d}{d-1} + \binom{d-1}{d-2} + \binom{d-3}{d-3} + \binom{d-4}{d-4},$$

and the Gotzmann number of $P_X(d)$ is 5.

Gotzmann's Regularity Theorem bounds the Castelnuovo-Mumford regularity by the Gotzmann number. The proof of the generalization in Chapter 3 will be similar to the proof of this result cited below, and is omitted here.

Theorem 2.4. *(Gotzmann's Regularity Theorem [14]) If X is a projective k-scheme with Gotzmann number s, then $\mathcal{I}_X$ is s-regular.*

Proof. See [34, Theorem B.33]. □

The next result determines what happens when equality is reached in Macaulay's growth bound from the last section. The proof follows [16], and contrasts with a different proof needed to generalize this result in Chapter 4.

Theorem 2.5. *(Gotzmann's Persistence Theorem) [14] Suppose I is a graded ideal of S generated in degree at most d. If $H(S/I, d+1) = H(S/I, d)^{\langle d\rangle}$, then*

$H(S/I, s+1) = H(S/I, s)^{\langle s\rangle}$ for all $s \geq d$.

Proof. Following [16]: By induction on dimension, where dimension 1 is clear. With the same notation as in Macaulay's Theorem, the restriction sequence for W_1 and the containment

$W \subseteq W_1(-H)$ implies $c_{1,H} \geq c_1 - c$. Note that

$$(c_{\langle d \rangle})^{\langle d \rangle} \geq c_H^{\langle d \rangle} \geq c_{1,H} \geq c_1 - c = (c_{\langle d \rangle})^{\langle d \rangle},$$

where the first inequality holds since $c_H \leq c_{\langle d \rangle}$ by Green's Theorem, the second inequality holds by Macaulay's Theorem, the third inequality holds by the observation on the restriction sequence, and the last equality is by hypothesis. Thus

$$c_H^{\langle d \rangle} = c_{1,H}.$$

By the inductive hypothesis, $\operatorname{codim}(W_H)_{s+1} = \operatorname{codim}(W_H)_s^{\langle s \rangle}$ for all $s \geq d$. By Gotzmann's Regularity Theorem, the saturation of I/H is d-regular in S/H. The same argument as in Gotzmann's Regularity Theorem shows that the saturation of I is also d-regular. Thus for all $s \geq d$, we have

$$H(S/\operatorname{sat}(I), s) \leq H(S/I, s) \leq P_{S/I}(s) = P_{S/\operatorname{sat}(I)}(s) = H(S/\operatorname{sat}(I), s);$$

so equality holds, and persistence holds by the Gotzmann representation of the Hilbert function. $\square$

Example 2.4. *Consider an ideal I in $S = \mathbb{C}[x, y, z, w]$ such that the Hilbert function is $H(S/I, 2) = 2 = \binom{2}{2} + \binom{1}{1}$. By Macaulay's Theorem, $H(S/I, 3) \leq \binom{3}{3} + \binom{2}{2} = 2$, and similarly in higher degrees; therefore, the Hilbert polynomial of S/I is a constant, either 2, 1, or 0. If $H(S/I, 3) = 2$, then by Gotzmann's Persistence Theorem, $P_{S/I} = 2$. If $H(S/I, 3) = 1$, then either $H(S/I, 4) = 1$ and $P_{S/I} = 1$ by Persistence, or $H(S/I, 4) = 0$ and $P_{S/I} = 0$. Finally, if $H(S/I, 3) = 0$, then $P_{S/I} = 0$. There is a lex ideal with each of these possible Hilbert functions.*

Gotzmann used the theorems on Regularity and Persistence to construct the Hilbert scheme.

Let

$$\mathcal{G}_s = \mathrm{Gr}(S_s, P(s)) \tag{2.1}$$

be the Grassmannian of $P(s)$-codimensional vector subspaces of S_s, and similarly for $\mathrm{Gr}(S_{s+1}, P(s+1))$; and

$$W = \{(F, G) \in \mathcal{G}_s \times \mathcal{G}_{s+1} | F \cdot S_1 = G\}. \tag{2.2}$$

Theorem 2.6. *The Hilbert scheme* $\mathrm{Hilb}^P_{\mathbb{P}^n}$ *is isomorphic to* W.

Proof. See [14, Satz 3.4]. □

Chapter 3

Gotzmann regularity for globally generated coherent sheaves

One would be interested in extending Gotzmann's construction of the Hilbert scheme to a construction of the Quot scheme. In order to do so, Gotzmann's Regularity and Persistence theorems need to be extended to modules. Gasharov extended the Persistence theorem; however, Gotzmann's Regularity theorem does not extend to all modules, because a Gotzmann representation does not always exist, which will be shown below. However, a Gotzmann representation exists for the Hilbert polynomial of a globally generated coherent sheaf. This allows the extension of Gotzmann's Regularity Theorem to this class of sheaves. This will be used to extend Gotzmann's construction to the Quot scheme $\mathrm{Quot}_P(\mathcal{O}_{\mathbb{P}^n}^r)$ of quotients of $\mathcal{O}_{\mathbb{P}^n}^r$ with Hilbert polynomial P.

The following is Gasharov's extension of Macaulay's Theorem (part 1) and Gotzmann's Persistence Theorem (part 2).

Theorem 3.1. *Assume F is a free graded S-module with l the maximal degree of its generators, N a submodule of F and $M = F/N$. For each pair (p, d) such that $p \geq 0$ and*

$d \geq p + l + 1$, *we have:*

1. $H(M, d+1) \leq H(M, d)^{\langle d-l-p \rangle}$;

2. *If* N *is generated in degree at most* d *and* $H(M, d+1) = H(M, d)^{\langle d-l-p \rangle}$, *then* $H(M, d+2) = H(M, d+1)^{\langle d+1-l-p \rangle}$.

Proof. See [12, Theorem 4.2]. □

However, Gotzmann's Regularity Theorem does not extend directly to all modules, because not all Hilbert polynomials of graded S-modules have Gotzmann representations, as the next example shows.

Example 3.1. *Consider* $\mathcal{O}_{\mathbb{P}^1}(-1)$. *It has Hilbert polynomial* $P(d) = d$. *By reason of degree, the Gotzmann representation would have to have* $\binom{d+1}{1} = d + 1$ *as its first term; and the Gotzmann representation can only add more positive terms, so no Gotzmann representation exists.*

In the next section, we will see that when the generating degrees of a module are at most zero, then the module has a Gotzmann representation. These modules correspond to globally generated coherent sheaves. Subsequently, I will prove Gotzmann's Regularity Theorem for this class of modules, and extend Gotzmann's construction to the Quot scheme.

3.1 Gotzmann representations

As before, assume a graded module $F = Se_1 + \cdots + Se_r$ with $\deg(e_i) = f_i$ and $f_1 \leq \cdots \leq f_r$; furthermore, assume $f_r \leq 0$. Note that a quotient module of F corresponds to a globally generated coherent sheaf on $\mathbb{P}^r$. This section will establish the following proposition.

Proposition 3.1. *If M is a submodule of F, then $P_{F/M}(d)$ has a unique Gotzmann representation.*

To prove this proposition, some lemmas are necessary. The first is a restatement of an earlier result in the framework of schemes.

Theorem 3.2 ([26])**.** *Let $H : \mathbb{N} \to \mathbb{N}$ and let k be a field. The following are equivalent.*

1. *H is the Hilbert function of a projective k-scheme.*

2. *$H(0) = 1$ and $H(d)^{\langle d \rangle} \leq H(d+1)$ for all d.* $\square$

Lemma 3.1. *The following are equivalent.*

1. *P is the Hilbert polynomial of a projective k-scheme.*

2. *P has a Gotzmann representation.*

Proof. $1 \Rightarrow 2$ is given by Gotzmann's Regularity Theorem. Conversely, if

$$P(d) = \binom{d+a_1}{a_1} + \binom{d+a_2-1}{a_2} + \cdots + \binom{d+a_s-(s-1)}{a_s},$$

then define $H(d)$ as

$$H(d) = \binom{d+a_1}{a_1} + \binom{d+a_2-1}{a_2} + \cdots + \binom{a_{d+1}}{a_{d+1}}$$

for $d \leq s-2$, and $H(d) = P(d)$ for $d \geq s-1$. Then

$$H(0) = \binom{a_1}{a_1} = 1.$$

If $d \leq s-2$, then

$$\begin{aligned} H(d+1) &= \binom{d+1+a_1}{a_1} + \cdots + \binom{a_{d+1}+1}{a_{d+1}} \\ &\quad + \binom{d+1+a_{d+2}-(d+1)}{a_{d+2}} \\ &= \binom{d+a_1+1}{d+1} + \cdots + \binom{a_{d+1}+1}{1} + \binom{a_{d+2}}{0} \\ &= H(d)^{\langle d \rangle} + \binom{a_{d+2}}{0} \\ &\geq H(d)^{\langle d \rangle}; \end{aligned}$$

and if $d \geq s-1$, then

$$H(d+1) = H(d)^{\langle d \rangle}.$$

By Macaulay's criterion, H is the Hilbert function of a projective k-scheme, and H has Hilbert polynomial P. □

Lemma 3.2. *If P and Q are polynomials with Gotzmann representations, then $P+Q$ has a Gotzmann representation.*

Proof. By Lemma 3.1, P and Q are Hilbert polynomials of projective k-schemes X and Y respectively. There exists a projective space $\mathbb{P}^N$ such that X and Y can be embedded disjointly. Then $X \cup Y$ is a projective k-scheme with Hilbert polynomial $P+Q$, so by Lemma 3.1, $P+Q$ has a Gotzmann representation. □

Lemma 3.3. *The polynomial*

$$\binom{d+n+1}{n}$$

has a Gotzmann representation.

Proof. One easily sees that

$$\binom{d+n+1}{n} = 1 + \binom{d+1}{1} + \binom{d+2}{2} + \cdots + \binom{d+n}{n}.$$

Lemma 3.2 then gives the result. $\square$

Proof. (of Proposition 3.1): By Theorem 2.1, we can assume $F/M = S/I_1e_1 + \cdots + S/I_re_r$. There exist Gotzmann representations

$$P_{S/I_j}(d) = \binom{d+a_{j,1}}{a_{j,1}} + \binom{d+a_{j,2}-1}{a_{j,2}} + \cdots + \binom{d+a_{j,s_j}-(s_j-1)}{a_{j,s_j}}.$$

It is sufficient to show that $P_{S/I_j}(d+1)$ has a Gotzmann representation. Note that

$$P_{S/I_j}(d+1) = \binom{d+a_{j,1}+1}{a_{j,1}} + \binom{d+a_{j,2}}{a_{j,2}} + \cdots + \binom{d+a_{j,s_j}-(s_j-2)}{a_{j,s_j}};$$

The first term has a Gotzmann representation by Lemma 3.3; the remaining terms are a Gotzmann representation; the sum of these two polynomials has a Gotzmann representation by Lemma 3.2. By Lemma 3.1, this is the Hilbert polynomial of some projective k-scheme, and by Proposition 3.1 the Gotzmann representation is unique. $\square$

3.2 Gotzmann regularity

We can now prove Gotzmann regularity for globally generated sheaves. The argument is very similar to the one given for the case of ideals from [34].

Proposition 3.2. *If F is a rank-r free S-module with module generators having degree at* *most zero, N a graded submodule of F and $M = F/N$, with Gotzmann representation*

$$P_M(d) = \binom{d+a_1}{a_1} + \binom{d+a_2-1}{a_2} + \cdots + \binom{d+a_s-(s-1)}{a_s}, \tag{3.1}$$

then the associated sheaf $\tilde{N}$ is s-regular.

Proof. As in [12], we can assume that the largest degree of a module generator of F is 0; otherwise, twist F and N by the same amount to cause it, and such twist only makes the Castelnuovo-Mumford regularity smaller by the amount of the twist. We can also assume N is saturated, and that $N \neq F$.

Induct on the number $n = \dim S - 1$. The case $n = 0$ is trivial.

In the case $n \geq 1$, there exists $h \in S_1$ such that h is M-regular. Indeed, if no $h \in S_1$ is M-regular, then $(S_1) = \cup_{h \in S_1} h$ is a zero-divisor on M, so there exists a non-zero $z \in F \setminus N$ such that $S_1 z \subset N$, contradicting N saturated.

Set

$$S' = S/hS(-1),\ F' = F/hF(-1),\ N' = N/hN(-1),\ M' = M/hM(-1).$$

Then M' satisfies the hypotheses of Proposition 3.1, so we can write

$$P_{M'}(d) = \binom{d+b_1}{b_1} + \binom{d+b_2-1}{b_2} + \cdots + \binom{d+b_r-(r-1)}{b_r}, \tag{3.2}$$

and $\tilde{N'}$ is r-regular.

Also, from the exact sequence $0 \to M(-1) \xrightarrow{h} M \to M' \to 0$, it follows that

$$P_{M'}(d) = P_M(d) - P_M(d-1). \tag{3.3}$$

Equations (3.1), (3.2) and (3.3) give

$$P_{M'}(d) = \binom{d+a_1-1}{a_1-1} + \binom{d+a_2-2}{a_2-1} + \cdots + \binom{d+a_t-t}{a_t-1}, \tag{3.4}$$

where $t \leq s$ is largest such that $a_t \neq 0$.

There exists a projective k-scheme with Gotzmann representations (3.2) and (3.4), so $t = r$ and $b_i = a_i - 1$ for $i = 1, \ldots, r$.

From the induction hypothesis, $\tilde{N}'$ is r-regular, so for all $q \geq 1$, we have

$$H^q(\tilde{N}'(d-q)) = 0$$

for all $d \geq r$, so

$$H^q(\tilde{N}(d-q)) = 0$$

for all $q \geq 2$ and all $d \geq s$. To show that this also holds for $q = 1$ and all $d = s$, suppose to the contrary that $H^1(\tilde{N}(s-1)) \neq 0$.

Let $\mathfrak{m}$ be the maximal homogeneous ideal of S. Recall that the Hilbert function and Hilbert polynomial can be related to the local cohomology [9, Corollary A1.15]:

$$H(M,d) - P_M(d) = \sum_i (-1)^i H^i_{\mathfrak{m}} M_d. \tag{3.5}$$

Since N is saturated and F is 0-regular, $H^0_{\mathfrak{m}}(M) = 0$ and $H^i_{\mathfrak{m}}(M) = H^{i+1}_{\mathfrak{m}}(N)$ for all $i \geq 1$. From the relations $H^i_{\mathfrak{m}}(N) = \bigoplus_d H^{i-1}(\tilde{N}(d))$ for $i \geq 2$, and equation 3.5, $H(M, s-1) <$

$P_M(s-1)$. So we can write

$$H(M, s-1) \le \binom{s-1+a_1}{s-1} + \cdots + \binom{1+a_{s-1}}{1},$$

and so

$$\begin{aligned} H(M,s) &\le H(M,s-1)^{\langle s-1 \rangle} \\ &\le \binom{s+a_1}{a_1} + \cdots + \binom{2+a_{s-1}}{a_{s-1}} \\ &< P_M(s). \end{aligned}$$

By repeating the last step, we have $H(M,d) < P_M(d)$ for all $d \ge s$, a contradiction. Therefore, $\tilde{N}$ is s-regular. □

The following corollary on regularity for arbitrary coherent sheaves is immediate.

Corollary 3.1. *Suppose $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$ and $a \in \mathbb{Z}^{\ge 0}$ such that $\mathcal{F}(a)$ is generated by global sections and $\mathcal{F}(a)$ has Gotzmann regularity s. Then $\mathcal{F}$ is $s+a$-regular.* □

3.3 An explicit construction of the Quot scheme

Let us use the above results and those of Gasharov to extend Gotzmann's construction of the Hilbert scheme to a construction of the Quot scheme. Write $\mathcal{O} = \mathcal{O}_{\mathbb{P}^n}$. For simplicity, consider $\mathrm{Quot}_P(\mathcal{O}^r)$, the quotients of $\mathcal{O}^r$ with Hilbert polynomial P. By Proposition 3.1, P has a Gotzmann representation, say with Gotzmann number s. By Proposition 3.2, any coherent sheaf which is a quotient of $\mathcal{O}^r$ with Hilbert polynomial P is generated in degrees at most s, and all cohomology vanishes in degrees at least s.

The construction of the Quot scheme follows as in the construction of the Hilbert scheme.

Let

$$\mathcal{G}_s = \mathrm{Gr}(S^r_s, P(s)) \tag{3.6}$$

be the Grassmannian of $P(s)$-codimensional vector subspaces of S^r_s, and similarly

$$\mathcal{G}_{s+1} = \mathrm{Gr}(S^r_{s+1}, P(s+1));$$

and

$$W = \{(F, G) \in \mathcal{G}_s \times \mathcal{G}_{s+1} | F \cdot S_1 = G\}. \tag{3.7}$$

Theorem 3.3. *The Quot scheme* $\mathrm{Quot}_P(\mathcal{O}^r)$ *is isomorphic to* W*.*

The proof will follow [14, Satz 3.4]. Before beginning the proof, let us fix some notation. Write Quot^r_P for $\mathrm{Quot}_P(\mathcal{O}^r)$. Write $\mathbb{W}$ for the functor that will be represented by W, and $\mathbb{G}_s$ for the functor of $\mathcal{G}_s$. For an affine scheme $T = \mathrm{Spec}A$ and the projection $\pi : T \times \mathbb{P}^n \to T$, recall that an element of $\mathbb{W}(A)$ is a pair (F, G) of locally free direct summands of $S^r_s \otimes A$ and $S^r_{s+1} \otimes A$ of co-ranks $P(s)$ and $P(s+1)$ respectively, such that $F \cdot S_1 = G$.

Two lemmas will be given first. Their proofs are the same as the case of Hilbert schemes and ideals as given in their sources.

Lemma 3.4. *For any* $d \geq s$ *with* s *being the Gotzmann number of* P*, the subfunctor* $\mathbb{W}(A)$ *of* $\mathbb{G}_s(A)$ *is representable by the scheme* W*, and the first projection embeds* W *in* $\mathcal{G}_s$*; and* W *contains the image of* Quot^r_P *under the closed diagonal embedding.*

Proof. The same argument from [24, Proposition C.28] for the Hilbert scheme will be used. It is sufficient to consider an affine open subscheme U of $\mathbb{G}_s(A)$ with coordinate ring A. Let F be the universal submodule of S^r_d corresponding to the inclusion of U. Let $u : F^{n+1} \to S^r_{d+1}$

be the multiplication map by $x_0, \ldots, x_n$. Set $q = r\binom{d+n}{n} - P(d+1)$. We obtain a subscheme V of U by combining the closed condition $\operatorname{rank}(u) \leq q$ and the open condition $\operatorname{rank}(u) > q-1$.

Let B be an A-algebra. An element of $\mathbb{W}(B)$ can be written $(F \otimes B, G \otimes B)$; the map $u \otimes B$ satisfies the two rank conditions above, so we have an injection

$$i : \mathbb{W}(B) \to V(B)$$

via the projection onto the first coordinate. It remains to show that i is a bijection.

Let $F' = F \otimes B \in V(B)$. Set $C = \operatorname{coker}(u) \otimes B$. From the assumption on the rank of u, the Fitting ideals of C satisfy $\operatorname{Fitt}_q(C) = A$ and $\operatorname{Fitt}_{q-1}(C) = 0$, so C is locally free by [9, Proposition 20.8]. Set $S' = S \otimes B$ and $G' = F' \cdot S'_1$. Then G' is a direct summand of $(S'_{d+1})^r$ of corank q, so $(F', G') \in \mathbb{W}(B)$, therefore i is a bijection, and V represents the restricted functor $\mathbb{W}|_U$.

Since $V(A) \cong W(V) \subseteq \mathcal{G}_s(A)$, we can lift to an embedding of V into $\mathcal{G}_s$. $\square$

Thus we can use

$$W' = \{F \in \mathcal{G}_s | \operatorname{corank}(F \cdot S_1) = P(s+1)\}$$

interchangeably with W.

Lemma 3.5. *If A is a local Noetherian ring with maximal ideal $\mathfrak{m}$ and $A/\mathfrak{m} = k$, $S = A[x_0, \ldots, x_n]$, N is a graded submodule of S^r generated by N_d such that* $\operatorname{sat}(N \otimes k)$ *is d-regular, and $M = F/N$ with* $\dim M_d \otimes k = P(d)$*, then the following are equivalent:*

1. *M_n is free of rank $P(n)$ for all $n \geq d$*
2. *M_n is free of rank $P(n)$ for all $n \gg 0$*

3. $M \otimes k$ has Hilbert polynomial P, and M_{d+1} is flat with rank $P(d+1)$.

Proof. The argument is the same as [14, Satz 1.5]. The equivalence of 1 and 2 is by [17, Corollaire 7.9.9], and clearly $1 \Rightarrow 3$.

To show $3 \Rightarrow 2$, assume that M_{d+1} is flat. Writing R for the syzygy module of N (with an exact sequence $0 \to R \to S^m \to N \to 0$), one may argue as in Section 1.4 of [14] to show that $R \otimes k$ is generated by $R_1 \otimes k$, or we may note that $\operatorname{sat}(N \otimes k)_{\geq d}$ has a linear resolution by [6, Proposition 1.3.1], and that $\operatorname{sat}(N \otimes k)$ and $N \otimes k$ agree in degree d and larger, since they have the same Hilbert function for $n \geq d$.

Now if $v \gg 0$ and $x \otimes 1 \in \ker(N_{d+v} \otimes k \to S^r_{d+v} \otimes k)$, then writing $x = fy$ with $f \in N_d$ and $y \in S^m_v$, we have $y \otimes 1 \in R_v \otimes k = (S_{v-1} \otimes k)R_1$, hence $y \in S_{v-1}R_1 + \mathfrak{m}S^m_v$. It follows that $x \in \mathfrak{m}N_{d+v}$, thus $N_{d+v} \otimes k \to S^r_{d+v} \otimes k$ is an injection; therefore a basis for $M_{d+v} \otimes k$ can be lifted to a basis for M_{d+v} and M_{d+v} is free. □

Proof. (of Theorem 3.3)

Let (F, G) be the universal element of $\mathbb{W}(W)$ and set $M = \oplus_d (S^r_{s+d}/S_d \cdot F)$. Note that Quot^r_P is the maximal subscheme of W such that $M \otimes_W \mathcal{O}_{\mathrm{Quot}^r_P}$ is flat over Quot^r_P with Hilbert polynomial P. For each $d \geq s$, take Z_d to be the maximal subscheme of W such that $M_d \otimes \mathcal{O}_{Z_d}$ is flat over $\mathcal{O}_{Z_d}$ with rank $P(d)$. By Theorem 3.1 #1, $\dim M_d \otimes k(p) \leq P(d)$ for all $p \in W$ and $d \geq s$, hence Z_d is closed. There exists d_0 such that $Z_{d_0} = Z_d$ for all $d \geq d_0$, since the Z_d form an ascending chain, so $\mathrm{Quot}^r_P = Z_{d_0}$ is closed.

Now consider the set Z of points $p \in W$ such that $M \otimes_W k(p)$ has Hilbert polynomial P. Note that $Z_{d_0} \subseteq Z$. By Lemma 3.5, for any $p \in Z$, we have $M_d \otimes_W \mathcal{O}_p$ is free of rank $P(d)$ over $\mathcal{O}_p$ for all $d \geq s$, so $M \otimes_W \mathcal{O}_p$ is free over $\mathcal{O}_p$ with Hilbert polynomial P, thus $p \in Z_{d_0}$. It follows that Z is the set of points for which $M_{d_0} \otimes_W \mathcal{O}_p$ is free of rank $P(d_0)$ over $\mathcal{O}_p$, so Z is an open subset of W, and thus Quot^r_P is an open subscheme of W.

Since Quot_P^r is an open and closed subscheme of W, it remains to show that all connected components of W are contained in Quot_P^r, for which it is sufficient to check geometric points. Let $(F, G) \in \mathbb{W}(K)$ be a geometric point. Then $M = \oplus_d (S_{s+d}^r / S_d \cdot F)$ has Hilbert function $P(s)$ and $P(s+1)$ in degrees s and $s+1$, so by Theorem 3.1 #2, M has Hilbert polynomial P, so (F, G) is in the image of Quot_P^r. Therefore, $\mathrm{Quot}_P^r \cong W$. $\square$

3.4 Quot schemes on $\mathbb{P}^1$

Consider this construction for the Quot scheme

$$\mathcal{Q} = \mathrm{Quot}_P(\mathcal{O}_{\mathbb{P}^1}^r),$$

with $P(d) = k(d+1) + m$. The Gotzmann representation of P is

$$\binom{d+1}{1} + \binom{d}{1} + \cdots + \binom{d-k+2}{1} + \frac{k(k-1)}{2} + m,$$

so the Gotzmann number of P is

$$s = \frac{k(k+1)}{2} + m.$$

Thus,

$$\mathcal{Q} \cong \{F \in \mathrm{Gr}(r(s+1), k(s+1)+m) | \mathrm{codim} F \cdot S_1 = k(s+2) + m\}.$$

Let $\mathcal{F} \in \mathcal{Q}$, and let l be an integer at least the Castelnuovo-Mumford regularity of $\mathcal{F}$. We have an exact sequence

$$0 \to \mathcal{K}(l) \to \mathcal{O}^r(l) \to \mathcal{F}(l) \to 0.$$

Set $V = H^0(\mathcal{K}(l)) \subset S_l^r$. Then $\dim V = (r-k)(l+1)-m$. If $\dim V \cdot S_1 = (r-k)(l+2)-m$, one expects an element F in $\mathrm{Gr}((r-k)(l+1)-m, r(l+1))$ and an element $G = F \cdot S_1$ in $\mathrm{Gr}((r-k)(l+2)-m, r(l+2))$. The Porteous formula gives a degeneracy locus with codimension

$$\begin{aligned} &[2(r-k)(l+1)-2m-(r-k)(l+2)+m][r(l+2)-(r-k)(l+2)+m] \\ =&[(r-k)l-m][k(l+2)+m]. \end{aligned}$$

So the expected dimension of $\mathcal{Q}$ is

$$[(r-k)(l+1)-m][k(l+1)+m]-[(r-k)l-m][k(l+2)+m]=(r-k)k+rm.$$

On the other hand, given

$$0 \to \mathcal{K} \to \mathcal{O}^r \to \mathcal{F} \to 0$$

we can write $\mathcal{K} = \oplus\mathcal{O}(-t_i)$, and since the Hilbert polynomial of $\mathcal{F}$ is $k(d+1)+m$, we have $r-k$ summands of $\mathcal{K}$, each t_i is non-negative, and $\sum t_i = m$. In order to compute $\dim Hom(\oplus\mathcal{O}(-t_i), \mathcal{O}^r)/Aut(\oplus\mathcal{O}(-t_i))$, we need a bound on $\dim Aut(\oplus\mathcal{O}(-t_i))$.

Rewrite $\mathcal{K} = \oplus\mathcal{O}(-t_i)$ by gathering common twists, as

$$\mathcal{K} = \oplus_{i=0}^{s}\mathcal{O}(i)^{e_i}.$$

Then $\dim AutK = \sum_{0\le i\le j\le s}(j-i+1)e_ie_j$. This dimension is bounded as follows.

Lemma 3.6. *Assume m and n are integers such that $0 \le m \le n$ and $e_0, e_1, \ldots, e_n$ are such that $\sum e_i = m$ and $\sum ie_i = n$. Then*

$$\sum_{0 \le i \le j \le n} (j - i + 1)e_i e_j \ge m^2,$$

with a unique solution giving equality.

Proof. Observe that

$$\begin{aligned} \sum_{0 \le i \le j \le n} (j - i + 1)e_i e_j &= \sum_{i=0}^{n} e_i^2 + \sum_{0 \le i < j \le n} (j - i + 1)e_i e_j \\ &\ge \sum_{i=0}^{n} e_i^2 + \sum_{0 \le i < j \le n} 2e_i e_j \\ &= m^2, \end{aligned}$$

and equality occurs only if there are at most two non-zero exponents. Assume $0 < e_i \le m$, and $e_i + e_{i+1} = m$ and $ie_i + (i+1)e_{i+1} = n$. Then $n = i(e_i + e_{i+1}) + e_{i+1} = im + e_{i+1}$, with $0 \le e_{i+1} < m$. By the Division Algorithm, such a solution exists and is unique. □

So

$$\begin{aligned} &\dim Hom(\oplus\mathcal{O}(-t_i), \mathcal{O}^r)/Aut(\oplus\mathcal{O}(-t_i)) \\ &\quad = \dim H^0(\oplus\mathcal{O}(e_i)^r) - \dim Aut(\oplus\mathcal{O}(-t_i)) \\ &\quad \le r(r - k) - rm - (r - k)^2 \\ &\quad = (r - k)k + rm, \end{aligned}$$

agreeing with the previous computation; and the uniqueness makes $\mathcal{Q}$ irreducible of the expected dimension. The result of this computation gives the following:

Proposition 3.3. *The Quot Scheme $\mathrm{Quot}_P(\mathcal{O}^r_{\mathbb{P}^1})$ with $P(d) = k(d+1)+m$ having Gotzmann* *number*

$$\frac{r(r+1)}{2}+m,$$

is the irreducible scheme given by the degeneracy locus of

$$\mathrm{Gr}((r-k)(\frac{r(r+1)}{2}+m+1)-m, r(\frac{r(r+1)}{2}+m+1))$$

with rank $(r-k)k+rm$. $\square$

Iarrobino and Kleiman [24] showed that the maximal Castelnuovo-Mumford regularity of a scheme with Hilbert polynomial P is equal to the Gotzmann number of P. However, this is not the case for globally generated sheaves. In fact, the following example shows that we can use the Gotzmann construction to embed a Quot scheme into a smaller Grassmannian than the Gotzmann number gives.

Example 3.2. *Consider* $\mathcal{Q} = \mathrm{Quot}_P(\mathcal{O}^3_{\mathbb{P}^1})$, *where* $P(d) = 2(d+1)$. *The Gotzmann number of* P *is 3, and the Castelnuovo-Mumford regularity of a sheaf* $\mathcal{F} \in \mathcal{Q}$ *is 0. Set*

$$\begin{aligned} W_0 &= \{F \in \mathrm{Gr}(S^3_0, P(0)) | \mathrm{codim} F \cdot S_1 = P(1)\} \\ &= \{F \in \mathrm{Gr}(3,2) | \mathrm{codim} F \cdot S_1 = 4\} \\ &\cong \mathbb{P}^2 \end{aligned}$$

and

$$\begin{aligned} W_3 &= \{F \in \mathrm{Gr}(S^3_3, P(3)) | \mathrm{codim} F \cdot S_1 = P(4)\} \\ &= \{F \in \mathrm{Gr}(12,8) | \mathrm{codim} F \cdot S_1 = 10\}. \end{aligned}$$

Note that $\mathcal{Q} \cong W_3$, *but that by the last section,* $\mathcal{Q}$ *embeds into* W_0. *Given a point* $F \in W_0$, *one shows that* $F \cdot S_3 \in W_3$*, so* $Q \cong W_0$*.*

3.5 Bounding Chern classes

Given a rank 2 globally generated vector bundle $\mathcal{F}$ with Chern classes c_1 and c_2, [7] gives a bound for c_2 in terms of c_1, using the vanishing of a section of $\mathcal{F}$ along a smooth curve. Recall that a vector bundle is *split* if it can be written as a direct sum of line bundles.

Proposition 3.4. *If* $\mathcal{E}$ *is a non-split rank-2 globally generated vector bundle on* $\mathbb{P}^3$ *with* $c_1 \geq 4$*, then*

$$c_2 \leq \frac{2c_1^3 - 4c_1^2 + 2}{3c_1 - 4}.$$

Proof. See [7], Lemma 1.5. □

We can use the results herein to give a larger bound, but a bound that applies to any globally generated coherent sheaf.

Let $\mathcal{E}$ be a rank 2 globally generated coherent sheaf on $\mathbb{P}^3$ with Chern classes c_1, c_2 and c_3. Note that c_3 may be non-zero, since we are not restricting attention to vector bundles. The Hilbert polynomial was computed in example 1.3:

$$P_{\mathcal{E}}(d) = \frac{1}{3}d^3 + (2 + \frac{1}{2}c_1)d^2 + (\frac{1}{2}c_1^2 + 2c_1 + \frac{11}{3} - c_2)d + \overline{c}, \tag{3.8}$$

where

$$\overline{c} = c_1^3/6 + c_1^2 + 11c_1/6 - c_1c_2/2 - 2c_2 + c_3/2 + 2$$

is given for completeness; although it has no bearing on the following computation, it will be used again in the next chapter.

Since $\mathcal{E}$ is globally generated, $P_{\mathcal{E}}(d)$ has a Gotzmann representation; write

$$P_{\mathcal{E}}(d) = P_3 + P_2 + P_1 + P_0,$$

where P_i is the sum of binomial coefficients in the Gotzmann representation of degree i. Since $\mathcal{E}$ is rank 2, it follows that

$$P_3 = \binom{d+3}{3} + \binom{d+2}{3} = \frac{1}{3}d^3 + \frac{3}{2}d^2 + \frac{13}{6}d + 1,$$

so

$$P_{\mathcal{E}} - P_3 = \frac{1}{2}(1+c_1)d^2 + (\frac{1}{2}c_1^2 + 2c_1 + \frac{3}{2} - c_2)d + \overline{c} - 1.$$

This tells us that $c_1 \geq -1$, but in fact $c_1 \geq 0$ for any globally generated coherent sheaf on $\mathbb{P}^n$. To see this, assume $\mathcal{F}$ is a rank r globally generated sheaf on $\mathbb{P}^n$ with Hilbert polynomial $P(d)$; the top two terms of $P(d)$ are $rd^n/n! + (r(n+1)/2 + c_1(\mathcal{F}))d^{n-1}/(n-1)!$. By reason of rank, the torsion subsheaf $\text{tors}(\mathcal{F})$ has a Hilbert polynomial of degree at most $n-1$; so the Hilbert polynomial of the torsion-free quotient $\overline{\mathcal{F}}$ shows that $c_1(\mathcal{F}) \geq c_1(\overline{\mathcal{F}})$.

So now assume in addition that $\mathcal{F}$ is torsion-free. For a hyperplane $i : H \hookrightarrow \mathbb{P}^n$, there is an exact sequence

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_H \to 0,$$

and $c_1(\mathcal{F}) = c_1(\mathcal{F}_H)$. Since there exists a line L such that $\mathcal{F}_L$ is a vector bundle, the result $c_1 \geq 0$ follows from induction on n.

Next, it follows that P_2 has $c_1 + 1$ terms:

$$P_2 = \binom{d}{2} + \cdots + \binom{d - c_1}{2} = \frac{1}{2}(1 + c_1)d^2 - \frac{1}{2}(c_1 + 1)^2 d + b,$$

where $b = \frac{1}{6}(c_1^3 + 3c_1^2 + 2c_1)$. So

$$P_{\mathcal{E}} - P_3 - P_2 = (c_1^2 + 3c_1 + 2 - c_2)d + \overline{c} - 1 - b.$$

This leading coefficient must be non-negative for there to be a Gotzmann representation. Therefore we have shown:

Proposition 3.5. *Let $\mathcal{E}$ be a rank 2 globally generated coherent sheaf on $\mathbb{P}^3$. Then its first and second Chern classes satisfy the inequality*

$$c_2 \leq c_1^2 + 3c_1 + 2. \quad \square$$

This bound will be improved in the next chapter, and an example will be given there.

Chapter 4

Hilbert functions of modules with known rank and generating degrees

We saw in the last chapter that the Gotzmann regularity bound may not be achieved for globally generated coherent sheaves. In this chapter, the Macaulay and Gotzmann representations will be adjusted to account for the rank and generating degrees of a module. The representation for the Hilbert function is in a sense less granular than that given by Hulett in [23], but which uses the same number of repeated terms in all degrees. This allows generalizations of the theorems of Macaulay, Green, and Gotzmann. The generalized Gotzmann regularity bound is strict for a module generated in degree zero, achieved by the lex submodule with the same Hilbert polynomial, which was not necessarily so for the standard Gotzmann number.

These results can be used to embed the Quot scheme $\mathrm{Quot}^{P}_{\mathcal{O}_{\mathbb{P}^n}}$ into a smaller dimensional Grassmannian using the construction given in the last chapter. Additionally, I will show that any such rank $r \geq 1$ sheaf on $\mathbb{P}^n$ satisfies an inequality of its first 2 Chern classes, $c_2 \leq c_1^2$, which improves the bound given in the last chapter; and this new bound is achieved for any

$r \geq 1$, and $n \geq 2$, and any $c_1 \geq 0$.

4.1 Macaulay and Gotzmann representations adjusted for rank and degree

For this section, take $F = Se_1 + \ldots + Se_m$, with $\deg(e_i) = f_i$ for $i = 1, \ldots, m$, satisfying $f_1 \leq \cdots \leq f_m$, and M a quotient module of F of rank r.

The following proposition establishes a Macaulay representation of a Hilbert function adjusted by the rank and generating degrees of a module.

Proposition 4.1. *We can write the Hilbert function of M as*

$$H(M, d) = \sum_{i=m-r+1}^{m} \binom{d - f_i + n}{n} + \rho_d,$$

with $0 \leq \rho_d \leq \sum_{i=1}^{m-r} \binom{d-f_i+n}{n}$ for each d.

Proof. By Theorem 2.1, there exists a lex submodule $L \subset F$ such that $H(M, d) = H(F/L, d)$. Note that the rank of F/L is the same as the rank of M. It follows that

$$L = \bigoplus_{i=1}^{m-r} I_i e_i$$

for some monomial ideals $I_1, \ldots, I_r$ of S. The existence of the representation is immediate. □

Next, we establish an adjusted Gotzmann representation of a Hilbert polynomial in two ways; the second will be a coarser version of the first.

Proposition 4.2. *We can write the Hilbert polynomial of M as follows:*

1. *If $f_{m-r} \le 0$, then the Hilbert polynomial of M can be written*

$$P_M(d) = \sum_{i=m-r+1}^{m} \binom{d - f_i + n}{n} + Q_M(d),$$

with $Q_M(d)$ having a unique Gotzmann representation.

2. *If $f_m \le 0$, then the Hilbert polynomial of M can also be written*

$$P_M(d) = r\binom{d+n}{n} + Q'_M(d),$$

with $Q'_M(d)$ having a unique Gotzmann representation.

Proof. Again writing

$$F/L = S(-f_1)/I_1 \oplus \cdots \oplus S(-f_{m-r})/I_{m-r} \oplus \bigoplus_{i=m-r+1}^{m} S(-f_i)$$

with $P_M(d) = P_{F/L}(d)$, apply Proposition 3.1 to

$$S(-f_{r+1})/I_{r+1} \oplus \cdots \oplus S(-f_m)/I_m$$

to give the first result. For the second result, since $f_1, \ldots, f_m \le 0$, we can repeatedly apply the formula

$$\binom{d+n+1}{n} = 1 + \binom{d+1}{1} + \binom{d+2}{2} + \cdots + \binom{d+n}{n}$$

from the proof of Lemma 3.3 to the binomial coefficients

$$\bigoplus_{i=m-r+1}^{m} S(-f_i)$$

until we are left with

$$P_M(d) = r\binom{d+n}{n} + Q'_M(d),$$

where $Q'_M(d)$ is the sum of $Q_M(d)$ and the remaining binomial coefficients from the previous step. Apply Lemma 3.2 to the terms in $Q'_M(d)$ to give a Gotzmann representation. □

Definition 4.1. *The first representation in the last proposition will be called a* rank-and-degree adjusted Gotzmann representation, *and the second a* rank-adjusted Gotzmann representation.

Remark 4.1. *If $M = S/I$ is a quotient of S, then $r = 1$ and $m = f_1 = 0$, and in this case both representations specialize to the Gotzmann representation for the Hilbert polynomial of S/I. If the last r generating degrees of F are all zero, then the rank-and-degree adjusted Gotzmann representation and the rank-adjusted Gotzmann representation are the same.*

Also, this allows a rank-and-degree adjusted Gotzmann representation for a module that does not correspond to a globally generated coherent sheaf, as long as the rank of the module is at least as large as the number of generators of positive degree.

Example 4.1. *Set $S = \mathbb{C}[x, y, z]$ and $F = Se_1 + Se_2 + Se_3$, where the generating degrees of F are $f_1 = f_2 = -1, f_3 = 0$. Assume $M = F/Se_3$. The Hilbert polynomial of M is $P_M(d) = 2\binom{d+3}{2}$. The rank-and-degree adjusted Gotzmann representation is*

$$P_M(d) = \binom{d+3}{2} + \binom{d+2}{2} + \binom{d+1}{d} + 1,$$

and the rank-adjusted Gotzmann representation is

$$P_M(d) = 2\binom{d+2}{2} + \binom{d+1}{d} + \binom{d}{d-1} + 3.$$

The standard Gotzmann representation is

$$P_M(d) = \binom{d+2}{d} + \binom{d+1}{d-1} + \binom{d-1}{d-2} + \binom{d-2}{d-3} + \binom{d-3}{d-4} + 11.$$

Note that $Q_M(d)$ has Gotzmann number 2, and $Q'_M(d)$ has Gotzmann number 5. The standard Gotzmann number is 16.

4.2 Strict bounds on first and second Chern classes

As a first application, we will refine the inequality on Chern classes of a rank $r = 2$ globally generated coherent sheaf on $\mathbb{P}^3$ given in the last chapter. This bound will be strict, and applies to all positive ranks r and all dimensions n of $\mathbb{P}^n$.

Theorem 4.1. *Let E be a rank $r \geq 1$ globally generated coherent sheaf on $\mathbb{P}^n$ for an integer $n \geq 0$. Then the first and second Chern classes of E satisfy the inequality*

$$c_2 \leq c_1^2.$$

Proof. If $n \leq 1$, then $c_2 = 0$, and $c_1 \geq 0$ for any such globally generated sheaf on $\mathbb{P}^n$, so assume that $n \geq 2$. We will only use the top 3 terms of the Hilbert polynomial, so it is sufficient to write the Todd class and Chern character as

$$Td(\mathbb{P}^n) = 1 + \frac{n+1}{2}H + \frac{(n+1)(3n+2)}{24}H^2 + \cdots$$

where H is the class of a hyperplane, and

$$ch(E(d)) = \sum_{i=0}^{m} (-1)^i ch(F_i(d)),$$

where

$$0 \to F_m \to \cdots \to F_0 \to E$$

is a resolution of E by locally free sheaves, and

$$\begin{aligned} ch(F_i(d)) =& \frac{(\alpha_{i,1}+d)^n + \cdots + (\alpha_{i,r_i}+d)^n}{n!} H^n \\ &+ \frac{(\alpha_{i,1}+d)^{n-1} + \cdots + (\alpha_{i,r_i}+d)^{n-1}}{(n-1)!} H^{n-1} \\ &+ \frac{(\alpha_{i,1}+d)^{n-2} + \cdots + (\alpha_{i,r_i}+d)^{n-2}}{(n-2)!} H^{n-2} + \cdots \end{aligned}$$

Now, we compute the top 3 terms of the Hilbert polynomial using Hirzebruch-Riemann-Roch:

$$\begin{aligned} P_E(d) = r\frac{d^n}{n!} &+ \left(\frac{r(n+1)}{2} + c_1\right)\frac{d^{n-1}}{(n-1)!} \\ &+ \left(\frac{c_1^2 - 2c_2 + (n+1)c_1}{2} + \frac{r(n+1)(3n+2)}{24}\right)\frac{d^{n-2}}{(n-2)!} + \cdots, \end{aligned}$$

where $c_i = c_i(E)$.

Write the rank-adjusted Gotzmann representation of P_E as

$$P_E(d) = P_n + P_{n-1} + \cdots + P_0,$$

where P_i is the sum of binomial coefficients in the Gotzmann representation of degree i. Note that

$$\begin{aligned} P_n &= r\binom{d+n}{n} \\ &= r\frac{d^n}{n!} + \frac{r(n+1)}{2}\frac{d^{n-1}}{(n-1)!} + \frac{r(n+1)(3n+2)}{24}\frac{d^{n-2}}{(n-2)!} + \cdots, \end{aligned}$$

where the third term is determined by

$$\sum_{1\le i<j\le n} ij = \frac{(n-1)n(n+1)(3n+2)}{24}$$

using a straightforward induction argument.

Then

$$P_E - P_n = c_1\frac{d^{n-1}}{(n-1)!} + \left(\frac{c_1^2 - 2c_2 + (n+1)c_1}{2}\right)\frac{d^{n-2}}{(n-2)!} + \cdots$$

If $c_1 = 0$, then $P_E - P_n = -c_2\frac{d^{n-2}}{(n-2)!} + \cdots$, which requires $c_2 \le 0$ for there to be a Gotzmann representation; otherwise,

$$\begin{aligned} P_{n-1} &= \binom{d+n-1}{n-1} + \binom{d+n-2}{n-1} + \cdots + \binom{d+n-c_1}{n-1} \\ &= c_1\frac{d^{n-1}}{(n-1)!} + \left(\frac{(n+1)c_1 - c_1^2}{2}\right)\frac{d^{n-2}}{(n-2)!} + \cdots \end{aligned}$$

So we have

$$P_E - P_n - P_{n-1} = (c_1^2 - c_2)\frac{d^{n-2}}{(n-2)!} + \cdots ,$$

with a non-negative leading coefficient for there to be a rank-adjusted Gotzmann representation, therefore $c_2 \le c_1^2$. □

Example 4.2. *Let $S = \mathbb{C}[x_0, \ldots, x_n]$, with $n \ge 2$, let $a > 0$ be an integer, and take*

$$N = (x_0^a, x_1^a)S,$$

and let $\mathcal{E}$ be the sheaf associated to N. Then $\mathcal{E}(a)$ is globally generated of rank 1, and from *the resolution*

$$0 \to S(-a) \to S^2 \to N(a) \to 0$$

it follows that $\mathcal{E}(a)$ *has Chern polynomial*

$$c_t(\mathcal{E}(a)) = \frac{1}{1-at} = 1 + at + a^2t^2 + \cdots ,$$

with $c_1 = a$ *and* $c_2 = a^2$ *achieving the strict bound in the theorem. For* $r > 1$*, take a direct sum of* $\mathcal{E}$ *with* $\mathcal{O}^{r-1}$.

Bounds can also be computed for higher Chern classes. For example, consider rank-2 vector bundles on $\mathbb{P}^3$, discussed in section 3.5, with Hilbert polynomial in equation (3.8).

From the work above, we consider two cases:

Case 1: $c_1 = 0$: Then

$$P_{\mathcal{E}} - P_3 = -c_2 d - 2c_2 + c_3/2.$$

Next compute

$$\begin{aligned} P_1 &= (d+1) + d + \cdots + (d - (-c_2 - 2)) \\ &= -c_2 d + 1 - \frac{(-c_2-2)(-c_2-1)}{2} \\ &= -c_2 d - \frac{c_2^2 + 3c_2}{2} \end{aligned}$$

so

$$P_{\mathcal{E}} - P_3 - P_1 = \frac{c_2^2 - c_2 + c_3}{2} \geq 0.$$

Case 2: $c_1 > 0$: Then the top 2 terms of P_2 come from above and the constant term is

$$c_1^3/6 - c_1^2 + 11c_1/6,$$

so

$$P_{\mathcal{E}} - P_3 - P_2 = (c_1^2 - c_2)d + 2c_1^2 - c_1c_2/2 - 2c_2 + c_3/2.$$

Next compute

$$\begin{aligned} P_1 &= (d + 1 - c_1) + (d - c_1) + \cdots + (d - c_1 - (c_1^2 - c_2 - 2)) \\ &= (c_1^2 - c_2)d - c_1^4/2 - c_1^3 + c_1^2c_2 + 3c_1^2/2 - 3c_2/2 + c_1c_2 - c_2^2/2. \end{aligned}$$

so

$$P_{\mathcal{E}} - P_3 - P_2 - P_1 = \frac{c_1^4 + 2c_1^3 + c_1^2 - 2c_1^2c_2 - 3c_1c_2 + c_2^2 - c_2 + c_3}{2} \geq 0.$$

Thus in general we have

$$c_3 \geq -c_1^4 - 2c_1^3 - c_1^2 + 2c_1^2c_2 + 3c_1c_2 - c_2^2 + c_2.$$

When the first and second Chern classes achieve the equality $c_2 = c_1^2$ as in the last example, we have $c_3 \geq c_1^3$.

4.3 Rank and degree-adjusted Macaulay, Green, and Gotzmann Regularity

We would like to extend Macaulay's theorem, Green's Theorem, and Gotzmann's Regularity and Persistence Theorems to these representations. I will prove persistence in the next section; the others are proven in this section. As before, assume $F = Se_1 + \cdots + Se_m$ with $\deg(e_i) = f_i$ and $f_1 \le \cdots \le f_m$.

First we will require a lemma giving more properties of Green and Macaulay transformations.

Lemma 4.1. *Assume a, b and d are positive integers.*

1. $a_{\langle d\rangle} + b_{\langle d\rangle} \le (a+b)_{\langle d\rangle}$
2. $a^{\langle d\rangle} + b^{\langle d\rangle} \le (a+b)^{\langle d\rangle}$
3. $a_{\langle d+1\rangle} \le a_{\langle d\rangle}$
4. $a^{\langle d+1\rangle} \le a^{\langle d\rangle}$

Proof. See [12, Lemmas 4.4 and 4.5]. □

Proposition 4.3. *If N is a submodule of F such that $M = F/N$ has rank r, and Hilbert function*

$$H(M,d) = \sum_{i=m-r+1}^{m} \binom{d-f_i+n}{n} + \rho_d.$$

1. *For all $d \ge f_{m-r}+1$,*

$$H(M,d+1) \le \sum_{i=m-r+1}^{m} \binom{d+1-f_i+n}{n} + \rho_d^{\langle d-f_{m-r}\rangle}.$$

2. *For a general element $h \in S_1$, writing $F' = F/hF$, $M' = F/(N + hF)$ and $S' = S/hS$, we have*

$$H(M', d) \leq \sum_{i=m-r+1}^{m} \binom{d - f_i + n - 1}{n - 1} + (\rho_d)_{\langle d - f_{m-r} \rangle}$$

for all $d \geq f_{m-r} + 1$, where $H(M', d) = \dim_{S'}(M'_d)$.

Proof. As before, the lex module with the same Hilbert function can be written

$$L = \bigoplus_{i=1}^{m-r} I_i e_i$$

for some lex ideals $I_1, \ldots, I_r$ of S. Note that

$$\begin{aligned} H(M, d) &= H(F/L, d) \\ &= \sum_{i=m-r+1}^{m} \binom{d - f_i + n}{n} + H(S/I_1, d - f_1) + \cdots H(S/I_{m-r}, d - f_{m-r}). \end{aligned}$$

Then for $d \geq f_{m-r} + 1$,

$$\begin{aligned} H(M, d+1) &= H(F/L, d+1) \\ &\leq \sum_{i=m-r+1}^{m} \binom{d + 1 - f_i + n}{n} \\ &\quad + H(S/I_1, d - f_1)^{\langle d - f_1 \rangle} + \cdots H(S/I_{m-r}, d - f_{m-r})^{\langle d - f_{m-r} \rangle} \\ &\leq \sum_{i=m-r+1}^{m} \binom{d + 1 - f_i + n}{n} \\ &\quad + H(S/I_1, d - f_1)^{\langle d - f_{m-r} \rangle} + \cdots H(S/I_{m-r}, d - f_{m-r})^{\langle d - f_{m-r} \rangle} \\ &\leq \sum_{i=m-r+1}^{m} \binom{d + 1 - f_i + n}{n} + \rho_d^{\langle d - f_{m-r} \rangle} \end{aligned}$$

where the first inequality is by Macaulay's Theorem, the second and third inequalities are by Lemma 4.1.

For the second statement, write I_i' and for the image of I_i in S' and L' for the image of L in F'. By [15, Theorem 3.18], we have $H(M',d) \leq H(F'/L',d)$, and $H(S'/I_i',d) = H(S/I,d)_{\langle d \rangle}$ for each lex ideal I_i and for all $d > 0$ by [25, Proposition 5.5.23]. Then for $d \geq f_{m-r}+1$, with the last two inequalities by Lemma 4.1,

$$\begin{aligned}
H(M',d) &\leq H(F'/L',d) \\
&= \sum_{i=m-r+1}^{m} \binom{d-f_i+n-1}{n-1} \\
&\quad + H(S/I_1, d-f_1)_{\langle d-f_1 \rangle} + \cdots H(S/I_{m-r}, d-f_{m-r})_{\langle d-f_{m-r} \rangle} \\
&\leq \sum_{i=m-r+1}^{m} \binom{d-f_i+n-1}{n-1} \\
&\quad + H(S/I_1, d-f_1)_{\langle d-f_{m-r} \rangle} + \cdots H(S/I_{m-r}, d-f_{m-r})_{\langle d-f_{m-r} \rangle} \\
&\leq \sum_{i=m-r+1}^{m} \binom{d-f_i+n-1}{n-1} + (\rho_d)_{\langle d-f_{m-r} \rangle}. \quad \square
\end{aligned}$$

Next, the rank and degree-adjusted form of Gotzmann regularity is proved.

Theorem 4.2. *Assume $F = Se_1 + \cdots + Se_m$ with $\deg(e_i) = f_i$ and $f_1 \leq \cdots \leq f_m$, such that $f_{m-r} \leq 0$, and N is a submodule of F such that $M = F/N$ has rank r, and Hilbert polynomial*

$$P_M(d) = \sum_{i=m-r+1}^{m} \binom{d-f_i+n}{n} + Q(d)$$

where $Q_M(d)$ has Gotzmann number s. Then the saturation of N is $\max(s, f_m)$-regular. Furthermore, if $f_{m-r} = 0$ and $s \geq f_m$, then this bound is achieved by the lex submodule.

Proof. The proof of regularity is essentially the same as the previous proof in the last chapter, by induction on $\dim(S) - 1$, and will be sketched. From a saturated module N with $M =$

F/N, we obtain $M' = M/hM(-1)$ which still has rank r since h is M-regular, and satisfies the hypotheses of Proposition 4.2 (1), so the uniqueness of the Gotzmann representation for $Q_{M'}(d)$ gives

$$P_{M'}(d) = \sum_{i=m-r+1}^{m} \binom{d-f_i+n-1}{n-1} + \binom{d+a_1-1}{a_1-1} + \cdots + \binom{d+a_t-(t-1)}{a_t-1}$$

for some $t \leq s$, where

$$Q_M(d) = \binom{d+a_1}{a_1} + \cdots + \binom{d+a_s-(a_s-1)}{a_s}$$

is the Gotzmann representation of $Q_M(d)$. The same concluding argument as in the last chapter, using Proposition 4.3 in place of Theorem 3.1, shows that M is $(s-1)$-regular, and since F is f_m-regular, by Lemma 1.1 (2), N is $\max(s, f_m)$-regular.

To show strictness when $f_{m-r} = 0$ and $s \geq f_m$, consider a given Hilbert polynomial

$$P(d) = \sum_{i=m-r+1}^{m} \binom{d-f_i+n}{n} + Q(d)$$

with $Q(d)$ having Gotzmann number s. The saturated lex module L with F/L having the same Hilbert polynomial is

$$L = S(-f_1) \oplus \cdots \oplus S(-f_{m-r-1}) \oplus L_{m-r}.$$

for a lex ideal L_{m-r} with $P_{S/L_{m-r}}(d) = Q(d)$.

Since $f_{m-r} = 0$ and $\text{reg}(L_{m-r}) = s$, it follows that $\text{reg}(L) = s$. $\square$

Remark 4.2. *In particular, rank-adjusted Gotzmann regularity is strict for modules generated in degree zero, just as standard Gotzmann regularity is strict for ideals. This is in contrast to example 3.2, where no submodule achieves the regularity given by the standard*

Gotzmann number.

The following proposition establishes an ordering on the different Gotzmann numbers that can be seen in example 4.1.

Proposition 4.4. *Assume P is the Hilbert polynomial of a quotient module M with rank r, and generating degrees $f_1, \dots, f_m$, such that $f_1 \leq \dots \leq f_m \leq 0$. Write G_{rd} for the rank-and-degree adjusted Gotzmann number (that is, the Gotzmann number of $Q_M(d)$ from Proposition 4.2 (1)), G_r for the rank-adjusted Gotzmann number (the Gotzmann number of $Q'_M(d)$ from Proposition 4.2 (2)), and G for the Gotzmann number (from chapter 3). Then*

$$G_{rd} \leq G_r \leq G.$$

Proof. We will construct saturated modules with the same Hilbert polynomial that have known regularity. Write $F_1 = Se_1 + \dots + Se_{r+1}$ with degrees $\deg(e_1) = f_{m-r+1}, \dots, \deg(e_r) = f_m, \deg(e_{r+1}) = 0$, and $F_2 = Se_{r+2} + \cdots Se_{2r+2}$, with all degrees zero. There exists a saturated lex ideal I_1 of S with Hilbert polynomial $P(S/I_1) = Q_M$; set $L_1 = I_1 e_{r+1}$, and note that $P(F_1/L_1) = P$. There is also a saturated lex submodule $L_2 = I_2 e_{2r+2}$ of F_2 with Hilbert polynomial $P(F_2/L_2) = P$ (so that $P(S/I_{r+2}e_{r+2}) = Q'_M$). We have $\operatorname{reg}(L_1) = G_{rd}$; similarly, $\operatorname{reg}(L_2) = G_r$. By Proposition 3.2, L_2 is G-regular, so $G_r \leq G$. Now consider the saturated submodule $L_1 + F_2$ of $F_1 + F_2$. We still have $\operatorname{reg}(L_1 + F_2) = G_{rd}$ by Lemma 1.1 (1), and $L_1 + F_2$ is G_r-regular by Theorem 4.2, so $G_{rd} \leq G_r$. □

4.4 Rank and degree-adjusted Gotzmann Persistence

It remains to prove rank-and-degree adjusted Gotzmann Persistence. This is not a straightforward application of [12], because the deformation to a monomial module does not a priori preserve the hypotheses. Instead, we will prove this similar to the proof for ideals given in [13].

Theorem 4.3. *Assume (in addition to previous assumptions on F, N, and $M = F/N$) that N is generated in degree at most d and $d \geq f_{m-r} + 1$. If*

$$H(M, d+1) = \sum_{i=m-r+1}^{m} \binom{d+1-f_i+n}{n} + \rho_d^{\langle d-f_{m-r} \rangle},$$

then

$$H(M, d+2) = \sum_{i=m-r+1}^{m} \binom{d+2-f_i+n}{n} + \rho_{d+1}^{\langle d-f_{m-r}+1 \rangle}.$$

The proof follows from extensions of some other results to modules, which will be given in the following lemmas.

Definition 4.2. *Suppose the module M has Betti numbers $\beta_{i,j}$. A* consecutive cancellation *is the process of choosing i, j such that $\beta_{i,j}$ and $\beta_{i+1,j}$ are positive, and replacing $\beta_{i,j}, \beta_{i+1,j}$ with $\beta_{i,j} - 1, \beta_{i+1,j} - 1$ respectively.*

The following Lemma is the extension of [30, Theorem 1.1] to modules.

Lemma 4.2. *If N is a submodule of F and $M = F/N$, then the graded Betti numbers of F/N are obtained from those of $F/\mathrm{in}(N)$ by consecutive cancellations.*

Proof. By Proposition 2.2, there exists a weight vector $(w_0, \ldots w_n, v_1, \ldots, v_m)$ with positive entries such that $x^{\alpha_1}e_i > x^{\alpha_2}e_j$ if and only if $\alpha_{1,0}w_0 + \cdots + \alpha_{1,n}w_n + f_iv_i > \alpha_{2,0}w_0 + \cdots + \alpha_{2,n}w_n + f_jv_j$. Let $\tilde{N}$ be the homogenization of N as a $\tilde{S} = S[t]$-module with respect to the grading $\deg(x_i) = w_i$ and $\deg(t) = 1$ on $\tilde{S}$, and extending to $\tilde{F} = \oplus\tilde{S}(-f_j)$. Note that every generator of $\tilde{N}$ comes from an element in N with every term other than the initial term multiplied by a positive power of t. Also t and $t-1$ are $\tilde{F}/\tilde{N}$-regular.

Write $\mathbf{F}$ for a minimal graded free resolution of $\tilde{F}/\tilde{N}$. By [9, Theorem 15.17], $\mathbf{F} \otimes \tilde{S}/(t)$ is a minimal free graded resolution of $\tilde{F}/\tilde{N} \otimes \tilde{S}/(t) \cong F/\text{in}(N)$, and $\mathbf{F} \otimes \tilde{S}/(t-1)$ is a (not necessarily minimal) graded free resolution of F/N. Thus, we can remove a trivial complex from $\mathbf{F} \otimes \tilde{S}/(t-1)$ to get a minimal free resolution of F/N, resulting in consecutive cancellation of the Betti numbers from $F/\text{in}(N)$ to F/N. $\square$

Example 4.3. *In example 1.2, we saw* $I = (x^2 - y^2, xy - z^2)$ *and* $\text{in}(I) = (x^2, xy, y^3)$ *in* $S = \mathbb{C}[x, y, z, w]$, *with the graded Betti numbers of* S/I

$$\beta_{0,0} = 1,\ \beta_{1,2} = 2,\ \beta_{2,4} = 1$$

and the graded Betti numbers of $S/\text{in}(I)$

$$\beta_{0,0} = 1,\ \beta_{1,2} = 2,\ \beta_{1,3} = 1,\ \beta_{2,3} = 1,\ \beta_{2,4} = 1$$

differing by the consecutive cancellation of $\beta_{1,3}$ *and* $\beta_{2,3}$.

The following lemma uses the construction from [29] given in section 2.1.

Lemma 4.3. *If* N *is a submodule of* F *and* L *is the lex submodule of* F *such that the Hilbert functions of* F/N *and* F/L *agree, then* N *can be deformed to* L *using a finite sequence of the following operations:*

1. *general change of coordinates*
2. *taking the initial module*
3. *the operation* $\sigma'_L(N^{(p)})$ *from Proposition 2.3.*

Proof. See [29, Proposition 30]. $\square$

Proof. (of Theorem 4.3): Given N, we deform to the the lex submodule L using the operations above. Clearly the first operation preserves Betti numbers; by Proposition 2.3, the third operation preserves Betti numbers also. By Lemma 4.2, the second operation can only increase Betti numbers in such a way that the Betti numbers of F/N can be recovered from those of F/L by consecutive cancellations.

By the assumption on $H(F/L, d)$ and $H(F/L, d+1)$, it follows that L has no generators in degree $d+1$. Assume that

$$H(M, d+2) < \sum_{i=m-r+1}^{m} \binom{d+2-f_i+n}{n} + \rho_{d+1}^{\langle d-f_{m-r}+1\rangle}.$$

Then L has a generator in degree $d+2$. But N is generated in degrees at most d. This must be explained by consecutive cancellation with a non-zero Betti number $\beta_{1,d+2}$ of L.

Since F/L can be written $\oplus_m S/L_i(-f_i)$ and lex ideals have linear resolutions by Proposition 2.1, it follows that the non-zero first syzygy of degree $d+2$ must result from a generator of degree $d+1$, which is a contradiction. □

Since we have rank-adjusted regularity and persistence, we can use the rank-adjusted Gotzmann number to determine the Grassmannian into which to embed the Quot scheme from section 3.3, $\mathrm{Quot}^P(\mathcal{O}^r_{\mathbb{P}^n})$.

Example 4.4. *In section 3.4, we saw that the Hilbert polynomial $P(d) = k(d+1) + m$ has Gotzmann number*

$$s = \frac{k(k+1)}{2} + m;$$

however, when this is being used for $\mathrm{Quot}^P(\mathcal{O}^r_{\mathbb{P}_1})$, we can use the rank-adjusted Gotzmann number of m. This agrees with the lower number used to embed the Quot scheme $\mathrm{Quot}^P(\mathcal{O}^3_{\mathbb{P}^1})$ when $P(d) = 2(d+1)$, as given in example 3.2.

The next example shows the improvement in the regularity bound for a rank-k sheaf on $\mathbb{P}^2$ with Hilbert polynomial

$$P(d) = k\binom{d+2}{2} + m_1(d+1) + m_2.$$

The standard Gotzmann number is

$$s = \frac{1}{24}k(k+1)(3k^2 - k + 10 + 12m_1) + \frac{1}{2}m_1(m_1+1) + m_2,$$

and the rank-adjusted Gotzmann number is

$$s = \frac{1}{2}m_1(m_1+1) + m_2.$$

4.5 Future work

In this section, I will mention two opportunities for future work. The first opportunity is to consider Hilbert and Quot schemes by fixing a sheaf $\mathcal{F}$ other than $\mathcal{O}^r_{\mathbb{P}^n}$ from which to take quotients. After an appropriate twist, $\mathcal{F}$ can be embedded into some $\mathcal{O}^r_{\mathbb{P}^n}$, so we have a subscheme of the Grassmannian constructed herein. However, it is possible that we can construct an embedding into a smaller-dimensional Grassmannian. The question is: to what extent can we generalize the theorems of Macaulay, Green, and Gotzmann in rings other than the polynomial ring? For questions about regularity, it is natural to restrict attention to Koszul k-algebras, since these are the rings R for which all graded R-modules have finite regularity [1] (with respect to minimal free graded R-resolutions). McCullough and Peeva say that it is an interesting open problem to find analogues over Koszul rings of conjectures or results on regularity over a polynomial ring, and Conca, de Negri and Rossi pose some specific questions in this vein in [8, Section 5]. Another important generalization is to explicitly construct the Quot scheme of a trivial bundle over a smooth projective curve, which is studied for its intersection theory.

The second opportunity for future work is to apply these results to classes of finitely generated modules such as the cohomology modules of a vector bundle. For example, the conjecture of Hartshorne on the indecomposability of a rank-2 vector bundle on $\mathbb{P}^5$ [19] has been investigated in spacial cases by explicit methods, such as the consideration of globally generated vector bundles with small first Chern class [7]. It would be interesting to consider growth conditions on the cohomology modules to limit the Hilbert functions that such modules can take.

Bibliography

[1] L. L. Avramov and I. Peeva. Finite regularity and Koszul algebras. *Amer. J. Math.*, 123(2):275–281, 2001.

[2] D. A. Bayer. *The division algorithm and the Hilbert scheme*. ProQuest LLC, Ann Arbor, MI, 1982. Thesis (Ph.D.)–Harvard University.

[3] A. M. Bigatti. Upper bounds for the Betti numbers of a given Hilbert function. *Communications in Algebra*, 21(7):2317–2334, 1993.

[4] A. Borel and J.-P. Serre. Le théorème de Riemann-Roch. *Bull. Soc. Math. France*, 86:97–136, 1958.

[5] W. Bruns and J. Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[6] M. Chardin. Some results and questions on Castelnuovo-Mumford regularity. In *Syzygies and Hilbert functions*, volume 254 of *Lect. Notes Pure Appl. Math.*, pages 1–40. Chapman & Hall/CRC, Boca Raton, FL, 2007.

[7] L. Chiodera and P. Ellia. Rank two globally generated vector bundles with $c_1 \leq 5$. *Rend. Istit. Mat. Univ. Trieste*, 44:1–10, 2012.

[8] A. Conca, E. De Negri, and M. E. Rossi. Koszul algebras and regularity. In *Commutative algebra*, pages 285–315. Springer, New York, 2013.

[9] D. Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[10] D. Eisenbud. *The geometry of syzygies*, volume 229 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005. A second course in commutative algebra and algebraic geometry.

[11] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[12] V. Gasharov. Extremal properties of Hilbert functions. *Illinois J. Math.*, 41(4):612–629, 1997.

[13] V. Gasharov, S. Murai, and I. Peeva. Hilbert schemes and maximal Betti numbers over Veronese rings. *Math. Z.*, 267(1-2):155–172, 2011.

[14] G. Gotzmann. Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes. *Math. Z.*, 158(1):61–70, 1978.

[15] O. Greco. Green's Hyperplane Restriction Theorem: an extension to modules. *J. Pure Appl. Algebra*, 219(8):3506–3517, 2015.

[16] M. Green. Restrictions of linear series to hyperplanes, and some results of Macaulay and Gotzmann. In *Algebraic curves and projective geometry (Trento, 1988)*, volume 1389 of *Lecture Notes in Math.*, pages 76–86. Springer, Berlin, 1989.

[17] A. Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. II. *Inst. Hautes Études Sci. Publ. Math.*, (17):91, 1963.

[18] R. Hartshorne. *Algebraic geometry*, volume 52. Springer Verlag, 1977.

[19] R. Hartshorne. Algebraic vector bundles on projective spaces: a problem list. *Topology*, 18(2):117–128, 1979.

[20] R. Hartshorne. Stable reflexive sheaves. *Math. Ann.*, 254(2):121–176, 1980.

[21] D. Hilbert. Über die Theorie der algebraischen Formen. *Mathematische Annalen*, 36(4):473–534, 1890.

[22] H. A. Hulett. Maximum Betti numbers of homogeneous ideals with a given Hilbert function. *Communications in Algebra*, 21(7):2335–2350, 1993.

[23] H. A. Hulett. A generalization of Macaulay's theorem. *Communications in Algebra*, 23(4):1249–1263, 1995.

[24] A. Iarrobino and S. Kleiman. The Gotzmann theorems and the Hilbert scheme. In *A. Iarrobino and V Kanev,Power Sums, Gorenstein Algebras, and Determinantal Loci*, pages 289–312. Springer-Verlag, Berlin, 1999.

[25] M. Kreuzer and L. Robbiano. *Computational commutative algebra. 2.* Springer-Verlag, Berlin, 2005.

[26] F. S. Macaulay. Some Properties of Enumeration in the Theory of Modular Systems. *Proc. London Math. Soc.*, 26(1):531–555, 1927.

[27] D. Mumford. *Lectures on curves on an algebraic surface.* With a section by G. M. Bergman. Annals of Mathematics Studies, No. 59. Princeton University Press, Princeton, N.J., 1966.

[28] N. Nitsure. Construction of Hilbert and Quot schemes. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 105–137. Amer. Math. Soc., Providence, RI, 2005.

[29] K. Pardue. Deformation classes of graded modules and maximal Betti numbers. *Illinois J. Math.*, 40(4):564–585, 1996.

[30] I. Peeva. Consecutive cancellations in Betti numbers. *Proc. Amer. Math. Soc.*, 132(12):3503–3507, 2004.

[31] I. Peeva. *Syzygies and Hilbert functions*. CRC Press, 2007.

[32] I. Peeva and M. Stillman. The minimal free resolution of a Borel ideal. *Expositiones Mathematicae*, 26(3):237–247, 2008.

[33] I. Peeva and M. Stillman. Open problems on syzygies and Hilbert functions. *J. Commut. Algebra*, 1(1):159–195, 2009.

[34] W. V. Vasconcelos. *Computational methods in commutative algebra and algebraic geometry*, volume 2 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, 1998. With chapters by David Eisenbud, Daniel R. Grayson, Jürgen Herzog and Michael Stillman.