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TWO-PLAYER ZERO-SUM HYBRID GAMES

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Abstract

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Optimizing cost functions under dynamic constraints has been widely studied for over 70 years, with applications across engineering, medicine, and biology. A key challenge arises when adversarial agents, designed to oppose the main control objective, are involved. This scenario is often modeled using differential games, where constraints are governed by differential equations. Dynamic constraints in modern applications that combine physics, computing, and networks often exhibit both continuous and discrete behavior, influenced by nonsmooth factors like intermittent information and resets of variables. These constraints are well-suited to hybrid system models, which combine continuous and discrete dynamics. However, designing algorithms that ensure optimality under these hybrid constraints requires new methods, as existing tools from differential games may lead to suboptimal solutions. This dissertation aims to address the lack of tools for designing algorithms for hybrid games with dynamic constraints, specifically beyond those modeled by finite-state automata or switched systems. First, we formulate a framework for the study of two-player zero-sum games under dynamic constraints given in terms of hybrid dynamical systems. We employ our framework to study games with different types of termination conditions. Analyzing the case in which solutions to the hybrid system are complete allows us to propose results on optimality and asymptotic stability for games over the infinite horizon with applications to security and disturbance rejection problems. By considering the more general case of games over a finite horizon, we employ existing tools in hybrid systems to design optimal strategies upon appropriate specifications of terminal hybrid time and terminal state sets. We study input-to-state stability and safety of hybrid systems under disturbances as inverse-optimal two-player zero-sum games. We propose QP-based controls in terms of Lyapunov and barrier functions to construct a meaningful cost functional that is minimized under the worst-case disturbance. For multi-stage hybrid games, an optimality analysis is proposed with applications to capture-the-flag games. Finally, imperfect state information motivates the study of optimal designs of control strategies together with state observers.

To King Jesus, my mother Olga, and my father Fernando.

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I would like to extend my heartfelt thanks to my coauthors, for their collaboration and contributions to the research presented in this dissertation. Working with each of you has been a rewarding experience, and I deeply appreciate your intellectual input, creativity, and dedication. The contribution by the candidate to all multiauthored publications is included as follows.

Chapter 3 builds upon the material from our work on infinite-horizon hybrid games. Sections 3.1 and 3.2, in full, are a reprint of the material as it appears in "Sufficient Conditions for Optimality and Asymptotic Stability in Two-Player Zero-Sum Hybrid Games." [1]. Sections 3.3 and 3.4 are partially a reprint of the material therein. The dissertation author was the first author of this paper.

Chapter 4 includes material from our work on finite-horizon hybrid games. Sections 4.1-4.3, in full, are a reprint of the material as it appears in "Sufficient Conditions for Optimality in Finite-Horizon Two-Player Zero-Sum Hybrid Games." [2]. The dissertation author was the first author of this paper.

Chapter 5 contains the unpublished material from our work on hybrid games with terminal state specifications. The dissertation author was the first author of this material.

Chapter 6 includes material from our work on hybrid games with set-valued dynamics. Sections 6.1 and 6.3-6.5 are a partial reprint of the material as it appears in "On the Optimal Cost and Asymptotic Stability in Two-Player Zero-Sum Set-Valued Hybrid Games." [3]. The dissertation author was the first author of this paper.

Chapters 7 and 8 containts unpublished material of our work on inverse-optimality approaches to solve zero-sum two-player hybrid games while endowing a system with input-to-state stability [4] and safety properties with respect to disturbances [5]. The dissertation author was on of the primary authors of this work.

Chapter 9 includes material from our work on capture-the-flag hybrid dynamical modeling and control. Sections 9.1, 9.2, and 9.5, in full, are a reprint of the material as it appears in "A Hybrid Systems Formulation for a Capture-the-Flag Game." [6]. The dissertation author was the first author of this paper.

Chapter 10 presents material from our work on observers and control design for hybrid games under imperfect information. Sections 10.1-10.6, excluding the proofs, are a reprint of the material as it appears in "An Observer-based Switching Algorithm for Safety under Sensor Denial-of-Service Attacks." [7]. The dissertation author was the first author of this paper.

In each of these chapters, I participatd as one of the leading authors in both research and writting. I want to thank to Francesco Ferrante for his contributions on the work on set-valued hybrid games. I am grateful with Philipp Braun for his great guidance and collaboration on the work on capture-the-flag games. I am also thanful with Kunal Garg and Alvaro Cardenas for their contribution and supervision in the work on control design for safe recovery under imperfect information. Special thanks go to Carlos Montenegro for playing a co-leading role in the work on inverse-opimality methods to solve two-player zero-sum hybrid games while endowing hybrid systems with input-to-state properties with respect to disturbances. His friendship, resilience, and research curiosity to explore novel approaches provided a great encouragement during the challenging times of this journey.

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Chapter 1

Introduction

Games involving multiple players with potentially different interests emerge in multi-agent systems, both in benign (or cooperative) and contested (or noncooperative) settings. A list of examples includes and is not limited to rout selection in a road network [8], heavy duty vehicle platooning [9], control of smart grids [10], trading modeling in the stock market [11], and control of large populations of systems [12]. Generally speaking, a game is an optimization problem with multiple decision makers (players), a set of constraints (potentially dynamic) that enforces the "rules" of the game, and a set of payoff functions to be optimized by selecting decision variables. Constraints on the state and decision variables formulated as dynamic relationships lead to *dynamic games*; see [13] and the references therein. Of particular interest is the contested setting, which occurs when the players have independent objectives, such as when one agent aims at minimizing a cost function and another agent aims at maximizing it under dynamic constraints. Dynamic noncooperative games focus on the case in which the players select their actions in a competitive manner, such that an individual benefit potentially implies a detriment to the other players [14–19]. This type of dynamic games has been thoroughly studied in the literature, when the dynamic constraints are given in terms of difference equations or differential equations – in general, referred to as differential games.

Interestingly, the combination of physics, computing, and networks leads to dynamic constraints that exhibit both continuous and discrete behavior. In particular, intermittent information availability, resets of variables, such as expiring timers, and other nonsmooth and instantaneous changes lead to dynamic constraints that can conveniently be captured using hybrid system models [20,21]. Under certain assumptions, differential algebraic equation (DAEs) [22] – also known as descriptor systems – can be recast as hybrid equations, see [23, Lemma 2]. In specific, when the initial condition to a DAE is *consistent* and the data pair of the system is *regular* (for each subsystem, in the case of switched DAEs [24]), a solution to the DAE is also a solution to the equivalent hybrid system defined as in [20]. However, when designing algorithms that make optimal choices of the decision variables under constraints given by hybrid dynamics, relying only on continuous-time or discrete-time approaches potentially results in suboptimal solutions. Unfortunately, tools for the design of algorithms for games with such hybrid dynamic constraints, which we refer to as *hybrid games*, are not fully developed.

Particular classes of dynamic games involving hybrid dynamic constraints have been recently studied in the literature. A game-theory-based control design approach is presented for timed automata in [25, 26], for hybrid finite-state automata in [27, 28], and for o-minimal hybrid systems in [29]. In these articles, the specifications to be guaranteed by the system are defined in terms of temporal logic formulae. When the payoff is defined in terms of a terminal cost, such approach allows designing reachability-based controllers through the satisfaction of Hamilton-Jacobi conditions to certify safety or obstace-avoidance of hybrid finite-state automata [30, 31]. Following an approach that allows for richer dynamics, [32] studies a class of reachability games between a controller and the environment, under constraints defined by hybrid automata (STORMED games) for which at each decision step, the players can choose either to have their variables evolve continuously or discretely, following predefined rules. For continuous-time systems with state resets, tools for the computation of the region of attraction for hybrid limit cycles under the presence of disturbances are provided in [33], where the inputs only affect the flow.

Efforts pertaining to differential games with impulsive elements include [34,35], where the interaction between the players is modeled similarly to switched systems, [36], which establishes continuity of bounds on value functions and viscosity solutions, [37], which formulates necessary and sufficient conditions for optimality in bimodal linear-quadratic differential games, [38], which studies conditional viability for impulsive systems with two competing input actions as an evolutionary game, and [39], which studies a class of stochastic two-player differential games in match race problems.

Motivated by the lack of tools for the design of algorithms for general hybrid games, we formulate a framework for the study of two-player zero-sum games with hybrid dynamic constraints. Specifically, we formulate an optimization problem with cost functional including

- a stage cost that penalizes the evolution of the state and the input during flow,
- a stage cost that penalizes the evolution of the state and the input at jumps, and
- a terminal cost to penalize the final value of the variables.

Following the framework in [20, 40], we model the hybrid dynamic constraints as a hybrid dynamical system, which allows to cover contested scenarios with continuous-time dynamics with logical modes, switching systems, hybrid automata, impulsive differential equations, and dynamics described by algebraic differential equations (DAEs).

Zero-sum games for DAEs have been studied in the literature [41]. A min-max principle built upon Pontryagin's Maximum Principle is provided in [42]. Linear dynamics and quadratic costs result in coupled Riccati differential equations, and conditions for their solvability are provided in [43] and [44]. In [45], noncooperative games for Markov switching DAEs are studied and Hamilton–Jacobi–Bellman-Isaacs equations are derived. When the initial condition to a switching DAE is consistent and the data pair of the system is regular for each subsystem, a deterministic version of the problem solved in [45] can be addressed employing the modeling framework in [20].

Several applications exhibit this type of games, including robust control problems and security-critical scenarios. On the one hand, a disturbance rejection problem for hybrid systems can be formulated as a zero-sum game in which a player selects a control input that minimizes a cost function in the presence of a disturbance assigned by an adversarial player. Solving such a problem as a zero-sum game can be too conservative, nevertheless, such approach can be employed only at the boundary of a set where critical properties are guaranteed, e.g., forward invariance of a set of interest. On the other hand, consider a security problem that consists of finding conditions such that the control input guarantees the performance of a hybrid system via minimizing a cost functional (representing damage or effect of attacks) under the action of an attacker that is designed to maximize it. A cost-free preliminary instance of this problem is studied in [7] following an attack-recovery approach [46], where the composition of a continuous-time dynamical system with a switched-observer-controller scheme results in a hybrid closed-loop system, for which conditions are provided to guarantee safety under Denial-of-Service attacks.

The solution to the type of games formulated in this work, known as a *saddle-point equilibrium*, is given in terms of the actions of the players. Informally, when a player unilaterally deviates from the equilibrium action, it does not improve its individual outcome. Thus, by formulating the applications above as two-player zero-sum hybrid games, we can synthesize the saddle-point equilibrium and determine the control action that minimizes the cost for the maximizing adversarial action. To the best of our knowledge, there are no results in the literature that can be used to solve two-player zero-sum games with hybrid dynamics modeled as in [20].

In recent works, optimality for hybrid systems modeled as in [20] is certified via Lyapunov-like conditions [47], providing cost evaluation results for the case in which the data is given in terms of set-valued maps. The work in [48] provides sufficient conditions to guarantee the existence of optimal solutions. A receding-horizon algorithm to implement these ideas is presented in [49]. Scenarios with terminal specifications are studied, favoring the development of results over both the infinite and the finite horizon. Cost evaluation results and conditions to guarantee asymptotic stability of a set of interest are established for a discrete-time system under adversarial scenarios in [50]. A fixed finite-horizon hybrid game is studied in [2]. The conditions on the optimization problem formulated therein are similar to their counterparts in the differential/dynamic game theory literature. The end of the game therein is attained when the time of solutions to $\mathcal{H}$ reach a terminal set $\mathcal{T}$. To account for hybrid time domains, which are introduced in Chapter 2, a hybrid time domain-like geometry is assumed for $\mathcal{T}$ as in [49]. This results in optimality conditions in terms of PDEs, and the optimal feedback laws are not stationary. The work in [51] provides a preliminary step towards a learning-based approach to certify cost evaluation and asymptotic stability for hybrid games. The method employs neural networks to learn a Lyapunov function and a value-like function to guarantee the extension of pointwise conditions from finitely many points to a entire set of interest. In [3], a two-player zero-sum game under dynamic constraints is formulated in terms of a hybrid inclusion, which requires an appropriate definition of the cost function to characterize the solution to the game. A hybrid system model for capture-the-flag games with a corresponding zero-sum game formulation is proposed in [6].

Given that the computation of the optimal strategy and the optimal cost via solving Hamilton-Jacobi-Bellman-Isaacs represents a nontrivial task, an inverse optimality approach has been employed in the literature, in combination with input-to-state notions. Input-to-state stability notions trace back to [52, 53] for continuous-time systems, [54] for discrete-time systems, and [55, 56] for hybrid systems modeled as in [57], [58] for impulsive systems, and [59] for switched systems. In addition, input-to-state safety notions trace back to [60] and [61] for continuous-time systems, and [62] considers the case of compositional input-to-state safety for nonlinear systems given as an interconnection of subsystems. Inverse optimal design of stabilizing controllers for nonlinear continuous-time systems with disturbances was studied in [63, 64] with a two-player zero-sum formulation of a differential game. Similarly, the inverse optimal design of CBF-based safety filters for continuous-time systems with disturbances was studied in [65]. The inverse-optimal design of control strategies under disturbances as a two-player zero-sum game is studied in [4] to input-to-state stabilize hybrid systems and in [5] to provide input-to-state safety guarantees.

Chapter 2

Preliminaries

2.1 Notation

Given two vectors x, y, we use the equivalent notation $(x, y) = [x^\top y^\top]^\top$ and $\langle x, y \rangle$ denotes the Euclidean inner product. We denote by $|x|$ a vector (e.g., Euclidean) norm of x. The symbol $\mathbb{N}$ denotes the set of natural numbers including zero and $\mathbb{N}_{>0}$ denotes the set of positive naturals. The symbol $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative reals. Given a vector x and a nonempty set $\mathcal{A}$, the distance from x to $\mathcal{A}$ is defined as $|x|_{\mathcal{A}} = \inf_{y \in \mathcal{A}} |x - y|$. We denote with $\mathbb{S}^n_+$ the set of real positive definite matrices of dimension n, and with $\mathbb{S}^n_{0+}$ the set of real positive semidefinite matrices of dimension n. We denote by $\mathrm{card}(\mathcal{A})$ the cardinality of $\mathcal{A}$, by $\mathrm{int}\,\mathcal{A}$ its interior, by $\overline{\mathcal{A}}$ its closure, by $\mathrm{vol}(\mathcal{A})$ its Lebesgue measure, by $z \sim \mathbb{U}(\mathcal{A})$ that an element z is sampled from the uniform probability distribution over the set $\mathcal{A}$, and by $\overline{\mathrm{con}}\mathcal{A}$ the closure of the convex hull of $\mathcal{A}$. We denote with $\mathbb{1}_{\mathcal{A}} : \mathbb{R}^n \to \{0, 1\}$ the indicator function of the set $\mathcal{A}$. The n-dimensional identity matrix is denoted by I_n. Given a matrix $A \in \mathbb{R}^{n \times n}$, let $|A|$ be its induced matrix $2-$norm, $\mathrm{rank}(A)$ denote its rank, and $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the eigenvalues with minimum and largest real part, respectively. We represent by $\mathbb{B}$ the closed Euclidean unit ball and by $x + \varepsilon\mathbb{B}$ the closed ball of radius ε centered at x and by $\mathcal{A} + \varepsilon\mathbb{B} := \{a + b : a \in \mathcal{A}, b \in \varepsilon\mathbb{B}\}$ the Minkowski sum of $\mathcal{A}$ and $\varepsilon\mathbb{B}$. Given an open set U, the function $f : U \to \mathbb{R}^n$ is said to be of differentiability class $\mathcal{C}^k$ if the derivatives $f', f'', \ldots, f^{(k)}$ exist and are continuous on U. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}$ function, also written as $\alpha \in \mathcal{K}$, if α is zero at zero, continuous, and strictly increasing. Similarly, a function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

is said to belong to class-$\mathcal{KL}$ if it is continuous, nondecreasing in its first argument, nonincreasing in its second argument and $\lim_{r\to 0^+0}\beta(r,s)=0$ for each $s\in\mathbb{R}_{\geq 0}$, and $\lim_{s\to\infty}\beta(r,s)=0$ for each $r\in\mathbb{R}_{\geq 0}$. We denote by $\tilde{\mathcal{O}}(C,A)$ the observability matrix of the pair (C,A) and by $\tilde{\mathcal{C}}(A,B)$ the controllability matrix of the pair (A,B) .

2.2 Hybrid System with Inputs

We consider hybrid systems that will be modeled based on the framework in [20]. In this framework, the continuous dynamics of the system are modeled by differential inclusions, while the discrete dynamics are modeled by difference inclusions. Based on this, a hybrid dynamical inclusion $\mathcal{H}$ with input $u=(u_C,u_D)\in\mathbb{R}^{m_C}\times\mathbb{R}^{m_D}$ can be represented by

$$\mathcal{H}_s \begin{cases} \dot{x} & \in & F(x,u_C) & (x,u_C)\in C \\ x^+ & \in & G(x,u_D) & (x,u_D)\in D \end{cases} \tag{2.1}$$

where $x\in\mathbb{R}^n$ is the state. The *flow map* $F:\mathbb{R}^n\times\mathbb{R}^{m_C}\rightrightarrows\mathbb{R}^n$ captures the continuous evolution of the system, when the state is in the *flow set* C. The *jump map* $G:\mathbb{R}^n\times\mathbb{R}^{m_D}\rightrightarrows\mathbb{R}^n$ describes the discrete evolution of the system when the state is in the *jump set* D. Based on this framework, we can model the players behavior enclosing the richness of their dynamics.

Since solutions to the dynamical system $\mathcal{H}_s$ can exhibit both continuous and discrete behavior, we use ordinary time t to determine the amount of flow, and a counter $j\in\mathbb{N}$ that counts the number of jumps. Based on this, the concept of hybrid time domain, in which solutions are fully described, is proposed.

Definition 2.2.1. (Hybrid time domain) *A set $E\subset\mathbb{R}_{\geq 0}\times\mathbb{N}$ is a hybrid time domain if, for each $(T,J)\in E$, the set $E\cap([0,T]\times\{0,1,\ldots,J\})$ is a compact hybrid time domain, i.e., it can be written in the form*

$$\bigcup_{j=0}^{J}([t_j,t_{j+1}]\times\{j\})$$

for some finite nondecreasing sequence of times $\{t_j\}_{j=0}^{J+1}$ with $t_{J+1}=T$. Each element $(t,j)\in E$ denotes the elapsed hybrid time, which indicates that t seconds of flow time and j jumps have occurred.

A hybrid signal is a function defined on a hybrid time domain. Given a hybrid signal ϕ and $j \in \mathbb{N}$, we define $I_\phi^j = \{t : (t, j) \in \operatorname{dom} \phi\}$ as the interval of flow after jump j.

Definition 2.2.2. (Hybrid arc) *A hybrid signal $\phi : \operatorname{dom} \phi \to \mathbb{R}^n$ is called a hybrid arc if for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on the interval I_ϕ^j. A hybrid arc ϕ is compact if $\operatorname{dom} \phi$ is compact.*

Definition 2.2.3. (Hybrid Input) *A hybrid signal u is a hybrid input if for each $j \in \mathbb{N}$, the function $t \mapsto u(t, j)$ is Lebesgue measurable and locally essentially bounded on the interval I_u^j.*

Let $\mathcal{X}$ be the set of hybrid arcs $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$, and $\mathcal{U} = \mathcal{U}_C \times \mathcal{U}_D$ the set of hybrid inputs $u = (u_C, u_D) : \operatorname{dom}u \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$. A solution to the hybrid system with input $\mathcal{H}_s$ is defined as follows.

Definition 2.2.4. (Solution to the hybrid system $\mathcal{H}$) *A hybrid signal (ϕ, u) defines a solution pair to the hybrid system (2.1) if $\phi \in \mathcal{X}$, $u = (u_C, u_D) \in \mathcal{U}$, $\operatorname{dom}\phi = \operatorname{dom}u$, and*

- $(\phi(0,0), u_C(0,0)) \in \overline{C}$ *or* $(\phi(0,0), u_D(0,0)) \in D$,
- *For each $j \in \mathbb{N}$ such that I_ϕ^j has a nonempty interior $\operatorname{int}I_\phi^j$, we have, for all $t \in \operatorname{int}I_\phi^j$,*

  $$(\phi(t, j), u_C(t, j)) \in C$$

  and, for almost all $t \in I_\phi^j$,

  $$\frac{d}{dt}\phi(t, j) \in F(\phi(t, j), u_C(t, j))$$

- *For all $(t, j) \in \operatorname{dom} \phi$ such that $(t, j + 1) \in \operatorname{dom} \phi$,*

  $$(\phi(t, j), u_D(t, j)) \in D$$
  $$\phi(t, j + 1) \in G(\phi(t, j), u_D(t, j))$$

A solution pair (ϕ, u) is a compact solution pair if ϕ is a compact hybrid arc; see Definition 2.2.2.

Given a solution pair (ϕ, u), the component ϕ is referred to as the state trajectory. In this work, the same symbols are used to denote input actions and their values. The context clarifies the meaning of u, as follows: "the function u," "the signal u," or "the hybrid signal u" that appears in "the solution pair (ϕ, u)" refer to the input action, whereas "u" refers to the input value as a point in $\mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ in any other case. The reader can replace "the function u" by "u_ϕ", that is the input action yielding the system to a state trajectory ϕ.

The $\mathcal{L}^\infty$ norm of a hybrid signal $r = (r_C, r_D)$ is given by

$$\|r\|_{(t,j)} := \max\left\{ \|r_C\|_{(t,j)}, \|r_D\|_{(t,j)} \right\} \tag{2.2a}$$

$$\|r_C\|_{(t,j)} := \max_{j' \leq j} \operatorname*{ess\,sup}_{t' \text{s.t.} (t',j') \in \operatorname{dom} r} |r(t', j')|, \tag{2.2b}$$

$$\|r_D\|_{(t,j)} := \sup_{(t',j') \in \Gamma(r), t'+j' \leq t+j} |r(t', j')| \tag{2.2c}$$

where $\Gamma(r) := \{(t, j) \in \operatorname{dom} r : (t, j+1) \in \operatorname{dom} r\}$. For notational convenience, $\|r\|_\#$ denotes $\lim_{t+j \to N} \|r\|_{(t,j)}$, where $N = \sup_{(t,j) \in \operatorname{dom} r} t + j \in [0, \infty)$.

A solution pair (ϕ, u) to $\mathcal{H}_s$ from $\xi \in \mathbb{R}^n$ is complete if $\operatorname{dom}(\phi, u)$ is unbounded. It is maximal if there is no solution (ψ, w) from ξ such that $\phi(t, j) = \psi(t, j)$ and $u(t, j) = w(t, j)$ for all $(t, j) \in \operatorname{dom}(\phi, u)$ and $\operatorname{dom}(\phi, u)$ is a proper subset of $\operatorname{dom}(\psi, w)$. We denote by $\hat{\mathcal{S}}_{\mathcal{H}_s}(M)$ the set of solution pairs (ϕ, u) to $\mathcal{H}_s$ as in (2.1) such that $\phi(0, 0) \in M$. The set $\mathcal{S}_{\mathcal{H}_s}(M) \subset \hat{\mathcal{S}}_{\mathcal{H}_s}(M)$ denotes all maximal solution pairs from M and $\mathcal{S}^\infty_{\mathcal{H}_s}(M) \subset \mathcal{S}_{\mathcal{H}_s}(M)$ the set of complete solutions from M. Given $\xi \in \mathbb{R}^n$, we define the set of input actions that yield maximal solutions to $\mathcal{H}_s$ from ξ as $\mathcal{U}_{\mathcal{H}_s}(\xi) := \{u : \exists (\phi, u) \in \hat{\mathcal{S}}_{\mathcal{H}_s}(\xi), (\phi, u) \in \mathcal{S}_{\mathcal{H}_s}(\xi)\}$ and the set of input actions that yield complete solutions as $\mathcal{U}^\infty_{\mathcal{H}_s}(\xi) := \{u : \forall (\phi, u) \in \mathcal{S}_{\mathcal{H}_s}(\xi), (\phi, u) \in \mathcal{S}^\infty_{\mathcal{H}_s}(\xi)\}$. For a given $u \in \mathcal{U}$, we denote the set of maximal state trajectories to $\mathcal{H}_s$ from ξ for u by $\mathcal{R}(\xi, u) = \{\phi : (\phi, u) \in \mathcal{S}_{\mathcal{H}_s}(\xi)\}$. We say u renders a maximal trajectory ϕ to $\mathcal{H}_s$ from ξ if $\phi \in \mathcal{R}(\xi, u)$. A complete solution (ϕ, u) is discrete if $\operatorname{dom}(\phi, u) \subset \{0\} \times \mathbb{N}$ and continuous if $\operatorname{dom}(\phi, u) \subset \mathbb{R}_{\geq 0} \times \{0\}$.

We define the projections of $C \subseteq \mathbb{R}^n \times \mathbb{R}^{m_C}$ and $D \subseteq \mathbb{R}^n \times \mathbb{R}^{m_D}$ onto $\mathbb{R}^n$, respectively, as

$$\Pi(C) := \{\xi \in \mathbb{R}^n : \exists u_C \in \mathbb{R}^{m_C} \text{ s.t. } (\xi, u_C) \in C\}$$

$$\Pi(D) := \{\xi \in \mathbb{R}^n : \exists u_D \in \mathbb{R}^{m_D} \text{ s.t. } (\xi, u_D) \in D\}.$$

We also define the set-valued maps that output the allowed input values at a given state

x as

$$\Pi_u^C(x) = \{u_C \in \mathbb{R}^{m_C} : (x, u_C) \in C\},$$
$$\Pi_u^D(x) = \{u_D \in \mathbb{R}^{m_D} : (x, u_D) \in D\}.$$

Moreover, $\sup_t \operatorname{dom} \phi := \sup\{t \in \mathbb{R}_{\geq 0} : \exists j \text{ s.t. } (t, j) \in \operatorname{dom} \phi\}$, $\sup_j \operatorname{dom} \phi := \sup\{j \in \mathbb{N} : \exists t \text{ s.t. } (t, j) \in \operatorname{dom} \phi\}$, and $\sup \operatorname{dom} \phi := (\sup_t \operatorname{dom} \phi, \sup_j \operatorname{dom} \phi)$. Whenever $\operatorname{dom} \phi$ is compact, $\operatorname{dom} \phi \supset \max \operatorname{dom} \phi := \sup \operatorname{dom} \phi$.

As a special case of (2.1), consider the single-valued system

$$\mathcal{H} \begin{cases} \dot{x} &= F(x, u_C) \quad (x, u_C) \in C \\ x^+ &= G(x, u_D) \quad (x, u_D) \in D \end{cases} \tag{2.3}$$

The following conditions guarantee uniqueness of solutions to $\mathcal{H}$ as in (2.3) [20, Proposition 2.11].

Proposition 2.2.5. (Uniqueness of Solutions) *Consider the hybrid system $\mathcal{H}$ as in (2.3). For every $\xi \in \Pi(\overline{C}) \cup \Pi(D)$ and each $u \in \mathcal{U}$ there exists a unique maximal solution (ϕ, u) with $\phi(0, 0) = \xi$ provided that the following holds:*

1) *for every $\xi \in \Pi(\overline{C}) \setminus \Pi(D)$ and $T > 0$, if two locally absolutely continuous functions $z_1, z_2 : I_z \to \mathbb{R}^n$ and a Lebesgue measurable function $u_z : I_z \to \mathbb{R}^{m_C}$ with I_z of the form $I_z = [0, T)$ or $I_z = [0, T]$, are such that, for each $i \in \{1, 2\}$, $\dot{z}_i(t) \in F(z_i(t), u_z(t))$ for almost all $t \in I_z$, $(z_i(t), u_z(t)) \in C$ for all $t \in \operatorname{int} I_z$, and $z_i(0) = \xi$, then $z_1(t) = z_2(t)$ for every $t \in I_z$;*

2) *for every $(\xi, u_D) \in D$, $G(\xi, u_D)$ consists of one point.*

Proof. Proceeding by contradiction, suppose there exist two maximal solutions to $\mathcal{H}$, (ϕ_1, u) and (ϕ_2, u), with $u = (u_C, u_D)$, $\operatorname{dom} \phi_1 = \operatorname{dom} \phi_2 = \operatorname{dom} u$, and $\phi_1(0, 0) = \phi_2(0, 0)$ such that ϕ_1 and ϕ_2 are not identical over $\operatorname{dom} u$, namely, there exists $(t^*, j^*) \in \operatorname{dom} u$ such that $\phi_1(t^*, j^*) \neq \phi_2(t^*, j^*)$. We have the following three cases:

a) If $(t^*, j^*) \in [t', t'') \times \{j^*\} \subset \operatorname{dom} u$ for some $t'' > t' \geq 0$, then, the functions $z_1, z_2 : I_z \to \mathbb{R}^n$, and $u_z : I_z \to \mathbb{R}^{m_C}$, with $T = t'' - t'$ and I_z of the form $I_z = [0, T)$, defined for each $i \in \{1, 2\}$ as $z_i(t) = \phi_i(t' + t, j)$ for all $t \in I_z$, and as $u_z(t) = u_C(t' + t, j)$ for all $t \in I_z$, satisfy, by Definition 2.2.4, $\dot{z}_i(t) \in F(z_i(t), u_z(t))$ for almost all $t \in I_z$, and $(z_i(t), u_z(t)) \in C$ for all $t \in \operatorname{int} I_z$. However, at $t^* \in I_z$, $z_1(t^*) \neq z_2(t^*)$, which contradicts item 1.

b) If $(t^*, j^*) \in [t', t''] \times \{j^*\} \subset \operatorname{dom} u$ for some $t'' > t' \geq 0$, then, the functions $z_1, z_2 : I_z \to \mathbb{R}^n$, and $u_z : I_z \to \mathbb{R}^{m_C}$, with $T = t'' - t'$ and I_z of the form $I_z = [0, T]$, defined for each $i \in \{1, 2\}$ as $z_i(t) = \phi_i(t' + t, j)$ for all $t \in I_z$, and as $u_z(t) = u_C(t' + t, j)$ for all $t \in I_z$, satisfy, by Definition 2.2.4, $\dot{z}_i(t) \in F(z_i(t), u_z(t))$ for almost all $t \in [0, T)$, and $(z_i(t), u_z(t)) \in C$ for all $t \in \text{int} I_z$. However, at $t^* \in I_z$, $z_1(t^*) \neq z_2(t^*)$, which contradicts item 1.

c) If (t^*, j^*) is such that $(t^*, j^* - 1) \in \operatorname{dom} u$, then $(t^*, j^* - 1)$ is a jump time of ϕ_1 and ϕ_2. Then, by Definition 2.2.4, for each $i \in \{1, 2\}$, $\phi_i(t^*, j^*) = G(\phi_i(t^*, j^* - 1), u_D(t^*, j^* - 1))$. Since, $\phi_1(t^*, j^*) \neq \phi_2(t^*, j^*)$, and $\phi_1(t^*, j^* - 1) = \phi_2(t^*, j^* - 1), G(\phi_1(t^*, j^* - 1), u_D(t^*, j^* - 1))$ takes more than one value, which contradicts item 2.

Thus, provided items 1 and 2 hold, for every $\xi \in \Pi(\overline{C}) \cup \Pi(D)$ and each $u \in \mathcal{U}$ there exists a unique maximal solution (ϕ, u) with $\phi(0, 0) = \xi$. □

2.3 Closed-loop Hybrid Systems

Given a hybrid system $\mathcal{H}_s$ and a function $\kappa := (\kappa_C, \kappa_D)$ with $\kappa : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$, the autonomous hybrid system resulting from assigning $u = \kappa(x)$, namely the hybrid closed-loop system is given by

$$\mathcal{H}_\kappa \begin{cases} \dot{x} & \in & F(x, \kappa_C(x)) & x \in C_\kappa \\ x^+ & \in & G(x, \kappa_D(x)) & x \in D_\kappa \end{cases} \tag{2.4}$$

where $C_\kappa := \{x \in \mathbb{R}^n : (x, \kappa_C(x)) \in C\}$ and $D_\kappa := \{x \in \mathbb{R}^n : (x, \kappa_D(x)) \in D\}$.

A solution to the closed-loop hybrid system $\mathcal{H}_\kappa$ is defined as follows.

Definition 2.3.1. (Solution to the hybrid system $\mathcal{H}_\kappa$) *A hybrid arc ϕ defines a solution to the hybrid system $\mathcal{H}_\kappa$ in (2.4) if*

- $\phi(0, 0) \in \overline{C_\kappa} \cup D_\kappa$,
- *For each $j \in \mathbb{N}$ such that I_ϕ^J has a nonempty interior* $\text{int} I_\phi^J$, *we have, for all* $t \in \text{int} I_\phi^J$,

$$\phi(t, j) \in C_\kappa$$

and, for almost all $t \in I^J_\phi$,

$$\frac{d}{dt}\phi(t,j) \in F(\phi(t,j), \kappa_C(\phi(t,j)))$$

- *For all $(t,j) \in \operatorname{dom}\phi$ such that $(t, j+1) \in \operatorname{dom}\phi$,*

$$\begin{aligned} \phi(t,j) &\in D_\kappa \\ \phi(t,j+1) &\in G(\phi(t,j), \kappa_D(\phi(t,j))) \end{aligned}$$

A solution ϕ is a compact solution if ϕ is a compact hybrid arc.

We denote by $\hat{\mathcal{S}}_{\mathcal{H}_\kappa}(M)$ the set of solutions ϕ to (2.4) such that $\phi(0,0) \in M$. The set $\mathcal{S}_{\mathcal{H}_\kappa}(M) \subset \hat{\mathcal{S}}_{\mathcal{H}_\kappa}(M)$ denotes all maximal solutions and $\hat{\mathcal{S}}^\infty_{\mathcal{H}_\kappa}(M) \subset \hat{\mathcal{S}}_{\mathcal{H}_\kappa}(M)$ the set of complete solutions.

Definition 2.3.2. (Hybrid Basic Conditions) *Given a system $\mathcal{H}_s$ and a feedback law $\kappa := (\kappa_C, \kappa_D)$ with $\kappa : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$, the resulting closed-loop system from their composition, denoted $\mathcal{H}_\kappa$ and defined as in (2.4), is said to satisfy the hybrid basic conditions if its data $(C_\kappa, F, D_\kappa, G)$ satisfies the following properties:*

1. *C_κ and D_κ are closed subsets of $\mathbb{R}^n$;*
2. *$x \mapsto F(x, \kappa_C(x))$ is outer semicontinuous and locally bounded relative to C_κ, every $x \in C_\kappa$ is such that $(x, \kappa(x)) \in \operatorname{dom} F$, and $x \mapsto F(x, \kappa_C(x))$ is convex for each $x \in C_\kappa$.*
3. *$x \mapsto G(x, \kappa_D(x))$ is outer semicontinuous and locally bounded relative to D_κ, and every $x \in D_\kappa$ is such that $(x, \kappa_D(x)) \in \operatorname{dom} G$.*

Definition 2.3.3. (Tangent cone) *The tangent cone to a set $S \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$, denoted $T_S(x)$, is the set of all vectors $w \in \mathbb{R}^n$ for which there exist sequences $x_i \in S, \tau_i > 0$ with $x_0 \to x, \tau_i \searrow 0$, and $w = \lim_{i\to\infty} \frac{x_i - x}{\tau_i}$.*

The following result provides conditions that guarantee that solutions to a hybrid closed-loop system exist. Below, $T_C(x)$ denotes the tangent cone of C_κ at x

Proposition 2.3.4. (Existence of solutions to $\mathcal{H}_s$) *Consider a system $\mathcal{H}_s$ and a feedback law $\kappa := (\kappa_C, \kappa_D)$ with $\kappa : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$, and suppose the resulting closed-loop system from their composition, denoted $\mathcal{H}_\kappa$ and defined as in (2.4) with data*

$(C_\kappa, F, D_\kappa, G)$, satisfies the hybrid basic conditions in Definition 2.3.2. Let $\xi \in C_\kappa \cup D_\kappa$ be arbitrary. If $\xi \in D_\kappa$ or

(VC) there exists a neighborhood X_n of ξ such that for every $x \in X_n \cap C_\kappa$,

$$F(x, \kappa_C(x)) \cap T_C(x) \neq \emptyset$$

then there exists a nontrivial solution ϕ to $\mathcal{H}_\kappa$ with $\phi(0,0) = \xi$. If (VC) holds for every $\xi \in C_\kappa \setminus D_\kappa$, then there exists a nontrivial solution to $\mathcal{H}_\kappa$ for every initial point in $C_\kappa \cup D_\kappa$.

Part I

Two-Player Zero-Sum Hybrid Games

Chapter 3

Infinite-Horizon Hybrid Games

In this chapter, we present a framework for the study of two-player zero-sum games with hybrid dynamic constraints. We present in Theorem 3.2.1 sufficient conditions based on Hamilton–Jacobi–Bellman-Isaacs-like equations to design a saddle-point equilibrium and evaluate the game value function without computing solutions to the hybrid system. Connections between optimality and asymptotic stability of a set are proposed in Section 3.2 and framed in the game theoretical approach employed. We present in Section 3.4 applications to robust and security scenarios by formulating and solving them as two-player zero-sum hybrid dynamic games.

For the broad class of systems covered in Chapter 2.2, consider the case in which $u_C = (u_{C1}, u_{C2})$ and $u_D = (u_{D1}, u_{D2})$, where (u_{C1}, u_{D1}) is the input selected by player P_1 and (u_{C2}, u_{D2}) is the input chosen by player P_2. When solutions are unique, we consider a cost functional $\mathcal{J} : \mathbb{R}^n \times \mathbb{R}^{m_C} \times \mathbb{R}^{m_D} \to \mathbb{R}$ associated to the solution to $\mathcal{H}$ as in (2.3) from ξ and study the problem

$$\min_{(u_{C1},u_{D1})} \max_{(u_{C2},u_{D2})} \mathcal{J}(\xi, u_{C1}, u_{C2}, u_{D1}, u_{D2}) \tag{3.1}$$

as a two-player zero-sum hybrid game. We introduce formally some of the applications mentioned in the introduction in which thus type of hybrid games emerges.

Application 1. (Robust Control) *Given the system $\mathcal{H}$ as in (2.3) with state x, the disturbance rejection problem consists of establishing conditions such that player P_1 selects a control input (u_{C1}, u_{D1}) that minimizes the cost of solutions to $\mathcal{H}$ in the presence of a disturbance (u_{C2}, u_{D2}) chosen by P_2.*

Application 2. (Security) *Given the system $\mathcal{H}$ as in (2.3) with state x and*

$$\begin{aligned} F(x, u_{C1}, u_{C2}) &= f_d(x, u_{C1}) + f_a(u_{C2}) \\ G(x, u_{D1}, u_{D2}) &= g_d(x, u_{D1}) + g_a(u_{D2}) \end{aligned}$$

the security problem consists of finding conditions such that the control input (u_{C1}, u_{D1}) guarantees the performance of the system via minimizing a cost functional $\mathcal{J}$ (representing damage or effect of attacks) under the action of an attacker (u_{C2}, u_{D2}) that knows f_d and g_d, and is designed to maximize $\mathcal{J}$.

3.1 Formulation of Two-player Zero-sum Hybrid Games

3.1.1 Elements of a Hybrid Game

Following the formulation in [13], for each $i \in \{1, 2\}$, consider the i-th player P_i with dynamics described by $\mathcal{H}_i$ as in (2.3) with data (C_i, F_i, D_i, G_i), state $x_i \in \mathbb{R}^{n_i}$, and input $u_i = (u_{Ci}, u_{Di}) \in \mathbb{R}^{m_{Ci}} \times \mathbb{R}^{m_{Di}}$, where $C_i \subset \mathbb{R}^n \times \mathbb{R}^{m_C}$, $F_i : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}^{n_i}$, $D_i \subset \mathbb{R}^n \times \mathbb{R}^{m_D}$ and $G_i : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}^{n_i}$, with $n_1 + n_2 = n$. We denote by $\mathcal{U}_i = \mathcal{U}_{Ci} \times \mathcal{U}_{Di}$ the set of hybrid inputs for $\mathcal{H}_i$; see Definition 2.3.

Notice that each player's dynamics are described in terms of maps and sets defined in the entire state and input space rather than the individual spaces ($\mathbb{R}^n$ and $\mathbb{R}^m$ rather than $\mathbb{R}^{n_i}$ and $\mathbb{R}^{m_i}$, respectively). This allows to model the ability of each player's state to evolve according to the state variables and input of the other players.

Definition 3.1.1. (Elements of a two-player zero-sum hybrid game) *A two-player zero-sum hybrid game is composed by*

1) *The state $x = (x_1, x_2) \in \mathbb{R}^n$, where, for each $i \in \{1, 2\}$, $x_i \in \mathbb{R}^{n_i}$ is the state of player P_i.*

2) *The set of joint input actions $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ with elements $u = (u_1, u_2)$, where, for each $i \in \{1, 2\}$, $u_i = (u_{Ci}, u_{Di})$ is a hybrid input. For each $i \in \{1, 2\}$, P_i selects u_i independently of P_{3-i}, who selects u_{3-i}, namely, the joint input action u has components u_i that are independently chosen by each player.*

3) *The dynamics of the game, described as in* (2.3) *and denoted by* $\mathcal{H}$, *with data*

$$\begin{aligned} C &:= C_1 \cap C_2 \\ F(x,u_C) &:= (F_1(x,u_C), F_2(x,u_C)) \qquad \forall (x,u_C) \in C \\ D &:= D_1 \cup D_2 \\ G(x,u_D) &:= \{\hat{G}_i(x,u_D) : (x,u_D) \in D_i, i \in \{1,2\}\} \qquad \forall (x,u_D) \in D \end{aligned}$$

where $\hat{G}_1(x,u_D) = (G_1(x,u_D), I_{n_2})$, $\hat{G}_2(x,u_D) = (I_{n_1}, G_2(x,u_D))$, $u_C = (u_{C1}, u_{C2})$, *and* $u_D = (u_{D1}, u_{D2})$.

4) *For each* $i \in \{1,2\}$, *a strategy space* $\mathcal{K}_i$ *of* P_i *defined as a collection of mappings* $\kappa_i : \mathbb{R}^n \to \mathbb{R}^{m_{Ci}} \times \mathbb{R}^{m_{Di}}$. *The strategy space of the game, namely* $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$, *is the collection of mappings with elements* $\kappa = (\kappa_1, \kappa_2)$, *where* $\kappa_i \in \mathcal{K}_i$ *for each* $i \in \{1,2\}$, *such that every maximal solution* (ϕ, u) *to* $\mathcal{H}$ *with input assigned as* $\operatorname{dom}\phi \ni (t,j) \mapsto u_i(t,j) = \kappa_i(\phi(t,j))$ *for each* $i \in \{1,2\}$ *is complete. Each* $\kappa_i \in \mathcal{K}_i$ *is said to be a permissible pure*[1] *strategy for* P_i.

5) *A scalar-valued functional* $(\xi, u) \mapsto \mathcal{J}_i(\xi, u)$ *defined for each* $i \in \{1,2\}$, *and called the cost associated to* P_i. *For each* $u \in \mathcal{U}$, *we refer to a single cost functional* $\mathcal{J} := \mathcal{J}_1 = -\mathcal{J}_2$ *as the cost associated to the unique solution to* $\mathcal{H}$ *from* ξ *for* u, *and its structure is defined for each type of game.*

Remark 3.1.2. (Players' state) *In scenarios where each player has its own dynamics, as in pursue-evasion [66], or target defense [67] games, it is common to have a state associated to each player, namely* x_1 *for* P_1 *and* x_2 *for* P_2, *justifying the partition of the state* x *in* x_1 *and* x_2. *When the players do not have their own dynamics but can independently select an input, e.g.,* P_1 *selects* u_1 *and* P_2 *selects* u_2 *to control a common state* x, *such state can be associated, without loss of generality, to either of the players, e.g.,* $x = x_1$ *with* $n = n_1$ *and* $n_2 = 0$. *This is illustrated in Example 6.5.3.*

Notice that Definition 3.1.1 is general enough to cover games with a finite horizon, for which additional conditions specify the end of the game, e.g., a terminal set in the state space or fixed duration specifications [2] as in Chapter 4.

We say that a game formulation is in normal (or matrix) form when it describes only the correspondences between strategies and costs. On the other hand, we refer

[1] This is in contrast to when $\mathcal{K}_i$ is defined as a probability distribution, in which case $\kappa_i \in \mathcal{K}_i$ is referred to as a mixed strategy.

to the mathematical description of a game to be in the Kuhn's extensive form if the formulation describes:

- the evolution of the game defined by its dynamics,
- the decision-making process defined by the strategies,
- the sharing of information between the players defined by the communication network, and
- their outcome defined by the cost associated to each player.

If a game is formulated in a Kuhn's extensive form, then it admits a solution [13]. From a given initial condition ξ, a given strategy $\kappa \in \mathcal{K}$ potentially leads to nonunique solutions[2] $(\phi^1, u^1), (\phi^2, u^2), \ldots, (\phi^k, u^k)$ to $\mathcal{H}$, where $u^l = \kappa(\phi^l)$ and $\phi^l(0,0) = \xi$ for each $l \in \{1, 2, \ldots k\}$. Thus, for the formulation in Definition 3.1.1 to be in Kuhn's extensive form , an appropriate cost definition is required so each strategy $\kappa \in \mathcal{K}$ has a unique cost correspondence, namely, every solution (ϕ^l, u^l) with $u^l = \kappa(\phi^l)$, $l \in \{1, 2, \ldots, k\}$ is assigned the same cost.

3.1.2 Equilibrium Solution Concept

Given the formulation of the elements of a zero-sum hybrid game in Definition 3.1.1, its solution is defined as follows.

Definition 3.1.3. (Saddle-point equilibrium) *Consider a two-player zero-sum game, with dynamics $\mathcal{H}$ as in (2.3) with $\mathcal{J}_1 = \mathcal{J}$, $\mathcal{J}_2 = -\mathcal{J}$, for a given cost functional $\mathcal{J} : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$. We say that a strategy $\kappa = (\kappa_1, \kappa_2) \in \mathcal{K}$ is a saddle-point equilibrium if for each $\xi \in \Pi(\overline{C}) \cup \Pi(D)$, every hybrid input $u^* = (u_1^*, u_2^*)$ such that there exists $\phi^* \in \mathcal{R}(\xi, u^*)$, with components defined as $\operatorname{dom} \phi^* \ni (t, j) \mapsto u_i^*(t, j) = \kappa_i(\phi^*(t, j))$, $i \in \{1, 2\}$, satisfies*

$$\mathcal{J}(\xi, (u_1^*, u_2)) \leq \mathcal{J}(\xi, u^*) \leq \mathcal{J}(\xi, (u_1, u_2^*)) \tag{3.2}$$

for all hybrid inputs u_1 and u_2 such that $\mathcal{R}(\xi, (u_1^*, u_2))$ and $\mathcal{R}(\xi, (u_1, u_2^*))$ are nonempty.

[2] A given strategy κ can lead to multiple input actions due to a nonempty $\Pi(C) \cap \Pi(D)$.

Definition 3.1.3 is a generalization of the classical pure strategy Nash equilibrium [13, (6.3)] to the case where the players exhibit hybrid dynamics and opposite optimization goals. In words, we refer to the strategy $\kappa^* = (\kappa_1^*, \kappa_2^*)$ as a saddle-point when a player P_i cannot improve the cost $\mathcal{J}_i$ by playing any strategy different from κ_i^* when the player P_{3-i} is playing the strategy of the saddle-point, κ_{3-i}^*. Condition (3.2) is verified over the set of inputs that define joint input actions (u_1^*, u_2) and (u_1, u_2^*), yielding at least one nontrivial solution to $\mathcal{H}$ from ξ. Notice that the saddle-point, as a solution to the zero-sum two-player game, is a strategy in $\mathcal{K}$, though the concept of a solution to a hybrid system $\mathcal{H}$, as in Definition 2.2.4, is a hybrid arc.

3.1.3 Problem Statement

We formulate an infinite-horizon optimization problem to solve the two-player zero-sum hybrid game and provide the sufficient conditions to characterize the solution. Following the formulation in Definition 3.1.1, consider a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ described by (2.3) for given (C, F, D, G). Uniqueness of solutions for a given input implies a unique correspondence from cost to control action, which allows this type of games to be *well-defined*, so that an equilibrium solution is defined [13, Remark 5.3]. This justifies the following assumption.

Assumption 3.1.4. *The flow map F and the flow set C are such that solutions to $\dot{x} = F(x, u_C)$ $(x, u_C) \in C$ are unique for each input u_C. The jump map G is single valued, i.e., the jump set for each player satisfies $D_1 = D_2$.*

Sufficient conditions to guarantee that Assumption 3.1.4 holds include Lipschitz continuity of the flow map F, provided it is a single-valued function. Under Assumption 3.1.4, the conditions in Proposition 2.2.5 are satisfied, so solutions to $\mathcal{H}$ are unique[3] for each $u \in \mathcal{U}$.

Given $\xi \in \Pi(C \cup D)$, a joint input action $u = (u_C, u_D) \in \mathcal{U}$ such that maximal solutions to $\mathcal{H}$ from ξ for u are complete, the stage cost for flows $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$, the stage cost for jumps $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, and the terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, we define the cost associated to the solution (ϕ, u) to $\mathcal{H}$ from ξ, under Assumption 3.1.4,

[3] Under Assumption 3.1.4, the domain of the input u specifies whether from points in $\Pi(C) \cap \Pi(D)$ a jump or flow occur.

as

$$\mathcal{J}(\xi,u) := \sum_{j=0}^{\sup_j \operatorname{dom}\phi} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j),u_C(t,j))dt + \sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} L_D(\phi(t_{j+1},j),u_D(t_{j+1},j)) + \limsup_{\substack{t+j\to\infty \\ (t,j)\in \operatorname{dom}\phi}} q(\phi(t,j)) \tag{3.3}$$

where $\{t_j\}_{j=0}^{\sup_j \operatorname{dom}\phi}$ is a nondecreasing sequence associated to the definition of the hybrid time domain of ϕ; see Definition 2.2.2.

We are ready to formulate the two-player zero-sum game.

Problem ($\diamond$): Given $\xi \in \mathbb{R}^n$, under Assumption 3.1.4, solve

$$\underset{\substack{u_1 \\ u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^{\infty}(\xi)}}{\text{minimize}} \; \underset{u_2}{\text{maximize}} \; \mathcal{J}(\xi,u) \tag{3.4}$$

where $\mathcal{U}_{\mathcal{H}}^{\infty}$ is the set of joint input actions yielding maximal complete solutions to $\mathcal{H}$, as defined in Section 2.2.

Remark 3.1.5. (Saddle-point equilibrium and min-max control) *A solution to Problem ($\diamond$), when it exists, can be expressed in terms of the pure strategy saddle-point equilibrium $\kappa = (\kappa_1,\kappa_2)$) for the two-player zero-sum infinite-horizon game. Each $u^* = (u_1^*, u_2^*)$ rendering a state trajectory ϕ^* such that $(\phi^*, u^*) \in \mathcal{S}_{\mathcal{H}}^{\infty}(\xi)$, with components defined as* $\operatorname{dom}\phi^* \ni (t,j) \mapsto u_i^*(t,j) = \kappa_i(\phi^*(t,j))$ *for each $i \in \{1,2\}$, satisfies*

$$u^* = \arg \min_{\substack{u_1 \\ u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^{\infty}(\xi)}} \max_{u_2} \; \mathcal{J}(\xi,u) = \arg \max_{\substack{u_2 \\ u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^{\infty}(\xi)}} \min_{u_1} \; \mathcal{J}(\xi,u)$$

and it is referred to as an min-max control at ξ.

Definition 3.1.6. (Value function) *Given $\xi \in \Pi(\overline{C}) \cup \Pi(D)$, under Assumption 3.1.4, the value function at ξ is given by*

$$\mathcal{J}^*(\xi) := \min_{\substack{u_1 \\ u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^{\infty}(\xi)}} \max_{u_2} \; \mathcal{J}(\xi,u) = \max_{\substack{u_2 \\ u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^{\infty}(\xi)}} \min_{u_1} \; \mathcal{J}(\xi,u) \tag{3.5}$$

Next, we position the setting studied here in terms of the existing literature of dynamic games. Let $u_{[(s,k),(r,l)]}$ denote the truncation of $u \in \mathcal{U}, \sup \operatorname{dom} u = (T,J)$, to the hybrid time interval $[(s,k),(r,l)] \subset \operatorname{dom} u$, and

$$(\phi, u)^{\kappa}_{[(s,k),(r,l)]} := \big\{(\phi,u) \in \mathcal{S}_{\mathcal{H}} : u_{[(0,0),(s,k)]} = \kappa(\phi_{[(0,0),(s,k)]}), \\ u_{[(r,l),(T,J)]} = \kappa(\phi_{[(r,l),(T,J)]})\big\} \quad (3.6)$$

denotes the set of solutions with input defined in terms of $\kappa \in \mathcal{K}$ in the intervals $[(0,0),(s,k)]$ and $[(r,l),(T,J)]$. Then, we define time consistency as follows.

Definition 3.1.7. *An input action u^* that solves Problem ($\diamond$) is* strongly time consistent *(STC) if its truncation to the interval $[(s,k),(T,J)]$, namely $u^*_{[(s,k),(T,J)]}$, solves the truncated version of Problem ($\diamond$) over the set of solutions $(\phi,u)^{\kappa}_{[(s,k),(T,J)]}$, for every $\kappa \in \mathcal{K}$ and every $(s,k) \in ((0,0),(T,J)] \cap \operatorname{dom}(\phi,u)$.*

Remark 3.1.8. (Time consistency and subgame perfection) *The permissible strategies considered in this work have a feedback information structure, in the sense that they depend only on the current value of the state, and not on any past history of the values of the state or hybrid time. Given $\xi \in \mathbb{R}^n$, we say that an input action u^* is* strongly time consistent *if even when the past history of input values that led $\mathcal{H}$ as in* (2.3) *to ξ were not optimal, the action u^* is still a solution for the remaining of the game (subgame) defined by Problem ($\diamond$), starting from ξ. When this property holds for every state ξ in $\Pi(C) \cup \Pi(D)$, we say that u^* is* subgame perfect*, see [68]. Then, under a strategy space that does not impose structural restrictions on the permissible strategies, (e.g., a linear dependence on the state) the saddle-point equilibrium strategy, when it exists, is said to be strongly time consistent if its components κ_C and κ_D lead to input actions that are strong time consistent for each ξ in $\Pi(C) \cup \Pi(D)$. Notice that given the hybrid time horizon structure of the input actions considered in this work, the saddle-point equilibrium is time independent. This results in truncations of input actions not keeping a record of previous hybrid time values, i.e., if there exists any past history of strategies that led to the current state, this is hidden for the evaluation of the saddle-point equilibrium at the current state, which results in preservation of optimality in the subgame, property known as* permanent optimality *[13, Section 5.6].*

We revisit the applications presented in the introduction and reformulate them according to the mathematical framework provided in this chapter as follows.

Application 1. (Robust Control) *Given the system $\mathcal{H}$ as in (2.3) and $\xi \in \mathbb{R}^n$, the disturbance rejection problem consists of finding the control input of P_1*

$$u_1 = (u_{C1}, u_{D1}) = \underset{\substack{(u_{C1},u_{D1}) \\ u=(u_1,u_2)\in\mathcal{U}^{\infty}_{\mathcal{H}}(\xi)}}{\arg\min} \mathcal{J}(\xi, u) \tag{3.7}$$

in the presence of a disturbance $u_2 = (u_{C2}, u_{D2})$ chosen by P_2. To account for the worst-case disturbance, (6.5) *is addressed by solving Problem ($\diamond$).*

Application 2. (Security) *Given the system $\mathcal{H}$ as in (2.3), $F(x, u_{C1}, u_{C2}) = f_d(x, u_{C1}) + f_a(u_{C2})$, and $G(x, u_{D1}, u_{D2}) = g_d(x, u_{D1}) + g_a(u_{D2})$, the security problem consists of finding the control input (u_{C1}, u_{D1}) that guarantees the performance of the system in spite of the action*

$$u_2 = (u_{C2}, u_{D2}) = \underset{\substack{(u_{C2},u_{D2}) \\ u=(u_1,u_2)\in\mathcal{U}^{\infty}_{\mathcal{H}}(\xi)}}{\arg\max} \mathcal{J}(\xi, u) \tag{3.8}$$

chosen by an attacker P_2 that knows f_d and g_d. The cost functional $\mathcal{J}$ represents the damage caused by attacks. To account for the best-scenario, this problem is addressed by solving Problem ($\diamond$).

3.2 Design of Saddle-Point Equilibrium for Infinite-Horizon Hybrid Games

The following result provides sufficient conditions to characterize the value function, and the feedback law that attains it. It addresses the solution to Problem ($\diamond$) showing that the optimizer is the saddle-point equilibrium.

Theorem 3.2.1. (Hamilton-Jacobi-Isaacs (HJI) for Problem ($\diamond$)) *Given a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ as in (2.3) with data (C, F, D, G) satisfying Assumption 3.1.4, stage costs $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}, L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, and terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, suppose the following hold:*

1) *There exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on a neighborhood of* $\Pi(C)$ *that satisfies the Hamilton-Jacobi-Isaacs (HJBI) hybrid*

equations given as

$$\begin{aligned} 0 &= \min_{u_{C1}} \max_{u_{C2}}_{u_C=(u_{C1},u_{C2})\in\Pi_u^C(x)} \{L_C(x,u_C) + \langle \nabla V(x), F(x,u_C)\rangle\} \\ &= \max_{u_{C2}} \min_{u_{C1}}_{u_C=(u_{C1},u_{C2})\in\Pi_u^C(x)} \{L_C(x,u_C) + \langle \nabla V(x), F(x,u_C)\rangle\} \qquad \forall x \in \Pi(C), \end{aligned} \tag{3.9}$$

$$\begin{aligned} V(x) &= \min_{u_{D1}} \max_{u_{D2}}_{u_D=(u_{D1},u_{D2})\in\Pi_u^D(x)} \{L_D(x,u_D) + V(G(x,u_D))\} \\ &= \max_{u_{D2}} \min_{u_{D1}}_{u_D=(u_{D1},u_{D2})\in\Pi_u^D(x)} \{L_D(x,u_D) + V(G(x,u_D))\} \qquad \forall x \in \Pi(D) \end{aligned} \tag{3.10}$$

2) For each $\xi \in \Pi(\overline{C}) \cup \Pi(D)$*, each* $(\phi, u) \in \mathcal{S}_{\mathcal{H}}^{\infty}(\xi)$ *satisfies*[4]

$$\limsup_{\substack{t+j\to\infty \\ (t,j)\in\operatorname{dom}\phi}} V(\phi(t,j)) = \limsup_{\substack{t+j\to\infty \\ (t,j)\in\operatorname{dom}\phi}} q(\phi(t,j)) \tag{3.11}$$

Then

$$\mathcal{J}^*(\xi) = V(\xi) \qquad \forall \xi \in \Pi(\overline{C}) \cup \Pi(D), \tag{3.12}$$

and any stationary feedback law $\kappa := (\kappa_C, \kappa_D) = ((\kappa_{C1}, \kappa_{C2}), (\kappa_{D1}, \kappa_{D2})) : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ *with values*

$$\kappa_C(x) \in \arg\min_{u_{C1}} \max_{u_{C2}}_{u_C=(u_{C1},u_{C2})\in\Pi_u^C(x)} \{L_C(x,u_C) + \langle \nabla V(x), F(x,u_C)\rangle\} \qquad \forall x \in \Pi(C) \tag{3.13}$$

and

$$\kappa_D(x) \in \arg\min_{u_{D1}} \max_{u_{D2}}_{u_D=(u_{D1},u_{D2})\in\Pi_u^D(x)} \{L_D(x,u_D) + V(G(x,u_D))\} \qquad \forall x \in \Pi(D) \tag{3.14}$$

is a pure strategy saddle-point equilibrium for Problem $(\diamond)$ *with* $\mathcal{J}_1 = \mathcal{J}$*,* $\mathcal{J}_2 = -\mathcal{J}$*, where* $\mathcal{J}$ *is as in* (3.3).

Notice that when the players select the optimal strategy, the value function equals the function V evaluated at the initial condition. This makes evident the independence of the result from needing to compute solutions, at the price of finding the function V satisfying the conditions therein.

[4]The boundary condition (3.11) matches the value of V to the terminal cost q at the final value of ϕ.

To establish the proof of Theorem 3.2.1, we first present the following results providing sufficient conditions to bound and exactly evaluate the cost of the game. These results are instrumental on guaranteeing that the saddle-point equilibrium is attained and in evaluating the value function of the game.

Proposition 3.2.2. (Time-dependent conditions for upper bound) *Consider* $(\phi, u) \in \mathcal{S}_{\mathcal{H}}^{\infty}(\xi)$ *with* $u = (u_C, u_D)$*, such that*

1) for each $j \in \mathbb{N}$ *such that* I_ϕ^j *has a nonempty interior*[5] $\mathrm{int} I_\phi^j$,

$$L_C(\phi(t,j), u_C(t,j)) + \frac{d}{dt} V(\phi(t,j)) \leq 0 \qquad \forall t \in \mathrm{int} I_\phi^j \tag{3.15}$$

and

2) for every $(t_{j+1}, j) \in \mathrm{dom}\,\phi$ *such that* $(t_{j+1}, j+1) \in \mathrm{dom}\,\phi$,

$$L_D(\phi(t_{j+1}, j), u_D(t_{j+1}, j)) + V(\phi(t_{j+1}, j+1)) - V(\phi(t_{j+1}, j)) \leq 0. \tag{3.16}$$

Then

$$\sum_{j=0}^{\sup_j \mathrm{dom}\,\phi} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j), u_C(t,j)) dt + \sum_{j=0}^{\sup_j \mathrm{dom}\,\phi - 1} L_D(\phi(t_{j+1}, j), u_D(t_{j+1}, j)) + \limsup_{\substack{t+j \to \infty \\ (t,j) \in \mathrm{dom}\phi}} V(\phi(t,j)) \leq V(\xi). \tag{3.17}$$

The following corollary is immediate from the proof of Proposition 3.2.2.

Corollary 3.2.3. (Change of Signs) *If the inequalities in the conditions in Proposition 3.2.2 are inverted, namely, if "*$\leq$*" in* (3.15) *and* (3.16) *is replaced with "*$\geq$*", then* (5.13) *holds with the inequality inverted. Likewise, if the conditions in Proposition 3.2.2 hold with equalities, then* (5.13) *holds with equality.*

Remark 3.2.4. (Connections between Theorem 3.2.1 and Problem ($\diamond$)) *Given* $\xi \in \Pi(\overline{C}) \cup \Pi(D)$*, if there exist a function* V *satisfying the conditions in Theorem 3.2.1, then a solution to Problem* ($\diamond$) *exists, namely there is an optimizer input action* $u^* =$

[5] When $j = \sup_j \mathrm{dom}\,\phi \in \mathbb{N}$ and $\sup_t \mathrm{dom}\,\phi = \infty$, we define $t_{j+1} := \infty$.

$(u_C^*, u_D^*) = ((u_{C1}^*, u_{C2}^*), (u_{D1}^*, u_{D2}^*)) \in \mathcal{U}_{\mathcal{H}}^{\infty}(\xi)$ *that satisfies* (3.2), *and* V *is the value function as in Definition 3.1.6. In addition, notice that the strategy* $\kappa = (\kappa_C, \kappa_D) \in \mathcal{K}$ *with elements as in (3.13) and (3.14) is such that every maximal solution to the closed-loop system* $\mathcal{H}_\kappa$ *from* ξ *has a cost that is equal to the min-max in (6.3), which is equal to the max-min.*

Remark 3.2.5. (Existence of a value function) *Theorem 3.2.1 does not explicitly rely on regularity conditions over the stage costs, flow and jump maps, convexity of* $\mathcal{J}$, *or compactness of the set of inputs* $\mathcal{U}_{\mathcal{H}}^{\infty}$. *Sufficient conditions to guarantee the existence of a solution to Problem* ($\diamond$) *are not currently available in the literature. One could expect that, as in any converse results, guaranteeing the existence of a value function satisfying* (3.9) *and* (3.10) *would require the data of the system and the game to satisfy certain regularity properties. In the context of optimal control such regularity is required to guarantee existence [48].*

Remark 3.2.6. (Computation of the function V) *In some cases, computing the saddle-point equilibrium strategy and the function* V *satisfying the HJBI hybrid equations is difficult. This is a challenge already present in the certification of asymptotic stability. However, the complexity associated to the computation of a Lyapunov function does not diminish the contribution that the sufficient conditions for stability have had in the field. In the same spirit, a contribution of Theorem 3.2.1, as an important step in games with dynamics defined as in [69], is in providing sufficient conditions that characterize value functions and saddle-point equilibria for such systems, similar to the results for continuous-time and discrete-time systems already available in the literature; see,e.g., [13].*

3.3 Asymptotic Stability for Hybrid Games with Infinite Horizon

We present a result that connects optimality and asymptotic stability for two-player zero-sum hybrid games. First, we introduce definitions of some classes of functions.

Definition 3.3.1. (Class-$\mathcal{K}_\infty$ functions) *A function* $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is a class-*$\mathcal{K}_\infty$ *function, also written as* $\alpha \in \mathcal{K}_\infty$, *if* α *is zero at zero, continuous, strictly increasing, and unbounded.*

Definition 3.3.2. (Positive definite functions) *A function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is positive definite, also written as $\rho \in \mathcal{PD}$, if $\rho(s) > 0$ for all $s > 0$ and $\rho(0) = 0$. A function $\rho : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ is positive definite with respect to a set $\mathcal{A} \subset \mathbb{R}^n$, in composition with $\kappa : \mathbb{R}^n \to \mathbb{R}^m$, also written as $\rho \in \mathcal{PD}_\kappa(\mathcal{A})$, if $\rho(x, \kappa(x)) > 0$ for all $x \in \mathbb{R}^n \setminus \mathcal{A}$ and $\rho(\mathcal{A}, \kappa(\mathcal{A})) = \{0\}$.*

Definition 3.3.3. (Uniform global asymptotic stability) *A closed set $\mathcal{A} \subset \mathbb{R}^n$ is uniformly globally asymptotically stable for a hybrid closed-loop system $\mathcal{H}_\kappa$ as in (2.4) if it is*

- *uniformly globally stable for $\mathcal{H}_\kappa$, i.e., there exists a class-$\mathcal{K}_\infty$ function α such that any solution ϕ to $\mathcal{H}_\kappa$ satisfies $|\phi(t,j)|_\mathcal{A} \leq \alpha(|\phi(0,0)|_\mathcal{A})$ for all $(t,j) \in \operatorname{dom}\phi$; and*
- *uniformly globally attractive for $\mathcal{H}_\kappa$, i.e., for each $\varepsilon > 0$ and $r > 0$ there exists $T > 0$ such that, for any solution ϕ to $\mathcal{H}_\kappa$ with $|\phi(0,0)|_\mathcal{A} \leq r$, $(t,j) \in \operatorname{dom}\phi$ and $t + j \geq T$ imply $|\phi(t,j)|_\mathcal{A} \leq \varepsilon$.*

In the next result, we provide alternative conditions to those in Theorem 3.2.1 for the solution to Problem ($\diamond$).

Lemma 3.3.4. (Equivalent conditions) *Given $\mathcal{H}_\kappa$ as in (2.4) with data (C, F, D, G) and feedback $\kappa := (\kappa_C, \kappa_D) = ((\kappa_{C1}, \kappa_{C2}), (\kappa_{D1}, \kappa_{D2})) : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ that satisfies* (3.13) *and* (3.14)*, if there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on a neighborhood of $\Pi(C)$ such that*[6] *$C_\kappa = \Pi(C)$, $D_\kappa = \Pi(D)$, then (3.9), (3.10), (3.13), and (3.14) are satisfied if and only if*

$$L_C(x, \kappa_C(x)) + \langle \nabla V(x), F(x, \kappa_C(x)) \rangle = 0 \quad \forall x \in C_\kappa, \tag{3.18}$$

$$L_C(x, (u_{C1}, \kappa_{C2}(x))) + \langle \nabla V(x), F(x, (u_{C1}, \kappa_{C2}(x))) \rangle \geq 0 \qquad \forall (x, u_{C1}) : \ (x, (u_{C1}, \kappa_{C2}(x))) \in C, \tag{3.19}$$

$$L_C(x, (\kappa_{C1}(x), u_{C2})) + \langle \nabla V(x), F(x, (\kappa_{C1}(x), u_{C2})) \rangle \leq 0 \qquad \forall (x, u_{C2}) : \ (x(\kappa_{C1}(x), u_{C2})) \in C, \tag{3.20}$$

[6] Notice that $C_\kappa = \Pi(C)$ and $D_\kappa = \Pi(D)$ when $\kappa_C(x) \in \Pi_u^C(x)$ for all $x \in \Pi(C)$ and $\kappa_D(x) \in \Pi_u^D(x)$ for all $x \in \Pi(D)$. In words, the feedback law κ defining the hybrid closed-loop system $\mathcal{H}_\kappa$ does not render input actions outside C or D.

$$L_D(x, \kappa_D(x)) + V(G(x, \kappa_D(x))) = V(x) \quad \forall x \in D_\kappa, \tag{3.21}$$

$$L_D(x, (u_{D1}, \kappa_{D2}(x))) + V(G(x, (u_{D1}, \kappa_{D2}(x)))) \geq V(x)$$
$$\forall (x, u_{D1}) : \ (x, (u_{D1}, \kappa_{D2}(x))) \in D, \tag{3.22}$$

$$L_D(x, (\kappa_{D1}(x), u_{D2})) + V(G(x, (\kappa_{D1}(x), u_{D2}))) \leq V(x)$$
$$\forall (x, u_{D2}) : \ (x, (\kappa_{D1}(x), u_{D2})) \in D. \tag{3.23}$$

Theorem 3.3.5. (Saddle-point equilibrium under the existence of a Lyapunov function)

Consider a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ as in (2.3) *with data (C, F, D, G) satisfying Assumption 3.1.4, and $\kappa := (\kappa_C, \kappa_D) : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ defining the closed-loop dynamics $\mathcal{H}_\kappa$ as in (2.4) such that $C_\kappa = \Pi(C)$ and $D_\kappa = \Pi(D)$, and every maximal solution to $\mathcal{H}_\kappa$ from $\overline{C_\kappa} \cup D_\kappa$ is complete. Given a closed set $\mathcal{A} \subset \Pi(C) \cup \Pi(D)$, continuous functions $L_C : C \to \mathbb{R}_{\geq 0}$ and $L_D : D \to \mathbb{R}_{\geq 0}$ defining the stage costs for flows and jumps, respectively, and $q : \mathbb{R}^n \to \mathbb{R}$ defining the terminal cost, suppose there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on an open set containing $\overline{C_\kappa}$, satisfying (3.18)-(3.23), and such that for each $\xi \in \overline{C_\kappa} \cup D_\kappa$, each $\phi \in \mathcal{S}^\infty_{\mathcal{H}_\kappa}(\xi)$ satisfies (3.11). If there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that*

$$\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A}) \qquad \forall x \in \overline{C_\kappa} \cup D_\kappa \tag{3.24}$$

and one of the following conditions holds

1) *$L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A})$ and $L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$;*

2) *$L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$ and there exists a continuous function $\eta \in \mathcal{PD}$ such that $L_C(x, \kappa_D(x)) \geq \eta(|x|_\mathcal{A})$ for all $x \in C_\kappa$;*

3) *$L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A})$ and there exists a continuous function $\eta \in \mathcal{PD}$ such that $L_D(x, \kappa_D(x)) \geq \eta(|x|_\mathcal{A})$ for all $x \in D_\kappa$;*

4) *$L_C \equiv 0, L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$, and for each $r > 0$, there exist $\gamma_r \in \mathcal{K}_\infty$ and $N_r \geq 0$ such that for every solution $\phi \in \mathcal{S}^\infty_{\mathcal{H}_\kappa}(\xi)$, $|\phi(0,0)|_\mathcal{A} \in (0, r]$, $(t,j) \in \operatorname{dom}\phi$, $t + j \geq T$ imply $j \geq \gamma_r(T) - N_r$;*

5) $L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A}), L_D \equiv 0$, and for each $r > 0$, there exist $\gamma_r \in \mathcal{K}_\infty$ and $N_r \geq 0$ such that for every solution $\phi \in \mathcal{S}^\infty_{\mathcal{H}_\kappa}(\xi)$, $|\phi(0,0)|_\mathcal{A} \in (0, r]$, $(t,j) \in \operatorname{dom}\phi$, $t + j \geq T$ imply $t \geq \gamma_r(T) - N_r$;

6) $L_C(x, \kappa_C(x)) \geq -\lambda_C V(x)$ for all $x \in C_\kappa$, $L_D(x, \kappa_D(x)) \geq (1 - e^{\lambda_D})V(x)$ for all $x \in D_\kappa$, and there exist $\gamma > 0$ and $M > 0$ such that, for each solution $\phi \in \mathcal{S}^\infty_{\mathcal{H}_\kappa}(\xi)$, $(t,j) \in \operatorname{dom}\phi$ implies $\lambda_C t + \lambda_D j \leq M - \gamma(t + j)$;

then

$$\mathcal{J}^*(\xi) = V(\xi), \tag{3.25}$$

for all $\xi \in \overline{C_\kappa} \cup D_\kappa$. Furthermore, the feedback law κ is the saddle-point equilibrium (see Definition 3.1.3) and it renders $\mathcal{A}$ uniformly globally asymptotically stable for $\mathcal{H}_\kappa$.

3.4 Applications

We illustrate in the following applications with hybrid dynamics and quadratic costs how Theorem 3.2.1 provides conditions to solve the disturbance rejection and security problems introduced above by addressing them as zero-sum hybrid games.

3.4.1 Application 1.1: Robust Hybrid LQR with Aperiodic Jumps

In this section, we study a special case of Application 1 that emerges in practical scenarios with hybrid systems with linear flow and jump maps and aperiodic jumps, as in noise attenuation of cyber-physical systems, see, e.g., [47, 70, 71]. We introduce a state variable τ that plays the role of a timer. Once τ reaches an element in a threshold set $\{T_1, T_2\}$ with $0 \leq T_1 \leq T_2$, it potentially[7] triggers a jump in the state and resets τ to zero. More precisely, given $\bar{T} \in \mathbb{R}$, we consider a hybrid system with state $x = (x_p, \tau) = ((x_{p1}, x_{p2}), \tau) \in \mathbb{R}^n \times [0, T_2]$, input $u = (u_C, u_D) = ((u_{C1}, u_{C2}),$

[7] When $T_1 < T_2$, solutions can either evolve via flow or jump when $\tau = T_1$. A sequence $\{T_i\}_{i=1}^N$ can be handled similarly.

$(u_{D1}, u_{D2})) \in \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$, and dynamics $\mathcal{H}$ as in (2.3), defined by

$$\begin{aligned} C &:= \mathbb{R}^n \times [0, T_2] \times \mathbb{R}^{m_C} \\ F(x, u_C) &:= (A_C x_p + B_C u_C, 1) \quad \forall (x, u_C) \in C \\ D &:= \mathbb{R}^n \times \{T_1, T_2\} \times \mathbb{R}^{m_D} \\ G(x, u_D) &:= (A_D x_p + B_D u_D, 0) \quad \forall (x, u_D) \in D \end{aligned} \tag{3.26}$$

with $A_C = \begin{bmatrix} A_{C1} & 0 \\ 0 & A_{C2} \end{bmatrix}, B_C = [\, B_{C1} \; B_{C2} \,], A_D = \begin{bmatrix} A_{D1} & 0 \\ 0 & A_{D2} \end{bmatrix}$, and $B_D = [\, B_{D1} \; B_{D2} \,]$. Following Application 1, the input $u_1 := (u_{C1}, u_{D1})$ plays the role of the control and is assigned by player P_1, and $u_2 := (u_{C2}, u_{D2})$ is the disturbance input, which is assigned by player P_2. The problem of upper bounding the effect of the disturbance u_2 in the cost of complete solutions to $\mathcal{H}$ is formulated as a two-player zero-sum game as in Section 3.1.1. Thus, by solving Problem ($\diamond$) for every $\xi \in \Pi(C) \cup \Pi(D)$, the control objective is achieved.

The following result presents a tool for the solution of optimal control problems for hybrid systems with linear maps and aperiodic jumps under an adversarial action.

Proposition 3.4.1. (Hybrid Riccati equation for disturbance rejection with aperiodic jumps) *Given a hybrid system $\mathcal{H}$ as in (2.3) defined by (C, F, D, G) as in (3.26), let $\bar{T} \in \mathbb{R}$, and, with the aim of pursuing minimum energy and distance to the origin, consider the cost functions $L_C(x, u_C) := x_p^\top Q_C x_p + u_{C1}^\top R_{C1} u_{C1} + u_{C2}^\top R_{C2} u_{C2}$, $L_D(x, u_D) := x_p^\top Q_D x_p + u_{D1}^\top R_{D1} u_{D1} + u_{D2}^\top R_{D2} u_{D2}$, and terminal cost $q(x) := x_p^\top P(\tau) x_p$ defining $\mathcal{J}$ as in (3.3), with $Q_C, Q_D \in \mathbb{S}_+^n$, $R_{C1} \in \mathbb{S}_+^{m_{C_1}}$, $-R_{C2} \in \mathbb{S}_+^{m_{C_2}}$, $R_{D1} \in \mathbb{S}_+^{m_{D_1}}$, and $-R_{D2} \in \mathbb{S}_+^{m_{D_2}}$. Suppose there exists a matrix function $P : [0, T_2] \to \mathbb{S}_+^n$ that is continuously differentiable and such that*

$$-\frac{d}{d\tau} P(\tau) = - P(\tau)(B_{C2} R_{C2}^{-1} B_{C2}^\top + B_{C1} R_{C1}^{-1} B_{C1}^\top) P(\tau) + Q_C + P(\tau) A_C + A_C^\top P(\tau) \quad \forall \tau \in (0, T_2), \tag{3.27}$$

$$-R_{D2} - B_{D2}^\top P(0) B_{D2}, \quad R_{D1} + B_{D1}^\top P(0) B_{D1} \quad \in \mathbb{S}_{0+}^{m_D}, \tag{3.28}$$

the matrix $R_v = \begin{bmatrix} R_{D1} + B_{D1}^\top P(0) B_{D1} & B_{D1}^\top P(0) B_{D2} \\ B_{D2}^\top P(0) B_{D1} & R_{D2} + B_{D2}^\top P(0) B_{D2} \end{bmatrix}$ *is invertible, and*

$$P(\bar{T}) = Q_D + A_D^\top P(0) A_D - [\, A_D^\top P(0) B_{D1} \; A_D^\top P(0) B_{D2} \,] R_v^{-1} \begin{bmatrix} B_{D1}^\top P(0) A_D \\ B_{D2}^\top P(0) A_D \end{bmatrix} \tag{3.29}$$

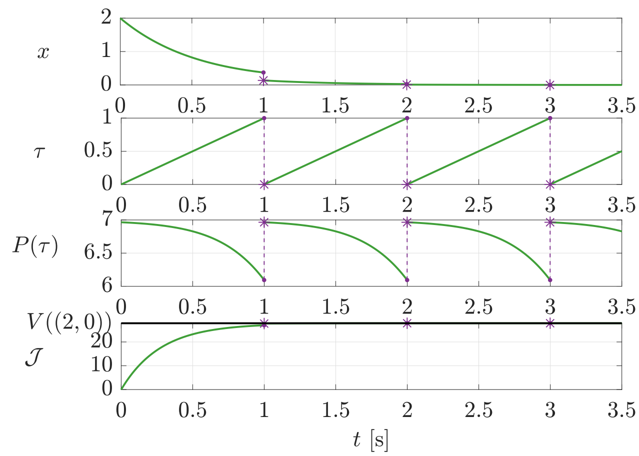

Figure 3.1: 1D Robust hybrid LQR with periodic jumps. Dynamics as in (3.26) with $A_C = 1.8, B_C = [1,1], A_D = 2, B_D = [1,1], Q_C = 0.1,\ R_C = \text{diag}(1.304, -4), Q_D = 1, R_D = \text{diag}(1.304, -8),\ P(0) = 6.9653, T_1 = T_2 = 1$.

at each $\bar{T} \in \{T_1, T_2\}$, where $A_C, B_{C1}, B_{C2}, A_D, B_{D1}$, and B_{D2} are defined below (3.26)*. Then, the feedback law $\kappa := (\kappa_C, \kappa_D)$, with values*

$$\kappa_C(x) = (-R_{C1}^{-1} B_{C1}^\top P(\tau) x_p, -R_{C2}^{-1} B_{C2}^\top P(\tau) x_p) \quad \forall x \in \Pi(C), \tag{3.30}$$

$$\kappa_D(x) = -R_v^{-1} \begin{bmatrix} B_{D1}^\top P(0) A_D \\ B_{D2}^\top P(0) A_D \end{bmatrix} x_p \quad \forall x \in \Pi(D) \tag{3.31}$$

is the pure strategy saddle-point equilibrium for the two-player zero-sum hybrid game with periodic jumps. In addition, for each $x = (x_p, \tau) \in \Pi(\overline{C}) \cup \Pi(D)$, the value function is equal to $V(x) := x_p^\top P(\tau) x_p$.

Notice that the saddle-point equilibrium $\kappa = (\kappa_1, \kappa_2)$ is composed by P_1 playing the upper bounding strategy κ_1, and P_2 playing the maximizing disturbance κ_2 with values as in (3.30) and (3.31).

Furthermore, notice that when $T_1 < T_2$ are finite, the jumps are not necessarily periodic, since they can occur when $\tau = T_1$ or when $\tau = T_2$. When $T_1 = T_2 = 0$ we recover the discrete-time LQR robust problem, when $T_1 = T_2 = \infty$ we recover the continuous-time LQR robust problem, and when $T_1 = T_2$ are finite, we have a hybrid game with periodic jumps as in in Figure 3.1.

3.4.2 Application 1.2: Robust Control with Flows-Actuated Nonunique Solutions

As illustrated next, there are useful families of hybrid systems for which a pure strategy saddle-point equilibrium exists. The following problem (with nonunique solutions to $\mathcal{H}$ for a given feedback law) characterizes both the pure strategy saddle-point equilibrium and the value function in a two-player zero-sum game with a one-dimensional state, that is associated to player P_1, i.e., $n_1 = 1, n_2 = 0$.

Consider a hybrid system $\mathcal{H}$ with state $x \in \mathbb{R}$, input $u_C := (u_{C1}, u_{C2}) \in \mathbb{R}^2$, and dynamics

$$\begin{aligned} \dot{x} = &\quad F(x, u_C) := ax + \langle B, u_C \rangle & x \in [0, \delta] \\ x^+ = &\quad G(x) := \sigma & x = \mu \end{aligned} \tag{3.32}$$

where $a < 0, B = (b_1, b_2) \in \mathbb{R}^2$ and $\mu > \delta > \sigma > 0$[8]. Consider the cost functions $L_C(x, u_C) := x^2 Q_C + u_C^\top R_C u_C$, $L_D(x) := P(x^2 - \sigma^2)$, and terminal cost $q(x) := Px^2$, defining $\mathcal{J}$ as in (3.3), with $R_C := \begin{bmatrix} R_{C1} & 0 \\ 0 & R_{C2} \end{bmatrix}$, Q_C, R_{C1}, $-R_{C2}$, $P \in \mathbb{R}_{>0}$, such that

$$Q_C + 2Pa - P^2(b_1^2 R_{C1}^{-1} + b_2^2 R_{C2}^{-1}) = 0. \tag{3.33}$$

Following Application 1, the input $u_1 := (u_{C1}, u_{D1})$ designed by player P_1 plays the role of the control and $u_2 := (u_{C2}, u_{D2})$ is the disturbance input assigned by player P_2. This is formulated as a two-player zero-sum hybrid game via solving Problem ($\diamond$) in Section 3.1.3. The function $V(x) := Px^2$ is such that

$$\begin{aligned} &\min_{\substack{u_{C1} \\ u_C = (u_{C1}, u_{C2}) \in \mathbb{R}^2}} \max_{u_{C2}} \quad \{L_C(x, u_C) + \langle \nabla V(x), F(x, u_C) \rangle\} \\ &= \min_{u_{C1} \in \mathbb{R}} \max_{u_{C2} \in \mathbb{R}} \left\{ (Q_C + 2Pa)x^2 + R_{C1} u_{C1}^2 + R_{C2} u_{C2}^2 + 2xP(b_1 u_{C1} + b_2 u_{C2}) \right\} \\ &= 0 \end{aligned} \tag{3.34}$$

holds for all $x \in [0, \delta]$. In fact, the min-max in (6.46) is attained by $\kappa_C(x) = (-R_{C1}^{-1} b_1 Px, -R_{C2}^{-1} b_2 Px)$. In particular, thanks to (6.45), we have $\mathcal{L}_C(x, \kappa_C(x)) = 0$. Then, $V(x) = Px^2$ is a solution to (3.9). In addition, the function V is such that

$$\min_{\substack{u_{D1} \\ (u_{D1}, u_{D2}) \in \mathbb{R}^2}} \max_{u_{D2}} \quad \{L_D(x) + V(G(x))\} = Px^2 \tag{3.35}$$

[8]Given that $\mu > \delta$, flow from μ is not possible.

at $x = \mu$, which makes $V(x) = Px^2$ a solution to (3.10) with saddle-point equilibrium κ_C. Given that V is continuously differentiable on $\mathbb{R}$, and that (3.9) and (3.10) hold thanks to (6.46) and (6.47), from Theorem 3.2.1 we have that the value function is $\mathcal{J}^*(\xi) := P\xi^2$ for any $\xi \in [0, \delta] \cup \{\mu\}$.

To investigate the case of nonunique solutions yielded by the feedback law κ_C, now let $\delta \geq \mu > \sigma > 0$ and notice that solutions can potentially flow or jump at $x = \mu$. The set of all maximal solutions from $\xi = \delta$ is denoted $\mathcal{R}_\kappa(\xi) = \{\phi_\kappa, \phi_h\}$. The continuous solution ϕ_κ is such that $\operatorname{dom} \phi_\kappa = \mathbb{R}_{\geq 0} \times \{0\}$, and is given by $\phi_\kappa(t, 0) = \delta \exp((a - R_{C1}^{-1} b_1 P - R_{C2}^{-1} b_2 P)t)$ for all $t \in [0, \infty)$. In simple words, ϕ_κ flows from δ, and converges (exponentially fast) to 0. The maximal solution ϕ_h has domain $\operatorname{dom} \phi_h = ([0, t^h] \times \{0\}) \cup ([t^h, \infty) \times \{1\})$, and is given by $\phi_h(t, 0) = \delta \exp((a - R_{C1}^{-1} b_1 P - R_{C2}^{-1} b_2 P)t)$, $\quad \phi_h(t, 1) = \sigma \exp((a - R_{C1}^{-1} b_1 P - R_{C2}^{-1} b_2 P)(t - t^h))$. In simple words, ϕ_h flows from δ to μ in t^h seconds, then it jumps to σ, and flows converging (exponentially fast) to zero. Figure 6.2 illustrates this behavior. By denoting the corresponding input signals as $u_\kappa = \kappa(\phi_\kappa)$ and $u_h = \kappa(\phi_h)$, we show in the bottom of Figure 6.2 that the cost of the solutions ϕ_κ and ϕ_h, yielded by κ_C, equal $P\delta^2$. This corresponds to the optimal value with every

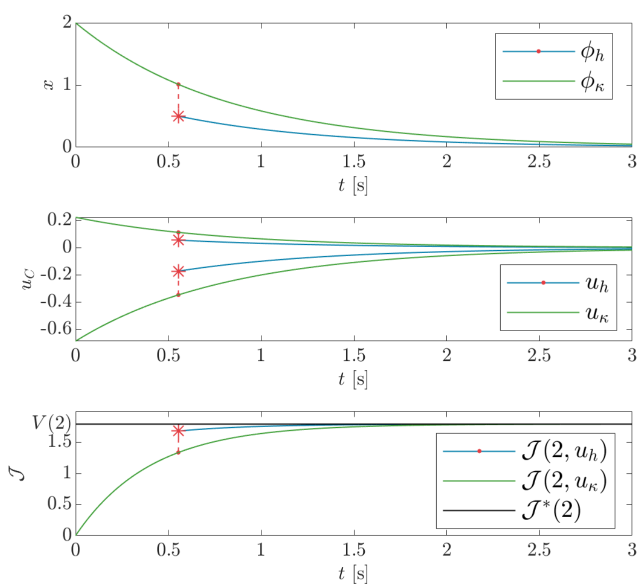

Figure 3.2: Nonunique solutions attaining min-max optimal cost for $a = -1, b_1 = b_2 = 1, \delta = \xi = 2, \mu = 1, \sigma = 0.5, Q_C = 1$, $R_{C1} = 1.304, R_{C2} = -4$, and $P = 0.4481$. Continuous solution (green). Hybrid solution (blue and red).

maximal solution rendered by the equilibrium κ_C from $\xi = 2$ attaining it.

The next example illustrates Theorem 3.3.5 and shows that our results, in the spirit of the Lyapunov theorem, only require that the conditions in Corollary 3.3.5 hold.

Example 3.4.2. (Hybrid game with nonunique solutions) Let $\mathcal{A} = \{0\}$ and given that $L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A})$, (3.18)-(3.23) hold, and the function $s \mapsto \eta(s) =: P\frac{s^2}{2}$ is such that $L_D(x, \kappa_D(x)) \geq \eta(|x|_{\mathcal{A}})$ for all $x \in D_\kappa$, by setting $\alpha_1(|x|_{\mathcal{A}}) = \lambda_{\min}(P)|x|^2$ and $\alpha_2(|x|_{\mathcal{A}}) = \overline{\lambda}(P)|x|^2$, from Corollary 3.3.5 we have that κ_C is the saddle-point equilibrium and renders $\mathcal{A}$ uniformly globally asymptotically stable for $\mathcal{H}$ as in (3.32). □

3.4.3 Application 2: Security Jumps-Actuated Hybrid Game

We study a special case of Application 2 and apply Theorem 3.2.1 in this section. Consider a hybrid system with state $x \in \mathbb{R}^n$, input $u_D = (u_{D1}, u_{D2}) \in \mathbb{R}^{m_D}$, and dynamics $\mathcal{H}$ as in (2.3), described by

$$\begin{aligned} \dot{x} = &\quad F(x) & x \in C \\ x^+ = &\quad A_D x + \begin{bmatrix} B_{D1} \; B_{D2} \end{bmatrix} \begin{bmatrix} u_{D1} \\ u_{D2} \end{bmatrix} & (x, u_D) \in D \end{aligned} \tag{3.36}$$

with Lipschitz continuous $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}, A_D \in \mathbb{R}^{n\times n}$, and $C \subset \mathbb{R}^n, D \subset \mathbb{R}^n \times \mathbb{R}^{m_D}$, such that $C \cup \Pi(D)$ is nonempty. The input u_{D1} plays the role of the control and u_{D2} the disturbance input. Following Application 2, the problem of minimizing a cost functional $\mathcal{J}$ in the presence of the maximizing attack u_{D2} is formulated as a two-player zero-sum game. Thus, by solving Problem ($\diamond$) for every $\xi \in \Pi(C) \cup \Pi(D)$,the control objective is achieved.

The following result presents a tool for the solution of optimal control problems for jumps-actuated hybrid systems and state-affine flow maps under a malicious input attack designed to cause as much damage as possible.

Corollary 3.4.3. (Hybrid Riccati equation for security) *Given a hybrid system $\mathcal{H}$ as in* (2.3) *defined by* (C, F, D, G) *as in* (3.36), *and, with the aim of pursuing minimum energy and distance to the origin, consider the cost functions* $L_C(x, u_C) := 0$, $L_D(x, u_D) := x^\top Q_D x + u_{D1}^\top R_{D1} u_{D1} + u_{D2}^\top R_{D2} u_{D2}$, *and terminal cost* $q(x) := x^\top P x$, *defining $\mathcal{J}$ as in (3.3), with $Q_D \in \mathbb{S}^n_+$, $R_{D1} \in \mathbb{S}^{m_{D_1}}_+$, $-R_{D2} \in \mathbb{S}^{m_{D_2}}_+$ and $P \in \mathbb{S}^n_+$. Suppose there exists a* *matrix* $P \in \mathbb{S}^n_+$ *such that*

$$0 = 2x^\top PF(x) \qquad \forall x \in \Pi(C), \tag{3.37}$$

$$-R_{D2} - B_{D2}^\top PB_{D2}, R_{D1} + B_{D1}^\top PB_{D1} \quad \in \mathbb{S}^{m_D}_{0+}, \tag{3.38}$$

the matrix $R_v = \begin{bmatrix} R_{D1}+B_{D1}^\top PB_{D1} & B_{D1}^\top PB_{D2} \\ B_{D2}^\top PB_{D1} & R_{D2}+B_{D2}^\top PB_{D2} \end{bmatrix}$ *is invertible, and*

$$0 = -P + Q_D + A_D^\top PA_D - \begin{bmatrix} A_D^\top PB_{D1} & A_D^\top PB_{D2} \end{bmatrix} R_v^{-1} \begin{bmatrix} B_{D1}^\top PA_D \\ B_{D2}^\top PA_D \end{bmatrix} \tag{3.39}$$

Then, the feedback law

$$\kappa_{D1}(x) = -[R_v^{-1}(1,1)\; R_v^{-1}(1,2)] \begin{bmatrix} B_{D1}^\top PA_D \\ B_{D2}^\top PA_D \end{bmatrix} x \quad \forall x \in \Pi(D) \tag{3.40}$$

minimizes the cost functional $\mathcal{J}$ *in the presence of the maximizing attack* u_2*, given by*

$$\kappa_{D2}(x) = -[R_{v_{21}}^{-1}\; R_{v_{22}}^{-1}] \begin{bmatrix} B_{D1}^\top PA_D \\ B_{D2}^\top PA_D \end{bmatrix} x \quad \forall x \in \Pi(D) \tag{3.41}$$

In addition, for each $x \in \Pi(\overline{C}) \cup \Pi(D)$*, the value function is equal to* $V(x) := x^\top Px$*.*

Notice that the saddle-point equilibrium $\kappa_D := (\kappa_{D1}, \kappa_{D2})$ is composed by P_1 playing the minimizer strategy κ_{D1} as in (3.40), and P_2 playing the maximizing attack κ_{D2} as in (3.41).

Example 3.4.4. (Bouncing ball hybrid game with infinite horizon) Inspired by the problem in [72], consider a simplified model of a juggling system as in [73], with state $x = (x_1, x_2) \in \mathbb{R}^2$, input $u_D := (u_{D1}, u_{D2}) \in \mathbb{R}^2$, and dynamics $\mathcal{H}$ as in (2.3), with data

$$\begin{aligned} C = \mathbb{R}_{\geq 0} \times \mathbb{R}, \qquad F(x) = (x_2, -1) \quad \forall x \in C \\ D = \{0\} \times \mathbb{R}_{\leq 0} \times \mathbb{R}^2, \\ G(x, u_D) = (0, -\lambda x_2 + u_{D1} + u_{D2}) \quad \forall (x, u_D) \in D \end{aligned} \tag{3.42}$$

where u_{D1} is the control input, u_{D2} is the action of an attacker, and $\lambda \in (0, 1)$ is the coefficient of restitution of the ball. As an instance of Application 2, the scenario in which u_{D1} is designed to minimize a cost functional $\mathcal{J}$ under the presence of the worst-case is formulated as a two-player zero-sum game. With the aim of pursuing minimum velocity and control effort at jumps, consider the cost functions $L_C(x, u_C) :=$

0, $L_D(x,u_D) := x_2^2 Q_D + u_D^\top R_D u_D$, and terminal cost $q(x) := \frac{1}{2}x_2^2 + x_1$ defining $\mathcal{J}$ as in (3.3), with $R_D := \begin{bmatrix} R_{D1} & 0 \\ 0 & R_{D2} \end{bmatrix}$ and Q_D, R_{D1}, $-R_{D2} > 0$. Here, u_{D1} is designed by player P_1, which aims to minimize $\mathcal{J}$, while player P_2 seeks to maximize it by choosing u_{D2}.

The function $V(x) := x_1 + \frac{1}{2}x_2^2$ is such that $\langle \nabla V(x), F(x) \rangle = 0$ for all $x \in C$, making V a solution to (3.9). In addition, the function V is such that

$$\min_{\substack{u_{D1} \\ u_D=(u_{D1},u_{D2})\in\mathbb{R}^2}} \max_{u_{D2}} \{L_D(x,u_D) + V(G(x,u_D))\} = \frac{1}{2}x_2^2 \tag{3.43}$$

for all $(x,u_D) \in D$. Equality (3.43) is attained by $\kappa_D(x) = (\kappa_{D1}(x), \kappa_{D2}(x))$ with $\kappa_{D1}(x) = \frac{R_{D2}\lambda}{R_{D1}+R_{D2}+2R_{D1}R_{D2}} x_2$ and $\kappa_{D2}(x) = \frac{R_{D1}\lambda}{R_{D1}+R_{D2}+2R_{D1}R_{D2}} x_2$ when

$$Q_D = \frac{-2R_{D1}R_{D2}\lambda^2 + R_{D1} + R_{D2} + 2R_{D1}R_{D2}}{2R_{D1} + 2R_{D2} + 4R_{D1}R_{D2}}, \tag{3.44}$$

which yields maximal solutions complete given that $G(x,\kappa_D(x)) \in \; C \cap \Pi(D)$, and makes V a solution to (3.10) with saddle-point equilibrium κ_D. Thus, given that V is continuously differentiable on $\mathbb{R}^2$, and that (3.9) and (3.10) hold thanks to (3.43) and (3.44), from Theorem 3.2.1, the value function is $\mathcal{J}^*(\xi) = \frac{\xi_2^2}{2} + \xi_1$. Figure 3.3 displays this behavior with both players playing the saddle point equilibrium. The cost of the displayed solution is $V(\xi)$. Figure 3.4 displays a solution, its associated cost over time, and the value function. Notice that the cost of such solution from ξ, under both players playing the saddle point equilibrium, is equal to $V(\xi)$.

Consider the case in which $\mathcal{A} = \{0\}$, encoding the goal of stabilizing the ball to rest under the effect of an attacker. Notice that, given that $L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$, and (3.18)-(3.23) hold, by setting $\alpha_1(s) = \min\left\{\frac{1}{2}\left(\frac{s}{\sqrt{2}}\right)^2, \frac{s}{\sqrt{2}}\right\}$ and $\alpha_2(s) = \frac{1}{2}s^2 + s$, from Theorem 3.3.5, we have that κ_D is the saddle-point equilibrium and renders $\mathcal{A} = \{0\}$ uniformly globally asymptotically stable for $\mathcal{H}$.

In Figure 3.5, we let the players select feedback laws close to the Nash equilibrium and calculate the cost associated to the new laws. The variation of the cost along the changes in the feedback laws makes evident the saddle-point geometry. This example illustrates how our results apply to Zeno systems. ◻

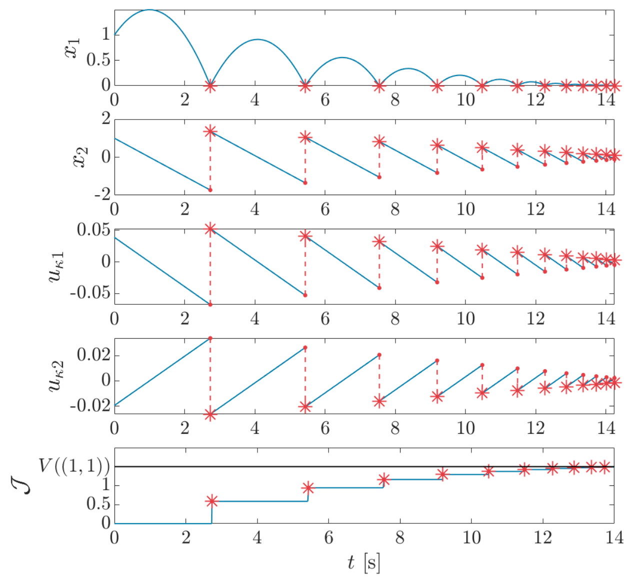

Figure 3.3: Bouncing ball solutions attaining minimum cost under worst-case u_2, with $\lambda = 0.8$, $R_{D1} = 10, R_{D2} = -20$, and $Q_D = 0.189$.

3.5 Relationship with the Literature

Some results provided in this dissertation have direct counterparts in the continuous-time and discrete-time game theory literature. The definition of a game in terms of its elements can bedirectly traced back to [13], as explained below.

Given a discrete-time two-player zero-sum game with final time[9] "J", f_k and X defining the single-valued jump map and jump set, respectively, as in [13], setting the data of $\mathcal{H}$ as $C = \emptyset$, $G = f_k$ for $k \in \mathbb{N}_{\leq J}$, and $D = X$ reduces Definition 3.1.1 to [13, Def. 5.1] for the case in which the output of each player is equal to its state and there is a feedback information structure as in [13, Def. 5.2]. Thus, items $(vi) - (ix)$ in [13, Def. 5.1] are omitted in the formulation herein and items $(i) - (v)$ and $(x) - (xi)$ are covered by Definition 3.1.1, the definition of the hybrid time domain with final time $(0, J)$, and the set $\mathcal{S}_{\mathcal{H}}$.

Given a continuous-time two-player zero-sum game with final time[10] "T", f and $\mathcal{S}^0$ defining the single-valued flow map and flow set, respectively, as in [13], setting the data of $\mathcal{H}$ as $D = \emptyset$, $F = f$, and $C = \mathcal{S}^0$ reduces Definition 3.1.1 to [13, Def. 5.5] for

[9]This corresponds to the hybrid time $(0, J)$ for $\mathcal{H}$.
[10]This corresponds to the hybrid time $(T, 0)$ for $\mathcal{H}$.

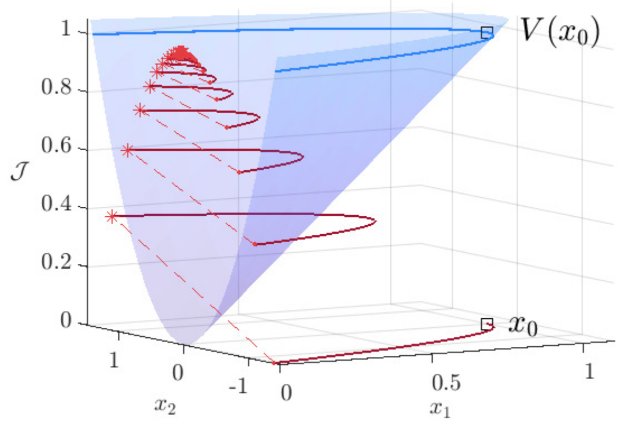

Figure 3.4: Bouncing ball cost. Initial condition (square). Value function and saddle-point equilibrium trajectory attaining evaluated cost at initial condition.

the case in which the output of each player is equal to its state and there is a feedback information structure as in [13, Def. 5.6]. Thus, items $(vi) - (vii)$ in [13, Def. 5.5] are omitted in the formulation herein and items $(i) - (v)$ and $(viii), (ix)$ are covered by Definition 3.1.1, the definition of the hybrid time domain with final time $(0, T)$, and the set $\mathcal{S}_{\mathcal{H}}$.

Remark 3.5.1. (Equivalent costs) *Given* $\xi \in \Pi(\overline{C}) \cup \Pi(D)$ *and a strategy* $\kappa^* = (\kappa_1^* \kappa_2^*) \in \mathcal{K}$, *denote by* $\mathcal{U}^*(\xi, \kappa^*)$ *the set of joint actions* $u = (u_1, u_2)$ *rendering a maximal trajectory* ϕ *to* $\mathcal{H}$ *from* ξ *with components defined as* $\operatorname{dom} \phi \ni (t, j) \mapsto u_i(t, j) = \kappa_i^*(\phi(t, j))$ *for each* $i \in \{1, 2\}$. *By expressing the largest cost associated to the solutions to* $\mathcal{H}$ *from* ξ *under the strategy* κ^* *as* $\hat{\mathcal{J}}(\xi, \kappa^*) := \sup_{u \in \mathcal{U}^*(\xi, \kappa^*)} \mathcal{J}(\xi, u)$, *an equivalent condition to (3.2) for when* $\mathcal{J}(\xi, u) = \hat{\mathcal{J}}(\xi, \kappa^*)$ *for every* $u \in \mathcal{U}^*(\xi, \kappa^*)$ *is*

$$\hat{\mathcal{J}}(\xi, (\kappa_1^*, \kappa_2)) \leq \hat{\mathcal{J}}(\xi, \kappa^*) \leq \hat{\mathcal{J}}_1(\xi, (\kappa_1, \kappa_2^*))$$

for all $\kappa_i \in \mathcal{K}_i$, $i \in \{1, 2\}$.

Remark 3.5.2. (Relation of definition of solution to literature) *By considering a discrete-time system with the single-valued function* G *or by considering a continuous-time system with* F *Lipschitz continuous in* $\overline{C}$, *and by removing the initial condition as an argument of the cost functionals and specifying it in the state equation, Remark 3.5.1 presents equivalent conditions to those in [13, (6.3)]. Thus, Definition 3.1.3 covers*

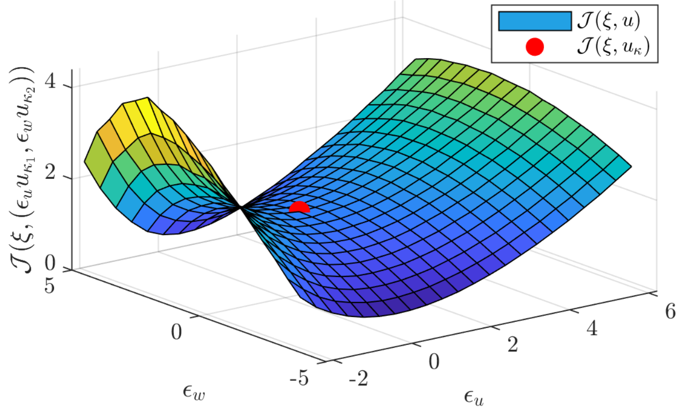

Figure 3.5: Saddle point behavior in the cost of solutions to bouncing ball from $\xi = (1,1)$ when varying the feedback gains around the optimal value. The cost is evaluated on solutions $(\phi, u) \in \mathcal{S}_{\mathcal{H}}^{\infty}(\xi)$ with feedback law variations specified by ϵ_u and ϵ_w in $u = (\epsilon_u \kappa_1(\phi), \epsilon_w \kappa_2(\phi))$.

the definitions of a pure strategy Nash equilibrium in [13, Sec. 6.2, 6.5] for the zero-sum case.

Conditions for computing value functions for linear quadratic problems have been widely studied, concerning solving differential and algebraic Riccati equations. The computation of value functions for systems with nonlinear dynamics is an open research problem and has seen interesting learning-based contributions in the last years, e.g., [51]. The computation of value functions for DAEs is discussed in [74], [75], for the case of linear differential games under algebraic constraints. Such value functions have a similar structure to the ones provided herein for hybrid systems with linear jump and flow maps and algebraic constraints encoded by the flow set C.

The design of value functions for switched DAEs imposes additional challenges that follow the discussion in [24] on the existence of Lyapunov functions and asymptotic stability. In some cases, a common Lyapunov function for all the subsystems of a switched DAE does not exist and even when it exists, it is not enough to guarantee asymptotic stability due to arbitrary switching. To solve this, conditions over switching are provided in [24, Theorem 4.1], and for the optimality of hybrid systems, such conditions are resembled by the point-wise conditions on the change of V along jumps. In [45], there are coupled value functions associated to each subsystem of a switched DAE in a zero-sum game, which result in coupled Riccati differential equations with optimal feedback strategies described by linear-time-varying functions of the state. Note that both scenarios are accounted for in the design of a value function for hybrid games based on optimality pointwise conditions provided in this work.

Acknowledgements

Sections 3.1 and 3.2, in full, are a reprint of the material as it appears in "Sufficient Conditions for Optimality and Asymptotic Stability in Two-Player Zero-Sum Hybrid Games." [1]. Sections 3.3 and 3.4 are partially a reprint of the material therein. The dissertation author was the first author of this paper.

Chapter 4

Fixed Terminal Time Hybrid Games

Following the hybrid games framework presented in Chapter 3, we generate results on sufficient conditions for optimality and design of saddle-point equilibrium feedback laws for two-player zero-sum hybrid games with fixed terminal time. To invoke the elements of a two-player zero-sum hybrid game in Definition 3.1.1, we redefine the strategy space as,

Definition 4.0.1. (Finite-horizon strategy space)

4) For each $i \in \{1, 2\}$, a strategy space K_i of P_i defined as a collection of mappings $\gamma_i : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^{m_{Ci}} \times \mathbb{R}^{m_{Di}}$. The strategy space of the game K is the collection of mappings with elements $\gamma = (\gamma_1, \gamma_2)$, where $\gamma_i \in K_i$ for each $i \in \{1, 2\}$. Each $\gamma_i \in K_i$ is said to be a permissible pure[1] strategy for P_i.

Additional constraints arise when a hybrid game is to be solved in a finite horizon, which is typically studied using backward induction tools [76]. Nevertheless, setting a priori a specific combination of the amount of continuous evolution and discrete evolution allowed to a hybrid system significantly restricts the set in which the optimization problem is solved. Therefore, in this chapter, we formulate a finite horizon optimization problem in which the terminal set $\mathcal{T}$ is properly designed to define the end of the game

[1] This is in contrast to when K_i is defined as a probability distribution, in which case $\gamma_i \in K_i$ is referred to as a mixed strategy.

constraint. The conditions on this problem formulation are similar to their counterparts in the differential/dynamic game theory literature. Nevertheless, in contrast to Chapter 3 and conventional finite-horizon game theory, the notion of terminal time herein allows for state trajectories with terminal times belonging to $\mathcal{T}$, which is endowed in a hybrid time domain-like geometry to account for hybrid time domains, as in [49]. We present sufficient conditions based on Hamilton-Jacobi-Isaacs-like equations to attain a finite-horizon saddle-point equilibrium and evaluate the game value function without computation of solutions.

As a motivational example, consider a system $\mathcal{H}$ as in (2.3), with state $x \in \mathbb{R}$, input $u_C := (u_{C1}, u_{C2}) \in \mathbb{R}^2$, and dynamics

$$\begin{aligned} \dot{x} &= F(x, u_C) := ax + Bu_C & x &\in [0, \delta] \\ x^+ &= G(x) := \sigma & x &= \mu \end{aligned} \tag{4.1}$$

where $a < 0, B = [b_1\ b_2]$, with $b_1, b_2 \in \mathbb{R}$, and $\delta \geq \mu > \sigma > 0$. Here, u_{C1} is designed by player P_1, which aims to minimize a cost functional $\mathcal{J}$, while player P_2 seeks to maximize it by choosing u_{C2}. The terminal set $\mathcal{T}$ describes the hybrid time domain of the set of solutions over which the optimization problem is solved and is defined as

$$\mathcal{T} := \{(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N} : \max\{T/\delta_p, J\} = \tau_p\} \tag{4.2}$$

where $\tau_p \in \mathbb{N}\setminus\{0\}$ defines the number of jumps and $\delta_p > 0$ determines the ordinary time t allowed by $\mathcal{T}$. Applying classical continuous-time or discrete-time game theoretical tools to solve this optimization problem might lead to suboptimal input actions due to solutions to $\mathcal{H}$ potentially exhibiting both continuous and discrete behavior. Indeed, solutions starting from δ can either jump or flow at μ[2]. In Figure 4.1, the response ϕ_h to the hybrid system $\mathcal{H}$ from $\xi = \delta = 2$ for a certain input action displays this behavior. For the case in which the cost functional $\mathcal{J}$ penalizes both the continuous evolution and the discrete evolution, the associated cost to ϕ_h, denoted J_h, is calculated using the hybrid methods developed in this work. In contrast, the costs J_c and J_d are computed using continuous-time methods and discrete-time methods, respectively. As the plot shows, existing tools are incapable of properly evaluating the cost of solutions to hybrid systems. [3]

[2] The domain of u_C determines whether jump or flow occurs from μ.

[3] Code at https://github.com/HybridSystemsLab/HybridGames-FiniteHorizon

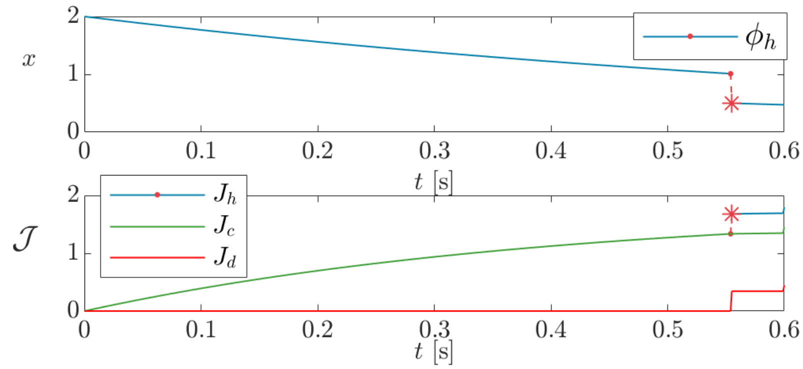

Figure 4.1: A solution to (4.1) (blue) and its cost with time horizon of 2 jumps or 0.6 seconds. The cost computed with continuous-time methods is displayed in green, and with discrete-time methods is displayed in red. The parameters used are $a = -1, b_1 = b_2 = 1, \delta = \xi = 2, \mu = 1, \sigma = 0.5, Q_C = 1, R_{C1} = 1.304$, and $R_{C2} = -4$.

We are interested in designing feedback laws, potentially time-dependent, to solve finite-horizon two-player hybrid games. This motivates the need for a hybrid zero-sum game formulation for scenarios with finite horizon and results providing sufficient conditions to certify optimality in a min-max sense of feedback laws for hybrid systems. In addition, we are interested in solutions that guarantee optimality without the need of computing solutions.

4.1 Problem Statement

We formulate a finite-horizon optimization problem to solve a two-player zero-sum hybrid game and provide the sufficient conditions to characterize the solution. Consider a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ described by (2.3) with data (C, F, D, G). Uniqueness of solutions for a given input implies a unique correspondence from cost to control action, which allows this type of games to be *well-defined*, so that an equilibrium solution is defined [13, Remark 5.3]. Under Assumption 3.1.4, the conditions in Proposition 2.2.5 are satisfied, so solutions to $\mathcal{H}$ are unique

We relax the requirement on maximal solutions being complete in Chapter 3 and consider solutions to $\mathcal{H}$ with terminal time that belongs to a given set. With this purpose, we introduce the following set definitions.

Given a solution (ϕ, u) to $\mathcal{H}$, $(T, J) \in \text{dom}(\phi, u)$ is refered to the terminal time of

(ϕ, u) if $T \geq t$ and $J \geq j$ for all $(t, j) \in \text{dom}(\phi, u)$. Given $\mathcal{T} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$, let us denote by $\hat{\mathcal{S}}_{\mathcal{H}}^{\mathcal{T}}(M) \subset \hat{\mathcal{S}}_{\mathcal{H}}(M)$ the set of compact solutions to $\mathcal{H}$ from M, with terminal time in $\mathcal{T}$, i.e., if $(\phi, u) \in \hat{\mathcal{S}}_{\mathcal{H}}^{\mathcal{T}}(M)$ and $\max \text{dom}(\phi, u) = (T, J)$, then $(T, J) \in \mathcal{T}$. We denote by $\mathcal{U}_{\mathcal{H}}^{\mathcal{T}}(M)$ the set of input actions u such that compact solutions to $\mathcal{H}$ from M for u have terminal time in $\mathcal{T}$.

Given $\xi \in \Pi(C \cup D)$, a joint input action $u = (u_C, u_D) \in \mathcal{U}$, the stage cost for flows $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$, the stage cost for jumps $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, and the terminal cost $q : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$, we define the cost associated to the solution (ϕ, u) to $\mathcal{H}$ from ξ with terminal time $(T, J) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, under Assumption 3.1.4, as

$$\mathcal{J}(\xi, u) := \sum_{j=0}^{J} \int_{t_j}^{t_{j+1}} L_C(\phi(t, j), u_C(t, j))dt + \sum_{j=0}^{J-1} L_D(\phi(t_{j+1}, j), u_D(t_{j+1}, j)) + q(T, J, \phi(T, J)) \quad (4.3)$$

where $t_{J+1} = T$ and $\{t_j\}_{j=0}^{\sup_j \text{dom}\, \phi}$ is a nondecreasing sequence associated to the definition of the hybrid time domain of ϕ; see Definition 2.2.2. For this scenario, the terminal set is defined as in (4.2). Let us also denote the set of points contained by the box described by $\mathcal{T}$ and the coordinate axes as

$$\mathcal{T}_{\leq \tau_p} := \{(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N} : \max\{t/\delta_p, j\} \leq \tau_p\} \quad (4.4)$$

Using the formulation above, the two-player zero-sum game consists of solving the following problem.

Problem (⋆): Given $\xi \in \mathbb{R}^n$, $\mathcal{T} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$, under Assumption 3.1.4, solve

$$\underset{\substack{u_1 \\ u=(u_1,u_2) \in \mathcal{U}_{\mathcal{H}}^{\mathcal{T}}(\xi)}}{\text{minimize}} \; \underset{u_2}{\text{maximize}} \;\; \mathcal{J}(\xi, u) \quad (4.5)$$

where $\mathcal{U}_{\mathcal{H}}^{\mathcal{T}}$ is the set of joint input actions yielding solutions with terminal time in $\mathcal{T}$.

Remark 4.1.1. (Finite-horizon saddle-point equilibrium and min-max control) *A solution to Problem (⋆), when it exists, can be expressed in terms of the pure strategy saddle-point equilibrium γ for the two-player zero-sum finite-horizon game. Each $u^* = (u_1^*, u_2^*)$ rendering a response ϕ^* such that $(\phi^*, u^*) \in \mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$, defined as $\text{dom}\, \phi^* \ni (t, j) \mapsto u_i^*(t, j) = \gamma_i(t, j, \phi^*(t, j))$ for each $i \in \{1, 2\}$, satisfies*

$$u^* = \underset{\substack{u_1 \;\; u_2 \\ u=(u_1,u_2) \in \mathcal{U}_{\mathcal{H}}^{\mathcal{T}}(\xi)}}{\arg\min\max} \; \mathcal{J}(\xi, u) = \underset{\substack{u_2 \;\; u_1 \\ u=(u_1,u_2) \in \mathcal{U}_{\mathcal{H}}^{\mathcal{T}}(\xi)}}{\arg\max\min} \; \mathcal{J}(\xi, u)$$

and it is referred to as a min-max control at ξ.

Definition 4.1.2. (Value function) *Given $\xi \in \Pi(\overline{C}) \cup \Pi(D)$, and parameters $\delta_p > 0$ and $\tau_p \in \mathbb{N} \setminus \{0\}$ defining the set $\mathcal{T} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ as in (4.2), under Assumption 3.1.4, the value function at ξ is given by*

$$\mathcal{J}^*_{\mathcal{T}}(\xi) := \min_{\substack{u_1 \\ u=(u_1,u_2)\in\hat{\mathcal{S}}^{\mathcal{T}}_{\mathcal{H}}(\xi)}} \max_{u_2} \mathcal{J}(\xi, u) = \max_{\substack{u_2 \\ u=(u_1,u_2)\in\hat{\mathcal{S}}^{\mathcal{T}}_{\mathcal{H}}(\xi)}} \min_{u_1} \mathcal{J}(\xi, u) \tag{4.6}$$

If there does not exist (ϕ, u) from ξ such that $\operatorname{dom}\phi$ ever enters $\mathcal{T}$, i.e. if $\hat{\mathcal{S}}^{\mathcal{T}}_{\mathcal{H}}(\xi)$ is empty, then $\mathcal{J}^*_{\mathcal{T}}(\xi) := \infty$.

4.2 Design of Saddle-Point Equilibrium for Finite-Horizon Hybrid Games

The following theorem provides sufficient conditions to characterize the value function $\mathcal{J}^*_{\mathcal{T}}$ and the feedback law that attains it. It addresses the solution to Problem ($\star$) for each $\xi \in \Pi(\overline{C}) \cup \Pi(D)$, showing that the optimizer is the saddle-point equilibrium.

Theorem 4.2.1. (Hamilton-Jacobi-Isaacs (HJI) equations for Problem ($\star$)) *Given a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ as in (2.3) described by (C, F, D, G) satisfying Assumption 3.1.4, stage costs $L_C : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$ and $L_D : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, terminal cost $q : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$, and parameters $\delta_p > 0$ and $\tau_p \in \mathbb{N} \setminus \{0\}$ defining the sets $\mathcal{T}, \mathcal{T}_{\leq \tau_p} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ as in (4.2) and (4.4), respectively, suppose the following hold:*

1. *There exists a function $V : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on a neighborhood of $\mathcal{T}_{\leq \tau_p} \times \Pi(C)$ satisfying the Hamilton-Jacobi-Isaacs hybrid PDEs given as*

$$\begin{aligned} 0 &= \min_{\substack{u_{C1} \\ u_C=(u_{C1},u_{C2})\in\Pi^C_u(x)}} \max_{u_{C2}} \left\{ L_C(t,j,x,u_C) + \frac{\partial V}{\partial t}(t,j,x) + \frac{\partial V}{\partial x}(t,j,x) F(x,u_C) \right\} \\ &= \max_{\substack{u_{C2} \\ u_C=(u_{C1},u_{C2})\in\Pi^C_u(x)}} \min_{u_{C1}} \left\{ L_C(t,j,x,u_C) + \frac{\partial V}{\partial t}(t,j,x) + \frac{\partial V}{\partial x}(t,j,x) F(x,u_C) \right\} \\ &\qquad \forall ((t,j),x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(C) \end{aligned} \tag{4.7}$$

$$V(t,j,x) = \min_{u_{D1}} \max_{u_{D2}} \{L_D(t,j,x,u_D) + V(t,j+1,G(x,u_D))\}$$
$$u_D=(u_{D1},u_{D2})\in\Pi_u^D(x)$$
$$= \max_{u_{D2}} \min_{u_{D1}} \{L_D(t,j,x,u_D) + V(t,j+1,G(x,u_D))\}$$
$$u_D=(u_{D1},u_{D2})\in\Pi_u^D(x)$$
$$\forall((t,j),x) \in \mathcal{T}_{\leq\tau_p} \times \Pi(D) \tag{4.8}$$

2. For each $\xi \in \Pi(\overline{C}) \cup \Pi(D)$, each $(\phi,u) \in \mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ satisfies

$$V(t,j,\phi(t,j)) = q(t,j,\phi(t,j)) \qquad \forall(t,j) \in \operatorname{dom}\phi \cap \mathcal{T} \tag{4.9}$$

Then

$$\mathcal{J}_{\mathcal{T}}^*(\xi) = V(0,0,\xi) \qquad\qquad \forall\xi \in \Pi(\overline{C}) \cup \Pi(D), \tag{4.10}$$

and any feedback law $\gamma := (\gamma_C,\gamma_D) : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ with values

$$\gamma_C(t,j,x) \in \arg\min_{u_{C1}} \max_{u_{C2}} \left\{L_C(t,j,x,u_C) + \frac{\partial V}{\partial x}(t,j,x)F(x,u_C)\right\}$$
$$u_C=(u_{C1},u_{C2})\in\Pi_u^C(x)$$
$$\forall((t,j),x) \in \mathcal{T}_{\leq\tau_p} \times \Pi(C)$$

and

$$\gamma_D(t,j,x) \in \arg\min_{u_{D1}} \max_{u_{D2}} \{L_D(t,j,x,u_D) + V(t,j+1,G(x,u_D))\}$$
$$u_D=(u_{D1},u_{D2})\in\Pi_u^D(x)$$
$$\forall((t,j),x) \in \mathcal{T}_{\leq\tau_p} \times \Pi(D)$$

is a pure strategy saddle-point equilibrium for the two-player zero-sum finite-horizon hybrid game with $\mathcal{J}_1 = \mathcal{J}$, $\mathcal{J}_2 = -\mathcal{J}$.

4.3 Examples

We characterize the pure strategy saddle-point equilibrium and the value function for the example introduced in Section III.A.

Example 4.3.1. (Bouncing ball) Inspired by the problem in [72], consider a simplified model of a juggling system as in [73], with state $x \in \mathbb{R}^2$, input $u_D := (u_{D1}, u_{D2}) \in \mathbb{R}^2$, and dynamics

$$\begin{aligned} \dot{x} &= F(x) := (x_2, -1) & x &\in \mathbb{R}_{\geq 0} \times \mathbb{R} \\ x^+ &= G(x,u_D) := (0, -\lambda x_2 + u_{D1} + u_{D2}) & (x,u_D) &\in \{0\} \times \mathbb{R}_{\leq 0} \times \mathbb{R}^2 \end{aligned} \tag{4.11}$$

where u_{D1} is the control input, u_{D2} is the action of an attacker, and $\lambda \in (0,1)$ is the coefficient of restitution of the ball. The scenario in which u_{D1} is designed to minimize a cost functional $\mathcal{J}$ under the presence of the worst-case attack u_{D2} is formulated as a two-player zero-sum finite-horizon hybrid game. With the aim of pursuing minimum energy and distance to the origin at jumps, consider the cost functions $L_C(x,u_C) := 0$, $L_D(x,u_D) := x_2^2 Q_D + u_D^\top R_D u_D$, and terminal cost $q(x) := \frac{1}{2}x_2^2 + x_1$ defining $\mathcal{J}$ as in (4.3), with $R_D := \begin{bmatrix} R_{D1} & 0 \\ 0 & R_{D2} \end{bmatrix}$ and Q_D, R_{D1}, $-R_{D2} > 0$. Here, u_{D1} is designed by player P_1, which aims to minimize $\mathcal{J}$, while player P_2 seeks to maximize it by choosing u_{D2}. The function $V(x) := x_1 + \frac{1}{2}x_2^2$ is such that $\frac{\partial V}{\partial x}(x)F(x) = 0$ for all $x \in \mathbb{R}_{\geq 0} \times \mathbb{R}$, making V a solution to (4.7). In addition, the function V is such that

$$\min_{u_{D1}} \max_{u_{D2}} \underset{u_D=(u_{D1},u_{D2})\in\mathbb{R}^2}{} \{L_D(x,u_D) + V(G(x,u_D))\} = \frac{1}{2}x_2^2 \tag{4.12}$$

for all $(x, u_D) \in \{0\} \times \mathbb{R}_{\leq 0} \times \mathbb{R}^2$, and attained by $\gamma_D(x) = (\gamma_{D1}(x), \gamma_{D2}(x))$ with $\gamma_{D1}(x) = \frac{R_{D2}\lambda}{R_{D1}+R_{D2}+2R_{D1}R_{D2}} x_2$ and $\gamma_{D2}(z) = \frac{R_{D1}\lambda}{R_{D1}+R_{D2}+2R_{D1}R_{D2}} x_2$ when

$$Q_D = \frac{-2R_{D1}R_{D2}\lambda^2 + R_{D1} + R_{D2} + 2R_{D1}R_{D2}}{2R_{D1} + 2R_{D2} + 4R_{D1}R_{D2}}, \tag{4.13}$$

which makes V a solution to (4.8). Thus, given that V is continuously differentiable on $\mathbb{R}^2$, and that (4.7) and (4.8) hold thanks to (4.12) and (4.13), from Theorem 4.2.1, the value function is $\mathcal{J}^*_{\mathcal{T}}(\xi_1,\xi_2) := \frac{\xi_2^2}{2} + \xi_1$. Figure 4.2 displays this behavior.

In Figure 4.3, we let the players select feedback laws close with the Nash equilibrium and calculate the cost associated to the new laws. The variation of the cost along the changes in the feedback laws makes evident the saddle-point geometry.

□

4.3.1 Periodic-Jumps Finite-Horizon Hybrid Games

Next, we consider a special case of our result that emerges in hybrid systems with linear flow and jump maps and periodic jumps. We introduce a state variable τ that plays the role of a timer. Once τ reaches a fixed threshold $\bar{T} \in \mathbb{R}_{\geq 0}$, it triggers a jump in the state and resets τ to 0.

Consider a hybrid system with state $x = (x_p, \tau) = (x_{p1}, x_{p_2}, \tau) \in \mathbb{R}^n \times [0, \bar{T}]$, input $u = (u_C, u_D) = ((u_{C1}, u_{C2}),\ (u_{D1}, u_{D2})) \in \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$, and dynamics $\mathcal{H}$ as in (2.3),

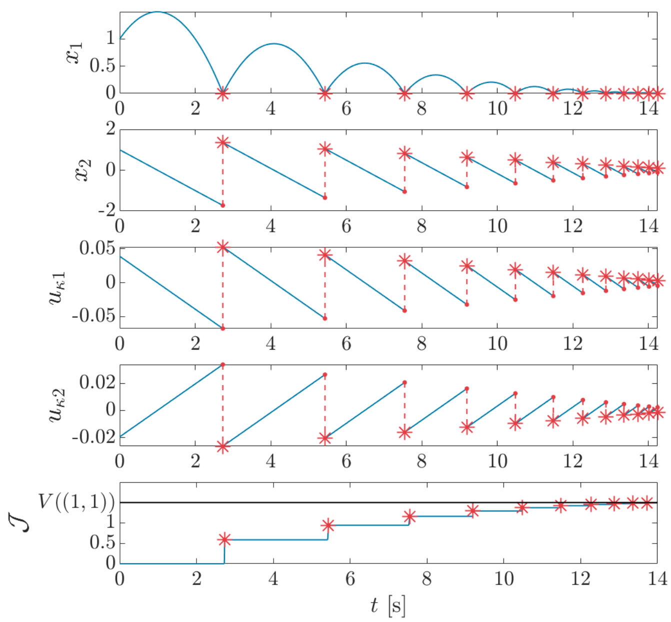

Figure 4.2: Bouncing ball solutions attaining minimum cost under worst-case u_2, with $\tau_p = 100, \delta_p = 2/25, \lambda = 0.8, R_{D1} = 10, R_{D2} = -20$, and $Q_D = 0.189$.

described by

$$
\begin{aligned}
C &= \mathbb{R}^n \times [0, \bar{T}] \times \mathbb{R}^{m_C} \\
F(x, u_C) &= (A_C x_p + B_C u_C, 1) \\
&=: \left(\begin{bmatrix} A_{C1} & 0 \\ 0 & A_{C2} \end{bmatrix} \begin{bmatrix} x_{p1} \\ x_{p2} \end{bmatrix} + \begin{bmatrix} B_{C1} & B_{C2} \end{bmatrix} \begin{bmatrix} u_{C1} \\ u_{C2} \end{bmatrix}, 1 \right) \\
D &= \mathbb{R}^n \times \{\bar{T}\} \times \mathbb{R}^{m_D} \\
G(x, u_D) &= (A_D x_p + B_D u_D, 0) \\
&=: \left(\begin{bmatrix} A_{D1} & 0 \\ 0 & A_{D2} \end{bmatrix} \begin{bmatrix} x_{p1} \\ x_{p2} \end{bmatrix} + \begin{bmatrix} B_{D1} & B_{D2} \end{bmatrix} \begin{bmatrix} u_{D1} \\ u_{D2} \end{bmatrix}, 0 \right)
\end{aligned}
$$

The input $u_1 = (u_{C1}, u_{D1})$ is assigned by P_1 and the input $u_2 = (u_{C2}, u_{D2})$ is assigned by P_2. The problem of finding conditions for u_1 to minimize a cost functional $\mathcal{J}$ in the presence of the action u_2 that seeks to maximize it, is formulated as a two-player zero-sum game. Given $\mathcal{T} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ as in (4.2), we synthesize a feedback law that solves Problem ($\star$) through the solution of the HJI hybrid PDEs in (4.7) and (4.8).

With the aim of pursuing minimum energy and distance to the origin, consider the cost functions $L_C(x, u_C) := x_p^\top Q_C x_p + u_{C1}^\top R_{C1} u_{C1} + u_{C2}^\top R_{C2} u_{C2}$, $L_D(x, u_D) := x_p^\top Q_D x_p + u_{D1}^\top R_{D1} u_{D1} + u_{D2}^\top R_{D2} u_{D2}$, and terminal cost $q(x) := x_p^\top P(\tau) x_p$ where

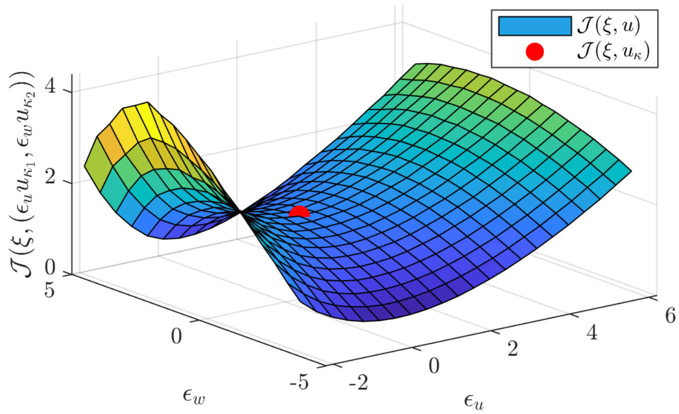

Figure 4.3: Saddle point behavior in the cost of solutions to bouncing ball from $\xi = (1,1)$ when the feedback gains vary around the optimal value. The cost is evaluated on solutions $(\phi,u) \in \mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ with feedback law variations specified by ϵ_u and ϵ_u in $u = (\epsilon_u \gamma_1(t,j,\phi), \epsilon_w \gamma_2(t,j,\phi))$.

$Q_C, Q_D \in \mathbb{S}_+^n$, $R_{C1} \in \mathbb{S}_+^{m_{C_1}}$, $-R_{C2} \in \mathbb{S}_+^{m_{C_2}}$, $R_{D1} \in \mathbb{S}_+^{m_{D_1}}$, $-R_{D2} \in \mathbb{S}_+^{m_{D_2}}$ and $P(\tau) \in \mathbb{S}_+^n$ for all $\tau \in [0,\bar{T}]$. These functions define $\mathcal{J}$ as in (4.3). Inspired by [47] and [77], the following result presents a tool for the solution of the finite horizon optimal control problem for hybrid systems with linear maps and periodic jumps under the adversarial action provided by P_2.

Corollary 4.3.2. (Hybrid Riccati equation for periodic jumps with finite horizon) *Given $\bar{T} \in \mathbb{R}_{\geq 0}$, $A_C, A_D \in \mathbb{R}^{n\times n}$, $B_C := [B_{C1}\ B_{C2}] \in \mathbb{R}^{n\times m_C}$, $B_D := [B_{D1}\ B_{D2}] \in \mathbb{R}^{n\times m_D}$, Q_C, $Q_D \in \mathbb{S}_+^n$, $R_{C1} \in \mathbb{S}_+^{m_{C_1}}$, $-R_{C2} \in \mathbb{S}_+^{m_{C_2}}$, $R_{D1} \in \mathbb{S}_+^{m_{D_1}}$, $-R_{D2} \in \mathbb{S}_+^{m_{D_2}}$, and $\delta_p > 0$ and $\tau_p \in \mathbb{N}\setminus\{0\}$ defining the set $\mathcal{T}_{\leq \tau_p} \subset \mathbb{R}_{\geq 0}\times\mathbb{N}$ as in (4.4), suppose there exists a continuously differentiable matrix function $P : [0,\bar{T}] \to \mathbb{S}_+^n$ such that*

$$\begin{aligned} -\frac{\mathrm{d}P}{\mathrm{d}\tau}(\tau) = &-P(\tau)(B_{C2}R_{C2}^{-1}B_{C2}^\top + B_{C1}R_{C1}^{-1}B_{C1}^\top)P(\tau) \\ &+ Q_C + P(\tau)A_C + A_C^\top P(\tau) \quad \forall \tau \in (0,\bar{T}), \end{aligned} \tag{4.14}$$

$$\begin{aligned} -R_{D2} - B_{D2}^\top P(0)B_{D2} &\in \mathbb{S}_{0+}^{m_D}, \\ R_{D1} + B_{D1}^\top P(0)B_{D1} &\in \mathbb{S}_{0+}^{m_D}, \end{aligned} \tag{4.15}$$

the matrix $R_v = \begin{bmatrix} R_{D1}+B_{D1}^\top P(0)B_{D1} & B_{D1}^\top P(0)B_{D2} \\ B_{D2}^\top P(0)B_{D1} & R_{D2}+B_{D2}^\top P(0)B_{D2} \end{bmatrix}$ is invertible, and

$$P(\bar{T}) = Q_D + A_D^\top P(0)A_D - \begin{bmatrix} A_D^\top P(0)B_{D1} & A_D^\top P(0)B_{D2} \end{bmatrix} R_v^{-1} \begin{bmatrix} B_{D1}^\top P(0)A_D \\ B_{D2}^\top P(0)A_D \end{bmatrix} \tag{4.16}$$

Then, the feedback law $\gamma := (\gamma_C, \gamma_D)$, *with values*

$$\gamma_C(x) = (-R_{C1}^{-1}B_{C1}^\top P(\tau)x_p, -R_{C2}^{-1}B_{C2}^\top P(\tau)x_p) \quad \forall x : \tau \in (0, \bar{T}) \tag{4.17}$$

$$\gamma_D(x) = -R_v^{-1}\begin{bmatrix} B_{D1}^\top P(0)A_D \\ B_{D2}^\top P(0)A_D \end{bmatrix} x_p \qquad \forall x : \tau = 0 \tag{4.18}$$

is a pure strategy saddle-point equilibrium for the two-player zero-sum finite-horizon hybrid game with periodic jumps. In addition, for each (τ, x_p) *and each* $(t, j) \in \mathcal{T}_{\leq \tau_p}$, *the value function is equal to* $V(x) := x_p^\top P(\tau)x_p$.

4.3.2 Jumps-Actuated Finite-Horizon Hybrid Game

Inspired by Example 4.3.1, we consider the class of hybrid systems with state $x \in \mathbb{R}^n$, input $u_D = (u_{D1}, u_{D2}) \in \mathbb{R}^{m_D}$, and dynamics $\mathcal{H}$ as in (2.3), described by

$$\begin{aligned} \dot{x} &= F(x) && x \in C \\ x^+ &= A_D x + \begin{bmatrix} B_{D1} \; B_{D2} \end{bmatrix} \begin{bmatrix} u_{D1} \\ u_{D2} \end{bmatrix} && (x, u_D) \in D \end{aligned} \tag{4.19}$$

with Lipschitz continuous $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $A_D \in \mathbb{R}^{n\times n}$, and $C \subset \mathbb{R}^n, D \subset \mathbb{R}^n \times \mathbb{R}^{m_D}$, such that $C \cup D$ is nonempty. The input u_{D1} plays the role of the control and u_{D2} is the disturbance input. The problem of minimizing a cost functional $\mathcal{J}$ in the presence of the worst-case disturbance u_2 under a finite horizon defined by $\mathcal{T}$ as in (4.2) is formulated as a two-player zero-sum game. Thus, by solving Problem ($\star$) for every $\xi \in \Pi(C) \cup \Pi(D)$, the control objective is achieved.

With the aim of pursuing minimum energy and distance to the origin during jumps, consider the cost functions $L_C(t, j, x, u_C) := 0$, $L_D(t, j, x, u_D) := x^\top Q_D x + u_{D1}^\top R_{D1} u_{D1} + u_{D2}^\top R_{D2} u_{D2}$, and terminal cost $q(t, j, x) := x^\top P(t)x$, where $Q_D \in \mathbb{S}_+^n$, $R_{D1} \in \mathbb{S}_+^{m_{D_1}}$, $-R_{D2} \in \mathbb{S}_+^{m_{D_2}}$ and $t \mapsto P(t) \in \mathbb{S}_+^n$. These functions define $\mathcal{J}$ as in (4.3). The following result presents a tool for the solution of the optimal control problem for jumps-actuated finite-horizon hybrid systems with state-affine flow maps under a worst-case disturbance.

Corollary 4.3.3. (Hybrid Riccati equation for jumps-actuated game) *Given* $F : \mathbb{R}^n \to \mathbb{R}^n$, $A_D \in \mathbb{R}^{n\times n}$, $B_D := [B_{D1} \; B_{D2}] \in \mathbb{R}^{n\times m_D}$, $Q_D \in \mathbb{S}_+^n$, $R_{D1} \in \mathbb{S}_+^{m_{D_1}}$, $-R_{D2} \in \mathbb{S}_+^{m_{D_2}}$, $\zeta \in \mathbb{R}^n$, *and* $\delta_p > 0$ *and* $\tau_p \in \mathbb{N} \setminus \{0\}$ *defining the set* $\mathcal{T}_{\leq \tau_p} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ *as in (4.4),* *suppose there exists a continuously differentiable matrix function $t \mapsto P \in \mathbb{S}^n_+$ defined on $(0, \delta_p\tau_p)$ such that*

$$0 = 2P(t)F(x) + \frac{\mathrm{d}P}{\mathrm{d}t}(t)x \qquad \forall((t,j),x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(C), \tag{4.20}$$

$$\begin{aligned} -R_{D2} - B_{D2}^\top P(t) B_{D2} \in \mathbb{S}^{m_D}_{0+} \qquad & \forall t \leq \tau_p\delta_p, \\ R_{D1} + B_{D1}^\top P(t) B_{D1} \in \mathbb{S}^{m_D}_{0+} \qquad & \forall t \leq \tau_p\delta_p, \end{aligned} \tag{4.21}$$

the matrix $R_v(t) = \begin{bmatrix} R_{D1}+B_{D1}^\top P(t)B_{D1} & B_{D1}^\top P(t)B_{D2} \\ B_{D2}^\top P(t)B_{D1} & R_{D2}+B_{D2}^\top P(t)B_{D2} \end{bmatrix}$ is invertible for all $t \in [0, \tau_p\delta_p]$, and

$$0 = -P(t) + Q_D + A_D^\top P(t)A_D - \begin{bmatrix} A_D^\top P(t)B_{D1} & A_D^\top P(t)B_{D2} \end{bmatrix} R_v(t)^{-1} \begin{bmatrix} B_{D1}^\top P(t)A_D \\ B_{D2}^\top P(t)A_D \end{bmatrix}$$
$$\forall((t,j),x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(D), \tag{4.22}$$

then, the feedback law [4]

$$\gamma_{D1}(t,j,x) = -[R_v^{-1}(1,1)\ R_v^{-1}(1,2)] \begin{bmatrix} B_{D1}^\top P(t)A_D \\ B_{D2}^\top P(t)A_D \end{bmatrix} x \quad \forall((t,j),x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(D) \tag{4.23}$$

minimizes the cost functional $\mathcal{J}$ in the presence of the worst-case disturbance u_2, given by

$$\gamma_{D2}(t,j,x) = -[R_v^{-1}(2,1)\ R_v^{-1}(2,2)] \begin{bmatrix} B_{D1}^\top P(t)A_D \\ B_{D2}^\top P(t)A_D \end{bmatrix} x \quad \forall((t,j),x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(D) \tag{4.24}$$

In addition, for each $(t,j,x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(\overline{C}) \cup \Pi(D)$, the value function is equal to $V(t,j,x) := x^\top P(t)x$.

4.4 Approximate Min-Max Optimality

Inspired by [19, Theorem 15.2], the following results allow the construction of a state-feedback policy based on a function V that satisfies the Hamilton Jacobi Isaacs PDEs (4.7),(4.8) only approximately.

[4] The notation $R_v^{-1}(a,b)$ denotes the (a,b) entry of matrix R_v^{-1}.

Lemma 4.4.1. (Upper bound) *Given a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ as in (2.3) with $N = 2$, and data (C, F, D, G), satisfying Assumption 3.1.4, stage costs $L_C : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$ and $L_D : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, terminal cost $q : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$, and parameters $\delta_p > 0$ and $\tau_p \in \mathbb{N} \setminus \{0\}$ defining the the sets $\mathcal{T}, \mathcal{T}_{\leq \tau_p} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ as in (4.2) and (4.4), respectively, if there exist $\varepsilon \geq 0$, and a function $V : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on a neighborhood of $\mathcal{T}_{\leq \tau_p} \times \Pi(C)$ satisfying*

$$-\varepsilon \leq \min_{\substack{u_{C1} \\ u_C = (u_{C1}, u_{C2}) \in \Pi_u^C(x)}} \max_{u_{C2}} \left\{ L_C(t, j, x, u_C) + \frac{\partial V}{\partial x}(t, j, x) F(x, u_C) \right\} + \frac{\partial V}{\partial t}(t, j, x) \qquad \forall ((t, j), x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(C) \tag{4.25}$$

$$-\varepsilon \leq V(t, j, x) - \min_{\substack{u_{D1} \\ u_D = (u_{D1}, u_{D2}) \in \Pi_u^D(x)}} \max_{u_{D2}} \{ L_D(t, j, x, u_D) + V(t, j+1, G(x, u_D)) \} \qquad \forall ((t, j), x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(D), \tag{4.26}$$

and, for $\xi \in \Pi(\overline{C}) \cup \Pi(D)$, if any $(\phi_s, u^s) \in \mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ with $u^s = (u_1^s, u_2^s)$, $\operatorname{dom} \phi_s \ni (t, j) \mapsto u_1^s(t, j) = \bar{\gamma}_1(t, j, \phi_s(t, j))$ for some $\bar{\gamma}_1 \in K_1$, $\operatorname{dom} \phi_s \ni (t, j) \mapsto u_2^s(t, j) = \gamma_2^*(t, j, \phi_s(t, j))$ for $\gamma_2^* := (\gamma_{C2}^*, \gamma_{D2}^*)$ attaining the supremum in (4.25) and (4.26), is such that

$$V(\sup \operatorname{dom} \phi_s, \phi_s(\sup \operatorname{dom} \phi_s)) = q(\sup \operatorname{dom} \phi_s, \phi_s(\sup \operatorname{dom} \phi_s)) \tag{4.27}$$

then,

$$V(0, 0, \xi) \leq \mathcal{J}(\xi, u^s) + \tau_p(1 + \delta_p)\varepsilon \tag{4.28}$$

Lemma 4.4.2. (Lower bound) *Given a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ as in (2.3) with $N = 2$, described by (C, F, D, G), satisfying Assumption 3.1.4, stage costs $L_C : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$ and $L_D : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, terminal cost $q : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$, and parameters $\delta_p > 0$ and $\tau_p \in \mathbb{N} \setminus \{0\}$ defining the sets $\mathcal{T}, \mathcal{T}_{\leq \tau_p} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ as in (4.2) and (4.4), respectively, if there exist $\varepsilon \geq 0$, and a function $V : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on a neighborhood* *of* $\mathcal{T}_{\leq \tau_p} \times \Pi(C)$ *satisfying*

$$\varepsilon \geq \min_{\substack{u_{C1} \\ u_C=(u_{C1},u_{C2})\in\Pi_u^C(x)}} \max_{u_{C2}} \left\{ L_C(t,j,x,u_C) + \frac{\partial V}{\partial x}(t,j,x)F(x,u_C) \right\} + \frac{\partial V}{\partial t}(t,j,x) \quad \forall((t,j),x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(C) \tag{4.29}$$

$$\varepsilon \geq V(t,j,x) - \min_{\substack{u_{D1} \\ u_D=(u_{D1},u_{D2})\in\Pi_u^D(x)}} \max_{u_{D2}} \{ L_D(t,j,x,u_D) + V(t,j+1,G(x,u_D)) \} \quad \forall((t,j),x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(D), \tag{4.30}$$

and, given $\xi \in \Pi(\overline{C}) \cup \Pi(D)$, *if any* $(\phi_w, u^w) \in \mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ *with* $u^w = (u_1^w, u_2^w)$, $\operatorname{dom}\phi_w \ni (t,j) \mapsto u_1^w(t,j) = \gamma_1^*(t,j,\phi_w(t,j))$ *for* $\gamma_1^* = (\gamma_{C1}^*, \gamma_{D1}^*)$ *attaining the infimum in (4.29) and (4.30),* $\operatorname{dom}\phi_w \ni (t,j) \mapsto u_2^w(t,j) = \bar{\gamma}_2(t,j,\phi_w(t,j))$ *for some* $\bar{\gamma}_2 \in K_2$, *is such that*

$$V(\sup\operatorname{dom}\phi_w, \phi_w(\sup\operatorname{dom}\phi_w)) = q(\sup\operatorname{dom}\phi_w, \phi_w(\sup\operatorname{dom}\phi_w)) \tag{4.31}$$

then,

$$\mathcal{J}(\xi, u^w) - \tau_p(1+\delta_p)\varepsilon \leq V(0,0,\xi) \tag{4.32}$$

Theorem 4.4.3. (Approximate Hamilton-Jacobi-Isaacs (HJI) for Problem ($\star$)) *Given a two-player zero-sum hybrid game with dynamics* $\mathcal{H}$ *as in (2.3) with* $N = 2$, *described by* (C, F, D, G), *satisfying Assumption 3.1.4, stage costs* $L_C : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$ *and* $L_D : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, *terminal cost* $q : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$, *and parameters* $\delta_p > 0$ *and* $\tau_p \in \mathbb{N} \setminus \{0\}$ *defining the sets* $\mathcal{T}, \mathcal{T}_{\leq \tau_p} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ *as in (4.2) and (4.4), respectively, if there exist constants* $\varepsilon, \delta \geq 0$, *and a function* $V : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$ *that is continuously differentiable on a neighborhood of* $\mathcal{T}_{\leq \tau_p} \times \Pi(C)$ *satisfying the approximate Hamilton-Jacobi-Isaacs hybrid PDEs given as*

$$\begin{aligned}\varepsilon &\geq \left| \min_{\substack{u_{C1} \\ u_C=(u_{C1},u_{C2})\in\Pi_u^C(x)}} \max_{u_{C2}} \left\{ L_C(t,j,x,u_C) + \frac{\partial V}{\partial x}(t,j,x)F(x,u_C) \right\} + \frac{\partial V}{\partial t}(t,j,x) \right| \\ &= \left| \max_{\substack{u_{C2} \\ u_C=(u_{C1},u_{C2})\in\Pi_u^C(x)}} \min_{u_{C1}} \left\{ L_C(t,j,x,u_C) + \frac{\partial V}{\partial x}(t,j,x)F(x,u_C) \right\} + \frac{\partial V}{\partial t}(t,j,x) \right| \\ &\qquad \forall((t,j),x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(C)\end{aligned} \tag{4.33}$$

$$\begin{aligned}
\varepsilon \geq &\left| V(t,j,x) - \min_{\substack{u_{D1} \\ u_D=(u_{D1},u_{D2})\in\Pi_u^D(x)}} \max_{u_{D2}} \{L_D(t,j,x,u_D) + V(t,j+1,G(x,u_D))\} \right| \\
= &\left| V(t,j,x) - \max_{\substack{u_{D2} \\ u_D=(u_{D1},u_{D2})\in\Pi_u^D(x)}} \min_{u_{D1}} \{L_D(t,j,x,u_D) + V(t,j+1,G(x,u_D))\} \right| \\
&\qquad \forall((t,j),x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(D),
\end{aligned} \tag{4.34}$$

such that for each $\xi \in \Pi(\overline{C}) \cup \Pi(D)$, *each* $(\phi, u) \in \mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ *satisfies*

$$V(t,j,\phi(t,j)) = q(t,j,\phi(t,j)) \qquad \forall (t,j) \in \operatorname{dom}\phi \cap \mathcal{T} \tag{4.35}$$

and a feedback law $\gamma := (\gamma_C, \gamma_D) := ((\gamma_{C1}, \gamma_{C2}), (\gamma_{D1}, \gamma_{D2})) : \mathbb{R}_{\geq 0} \times \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ *for which*

$$\begin{aligned}
-\delta \leq &\min_{\substack{u_{C1} \\ u_C=(u_{C1},u_{C2})\in\Pi_u^C(x)}} \max_{u_{C2}} \left\{ L_C(t,j,x,u_C) + \frac{\partial V}{\partial x}(t,j,x) F(x,u_C) \right\} \\
&- L_C(t,j,x,\gamma_C(t,j,x)) + \frac{\partial V}{\partial x}(t,j,x) F(x,\gamma_C(t,j,x)) \leq \delta \\
&\qquad \forall((t,j),x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(C)
\end{aligned} \tag{4.36}$$

and

$$\begin{aligned}
-\delta \leq &\min_{\substack{u_{D1} \\ u_D=(u_{D1},u_{D2})\in\Pi_u^D(x)}} \max_{u_{D2}} \{L_D(t,j,x,u_D) + V(t,j+1,G(x,u_D))\} \\
&- L_D(t,j,x,\gamma_D(t,j,x)) + V(t,j+1,G(x,\gamma_D(t,j,x))) \leq \delta \\
&\qquad \forall((t,j),x) \in \mathcal{T}_{\leq \tau_p} \times \Pi(D)
\end{aligned} \tag{4.37}$$

Then, any $(\phi^*, u^*) \in \hat{\mathcal{S}}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ *with* $\operatorname{dom}\phi^* \ni (t,j) \mapsto u^*(t,j) = \gamma(t,j,\phi^*(t,j,))$, *has a cost that satisfies*

$$\mathcal{J}(\xi,u^w) - \tau_p(1+\delta_p)(2\varepsilon+\delta) \leq \mathcal{J}(\xi,u^*) \leq \mathcal{J}(\xi,u^s) + \tau_p(1+\delta_p)(2\varepsilon+\delta) \qquad \forall \xi \in \Pi(\overline{C}) \cup \Pi(D), \tag{4.38}$$

for any $(\phi_s, u^s) \in \hat{\mathcal{S}}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ *with* $u^s = (u_1^s, u_2^s)$, $\operatorname{dom}\phi_s \ni (t,j) \mapsto u_1^s(t,j) = \bar{\gamma}_1(t,j,\phi_s(t,j,))$ *for some* $\bar{\gamma}_1 \in K_1$ *and* $\operatorname{dom}\phi_s \ni (t,j) \mapsto u_2^s(t,j) = \gamma_2^*(t,j,\phi_s(t,j))$ *for* $\gamma_2^* := (\gamma_{C2}^*, \gamma_{D2}^*)$ *attining the supremum in (4.33) and (4.34), and for any* $(\phi_w, u^w) \in \hat{\mathcal{S}}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ *with* $u^w = (u_1^w, u_2^w)$, $\operatorname{dom}\phi_w \ni (t,j) \mapsto u_1^w(t,j) = \gamma_1^*(t,j,\phi_w(t,j))$ *for* $\gamma_1^* = (\gamma_{C1}^*, \gamma_{D1}^*)$ *attaining the infimum in (4.33) and (4.34),* $\operatorname{dom}\phi_w \ni (t,j) \mapsto u_2^w(t,j) = \bar{\gamma}_2(t,j,\phi_w(t,j,))$ *for some* $\bar{\gamma}_w \in K_w$.

Note that for $\varepsilon = 0$, the equations (4.33) and (4.34) are equivalent to (4.7) and (4.8). If, in addition, $\delta = 0$, (4.36) and (4.37) show that the saddle point is attained by $\gamma = (\gamma_C, \gamma_D)$. When $\varepsilon = \delta = 0$, the bound (4.38) implies that no control action u^s can lead to a cost lower than that of u^*, nor any control action u^w can lead to a cost greater than that of u^*, which we already concluded from Theorem 4.2.1. When $\delta > 0$ and $\varepsilon > 0$, there may exist strategies of player P_1 that when the adversary player P_2 plays optimally, render actions u^s that improve upon u^*. Likewise, there may exist strategies of the adversarial player P_2 that when player P_1 plays optimally, render actions u^w attaining a cost greater that that of u^*, but never by more than $\tau_p(1 + \delta_p)(2\varepsilon + \delta)$, which can be very small if ε and δ are both very small.

Acknowledgements

Sections 4.1-4.3, in full, are a reprint of the material as it appears in "Sufficient Conditions for Optimality in Finite-Horizon Two-Player Zero-Sum Hybrid Games." [2]. The dissertation author was the first author of this paper.

Chapter 5

Fixed Terminal State Hybrid Games

Using the hybrid games framework presented in Chapter 3, in this Chapter we study hybrid two-player zero-sum games with a terminal state sets conditions. The specification of this type of conditions results in a free terminal time nature in the set of inputs over which the optimization problem is solved.

5.1 Problem Statement

We formulate a finite-horizon optimization problem to solve a two-player zero-sum hybrid game with free terminal time and a fixed terminal set, and provide sufficient conditions to characterize its solution. A special case of this setting allows to cover the infinite horizon games in Chapter 3.

Following the formulation in Definition 3.1.1, consider a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ described by (2.3) with data (C, F, D, G). Let the closed set $X \subset \Pi(C) \cup \Pi(D)$ be the terminal constraint set. We say that a solution (ϕ, u) to $\mathcal{H}$ is *feasible* if there exists a finite $(T, J) \in \operatorname{dom}(\phi, u)$ such that $\phi(T, J) \in X$. In addition, we make (T, J) to be both the terminal time of (ϕ, u) and the first time at which ϕ reaches X, i.e., there does not exist $(t, j) \in \operatorname{dom}\phi$ with $t + j < T + J$ such that $\phi(t, j) \in X$ and $(T, J) = \max \operatorname{dom}(\phi, u)$; hence $\operatorname{dom}\phi$ is compact.[1] Uniqueness of solutions for a given

[1]When $X = \emptyset$, the requirement that ϕ belongs to X is not enforced, hence, there is no terminal constraint and the two-player zero-sum hybrid game evolves over an infinite (hybrid) horizon when

input implies a unique correspondence from cost to control action, which allows this type of games to be *well-defined*, so that an equilibrium solution is defined [13, Remark 5.3]. Sufficient conditions to guarantee that Assumption 3.1.4 holds include Lipschitz continuity of the flow map F, provided it is a single-valued function. Under Assumption 3.1.4, the conditions in Proposition 2.2.5 are satisfied, so solutions to $\mathcal{H}$ are unique[2] for each $u \in \mathcal{U}$. Given $\xi \in \Pi(C) \cup \Pi(D)$, a joint input action $u = (u_C, u_D) \in \mathcal{U}$, the stage cost for flows $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$, the stage cost for jumps $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, and the terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, we define the cost associated to the solution (ϕ, u) to $\mathcal{H}$ from ξ, under Assumption 3.1.4, as

$$\begin{aligned}\mathcal{J}(\xi, u) := &\sum_{j=0}^{\sup_j \operatorname{dom}\phi} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j), u_C(t,j))dt + \sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} L_D(\phi(t_{j+1}, j), u_D(t_{j+1}, j)) \\ &+ \limsup_{\substack{t+j \to \sup_t \operatorname{dom}\phi + \sup_j \operatorname{dom}\phi \\ (t,j)\in \operatorname{dom}\phi}} q(\phi(t,j))\end{aligned} \tag{5.1}$$

where $t_{\sup_j \operatorname{dom}\phi + 1} = \sup_t \operatorname{dom}\phi$ defines the upper limit of the last integral, and $\{t_j\}_{j=0}^{\sup_j \operatorname{dom}\phi}$ is a nondecreasing sequence associated to the definition of the hybrid time domain of (ϕ, u); see Definition 2.2.2.

When X is nonempty, the set $\mathcal{S}_{\mathcal{H}}^X(\xi) \subset \mathcal{S}_{\mathcal{H}}(\xi)$ denotes all maximal solutions from ξ that reach X at their terminal time. When X is empty, $\mathcal{S}_{\mathcal{H}}^X(\xi)$ reduces to the set of complete solutions from ξ. We define the set of input actions that yield maximal solutions to $\mathcal{H}$ from ξ entering X as $\mathcal{U}_{\mathcal{H}}^X(\xi) := \{u : \exists(\phi, u) \in \mathcal{S}_{\mathcal{H}}^X(\xi)\}$. The feasible set $\mathcal{M} \subset \Pi(C) \cup \Pi(D)$ is the set of states ξ such that there exists $(\phi, u) \in \hat{\mathcal{S}}_{\mathcal{H}}^X(\xi)$ with $\phi(T, J) \in X$, where (T, J) is the terminal time of $\operatorname{dom}(\phi, u)$, namely, $(T, J) = \max \operatorname{dom}\phi$.

We are ready to formulate the two-player zero-sum game.

Problem ($\diamond_X$): Given the terminal set X, the feasible set $\mathcal{M} \subset \Pi(\overline{C}) \cup \Pi(D)$, and $\xi \in \mathcal{M}$, under Assumption 3.1.4, solve

$$\underset{\substack{u_1 \\ u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^X(\xi)}}{\text{minimize}} \ \underset{u_2}{\text{maximize}} \ \mathcal{J}(\xi, u) \tag{5.2}$$

where ϕ is the maximal state trajectory rendered by u to $\mathcal{H}$ from ξ.

$\operatorname{dom}\phi$ is unbounded.

[2]Under Assumption 3.1.4, the domain of the input u specifies whether from points in $\Pi(C) \cap \Pi(D)$ a jump or flow occur.

Remark 5.1.1. (Infinite horizon games) *When the terminal set X is empty and maximal solutions are complete, Problem $(\diamond_X)$ reduces to an infinite horizon hybrid game as in Chapter 3, as stated in footnote 4. In this case, the feasible set satisfies $\mathcal{M} = \Pi(C) \cup \Pi(D)$ and, for each $\xi \in \mathcal{M}$, the set of complete solutions $\mathcal{S}_{\mathcal{H}}^X(\xi)$ is nonempty. For infinite horizon games, the set of decision variables $\mathcal{U}_{\mathcal{H}}^X$ in Problem $(\diamond_X)$ denotes all joint input actions yielding maximal complete solutions to $\mathcal{H}$.*

Remark 5.1.2. (Saddle-point equilibrium and min-max control) *A solution to Problem $(\diamond_X)$, when it exists, can be expressed in terms of the saddle-point equilibrium $\kappa = (\kappa_1, \kappa_2)$ for the two-player zero-sum game. Each $u^* = (u_1^*, u_2^*)$ that renders a state trajectory $\phi^* \in \mathcal{R}(\xi, u^*)$, with components defined as $\operatorname{dom}\phi^* \ni (t,j) \mapsto u_i^*(t,j) = \kappa_i(\phi^*(t,j))$ for each $i \in \{1,2\}$, satisfies*

$$u^* = \arg \min_{\substack{u_1 \\ u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^X(\xi)}} \max_{u_2} \mathcal{J}(\xi, u) = \arg \max_{\substack{u_2 \\ u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^X(\xi)}} \min_{u_1} \mathcal{J}(\xi, u)$$

and it is referred to as a min-max control at ξ.

Definition 5.1.3. (Value function) *Given the terminal set X, the feasible set $\mathcal{M} \subset \Pi(\overline{C}) \cup \Pi(D)$, and $\xi \in \mathcal{M}$, under Assumption 3.1.4, the value function at ξ is given by*

$$\mathcal{J}_X^*(\xi) := \min_{\substack{u_1 \\ u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^X(\xi)}} \max_{u_2} \mathcal{J}(\xi, u) = \max_{\substack{u_2 \\ u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^X(\xi)}} \min_{u_1} \mathcal{J}(\xi, u) \tag{5.3}$$

We revisit the security application presented in the introduction and reformulate it according to the mathematical framework provided in this chapter as follows.

Application 2. (Security) *Given the system $\mathcal{H}$ as in (2.3), $F(x, u_{C1}, u_{C2}) = f_d(x, u_{C1}) + f_a(u_{C2})$, and $G(x, u_{D1}, u_{D2}) = g_d(x, u_{D1}) + g_a(u_{D2})$, the security problem consists of finding the control input (u_{C1}, u_{D1}) that guarantees the performance of the system until the game ends, which occurs when the state enters a set X, in spite of the action*

$$u_2 = (u_{C2}, u_{D2}) = \arg \max_{\substack{(u_{C2}, u_{D2}) \\ u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^X(\xi)}} \mathcal{J}(\xi, u) \tag{5.4}$$

chosen by an attacker P_2 that knows f_d and g_d. The cost functional $\mathcal{J}$ represents the damage caused by attacks. To account for the best-scenario, this problem is addressed by solving Problem $(\diamond_X)$.

5.2 Design of Saddle-Point Equilibrium for Hybrid Games with Teminal State Set

The following result provides sufficient conditions to characterize the value function, and the feedback law that attains it. It addresses the solution to Problem ($\diamond_X$) showing that the optimizer is the saddle-point equilibrium. It involves the feasible set $\mathcal{M}$ which potentially reduces the set over which the sufficient conditions need to be checked in comparison to the results in Chapter 3. When $\mathcal{M}$ is not known, it could just be replaced by $\mathbb{R}^n$.

Theorem 5.2.1. (Hamilton–Jacobi–Bellman-Isaacs (HJBI) for Problem ($\diamond_X$)) *Given a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ as in (2.3) with data (C, F, D, G) satisfying Assumption 3.1.4, stage costs $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}, L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, (potentially empty) terminal set X, and feasible set $\mathcal{M}$, suppose the following hold:*

1) *There exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on a neighborhood of $\Pi(C)$ and that satisfies the Hamilton–Jacobi–Bellman-Isaacs (HJBI) hybrid equations given as*

$$0 = \min_{\substack{u_{C1} \\ u_C=(u_{C1},u_{C2})\in\Pi_u^C(x)}} \max_{u_{C2}} \mathcal{L}_C(x, u_C) = \max_{\substack{u_{C2} \\ u_C=(u_{C1},u_{C2})\in\Pi_u^C(x)}} \min_{u_{C1}} \mathcal{L}_C(x, u_C) \quad \forall x \in \Pi(C)\cap\mathcal{M}, \tag{5.5}$$

where $\mathcal{L}_C(x, u_C) := L_C(x, u_C) + \langle \nabla V(x), F(x, u_C)\rangle$,

$$V(x) = \min_{\substack{u_{D1} \\ u_D=(u_{D1},u_{D2})\in\Pi_u^D(x)}} \max_{u_{D2}} \mathcal{L}_D(x, u_D) = \max_{\substack{u_{D2} \\ u_D=(u_{D1},u_{D2})\in\Pi_u^D(x)}} \min_{u_{D1}} \mathcal{L}_D(x, u_D) \quad \forall x \in \Pi(D)\cap\mathcal{M}, \tag{5.6}$$

where $\mathcal{L}_D(x, u_D) := L_D(x, u_D) + V(G(x, u_D))$.

2) *For each $\xi \in \mathcal{M}$, each $(\phi, u) \in \mathcal{S}_{\mathcal{H}}^X(\xi)$ satisfies*[3]

$$\limsup_{\substack{t+j\to\sup_t \operatorname{dom}\phi+\sup_j \operatorname{dom}\phi \\ (t,j)\in\operatorname{dom}\phi}} V(\phi(t,j)) = \limsup_{\substack{t+j\to\sup_t \operatorname{dom}\phi+\sup_j \operatorname{dom}\phi \\ (t,j)\in\operatorname{dom}\phi}} q(\phi(t,j)) \tag{5.7}$$

Then

$$\mathcal{J}_X^*(\xi) = V(\xi) \qquad \forall \xi \in \Pi(\overline{C}) \cup \Pi(D), \tag{5.8}$$

[3]The boundary condition (5.7) matches the value of V to the terminal cost q at the final value of ϕ.

and any feedback law $\kappa := (\kappa_C, \kappa_D) = ((\kappa_{C1}, \kappa_{C2}), (\kappa_{D1}, \kappa_{D2})) : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ *with values*

$$\kappa_C(x) \in \arg \min_{\substack{u_{C1} \\ u_C = (u_{C1}, u_{C2}) \in \Pi_u^C(x)}} \max_{u_{C2}} \mathcal{L}_C(x, u_C) \qquad \forall x \in \Pi(C) \cap \mathcal{M} \tag{5.9}$$

and

$$\kappa_D(x) \in \arg \min_{\substack{u_{D1} \\ u_D = (u_{D1}, u_{D2}) \in \Pi_u^D(x)}} \max_{u_{D2}} \mathcal{L}_D(x, u_D) \qquad \forall x \in \Pi(D) \cap \mathcal{M} \tag{5.10}$$

is a pure strategy saddle-point equilibrium for Problem $(\diamond_X)$ *with* $\mathcal{J}_1 = \mathcal{J}$, $\mathcal{J}_2 = -\mathcal{J}$, *where* $\mathcal{J}$ *is as in* (5.1).

Notice that when the players select the optimal strategy, the value function equals the function V evaluated at the initial condition. This makes evident the independence of the result from needing to computesolutions, at the price of finding the function V satisfying the conditions therein.

The terminal set X determines the size of the compact hybrid time domain of the solutions considered in Theorem 5.2.1. Based on reachability tools, given a terminal set X, the feasible set can be numerically computed for certain family of systems.

When the feasible set $\mathcal{M}$ is known a priori, the set of states for which equations (5.5) and (5.6) need to be enforced could potentially be smaller than the sets of states studied in infinite horizon games.

We present the following results providing sufficient conditions to bound and exactly evaluate the cost of the game. These results are instrumental on guaranteeing that the saddle-point equilibrium is attained and in evaluating the value function of the game.

Proposition 5.2.2. (Time-dependent conditions for upper bound) *Consider* $(\phi, u) \in \mathcal{S}_{\mathcal{H}}^X(\xi)$ *with* $u = (u_C, u_D)$, *such that*

1) *for each* $j \in \mathbb{N}$ *such that* I_ϕ^j *has a nonempty interior*[4] *int*I_ϕ^j,

$$L_C(\phi(t,j), u_C(t,j)) + \frac{d}{dt} V(\phi(t,j)) \leq 0 \qquad \forall t \in \text{int} I_\phi^j \tag{5.11}$$

and

2) *for every* $(t_{j+1}, j) \in \operatorname{dom} \phi$ *such that* $(t_{j+1}, j+1) \in \operatorname{dom} \phi$,

$$L_D(\phi(t_{j+1}, j), u_D(t_{j+1}, j)) + V(\phi(t_{j+1}, j+1)) - V(\phi(t_{j+1}, j)) \leq 0. \tag{5.12}$$

[4]When $j = \sup_j \operatorname{dom} \phi \in \mathbb{N}$ and $\sup_t \operatorname{dom} \phi = \infty$, we define $t_{j+1} := \infty$.

Then

$$\sum_{j=0}^{\sup_j \operatorname{dom}\phi} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j), u_C(t,j))dt + \sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} L_D(\phi(t_{j+1},j), u_D(t_{j+1},j)) + \limsup_{\substack{t+j\to\sup_t \operatorname{dom}\phi + \sup_j \operatorname{dom}\phi \\ (t,j)\in\operatorname{dom}\phi}} V(\phi(t,j)) \le V(\xi). \tag{5.13}$$

The proof of Proposition 5.2.2 follows the same arguments as in the proof of Proposition 3.2.2 in the Appendix A.1. The following corollary is immediate from the proof of Proposition 5.2.2.

Corollary 5.2.3. (Change of Signs) *If the inequalities in the conditions in Proposition 5.2.2 are inverted, namely, if "$\le$" in* (5.11) *and* (5.12) *is replaced with "$\ge$", then* (5.13) *holds with the inequality inverted. Likewise, if the conditions in Proposition 5.2.2 hold with equalities, then* (5.13) *holds with equality.*

Remark 5.2.4. (Connections between Theorem 5.2.1 and Problem ($\diamond_X$)) *Given $\xi \in (\Pi(\overline{C})\cup\Pi(D))\cap\mathcal{M}$, if there exist a function V satisfying the conditions in Theorem 5.2.1, then a solution to Problem ($\diamond_X$) exists, namely there is an optimizer input action $u^* = (u^*_C, u^*_D) = ((u^*_{C1}, u^*_{C2}), (u^*_{D1}, u^*_{D2})) \in \mathcal{U}^X_{\mathcal{H}}(\xi)$ that satisfies* (3.2)*, and V is the value function as in Definition 5.1.3.*

5.3 pre-Asymptotic Stability for Hybrid Games with Terminal State Set

We present a result that connects optimality and asymptotic stability for two-player zero-sum hybrid games with terminal state set.

Definition 5.3.1. (Pre-asymptotic stability) *A closed set $\mathcal{A} \subset \mathbb{R}^n$ is locally pre-asymptotically stable for a hybrid closed-loop system $\mathcal{H}_\kappa$ as in (2.4) if it is*

- *stable for $\mathcal{H}_\kappa$, i.e., if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every solution ϕ to $\mathcal{H}_\kappa$ with $|\phi(0,0)|_{\mathcal{A}} \le \delta$ satisfies $|\phi(0,0)|_{\mathcal{A}} \le \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$; and*
- *locally pre-attractive for $\mathcal{H}_\kappa$, i.e., there exists $\mu > 0$ such that every solution ϕ to $\mathcal{H}_\kappa$ with $|\phi(0,0)|_{\mathcal{A}} \le \mu$ is bounded and, if ϕ is complete, then also $\lim_{t+j\to\infty} |\phi(t,j)|_{\mathcal{A}} = 0$.*

In the next result, we provide alternative conditions to those in Theorem 5.2.1 for the solution to Problem $(\diamond_X)$.

Lemma 5.3.2. (Equivalent conditions) *Given $\mathcal{H}_\kappa$ as in (2.4) with data (C, F, D, G), the terminal set X, the feasible set $\mathcal{M} \subset \Pi(\overline{C}) \cup \Pi(D)$, and feedback $\kappa := (\kappa_C, \kappa_D) = ((\kappa_{C1}, \kappa_{C2}), (\kappa_{D1}, \kappa_{D2})) : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ that satisfies* (5.9) *and* (5.10), *if there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on a neighborhood of* $\Pi(C)$ *such that*[5] $C_\kappa = \Pi(C)$ *and $D_\kappa = \Pi(D)$, then (5.5), (5.6), (5.9), and (5.10) are satisfied if and only if*

$$\mathcal{L}_C(x, \kappa_C(x)) = 0 \quad \forall x \in C_\kappa \cap \mathcal{M}, \tag{5.14}$$

$$\mathcal{L}_C(x, (u_{C1}, \kappa_{C2}(x))) \geq 0 \qquad \forall (x, u_{C1}) : \ (x, (u_{C1}, \kappa_{C2}(x))) \in C \cap \mathcal{M}, \tag{5.15}$$

$$\mathcal{L}_C(x, (\kappa_{C1}(x), u_{C2})) \leq 0 \qquad \forall (x, u_{C2}) : \ (x, (\kappa_{C1}(x), u_{C2})) \in C \cap \mathcal{M}, \tag{5.16}$$

$$\mathcal{L}_D(x, \kappa_D(x)) = V(x) \quad \forall x \in D_\kappa \cap \mathcal{M}, \tag{5.17}$$

$$\mathcal{L}_D(x, (u_{D1}, \kappa_{D2}(x))) \geq V(x) \qquad \forall (x, u_{D1}) : \ (x, (u_{D1}, \kappa_{D2}(x))) \in D \cap \mathcal{M}, \tag{5.18}$$

$$\mathcal{L}_D(x, (\kappa_{D1}(x), u_{D2})) \leq V(x) \qquad \forall (x, u_{D2}) : \ (x, (\kappa_{D1}(x), u_{D2})) \in D \cap \mathcal{M}. \tag{5.19}$$

The proof of Lemma 5.19 follows the same arguments as in the proof of Lemma 3.23 in the Appendix A.1.

Theorem 5.3.3. (Saddle-point equilibrium under the existence of a Lyapunov function) *Consider a two-player zero-sum hybrid game with dynamics $\mathcal{H}$ as in* (2.3) *with data* (C, F, D, G) *satisfying Assumption 3.1.4, and $\kappa := (\kappa_C, \kappa_D) : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ defining the closed-loop dynamics $\mathcal{H}_\kappa$ as in (2.4) such that $C_\kappa = \Pi(C)$ and $D_\kappa = \Pi(D)$. Given the terminal set X, the feasible set $\mathcal{M} \subset \Pi(\overline{C}) \cup \Pi(D)$, and a closed set $\mathcal{A} \subset \Pi(C) \cup \Pi(D)$, continuous functions $L_C : C \to \mathbb{R}_{\geq 0}$ and $L_D : D \to \mathbb{R}_{\geq 0}$ defining the stage costs for flows and jumps, respectively, and $q : \mathbb{R}^n \to \mathbb{R}$ defining the terminal cost, suppose there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on an open set containing $\overline{C_\kappa}$, satisfying (5.14)-(5.19), and such that for each $\xi \in (\overline{C_\kappa} \cup D_\kappa) \cap \mathcal{M}$, each $\phi \in \mathcal{S}^X_{\mathcal{H}_\kappa}(\xi)$ satisfies (5.7). If one of the following conditions*[6] *holds*

[5] Notice that $C_\kappa = \Pi(C)$ and $D_\kappa = \Pi(D)$ when $\kappa_C(x) \in \Pi^C_u(x)$ for all $x \in \Pi(C)$ and $\kappa_D(x) \in \Pi^D_u(x)$ for all $x \in \Pi(D)$. In words, the feedback law κ defining the hybrid closed-loop system $\mathcal{H}_\kappa$ does not render input actions outside C or D.

[6] The subindex in the set of positive definite functions $\mathcal{PD}_*$ denotes the feedback law that the functions in the set are composed with to satisfy the properties in Definition 3.3.2.

1) $L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A})$ *and* $L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$*;*

2) $L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$ *and there exists a continuous function* $\eta \in \mathcal{PD}$ *such that* $L_C(x, \kappa_D(x)) \geq \eta(|x|_{\mathcal{A}})$ *for all* $x \in C_\kappa \cap \mathcal{M}$*;*

3) $L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A})$ *and there exists a continuous function* $\eta \in \mathcal{PD}$ *such that* $L_D(x, \kappa_D(x)) \geq \eta(|x|_{\mathcal{A}})$ *for all* $x \in D_\kappa \cap \mathcal{M}$*;*

4) $L_C \equiv 0, L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$*, and for each* $r > 0$*, there exist* $\gamma_r \in \mathcal{K}_\infty$ *and* $N_r \geq 0$ *such that for every solution* $\phi \in \mathcal{S}^X_{\mathcal{H}_\kappa}(\xi)$*,* $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$*,* $(t,j) \in \operatorname{dom}\phi$*,* $t + j \geq T$ *imply* $j \geq \gamma_r(T) - N_r$*;*

5) $L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A}), L_D \equiv 0$*, and for each* $r > 0$*, there exist* $\gamma_r \in \mathcal{K}_\infty$ *and* $N_r \geq 0$ *such that for every solution* $\phi \in \mathcal{S}^X_{\mathcal{H}_\kappa}(\xi)$*,* $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$*,* $(t,j) \in \operatorname{dom}\phi$*,* $t + j \geq T$ *imply* $t \geq \gamma_r(T) - N_r$*;*

6) $L_C(x, \kappa_C(x)) \geq -\lambda_C V(x)$ *for all* $x \in C_\kappa$*,* $L_D(x, \kappa_D(x)) \geq (1 - e^{\lambda_D})V(x)$ *for all* $x \in D_\kappa$*, and there exist* $\gamma > 0$ *and* $M > 0$ *such that, for each solution* $\phi \in \mathcal{S}^X_{\mathcal{H}_\kappa}(\xi)$*,* $(t,j) \in \operatorname{dom}\phi$ *implies* $\lambda_C t + \lambda_D j \leq M - \gamma(t + j)$*;*

then

$$\mathcal{J}^*_X(\xi) = V(\xi) \qquad \forall \xi \in (\overline{C_\kappa} \cup D_\kappa) \cap \mathcal{M} \tag{5.20}$$

Furthermore, the feedback law κ *is the saddle-point equilibrium (see Definition 3.1.3) and it renders* $\mathcal{A}$ *pre-asymptotically stable for* $\mathcal{H}_\kappa$ *with basin of attraction containing the largest sublevel set of* V *contained in* $\mathcal{M}$*.*

5.4 Applications

We illustrate in the following applications with hybrid dynamics and quadratic costs how Theorem 5.2.1 provides conditions to solve the security problem discussed in the introduction by addressing it as zero-sum hybrid game with terminal state set.

Example 5.4.1. (Bouncing ball with terminal set) As an instance of Application 2, the Bouncing Ball system from Example 4.3.1 in which u_{D1} is designed to minimize a cost functional $\mathcal{J}$ until the game ends, which occurs when the state enters a set X under the presence of the worst-case disturbance u_{D2} is formulated as a two-player zero-sum game with terminal state set. With the aim of pursuing minimum velocity and control effort at jumps, consider the cost functions $L_C(x, u_C) := 0$, $L_D(x, u_D) := x_2^2 Q_D + u_D^\top R_D u_D$, and terminal cost $q(x) := \frac{1}{2}x_2^2 + x_1$ defining $\mathcal{J}$ as in (5.1), with $R_D := \begin{bmatrix} R_{D1} & 0 \\ 0 & R_{D2} \end{bmatrix}$ and Q_D, R_{D1}, $-R_{D2} > 0$.

A game of kind [13, Section 5.2] arises and its solution characterizes a division of the state space into two dominance regions $\mathcal{M}, \psi \subset \Pi(C) \cup \Pi(D)$, in which, under optimal play, it can be determined whether the terminal set X is reached or not as a function of the initial condition. If the initial state satisfies $\xi \in \mathcal{M}$ (the feasible set), then, under optimal play, the ball reaches the terminal set X at some time (T, J) and the game ends. On the other hand, if $\xi \in \psi$, under optimal play, we have an infinite horizon game (if maximal solutions are complete after the inputs are assigned). The function $V(x) := x_1 + \frac{1}{2}x_2^2$ is such that

$$\langle \nabla V(x), F(x) \rangle = 0$$

for all $x \in C$, making V a solution to (5.5). In addition, the function V is such that

$$\min_{\substack{u_{D1} \\ u_D = (u_{D1}, u_{D2}) \in \mathbb{R}^2}} \max_{u_{D2}} \{L_D(x, u_D) + V(G(x, u_D))\} = \frac{1}{2}x_2^2 \tag{5.21}$$

for all $(x, u_D) \in D$. Equality (5.21) is attained by $\kappa_D(x) = (\kappa_{D1}(x), \kappa_{D2}(x))$ with

$$\kappa_{D1}(x) = \frac{R_{D2}\lambda}{R_{D1} + R_{D2} + 2R_{D1}R_{D2}} x_2$$

and

$$\kappa_{D2}(x) = \frac{R_{D1}\lambda}{R_{D1} + R_{D2} + 2R_{D1}R_{D2}} x_2$$

when

$$Q_D = \frac{-2R_{D1}R_{D2}\lambda^2 + R_{D1} + R_{D2} + 2R_{D1}R_{D2}}{2R_{D1} + 2R_{D2} + 4R_{D1}R_{D2}}, \tag{5.22}$$

which makes V a solution to (5.6) with saddle-point equilibrium κ_D. Thus, given that V is continuously differentiable on $\mathbb{R}^2$, and that (5.5) and (5.6) hold thanks to (5.21) and (5.22), from Theorem 5.2.1, the value function is $\mathcal{J}_X^*(\xi) = \frac{\xi_2^2}{2} + \xi_1$. Figure 5.1 displays this behavior with $\xi \in \mathcal{M}$ and both players playing the saddle point equilibrium. The terminal set X is reached at $t = 8$s and the cost of the displayed solution is $V(\xi)$.

Let $\mathcal{A} = \{0\}$, encoding the goal of stabilizing the ball to rest under the effect of an attacker. Furthermore, given that $L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$, and (5.14)-(5.19) hold, by setting

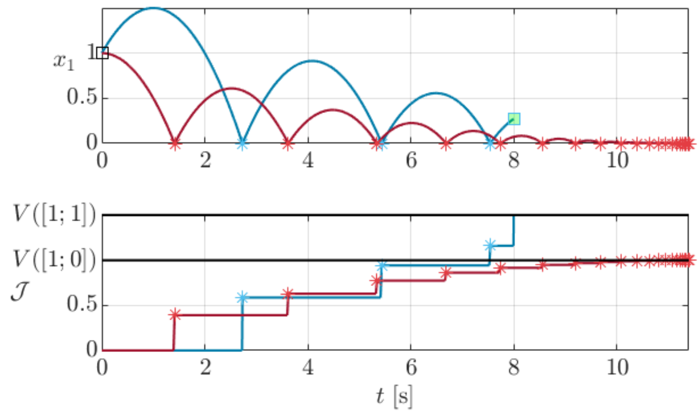

Figure 5.1: Bouncing ball solutions attaining minimum cost under worst-case u_2, with $X = \{x \in \mathbb{R}^2 : 0 \leq x_1 \leq 0.3, -0.37 \leq x_2 \leq 0.37\}$, $\lambda = 0.8$, $R_{D1} = 10, R_{D2} = -20$, and $Q_D = 0.189$.

$\alpha_1(s) = \min\left\{\frac{1}{2}\left(\frac{s}{\sqrt{2}}\right)^2, \frac{s}{\sqrt{2}}\right\}$ and $\alpha_2(s) = \frac{1}{2}s^2 + s$, from Theorem 5.3.3, we have that κ_D is the saddle-point equilibrium and renders $\mathcal{A} = \{0\}$ pre-asymptotically stable for $\mathcal{H}$. □

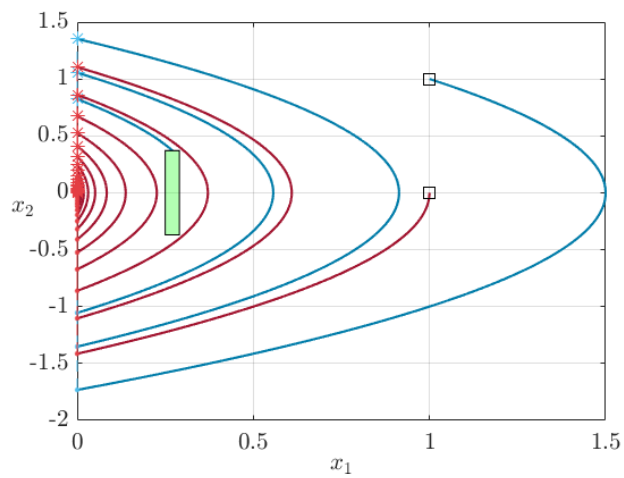

Figure 5.2: Bouncing ball phase portrait. Terminal set (green) and initial condition (square).

Chapter 6

Set-Valued Hybrid Games

In this Chapter, we present a less conservative approach by relaxing the assumption on uniqueness of solutions to the hybrid system defining the constraints of the game and allow for set-valued flow and jump maps. Although, it might not be possible to construct a saddle-point equilibrium as the solution to the game when the dynamics admit nonunique solutions (due to the game being ill-defined by the nouniqueness of costs associated to a given input), a weak saddle-point equilibrium and an upper value function are provided based on a suitable definition of the cost. Specifically, we optimize the worst-case cost to associated to a given input, which is still conveniently defined to penalize the evolution of the state and the input during flow, at jumps, and at their final value. To invoke the elements of a two-player zero-sum hybrid game in Definition 3.1.1, we redefine the cost as,

Definition 6.0.1. (Cost for set-valued games)

5) A scalar-valued functional[1] $(\xi, u) \mapsto \mathcal{J}_i(\xi, u)$ *defined for each* $i \in \{1, 2\}$*, and called the cost associated to* P_i*. For each* $u \in \mathcal{U}$*, we refer to* $\mathcal{J} := \mathcal{J}_1 = -\mathcal{J}_2$ *as the worst-case cost due to nonuniqueness of solutions to* $\mathcal{H}_s$ *for the hybrid input* u *from the initial condition* ξ*.*

[1] Given that we do not insist on having unique solutions, the cost $\mathcal{J}$ measures the largest cost of the solutions yielded to $\mathcal{H}_s$ from ξ by u. Thus, its arguments are hybrid inputs as in Definition 2.2.3 and not solution pairs.

6.1 Problem Statement

We formulate an optimization problem to solve a two-player zero-sum hybrid game with set-valued dynamics, and provide sufficient conditions to characterize its solution. Following Definition 3.1.1, consider a two-player zero-sum hybrid game with dynamics $\mathcal{H}_s$ as in (2.1) for given (C, F, D, G). Given $\xi \in C \cup D$, a joint input action $u = (u_C, u_D) \in \mathcal{U}$, the stage cost for flows $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$, the stage cost for jumps $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, and the terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, we define the cost associated to the solutions to $\mathcal{H}_s$ from the initial condition ξ and for the hybrid input u, as

$$\mathcal{J}(\xi, u) := \sup_{\phi \in \mathcal{R}(\xi, u)} \widetilde{\mathcal{J}}(\phi, u) \tag{6.1}$$

where[2]

$$\widetilde{\mathcal{J}}(\phi, u) := \sum_{j=0}^{\sup_j \operatorname{dom} \phi} \int_{t_j}^{t_{j+1}} L_C(\phi(t, j), u_C(t, j)) dt + \sum_{j=0}^{\sup_j \operatorname{dom} \phi - 1} L_D(\phi(t_{j+1}, j), u_D(t_{j+1}, j)) + \limsup_{\substack{(t,j) \to \sup \operatorname{dom} \phi \\ (t,j) \in \operatorname{dom} \phi}} q(\phi(t, j)), \tag{6.2}$$

$\{t_j\}_{j=0}^{\sup_j \operatorname{dom} \phi}$ is a nondecreasing sequence associated to the definition of the hybrid time domain of (ϕ, u) – see Definition 2.2.2 – and $\mathcal{R}(\xi, u)$ is the set of maximal state trajectories to $\mathcal{H}_s$ for the hybrid input u from the initial condition ξ, as defined in Section 2.2. The cost $\mathcal{J}$ is defined as the worst-case cost over all solutions from ξ.

A solution to the two-player zero-sum game can be obtained by solving the following problem.

Problem ($\diamond_s$): Given $\xi \in \mathbb{R}^n$, solve

$$\underset{\substack{u_1 \\ u=(u_1,u_2) \in \mathcal{U}_{\mathcal{H}_s}(\xi)}}{\text{minimize}} \; \underset{u_2}{\text{maximize}} \; \mathcal{J}(\xi, u) \tag{6.3}$$

where $\mathcal{U}_{\mathcal{H}_s}$ is the set of joint input actions yielding maximal solutions to $\mathcal{H}_s$, as defined in Section II.A.

Remark 6.1.1. (Saddle-point equilibrium and min-max control) *A solution to Problem ($\diamond_s$), when it exists, can be expressed in terms of the saddle-point equilibrium strategy*

[2] Notice that $\mathcal{J}$ depends on the initial condition ξ and input u, while $\widetilde{\mathcal{J}}$ depends on the solution pair (ϕ, u) with $\phi(0, 0) = \xi$.

$\kappa = (\kappa_1, \kappa_2)$ for the two-player zero-sum game, as in Definition 3.1.3. Each $u^* = (u_1^*, u_2^*)$ that renders a worst-case cost state trajectory $\phi^* \in \mathcal{R}(\xi, u^*)$, with components defined as $\operatorname{dom}\phi^* \ni (t,j) \mapsto u_i^*(t,j) = \kappa_i(\phi^*(t,j))$ for each $i \in \{1,2\}$, satisfies

$$u^* \in \underset{u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}_s}(\xi)}{\arg\min_{u_1}\max_{u_2}} \mathcal{J}(\xi,u) = \underset{u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}_s}(\xi)}{\arg\max_{u_2}\min_{u_1}} \mathcal{J}(\xi,u)$$

and it is referred to as a min-max control at ξ.

Definition 6.1.2. (Value function) *Given $\xi \in \Pi(\overline{C}) \cup \Pi(D)$, the value function at ξ, when it exists, is given by*

$$\mathcal{J}^*(\xi) := \underset{u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}_s}(\xi)}{\min_{u_1}\max_{u_2}} \mathcal{J}(\xi,u) = \underset{u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}_s}(\xi)}{\max_{u_2}\min_{u_1}} \mathcal{J}(\xi,u) \tag{6.4}$$

We revisit the robust control application presented in the introduction and reformulate it according to the mathematical framework provided in this chapter as follows.

Application 1. (Robust Control) *Given the system $\mathcal{H}_s$ as in (2.3) and $\xi \in \mathbb{R}^n$, the disturbance rejection problem consists of finding the control input*

$$u_1 = (u_{C1}, u_{D1}) = \underset{\substack{(u_{C1},u_{D1}) \\ u=(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}(\xi)}}{\arg\min} \mathcal{J}(\xi,u) \tag{6.5}$$

in the presence of a disturbance $u_2 = (u_{C2}, u_{D2})$ chosen by P_2. To account for the worst-case disturbance, (6.5) *is addressed by solving Problem ($\diamond_s$).*

As a preliminary step, following the approach in [78], we present a framework for cost evaluation for hybrid set-valued systems as in (2.1), under the special case of no inputs present.

6.2 Cost Evaluation for Autonomous Hybrid Inclusions

6.2.1 Upper Bounds

By following the general ideas proposed in [79], in this section we investigate how a Lyapunov-like function can be used to provide estimates of nonlinear cost functionals for a given hybrid inclusion.

For each initial condition $\xi \in \overline{C} \cup D$ to $\mathcal{H}_s$ in (2.1), with $u_C = 0$ and $u_D = 0$, the costs in (6.1) and (6.2) can be expressed as:

$$\mathcal{J}(\xi) := \sup_{\phi \in \mathcal{S}_{\mathcal{H}_s}(\xi)} \widetilde{\mathcal{J}}(\phi) \tag{6.6}$$

where

$$\widetilde{\mathcal{J}}(\phi) := \sum_{j=0}^{\sup_j \operatorname{dom} \phi} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j))dt + \sum_{j=0}^{\sup_j \operatorname{dom} \phi - 1} L_D(\phi(t_{j+1}, j)) + \limsup_{\substack{(t,j) \to \sup \operatorname{dom} \phi \\ (t,j) \in \operatorname{dom} \phi}} q(\phi(t,j)), \tag{6.7}$$

where we denote $L_C(x) \equiv L_C(x,0)$ and $L_D(x) \equiv L_D(x,0)$. The following result can be established.

Proposition 6.2.1. (Upper bound for a given trajectory) *Let $\xi \in C \cup D$, $L_C \colon C \to \mathbb{R}_{\geq 0}$, $L_D \colon D \cup G(D) \to \mathbb{R}_{\geq 0}$, and $q : C \cup D \cup G(D) \to \mathbb{R}$. Let $V \colon \operatorname{dom} V \to \mathbb{R}$ with $\operatorname{dom} V \supset C \cup D \cup G(D)$ be continuously differentiable on an open set containing $\overline{C}$. Assume that*

$$\sup_{f \in F(x)} \langle \nabla V(x), f \rangle + L_C(x) \leq 0 \qquad \forall x \in C \tag{6.8a}$$

$$\sup_{g \in G(x)} V(g) - V(x) + L_D(x) \leq 0 \qquad \forall x \in D \tag{6.8b}$$

where we denote $F(x) \equiv F(x,0)$ and $G(x) \equiv G(x,0)$. Let $\phi \colon \operatorname{dom} \phi \to \mathbb{R}^n$ be a solution to $\mathcal{H}_s$ as in (2.1) from ξ with no inputs. Assume that $(t,j) \mapsto V \circ \phi(t,j)$ is bounded and

$$\limsup_{\substack{(t,j) \to \sup \operatorname{dom} \phi \\ (t,j) \in \operatorname{dom} \phi}} V(\phi(t,j)) = \limsup_{\substack{(t,j) \to \sup \operatorname{dom} \phi \\ (t,j) \in \operatorname{dom} \phi}} q(\phi(t,j)). \tag{6.9}$$

Then, $\widetilde{\mathcal{J}}(\phi)$ is a finite number and in particular

$$\widetilde{\mathcal{J}}(\phi) \leq V(\xi) \tag{6.10}$$

Proposition 6.2.1, by building on a suitable function V, provides an upper bound on the cost $\widetilde{\mathcal{J}}(\phi)$ that depends on the solution chosen from ξ.

Remark 6.2.2. (Upper bound for an initial condition) *Notice that since Proposition 6.2.1 provides an upper bound for the cost of any solution to $\mathcal{H}_s$ from ξ, it follows that under the conditions stated therein, we have*

$$\sup_{\phi \in \mathcal{S}_{\mathcal{H}_s}(\xi)} \widetilde{\mathcal{J}}(\phi) \leq V(\xi) \tag{6.11}$$

which, thanks to (6.6)*, implies*

$$\mathcal{J}(\xi) \leq V(\xi). \tag{6.12}$$

6.2.2 Exact Cost Evaluation

In this section, our main objective is to obtain the exact value of the cost $\mathcal{J}(\xi)$ in (6.6) for a given initial condition ξ, without explicitly computing it. To that end, next, under further assumptions on the system data and a stronger condition than (6.8), we provide a result on exact cost evaluation.

Corollary 6.2.3. () *Let $\xi \in \overline{C} \cup D$, $L_C\colon C \to \mathbb{R}_{\geq 0}$, $L_D\colon D \cup G(D) \to \mathbb{R}_{\geq 0}$, $q : C \cup D \cup G(D) \to \mathbb{R}$, and $F(x)$ and $G(x)$ be compact, respectively, for each $x \in C$ and each $x \in D$. Assume that there exists a continuous function $V\colon \operatorname{dom} V \to \mathbb{R}$, $\operatorname{dom} V \supset C \cup D \cup G(D)$, that is continuously differentiable on an open set containing $\overline{C}$ such that*

$$\max_{f \in F(x)} \langle \nabla V(x), f \rangle + L_C(x) = 0 \qquad \forall x \in C \tag{6.13a}$$

$$\max_{g \in G(x)} V(g) - V(x) + L_D(x) = 0 \qquad \forall x \in D \tag{6.13b}$$

Furthermore, assume that for any solution ϕ to $\mathcal{H}_s$ as in (2.1)*, with no inputs, from ξ, $V \circ \phi$ is bounded and* (6.9) *holds. Pick any solution ϕ^* to the maximal hybrid system*

$$\begin{aligned} \dot{x} &\in \arg\max_{f \in F(x)} \langle \nabla V(x), f \rangle & x \in C \\ x^+ &\in \arg\max_{g \in G(x)} V(g) & x \in D \end{aligned} \tag{6.14}$$

with $\phi^*(0) = \xi$ and let ϕ be any solution to (2.1) *with no inputs from ξ. Then, one has that $\mathcal{J}(\phi)$ and $\mathcal{J}(\phi^*)$ are finite and in particular*

$$\widetilde{\mathcal{J}}(\phi) \leq \widetilde{\mathcal{J}}(\phi^*) = \mathcal{J}(\xi) = V(\xi) \tag{6.15}$$

The results given in this section extend previous results on cost evaluation for continuous-time nonlinear systems [79] and constrained difference inclusions [80] to hybrid inclusions. Similarly as in [78,79], our results have strong connections to Lyapunov analysis. More specifically, the applicability of our results to specific examples requires the search of a suitable Lyapunov-like function, which is in general a challenging task. In the subsequent section, we provide an analysis for the case of nonautonomous hybrid inclusions as in (2.1).

6.3 Design of Weak Saddle-Point Equilibrium for Set-Valued Hybrid Games

In general, the cost evaluation tools employed in approaches based on dynamic programming fall short to characterize strategies to attain a saddle-point equilibrium solution for a two-player zero-sum game with dynamics given by hybrid inclusions. The classical conditions involved therein do not guarantee the existence of a lower bound in the costs for a given input action. Nevertheless, conditions can still be established to characterize the worst-case cost (due to the set-valued dynamics) associated to it. Thus, in this section, we provide sufficient conditions to solve Problem ($\diamond_s$) via finding a control strategy that minimizes the worst-case cost under the maximizing adversarial action, in this case, leading to a solution of a min-max problem with nonunique solutions due to F or G being possibly setvalued, or $C \cap D$ being nonempty. In addition, such conditions allow to evaluate the value function without computing solutions. First, we provide pointwise conditions that allow to upper bound the cost for an initial condition and input action.

Proposition 6.3.1. (Upper bound for a given control action) *Given a system with dynamics $\mathcal{H}_s$ as in (2.1) with data (C, F, D, G), stage costs $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$ and $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, and terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, suppose there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on a neighborhood of $\Pi(C)$ such that*

$$L_C(x, u_C) + \sup_{f \in F(x,u_C)} \langle \nabla V(x), f \rangle \leq 0 \qquad \forall (x, u_C) \in C, \tag{6.16}$$

$$L_D(x, u_D) + \sup_{g \in G(x,u_D)} V(g) - V(x) \leq 0 \quad \forall (x, u_D) \in D. \tag{6.17}$$

Let (ϕ, u) be any solution to $\mathcal{H}_s$ from $\xi \in \Pi(\overline{C}) \cup \Pi(D)$. Then,

$$\widetilde{\mathcal{J}}(\phi, u) \leq V(\xi) \tag{6.18}$$

where $\widetilde{\mathcal{J}}$ is defined as in (6.2).

In the following result we study a special hybrid system, whose solutions are a subset of the solutions to $\mathcal{H}_s$ as in (2.1) and attain the worst-case cost due to nonuniqueness of solutions to $\mathcal{H}_s$. Following [81], we provide conditions to exactly evaluate such a cost and show how it is an upper bound for the cost of any other solution to $\mathcal{H}_s$.

Proposition 6.3.2. (Maximal System) *Consider a system with dynamics $\mathcal{H}_s$ as in (2.1) with data (C, F, D, G), where F and G are compact for each $(x, u_C) \in C$ and each $(x, u_D) \in D$, respectively, stage costs $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$ and $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, and suppose that there exists a continuous function $V : \operatorname{dom} V \to \mathbb{R}$, $\operatorname{dom} V \supset \Pi(\overline{C}) \cup \Pi(D) \cup G(D)$, that is continuously differentiable on a neighborhood of $\Pi(C)$. Given $\xi \in \Pi(\overline{C}) \cup \Pi(D)$ and a solution[3] (ϕ^*, u) to*

$$\mathcal{H}_{\max} \begin{cases} \dot{x} & \in \quad \underset{f \in F(x, u_C)}{\operatorname{argmax}} \ \langle \nabla V(x), f \rangle \quad (x, u_C) \in C \\ x^+ & \in \quad \underset{g \in G(x, u_D)}{\operatorname{argmax}} \ V(g) \qquad\quad (x, u_D) \in D \end{cases} \tag{6.19}$$

from ξ with $u = (u_C, u_D)$, if

$$0 = L_C(x, u_C) + \sup_{f \in F(x, u_C)} \langle \nabla V(x), f \rangle \qquad \forall (x, u_C) \in C, \tag{6.20}$$

$$0 = L_D(x, u_D) + \sup_{g \in G(x, u_D)} V(g) \qquad \forall (x, u_D) \in D, \tag{6.21}$$

and

$$\limsup_{\substack{(t,j) \to \sup \operatorname{dom} \phi^* \\ (t,j) \in \operatorname{dom} \phi^*}} V(\phi^*(t, j)) = \limsup_{\substack{(t,j) \to \sup \operatorname{dom} \phi^* \\ (t,j) \in \operatorname{dom} \phi^*}} q(\phi^*(t, j)), \tag{6.22}$$

then

$$\mathcal{J}(\xi, u) = \tilde{\mathcal{J}}(\phi^*, u) \tag{6.23}$$

and

$$V(\xi) = \mathcal{J}(\xi, u). \tag{6.24}$$

A solution to (6.19) attains the worst-case cost among the potential nonunique solutions to (2.1) due to the set-valuedness of the maps. Furthermore, the worst-case cost associated to the input action that satisfies (6.20) and (6.21), can be evaluated without computing solutions and equates V evaluated at the initial state ξ.

Corollary 6.3.3. (Change of Signs) *If the conditions in Proposition 6.3.2 hold with inequality, namely, if "$=$" in* (6.20) *and* (6.21) *is replaced with "$\leq$" (or "$\geq$"), then* (6.24) *holds with "$\leq$" (or "$\geq$", respectively).*

Lemma 6.3.4. (Solutions to $\mathcal{H}_{\max}$) *Any solution to $\mathcal{H}_{\max}$ as in* (6.19) *is a solution to $\mathcal{H}_s$ as in* (2.1).

[3]Solutions to the "maximal system" in (6.19) exist under the conditions in Proposition 6.3.6.

Lemma 6.3.5. (Existence of Solutions to Optimization Problems in Maximal System) *Consider the maximal system* $\mathcal{H}_{\max}$ *as in* (6.19) *with* (C, F, D, G) *and* $u = (u_1, u_2)$*. If the maps* $F(x, u_C)$ *and* $G(x, u_D)$ *are compact for each* $(x, u_C) \in C$ *and each* $(x, u_D) \in D$*, respectively, and the function* $V : \operatorname{dom} V \to \mathbb{R}$*,* $\operatorname{dom} V \supset \Pi(\overline{C}) \cup \Pi(D) \cup G(D)$ *is continuous on* $\operatorname{dom} V$ *and continuously differentiable on a neighborhood of* $\Pi(C)$*, then there exists a solution to*

$$\max_{f \in F(x, u_C)} \langle \nabla V(x), f \rangle \tag{6.25}$$

for each $(x, u_C) \in C$*, and to*

$$\max_{g \in G(x, u_D)} V(g) \tag{6.26}$$

for each $(x, u_D) \in D$*.*

Proposition 6.3.6. (Existence of solutions to $\mathcal{H}_{\max}$) *Consider a feedback law* $\kappa := (\kappa_C, \kappa_D)$ *with* $\kappa : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$*, a function* $V : \operatorname{dom} V \to \mathbb{R}$*,*$\operatorname{dom} V \supset \Pi(\overline{C}) \cup \Pi(D) \cup G(D)$*, that is continuously differentiable on a neighborhood of* $\Pi(C)$*, and the closed-loop system*

$$\mathcal{H}_{\max_\kappa} \begin{cases} \dot{x} \in \underset{f \in F(x, \kappa_C(x))}{\operatorname{argmax}} \langle \nabla V(x), f \rangle & (x, \kappa_C(x)) \in C_\kappa \\ x^+ \in \underset{g \in G(x, \kappa_D(x))}{\operatorname{argmax}} V(g) & (x, \kappa_D(x)) \in D_\kappa \end{cases} \tag{6.27}$$

Suppose $\mathcal{H}_{\max_\kappa}$ *satisfies the hybrid basic conditions in Definition 2.3.2, the function* V *is continuous, and* $F(x, \kappa_C(x))$ *and* $G(x, \kappa_D(x))$ *are compact for each* $x \in C_\kappa$ *and each* $x \in D_\kappa$*, respectively. Let* $\xi \in C_\kappa \cup D_\kappa$ *be arbitrary. If* $\xi \in D_\kappa$ *or*

(VCM) there exists a neighborhood X_n *of* ξ *such that for every* $x \in X_n \cap C_\kappa$*,*

$$F(x, \kappa_C(x)) \cap T_C(x) \neq \emptyset$$

then there exists a nontrivial solution ϕ *to* $\mathcal{H}_{\max_\kappa}$ *with* $\phi(0, 0) = \xi$*. If* (VC) *holds for every* $\xi \in C_\kappa \setminus D_\kappa$*, then there exists a nontrivial solution to* $\mathcal{H}_{\max_\kappa}$ *for every initial point in* $C_\kappa \cup D_\kappa$*.*

Based on Proposition 6.3.1, that provides an upper bound on the cost $\widetilde{\mathcal{J}}$, and the exact cost evaluation in Proposition 6.3.2, we introduce the main result of the section with sufficient conditions to characterize the saddle-point equilibrium strategy and evaluate the value function without computing solutions.

Theorem 6.3.7. (Sufficient conditions to solve Problem $(\diamond_s)$) *Given a system with dynamics $\mathcal{H}_s$ as in (2.1) with data (C, F, D, G), stage costs $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$ and $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, and terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, suppose the following hold:*

1. *There exists a continuous function $V : \operatorname{dom} V \to \mathbb{R}$, $\operatorname{dom} V \supset \Pi(\overline{C}) \cup \Pi(D) \cup G(D)$, that is continuously differentiable on a neighborhood of $\Pi(C)$ and a feedback law $\kappa := (\kappa_C, \kappa_D) = ((\kappa_{C1}, \kappa_{C2}), (\kappa_{D1}, \kappa_{D2})) : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ such that $F(x, \kappa_C(x))$ and $G(x, \kappa_D(x))$ are compact for every x such that $(x, \kappa_C(x)) \in C$ and $(x, \kappa_D(x)) \in D$, respectively, and such that the functions $\mathcal{L}_C(x, u_C) := L_C(x, u_C) + \sup_{f \in F(x,u_C)} \langle \nabla V(x), f \rangle$, and $\mathcal{L}_D(x, u_D) := L_D(x, u_D) + \sup_{g \in G(x,u_D)} V(g)$ satisfy*

$$0 = \mathcal{L}_C(x, \kappa_C(x)) \qquad \forall x : (x, \kappa_C(x)) \in C, \tag{6.28}$$

$$0 \leq \mathcal{L}_C(x, (u_{C1}, \kappa_{C2}(x))) \qquad \forall (x, u_{C1}) : (x, (u_{C1}, \kappa_{C2}(x))) \in C, \tag{6.29}$$

$$0 \geq \mathcal{L}_C(x, (\kappa_{C1}(x), u_{C2})) \qquad \forall (x, u_{C2}) : (x, (\kappa_{C1}(x), u_{C2})) \in C, \tag{6.30}$$

$$V(x) = \mathcal{L}_D(x, \kappa_D(x)) \qquad \forall x : (x, \kappa_D(x)) \in D, \tag{6.31}$$

$$V(x) \leq \mathcal{L}_D(x, (u_{D1}, \kappa_{D2}(x))) \quad \forall (x, u_{D1}) : (x, (u_{D1}, \kappa_{D2}(x)))) \in D, \tag{6.32}$$

$$V(x) \geq \mathcal{L}_D(x, (\kappa_{D1}(x), u_{D2})) \quad \forall (x, u_{D2}) : (x, (\kappa_{D1}(x), u_{D2})) \in D, \tag{6.33}$$

2. *For each $\xi \in \Pi(\overline{C}) \cup \Pi(D)$, each $\phi \in \mathcal{S}_{\mathcal{H}_\kappa}(\xi)$ satisfies*

$$\limsup_{\substack{(t,j) \to \sup \operatorname{dom} \phi \\ (t,j) \in \operatorname{dom} \phi}} V(\phi(t,j)) = \limsup_{\substack{(t,j) \to \sup \operatorname{dom} \phi \\ (t,j) \in \operatorname{dom} \phi}} q(\phi(t,j)), \tag{6.34}$$

Then

$$\mathcal{J}^*(\xi) = V(\xi) \quad \forall \xi \in \Pi(\overline{C}) \cup \Pi(D). \tag{6.35}$$

Remark 6.3.8. (Weak optimality of the saddle-point equilibrium) *When both players play the saddle-point equilibrium strategy, due to nonuniqueness of solutions, there is no reason to assume that the worst-cost is attained, implying that such a strategy is not necessarily optimal in the min-max sense. Nevertheless, by playing the saddle-point equilibrium, the worst-case cost is minimized under the adversarial action that aims to maximize it. We illustrate this in the examples in Section 6.5.*

6.4 Asymptotic Stability for Set-Valued Hybrid Games

We present a result that connects optimality and asymptotic stability for two-player zero-sum hybrid games with set-valued dynamics.

Definition 6.4.1. (Uniform global pre-asymptotic stability) *A closed set $\mathcal{A} \subset \mathbb{R}^n$ is uniformly globally pre-asymptotically stable for a hybrid closed-loop system $\mathcal{H}_\kappa$ as in (2.4) if it is*

- *uniformly globally stable for $\mathcal{H}_\kappa$, i.e., there exists a class-$\mathcal{K}_\infty$ function α such that any solution ϕ to $\mathcal{H}_\kappa$ satisfies $|\phi(t,j)|_\mathcal{A} \leq \alpha(|\phi(0,0)|_\mathcal{A})$ for all $(t,j) \in \operatorname{dom} \phi$; and*
- *uniformly globally pre-attractive for $\mathcal{H}_\kappa$, i.e., for each $\varepsilon > 0$ and $r > 0$ there exists $T > 0$ such that, for any solution ϕ to $\mathcal{H}_\kappa$ with $|\phi(0,0)|_\mathcal{A} \leq r$, $(t,j) \in \operatorname{dom} \phi$ and $t + j \geq T$ imply $|\phi(t,j)|_\mathcal{A} \leq \varepsilon$.*

Theorem 6.4.2. (Saddle-point equilibrium under the existence of a Lyapunov function) *Consider a two-player zero-sum hybrid game with closed-loop dynamics $\mathcal{H}_\kappa$ as in (2.4) with data (C, F, D, G), and $\kappa := (\kappa_C, \kappa_D) : \mathbb{R}^n \to \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$ such that $C_\kappa = \Pi(C)$ and $D_\kappa = \Pi(D)$. Given a closed set $\mathcal{A} \subset \mathbb{R}^n$, continuous functions $L_C : C \to \mathbb{R}_{\geq 0}$ and $L_D : D \to \mathbb{R}_{\geq 0}$ defining the stage costs for flows and jumps, respectively, and $q : \mathbb{R}^n \to \mathbb{R}$ defining the terminal cost, suppose there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ that is continuously differentiable on an open set containing $\overline{C_\kappa}$, satisfying (6.28)-(6.33), and such that for each $\xi \in \overline{C_\kappa} \cup D_\kappa$, each $\phi \in \mathcal{S}_{\mathcal{H}_\kappa}(\xi)$ satisfies* (6.34)*. Furthermore, suppose that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that*

$$\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A}) \qquad \forall x \in \overline{C_\kappa} \cup D_\kappa \cup G(D_\kappa) \tag{6.36}$$

and one of the following conditions[4] *holds*

1. *$L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A})$ and $L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$;*
2. *$L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A})$ and there exists a continuous function $\eta \in \mathcal{PD}$ such that $L_C(x, \kappa_D(x)) \geq \eta(|x|_\mathcal{A})$ for all $x \in C_\kappa$;*

[4] The subindexes in the sets of positive definite functions $\mathcal{PD}_*$ denote the feedback law that they are composed with as in Definition 3.3.2.

3. $L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A})$ *and there exists a continuous function* $\eta \in \mathcal{PD}$ *such that* $L_D(x, \kappa_D(x)) \geq \eta(|x|_{\mathcal{A}})$ *for all* $x \in D_\kappa$*.*

Then

$$\mathcal{J}^*(\xi) = V(\xi) \qquad \forall \xi \in \overline{C_\kappa} \cup D_\kappa \tag{6.37}$$

Furthermore, the feedback law κ *is the saddle-point equilibrium (see Definition 3.1.3) and it renders* $\mathcal{A}$ *uniformly globally pre-asymptotically stable for* $\mathcal{H}_\kappa$ *as in Definition 6.4.1.*

6.5 Applications

As an instance of Application 1, we illustrate in the following scenario with hybrid inclusions and quadratic costs how Theorem 6.3.7 provides conditions to solve a disturbance rejection problem by addressing it as zero-sum hybrid game.

6.5.1 Robust Hybrid Linear Quadratic Problems with Sponteneous Jumps

In this section, we study a special case that emerges in practical scenarios with hybrid systems with linear flow and jump maps and sponteneous jumps, as in noise attenuation of cyber-physical systems with intermitent communcation, see, e.g., [47, 70, 71]. We introduce a state variable τ that plays the role of a timer. Once τ reaches a fixed threshold $\bar{T}$, it triggers a jump in the state and resets τ to a random number in $[0, \bar{T}]$. More precisely, given $\bar{T} \in \mathbb{R}$, we consider a hybrid system with state $x = (x_p, \tau) = ((x_{p1}, x_{p2}), \tau) \in \mathbb{R}^n \times [0, \bar{T}]$, input $u = (u_C, u_D) = ((u_{C1}, u_{C2}), (u_{D1}, u_{D2})) \in \mathbb{R}^{m_C} \times \mathbb{R}^{m_D}$, and dynamics $\mathcal{H}_s$ as in (2.1), defined by

$$\begin{aligned} C &:= \mathbb{R}^n \times [0, \bar{T}] \times \mathbb{R}^{m_C} \\ F(x, u_C) &:= (A_C x_p + B_C u_C, 1) \quad \forall (x, u_C) \in C \\ D &:= \mathbb{R}^n \times \{\bar{T}\} \times \mathbb{R}^{m_D} \\ G(x, u_D) &:= (A_D x_p + B_D u_D, [0, \bar{T}]) \quad \forall (x, u_D) \in D \end{aligned} \tag{6.38}$$

with $A_C = \begin{bmatrix} A_{C1} & 0 \\ 0 & A_{C2} \end{bmatrix}$, $B_C = \begin{bmatrix} B_{C1} & B_{C2} \end{bmatrix}$, $A_D = \begin{bmatrix} A_{D1} & 0 \\ 0 & A_{D2} \end{bmatrix}$, and $B_D = \begin{bmatrix} B_{D1} & B_{D2} \end{bmatrix}$. In this case, the input $u_1 := (u_{C1}, u_{D1})$ plays the role of the control and is assigned by player P_1, and $u_2 := (u_{C2}, u_{D2})$ is the disturbance input, which is assigned by player P_2.

The problem of upper bounding the effect of the disturbance u_2 in the cost of solutions to $\mathcal{H}_s$ is formulated as a two-player zero-sum set-valued hybrid game as in Definitions 3.1.1 and 6.0.1 . Thus, by solving Problem $(\diamond_s)$ for every $\xi \in \Pi(C) \cup \Pi(D)$, the control objective is achieved.

The following result presents a tool for the solution of the optimal control problem for hybrid systems with linear maps and sponteneous jumps under an adversarial action.

Corollary 6.5.1. (Hybrid Riccati equation for disturbance rejection with sponteneous jumps) *Given a hybrid system $\mathcal{H}_s$ as in (2.1) with data (C, F, D, G) as in* (6.38)*, let $\bar{T} \in \mathbb{R}$, and, $L_C(x, u_C) := x_p^\top Q_C x_p + u_C^\top R_C u_C$, $L_D(x, u_D) := x_p^\top Q_D x_p + u_D^\top R_D u_D$, and terminal cost $q(x) := x_p^\top P(\tau) x_p$ defining $\mathcal{J}$ as in (6.2), with $Q_C, Q_D \in \mathbb{S}_+^n$, $R_C = \begin{bmatrix} R_{C1} & 0 \\ 0 & R_{C2} \end{bmatrix}$, $R_D = \begin{bmatrix} R_{D1} & 0 \\ 0 & R_{D2} \end{bmatrix}$, $R_{C1} \in \mathbb{S}_+^{m_{C_1}}$, $-R_{C2} \in \mathbb{S}_+^{m_{C_2}}$, $R_{D1} \in \mathbb{S}_+^{m_{D_1}}$, and $-R_{D2} \in \mathbb{S}_+^{m_{D_2}}$. Suppose there exists a matrix function $P : [0, \bar{T}] \to \mathbb{S}_+^n$ that is nonincreasing and continuously differentiable and such that*

$$-\frac{\mathrm{d}P(\tau)}{\mathrm{d}\tau} = - P(\tau) B_C R_C^{-1} B_C^\top P(\tau) + Q_C + P(\tau) A_C + A_C^\top P(\tau) \qquad \forall \tau \in (0, \bar{T}), \tag{6.39}$$

$$-R_{D2} - B_{D2}^\top P(0) B_{D2}, \quad R_{D1} + B_{D1}^\top P(0) B_{D1} \in \mathbb{S}_{0+}^{m_D}, \tag{6.40}$$

the matrix $R_v = R_D + B_D^\top P(0) B_D$ is invertible, and

$$P(\bar{T}) = Q_D + A_D^\top P(0) A_D - A_D^\top P(0) B_D R_v^{-1} B_D^\top P(0) A_D \tag{6.41}$$

where $A_C, B_{C1}, B_{C2}, A_D, B_{D1}$, and B_{D2} are defined below (6.38)*. Then, the feedback law $\kappa := (\kappa_C, \kappa_D)$, defined as*

$$\kappa_C(x) = -R_C^{-1} B_C^\top P(\tau) x_p \qquad \forall x \in \Pi(C), \tag{6.42}$$

$$\kappa_D(x) = -R_v^{-1} B_D^\top P(0) A_D x_p \qquad \forall x \in \Pi(D) \tag{6.43}$$

is the saddle-point equilibrium for the two-player zero-sum hybrid game with sponteneous jumps. In addition, for each $x = (x_p, \tau) \in \Pi(\overline{C}) \cup \Pi(D)$, the value function is equal to $V(x) := x_p^\top P(\tau) x_p$.

Notice that the saddle-point equilibrium $\kappa = (\kappa_1, \kappa_2)$ is composed by P_1 playing the upper bounding strategy κ_1, and P_2 playing the maximizing disturbance κ_2 with values as in (6.42) and (6.43).

Example 6.5.2. (Sporadic Jump Times and Linear Dynamics) Consider the following data for (6.38):

$$A_C = 1.8, B_C = B_D = [1, 1], A_D = 2, \bar{T} = 1$$

We solve (6.39) under (6.41) and (6.40), obtaining a nonincreasing $P(\tau)$. In Figure 6.1 we report the evolution of two solutions, with a worst-case scenario depicting periodic jumps, an arbitrary scenario with sporadic jumps. As expected, as $t+j$ goes to infinity, the costs approach $V(\xi) = \xi_p^\top P(\xi_\tau)\xi_p$, and the periodic solution attains the upper bound.

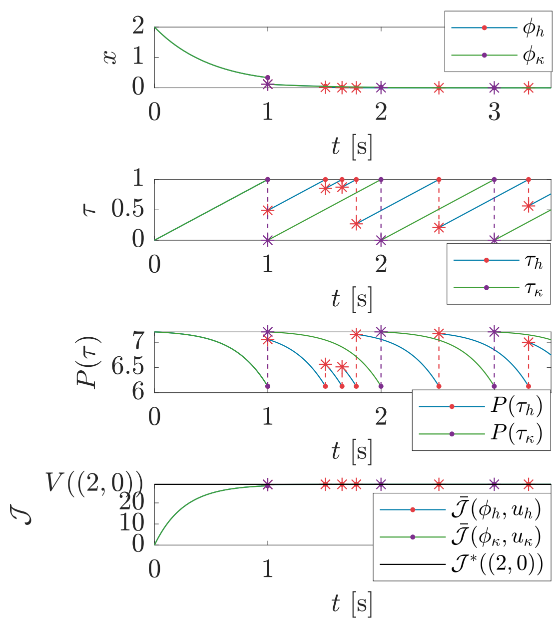

Figure 6.1: Nonnunique solutions to (6.38) due to set-valued dynamics for $\xi = [2, 0], Q_C = Q_D = 1$, $R_{C1} = R_{D1} = 1.304$, and $2R_{C2} = R_{D2} = -8$. Worst-case cost solution (with periodic jump times) in green and pruple. Arbitrary solution (with spontaneous jump times) in blue and red.

□

6.5.2 Set-Valued Flow and Jump Maps

As illustrated next, there are useful families of hybrid set-valued systems for which a weak saddle-point equilibrium exists. The following example characterizes both the saddle-point equilibrium and the value function in a two-player zero-sum game with a scalar state associated to player P_1. Thus, $n_1 = 1, n_2 = 0$, and the role of player P_2 reduces to select the action u_{C2}.

Example 6.5.3. (1D Set-Valued Hybrid Game) Consider a hybrid system $\mathcal{H}_s$ as in (2.1) with state $x \in \mathbb{R}$, input $u_C := (u_{C1}, u_{C2}) \in \mathbb{R}^2$, and dynamics

$$\begin{aligned} \dot{x} \in \quad & F(x, u_C) := [\underline{a}, \overline{a}]x + Bu_C \quad & x \in [0, \overline{\sigma}] \cup [\mu, \delta] \\ x^+ \in \quad & G(x) := [\underline{\sigma}, \overline{\sigma}] \quad & x = \mu \end{aligned} \tag{6.44}$$

where $\underline{a} < \overline{a} < 0, B = [b_1\ b_2]$ and $\delta \geq \mu > \overline{\sigma} > \underline{\sigma} > 0$[5]. Consider the cost functions $L_C(x, u_C) := x^2 Q_C + u_C^\top R_C u_C$, $L_D(x) := P(x^2 - \overline{\sigma}^2)$, and terminal cost $q(x) := Px^2$, defining $\mathcal{J}$ as in (6.2), with $R_C := \begin{bmatrix} R_{C1} & 0 \\ 0 & R_{C2} \end{bmatrix}$, Q_C, R_{C1}, $-R_{C2}$, $P \in \mathbb{R}_{>0}$, such that

$$Q_C + 2P\overline{a} - P^2(b_1^2 R_{C1}^{-1} + b_2^2 R_{C2}^{-1}) = 0. \tag{6.45}$$

Here, u_{C1} is designed by player P_1, which aims to minimize a cost functional $\mathcal{J}$, while player P_2 seeks to maximize it by means of u_{C2}. This is formulated as a two-player zero-sum hybrid game via solving Problem $(\diamond_s)$ in Section 3.1.3. The function $V(x) := Px^2$ satisfies the sufficient condition for (6.28)-(6.30) in Theorem 6.3.7 given as

$$\begin{aligned} & \min_{\substack{u_{C1} \\ u_C = (u_{C1}, u_{C2}) \in \mathbb{R}^2}} \max_{u_{C2}} \left\{ L_C(x, u_C) + \sup_{f \in F(x, u_C)} \langle \nabla V(x), f \rangle \right\} \\ & = \min_{u_{C1} \in \mathbb{R}} \max_{u_{C2} \in \mathbb{R}} \left\{ (Q_C + 2P\overline{a})x^2 + R_{C1} u_{C1}^2 + R_{C2} u_{C2}^2 + 2xP(b_1 u_{C1} + b_2 u_{C2}) \right\} = 0 \end{aligned} \tag{6.46}$$

which holds for all $x \in [0, \overline{\sigma}] \cup [\mu, \delta]$.In fact, the min-max in (6.46) is attained by $\kappa_C(x) = (-R_{C1}^{-1} b_1 Px, -R_{C2}^{-1} b_2 Px)$. In particular, thanks to (6.45), we have

$$-L_C(x, \kappa_C(x)) = \sup_{f \in F(x, \kappa_C(x))} \langle \nabla V(x), f \rangle$$

Then, $V(x) = Px^2$ is a solution to (6.28)-(6.30). In addition, the function V satisfies the sufficient condition for (6.31)-(6.33) in Theorem 6.3.7 given as

$$L_D(x) + \sup_{g \in G(x)} V(g(x)) = Px^2 \tag{6.47}$$

at $x = \mu$, which makes $V(x) = Px^2$ a solution to (6.31)-(6.33) with saddle-point equilibrium κ_C. Given that V is continuously differentiable on $\mathbb{R}$, and that (6.28)-(6.33)

[5]Given that $\mu > \delta$, flow from μ is not possible.

hold thanks to (6.46) and (6.47), from Theorem 6.3.7 we have that the value function is $\mathcal{J}^*(\xi) := P\xi^2$ for any $\xi \in [0, \overline{\sigma}] \cup [\mu, \delta]$.

To study in detail the nonunique solutions yielded by the feedback law κ_C, notice that solutions jump at $x = \mu$ to $\sigma_s \in [\underline{\sigma}, \overline{\sigma}]$. Consider a solution ϕ_h with domain $\operatorname{dom}\phi_h = ([0, t^h] \times \{0\}) \cup ([t^h, \infty) \times \{1\})$, and given by $\phi_h(t, 0) = \delta \exp((a_s - R_{C1}^{-1} b_1 P - R_{C2}^{-1} b_2 P)t)$, $\phi_h(t, 1) = \sigma_s \exp((a_s - R_{C1}^{-1} b_1 P - R_{C2}^{-1} b_2 P)(t - t^h))$ with $a_s \in [\underline{a}, \overline{a}]$. In simple words, ϕ_h flows from δ to μ in t^h seconds, then it jumps to σ_s, and flows converging (exponentially fast) to zero. Notice that κ_C as defined above also yields a solution ϕ_κ with domain $\operatorname{dom}\phi_\kappa = ([0, t^\kappa] \times \{0\}) \cup ([t^\kappa, \infty) \times \{1\})$, and given by $\phi_\kappa(t, 0) = \delta \exp((\overline{a} - R_{C1}^{-1} b_1 P - R_{C2}^{-1} b_2 P)t)$, $\phi_\kappa(t, 1) = \overline{\sigma} \exp((\overline{a} - R_{C1}^{-1} b_1 P - R_{C2}^{-1} b_2 P)(t - t^\kappa))$ attaining the worst-case cost. Figure 6.2 illustrates the similar behavior of the solutions ϕ_h and ϕ_κ, yielded by κ_C, with the cost of the latter equating $P\delta^2$. Notice that when

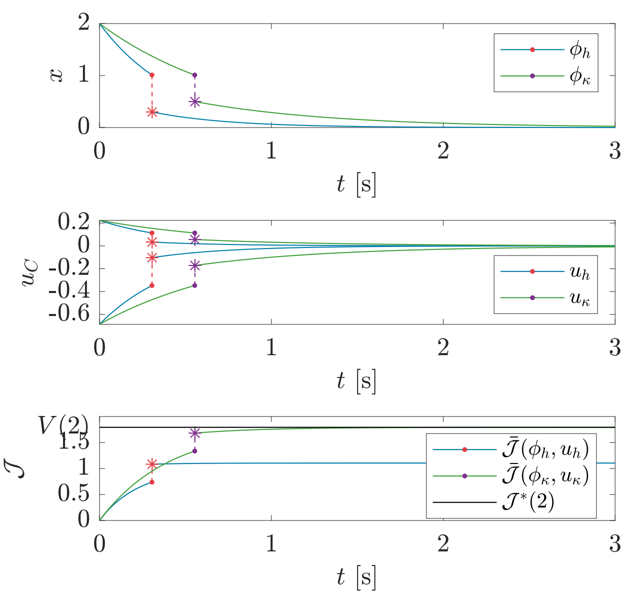

Figure 6.2: Nonunique solutions due to set-valued dynamics for $\underline{a} = -2, \overline{a} = -1, b_1 = b_2 = 1, \delta = \xi = 2, \mu = 1, \underline{\sigma} = 0.3, \overline{\sigma} = 0.5, Q_C = 1$, $R_{C1} = 1.304, R_{C2} = -4$, and $P = 0.4481$. Worst-case cost solution (green and pruple). Arbitrary solution (blue and red).

both players play the weak saddle-point equilibrium strategy ($\epsilon_u = 1$ and $\epsilon_w = 1$ in Figure 6.3), the worst-cost (in red) is not necessarily attained, which implies that such

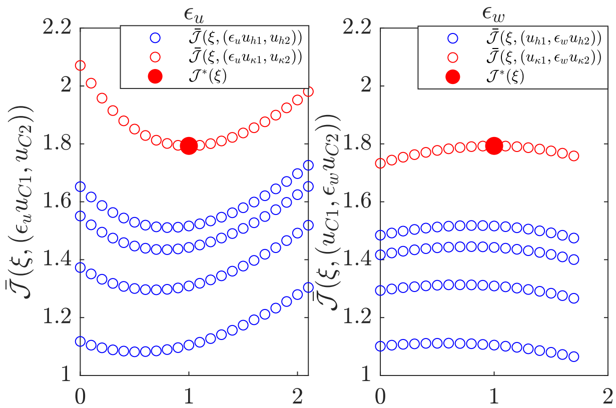

Figure 6.3: Nonunique costs due to set-valued dynamics when varying the feedback gains around the optimal value. The cost is evaluated on solutions to $\mathcal{H}_s$ from ξ with feedback law variations specified by ϵ_u and ϵ_w in $u = (\epsilon_u \kappa_{C1}, \epsilon_w \kappa_{C2})$. Worst-case costs (red). Costs of arbitrary solutions (blue). Value function (filled red).

a strategy is not necessarily optimal in the min-max sense. Indeed, notice that other strategies (e.g. $\epsilon_u = 1.5$) attain lower costs (in blue) when player P_2 sticks to the weak saddle-point equilibrium strategy. Nevertheless, by playing the weak saddle-point equilibrium, player P_1 minimizes the worst-case cost (red) under the maximizing adversarial action.

Let $\mathcal{A} = \{0\}$ and given that $L_C \in \mathcal{PD}_{\kappa_C}(\mathcal{A} \cap C_\kappa)$, $L_D \in \mathcal{PD}_{\kappa_D}(\mathcal{A} \cap D_\kappa)$ vacuously, and (6.28)-(6.33) hold, by setting $\alpha_1(|x|_\mathcal{A}) = x^\top(P - I)x$ and $\alpha_2(|x|_\mathcal{A}) = x^\top(P + I)x$, and for $|x|_\mathcal{A} \mapsto \eta(|x|_\mathcal{A}) = P\frac{x^2}{2}$, $L_D(x, \kappa_D(x)) \geq \eta(|x|_\mathcal{A})$ for all $x \in D_\kappa$ when $\mu^2 \geq 2\bar{\sigma}^2$, from Theorem 6.4.2, we have that κ_C is the weak saddle-point equilibrium and renders $\mathcal{A}$ uniformly globally asymptotically stable for (6.44). This corresponds to the Definition 3.1.3, with every maximal solution rendered by κ from $\xi = 2$ attaining the optimal cost. ◻

Acknowledgements

Sections 6.1 and 6.3-6.5 are a partial reprint of the material as it appears in “On the Optimal Cost and Asymptotic Stability in Two-Player Zero-Sum Set-Valued Hybrid Games.” [3]. The dissertation author was the first author of this paper.

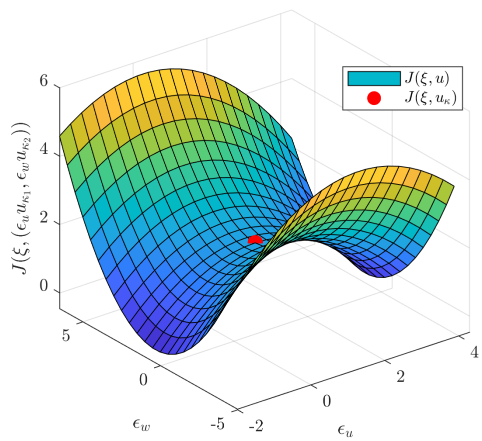

Figure 6.4: Saddle point behavior in the cost of continuous solutions from $\xi = 2$ when varying the feedback gains around the optimal value. The cost is evaluated on solutions $(\phi, u) \in \mathcal{S}_{\mathcal{H}}^{\infty}(\xi)$ with feedback law variations specified by ϵ_u and ϵ_u in $u = (\epsilon_u \kappa_1(\phi), \epsilon_w \kappa_2(\phi))$.

Part II

Saddle-Point Equilibrium via Inverse Optimality

Chapter 7

Input-to-State Stabilizing Control for Hybrid Systems

In the previous chapters, given a cost function, we presented results on sufficient conditions in terms of the stages costs to design the saddle-point equilibrium of a hybrid game. These results rely on the existence of a Lyapunov-like function V satisfying the optimality conditions (HJBI). Finding such function and designing the equilibrium strategies in most cases is not a trivial task. In the infinite horizon, it requires the solution of differential (difference) equations along flows (jumps) whose analytical complexity increases when the dynamics are nonlinear or the cost is nonquadratic and when the finite horizon is considred, turn into partial differential (difference) equations, most of which lack analytical solutions. An alternative approach in the literature to address optimal contro problems, is to start from a given feedback strategie that satisfies desired conditions ,e.g., stability, and the problem is to find the cost function that it is optimal with respect to. This is known as a invere optimal control problem.

In this Chapter, we address a two-player zero-sum hybrid game as an inverse optimal control problem with stabilizing controllers under the presence of a disturbance. We present results on sufficient conditions to guarantee input-to-state stability with respect to disturbances for hybrid systems. First, the control feedback laws are considered as solutions to QP problems, and to Sontag's formula. Under additional conditions, non-QP controllers are formulated and the cost functional that the feeback laws optimize is constructed via inverse optimality.

We define a hybrid dynamical affine system $\mathcal{H}$ with input $u = (u_C, u_D) = ((u_{C1}, u_{C2}), (u_{D1}, u_{D2})) \in \mathbb{R}^{m_C} \times \mathbb{R}^{m_D} = \mathbb{R}^m$, where $u_1 := (u_{C1}, u_{D1}) \in \mathbb{R}^{m_{C1}} \times \mathbb{R}^{m_{D1}}$ is a control input and $u_2 := (u_{C2}, u_{D2}) \in \mathbb{R}^{m_{C2}} \times \mathbb{R}^{m_{D2}}$ is a disturbance, as

$$\mathcal{H} \; : \; \begin{cases} \dot{x} = F(x, u_C) := f(x) + f_{u1}(x)u_{C1} + f_{u2}(x)u_{C2} & (x, u_C) \in C \\ x^+ = G(x, u_D) := g(x) + g_{u1}(x)u_{D1} + g_{u2}(x)u_{D2} & (x, u_D) \in D \end{cases} \tag{7.1}$$

where $x \in \mathbb{R}^n$ is the state.

Well-posed dynamical systems refer to a class of dynamical systems where the solutions enjoy very useful structural properties [20]. A hybrid system $\mathcal{H}$ as in (7.1) is well-posed if the basic conditions hold.

Assumption 7.0.1. (Hybrid Basic Conditions for Input Affine System) *For a hybrid system $\mathcal{H}$ as in* (7.1), *suppose that i) the sets C and D are closed subsets of $\mathbb{R}^n$, and ii) the flow map $F : \mathbb{R}^n \to \mathbb{R}^n$ and the jump map $G : \mathbb{R}^n \to \mathbb{R}^n$ are continuous.*

Consider the hybrid system resulting from assigning the control input u_1 of $\mathcal{H}$ as in (7.1) to the feedback law $\kappa_1 := (\kappa_{C1}, \kappa_{D1}) : \mathbb{R}^n \to \mathbb{R}^{m_{C1}} \times \mathbb{R}^{m_{D1}}$, and with disturbance input u_2, as

$$\mathcal{H}_{\kappa 1} \; : \; \begin{cases} \dot{x} &= F(x, (\kappa_{C1}(x), u_{C2})) \;\; (x, u_{C2}) \in C_{\kappa 1} \\ x^+ &= G(x, (\kappa_{D1}(x), u_{D2})) \;\; (x, u_{D2}) \in D_{\kappa 1} \end{cases} \tag{7.2}$$

where $C_{\kappa 1} := \{(x, u_{C2}) \in \mathbb{R}^n \times \mathbb{R}^{m_{C2}} : (x, (\kappa_{C1}(x), u_{C2})) \in C\}$ and $D_{\kappa 1} := \{(x, u_{D2}) \in \mathbb{R}^n \times \mathbb{R}^{m_{D2}} : (x, (\kappa_{D1}(x), u_{D2})) \in D\}$.

Definition 7.0.2. (Solution to $\mathcal{H}_{\kappa 1}$) *A pair (ϕ, u_2) defines a solution to $\mathcal{H}_{\kappa 1}$ as in (7.2) if $\phi \in \mathcal{X}$, $u_2 = (u_{C2}, u_{D2}) \in \mathcal{U}_2$, $\operatorname{dom}\phi = \operatorname{dom} u_2$, and*

- $(\phi(0,0), u_{C2}(0,0)) \in \overline{C_{\kappa 1}}$ *or*
  $(\phi(0,0), u_{D2}(0,0)) \in D_{\kappa 1}$,

- *For each $j \in \mathbb{N}$ such that I^j_ϕ has a nonempty interior $\operatorname{int} I^j_\phi$, we have, for all $t \in \operatorname{int} I^j_\phi$,*

  $$(\phi(t,j), u_{C2}(t,j)) \in C_{\kappa 1}$$

  and, for almost all $t \in I^j_\phi$,

  $$\frac{d}{dt}\phi(t,j) = F(\phi(t,j), (\kappa_{C1}(\phi(t,j)), u_{C2}(t,j)))$$

- *For all* $(t,j) \in \operatorname{dom}\phi$ *such that* $(t,j+1) \in \operatorname{dom}\phi$,

$$
\begin{aligned}
&(\phi(t,j), u_{D2}(t,j)) \in D_{\kappa 1} \\
&\qquad \phi(t,j+1) = G(\phi(t,j),(\kappa_{D1}(\phi(t,j)), u_{D2}(t,j)))
\end{aligned}
$$

A solution pair (ϕ, u_2) *is a compact solution if* ϕ *is a compact hybrid arc; see Definition 2.2.2.*

We denote by $\hat{\mathcal{S}}_{\mathcal{H}_{\kappa 1}}(M)$ the set of solution pairs (ϕ, u_2) to $\mathcal{H}_{\kappa 1}$ as in (7.2) such that $\phi(0,0) \in M$, and by $\mathcal{S}_{\mathcal{H}_{\kappa 1}}(M) \subset \hat{\mathcal{S}}_{\mathcal{H}_{\kappa 1}}(M)$ the set of all maximal solution from M. We refer to the input action resulting from the composition of a feedback law $\kappa_1 \in \mathcal{K}_1$ and a solution ϕ as a *feedback control action.*

7.1 Input-to-State Stability for Hybrid Systems with Disturbances

Given a feedback law $\kappa_1 \in \mathcal{K}_1$, we are interested in studying conditions to guarantee the stability of a closed set $\mathcal{A} \subset \mathbb{R}^n$ with respect to $\mathcal{H}_{\kappa_1}$ as in (7.2) under the presence of a disturbance $u_2 = (u_{C2}, u_{D2})$, in the following sense.

Definition 7.1.1. (Input-to-State pre-Stability with respect to disturbances) *Given a compact set* $\mathcal{A} \subset \mathbb{R}^n$ *nand a feedback law* $\kappa_1 \in \mathcal{K}_1$*, the system* $\mathcal{H}_{\kappa_1}$ *in* (7.2) *is input-to-state pre-stable (ISpS) with respect to the disturbance* u_2 *and the set* $\mathcal{A}$ *if there exist* $\beta \in \mathcal{KL}$ *and* $\sigma \in \mathcal{K}$ *such that, for each* $\xi \in \mathbb{R}^n$*, each* $(\phi, u_2) \in \mathcal{S}_{\mathcal{H}_{\kappa_1}}(\xi)$ *satisfies, for each* $(t,j) \in \operatorname{dom}\phi$,

$$|\phi(t,j)|_{\mathcal{A}} \leq \max\left\{\beta(|\xi|_{\mathcal{A}}, t+j), \sigma\left(\|u_2\|_{(t,j)}\right)\right\} \tag{7.3}$$

Definition 7.1.1 is essentially the definition of ISS in [56] for the case of disturbances where the distance to $\mathcal{A}$ is employed as the proper indicator for $\mathcal{A}$ oand maximal solutions are not required to be complete. This subsumes the standard notion of ISS for continuous-time and discrete-time systems.

Definition 7.1.2. (ISpS Lyapunov function candidate) *Given a compact set* $\mathcal{A} \subset \mathbb{R}^n n$*, a feedback law* $\kappa_1 \in \mathcal{K}_1$*, and a hybrid system* $\mathcal{H}_{\kappa 1}$ *as in* (7.2) *with disturbance* $u_2 = (u_{C2}, u_{D2})$*, the function* $V : \operatorname{dom} V \to \mathbb{R}$ *defines an ISpS-Lyapunov function (ISpS-LF) candidate for* $\mathcal{H}_{\kappa 1}$ *with respect to* $\mathcal{A}$ *if the following conditions hold:*

1. $\Pi(\overline{C_{\kappa 1}}) \cup \Pi(D_{\kappa 1}) \cup G(D) \subset \operatorname{dom} V$ *;*

2. V *is continuous and, on an open set containing* $\Pi(\overline{C_{\kappa 1}})$*, locally Lipschitz ;*

3. there exist $\alpha_1, \alpha_2 \in \mathcal{K}$ *such that* $\alpha_1(|x|_{\mathcal{A}}) \le V(x) \le \alpha_2(|x|_{\mathcal{A}})$ *for all* $x \in \Pi(C_{\kappa 1}) \cup \Pi(D_{\kappa 1})$.

The following theorem establishes the connection between the existence of a Lyapunov function and input-to-state stability.

Theorem 7.1.3. (ISpS under a Lyapunov function candidate) *Given a compact set* $\mathcal{A} \subset \mathbb{R}^n$*, a feedback law* $\kappa_1 \in \mathcal{K}_1$*, and a hybrid system* $\mathcal{H}_{\kappa 1} = (C_{\kappa 1}, F, D_{\kappa 1}, G)$ *as in* (7.2)*, satisfying Assumption 7.0.1, with disturbance* $u_2 = (u_{C2}, u_{D2})$*, suppose there exists an ISpS-Lyapunov function candidate* $V : \operatorname{dom} V \to \mathbb{R}_{\ge 0}$ *for* $\mathcal{H}_{\kappa 1}$ *with respect to* $\mathcal{A}$ *such that*

$$\begin{aligned} &|x|_{\mathcal{A}} \ge \rho(|u_{C2}|), (x, u_{C2}) \in C_{\kappa 1} \\ &\qquad\qquad \Rightarrow \langle \nabla V(x), F(x, (\kappa_{C1}(x), u_{C2})) \rangle \le -\alpha_C(|x|_{\mathcal{A}}) \end{aligned} \tag{7.4a}$$

$$\begin{aligned} &|x|_{\mathcal{A}} \ge \rho(|u_{D2}|), (x, u_{D2}) \in D_{\kappa 1} \\ &\qquad\qquad \Rightarrow V(G(x, (\kappa_{D1}(x), u_{D2}))) - V(x) \le -\alpha_D(|x|_{\mathcal{A}}) \end{aligned} \tag{7.4b}$$

where $\rho \in \mathcal{K}_\infty$ *and* $\alpha_C, \alpha_D \in \mathcal{K}$*. Then, the system* $\mathcal{H}_{\kappa 1}$ *is ISpS with respect to the disturbance* u_2 *and the set* $\mathcal{A}$ *for some* $\beta \in \mathcal{KL}$ *and* $\sigma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

Proof. The proof is developed following the arguments of [56, Proposition 2.7] and [53, Lemma 2.1]. Pick $\xi \in \mathbb{R}^n$, and a solution pair $(\phi, u_2) \in \mathcal{S}_{\mathcal{H}_{\kappa 1}}(\xi)$. Consider the set $S := \{\eta \in \Pi(C_{\kappa 1}) \cup \Pi(D_{\kappa 1}) : V(\eta) \le \alpha_2 \circ \rho(\|u_2\|_\#)\}$.

Notice that S is compact and contains $\mathcal{A}$ and that any $\xi \notin S$ satisfies $\alpha_2(|\xi|_{\mathcal{A}}) \ge V(\xi) > \alpha_2 \circ \rho(\|u_2\|_\#)$ and hence satisfies $|\xi|_{\mathcal{A}} > \rho(\|u_2\|_\#)$.

Claim 1 (Forward invariance of S): If there exists some $(t, j) \in \operatorname{dom} \phi$ such that $\phi(t, j) \in S$, then $\phi(t', j') \in S$ for all $t' + j' \ge t + j$.

Proof of Claim 1: Proceeding by contradiction, suppose that ϕ leaves the set S. The following cases are possible:

- The state trajectory ϕ leaves S after a jump, that is, $\phi(t, j) \in S$ and $\phi(t, j+1) \notin S$. This implies $V(\phi(t, j)) \le \alpha_2 \circ \rho(\|u_2\|_\#) < V(\phi(t, j+1))$. Then, from (8.8b) we have that if $|\phi(t, j)|_{\mathcal{A}} \ge \rho(\|u_2\|_\#)$, then $V(\phi(t, j+1)) \le V(\phi(t, j)) - \alpha_D(|\phi(t, j)|_{\mathcal{A}}) \le V(\phi(t, j))$, which implies that $\phi(t, j+1) \in S$.

- The state trajectory ϕ leaves the set S by flow: there exist $(\tau, k), (\tau', k) \in \operatorname{dom}\phi$ such that $V(\phi(t,j)) \leq \alpha_2 \circ \rho(\|u_2\|_{\#})$ for all $(t,j) \in \operatorname{dom}\phi$, $t+j \leq \tau + k$, and $\phi(t,k) \in \Pi(C_{\kappa 1})$ for all $(t,k) \in \operatorname{dom}\phi$, $\tau < t \leq \tau'$. Using continuous differentiability of V and absolute continuity of $t \mapsto \phi(t,k)$ on $[\tau, \tau']$, $t \mapsto V(\phi(t,k))$ is also absolutely continuous on $[\tau, \tau']$ and, via integration, satisfies

$$V(\phi(\tau', k)) - V(\phi(\tau, k)) = \int_{\tau}^{\tau'} \langle \nabla V(\phi(t,k)), \dot{\phi}(t,k) \rangle dt \tag{7.5}$$

  Since $V(\phi(t,k)) > \alpha_2 \circ \rho(\|u_2\|_{\#})$ for all $t \in (\tau, \tau']$ and $V(\phi(\tau,k)) = \alpha_2 \circ \rho(\|u_2\|_{\#})$, the expression in (8.9) is positive. On the other hand, since $\phi((\tau,\tau'],k) \subset \Pi(C_{\kappa 1})$, (8.8a) implies that $\langle \nabla V(\phi(t,k)), F_d(\phi(t,k), u_{C2}(t,k)) \rangle \leq -\alpha_C(|\phi(t,k)|_{\mathcal{A}})$ for almost all $t \in (\tau, \tau')$. Hence, via integration again, the expression in (8.9) is less than or equal to zero.

Since a contradiction is reached in both cases, the result is established.

□

Proof of Theorem 7.1.3 (Continued). Now let $(t^*, j^*) \in \operatorname{dom}(\phi, u_2)$ be such that $t^* + j^* = \inf\{t + j \geq 0 : \phi(t,j) \in S\} \leq \infty$. Then, it follows from the above argument that

$$V(\phi(t,j)) \leq \alpha_2 \circ \rho(\|u_2\|_{\#}) \tag{7.6}$$

for all $(t,j) \in \operatorname{dom}(\phi, u_2)$ such that $t + j \geq t^* + j^*$.

For all $(t,j) \in \operatorname{dom}(\phi, u_2)$ such that $t + j < t^* + j^*$, $\phi(t,j) \notin S$, which implies that $|\phi(t,j)|_{\mathcal{A}} > \rho(\|u_2\|_{\#})$. Consequently, for each $j \in \mathbb{N}$ such that I_ϕ^j has a nonempty interior $\operatorname{int} I_\phi^j$ and such that $t + j < t^* + j^*$, we have, for all $t \in \operatorname{int} I_\phi^j$

$$L_f V(\phi(t,j)) + L_{f_{u2}} V(\phi(t,j)) u_{C2}(t,j) \leq -\alpha_C(|\phi(t,j)|_{\mathcal{A}})$$

and for all $(t,j) \in \operatorname{dom}(\phi, u_2)$ such that $(t, j+1) \in \operatorname{dom}(\phi, u_2)$ and $t + j < t^* + j^*$, we have

$$V(G(\phi(t,j), u_{D2}(t,j))) - V(\phi(t,j)) \leq -\alpha_D(|\phi(t,j)|_{\mathcal{A}}).$$

Thanks to the hybrid comparison principle in [56, Lemma C.1], there exists $\beta \in \mathcal{KL}$ such that

$$V(\phi(t,j)) \leq \beta(V(\xi), t + j) \qquad \forall (t,j) \in \operatorname{dom}(\phi, u_2) : t + j \leq t^* + j^* \tag{7.7}$$

Combining (7.6) and (7.7), we conclude that

$$|\phi(t,j)|_{\mathcal{A}} \leq \max\left\{\beta(|\xi|_{\mathcal{A}}, t+j), \alpha_1^{-1} \circ \alpha_2 \circ \rho\left(\|u_2\|_{(t,j)}\right)\right\}$$

for all $(t,j) \in \operatorname{dom}(\phi, u_2)$. □

7.2 Problem Statement

Consider the system $\mathcal{H}$ in (7.1), with control input u_1 assigned to a feedback law $\kappa_1 = (\kappa_{1C}, \kappa_{1D}) \in \mathcal{K}_1$, and the disturbance input $u_2 \in \mathcal{U}_2$. Given a set $\mathcal{A}$, we say $\mathcal{H}$ is input-to-state controlled pre-stable when the closed-loop system resulting from assigning u_1 to κ_1 is input-to-state pre-stable with respect to disturbances and $\mathcal{A}$.

In this chapter, we address the problem of designing the feedback law κ_1 that not only renders $\mathcal{H}$ input-to-state controlled pre-stable with respect to $\mathcal{A}$ but also solves a zero-sum hybrid game. Specifically, we seek the existence of $\beta \in \mathcal{KL}$ and $\sigma \in \mathcal{K}$ such that every $(\phi, u) \in \mathcal{S}_{\mathcal{H}}(\Pi(C) \cup \Pi(D))$ with input u_1 defined as $\operatorname{dom}\phi \ni (t,j) \mapsto u_1(t,j) = \kappa_1(\phi(t,j))$ satisfies (7.3) for all $(t,j) \in \operatorname{dom}\phi$. This objective is attained by considering the closed-loop system $\mathcal{H}_{\kappa 1}$ resulting from assigning the input u_1 of $\mathcal{H}$ to κ_1 and solving the following problem.

Problem 7.2.1. (Inverse-optimal ISpS) *Given a compact set $\mathcal{A} \subset \mathbb{R}^n$, design a feedback law $\kappa_1 \in \mathcal{K}_1$ that renders the system $\mathcal{H}_{\kappa 1}$ input-to-state pre-stable with respect to the disturbance u_2 and the set $\mathcal{A}$. In addition, determine the cost functional that κ_1 minimizes under the worst-case disturbance u_2.*

Remark 7.2.2. (Relation to the literature) *A version of Problem 7.2.1 was solved in [82] for continuous-time systems without constraints, i.e., the case in which $\mathcal{H} = (\mathbb{R}^n, F, \emptyset, \star)$ and $\mathcal{A} = \{0\}$.*

7.3 Input-to-Sate Stabilizability

In this section, we address the first part of Problem 7.2.1 by using control Lyapunov functions as a synthesis tool to stabilize a hybrid system. First, we introduce definitions and preliminary results on control Lyapunov functions for hybrid systems with disturbances. We say that a system $\mathcal{H}$ as in (7.1) is input-to-state pre-stabilizable with respect to a set $\mathcal{A}$ if there exists a control feedback law κ_1 such that the resulting closed-loop system of assigning u_1 to κ_1 is ISpS with respect to the disturbance u_2 and the set $\mathcal{A}$.

7.3.1 Input-to-State Stability Control Lyapunov Functions

Definition 7.3.1. (ISpS-CLF with respect to disturbances) *Given a system $\mathcal{H} = (C, F, D, G)$ as in (7.1) and a compact set $\mathcal{A} \subset \mathbb{R}^n$, an ISpS-LF candidate V is an ISpS-control Lyapunov function (ISpS-CLF) for $\mathcal{H}$ with respect to disturbance u_2 and $\mathcal{A}$ if V is continuously differentiable on an open neighborhood of $\Pi(C)$ and there exist $\rho, \alpha_C, \alpha_D \in \mathcal{K}$ such that*

$$|x|_{\mathcal{A}} \geq \rho(|u_{C2}|), (x, u_{C2}) \in \Pi_{u_{C1}}(C) \Rightarrow \inf_{u_{C1} \in \Psi_C(x, u_{C2})} \langle \nabla V(x), F(x, u_C) \rangle \leq -\alpha_C(|x|_{\mathcal{A}}) \tag{7.8a}$$

$$|x|_{\mathcal{A}} \geq \rho(|u_{D2}|), (x, u_{D2}) \in \Pi_{u_{D1}}(D) \Rightarrow \inf_{u_{D1} \in \Psi_D(x, u_{D2})} V(G(x, u_D)) - V(x) \leq -\alpha_D(|x|_{\mathcal{A}}) \tag{7.8b}$$

where $\Pi_{u_{\star 1}}(\star) = \{(x, u_{\star 2}) : \exists u_{\star 1} \, \text{s.t.} \, (x, (u_{\star 1}, u_{\star 2})) \in \star\}$, and $\Psi_\star(x, u_{\star 2}) := \{u_{\star 1} \in \mathbb{R}^{m_{\star 1}} : (x, (u_{\star 1}, u_{\star 2})) \in \star\}$ for $\star \in \{C, D\}$.

Theorem 7.3.2. (Characterization of ISpS [56, Thm. 3.1]) *Consider a compact set $\mathcal{A} \subset \mathbb{R}^n$ and a system $\mathcal{H} = (C, F, D, G)$ as in (7.1). Suppose that F is convex-valued with respect to disturbances on C. The hybrid system $\mathcal{H}$ is input-to-state pre-stabilizable with respect to $\mathcal{A}$ if and only if there exists an ISpS-CLF for $\mathcal{H}$ with respect to the disturbance u_2 and $\mathcal{A}$.*

The following results are used to establish a connection between the existence of an ISpS-CLF and a feedback law that input-to-state pre-stabilizes the system.

Assumption 7.3.3. *Given a system $\mathcal{H} = (C, F, D, G)$ as in (7.1) and a function $V : \mathbb{R}^n \to \mathbb{R}$, suppose there exist functions $\widehat{V}_{L1} : \mathbb{R}^n \to \mathbb{R}^{m_{D1}}$ and $\widehat{V}_{L2} : \mathbb{R}^n \to \mathbb{R}^{m_{D2}}$ such that for all $(x, u_D) \in D$,*

$$\begin{aligned} V(G(x, u_D)) &= V(g(x) + g_{u1}(x) u_{D1} + g_{u2}(x) u_{D2}) \\ &= V(g(x)) + \widehat{V}_{L1}(x) u_{D1} + \widehat{V}_{L2}(x) u_{D2} \end{aligned} \tag{7.9}$$

Lemma 7.3.4. (Equivalent ISpS conditions) *Under Assumption 7.3.3, a pair* (V, ρ) *satisfies* (7.8a) *and* (7.8b) *if and only if*

$$L_{f_{u1}}V(x) = 0, x \in \Pi(C) \Rightarrow \omega_C(x) \leq 0 \tag{7.10a}$$

where

$$\omega_C(x) = L_f V(x) + \alpha_C(|x|_{\mathcal{A}}) + |L_{f_{u2}}V(x)|\rho^{-1}(|x|_{\mathcal{A}}) \tag{7.10b}$$

and

$$\widehat{V}_{L1}(x) = 0, x \in \Pi(D) \Rightarrow \omega_D(x) \leq 0 \tag{7.11a}$$

where

$$\omega_D(x) = V(g(x)) - V(x) + \alpha_D(|x|_{\mathcal{A}}) + |\widehat{V}_{L2}(x)|\rho^{-1}(|x|_{\mathcal{A}}) \tag{7.11b}$$

The proof follows similar arguments as in [82, Lemma 2.1].

Proof. ($\Leftarrow$) By Definition 7.3.1, if $L_{f_{u1}}V(x) = 0$, then

$$|x|_{\mathcal{A}} \geq \rho(|u_{C2}|), (x, u_{C2}) \in \Pi_{u_{C1}}(C) \Rightarrow L_f V(x) + L_{f_{u2}}V(x)u_{C2} \leq -\alpha_C(|x|_{\mathcal{A}}) \tag{7.12}$$

Now consider the particular input u_{C2} defined by the feedback law

$$\kappa_{C2}(x) = \begin{cases} \dfrac{L_{f_{u2}}V(x)}{|L_{f_{u2}}V(x)|}\rho^{-1}(|x|_{\mathcal{A}}) & \text{if} \quad L_{f_{u2}}V(x) \neq 0 \\ 0 & \text{if} \quad L_{f_{u2}}V(x) = 0 \end{cases} \tag{7.13}$$

for all $x \in \Pi(C)$. Note that if $u_{C2} = \kappa_{C2}(x)$, then $\rho(|u_{C2}|) \leq |x|_{\mathcal{A}}$. Therefore, substituting (7.13) in (7.12), we have that if $L_{f_{u1}}V(x) = 0$ and $x \in \Pi(C)$, when $u_{C2} = \kappa_{C2}(x)$,

$$L_f V(x) + |L_{f_{u2}}V(x)|\rho^{-1}(|x|_{\mathcal{A}}) \leq -\alpha_C(|x|_{\mathcal{A}}) \tag{7.14}$$

that is, (7.10) is satisfied.

In addition, by Definition 7.3.1, and thanks to Assumption 7.3.3, if $\widehat{V}_{L1}(x) = 0$, then

$$|x|_{\mathcal{A}} \geq \rho(|u_{D2}|), (x, u_{D2}) \in \Pi_{u_{D1}}(D) \Rightarrow V(g(x)) + \widehat{V}_{L2}(x)u_{D2} - V(x) \leq -\alpha_D(|x|_{\mathcal{A}}) \tag{7.15}$$

Now consider the particular input u_{D2} defined by the feedback law

$$\kappa_{D2}(x) = \begin{cases} \dfrac{\widehat{V}_{L2}(x)}{|\widehat{V}_{L2}(x)|}\rho^{-1}(|x|_{\mathcal{A}}) & \text{if} \quad \widehat{V}_{L2}(x) \neq 0 \\ 0 & \text{if} \quad \widehat{V}_{L2}(x) = 0 \end{cases} \tag{7.16}$$

for all $x \in \Pi(D)$. Note that if $u_{D2} = \kappa_{D2}(x)$, then $\rho(|u_{D2}|) \leq |x|_{\mathcal{A}}$. Substituting (7.16) in (7.15), we have that if $\widehat{V}_{L1}(x) = 0$ and $x \in \Pi(D)$, when $u_{D2} = \kappa_{D2}(x)$,

$$V(g(x)) + |\widehat{V}_{L2}(x)|\rho^{-1}(|x|_{\mathcal{A}}) - V(x) \leq -\alpha_D(|x|_{\mathcal{A}}) \tag{7.17}$$

that is, (7.11) is satisfied.

($\Rightarrow$) If $|x|_{\mathcal{A}} \geq \rho(|u_{C2}|), (x, u_{C2}) \in \Pi_{u_{C1}}(C)$, using (7.10), we have

$$\begin{aligned} &\inf_{u_{C1} \in \Psi_C(x, u_{C2})} \{L_f V(x) + L_{f_{u1}} V(x) u_{C1} + L_{f_{u2}} V(x) u_{C2}\} \\ &\qquad \leq \inf_{u_{C1} \in \Psi_C(x, u_{C2})} \{L_f V(x) + L_{f_{u1}} V(x) u_{C1} + |L_{f_{u2}} V(x)||u_{C2}|\} \\ &\qquad \leq \inf_{u_{C1} \in \Psi_C(x, u_{C2})} \{L_f V(x) + L_{f_{u1}} V(x) u_{C1} + |L_{f_{u2}} V(x)|\rho^{-1}(|x|_{\mathcal{A}})\} \\ &\qquad \leq -\alpha_C(|x|_{\mathcal{A}}) \end{aligned}$$

In addition, if $|x|_{\mathcal{A}} \geq \rho(|u_{D2}|), (x, u_{D2}) \in \Pi_{u_{D1}}(D)$, using (7.11), we have

$$\begin{aligned} &\inf_{u_{D1} \in \Psi_D(x, u_{D2})} \{V(g(x)) + \widehat{V}_{L1}(x) u_{D1} + \widehat{V}_{L2}(x) u_{D2}\} \\ &\qquad \leq \inf_{u_{D1} \in \Psi_D(x, u_{D2})} \{V(g(x)) + \widehat{V}_{L1}(x) u_{D1} + |\widehat{V}_{L2}(x)||u_{D2}|\} \\ &\qquad \leq \inf_{u_{D1} \in \Psi_D(x, u_{D2})} \{V(g(x)) + \widehat{V}_{L1}(x) u_{D1} + |\widehat{V}_{L2}(x)|\rho^{-1}(|x|_{\mathcal{A}})\} \\ &\qquad \leq -\alpha_D(|x|_{\mathcal{A}}) \end{aligned}$$

□

Theorem 7.3.5. (ISpS CLF Sontag-like formula) *Under Assumption 7.3.3, if there exists a ISpS-CLF with respect to disturbance u_2 and $\mathcal{A}$, the system $\mathcal{H} = (C, F, D, G)$ in (7.1) is input-to-state pre-stabilizable using the following Sontag-type control law, in which we assign the input u_{C1} to*

$$\tilde{\kappa}_{SC1}(x) := \begin{cases} L_{f_{u1}} V(x) \kappa_{SC1}(x) & \text{if} \quad L_{f_{u1}} V(x) \neq 0 \\ 0, & \text{if} \quad L_{f_{u1}} V(x) = 0 \end{cases} \tag{7.18a}$$

where

$$\kappa_{SC1}(x) := \frac{-\omega_C(x) - \sqrt{\omega_C^2(x) + |L_{f_{u1}}V(x)|^4}}{|L_{f_{u1}}V(x)|^2} \tag{7.18b}$$

and we assign the input u_{D1} *to*

$$\tilde{\kappa}_{SD1}(x) := \begin{cases} \widehat{V}_{L_1}(x)\kappa_{SD1}(x) & \text{if} \quad \widehat{V}_{L_1}(x) \neq 0 \\ 0 & \text{if} \quad \widehat{V}_{L_1}(x) = 0 \end{cases} \tag{7.19a}$$

where

$$\kappa_{SD1}(x) := \frac{-\omega_D(x) - \sqrt{\omega_D^2(x) + |\widehat{V}_{L1}(x)|^4}}{|\widehat{V}_{L1}(x)|^2} \tag{7.19b}$$

with $\omega_C(x)$ *and* $\omega_D(x)$ *defined in* (7.10b) *and* (7.11b), *respectively.*

Proof. We substitute (7.18) and (7.19) into $\mathcal{H}$ to obtain the closed-loop system $\mathcal{H}_{\kappa 1} = (C_{\kappa 1}, F, D_{\kappa 1}, G)$ as in (7.2). Then, for each $(x, u_{C2}) \in C_{\kappa 1}$, we have

- if $L_{f_{u1}}V(x) = 0$,

$$\begin{aligned}
&\langle \nabla V(x), F(x, (\tilde{\kappa}_{SC1}(x), u_{C2})) \rangle \\
&\qquad = L_f V(x) + L_{f_{u2}}V(x)u_{C2} \\
&\qquad = \omega_C(x) - \alpha_C(|x|_{\mathcal{A}}) - |L_{f_{u2}}V(x)|\rho^{-1}(|x|_{\mathcal{A}}) + L_{f_{u2}}V(x)u_{C2} \\
&\qquad \leq -\alpha_C(|x|_{\mathcal{A}}) + |L_{f_{u2}}V(x)|(|u_{C2}| - \rho^{-1}(|x|_{\mathcal{A}}))
\end{aligned}$$

  and if $|x|_{\mathcal{A}} \geq \rho(|u_{C2}|)$, we have

$$\langle \nabla V(x), F(x, (\tilde{\kappa}_{SC1}(x), u_{C2})) \rangle \leq -\alpha_C(|x|_{\mathcal{A}})$$

- if $L_{f_{u1}}V(x) \neq 0$,

$$\begin{aligned}
&\langle \nabla V(x), F(x, (\tilde{\kappa}_{SC1}(x), u_{C2})) \rangle \\
&\qquad = L_f V(x) - \omega_C(x) - \sqrt{\omega_C^2(x) + |L_{f_{u1}}V(x)|^4} + L_{f_{u2}}V(x)u_{C2} \\
&\qquad \leq -\alpha_C(|x|_{\mathcal{A}}) - |L_{f_{u2}}V(x)|\rho^{-1}(|x|_{\mathcal{A}}) + L_{f_{u2}}V(x)u_{C2} \\
&\qquad \leq -\alpha_C(|x|_{\mathcal{A}}) + |L_{f_{u2}}V(x)|(|u_{C2}| - \rho^{-1}(|x|_{\mathcal{A}}))
\end{aligned}$$

  and if $|x|_{\mathcal{A}} \geq \rho(|u_{C2}|)$, we have

$$\langle \nabla V(x), F(x, (\tilde{\kappa}_{SC1}(x), u_{C2})) \rangle \leq -\alpha_C(|x|_{\mathcal{A}}).$$

For each $(x, u_{D2}) \in D_{\kappa 1}$, we obtain

- if $\widehat{V}_{L1}(x) = 0$,

$$\begin{aligned} V(G(x,(\tilde{\kappa}_{SD1}(x),u_{D2}))) - V(x) \\ &= V\big(g(x) + g_{u2}(x)u_{D2}\big) - V(x) \\ &= V(g(x)) + \widehat{V}_{L2}(x)u_{D2} - V(x) \\ &= \omega_D(x) - \alpha_D(|x|_{\mathcal{A}}) - |\widehat{V}_{L2}(x)|\rho^{-1}(|x|_{\mathcal{A}}) + \widehat{V}_{L2}(x)u_{D2} \\ &\leq -\alpha_D(|x|_{\mathcal{A}}) + |\widehat{V}_{L2}(x)|(|u_{D2}| - \rho^{-1}(|x|_{\mathcal{A}})) \end{aligned}$$

  and if $|x|_{\mathcal{A}} \geq \rho(|u_{D2}|)$, we have

$$V(G(x,(\tilde{\kappa}_{SD1}(x),u_{D2}))) - V(x) \leq -\alpha_D(|x|_{\mathcal{A}})$$

- if $\widehat{V}_{L1}(x) \neq 0$,

$$\begin{aligned} V(G(x,(\tilde{\kappa}_{SD1}(x),u_{D2}))) - V(x) \\ &= V\big(g(x) + g_{u1}(x)\tilde{\kappa}_{SD} + g_{u2}(x)u_{D2}\big) - V(x) \\ &\leq V(g(x)) - \omega_D(x) - \sqrt{\omega_D^2(x) + |\widehat{V}_{L1}(x)|^4} + \widehat{V}_{L2}(x)u_{D2} - V(x) \\ &\leq -\alpha_D(|x|_{\mathcal{A}}) - |\widehat{V}_{L2}(x)|\rho^{-1}(|x|_{\mathcal{A}}) + \widehat{V}_{L2}(x)u_{D2} \\ &\leq -\alpha_D(|x|_{\mathcal{A}}) + |\widehat{V}_{L2}(x)|(|u_{D2}| - \rho^{-1}(|x|_{\mathcal{A}})) \end{aligned}$$

  and if $|x|_{\mathcal{A}} \geq \rho(|u_{D2}|)$, we have

$$V(G(x,(\tilde{\kappa}_{SD1}(x),u_{D2}))) - V(x) \leq -\alpha_D(|x|_{\mathcal{A}}).$$

Finally, we invoke Theorem 7.1.3 to establish input-to-state pre-stability of $\mathcal{H}_{\kappa 1}$ with respect to the disturbance u_2 and the set $\mathcal{A}$. □

7.3.2 Input-to-State Stabilizing QP Control

We endow a system $\mathcal{H}$ with an input-to-state-stability property by solving a quadratic program (QP) in terms of an ISpS control Lyapunov function V.

The definitions of ISS and of a Lyapunov function candidate for the closed-loop system $\mathcal{H}_{\kappa 1}$ as in (7.2) follow from Definitions 7.1.1 and 7.1.2, respectively. The result to guarantee that $\mathcal{H}_{\kappa 1}$ is ISpS with respect to the disturbance u_2 and $\mathcal{A}$ given a Lyapunov function follows from Theorem 7.1.3.

Given a function α_C, we define ω_C as in (7.10b) for all $x \in \Pi(C)$, and the following QP:

$$\begin{aligned} \kappa_{C1_{QP}}(x) := \underset{v \in \mathbb{R}^{m_{C1}}}{\arg\min} \quad & |v|^2 \\ \text{subject to} \quad & L_{f_{u1}} V(x) v \leq -\omega_C(x) \end{aligned} \tag{7.20}$$

Since the cost function and constraint defining (7.20) are both convex and continuously differentiable with respect to the decision variable v, (7.20) is a convex optimization problem, and the Karush-Kuhn-Tucker (KKT) conditions [83, Sec. 5.5.3] provide necessary and sufficient[1] conditions for optimality. In particular, for an optimal solution $\Pi(C) \ni x \mapsto \kappa_{C1_{QP}}(x)$ to (7.20), there exists $\theta^* : \Pi(C) \to \mathbb{R}_{\geq 0}$ such that

$$2\kappa_{C1_{QP}}(x) + \theta^*(x) L_{f_{u1}} V(x) = 0, \tag{7.21}$$

$$\theta^*(x) \left(L_{f_{u1}} V(x) \kappa_{C1_{QP}}(x) + \omega_C(x) \right) = 0, \tag{7.22}$$

$$L_{f_{u1}} V(x) \kappa_{C1_{QP}}(x) \leq -\omega_C(x), \tag{7.23}$$

We consider the following two cases:

- If, for $x \in \Pi(C)$, we have that the constraint is not active, namely

  $$L_{f_{u1}} V(x) \kappa_{C1_{QP}}(x) < -\omega_C(x)$$

  then, from (7.22) it follows that $\theta^*(x) = 0$; thus, from (7.21) we have that $\kappa_{C1_{QP}}(x) = 0$.

- If, for $x \in \Pi(C)$, we have that the constraint is binding, that is,

  $$L_{f_{u1}} V(x) \kappa_{C1_{QP}}(x) = -\omega_C(x)$$

  then, from (7.21)-(7.22) we have that

  $$\begin{bmatrix} 2I & L_{f_{u1}} V(x)^\top \\ L_{f_{u1}} V(x) & 0 \end{bmatrix} \begin{bmatrix} \kappa_{C1_{QP}}(x) \\ \theta^*(x) \end{bmatrix} = \begin{bmatrix} 0 \\ -\omega_C(x) \end{bmatrix}$$

[1] An additional *constraint qualification* is necessary for the KKT conditions to be necessary and sufficient conditions for optimality. One such condition is *Slater's Condition* [83, Sec. 5.2.3], which for (7.20) always holds as it is feasible for all $x \in \Pi(C)$, because it is a convex program with a single affine constraint.

and using block matrix inversion, it follows

$$\kappa_{C1_{QP}}(x) = -\frac{\omega_C(x)}{|L_{f_{u1}}V(x)|^2} L_{f_{u1}}V(x)$$
$$\theta^*(x) = \frac{\omega_C(x)}{2|L_{f_{u1}}V(x)|^2}$$

Thus, by combining the two cases[2] , the closed-form solution to (7.20) is given by

$$\kappa_{C1_{QP}}(x) := \begin{cases} -\dfrac{\max\{0, \omega_C(x)\}}{|L_{f_{u1}}V(x)|^2} L_{f_{u1}}V(x) & \text{if } L_{f_{u1}}V(x) \neq 0 \\ 0 & \text{if } L_{f_{u1}}V(x) = 0. \end{cases} \tag{7.24}$$

Similarly, given a function $\alpha_D \in \mathcal{K}$, under Assumption 7.3.3, we define ω_D as in (7.11b) for all $x \in \Pi(D)$, and introduce the following QP:

$$\begin{aligned} \kappa_{D1_{QP}}(x) = \underset{v \in \mathbb{R}^{m_{D1}}}{\arg\min} \quad & |v|^2 \\ \text{subject to} \quad & \widehat{V}_{L1}(x)v \leq -\omega_D(x) \end{aligned} \tag{7.25}$$

where the KKT conditions allow to express the solution explicitly as

$$\kappa_{D1_{QP}}(x) := \begin{cases} -\dfrac{\max\{0, \omega_D(x)\}}{|\widehat{V}_{L1}(x)|^2} \widehat{V}_{L1}(x) & \text{if } \widehat{V}_{L1}(x) \neq 0 \\ 0 & \text{if } \widehat{V}_{L1}(x) = 0. \end{cases} \tag{7.26}$$

With the QP ISpS feedback control law with components as in (7.20) and (7.25) we establish the following result.

Theorem 7.3.6. (ISpS control via QP CLF) *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ as in (7.1), a compact set $\mathcal{A} \subset \mathbb{R}^n$, suppose there exist $\rho \in \mathcal{K}$ and an ISpS-CLF V for $\mathcal{H}$ with respect to the disturbance u_2 and $\mathcal{A}$ such that Assumption 7.3.3 holds. The feedback law $\kappa_1 = (\kappa_{C1_{QP}}, \kappa_{D1_{QP}})$, with $\kappa_{C1_{QP}}$ as in (7.24) and $\kappa_{D1_{QP}}$ as in (7.26), renders the resulting closed-loop system $\mathcal{H}_{\kappa 1} = (C_{\kappa 1}, F, D_{\kappa 1}, G)$ as in (7.2) ISpS with respect to the disturbance u_2 and $\mathcal{A}$.*

[2]Notice that when the constraint in (7.20) is not active for each $x \in \Pi(C)$, namely $L_{f_{u1}}V(x)\kappa_{C1_{QP}}(x) < -\omega_C(x)$, we have that $\kappa_{C1_{QP}}(x) = 0$. Thus, from (7.23) it follows that $\omega_C(x) \leq 0$. When the constraint is active, it must follow that $\omega_C(x) > 0$.

Proof. For each $(x, u_{C2}) \in C_{\kappa 1}$, we have

$$\begin{aligned}
&\langle \nabla V(x), F(x, (\kappa_{C1_{QP}}(x), u_{C2}))\rangle \\
&= L_f V(x) + L_{f_{u2}} V(x) u_{C2} - \max\{0, \omega_C(x)\} \\
&= \omega_C(x) - \alpha_C(|x|_{\mathcal{A}}) - |L_{f_{u2}} V(x)|\rho^{-1}(|x|_{\mathcal{A}}) + L_{f_{u2}} V(x) u_{C2} - \max\{0, \omega_C(x)\} \\
&\leq \min\{\omega_C(x), 0\} - \alpha_C(|x|_{\mathcal{A}}) + |L_{f_{u2}} V(x)|\big(|u_{C2}| - \rho^{-1}(|x|_{\mathcal{A}})\big) \\
&\leq -\alpha_C(|x|_{\mathcal{A}}) + |L_{f_{u2}} V(x)|\big(|u_{C2}| - \rho^{-1}(|x|_{\mathcal{A}})\big)
\end{aligned}$$

and if $|x|_{\mathcal{A}} \geq \rho(|u_{C2}|)$, it follows that

$$\langle \nabla V(x), F(x, (\kappa_{C1_{QP}}(x), u_{C2}))\rangle \leq -\alpha_C(|x|_{\mathcal{A}}). \tag{7.27}$$

Similarly, for each $(x, u_{D2}) \in D_{\kappa 1}$, we obtain

$$\begin{aligned}
&V\big(G(x, (\kappa_{D1_{QP}}(x), u_{D2})\big) - V(x) \\
&= V\big(g(x)\big) - \max\{0, \omega_D(x)\} + \widehat{V}_{L2}(x) u_{D2} - V(x) \\
&= \omega_D(x) - \alpha_D(|x|_{\mathcal{A}}) - |\widehat{V}_{L2}(x)|\rho^{-1}(|x|_{\mathcal{A}}) - \max\{0, \omega_D(x)\} + \widehat{V}_{L2}(x) u_{D2} \\
&\leq \min\{\omega_D(x), 0\} - \alpha_D(|x|_{\mathcal{A}}) + |\widehat{V}_{L2}(x)|\big(|u_{D2}| - \rho^{-1}(|x|_{\mathcal{A}})\big) \\
&\leq -\alpha_D(|x|_{\mathcal{A}}) + |\widehat{V}_{L2}(x)|\big(|u_{D2}| - \rho^{-1}(|x|_{\mathcal{A}})\big)
\end{aligned}$$

and if $|x|_{\mathcal{A}} \geq \rho(|u_{D2}|)$, then

$$V\big(G(x, (\kappa_{D1_{QP}}(x), u_{D2}))\big) - V(x) \leq -\alpha_D(|x|_{\mathcal{A}}) \tag{7.28}$$

Finally[3], we invoke Theorem 7.1.3 to establish input-to-state pre-stability of $\mathcal{H}_{\kappa 1}$ with respect to the disturbance u_2 and the set $\mathcal{A}$. □

Remark 7.3.7. (Noncompleteness of solutions under QP control) *Notice that the optimization in* (7.20) *and* (7.25) *is carried over* $\mathbb{R}^{m_{C1}}$ *and* $\mathbb{R}^{m_{D1}}$*, respectively, instead of over the constrain sets* $\Psi_\star$, $\star \in \{C, D\}$*, as in Definition 7.3.1. This allows to compute the closed-form feedback law* $\kappa_1 = (\kappa_{C1_{QP}}, \kappa_{D1_{QP}})$ *which may potentially lead to maximal solution to* $\mathcal{H}_{\kappa 1}$ *that are not complete. The "pre" term in the results accounts for this trade-off.*

[3]The cases where $L_{f_{u1}} V(x) = 0$ for each $x \in \Pi(C)$ and $\widehat{V}_{L1}(x) = 0$ for each $x \in \Pi(D)$ follow the approach in the proof of Theorem 7.3.5.

The following example illustrates the control synthesis to input-to-state pre-stabilize a compact set for a one-dimensional linear hybrid system using QPs, as in (7.20) and (7.25).

Example 7.3.8. (()Linear hybrid system) Consider the set $\mathcal{A} = \{0\}$ and a system with state $x \in \mathbb{R}$, inputs $u_C = (u_{C1}, u_{C2}) \in \mathbb{R}^2$ and $u_D = (u_{D1}, u_{D2}) \in \mathbb{R}^2$, and dynamics $\mathcal{H}$ as in (7.1) described by

$$\begin{aligned} \dot{x} &= ax + Bu_C \qquad x \in [\mu, \delta] \\ x^+ &= \sigma + Pu_D \qquad x = \mu \end{aligned} \tag{7.29}$$

where $a < 0$, $B = \begin{bmatrix} b_1 & b_2 \end{bmatrix}$, $P = \begin{bmatrix} p_1 & p_2 \end{bmatrix}$, $(b_1, p_1) \neq (0, 0)$, and $\delta > \mu > \sigma > 0$. Let $V(x) = \eta x$, for a given $\eta > 0$. To show that V is a ISpS-LF candidate as in Definition 7.1.2, notice that: i) $[\mu, \delta] \cup \{\sigma + Pu_D\} \subset \mathbb{R}$ for all $u_D \in \mathbb{R}^2$, ii) V is continuous and locally Lipschitz (with constant η) on $(\mu - 1, \delta + 1)$, and iii) $\frac{\eta}{2}x \leq V(x) \leq 2\eta x$. Then, V is an ISpS Lyapunov function candidate for $\mathcal{H}$ with respect to $\mathcal{A}$.

Additionally, consider $\rho(r) = \alpha_\star(r) = r$ for $\star \in \{C, D\}$. From (7.10b) and (7.11b), we have that

$$\begin{aligned} \omega_C(x) &= x\,(1 + \eta(a + |b_2|)) \\ \omega_D(x) &= \eta(\sigma - x) + x(1 + \eta|p_2|) \end{aligned}$$

Then, the pointwise minimum norm state-feedback law is given by

$$\kappa_1(x) = (\kappa_{C1_{QP}}(x), \kappa_{D1_{QP}}(x)) = -\left(\frac{\max\{0, \omega_C(x)\}}{\eta b_1}, \frac{\max\{0, \omega_D(x)}{\eta p_1}\right)$$

As a result, the closed-loop system $\mathcal{H}_{\kappa 1}$, resulting from assigning the control input u_1 to κ_1 is described by

$$\begin{aligned} \dot{x} &= ax - \frac{\max\{0, \omega_C(x)\}}{\eta} + b_2 u_{C2} \qquad x \in [\mu, \delta] \\ x^+ &= \sigma - \frac{\max\{0, \omega_D(x)\}}{\eta} + p_2 u_{D2} \qquad x = \mu \end{aligned} \tag{7.30}$$

Notice that, for all $(x, u_{C2}) \in [\mu, \delta] \times \mathbb{R}$, we get

$$\begin{aligned} \left\langle \nabla V(x), ax - \frac{\max\{0, \omega_C(x)\}}{\eta} + b_2 u_{C2} \right\rangle &= \eta a x - \max\{0, x\,(1 + \eta(a + |b_2|))\} + \eta b_2 u_{C2} \\ &\leq \min\{0, x\,(1 + \eta(a + |b_2|))\} - x + \eta|b_2|(|u_{C2}| - x) \\ &\leq -x \qquad \text{if } x \geq |u_{C2}| \end{aligned}$$

Similarly, for all $(x, u_{D2}) \in \{\mu\} \times \mathbb{R}$, it follows that

$$
\begin{aligned}
V\left(\sigma - \frac{\max\{0, \omega_D(x)\}}{\eta} + p_2 u_{D2}\right) - V(x) \\
&= -\max\{0, \eta(\sigma - x) + x(1 + \eta|p_2|)\} + \eta(\sigma - x) + \eta p_2 u_{D_2} \\
&\leq \min\{0, \eta(\sigma - x) + x(1 + \eta|p_2|)\} - x + \eta|p_2|(|u_{D2}| - x) \\
&\leq -x \qquad \text{if } x \geq |u_{D2}|
\end{aligned}
$$

Therefore, invoking Theorem 7.1.3 we conclude that the closed-loop system $\mathcal{H}_{\kappa 1}$ is ISpS with respect to the disturbance u_2 and the set $\mathcal{A} = \{0\}$. Notice that the "pre" term accounts for solutions that jump from μ outside of $[\mu, \delta]$.

◻

Theorem 7.3.9. (Half Sontag) *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ as in (7.1), a closed set $\mathcal{A} \subset \mathbb{R}^n$, suppose there exist $\rho \in \mathcal{K}$ and an ISpS-CLF V for $\mathcal{H}$ with respect to disturbances and $\mathcal{A}$ such that Assumption 7.3.3 holds. The feedback law $\tilde{\kappa} = \frac{1}{2}(\tilde{\kappa}_{SC1}, \tilde{\kappa}_{SD1})$, with $\tilde{\kappa}_{SC1}$ as in (7.18) and $\tilde{\kappa}_{SD1}$ as in (7.19), renders the resulting closed-loop system $\mathcal{H}_{\kappa 1}$ as in (7.2) ISpS with respect to disturbances and $\mathcal{A}$. In addition, for all $x \in \Pi(C)$, the feedback law $\frac{1}{2}\tilde{\kappa}_{SC1}$ is the pointwise minimizer of*

$$
\begin{aligned}
&\operatorname*{arg\,min}_{v \in \mathbb{R}^{m_{C1}}} \quad |v|^2 \\
&\textit{subject to} \quad L_{f_{u1}} V(x) v \leq \frac{1}{2} |L_{f_{u1}} V(x)|^2 \kappa_{SC1}(x)
\end{aligned}
\tag{7.31}
$$

Similarly, for all $x \in \Pi(D)$, the feedback law $\frac{1}{2}\tilde{\kappa}_{SD1}$ is the pointwise minimizer of

$$
\begin{aligned}
&\operatorname*{arg\,min}_{v \in \mathbb{R}^{m_{D1}}} \quad |v|^2 \\
&\textit{subject to} \quad \widehat{V}_{L1}(x) v \leq \frac{1}{2} |\widehat{V}_{L1}(x)|^2 \kappa_{SD1}(x)
\end{aligned}
\tag{7.32}
$$

Proof. To show that $\Pi(C) \ni x \mapsto \frac{1}{2}\tilde{\kappa}_{SC1}(x)$ as in (7.18) is the pointwise minimizer of (7.31) Karush-Kuhn-Tucker (KKT) conditions [83, Sec. 5.5.3] provide necessary and sufficient conditions for optimality. Namely, it is sufficient to show that there exists some $\theta_C^* : \Pi(C) \to \mathbb{R}_{\geq 0}$ such that

$$\tilde{\kappa}_{SC1}(x) + \theta_C^*(x) L_{f_{u1}} V(x) = 0, \tag{7.33}$$

$$\theta_C^*(x) L_{f_{u1}} V(x) \tilde{\kappa}_{SC1}(x) = \theta_C^*(x) |L_{f_{u1}} V(x)|^2 \kappa_{SC1}(x), \tag{7.34}$$

$$L_{f_{u1}} V(x) \tilde{\kappa}_{SC1}(x) \leq |L_{f_{u1}} V(x)|^2 \kappa_{SC1}(x). \tag{7.35}$$

Using (7.37), it readily follows that (7.34) and (7.35) hold, and it can easily be shown that (7.33) are satisfied under

$$\theta_C^*(x) := \begin{cases} -\kappa_{SC1}(x) & \text{if} \quad L_{f_{u1}}V(x) \neq 0 \\ p_C, & \text{if} \quad L_{f_{u1}}V(x) = 0 \end{cases} \tag{7.36}$$

where $p_C > 0$ is any arbitrary value. A similar approach can be used to show that $\frac{1}{2}\tilde{\kappa}_{SD1}$ as in (7.19) is the pointwise minimizer of (7.32) with

$$\theta_D^*(x) := \begin{cases} -\kappa_{SD1}(x) & \text{if} \quad \widehat{V}_{L1}(x) \neq 0 \\ p_D, & \text{if} \quad V_{L1}(x) = 0 \end{cases} \tag{7.37}$$

where $p_D > 0$ is any arbitrary value. The resulting closed-loop system $\mathcal{H}_{\kappa 1}$ with feedback law $\kappa_1 = \frac{1}{2}(\tilde{\kappa}_{SC1}, \tilde{\kappa}_{SD1})$ satisfies, for all $(x, u_{C2}) \in C_{\kappa 1}$

$$\begin{aligned}
&\left\langle \nabla V(x), F\left(x, \left(\frac{1}{2}\tilde{\kappa}_{SC1}(x), u_{C2}\right)\right) \right\rangle \\
&= L_fV(x) - \frac{1}{2}\omega_C(x) - \frac{1}{2}\sqrt{\omega_C^2(x) + |L_{f_{u1}}V(x)|^4} + L_{f_{u2}}V(x)u_{C2} \\
&\leq \frac{1}{2}(L_fV(x) - \alpha_C(|x|_{\mathcal{A}}) - |L_{f_{u2}}V(x)|\rho^{-1}(|x|_{\mathcal{A}})) - \frac{1}{2}\sqrt{\omega_C^2(x) + |L_{f_{u1}}V(x)|^4} \\
&\qquad + L_{f_{u2}}V(x)u_{C2} \\
&= \frac{1}{2}(L_fV(x) + \alpha_C(|x|_{\mathcal{A}}) + |L_{f_{u2}}V(x)|\rho^{-1}(|x|_{\mathcal{A}})) + L_{f_{u2}}V(x)u_{C2} - \alpha_C(|x|_{\mathcal{A}}) \\
&\quad - |L_{f_{u2}}V(x)|\rho^{-1}(|x|_{\mathcal{A}}) - \frac{1}{2}\sqrt{\omega_C^2(x) + |L_{f_{u1}}V(x)|^4} \\
&\leq \frac{1}{2}\omega_C(x) - \frac{1}{2}\sqrt{\omega_C^2(x) + |L_{f_{u1}}V(x)|^4} - \alpha_C(|x|_{\mathcal{A}}) \\
&\qquad - |L_{f_{u2}}V(x)|\rho^{-1}(|x|_{\mathcal{A}}) + |L_{f_{u2}}V(x)||u_{C2}| \\
&\leq -\frac{1}{2}\Big(- \omega_C(x) + \sqrt{\omega_C^2(x) + |L_{f_{u1}}V(x)|^4}\Big) - \alpha_C(|x|_{\mathcal{A}}) + |L_{f_{u2}}V(x)|(|u_{C2}| - \rho^{-1}(|x|_{\mathcal{A}}))
\end{aligned}$$

Given that $\omega_C(x) \leq \sqrt{\omega_C^2(x) + |L_{f_{u1}}V(x)|^4}$ because of (7.10a), for $|x|_{\mathcal{A}} \geq \rho(|u_{C2}|)$ we have

$$\left\langle \nabla V, F\left(x, \left(\frac{1}{2}\tilde{\kappa}_{SC1}(x), u_{C2}\right)\right) \right\rangle \leq -\alpha_C(|x|_{\mathcal{A}}) \tag{7.38}$$

For each $(x, u_{D2}) \in D_{\kappa 1}$

$$
\begin{aligned}
&V\Big(G\Big(x, \Big(\frac{1}{2}\tilde{\kappa}_{SD1}(x), u_{D2}\Big)\Big)\Big) - V(x) \\
&= V\big(g(x) + \frac{1}{2} g_{u1}(x)\tilde{\kappa}_{SD} + g_{u2}(x)u_{D2}\big) - V(x) \\
&= V(g(x)) - \frac{1}{2}\omega_D(x) - \frac{1}{2}\sqrt{\omega_D^2(x) + |\widehat{V}_{L1}(x)|^4} + \widehat{V}_{L2}(x)u_{D2} - V(x) \\
&\leq \frac{1}{2}(V(g(x)) - \alpha_D(|x|_{\mathcal{A}}) - |\widehat{V}_{L2}(x)|\rho^{-1}(|x|_{\mathcal{A}})) \\
&\qquad\qquad - \frac{1}{2}\sqrt{\omega_D^2(x) + |\widehat{V}_{L1}(x)|^4} + \widehat{V}_{L2}(x)u_{D2} - \frac{1}{2}V(x) \\
&\leq \frac{1}{2}(V(g(x)) - V(x) + \alpha_D(|x|_{\mathcal{A}}) + |\widehat{V}_{L2}(x)|\rho^{-1}(|x|_{\mathcal{A}})) \\
&\quad - \frac{1}{2}\sqrt{\omega_D^2(x) + |\widehat{V}_{L1}(x)|^4} - \alpha_D(|x|_{\mathcal{A}}) - |\widehat{V}_{L2}(x)|\rho^{-1}(|x|_{\mathcal{A}}) + |\widehat{V}_{L2}(x)||u_{D2}| \\
&= -\frac{1}{2}\Big(- \omega_D(x) + \sqrt{\omega_D^2(x) + |\widehat{V}_{L1}(x)|^4}\Big) - \alpha_D(|x|_{\mathcal{A}}) + |\widehat{V}_{L2}(x)|(|u_{D2}| - \rho^{-1}(|x|_{\mathcal{A}}))
\end{aligned}
$$

Given that $\omega_D(x) \leq \sqrt{\omega_D^2(x) + |\widehat{V}_{L1}(x)|^4}$ because of (7.11a), for $|x|_{\mathcal{A}} \geq \rho(|u_{D2}|)$ we have

$$
V\left(G\left(x, \left(\frac{1}{2}\tilde{\kappa}_{SD1}(x), u_{D2}\right)\right)\right) - V(x) \leq -\alpha_D(|x|_{\mathcal{A}}) \tag{7.39}
$$

Finally[4], with (7.38) and (7.39) we invoke Theorem 7.1.3 to establish input-to-state pre-stability of $\mathcal{H}_{\kappa 1}$ with respect to the disturbance u_2 and the set $\mathcal{A}$.

□

7.4 Inverse-Optimal Stabilizing Control

Since the control input u_1 assigned to a feedback law aims to stabilize $\mathcal{H}$ to $\mathcal{A}$ but the disturbance u_2 seeks to prevent it, we formulate a zero-sum hybrid game that captures such setting. For this game, we study the following inverse optimality problem: given a control law κ_1 that input-to-state pre-stabilizes $\mathcal{H}$ with respect to the disturbance u_2 and the set $\mathcal{A}$, we determine the cost functional that renders the feedback control action κ_1 optimal.

For starters, following Chapter 3, we formulate a zero-sum hybrid game. Given $\xi \in \Pi(\overline{C}) \cup \Pi(D)$, an input action $u = (u_1, u_2) = ((u_{C1}, u_{D1}), (u_{C2}, u_{D2})) \in \mathcal{U}$, the stage cost for flows $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$, the stage cost for jumps $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$,

[4] The cases where $L_{f_{u1}}V(x) = 0$ for each $x \in \Pi(C)$ and $\widehat{V}_{L1}(x) = 0$ for each $x \in \Pi(D)$ follow the approach in the proof of Theorem 7.3.5.

and the terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, we define the cost associated to the solution (ϕ, u) to $\mathcal{H}$ from ξ, as

$$\mathcal{J}(\xi, u) := \sum_{j=0}^{\sup_j \operatorname{dom} \phi} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j), u_C(t,j))dt + \sum_{j=0}^{\sup_j \operatorname{dom} \phi - 1} L_D(\phi(t_{j+1}, j), u_D(t_{j+1}, j)) + \limsup_{\substack{(t,j)\to \sup \operatorname{dom} \phi \\ (t,j)\in \operatorname{dom}\phi}} q(\phi(t,j)) \quad (7.40)$$

where $t_{\sup_j \operatorname{dom} \phi + 1} := \sup_t \operatorname{dom} \phi$ defines the upper limit of the last integral, and $\{t_j\}_{j=0}^{\sup_j \operatorname{dom} \phi}$ is a nondecreasing sequence associated to the definition of the hybrid time domain of ϕ; see Definition 2.2.2. The terminal cost in (7.40) is evaluated at the final value of the state trajectory ϕ via the third term therein.

Given $\xi \in \Pi(\overline{C}) \cup \Pi(D)$ and $\kappa_1 \in \mathcal{K}_1$, we consider the following optimization problem

$$\underset{\substack{u_1 \\ u=(u_1,u_2)\in\mathcal{U}}}{\text{minimize}} \ \underset{u_2}{\text{maximize}} \ \mathcal{J}(\xi, u) \quad (7.41)$$

Definition 7.4.1. (Value function) *Given $\xi \in \Pi(\overline{C}) \cup \Pi(D)$ and $\kappa_1 \in \mathcal{K}_1$, the value function at ξ is given by*

$$\mathcal{J}^*(\xi) := \min_{\substack{u_1 \\ u=(u_1,u_2)\in\mathcal{U}}} \max_{u_2} \mathcal{J}(\xi, u) \quad (7.42)$$

7.4.1 Inverse Optimal Non-QP Control

The problem of designing a feedback law κ_1 using a Lyapunov function to stabilize a system $\mathcal{H}$ under disturbances can be addressed by solving a QP as in Section 7.3.2. This approach is myopic [63] because it may sacrifice future performance to guarantee a desired behavior in the present time. To compensate for this, following the ideas in [63], in this section we propose a non-QP version of the ISpS control design via inverse optimality. Specifically, the cost associated to the solution (ϕ, u) to $\mathcal{H}$ from ξ is in (7.40) with

$$L_C(x, u_C) := L_{1C}(x) + L_{2C}(x)u_{C1} + u_{C1}^\top R_C(x)u_{C1} - \lambda\gamma\Big(\frac{|u_{C2}|}{\lambda}\Big) \quad \forall (x, u_C) \in C \quad (7.43a)$$

$$L_D(x, u_D) := L_{1D}(x) + L_{2D}(x)u_{D1} + u_{D1}^\top R_D(x)u_{D1} - \lambda\gamma\Big(\frac{u_{D2}|}{\lambda}\Big) \quad \forall (x, u_D) \in D \quad (7.43b)$$

$$q(x) = V(x) \qquad \forall x \in \Pi(C) \cup \Pi(D) \tag{7.43c}$$

where $\gamma \in \mathcal{K}_\infty$ and $\lambda \in (0, 1]$. The inverse optimality approach allows us to design the optimal feedback law κ_1, the stage costs L_{1C}, L_{1D}, and the matrix functions R_C and R_D in (7.43).

Definition 7.4.2. ([63, Lemma A.1]) *For a class $\mathcal{K}_\infty$ function γ whose derivative exists and is also a class $\mathcal{K}_\infty$ function, the Legendre–Fenchel transform is defined as*

$$\ell\gamma(r) = r(\gamma')^{-1}(r) - \gamma((\gamma')^{-1}(r)) \quad \forall r \geq 0 \tag{7.44}$$

where $(\gamma')^{-1}(r)$ stands for the inverse function of $\dfrac{d\gamma(r)}{dr}$.

Theorem 7.4.3. (Non-QP ISpS Control) *Consider the hybrid system $\mathcal{H} = (C, F, D, G)$ as in (7.1) and a compact set $\mathcal{A} \subset \mathbb{R}^n$, suppose there exist $\rho \in \mathcal{K}$, and an ISpS-CLF V for $\mathcal{H}$ with respect to disturbances and $\mathcal{A}$. In addition, suppose there exist functions $\widehat{V}_{L1} : \mathbb{R}^n \to \mathbb{R}^{m_{D1}}, \widehat{V}_{L2} : \mathbb{R}^n \to \mathbb{R}^{m_{D2}}$ and $\widehat{V}_Q : \mathbb{R}^n \to \mathbb{S}^{m_D}_{>0}$ such that for all $(x, u_D) \in D$,*

$$\begin{aligned} V\left(G(x, u_D)\right) =& V\Big(g(x) + g_{u1}(x)u_{D1} + g_{u2}(x)u_{D2}\Big) \\ \leq& V(g(x)) + \widehat{V}_{L1}(x)u_{D1} + u_{D1}^\top \widehat{V}_Q(x)u_{D1} + \widehat{V}_{L2}(x)u_{D2}, \end{aligned} \tag{7.45}$$

and there exist functions $R_C : \Pi(C) \to \mathbb{S}^{m_{C1}}_{>0}$ and $R_D : \Pi(D) \to \mathbb{S}^{m_{D1}}_{>0}$, and functions $L_{2C} : \Pi(C) \to \mathbb{R}^{m_{C2}}$ and $L_{2D} : \Pi(D) \to \mathbb{R}^{m_{D2}}$ such that for the resulting closed-loop system $\mathcal{H}_{\kappa1} = (C_{\kappa1}, F, D_{\kappa1}, G)$ as in (7.2) from assigning u_1 to the feedback law $\kappa_1 = (\kappa_{C1}, \kappa_{D1})$, with values

$$\kappa_{C1}(x) := -\frac{1}{2}R_C^{-1}(x)(L_{2C}(x) + L_{f_{u1}}V(x)) \tag{7.46}$$

$$\kappa_{D1}(x) := -\frac{1}{2}(R_D(x) + \widehat{V}_Q(x))^{-1}(L_{2D}(x) + \widehat{V}_{L1}(x)) \tag{7.47}$$

the following holds

$$L_f V(x) + L_{f_{u1}}V(x)\kappa_{C1}(x) + \ell\gamma(|L_{f_{u2}}V(x)|) \leq -\alpha_C(|x|_\mathcal{A}) \qquad \forall x \in \Pi(C_{\kappa1}) \tag{7.48}$$

$$\begin{aligned} &V(g(x)) + \widehat{V}_{L1}(x)\kappa_{D1}(x) + \kappa_{D1}(x)^\top \widehat{V}_Q(x)\kappa_{D1}(x) \\ &\qquad - V(x) + \ell\gamma\Big(|\widehat{V}_{L2}(x)|\Big) \leq -\alpha_D(|x|_\mathcal{A}) \qquad \forall x \in \Pi(D_{\kappa1}) \end{aligned} \tag{7.49}$$

where $\alpha_C, \alpha_D \in \mathcal{K}$, and $\gamma \in \mathcal{K}_\infty$ has a derivative that is also a class-$\mathcal{K}_\infty$ function. Then, for any $\xi \in \Pi(\overline{C}) \cup \Pi(D)$, κ_1 solves the inverse optimal problem by minimizing the cost $\mathcal{J}$ as in (7.40) with

$$L_{1C}(x) := -\left(L_f V(x) - \kappa_{C1}^\top(x)R_C(x)\kappa_{C1}(x) + \lambda\ell\gamma(|L_{f_{u2}}V(x)|)\right) \tag{7.50}$$

and

$$L_{1D}(x) := -\Big(V(g(x)) - V(x) - \kappa_{D1}^{\top}(x)(R_D(x) + \widehat{V}_Q(x))\kappa_{D1}(x) + \lambda\ell\gamma(|\widehat{V}_{L2}(x)|)\Big) \tag{7.51}$$

Proof. The feedback law $\kappa_1 = (\kappa_{C1}, \kappa_{D1})$ is obtained by solving[5]

$$0 = \sup_{u_{C1}}\{L_C(x,(u_{C1},u_{C2})) + L_f V(x) + L_{f_{u1}}V(x)u_{C1}\} \quad \forall (x,u_{C2}) \in \Pi_{u_{C1}}(C) \tag{7.52}$$

and

$$V(x) = \sup_{u_{D1}}\{L_D(x,(u_{D1},u_{D2})) + V(G(x,(u_{D1},u_{D2})))\} \quad \forall (x,u_{D2}) \in \Pi_{u_{D1}}(D) \tag{7.53}$$

Using (7.50) and (7.51) in (7.43a) and (7.43b), respectively, we express the cost $\mathcal{J}$ associated to a solution (ϕ, u) as[6]

$$\begin{aligned}
\mathcal{J}(\xi,u) = & \sum_{j=0}^{\sup_j \operatorname{dom}\phi} \int_{t_j}^{t_{j+1}} \Big(- L_f V(\phi) + \kappa_{C1}^{\top}(\phi) R_C(\phi)\kappa_{C1}(\phi) \\
& \qquad - \lambda\ell\gamma(|L_{f_{u1}}V(\phi)|) + L_{2C}(\phi)u_{C1} + u_{C1}^{\top}R_C(\phi)u_{C1} - \lambda\gamma\Big(\frac{|u_{C2}|}{\lambda}\Big)\Big)dt \\
& + \sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} \Big(- V(g(\phi)) + V(\phi) + \kappa_{D1}^{\top}(\phi)(R_D(\phi) + \widehat{V}_Q(\phi))\kappa_{D1}(\phi) - \lambda\ell\gamma(2|\widehat{V}_{L1}(x)|) \\
& \qquad + L_{2D}(\phi)u_{D1} + u_{D1}^{\top}R_D(\phi)u_{D1} - \lambda\gamma\Big(\frac{|u_{D2}|}{\lambda}\Big)\Big) \\
& + \limsup_{\substack{t+j \to \sup_t \operatorname{dom}\phi + \sup_j \operatorname{dom}\phi \\ (t,j)\in \operatorname{dom}\phi}} V(\phi(t,j))
\end{aligned} \tag{7.54}$$

where $\{t_j\}_{j=0}^{\sup_j \operatorname{dom}\phi}$ is a nondecreasing sequence associated to the definition of the hybrid time domain of (ϕ, u), and κ_{C1} and κ_{D1} are as in (7.46) and (7.47), respectively.

[5]See Remark 7.3.7.

[6]For ease of notation, where needed, we will drop the arguments of the pair (ϕ, u), which are (t, j), unless they are specified.

For each $j \in \mathrm{dom}_j\, \phi$ notice that

$$\int_{t_j}^{t_{j+1}} \Big(- L_f V(\phi) + \kappa_{C1}^\top(\phi) R_C(\phi) \kappa_{C1}(\phi) - \lambda \ell \gamma(|L_{f_{u2}} V(\phi)|) + L_{2C}(\phi) u_{C1} + u_{C1}^\top R_C(\phi) u_{C1} - \lambda \gamma\Big(\frac{|u_{C2}|}{\lambda}\Big)\Big) dt$$

$$= - \int_{t_j}^{t_{j+1}} \Big(L_f V(\phi) + L_{f_{u1}} V(\phi) u_{C1} + L_{f_{u2}} V(\phi) u_{C2}\Big) dt$$

$$- \int_{t_j}^{t_{j+1}} \Big(- L_{2C}(\phi) u_{C1} - u_{C1}^\top R_C(\phi) u_{C1} - L_{f_{u1}} V(\phi) u_{C1} - \kappa_{C1}^\top(\phi) R_C(\phi) \kappa_{C1}(\phi)\Big) dt$$

$$- \int_{t_j}^{t_{j+1}} \Big(\lambda \gamma\Big(\frac{|u_{C2}|}{\lambda}\Big) + \lambda \ell \gamma(|L_{f_{u2}} V(\phi)|) - L_{f_{u2}} V(\phi) u_{C2}\Big) dt$$

$$= - \int_{t_j}^{t_{j+1}} \frac{dV}{dt}(\phi(t,j)) dt + \int_{t_j}^{t_{j+1}} \Big(u_{C1}^\top R_C(\phi) u_{C1} + \kappa_{C1}^\top(\phi) R_C(\phi) \kappa_{C1}(\phi) + \big((L_{2C}(\phi) + L_{f_{u1}} V(\phi)) R_C^{-1}(\phi)\big) R_C(\phi) u_{C1}\Big) dt$$

$$- \int_{t_j}^{t_{j+1}} \left(\lambda \gamma \left(\frac{|u_{C2}|}{\lambda}\right) - \lambda \gamma((\gamma')^{-1}(|L_{f_{u2}} V(\phi)|)) + \lambda |L_{f_{u2}} V(\phi)| (\gamma')^{-1}(|L_{f_{u2}} V(\phi)|) - L_{f_{u2}} V(\phi) u_{C2}\right) dt$$

$$= - \int_{t_j}^{t_{j+1}} \frac{dV}{dt}(\phi(t,j)) dt + \int_{t_j}^{t_{j+1}} \Big(u_{C1}^\top R_C(\phi) u_{C1} + \kappa_{C1}^\top(\phi) R_C(\phi) \kappa_{C1}(\phi) - 2 \kappa_{C1}^\top(\phi) R_C(\phi) u_{C1}\Big) dt - \lambda \int_{t_j}^{t_{j+1}} \Gamma(u_{C2}, \kappa_{C2}(\phi)) dt$$

$$= - \Big(V(\phi(t_{j+1}, j) - V(\phi(t_j, j))\Big) + \int_{t_j}^{t_{j+1}} (u_{C1} - \kappa_{C1}(\phi))^\top R_C(\phi) (u_{C1} - \kappa_{C1}(\phi)) dt - \lambda \int_{t_j}^{t_{j+1}} \Gamma(u_{C2}, \kappa_{C2}(\phi)) dt \quad (7.55)$$

where

$$(x, u) \mapsto \Gamma(u, \kappa(x)) := \gamma\Big(\frac{|u|}{\lambda}\Big) - \gamma\Big(\frac{|\kappa(x)|}{\lambda}\Big) + \gamma'\Big(\frac{|\kappa(x)|}{\lambda}\Big) \frac{\kappa(x)}{\lambda |\kappa(x)|} (\kappa(x) - u) \quad (7.56)$$

and

$$x \mapsto \kappa_{C2}(x) := \lambda (\gamma')^{-1}(|L_{f_{u2}} V(x)|) \frac{L_{f_{u2}} V(x)}{|L_{f_{u2}} V(x)|} \quad (7.57)$$

In addition, for all $(t,j) \in \operatorname{dom}\phi$ such that $(t, j+1) \in \operatorname{dom}\phi$, we have[7]

$$\begin{aligned}
&- V(g(\phi(t,j))) + V(\phi(t,j)) + \kappa_{D1}^\top(\phi)(R_D(\phi) + \widehat{V}_Q(\phi))\kappa_{D1}(\phi) - \lambda\ell\gamma(|\widehat{V}_{L2}(\phi)|) \\
&\qquad\qquad + L_{2D}(\phi)u_{D1} + u_{D1}^\top R_D(\phi)u_{D1} - \lambda\gamma\Big(\frac{|u_{D2}|}{\lambda}\Big) \\
&= -\Big(V(g(\phi)) + \widehat{V}_{L1}(\phi)u_{D1} + u_{D1}^\top \widehat{V}_Q(\phi)u_{D1} - V(\phi) + \widehat{V}_{L2}(\phi)u_{D2}\Big) \\
&\quad - \Big(- \kappa_{D1}^\top(\phi)(R_D(\phi) + \widehat{V}_Q(\phi))\kappa_{D1}(\phi) \\
&\qquad\qquad - L_{2D}(\phi)u_{D1} - \widehat{V}_{L1}(\phi)u_{D1} - u_{D1}^\top(R_D(\phi) + \widehat{V}_Q(\phi))u_{D1}\Big) \\
&\qquad\qquad - \Big(\lambda\gamma\Big(\frac{|u_{D2}|}{\lambda}\Big) + \lambda\ell\gamma(|\widehat{V}_{L2}(\phi)|) - \widehat{V}_{L2}(\phi)u_{D2}\Big) \\
&= -\Big(V(G(\phi,u_D)) - V(\phi)\Big) + u_{D1}^\top(R_D(\phi) + \widehat{V}_Q(\phi))u_{D1} + \kappa_{D1}^\top(R_D(\phi) + \widehat{V}_Q(\phi))\kappa_{D1} + \\
&\quad (L_{2D}(\phi) + \widehat{V}_{L1}(\phi))(R_D(\phi) + \widehat{V}_Q(\phi))^{-1}(R_D(\phi) + \widehat{V}_Q(\phi))u_{D1} - \lambda\Gamma(u_{D2}, \kappa_{D2}(\phi)) \\
&= -\Big(V(G(\phi,u_D)) - V(\phi)\Big) + (u_{D1} - \kappa_{D1}(\phi))^\top(R_D(\phi) + \widehat{V}_Q(\phi))(u_{D1} - \kappa_{D1}(\phi)) \\
&\qquad\qquad - \lambda\Gamma(u_{D2}, \kappa_{D2}(\phi))
\end{aligned} \tag{7.58}$$

where Γ is defined as in (7.56) and

$$x \mapsto \kappa_{D2}(x) := \lambda(\gamma')^{-1}(|\widehat{V}_{L2}(x)|)\frac{\widehat{V}_{L2}(x)}{|\widehat{V}_{L2}(x)|} \tag{7.59}$$

Thus, by replacing (7.55) and (7.58) in (7.54), we obtain

$$
\begin{aligned}
\mathcal{J}(\xi,u) = & \sum_{j=0}^{\sup_j \operatorname{dom}\phi} \Big(-\big(V(\phi(t_{j+1},j)) - V(\phi(t_j,j))\big) \\
& \qquad + \int_{t_j}^{t_{j+1}} (u_{C1} - \kappa_{C1}(\phi))^\top R_C(\phi)(u_{C1} - \kappa_{C1}(\phi))dt \\
& \qquad\qquad - \lambda \int_{t_j}^{t_{j+1}} \Gamma(u_{C2}, \kappa_{C2}(\phi))dt \Big) \\
& - \sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} \Big(V(\phi(t_{j+1}, j+1)) - V(\phi(t_{j+1}, j)) \\
& \qquad - (u_{D1} - \kappa_{D1}(\phi))^\top (R_D(\phi) + \widehat{V}_Q(\phi))(u_{D1} - \kappa_{D1}(\phi)) \\
& \qquad + \lambda\Gamma(u_{D2}, \kappa_{D2}(\phi)) \Big) \\
& + \limsup_{\substack{t+j \to \sup_t \operatorname{dom}\phi + \sup_j \operatorname{dom}\phi \\ (t,j)\in\operatorname{dom}\phi}} V(\phi(t,j)) \\
= & V(\xi) + \sum_{j=0}^{\sup_j \operatorname{dom}\phi} \int_{t_j}^{t_{j+1}} (u_{C1} - \kappa_{C1}(\phi))^\top R_C(\phi)(u_{C1} - \kappa_{C1}(\phi))dt \\
& \qquad - \lambda \int_{t_j}^{t_{j+1}} \Gamma(u_{C2}, \kappa_{C2}(\phi))dt + \\
& \sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} (u_{D1} - \kappa_{D1}(\phi))^\top (R_D(\phi) + \widehat{V}_Q(\phi))(u_{D1} - \kappa_{D1}(\phi)) - \lambda\Gamma(u_{D2}, \kappa_{D2}(\phi))
\end{aligned} \tag{7.60}
$$

Given that $R_C(x) \in \mathbb{S}_{>0}^{m_{C1}}$ for all $x \in \Pi(C)$, and $R_D(x), \widehat{V}_Q(x) \in \mathbb{S}_{>0}^{m_{D1}}$ for all $x \in \Pi(D)$, the cost $\mathcal{J}(\xi,u)$ is minimized under $\kappa_1 = (\kappa_{C1}, \kappa_{D1})$ and the value function is $\mathcal{J}^*(\xi) = V(\xi)$. Furthermore, since $\Gamma(u, \kappa(x))$ vanishes when $u = \kappa(x)$, and, for any other u, it is positive (see Lemma 7.4.6), the second term in each sum in (7.60) is maximized under $\kappa_2 = (\kappa_{C2}, \kappa_{D2})$ with values as in (7.57) and (7.59). □

Theorem 7.4.4. (Solvability of Inverse Optimal Problem) *Consider the hybrid system $\mathcal{H} = (C, F, D, G)$ as in* (7.1) *and a compact set $\mathcal{A} \subset \mathbb{R}^n$. If there exists an ISpS CLF for $\mathcal{H}$ with respect to the disturbance u_2 and the set $\mathcal{A}$ satisfying Assumption 7.3.3, and $\pi_C, \pi_D \in \mathcal{K}_\infty$ such that*

$$|L_{f_{u_2}} V(x)| \le \pi_C(|x|_{\mathcal{A}}) \quad \forall x \in \Pi(C) \tag{7.61a}$$

$$|\widehat{V}_{L2}(x)| \le \pi_D(|x|_{\mathcal{A}}) \quad \forall x \in \Pi(D) \tag{7.61b}$$

then, there exist a class-$\mathcal{K}_\infty$ function γ whose derivative γ' is also a class-$\mathcal{K}_\infty$ function, matrix functions $R_C : \Pi(C) \to \mathbb{S}_{>0}^{m_{C1}}$ and $R_D : \Pi(C) \to \mathbb{S}_{>0}^{m_{D1}}$, functions $L_{1C} : \Pi(C) \to \mathbb{R}_{>0}$, $L_{1D} : \Pi(D) \to \mathbb{R}_{>0}$, and $q : \Pi(C) \cup \Pi(D) \to \mathbb{R}_{>0}$, and a continuous feedback law κ_1 that not only renders $\mathcal{H}$ input-to-state controlled pre-stable, but also minimizes the cost functional $\mathcal{J}$ as in (7.40) under the worst-case disturbance u_2.

Proof. From Definition 7.3.1 and Theorem 7.3.2, there exists $\rho \in \mathcal{K}_\infty$ satisfying (7.8), and $\mathcal{H}$ is input-to-state pre-stabilizable with respect to the disturbance u_2 and the set $\mathcal{A}$. In particular, from Theorem 7.3.9, consider the feedback law $\tilde{\kappa}_1 = \frac{1}{2}(\tilde{\kappa}_{SC1}, \tilde{\kappa}_{SC2})$, with values as in (7.18) and (7.19), and the resulting closed-loop system $\mathcal{H}_{\tilde{\kappa}1}$ with disturbances. Then, for all $x \in \Pi(C_{\tilde{\kappa}1})$,

$$L_{f+f_{u1}\frac{\tilde{\kappa}_{SC}}{2}} V(x) + |L_{f_{u2}} V(x)| \rho^{-1}(|x|_{\mathcal{A}}) \leq -\alpha_C(|x|_{\mathcal{A}}).$$

Since $\rho^{-1} \circ \pi_C^{-1} \in \mathcal{K}_\infty$, there exists $\zeta \in \mathcal{K}_\infty$ such that $\zeta' \in \mathcal{K}_\infty$ and

$$\zeta(r) \leqslant r\rho^{-1}(\pi_C^{-1}(r)) \qquad \forall r \geq 0$$

Let us define: $\gamma_C = \ell\zeta$ as in Definition 7.4.2. From [82, Lemma A1-2], it follows that $\ell\ell\zeta = \zeta$, which implies that:

$$\ell\gamma_C(r) \leqslant r\rho^{-1}(\pi_C^{-1}(r)) \qquad \forall r \geq 0$$

Then, with (7.18), we have

$$\begin{aligned}
& L_{f+f_{u_1}\frac{\tilde{\kappa}_{SC}}{2}} V(x) + \ell\gamma_C\left(|L_{f_{u2}} V(x)|\right) \\
& \leq L_{f+f_{u_1}\frac{\tilde{\kappa}_{SC}}{2}} V(x) + |L_{f_{u2}} V(x)| \rho^{-1} \circ \pi_C^{-1}\left(|L_{f_{u2}} V(x)|\right). \\
& \leq L_{f+f_{u_1}\frac{\tilde{\kappa}_{SC}}{2}} V(x) + |L_{f_{u2}} V(x)| \rho^{-1}(|x|_{\mathcal{A}}) \\
& \leq -\alpha_C(|x|_{\mathcal{A}})
\end{aligned}$$

On the other hand, from (7.39) for all $x \in \Pi(D)$, it follows that

$$V(g(x)) + \frac{1}{2}\widehat{V}_{L1}(x)\tilde{\kappa}_{SD}(x) - V(x) + |\widehat{V}_{L2}(x)| \rho^{-1}(|x|_{\mathcal{A}}) \leq -\alpha_D(|x|_{\mathcal{A}}).$$

Since $\rho^{-1} \circ \pi_D^{-1} \in \mathcal{K}_\infty$, there exists $\zeta \in \mathcal{K}_\infty$ such that $\zeta' \in \mathcal{K}_\infty$ and

$$\zeta(r) \leqslant r\rho^{-1}(\pi_D^{-1}(r))$$

Let us define: $\gamma_D = \ell\zeta$ as in Definition 7.4.2. From [82, Lemma A1-2], it follows that $\ell\ell\zeta = \zeta$, which implies that:

$$\ell\gamma_D(r) \leqslant r\rho^{-1}(\pi_D^{-1}(r)).$$

Then, with (7.19), we have

$$\begin{aligned}
&V(g(x)) + \frac{1}{2}\widehat{V}_{L1}(x)\tilde{\kappa}_{SD}(x) - V(x) + \ell\gamma_D\left(|\widehat{V}_{L2}(x)|\right)\\
&V(g(x)) + \frac{1}{2}\widehat{V}_{L1}(x)\tilde{\kappa}_{SD}(x) - V(x) + |\widehat{V}_{L2}(x)|\rho^{-1}\circ\pi_D^{-1}\left(|\widehat{V}_{L2}(x)|\right)\\
&\leq V(g(x)) + \frac{1}{2}\widehat{V}_{L1}(x)\tilde{\kappa}_{SD}(x) - V(x) + |\widehat{V}_{L2}(x)|\rho^{-1}(|x|_{\mathcal{A}})\\
&\leq -\alpha_D(|x|_{\mathcal{A}})
\end{aligned}$$

Thus,

$$L_f V(x) + \frac{1}{2}L_{f_{u1}}V(x)\tilde{\kappa}_{SC1}(x) + \ell\gamma(|L_{f_{u2}}V(x)|) \leq -\alpha_C(|x|_{\mathcal{A}}) \qquad \forall x \in \Pi(C_{\tilde{\kappa}1}) \tag{7.62}$$

$$V(g(x)) + \frac{1}{2}\widehat{V}_{L1}(x)\tilde{\kappa}_{SD1}(x) - V(x) + \ell\gamma(|\widehat{V}_{L2}(x)|) \leq \alpha_D(|x|_{\mathcal{A}}) \qquad \forall x \in \Pi(D_{\tilde{\kappa}1}) \tag{7.63}$$

with $\gamma := \min\{\gamma_C, \gamma_D\}$. In addition, notice that the control law $\frac{1}{2}\tilde{\kappa}_{SC1}$ is of the form (7.46) with $L_{2C}(x) = 0$ for all $x \in \Pi(C_{\tilde{\kappa}1})$ and

$$R_C(x) := \begin{cases} \dfrac{|L_{f_{u1}}V(x)|^2}{\omega_C(x) + \sqrt{\omega_C^2(x) + |L_{f_{u1}}V(x)|^4}} & \text{if } L_{f_{u1}}V(x) \neq 0 \\ p_C & \text{if } L_{f_{u1}}V(x) = 0. \end{cases}$$

where $p_C > 0$ is arbitrary, and $\frac{1}{2}\tilde{\kappa}_{SD1}$ is of the form (7.47) with $L_{2D}(x) = 0$ and $\widehat{V}_Q(x) = 0$ for all $x \in \Pi(D_{\tilde{\kappa}1})$, and

$$R_D(x) := \begin{cases} \dfrac{|\widehat{V}_{L1}(x)|^2}{\omega_D(x) + \sqrt{\omega_D^2(x) + |\widehat{V}_{L1}(x)|^4}} & \text{if } \widehat{V}_{L1}(x) \neq 0 \\ p_D & \text{if } \widehat{V}_{L1}(x) = 0. \end{cases}$$

where $p_D > 0$ is arbitrary. As a result, by Theorem 7.4.3, for any $\xi \in \Pi(C) \cup \Pi(D)$ the feedback law κ_1 minimizes the cost $\mathcal{J}$ as in (7.40) with stage costs (7.43), (7.50), and (7.51). □

7.4.2 Inverse Optimal QP Filter

In this section, as a special case of section 7.4.1, we provide a result with sufficient conditions to solve Problem 7.2.1 when the controller is expressed as the pointwise solution to a QP.

Based on this, we consider the problem of finding the min-norm feedback law $\kappa_{1_{QP}} = (\kappa_{C1_{QP}}, \kappa_{D1_{QP}})$, with values as in (7.24) and (7.26), that input-to-state pre-stabilizes the system with respect to the disturbance u_2.

Given $\xi \in \Pi(\overline{C}) \cup \Pi(D)$, an input action $u = (u_C, u_D) \in \mathcal{U}$, the stage cost for flows $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$, the stage cost for jumps $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, and the terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, we define the cost associated to the solution (ϕ, u) to $\mathcal{H}$ from ξ, as in (7.40) with

$$L_C(x, u_C) := L_{1C}(x) + R_C(x)|u_{C1}|^2 - \lambda\gamma\Big(\frac{|u_{C2}|}{\lambda}\Big) \qquad \forall (x, u_C) \in C \tag{7.64a}$$

$$L_D(x, u_D) := L_{1D}(x) + R_D(x)|u_{D1}|^2 - \lambda\gamma\Big(\frac{|u_{D2}|}{\lambda}\Big) \qquad \forall (x, u_D) \in D \tag{7.64b}$$

$$q(x) = V(x) \qquad \forall x \in \Pi(C) \cup \Pi(D) \tag{7.64c}$$

We approach the optimization problem in (7.41) as an inverse problem: we design the optimal feedback law κ_1, and the stage costs L_{1C} and L_{1D} in (7.64).

Corollary 7.4.5. (QP Safety Filter) *Consider the hybrid system $\mathcal{H}$ as in* (7.1) *and a compact set $\mathcal{A} \subset \mathbb{R}^n$, suppose there exist $\rho \in \mathcal{K}$ and a ISpS-CLF V for $\mathcal{H}$ with respect to the disturbance u_2 and $\mathcal{A}$. Suppose there exist functions $\widehat{V}_{L1} : \mathbb{R}^n \to \mathbb{R}^{m_{D1}}$ and $\widehat{V}_{L2} : \mathbb{R}^n \to \mathbb{R}^{m_{D2}}$ such that Assumption 7.3.3 holds for all $(x, u_D) \in D$, and for the resulting closed-loop system $\mathcal{H}_{\kappa 1} = (C_{\kappa 1}, F, D_{\kappa 1}, G)$ as in* (7.2) *from assigning u_1 to the feedback law $\kappa_{1_{QP}} := (\kappa_{C1_{QP}}, \kappa_{D1_{QP}})$ with values as in* (7.24) *and* (7.26)*, where $R_C(x) := \frac{1}{2}\frac{|L_{f_{u1}}V(x)|^2}{\max\{0, \omega_C(x)\}}$ with $\omega_C(x)$ as in* (7.10b) *and $R_D(x) := \frac{1}{2}\frac{|\widehat{V}_{L1}(x)|^2}{\max\{0, \omega_D(x)\}}$ with $\omega_D(x)$ as in* (7.11b)*, the following holds*

$$L_f V(x) + L_{f_{u1}} V(x)\kappa_{C1_{QP}}(x) + \ell\gamma(|L_{f_{u2}} V(x)|) \leq -\alpha_C(|x|_{\mathcal{A}}) \qquad \forall x \in C_{\kappa 1} \tag{7.65}$$

$$V(g(x)) + \widehat{V}_{L1}(x)\kappa_{D1_{QP}}(x) - V(x) + \ell\gamma(2|\widehat{V}_{L2}(x)|) \leq \alpha_D(|x|_{\mathcal{A}}) \qquad \forall x \in D_{\kappa 1} \tag{7.66}$$

for some $\alpha_C, \alpha_D \in \mathcal{K}$, *and* $\gamma \in \mathcal{K}_\infty$ *whose derivative* γ' *is also a class* $\mathcal{K}_\infty$ *function. Then,* $\kappa_{1_{QP}}$ *solves, for any* $\xi \in \Pi(\bar{C}) \cup \Pi(D)$, *the inverse optimal problem by minimizing the cost* $\mathcal{J}$ *in* (7.40) *with*

$$L_{1C}(x) := -\Big(L_f V(x) - \frac{1}{4} R_C^{-1}(x) |L_{f_{u1}} V(x)|^2 + \lambda \ell \gamma(|L_{f_{u2}} V(x)|)\Big) \tag{7.67}$$

and

$$L_{1D}(x) := -\Big(V(g(x)) - V(x) - \frac{1}{4} R_D^{-1}(x) |\widehat{V}_{L1}(x)|^2 + \lambda \ell \gamma(|\widehat{V}_{L2}(x)|)\Big) \tag{7.68}$$

Proof. Following Theorem 7.3.6, the feedback law $\kappa_{1_{QP}} := (\kappa_{C1_{QP}}, \kappa_{D1_{QP}})$ with values as in (7.24) and (7.26) input-to-state pre-stabilizes the closed-loop system $\mathcal{H}_{\kappa 1}$ with respect to the disturbance u_2 and $\mathcal{A}$.

Thanks to (7.67) and (7.68), we denote the cost $\mathcal{J}$ associated to a solution (ϕ, u) to $\mathcal{H}$ as in (7.1) as

$$\begin{aligned} \mathcal{J}(\xi, u) = & \sum_{j=0}^{\sup_j \operatorname{dom} \phi} \int_{t_j}^{t_{j+1}} \Big(- L_f V(\phi) + \frac{1}{2} \max\{0, \omega_C(\phi)\} \\ & \qquad - \lambda \ell \gamma(|L_{f_{u2}} V(\phi)|) + \frac{1}{2} \frac{|L_{f_{u1}} V(\phi)|^2}{\max\{0, \omega_C(\phi)\}} |u_{C1}|^2 - \lambda \gamma\Big(\frac{|u_{C2}|}{\lambda}\Big)\Big) dt \\ & + \sum_{j=0}^{\sup_j \operatorname{dom} \phi - 1} \Big(- V(g(\phi)) + V(\phi) + \frac{1}{2} \max\{0, \omega_D(\phi)\} \\ & \qquad - \lambda \ell \gamma(|\widehat{V}_{L2}(\phi)|) + \frac{1}{2} \frac{|\widehat{V}_{L1}(\phi)|^2}{\max\{0, \omega_D(\phi)\}} |u_{D1}(\phi)|^2 - \lambda \gamma\Big(\frac{|u_{D2}|}{\lambda}\Big)\Big) \\ & + \limsup_{\substack{(t,j) \to \sup_t \operatorname{dom} \phi + \sup_j \operatorname{dom} \phi \\ (t,j) \in \operatorname{dom} \phi}} V(\phi(t,j)) \end{aligned} \tag{7.69}$$

where $\{t_j\}_{j=0}^{\sup_j \operatorname{dom} \phi}$ is a nondecreasing sequence associated to the definition of the hybrid time domain of (ϕ, u).

For each $j \in \operatorname{dom}_j \phi$ notice that

$$
\begin{aligned}
&\int_{t_j}^{t_{j+1}} \Big(-L_f V(\phi) + \frac{1}{2}\max\{0, \omega_C(\phi)\} - \lambda\ell\gamma(|L_{f_{u2}}V(\phi)|) \\
&\qquad\qquad + \frac{1}{2}\frac{|L_{f_{u1}}V(\phi)|^2}{\max\{0,\omega_C(\phi)\}}|u_{C1}|^2 - \lambda\gamma\Big(\frac{|u_{C2}|}{\lambda}\Big)\Big)dt \\
&= -\int_{t_j}^{t_{j+1}} \Big(L_f V(\phi) + L_{f_{u1}}V(\phi)u_{C1} + L_{f_{u2}}V(\phi)u_{C2}\Big)dt \\
&\qquad + \int_{t_j}^{t_{j+1}} \Big(\frac{1}{2}\frac{|L_{f_{u1}}V(\phi)|^2}{\max\{0,\omega_C(\phi)\}}|u_{C1}|^2 + L_{f_{u1}}V(\phi)u_{C1} + \frac{1}{2}\max\{0,\omega_C(\phi)\}\Big)dt \\
&\qquad - \int_{t_j}^{t_{j+1}} \Big(\lambda\gamma\Big(\frac{|u_{C2}|}{\lambda}\Big) + \lambda\ell\gamma(L_{f_{u2}}V(\phi)|) - L_{f_{u2}}V(\phi)u_{C2}\Big)dt \\
&= -\int_{t_j}^{t_{j+1}} \frac{dV}{dt}(\phi(t,j))dt + \int_{t_j}^{t_{j+1}} \Big(\frac{1}{2}\frac{|L_{f_{u1}}V(\phi)|^2}{\max\{0,\omega_C(\phi)\}}|u_{C1} - \kappa_{C1_{QP}}(\phi)|^2\Big)dt \\
&\qquad - \int_{t_j}^{t_{j+1}} \Big(\lambda\gamma\Big(\frac{|u_{C2}|}{\lambda}\Big) - \lambda\gamma((\gamma')^{-1}(|L_{f_{u2}}V(\phi)|)) \\
&\qquad\qquad + \lambda|L_{f_{u2}}V(\phi)|(\gamma')^{-1}(|L_{f_{u2}}V(\phi)|) - L_{f_{u2}}V(\phi)u_{C2}\Big)dt \\
&= -\Big(V(\phi(t_{j+1},j) - V(\phi(t_j,j)))\Big) + \int_{t_j}^{t_{j+1}} \Big(\frac{1}{2}\frac{|L_{f_{u1}}V(\phi)|^2}{\max\{0,\omega_C(\phi)\}}|u_{C1} - \kappa_{C1_{QP}}(\phi)|^2\Big)dt \\
&\qquad - \lambda\int_{t_j}^{t_{j+1}} \Gamma(u_{C2}, \kappa_{C2}(\phi))dt \qquad (7.70)
\end{aligned}
$$

where Γ is defined as in (7.56) and κ_{C2} as in (7.57). In addition, for all $(t, j) \in \operatorname{dom}\phi$ such that $(t, j+1) \in \operatorname{dom}\phi$, we have

$$
\begin{aligned}
&-V(g(\phi)) + V(\phi) + \frac{1}{2}\max\{0,\omega_D(\phi)\} - \lambda\ell\gamma(\widehat{V}_{L2}(\phi)|) \\
&\qquad\qquad + \frac{1}{2}\frac{|\widehat{V}_{L1}(\phi)|^2}{\max\{0,\omega_D(\phi)\}}|u_{D1}|^2 - \lambda\gamma\Big(\frac{|u_{D2}|}{\lambda}\Big) \\
&= -\Big(V(g(\phi)) + \widehat{V}_{L1}(\phi)u_{D1} - V(\phi) + \widehat{V}_{L2}(\phi)u_{D2}\Big) \\
&\qquad + \Big(\frac{1}{2}\frac{|\widehat{V}_{L1}(\phi)|^2}{\max\{0,\omega_D(\phi)\}}|u_{D1}|^2 + \widehat{V}_{L1}(\phi)u_{D1} + \frac{1}{2}\max\{0,\omega_D(\phi)\}\Big) \\
&\qquad - \Big(\lambda\ell\gamma(\widehat{V}_{L2}(\phi)|) + \lambda\gamma\Big(\frac{|u_{D2}|}{\lambda}\Big) - \widehat{V}_{L2}(\phi)u_{D2}\Big) \\
&= -\big(V(G(\phi,u)) - V(\phi)\big) + \Big(\frac{1}{2}\frac{|\widehat{V}_{L1}(\phi)|^2}{\max\{0,\omega_D(\phi)\}}|u_{D1} - \kappa_{D1_{QP}}(\phi)|^2\Big) \\
&\qquad - \lambda\Gamma(u_{D2}, \kappa_{D2}(\phi)) \qquad (7.71)
\end{aligned}
$$

where Γ is defined as in (7.56) and κ_{D2} as in (7.59). Thus, substituting (7.70) and (7.71) in (7.69), we obtain

$$\begin{aligned}
\mathcal{J}(\xi,u) = &\sum_{j=0}^{\sup_j \operatorname{dom}\phi} \Bigg(-\Big(V(\phi(t_{j+1},j) - V(\phi(t_j,j))\Big) \\
&+ \int_{t_j}^{t_{j+1}} \bigg(\frac{1}{2}\frac{|L_{f_{u1}}V(\phi)|^2}{\max\{0,\omega_C(\phi)\}} |u_{C1} - \kappa_{C1_{QP}}(\phi)|^2 \bigg) dt \\
&- \lambda \int_{t_j}^{t_{j+1}} \Gamma(u_{C2},\kappa_{C2}(\phi))dt \Bigg) \\
- &\sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} \Bigg(\Big(V(\phi(t_{j+1},j+1)) - V(\phi(t_{j+1},j))\Big) - \bigg(\frac{1}{2}\frac{|\widehat{V}_{L1}(\phi)|^2}{\max\{0,\omega_D(\phi)\}} |u_{D1} - \kappa_{D1_{QP}}(\phi)|^2 \bigg) \\
&+ \lambda \Gamma(u_{D2},\kappa_{D2}(\phi)) \Bigg) \\
&+ \limsup_{\substack{(t,j)\to \sup \operatorname{dom}\phi \\ (t,j)\in \operatorname{dom}\phi}} V(\phi(t,j)) \\
= V(\xi) + &\sum_{j=0}^{\sup_j \operatorname{dom}\phi} \int_{t_j}^{t_{j+1}} \bigg(\frac{1}{2}\frac{|L_{f_{u1}}V(\phi)|^2}{\max\{0,\omega_C(\phi)\}} |u_{C1} - \kappa_{C1_{QP}}(\phi)|^2 - \lambda\Gamma(u_{C2},\kappa_{C2}(\phi)) \bigg) dt \\
+ &\sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} \bigg(\frac{1}{2}\frac{|\widehat{V}_{L1}(\phi)|^2}{\max\{0,\omega_D(\phi)\}} |u_{D1} - \kappa_{D1_{QP}}(\phi)|^2 - \lambda\Gamma(u_{D2},\kappa_{D2}(\phi)) \bigg) \qquad (7.72)
\end{aligned}$$

Given that $\max\{0,\omega_\star(\phi(t,j))\} \geq 0$, $(t,j) \in \operatorname{dom}\phi$, for $\star \in \{C,D\}$, the cost $\mathcal{J}(\xi,u)$ is minimized under $\kappa_1^* = (\kappa_{C1}^*, \kappa_{D1}^*) = (\kappa_{C1_{QP}}, \kappa_{D1_{QP}})$ and the value function is $\mathcal{J}^*(\xi) = V(\xi)$. □

Lemma 7.4.6. () *Given a class $\mathcal{K}_\infty$ function γ whose derivative exists and is also a class $\mathcal{K}_\infty$ function, $\lambda > 0$, and a feedback law $\kappa \in \mathcal{K}$, define*

$$(x,u) \mapsto \Gamma(u,\kappa(x)) := \gamma\Big(\frac{|u|}{\lambda}\Big) - \gamma\Big(\frac{|\kappa(x)|}{\lambda}\Big) + \gamma'\Big(\frac{|\kappa(x)|}{\lambda}\Big)\frac{\kappa(x)}{\lambda|\kappa(x)|}(\kappa(x) - u) \qquad (7.73)$$

where $(\gamma')^{-1}(r)$ stands for the inverse function of $\frac{d\gamma}{dr}(r)$. Then

$$\Gamma(u,\kappa(x)) \geq 0 \qquad \forall (x,u) \in \operatorname{dom}\Gamma \qquad (7.74)$$

and

$$\Gamma(u,\kappa(x)) = 0 \quad \text{iff} \quad u = \kappa(x) \qquad (7.75)$$

7.5 Applications & Examples

7.5.1 Inverse Optimal Non-QP Control for a Hybrid Oscillator

Consider the following oscillator with impacts with dynamics given by

$$\mathcal{H}\begin{cases}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}x_2\\ -\zeta_C\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u_{C1}+\begin{bmatrix}0\\ 1\end{bmatrix}u_{C2} & (x,u_C)\in C\\ \\ \begin{bmatrix}x_1^+\\ x_2^+\end{bmatrix}=\begin{bmatrix}x_1\\ 0\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u_{D1}+\begin{bmatrix}1\\ 0\end{bmatrix}u_{D2} & (x,u_D)\in D\end{cases} \tag{7.76}$$

where $\zeta_C \geq 0$, and

$$C := \big\{(x,u_C)\in\mathbb{R}^2\times\mathbb{R}^2 : x_1\geq 0\big\}$$
$$D := \big\{(x,u_D)\in\mathbb{R}^2\times\mathbb{R}^2 : x_1 = 0,\ x_2\leq 0\big\}$$

with $u_\star = (u_{\star 1}, u_{\star 2}) \in \mathbb{R}^2$, for $\star \in \{C, D\}$. Now, let $\mathcal{A} = \{(0,0)\} \subset \mathbb{R}^2$ and consider the energy function given by

$$V(x) = \zeta_C x_1 + \frac{1}{2}x_2^2.$$

To show that V is a ISpS-LF candidate as in Definition 7.1.2, notice that: i) $\operatorname{dom} V = \mathbb{R}^2$, ii) V is continuous and locally Lipschitz on any open set containing $x_1 \geq 0$, and iii) $\min\left\{\frac{1}{2}\left(\frac{|x|}{\sqrt{2}}\right)^2, \zeta_C\left(\frac{|x|}{\sqrt{2}}\right)\right\} \leq V(x) \leq \frac{1}{2}|x|^2 + \zeta_C|x|$ for all $x \in \{z \in \mathbb{R}^2 : z_1 \geq 0\}$. Then, V is an ISpS Lyapunov function candidate for $\mathcal{H}$ with respect to $\mathcal{A}$. In addition, consider the feedback law $\kappa_1 = (\kappa_{C1}, \kappa_{D1})$ with values[7]

$$\kappa_1(x) = (\kappa_{C1}(x), \kappa_{D1}(x)) = -\left(x_2, \frac{1}{2}(x_2 + \zeta_C)\right)$$

and the corresponding closed-loop hybrid oscillator $\mathcal{H}_{\kappa 1}$. Since (7.48) and (7.49) are satisfied with $\alpha_C(r) = \alpha_D(r) = r$, and $\gamma(r) = r^2$ for all $r \geq 0$, by invoking Theorem 7.4.3, we have that κ_1 solves the inverse optimal control problem with cost functional as in (7.40) [8] when u_2 is assigned to κ_2 with values as in (7.57) and (7.59).

[7]Notice that κ_{C1} is of the form (7.46) with $R_C(x) = 1$ and $L_{2C}(x) = L_{f_{u1}}V(x) = x_2$, and κ_{D1} is of the form (7.47) with $R_D(x) = \widehat{V}_Q(x) = \frac{1}{2}$, $L_{2D}(x) = x_2$ and $\widehat{V}_{L1}(x) = \zeta_C$.

[8]L_C and L_D as in (7.64), with L_{C1} and L_{D1} are as in (7.50) and (7.51), respectively, and $\lambda = 0.81$.

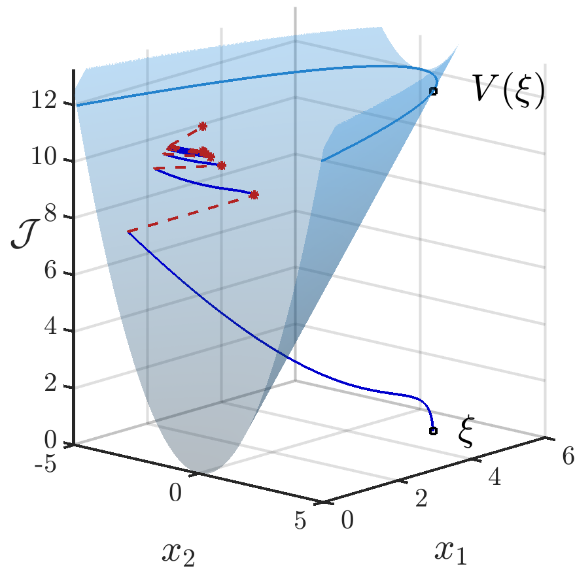

Figure 7.1: Phase portrait for the hybrid oscillator. Initial condition ξ. Value function (light blue) and cost of solution (blue-red) rendered by the saddle-point equilibrium strategy, attaining the value evaluated at the initial condition, $V(\xi)$.

7.5.2 Inverse Optimal QP Control

Consider the linear hybrid system in Example 7.3.8 with dynamics $\mathcal{H}$ described by (7.29), the feedback law $\kappa_1 = (\kappa_{C1_{QP}}, \kappa_{D1_{QP}})$ as in (7.30), and the ISpS LF candidate $V(x) = \eta x$, for a given $\eta > 0$. Let $\mu = \frac{1}{4}\eta|b_2|$, $|b_2| \geq \eta p_2^2$, and pick $r \mapsto \gamma(r) = r^2$. Then it follows that, for all $x \in [\mu, \delta]$,

$$\begin{aligned}
&\left\langle \nabla V(x), ax - \frac{\max\{0, \omega_C(x)\}}{\eta} \right\rangle + \ell\gamma(\eta|b_2|) \\
&\qquad = \eta a x - \max\{0, x\,(1 + \eta(a + |b_2|))\} + \frac{1}{4}\eta^2|b_2|^2 \\
&\qquad = \min\{0, x\,(1 + \eta(a + |b_2|))\} - x(1 + \eta|b_2|) + \frac{1}{4}\eta^2|b_2|^2 \\
&\qquad \leq -x(1 + \eta|b_2|) + \frac{1}{4}\eta^2|b_2|^2 \\
&\qquad \leq -x
\end{aligned}$$

and (7.65) is satisfied. Likewise, for $x = \mu$, we get

$$\begin{aligned}\eta(\sigma - x) - \max\{0, \eta(\sigma - x) + x(1 + \eta|p_2|)\} + \ell\gamma(\eta|p_2|)&\\ &= \min\{0, \eta(\sigma - x) + x(1 + \eta|p_2|)\} - x(1 + \eta|p_2|) + \frac{1}{4}\eta^2 p_2^2\\ &\leq -x + \frac{1}{4}\eta^2 p_2^2\\ &\leq -x\end{aligned}$$

and (7.66) is also satisfied. Then, the feedback law $\kappa_1 = (\kappa_{C1_{QP}}, \kappa_{D1_{QP}})$ as in (7.30), not only renders the closed-loop system $\mathcal{H}_{\kappa 1}$ ISpS with respect to the disturbance u_2 and the set $\mathcal{A} = \{0\}$, but also, by Corollary 7.4.5, minimizes the cost $\mathcal{J}$ as in (7.40) with

$$\begin{aligned}L_C(x, u_C) &= -\left(\eta a x - \frac{1}{2}\max\{0, \omega_C(x)\} + \frac{\lambda}{4}\eta^2 b_2^2\right) + \frac{\eta^2 b_1^2}{2}\frac{u_{C1}^2}{\max\{0, \omega_C(x)\}} - \frac{u_{C2}^2}{\lambda}\\ L_D(x, u_D) &= -\left(\eta(\sigma - x) - \frac{1}{2}\max\{0, \omega_D(x)\} + \frac{\lambda}{4}\eta^2 p_2^2\right) + \frac{\eta^2 p_1^2}{2}\frac{u_{D1}^2}{\max\{0, \omega_D(x)\}} - \frac{u_{D2}^2}{\lambda}\end{aligned}$$

for some $\lambda \in (0, 1]$, under the maximizing disturbance u_2, where the value function is $\mathcal{J}^*(\xi) = \eta\xi$ with $\xi \in [\mu, \delta]$.

Chapter 8

Input-to-State Safety Control for Hybrid Systems

Following the approach in Chapter 7, in this Chapter, we address a two-player zero-sum hybrid game as an inverse optimal control problem with safe controllers under the presence of a disturbance. We present results on sufficient conditions to guarantee input-to-state safety with respect to disturbances for hybrid systems. First, the control feedback laws are considered as solutions to QP problems, and to Sontag's formula. Under additional conditions, non-QP controllers are formulated and the cost functional that the feeback laws optimize is constructed via inverse optimality.

We define a hybrid dynamical affine system $\mathcal{H}$ with input $(u, w) = ((u_C, w_C), (u_D, w_D)) \in \mathbb{R}^{m_C} \times \mathbb{R}^{m_D} = \mathbb{R}^m$, where $u := (u_C, u_D) \in \mathbb{R}^{m_{Cu}} \times \mathbb{R}^{m_{Du}} = \mathbb{R}^{m_u}$ is a control input and $w := (w_C, w_D) \in \mathbb{R}^{m_{Cw}} \times \mathbb{R}^{m_{Dw}} = \mathbb{R}^{m_w}$ is a disturbance, as

$$\mathcal{H} \,:\, \begin{cases} \dot{x} = F(x, (u_C, w_C)) := f(x) + f_u(x)u_C + f_w(x)w_C & (x, (u_C, w_C)) \in C \\ x^+ = G(x, (u_D, w_D)) := g(x) + g_u(x)u_D + g_w(x)w_D & (x, (u_D, w_D)) \in D \end{cases} \tag{8.1}$$

where $x \in \mathbb{R}^n$ is the state.

Consider the hybrid system resulting from assigning the control input u of $\mathcal{H}$ as in (8.1) to the sum of a nominal feedback law $\bar{\kappa} := (\bar{\kappa}_C, \bar{\kappa}_D) : \mathbb{R}^n \to \mathbb{R}^{m_{Cu}} \times \mathbb{R}^{m_{Du}}$ and a safeguarding feedback law $\widehat{\kappa} := (\widehat{\kappa}_C, \widehat{\kappa}_D) : \mathbb{R}^n \to \mathbb{R}^{m_{Cu}} \times \mathbb{R}^{m_{Du}}$, and with disturbance input w, as

$$\mathcal{H}_\kappa : \begin{cases} \dot{x} = F(x, (\bar{\kappa}_C(x) + \widehat{\kappa}_C(x), w_C)) =: F_\kappa(x, w_C) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (x, w_C) \in C_\kappa \\ x^+ = G(x, (\bar{\kappa}_D(x) + \widehat{\kappa}_D(x), w_D)) =: G_\kappa(x, w_D) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (x, w_D) \in D_\kappa \end{cases} \tag{8.2}$$

where $C_\kappa := \{(x, w_C) \in \mathbb{R}^n \times \mathbb{R}^{m_{C_w}} : (x, (\bar{\kappa}_C(x) + \widehat{\kappa}_C(x), w_C)) \in C\}$ and $D_\kappa := \{(x, w_D) \in \mathbb{R}^n \times \mathbb{R}^{m_{D_w}} : (x, (\bar{\kappa}_D(x) + \widehat{\kappa}_D(x), w_D)) \in D\}$.

Definition 8.0.1. (Solution to $\mathcal{H}_\kappa$) *A pair (ϕ, w) defines a solution to $\mathcal{H}_\kappa$ as in (8.2) if $\phi \in \mathcal{X}$, $w = (w_C, w_D) \in \mathcal{W}$, $\operatorname{dom}\phi = \operatorname{dom} w$, and*

- $(\phi(0,0), w_C(0,0)) \in \overline{C_\kappa}$ *or*
  $(\phi(0,0), w_D(0,0)) \in D_\kappa$,
- *For each $j \in \mathbb{N}$ such that I^j_ϕ has a nonempty interior $\operatorname{int} I^j_\phi$, we have, for all $t \in \operatorname{int} I^j_\phi$,*
  $$(\phi(t,j), w_C(t,j)) \in C_\kappa$$
  and, for almost all $t \in I^j_\phi$,
  $$\frac{d\phi}{dt}(t,j) = F(\phi(t,j), (\kappa_C(\phi(t,j)), w_C(t,j)))$$
- *For each $(t,j) \in \operatorname{dom}\phi$ such that $(t, j+1) \in \operatorname{dom}\phi$,*
  $$\begin{aligned} (\phi(t,j), w_D(t,j)) &\in D_\kappa \\ \phi(t,j+1) &= G(\phi(t,j), (\kappa_D(\phi(t,j)), w_D(t,j))) \end{aligned}$$

A solution pair (ϕ, w) is a compact solution if ϕ is a compact hybrid arc; see Definition 2.2.2.

We denote by $\hat{\mathcal{S}}_{\mathcal{H}_\kappa}(M)$ the set of solution pairs (ϕ, w) to $\mathcal{H}_\kappa$ as in (8.2) such that $\phi(0,0) \in M$, and by $\mathcal{S}_{\mathcal{H}_\kappa}(M) \subset \hat{\mathcal{S}}_{\mathcal{H}_\kappa}(M)$ the set of all maximal solutions from M. We say that the hybrid closed-loop system with disturbances, namely $\mathcal{H}_\kappa$ as in (8.2), results from assigning the input u of $\mathcal{H}$ in (8.1) to a feedback law κ.

8.1 Input-to-State Safety for Hybrid Systems with Disturbances

Given a feedback law κ, we formulate conditions guaranteeing that every solution to $\mathcal{H}_\kappa$ that starts in a closed set $K \subset \mathbb{R}^n$ remains close to K under the presence of a disturbance $w = (w_C, w_D)$, where the closeness to K depends on the size of w. For this purpose, we use the notion of input-to-state safety (ISSf) to guarantee that a larger set containing K is conditionally invariant for $\mathcal{H}_\kappa$ with respect to w and K. We introduce the following definitions of invariance and safety.

Definition 8.1.1. (Conditional pre-invariance with disturbances) *Given a feedback law κ, a set $S \subset \mathbb{R}^n$ is said to be conditionally pre-invariant for $\mathcal{H}_\kappa$ in* (8.2) *with respect to the disturbance w and the set $K \subset S$ if each $(\phi, w) \in \mathcal{S}_{\mathcal{H}_\kappa}(K)$ is such that $\phi(t,j) \in S$ for all $(t,j) \in \operatorname{dom} \phi$.*

Barrier functions (BFs) serve as a synthesis tool to guarantee invariance of a set of interest, see, e.g., [84] and [85]. In the context of safety, given an unsafe set $X_u \subset \Pi(C_\kappa) \cup \Pi(D_\kappa) \cup G_\kappa(D_\kappa)$ and a continuous function $B : \mathbb{R}^n \to \mathbb{R}$ such that $B(x) > 0$ for all $x \in X_u$, we define a set K as the zero-sublevel set of B restricted to $\Pi(C_\kappa) \cup \Pi(D_\kappa)$, i.e.,

$$K := \{x \in (\Pi(C_\kappa) \cup \Pi(D_\kappa)) \mid B(x) \leq 0\}, \tag{8.3}$$

which is closed when $\Pi(C_\kappa) \cup \Pi(D_\kappa)$ is closed.

The following definition introduces the notion of safety for hybrid systems with disturbance inputs.

Definition 8.1.2. (Input-to-state safety) *Given a closed set $K \subset \mathbb{R}^n$ defined by a function $B : \operatorname{dom} B \to \mathbb{R}$ as in* (8.3), *and a feedback law κ, the system $\mathcal{H}_\kappa$ in* (8.2) *is $\bar{w}$-small-input input-to-state safe ($\bar{w}$-small-input ISSf) with respect to the disturbance w and the set K if there exist $\bar{w} > 0$ and $\rho \in \mathcal{K}_\infty$ such that*

$$\begin{aligned} &(\phi, w) \in \mathcal{S}_{\mathcal{H}_\kappa}(K), \|w\|_\# \leq \bar{w} \\ &\qquad \Rightarrow B(\phi(t,j)) \leq \rho(\bar{w}) \quad \forall (t,j) \in \operatorname{dom} \phi \end{aligned} \tag{8.4}$$

where the function ρ is referred to as the ISSf gain function.

Notice that Definition 8.1.1 and Definition 8.1.2 do not require maximal solutions to be complete, for which we employ the prefix 'pre-', for more details see [86–88]. In addition, observe that, from the construction of K in (8.3) and the properties of the barrier function B, it follows that

$$K \subset (\Pi(C_\kappa) \cup \Pi(D_\kappa) \cup G(D)) \setminus X_u \tag{8.5}$$

Small-input ISSf is strengthened to ISSf if (8.4) holds for arbitrary large $\bar{w}$. In addition, small-input ISSf resembles the notion of safety in [86] when $\bar{w} = 0$.

Remark 8.1.3. (Safety and invariance) *It is immediate that the system $\mathcal{H}_\kappa$ is $\bar{w}$-small-input ISSf with respect to w and K if and only if there exists $\bar{w} > 0$ and $\rho \in \mathcal{K}_\infty$ such that the set $K_d(\bar{w}) \supset K$ defined as*

$$K_d(\bar{w}) := \{x \in \Pi(C_\kappa) \cup \Pi(D_\kappa) \mid B(x) - \rho(\bar{w}) \leq 0\}, \tag{8.6}$$

is conditionally pre-invariant for $\mathcal{H}_\kappa$ with respect to w and K.

Remark 8.1.4. (Connection with literature) *In this work, we are interested in characterizing the ISSf property in Definition 8.1.2 for hybrid systems so that we can guarantee that, under the worst-case disturbance w, trajectories starting from K do not reach the unsafe set X_u.*

- *Selection of a finite $\bar{w}$: following [61], in which a connection between safety and conditional invariance of a set is established in terms of an upper bound of the disturbances, the notion of input-to-state safety herein relies on a similar approach. In the context of robust safety for continuous-time systems, previous work, such as [89], considers disturbances bounded by a known constant to design feedback laws that robustly stabilize the system while rendering a set of interest forward invariant. Notice that for $\mathcal{H}_\kappa$ to be w-robustly safe[1] with respect to (K, X_u), it is sufficient to find a finite*

$$\begin{aligned} \bar{w} \leq v^* := \arg\sup_{v>0} \quad & v \\ \textit{subject to} \quad & K_d(v) \cap X_u = \emptyset \end{aligned} \tag{8.7}$$

[1]Following [21, 90], the system $\mathcal{H}_\kappa$ is said to be w-robustly safe with respect to (K, X_u) if each $(\phi, w) \in \mathcal{S}_{\mathcal{H}_\kappa}(K)$ is such that $\phi(t,j) \in \mathbb{R}^n \setminus X_u$ for all $(t,j) \in \operatorname{dom} \phi$.

and κ such that $K_d(\bar{w}) \supset K$ in (8.6) is conditionally pre-invariant for $\mathcal{H}_\kappa$ with respect to w and K. Thus, if $\mathcal{H}_\kappa$ is w-robustly safe with respect to (K, X_u), then it is $\bar{w}$-small-input ISSf with respect to the disturbance w and the set K satisfying (8.5). Furthermore, when $\bar{w} = 0$, $\bar{w}-$small-input ISSf of the system $\mathcal{H}_\kappa$ with no disturbances implies that each $\phi \in \mathcal{S}_{\mathcal{H}_\kappa}(K)$ is such that $\phi(t, j) \in K$ for all $(t, j) \in \operatorname{dom} \phi$.

- *Existing notions of ISSf with respect to disturbances: a version of Definition 8.1.2 was presented in [65] for continuous-time systems. The $\mathcal{KL}$ bound therein accounts for solutions that start outside of K, case we do not consider in this work. The set K in (8.5) is defined following an opposite sign convention, namely, K is defined as the zero-sublevel set of B (contrary to being defined as a zero-superlevel set of h in [65]). Without loss of generality, (5) relies on an upper bound $\bar{w}$ on disturbances, which can be conveniently chosen to resemble (7) in [65].*

Definition 8.1.5. (ISSf barrier function candidate) *Given a hybrid system $\mathcal{H}_\kappa = (C_\kappa, F_\kappa, D_\kappa, G_\kappa)$ as in (8.2) with disturbance $w = (w_C, w_D)$, the function $B : \operatorname{dom} B \to \mathbb{R}$ and the sets $K \subset K_i \subset \mathbb{R}^n$ define an ISSf barrier function (ISSf-BF) candidate for $\mathcal{H}_\kappa$ with respect to (K, K_i) if the following conditions hold:*

1) $\Pi(\overline{C_\kappa}) \cup \Pi(D_\kappa) \cup G_\kappa(D_\kappa) \subset \operatorname{dom} B$ *and* $K_i \subset \Pi(C_\kappa) \cup \Pi(D_\kappa)$;

2) *for some open set $\mathcal{V}$ containing an open neighborhood of K_i, B is continuously differentiable on $(\mathcal{V} \setminus K_i) \cap \Pi(\overline{C_\kappa})$;*

3) $B(x) > 0$ *for all* $x \in \big(\Pi(\overline{C_\kappa}) \cup \Pi(D_\kappa)\big) \setminus K$;

4) $B(x) \leq 0$ *for all* $x \in K$.

Notice that $K_i \supset K$ in Definition 8.1.5 is the set we aim to render invariant, whose role will be played by $K_d(\bar{w})$ in the following results .

Theorem 8.1.6. (ISSf under a barrier function candidate) *Given a closed set $K \subset \mathbb{R}^n$ and a feedback law $\kappa = (\kappa_C, \kappa_D)$ defining a hybrid system $\mathcal{H}_\kappa = (C_\kappa, F_\kappa, D_\kappa, G_\kappa)$ as in (8.2) with disturbance $w = (w_C, w_D)$, suppose B is an ISSf-BF candidate for $\mathcal{H}_\kappa$ with respect to $(K, K_d(\bar{w}))$, where $K_d(\bar{w})$ is defined as in (8.6) for some $\rho \in \mathcal{K}_\infty$ and*

$\bar{w} > 0$, and let $\mathcal{V}$ be an open set containing an open neighborhood of $K_d(\bar{w})$. If there exist $\alpha_C \geq 0, \alpha_D \in [0,1]$, such that

$$(x, w_C) \in C_\kappa : x \in \mathcal{V}\backslash K_d(\bar{w}), |w_C| \leq \bar{w} \Rightarrow \langle \nabla B(x), F(x, (\kappa_C(x), w_C)) \rangle \leq -\alpha_C B(x) \tag{8.8a}$$

$$(x, w_D) \in D_\kappa : x \in K_d(\bar{w}), |w_D| \leq \bar{w} \Rightarrow B(G(x, (\kappa_D(x), w_D))) - B(x) \leq -\alpha_D(B(x) - \rho(\bar{w})) \tag{8.8b}$$

$$G_\kappa(D_{K_d}) \subset \Pi(C_\kappa) \cup \Pi(D_\kappa) \tag{8.8c}$$

where $D_{K_d} := \{(x, w_D) \in D_\kappa : x \in K_d(\bar{w})\}$, then $\mathcal{H}_\kappa$ is $\bar{w}$-small-input ISSf with respect to the disturbance w and the set K, as in Definition 8.1.2.

Proof. The proof is developed following the arguments of [56, Proposition 2.7], [53, Lemma 2.1], and [65, Lemma 1]. Pick $\xi \in K$ and a solution pair $(\phi, w) \in \mathcal{S}_{\mathcal{H}_\kappa}(\xi)$ such that $\|w\|_\# \leq \bar{w}$. Notice that $K_d(\bar{w})$ is closed relative to $\Pi(C) \cup \Pi(D)$ because B is continuous, and that any $x \in (\Pi(C) \cup \Pi(D)) \setminus K_d(\bar{w})$ satisfies $B(x) > \rho(\bar{w})$.

We show that $K_d(\bar{w})$ is conditionally invariant with respect to K, namely, if there exists some $(t, j) \in \operatorname{dom} \phi$ such that $\phi(t, j) \in K$, then $\phi(t', j') \in K_d(\bar{w})$ for all $t' + j' \geq t + j$.

Proceeding by contradiction, suppose that ϕ leaves $K_d(\bar{w})$. The following cases are possible:

- The state trajectory ϕ leaves $K_d(\bar{w})$ after a jump, that is $\phi(t, j) \in K_d(\bar{w}) \cap \Pi(D_\kappa)$ and $\phi(t, j+1) \notin K_d(\bar{w})$. Using the definition of B and $K_d(\bar{w})$, $B(\phi(t, j)) \leq \rho(\bar{w})$ and $B(\phi(t, j+1)) > \rho(\bar{w})$, and, from the definition of a solution to $\mathcal{H}_\kappa$, $\phi(t, j+1) = G_\kappa(\phi(t, j), w_D(t, j))$. Since (8.8b) implies that $B(\phi(t, j+1)) \leq (1 - \alpha_D) B(\phi(t, j)) + \alpha_D \rho(\bar{w})$ with $\alpha_D \in [0, 1]$ and (8.8c) implies that $\phi(t, j+1) \in \Pi(C_\kappa) \cup \Pi(D_\kappa)$, then $\phi(t, j+1) \in K_d(\bar{w})$.

- The state trajectory ϕ leaves the set $K_d(\bar{w})$ by flow: there exist $(\tau, k), (\tau', k) \in \operatorname{dom} \phi$ such that $B(\phi(t, j)) \leq \rho(\bar{w})$ for all $(t, j) \in \operatorname{dom} \phi$, $t + j \leq \tau + k$, and $\phi(t, k) \in (\mathcal{V} \setminus K_d(\bar{w})) \cap \Pi(C_\kappa)$ for all $(t, k) \in \operatorname{dom} \phi$, $\tau < t \leq \tau'$. Using continuous differentiability of B and absolute continuity of $t \mapsto \phi(t, k)$ on $[\tau, \tau']$,

$t \mapsto B(\phi(t,k))$ is also absolutely continuous on $[\tau, \tau']$ and, via integration, satisfies

$$B(\phi(\tau', k)) - B(\phi(\tau, k)) = \int_{\tau}^{\tau'} \langle \nabla B(\phi(t,k)), \dot{\phi}(t,k) \rangle dt \tag{8.9}$$

Since $B(\phi(t,k)) > \rho(\bar{w})$ for all $t \in (\tau, \tau']$ and $B(\phi(\tau,k)) = \rho(\bar{w})$, the expression in (8.9) is positive. On the other hand, since $\phi((\tau,\tau'],k) \subset (\mathcal{V} \setminus K_d(\bar{w})) \cap \Pi(C_\kappa)$, (8.8a) implies that $\langle \nabla B(\phi(t,k)), F_\kappa(\phi(t,k), w_C(t,k))) \rangle \leq -\alpha_C B(\phi(t,k))$ with $\alpha_C \geq 0$ for almost all $t \in (\tau, \tau')$. Hence, via integration again, the expression in (8.9) is less than or equal to zero.

Since a contradiction is reached in both cases, it follows that, for every $(\phi, w) \in \mathcal{S}_{\mathcal{H}_\kappa}(K)$,

$$B(\phi(t,j)) \leq \rho(\bar{w}) \qquad \forall (t,j) \in \operatorname{dom}(\phi, w)$$

□

8.2 Problem Statement

Consider the system $\mathcal{H}$ in (8.1), with control input u assigned to a feedback law $\kappa = (\kappa_C, \kappa_D)$, the disturbance input $w \in \mathcal{W}$, and an unsafe set $X_u \subset \mathbb{R}^n$. The feedback κ is the sum of a given nominal feedback law $\bar{\kappa} = (\bar{\kappa}_C, \bar{\kappa}_D)$ (that captures desired properties, referred to as *uncertified* objectives, such as rendering a set asymptotically stable for $\mathcal{H}$), and a safeguarding feedback law $\widehat{\kappa}$. We say $\mathcal{H}$ is $\bar{w}$-small-input input-to-state controlled safe when the corresponding closed-loop system $\mathcal{H}_\kappa$ as in (8.2) is $\bar{w}$-small-input ISSf.In this chapter, we address the problem of designing the safeguarding feedback law $\widehat{\kappa}$ that not only renders $\mathcal{H}$ $\bar{w}$-small-input input-to-state controlled safe but also solves a zero-sum hybrid game. We use a continuous function $x \mapsto B(x)$ defining a barrier function candidate, with zero-sublevel set K satisfying (8.3) and (8.5), to guarantee that state trajectories starting in K never enter X_u. Specifically, we seek the existence of $\rho \in \mathcal{K}_\infty$ such that every $(\phi, (u,w)) \in \mathcal{S}_{\mathcal{H}}(\Pi(C) \cup \Pi(D))$, with input u given by $\operatorname{dom}\phi \ni (t,j) \mapsto u(t,j) = \kappa(\phi(t,j)) = \bar{\kappa}(\phi(t,j)) + \widehat{\kappa}(\phi(t,j))$, satisfies (8.4) for all $(t,j) \in \operatorname{dom}\phi$. This objective[2] is attained by considering the closed-loop system $\mathcal{H}_\kappa$ resulting from assigning the input u of $\mathcal{H}$ to κ and solving the following problem.

[2]Notice that the safeguarding map κ plays the role of a filter that shall be null when (8.4) is satisfied by the nominal feedback law $\bar{\kappa}$.

Problem 8.2.1. (Inverse-optimal Safety Filter) *Given a closed set $K \subset \mathbb{R}^n$, and an uncertified nominal feedback law $\bar{\kappa}$, design a safeguarding feedback law $\widehat{\kappa}$ that renders the corresponding closed-loop system $\mathcal{H}_\kappa$ $\bar{w}$-small-input input-to-state safe with respect to the disturbance w and the set K. In addition, determine the cost functional that $\widehat{\kappa}$ minimizes under the worst-case disturbance w.*

Remark 8.2.2. (Relation to the literature) *A version of Problem 8.2.1 was solved in [65] for continuous-time systems without constraints, i.e., the case in which $\mathcal{H} = (\mathbb{R}^n \times \mathbb{R}^m, F, \emptyset, \star)$, where $\star$ denotes an arbitrary jump map, and K in* (8.3) *is defined as $K := \{x \in \mathbb{R}^n \mid B(x) \geq 0\}$.*

8.3 Input-to-Sate Safety Filters

In this section, we address the first part of Problem 8.2.1 by using ISSf control barrier functions (ISSf-CBFs) as a synthesis tool to guarantee safety of a hybrid system. First, we introduce definitions and preliminary results on ISSf-CBFs for hybrid systems with disturbances.

8.3.1 Input-to-State Safety Control Barrier Functions

Definition 8.3.1. (ISSf-CBF with respect to disturbances) *Given a system $\mathcal{H} = (C, F, D, G)$ as in* (8.1) *and a closed set K, suppose B is an ISSf-BF candidate for $\mathcal{H}$ with respect to $(K, K_d(\bar{w}))$, where $K_d(\bar{w})$ is defined as in* (8.6) *for some $\rho \in \mathcal{K}_\infty$ and $\bar{w} > 0$. Let $\mathcal{V}$ be an open set containing an open neighborhood of $K_d(\bar{w})$. We say that B is an ISSf-control barrier function (ISSf-CBF) for $\mathcal{H}$ with respect to $(K, K_d(\bar{w}))$ if there exist $\alpha_C \geq 0$, $\alpha_D \in [0, 1]$ such that*

$$(x, w_C) \in \Pi_{u_C}(C) : x \in \mathcal{V} \backslash K_d(\bar{w}), |w_C| \leq \bar{w}$$
$$\Rightarrow \inf_{u_C \in \Psi_C(x, w_C)} \langle \nabla B(x), F(x, (u_C, w_C)) \rangle \leq -\alpha_C B(x) \tag{8.10a}$$

$$(x, w_D) \in \Pi_{u_D}(D) : x \in K_d(\bar{w}), |w_D| \leq \bar{w}$$
$$\Rightarrow \inf_{u_D \in \Psi_D(x, w_D)} B(G(x, (u_D, w_D))) - B(x)$$
$$\leq -\alpha_D (B(x) - \rho(\bar{w})) \tag{8.10b}$$

where $\Pi_{u_\star}(\star) = \{(x, w_\star) : \exists u_\star \text{ s.t. } (x, (u_\star, w_\star)) \in \star\}$, and $\Psi_\star(x, w_\star) := \{u_\star \in \mathbb{R}^{m_{\star u}} : (x, (u_\star, w_\star)) \in \star\}$ for $\star \in \{C, D\}$.

The following results are used to establish a connection between the existence of an ISSf-CBF and a feedback law that renders the closed-loop system ISSf. To characterize the effect of inputs in the safety conditions at jumps, we restrict our attention to a family of systems and barrier functions that obey the following assumption.

Assumption 8.3.2. *Given a system $\mathcal{H} = (C, F, D, G)$ as in* (8.1) *and a function $B : \mathbb{R}^n \to \mathbb{R}$, suppose there exist functions $\widehat{B}_{Lu} : \mathbb{R}^n \to \mathbb{R}^{m_{D_u}}$ and $\widehat{B}_{Lw} : \mathbb{R}^n \to \mathbb{R}^{m_{D_w}}$ such that, for all $(x, (u_D, w_D)) \in D$,*

$$\begin{aligned} B\left(G(x, u_D)\right) &= B\left(g(x) + g_u(x)u_D + g_w(x)w_D\right) \\ &\leq B(g(x)) + \widehat{B}_{Lu}(x)u_D + \widehat{B}_{Lw}(x)w_D \end{aligned} \tag{8.11}$$

Lemma 8.3.3. (Equivalent ISSf conditions) *Given a system $\mathcal{H} = (C, F, D, G)$ as in* (8.1) *and a closed set $K \subset \mathbb{R}^n$, suppose B is an ISSf-CBF for $\mathcal{H}$ with respect to $(K, K_d(\bar{w}))$, where $K_d(\bar{w})$ is defined as in* (8.6) *for some $\rho \in \mathcal{K}_\infty$ and $\bar{w} > 0$, $\mathcal{V}$ is an open set containing an open neighborhood of $K_d(\bar{w})$, and $\alpha_C \geq 0$, $\alpha_D \in [0, 1]$. The tuple $(B, \rho, \bar{w})$ satisfies* (8.10a) *if and only if*

$$L_{f_u}B(x) = 0, x \in (\mathcal{V} \setminus K_d(\bar{w})) \cap \Pi(C) = 0 \Rightarrow \omega_C(x) \leq 0 \tag{8.12a}$$

where

$$\omega_C(x) := L_f B(x) + \alpha_C B(x) + |L_{f_w}B(x)|\rho^{-1}(B(x)) \tag{8.12b}$$

and, under Assumption 8.3.2, satisfies (8.10b) *if and only if*

$$\widehat{B}_{Lu}(x) = 0, x \in \Pi(D) \cap K_d(\bar{w}) \Rightarrow \omega_D(x) \leq 0 \tag{8.13a}$$

where

$$\omega_D(x) := B(g(x)) - B(x) + \alpha_D(B(x) - \rho(\bar{w})) + |\widehat{B}_{Lw}(x)|\bar{w} \tag{8.13b}$$

Proof. The proof is developed following similar arguments as in [82, Lemma 2.1] and [65, Lemma 2].

($\Leftarrow$) By Definition 8.3.1, if $L_{f_u}B(x) = 0$, then

$$\begin{aligned} &(x, w_C) \in \Pi_{u_C}(C) : x \in \mathcal{V}\backslash K_d(\bar{w}), |w_C| \leq \bar{w} \\ &\qquad \Rightarrow L_f B(x) + L_{f_w}B(x)w_C \leq -\alpha_C B(x) \end{aligned} \tag{8.14}$$

Now consider the particular input w_C defined by the feedback law

$$\pi_C(x) = \begin{cases} \dfrac{L_{f_w}B(x)}{|L_{f_w}B(x)|}\rho^{-1}(B(x)) & \text{if} \quad L_{f_w}B(x) \neq 0 \\ 0 & \text{if} \quad L_{f_w}B(x) = 0 \end{cases} \tag{8.15}$$

for all $x \in \Pi(C)$. Note that if $w_C = \pi_C(x)$, then $\rho(|w_C|) \leq B(x)$. Therefore, substituting (8.15) in (8.14), we have that if $L_{f_u}B(x) = 0$, then

$$L_f B(x) + |L_{f_w}B(x)|\rho^{-1}(B(x)) \leq -\alpha_C B(x) \tag{8.16}$$

that is, (8.12a) is satisfied.

In addition, by Definition 8.3.1, and thanks to Assumption 8.3.2, if $\widehat{B}_{Lu}(x) = 0$, then

$$\begin{aligned} (x, w_D) \in \Pi_{u_D}(D) : & x \in K_d(\bar{w}), |w_D| \leq \bar{w} \\ & \Rightarrow B(g(x)) + \widehat{B}_{Lw}(x)w_D - B(x) \leq -\alpha_D(B(x) - \rho(\bar{w})) \end{aligned} \tag{8.17}$$

Now consider the particular input w_D defined by the feedback law

$$\pi_D(x) = \begin{cases} \dfrac{\widehat{B}_{Lw}(x)}{|\widehat{B}_{Lw}(x)|}\bar{w} & \text{if} \quad \widehat{B}_{Lw}(x) \neq 0 \\ 0 & \text{if} \quad \widehat{B}_{Lw}(x) = 0 \end{cases} \tag{8.18}$$

for all $x \in \Pi(D)$. Note that if $w_D = \pi_D(x)$, then $|w_D| \leq \bar{w}$. Therefore, substituting (8.18) in (8.17), we have that if $\widehat{B}_{Lu}(x) = 0$, then

$$B(g(x)) + |\widehat{B}_{Lw}(x)|\bar{w} - B(x) \leq -\alpha_D(B(x) - \rho(\bar{w})) \tag{8.19}$$

that is, (8.13a) is satisfied.

($\Rightarrow$) If $(x, w_C) \in \Pi_{u_C}(C) : x \in \mathcal{V} \backslash K_d(\bar{w}), |w_C| \leq \bar{w}$, using (8.12), we have

$$\begin{aligned} \inf_{u_C \in \Psi_C(x, w_C)} & \{L_f B(x) + L_{f_u}B(x)u_C + L_{f_w}B(x)w_C\} \\ & \leq \inf_{u_C \in \Psi_C(x, w_C)} \{L_f B(x) + L_{f_u}B(x)u_C + |L_{f_w}B(x)||w_C|\} \\ & \leq \inf_{u_C \in \Psi_C(x, w_C)} \{L_f B(x) + L_{f_u}B(x)u_C + |L_{f_w}B(x)|\rho^{-1}(B(x))\} \\ & \leq -\alpha_C B(x) \end{aligned}$$

In addition, if $(x, w_D) \in \Pi_{u_D}(D) : x \in K_d(\bar{w}), |w_D| \leq \bar{w}$, using (8.13a), we have

$$\begin{aligned}
&\inf_{u_D \in \Psi_D(x,w_D)} \{B(g(x)) + \widehat{B}_{Lu}(x)u_D + \widehat{B}_{Lw}(x)w_D\} \\
&\qquad \leq \inf_{u_D \in \Psi_D(x,w_D)} \{B(g(x)) + \widehat{B}_{Lu}(x)u_D + |\widehat{B}_{Lw}(x)||w_D|\} \\
&\qquad \leq \inf_{u_D \in \Psi_D(x,w_D)} \{B(g(x)) + \widehat{B}_{Lu}(x)u_D + |\widehat{B}_{Lw}(x)|\bar{w}\} \\
&\qquad \leq -\alpha_D(B(x) - \rho(\bar{w})) + B(x)
\end{aligned}$$

□

Theorem 8.3.4. (ISSf CBF Sontag-like formula) *Under Assumption 8.3.2, if there exists a ISSf-CBF B for $\mathcal{H} = (C, F, D, G)$ with respect to $(K,\ K_d(\bar{w}))$, where $K_d(\bar{w})$ is defined as in* (8.6) *for some $\rho \in \mathcal{K}_\infty$ and $\bar{w} > 0$, the system $\mathcal{H}$ is rendered $\bar{w}$-small-input input-to-state controlled safe with respect to the disturbance w and K (see Remark 8.1.3) using the following Sontag-type control law, in which we assign the input u_C to*

$$\tilde{\kappa}_{SC}(x) := \begin{cases} L_{f_u}B(x)\kappa_{SC}(x) & \text{if} \quad L_{f_u}B(x) \neq 0 \\ 0 & \text{if} \quad L_{f_u}B(x) = 0 \end{cases} \tag{8.20a}$$

where

$$\kappa_{SC}(x) := \frac{-\omega_C(x) - \sqrt{\omega_C^2(x) + |L_{f_u}B(x)|^4}}{|L_{f_u}B(x)|^2} \tag{8.20b}$$

and we assign the input u_D to

$$\tilde{\kappa}_{SD}(x) := \begin{cases} \widehat{B}_{Lu}(x)\kappa_{SD}(x) & \text{if} \quad \widehat{B}_{Lu}(x) \neq 0 \\ 0 & \text{if} \quad \widehat{B}_{Lu}(x) = 0 \end{cases} \tag{8.21a}$$

where

$$\kappa_{SD}(x) := \frac{-\omega_D(x) - \sqrt{\omega_D^2(x) + |\widehat{B}_{Lu}(x)|^4}}{|\widehat{B}_{Lu}(x)|^2} \tag{8.21b}$$

with ω_C and ω_D defined in (8.12b) *and* (8.13b)*, respectively.*

Proof. We substitute (8.20) and (8.21) into $\mathcal{H}$ to obtain the closed-loop system $\mathcal{H}_\kappa = (C_\kappa, F, D_\kappa, G)$ as in (8.2). Then, for each $(x, w_C) \in C_\kappa : x \in \mathcal{V} \setminus K_d(\bar{w})$, we have

- if $L_{f_u}B(x) = 0$,

$$\begin{aligned}\langle \nabla B(x), F(x, (\tilde{\kappa}_{SC}(x), w_C))\rangle &= L_f B(x) + L_{f_w} B(x) w_C \\ &= \omega_C(x) - \alpha_C(B(x)) - |L_{f_w} B(x)| \rho^{-1}(B(x)) + L_{f_w} B(x) w_C \\ &\leq -\alpha_C(B(x)) + |L_{f_w} B(x)|(|w_C| - \rho^{-1}(B(x)))\end{aligned}$$

and if $|w_C| \leq \bar{w}$, which implies that $B(x) \geq \rho(|w_C|)$, we have

$$\langle \nabla B(x), F(x, (\tilde{\kappa}_{SC}(x), w_C))\rangle \leq -\alpha_C(B(x))$$

- if $L_{f_u}B(x) \neq 0$,

$$\begin{aligned}&\langle \nabla B(x), F(x, (\tilde{\kappa}_{SC}(x), w_C))\rangle \\ &\qquad = L_f B(x) - \omega_C(x) - \sqrt{\omega_C^2(x) + |L_{f_u} B(x)|^4} + L_{f_w} B(x) w_C \\ &\qquad \leq -\alpha_C B(x) - |L_{f_w} B(x)| \rho^{-1}(B(x)) + L_{f_w} B(x) w_C \\ &\qquad \leq -\alpha_C B(x) + |L_{f_w} B(x)|(|w_C| - \rho^{-1}(B(x)))\end{aligned}$$

and if $x \in \mathcal{V} \setminus K_d(\bar{w})$ and $|w_C| \leq \bar{w}$, which implies that $B(x) \geq \rho(|w_C|)$, we have

$$\langle \nabla B(x), F(x, (\tilde{\kappa}_{SC}(x), w_C))\rangle \leq -\alpha_C B(x).$$

For each $(x, w_D) \in D_\kappa$, we obtain

- if $\widehat{B}_{Lu}(x) = 0$,

$$\begin{aligned}&B(G(x, (\tilde{\kappa}_{SD}(x), w_D))) - B(x) \\ &\qquad = B\big(g(x) + g_w(x) w_D\big) - B(x) \\ &\qquad = B(g(x)) + \widehat{B}_{Lw}(x) w_D - B(x) \\ &\qquad = \omega_D(x) - \alpha_D(B(x) - \rho(\bar{w})) - |\widehat{B}_{Lw}(x)|\bar{w} + \widehat{B}_{Lw}(x) w_D \\ &\qquad \leq -\alpha_D(B(x) - \rho(\bar{w})) + |\widehat{B}_{Lw}(x)|(|w_D| - \bar{w})\end{aligned}$$

and if $|w_D| \leq \bar{w}$, we have

$$B(G(x, (\tilde{\kappa}_{SD}(x), w_D))) - B(x) \leq -\alpha_D(B(x) - \rho(\bar{w}))$$

- if $\widehat{B}_{Lu}(x) \neq 0$,

$$
\begin{aligned}
B(G(x,(\tilde{\kappa}_{SD}(x),&w_D))) - B(x) \\
&= B\big(g(x) + g_u(x)\tilde{\kappa}_{SD}(x) + g_w(x)w_D\big) - B(x) \\
&\leq B(g(x)) - \omega_D(x) - \sqrt{\omega_D^2(x) + |\widehat{B}_{Lu}(x)|^4} + \widehat{B}_{Lw}(x)w_D - B(x) \\
&\leq -\alpha_D(B(x) - \rho(|w_D|)) - |\widehat{B}_{Lw}(x)|\bar{w} + \widehat{B}_{Lw}(x)w_D \\
&\leq -\alpha_D(B(x) - \rho(|w_D|)) + |\widehat{B}_{Lw}(x)|(|w_D| - \bar{w})
\end{aligned}
$$

and if $|w_D| \leq \bar{w}$, we have

$$B(G(x,(\tilde{\kappa}_{SD}(x), w_D))) - B(x) \leq -\alpha_D(B(x) - \rho(\bar{w}))$$

Finally, we invoke Theorem 8.1.6 to establish $\bar{w}$-small-input input-to-state safety of $\mathcal{H}_\kappa$ with respect to the disturbance w and the set K. □

8.3.2 Input-to-tate Safety QP Filter

Given a nominal feedback law $\bar{\kappa}$, we endow a system $\mathcal{H}$ with an input-to-state safety property by solving a quadratic program (QP) problem in terms of an ISSf control barrier function B.

Let $\mathcal{V}$ be an open set containing an open neighborhood of $K_d(\bar{w})$. Given $\alpha_C \geq 0$, we define

$$\omega_C(x) := L_{f+f_u\bar{\kappa}_C}B(x) + |L_{f_w}(x)|\rho^{-1}(B(x)) + \alpha_C B(x) \tag{8.22}$$

for all $x \in (\mathcal{V} \setminus K_d(\bar{w})) \cap \Pi(C)$ and introduce the following QP:

$$
\begin{aligned}
\kappa_{C_{QP}}(x) = \underset{v\in\mathbb{R}^{m_{C_u}}}{\arg\min} \quad & |v|^2 \\
\text{subject to} \quad & L_{f_u}B(x)v \leq -\omega_C(x)
\end{aligned} \tag{8.23}
$$

Since the cost function and constraint defining (8.23) are both convex and continuously differentiable with respect to the decision variable v, (8.23) is a convex optimization problem, and the Karush-Kuhn-Tucker (KKT) conditions [83, Sec. 5.5.3] provide necessary and sufficient[3] conditions for optimality. In particular, for an optimal solution

[3] An additional condition is necessary for the KKT conditions to be necessary and sufficient conditions for optimality. One such condition is *Slater's condition* [83, Sec. 5.2.3], which, in our setting, holds at x only if there exists $v \in \mathbb{R}^{m_{C_u}}$ such that $L_{f_u}B(x)v \leq -\omega_C(x)$. Clearly, this condition holds for (8.23) as it is feasible for all $x \in (\mathcal{V} \setminus K_d(\bar{w})) \cap \Pi(C)$, because it is a convex program with a single affine constraint.

$x \mapsto \kappa_{C_{QP}}(x)$ to (8.23), there exists $\theta^* : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$2\kappa_{C_{QP}}(x) + \theta^*(x) L_{f_u} B(x)^\top = 0, \tag{8.24}$$

$$\theta^*(x) \left(L_{f_u} B(x) \kappa_{C_{QP}}(x) + \omega_C(x) \right) = 0, \tag{8.25}$$

$$L_{f_u} B(x) \kappa_{C_{QP}}(x) \leq -\omega_C(x) \tag{8.26}$$

We consider the following two cases:

- If, for $x \in (\mathcal{V} \setminus K_d(\bar{w})) \cap \Pi(C)$, we have that the constraint is not active, namely

$$L_{f_u} B(x) \kappa_{C_{QP}}(x) < -\omega_C(x)$$

  then, from from (8.25) it follows that $\theta^*(x) = 0$; thus, from (8.24) we have that $\kappa_{C_{QP}}(x) = 0$.

- If, for $x \in (\mathcal{V} \setminus K_d(\bar{w})) \cap \Pi(C)$, we have that the constraint is active, that is,

$$L_{f_u} B(x) \kappa_{C_{QP}}(x) = -\omega_C(x)$$

  then, from (8.24)-(8.25) we have that

$$\begin{bmatrix} 2I & L_{f_u} B(x)^\top \\ L_{f_u} B(x) & 0 \end{bmatrix} \begin{bmatrix} \kappa_{C_{QP}}(x) \\ \theta^*(x) \end{bmatrix} = \begin{bmatrix} 0 \\ -\omega_C(x) \end{bmatrix}$$

  and using block matrix inversion, it follows that

$$\kappa_{C_{QP}}(x) = -\frac{\omega_C(x)}{|L_{f_u} B(x)|^2} L_{f_u} B(x)$$
$$\theta^*(x) = \frac{\omega_C(x)}{2|L_{f_u} B(x)|^2}$$

Thus, by combining the two cases[4], the closed-form solution to (8.23) is given by

$$\kappa_{C_{QP}}(x) := \begin{cases} -\dfrac{\max\{0, \omega_C(x)\}}{|L_{f_u} B(x)|^2} L_{f_u} B(x) & \text{if } L_{f_u} B(x) \neq 0 \\ 0 & \text{if } L_{f_u} B(x) = 0. \end{cases} \tag{8.27}$$

Similar to Assumption 8.3.2, to characterize the effect of the QP filter and the disturbance in the safety conditions at jumps, we impose the following assumption.

[4]Notice that when the constraint in (8.23) is not active for each $x \in (\mathcal{V} \setminus K_d(\bar{w})) \cap \Pi(C)$, namely $L_{f_u} B(x) \kappa_{C_{QP}}(x) < -\omega_C(x)$, we have that $\kappa_{C_{QP}}(x) = 0$. Thus, from (8.26) it follows that $\omega_C(x) \leq 0$. When the constraint is active, it must follow that $\omega_C(x) > 0$.

Assumption 8.3.5. *Given a feedback law $\kappa = (\bar{\kappa}_D + \widehat{\kappa}_D, \bar{\kappa}_C + \widehat{\kappa}_C)$, a system $\mathcal{H}_\kappa = (C_\kappa, F_\kappa, D_\kappa, G_\kappa)$ as in* (8.2)*, and a function $B : \mathbb{R}^n \to \mathbb{R}$, suppose there exist functions $\widehat{B}_{Lu} : \mathbb{R}^n \to \mathbb{R}^{m_{D_u}}$ and $\widehat{B}_{Lw} : \mathbb{R}^n \to \mathbb{R}^{m_{D_w}}$ such that, for all $(x, (u_D, w_D)) \in D$,*

$$\begin{aligned} B\left(G(x, u_D)\right) &= B\big(g(x) + g_u(x)\kappa_D(x) + g_w(x)w_D\big) \\ &= B\big(g(x) + g_u(x)\bar{\kappa}_D(x)\big) + \widehat{B}_{Lu}(x)\widehat{\kappa}_D(x) + \widehat{B}_{Lw}(x)w_D \end{aligned} \tag{8.28}$$

Similarly, given $\alpha_D \in [0, 1]$, under Assumption 8.3.5, we define

$$\omega_D(x) := B\big(g(x) + g_u(x)\bar{\kappa}_D(x)\big) - B(x) + |\widehat{B}_{Lw}(x)|\bar{w} + \alpha_D(B(x) - \rho(\bar{w})) \tag{8.29}$$

for all $x \in \Pi(D) \cap K_d(\bar{w})$, and introduce the following QP:

$$\begin{aligned} \kappa_{D_{QP}}(x) = \underset{v \in \mathbb{R}^{m_{D_u}}}{\arg\min} \quad & |v|^2 \\ \text{subject to} \quad & \widehat{B}_{Lu}(x)v \leq -\omega_D(x) \end{aligned} \tag{8.30}$$

where the KKT conditions allow to express the solution explicitly as

$$\kappa_{D_{QP}}(x) := \begin{cases} -\dfrac{\max\{0, \omega_D(x)\}}{|\widehat{B}_{Lu}(x)|^2}\widehat{B}_{Lu}(x) & \text{if } \widehat{B}_{Lu}(x) \neq 0 \\ 0 & \text{if } \widehat{B}_{Lu}(x) = 0. \end{cases} \tag{8.31}$$

With the QP safety filters (8.23) and (8.30) we establish the following result.

Theorem 8.3.6. (ISSf filter via CBF) *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ as in* (8.1)*, an nominal feedback law $\bar{\kappa} = (\bar{\kappa}_C, \bar{\kappa}_D)$, and a closed set $K \subset \mathbb{R}^n$. Suppose there exists an ISSf-CBF B for $\mathcal{H}$ with respect to $(K, K_d(\bar{w}))$, where $K_d(\bar{w})$ is defined as in* (8.6) *for some $\rho \in \mathcal{K}_\infty$ and $\bar{w} > 0$, such that Assumption 8.3.5 holds. Let $\mathcal{V}$ be an open set containing $K_d(\bar{w})$ and suppose B is continuously differentiable on an open neighborhood of $(\mathcal{V} \setminus K_d(\bar{w})) \cap \Pi(\overline{C})$. The feedback law $\kappa = (\bar{\kappa}_C + \kappa_{C_{QP}}, \bar{\kappa}_D + \kappa_{D_{QP}})$, with $\kappa_{C_{QP}}$ as in* (8.27) *and $\kappa_{D_{QP}}$ as in* (8.31)*, renders the resulting closed-loop system $\mathcal{H}_\kappa = (C_\kappa, F_\kappa, D_\kappa, G_\kappa)$ as in* (8.2) *$\bar{w}$-small-input ISSf with respect to the disturbance w and the set K (see Remark 8.1.3), with $\alpha_C \geq 0, \alpha_D \in [0, 1]$.*

Proof. For each $(x, w_C) \in C_\kappa : x \in \mathcal{V} \setminus K_d(\bar{w})$

$$\begin{aligned}
\langle \nabla B(x), &F(x, (\bar{\kappa}_C(x) + \kappa_{C_{QP}}(x), w_C) \rangle \\
&= L_{f+f_u\bar{\kappa}_C} B(x) + L_{f_w} B(x) w_C - \max\{0, \omega_C(x)\} \\
&= -\alpha_C B(x) + \omega_C(x) - \max\{0, \omega_C(x)\} + L_{f_w} B(x) w_C - |L_{f_w} B(x)| \rho^{-1}(B(x)) \\
&\leq -\alpha_C B(x) + \min\{\omega_C(x), 0\} + |L_{f_w} B(x)| \big(|w_C| - \rho^{-1}(B(x))\big) \\
&\leq -\alpha_C B(x) + |L_{f_w} B(x)| \big(|w_C| - \rho^{-1}(B(x))\big) \\
&\leq -\alpha_C B(x) + |L_{f_w} B(x)| \big(|w_C| - \bar{w}\big)
\end{aligned}$$

For $|w_C| \leq \bar{w}$ we have

$$\langle \nabla B(x), F(x, (\bar{\kappa}_C(x) + \kappa_{C_{QP}}(x), w_C)) \rangle \leq -\alpha_C B(x) \tag{8.32}$$

Similarly, for all $(x, w_D) \in D_\kappa : x \in K_d(\bar{w})$

$$\begin{aligned}
B\big(G(x, &(\bar{\kappa}_D(x) + \kappa_{D_{QP}}(x), w_D)\big) - B(x) \\
&= B\big(g(x) + g_u(x)\bar{\kappa}_D(x)\big) + \widehat{B}_{Lw}(x) w_D - B(x) - \max\{0, \omega_D(x)\} \\
&= -\alpha_D (B(x) - \rho(\bar{w})) + \omega_D(x) - \max\{0, \omega_D(x)\} + \widehat{B}_{Lw}(x) w_D - |\widehat{B}_{Lw}(x)| \bar{w} - B(x) \\
&\leq -\alpha_D (B(x) - \rho(\bar{w})) + \min\{\omega_D(x), 0\} + |\widehat{B}_{Lw}(x)| \big(|w_D| - \bar{w}\big) \\
&\leq -\alpha_D (B(x) - \rho(\bar{w})) + |\widehat{B}_{Lw}(x)| \big(|w_D| - \bar{w}\big)
\end{aligned}$$

and if $|w_D| \leq \bar{w}$, then

$$B\big(G(x, (\bar{\kappa}_D(x) + \kappa_{D_{QP}}(x), w_D))\big) - B(x) \rangle \leq -\alpha_D (B(x) - \rho(\bar{w})) \tag{8.33}$$

Finally, with (8.32) and (8.33) we invoke Theorem 8.1.6. to establish $\bar{w}$-small-input input-to-state safety of $\mathcal{H}_\kappa$ with respect to the disturbance w and the set K. □

Remark 8.3.7. (Noncompleteness of solutions under QP control) *Notice that the optimization in* (8.23) *and* (8.30) *is carried over* $\mathbb{R}^{m_{C_u}}$ *and* $\mathbb{R}^{m_{D_u}}$*, respectively, instead of over the constrain sets* $\Psi_\star$*,* $\star \in \{C, D\}$*, as in Definition 8.3.1. This allows to compute the closed-form safeguarding feedback law* $\widehat{\kappa} = (\kappa_{C_{QP}}, \kappa_{D_{QP}})$*, which may potentially lead to maximal solutions to* $\mathcal{H}_\kappa$ *that are not complete. The "pre" term in the results accounts for this trade-off. We refer the reader to [21, Prop. 2.34] for sufficient conditions to assure completeness of solutions for the hybrid closed-loop system* $\mathcal{H}_\kappa$*.*

Theorem 8.3.8. (Half Sontag Control) *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ as in (8.1), an uncertified nominal feedback law $\bar{\kappa} = (\bar{\kappa}_C, \bar{\kappa}_D)$, and a closed set $K \subset \mathbb{R}^n$, suppose there exist an ISSf-CBF B for $\mathcal{H}$ with respect to $(K, K_d(\bar{w}))$, where $K_d(\bar{w})$ is defined as in (8.6) for some $\rho \in \mathcal{K}_\infty$ and $\bar{w} > 0$, such that Assumption 8.3.5 holds. Let $\mathcal{B}$ be an open set containing $K_d(\bar{w})$ and suppose B is continuously differentiable on an open neighborhood of $(\mathcal{V} \setminus K_d(\bar{w})) \cap \Pi(\overline{C})$. The feedback law $\tilde{\kappa} = (\bar{\kappa}_C + \frac{1}{2}\tilde{\kappa}_{SC}, \bar{\kappa}_D + \frac{1}{2}\tilde{\kappa}_{SD})$, with $\tilde{\kappa}_{SC}$ as in (8.20) and $\tilde{\kappa}_{SD}$ as in (8.21), renders the resulting closed-loop system $\mathcal{H}_\kappa = (C_\kappa, F, D_\kappa, G)$ as in (8.2) $\bar{w}$-small-input ISSf with respect to the disturbance w and the set K (see Remark 8.1.3), with $\alpha_C \geq 0$ and $\alpha_D \in [0,1]$. In addition, for all $x \in (\mathcal{V} \setminus K_d(\bar{w})) \cap \Pi(C)$, the feedback law $\frac{1}{2}\tilde{\kappa}_{SC}$ is the pointwise minimizer of*

$$\begin{aligned} \underset{v \in \mathbb{R}^{m_{Cu}}}{\arg\min} \quad & |v|^2 \\ \textit{subject to} \quad & L_{f_u}B(x)v \leq \frac{1}{2}|L_{f_u}B(x)|^2 \kappa_{SC}(x) \end{aligned} \tag{8.34}$$

Similarly, for all $x \in \Pi(D) \cap K_d(\bar{w})$, the feedback law $\frac{1}{2}\tilde{\kappa}_{SD}$ is the pointwise minimizer of

$$\begin{aligned} \underset{v \in \mathbb{R}^{m_{Du}}}{\arg\min} \quad & |v|^2 \\ \textit{subject to} \quad & \widehat{B}_{Lu}(x)v \leq \frac{1}{2}|\widehat{B}_{Lu}(x)|^2 \kappa_{SD}(x) \end{aligned} \tag{8.35}$$

Proof. To show that $\Pi(C) \ni x \mapsto \frac{1}{2}\tilde{\kappa}_{SC}(x)$ as in (8.20) is the pointwise minimizer of (8.34), Karush-Kuhn-Tucker (KKT) conditions [83, Sec. 5.5.3] provide necessary and sufficient conditions for optimality. Namely, it is sufficient to show that there exists some $\theta_C^* : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$\tilde{\kappa}_{SC}(x) + \theta_C^*(x) L_{f_u}B(x) = 0, \tag{8.36}$$

$$\theta_C^*(x) L_{f_u}B(x)\tilde{\kappa}_{SC}(x) = \theta_C^*(x)|L_{f_u}B(x)|^2 \kappa_{SC}(x), \tag{8.37}$$

$$L_{f_u}B(x)\tilde{\kappa}_{SC}(x) \leq |L_{f_u}B(x)|^2 \kappa_{SC}(x). \tag{8.38}$$

Using (8.40), it readily follows that (8.37) and (8.38) hold, and it can easily be shown that (8.36) are satisfied under

$$\theta_C^*(x) := \begin{cases} -\kappa_{SC}(x) & \text{if} \quad L_{f_u}B(x) \neq 0 \\ p_C & \text{if} \quad L_{f_u}B(x) = 0 \end{cases} \tag{8.39}$$

where $p_C > 0$ is any arbitrary value. A similar approach can be used to show that $\frac{1}{2}\tilde{\kappa}_{SD}$ as in (8.21) is the pointwise minimizer of (8.35) with

$$\theta_D^*(x) := \begin{cases} -\kappa_{SD}(x) & \text{if} \quad \widehat{B}_{Lu}(x) \neq 0 \\ p_D & \text{if} \quad \widehat{B}_{Lu}(x) = 0 \end{cases} \tag{8.40}$$

where $p_D > 0$ is any arbitrary value. The resulting closed-loop system $\mathcal{H}_\kappa$ with safeguarding feedback law $\widehat{\kappa} = \frac{1}{2}(\tilde{\kappa}_{SC}, \tilde{\kappa}_{SD})$ satisfies, for each $(x, w_C) \in C_\kappa : x \in \mathcal{V} \setminus K_d(\bar{w})$,

$$\begin{aligned}
&\left\langle \nabla B(x), F\left(x, \left(\bar{\kappa}_C(x) + \frac{1}{2}\tilde{\kappa}_{SC}(x), w_C\right)\right) \right\rangle \\
&= L_{f+f_u\bar{\kappa}_C} B(x) - \frac{1}{2}\omega_C(x) + L_{f_w}B(x)w_C - \frac{1}{2}\sqrt{\omega_C^2(x) + |L_{f_u}B(x)|^4} \\
&\leq \frac{1}{2}\Big(L_{f+f_u\bar{\kappa}_C}B(x) - \alpha_C B(x) - |L_{f_w}B(x)|\rho^{-1}(B(x))\Big) \\
&\qquad\qquad - \frac{1}{2}\sqrt{\omega_C^2(x) + |L_{f_u}B(x)|^4} + L_{f_w}B(x)w_C \\
&\leq \frac{1}{2}\Big(L_{f+f_u\bar{\kappa}_C}B(x) + \alpha_C B(x) + |L_{f_w}B(x)|\rho^{-1}(B(x))\Big) \\
&\quad + L_{f_w}B(x)w_C - \alpha_C B(x) - |L_{f_w}B(x)|\rho^{-1}(B(x)) - \frac{1}{2}\sqrt{\omega_C^2(x) + |L_{f_u}B(x)|^4} \\
&\leq \frac{1}{2}\omega_C(x) - \frac{1}{2}\sqrt{\omega_C^2(x) + |L_{f_u}B(x)|^4} - \alpha_C B(x) - |L_{f_w}B(x)|\rho^{-1}(B(x)) + |L_{f_w}B(x)||w_C| \\
&\leq -\frac{1}{2}\Big(- \omega_C(x) + \sqrt{\omega_C^2(x) + |L_{f_u}B(x)|^4}\Big) - \alpha_C B(x) + |L_{f_w}B(x)|(|w_C| - \bar{w})
\end{aligned}$$

Given that $\omega_C(x) \leq \sqrt{\omega_C^2(x) + |L_{f_u}B(x)|^4}$ because of (8.12a), for $|w_C| \leq \bar{w}$ we have

$$\left\langle \nabla B, F\left(x, \left(\bar{\kappa}_C(x) + \frac{1}{2}\tilde{\kappa}_{SC}(x), w_C\right)\right) \right\rangle \leq -\alpha_C B(x) \tag{8.41}$$

For each $(x, w_D) \in D_\kappa : x \in K_d(\bar{w})$

$$
\begin{aligned}
&B\Big(G\Big(x, \Big(\bar{\kappa}_D(x) + \frac{1}{2}\tilde{\kappa}_{SD}(x), w_D\Big)\Big)\Big) - B(x) \\
&\quad = B\Big(g(x) + g_u(x)(\bar{\kappa}_D(x) + \frac{1}{2}\tilde{\kappa}_{SD}(x)) + g_w(x)w_D\Big) - B(x) \\
&\quad \leq B(g(x)+g_u(x)\bar{\kappa}_D(x)) - \frac{1}{2}\omega_D(x) + \widehat{B}_{Lw}(x)w_D - \frac{1}{2}\sqrt{\omega_D^2(x) + |\widehat{B}_{Lu}(x)|^4} - B(x) \\
&\quad \leq \frac{1}{2}\Big(B(g(x) + g_u(x)\bar{\kappa}_D(x)) - \alpha_D(B(x) - \rho(\bar{w})) - |\widehat{B}_{Lw}(x)|\bar{w}\Big) \\
&\qquad\qquad - \frac{1}{2}\sqrt{\omega_D^2(x) + |\widehat{B}_{Lu}(x)|^4} + \widehat{B}_{Lw}(x)w_D - \frac{1}{2}B(x) \\
&\quad \leq \frac{1}{2}\Big(B(g(x) + g_u(x)\bar{\kappa}_D(x)) - B(x) + |\widehat{B}_{Lw}(x)|\bar{w} + \alpha_D(B(x) - \rho(\bar{w}))\Big) \\
&\qquad - \frac{1}{2}\sqrt{\omega_D^2(x) + |\widehat{B}_{Lu}(x)|^4} - \alpha_D(B(x) - \rho(\bar{w})) + |\widehat{B}_{Lw}(x)||w_D| - |\widehat{B}_{Lw}(x)|\bar{w} \\
&\quad \leq -\frac{1}{2}\Big(-\omega_D(x) + \sqrt{\omega_D^2(x) + |\widehat{B}_{Lu}(x)|^4}\Big) + |\widehat{B}_{Lw}(x)|(|w_D| - \bar{w}) \\
&\qquad\qquad\qquad - \alpha_D(B(x) - \rho(\bar{w}))
\end{aligned}
$$

Given that $\omega_D(x) \leq \sqrt{\omega_D^2(x) + |\widehat{B}_{Lu}(x)|^4}$ because of (8.13a), for $|w_D| \leq \bar{w}$ we have

$$
B\left(G\left(x, \left(\bar{\kappa}_D(x) + \frac{1}{2}\tilde{\kappa}_{SD}(x), w_D\right)\right)\right) - B(x) \leq -\alpha_D(B(x) - \rho(\bar{w})) \tag{8.42}
$$

Finally[5], with (8.41) and (8.42) we invoke Theorem 8.1.6 to establish $\bar{w}$-small-input input-to-state safety of $\mathcal{H}_\kappa$ with respect to the disturbance w and the set K.

□

8.4 Inverse-Optimal Safety Filters

Given that the control input u defined in terms of a safeguarding feedback law $\widehat{\kappa}$ aims to keep state trajectories to $\mathcal{H}$ from the set K close to K, but the disturbance w seeks to prevent it, we formulate a zero-sum hybrid game that captures such a setting. For this game, we study the following inverse optimality problem: given a feedback law κ, which is the sum of a nominal feedback law $\bar{\kappa}$ and a safeguarding feedback law $\widehat{\kappa}$, that renders $\mathcal{H}_\kappa$ $\bar{w}$-small-input input-to-state safe with respect to the disturbance w and the set K, determine the cost functional that makes the feedback control action κ optimal.

[5]The cases where $L_{f_u}B(x) = 0$ for each $x \in \Pi(C)$ and $\widehat{B}_{Lu}(x) = 0$ for each $x \in \Pi(D)$ follow the approach in the proof of Theorem 8.3.4.

For starters, following Chapter 3, we formulate a zero-sum hybrid game. Given $\xi \in K$, an input action $(u,w) = ((u_C,u_D),(w_C,w_D)) \in \mathcal{U}\times\mathcal{W}$, the stage cost for flows $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$, the stage cost for jumps $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, and the terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, we define the cost associated to the solution $(\phi,(u,w))$ to $\mathcal{H}$ from ξ as as

$$\begin{aligned}\mathcal{J}(\xi,(u,w)) := &\sum_{j=0}^{\sup_j \operatorname{dom}\phi} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j),(u_C(t,j),w_C(t,j)))dt \\ &+ \sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} L_D(\phi(t_{j+1},j),(u_D(t_{j+1},j),w_D(t_{j+1},j))) \\ &+ \limsup_{\substack{t+j \to \sup_t \operatorname{dom}\phi + \sup_j \operatorname{dom}\phi \\ (t,j)\in \operatorname{dom}\phi}} q(\phi(t,j))\end{aligned} \tag{8.43}$$

where $t_{\sup_j \operatorname{dom}\phi+1} := \sup_t \operatorname{dom}\phi$ defines the upper limit of the last integral, and $\{t_j\}_{j=0}^{\sup_j \operatorname{dom}\phi}$ is a nondecreasing sequence associated to the definition of the hybrid time domain of ϕ; see Definition 2.2.2. The terminal cost in (8.43) is evaluated at the value of the state trajectory ϕ at the terminal time via the third term therein.

Given a system $\mathcal{H} = (C,F,D,G)$ as in (8.1), a closed set $K \subset \mathbb{R}^n$, an ISSf-CBF B for $\mathcal{H}$ with respect to $(K,\ K_d(\bar{w}))$, where $K_d(\bar{w})$ is defined as in (8.6) for some $\rho \in \mathcal{K}_\infty$, $\bar{w} > 0$, a nominal feedback law $\bar{\kappa}$, and $\xi \in K$, we consider the following optimization problem:

$$\underset{\substack{u \\ u=(u,w)\in\mathcal{U}_{\mathcal{H}}(\bar{\kappa},\rho(\bar{w}))}}{\text{minimize}}\ \underset{w}{\text{maximize}}\ \mathcal{J}(\xi,(u,w)) \tag{8.44}$$

where $\mathcal{U}_{\mathcal{H}}(\bar{\kappa},\rho(\bar{w})) := \{u \in \mathcal{U} : \exists \widehat{\kappa}, (\phi,(u,w)) \in \mathcal{S}_{\mathcal{H}}(\xi),$
$\phi(t,j) \in K_d(\bar{w})$ for all $(t,j) \in \operatorname{dom}\phi, \operatorname{dom}\phi \ni (t,j) \mapsto u(t,j) = \bar{\kappa}(\phi(t,j)) + \widehat{\kappa}(\phi(t,j))\}$.

Definition 8.4.1. (Value function) *Given $\xi \in K$ and a nominal feedback law $\bar{\kappa}$, the value function at ξ is given by*

$$\mathcal{J}^*(\xi) := \min_{\substack{u \\ u=(u,w)\in\mathcal{U}_{\mathcal{H}}(\bar{\kappa},\rho(\bar{w}))}} \max_{w} \mathcal{J}(\xi,(u,w)) \tag{8.45}$$

8.4.1 Inverse Optimal Non-QP Control

Given a nominal uncertified feedback law $\bar{\kappa} = (\bar{\kappa}_C, \bar{\kappa}_D)$, the problem of designing a safeguarding feedback law $\widehat{\kappa}$ using a barrier function candidate to keep state trajectories to $\mathcal{H}$ starting from the set K close to K under disturbances can be addressed solving a QP problem as in Section 8.3.2. This approach is myopic [65] because it may sacrifice future performance to guarantee a desired behavior at the current time. To compensate for this limitation, following the ideas in [65], in this section, we propose a non-QP version of the safety filter design via inverse optimality. Specifically, the cost associated to the solution $(\phi, (u, w))$ to $\mathcal{H}$ from ξ is in (8.43) with

$$L_C(x,(u_C,w_C)) := L_{1C}(x) + L_{2C}(x)(u_C - \bar{\kappa}_C(x)) + (u_C - \bar{\kappa}_C(x))^\top R_C(x)(u_C - \bar{\kappa}_C(x)) - \lambda\gamma\left(\frac{|w_C|}{\lambda}\right) \qquad \forall (x,(u_C,w_C)) \in C\colon x \in \mathcal{V}\backslash K_d(\bar{w}) \tag{8.46a}$$

$$L_D(x,(u_D,w_D)) := L_{1D}(x) + L_{2D}(x)(u_D - \bar{\kappa}_D(x)) + (u_D - \bar{\kappa}_D(x))^\top R_D(x)(u_D - \bar{\kappa}_D(x)) - \lambda\gamma\left(\frac{|w_D|}{\lambda}\right) \qquad \forall (x,(u_D,w_D)) \in D\colon x \in K_d(\bar{w}) \tag{8.46b}$$

$$q(x) = B(x) \qquad \forall x \in (\Pi(C) \cup \Pi(D)) \cap \mathcal{V} \tag{8.46c}$$

where $\gamma \in \mathcal{K}_\infty$, $\lambda \in (0,1]$, and $K_d(\bar{w})$ is defined as in (8.6) for some $\rho \in \mathcal{K}_\infty$ and $\bar{w} > 0$, and $\mathcal{V}$ is an open set containing and open neighborhood of $K_d(\bar{w})$. The inverse optimality approach allows us to design the optimal safeguarding feedback law $\widehat{\kappa}$, the stage costs L_{1C}, L_{1D}, and the matrix functions R_C and R_D in (8.46).

Now, we are ready to present our main result to solve the inverse optimality problem for the case of non-QP safety filters.

Theorem 8.4.2. (Non-QP safety filter) *Given the hybrid system $\mathcal{H} = (C, F, D, G)$ as in (8.1), an uncertified nominal feedback law $\bar{\kappa} = (\bar{\kappa}_C, \bar{\kappa}_D)$, and a closed set $K \subset \mathbb{R}^n$, suppose there exists an ISSf-CBF B for $\mathcal{H}$ with respect to $(K, K_d(\bar{w}))$, and an open set $\mathcal{V}$ containing an open neighborhood of $K_d(\bar{w})$, where $K_d(\bar{w})$ is defined as in (8.6) for some $\rho \in \mathcal{K}_\infty$ and $\bar{w} > 0$. In addition, suppose there exist functions $\widehat{B}_{Lu} : \Pi(D) \to \mathbb{R}^{m_{Du}}$, $\widehat{B}_{Lw} : \Pi(D) \to \mathbb{R}^{m_{Dw}}$ and $\widehat{B}_Q : \Pi(D) \to \mathbb{S}^{m_{Du}}_{>0}$ such that, for all $(x, u_D) \in D$,*

$$\begin{aligned} B\left(G(x,u_D)\right) &= B\Big(g(x) + g_u(x)\bar{\kappa}_D(x) + g_w(x)w_D + g_u(x)(u_D - \bar{\kappa}_D(x))\Big) \\ &\le B\big(g(x) + g_u(x)\bar{\kappa}_D(x)\big) + \widehat{B}_{Lu}(x)(u_D - \bar{\kappa}_D(x)) \\ &\quad + (u_D - \bar{\kappa}_D(x))^\top \widehat{B}_Q(x)(u_D - \bar{\kappa}_D(x)) + \widehat{B}_{Lw}(x)w_D \end{aligned} \tag{8.47}$$

and there exist functions $R_C : \Pi(C) \to \mathbb{S}^{m_{Cu}}_{>0}$ *and* $R_D : \Pi(D) \to \mathbb{S}^{m_{Du}}_{>0}$, *and functions* $L_{2C} : \Pi(C) \to \mathbb{R}^{m_{Cw}}$ *and* $L_{2D} : \Pi(D) \to \mathbb{R}^{m_{Dw}}$ *such that for the resulting closed-loop system* $\mathcal{H}_\kappa = (C_\kappa, F, D_\kappa, G)$ *as in* (8.2) *from assigning* u *to the feedback law* $\kappa = (\kappa_C, \kappa_D) = (\bar{\kappa}_C + \widehat{\kappa}_C, \bar{\kappa}_D + \widehat{\kappa}_D)$, *where*

$$\widehat{\kappa}_C(x) := \begin{cases} 0 & \text{if} \quad x \in \Pi(C) \cap K_d(\bar{w}) \\ -\dfrac{1}{2} R_C^{-1}(x)(L_{C2}(x) + L_{f_u} B(x)) & \text{if} \quad x \in \Pi(C) \cap (\mathcal{V} \backslash K_d(\bar{w})) \end{cases} \tag{8.48a}$$

$$\widehat{\kappa}_D(x) := -\frac{1}{2}(R_D(x) + \widehat{B}_Q(x))^{-1}(L_{2D}(x) + \widehat{B}_{Lu}(x)) \qquad \forall x \in \Pi(D) \cap K_d(\bar{w}) \tag{8.48b}$$

the following holds:

$$\begin{aligned} L_{f+L_{f_u}\bar{\kappa}_C} B(x) + L_{f_u} B(x) \widehat{\kappa}_C(x) + \bar{\gamma}(|L_{L_{f_w}} B(x)|) &\leq -\alpha_C B(x) \\ &\forall x \in \Pi(C_\kappa) \cap (\mathcal{V} \setminus K_d(\bar{w})) \end{aligned} \tag{8.49}$$

$$\begin{aligned} B\big(g(x) + g_u(x)\bar{\kappa}_D(x)\big) + \widehat{B}_{Lu}(x)\widehat{\kappa}_D(x) - B(x) + \bar{\gamma}\big(|\widehat{B}_{Lw}(x)|\big) &\leq -\alpha_D(B(x) - \rho(\bar{w})) \\ &\forall x \in \Pi(D_\kappa) \cap K_d(\bar{w}) \end{aligned} \tag{8.50}$$

where $\alpha_C \geq 0$, $\alpha_D \in [0, 1]$, $\gamma \in \mathcal{K}_\infty$ *has a derivative that is also a class-*$\mathcal{K}_\infty$ *function, and* $\bar{\gamma} \in \mathcal{K}_\infty$ *is defined as in* (7.44). *Then, for any* $\xi \in K$, κ *solves the inverse optimal problem by minimizing the cost* $\mathcal{J}$ *as in* (8.43) *with*

$$\begin{aligned} L_{1C}(x) := -\big(L_{f+L_{f_u}\bar{\kappa}_C} B(x) - \widehat{\kappa}_C(x)^\top R_C(x) \widehat{\kappa}_C(x) + \lambda \bar{\gamma}(|L_{L_{f_w}} B(x)|)\big) \\ \forall x \in \Pi(C_\kappa) \cap \mathcal{V} \end{aligned} \tag{8.51}$$

and

$$\begin{aligned} L_{1D}(x) := -\Big(B(g(x) + g_u(x)\bar{\kappa}_D(x)) - B(x)\Big) \\ + \widehat{\kappa}_D(x)^\top \big(R_D(x) + \widehat{B}_Q(x)\big)\widehat{\kappa}_D(x) - \lambda\bar{\gamma}(|\widehat{B}_{Lw}(x)|) \\ \forall x \in \Pi(D_\kappa) \cap K_d(\bar{w}) \end{aligned} \tag{8.52}$$

Proof. The feedback law $\kappa = (\kappa_C, \kappa_D)$ is obtained by solving[6]

$$0 = \sup_{u_C \in \mathbb{R}^{m_{C_u}}} \left\{ L_C(x,(u_C,w_C)) + L_f B(x) + L_{f_u} B(x) u_C \right\} \qquad \forall (x,w_C) \in \Pi_{u_C}(C) : x \in \mathcal{V} \tag{8.53}$$

and

$$B(x) = \sup_{u_D \in \mathbb{R}^{m_{D_u}}} \Big\{ L_D(x,(u_D,w_D)) + B\big(G(x,(u_D,w_D))\big) \Big\} \qquad \forall (x,w_D) \in \Pi_{u_D}(D) : x \in K_d(\bar{w}) \tag{8.54}$$

Using (8.51) and (8.52) in (8.46a) and (8.46b), respectively, we express the cost $\mathcal{J}$ associated to a solution (ϕ, u) as[7]

$$\begin{aligned}
\mathcal{J}(\xi,(u,w)) = &\sum_{j=0}^{\sup_j \operatorname{dom}\phi} \int_{t_j}^{t_{j+1}} \Big(- L_{f+f_u\bar{\kappa}_C} B(\phi) + \widehat{\kappa}_C(\phi)^\top R_C(\phi) \widehat{\kappa}_C(\phi) \\
&-\lambda\bar{\gamma}(|L_{f_w}B(\phi)|) + L_{2C}(\phi)(u_C - \bar{\kappa}_C(\phi)) + (u_C - \bar{\kappa}_C(\phi))^\top R_C(\phi)(u_C - \bar{\kappa}_C(\phi)) - \lambda\gamma\Big(\frac{|w_C|}{\lambda}\Big)\Big) dt \\
&+ \sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} \Big(- B\big(g(\phi) + g_u(\phi)\bar{\kappa}_D(\phi)\big) + B(\phi) - \lambda\bar{\gamma}(|\widehat{B}_{L2}(x)) \\
&\qquad + \widehat{\kappa}_D(\phi)^\top \big(R_D(\phi) + \widehat{B}_Q(\phi)\big)\widehat{\kappa}_D(\phi) + L_{2D}(\phi)(u_D - \bar{\kappa}_D(\phi)) \\
&\qquad + (u_D - \bar{\kappa}_D(\phi))^\top R_D(\phi)(u_D - \bar{\kappa}_D(\phi)) - \lambda\gamma\Big(\frac{|w_D|}{\lambda}\Big)\Big) \\
&+ \limsup_{\substack{t+j \to \sup_t \operatorname{dom}\phi + \sup_j \operatorname{dom}\phi \\ (t,j) \in \operatorname{dom}\phi}} B(\phi(t,j))
\end{aligned} \tag{8.55}$$

where $\{t_j\}_{j=0}^{\sup_j \operatorname{dom}\phi}$ is a nondecreasing sequence associated to the definition of the hybrid time domain of $(\phi,(u,w))$, and $\widehat{\kappa}_C$ and $\widehat{\kappa}_D$ are as in (8.48a) and (8.48b), respectively.

---

[6] See Remark 8.3.7.

[7] For ease of notation, where needed, we will drop the arguments of the solution $(\phi,(u,w))$, which are (t,j), unless they are specified

For each $j \in \operatorname{dom}_j \phi$ notice that

$$\int_{t_j}^{t_{j+1}} \Big(- L_{f+f_u\bar{\kappa}_C} B(\phi) + \widehat{\kappa}_C(\phi)^\top R_C(\phi)\widehat{\kappa}_C(\phi) - \lambda\bar{\gamma}(|L_{f_w}B(\phi)|) + L_{2C}(\phi)(u_C - \bar{\kappa}_C(\phi)) + (u_C - \bar{\kappa}_C(\phi))^\top R_C(\phi)(u_C - \bar{\kappa}_C(\phi)) - \lambda\gamma\Big(\frac{|w_C|}{\lambda}\Big)\Big)dt$$

$$= -\int_{t_j}^{t_{j+1}} \Big(L_{f+f_u\bar{\kappa}_C} B(\phi) + L_{f_u}B(\phi)(u_C - \bar{\kappa}_C(\phi)) + L_{f_w}B(\phi)w_C\Big)dt$$
$$- \int_{t_j}^{t_{j+1}} \Big(- L_{2C}(\phi)(u_C - \bar{\kappa}_C(\phi)) - L_{f_u}B(\phi)(u_C - \bar{\kappa}_C(\phi)) - (u_C - \bar{\kappa}_C(\phi))^\top R_C(\phi)(u_C - \bar{\kappa}_C(\phi)) - \widehat{\kappa}_C(\phi)^\top R_C(\phi)\widehat{\kappa}_C(\phi)\Big)dt$$
$$- \int_{t_j}^{t_{j+1}} \Big(\lambda\gamma\Big(\frac{|w_C|}{\lambda}\Big) + \lambda\bar{\gamma}(|L_{f_w}B(\phi)|) - L_{f_w}B(\phi)w_C\Big)dt$$
$$= -\int_{t_j}^{t_{j+1}} \frac{dB}{dt}(\phi(t,j))dt$$
$$+ \int_{t_j}^{t_{j+1}} \Big((u_C - \bar{\kappa}_C(\phi))^\top R_C(\phi)(u_C - \bar{\kappa}_C(\phi)) + \widehat{\kappa}_C(\phi)^\top R_C(\phi)\widehat{\kappa}_C(\phi) + \big((L_{2C}(\phi) + L_{f_u}B(\phi))R_C^{-1}(\phi)\big)R_C(\phi)(u_C - \bar{\kappa}_C(\phi))\Big)dt$$
$$- \int_{t_j}^{t_{j+1}} \left(\lambda\gamma\left(\frac{|w_C|}{\lambda}\right) - \lambda\gamma((\gamma')^{-1}(|L_{f_w}B(\phi)|)) + \lambda|L_{f_w}B(\phi)|(\gamma')^{-1}(|L_{f_w}B(\phi)|) - L_{f_w}B(\phi)w_C\right)dt$$
$$= -\int_{t_j}^{t_{j+1}} \frac{dB}{dt}(\phi(t,j))dt$$
$$+ \int_{t_j}^{t_{j+1}} \Big((u_C - \bar{\kappa}_C(\phi))^\top R_C(\phi)(u_C - \bar{\kappa}_C(\phi)) + \widehat{\kappa}_C(\phi)^\top R_C(\phi)\widehat{\kappa}_C(\phi) - 2\widehat{\kappa}_C R_C(\phi)(u_C - \bar{\kappa}_C(\phi))\Big)dt$$
$$- \lambda \int_{t_j}^{t_{j+1}} \Gamma(w_C, \pi_C(\phi))dt$$
$$= -\Big(B(\phi(t_{j+1},j)) - B(\phi(t_j,j))\Big) + \int_{t_j}^{t_{j+1}} (u_C - \bar{\kappa}_C(\phi) - \widehat{\kappa}_C(\phi))^\top R_C(u_C - \bar{\kappa}_C(\phi) - \widehat{\kappa}_C(\phi))dt$$
$$- \lambda \int_{t_j}^{t_{j+1}} \Gamma(w_C, \pi_C(\phi))dt \tag{8.56}$$

where

$$(x,u) \mapsto \Gamma(u,\pi(x)) := \gamma\Big(\frac{|u|}{\lambda}\Big) - \gamma\Big(\frac{|\pi(x)|}{\lambda}\Big) + \gamma'\Big(\frac{|\pi(x)|}{\lambda}\Big)\frac{\pi(x)}{\lambda|\pi(x)|}(\pi(x) - u) \tag{8.57}$$

and

$$x \mapsto \pi_C(x) = \lambda(\gamma')^{-1}(|L_{f_w}B(\phi)|)\frac{L_{f_w}B(\phi)}{|L_{f_{u2}}B(\phi)|} \tag{8.58}$$

In addition, for all $(t, j) \in \operatorname{dom}\phi$ such that $(t, j+1) \in \operatorname{dom}\phi$, and $\phi(t, j) \in K_d(\bar{w})$, we have[7]

$$\begin{aligned}
&- B(g(\phi) + g_u(\phi)\bar{\kappa}_D(\phi)) + B(\phi) + \widehat{\kappa}_D(\phi)^\top(R_D(\phi) + \widehat{B}_Q(\phi))\widehat{\kappa}_D(\phi) - \lambda\bar{\gamma}(|\widehat{B}_{Lu}(x)|) \\
&\qquad + L_{2D}(\phi)(u_D - \bar{\kappa}_D(\phi)) + (u_D - \bar{\kappa}_D(\phi))^\top R_D(\phi)(u_D - \bar{\kappa}_D(\phi)) - \lambda\gamma\Big(\frac{|w_D|}{\lambda}\Big) \\
&= -\Big(B(g(\phi) + g_u(\phi)\bar{\kappa}_D(\phi)) + \widehat{B}_{Lu}(\phi)(u_D - \bar{\kappa}_D(\phi)) \\
&\qquad + (u_D - \bar{\kappa}_D(\phi))^\top \widehat{B}_Q(\phi)(u_D - \bar{\kappa}_D(\phi)) - B(\phi) + \widehat{B}_{Lw}(\phi)w_D\Big) \\
&\quad - \Big(- \widehat{\kappa}_D(\phi)^\top(R_D(\phi) + \widehat{B}_Q(\phi))\widehat{\kappa}_D(\phi) - L_{2D}(\phi)(u_D - \bar{\kappa}_D(\phi)) - \widehat{B}_{Lu}(\phi)(u_D - \bar{\kappa}_D(\phi)) \\
&\qquad - (u_D - \bar{\kappa}_D(\phi))^\top(R_D(\phi) + \widehat{B}_Q(\phi))(u_D - \bar{\kappa}_D(\phi))\Big) \\
&\quad - \Big(\lambda\gamma\Big(\frac{|w_D|}{\lambda}\Big) + \lambda\bar{\gamma}(|\widehat{B}_{L2}(x)|) - \widehat{B}_{Lw}(\phi)w_D\Big) \\
&= -\Big(B(G(\phi, (u_D, w_D))) - B(\phi)\Big) + (u_D - \bar{\kappa}_D(\phi))^\top(R_D(\phi) + \widehat{B}_Q(\phi))(u_D - \bar{\kappa}_D(\phi)) \\
&\quad + \widehat{\kappa}_D(\phi)^\top(R_D(\phi) + \widehat{B}_Q(\phi))\widehat{\kappa}_D(\phi) \\
&\quad + (L_{2D}(\phi)+\widehat{B}_{Lu}(\phi))(R_D(\phi)+\widehat{B}_Q(\phi))^{-1}(R_D(\phi)+\widehat{B}_Q(\phi))(u_D-\bar{\kappa}_D(\phi)) - \lambda\Gamma(w_D, \pi_D(\phi)) \\
&= -\Big(B(G(\phi, (u_D, w_D)))-B(\phi)\Big)+(u_D-\bar{\kappa}_D(\phi)-\widehat{\kappa}_D(\phi))^\top(R_D+\widehat{B}_Q)(u_D-\bar{\kappa}_D(\phi)-\widehat{\kappa}_D(\phi)) \\
&\quad - \lambda\Gamma(w_D, \pi_D(\phi))
\end{aligned} \tag{8.59}$$

where Γ is defined as in (8.57) and

$$x \mapsto \pi_D(x) := \lambda(\gamma')^{-1}(|\widehat{B}_{L2}(\phi)|)\frac{\widehat{B}_{L2}(\phi)}{|\widehat{B}_{L2}(\phi)|} \tag{8.60}$$

Thus, by replacing (8.56) and (8.59) in (8.55), we obtain

$$
\begin{aligned}
\mathcal{J}(\xi,(u,w)) = \sum_{j=0}^{\sup_j \operatorname{dom}\phi} &\Big(- \big(B(\phi(t_{j+1},j)) - B(\phi(t_j,j))\big) \\
&+ \int_{t_j}^{t_{j+1}} (u_C - \bar{\kappa}_C(\phi) - \widehat{\kappa}_C(\phi))^\top R_C(\phi)(u_C - \bar{\kappa}_C(\phi) - \widehat{\kappa}_C(\phi))dt \\
&\qquad - \lambda \int_{t_j}^{t_{j+1}} \Gamma(w_C, \pi_C(\phi))dt \Big) \\
- \sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} &\Big(B(\phi(t_{j+1}, j+1)) - B(\phi(t_{j+1},j)) \\
&- (u_D - \bar{\kappa}_D(\phi) - \widehat{\kappa}_D(\phi))^\top (R_D(\phi) + \widehat{B}_Q(\phi))(u_D - \bar{\kappa}_D(\phi) - \widehat{\kappa}_D(\phi)) \\
&\qquad + \lambda\Gamma(w_D, \pi_D(\phi)) \Big) \\
&+ \limsup_{\substack{t+j \to \sup_t \operatorname{dom}\phi + \sup_j \operatorname{dom}\phi \\ (t,j)\in \operatorname{dom}\phi}} B(\phi(t,j)) \\
= B(\xi) + \sum_{j=0}^{\sup_j \operatorname{dom}\phi} &\int_{t_j}^{t_{j+1}} (u_C - \bar{\kappa}_C - \widehat{\kappa}_C)^\top R_C (u_C - \bar{\kappa}_C - \widehat{\kappa}_C)dt - \lambda \int_{t_j}^{t_{j+1}} \Gamma(w_C, \pi_C(\phi))dt \\
- \lambda\Gamma(w_D, \pi_D(\phi)) + &\sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} (u_D - \bar{\kappa}_D - \widehat{\kappa}_D)^\top (R_D + \widehat{B}_Q)(u_D - \bar{\kappa}_D - \widehat{\kappa}_D)
\end{aligned}
\tag{8.61}
$$

Given that $R_C(x) \in \mathbb{S}_{>0}^{m_{C_u}}$ for all $x \in \Pi(C)$ and $R_D(x), \widehat{B}_Q(x) \in \mathbb{S}_{>0}^{m_{D_u}}$ for all $x \in \Pi(D)$, the cost $\mathcal{J}(\xi,(u,w))$ is minimized under $\kappa = (\kappa_C, \kappa_D) = (\bar{\kappa}_C, \bar{\kappa}_D) + (\widehat{\kappa}_C, \widehat{\kappa}_D)$ and the value function is $\mathcal{J}^*(\xi) = B(\xi)$. Furthermore, since $\Gamma(u, \kappa(x))$ vanishes when $u = \kappa(x)$, and, for any other u, it is positive (see Lemma 7.4.6), the second term in each sum in (8.61) is maximized under $\pi = (\pi_C, \pi_D)$ with values as in (8.58) and (8.60). □

8.4.2 Inverse Optimal QP Filter

In this section, as a special case of Section 8.4.1, we provide a result with sufficient conditions to solve Problem 8.2.1 when the safeguarding controller is expressed as the pointwise solution to a QP.

We consider the problem of finding the min-norm safeguarding feedback law $\kappa_{QP} = (\kappa_{C_{QP}}, \kappa_{D_{QP}})$, with values as in (8.27) and (8.31), that guarantees safety and makes the feedback law $\kappa = \bar{\kappa} + \kappa_{QP}$ deviate as little as possible from the given nominal uncertified feedback law $\bar{\kappa} = (\bar{\kappa}_C, \bar{\kappa}_D)$.

Given $\xi \in \Pi(\overline{C}) \cup \Pi(D)$, an input action $u = (u_C, u_D) \in \mathcal{U}$, a nominal feedback law $\bar{\kappa} = (\bar{\kappa}_C, \bar{\kappa}_D)$, the stage cost for flows $L_C : \mathbb{R}^n \times \mathbb{R}^{m_C} \to \mathbb{R}_{\geq 0}$, the stage cost for jumps $L_D : \mathbb{R}^n \times \mathbb{R}^{m_D} \to \mathbb{R}_{\geq 0}$, and the terminal cost $q : \mathbb{R}^n \to \mathbb{R}$, we define the cost associated to the solution $(\phi, (u, w))$ to $\mathcal{H}$ from ξ, as in (8.43) with

$$L_C(x, (u_C, w_C)) := L_{1C}(x) + R_C(x)|u_C - \bar{\kappa}_C(x)|^2 - \lambda\gamma\Big(\frac{|w_C|}{\lambda}\Big) \qquad \forall (x, (u_C, w_C)) \in C\text{: } x \in \mathcal{V}\backslash K_d(\bar{w}) \tag{8.62a}$$

$$L_D(x, (u_D, w_D)) := L_{1D}(x) + R_D(x)|u_D - \bar{\kappa}_D(x)|^2 - \lambda\gamma\Big(\frac{|w_D|}{\lambda}\Big) \qquad \forall (x, (u_D, w_D)) \in D\text{: } x \in K_d(\bar{w}) \tag{8.62b}$$

$$q(x) = B(x) \qquad \forall x \in (\Pi(C) \cup \Pi(D)) \cap \mathcal{V} \tag{8.62c}$$

where $\gamma \in \mathcal{K}_\infty$, $\lambda \in (0, 1]$, and $K_d(\bar{w})$ is defined as in (8.6) for some $\rho \in \mathcal{K}_\infty$ and $\bar{w} > 0$, and $\mathcal{V}$ is an open set containing and open neighborhood of $K_d(\bar{w})$. We approach the optimization problem in (8.44) as an inverse problem: we design the optimal safeguarding feedback law $\widetilde{\kappa}$, and the stage costs L_{1C} and L_{1D} in (8.62).

Corollary 8.4.3. (QP Safety Filter) *Consider the hybrid system $\mathcal{H}$ as in (8.1), a nominal feedback law $\bar{\kappa} = (\bar{\kappa}_C, \bar{\kappa}_D)$, and a closed set $K \subset \mathbb{R}^n$, suppose there exists an ISSf-CBF B for $\mathcal{H}$ with respect to $(K, K_d(\bar{w}))$, and an open set $\mathcal{V}$ containing an open neighborhood of $K_d(\bar{w})$, where $K_d(\bar{w})$ is defined as in (8.6) for some $\rho \in \mathcal{K}_\infty$ and $\bar{w} > 0$. In addition, suppose there exist functions $\widehat{B}_{Lu} : \mathbb{R}^n \to \mathbb{R}^{m_D}$ and $\widehat{B}_{Lw} : \mathbb{R}^n \to \mathbb{R}^{m_D}$ such that Assumption 8.3.5 holds for all $(x, u_D) \in D$, and for the resulting closed-loop system $\mathcal{H}_\kappa = (C_\kappa, F_\kappa, D_\kappa, G_\kappa)$ as in (8.2) from assigning u to the feedback law $\kappa = \bar{\kappa} + \kappa_{QP}$ with $\kappa_{QP} = (\kappa_{C_{QP}}, \kappa_{D_{QP}})$ with values as in (8.27) and (8.31), with $\omega_C(x)$ as in (8.22) and $\omega_D(x)$ as in (8.29), the following holds:*

$$L_{f + L_{f_u}\bar{\kappa}_C} B(x) + L_{f_u} B(x) \kappa_{C_{QP}}(x) + \bar{\gamma}(|L_{L_{f_w}} B(x)|) \leq -\alpha_C B(x) \qquad \forall x \in \Pi(C_\kappa) \cap (\mathcal{V}\backslash K_d(\bar{w})) \tag{8.63}$$

$$B(G(x, \bar{\kappa}_D(x) + \kappa_{D_{QP}}(x))) - B(x) + \bar{\gamma}(|\widehat{B}_{Lw}(x)|) \leq -\alpha_D(B(x) - \rho(\bar{w})) \qquad \forall x \in \Pi(D_\kappa) \cap K_d(\bar{w}) \tag{8.64}$$

where $\alpha_C \geq 0$, $\alpha_D \in [0,1]$, *and* $\gamma \in \mathcal{K}_\infty$ *has a derivative that is also a class* $\mathcal{K}_\infty$ *function, and* $\bar{\gamma} \in \mathcal{K}_\infty$ *is defined as in* (7.44)*. Then,* κ *renders* $\mathcal{H}_\kappa$ $\bar{w}$*-small-input ISSf with respect to the disturbance* w *and the set* K *and solves, for any* $\xi \in K$*, the inverse optimal problem by minimizing the cost* $\mathcal{J}$ *in* (8.43) *with*

$$L_{1C}(x) := -\Big(L_{f+L_{f_u}\bar{\kappa}_C}B(x) - \frac{1}{4}R_C^{-1}(x)|L_{f_u}B(x)|^2 + \lambda\bar{\gamma}(|L_{L_{f_w}}B(x)|)\Big) \quad \forall x \in \Pi(C_\kappa) \cap \mathcal{V} \tag{8.65}$$

and

$$L_{1D}(x) :=- \Big(B(g(x) + g_u(x)\bar{\kappa}_D(x)) - B(x) - \frac{1}{4}R_D^{-1}(x)|\widehat{B}_{Lu}(x)|^2 + \lambda\bar{\gamma}(|\widehat{B}_{Lw}(x)|)\Big) \quad \forall x \in \Pi(D_\kappa) \cap K_d(\bar{w}) \tag{8.66}$$

where $R_C(x) = \dfrac{1}{2}\dfrac{|L_{f_u}B(x)|^2}{\max\{0, \omega_C(x)\}}$ *and* $R_D(x) = \dfrac{1}{2}\dfrac{|\widehat{B}_{Lu}(x)|^2}{\max\{0, \omega_D(x)\}}$.

Proof. From Theorem 8.3.6, we have that the feedback law $\kappa_{1_{QP}} := (\kappa_{C_{QP}}, \kappa_{D_{QP}})$ with values as in (8.27) and (8.31) renders the closed-loop system $\mathcal{H}_\kappa$ $\bar{w}$-small-input input-to-state safe with respect to the disturbance w and K.

Thanks to (8.65) and (8.66), we denote the cost $\mathcal{J}$ associated to a solution (ϕ, u) to $\mathcal{H}$ as in (8.1) as

$$\begin{aligned}
\mathcal{J}(\xi, (u,w)) = &\sum_{j=0}^{\sup_j \operatorname{dom}\phi} \int_{t_j}^{t_{j+1}} \Big(- L_{f+f_u\bar{\kappa}_C}B(\phi) + \frac{1}{2}\max\{0, \omega_C(\phi)\} \\
&\qquad - \lambda\bar{\gamma}(|L_{f_w}B(\phi)|) + \frac{1}{2}\frac{|L_{f_u}B(\phi)|^2}{\max\{0, \omega_C(\phi)\}}|u_C - \bar{\kappa}_C(\phi)|^2 - \lambda\gamma\Big(\frac{|w_C|}{\lambda}\Big)\Big)dt \\
&+ \sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} \Big(B(\phi) - B\big(g(\phi) + g_u(\phi)\bar{\kappa}_D(\phi)\big) + \frac{1}{2}\max\{0, \omega_D(\phi)\} \\
&\qquad - \lambda\bar{\gamma}(|\widehat{B}_{L2}(\phi)|) + \frac{1}{2}\frac{|\widehat{B}_L(\phi)|^2}{\max\{0, \omega_D(\phi)\}}|u_D - \bar{\kappa}_D(\phi)|^2 - \lambda\gamma\Big(\frac{|w_D|}{\lambda}\Big)\Big) \\
&+ \limsup_{\substack{(t,j)\to \sup_t \operatorname{dom}\phi + \sup_j \operatorname{dom}\phi \\ (t,j)\in \operatorname{dom}\phi}} B(\phi(t,j))
\end{aligned} \tag{8.67}$$

where $\{t_j\}_{j=0}^{\sup_j \operatorname{dom}\phi}$ is a nondecreasing sequence associated to the definition of the hybrid time domain of $(\phi, (u,w))$.

For each $j \in \mathrm{dom}_j\, \phi$ notice that

$$\int_{t_j}^{t_{j+1}} \Big(- L_{f+f_u\bar{\kappa}_C} B(\phi) + \frac{1}{2}\max\{0, \omega_C(\phi)\} - \lambda\bar{\gamma}(|L_{f_w}B(\phi)|) + \frac{1}{2}\frac{|L_{f_u}B(\phi)|^2}{\max\{0,\omega_C(\phi)\}}|u_C - \bar{\kappa}_C(\phi)|^2 - \lambda\gamma\Big(\frac{|w_C|}{\lambda}\Big)\Big)dt$$

$$= -\int_{t_j}^{t_{j+1}} \Big(L_{f+f_u\bar{\kappa}_C}B(\phi) + L_{f_u}B(\phi)(u_C - \bar{\kappa}_C(\phi)) + L_{f_w}B(\phi)w_C\Big)dt$$

$$+\int_{t_j}^{t_{j+1}} \Big(\frac{1}{2}\frac{|L_{f_u}B(\phi)|^2}{\max\{0,\omega_C(\phi)\}}|u_C - \bar{\kappa}_C(\phi)|^2 + \frac{1}{2}\max\{0,\omega_C(\phi)\} + L_{f_u}B(\phi)(u_C - \bar{\kappa}_C(\phi))\Big)dt$$

$$-\int_{t_j}^{t_{j+1}} \Big(\lambda\gamma\Big(\frac{|w_C|}{\lambda}\Big) + \lambda\bar{\gamma}(L_{f_w}B(\phi)|) - L_{f_w}B(\phi)w_C\Big)dt$$

$$= -\int_{t_j}^{t_{j+1}} \frac{dB}{dt}(\phi(t,j))dt + \int_{t_j}^{t_{j+1}} \Big(\frac{1}{2}\frac{|L_{f_u}B(\phi)|^2}{\max\{0,\omega_C(\phi)\}}|u_C - \bar{\kappa}_C(\phi) - \kappa_{C_{QP}}(\phi)|^2\Big)dt$$

$$-\int_{t_j}^{t_{j+1}} \Big(\lambda\gamma\Big(\frac{|w_C|}{\lambda}\Big) - \lambda\gamma\big((\gamma')^{-1}(|L_{f_w}B(\phi)|)\big) + \lambda|L_{f_w}B(\phi)|(\gamma')^{-1}(|L_{f_w}B(\phi)|) - L_{f_w}B(\phi)w_C\Big)dt$$

$$= -\Big(B(\phi(t_{j+1},j) - B(\phi(t_j,j))\Big) + \int_{t_j}^{t_{j+1}} \Big(\frac{1}{2}\frac{|L_{f_u}B(\phi)|^2}{\max\{0,\omega_C(\phi)\}}|u_C - \bar{\kappa}_C(\phi) - \kappa_{C_{QP}}(\phi)|^2\Big)dt - \lambda\int_{t_j}^{t_{j+1}} \Gamma(w_C, \pi_C(\phi))dt \quad (8.68)$$

where Γ is defined as in (8.57) and π_C as in (8.58). In addition, for all $(t,j) \in \mathrm{dom}\,\phi$ such that $(t, j+1) \in \mathrm{dom}\,\phi$, we have

$$- B\big(g(\phi) + g_u(\phi)\bar{\kappa}_D(\phi)\big) + B(\phi) + \frac{1}{2}\max\{0,\omega_D(\phi)\} - \lambda\bar{\gamma}(\widehat{B}_{Lw}(\phi)|) + \frac{1}{2}\frac{|\widehat{B}_{Lu}(\phi)|^2}{\max\{0,\omega_D(\phi)\}}|u_D - \bar{\kappa}_D(\phi)|^2 - \lambda\gamma\Big(\frac{|w_D|}{\lambda}\Big)$$

$$= -\Big(B\big(g(\phi) + g_u(\phi)\bar{\kappa}_D(\phi)\big) + \widehat{B}_{Lu}(\phi)(u_D - \bar{\kappa}_D(\phi)) - B(\phi) + \widehat{B}_{L2}(\phi)w_D\Big)$$

$$+\Big(\frac{1}{2}\frac{|\widehat{B}_L(\phi)|^2}{\max\{0,\omega_D(\phi)\}}|u_D - \bar{\kappa}_D(\phi)|^2 + \widehat{B}_{Lu}(\phi)(u_D - \bar{\kappa}_D(\phi)) + \frac{1}{2}\max\{0,\omega_D(\phi)\}\Big)$$

$$-\Big(\lambda\bar{\gamma}(|\widehat{B}_{Lw}(\phi)|) + \lambda\gamma\Big(\frac{|w_D|}{\lambda}\Big) - \widehat{B}_{Lw}(\phi)w_D\Big)$$

$$= -\big(B(G(\phi,(u_D,w_D))) - B(\phi)\big) + \Big(\frac{1}{2}\frac{|\widehat{B}_L(\phi)|^2}{\max\{0,\omega_D(\phi)\}}|u_D - \bar{\kappa}_D(\phi) - \kappa_{D_{QP}}(\phi)|^2\Big) - \lambda\Gamma(w_D, \pi_D(\phi)) \quad (8.69)$$

where Γ is defined as in (8.57) and π_D as in (8.60). Thus, substituting (8.68) and (8.69) in (8.67), we obtain

$$
\begin{aligned}
\mathcal{J}(\xi,(u,w)) = & \sum_{j=0}^{\sup_j \operatorname{dom}\phi} \Bigg(- \Big(B(\phi(t_{j+1},j) - B(\phi(t_j,j)) \Big) \\
& + \int_{t_j}^{t_{j+1}} \left(\frac{1}{2} \frac{|L_{f_u} B(\phi)|^2}{\max\{0,\omega_C(\phi)\}} |u_C - \bar{\kappa}_C(\phi) - \kappa_{C_{QP}}(\phi)|^2 \right) dt - \lambda \int_{t_j}^{t_{j+1}} \Gamma(w_C, \pi_C(\phi)) dt \Bigg) \\
& - \sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} \Bigg(\Big(B(\phi(t_{j+1}, j+1)) - B(\phi(t_{j+1},j)) \Big) \\
& \qquad - \left(\frac{1}{2} \frac{|\widehat{B}_L(\phi)|^2}{\max\{0,\omega_D(\phi)\}} |u_D - \bar{\kappa}_D(\phi) - \kappa_{D_{QP}}(\phi)|^2 \right) + \lambda \Gamma(w_D, \pi_D(\phi)) \Bigg) \\
= & B(\xi) + \sum_{j=0}^{\sup_j \operatorname{dom}\phi} \int_{t_j}^{t_{j+1}} \left(\frac{1}{2} \frac{|L_{f_u} B(\phi)|^2}{\max\{0,\omega_C(\phi)\}} |u_C - \bar{\kappa}_C(\phi) - \kappa_{C_{QP}}(\phi)|^2 - \lambda \Gamma(w_C, \pi_C(\phi)) \right) dt \\
& + \sum_{j=0}^{\sup_j \operatorname{dom}\phi - 1} \left(\frac{1}{2} \frac{|\widehat{B}_L(\phi)|^2}{\max\{0,\omega_D(\phi)\}} |u_D - \bar{\kappa}_D(\phi) - \kappa_{D_{QP}}(\phi)|^2 - \lambda \Gamma(w_D, \pi_D(\phi)) \right) \quad (8.70)
\end{aligned}
$$

Given that $\max\{0, \omega_\star(\phi(t,j))\} \geq 0$, $(t,j) \in \operatorname{dom}\phi$, for $\star \in \{C, D\}$, the cost $\mathcal{J}(\xi,(u,w))$ is minimized under $\kappa^* = (\kappa_C^*, \kappa_D^*) = (\bar{\kappa}_C + \kappa_{C_{QP}}, \bar{\kappa}_D + \kappa_{D_{QP}})$ and the value function is $\mathcal{J}^*(\xi) = B(\xi)$. □

8.5 Illustrative Example

To illustrate our results, consider the following oscillator with impacts with dynamics given by

$$
\mathcal{H} \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\zeta_C \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_C + \begin{bmatrix} 0 \\ 2 \end{bmatrix} w_C & (x,(u_C,w_C)) \in C \\ \\ \begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \begin{bmatrix} x_1 \\ -\zeta_D x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \eta_u \end{bmatrix} u_D + \begin{bmatrix} 0 \\ \eta_w \end{bmatrix} w_D & (x,(u_D,w_D)) \in D \end{cases} \quad (8.71)
$$

where $\zeta_C, \eta_u, \eta_w \geq 0$, $\zeta_D \in (0,1]$, and

$$
\begin{aligned}
C &:= \left\{ (x,(u_C,w_C)) \in \mathbb{R}^2 \times \mathbb{R}^2 : x_1 \geq 0 \right\} \\
D &:= \left\{ (x,(u_D,w_D)) \in \mathbb{R}^2 \times \mathbb{R}^2 : x_1 = 0,\ x_2 \leq 0 \right\}
\end{aligned}
$$

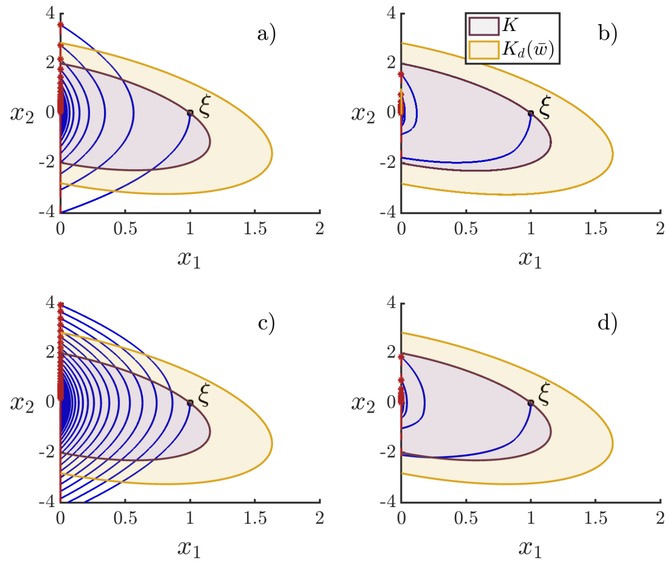

Figure 8.1: Phase portraits for the hybrid oscillator under different settings: a) with no disturbance and no ISSf QP filter, b) with no disturbance and ISSf QP filter, c) with disturbance and no ISSf QP filter, and d) with disturbance and ISSf QP filter.

with $(u_\star, w_\star) \in \mathbb{R}^2$, for $\star \in \{C, D\}$. Now, consider the following nominal feedback law

$$\bar{\kappa}(x) = (\bar{\kappa}_C(x), \bar{\kappa}_D(x)) := \left(-\frac{1}{2} r_C x_2, \frac{\zeta_D x_2}{1 + 2r_D}\right) \tag{8.72}$$

where $r_C > 0$ and $r_D \in \left(-\infty, \frac{1}{2\zeta_D - 2}\right) \cup \left(-\frac{1}{2\zeta_D + 2}, \infty\right)$. Next, consider the following set

$$K = \left\{ x \in \mathbb{R}^2 : x_1 \geq 0 \text{ or } x_2 \leq 0 \;\middle|\; \left(\frac{x_1}{a}\right)^2 + \frac{x_1 x_2}{ab} + \left(\frac{x_2}{b}\right)^2 \leq 1 \right\}$$

for some $a \neq 0$ and $b \neq 0$. Pick $r \mapsto \rho(r) = r^3$ and $\bar{w} = 1$, then

$$K_d(\bar{w}) = \left\{ x \in \mathbb{R}^2 : x_1 \geq 0 \text{ or } x_2 \leq 0 \;\middle|\; \left(\frac{x_1}{a}\right)^2 + \frac{x_1 x_2}{ab} + \left(\frac{x_2}{b}\right)^2 \leq 2 \right\}$$

which are depicted in Figure 8.1 and Figure 8.2. From this choice, notice that

$$B(x) = \left(\frac{x_1}{a}\right)^2 + \frac{x_1 x_2}{ab} + \left(\frac{x_2}{b}\right)^2 - 1 \tag{8.73}$$

is an ISSf barrier function candidate[8] for for $\mathcal{H}$ with respect to $(K, K_d(\bar{w}))$. Then, following Section 8.3.2, we can define the pointwise min-norm QP safeguarding feedback

[8] Notice that the dynamics $\mathcal{H}$ and B satisfy Assumption 8.3.5, for each $\star \in \{u, w\}$, with

$$\widehat{B}_{L\star}(x) = \frac{2\eta_\star x_2}{b^2} \qquad \forall x \in \mathbb{R}^2 : x_1 = 0, x_2 \leq 0.$$

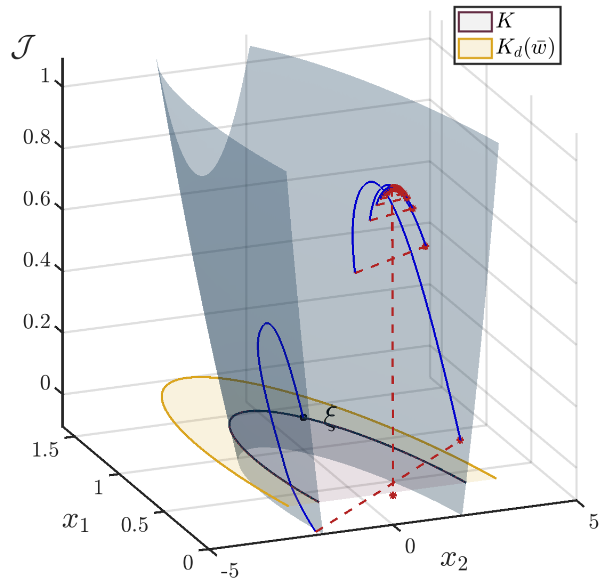

Figure 8.2: Phase portrait for a solution to the hybrid oscillator and plot of the cost. Initial condition ξ. Value function (dark gray) and cost of solution (blue-red) rendered by the saddle-point equilibrium strategy, attaining the value evaluated at the initial condition, $B(\xi)$.

law $\kappa_{QP} = (\kappa_{C_{QP}}, \kappa_{D_{QP}})$ with values as in (8.27) and (8.31) using $\alpha_C = 1$, and $\alpha_D = \frac{1}{2}$. Therefore, from Theorem 8.3.6, we conclude that the feedback law $\kappa = (\hat{\kappa}_C + \kappa_{C_{QP}}, \hat{\kappa}_D + \kappa_{D_{QP}})$ renders the resulting closed-loop system $\mathcal{H}_\kappa$ $\bar{w}$-small-input ISSf with respect to the disturbance w and the set K. In particular, we can see that in Figure 8.1d) the set $K_d(\bar{w})$ is conditionally invariant for $\mathcal{H}_\kappa$ with respect to w and K, as opposed to Figure 8.1c) where the ISSf QP filter is not active. Also, notice that from Figure 8.1a) and Figure 8.1b), when disturbances are not considered, the set K is forward invariant for $\mathcal{H}_\kappa$, as discussed in Remark 8.1.4.

In addition, pick $r \mapsto \gamma(r) = r^2$, and conditions (8.63) and (8.64) are verified numerically. Therefore, invoking Corollary 8.4.3, we have that κ solves, for any $\xi \in K$, the inverse optimal problem, under the maximizing disturbance w, by minimizing the cost $\mathcal{J}$ in (8.43). This can also be seen from Figure 8.2, where $\mathcal{J}$ attains the value of the value function evaluated at the initial condition. In this particular example, we chose the initial condition $\xi \in \partial K$, therefore, it follows that $J^*(\xi) = B(\xi) = 0$.

Part III

Study Cases of Hybrid Systems under Contested Scenarios

Chapter 9

Capture-the-Flag as a Hybrid Game

In this chapter, we derive a comprehensive hybrid system representation for capture-the-flag games and a corresponding zero-sum game formulation following the framework in Chapter 3. First, we introduce the rules of a capture-the-flag game, the derivation of a hybrid system formulation, and the objective functions that define a zero-sum hybrid game. Additionally, we provide a constructive scenario-based switching control design to test the model in a simulation tool where the rules of the game are encoded. By considering the dynamics of capture-the-flag games, this chapter is meant to be a stepping stone to investigate the foundations of multi-player decision making with constraints given by hybrid dynamical systems and for the analysis and design of (sub)optimal control laws for the capture-the-flag hybrid game.

9.1 Capture-The-Flag Games

Capture the flag is a rule-based game allegedly dating back to the book *"Scouting For Boys"*, by [91], where two (or more) teams of players compete against each other, trying to capture the opponents flag and return it to their own base. This game describes a rich family of sub-problems, where team members (representing robots and/or humans) cooperate with each other to maximize their profits, while teams compete to outperform each other. While humans and robots operate in continuous time, the capture-the-flag game is governed by discrete-time events at unknown and possibly periodic time instances, making it a *hybrid dynamical game*.

In this section, we introduce the rules of the game before they are translated into a hybrid system formulation in Section 9.2. While we use specific parameters used in the Aquaticus competition[1] for illustration purposes, the description in Section 9.2 is sufficiently general to model a large class of capture-the-flag games with slight variations. The Aquaticus competition[1] consists of two teams, a blue team (B) and a red team (R). The blue team has $b \in \mathbb{N}$ robots and the red team has $r \in \mathbb{N}$ robots. The k-th robot in either team is modeled as the dynamical system a dynamical system, specified by

$$\dot{p}_k = \begin{bmatrix} \dot{p}_{k,1} \\ \dot{p}_{k,2} \end{bmatrix} = \begin{bmatrix} v_k \cos u_k \\ v_k \sin u_k \end{bmatrix} =: f(p_k, u_k), \tag{9.1}$$

where $p_k = (p_{k,1},\ p_{k,2}) \in \mathbb{R}^2$ is the position, $u_k \in \mathcal{U}_k \subseteq [-\pi, \pi]$ is an input representing the instantaneous heading angle, and v_k is the input velocity.

The k-th robot in team B is denoted k_{B}, with $k \in N_{\mathrm{B}} := \{1, 2, \ldots, b\}$, and the i-th robot in team R is denoted i_{R}, with $i \in N_{\mathrm{R}} := \{1, 2, \ldots, r\}$. Instead of the dynamical system (9.1), we can alternatively consider a differential inclusion

$$\dot{p}_k \in \bar{f}(p_k), \quad \bar{f}(p_k) = \overline{\mathrm{con}}\{f(p_k, u_k)|\ u_k \in \mathcal{U}_k\} \tag{9.2}$$

to suppress the dependence on u_k and to simplify the notation in the following.

The playing field[2] $\mathbb{X} := [-\mathbb{X}_x, \mathbb{X}_x] \times [-\mathbb{X}_y, \mathbb{X}_y] \subset \mathbb{R}^2$ is divided into the regions $\mathbb{X}_{\mathrm{B}} := [-\mathbb{X}_x, -\varepsilon] \times [-\mathbb{X}_y, \mathbb{X}_y]$, $\mathbb{X}_{\mathrm{R}} := [\varepsilon, \mathbb{X}_x] \times [-\mathbb{X}_y, \mathbb{X}_y]$, where $\varepsilon > 0$, and an arbitrarily small *neutral zone* $(-\varepsilon, \varepsilon) \times [-\mathbb{X}_y, \mathbb{X}_y]$. The neutral zone is introduced to ensure that $\mathbb{X}_{\mathrm{B}} \cap \mathbb{X}_{\mathrm{R}} = \emptyset$. This simplifies the presentation in the following by excluding situations that might occur on the zero-measure set defined by the intersection of $\mathbb{X}_{\mathrm{B}}$ and $\mathbb{X}_{\mathrm{R}}$. Each team has a flag that it protects from being captured by the opponent team. The flags' bases are located at $F_{\mathrm{B}} = (-\mathbb{X}_F, 0) \in \mathbb{X}$ and $F_{\mathrm{R}} = (\mathbb{X}_F, 0) \in \mathbb{X}$, where $\mathbb{X}_F > 0$. The setting with six arbitrarily positioned robots is shown in Figure 9.1).

The parameters of the game are the tagging radius $\gamma_c > 0$, the capturing radius $\gamma_F > 0$, and a timeout parameter $\bar{T} > 0$. Without loss of generality, considering the perspective of a robot $k_{\mathrm{B}} \in N_{\mathrm{B}}$ that competes with a robot $i_{\mathrm{R}} \in N_{\mathrm{R}}$, the rules of the game are as follows:

[1] Aquaticus competion: https://oceanai.mit.edu/aquaticus

[2] In Aquaticus, $\mathbb{X}_x := 80$m, $\mathbb{X}_y := 40$m, and $\mathbb{X}_F := 60$m.

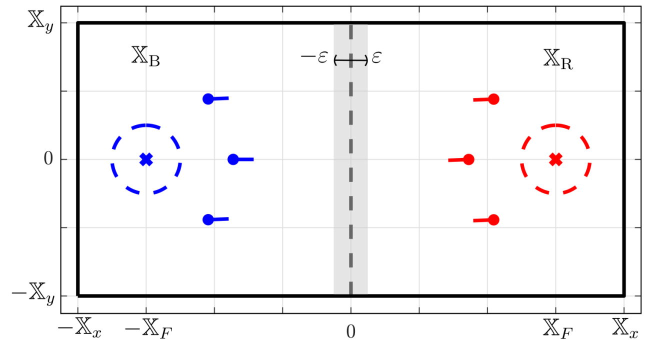

Figure 9.1: Playing field of the Aquaticus competition with three blue and three red robots arbitrarily positioned.

(a) *Tagging:* If k_{B} and i_{R} are in the blue region, namely, $p_{k_{\mathrm{B}}}, p_{i_{\mathrm{R}}} \in \mathbb{X}_{\mathrm{B}}$, and if $|p_{i_{\mathrm{R}}} - p_{k_{\mathrm{B}}}| \leq \gamma_c$, then i_{R} is tagged by k_{B}, i.e., i_{R} is temporarily deactivated and the blue flag is instantaneously returned to F_{B} if i_{R} was carrying it.

(b) *Reactivation:* A robot i_{R}, which is temporarily deactivated by being tagged or by leaving the playing field, which corresponds to $p_{i_{\mathrm{R}}} \notin \mathbb{X}$, needs to satisfy $|p_{i_{\mathrm{R}}} - F_{\mathrm{R}}| \leq \gamma_F$ to be reactivated. A robot leaving the playing field loses the flag if it was carrying it.

(c) *Disabled Tagging:* After tagging i_{R}, k_{B} loses its ability to tag another robot for time $\bar{T} > 0$.

(d) *Flag Capturing:* If k_{B} is not temporarily deactivated, and if $|p_{k_{\mathrm{B}}} - F_{\mathrm{R}}| \leq \gamma_F$, then k_{B} grabs the red flag. Only blue robots can grab the red flag.

(e) Only one robot can carry the flag at a time.

(f) *Flag Return:* If k_{B} satisfies $|p_{k_{\mathrm{B}}} - F_{\mathrm{B}}| \leq \gamma_F$ while carrying the red flag, then the blue team scores and the red flag is instantaneously returned to F_{R}.

It might be counterintuitive why a strategy would lead to a robot leaving the playing field. However, depending on the maneuverability defined through the input constraints $\mathcal{U}_k$, it might be the best strategy to tag a robot close to the boundary of the playing field even if this forces the robot to leave the playing field.

The goal of team B is to capture the red flag and to successfully return it to its own base F_B as many times as possible. Thus, points can be awarded in the following situations, for example:

(g) k_B successfully returns the red team's flag to F_B;

(h) k_B grabs the red team's flag;

(i) k_B tags i_R;

(j) $p_{k_\mathrm{B}} \in \mathbb{X}_\mathrm{R}$ while not being deactivated.

Notice that negative points can be awarded, in particular, in the following situation:

(k) k_B leaves the playing field, i.e., $p_{k_\mathrm{B}} \notin \mathbb{X}$.

In the next section, we translate the descriptive rules of the game to the hybrid system framework in [69].

9.2 Hybrid Systems Game Formulation

In this section, we formulate a hybrid system model of capture-the-flag games.

For the purpose of this chapter, a hybrid dynamical system $\mathcal{H}_s$ as in (2.1) is defined as

$$\mathcal{H}_s \begin{cases} \dot{x} &= F(x, u_\mathrm{B}, u_\mathrm{R}) \qquad x \in C \\ x^+ &\in G(x) \qquad\qquad\quad x \in D \end{cases} \tag{9.3}$$

where x is the state, and u_B and u_R are the inputs of the two teams. Here, we consider a special case where the function G does not depend on the input.

9.2.1 Implementation of the Rules of the Game

From the description of the competition above, to encode the rules of the game given in (a)-(f), we propose a state vector that includes the position of the $k-$th robot, introduced in (9.1). For each $k_\mathrm{B} \in N_\mathrm{B}$ and each $i_\mathrm{R} \in N_\mathrm{R}$, the additional states with associated dynamics are described, without loss of generality, taking the perspective of an arbitrary robot $k_\mathrm{B} \in N_\mathrm{B}$, as follows. In addition to p_{k_B}, the state of the robot k_B has a timer τ_{k_B} to model its tagging ability and two logic variables, $(q_{k_\mathrm{B}}, \eta_{k_\mathrm{B}}) \in \{0,1\}^2$ to model whether k_B is tagged and whether it carries the flag, respectively. The timer decreases according to $\dot{\tau}_{k_\mathrm{B}} = -1$, and takes values in $(-\infty, \bar{T}]$, where $\bar{T} > 0$ is the parameter introduced in Section 9.1 defining the length of the timeout after k_{B} has lost its tagging ability (see rule (c)). Thus, the state of robot k_{B} is

$$x_{k_{\mathrm{B}}} := (p_{k_{\mathrm{B}}}, \tau_{k_{\mathrm{B}}}, q_{k_{\mathrm{B}}}, \eta_{k_{\mathrm{B}}}) \in X_{k_{\mathrm{B}}} := \mathbb{R}^2 \times (-\infty, \bar{T}] \times \{0,1\}^2. \tag{9.4}$$

We model the dynamics of k_{B} as a hybrid system, for which the flow and jump sets are defined to constrain the evolution of the state $p_{k_{\mathrm{B}}}$, the timer $\tau_{k_{\mathrm{B}}}$, and the logic variables $(q_{k_{\mathrm{B}}}, \eta_{k_{\mathrm{B}}})$.

To model the tagging ability of the robot k_{B} (item (c)), we define

$$\tau_{k_{\mathrm{B}}} \begin{cases} > 0: & \text{robot } k_{\mathrm{B}} \text{ does not have tagging ability,} \\ \leq 0: & \text{robot } k_{\mathrm{B}} \text{ has tagging ability.} \end{cases} \tag{9.5}$$

To model whether a robot is tagged or carrying a flag, the logic variables take the following values:

$$q_{k_{\mathrm{B}}} = \begin{cases} 0: & \text{robot } k_{\mathrm{B}} \text{ is active,} \\ 1: & \text{robot } k_{\mathrm{B}} \text{ is deactivated,} \end{cases} \tag{9.6}$$

$$\eta_{k_{\mathrm{B}}} = \begin{cases} 0: & \text{robot } k_{\mathrm{B}} \text{ does not carry the flag,} \\ 1: & \text{robot } k_{\mathrm{B}} \text{ carries the flag.} \end{cases} \tag{9.7}$$

The logic variables remain constant while the other states evolve in continuous time. In addition to the states introduced above for robot k_{B}, a flag state μ_{B} is introduced for each team, as follows:

$$\mu_{\mathrm{B}} = \begin{cases} 0: & \text{the blue flag is not at its base,} \\ 1: & \text{the blue flag is at its base.} \end{cases} \tag{9.8}$$

With these definitions, the rules outlined in Section 9.1 can be summarized through the following update laws.

The dynamics of the 'tagging ability' state: According to rule (c), a robot loses its tagging ability after tagging another robot. Hence, if

$$\begin{matrix} \exists i_{\mathrm{R}} \in N_{\mathrm{R}}, \\ \exists k_{\mathrm{B}} \in N_{\mathrm{B}} \end{matrix} \text{ s.t.} \begin{cases} p_{i_{\mathrm{R}}} \in \mathbb{X}_{\mathrm{B}},\ q_{i_{\mathrm{R}}} = 0, |p_{i_{\mathrm{R}}} - p_{k_{\mathrm{B}}}| \leq \gamma_c, \tau_{k_{\mathrm{B}}} \leq 0, \\ p_{k_{\mathrm{B}}} \in \mathbb{X}_{\mathrm{B}},\ \text{and } q_{k_{\mathrm{B}}} = 0, \end{cases} \text{ then } \tau_{k_{\mathrm{B}}}^{+} = \bar{T}. \tag{9.9a}$$

Here, (9.9a) encodes that robot k_{B} is in the position to tag robot i_{R}, and thus, $\tau_{k_{\mathrm{B}}}$ is updated. By setting the flow of $\tau_{k_{\mathrm{B}}}$ as

$$\dot{\tau}_{k_{\mathrm{B}}} = -1, \tag{9.9b}$$

we can ensure that a robot regains its tagging ability after $\bar{T}$ seconds by checking when $\tau_{k_{\mathrm{B}}} \leq 0$.

The dynamics of the 'tagged' state: To implement rule (a), we consider the following scenario and update law. If

$$\begin{matrix} \exists i_{\mathrm{R}} \in N_{\mathrm{R}}, \\ \exists k_{\mathrm{B}} \in N_{\mathrm{B}} \end{matrix} \text{ s.t. } \begin{cases} p_{i_{\mathrm{R}}} \in \mathbb{X}_{\mathrm{R}},\ q_{i_{\mathrm{R}}} = 0, |p_{i_{\mathrm{R}}} - p_{k_{\mathrm{B}}}| \leq \gamma_c, \tau_{i_{\mathrm{R}}} \leq 0, \\ p_{k_{\mathrm{B}}} \in \mathbb{X}_{\mathrm{R}},\ \text{and } q_{k_{\mathrm{B}}} = 0 \end{cases} \text{ then } \ q_{k_{\mathrm{B}}}^{+} = 1 - q_{k_{\mathrm{B}}}. \tag{9.10a}$$

Here, (9.10a) reflects that robot i_{R} is in the position to tag robot k_{B}, and thus, $q_{k_{\mathrm{B}}}$ is updated to 1.

Two additional cases are considered for the update of $q_{k_{\mathrm{B}}}$. In particular,

$$\text{if } \Big\{ p_{k_{\mathrm{B}}} \notin \mathbb{X} \text{ and } q_{k_{\mathrm{B}}} = 0, \quad \text{then} \quad q_{k_{\mathrm{B}}}^{+} = 1 - q_{k_{\mathrm{B}}}, \tag{9.10b}$$

and

$$\text{if } \Big\{ |p_{k_{\mathrm{B}}} - F_{\mathrm{B}}| \leq \gamma_F \text{ and } q_{k_{\mathrm{B}}} = 1, \quad \text{then} \quad q_{k_{\mathrm{B}}}^{+} = 1 - q_{k_{\mathrm{B}}}, \tag{9.10c}$$

cover rule (b), i.e., a robot leaving the playing field is deactivated and a deactivated robot returning to its flag base is reactivated. If none of these events occur, $q_{k_{\mathrm{B}}}$ remains constant and, thus,

$$\dot{q}_{k_{\mathrm{B}}} = 0. \tag{9.10d}$$

The dynamics of the 'carrying-the-flag' state: If a robot $k_{\mathrm{B}} \in N_{\mathrm{B}}$ carries the flag when it is tagged, then it loses the flag. This (see rule (a)) is encoded through the following update law. If

$$\exists i_{\mathrm{R}} \in N_{\mathrm{R}} \text{ s.t. } \begin{cases} p_{i_{\mathrm{R}}} \in \mathbb{X}_{\mathrm{R}},\ q_{i_{\mathrm{R}}} = 0, \tau_{i_{\mathrm{R}}} \leq 0,\ \mu_{\mathrm{R}} = 0, \\ |p_{i_{\mathrm{R}}} - p_{k_{\mathrm{B}}}| \leq \gamma_c, q_{k_{\mathrm{B}}} = 0,\ \text{and } \eta_{k_{\mathrm{B}}} = 1, \end{cases} \text{ then } \ \eta_{k_{\mathrm{B}}}^{+} = 1 - \eta_{k_{\mathrm{B}}}. \tag{9.11a}$$

Similarly,

$$\text{if } \Big\{ p_{k_{\mathrm{B}}} \notin \mathbb{X},\ q_{k_{\mathrm{B}}} = 0, \eta_{k_{\mathrm{B}}} = 1,\ \mu_{\mathrm{R}} = 0, \quad \text{then} \quad \eta_{k_{\mathrm{B}}}^{+} = 1 - \eta_{k_{\mathrm{B}}} \tag{9.11b}$$

encodes that $k_{\mathrm{B}} \in \mathbb{N}_{\mathrm{B}}$ loses the flag when leaving the playing field (see rule (b)). The update law

$$\text{if } \Big\{ |p_{k_{\mathrm{B}}} - F_{\mathrm{R}}| \leq \gamma_F, q_{k_{\mathrm{B}}} = 0,\ \eta_{k_{\mathrm{B}}} = 0, \mu_{\mathrm{R}} = 1, \quad \text{then} \quad \eta_{k_{\mathrm{B}}}^{+} = 1 - \eta_{k_{\mathrm{B}}} \tag{9.11c}$$

encodes that k_{B} captures the red team's flag (see rule (d)). If k_{B} returns the red flag to the blue flag base, then $\eta_{k_{\mathrm{B}}}$ is updated according to rule (f), which is

$$\text{if } \Big\{ |p_{k_{\mathrm{B}}} - F_{\mathrm{B}}| \leq \gamma_F, q_{k_{\mathrm{B}}} = 0, \eta_{k_{\mathrm{B}}} = 1, \mu_{\mathrm{R}} = 0 \quad \text{then} \quad \eta_{k_{\mathrm{B}}}^{+} = 1 - \eta_{k_{\mathrm{B}}}. \tag{9.11d}$$

A final rule is implemented to ensure that only one robot carries the flag at a time. According to (9.11c), more than one robot is able to update their η logic state at the same time, encoding that they all have "captured the flag." To address this, and ensure that only one blue robot is carrying the red flag according to rule (e), for all $k_{\mathrm{B}} \in N_{\mathrm{B}}$, the update law

$$\text{if } \left(\eta_{k_{\mathrm{B}}} \sum_{m_{\mathrm{B}}=1}^{k_{\mathrm{B}}} \eta_{m_{\mathrm{B}}} \right) \geq 2 \quad \text{then} \quad \eta_{k_{\mathrm{B}}}^{+} = 1 - \eta_{k_{\mathrm{B}}} \tag{9.11e}$$

ensures that only one robot (by convention the robot with the smallest index) has its η logic state equal to 1 after the update. As before, if none of these events occur, $\eta_{k_{\mathrm{B}}}$ remains constant and thus,

$$\dot{\eta}_{k_{\mathrm{B}}} = 0. \tag{9.11f}$$

The dynamics of the flag state: If a red robot grabs the blue flag according to rule (d), μ_{B} needs to be updated from 1 to 0. This is modeled by the following update. If

$$\exists i_{\mathrm{R}} \in N_{\mathrm{R}} \text{ s.t.} \Big\{ q_{i_{\mathrm{R}}} = 0,\ \eta_{i_{\mathrm{R}}} = 0, |p_{i_{\mathrm{R}}} - F_{\mathrm{B}}| \leq \gamma_F, \mu_{\mathrm{B}} = 1, \quad \text{then } \mu_{\mathrm{B}}^{+} = 1 - \mu_{\mathrm{B}}. \tag{9.12a}$$

Similarly, if the red flag is successfully carried to the blue flag base F_{R} (see rule (f)), then, the flag is instantaneously returned to its base and μ_{B} is updated from 0 to 1, which is described by

$$\text{if } \exists i_{\mathrm{R}} \in N_{\mathrm{R}} \text{ s.t.} \Big\{ q_{i_{\mathrm{R}}} = 0, \eta_{i_{\mathrm{R}}} = 1, |p_{i_{\mathrm{R}}} - F_{\mathrm{R}}| \leq \gamma_F, \mu_{\mathrm{B}} = 0, \ \text{then} \ \ \mu_{\mathrm{B}}^{+} = 1 - \mu_{\mathrm{B}}. \tag{9.12b}$$

If the robot carrying the flag is deactivated by leaving the playing field, then the flag is instantaneously returned to its base, i.e.,

$$\text{if } \exists i_{\mathrm{R}} \in N_{\mathrm{R}} \text{ s.t.} \Big\{ q_{i_{\mathrm{R}}} = 0, \eta_{i_{\mathrm{R}}} = 1, p_{i_{\mathrm{R}}} \notin \mathbb{X}, \mu_{\mathrm{B}} = 0, \quad \text{then } \ \mu_{\mathrm{B}}^{+} = 1 - \mu_{\mathrm{B}}, \tag{9.12c}$$

(see rule (b)). Similarly, if a red robot is tagged while carrying the blue flag, then the flag is instantaneously returned to its base (see rule (a)), as follows. If

$$\begin{matrix} \exists i_{\mathrm{R}} \in N_{\mathrm{R}}, \\ \exists k_{\mathrm{B}} \in N_{\mathrm{B}} \end{matrix} \text{ s.t. } \begin{cases} q_{i_{\mathrm{R}}} = 0, \eta_{i_{\mathrm{R}}} = 1, p_{i_{\mathrm{R}}} \in \mathbb{X}_{\mathrm{B}}, \mu_{\mathrm{B}} = 0, \\ |p_{i_{\mathrm{R}}} - p_{k_{\mathrm{B}}}| \leq \gamma_c, q_{k_{\mathrm{B}}} = 0, p_{k_{\mathrm{B}}} \in \mathbb{X}_{\mathrm{B}}, \tau_{k_{\mathrm{B}}} = 0, \end{cases} \text{ then } \mu_{\mathrm{B}}^{+} = 1 - \mu_{\mathrm{B}}. \tag{9.12d}$$

If none of these events occur, μ_{B} remains constant, i.e.,

$$\dot{\mu}_{\mathrm{B}} = 0. \tag{9.12e}$$

9.2.2 Flow and Jump Sets of the Hybrid Game

Using the scenario-based update laws in the previous section, we describe the game as a hybrid dynamical system $\mathcal{H}_s$. The state and input of the system are

$$\begin{aligned} x &= (x_{\mathrm{B}}, x_{\mathrm{R}}), & u &= (u_{\mathrm{B}}, u_{\mathrm{R}}), \\ x_{\mathrm{B}} &= (x_{1_{\mathrm{B}}}, \ldots, x_{b_{\mathrm{B}}}, \mu_{\mathrm{B}}), & u_{\mathrm{B}} &= (u_{1_{\mathrm{B}}}, \ldots, u_{b_{\mathrm{B}}}), \\ x_{\mathrm{R}} &= (x_{1_{\mathrm{R}}}, \ldots, x_{r_{\mathrm{R}}}, \mu_{\mathrm{R}}), & u_{\mathrm{R}} &= (u_{1_{\mathrm{R}}}, \ldots, u_{r_{\mathrm{R}}}), \end{aligned}$$

where each $x_{k_{\mathrm{B}}}$ and $x_{i_{\mathrm{R}}}$ is defined as in (9.4), and the state space and input space are defined as

$$\begin{aligned} X &= \left\{ x \;\middle|\; \begin{matrix} x_{k_{\mathrm{B}}} \in X_{k_{\mathrm{B}}} \ \forall \ k_{\mathrm{B}} \in N_{\mathrm{B}} \\ x_{i_{\mathrm{R}}} \in X_{i_{\mathrm{R}}} \ \forall \ i_{\mathrm{R}} \in N_{\mathrm{R}} \\ \mu_{\mathrm{B}}, \mu_{\mathrm{R}} \in \{0, 1\} \end{matrix} \right\}, \\ U &= \left\{ u \;\middle|\; \begin{matrix} u_{k_{\mathrm{B}}} \in \mathcal{U}_{k_{\mathrm{B}}} \ \forall \ k_{\mathrm{B}} \in N_{\mathrm{B}} \\ u_{i_{\mathrm{R}}} \in \mathcal{U}_{i_{\mathrm{R}}} \ \forall \ i_{\mathrm{R}} \in N_{\mathrm{R}} \end{matrix} \right\}, \end{aligned} \tag{9.13}$$

respectively, with $X_{k_{\mathrm{B}}}$ defined as in (9.4). The definitions of the flow map F, the jump map G, the flow set C, and the jump set D are presented next. With that aim, we introduce first the following case-based jump sets.

Define the set where robot $k_{\mathrm{B}} \in N_{\mathrm{B}}$ tags robot $i_{\mathrm{R}} \in N_{\mathrm{R}}$

$$D^{\mathrm{tag}}_{k_{\mathrm{B}}, i_{\mathrm{R}}} := \left\{ x \in X \;\middle|\; p_{k_{\mathrm{B}}} \in \mathbb{X}_{\mathrm{B}},\ q_{k_{\mathrm{B}}} = 0, p_{i_{\mathrm{R}}} \in \mathbb{X}_{\mathrm{B}},\ q_{i_{\mathrm{R}}} = 0, |p_{i_{\mathrm{R}}} - p_{k_{\mathrm{B}}}| \leq \gamma_c,\ \tau_{k_{\mathrm{B}}} \leq 0 \right\}. \tag{9.14a}$$

This set corresponds to the update (9.9a). Similarly, we denote the set where a robot $i_{\mathrm{R}} \in N_{\mathrm{R}}$, tags a robot $k_{\mathrm{B}} \in N_{\mathrm{B}}$, as

$$D^{\mathrm{tag}}_{i_{\mathrm{R}},k_{\mathrm{B}}} := \left\{ x \in X \;\middle|\; p_{i_{\mathrm{R}}} \in \mathbb{X}_{\mathrm{R}},\ q_{i_{\mathrm{R}}} = 0, p_{k_{\mathrm{B}}} \in \mathbb{X}_{\mathrm{R}},\ q_{k_{\mathrm{B}}} = 0, |p_{i_{\mathrm{R}}} - p_{k_{\mathrm{B}}}| \le \gamma_c,\ \tau_{i_{\mathrm{R}}} \le 0 \right\}. \tag{9.14b}$$

These sets $D^{\mathrm{tag}}_{i_{\mathrm{R}},k_{\mathrm{B}}}$ additionally encode (9.10a) and (9.11a), and trigger a jump in $q_{k_{\mathrm{B}}}$ and $\eta_{k_{\mathrm{B}}}$. In addition, to encompass the states where the red robot i_{R} is tagged while carrying the blue flag, we define

$$D^{\mathrm{tag}f}_{k_{\mathrm{B}},i_{\mathrm{R}}} := \left\{ x \in D^{\mathrm{tag}}_{k_{\mathrm{B}},i_{\mathrm{R}}} | \eta_{i_{\mathrm{R}}} = 1, \mu_{\mathrm{B}} = 0 \right\}, \tag{9.15}$$

encoding the set for the jump of μ_{B} in (9.12d). For any $i_{\mathrm{R}} \in N_{\mathrm{R}}$ and $k_{\mathrm{B}} \in N_{\mathrm{B}}$, the set $D^{\mathrm{tag}f}_{i_{\mathrm{R}},k_{\mathrm{B}}}$ is defined similarly.

Combining these definitions, denote

$$D^{\mathrm{tag}}_{k_{\mathrm{B}}} := \bigcup_{i_{\mathrm{R}} \in N_{\mathrm{R}}} D^{\mathrm{tag}}_{k_{\mathrm{B}},i_{\mathrm{R}}}, \quad D^{\mathrm{tag}}_{i_{\mathrm{R}}} := \bigcup_{k_{\mathrm{B}} \in N_{\mathrm{B}}} D^{\mathrm{tag}}_{i_{\mathrm{R}},k_{\mathrm{B}}}, \tag{9.16}$$

$$D^{\mathrm{tag}}_{\mathrm{B}} := \bigcup_{k_{\mathrm{B}} \in N_{\mathrm{B}}} D^{\mathrm{tag}}_{k_{\mathrm{B}}}, \quad D^{\mathrm{tag}}_{\mathrm{R}} := \bigcup_{i_{\mathrm{R}} \in N_{\mathrm{R}}} D^{\mathrm{tag}}_{i_{\mathrm{R}}}, \tag{9.17}$$

$$D^{\mathrm{tag}f}_{k_{\mathrm{B}}} := \bigcup_{i_{\mathrm{R}} \in N_{\mathrm{R}}} D^{\mathrm{tag}f}_{k_{\mathrm{B}},i_{\mathrm{R}}}, \quad D^{\mathrm{tag}f}_{i_{\mathrm{R}}} := \bigcup_{k_{\mathrm{B}} \in N_{\mathrm{B}}} D^{\mathrm{tag}f}_{i_{\mathrm{R}},k_{\mathrm{B}}}, \tag{9.18}$$

$$D^{\mathrm{tag}f}_{\mathrm{B}} := \bigcup_{k_{\mathrm{B}} \in N_{\mathrm{B}}} D^{\mathrm{tag}f}_{k_{\mathrm{B}}}, \quad D^{\mathrm{tag}f}_{\mathrm{R}} := \bigcup_{i_{\mathrm{R}} \in N_{\mathrm{R}}} D^{\mathrm{tag}f}_{i_{\mathrm{R}}}. \tag{9.19}$$

Here, (9.16) characterizes the set where the robot k_{B} and i_{R} can tag, respectively, and (9.17) characterizes the sets where the blue team and the red team can tag, respectively. Likewise, (9.18) and (9.19) are subsets of (9.16) and (9.17), respectively, where the tagged robot carries the flag.

Remark 9.2.1. () *Due to the definitions of $\mathbb{X}_B$ and $\mathbb{X}_R$, it is not possible that $k_B \in N_B$ and $i_R \in N_R$ tag each other at the same time since $\varepsilon > 0$ and $\mathbb{X}_B \cap \mathbb{X}_R = \emptyset$.*

A robot $k_{\mathrm{B}} \in N_{\mathrm{B}}$, or a robot $i_{\mathrm{R}} \in N_{\mathrm{R}}$, captures the flag, respectively, in the following sets

$$D^{\mathrm{flag}}_{k_{\mathrm{B}}} := \{x \in X |\ q_{k_{\mathrm{B}}} = 0, \eta_{k_{\mathrm{B}}} = 0,\ |p_{k_{\mathrm{B}}} - F_{\mathrm{R}}| \le \gamma_F, \mu_{\mathrm{R}} = 1\},$$
$$D^{\mathrm{flag}}_{i_{\mathrm{R}}} := \{x \in X |\ q_{i_{\mathrm{R}}} = 0, \eta_{i_{\mathrm{R}}} = 0,\ |p_{i_{\mathrm{R}}} - F_{\mathrm{B}}| \le \gamma_F, \mu_{\mathrm{B}} = 1\},$$

which corresponds to the updates in (9.11c) and (9.12a). The sets at which the teams can capture the flag are defined as the union of individual sets, namely,

$$D_{\mathrm{B}}^{\mathrm{flag}} := \bigcup_{k_{\mathrm{B}} \in N_{\mathrm{B}}} D_{k_{\mathrm{B}}}^{\mathrm{flag}}, \qquad D_{\mathrm{R}}^{\mathrm{flag}} := \bigcup_{i_{\mathrm{R}} \in N_{\mathrm{R}}} D_{i_{\mathrm{R}}}^{\mathrm{flag}}. \tag{9.20}$$

As modeled by (9.11e), multiple blue robots potentially could capture the red flag at the same time. To rule out having multiple blue robots carrying the red flag simultaneously, we define the sets

$$\begin{aligned} D_{k_{\mathrm{B}}}^{\mathrm{flag},\mu} &:= \left\{ x \in X | \eta_{k_{\mathrm{B}}} \sum_{m_{\mathrm{B}}=1}^{k_{\mathrm{B}}} \eta_{m_{\mathrm{B}}} \geq 2 \right\}, D_{\mathrm{B}}^{\mathrm{flag},\mu} := \bigcup_{k_{\mathrm{B}} \in N_{\mathrm{B}}} D_{k_{\mathrm{B}}}^{\mathrm{flag},\mu}, \\ D_{i_{\mathrm{R}}}^{\mathrm{flag},\mu} &:= \{ x \in X \Big| \eta_{i_{\mathrm{R}}} \sum_{m_{\mathrm{R}}=1}^{i_{\mathrm{R}}} \eta_{m_{\mathrm{R}}} \geq 2 \}, D_{\mathrm{R}}^{\mathrm{flag},\mu} := \bigcup_{i_{\mathrm{R}} \in N_{\mathrm{R}}} D_{i_{\mathrm{R}}}^{\mathrm{flag},\mu} \end{aligned} \tag{9.21}$$

which trigger a jump if multiple blue robots update their η state at the same time and ensure that only one has its η logic state equal to 1 at the same time.

The flag has been successfully carried to the capturing team's base if the state is in

$$D_{\mathrm{B}}^{\mu} := \bigcup_{k_{\mathrm{B}} \in N_{\mathrm{B}}} D_{k_{\mathrm{B}}}^{\mu}, \qquad \text{or} \qquad D_{\mathrm{R}}^{\mu} := \bigcup_{i_{\mathrm{R}} \in N_{\mathrm{R}}} D_{i_{\mathrm{R}}}^{\mu}, \tag{9.22a}$$

respectively, where

$$\begin{aligned} D_{k_{\mathrm{B}}}^{\mu} &:= \{ x \in X | \ |p_{k_{\mathrm{B}}} - F_{\mathrm{B}}| \leq \gamma_F, \eta_{k\mathrm{B}} = 1, \mu_{\mathrm{R}} = 0, q_{k_{\mathrm{B}}} = 0 \}, \\ D_{i_{\mathrm{R}}}^{\mu} &:= \{ x \in X | \ |p_{i_{\mathrm{R}}} - F_{\mathrm{R}}| \leq \gamma_F, \eta_{i_{\mathrm{R}}} = 1, \mu_{\mathrm{B}} = 0, \ q_{i_{\mathrm{R}}} = 0 \}. \end{aligned} \tag{9.22b}$$

The corresponding jump is encoded by the updates (9.11d) and (9.12b), respectively, modeling that the robot no longer carries the flag and the flag is instantaneously returned to its base.

If a robot $k_{\mathrm{B}} \in N_{\mathrm{B}}$, or a robot $i_{\mathrm{R}} \in N_{\mathrm{R}}$, is active and leaves the playing field, i.e., the state is in either

$$\begin{aligned} D_{k_{\mathrm{B}}}^{\mathbb{X}} &:= \left\{ x \in X | \ p_{k_{\mathrm{B}}} \in \overline{\mathbb{R}^2 \backslash \mathbb{X}}, q_{k_{\mathrm{B}}} = 0 \right\}, \\ D_{i_{\mathrm{R}}}^{\mathbb{X}} &:= \left\{ x \in X | \ p_{i_{\mathrm{R}}} \in \overline{\mathbb{R}^2 \backslash \mathbb{X}}, q_{i_{\mathrm{R}}} = 0 \right\}, \end{aligned} \tag{9.23}$$

then, $q_{k_{\mathrm{B}}}, q_{i_{\mathrm{R}}}$ is updated according to (9.10b). The closure of $\mathbb{R}^2 \backslash \mathbb{X}$ is used so that $D_{k_{\mathrm{B}}}^{\mathbb{X}}$ is closed.

If in addition, the corresponding robot is carrying the flag when leaving the playing field, that is, the state is in either

$$D_{k_{\mathrm{B}}}^{\mathbb{X}f} := \left\{ x \in D_{k_{\mathrm{B}}}^{\mathbb{X}} \,|\eta_{k_{\mathrm{B}}} = 1, \mu_{\mathrm{R}} = 0 \right\},$$
$$D_{i_{\mathrm{R}}}^{\mathbb{X}f} := \left\{ x \in D_{i_{\mathrm{R}}}^{\mathbb{X}} \,|\eta_{i_{\mathrm{R}}} = 1, \mu_{\mathrm{B}} = 0 \right\},$$

then the states $\eta_{k_{\mathrm{B}}}$ and $\mu_{k_{\mathrm{B}}}$ are updated according to (9.11b) and (9.12c), respectively. From the local jump sets, we define the sets at which B leaves the playing field as

$$\begin{aligned} D_{\mathrm{B}}^{\mathbb{X}} &:= \bigcup_{k_{\mathrm{B}} \in N_{\mathrm{B}}} D_{k_{\mathrm{B}}}^{\mathbb{X}}, \quad D_{\mathrm{R}}^{\mathbb{X}} := \bigcup_{i_{\mathrm{R}} \in N_{\mathrm{R}}} D_{i_{\mathrm{R}}}^{\mathbb{X}}, \\ D_{\mathrm{B}}^{\mathbb{X}f} &:= \bigcup_{k_{\mathrm{B}} \in N_{\mathrm{B}}} D_{k_{\mathrm{B}}}^{\mathbb{X}f}, \quad D_{\mathrm{R}}^{\mathbb{X}f} := \bigcup_{i_{\mathrm{R}} \in N_{\mathrm{R}}} D_{i_{\mathrm{R}}}^{\mathbb{X}f}, \end{aligned} \tag{9.24}$$

If a robot k_{B} is tagged or has left the playing field, it needs to reach the set

$$D_{k_{\mathrm{B}}}^{\mathrm{utag}} := \{ x \in X | \; q_{k_{\mathrm{B}}} = 1, |p_{k_{\mathrm{B}}} - F_{\mathrm{B}}| \leq \gamma_F \} \tag{9.25a}$$

to be reactivated, as encoded through (9.10c). Similarly, if a robot i_{R} is tagged or has left the playing field, it needs to reach the set

$$D_{i_{\mathrm{R}}}^{\mathrm{utag}} := \{ x \in X | \; q_{i_{\mathrm{R}}} = 1, |p_{i_{\mathrm{R}}} - F_{\mathrm{R}}| \leq \gamma_F \} \tag{9.25b}$$

to be reactivated.

We define the sets at which the teams regain their tagging ability as

$$D_{\mathrm{B}}^{\mathrm{utag}} := \bigcup_{k_{\mathrm{B}} \in N_{\mathrm{B}}} D_{k_{\mathrm{B}}}^{\mathrm{utag}}, \qquad D_{\mathrm{R}}^{\mathrm{utag}} := \bigcup_{i_{\mathrm{R}} \in N_{\mathrm{R}}} D_{i_{\mathrm{R}}}^{\mathrm{utag}}.$$

The sets above define the events that encode the rules of the game, which we use to define the jump set of $\mathcal{H}_s$ as

$$\begin{aligned} D := D_{\mathrm{B}}^{\mathrm{tag}} \cup D_{\mathrm{R}}^{\mathrm{tag}} \cup D_{\mathrm{B}}^{\mathrm{flag}} \cup D_{\mathrm{R}}^{\mathrm{flag}} \cup D_{\mathrm{B}}^{\mathrm{flag},\mu} \cup D_{\mathrm{R}}^{\mathrm{flag},\mu} \\ \cup D_{\mathrm{B}}^{\mu} \cup D_{\mathrm{R}}^{\mu} \cup D_{\mathrm{B}}^{\mathbb{X}} \cup D_{\mathrm{R}}^{\mathbb{X}} \cup D_{\mathrm{B}}^{\mathrm{utag}} \cup D_{\mathrm{R}}^{\mathrm{utag}}. \end{aligned} \tag{9.26}$$

Note that $D \subset X$ is closed since it is the union of closed sets. Correspondingly, the flow set is defined as

$$C := \overline{X \backslash D}. \tag{9.27}$$

The closure is used to ensure that C is closed, which is needed to guarantee that $\mathcal{H}_s$ is well-posed.

Remark 9.2.2. () *Note that the proposed modeling approach allows a robot $k_B \in N_B$ to tag multiple robots of R at the same time if the state is in multiple tagging jump sets simultaneously, e.g., $x \in D^{tag}_{k_B,1_R} \cap D^{tag}_{k_B,2_R}$. Likewise, a robot k_B can be tagged by multiple robots of R at the same time (if $x \in D^{tag}_{1_R,k_B} \cap D^{tag}_{2_R,k_B}$, for example), removing multiple tagging abilities for $\bar{T}$ seconds.*

9.2.3 Hybrid Game Dynamical Model

Now, we use the expressions in Sections 9.2.1 and 9.2.2 to define a hybrid dynamical system $\mathcal{H}_s$ to model the capture-the-flag game. For each x in the flow set C as in (9.27) and each u in the set of inputs U as in (9.13), the state of a robot $k_\text{B} \in N_\text{B}$ evolves continuously according to

$$\dot{x}_{k_\text{B}} = \begin{bmatrix} \dot{p}_{k_\text{B}} \\ \dot{\tau}_{k_\text{B}} \\ \dot{q}_{k_\text{B}} \\ \dot{\eta}_{k_\text{B}} \end{bmatrix} = F_{k_\text{B}}(x_{k_\text{B}}, u_{k_\text{B}}) := \begin{bmatrix} f(p_{k_\text{B}}, u_{k_\text{B}}) \\ -1 \\ 0 \\ 0 \end{bmatrix},$$

and the flag state evolves according to $\dot{\mu}_\text{B} = 0$. The maps F_{k_B} are defined based on (9.1), (9.9b), (9.10d), (9.11f), and (9.12e). The maps $F_{i_\text{R}}(x_{i_\text{R}}, u_{i_\text{R}})$, $i_\text{R} \in N_\text{R}$, are defined similarly.

Remark 9.2.3. () *Instead of the functions F_{k_B}, set-valued maps relying on* (9.2) *can be used to remove the explicit dependency on u_{k_B}.*

Based on the robots individual flow maps, the game evolves continuously for all $x \in C$ according to

$$\dot{x} = F(x, u_\text{B}, u_\text{R}) := \begin{bmatrix} F_{1_\text{B}}(x_{1_\text{B}}, u_{1_\text{B}}) \\ \vdots \\ F_{b_\text{B}}(x_{b_\text{B}}, u_{b_\text{B}}) \\ 0 \\ F_{1_\text{R}}(x_{1_\text{R}}, u_{1_\text{R}}) \\ \vdots \\ F_{r_\text{R}}(x_{r_\text{R}}, u_{r_\text{R}}) \\ 0 \end{bmatrix} \quad x \in C. \tag{9.28}$$

The definition of the jump map is more complicated since it requires the union of case-based individual jump maps. Again, we focus on the derivation from the perspective of B, while the definitions for R follow analogously. For $x \in D$, $k_{\rm B} \in N_{\rm B}$, we consider the following definitions. First, notice that the position of a robot does not change at a jump, i.e., $p_{k_{\rm B}}^{+} = p_{k_{\rm B}}$ for each $x \in D$. For the remaining state variables recall the set $D_{k_{\rm B}}^{\rm tag}$ in (9.16), where a robot $k_{\rm B}$ tags a robot $i_{\rm R}$, and define the sets

$$\begin{aligned}
D_{k_{\rm B}}^{(1)} :=& D_{k_{\rm B}}^{\rm tag},\\
D_{k_{\rm B}}^{(2)} :=& \overline{\left(\left(\bigcup_{i_{\rm R}\in N_{\rm R}} D_{i_{\rm R},k_{\rm B}}^{\rm tag}\right) \cup D_{k_{\rm B}}^{\mathbb{X}} \cup D_{k_{\rm B}}^{\rm utag}\right) \setminus D_{k_{\rm B}}^{(1)}},\\
D_{k_{\rm B}}^{(3)} :=& \overline{\left(\left(\bigcup_{i_{\rm R}\in N_{\rm R}} D_{i_{\rm R},k_{\rm B}}^{{\rm tag}f}\right) \cup D_{k_{\rm B}}^{\rm flag} \cup D_{k_{\rm B}}^{{\rm flag},\mu} \cup D_{k_{\rm B}}^{\mu} \cup D_{k_{\rm B}}^{\mathbb{X}f}\right) \setminus D_{k_{\rm B}}^{(1)} \cup D_{k_{\rm B}}^{(2)}},\\
D_{\mu_{\rm R}} :=& D_{\rm R}^{{\rm tag}f} \cup D_{\rm B}^{\rm flag} \cup D_{\rm B}^{\mu} \cup D_{\rm B}^{\mathbb{X}f},
\end{aligned}$$

where a jump is triggered to update the variables $\tau_{k_{\rm B}}$, $q_{k_{\rm B}}$, $\eta_{k_{\rm B}}$, and $\mu_{\rm R}$. This construction gives a sequential priority to the jumps that occur on $D_{k_{\rm B}}^{(1)}$, then on $D_{k_{\rm B}}^{(2)}$, and lastly on $D_{k_{\rm B}}^{(3)}$. The definition of $D_{k_{\rm B}}^{(2)}$ follows from the local jump sets (9.14), (9.23), and (9.25). The justification for the form of $D_{k_{\rm B}}^{(3)}$ stems from (9.18), (9.20), (9.21), (9.22), and (9.24). The definition of $D_{\mu_{\rm R}}$ is consistent with the definition of the sets (9.19), (9.20), (9.22), and (9.24).

Based on these sets, we define the corresponding local jump maps $\hat{g}_{k_{\rm B},m} : D_{k_{\rm B}}^{(m)} \to X$, $m \in \{1,2,3\}$, where

$$\begin{aligned}
\hat{g}_{k_{\rm B},1}(x) &:= (p_{k_{\rm B}}, \bar{T}, q_{k_{\rm B}}, \eta_{k_{\rm B}}) && \text{if } x \in D_{k_{\rm B}}^{(1)},\\
\hat{g}_{k_{\rm B},2}(x) &:= (p_{k_{\rm B}}, \tau_{k_{\rm B}}, 1 - q_{k_{\rm B}}, \eta_{k_{\rm B}}) && \text{if } x \in D_{k_{\rm B}}^{(2)},\\
\hat{g}_{k_{\rm B},3}(x) &:= (p_{k_{\rm B}}, \tau_{k_{\rm B}}, q_{k_{\rm B}}, 1 - \eta_{k_{\rm B}}) && \text{if } x \in D_{k_{\rm B}}^{(3)},\\
\hat{g}_{\mu_{\rm R}}(x) &:= 1 - \mu_{\rm R} && \text{if } x \in D_{\mu_{\rm R}}.
\end{aligned}$$

Depending on the local jump maps, the corresponding states are updated.

Simplifying the jump sets above by eliminating the disjoint sets via a scenario-based analysis, we define the sets

$$\begin{aligned}
\Delta^{\tau}_{k_{\mathrm{B}}} &:= D^{\mathrm{tag}}_{k_{\mathrm{B}}},\\
\Delta^{q}_{k_{\mathrm{B}}} &:= \overline{(D^{\mathbb{X}}_{k_{\mathrm{B}}} \setminus D^{\mathrm{tag}}_{k_{\mathrm{B}}}) \cup D^{\mathrm{utag}}_{k_{\mathrm{B}}}},\\
\Delta^{\eta}_{k_{\mathrm{B}}} &:= \overline{\left(D^{\mathrm{flag}}_{k_{\mathrm{B}}} \setminus \bigcup_{i_{\mathrm{R}} \in N_{\mathrm{R}}} D^{\mathrm{tag}f}_{i_{\mathrm{R}},k_{\mathrm{B}}} \right) \cup (D^{\mu}_{k_{\mathrm{B}}} \setminus D^{\mathrm{tag}}_{k_{\mathrm{B}}})},\\
\Delta^{\mu}_{k_{\mathrm{B}}} &:= \overline{D^{\mathrm{flag},\mu}_{k_{\mathrm{B}}} \setminus \bigcup_{i_{\mathrm{R}} \in N_{\mathrm{R}}} D^{\mathrm{tag}}_{i_{\mathrm{R}},k_{\mathrm{B}}}}.
\end{aligned}$$

To construct the jump map of the tagging ability states, we define, for $k_{\mathrm{B}} \in N_{\mathrm{B}}, i_{\mathrm{R}} \in N_{\mathrm{R}}$, the maps

$$\hat{G}^{\tau}_{k_{\mathrm{B}},i_{\mathrm{R}}}(x) := (x_{1_{\mathrm{B}}}, \ldots, \hat{g}_{k_{\mathrm{B}},1}(x), \ldots, x_{b_{\mathrm{B}}}, \mu_{\mathrm{B}}, x_{1_{\mathrm{R}}}, \ldots, \hat{g}_{i_{\mathrm{R}},2}(x), \ldots, x_{r_{\mathrm{R}}}, \mu_{\mathrm{R}}) \quad \text{if } x \in D^{\mathrm{tag}}_{k_{\mathrm{B}},i_{\mathrm{R}}}$$

Putting the individual jump maps together, we define

$$\hat{G}_{k_{\mathrm{B}},\tau}(x) := \bigcup_{i_{\mathrm{R}} \in N_{\mathrm{R}}} \hat{G}^{\tau}_{k_{\mathrm{B}},i_{\mathrm{R}}}(x) \qquad \text{if } x \in \Delta^{\tau}_{k_{\mathrm{B}}}.$$

Consider the jump map of the tagged states for which we define, for $k_{\mathrm{B}} \in N_{\mathrm{B}}$, the mappings

$$\hat{G}_{k_{\mathrm{B}},q}(x) := (x_{1_{\mathrm{B}}}, \ldots, \hat{g}_{k_{\mathrm{B}},2}(x), \ldots, x_{b_{\mathrm{B}}}, \mu_{\mathrm{B}}, x_{1_{\mathrm{R}}}, \ldots, x_{r_{\mathrm{R}}}, \mu_{\mathrm{R}}) \qquad \text{if } x \in \Delta^{q}_{k_{\mathrm{B}}}.$$

Consider the jump map of the carrying-the-flag states for which we define, for $k_{\mathrm{B}} \in N_{\mathrm{B}}$, the mappings

$$\hat{G}_{k_{\mathrm{B}},\eta}(x) := (x_{1_{\mathrm{B}}}, \ldots, \hat{g}_{k_{\mathrm{B}},3}(x), \ldots, x_{b_{\mathrm{B}}}, \mu_{\mathrm{B}}, x_{1_{\mathrm{R}}}, \ldots, x_{r_{\mathrm{R}}}, \hat{g}_{\mu_{\mathrm{R}}}(x)) \qquad \text{if } x \in \Delta^{\eta}_{k_{\mathrm{B}}},$$

and for the case of multiple robots capturing the flag, the mappings

$$\hat{G}_{k_{\mathrm{B}},\mu}(x) := (x_{1_{\mathrm{B}}}, \ldots, \hat{g}_{k_{\mathrm{B}},3}(x), \ldots, x_{b_{\mathrm{B}}}, \mu_{\mathrm{B}}, x_{1_{\mathrm{R}}}, \ldots, x_{r_{\mathrm{R}}}, \mu_{\mathrm{R}}) \qquad \text{if } x \in \Delta^{\mu}_{k_{\mathrm{B}}}.$$

The constructions above lead to the jump map

$$G(x) := \left\{ \hat{G}_{s_{\star},z}(x) \,\middle|\, x \in \Delta^{z}_{s_{\star}}, \star \in \{\mathrm{B},\mathrm{R}\}, s \in N_{\star}, z \in \{\tau, q, \eta, \mu\} \right\}. \tag{9.29}$$

The discrete evolution of the game is governed by

$$x^{+} \in G(x), \quad x \in D. \tag{9.30}$$

The overall game is modeled by $\mathcal{H}_s$ given in (9.26)–(9.30).

Before we introduce the objective function of the game in the next section, we highlight some important properties of the hybrid system.

Lemma 9.2.4. () *Consider the hybrid system defined in* (9.26)–(9.30). *Suppose f is continuous. Then, $\mathcal{H}_s$ satisfies the hybrid basic conditions [40, Def. 2.20], that is, F is continuous, $G : D \rightrightarrows X$ in* (9.29) *is outer semicontinuous and locally bounded, and C and D are closed.*

Proof. Since f in (9.1) is continuous, then F is continuous, by construction. Following the same arguments as in Lemma A.33 in [21], given that for each $k_{\rm B} \in N_{\rm B}$ the sets $\Delta^{\tau}_{k_{\rm B}}, \Delta^{q}_{k_{\rm B}}, \Delta^{\eta}_{k_{\rm B}}$, and $\Delta^{\mu}_{k_{\rm B}}$ are closed by construction, and $\hat{G}_{k_{\rm B},\tau}(x) : \Delta^{\tau}_{k_{\rm B}} \to X, \hat{G}_{k_{\rm B},q}(x) : \Delta^{q}_{k_{\rm B}} \to X, \hat{G}_{k_{\rm B},\eta}(x) : \Delta^{\eta}_{k_{\rm B}} \to X, \hat{G}_{k_{\rm B},\mu}(x) : \Delta^{\tau}_{k_{\rm B}} \to X$ are continuous and locally bounded relative to $\Delta^{\tau}_{k_{\rm B}}, \Delta^{q}_{k_{\rm B}}, \Delta^{\eta}_{k_{\rm B}}, \Delta^{\mu}_{\rm R}$, respectively, (and using the same arguments for R,) then the set-valued map G as in (9.29) is outer semicontinuous and locally bounded. The flow set C as in (9.27) is closed by construction since it is the closure of the complement of D. Notice that $D^{\rm tag}_{k_{\rm B},i_{\rm R}}$ as in (9.14b) is closed by definition, which implies that $D^{\rm tag}_{\rm B}$ is closed, since it is defined as the union of closed sets. The set $D^{\rm flag}_{\rm B}$ as in (9.20) is closed by following a similar argument. According to the definition of $D^{\mu}_{k_{\rm B}}$ in (9.22), it follows that $D^{\mu}_{\rm B}$ is closed. The set $D^{\mathbb{X}}_{\rm B}$ is closed by being the union of closed sets as defined in (9.23). The set $D^{\rm utag}_{\rm B}$ is the union of closed sets as defined in (9.25), and thus, closed. Similar arguments apply to R, which leads to the jump set D as in (9.26) being closed by construction. □ □

Following [40, Definition 2.18], we say that a system $\mathcal{H}_s$ that satisfies the hybrid basic conditions is well posed. Well posedness of hybrid closed-loop systems guarantees key structural properties of solutions.

Lemma 9.2.5. () *Consider the hybrid system defined in* (9.26)–(9.30). *For each given input $t \mapsto (u_B(t), u_R(t))$ with domain* $[0, \infty)$, *and for each initial condition $x_0 \in C \cup D$, there exists a maximal solution [40, Definition 2.29] to $\mathcal{H}_s$ that is complete and its domain is unbounded in the ordinary time variable.* ⌟

Proof. Forward completeness of solutions follows from the design of the domain X, in (9.13), in combination with the definition of the hybrid system (C, F, D, G) in (9.26)–(9.30). Namely, following [92], given that for an input $t \mapsto (u_{\rm B}(t), u_{\rm R}(t))$ with domain

$[0, \infty)$, every solution $(\phi, u_{\mathrm{B}}(\cdot), u_{\mathrm{R}}(\cdot))$ to $\mathcal{H}_s$ with $\phi(0,0) \in C \cup D$ can, at time $(0,0)$, either flow, jump or both, and given that such solution cannot end with flow, nor with jump, then it is forward complete. Similarly, the second assertion of the lemma follows from the definition of the hybrid system. While multiple consecutive jumps are possible, Zeno behavior cannot occur. In particular, regaining the tagging ability and reactivating a robot ensure that for all $t \in \mathbb{R}_{\geq 0}$, there exists $j \in \mathbb{N}$ such that $(t, j) \in \mathrm{dom}(\phi, u_{\mathrm{B}}, u_{\mathrm{R}})$. □ □

9.3 Two-Team Zero-Sum Formulation

This section discusses the objective of the game that was described in Section 9.1 and it proposes a meaningful cost function to define a two-tem zero-sum game. The cost function uses the indicator function $\mathbb{1}_{\mathcal{A}} : \mathbb{R}^n \to \{0,1\}$, which is defined for a set $\mathcal{A} \subset \mathbb{R}^n$, $n \in \mathbb{N}$, as

$$\mathbb{1}_{\mathcal{A}}(y) = \begin{cases} 1 & \text{if } y \in \mathcal{A}, \\ 0 & \text{if } y \notin \mathcal{A}. \end{cases} \tag{9.31}$$

Based on the indicator function and the definition of the sets D^{μ}_{B} and D^{μ}_{R} in (9.22), the overall goal of capturing the opponent's flag and returning it to the team's own base (see item (g)) is encoded by the cost function

$$\mathcal{J}_g(x_0, (u_{\mathrm{R}}(\cdot), u_{\mathrm{B}}(\cdot))) := c_g \sum_{(t_{j+1}, j) \in \mathrm{dom}\, x(\cdot)} \Big(\mathbb{1}_{D^{\mu}_{\mathrm{B}}}(x(t_{j+1}, j)) - \mathbb{1}_{D^{\mu}_{\mathrm{R}}}(x(t_{j+1}, j)) \Big) \tag{9.32}$$

where $\{t_j\}_{j=0}^{\sup_j \mathrm{dom}\, x}$ is a nondecreasing sequence associated to the definition of the hybrid time domain of $(x(\cdot), (u_{\mathrm{R}}(\cdot), u_{\mathrm{B}}(\cdot)))$ as in [40] and $\mathcal{R}(x_0, u)$ is the set of maximal state trajectories to $\mathcal{H}_s$ from x_0 for $(u_{\mathrm{R}}(\cdot), u_{\mathrm{B}}(\cdot))$, as defined in Section 2.2. The cost $\mathcal{J}_g$ is defined as the worst-case cost over all solutions from x_0.

Based on this definition, the value function is given by

$$\mathcal{J}^{\star}_g(x_0) = \inf_{u_{\mathrm{R}}(\cdot)} \sup_{u_{\mathrm{B}}(\cdot)} \mathcal{J}_{\mathrm{g}}(x_0, (u_{\mathrm{R}}(\cdot), u_{\mathrm{B}}(\cdot))) = \sup_{u_{\mathrm{B}}(\cdot)} \inf_{u_{\mathrm{R}}(\cdot)} \mathcal{J}_{\mathrm{g}}(x_0, (u_{\mathrm{R}}(\cdot), u_{\mathrm{B}}(\cdot))) \tag{9.33}$$

where team B aims to maximize the cost while team R seeks to minimize it. If $\mathcal{J}^{\star}_{\mathrm{g}}(x_0) > 0$, team B wins, and if $\mathcal{J}^{\star}_{\mathrm{g}}(x_0) < 0$, team R wins.

As indicated through (h)–(k) in Section 9.1, additional objectives can be considered, depending on the specific version of the capture-the-flag game. In particular,

$$\mathcal{J}_h(x_0, u_\mathrm{R}(\cdot), u_\mathrm{B}(\cdot)) := c_h \sum_{(t_{j+1},j)\in\operatorname{dom}\phi} \Big(\mathbb{1}_{D_\mathrm{B}^\mathrm{flag}}(\phi(t_{j+1},j)) - \mathbb{1}_{D_\mathrm{R}^\mathrm{flag}}(\phi(t_{j+1},j))\Big)$$

awards points for grabbing the flag (item (h)),

$$\mathcal{J}_i(x_0, u_\mathrm{R}(\cdot), u_\mathrm{B}(\cdot)) := c_i \sum_{(t_{j+1},j)\in\operatorname{dom}\phi} \Big(\sum_{k_\mathrm{B}\in\mathbb{N}_\mathrm{B}} \mathbb{1}_{D_{k_\mathrm{B}}^\mathrm{tag}}(\phi(t_{j+1},j)) - \sum_{i_\mathrm{R}\in N_\mathrm{R}} \mathbb{1}_{D_{i_\mathrm{R}}^\mathrm{tag}}(\phi(t_{j+1},j))\Big)$$

awards points for tagging robots (item (i)), and

$$\mathcal{J}_k(x_0, u_\mathrm{R}(\cdot), u_\mathrm{B}(\cdot)) := c_k \sum_{(t_{j+1},j)\in\operatorname{dom}\phi} \Big(-\sum_{k_\mathrm{B}\in N_\mathrm{B}} \mathbb{1}_{D_{k_\mathrm{B}}^\mathbb{X}}(\phi(t_{j+1},j)) + \sum_{i_\mathrm{R}\in N_\mathrm{R}} \mathbb{1}_{D_{i_\mathrm{R}}^\mathbb{X}}(\phi(t_{j+1},j))\Big)$$

penalizes the teams for leaving the playing field (item (k)). Here, $c_h, c_i, c_k > 0$ denote weighting factors. To cover item (j), we first define the sets

$$D_{k_\mathrm{B}}^{\mathbb{X}_\mathrm{R}} = \{x \in X |\ p_{k_\mathrm{B}} \in \mathbb{X}_\mathrm{R}, q_{k_\mathrm{B}} = 0\},$$
$$D_{i_\mathrm{R}}^{\mathbb{X}_\mathrm{B}} = \{x \in X |\ p_{i_\mathrm{R}} \in \mathbb{X}_\mathrm{B}, q_{i_\mathrm{R}} = 0\},$$

for $k_\mathrm{B} \in N_\mathrm{B}$ and $i \in N_\mathrm{R}$. Then

$$\mathcal{J}_j(x_0, u_\mathrm{R}(\cdot), u_\mathrm{B}(\cdot)) := c_j \sum_{j\in\mathbb{N}} \int_{t_j}^{t_{j+1}} \Big(\sum_{k_\mathrm{B}\in N_\mathrm{B}} \mathbb{1}_{D_{k_\mathrm{B}}^{\mathbb{X}_\mathrm{R}}}(\phi(t,j)) - \sum_{i_\mathrm{R}\in N_\mathrm{R}} \mathbb{1}_{D_{i_\mathrm{R}}^{\mathbb{X}_\mathrm{B}}}(\phi(t,j))\Big) dt$$

awards points for exploring the opponent's half of the playing field, with $c_j > 0$.

By selecting different constants and by taking the sum over different cost functions, different versions of the capture-the-flag game can be obtained. Even though the cost function in (9.32) encodes the objective of each team, the synthesis of optimal control laws is an open problem of research. Consequently, the controller design discussed in the next section focuses on local objectives. While this necessarily leads to suboptimal control strategies with respect to (9.32), it is a first step towards a saddle-point equilibrium design.

9.4 Control Design

In this section, we present local control laws based on the presentation in [93].[3] Notice that the control design described here is specific to the dynamics in (9.1). The

[3]See [93] for details.

local control laws are combined into a global control law via switching rules to optimize the objective function in Section 9.3. In the following, we focus again on the perspective of B.

9.4.1 Local Controllers

1) Return to base: After $k_{\mathrm{B}} \in N_{\mathrm{B}}$ is temporarily deactivated (see rule (b)), it needs to return to F_{B} to be reactivated. This is achieved in minimal time through the feedback law

$$\kappa^1_{k_{\mathrm{B}}}(x) := \cos^{-1}\Big(\frac{-\mathbb{X}_F - p_{k_{\mathrm{B}},1}}{|F_{\mathrm{B}} - p_{k_{\mathrm{B}}}|}\Big). \tag{9.34}$$

2) Return the flag: If the opponent's flag has been captured by $k_{\mathrm{B}} \in N_{\mathrm{B}}$, the robot needs to return to its own half of the playing field $\mathbb{X}_{\mathrm{B}}$ (without being tagged) to be safe from team R and to ultimately score points according to (9.32). The optimal strategy to reach the safe zone while avoiding an opponent $i_{\mathrm{R}} \in N_{\mathrm{R}}$ is given by the feedback law

$$\kappa^2_{k_{\mathrm{B}}}(x) := \cos^{-1}\Big(\frac{m_x y^* + n_x - p_{k_{\mathrm{B}},1}}{|(m_x y^* + n_x, y^*) - p_{k_{\mathrm{B}}}|}\Big) \tag{9.35}$$

where $m_x = -\frac{p_{k_{\mathrm{B}},2} - p_{i_{\mathrm{R}},2}}{p_{k_{\mathrm{B}},1} - p_{i_{\mathrm{R}},1}}$, $n_x = \frac{1}{2}\frac{|p_{k_{\mathrm{B}}}|^2 - |p_{i_{\mathrm{R}}}|^2}{p_{k_{\mathrm{B}},1} - p_{i_{\mathrm{R}},1}}$, $y^* = \mathbb{X}_y$ if $m_x > 0$, $y^* = -\mathbb{X}_y$ if $m_x < 0$. This follows the attack strategy design in Phase II in [93].

3) Capture the flag: To capture the opponents flag (rule (d)), we consider the feedback law

$$\kappa^3_{k_{\mathrm{B}}}(x) := \cos^{-1}\Big(\frac{\mathbb{X}_F - p_{k_{\mathrm{B}},1}}{|F_{\mathrm{R}} - p_{k_{\mathrm{B}}}|}\Big). \tag{9.36}$$

4) Enter $\mathbb{X}_{\mathrm{B}}$ or $\mathbb{X}_{\mathrm{R}}$: To enter the opponent's half of the playing field (in order to gain points with respect to $\mathcal{J}_j$, for example), one can use the strategies

$$\kappa^4_{k_{\mathrm{B}}}(x) := 0, \qquad \kappa^4_{i_{\mathrm{R}}}(x) := \pi. \tag{9.37}$$

5) Defend the flag: To lead $k_{\mathrm{B}} \in \mathbb{N}_{\mathrm{B}}$ to defend the flag F_{B} from $i_{\mathrm{R}} \in \mathbb{N}_{\mathrm{R}}$, we consider the feedback law

$$\kappa^5_{k_{\mathrm{B}}}(x) := \begin{cases} \cos^{-1}\Big(\frac{p^*_1 - p_{k_{\mathrm{B}},1}}{|p^* - p_{k_{\mathrm{B}}}|}\Big) & \text{if} \quad x \in \mathcal{R}_D, \\ \cos^{-1}\Big(-\frac{p_{k_{\mathrm{B}},1}}{|p_{k_{\mathrm{B}}}|}\Big) & \text{if} \quad x \in \mathcal{R}_A. \end{cases} \tag{9.38}$$

Here, $\mathcal{R}_D \subset \mathbb{X}_\mathrm{B}$ denotes the set of initial conditions from where, under optimal strategies, robot k_B tags i_R before i_R reaches the flag, $\mathcal{R}_A \subset \mathbb{X}_\mathrm{B}$ denotes the set of initial conditions from where, under optimal strategies, i_R captures the flag before being tagged and

$$p^* := (p_1^*, p_2^*) = \frac{|p_{i_\mathrm{R}} - F_\mathrm{B}|^2 - |p_{k_\mathrm{B}} - F_\mathrm{B}|^2}{|p_{i_\mathrm{R}} - p_{k_\mathrm{B}}|^2} \frac{p_{i_\mathrm{R}} - p_{k_\mathrm{B}}}{2}$$

denotes the optimal interception point (see Phase I in [93] for details).

6) Defend midfield line: To avoid that a robot $i_\mathrm{R} \in \mathbb{N}_\mathrm{R}$ that has captured the flag crosses the half of the playing field, we define the strategy of $k_\mathrm{B} \in N_\mathrm{B}$ based on the feedback law

$$\kappa_{6,k}(x) := \cos^{-1}\left(\frac{m_x y^* + n_x - p_{k_\mathrm{B},1}}{|(m_x y^* + n_x, y^*) - p_{k_\mathrm{B}}|}\right) \tag{9.39}$$

where $m_x = -\frac{p_{i_\mathrm{R},2} - p_{k_\mathrm{B},2}}{p_{i_\mathrm{R},1} - p_{k_\mathrm{B},1}}$, $n_x = \frac{1}{2}\frac{|p_{i_\mathrm{R},1}|^2 - |z_{k_\mathrm{B},1}|^2}{p_{i_\mathrm{R},1} - p_{k_\mathrm{B},1}}$, $y^* = \mathbb{X}_y$ if $m_x > 0$, $y^* = -\mathbb{X}_y$ if $m_x < 0$. This follows the defense strategy design in Phase II in [93].

9.4.2 Global Controller

This section focuses on the combination of local control laws to obtain a rule-based global controller. With this aim, we introduce the logic variables

$$\delta^S_{k_\mathrm{B}} = \begin{cases} 0, & \text{defensive strategy,} \\ 1, & \text{attack strategy,} \end{cases} \tag{9.40}$$

$$\delta^D_{k_\mathrm{B}} = \begin{cases} 0, & \text{defend the flag,} \\ 1, & \text{defend the midfield line,} \end{cases} \tag{9.41}$$

$$\delta^A_{k_\mathrm{B}} = \begin{cases} 0, & \text{capture the flag,} \\ 1, & \text{cross the half line,} \end{cases} \tag{9.42}$$

and the global control law κ_{k_B} defined as follows:

$$\begin{aligned}\kappa_{k_\mathrm{B}}(x) :=& q_{k_\mathrm{B}}\kappa^1_{k_\mathrm{B}} + \eta_{k_\mathrm{B}}\kappa^2_{k_\mathrm{B}} + (1-\eta_{k_\mathrm{B}})\Big[(1-q_{k_\mathrm{B}})\big[\delta^S_{k_\mathrm{B}}\Big((1-\delta^A_{k_\mathrm{B}})\kappa^3_{k_\mathrm{B}} + \delta^A_{k_\mathrm{B}}\kappa^4_{k_\mathrm{B}}\Big) \\ &+ (1-\delta^S_{k_\mathrm{B}})\Big((1-\delta^D_{k_\mathrm{B}})\kappa^5_{k_\mathrm{B}} + \delta^D_{k_\mathrm{B}}\kappa^6_{k_\mathrm{B}}\Big)\big]\Big].\end{aligned} \tag{9.43}$$

With these definitions, the optimization objectives encoded in the cost function can be summarized through the following update rules.

The dynamics of $\delta^S_{k_\mathrm{B}} \in \{0,1\}$: To implement a defensive strategy, we consider the following scenario and the corresponding update rule, if

$$\exists i_\mathrm{R} \in N_\mathrm{R} \text{ s.t.} \begin{cases} p_{i_\mathrm{R}}, p_{k_\mathrm{B}} \in \mathbb{X}_\mathrm{B}, q_{i_\mathrm{R}} = 0, q_{k_\mathrm{B}} = 0, \\ \eta_{k_\mathrm{B}} = 0, \delta^S_{k_\mathrm{B}} = 1, \end{cases} \text{then } \delta^{S+}_{k_\mathrm{B}} = 1 - \delta^S_{k_\mathrm{B}}. \tag{9.44a}$$

Here, (9.44a) reflects that robot i_R is in $\mathbb{X}_\mathrm{B}$ and k_B has an attack strategy and thus $\delta^S_{k_\mathrm{B}}$ is updated. An additional case needs to be considered for the update of $\delta^S_{k_\mathrm{B}}$. In particular,

$$\text{if } \nexists i_\mathrm{R} \in N_\mathrm{R} \text{ s.t.} \big\{ p_{i_\mathrm{R}}, p_{k_\mathrm{B}} \in \mathbb{X}_\mathrm{B}, q_{i_\mathrm{R}} = 0, q_{k_\mathrm{B}} = 0, \delta^S_{k_\mathrm{B}} = 0, \text{ then } \delta^{S+}_{k_\mathrm{B}} = 1 - \delta^S_{k_\mathrm{B}}, \tag{9.44b}$$

covers when a robot takes an offensive strategy due to no threats of an opponent capturing the flag. If none of these events occurs, $\delta^S_{k_\mathrm{B}}$ remains constant and thus $\dot{\delta}^S_{k_\mathrm{B}} = 0$.

The dynamics of $\delta^A_{k_\mathrm{B}} \in \{0,1\}$: To implement a capture-the-flag strategy, we consider the following scenario and the corresponding update rule:

$$\text{if } \big\{ \delta^A_{k_\mathrm{B}} = 1, p_{k_\mathrm{B}} \in \mathbb{X}_\mathrm{R}, q_{k_\mathrm{B}} = 0, \eta_{k_\mathrm{B}} = 0, \mu_\mathrm{R} = 1 \quad \text{then} \quad \delta^{A+}_{k_\mathrm{B}} = 1 - \delta^A_{k_\mathrm{B}}. \tag{9.45a}$$

Here, (9.45a) reflects that robot k_B has entered the opponent's half, which triggers an update of $\delta^A_{k_\mathrm{B}}$. One additional case needs to be considered for the update of $\delta^A_{k_\mathrm{B}}$. In particular,

$$\text{if } \big\{ \delta^S_{k_\mathrm{B}} = 1, \delta^A_{k_\mathrm{B}} = 0, p_{k_\mathrm{B}} \in \mathbb{X}_\mathrm{B}, q_{k_\mathrm{B}} = 0, \eta_{t_\mathrm{B}} = 0 \quad \text{then} \quad \delta^{A+}_{k_\mathrm{B}} = 1 - \delta^A_{k_\mathrm{B}}, \tag{9.45b}$$

covers when a robot takes the strategy to cross into the opponents half of the playing field. If none of these events occurs, $\delta^A_{k_\mathrm{B}}$ remains constant and thus $\dot{\delta}^A_{k_\mathrm{B}} = 0$.

The dynamics of $\delta^D_{k_\mathrm{B}} \in \{0,1\}$: To implement a defend-the-flag strategy, we consider the following scenario and the corresponding update rule:

$$\exists i_\mathrm{R} \in N_\mathrm{R} \text{ s.t.} \begin{cases} p_{i_\mathrm{R}} \in \mathbb{X}_\mathrm{B}, \delta^S_{k_\mathrm{B}} = 0, p_{k_\mathrm{B}} \in \mathbb{X}_\mathrm{B}, \delta^D_{k_\mathrm{B}} = 1, \\ q_{k_\mathrm{B}} = 0, q_{i_\mathrm{R}} = 0, |p_{i_\mathrm{R}} - F_\mathrm{B}| > |p_{k_\mathrm{B}} - F_\mathrm{B}| \end{cases} \text{then } \delta^{D+}_{k_\mathrm{B}} = 1 - \delta^D_{k_\mathrm{B}} \tag{9.46a}$$

Here, (9.46a) reflects that robot k_B is closer to its own flag and i_R threatens to capture it so it will be intercepted before it happens. As a second case we consider

$$\text{if } \exists i_\mathrm{R} \in N_\mathrm{R} \text{ s.t.} \begin{cases} p_{i_\mathrm{R}} \in \mathbb{X}_\mathrm{B}, \delta^S_{k_\mathrm{B}} = 0, p_{k_\mathrm{B}} \in \mathbb{X}_\mathrm{B}, \delta^D_{k_\mathrm{B}} = 0, \\ q_{k_\mathrm{B}} = 0, q_{i_\mathrm{R}} = 0, \eta_{i_\mathrm{R}} = 1 |p_{i_\mathrm{R}} - F_\mathrm{B}| < |p_{k_\mathrm{B}} - F_\mathrm{B}| \end{cases} \text{then } \delta^{D+}_{k_\mathrm{B}} = 1 - \delta^D_{k_\mathrm{B}}. \tag{9.46b}$$

This update switches to boundary-protection strategy. If none of these events occurs, $\delta^D_{k_\mathrm{B}}$ remains constant, i.e., $\dot{\delta}^D_{k_\mathrm{B}} = 0$.

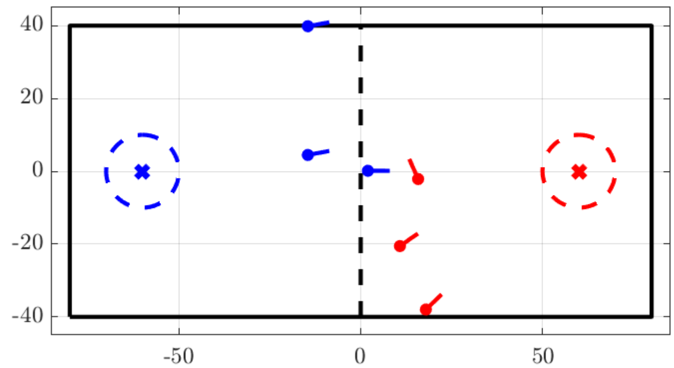

Figure 9.2: Blue robots getting tagged.

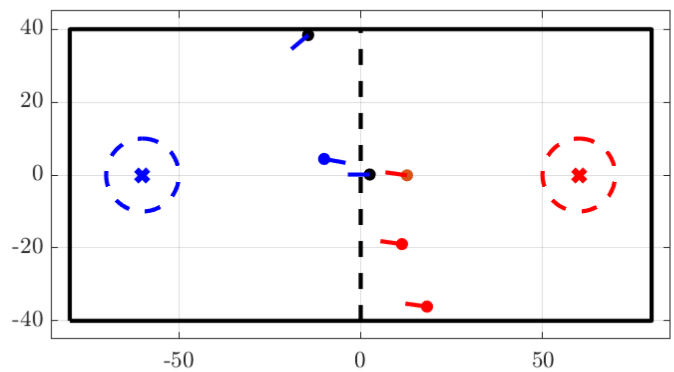

Figure 9.3: Blue robots are deactivated.

9.5 Simulation Testbed

Consider a scenario with three robots in each team, namely, $b = r = 3$, where each of them is initialized as in Figure 9.1. The constant speed in normalized units of each of the robots is $v = 1$. A simulation tool has been developed[4] where the rules of the game are encoded by implementing the hybrid model in Section 9.2. This tool allows one to test different controllers. Some of the robots have been endowed with a switched controller that combines the local laws in (9.43) as in Section 9.4. In Figure 9.2, a blue robot is about to be tagged and another blue robot is about to exit the playing field. Thus, $x \in D^{\text{tag}}_{i_R,k_B} \cap D^{\mathbb{X}}_{k_B}$ as in (9.14b) and (9.23), which trigger the state to jump according to (9.10). When the state q_{k_B} is updated, denoting a robot has been deactivated, its marker color changes to black as in Figure 9.3 and the controller leads them to head back

[4]Code at https://github.com/sjleudo/HybridCaptureTheFlag

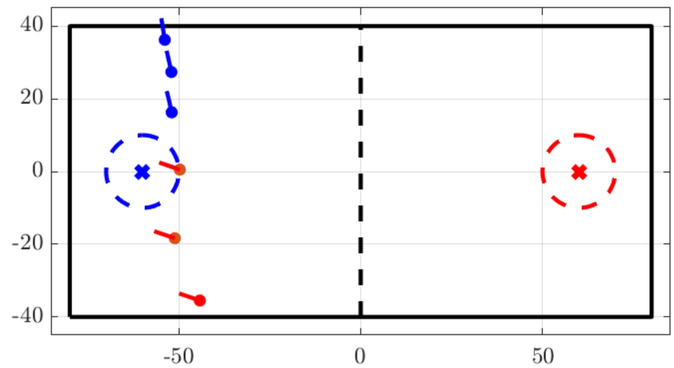

Figure 9.4: Red robot captures the flag.

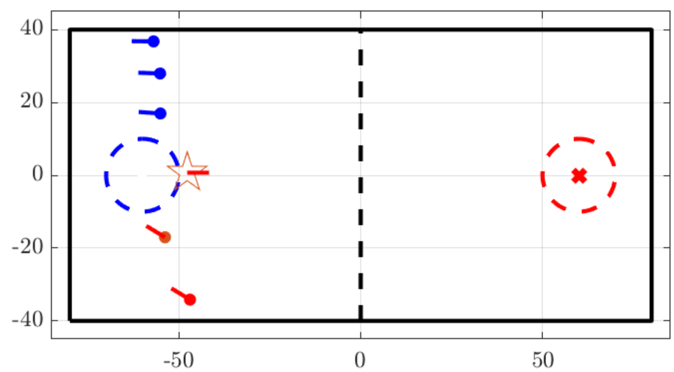

Figure 9.5: Red robot carries the flag.

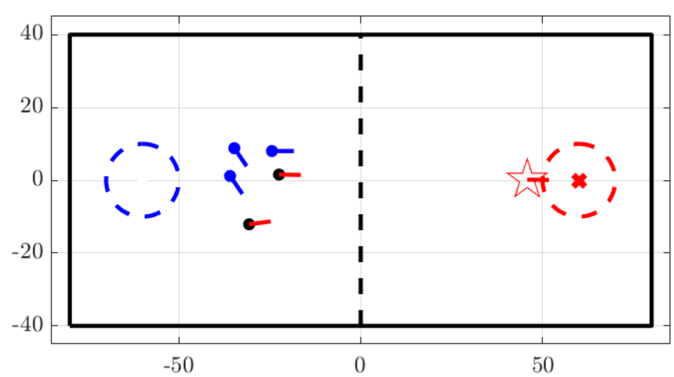

Figure 9.6: Red robot dropping the flag.

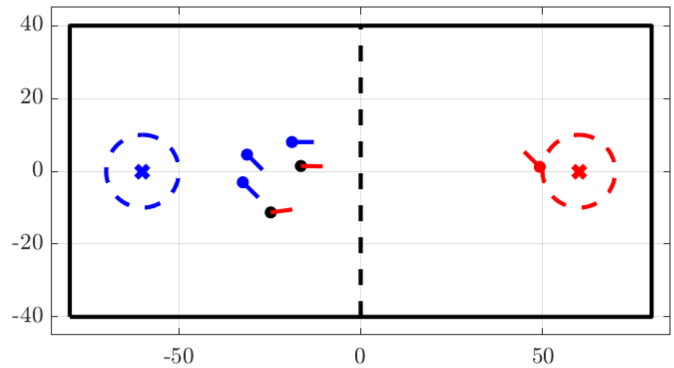

Figure 9.7: Flag back at blue home.

to the base according to (9.34). In addition, a robot that has a tagging timeout changes its color to green or orange, depending on whether it belongs to B or R, respectively. In Figure 9.4, a red robot is about to capture the blue flag, so $x \in D^{\text{flag}}_{i_{\text{R}}}$ as in (9.20), which triggers a jump in the state according to (9.11c) and (9.12a). When the states $\eta_{i_{\text{R}}}$ and μ_{B} are updated, denoting that a red robot has captured the blue flag, its marker changes to a star as in Figure 9.5, the marker denoting the position of the flag at its base turns white, and the controller leads the red robot to head back to its base according to (9.35). In Figure 9.6, the red robot carrying the flag arrives to its base, namely, $x \in D^{\mu}_{\text{R}}$ as in (9.22), which triggers a jump in the state according to (9.11d) and (9.12b). When the states $\eta_{i_{\text{R}}}$ and μ_{B} are updated, denoting the red team has scored because a red robot returns the blue flag to the red base, its marker changes back to a circle as in Figure 9.7, the marker denoting the position of the base turns blue, and the controller $\kappa_{i_{\text{R}}}(x) := \pi$ leads the red robot to cross the midfieldaccording to (9.37).

Acknowledgements

Sections 9.1, 9.2, and 9.5, in full, are a reprint of the material as it appears in "A Hybrid Systems Formulation for a Capture-the-Flag Game." [94]. The dissertation author was the first author of this paper.

Chapter 10

Hybrid Games under Imperfect Information

In this chapter, we study a contested scenario that arises under imperfect state information on a system with hybrid dynamical behavior. We formulate a finite-horizon optimization problem to model a two-player zero-sum hybrid game with limited access to the state. We consider an approach based on a state observers design. Consider a two-player zero-sum game with dynamics $\mathcal{H}$ described by (2.3) for given (C, F, D, G, h) where the *output function* $h : \mathbb{R}^n \to \mathbb{R}^p$ defines the system's output $y = h(x)$.

Observer Design

We propose to design an observer of the form

$$\hat{\mathcal{H}} \begin{cases} \dot{\hat{x}} & = & F(\hat{x}, u_C) + \mathcal{L}_{oC}(y, h(\hat{x})) & (\hat{x}, u_C) \in C \\ \hat{x}^+ & = & G(\hat{x}, u_D) + \mathcal{L}_{oD}(y, h(\hat{x})) & (\hat{x}, u_D) \in D \end{cases} \tag{10.1}$$

where $\hat{x} \in \mathbb{R}^n$ is an estimate of the state x, and the maps $\mathcal{L}_{oC} : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^n$ and $\mathcal{L}_{oD} : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^n$ are state estimate correctors based on comparisons between the output y and the output estimation $h(\hat{x})$ during flows and jumps, respectively. We are interested on the design of optimal feedback laws $\kappa := (\kappa_C, \kappa_D) : \mathbb{R}^n \to \mathbb{R}^{m_c} \times \mathbb{R}^{m_D}$ that map from the state estimates.

Definition 10.0.1. (Solution to the hybrid observer $\hat{\mathcal{H}}$) *A hybrid signal* $(\hat{\phi}, u, y)$ *defines a solution to the hybrid system (10.1) if* $\hat{\phi} \in \mathcal{X}$, $u = (u_C, u_D) \in \mathcal{U}$, $y \in h(\mathcal{X})$, $\mathrm{dom}\hat{\phi} = \mathrm{dom}u = \mathrm{dom}y$, *and*

- *$(\hat{\phi}(0,0), u_C(0,0)) \in \overline{C}$ or $(\hat{\phi}(0,0), u_D(0,0)) \in D$,*
- *For each $j \in \mathbb{N}$ such that $I^j_{\hat{\phi}}$ has a nonempty interior $\text{int}I^j_{\hat{\phi}}$, we have, for all $t \in \text{int}I^j_{\hat{\phi}}$,*

$$(\hat{\phi}(t,j), u_C(t,j)) \in C$$

and, for almost all $t \in I^j_{\hat{\phi}}$,

$$\frac{d}{dt}\hat{\phi}(t,j) \;=\; F(\hat{\phi}(t,j), u_C(t,j)) + \mathcal{L}_{oC}(y(t,j), h(\hat{\phi}(t,j)))$$

- *For all $(t,j) \in \text{dom}\,\hat{\phi}$ such that $(t, j+1) \in \text{dom}\,\hat{\phi}$,*

$$\begin{aligned}(\hat{\phi}(t,j), u_D(t,j)) &\in D \\ \hat{\phi}(t,j+1) &= G(\hat{\phi}(t,j), u_D(t,j)) + \mathcal{L}_{oD}(y(t,j), h(\hat{\phi}(t,j)))\end{aligned}$$

A solution $(\hat{\phi}, u, y)$ is a compact solution if $\hat{\phi}$ is a compact hybrid arc.

We say that a solution $(\hat{\phi}, u, y)$ to $\hat{\mathcal{H}}$ is maximal if it cannot be extended and we say it is complete when $\text{dom}\,\hat{\phi}$ is unbounded. We denote by $\mathcal{S}_{\hat{\mathcal{H}}}(M)$ the set of solutions $(\hat{\phi}, u, y)$ to (10.1) such that $\hat{\phi}(0,0) \in M$.

We define the state error $e := x - \hat{x}$ as the deviation of the estimator from the state. Notice that the dynamics in the error coordinates are given by

$$\mathcal{E}\begin{cases}\dot{e} &= F(x,u_C) - F(\hat{x},u_C) - \mathcal{L}_{oC}(y,h(\hat{x})) \quad (e,u_C) \in C_{\mathcal{E}} \\ e^+ &= G(x,u_D) - G(\hat{x},u_D) - \mathcal{L}_{oD}(y,h(\hat{x})) \quad (e,u_D) \in D_{\mathcal{E}}\end{cases} \tag{10.2}$$

where $C_{\mathcal{E}} := \{(x-\hat{x}, u_C) \in \mathbb{R}^n \times \mathbb{R}^{m_C} : (x,u_C) \text{ and } (\hat{x},u_C) \in C\}$ and $D_{\mathcal{E}} := \{(x-\hat{x}, u_D) \in \mathbb{R}^n \times \mathbb{R}^{m_D} : (x,u_D) \text{ or } (\hat{x},u_D) \in D\}$.

Thus, the input action $u = (u_1, u_2)$ rendering a response ϕ_e to $\mathcal{E}$ with components defined as $\text{dom}\,\phi_e \ni (t,j) \mapsto u_i(t,j) = \gamma_i(t,j,\phi_e(t,j))$, for each $i \in \mathcal{V}$, defines the closed-loop system

$$\mathcal{H}_{\mathcal{E}}\begin{cases}\dot{z} &= \begin{bmatrix} F(x,\kappa_C(\hat{x})) \\ F(x,\kappa_C(\hat{x})) - F(\hat{x},\kappa_C(\hat{x})) - \mathcal{L}_{oC}(y,h(\hat{x}))\end{bmatrix} \quad x \text{ and } \hat{x} \in C_\kappa \\ z^+ &= \begin{bmatrix} G(x,\kappa_D(\hat{x})) \\ G(x,\kappa_D(\hat{x})) - G(\hat{x},\kappa_D(\hat{x})) - \mathcal{L}_{oD}(y,h(\hat{x}))\end{bmatrix} \quad x \text{ or } \hat{x} \in D_\kappa\end{cases} \tag{10.3}$$

with state $z = (x, e)$.

To provide insight on the proposed approach, consider the linear case with maps

$$\begin{aligned}(x,u_C)&\mapsto F(x,u_C):=A_Cx+B_Cu_C\\(x,u_D)&\mapsto G(x,u_D):=A_Dx+B_Du_D\\x&\mapsto h(x):=Hx\\x&\mapsto \kappa_C(x):=-K_Cx\\x&\mapsto \kappa_D(x):=-K_Dx\\(y_1,y_2)&\mapsto \mathcal{L}_{oC}(y_1,y_2):=L_{oC}(y_1-y_2)\\(y_1,y_2)&\mapsto \mathcal{L}_{oD}(y_1,y_2):=L_{oD}(y_1-y_2),\end{aligned} \tag{10.4}$$

where $A_C, B_C, A_D, B_D, H, K_C, K_D, L_{oC}$, and L_{oD} are matrices with appropriate dimension. The closed-loop system (10.3) reduces to

$$\mathcal{H}_{\mathcal{E}}\begin{cases}\dot{z} &= \begin{bmatrix}A_C-B_CK_C & B_CK_C\\0 & A_C-L_{oC}H\end{bmatrix}z \quad \hat{x}\in C_\kappa\\ z^+ &= \begin{bmatrix}A_D-B_DK_D & B_DK_D\\0 & A_D-L_{oD}H\end{bmatrix}z \quad \hat{x}\in D_\kappa\end{cases} \tag{10.5}$$

Given $\xi\in\Pi(C\cup D)$, a joint input action $u=(u_C,u_D)\in\mathcal{U}$, the stage cost for flows $L_C:\mathbb{R}^n\times\mathbb{R}^{m_C}\to\mathbb{R}_{\geq 0}$, the stage cost for jumps $L_D:\mathbb{R}^n\times\mathbb{R}^{m_D}\to\mathbb{R}_{\geq 0}$, and the terminal cost $q:\mathbb{R}^n\to\mathbb{R}$, the cost associated to the solution (ϕ,u) to $\mathcal{H}$ from ξ with end time $(T,J)\in\mathbb{R}_{\geq 0}\setminus\{\infty\}\times\mathbb{N}$, under Assumption 3.1.4 is defined as in (5.1). We are ready to formulate the two-player zero-sum game with imperfect state information.

Problem ($\star_{\hat{x}}$): Given $\xi\in\mathbb{R}^n$, under Assumption 3.1.4, design $\kappa_C,\kappa_D,\mathcal{L}_{oC}$, $\mathcal{L}_{oD}$ so that we solve

$$\underset{\substack{u_1\\u=(u_1,u_2)\in\mathcal{U}}}{\text{minimize}}\ \underset{u_2}{\text{maximize}}\ \mathcal{J}(\xi,u) \tag{10.6}$$

while $e(t,j)\to 0$ as $(t,j)\to(T,J)$, where $(t,j)\in\operatorname{dom}u$ and (T,J) is the end time of u.

10.1 An Observer-based Switching Algorithm for Safety under Sensor Denial-of-Service Attacks

As a stepping stone, motivated by the study of contested scenarios with imperfect information patterns where hybrid dynamics emerge, in the reminder of this chapter we study the nonoptimal design of safe-critical switched-control algorithms for systems under Denial-of-Service (DoS) attacks on the system output. We aim to address scenarios where attack-mitigation approaches are not feasible, and the system needs to maintain safety under adversarial attacks as in [46]. We propose an attack-recovery strategy by designing a switching observer and characterizing bounds in the error of a state estimation scheme by specifying tolerable limits on the time length of attacks. Then, we propose a switching control algorithm that renders forward invariant a set for the observer. By satisfying the error bounds of the state estimation, we guarantee that the safe set is rendered conditionally invariant with respect to a set of initial conditions.

The proposed formulation is applicable to several use cases with objectives including obstacle avoidance and collision-free navigation for autonomous vehicles, reach-avoid control problems, surveillance, and convoy of multi-agent systems, among others. We consider scenarios in which every attack has finite duration, succeeded by an interval of time without attacks. We are interested in finding the set of initial conditions and the control action such that the state trajectory remains in a safe set at all times. During attacks, the controller relies only on the uncompromised outputs, from which we generate an estimate of the state, whereas the entire output is used when attacks are not present.

We provide sufficient conditions involving key properties of the system, such as the maximum tolerable length of the DoS attack and the minimum required length of the interval without an attack for recovery, guaranteeing that the state estimation error remains uniformly bounded. Furthermore, we design CBF-based observer-based feedback laws to render a properly defined set forward invariant for the observer so that with bounded estimation error, the system is safe. This is obtained provided conditional invariance of a set of interest with respect to a set of initial states.

10.2 Preliminaries

Consider the nonlinear system

$$\mathcal{F} : \begin{cases} \dot{z} = F(t, z), \\ y = H(t, z) \end{cases} \tag{10.7}$$

where $z \in \mathbb{R}^n$ is the system state, $y \in \mathbb{R}^p$ is the system output, $F : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is the (potentially nonsmooth) flow map and $H : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^p$ is the output map.

A solution to the system $\mathcal{F}$ is defined as follows.

Definition 10.2.1. (Solution to $\mathcal{F}$) *A locally absolutely continuous function $t \mapsto z(t)$ defines a solution to the system $\mathcal{F}$ in (10.7) if $\frac{d}{dt}z(t) = F(t, z(t))$ for almost all $t \in \mathbb{R}_{\geq 0}$.*

We say that a solution z to $\mathcal{F}$ is maximal if it cannot be extended and we say it is complete when $\operatorname{dom} z = [0, \infty)$.

Definition 10.2.2. (Safety) *The system (10.7) is said to be safe with respect to (X_0, X_u), with $X_0 \subset \mathbb{R}^n \setminus X_u$, if for each $z_0 \in X_0$, each solution $t \mapsto z(t)$ to (10.7) with $z(0) = z_0$ satisfies $z(t) \in \mathbb{R}^n \setminus X_u$ for all $t \in \operatorname{dom} z$.*

Definition 10.2.3. (Conditional invariance) *A closed set $S \subset \mathbb{R}^n$ is said to be conditionally invariant for system (10.7) with respect to $M \subset S$ if, for each $z_0 \in M$, any solution $t \mapsto z(t)$ to (10.7) from z_0 satisfies $z(t) \in S$ for all $t \in \operatorname{dom} z$.*

It is immediate that the system (10.7) is safe with respect to (X_0, X_u) if and only if the set $S := \mathbb{R}^n \setminus X_u$ is conditionally invariant for (10.7) with respect to X_0. For more details see [86].

10.3 Problem Formulation

10.3.1 System Model

Consider the linear time-invariant control system

$$\mathcal{S} : \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx \end{cases} \tag{10.8}$$

where $x \in \mathbb{R}^n$ is the system state, $y \in \mathbb{R}^p$ is the system output, $u \in \mathcal{U}$ is the control input, and $\mathcal{U} \subset \mathbb{R}^m$. Here, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times n}$.

10.3.2 Attack Model

In this chapter, we consider attacks on the system output y. In particular, we consider an attack where a subset of the components of the system output is compromised. Under such an attack model, the *measured* system output $\bar{y}$ takes the form

$$\bar{y} = (y_s, y_a) \tag{10.9}$$

where $y_s = \tilde{C}x$, and, for each solution $t \mapsto x(t)$ to (10.8),

$$y_a(t) = \begin{cases} \bar{C}x(t) & \text{if } t \notin \mathcal{T}_a, \\ Y(t, x(t)) & \text{if } t \in \mathcal{T}_a \end{cases} \tag{10.10}$$

The quantity $\tilde{C}x$ denotes the *secured* output components that *cannot* be attacked with $\tilde{C} \in \mathbb{R}^{\tilde{p}\times n}$ and $0 \leq \tilde{p} < p$, $\bar{C}x$ denotes the *vulnerable* output components that *can* be attacked with $\bar{C} \in \mathbb{R}^{(p-\tilde{p})\times n}$ such that $C = \begin{bmatrix} \tilde{C} \\ \bar{C} \end{bmatrix}$, and $Y : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^{p-\tilde{p}}$ denotes the attacked output signal. We denote with $\mathcal{T}_a \subset \mathbb{R}_{\geq 0}$ the set of times when an attack is present on the system output, which is assumed to be known provided a DoS attack detection mechanism. The attack model (10.9) captures Denial-of-Service (DoS) attacks on the system output. Let $[t_1^i, t_2^i)$ with $t_2^i > t_1^i \geq 0$ denote the interval of time over which the $i-$th DoS attack occurs, with $i \in \mathbb{N}_{>0}$. Define $\mathcal{T}_a := \bigcup_i [t_1^i, t_2^i)$, $\mathcal{T}_1 = \bigcup_i \{t_1^i\}$, and $\mathcal{T}_2 = \bigcup_i \{t_2^i\}$ as the intervals of attack, and the sets of the starting and ending time instants of attacks, respectively. To provide sufficient conditions to guarantee safety, we characterize the attacks by defining $T_a := \max_{i\in\{1,2,\dots\}}(t_2^i - t_1^i)$ and $T_{na} := \min_{i\in\{2,3,\dots\}}(t_1^i - t_2^{i-1})$ as the maximum length of the DoS attack and the minimum length of the interval without an attack, respectively. Notice that $t_2^0 := 0$, and when $t_1^1 > 0$, we have $t_1^1 \geq T_{na}$.

10.3.3 Problem Statement

Given a nonempty, closed set $S \subset \mathbb{R}^n$, referred to as the *safe* set, the problem to solve is the design of an algorithm such that the set S is conditionally invariant for (10.8) with respect to the set X_0. Formally, the control design problem studied in this chapter is stated as follows.

Problem (⊗): Given system (10.8), a closed set $S \subset \mathbb{R}^n$, and the attack model in (10.9),

1. Find a set of initial states $X_0 \subset S$, and

2. Design a control law κ assigning the input u of (10.8) using measurements of $\bar{y}$

such that, for each $x_0 \in X_0$, the solution to the resulting closed-loop system, namely $t \mapsto x(t)$, with $x(0) = x_0$, satisfies $x(t) \in S$ for all $t \geq 0$.

10.3.4 Proposed Solution

To solve Problem ($\otimes$), we propose the design of an observer-based feedback law that induces conditional invariance of S with respect to X_0. Most CBF-based methods for forward invariance rely on measurement of the entire state [95]. We propose to employ a state estimator that reconstructs the system state using the measured output $\bar{y}$. The observer is given as

$$\dot{\hat{x}} = A\hat{x} + Bu + g(\bar{y}, \hat{y}), \tag{10.11a}$$

$$\hat{y} = C\hat{x}, \tag{10.11b}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimate of x and $g : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^n$ is the innovation term to be designed such that $g(\bar{y}, \hat{y}) = 0$ at $\bar{y} = \hat{y}$. When the system output is under an attack according to the attack model (10.9), the actual output information is not available to the state observer. Thus, the observer needs to take into account the attacks on the system output. To this end, we design an observer that uses the *complete output* vector when there is no attack and only the *non-attacked output* components when the system output is under attack. More specifically, the proposed observer under the attack model (10.9) is given as

$$\dot{\hat{x}} = \begin{cases} A\hat{x} + Bu + g_1(Cx, C\hat{x}) & \text{if} \quad t \notin \mathcal{T}_a, \\ A\hat{x} + Bu + g_2(\tilde{C}x, C\hat{x}) & \text{if} \quad t \in \mathcal{T}_a \end{cases} \tag{10.12}$$

where $g_1, g_2 : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^n$ are to be designed. Given a set $\mathcal{T}_a \subset \mathbb{R}$, the feedback law κ assigning u is defined as

$$\kappa(t, \hat{x}, y) = \begin{cases} \kappa_1(\hat{x}, y) & \text{if} \quad t \notin \mathcal{T}_a, \\ \kappa_2(\hat{x}, y) & \text{if} \quad t \in \mathcal{T}_a, \end{cases} \tag{10.13}$$

where $\kappa_1, \kappa_2 : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ are functions to be designed under *nominal* operation (i.e., when the system is not under an attack) and under attack, respectively. Notice that the closed-loop system resulting from the composition of (10.8) and (10.12) with κ as in (10.13) can be expressed as in (10.7) with $z = (x, \hat{x})$.

We make the following assumption on $\mathcal{S}$ in (10.8).

Assumption 10.3.1. *The pair (A, B) is controllable and the pair (C, A) is detectable.*

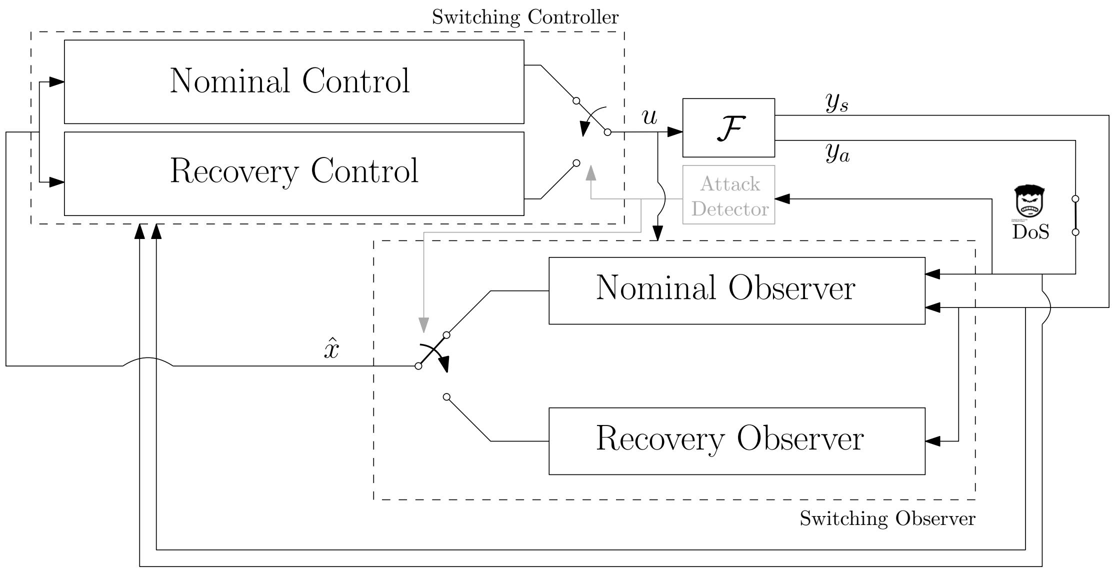

Figure 10.1: Closed-loop attack recovery scheme.

Based on the structure of the observer in (10.8) and the observer-based feedback law in (10.13), the approach followed in this chapter for safety under attacks for system (10.8) is as in Figure 10.1 with details as follows .

Approach: Given a closed set $S \subset \mathbb{R}^n$, the system (10.8), and the attack model (10.9), our approach is to compute sets $X_0, \hat{X}_0, \hat{S}_0 \subset S$ and design functions g_1, g_2 for the observer in (10.12) and functions κ_1, κ_2 for the observer-based feedback law κ as in (10.13) such that each solution pair $t \mapsto (x(t), \hat{x}(t))$ to the closed-loop system resulting from the composition of (10.8) and (10.12) with κ satisfies the following properties:

1) For each $t_0 \in \mathcal{T}_1$ such that $x(t_0) \in X_0$ and $\hat{x}(t_0) \in \hat{X}_0$, the x component of the resulting closed-loop solution satisfies $x(t) \in S$ for all $t \in [t_0, t_0 + T_a)$;

2) For each $t_0 \in \mathcal{T}_2$ such that $x(t_0) \in S$ and $\hat{x}(t_0) \in \hat{S}_0$, and for $\hat{t}_0 = \max\{t_0, \inf_{t \geq t_0} \mathcal{T}_1\}$, the x component of the resulting closed-loop solution satisfies $x(\hat{t}_0) \in X_0$ and $x(t) \in S$ for all $t \in [t_0, \hat{t}_0)$.

Remark 10.3.2. () *The sets $\hat{X}_0$ and $\hat{S}_0$ denote the sets of estimates before and after an attack, respectively. We will design these sets in the next section. Item 1 in our* *solution approach encodes conditional invariance of the set* S *for system* (10.8) *with respect to* X_0*, under an attack with maximum duration. Upon the requirement of the state to be in* S *at the end of every attack, item 2 encodes safety of system* (10.8) *with respect to* $(X_0, \mathbb{R}^n \setminus S)$ *during the time-intervals with no attacks, and the state to be in* X_0 *at the beginning of the next attack.*

In the next sections, we present the design of the observer in (10.12), and the design of the observer-based feedback law in (10.13).

10.4 Design of Switching Observer

Under an attack on the system output of the form (10.9), it might not be possible to reconstruct the state of (10.8) for a full-state feedback control design. Specifically, under the considered attack model, the rank of the observability matrix $\tilde{\mathcal{O}}$ for the pair $(\tilde{C}, A)$, namely, $\text{rank}(\tilde{\mathcal{O}}) = \tilde{n}$, potentially smaller than n. Thus, there might be $n - \tilde{n} > 0$ eigenvalues in the closed right-half plane for the dynamics of the estimation error resulting for any observer design under attack. Keeping this in mind, the switching observer in (10.12) is defined as

$$\dot{\hat{x}} = \begin{cases} A\hat{x} + Bu + L(Cx - C\hat{x}) & \text{if} \quad t \notin \mathcal{T}_a, \\ A\hat{x} + Bu + \tilde{L}(\tilde{C}x - \tilde{C}\hat{x}) & \text{if} \quad t \in \mathcal{T}_a, \end{cases} \tag{10.14}$$

where $L \in \mathbb{R}^{n \times p}$ and $\tilde{L} \in \mathbb{R}^{n \times \tilde{p}}$ is such that $\tilde{n}$ (with $\tilde{n} \leq n$) eigenvalues of the matrix $A - \tilde{L}\tilde{C}$ lie in the open left-half plane. On the other hand, since (C, A) is detectable under Assumption 10.3.1, we can design L such that all the eigenvalues of $(A - LC)$ are in the open left-half plane. Now, define $e = x - \hat{x}$ as the estimation error to obtain the error dynamics given as

$$\dot{e} = \begin{cases} (A - LC)e & \text{if} \quad t \notin \mathcal{T}_a, \\ (A - \tilde{L}\tilde{C})e & \text{if} \quad t \in \mathcal{T}_a \end{cases} \tag{10.15}$$

with $e(0) = x(0) - \hat{x}(0)$. Next, we analyze the error bounds when there is no attack, i.e., at each $t \notin \mathcal{T}_a$.

10.4.1 Analysis under no Attacks

Consider the starting instant of an interval during which there is no attack on the system output, namely $t_2^i \in \mathcal{T}_2 \cup \{0\}$, with $i \in \mathbb{N}$. The following result is the initial step to guarantee conditional invariance of S with respect to X_0 for the system (10.8) when there are no attacks.

Lemma 10.4.1. () *Given system (10.8), suppose Assumption 10.3.1 holds. For given $T_{na}, \bar{e}_0 > 0$, an associated observer (10.14), and corresponding error dynamics (10.15), if at the $i-th$ interval of no attacks with $i \in \mathbb{N}$, $|e(t_2^i)| \leq \bar{e}_0$ with $t_2^i \in \mathcal{T}_2$, then the state estimation error satisfies $|e(t)| \leq \gamma_1(t-t_2^i)\bar{e}_0$ for all $t \in [t_2^i, t_1^{i+1}]$, where*

$$\gamma_1(t) := c_1 \exp\left(-\bar{\lambda}_1 t\right) \tag{10.16}$$

with $\bar{\lambda}_1 = \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}$, $c_1 = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$, and L such that for some symmetric positive definite matrices P and Q, $-Q = (A - LC)^\top P + P(A - LC)$ holds.

Proof. Since the pair (C, A) is detectable, it follows that there exist positive definite matrices P, Q such that $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, defined as $V(e) := e^T P e$, satisfies

$$\begin{aligned}
\dot{V}(e) &= \dot{e}^\top P e + e^\top P \dot{e} \\
&= [(A - LC)e]^\top P e + e^\top P[(A - LC)e] \\
&= e^\top (A - LC)^\top P e + e^\top P(A - LC)e \\
&= -e^\top Q e.
\end{aligned}$$

Given that $\lambda_{\min}(Q)e^\top e \leq e^\top Q e$ for all $e \in \mathbb{R}^n$, we obtain

$$\dot{V}(e) \leq -|e|^2 \lambda_{\min}(Q). \tag{10.17}$$

Since P is symmetric, we have

$$\lambda_{\min}(P)|e|^2 \leq V(e) \leq \lambda_{\max}(P)|e|^2, \tag{10.18}$$

for all $e \in \mathbb{R}^n$, which, together with (10.17) implies that $\dot{V}(e) \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} V(e)$. Let $t \mapsto e(t)$ be a solution of (10.15) and consider the interval $[t_2^i, t_1^{i+1}]$ for $i \in \mathbb{N}_{>0}$. It follows that $V(e(t)) \leq V(e(t_2^i)) \exp\left(-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}(t - t_2^i)\right)$ for all $t \in [t_2^i, t_1^{i+1}]$. Since $V(e(t_2^i)) \leq \lambda_{\max}(P)|e(t_2^i)|^2$, using the left inequality in (10.18), it follows that

$$\lambda_{\min}(P)|e(t)|^2 \leq \lambda_{\max}(P) \exp\left(-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}(t-t_2^i)\right) |e(t_2^i)|^2$$

for all $t \in [t_2^i, t_1^{i+1}]$, which thanks to $|e(t_2^i)| \le \bar{e}_0$, implies that

$$|e(t)| \le c_1 \exp\left(-\bar{\lambda}_1(t-t_2^i)\right)|e(t_2^i)| \le \gamma_1(t-t_2^i)\bar{e}_0,$$

for all $t \in [t_2^i, t_1^{i+1}]$, completing the proof. □

Notice that the above analysis (withnal Luenberger observer) can be used to show that starting from $e(t_2^i)$ with $t_2^i \in \mathcal{T}_2 \cup \{0\}, i \in \mathbb{N}$, the error exponentially converges to $\delta\mathbb{B}$ in time T_{na}, where $\delta = \gamma_1(T_{na})|e(t_2^i)|$, and stays in that ball until the next attack starts at t_1^{i+1}.

Remark 10.4.2. () *The Luenberger observer used when there are no attacks is just one choice of a state estimator. It is also possible to use a finite-time stable state estimator [96], or any other observer that has faster convergence guarantees.*

10.4.2 Analysis under Attacks

During the attack on the output, we use a different observer gain designed for the pair $(\tilde{C}, A)$. Since it might not be possible to place all the eigenvalues of $A - \tilde{L}\tilde{C}$ in the open left-half plane, the matrix $\tilde{L}$ in (10.14) can be designed to minimize the maximum eigenvalue of $A - \tilde{L}\tilde{C}$, which minimizes the rate of growth of the error during attacks. Based on $\tilde{L}$, we compute the maximum growth rate possible in the estimation error e during intervals of attacks in the system output, assuming a worst-case attack.

Under the attack model (10.9), a subset of the state space may still be detectable for the pair $(\tilde{C}, A)$. Thus, under the observer (10.14) for $t \in \mathcal{T}_a$, it is possible that some of the eigenvalues of the matrix $A - \tilde{L}\tilde{C}$ are in the open left-half plane. To bound the error growth during the attack, we consider the general case in which we can decompose the matrix $A - \tilde{L}\tilde{C}$ into submatrices $\hat{A}_{11}$ and $\hat{A}_{22}$, such that the eigenvalues of $\hat{A}_{11}$ are in the open left-half plane. To this end, let $\Phi \in \mathbb{R}^{n\times n}$ be an invertible matrix consisting of the generalized eigenvectors of the matrix $A - \tilde{L}\tilde{C}$ such that

$$\Phi^{-1}(A - \tilde{L}\tilde{C})\Phi = \begin{bmatrix} \hat{A}_{11} & 0_{\tilde{n}\times(n-\tilde{n})} \\ 0_{(n-\tilde{n})\times\tilde{n}} & \hat{A}_{22} \end{bmatrix} \tag{10.19}$$

where $\hat{A}_{11}$ and $\hat{A}_{22}$ are Jordan blocks such that $\lambda_{\max}(\hat{A}_{11}) < 0$ and $0_{p\times q} \in \mathbb{R}^{p\times q}$ is a matrix consisting of zeros[1]. Also, let $\Phi^{-1} = \left[\hat{\Phi}_1^\top, \hat{\Phi}_2^\top\right]$, and define the change of

[1] Note that it is always possible to find the Jordan form of the matrix $A - \tilde{L}\tilde{C}$, even when it is not diagonalizable.

coordinates $z = \Phi^{-1}e$. Then, $e = \Phi z$, and in the new coordinates, the error dynamics are expressed as

$$\dot{z} = \Phi^{-1}\dot{e} = \begin{bmatrix} \hat{A}_{11} & 0_{\tilde{n}\times(n-\tilde{n})} \\ 0_{(n-\tilde{n})\times\tilde{n}} & \hat{A}_{22} \end{bmatrix} z.$$

Define $z = (z_{11}, z_{22})$, where $z_{11} \in \mathbb{R}^{\tilde{n}}$ and $z_{22} \in \mathbb{R}^{n-\tilde{n}}$ so that we have

$$\dot{z}_{11} = \hat{A}_{11} z_{11} \tag{10.20}$$

$$\dot{z}_{22} = \hat{A}_{22} z_{22}. \tag{10.21}$$

We can now state the following result providing a bound on the state estimation error under attacks.

Lemma 10.4.3. () *Given system (10.8), suppose Assumption 10.3.1 holds. For given* $T_a, \bar{e}_0 > 0,$ *an associated observer (10.14), and corresponding error dynamics (10.15), if at the* $i-$*th interval of attack with* $i \in \mathbb{N}_{>0}$ *and maximum length* T_a, $|e(t_1^i)| \leq \bar{e}_0$ *with* $t_1^i \in \mathcal{T}_1$*, then the state estimation error satisfies* $|e(t)| \leq \gamma_2(T_a)\bar{e}_0$ *for all* $t \in [t_1^i, t_2^i]$*, where*

$$\gamma_2(T_a) := \max_{t\in[0,T_a]} \hat{c}_1 \exp\left(-\hat{\lambda}_1 t\right) + \hat{c}_2 \exp\left(\hat{\lambda}_2 t\right) \tag{10.22}$$

with

$$\hat{c}_1 = |\Phi||\hat{\Phi}_1|\sqrt{\frac{\lambda_{\max}(\hat{P})}{\lambda_{\min}(\hat{P})}}, \quad \hat{c}_2 = |\Phi||\hat{\Phi}_2|, \quad \hat{\lambda}_1 = \frac{\lambda_{\min}(\hat{Q})}{2\lambda_{\max}(\hat{P})}, \quad \hat{\lambda}_2 = |\hat{A}_{22}|,$$

and $\tilde{L}$ *such that for some symmetric positive definite matrices* $\hat{P}$ *and* $\hat{Q}$, $-\hat{Q} = \hat{A}_{11}^\top \hat{P} + \hat{P}\hat{A}_{11}$ *holds.*

Proof. Consider the dynamics (10.20). Since by construction, the eigenvalues of the matrix $\hat{A}_{11}$ are in the open left-half plane, it follows that there exist positive definite matrices $\hat{P}, \hat{Q}$ such that $\hat{V} : \mathbb{R}^l \to \mathbb{R}_{\geq 0}$ defined as $\hat{V}(z_{11}) := z_{11}^\top \hat{P} z_{11}$ satisfies

$$\dot{\hat{V}}(z_{11}) = -z_{11}^\top \hat{Q} z_{11}$$

Given that $\lambda_{\min}(\hat{Q}) z_{11}^\top z_{11} \leq z_{11}^\top \hat{Q} z_{11}$ for all $z_{11} \in \mathbb{R}^{\tilde{n}}$, we obtain

$$\dot{\hat{V}}(z_{11}) \leq -|z_{11}|^2 \lambda_{\min}(\hat{Q}). \tag{10.23}$$

Let $t \mapsto (z_{11}(t), z_{22}(t))$ denote a solution of (10.20)-(10.21). Denote $\hat{\lambda}_1 = \frac{\lambda_{\min}(\hat{Q})}{2\lambda_{\max}(\hat{P})}$ and $\breve{c}_1 = \sqrt{\frac{\lambda_{\max}(\hat{P})}{\lambda_{\min}(\hat{P})}}$ to obtain that

$$|z_{11}(t)| \le \breve{c}_1 \exp\left(-\hat{\lambda}_1(t - t_1^i)\right)|z_{11}(t_1^i)|.$$

On the other hand, for (10.21), given that $|\hat{A}_{22}z_{22}| \le |\hat{A}_{22}||z_{22}|$, we have

$$|z_{22}(t)| \le |z_{22}(t_1^i)| \exp(|\hat{A}_{22}|(t - t_1^i))$$

Thus, by denoting $\hat{c}_1 = |\Phi||\hat{\Phi}_1|\breve{c}_1$, and $\hat{c}_2 = |\Phi||\hat{\Phi}_2|, \hat{\lambda}_2 = |\hat{A}_{22}|$, in the original coordinates we have

$$\begin{aligned}
|e(t)| \le & |\Phi|(|z_{11}(t)| + |z_{22}(t)|) \\
\le & |\Phi|\breve{c}_1|z_{11}(t_1^i)| \exp\left(-\hat{\lambda}_1(t - t_1^i)\right) + |\Phi||z_{22}(t_1^i)| \exp(|\hat{A}_{22}|(t - t_1^i)) \\
= & |\Phi|\breve{c}_1|\hat{\Phi}_1 e(t_1^i)| \exp\left(-\hat{\lambda}_1(t - t_1^i)\right) + |\Phi||\hat{\Phi}_2 e(t_1^i)| \exp(\hat{\lambda}_2(t - t_1^i)) \\
\le & \left(\hat{c}_1 \exp(-\hat{\lambda}_1(t - t_1^i)) + \hat{c}_2 \exp(\hat{\lambda}_2(t - t_1^i))\right)|e(t_1^i)|
\end{aligned}$$

Thus, thanks to $|e(t_1^i)| \le \bar{e}_0$, for all $t \in [t_1^i, t_2^i]$ it follows that

$$|e(t)| \le \max_{t \in [0, T_a]} \left(\hat{c}_1 \exp(-\hat{\lambda}_1 t) + \hat{c}_2 \exp(\hat{\lambda}_2 t)\right)\bar{e}_0$$

which completes the proof. □

Note that it is possible that $\tilde{n} = 0$, i.e., all the eigenvalues of the matrix $A - \tilde{L}\tilde{C}$ are in the closed right-half plane. In that case, $\hat{c}_1 = 0$ in (10.22).

10.4.3 Global Bound on Estimation Error

The following assumption on the initial state estimation error is used to establish a bound on the estimation error at all times.

Assumption 10.4.4. *The closed set $S \subset \mathbb{R}^n$ is such that there exists $\bar{E} > 0$ such that, for the initial state $x(0) \in S$ and initial estimate $\hat{x}(0) \in S$, the error satisfies $|e(0)| = |x(0) - \hat{x}(0)| \le \bar{E}$.*

A pre-defined initial error bound helps us guarantee the existence of a switching observer of the form (10.12) such that safety is guaranteed.

Now, we provide a result on bounds on the state estimation error under the proposed switching observer algorithm.

Theorem 10.4.5. () *Given system (10.8), suppose Assumptions 10.3.1 and 10.4.4 hold for* $\bar{E} > 0$. *For given* $T_{na}, T_a > 0$, *an associated observer (10.14), and corresponding error dynamics (10.15), let* $c_1, \bar{\lambda}_1, \hat{c}_1, \hat{c}_2, \hat{\lambda}_1, \hat{\lambda}_2 > 0$ *be defined as per Lemma 10.4.1 and Lemma 10.4.3. If* T_{na} *and* T_a *are such that* $\gamma_1(T_{na})\gamma_2(T_a) \leq 1$ *with* γ_1 *as in* (10.16) *and* γ_2 *as in* (10.22), *then* $|e(t)| \leq \gamma_1(0)\gamma_2(T_a)\bar{E}$ *for all* $t \geq 0$. *In addition,*

- *if there is an attack at time* $t = 0$, *then* $|e(t)| \leq \bar{E}$ *for all* $t \in \mathcal{T}_1 \cup \{0\}$, *and*
- *if the first attack is launched after at least* T_{na} *seconds, then* $|e(t)| \leq \bar{E}$ *for all* $t \in \mathcal{T}_2 \cup \{0\}$.

Proof. From Lemma 10.4.1, we have that for each interval without attacks starting at $t_2^i \in \mathcal{T}_2$ and ending at $t_1^{i+1} \in \mathcal{T}_1$, with $i \in \mathbb{N}$, the estimation error satisfies

$$\begin{aligned}|e(t_1^{i+1})| &\leq c_1 \exp\left(-\bar{\lambda}_1(t_1^{i+1} - t_2^i)\right)|e(t_2^i)| \\ &\leq c_1 \exp\left(-\bar{\lambda}_1 T_{na}\right)|e(t_2^i)| = \gamma_1(T_{na})|e(t_2^i)| \end{aligned} \quad (10.24)$$

The right-hand bound holds for any $t_1^{i+1} \geq T_{na}$ thanks to $c_1, \bar{\lambda}_1 > 0$ in Lemma 10.4.1, namely, thanks to the exponential being decrescent. For each attack interval starting at $t_1^i \in \mathcal{T}_1$ and ending at $t_2^i \in \mathcal{T}_2$, with $i \in \mathbb{N}_{>0}$, the estimation error satisfies

$$\begin{aligned}|e(t_2^i)| &\leq \max_{t \in [t_1^i, t_1^i + T_a]} \left\{\hat{c}_1 \exp(-\hat{\lambda}_1(t - t_1^i)) + \hat{c}_2 \exp(\hat{\lambda}_2(t - t_1^i))\right\}|e(t_1^i)| \\ &= \max_{t \in [0, T_a]} \left\{\hat{c}_1 \exp(-\hat{\lambda}_1(t)) + \hat{c}_2 \exp(\hat{\lambda}_2(t))\right\}|e(t_1^i)| = \gamma_2(T_a)|e(t_1^i)| \end{aligned} \quad (10.25)$$

Thus, if there is an attack at the initial time, namely $t_1^1 = 0$, from Assumption 10.4.4, we have $|e(t_1^1)| \leq \bar{E}$, from (10.25) we have $|e(t_2^1)| \leq \gamma_2(T_a)\bar{E}$, and from (10.24) we have $|e(t_1^2)| \leq \gamma_1(T_{na})\gamma_2(T_a)\bar{E}$. If T_{na} and T_a are such that $\gamma_1(T_{na})\gamma_2(T_a) \leq 1$, then $|e(t_1^2)| \leq \bar{E}$, namely, at the beginning of the second attack, the error will be bounded by $\bar{E}$. Recursively, this implies that $|e(t)| \leq \bar{E}$ at the beginning of every attack, namely, for all $t \in \mathcal{T}_1 \cup \{0\}$. Notice that the error is bounded during the first interval of attack by $|e(t)| \leq \gamma_2(T_a)\bar{E}$ for all $t \in [0, t_2^1]$. Given that $\max_{t \in [0, T_{na}]} \gamma_1(t) = \gamma_1(0)$, we have that $|e(t)| \leq \gamma_1(0)\gamma_2(T_a)\bar{E}$ for all $t \in [0, t_1^2]$. Recursively, this implies that $|e(t)| \leq \gamma_1(0)\gamma_2(T_a)\bar{E}$ for all $t \geq 0$.

On the other hand, if there is no attack at the beginning, we have at least T_{na} seconds before the first attack is launched, namely $t_1^1 > T_{na}$, and from Assumption 10.4.4, we have $|e(t_2^0)| \leq \bar{E}$, from (10.24) we have $|e(t_1^1)| \leq \gamma_1(T_{na})\bar{E}$, and from (10.25), $|e(t_2^1)| \leq \gamma_1(T_{na})\gamma_2(T_a)\bar{E}$. If T_{na} and T_a are such that $\gamma_1(T_{na})\gamma_2(T_a) \leq 1$, then $|e(t_2^1)| \leq \bar{E}$, i.e., at the end of the first attack, the error will be bounded by $\bar{E}$. Recursively, this implies that $|e(t)| \leq \bar{E}$ at the end of every attack, namely, for all $t \in \mathcal{T}_2 \cup \{0\}$. Notice that since $\max_{t\in[0,T_{na}]} \gamma_1(t) = \gamma_1(0)$, the error is bounded during the first interval without attacks by $|e(t)| \leq \gamma_1(0)\bar{E}$ for all $t \in [0, t_1^1]$. Given that the function $\hat{c}_1 \exp(-\hat{\lambda}_1 t) + \hat{c}_2 \exp(\hat{\lambda}_2 t)$ in (10.22) with $t \in [t_1^1, t_2^1]$ is convex, we have that $|e(t)| \leq \gamma_1(0)\gamma_2(T_a)\bar{E}$ for all $t \in [0, t_2^1]$. Recursively, this implies that $|e(t)| \leq \gamma_1(0)\gamma_2(T_a)\bar{E}$ for all $t \geq 0$. □

Remark 10.4.6. () *Consider a set X_0, and for a given $x_0 \in X_0$ such that $x(0) = x_0$, define the set $\hat{X}_0(x_0) := \{x \in \mathbb{R}^n : x \in x_0 + \bar{E}\mathbb{B}\}$. Notice that thanks to Theorem 10.4.5, for each $x_0 \in X_0$, and each $\hat{x}_0 \in \hat{X}_0(x_0)$ we have that each solution pair $t \mapsto (x(t), \hat{x}(t))$ to* (10.8) *from $x(0) = x_0, \hat{x}(0) = \hat{x}_0$ satisfies*

1) *Boundedness of error at all times: $|x(t) - \hat{x}(t)| \leq \bar{E}$ for all $t \geq 0$;*
2) *Maximum error at the beginning of each attack: $|x(t_1^i) - \hat{x}(t_1^i)| \leq \gamma_1(T_{na})$ for each $i \in \mathbb{N}_{>0}$.*

Under an attack, it is possible that the error grows, and when there is no attack, the error decreases. However, using the proposed observer, the norm of the error always remains bounded by $\gamma_1(0)\gamma_2(T_a)\bar{E}$, as long as Assumption 10.4.4 holds.

10.5 Design of Observer-based Feedback Control

In this section, we present a set construction process to solve part 1 of Problem (⊗) and, based on it, a control design scheme that solves part 2. First, we define the sets that are going to be used in the control design.

10.5.1 Construction of Sets of Initial Conditions

Consider a closed set $S \subset \mathbb{R}^n$, $T_a, T_{na} > 0$, maps γ_1 and γ_2 as in (10.16) and (10.22), and $\bar{E} > 0$ in Assumption 10.4.4. Pick $\varepsilon > (1 + \gamma_1(0)\gamma_2(T_a))\bar{E}$ Define the set of initial states as

$$X_0 := S \setminus (\partial S + \varepsilon\mathbb{B}). \tag{10.26}$$

Note that under Assumption 10.4.4, X_0 is nonempty. Now, given $x_0 \in X_0$, set $x(0) = x_0$ and define the set-valued map

$$\hat{X}_0(x_0) := x_0 + \bar{E}\mathbb{B}. \tag{10.27}$$

Thus, for each $x_0 \in X_0$ and $\hat{x}_0 \in \hat{X}_0(x_0)$, it holds that $|x_0 - \hat{x}_0| \leq \bar{E}$. Additionally, notice that $\hat{x}_0 \in \tilde{X}$, where

$$\tilde{X} := X_0 + \bar{E}\mathbb{B} \tag{10.28}$$

which is an inflation of X_0 by $\bar{E}$. This construction of the sets of initial conditions, namely, X_0 and $\hat{X}_0$, leads to conditional invariance of S, as shown below.

Lemma 10.5.1. () *Given the system* (10.8)*, the observer* (10.14)*, the observer-based feedback law* κ (10.13)*, a closed set* $S \subset \mathbb{R}^n$*,* X_0 *as in* (10.26)*, and* $\hat{X}_0$ *as in* (10.27)*, consider the solution* $t \mapsto (x(t), \hat{x}(t))$ *to the resulting closed-loop system from the composition of* (10.8) *and* (10.14) *with* κ *from* $x(0) \in X_0$*,* $\hat{x}(0) \in \hat{X}_0(x(0))$ *and* $T_a, T_{na}, \bar{E}$*, such that conditions of Theorem 10.4.5 are satisfied. If* $S \setminus (\partial S + (1 + \gamma_1(0)\gamma_2(T_a))\bar{E}\mathbb{B}) \neq \emptyset$ *and* $\hat{x}(t) \in \tilde{X}$ *for all* $t \geq 0$*, then* $x(t) \in S$ *for all* $t \geq 0$*.*

Proof. From the definition of the sets X_0 and $\tilde{X}$, and $\varepsilon \geq (1 + \gamma_1(0)\gamma_2(T_a))\bar{E}$ it follows that $X_0 \subset S \setminus (\partial S + (1 + \gamma_1(0)\gamma_2(T_a))\bar{E}\mathbb{B})$ and $\tilde{X} \subset S \setminus (\partial S + \gamma_1(0)\gamma_2(T_a)\bar{E}\mathbb{B})$. Thus, under the assumption that $\hat{x}(t) \in \tilde{X}$ for all $t \geq 0$ with $|\hat{x}(t) - x(t)| \leq \gamma_1(0)\gamma_2(T_a)\bar{E}$ for all $t \geq 0$, it follows that $x(t) \in S$ for all $t \geq 0$. □

In words, the set of initial states X_0 and the set of initial estimates $\hat{X}_0$ are defined such that the initial estimation error is upper bounded by $\bar{E}$. Furthermore, we define $\tilde{X}$ in (10.28) as the set resulting from an inflation of X_0 by $\bar{E}$. Under this construction, for the resulting closed-loop system from the composition of (10.8) and (10.14) with κ, forward invariance of $\tilde{X}$ for the observer (10.14) implies conditional invariance of the set S for the system (10.8) with respect to X_0. Thus, the control objective is to enforce the estimate $\hat{x}$ in the set $\tilde{X}$ at all times to guarantee safety of S.

10.5.2 QP-based Safety Feedback Control

We use a control barrier function (CBF)-based approach for guaranteeing forward invariance of a subset $\bar{X}$ of the set $\tilde{X}$ in (10.28) for (10.14) (see [95]). In order to use

CBF for forward invariance, we need a zero sublevel set representation of the set $\bar{X}$. To this end, consider the function $h : \mathbb{R}^n \to \mathbb{R}$ and define a set

$$\bar{X} \coloneqq \{\hat{x} \mid h(\hat{x}) \leq 0\} \subset \tilde{X}. \tag{10.29}$$

Given an observer-based feedback law κ assigning the input $u = \kappa(t, \hat{x}, \bar{y})$ of (10.14), consider a solution $t \mapsto \hat{x}(t)$ to (10.14) from $\hat{x}(0) \in \bar{X}$. For the given measurement $\bar{y}$, it is sufficient to ensure that for each $\hat{x}(0) \in \bar{X}$, the estimate satisfies $\hat{x}(t) \in \bar{X} \subset \tilde{X}$, for all $t \geq 0$. The CBF condition for guaranteeing this when there is no attack is:

$$\frac{\partial}{\partial \hat{x}} h(\hat{x}(t)) \left(A\hat{x}(t) + B\kappa_1(\hat{x}(t), \bar{y}(t)) + L(\bar{y}(t) - C\hat{x}(t))\right) \leq \alpha_1(-h(\hat{x}(t))), \tag{10.30}$$

for all $t \geq 0$, where $t \mapsto \bar{y}(t)$ is the measured output signal, and the CBF condition under attack is

$$\frac{\partial}{\partial \hat{x}} h(\hat{x}(t)) \left(A\hat{x}(t) + B\kappa_2(\hat{x}(t), \bar{y}(t)) + \tilde{L}(y_s(t) - \tilde{C}\hat{x}(t))\right) \leq \alpha_2(-h(\hat{x}(t))), \tag{10.31}$$

for all $t \geq 0$, where $t \mapsto y_s(t)$ is the secured output signal and α_1, α_2 are class-$\mathcal{K}$ functions. We can use a Quadratic Programming (QP) formulation to compute the input u in the respective cases.

Consider the following QP for each $\hat{x} \in \bar{X}$ and $\bar{y}$ such that $x \in S$ for input synthesis when there is no attack:

$$\min_{(v,\eta)} \quad \frac{1}{2}|v - K\hat{x}|^2 + \frac{1}{2}\eta^2 \tag{10.32a}$$

$$\text{s.t.} \quad \frac{\partial}{\partial \hat{x}} h(\hat{x}) \left(A\hat{x} + Bv + L(\bar{y} - C\hat{x})\right) \leq -\eta h(\hat{x}), \tag{10.32b}$$

where K is the optimal LQR gain for the pair (A, B). Next, we use a similar QP to compute the input under attack. Consider the following QP for each $\hat{x} \in \bar{X}$ and $y_s = \tilde{C}x$ such that $x \in S$:

$$\min_{(v_s,\zeta)} \quad \frac{1}{2}|v_s - K\hat{x}|^2 + \frac{1}{2}\zeta^2 \tag{10.33a}$$

$$\text{s.t.} \quad \frac{\partial}{\partial \hat{x}} h(\hat{x}) \left(A\hat{x} + Bv_s + \tilde{L}(y_s - \tilde{C}\hat{x})\right) \leq -\zeta h(\hat{x}). \tag{10.33b}$$

The objective functions in (10.32) and (10.33) set the convex minimization problem to obtain the closest control action to the LQR control that satisfies the constraints. The additional decision variables, namely (η, ζ), respectively, are slack variables. Denote the solutions to (10.32) and (10.33) as $t \mapsto u_1^*(\hat{x}(t), \bar{y}(t))$ and $t \mapsto u_2^*(\hat{x}(t), \bar{y}(t))$, respectively. To guarantee continuity of these solutions with respect to $\hat{x}$, we need to impose the strict complementary slackness condition (see [97]). In brief, if the $i-$th constraint of (10.32) (or (10.33)), with $i \in \{1, 2\}$, is written as $G_i(\hat{x}, \bar{y}, u_{QP}) \leq 0$ with $u_{QP} = (v, \eta)$ (respectively, $u_{QP} = (v_s, \zeta)$ for (10.33)), and the corresponding Lagrange multiplier is $\bar{\lambda}_i \in \mathbb{R}_{\geq 0}$, then strict complementary slackness requires that $\bar{\lambda}_i^* G(\hat{x}, \bar{y}, u_{QP}^*) < 0$, where u_{QP}^* and $\bar{\lambda}_i^*$ denote the optimal solution and the corresponding optimal Lagrange multiplier, respectively. We are now ready to state the second main result of the chapter.

Theorem 10.5.2. () *Given system* (10.8), *suppose that Assumptions 10.3.1 and 10.4.4 hold. For the attack model* (10.9), *the observer* (10.14), *and a closed set* $S \subset \mathbb{R}^n$, *let* X_0, $\hat{X}_0$, $\tilde{X}$ *and* $\bar{X}$ *be given as in* (10.26)-(10.29), *and assume that the strict complementary slackness holds for the QPs* (10.32) *and* (10.33) *for all* $\hat{x} \in \tilde{X}$. *The following holds:*

1. *If* $S \setminus (\partial S + (1+\gamma_1(0)\gamma_2(T_a))\bar{E}\mathbb{B}) \neq \emptyset$, *then, for each* $\hat{x} \in \text{int}(\bar{X})$, *the QPs* (10.32) *and* (10.33) *are feasible and their respective solutions* $t \mapsto u_1^*(\hat{x}(t), \bar{y}(t)), t \mapsto u_2^*(\hat{x}(t), \bar{y}(t))$ *are continuous on* $\text{int}(\bar{X})$.

2. *For each* $x_0 \in X_0$ *and* $\hat{x}_0 \in \bar{X} \cap \hat{X}_0(x_0)$, *each solution pair* $t \mapsto (x(t), \hat{x}(t))$ *to the closed-loop system resulting from assigning the input* u *of* (10.8) *and* (10.14) *to the observer-based feedback law* κ *in* (10.13) *with* $\kappa_1(\hat{x}, \bar{y}) = u_1^*(\hat{x}, \bar{y})$ *and* $\kappa_2(\hat{x}, \bar{y}) = u_2^*(\hat{x}, \bar{y})$, *satisfies* $\hat{x}(t) \in \bar{X}$ *and* $x(t) \in S$ *for all* $t \geq 0$.

Proof. Feasibility and continuity of the solutions of the QPs (10.32) and (10.33) follow from [97, Lemma 5] and [97, Theorem 1], respectively. From feasibility of the QPs and continuity of its solutions, there exists a continuous control input u such that the CBF condition (10.31) holds along the closed-loop trajectory $\hat{x}(t)$. Thus, it follows that the set $\tilde{X}$ is forward invariant for the observer (10.14), and hence $\hat{x}(t) \in \tilde{X}$ for all $t \geq 0$ and for each $\hat{x}(0) \in \tilde{X}_0$. Thanks to Theorem 10.4.5, for each $x(0) \in X_0$, one has $|e(t)| \leq \gamma_1(0)\gamma_2(T_a)\bar{E}$, which from Lemma 10.5.1 implies $x(t) \in S$ for all $t \geq 0$. □

Thus, the proposed observer-based feedback framework, based on a switching observer and a switching control scheme, can keep the system safe even under output attacks. Next, we evaluate our proposed scheme via numerical experiments.

10.6 Illustrative Example

Consider a system $\mathcal{S}$ as in (10.8), with state $x = (x_1, x_2) \in \mathbb{R}^2$, input $u \in \mathbb{R}$, and dynamics $\dot{x} = (0.5x_1 + x_2, u), y = (x_1, x_2)$ where $y_a = x_1$ is only available when there are no attacks. DoS attacks have maximum duration of $T_a = 1.6$ seconds and are launched only after at least $T_{na} = 0.047$ seconds without an attack. Here, u is designed such that every response $t \mapsto x(t)$ to $\mathcal{S}$ satisfies $x(t) \in S := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + 2x_2^2 + 2x_1x_2 - 35 \leq 0\}$ for all $t \geq 0$, given that $x(0) \in X_0 := S \setminus (\partial S + \varepsilon\mathbb{B})$, with $\varepsilon = 2.01$.

An observer as in (10.14) is designed. Given that Assumption 10.3.1 is satisfied, and by setting $L = \left[\begin{smallmatrix} 32 & 0.5 \\ 0.5 & 32 \end{smallmatrix}\right]$ and $\tilde{L} = \left[\begin{smallmatrix} 0.05 \\ 3.2 \end{smallmatrix}\right]$, we have $\lambda(A - LC) = -31.75 \pm i0.43$, and $\lambda(A - \tilde{L}\tilde{C}) = \{0.5, -3.2\}$. Given $x_0 = (5.3, -2.4)$, $\hat{x}_0 = (4.9, -2.1)$, and $\bar{E} = 0.55$, we have that $|e(0)| = 0.5 \leq \bar{E}$, so Assumption 10.4.4 holds.

Thus, by applying Lemma 10.4.1, with $P = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], Q = \left[\begin{smallmatrix} 63 & 0 \\ 0 & 64 \end{smallmatrix}\right]$, and given that every pair of subsequent attacks are separated by at least T_{na} seconds, the estimation error satisfies $|e(t)| \leq \gamma_1(t - t_2^i)e(t_2^i)$ for all $t \in [t_2^i, t_1^{i+1}]$, $i \in \mathbb{N}$, $\gamma_1(T_{na} = 0.047) = 0.226$, and is displayed in green[2] in Figure 10.2. Given that the growth rate of the exponential defining the function γ_1 is negative, the bound on the error norm decreases at each interval without attacks.

In addition, by applying Lemma 10.4.3 with $\hat{c}_1 = 1.12, \hat{c}_2 = 1.19, \hat{\lambda}_1 = 3.2, \hat{\lambda}_2 = 0.5$, $\hat{P} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \hat{Q} = \left[\begin{smallmatrix} 6.4 & 0 \\ 0 & 6.4 \end{smallmatrix}\right]$, $\Phi = \left[\begin{smallmatrix} 1 & -0.25 \\ 0 & 0.97 \end{smallmatrix}\right]$, and $\hat{A}_{22} = 0.5$, given that every attack has a maximum duration of T_a seconds, the estimation error satisfies $|e(t)| \leq \gamma_2(T_a)|e(t_1^i)|$ for all $t \in [t_1^i, t_2^i]$, $i \in \mathbb{N}_{>0}$ where $\gamma_2(T_a) = 2.65$, and is displayed in light blue in Figure 10.2. Thanks to Theorem 1, given that $\gamma_1(T_{na})\gamma_2(T_1) \leq 1$, the error satisfies $|e(t)| \leq c_1\gamma_2(T_a)\bar{E} = 1.46$ for all $t \geq 0$.

In Figure 10.3, the set X_0 is a deflation of the set S by ε, and the set $\tilde{X}$ is an inflation of the set X_0 by $\bar{E}$. The set of initial estimations, $\hat{X}_0(x_0)$, is defined as the ball of radius $\bar{E}$ centered at x_0. Thus, the estimator $\hat{x}$ is initialized at $X_0(x_0) \subset \tilde{X}$. The set $\bar{X} := \{(x_1, x_2) \in \mathbb{R}^2 : h(x) \leq 0\} \subset \tilde{X}$ is defined by the barrier function $h(x) = x_1^2 + 2x_2^2 + 2x_1x_2 - 12.5$. Given that the set $S \subset \mathbb{R}^n$ is such that $S \setminus (\partial S + (1 + \gamma_1(0)\gamma_2(T_a))\bar{E}\mathbb{B}) \neq \emptyset$, by assigning $K = [2.3016, \; 2.3671]$ and solving the QPs (10.32) and (10.33) at every point of the trajectory $\hat{x}(t) \in \bar{X}$ to assign the input action, thanks to Theorem 10.5.2, we ensure that $\hat{x}(t) \in \bar{X}$ for all t, and consequently, $x(t) \in S$ for all t.

[2]Code at https://github.com/HybridSystemsLab/SafeRecovery-DoSAttacks

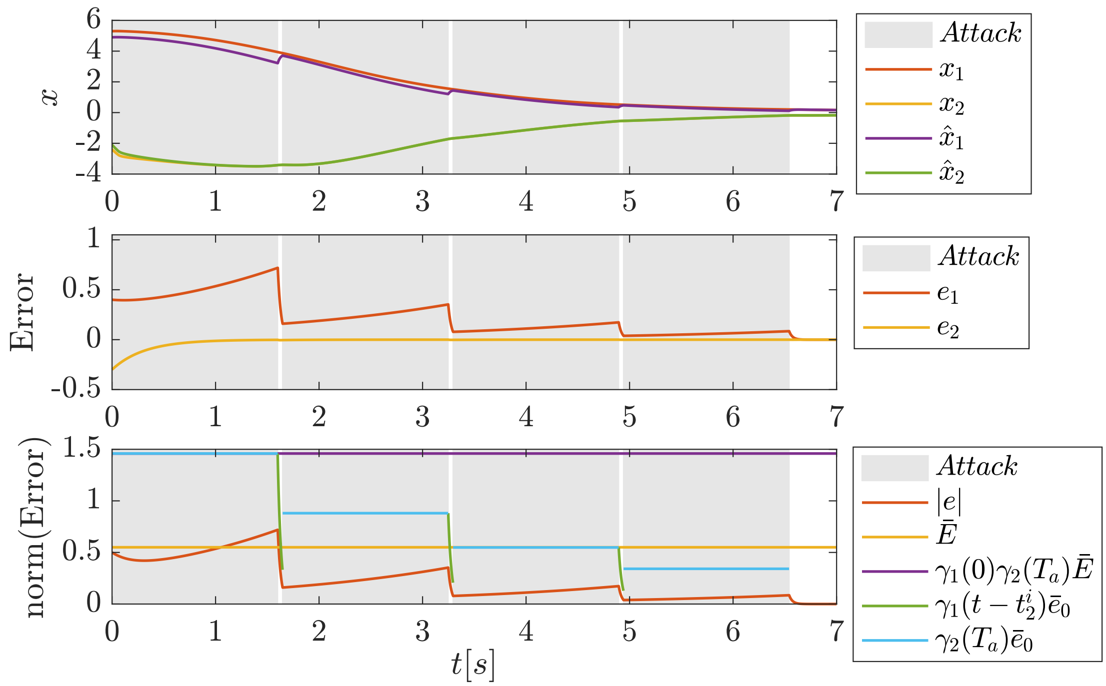

Figure 10.2: Solutions to the 2D system and state estimation error during worst-case attacks of $T_a = 1.6s$, for $x_0 = (5.3, -2.4)$, $\hat{x}_0 = (4.9, -2.1)$, and $\bar{E} = 0.55$. In the third plot, the bound (purple) is defined as in Theorem 1.

Acknowledgements

Sections 10.1-10.6, excluding the proofs, are a reprint of the material as it appears in "An Observer-based Switching Algorithm for Safety under Sensor Denial-of-Service Attacks." [7]. The dissertation author was the first author of this paper.

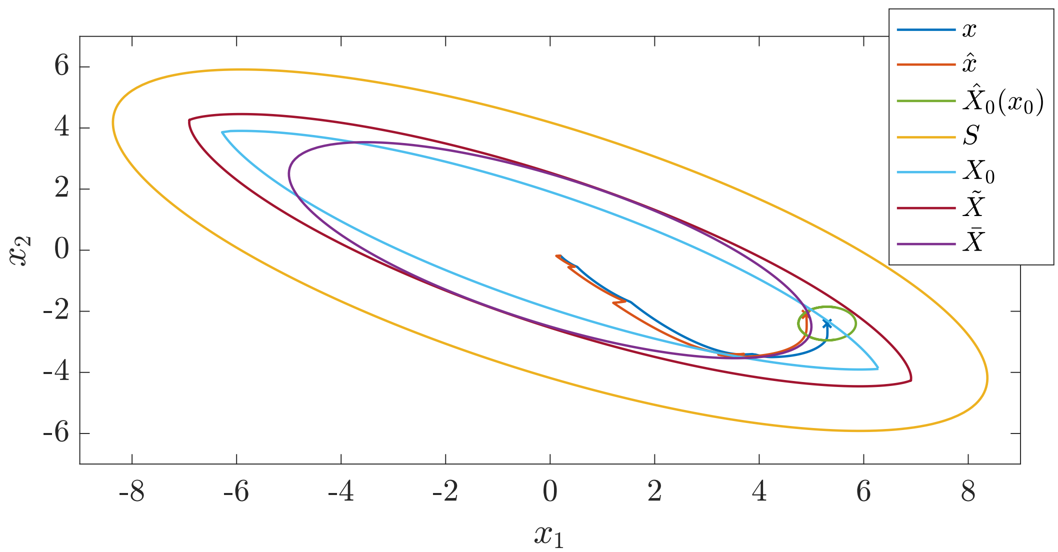

Figure 10.3: Phase portrait of $\dot{x} = (x_2, u), y = (x_1, x_2)$ with state estimation for safe recovery of DoS attacks in the measurements of x_1. By initializing the estimation $\hat{x}$ in the $\bar{E}-$ball (green) around $x(0)$, the set $\bar{X}$ (purple) is rendered forward invariant for $\hat{x}$ (orange), and the safe set S (yellow) conditionally invariant for x (dark blue) with respect to the set of allowed initial states, namely X_0 (light blue), via the control barrier function. The set $\tilde{X}$ (scarlet) denotes the allowed initial observer states.

Part IV

Conclusions and Future Work

Summary

In this dissertation we presented a framework to formulate two-player zero-sum games under dynamic constraints defined by hybrid dynamical equations as in [20]. Variations of the constraints structure and the termination conditions allow to consider different type of games for which results are provided with sufficient conditions to characterize their solution. Connections between optimality and stability conditions are revealed with the appropriate notion for each type of game.

In Chapter 3, we formulate the general framework of two-player zero-sum hybrid games. In section 3.1., we provide the elements of a game, and the solution concept in terms of a saddle-point equilibrium. Such framework is invoked in Chapters 4-6. Scenarios in which the control action to a hybrid system is selected by a player P_1 to accomplish an objective under the infinite horizon and countereffect the damage of an adversarial player P_2 are studied. By encoding the objectives of the players in the optimization of a cost functional that depends on the actions and resulting solutions to the hybrid system, defined as functions of hybrid time and, hence, can flow or jump, sufficient conditions in the form of Hamilton-Jacobi-Bellman-Isaacs equations are provided to characterize the solution of the game. The main result ensures that the optimal strategy of P_1 minimizes the cost under the worst-case scenario attack/disturbance. Additional conditions are proposed to allow the saddle-point strategy to render a set of interest asymptotically stable by letting the value function take the role of a Lyapunov function.

In Chapter 4, we consider two-player zero-sum hybrid games under terminal time conditions. A control action is designed by P_1 to accomplish an optimization objective within a finite hybrid horizon while P_2 has the goal to maximize it under the hybrid dynamic constraints. The general formulation of hybrid games is used as the basis to state sufficient conditions in terms of Hamilton-Jacobi-Isaacs hybrid PDEs to attain the solution of the game. When such conditions are only satisfied approximately, a result is established with an approximate optimality outcome.

In Chapter 5, we formulated a two-player zero-sum finite-horizon hybrid game with a set of the state space that defines the termination of the game once the solutions to the hybrid system eneter it. We present sufficient conditions given in terms of Hamilton-Jacobi-Bellman-Isaacs-like equations to guarantee to attain a solution to the game. It is shown that when the players select the optimal strategy, the value function can be evaluated without computing solutions to the hybrid system. Under additional conditions, we show that the optimal state-feedback laws render a set of interest pre-asymptotically stable for the resulting hybrid closed-loop system. A disturbance rejection scenario is studied for which the effect of the perturbation is upper bounded.

In Chapter 6, we formulated a two-player zero-sum game under dynamic constraints given in terms of a hybrid inclusion. The game consists of a min-max problem involving a properly defined cost functional associated to the actions and corresponding (potentially nonunique) solutions to the system. We presented sufficient conditions given in terms of Hamilton-Jacobi-Bellman-Isaacs-like equations to establish a bound on the worst-case cost under the optimal strategy and to exactly evaluate it. Under additional conditions, we show that the proposed optimal state feedback laws render a set of interest pre-asymptotically stable for the resulting hybrid closed-loop system.

In Chapter 7, we studied the problem of designing a stabilizing controller for a hybrid system as in [57] under disturbances as an inverse optimal problem. Via characterizing the stabilizing feedback law that assigns the control input, we formulated the problem as a two-player zero-sum hybrid game to minimize the effect of the worst-case disturbance. Instead of solving the game for a given cost functional, we design the cost functional that the stabilizing controller minimizes (inverse approach). A QP formulation is shown to solve the problem, with a nonQP variation that addresses the myopic nature of the former.

In Chapter 8, we studied the problem of designing safety filters for a hybrid system as in [57] under disturbances as an inverse optimal problem. Via characterizing the safeguarding feedback law that assigns the control input, we formulate the problem as a two-player zero-sum hybrid game to minimize the effect of the worst-case disturbance. We design the cost functional that the safeguarding feedback law minimizes.

In Chapter 9, we proposed a hybrid system formulation with a zero-sum hybrid game framework to describe exhaustively capture-the-flag games. While synthesizing a controller in an optimal fashion represents an unsolved challenge, a simulation tool is developed to implement the game model with a preliminary controller design.

In Chapter 10, we studied two-player hybrid games under imperfect dtate information. As a case of study, we designed a switched controller that, together with a switched observer, ensures a linear time-invariant system to recover safely from finite-time DoS attacks in some of the system outputs. Conditional invariance of a set is guaranteed with respect to a subset of initial conditions by employing a barrier function approach and bounding the estimation error at all times.

Future Directions

This dissertation is meant to be a stepping stone for the sutdy of hybrid games. The formulation provided herein provides several directions that can be explored as for future research areas.

- **Solution to the Min-Max Problem:** Structural conditions on the hybrid system that do not involve V and guarantee the existence of a solution to the min-max problems based on the smoothness and regularity of the data of the system, similar to those in [98] are to be established.

- **Different type of Hybrid Games:** An extension of the formulation provided in this dissertation to N-player non-zero-sum hybrid games is to be studied, together with Stackelberg hybrid games in which there is an advantage for playing first. We expect the results can be generalized to randomized strategies, in particular, through the connection between set-valued dynamics and nonuniqueness of solutions, which captures nondeterminism.

- **Recasting Finite-Horizon Games as Infinite-Horizon:** Designing projection tools to deal with stabilizing or safeguarding feedback laws that render maximal solutions noncomplete and force them to satisfy the constraints specified by the hybrid dynamics are to be studied under the inverse-optimal approach of hybrid games.

- **Capture-the-Flag Winning Strategy:** Provided the hybrid dynamical model of capture-the-flag games and the simulation tool that encodes the rules of the game, the design of near-optimal controllers with performance guarantees is an open research problem. Model predictive control, multi-stage, or learning approaches can be studied.

Part V

Proofs of Results for Two-Player Zero-Sum Hybrid Games

Appendix A

Proofs of Chapter 3

A.1 Proof of Theorem 3.2.1.

To show the claim we proceed as follows:

a) Pick an initial condition ξ and evaluate the cost associated to any solution yield by γ, with values as in (3.13) and (3.14), from ξ. Show that this cost coincides with the value of the function V at ξ.

b) Lower bound the cost associated to any solution from ξ when P_2 plays $\gamma_2 := (\gamma_{C2}, \gamma_{D2})$ by the value of the function V evaluated at ξ.

c) Upper bound the cost associated to any solution from ξ when P_1 plays $\gamma_1 := (\gamma_{C1}, \gamma_{D1})$ by the value of the function V evaluated at ξ.

d) By showing that the cost of any solution from ξ when P_1 plays γ_1 is not greater than the cost of any solution yield by γ from ξ, and by showing that the cost of any solution from ξ when P_2 plays γ_2 is not less than the cost of any solution yield by γ from ξ, we show optimality of γ in Problem ($\diamond$) in the min-max sense.

Proceeding as in item a above, pick any $\xi \in \Pi(\overline{C}) \cup \Pi(D)$ and any $(\phi^*, u^*) \in \mathcal{S}^{\infty}_{\mathcal{H}}(\xi)$ with $\operatorname{dom}\phi^* \ni (t,j) \mapsto u^*(t,j) = \gamma(t,j,\phi^*(t,j))$. We show that the cost of (ϕ^*, u^*) is optimal in the min-max sense. Given that V satisfies (3.9), and γ_C is as in (3.13), for each $j \in \mathbb{N}$ such that $I^j_{\phi^*} = [t_j, t_{j+1}]$ has a nonempty interior $\operatorname{int} I^j_{\phi^*}$, we have, for all

$t \in \mathrm{int} I^j_{\phi^*}$,

$$
\begin{aligned}
0 = & \min_{\substack{u_{C1} \\ u_C(t,j)=(u_{C1},u_{C2})\in\Pi_u(\phi^*(t,j),C)}} \max_{u_{C2}} \{L_C(\phi^*(t,j), u_C(t,j)) + \langle \nabla V(\phi^*(t,j)), F(\phi^*(t,j), u_C(t,j))\rangle\} \\
& = L_C(\phi^*(t,j), \gamma_C(\phi^*(t,j))) + \langle \nabla V(\phi^*(t,j)), F(\phi^*(t,j), \gamma_C(\phi^*(t,j)))\rangle
\end{aligned}
$$

and $\phi^*(t,j) \in C_\gamma$, as in (2.4). Given that V is continuously differentiable on a neighborhood of $\Pi(C)$, we can express its total derivative along ϕ^* as

$$
\frac{d}{dt} V(\phi^*(t,j)) = \langle \nabla V(\phi^*(t,j)), F(\phi^*(t,j), \gamma_C(\phi^*(t,j)))\rangle \tag{A.1}
$$

for every $(t,j) \in \mathrm{int}(I^j_{\phi^*}) \times \{j\}$ with $\mathrm{int}(I^j_{\phi^*})$ nonempty. Given that V satisfies (3.10) and γ_D is as in (3.14), for every $(t_{j+1}, j) \in \mathrm{dom}\, \phi^*$ such that $(t_{j+1}, j+1) \in \mathrm{dom}\, \phi^*$, we have that

$$
\begin{aligned}
V(\phi^*(t_{j+1}, j)) = & \min_{\substack{u_{D1} \\ u_D(t_{j+1},j)=(u_{D1},u_{D2})\in\Pi_u(\phi^*(t_{j+1},j),D)}} \max_{u_{D2}} \{L_D(\phi^*(t_{j+1}, j), u_D(t_{j+1}, j)) \\
& \qquad + V(G(\phi^*(t_{j+1}, j), u_D(t_{j+1}, j)\} \\
& = L_D(\phi^*(t_{j+1}, j), \gamma_D(\phi^*(t_{j+1}, j))) + V(G(\phi^*(t_{j+1}, j), \gamma_D(\phi^*(t_{j+1}, j)))) \\
& = L_D(\phi^*(t_{j+1}, j), \gamma_D(\phi^*(t_{j+1}, j))) + V(\phi^*(t_{j+1}, j+1))
\end{aligned} \tag{A.2}
$$

where $\phi^*(t_{j+1}, j) \in D_\gamma$ is defined in (2.4). Now, given that (ϕ^*, u^*) is maximal with $\mathrm{dom}\, \phi^* \ni (t,j) \mapsto u^*(t,j) = \gamma(t,j,\phi^*(t,j))$, thanks to (C.1) and (C.2), from Corollary 3.2.3 and (3.11), we have that

$$
V(\xi) = \mathcal{J}(\xi, u^*). \tag{A.3}
$$

Continuing with item b as above, pick any $(\phi_s, u^s) \in \mathcal{S}^s_{\mathcal{H}}(\xi)$ where $\mathcal{S}^s_{\mathcal{H}}(\xi)(\subset \mathcal{S}^\infty_{\mathcal{H}}(\xi))$ is the set of solutions (ϕ, u) with $u = (u_1, u_2)$, $\mathrm{dom}\, \phi \ni (t,j) \mapsto u_1(t,j) = \bar\gamma_1(t,j,\phi(t,j))$ for some $\bar\gamma_1 \in \mathcal{K}_1$, $\mathrm{dom}\, \phi \ni (t,j) \mapsto u_2(t,j) = \gamma_2(t,j,\phi(t,j))$ for $\gamma_2 := (\gamma_{C2}, \gamma_{D2})$ as in (3.13) and (3.14). Since $\bar\gamma_1$ does not necessarily attain the minimum in (3.9), then, for each $j \in \mathbb{N}$ such that $I^j_{\phi_s} = [t_j, t_{j+1}]$ has a nonempty interior $\mathrm{int} I^j_{\phi_s}$, we have that for every $t \in \mathrm{int} I^j_{\phi_s}$,

$$
0 \le L_C(\phi_s(t,j), u^s_C(t,j)) + \langle \nabla V(\phi_s(t,j)), F(\phi_s(t,j), u^s_C(t,j))\rangle .
$$

Similarly to (C.1), we have

$$
\frac{d}{dt} V(\phi_s(t,j)) := \langle \nabla V(\phi_s(t,j)), F(\phi_s(t,j), u^s_C(t,j))\rangle \tag{A.4}
$$

for every $(t, j) \in \text{int}(I^j_{\phi_s}) \times \{j\}$ with $\text{int}(I^j_{\phi_s})$ nonempty. In addition, since $\bar{\gamma}_1$ does not necessarily attain the minimum in (3.10), then for every $(t_{j+1}, j) \in \text{dom}\,\phi_s$ such that $(t_{j+1}, j+1) \in \text{dom}\,\phi_s$, we have

$$\begin{aligned} V(\phi_s(t_{j+1}, j)) &\leq L_D(\phi_s(t_{j+1}, j), u^s_D(t_{j+1}, j)) + V(G(\phi_s(t_{j+1}, j), u^s_D(t_{j+1}, j))) \\ &= L_D(\phi_s(t_{j+1}, j), u^s_D(t_{j+1}, j)) + V(\phi_s(t_{j+1}, j+1)) \end{aligned} \tag{A.5}$$

Now, given that (ϕ_s, u_s) is maximal, with $u^s = (u^s_1, u^s_2)$, u^s_1 defined by any $\bar{\gamma}_1 \in \mathcal{K}_1$, and u^s_2 defined by γ_2 as in (3.13) and (3.14), thanks to (C.4) and (C.5), from Proposition 3.2.2 and (3.11), we have

$$V(\xi) \leq \mathcal{J}(\xi, u^s). \tag{A.6}$$

Proceeding with item c as above, pick any $(\phi_w, u^w) \in \mathcal{S}^w_{\mathcal{H}}(\xi)$, where $\mathcal{S}^w_{\mathcal{H}}(\xi)(\subset \mathcal{S}^\infty_{\mathcal{H}}(\xi))$ is the set of solutions (ϕ, u) with $u = (u_1, u_2)$, $\text{dom}\,\phi \ni (t, j) \mapsto u_1(t, j) = \gamma_1(t, j, \phi(t, j))$ for $\gamma_1 := (\gamma_{C1}, \gamma_{D1})$ as in (3.13) and (3.14), $\text{dom}\,\phi \ni (t, j) \mapsto u_2(t, j) = \bar{\gamma}_2(t, j, \phi(t, j))$ for some $\bar{\gamma}_2 \in \mathcal{K}_2$. Since $\bar{\gamma}_w$ does not necessarily attain the maximum in (3.9), then, for each $j \in \mathbb{N}$ such that $I^j_{\phi_w} = [t_j, t_{j+1}]$ has a nonempty interior $\text{int}I^j_{\phi_w}$, we have that for every $t \in \text{int}I^j_{\phi_w}$,

$$0 \geq L_C(\phi_w(t, j), u^w_C(t, j)) + \langle \nabla V(\phi_w(t, j)), F(\phi_w(t, j), u^w_C(t, j)) \rangle$$

Similarly to (C.1), we have

$$\frac{d}{dt} V(\phi_w(t, j)) := \langle \nabla V(\phi_w(t, j)), F(\phi_w(t, j), u^w_C(t, j)) \rangle \tag{A.7}$$

for every $(t, j) \in \text{int}(I^j_{\phi_w}) \times \{j\}$ with $\text{int}(I^j_{\phi_w})$ nonempty. In addition, since $\bar{\gamma}_2$ does not necessarily attain the maximum in (3.10), then for every $(t_{j+1}, j) \in \text{dom}\,\phi_w$ such that $(t_{j+1}, j+1) \in \text{dom}\,\phi_w$, we have

$$\begin{aligned} V(\phi_w(t_{j+1}, j)) &\geq L_D(\phi_w(t_{j+1}, j), u^w_D(t_{j+1}, j)) + V(G(\phi_w(t_{j+1}, j), u^w_D(t_{j+1}, j))) \\ &= L_D(\phi_w(t_{j+1}, j), u^w_D(t_{j+1}, j)) + V(\phi_w(t_{j+1}, j+1)) \end{aligned} \tag{A.8}$$

Now, given that (ϕ_w, u^w) is maximal, with $u^w = (u^w_1, u^w_2)$, u^w_1 defined by γ_1 as in (3.13) and (3.14), and u^w_2 defined by any $\bar{\gamma}_2 \in \mathcal{K}_2$, thanks to (C.7) and (C.8), from Corollary 3.2.3 and (3.11), we have that

$$V(\xi) \geq \mathcal{J}(\xi, u^w). \tag{A.9}$$

Finally, by proceeding as in item d above, by applying the infimum on each side of (D.23) over the set $\mathcal{S}_{\mathcal{H}}^{\infty}(\xi)$, we obtain

$$V(\xi) \leq \inf_{u_1:(\phi_s,(u_1,\kappa_2(\phi_s)))\in\mathcal{S}_{\mathcal{H}}^{\infty}(\xi)} \mathcal{J}(\xi,(u_1,\kappa_2(\phi_s))) =: \overline{V}(\xi)$$

By applying the supremum on each side of (C.9) over the set $\mathcal{S}_{\mathcal{H}}^{\infty}(\xi)$, we obtain

$$V(\xi) \geq \sup_{u_2:(\phi_w,(\kappa_1(\phi_w),u_2))\in\mathcal{S}_{\mathcal{H}}^{\infty}(\xi)} \mathcal{J}(\xi,(\kappa_1(\phi_w),u_2)) =: \underline{V}(\xi).$$

Given that $V(\xi) = \mathcal{J}(\xi, u^*)$ from (A.3), we have that for any $\xi \in \Pi(\overline{C}) \cup \Pi(D)$, each $(\phi^*, u^*) \in \mathcal{S}_{\mathcal{H}}^{\infty}(\xi)$ with $u^* = (\kappa_1(\phi^*), \kappa_2(\phi^*))$ satisfies

$$\underline{V}(\xi) \leq \mathcal{J}(\xi, u^*) \leq \overline{V}(\xi) \tag{A.10}$$

Thanks to $(\phi^*, u^*) \in \mathcal{S}_{\mathcal{H}}^{s}(\xi) \cap \mathcal{S}_{\mathcal{H}}^{w}(\xi) (\subset \mathcal{S}_{\mathcal{H}}^{\infty}(\xi))$, we have

$$\underline{V}(\xi) = \sup_{(\phi^*,(\kappa_1(\phi^*),\kappa_2(\phi^*)))\in\mathcal{S}_{\mathcal{H}}^{\infty}(\xi)} \mathcal{J}(\xi,(\kappa_1(\phi^*),\kappa_2(\phi^*))) \tag{A.11}$$

and

$$\overline{V}(\xi) = \inf_{(\phi^*,(\kappa_1(\phi^*),\kappa_2(\phi^*)))\in\mathcal{S}_{\mathcal{H}}^{\infty}(\xi)} \mathcal{J}(\xi,(\kappa_1(\phi^*),\kappa_2(\phi^*))). \tag{A.12}$$

Given that the supremum and infimum are attained in (D.30) and (D.7) by $\underline{V}(\xi)$ and $\overline{V}(\xi)$, respectively, (C.10) leads to

$$\mathcal{J}(\xi, u^*) = \min_{u_1} \max_{u_2}{}_{(u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^{\infty}(\xi)} \mathcal{J}(\xi,(u_1,u_2)) \tag{A.13}$$

Thus, from (A.3) and (C.13), $V(\xi)$ is the value function for $\mathcal{H}$, as in Definition 3.1.6, and from (C.10), κ is the saddle-point equilibrium as in Definition 3.1.3.

A.2 Proof of Proposition 3.2.2.

Given a $(\phi, u) \in \mathcal{S}_{\mathcal{H}}^{\infty}(\xi)$, where $\{t_j\}_{j=0}^{\sup_j \operatorname{dom}\phi}$ is a nondecreasing sequence associated with the hybrid time domain of (ϕ, u) as in Definition 2.2.2, for each $j \in \mathbb{N}$ such that I_{ϕ}^{j} has a nonempty interior $\operatorname{int} I_{\phi}^{j}$, by integrating (3.15) over I_{ϕ}^{j}, we obtain

$$0 \geq \int_{t_j}^{t_{j+1}} \left(L_C(\phi(t,j), u_C(t,j)) + \frac{d}{dt} V(\phi(t,j)) \right) dt$$

from where we have

$$0 \geq \int_{t_j}^{t_{j+1}} L_C(\phi(t,j), u_C(t,j))dt + V(\phi(t_{j+1}, j)) - V(\phi(t_j, j))$$

Pick $(t^*, j^*) \in \mathrm{dom}(\phi, u)$. Summing from $j = 0$ to $j = j^*$ we obtain

$$0 \geq \sum_{j=0}^{j^*} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j), u_C(t,j))dt + \sum_{j=0}^{j^*} (V(\phi(t_{j+1}, j)) - V(\phi(t_j, j)))$$

Then, solving for V at the initial condition $\phi(0,0)$, we obtain

$$V(\phi(0,0)) \geq \\ \sum_{j=0}^{j^*} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j), u_C(t,j))dt + V(\phi(t_1, 0)) + \sum_{j=1}^{j^*} (V(\phi(t_{j+1}, j)) - V(\phi(t_j, j))) \tag{A.14}$$

In addition, if $j^* > 0$, adding (3.16) from $j = 0$ to $j = j^* - 1$, we obtain

$$\sum_{j=0}^{j^*-1} V(\phi(t_{j+1}, j)) \geq \sum_{j=0}^{j^*-1} L_D(\phi(t_{j+1}, j), u_D(t_{j+1}, j)) + \sum_{j=0}^{j^*-1} V(\phi(t_{j+1}, j+1))$$

Then, solving for V at the first jump time, we obtain

$$V(\phi(t_1, 0)) \geq V(\phi(t_1, 1)) + \sum_{j=0}^{j^*-1} L_D(\phi(t_{j+1}, j), u_D(t_{j+1}, j)) \\ + \sum_{j=1}^{j^*-1} (V(\phi(t_{j+1}, j+1)) - V(\phi(t_{j+1}, j)))$$

In addition, given that $\phi(0,0) = \xi$, lower bounding $V(\phi(t_1, 0))$ in (A.14) by the right-hand side of (A.15), we obtain

$$\begin{aligned} V(\xi) \geq & \sum_{j=0}^{j^*} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j), u_C(t,j))dt + V(\phi(t_1, 0)) + \sum_{j=1}^{j^*} (V(\phi(t_{j+1}, j)) - V(\phi(t_j, j))) \\ \geq & \sum_{j=0}^{j^*} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j), u_C(t,j))dt + \sum_{j=0}^{j^*-1} L_D(\phi(t_{j+1}, j), u_D(t_{j+1}, j)) \\ & + \sum_{j=1}^{j^*-1} (V(\phi(t_{j+1}, j+1)) - V(\phi(t_{j+1}, j))) + \sum_{j=1}^{j^*} (V(\phi(t_{j+1}, j)) - V(\phi(t_j, j))) \\ & + V(\phi(t_1, 1)) \end{aligned}$$

Since

$$V(\phi(t_1,1)) + \sum_{j=1}^{j^*-1} (V(\phi(t_{j+1}, j+1)) - V(\phi(t_{j+1}, j))) + \sum_{j=1}^{j^*} (V(\phi(t_{j+1}, j)) - V(\phi(t_j, j)))$$

$$= V(\phi(t_{j^*+1}, j^*)) + V(\phi(t_1,1)) + \sum_{j=1}^{j^*-1} (V(\phi(t_{j+1}, j+1))) - \sum_{j=1}^{j^*} (V(\phi(t_j, j)))$$

$$= V(\phi(t_{j^*+1}, j^*))$$

then we have

$$V(\xi) \geq V(\phi(t_{j^*+1}, j^*)) + \sum_{j=0}^{j^*} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j), u_C(t,j))dt + \sum_{j=0}^{j^*-1} L_D(\phi(t_{j+1}, j), u_D(t_{j+1}, j))$$

By taking the limit when $t_{j^*+1} + j^* \to \sup_t \operatorname{dom} \phi + \sup_j \operatorname{dom} \phi$, we establish (5.13). Notice that if $j^* = 0$, the solution (ϕ, u) is continuous and (5.13) reduces to

$$V(\xi) \geq \limsup_{t^* \to \sup_t \operatorname{dom} \phi} \int_{t_0}^{t^*} L_C(\phi(t,0), u_C(t,0))dt + V(\phi(t^*, 0)).$$

On the other hand, if $t_{j^*+1} = 0$ for all j^*, the solution (ϕ, u) is discrete and (5.13) reduces to

$$V(\xi) \geq \limsup_{j^* \to \sup_j \operatorname{dom} \phi} \sum_{j=0}^{j^*-1} L_D(\phi(0,j), u_D(0,j)) + V(\phi(0, j^*)).$$

□

A.3 Proof of Lemma 3.3.4.

($\to$) From (3.13) and (3.14) we have

$$\min_{u_{C1}} \max_{u_{C2}}_{u_C = (u_{C1}, u_{C2}) \in \Pi_u(x,C)} \{L_C(x, u_C) + \langle \nabla V(x), F(x, u_C) \rangle\}$$

$$= L_C(x, \kappa_C(x)) + \langle \nabla V(x), F(x, \kappa_C(x)) \rangle \quad \forall x \in \Pi(C) \quad \text{(A.15)}$$

and

$$\min_{u_{D1}} \max_{u_{D2}}_{u_D = (u_{D1}, u_{D2}) \in \Pi_u(x,D)} \{L_D(x, u_D) + V(G(x, u_D))\}$$

$$= L_D(x, \kappa_D(x)) + V(G(x, \kappa_D(x))) \quad \forall x \in \Pi(D) \quad \text{(A.16)}$$

Thus, (3.9) and (A.15) imply

$$L_C(x, \kappa_C(x)) + \langle \nabla V(x), F(x, \kappa_C(x)) \rangle = 0 \quad \forall x \in \Pi(C), \tag{A.17}$$

while, (3.10) and (A.16) imply

$$L_D(x, \kappa_D(x)) + V(G(x, \kappa_D(x))) = V(x) \quad \forall x \in \Pi(D). \tag{A.18}$$

From (A.17) and (3.9), we have

$$\min_{\substack{u_{C1} \\ u_{C1}:(u_{C1},\kappa_{C2}(x))\in\Pi_u(x,C)}} \{L_C(x, (u_{C1}, \kappa_{C2}(x))) + \langle \nabla V(x), F(x, (u_{C1}, \kappa_{C2}(x))) \rangle\} \geq 0$$
$$\forall x \in \Pi(C) \tag{A.19}$$

and

$$\max_{\substack{u_{C2} \\ u_{C2}:(\kappa_{C1}(x),u_{C2})\in\Pi_u(x,C)}} \{L_C(x, (\kappa_{C1}(x), u_{C2})) + \langle \nabla V(x), F(x, (\kappa_{C1}(x), u_{C2})) \rangle\} \leq 0$$
$$\forall x \in \Pi(C) \tag{A.20}$$

which imply (3.19) and (3.20), respectively. Likewise, From (A.18) and (3.10), we have

$$\min_{\substack{u_{D1} \\ u_{D1}:(u_{D1},\kappa_{D2}(x))\in\Pi_u(x,D)}} \{L_D(x, (u_{D1}, \kappa_{D2}(x))) + V(G(x, (u_{D1}, \kappa_{D2}(x))))\} \geq V(x)$$
$$\forall x \in \Pi(D) \tag{A.21}$$

and

$$\max_{\substack{u_{D2} \\ u_{D2}:(\kappa_{D1}(x),u_{D2})\in\Pi_u(x,D)}} \{L_D(x, (\kappa_{D1}(x), u_{D2})) + V(G(x, (\kappa_{D1}(x), u_{D2})))\} \leq V(x)$$
$$\forall x \in \Pi(D) \tag{A.22}$$

which imply (3.22) and (3.23), respectively.

($\leftarrow$) Given V and $\kappa := (\kappa_C, \kappa_D) = ((\kappa_{C1}, \kappa_{C2}), (\kappa_{D1}, \kappa_{D2}))$ such that (3.18)-(3.23) are satisfied, and such that $C_\kappa = \Pi(C), D_\kappa = \Pi(D)$, let us prove that V and κ satisfy (3.9), (3.10), (3.13), and (3.14). From (3.18) and (3.19) we have

$$\min_{\substack{u_{C1} \\ u_{C1}:(u_{C1},\kappa_{C2}(x))\in\Pi_u(x,C)}} \{L_C(x, (u_{C1}, \kappa_{C2}(x))) + \langle \nabla V(x), F(x, (u_{C1}, \kappa_{C2}(x))) \rangle\}$$
$$= L_C(x, \kappa_C(x)) + \langle \nabla V(x), F(x, \kappa_C(x)) \rangle = 0 \quad \forall x \in \Pi(C) \tag{A.23}$$

and from (3.18) and (3.20) we have

$$\max_{\substack{u_{C2} \\ u_{C2}:(\kappa_{C1}(x),u_{C2})\in\Pi_u(x,C)}} \{L_C(x,(\kappa_{C1}(x),u_{C2})) + \langle\nabla V(x), F(x,(\kappa_{C1}(x),u_{C2}))\rangle\}$$
$$= L_C(x,\kappa_C(x)) + \langle\nabla V(x), F(x,\kappa_C(x))\rangle = 0 \quad \forall x \in \Pi(C) \quad \text{(A.24)}$$

Thus, (A.23) and (A.24) imply (3.9) and (3.13). Similarly, from (3.21) and (3.22) we have

$$\min_{\substack{u_{D1} \\ u_{D1}:(u_{D1},\kappa_{D2}(x))\in\Pi_u(x,D)}} \{L_D(x,(u_{D1},\kappa_{D2}(x))) + V(G(x,(u_{D1},\kappa_{D2}(x))))\}$$
$$= L_D(x,\kappa_D(x)) + V(G(x,\kappa_D(x))) = V(x) \quad \forall x \in \Pi(D) \quad \text{(A.25)}$$

and from (3.21) and (3.23) we have

$$\max_{\substack{u_{D2} \\ u_{D2}:(\kappa_{D1}(x),u_{D2})\in\Pi_u(x,D)}} \{L_D(x,(\kappa_{D1}(x),u_{D2})) + V(G(x,(\kappa_{D1}(x),u_{D2})))\}$$
$$= L_D(x,\kappa_D(x)) + V(G(x,\kappa_D(x))) = V(x) \quad \forall x \in \Pi(D) \quad \text{(A.26)}$$

Thus, (A.25) and (A.26) imply (3.10) and (3.14). □

A.4 Proof of Theorem 3.3.5.

Since, by assumption, we have that $C_\kappa = \Pi(C), D_\kappa = \Pi(D)$, and $V, \kappa := (\kappa_C, \kappa_D) = ((\kappa_{C1}, \kappa_{C2}), (\kappa_{D1}, \kappa_{D2}))$ are such that (3.18)-(3.23) hold, then, thanks to Lemma 3.3.4, V and κ satisfy (3.9), (3.10), (3.13), and (3.14). Since in addition, for each $\xi \in \overline{C_\kappa} \cup D_\kappa$, each $\phi \in \mathcal{S}^\infty_{\mathcal{H}_\kappa}(\xi)$ satisfies (3.11), we have from Theorem 3.2.1 that V is the value function as in (6.4) for $\mathcal{H}_\kappa$ at $\overline{C_\kappa} \cup D_\kappa$ and the feedback law κ with values (3.13), (3.14) is the saddle-point equilibrium for this game. Given that maximal solutions to $\mathcal{H}_\kappa$ are complete by assumption, $G(D_\kappa) \subset \overline{C_\kappa} \cup D_\kappa$. Then, V is a Lyapunov candidate for $\mathcal{H}_\kappa$ [20, Definition 3.16] since $\overline{C_\kappa} \cup D_\kappa \subset \operatorname{dom} V = \mathbb{R}^n$ and V is continuously differentiable on an open set containing $\overline{C_\kappa}$. From (3.18) and (3.21), we have

$$\langle\nabla V(x), F(x,\kappa_C(x))\rangle \leq -L_C(x,\kappa_C(x)) \quad \forall x \in C_\kappa, \quad \text{(A.27)}$$
$$V(G(x,\kappa_D(x))) - V(x) \leq -L_D(x,\kappa_D(x)) \quad \forall x \in D_\kappa. \quad \text{(A.28)}$$

Moreover, if

a) Item 1, item 4, or item 5 above hold, define

$$\rho(x,\kappa(x)) := \begin{cases} L_C(x,\kappa_C(x)) & \text{if } x \in C_\kappa \setminus D_\kappa \\ \min\{L_C(x,\kappa_C(x)), L_D(x,\kappa_D(x))\} & \text{if } x \in C_\kappa \cap D_\kappa \\ L_D(x,\kappa_D(x)) & \text{if } x \in D_\kappa \setminus C_\kappa \end{cases}$$

b) Item 2 above holds, define

$$\rho(x,\kappa(x)) := \begin{cases} \eta(|x|_\mathcal{A}) & \text{if } x \in C_\kappa \setminus D_\kappa \\ \min\{\eta(|x|_\mathcal{A}), L_D(x,\kappa_D(x))\} & \text{if } x \in C_\kappa \cap D_\kappa \\ L_D(x,\kappa_D(x)) & \text{if } x \in D_\kappa \setminus C_\kappa \end{cases}$$

c) Item 3 above holds, define

$$\rho(x,\kappa(x)) := \begin{cases} L_C(x,\kappa_C(x)) & \text{if } x \in C_\kappa \setminus D_\kappa \\ \min\{L_C(x,\kappa_C(x)), \eta(|x|_\mathcal{A})\} & \text{if } x \in C_\kappa \cap D_\kappa \\ \eta(|x|_\mathcal{A}) & \text{if } x \in D_\kappa \setminus C_\kappa \end{cases}$$

d) Item 6 above holds, define

$$\rho(x,\kappa(x)) := \begin{cases} \lambda_C V(x) & \text{if } x \in C_\kappa \setminus D_\kappa \\ \min\{\lambda_C V(x), e^{\lambda_D} V(x)\} & \text{if } x \in C_\kappa \cap D_\kappa \\ e^{\lambda_D} V(x) & \text{if } x \in D_\kappa \setminus C_\kappa \end{cases}$$

Thus, given the functions α_1, α_2 satisfying (3.24), and given that from (D.31) and (D.32), for each case above the function ρ satisfies

$$\langle \nabla V(x), F(x,\kappa_C(x))\rangle \le -\rho(x,\kappa(x)) \quad \forall x \in C_\kappa, \tag{A.29}$$

$$V(G(x,\kappa_D(x))) - V(x) \le -\rho(x,\kappa(x)) \quad \forall x \in D_\kappa, \tag{A.30}$$

thanks to [69, Theorem 3.18], the set $\mathcal{A}$ is uniformly globally asymptotically stable for $\mathcal{H}_\kappa$. ◻

A.5 Proof of Proposition 3.4.1

We show that when conditions (3.27)-(3.29) hold, by using Theorem 3.2.1, the value function is equal to the function V and with the feedback law $\kappa := (\kappa_C, \kappa_D)$ with values as in (3.30) and (3.31), such a cost is attained. We can write (3.9) in Theorem 3.2.1 as

$$
\begin{aligned}
0 = &\min_{u_{C1}} \max_{u_{C2}} \quad \mathcal{L}_C(x, u_C), \\
&{}_{u_C=(u_{C1},u_{C2})\in\Pi_u^C(x)} \\
\mathcal{L}_C(x, u_C) = &x_p^\top Q_C x_p + u_{C1}^\top R_{C1} u_{C1} + u_{C2}^\top R_{C2} u_{C2} \\
&+2x_p^\top P(\tau)(A_C x_p + B_C u_C) + x_p^\top \frac{d}{d\tau} P(\tau) x_p \qquad \text{(A.31)}
\end{aligned}
$$

First, thanks to (3.27) and $x_p^\top(P(\tau)A_C + A_C^\top P(\tau))x_p = 2x_p^\top P(\tau)A_C x_p$, for every $x \in \Pi(C)$, one has

$$
\begin{aligned}
\mathcal{L}_C(x, u_C) = &x_p^\top P(\tau)(B_{C2}R_{C2}^{-1}B_{C2}^\top + B_{C1}R_{C1}^{-1}B_{C1}^\top)P(\tau)x_p \\
&+ u_{C1}^\top R_{C1} u_{C1} + u_{C2}^\top R_{C2} u_{C2} + 2x_p^\top P(\tau) B_C u_C
\end{aligned}
$$

The first-order necessary conditions for optimality

$$
\frac{\partial}{\partial u_{C1}} \mathcal{L}_C(x, u_C)\Big|_{u_C^*} = 0, \quad \frac{\partial}{\partial u_{C2}} \mathcal{L}_C(x, u_C)\Big|_{u_C^*} = 0
$$

for all $(x, u_C) \in C$ are satisfied by the point $u_C^* = (u_{C1}^*, u_{C2}^*)$, with values

$$
u_{C1}^* = -R_{C1}^{-1}B_{C1}^\top P(\tau)x_p, \quad u_{C2}^* = -R_{C2}^{-1}B_{C2}^\top P(\tau)x_p \qquad \text{(A.32)}
$$

for each $x = (x_p, \tau) \in \Pi(C)$. Since $R_{C1}, -R_{C2} \in \mathbb{S}_+^{m_D}$, the second-order sufficient conditions for optimality

$$
\frac{\partial^2}{\partial u_{C1}^2} \mathcal{L}_C(x, u_C)\Big|_{u_C^*} \succeq 0, \quad \frac{\partial^2}{\partial u_{C2}^2} \mathcal{L}_C(x, u_C)\Big|_{u_C^*} \preceq 0,
$$

hold for all $(x, u_C) \in C$, rendering u_C^* as in (D.36) as an optimizer of the min-max problem in (D.35). In addition, it satisfies $\mathcal{L}_C(x, u_C^*) = 0$, making $V(x) = x_p^\top P(\tau)x_p$ a solution to (3.9) in Theorem 3.2.1.

On the other hand, we can write (3.10) in Theorem 3.2.1as

$$
\begin{aligned}
x_p^\top P(\bar{T})x_p = &\min_{u_{D1}} \max_{u_{D2}} \quad \mathcal{L}_D(x, u_D), \\
&{}_{u_D=(u_{D1},u_{D2})\in\Pi_u^D(x)} \\
\mathcal{L}_D(x, u_D) = &x_p^\top Q_D x_p + u_{D1}^\top R_{D1} u_{D1} + u_{D2}^\top R_{D2} u_{D2} \\
&+(A_D x_p + B_D u_D)^\top P(0)(A_D x_p + B_D u_D) \qquad \text{(A.33)}
\end{aligned}
$$

Similar to the case along flows, the first-order necessary conditions for optimality are satisfied by the point $u_D^* = (u_{D1}^*, u_{D2}^*)$, such that, for each $x_p \in \Pi(D)$,

$$u_D^* = -\begin{bmatrix} R_{D1} + B_{D1}^\top P(0) B_{D1} & B_{D1}^\top P(0) B_{D2} \\ B_{D2}^\top P(0) B_{D1} & R_{D2} + B_{D2}^\top P(0) B_{D2} \end{bmatrix}^{-1} \begin{bmatrix} B_{D1}^\top P(0) A_D \\ B_{D2}^\top P(0) A_D \end{bmatrix} x_p \quad \text{(A.34)}$$

Thanks to (3.28), the second-order sufficient conditions for optimality are satisfied, rendering u_D^* as in (D.42) as an optimizer of the min-max problem in (D.40). In addition, u_D^* satisfies $\mathcal{L}_D(x, u_D^*) = x_p^\top P(\bar{T}) x_p$ with $\bar{T} \in \{T_1, T_2\}$ and $P(\bar{T})$ as in (3.29), making $V(x) = x_p^\top P(\tau) x_p$ a solution of (3.10) in Theorem 3.2.1.

Then, given that V is continuously differentiable on a neighborhood of $\Pi(C)$ and that Assumption 3.1.4 holds, by applying Theorem 3.2.1, in particular from (3.12), for every $\xi = (\xi_p, \xi_\tau) \in \Pi(\overline{C}) \cup \Pi(D)$ the value function is $\mathcal{J}^*(\xi)x = \xi_p^\top P(\xi_\tau)\xi_p$. From (3.13) and (3.14), when P_1 plays u_1^* defined by $\kappa_1 = (\kappa_{C1}, \kappa_{D1})$ with values as in (3.30) and (3.31), and P_2 plays any disturbance u_2 such that solutions to $\mathcal{H}$ with data as in (3.26) are complete, then the cost is upper bounded by $\mathcal{J}(\xi, u^*)$, satisfying (3.2). □

A.6 Proof of Corollary 3.4.3

We show that when conditions (3.37)-(3.39) hold, by using the result in Theorem 3.2.1, the value function is equal to the function V and under the feedback law as in (3.40) such a cost is attained in the presence of the maximizing attack given by (3.41). We can write (3.9) in Theorem 3.2.1 as $0 = 2x^\top P F(x)$ for all $x \in \Pi(C)$, which is satisfied thanks to (3.37). Likewise, we can write (3.10) in Theorem 3.2.1 as

$$\begin{aligned} x^\top P x &= \min_{\substack{u_{D1} \\ u_D = (u_{D1}, u_{D2}) \in \Pi_u^D(x)}} \max_{u_{D2}} \mathcal{L}_D(x, u_D), \\ \mathcal{L}_D(x, u_D) &= x^\top Q_D x + u_{D1}^\top R_{D1} u_{D1} + u_{D2}^\top R_{D2} u_{D2} \\ &\qquad + (A_D x + B_D u_D)^\top P (A_D x + B_D u_D) \end{aligned} \quad \text{(A.35)}$$

The first order necessary conditions for optimality are satisfied by $u_D^* = (u_{D1}^*, u_{D2}^*)$, defined for each $x \in \Pi(D)$ as

$$u_D^* = -\begin{bmatrix} R_{D1} + B_{D1}^\top P B_{D1} & B_{D1}^\top P B_{D2} \\ B_{D2}^\top P B_{D1} & R_{D2} + B_{D2}^\top P B_{D2} \end{bmatrix}^{-1} \begin{bmatrix} B_{D1}^\top P A_D \\ B_{D2}^\top P A_D \end{bmatrix} x \quad \text{(A.36)}$$

Given that (3.38) holds, the second-order sufficient conditions for optimality are satisfied, rendering u_D^* as in (A.36) as an optimizer of the min-max problem in (A.35). In addition, u_D^* satisfies $\mathcal{L}_D(x, u_D^*) = x^\top P x$, with P as in (3.39), leading $V(x) = x^\top P x$ as a solution of (3.10) in Theorem 3.2.1.

Thus, given that V is continuously differentiable in $\mathbb{R}^n$ and Assumption (3.1.4) holds, by applying Theorem 3.2.1, in particular from (3.12), for every $\xi \in \Pi(\overline{C}) \cup \Pi(D)$ the value function is $\mathcal{J}^*(\xi) = \mathcal{J}(\xi, (u_{D1}^*, u_{D2}^*)) = \xi^\top P \xi$. From (3.13) and (3.14), when P_2 plays u_2^* defined by κ_{D2} as in (3.41), P_1 minimizes the cost of complete solutions to $\mathcal{H}$ by playing u_1^* defined by κ_{D1} as in (3.40), attaining $\mathcal{J}(\xi, u^*)$, and satisfying (3.2). $\square$

Appendix B

Proofs of Chapter 4

B.1 Proof of Theorem 4.2.1

To show the claim we proceed as follows:

1. Pick an initial condition ξ and evaluate the cost associated to any solution yield by $\gamma = (\gamma_C, \gamma_D)$, with values as in (4.2.1) and (4.2.1), from ξ. Show that this cost coincides with the value of the function V at ξ.

2. Lower bound the cost associated to any solution from ξ when P_2 plays $\gamma_2 := (\gamma_{C2}, \gamma_{D2})$ by the value of the function V evaluated at ξ.

3. Upper bound the cost associated to any solution from ξ when P_1 plays $\gamma_1 := (\gamma_{C1}, \gamma_{D1})$ by the value of the function V evaluated at ξ.

4. By showing that the cost of any solution from ξ when P_1 plays γ_1 is not less than the cost of any solution yield by γ from ξ, and by showing that the cost of any solution from ξ when P_2 plays γ_2 is not greater than the cost of any solution yield by γ from ξ, we show optimality of γ in Problem ($\star$) in the min-max sense.

Proceeding as in item 1 above, pick any $\xi \in \Pi(\bar{C} \cup D)$ and any $(\phi^*, u^*) \in \hat{\mathcal{S}}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ with $\operatorname{dom} \phi^* \ni (t,j) \mapsto u^*(t,j) = \gamma(t,j,\phi^*(t,j))$, and $(T_{\phi^*}, J_{\phi^*}) = (t_{J_{\phi^*}+1}, J_{\phi^*}) = \sup \operatorname{dom}(\phi^*, u^*) \in \mathcal{T}$. We show that the cost of (ϕ^*, u^*) is optimal in the min-max sense. Given that V satisfies (4.7), and γ_C is as in (4.2.1), for each $j \in \mathbb{N}$ such that

$I^j_{\phi^*} = [t_j, t_{j+1}]$ has a nonempty interior $\mathrm{int} I^j_{\phi^*}$, we have, for all $t \in \mathrm{int} I^j_{\phi^*}$,

$$
\begin{aligned}
0 = & \min_{\substack{u_{C1} \\ u_C(t,j)=(u_{C1},u_{C2})\in\Pi_u(\phi^*(t,j),C)}} \max_{u_{C2}} \Big\{ L_C(t,j,\phi^*(t,j),u_C(t,j)) \\
& \qquad + \frac{\partial V}{\partial x}(t,j,\phi^*(t,j)) F(\phi^*(t,j),u_C(t,j)) \Big\} + \frac{\partial V}{\partial t}(t,j,\phi^*(t,j)) \\
= & L_C(t,j,\phi^*(t,j),\gamma_C(t,j,\phi^*(t,j))) + \frac{\partial V}{\partial t}(t,j,\phi^*(t,j)) \\
& \qquad + \frac{\partial V}{\partial x}(t,j,\phi^*(t,j)) F(\phi^*(t,j),\gamma_C(t,j,\phi^*(t,j)))
\end{aligned}
$$

and $\phi^*(t,j) \in \Pi(C)$. Given that the total derivative is defined as $\frac{\mathrm{d}V}{\mathrm{d}t}(t,j,\phi^*(t,j)) = \frac{\partial V}{\partial t}(t,j,\phi^*(t,j)) + \frac{\partial V}{\partial x}(t,j,\phi^*(t,j))F(\phi^*(t,j),\gamma_C(t,j,\phi^*(t,j)))$ for every (t,j) such that $t \in \mathrm{dom}_t\, \phi^*$, by integrating over the interval $[t_j, t_{j+1}]$, we obtain

$$
0 = \int_{t_j}^{t_{j+1}} \left(L_C(t,j,\phi^*(t,j),\gamma_C(t,j,\phi^*(t,j))) + \frac{\mathrm{d}V}{\mathrm{d}t}(t,j,\phi^*(t,j)) \right) dt
$$

from where we have

$$
0 = \int_{t_j}^{t_{j+1}} L_C(t,j,\phi^*(t,j),\gamma_C(t,j,\phi^*(t,j)))dt + V(t_{j+1},j,\phi^*(t_{j+1},j)) - V(t,j,\phi^*(t_j,j))
$$

Summing from $j = 0$ to $j = J_{\phi^*}$, we obtain

$$
\begin{aligned}
0 = & \sum_{j=0}^{J_{\phi^*}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi^*(t,j),\gamma_C(t,j,\phi^*(t,j)))dt \\
& \qquad + \sum_{j=0}^{J_{\phi^*}} \left(V(t_{j+1},j,\phi^*(t_{j+1},j)) - V(t,j,\phi^*(t_j,j)) \right)
\end{aligned}
$$

Then, solving for V at the initial condition $(0,0,\phi^*(0,0))$, since $t_0 = 0$, we obtain

$$
\begin{aligned}
V(0,0,\phi^*(0,0)) = & \sum_{j=0}^{J_{\phi^*}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi^*(t,j),\gamma_C(t,j,\phi^*(t,j)))dt + V(t_1,0,\phi^*(t_1,0)) \\
& + \sum_{j=1}^{J_{\phi^*}} \left(V(t_{j+1},j,\phi^*(t_{j+1},j)) - V(t,j,\phi^*(t_j,j)) \right)
\end{aligned}
\tag{B.1}
$$

Given that V satisfies (4.8), γ_D is as in (4.2.1), and ϕ^* is not complete, for every $(t_{j+1}, j) \in \mathrm{dom}\, \phi^*$ such that $(t_{j+1}, j+1) \in \mathrm{dom}\, \phi^*$, we have that

$$
V(t_{j+1},j,\phi^*(t_{j+1},j))
$$

$$= \min_{u_D(t_{j+1},j)=(u_{D1},u_{D2})\in\Pi_u(\phi^*(t_{j+1},j),D)} \min_{u_{D1}} \max_{u_{D2}} \{L_D(t_{j+1}, j, \phi^*(t_{j+1}, j), u_D(t_{j+1}, j)) + V(t_{j+1}, j+1, G(\phi^*(t_{j+1}, j), u_D(t_{j+1}, j)\}$$

$$= L_D(t_{j+1}, j, \phi^*(t_{j+1}, j), \gamma_D(t_{j+1}, j, \phi^*(t_{j+1}, j))) + V(t_{j+1}, j+1, G(\phi^*(t_{j+1}, j), \gamma_D(t_{j+1}, j, \phi^*(t_{j+1}, j))))$$

$$= L_D(t_{j+1}, j, \phi^*(t_{j+1}, j), \gamma_D(t_{j+1}, j, \phi^*(t_{j+1}, j))) + V(t_{j+1}, j+1, \phi^*(t_{j+1}, j+1))$$

where $\phi^*(t_{j+1}, j) \in \Pi(D)$. Summing both sides from $j = 0$ to $j = J_{\phi^*} - 1$, we obtain

$$\sum_{j=0}^{J_{\phi^*}-1} V(t_{j+1}, j, \phi^*(t_{j+1}, j)) = \sum_{j=0}^{J_{\phi^*}-1} L_D(t_{j+1}, j, \phi^*(t_{j+1}, j), \gamma_D(t_{j+1}, j, \phi^*(t_{j+1}, j))) + \sum_{j=0}^{J_{\phi^*}-1} V(t_{j+1}, j+1, \phi^*(t_{j+1}, j+1))$$

Then, solving for V at the first jump time, we obtain

$$V(t_1, 0, \phi^*(t_1, 0)) = V(t_1, 1, \phi^*(t_1, 1)) + \sum_{j=0}^{J_{\phi^*}-1} L_D(t_{j+1}, j, \phi^*(t_{j+1}, j), \gamma_D(t_{j+1}, j, \phi^*(t_{j+1}, j))) + \sum_{j=1}^{J_{\phi^*}-1} (V(t_{j+1}, j+1, \phi^*(t_{j+1}, j+1)) - V(t_{j+1}, j, \phi^*(t_{j+1}, j))) \tag{B.2}$$

$\phi^*(0,0) = \xi$, by substituting the right hand side of (B.2) in (B.1), we obtain

$$
\begin{aligned}
V(0,0,\xi) &= \sum_{j=0}^{J_{\phi^*}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi^*(t,j),\gamma_C(t,j,\phi^*(t,j)))dt + V(t_1,0,\phi^*(t_1,0)) \\
&\quad + \sum_{j=1}^{J_{\phi^*}} (V(t_{j+1},j,\phi^*(t_{j+1},j)) - V(t,j,\phi^*(t_j,j))) \\
&= \sum_{j=0}^{J_{\phi^*}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi^*(t,j),\gamma_C(t,j,\phi^*(t,j)))dt \\
&\quad + \sum_{j=0}^{J_{\phi^*}-1} L_D(t_{j+1},j,\phi^*(t_{j+1},j),\gamma_D(t_{j+1},j,\phi^*(t_{j+1},j))) + V((t_1,1,\phi^*(t_1,1)) \\
&\quad + \sum_{j=1}^{J_{\phi^*}-1} (V(t_{j+1},j+1,\phi^*(t_{j+1},j+1)) - V(t_{j+1},j,\phi^*(t_{j+1},j))) \\
&\quad + \sum_{j=1}^{J_{\phi^*}} (V(t_{j+1},j,\phi^*(t_{j+1},j)) - V(t,j,\phi^*(t_j,j)))
\end{aligned}
$$

Since

$$
\begin{aligned}
&\sum_{j=1}^{J_{\phi^*}-1} (V(t_{j+1},j+1,\phi^*(t_{j+1},j+1)) - V(t_{j+1},j,\phi^*(t_{j+1},j))) \\
&+V((t_1,1,\phi^*(t_1,1)) + \sum_{j=1}^{J_{\phi^*}} (V(t_{j+1},j,\phi^*(t_{j+1},j)) - V(t,j,\phi^*(t_j,j))) \\
&\qquad = V((t_{J_{\phi^*}+1},J_{\phi^*},\phi^*(t_{J_{\phi^*}+1},J_{\phi^*})) + V((t_1,1,\phi^*(t_1,1)) \\
&\qquad\quad + \sum_{j=1}^{J_{\phi^*}-1} (V(t_{j+1},j+1,\phi^*(t_{j+1},j+1))) - \sum_{j=1}^{J_{\phi^*}} (V(t,j,\phi^*(t_j,j))) \\
&\qquad = V(t_{J_{\phi^*}+1},J_{\phi^*},\phi^*(t_{J_{\phi^*}+1},J_{\phi^*}))
\end{aligned}
$$

then we have

$$
\begin{aligned}
V(0,0,\xi) &= \sum_{j=0}^{J_{\phi^*}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi^*(t,j),\gamma_C(t,j,\phi^*(t,j)))dt \\
&\quad + \sum_{j=0}^{J_{\phi^*}-1} L_D(t_{j+1},j,\phi^*(t_{j+1},j),\gamma_D(t_{j+1},j,\phi^*(t_{j+1},j))) \\
&\quad +V(t_{J_{\phi^*}+1},J_{\phi^*},\phi^*(t_{J_{\phi^*}+1},J_{\phi^*}))
\end{aligned}
$$

Given that (4.9) holds and $(T_{\phi^*}, J_{\phi^*}) \in \mathcal{T}$, from (4.3) we have

$$\begin{aligned} V(0,0,\xi) &= \sum_{j=0}^{J_{\phi^*}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi^*(t,j),\gamma_C(t,j,\phi^*(t,j)))dt \\ &\quad + \sum_{j=0}^{J_{\phi^*}-1} L_D(t_{j+1},j,\phi^*(t_{j+1},j),\gamma_D(t_{j+1},j,\phi^*(t_{j+1},j))) \\ &\quad + V(T_{\phi^*},J_{\phi^*},\phi^*(T_{\phi^*},J_{\phi^*})) \\ &= \mathcal{J}(\xi,u^*) \end{aligned} \tag{B.3}$$

for any $(\phi^*, u^*) \in \mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ with $\operatorname{dom}\phi^* \ni (t,j) \mapsto u^*(t,j) = \gamma(t,j,\phi^*(t,j))$.

Continuing with item 2 as above, pick any $(\phi_s, u^s) \in \mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ with $u^s = (u_1^s, u_2^s)$, $\operatorname{dom}\phi_s \ni (t,j) \mapsto u_1^s(t,j) = \bar{\gamma}_1(t,j,\phi_s(t,j))$ for some $\bar{\gamma}_1 \in K_1$, $\operatorname{dom}\phi_s \ni (t,j) \mapsto u_2^s(t,j) = \gamma_2(t,j,\phi_s(t,j))$ for $\gamma_2 := (\gamma_{C2}, \gamma_{D2})$ as in (4.2.1) and (4.2.1), and $(T_{\phi_s}, J_{\phi_s}) = (t_{J_{\phi_s}+1}, J_{\phi_s}) = \sup\operatorname{dom}(\phi_s, u^s) \in \mathcal{T}$. Since $\bar{\gamma}_1$ does not necessarily attain the minimum in (4.7), then, for each $j \in \mathbb{N}$ such that $I_{\phi_s}^j = [t_j, t_{j+1}]$ has a nonempty interior $\operatorname{int} I_{\phi_s}^j$, we have that for every $t \in \operatorname{int} I_{\phi_s}^j$,

$$0 \le L_C(t,j,\phi_s(t,j),u_C^s(t,j)) + \frac{\partial V}{\partial t}(t,j,\phi_s(t,j)) + \frac{\partial V}{\partial x}(t,j,\phi_s(t,j))F(\phi_s(t,j),u_C^s(t,j))$$

Given that the total derivative is defined as $\frac{\mathrm{d}V}{\mathrm{d}t}(t,j,\phi_s(t,j)) := \frac{\partial V}{\partial t}(t,j,\phi_s(t,j))$ $+\frac{\partial V}{\partial x}(t,j,\phi_s(t,j))F(\phi_s(t,j),u_C^s(t,j))$ for every $(t,j) : t \in \operatorname{dom}_t \phi_s$, by integrating over the interval $[t_j, t_{j+1}]$, we obtain

$$0 \le \int_{t_j}^{t_{j+1}} \left(L_C(t,j,\phi_s(t,j),u_C^s(t,j)) + \frac{\mathrm{d}V}{\mathrm{d}t}(t,j,\phi_s(t,j)) \right) dt$$

from which we have

$$V(t_j,j,\phi_s(t_j,j)) \le \int_{t_j}^{t_{j+1}} L_C(t,j,\phi_s(t,j),u_C^s(t,j))dt + V(t_{j+1},j,\phi_s(t_{j+1},j))$$

Summing both sides from $j=0$ to $j = J_{\phi_s}$, we obtain

$$\sum_{j=0}^{J_{\phi_s}} V(t_j,j,\phi_s(t_j,j)) \le \sum_{j=0}^{J_{\phi_s}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi_s(t,j),u_C^s(t,j))dt + \sum_{j=0}^{J_{\phi_s}} V(t_{j+1},j,\phi_s(t_{j+1},j))$$

Then, solving for V at the initial condition $(0, 0, \phi_s(0, 0))$, we obtain

$$V(0,0,\phi_s(0,0)) \leq \sum_{j=0}^{J_{\phi_s}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi_s(t,j),u_C^s(t,j))dt + V(t_1,0,\phi_s(t_1,0)) + \sum_{j=1}^{J_{\phi_s}} \left(V(t_{j+1},j,\phi_s(t_{j+1},j)) - V(t_j,j,\phi_s(t_j,j))\right) \quad \text{(B.4)}$$

In addition, since u^s, with u_1^s defined by $\bar{\gamma}_1$ does not necessarily attain the minimum in (4.8), then for every $(t_{j+1}, j) \in \operatorname{dom}\phi_s$ such that $(t_{j+1}, j+1) \in \operatorname{dom}\phi_s$, we have

$$\begin{aligned} &V(t_{j+1},j,\phi_s(t_{j+1},j)) \\ &\quad \leq L_D(t_{j+1},j,\phi_s(t_{j+1},j),u_D^s(t_{j+1},j)) + V(t_{j+1},j+1,G(\phi_s(t_{j+1},j),u_D^s(t_{j+1},j))) \\ &\quad = L_D(t_{j+1},j,\phi_s(t_{j+1},j),u_D^s(t_{j+1},j)) + V(t_{j+1},j+1,\phi_s(t_{j+1},j+1)) \end{aligned}$$

Summing both sides from $j = 0$ to $j = J_{\phi_s} - 1$, we obtain

$$\sum_{j=0}^{J_{\phi_s}-1} V(t_{j+1},j,\phi_s(t_{j+1},j)) \leq \sum_{j=0}^{J_{\phi_s}-1} L_D(t_{j+1},j,\phi_s(t_{j+1},j),u_D^s(t_{j+1},j)) + \sum_{j=0}^{J_{\phi_s}-1} V(t_{j+1},j+1,\phi_s(t_{j+1},j+1))$$

Then, solving for V at the first jump time, we obtain

$$V(t_1,0,\phi_s(t_1,0)) \leq V(t_1,1,\phi_s(t_1,1)) + \sum_{j=0}^{J_{\phi_s}-1} L_D(t_{j+1},j,\phi_s(t_{j+1},j),u_D^s(t_{j+1},j)) + \sum_{j=1}^{J_{\phi_s}-1} \left(V(t_{j+1},j+1,\phi_s(t_{j+1},j+1)) - V(t_{j+1},j,\phi_s(t_{j+1},j))\right) \quad \text{(B.5)}$$

In addition, $\phi_s(0,0) = \xi$, upper bounding $V(t_1, 0, \phi_s(t_1, 0))$ in (B.4) by the right hand-side of (B.5), we obtain

$$
\begin{aligned}
V(0,0,\xi) \leq & \sum_{j=0}^{J_{\phi_s}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi_s(t,j),u_C^s(t,j))dt + V(t_1,0,\phi_s(t_1,0)) \\
& + \sum_{j=1}^{J_{\phi_s}} (V(t_{j+1},j,\phi_s(t_{j+1},j)) - V(t_j,j,\phi_s(t_j,j))) \\
\leq & \sum_{j=0}^{J_{\phi_s}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi_s(t,j),u_C^s(t,j))dt \\
& + \sum_{j=0}^{J_{\phi_s}-1} L_D(t_{j+1},j,\phi_s(t_{j+1},j),u_D^s(t_{j+1},j)) \\
& + \sum_{j=1}^{J_{\phi_s}-1} (V(t_{j+1},j+1,\phi_s(t_{j+1},j+1)) - V(t_{j+1},j,\phi_s(t_{j+1},j))) \\
& + V(t_1,1,\phi_s(t_1,1)) + \sum_{j=1}^{J_{\phi_s}} (V(t_{j+1},j,\phi_s(t_{j+1},j)) - V(t_j,j,\phi_s(t_j,j)))
\end{aligned}
$$

Since

$$
\begin{aligned}
& V(t_1,1,\phi_s(t_1,1)) + \sum_{j=1}^{J_{\phi_s}-1} (V(t_{j+1},j+1,\phi_s(t_{j+1},j+1)) - V(t_{j+1},j,\phi_s(t_{j+1},j))) \\
& + \sum_{j=1}^{J_{\phi_s}} (V(t_{j+1},j,\phi_s(t_{j+1},j)) - V(t_j,j,\phi_s(t_j,j))) \\
& \qquad = V(t_{J_{\phi_s}+1},J_{\phi_s},\phi_s(t_{J_{\phi_s}+1},J_{\phi_s})) + V(t_1,1,\phi_s(t_1,1)) \\
& \qquad + \sum_{j=1}^{J_{\phi_s}-1} (V(t_{j+1},j+1,\phi_s(t_{j+1},j+1))) - \sum_{j=1}^{J_{\phi_s}} (V(t_j,j,\phi_s(t_j,j))) \\
& \qquad = V(t_{J_{\phi_s}+1},J_{\phi_s},\phi_s(t_{J_{\phi_s}+1},J_{\phi_s}))
\end{aligned}
$$

then we have

$$
\begin{aligned}
V(0,0,\xi) \leq & \sum_{j=0}^{J_{\phi_s}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi_s(t,j),u_C^s(t,j))dt \\
& + \sum_{j=0}^{J_{\phi_s}-1} L_D(t_{j+1},j,\phi_s(t_{j+1},j),u_D^s(t_{j+1},j)) \\
& + V(t_{J_{\phi_s}+1},J_{\phi_s},\phi_s(t_{J_{\phi_s}+1},J_{\phi_s}))
\end{aligned}
$$

Given that (4.9) holds and $(T_{\phi_s}, J_{\phi_s}) \in \mathcal{T}$, from (4.3)we have

$$\begin{aligned} V(0,0,\xi) \leq & \sum_{j=0}^{\sup_j \operatorname{dom} \phi_s} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi_s(t,j),u_C^s(t,j))dt \\ & + \sum_{j=0}^{\sup_j \operatorname{dom} \phi_s - 1} L_D(t_{j+1},j,\phi_s(t_{j+1},j),u_D^s(t_{j+1},j)) \\ & + V(T_{\phi_s}, J_{\phi_s}, \phi_s(T_{\phi_s}, J_{\phi_s})) \\ \leq & \mathcal{J}(\xi, u^s) \end{aligned} \tag{B.6}$$

for $u^s = (u_1^s, u_2^s)$, with u_1^s defined by any $\bar{\gamma}_1 \in K_1$ and u_2^s defined by γ_2 as in (4.2.1) and (4.2.1).

Proceeding with item 3 as above, pick any $(\phi_w, u^w) \in \mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ with $u^w = (u_1^w, u_2^w)$, $\operatorname{dom} \phi_w \ni (t,j) \mapsto u_1^w(t,j) = \gamma_1(t,j,\phi_w(t,j))$ for $\gamma_1 := (\gamma_{C1}, \gamma_{D1})$ as in (4.2.1) and (4.2.1), $\operatorname{dom} \phi_w \ni (t,j) \mapsto u_2^w(t,j) = \bar{\gamma}_2(t,j,\phi_w(t,j))$ for some $\bar{\gamma}_2 \in K_2$, and $(T_{\phi_w}, J_{\phi_w}) = (t_{J_{\phi_w}+1}, J_{\phi_w}) = \sup \operatorname{dom}(\phi_w, u^w) \in \mathcal{T}$. Since $\bar{\gamma}_w$ does not necessarily attain the maximum in (4.7), then, for each $j \in \mathbb{N}$ such that $I_{\phi_w}^j = [t_j, t_{j+1}]$ has a nonempty interior $\operatorname{int} I_{\phi_w}^j$, we have that for every $t \in \operatorname{int} I_{\phi_w}^j$,

$$0 \geq L_C(t,j,\phi_w(t,j),u_C^w(t,j)) + \frac{\partial V}{\partial t}(t,j,\phi_w(t,j)) + \frac{\partial V}{\partial x}(t,j,\phi_w(t,j))F(\phi_w(t,j),u_C^w(t,j))$$

Given that the total derivative is defined as

$$\frac{\mathrm{d}V}{\mathrm{d}t}(t,j,\phi_w(t,j)) := \frac{\partial V}{\partial t}(t,j,\phi_w(t,j)) + \frac{\partial V}{\partial x}(t,j,\phi_w(t,j))F(\phi_w(t,j),u_C^w(t,j))$$

for every $(t,j) : t \in \operatorname{dom}_t \phi_w$, by integrating over the interval $[t_j, t_{j+1}]$, we obtain

$$0 \geq \int_{t_j}^{t_{j+1}} \left(L_C(t,j,\phi_w(t,j),u_C^w(t,j)) + \frac{\mathrm{d}V}{\mathrm{d}t}(t,j,\phi_w(t,j)) \right) dt$$

from which we have

$$V(t_j,j,\phi_w(t_j,j)) \geq \int_{t_j}^{t_{j+1}} L_C(t,j,\phi_w(t,j),u_C^w(t,j))dt + V(t_{j+1},j,\phi_w(t_{j+1},j))$$

Summing both sides from $j = 0$ to $j = J_{\phi_w}$, we obtain

$$\sum_{j=0}^{J_{\phi_w}} V(t_j,j,\phi_w(t_j,j)) \geq \sum_{j=0}^{J_{\phi_w}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi_w(t,j),u_C^w(t,j))dt + \sum_{j=0}^{J_{\phi_w}} V(t_{j+1},j,\phi_w(t_{j+1},j))$$

Then, solving for V at the initial condition $(0, 0, \phi_w(0, 0))$, we obtain

$$\begin{aligned} V(0,0,\phi_w(0,0)) \geq \sum_{j=0}^{J_{\phi_w}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi_w(t,j),u_C^w(t,j))dt + V(t_1,0,\phi_w(t_1,0)) \\ + \sum_{j=1}^{J_{\phi_w}} \left(V(t_{j+1},j,\phi_w(t_{j+1},j)) - V(t_j,j,\phi_w(t_j,j))\right) \end{aligned} \tag{B.7}$$

In addition, since u^w, with u_2^w defined by $\bar{\gamma}_2$ does not necessarily attain the maximum in (4.8), then for every $(t_{j+1}, j) \in \operatorname{dom} \phi_w$ such that $(t_{j+1}, j+1) \in \operatorname{dom} \phi_w$, we have

$$\begin{aligned} V(t_{j+1},j,\phi_w(t_{j+1},j)) &\geq L_D(t_{j+1},j,\phi_w(t_{j+1},j),u_D^w(t_{j+1},j)) \\ &\qquad + V(t_{j+1},j+1,G(\phi_w(t_{j+1},j),u_D^w(t_{j+1},j))) \\ &= L_D(t_{j+1},j,\phi_w(t_{j+1},j),u_D^w(t_{j+1},j)) + V(t_{j+1},j+1,\phi_w(t_{j+1},j+1)) \end{aligned}$$

Summing both sides from $j = 0$ to $j = J_{\phi_w} - 1$, we obtain

$$\begin{aligned} \sum_{j=0}^{J_{\phi_w}-1} V(t_{j+1},j,\phi_w(t_{j+1},j)) \geq \sum_{j=0}^{J_{\phi_w}-1} L_D(t_{j+1},j,\phi_w(t_{j+1},j),u_D^w(t_{j+1},j)) \\ + \sum_{j=0}^{J_{\phi_w}-1} V(t_{j+1},j+1,\phi_w(t_{j+1},j+1)) \end{aligned}$$

Then, solving for V at the first jump time, we obtain

$$\begin{aligned} V(t_1,0,\phi_w(t_1,0)) \geq V(t_1,1,\phi_w(t_1,1)) + \sum_{j=0}^{J_{\phi_w}-1} L_D(t_{j+1},j,\phi_w(t_{j+1},j),u_D^w(t_{j+1},j)) \\ + \sum_{j=1}^{J_{\phi_w}-1} \left(V(t_{j+1},j+1,\phi_w(t_{j+1},j+1)) - V(t_{j+1},j,\phi_w(t_{j+1},j))\right) \end{aligned} \tag{B.8}$$

In addition, $\phi_w(0, 0) = \xi$, lower bounding $V(t_1, 0, \phi_w(t_1, 0))$ in (B.7) by the right-hand side of (B.8), we obtain

$$\begin{aligned}
V(0,0,\xi) \geq & \sum_{j=0}^{J_{\phi_w}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi_w(t,j),u_C^w(t,j))dt + V(t_1,0,\phi_w(t_1,0)) \\
& + \sum_{j=1}^{J_{\phi_w}} (V(t_{j+1},j,\phi_w(t_{j+1},j)) - V(t_j,j,\phi_w(t_j,j))) \\
\geq & \sum_{j=0}^{J_{\phi_w}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi_w(t,j),u_C^w(t,j))dt \\
& + \sum_{j=0}^{J_{\phi_w}-1} L_D(t_{j+1},j,\phi_w(t_{j+1},j),u_D^w(t_{j+1},j)) \\
& + \sum_{j=1}^{J_{\phi_w}-1} (V(t_{j+1},j+1,\phi_w(t_{j+1},j+1)) - V(t_{j+1},j,\phi_w(t_{j+1},j))) \\
& + V(t_1,1,\phi_w(t_1,1)) \\
& + \sum_{j=1}^{J_{\phi_w}} (V(t_{j+1},j,\phi_w(t_{j+1},j)) - V(t_j,j,\phi_w(t_j,j)))
\end{aligned}$$

Since

$$\begin{aligned}
& \sum_{j=1}^{J_{\phi_w}-1} (V(t_{j+1},j+1,\phi_w(t_{j+1},j+1)) - V(t_{j+1},j,\phi_w(t_{j+1},j))) \\
& + V(t_1,1,\phi_w(t_1,1)) + \sum_{j=1}^{J_{\phi_w}} (V(t_{j+1},j,\phi_w(t_{j+1},j)) - V(t_j,j,\phi_w(t_j,j))) \\
& \qquad = V(t_{J_{\phi_w}+1},J_{\phi_w},\phi_w(t_{J_{\phi_w}+1},J_{\phi_w})) + V(t_1,1,\phi_w(t_1,1)) \\
& \qquad + \sum_{j=1}^{J_{\phi_w}-1} (V(t_{j+1},j+1,\phi_w(t_{j+1},j+1))) - \sum_{j=1}^{J_{\phi_w}} (V(t_j,j,\phi_w(t_j,j))) \\
& \qquad = V(t_{J_{\phi_w}+1},J_{\phi_w},\phi_w(t_{J_{\phi_w}+1},J_{\phi_w}))
\end{aligned}$$

then we have

$$\begin{aligned}
V(0,0,\xi) \geq & \sum_{j=0}^{J_{\phi_w}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi_w(t,j),u_C^w(t,j))dt \\
& + \sum_{j=0}^{J_{\phi_w}-1} L_D(t_{j+1},j,\phi_w(t_{j+1},j),u_D^w(t_{j+1},j)) \\
& + V(t_{J_{\phi_w}+1},J_{\phi_w},\phi_w(t_{J_{\phi_w}+1},J_{\phi_w}))
\end{aligned}$$

Given that (4.9) holds and $(T_{\phi_w}, J_{\phi_w}) \in \mathcal{T}$, from (4.3) we have

$$\begin{aligned} V(0,0,\xi) \geq & \sum_{j=0}^{J_{\phi_w}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi_w(t,j),u_C^w(t,j))dt \\ & + \sum_{j=0}^{J_{\phi_w}-1} L_D(t_{j+1},j,\phi_w(t_{j+1},j),u_D^w(t_{j+1},j)) \\ & + V(T_{\phi_w}, J_{\phi_w}, \phi_w(T_{\phi_w}, J_{\phi_w})) \\ \geq & \mathcal{J}(\xi, u^w) \end{aligned} \tag{B.9}$$

for $u^w = (u_1^w, u_2^w)$, with u_1^w defined by γ_1 as in (4.2.1) and (4.2.1) and u_2^w defined by any $\bar{\gamma}_2 \in K_2$. Finally, given the set of solutions with finite horizon, namely $\mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$, by proceeding as in item 4 above, by applying the infimum on each side of (D.23) over the set $\mathcal{S}_{\mathcal{H}}^{\infty}(\xi)$, we obtain

$$V(0,0,\xi) \leq \inf_{(\phi_s,(u_1,\kappa_2(\phi_s)))\in\mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)} \mathcal{J}(\xi,(u_1,\kappa_2(\phi_s))$$

By applying the supremum on each side of (C.9) over the set $\mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$, we obtain

$$V(0,0,\xi) \geq \sup_{(\phi_w,(\kappa_1(\phi_w),u_2))\in\mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)} \mathcal{J}(\xi,(\kappa_1(\phi_w),u_2)).$$

Given that (B.3) leads to $V(0,0,\xi) = \mathcal{J}(\xi, u^*)$, we have that for any $\xi \in \Pi(\bar{C} \cup D)$, each $(\phi^*, u^*) \in \mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ with $u^* = (\kappa_1(\phi^*), \kappa_2(\phi^*))$ satisfies

$$\sup_{(\phi_w,(\kappa_1(\phi_w),u_2))\in\mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)} \mathcal{J}(\xi,(\kappa_1(\phi_w),u_2)) \leq \mathcal{J}(\xi,u^*) \leq \inf_{(\phi_s,(u_1,\kappa_2(\phi_s)))\in\mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)} \mathcal{J}(\xi,(u_1,\kappa_2(\phi_s))) \tag{B.10}$$

Since a response ϕ_w rendered by κ_1 or a response ϕ_s rendered by κ_2 can equate the response ϕ^* by properly choosing u_2 or u_1, respectively, we have

$$\sup_{(\phi_w,(\kappa_1(\phi_w),u_2))\in\mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)} \mathcal{J}(\xi,(\kappa_1(\phi_w),u_2)) = \sup_{(\phi^*,(\kappa_1(\phi^*),\kappa_2(\phi^*)))\in\mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)} \mathcal{J}(\xi,(\kappa_1(\phi^*),\kappa_2(\phi^*)) \tag{B.11}$$

and

$$\inf_{(\phi_s,(u_1,\kappa_2(\phi_s)))\in\mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)} \mathcal{J}(\xi,(u_1,\kappa_2(\phi_s))) = \inf_{(\phi^*,(\kappa_1(\phi^*),\kappa_2(\phi^*)))\in\mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)} \mathcal{J}(\xi,(\kappa_1(\phi^*),\kappa_2(\phi^*))) \tag{B.12}$$

leading to

$$\mathcal{J}(\xi, u^*) = \min_{\substack{u_1 \\ (u_1,u_2)\in\mathcal{U}_{\mathcal{H}}^{\mathcal{T}}(\xi)}} \max_{u_2} \mathcal{J}(\xi, (u_1, u_2)) \tag{B.13}$$

Thus, from (B.3) and (C.13), $V(0, 0, \xi)$ is the value function for $\mathcal{H}$, as in Definition 4.1.2 and from (C.10), γ is the saddle-point equilibrium as in Definition 3.1.3. □

B.2 Proof of Corollary 4.3.2

We show that when conditions (4.14)-(4.16) hold, by using the result in Theorem 4.2.1, the value function is equal to the function V and with the feedback law with values as in (4.17) and (4.18), such a cost is attained. We can write (4.7) in Theorem 4.2.1 as

$$\begin{aligned} 0 &= \min_{\substack{u_{C1} \\ u_C=(u_{C1},u_{C2})\in\Pi_u^C(x)}} \max_{u_{C2}} S_C(x, u_C), \\ S_C(x, u_C) &= x_p^\top Q_C x_p + u_{C1}^\top R_{C1} u_{C1} + u_{C2}^\top R_{C2} u_{C2} \\ &\qquad +2x_p^\top P(\tau)(A_C x_p + B_C u_C) + x_p^\top \frac{\mathrm{d}P(\tau)}{\mathrm{d}\tau} x_p \end{aligned} \tag{B.14}$$

First, given that (4.14) holds for all $\tau \in (0, \bar{T})$, and $x_p^\top(P(\tau))A_C + A_C^\top P(\tau)x_p = x_p^\top(2P(\tau)A_C)x_p$ for every (τ, x_p), one has

$$\begin{aligned} S_C(x, u_C) &= x_p^\top P(\tau)(B_{C2}R_{C2}^{-1}B_{C2}^\top + B_{C1}R_{C1}^{-1}B_{C1}^\top)P(\tau)x_p \\ &\qquad +u_{C1}^\top R_{C1} u_{C1} + u_{C2}^\top R_{C2} u_{C2} + 2x_p^\top P(\tau)B_C u_C \end{aligned} \tag{B.15}$$

The first order necessary conditions for optimality

$$\begin{aligned} &\frac{\partial}{\partial u_{C1}} \Big(x_p^\top P(\tau)(B_{C2}R_{C2}^{-1}B_{C2}^\top + B_{C1}R_{C1}^{-1}B_{C1}^\top)P(\tau)x_p \\ &\qquad +u_{C1}^\top R_{C1} u_{C1} + u_{C2}^\top R_{C2} u_{C2} + 2x_p^\top P(\tau)(B_{C1}u_{C1} + B_{C2}u_{C2}) \Big)\Big|_{u_C^*} = 0 \end{aligned}$$

$$\begin{aligned} &\frac{\partial}{\partial u_{C2}} \Big(x_p^\top P(\tau)(B_{C2}R_{C2}^{-1}B_{C2}^\top + B_{C1}R_{C1}^{-1}B_{C1}^\top)P(\tau)x_p \\ &\qquad +u_{C1}^\top R_{C1} u_{C1} + u_{C2}^\top R_{C2} u_{C2} + 2x_p^\top P(\tau)(B_{C1}u_{C1} + B_{C2}u_{C2}) \Big)\Big|_{u_C^*} = 0 \end{aligned}$$

are satisfied by the stationary point $u_C^* = (u_{C1}^*, u_{C2}^*)$, with values for each $\tau \in (0, \bar{T})$

$$u_{C1}^* = -R_{C1}^{-1}B_{C1}^\top P(\tau)x_p \tag{B.16}$$

$$u_{C2}^* = -R_{C2}^{-1} B_{C2}^\top P(\tau) x_p \tag{B.17}$$

Given that $R_{C1}, -R_{C2} \in \mathbb{S}_+^{m_D}$, the second-order sufficient conditions for optimality, namely,

$$\begin{aligned}\frac{\partial^2}{\partial u_{C1}^2} \Big(x_p^\top P(\tau)(B_{C2} R_{C2}^{-1} B_{C2}^\top + B_{C1} R_{C1}^{-1} B_{C1}^\top) P(\tau) x_p \\ + u_{C1}^\top R_{C1} u_{C1} + u_{C2}^\top R_{C2} u_{C2} + 2 x_p^\top P(\tau)(B_{C1} u_{C1} + B_{C2} u_{C2}) \Big)\Big|_{u_{C1}^*} \succeq 0\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial u_{C2}^2} \Big(x_p^\top P(\tau)(B_{C2} R_{C2}^{-1} B_{C2}^\top + B_{C1} R_{C1}^{-1} B_{C1}^\top) P(\tau) x_p \\ + u_{C1}^\top R_{C1} u_{C1} + u_{C2}^\top R_{C2} u_{C2} + 2 x_p^\top P(\tau)(B_{C1} u_{C1} + B_{C2} u_{C2}) \Big)\Big|_{u_{C2}^*} \preceq 0\end{aligned}$$

hold, rendering u_C^* as in (B.16) and (D.36) as an optimizer of the min-max problem in (D.35). In addition, it satisfies $S_C(x, u_C^*) = 0$, making $V(t, j, x) = x_p^\top P(\tau) x_p$ a solution of (4.7) in Theorem 4.2.1.

On the other hand, we can write (4.8) in Theorem 4.2.1 as

$$\begin{aligned} x_p^\top P(\bar{T}) x_p &= \min_{u_{D1}} \max_{u_{D2}}\limits_{u_D = (u_{D1}, u_{D2}) \in \Pi_u^D(x)} S_D(x, u_D), \\ S_D(x, u_D) &= x_p^\top Q_D x_p + u_{D1}^\top R_{D1} u_{D1} + u_{D2}^\top R_{D2} u_{D2} \\ &\quad + (A_D x_p + B_D u_D)^\top P(0)(A_D x_p + B_D u_D) \end{aligned} \tag{B.18}$$

which can be expanded as

$$\begin{aligned} S_D(x, u_D) &= x_p^\top (Q_D + A_D^\top P(0) A_D) x_p + 2 x_p^\top A_D^\top P(0) B_D u_D \\ &\quad + u_{D1}^\top (R_{D1} + B_{D1}^\top P(0) B_{D1}) u_{D1} + u_{D2}^\top (R_{D2} + B_{D2}^\top P(0) B_{D2}) u_{D2} \\ &\quad + u_{D1}^\top (B_{D1}^\top P(0) B_{D2}) u_{D2} + u_{D2}^\top (B_{D2}^\top P(0) B_{D1}) u_{D1} \end{aligned} \tag{B.19}$$

The first order necessary conditions for optimality

$$\frac{\partial}{\partial u_{D1}} S_D(x, u_D)|_{u_D^*} = 0$$
$$\frac{\partial}{\partial u_{D2}} S_D(x, u_D)|_{u_D^*} = 0$$

are satisfied by the stationary point $u_D^* = (u_{D1}^*, u_{D2}^*)$, such that for each $x_p \in \Pi(D)$

$$u_D^* = -\begin{bmatrix} R_{D1} + B_{D1}^\top P(0) B_{D1} & B_{D1}^\top P(0) B_{D2} \\ B_{D2}^\top P(0) B_{D1} & R_{D2} + B_{D2}^\top P(0) B_{D2} \end{bmatrix}^{-1} \begin{bmatrix} B_{D1}^\top P(0) A_D \\ B_{D2}^\top P(0) A_D \end{bmatrix} x_p \tag{B.20}$$

Given that (4.15) holds, the second-order sufficient conditions for optimality, namely,

$$\frac{\partial^2}{\partial u_{D1}^2}\ S_D(x, u_D)|_{u_D^*} \succeq 0,$$

$$\frac{\partial^2}{\partial u_{D2}^2}\ S_D(x, u_D)|_{u_D^*} \preceq 0,$$

are satisfied, rendering u_D^* as in (D.42) as an optimizer of the min-max problem in (D.40). In addition, u_D^* satisfies $S_D(x, u_D^*) = x_p^\top P(\bar{T})x_p$, with $P(\bar{T})$ as in (4.16), making $V(x) = x_p^\top P(\tau)x_p$ a solution of (4.8) in Theorem 4.2.1.

Then, given that V is continuously differentiable on a neighborhood of $\Pi(C)$ and Assumption 3.1.4 holds, by applying Theorem 4.2.1, in particular from (4.10), for every (t, j, ξ_p) such that $(t, j) \in \mathcal{T}_{\leq \tau_p}$ the value function is $\mathcal{J}_{\mathcal{T}}^*(\xi) = \mathcal{J}(\xi, ((u_{C1}^*, u_{D1}^*), (u_{C2}^*, u_{D2}^*))$ $= \xi_p^\top P(\tau)\xi_p$. From (4.2.1) and (4.2.1) the feedback law $\gamma = (\gamma_C, \gamma_D)$ with values as in (4.17) and (4.18) is a pure strategy saddle-point equilibrium.

□

B.3 Proof of Lemma 4.4.1

Pick any $(\phi_s, u^s) \in \mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ with $u^s = (u_1^s, u_2^s)$, $\operatorname{dom} \phi_s \ni (t, j) \mapsto u_1^s(t, j) = \bar{\gamma}_1(t, j, \phi_s(t, j))$ for some $\bar{\gamma}_1 \in K_1$, $\operatorname{dom} \phi_s \ni (t, j) \mapsto u_2^s(t, j) = \gamma_2^*(t, j, \phi_s(t, j))$ for $\gamma_2^* := (\gamma_{C2}^*, \gamma_{D2}^*)$ attining the supremum in (4.25) and (4.26), and $(T_{\phi_s}, J_{\phi_s}) = (t_{J_{\phi_s}+1}, J_{\phi_s}) = \sup \operatorname{dom}(\phi_s, u^s) \in \mathcal{T}$. Since $\bar{\gamma}_1$ does not necessarily attain the infimum in (4.25), then, for each $j \in \mathbb{N}$ such that $I_{\phi_s}^j = [t_j, t_{j+1}]$ has a nonempty interior $\operatorname{int} I_{\phi_s}^j$, we have that for every $t \in \operatorname{int} I_{\phi_s}^j$,

$$\begin{aligned}-\varepsilon \leq \min_{\substack{u_{C1} \\ u_C=(u_{C1},u_{C2})\in\Pi_u(\phi_s(t,j),C)}} \max_{u_{C2}} & \left\{ L_C(t, j, \phi(t, j), u_C(t, j)) + \frac{\partial V}{\partial x}(t, j, \phi_s(t, j))F(\phi_s(t, j), u_C(t, j)) \right\} \\ & + \frac{\partial V}{\partial t}(t, j, \phi_s(t, j)) \\ \leq L_C(t, j, \phi_s(t, j), u_C^s(t, j)) & + \frac{\partial V}{\partial x}(t, j, \phi_s(t, j))F(\phi_s(t, j), u_C^s(t, j)) + \frac{\partial V}{\partial t}(t, j, \phi_s(t, j))\end{aligned} \tag{B.21}$$

Given that the total derivative is defined as $\frac{\mathrm{d}V}{\mathrm{d}t}(t, j, \phi_s(t, j)) = \frac{\partial V}{\partial t}(t, j, \phi_s(t, j))$ $+ \frac{\partial V}{\partial x}(t, j, \phi_s(t, j))F(\phi_s(t, j), u^s(t, j))$ for every (t, j) such that $t \in \operatorname{dom}_t \phi_s$, by integrating

(B.21) over the interval $[t_j, t_{j+1}]$, and adding them up from $j = 0$ to $j = J_{\phi_s}$, we obtain

$$\sum_{j=0}^{J_{\phi_s}}(t_{j+1} - t_j)(-\varepsilon) \leq \sum_{j=0}^{J_{\phi_s}} \int_{t_j}^{t_{j+1}} \Big(L_C(t, j, \phi_s(t, j), u^s(t, j)) + \frac{\partial V}{\partial x}(t, j, \phi_s(t, j))F(\phi_s(t, j), u^s(t, j)) + \frac{\partial V}{\partial t}(t, j, \phi_s(t, j))\Big)\, dt$$

which yields

$$-\varepsilon T_{\phi_s} \leq \sum_{j=0}^{J_{\phi_s}} \int_{t_j}^{t_{j+1}} L_C(t, j, \phi_s(t, j), u^s(t, j))dt + V(t_1, 0, \phi_s(t_1, 0)) - V(0, 0, \xi) + \sum_{j=1}^{J_{\phi_s}} V(t_{j+1}, j, \phi_s(t_{j+1}, j)) - V(t_j, j, \phi_s(t_j, j)) \tag{B.22}$$

Since $\bar{\gamma}_1$ does not necessarily attain the infimum in (4.26), then, for every $(t_{j+1}, j) \in \operatorname{dom} \phi_s$ such that $(t_{j+1}, j + 1) \in \operatorname{dom} \phi_s$, we have that

$$\begin{aligned} -\varepsilon \leq & \min_{\substack{u_{D1} \\ u_D=(u_{D1},u_{D2})\in\Pi_u(\phi_s(t_{j+1},j),D)}} \max_{u_{D2}} \{L_D(t_{j+1}, j, \phi_s(t_{j+1}, j), u_D(t_{j+1}, j)) \\ & +V(t_{j+1}, j, G(\phi_s(t_{j+1}, j), u_D(t_{j+1}, j)))\} - V(t_{j+1}, j, \phi_s(t_{j+1}, j)) \\ \leq & L_D(t_{j+1}, j, \phi_s(t_{j+1}, j), u^s(t_{j+1}, j)) + V(t_{j+1}, j + 1, \phi_s(t_{j+1}, j + 1)) \\ & -V(t_{j+1}, j, \phi_s(t_{j+1}, j)) \end{aligned} \tag{B.23}$$

Summing both sides from $j = 0$ $j = J_{\phi_s} - 1$, we obtain

$$\begin{aligned} \sum_{j=0}^{J_{\phi_s}-1} (-\varepsilon) \leq & \sum_{j=0}^{J_{\phi_s}-1} (L_D(t_{j+1}, j, \phi_s(t_{j+1}, j), u^s(t_{j+1}, j)) \\ & +V(t_{j+1}, j + 1, \phi_s(t_{j+1}, j + 1)) - V(t_{j+1}, j, \phi_s(t_{j+1}, j))) \\ = & \sum_{j=0}^{J_{\phi_s}-1} L_D(t_{j+1}, j, \phi_s(t_{j+1}, j), u^s(t_{j+1}, j)) \\ & + \sum_{j=1}^{J_{\phi_s}-1} V(t_{j+1}, j + 1, \phi_s(t_{j+1}, j + 1)) - V(t_{j+1}, j, \phi_s(t_{j+1}, j)) \\ & +V(t_1, 1, \phi_s(t_1, 1)) - V(t_1, 0, \phi_s(t_1, 0)) \end{aligned} \tag{B.24}$$

from where we obtain

$$V(t_1,0,\phi_s(t_1,0)) \le J_{\phi_s}\varepsilon + V(t_1,1,\phi_s(t_1,1)) + \sum_{j=0}^{J_{\phi_s}-1} L_D(t_{j+1},j,\phi_s(t_{j+1},j),u^s(t_{j+1},j))$$
$$+ \sum_{j=1}^{J_{\phi_s-1}} V(t_{j+1},j+1,\phi_s(t_{j+1},j+1)) - V(t_{j+1},j,\phi_s(t_{j+1},j)) \quad \text{(B.25)}$$

By adding (B.22) and (B.25), we obtain

$$\begin{aligned}
&-\varepsilon T_{\phi_s} + V(t_1,0,\phi_s(t_1,0)) \\
&\le \sum_{j=0}^{J_{\phi_s}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi_s(t,j),u^s(t,j))dt + \sum_{j=0}^{J_{\phi_s-1}} L_D(t_{j+1},j,\phi_s(t_{j+1},j),u^s(t_{j+1},j)) \\
&+ \sum_{j=1}^{J_{\phi_s-1}} V(t_{j+1},j+1,\phi_s(t_{j+1},j+1)) - V(t_{j+1},j,\phi_s(t_{j+1},j)) \\
&+ V(t_1,1,\phi_s(t_1,1)) + \sum_{j=1}^{J_{\phi_s}} V(t_{j+1},j,\phi_s(t_{j+1},j)) - V(t_j,j,\phi_s(t_j,j)) \\
&+ J_{\phi_s}\varepsilon - V(0,0,\xi) + V(t_1,0,\phi_s(t_1,0))
\end{aligned} \quad \text{(B.26)}$$

which, thanks to (4.3) and (4.27), turns into

$$V(0,0,\xi) \le \mathcal{J}(\xi,u^s) + (T_{\phi_s} + J_{\phi_s})\varepsilon \quad \text{(B.27)}$$

□

B.4 Proof of Lemma 4.4.2

Pick any $(\phi_s,u^s) \in \mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ with $u^s = (u_1^s,u_2^s)$, $\operatorname{dom}\phi_s \ni (t,j) \mapsto u_1^s(t,j) = \gamma_1^*(t,j,\phi_s(t,j))$ for $\gamma_1^* := (\gamma_{C1}^*,\gamma_{D1}^*)$ attaining the infimum in (4.29) and (4.30), $\operatorname{dom}\phi_s \ni (t,j) \mapsto u_2^s(t,j) = \bar{\gamma}_2(t,j,\phi_s(t,j))$ for some $\bar{\gamma}_2 \in K_2$, and $(T_{\phi_s},J_{\phi_s}) = (t_{J_{\phi_s}+1},J_{\phi_s}) = \sup\operatorname{dom}(\phi_s,u^s) \in \mathcal{T}$. Since $\bar{\gamma}_2$ does not necessarily attain the supremum in (4.29), then, for each $j \in \mathbb{N}$ such that $I_{\phi_w}^j = [t_j,t_{j+1}]$ has a nonempty interior $\operatorname{int} I_{\phi_w}^j$, we have that for every $t \in \mathrm{int} I^j_{\phi_w}$,

$$\varepsilon \geq \min_{\substack{u_{C1} \\ u_C=(u_{C1},u_{C2})\in\Pi_u(\phi_w(t,j),C)}} \max_{u_{C2}} \left\{ L_C(t,j,\phi(t,j),u_C(t,j)) + \frac{\partial V}{\partial x}(t,j,\phi_w(t,j))F(\phi_w(t,j),u_C(t,j)) \right\}$$
$$+\frac{\partial V}{\partial t}(t,j,\phi_w(t,j))$$
$$\geq L_C(t,j,\phi_w(t,j),u^s_C(t,j)) + \frac{\partial V}{\partial x}(t,j,\phi_w(t,j))F(\phi_w(t,j),u^s_C(t,j)) + \frac{\partial V}{\partial t}(t,j,\phi_w(t,j)) \tag{B.28}$$

Given that the total derivative is defined as $\frac{\mathrm{d}V}{\mathrm{d}t}(t,j,\phi_w(t,j)) = \frac{\partial V}{\partial t}(t,j,\phi_w(t,j)) + \frac{\partial V}{\partial x}(t,j,\phi_w(t,j))F(\phi_w(t,j),u^w(t,j))$ for every (t,j) such that $t \in \mathrm{dom}_t\,\phi_w$, by integrating (B.28) over the interval $[t_j,t_{j+1}]$, and adding them up from $j=0$ to $j=J_{\phi_w}$, we obtain

$$\sum_{j=0}^{J_{\phi_w}}(t_{j+1}-t_j)\varepsilon \geq \sum_{j=0}^{J_{\phi_w}}\int_{t_j}^{t_{j+1}} \Big(L_C(t,j,\phi_w(t,j),u^w(t,j))$$
$$+\frac{\partial V}{\partial x}(t,j,\phi_w(t,j))F(\phi_w(t,j),u^w(t,j)) + \frac{\partial V}{\partial t}(t,j,\phi_w(t,j))\Big)\,dt$$

which yields

$$\varepsilon T_{\phi_w} \geq \sum_{j=0}^{J_{\phi_w}}\int_{t_j}^{t_{j+1}} L_C(t,j,\phi_w(t,j),u^w(t,j))dt + V(t_1,0,\phi_w(t_1,0)) - V(0,0,\xi)$$
$$+\sum_{j=1}^{J_{\phi_w}} V(t_{j+1},j,\phi_w(t_{j+1},j)) - V(t_j,j,\phi_w(t_j,j)) \tag{B.29}$$

Since $\bar{\gamma}_2$ does not necessarily attain the supremum in (4.30), then, for every $(t_{j+1},j) \in \mathrm{dom}\,\phi_w$ such that $(t_{j+1},j+1) \in \mathrm{dom}\,\phi_w$, we have that

$$\varepsilon \geq \min_{\substack{u_{D1} \\ u_D=(u_{D1},u_{D2})\in\Pi_u(\phi_w(t_{j+1},j),D)}} \max_{u_{D2}} \{L_D(t_{j+1},j,\phi_w(t_{j+1},j),u_D(t_{j+1},j))$$
$$+V(t_{j+1},j,G(\phi_w(t_{j+1},j),u_D(t_{j+1},j)))\} - V(t_{j+1},j,\phi_w(t_{j+1},j))$$
$$\geq L_D(t_{j+1},j,\phi_w(t_{j+1},j),u^w(t_{j+1},j)) + V(t_{j+1},j+1,\phi_w(t_{j+1},j+1))$$
$$-V(t_{j+1},j,\phi_w(t_{j+1},j)) \tag{B.30}$$

Summing both sides from $j = 0$ $j = J_{\phi_w} - 1$, we obtain

$$\begin{aligned}
\sum_{j=0}^{J_{\phi_w}-1} \varepsilon \geq & \sum_{j=0}^{J_{\phi_w}-1} \big(L_D(t_{j+1}, j, \phi_w(t_{j+1}, j), u^w(t_{j+1}, j)) \\
& + V(t_{j+1}, j+1, \phi_w(t_{j+1}, j+1)) - V(t_{j+1}, j, \phi_w(t_{j+1}, j))\big) \\
= & \sum_{j=0}^{J_{\phi_w}-1} L_D(t_{j+1}, j, \phi_w(t_{j+1}, j), u^w(t_{j+1}, j)) \\
& + \sum_{j=1}^{J_{\phi_w}-1} V(t_{j+1}, j+1, \phi_w(t_{j+1}, j+1)) - V(t_{j+1}, j, \phi_w(t_{j+1}, j)) \\
& + V(t_1, 1, \phi_w(t_1, 1)) - V(t_1, 0, \phi_w(t_1, 0))
\end{aligned} \tag{B.31}$$

from where we obtain

$$\begin{aligned}
V(t_1, 0, \phi_w(t_1, 0)) \geq & -J_{\phi_w}\varepsilon + V(t_1, 1, \phi_w(t_1, 1)) + \sum_{j=0}^{J_{\phi_w-1}} L_D(t_{j+1}, j, \phi_w(t_{j+1}, j), u^w(t_{j+1}, j)) \\
& + \sum_{j=1}^{J_{\phi_w-1}} V(t_{j+1}, j+1, \phi_w(t_{j+1}, j+1)) - V(t_{j+1}, j, \phi_w(t_{j+1}, j))
\end{aligned} \tag{B.32}$$

By adding (B.29) and (B.32), we obtain

$$\begin{aligned}
& \varepsilon T_{\phi_w} + V(t_1, 0, \phi_w(t_1, 0)) \\
\geq & \sum_{j=0}^{J_{\phi_w}} \int_{t_j}^{t_{j+1}} L_C(t, j, \phi_w(t, j), u^w(t, j))dt + \sum_{j=0}^{J_{\phi_w-1}} L_D(t_{j+1}, j, \phi_w(t_{j+1}, j), u^w(t_{j+1}, j)) \\
& + \sum_{j=1}^{J_{\phi_w-1}} V(t_{j+1}, j+1, \phi_w(t_{j+1}, j+1)) - V(t_{j+1}, j, \phi_w(t_{j+1}, j)) \\
& + V(t_1, 1, \phi_w(t_1, 1)) + \sum_{j=1}^{J_{\phi_w}} V(t_{j+1}, j, \phi_w(t_{j+1}, j)) - V(t_j, j, \phi_w(t_j, j)) \\
& - J_{\phi_w}\varepsilon - V(0, 0, \xi) + V(t_1, 0, \phi_w(t_1, 0))
\end{aligned} \tag{B.33}$$

which, thanks to (4.3) and (4.31), turns into

$$\mathcal{J}(\xi, u^w) \leq V(0, 0, \xi) + (T_{\phi_w} + J_{\phi_w})\varepsilon \tag{B.34}$$

□

B.5 Proof of Theorem 4.4.3

To show the claim we proceed as follows:

1. Pick an initial condition ξ and upper bound the cost associated to any solution yield by $\gamma = (\gamma_C, \gamma_D)$, with values satisfying (4.36) and (4.37), from ξ.

2. Lower bound the cost associated to any solution from ξ when P_2 plays $\gamma_2 := (\gamma_{C2}, \gamma_{D2})$ by the value of the function V evaluated at ξ plus a constant.

3. Upper bound the cost associated to any solution from ξ when P_1 plays $\gamma_1 := (\gamma_{C1}, \gamma_{D1})$ by the value of the function V evaluated at ξ plus a constant.

4. By showing that the cost of any solution from ξ when P_1 plays γ_1 is not less than the cost of any solution yield by γ plus a constant from ξ, and by showing that the cost of any solution from ξ when P_2 plays γ_2 is not greater than the cost of any solution yield by γ from ξ plus a constant, we show approximate optimality of γ in Problem $(\star)$ in the min-max sense.

Proceeding as in item 1 above, pick any $\xi \in \Pi(\bar{C}) \cup \Pi(D)$ and any $(\phi^*, u^*) \in \hat{\mathcal{S}}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ with $\operatorname{dom}\phi^* \ni (t,j) \mapsto u^*(t,j) = \gamma(t,j,\phi^*(t,j))$, and $(T_{\phi^*}, J_{\phi^*}) = (t_{J_{\phi^*}+1}, J_{\phi^*}) = \sup\operatorname{dom}(\phi^*, u^*) \in \mathcal{T}$. Given that V satisfies (4.33), and together with γ_C satisfy the right-hand side inequality in (4.36), for each $j \in \mathbb{N}$ such that $I_{\phi^*}^j = [t_j, t_{j+1}]$ has a nonempty interior $\operatorname{int} I_{\phi^*}^j$, we have, for all $t \in \operatorname{int} I_{\phi^*}^j$,

$$\begin{aligned}
\varepsilon \geq & \min_{\substack{u_{C1} \\ u_C = (u_{C1}, u_{C2}) \in \Pi_u(\phi^*(t,j), C)}} \max_{u_{C2}} \left\{ L_C(t,j,\phi^*(t,j), u_C(t,j)) + \frac{\partial V}{\partial x}(t,j,\phi^*(t,j)) F(\phi^*(t,j), u_C(t,j)) \right\} \\
& + \frac{\partial V}{\partial t}(t,j,\phi^*(t,j)) \\
\geq & -\delta + L_C(t,j,\phi^*(t,j), \gamma_C(t,j,\phi^*(t,j))) \\
& + \frac{\partial V}{\partial x}(t,j,\phi^*(t,j)) F(\phi^*(t,j), \gamma_C(t,j,\phi^*(t,j))) + \frac{\partial V}{\partial t}(t,j,\phi^*(t,j)) \qquad \text{(B.35)}
\end{aligned}$$

Given that the total derivative is defined as $\frac{\mathrm{d}V}{\mathrm{d}t}(t,j,\phi^*(t,j)) = \frac{\partial V}{\partial t}(t,j,\phi^*(t,j)) + \frac{\partial V}{\partial x}(t,j,\phi^*(t,j)) F(\phi^*(t,j), \gamma_C(t,j,\phi^*(t,j)))$ for every (t,j) such that $t \in \operatorname{dom}_t \phi^*$, by integrating (B.35) over the interval $[t_j, t_{j+1}]$, and adding them up from $j = 0$ to $j = J_{\phi^*}$, we obtain

$$\sum_{j=0}^{J_{\phi^*}}(t_{j+1}-t_j)(\varepsilon+\delta) \geq \sum_{j=0}^{J_{\phi^*}} \int_{t_j}^{t_{j+1}} \Big(L_C(t,j,\phi^*(t,j),\gamma_C(t,j,\phi^*(t,j))) + \frac{\partial V}{\partial x}(t,j,\phi^*(t,j))F(\phi^*(t,j),\gamma_C(t,j,\phi^*(t,j))) + \frac{\partial V}{\partial t}(t,j,\phi^*(t,j))\Big)\, dt$$

which yields

$$T_{\phi^*}(\varepsilon+\delta) \geq \sum_{j=0}^{J_{\phi^*}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi^*(t,j),\gamma_C(\phi^*(t,j)))dt + V(t_1,0,\phi^*(t_1,0)) - V(0,0,\xi) + \sum_{j=1}^{J_{\phi^*}} V(t_{j+1},j,\phi^*(t_{j+1},j)) - V(t_j,j,\phi^*(t_j,j)) \tag{B.36}$$

Since V satisfies (4.34), and together with γ_D satisfy the right-hand side inequality in (4.37), for every $(t_{j+1},j) \in \operatorname{dom}\phi^*$ such that $(t_{j+1},j+1) \in \operatorname{dom}\phi^*$, we have that

$$\begin{aligned}\varepsilon \geq & \min_{\substack{u_{D1} \\ u_D=(u_{D1},u_{D2})\in\Pi_u(\phi^*(t_{j+1},j),D)}} \max_{u_{D2}} \{L_D(t_{j+1},j,\phi^*(t_{j+1},j),u_D(t_{j+1},j)) \\ & +V(t_{j+1},j,G(\phi^*(t_{j+1},j),u_D(t_{j+1},j)))\} - V(t_{j+1},j,\phi^*(t_{j+1},j)) \\ \geq & -\delta + L_D(t_{j+1},j,\phi^*(t_{j+1},j),\gamma_D(t_{j+1},j,\phi^*(t_{j+1},j))) + V(t_{j+1},j+1,\phi^*(t_{j+1},j+1)) \\ & -V(t_{j+1},j,\phi^*(t_{j+1},j))\end{aligned} \tag{B.37}$$

Summing both sides from $j=0$ to $j=J_{\phi^*}-1$, we obtain

$$\begin{aligned}\sum_{j=0}^{J_{\phi^*}-1}(\varepsilon+\delta) \geq & \sum_{j=0}^{J_{\phi^*}-1} (L_D(t_{j+1},j,\phi^*(t_{j+1},j),\gamma_D(t_{j+1},j,\phi^*(t_{j+1},j))) \\ & +V(t_{j+1},j+1,\phi^*(t_{j+1},j+1)) - V(t_{j+1},j,\phi^*(t_{j+1},j))) \\ = & \sum_{j=0}^{J_{\phi^*}-1} L_D(t_{j+1},j,\phi^*(t_{j+1},j),\gamma_D(t_{j+1},j,\phi^*(t_{j+1},j))) \\ & + \sum_{j=1}^{J_{\phi^*}-1} V(t_{j+1},j+1,\phi^*(t_{j+1},j+1)) - V(t_{j+1},j,\phi^*(t_{j+1},j)) \\ & +V(t_1,1,\phi^*(t_1,1)) - V(t_1,0,\phi^*(t_1,0))\end{aligned} \tag{B.38}$$

from where we obtain

$$
\begin{aligned}
&V(t_1,1,\phi^*(t_1,1)) + \sum_{j=0}^{J_{\phi^*}-1} L_D(t_{j+1},j,\phi^*(t_{j+1},j),\gamma_D(t_{j+1},j,\phi^*(t_{j+1},j))) \\
&+ \sum_{j=1}^{J_{\phi^*}-1} V(t_{j+1},j+1,\phi^*(t_{j+1},j+1)) - V(t_{j+1},j,\phi^*(t_{j+1},j)) \\
&\leq J_{\phi^*}(\varepsilon+\delta) + V(t_1,0,\phi^*(t_1,0))
\end{aligned} \tag{B.39}
$$

By adding (B.36) and (B.39), we obtain

$$
\begin{aligned}
&\sum_{j=0}^{J_{\phi^*}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi^*(t,j),\gamma_C(\phi^*(t,j)))dt \\
&+ \sum_{j=0}^{J_{\phi^*}-1} L_D(t_{j+1},j,\phi^*(t_{j+1},j),\gamma_D(t_{j+1},j,\phi^*(t_{j+1},j))) \\
&+ \sum_{j=1}^{J_{\phi^*}-1} V(t_{j+1},j+1,\phi^*(t_{j+1},j+1)) - V(t_{j+1},j,\phi^*(t_{j+1},j)) \\
&+V(t_1,1,\phi^*(t_1,1)) + \sum_{j=1}^{J_{\phi^*}} V(t_{j+1},j,\phi^*(t_{j+1},j)) - V(t_j,j,\phi^*(t_j,j)) \\
&-V(0,0,\xi) + V(t_1,0,\phi^*(t_1,0)) \\
&\leq J_{\phi^*}(\varepsilon+\delta) + T_{\phi^*}(\varepsilon+\delta) + V(t_1,0,\phi^*(t_1,0))
\end{aligned} \tag{B.40}
$$

By straightforward simplifications, the reader can show that

$$
\begin{aligned}
V(T_{\phi^*},J_{\phi^*}) = &\sum_{j=1}^{J_{\phi^*}-1} (V(t_{j+1},j+1,\phi^*(t_{j+1},j+1)) - V(t_{j+1},j,\phi^*(t_{j+1},j))) \\
&+V(t_1,1,\phi^*(t_1,1)) + \sum_{j=1}^{J_{\phi^*}} V(t_{j+1},j,\phi^*(t_{j+1},j)) - V(t_j,j,\phi^*(t_j,j))
\end{aligned} \tag{B.41}
$$

Which, thanks to (4.3) and (4.35), turns (B.40) into

$$
\mathcal{J}(\xi,u^*) \leq (J_{\phi^*} + T_{\phi^*})(\varepsilon+\delta) + V(0,0,\xi) \tag{B.42}
$$

Likewise, given that V satisfies (4.33), and together with γ_C satisfy the left side inequality in (4.36), for each $j \in \mathbb{N}$ such that $I^j_{\phi^*} = [t_j, t_{j+1}]$ has a nonempty interior $\text{int} I^j_{\phi^*}$, we have, for all $t \in \mathrm{int} I^j_{\phi^*}$,

$$-\varepsilon \le \min_{u_{C1}} \max_{u_{C2}} \atop {u_C=(u_{C1},u_{C2})\in\Pi_u(\phi^*(t,j),C)} \left\{ L_C(t,j,\phi^*(t,j),u_C(t,j)) + \frac{\partial V}{\partial x}(t,j,\phi^*(t,j))F(\phi^*(t,j),u_C(t,j)) \right\}$$
$$+\frac{\partial V}{\partial t}(t,j,\phi^*(t,j))$$
$$\le \delta + L_C(t,j,\phi^*(t,j),\gamma_C(t,j,\phi^*(t,j))) + \frac{\partial V}{\partial x}(t,j,\phi^*(t,j))F(\phi^*(t,j),\gamma_C(t,j,\phi^*(t,j)))$$
$$+\frac{\partial V}{\partial t}(t,j,\phi^*(t,j)) \qquad \text{(B.43)}$$

Given that, $\frac{\mathrm{d}V}{\mathrm{d}t}(t,j,\phi^*(t,j)) = \frac{\partial V}{\partial t}(t,j,\phi^*(t,j)) + \frac{\partial V}{\partial x}(t,j,\phi^*(t,j))F(\phi^*(t,j),$ $\gamma_C(t,j,\phi^*(t,j)))$ for every (t,j) such that $t \in \mathrm{dom}_t\, \phi^*$, by integrating (B.43) over the interval $[t_j, t_{j+1}]$, and adding them up from $j = 0$ to $j = J_{\phi^*}$, we obtain

$$0 \le \sum_{j=0}^{J_{\phi^*}} (t_{j+1} - t_j)(\varepsilon + \delta) + \sum_{j=0}^{J_{\phi^*}} \int_{t_j}^{t_{j+1}} \Big(L_C(t,j,\phi^*(t,j),\gamma_C(t,j,\phi^*(t,j)))$$
$$+\frac{\partial V}{\partial x}(t,j,\phi^*(t,j))F(\phi^*(t,j),\gamma_C(t,j,\phi^*(t,j))) + \frac{\partial V}{\partial t}(t,j,\phi^*(t,j))\Big)\, dt$$

which yields

$$0 \le \sum_{j=0}^{J_{\phi^*}} \int_{t_j}^{t_{j+1}} L_C(t,j,\phi^*(t,j),\gamma_C(\phi^*(t,j)))dt + V(t_1,0,\phi^*(t_1,0)) - V(0,0,\xi)$$
$$+T_{\phi^*}(\varepsilon + \delta) + \sum_{j=1}^{J_{\phi^*}} V(t_{j+1},j,\phi^*(t_{j+1},j)) - V(t_j,j,\phi^*(t_j,j)) \qquad \text{(B.44)}$$

Since V satisfies (4.34), and together with γ_D satisfy the left side inequality in (4.37), for every $(t_{j+1}, j) \in \mathrm{dom}\, \phi^*$ such that $(t_{j+1}, j+1) \in \mathrm{dom}\, \phi^*$, we have that

$$-\varepsilon \le \min_{u_{D1}} \max_{u_{D2}} \atop {u_D=(u_{D1},u_{D2})\in\Pi_u(\phi^*(t_{j+1},j),D)} \{ L_D(t_{j+1},j,\phi^*(t_{j+1},j),u_D(t_{j+1},j))$$
$$+V(t_{j+1},j,G(\phi^*(t_{j+1},j),u_D(t_{j+1},j)))\} - V(t_{j+1},j,\phi^*(t_{j+1},j))$$
$$\le \delta + L_D(t_{j+1},j,\phi^*(t_{j+1},j),\gamma_D(t_{j+1},j,\phi^*(t_{j+1},j))) + V(t_{j+1},j+1,\phi^*(t_{j+1},j+1))$$
$$-V(t_{j+1},j,\phi^*(t_{j+1},j)) \qquad \text{(B.45)}$$

Summing both sides from $j = 0$ to $j = J_{\phi^*} - 1$, we obtain

$$0 \leq \sum_{j=0}^{J_{\phi^*}-1} (\varepsilon + \delta) + \sum_{j=0}^{J_{\phi^*}-1} (L_D(t_{j+1}, j, \phi^*(t_{j+1}, j), \gamma_D(t_{j+1}, j, \phi^*(t_{j+1}, j)))$$
$$+V(t_{j+1}, j+1, \phi^*(t_{j+1}, j+1)) - V(t_{j+1}, j, \phi^*(t_{j+1}, j)))$$
$$= \sum_{j=0}^{J_{\phi^*}-1} (\varepsilon + \delta) + \sum_{j=0}^{J_{\phi^*}-1} L_D(t_{j+1}, j, \phi^*(t_{j+1}, j), \gamma_D(t_{j+1}, j, \phi^*(t_{j+1}, j)))$$
$$+ \sum_{j=1}^{J_{\phi^*}-1} V(t_{j+1}, j+1, \phi^*(t_{j+1}, j+1)) - V(t_{j+1}, j, \phi^*(t_{j+1}, j))$$
$$+V(t_1, 1, \phi^*(t_1, 1)) - V(t_1, 0, \phi^*(t_1, 0)) \tag{B.46}$$

from where we obtain

$$V(t_1, 0, \phi^*(t_1, 0)) \leq J_{\phi^*}(\varepsilon + \delta) + V(t_1, 1, \phi^*(t_1, 1))$$
$$+ \sum_{j=0}^{J_{\phi^*-1}} L_D(t_{j+1}, j, \phi^*(t_{j+1}, j), \gamma_D(t_{j+1}, j, \phi^*(t_{j+1}, j)))$$
$$+ \sum_{j=1}^{J_{\phi^*-1}} V(t_{j+1}, j+1, \phi^*(t_{j+1}, j+1)) - V(t_{j+1}, j, \phi^*(t_{j+1}, j)) \tag{B.47}$$

By upper bounding the right hand side of (B.44) with the right hand side of (B.47), we obtain

$$0 \leq \sum_{j=0}^{J_{\phi^*}} \int_{t_j}^{t_{j+1}} L_C(t, j, \phi^*(t, j), \gamma_C(\phi^*(t, j)))dt$$
$$+ \sum_{j=0}^{J_{\phi^*-1}} L_D(t_{j+1}, j, \phi^*(t_{j+1}, j), \gamma_D(t_{j+1}, j, \phi^*(t_{j+1}, j)))$$
$$+ \sum_{j=1}^{J_{\phi^*-1}} V(t_{j+1}, j+1, \phi^*(t_{j+1}, j+1)) - V(t_{j+1}, j, \phi^*(t_{j+1}, j))$$
$$+V(t_1, 1, \phi^*(t_1, 1)) + \sum_{j=1}^{J_{\phi^*}} V(t_{j+1}, j, \phi^*(t_{j+1}, j)) - V(t_j, j, \phi^*(t_j, j))$$
$$-V(0, 0, \xi) + J_{\phi^*}(\varepsilon + \delta) + T_{\phi^*}(\varepsilon + \delta) \tag{B.48}$$

Thanks to (B.41), (4.35) and (4.3), the bound (B.48) turns into

$$V(0, 0, \xi) \leq \mathcal{J}(\xi, u^*) + (J_{\phi^*} + T_{\phi^*})(\varepsilon + \delta) \tag{B.49}$$

Continuing with item 2 as above, pick any $(\phi_s, u^s) \in \mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ with $u^s = (u_1^s, u_2^s)$, $\operatorname{dom}\phi_s \ni (t,j) \mapsto u_1^s(t,j) = \bar{\gamma}_1(t,j,\phi_s(t,j))$ for some $\bar{\gamma}_1 \in K_1$, $\operatorname{dom}\phi_s \ni (t,j) \mapsto u_2^s(t,j) = \gamma_2^*(t,j,\phi_s(t,j))$ for $\gamma_2^* := (\gamma_{C2}^*, \gamma_{D2}^*)$ attining the supremum in (4.33) and (4.34), and $(T_{\phi_s}, J_{\phi_s}) = (t_{J_{\phi_s}+1}, J_{\phi_s}) = \sup\operatorname{dom}(\phi_s, u^s) \in \mathcal{T}$. Thanks to Lemma 4.4.1, we have

$$V(0,0,\xi) \leq \mathcal{J}(\xi, u^s) + (T_{\phi_s} + J_{\phi_s})\varepsilon \tag{B.50}$$

Continuing with item 3 as above, pick any $(\phi_w, u^w) \in \mathcal{S}_{\mathcal{H}}^{\mathcal{T}}(\xi)$ with $u^w = (u_1^w, u_2^w)$, $\operatorname{dom}\phi_w \ni (t,j) \mapsto u_1^w(t,j) = \gamma_1^*(t,j,\phi_w(t,j))$ for $\gamma_1^* = (\gamma_{C1}^*, \gamma_{D1}^*)$ attaining the infimum in (4.33) and (4.34), $\operatorname{dom}\phi_w \ni (t,j) \mapsto u_2^w(t,j) = \bar{\gamma}_2(t,j,\phi_w(t,j))$ for some $\bar{\gamma}_2 \in K_2$, and $(T_{\phi_w}, J_{\phi_w}) = (t_{J_{\phi_w}+1}, J_{\phi_w}) = \sup\operatorname{dom}(\phi_w, u^w) \in \mathcal{T}$. Thanks to Lemma 4.4.2, we have

$$\mathcal{J}(\xi, u^w) \leq V(0,0,\xi) + (T_{\phi_w} + J_{\phi_w})\varepsilon \tag{B.51}$$

Finally, by proceeding as in item 4 above, by upperbounding the $V(0,0,\xi)$ term on the right hand side of (B.42) with (B.50), we obtain

$$\mathcal{J}(\xi, u^*) \leq \mathcal{J}(\xi, u^s) + (J_{\phi^*} + T_{\phi^*})(\varepsilon + \delta) + (J_{\phi_s} + T_{\phi_s})\varepsilon \qquad \forall \xi \in \Pi(\overline{C} \cup D), \tag{B.52}$$

By upperbounding the $V(0,0,\xi)$ term on the right hand side of (B.51) with (B.49), we obtain

$$\mathcal{J}(\xi, u^w) \leq \mathcal{J}(\xi, u^*) + (J_{\phi^*} + T_{\phi^*})(\varepsilon + \delta) + (J_{\phi_w} + T_{\phi_w})\varepsilon \qquad \forall \xi \in \Pi(\overline{C} \cup D), \tag{B.53}$$

Given that, $\max_{(T,J)\in\mathcal{T}}(T,J) = (\tau_p\delta_p, \tau_p)$, from (B.52) and (B.53), we have

$$\mathcal{J}(\xi, u^w) - \tau_p(1+\delta_p)(2\varepsilon + \delta) \leq \mathcal{J}(\xi, u^*) \leq \mathcal{J}(\xi, u^s) + \tau_p(1+\delta_p)(2\varepsilon + \delta) \tag{B.54}$$

for all $\xi \in \Pi(\overline{C} \cup D)$, which renders the feedback law γ approximately optimal in the min-max sense.

□

Appendix C

Proofs of Chapter 5

C.1 Proof of Theorem 5.2.1

To show the claim we proceed as follows:

a) Pick an initial condition $\xi \in \mathcal{M}$ and evaluate the cost associated to any solution from ξ yielded by $\kappa = (\kappa_C, \kappa_D)$, with values as in (5.9) and (5.10). Show that this cost coincides with the value of the function V at ξ.

b) Lower bound the cost associated to any solution from ξ when P_2 plays $\kappa_2 := (\kappa_{C2}, \kappa_{D2})$ by the value of the function V evaluated at ξ.

c) Upper bound the cost associated to any solution from ξ when P_1 plays $\kappa_1 := (\kappa_{C1}, \kappa_{D1})$ by the value of the function V evaluated at ξ.

d) By showing that the cost of any solution from ξ when P_1 plays κ_1 is not greater than the cost of any solution yield by κ from ξ, and by showing that the cost of any solution from ξ when P_2 plays κ_2 is not less than the cost of any solution yield by κ from ξ, we show optimality of κ in Problem $(\diamond)$ in the min-max sense.

Proceeding as in item a above, pick any $\xi \in \mathcal{M}$ and any $(\phi^*, u^*) \in \mathcal{S}_{\mathcal{H}}^X(\xi)$ with $\operatorname{dom}\phi^* \ni (t,j) \mapsto u^*(t,j) = \kappa(\phi^*(t,j))$. We show that (ϕ^*, u^*) is optimal in the min-max sense. Given that V satisfies (5.5), and κ_C is as in (5.9), for each $j \in \mathbb{N}$ such that $I_{\phi^*}^j = [t_j, t_{j+1}]$ has a nonempty interior $\operatorname{int} I_{\phi^*}^j$, we have, for all $t \in \operatorname{int} I_{\phi^*}^j$,

$$0 = L_C(\phi^*(t,j), \kappa_C(\phi^*(t,j))) + \langle \nabla V(\phi^*(t,j)), F(\phi^*(t,j), \kappa_C(\phi^*(t,j))) \rangle$$

and $\phi^*(t,j) \in C_\kappa$, as in (2.4). Given that V is continuously differentiable on a neighborhood of $\Pi(C)$, we can express its total derivative along ϕ^* as

$$\frac{d}{dt}V(\phi^*(t,j)) = \langle \nabla V(\phi^*(t,j)), F(\phi^*(t,j), \kappa_C(\phi^*(t,j)))\rangle \tag{C.1}$$

for every $(t,j) \in \text{int}(I^j_{\phi^*}) \times \{j\}$ with $\text{int}(I^j_{\phi^*})$ nonempty. Given that V satisfies (5.6) and κ_D is as in (5.10), for every $(t_{j+1}, j) \in \text{dom}\,\phi^*$ such that $(t_{j+1}, j+1) \in \text{dom}\,\phi^*$, we have that

$$\begin{aligned} V(\phi^*(t_{j+1},j)) &= L_D(\phi^*(t_{j+1},j), \kappa_D(\phi^*(t_{j+1},j))) + V(G(\phi^*(t_{j+1},j), \kappa_D(\phi^*(t_{j+1},j)))) \\ &= L_D(\phi^*(t_{j+1},j), \kappa_D(\phi^*(t_{j+1},j))) + V(\phi^*(t_{j+1},j+1)) \end{aligned} \tag{C.2}$$

where $\phi^*(t_{j+1},j) \in D_\kappa$ is defined in (2.4). Now, given that (ϕ^*, u^*) is maximal with $\text{dom}\,\phi^* \ni (t,j) \mapsto u^*(t,j) = \kappa(\phi^*(t,j))$, thanks to (C.1) and (C.2), from Corollary 5.2.3 and (5.7), we have that

$$V(\xi) = \mathcal{J}(\xi, u^*). \tag{C.3}$$

Continuing with item b as above, pick any $(\phi_s, u^s) \in \mathcal{S}^s_{\mathcal{H}}(\xi)$ where $\mathcal{S}^s_{\mathcal{H}}(\xi)(\subset \mathcal{S}^X_{\mathcal{H}}(\xi))$ is the set of solutions (ϕ, u) with $u = (u_1, u_2)$, $\text{dom}\,\phi \ni (t,j) \mapsto u_1(t,j) = \bar\kappa_1(\phi(t,j))$ for some $\bar\kappa_1 \in \mathcal{K}_1$, $\text{dom}\,\phi \ni (t,j) \mapsto u_2(t,j) = \kappa_2(t,j,\phi(t,j))$ for $\kappa_2 := (\kappa_{C2}, \kappa_{D2})$ as in (5.9) and (5.10). Since $\bar\kappa_1$ does not necessarily attain the minimum in (5.5), then, for each $j \in \mathbb{N}$ such that $I^j_{\phi_s} = [t_j, t_{j+1}]$ has a nonempty interior $\text{int}I^j_{\phi_s}$, we have for every $t \in \text{int}I^j_{\phi_s}$,

$$0 \le L_C(\phi_s(t,j), u^s_C(t,j)) + \langle \nabla V(\phi_s(t,j)), F(\phi_s(t,j), u^s_C(t,j))\rangle .$$

Similarly to (C.1), we have

$$\frac{d}{dt}V(\phi_s(t,j)) := \langle \nabla V(\phi_s(t,j)), F(\phi_s(t,j), u^s_C(t,j))\rangle \tag{C.4}$$

for every $(t,j) \in \text{int}(I^j_{\phi_s}) \times \{j\}$ with $\text{int}(I^j_{\phi_s})$ nonempty. In addition, since $\bar\kappa_1$ does not necessarily attain the minimum in (5.6), then for every $(t_{j+1}, j) \in \text{dom}\,\phi_s$ such that $(t_{j+1}, j+1) \in \text{dom}\,\phi_s$, we have

$$V(\phi_s(t_{j+1},j)) \le L_D(\phi_s(t_{j+1},j), u^s_D(t_{j+1},j)) + V(\phi_s(t_{j+1},j+1)). \tag{C.5}$$

Now, given that (ϕ_s, u_s) is maximal, with $u^s = (u_1^s, u_2^s)$, u_1^s defined by any $\bar{\kappa}_1 \in \mathcal{K}_1$, and u_2^s defined by κ_2 as in (5.9) and (5.10), thanks to (C.4) and (C.5), from Proposition 5.2.2 and (5.7), we have

$$V(\xi) \leq \mathcal{J}(\xi, u^s). \tag{C.6}$$

Proceeding with item c as above, pick any $(\phi_w, u^w) \in \mathcal{S}_{\mathcal{H}}^w(\xi)$, where $\mathcal{S}_{\mathcal{H}}^w(\xi)(\subset \mathcal{S}_{\mathcal{H}}^X(\xi))$ is the set of solutions (ϕ, u) with $u = (u_1, u_2)$, $\operatorname{dom}\phi \ni (t,j) \mapsto u_1(t,j) = \kappa_1(\phi(t,j))$ for $\kappa_1 := (\kappa_{C1}, \kappa_{D1})$ as in (5.9) and (5.10), $\operatorname{dom}\phi \ni (t,j) \mapsto u_2(t,j) = \bar{\kappa}_2(\phi(t,j))$ for some $\bar{\kappa}_2 \in \mathcal{K}_2$. Since $\bar{\kappa}_2$ does not necessarily attain the maximum in (5.5), then, for each $j \in \mathbb{N}$ such that $I_{\phi_w}^j = [t_j, t_{j+1}]$ has a nonempty interior $\operatorname{int} I_{\phi_w}^j$, we have for every $t \in \operatorname{int} I_{\phi_w}^j$,

$$0 \geq L_C(\phi_w(t,j), u_C^w(t,j)) + \langle \nabla V(\phi_w(t,j)), F(\phi_w(t,j), u_C^w(t,j)) \rangle .$$

Similarly to (C.1), we have

$$\frac{d}{dt} V(\phi_w(t,j)) := \langle \nabla V(\phi_w(t,j)), F(\phi_w(t,j), u_C^w(t,j)) \rangle \tag{C.7}$$

for every $(t,j) \in \operatorname{int}(I_{\phi_w}^j) \times \{j\}$ with $\operatorname{int}(I_{\phi_w}^j)$ nonempty. In addition, since $\bar{\kappa}_2$ does not necessarily attain the maximum in (5.6), then for every $(t_{j+1}, j) \in \operatorname{dom}\phi_w$ such that $(t_{j+1}, j+1) \in \operatorname{dom}\phi_w$, we have

$$V(\phi_w(t_{j+1}, j)) \geq L_D(\phi_w(t_{j+1}, j), u_D^w(t_{j+1}, j)) + V(\phi_w(t_{j+1}, j+1)). \tag{C.8}$$

Now, given that (ϕ_w, u^w) is maximal, with $u^w = (u_1^w, u_2^w)$, u_1^w defined by κ_1 as in (5.9) and (5.10), and u_2^w defined by any $\bar{\kappa}_2 \in \mathcal{K}_2$, thanks to (C.7) and (C.8), from Corollary 5.2.3 and (5.7), we have

$$V(\xi) \geq \mathcal{J}(\xi, u^w). \tag{C.9}$$

Finally, by proceeding as in item d above, by applying the infimum on each side of (D.23) over the set $\mathcal{S}_{\mathcal{H}}^X(\xi)$, we obtain

$$V(\xi) \leq \inf_{u_1 : (\phi_s, (u_1, \kappa_2(\phi_s))) \in \mathcal{S}_{\mathcal{H}}^X(\xi)} \mathcal{J}(\xi, (u_1, \kappa_2(\phi_s))) =: \overline{V}(\xi)$$

By applying the supremum on each side of (C.9) over the set $\mathcal{S}_{\mathcal{H}}^X(\xi)$, we obtain

$$V(\xi) \geq \sup_{u_2 : (\phi_w, (\kappa_1(\phi_w), u_2)) \in \mathcal{S}_{\mathcal{H}}^X(\xi)} \mathcal{J}(\xi, (\kappa_1(\phi_w), u_2)) =: \underline{V}(\xi).$$

Given that $V(\xi) = \mathcal{J}(\xi, u^*)$ from (C.3), we have that for any $\xi \in \mathcal{M}$, each $(\phi^*, u^*) \in \mathcal{S}^X_{\mathcal{H}}(\xi)$ with $u^* = (\kappa_1(\phi^*), \kappa_2(\phi^*))$ satisfies

$$\underline{V}(\xi) \leq \mathcal{J}(\xi, u^*) \leq \overline{V}(\xi) \tag{C.10}$$

Thanks to $(\phi^*, u^*) \in \mathcal{S}^s_{\mathcal{H}}(\xi) \cap \mathcal{S}^w_{\mathcal{H}}(\xi) (\subset \mathcal{S}^X_{\mathcal{H}}(\xi))$, we have

$$\underline{V}(\xi) = \sup_{(\phi^*, (\kappa_1(\phi^*), \kappa_2(\phi^*))) \in \mathcal{S}^X_{\mathcal{H}}(\xi)} \mathcal{J}(\xi, (\kappa_1(\phi^*), \kappa_2(\phi^*))), \tag{C.11}$$

and

$$\overline{V}(\xi) = \inf_{(\phi^*, (\kappa_1(\phi^*), \kappa_2(\phi^*))) \in \mathcal{S}^X_{\mathcal{H}}(\xi)} \mathcal{J}(\xi, (\kappa_1(\phi^*), \kappa_2(\phi^*))). \tag{C.12}$$

Since the supremum and infimum are attained in (D.30) and (D.7) by $\underline{V}(\xi)$ and $\overline{V}(\xi)$, respectively, (C.10) leads to

$$\mathcal{J}(\xi, u^*) = \min_{\substack{u_1 \\ (u_1,u_2) \in \mathcal{U}^X_{\mathcal{H}}(\xi)}} \max_{u_2} \mathcal{J}(\xi, (u_1, u_2)) \tag{C.13}$$

Thus, from (C.3) and (C.13), $V(\xi)$ is the value function for $\mathcal{H}$, as in Definition 5.1.3 and from (C.10), κ is the saddle-point equilibrium as in Definition 3.1.3. □

C.2 Proof of Theorem 5.3.3

Since, by assumption, we have that $C_\kappa = \Pi(C)$, $D_\kappa = \Pi(D)$, and $V, \kappa := (\kappa_C, \kappa_D) = ((\kappa_{C1}, \kappa_{C2}), (\kappa_{D1}, \kappa_{D2}))$ are such that (5.14)-(5.19) hold, then, thanks to Lemma 5.3.2, V and κ satisfy (5.5), (5.6), (5.9), and (5.10). Since in addition, for each $\xi \in (\overline{C_\kappa} \cup D_\kappa) \cap \mathcal{M}$, each $\phi \in \mathcal{S}^X_{\mathcal{H}_\kappa}(\xi)$ satisfies (5.7), we have from Theorem 5.2.1 that V is the value function as in (5.3) for $\mathcal{H}_\kappa$ at $(\overline{C_\kappa} \cup D_\kappa) \cap \mathcal{M}$ and the feedback law κ with values (5.9), (5.10) is the saddle-point equilibrium for this game. Then, V is a Lyapunov candidate for $\mathcal{H}_\kappa$ [69, Def. 3.16] since $\overline{C_\kappa} \cup D_\kappa \subset \operatorname{dom} V = \mathbb{R}^n$ and V is continuously differentiable on an open set containing $\overline{C_\kappa}$. From (5.14) and (5.17), we have

$$\langle \nabla V(x), F(x, \kappa_C(x)) \rangle \leq -L_C(x, \kappa_C(x)) \qquad \forall x \in C_\kappa \cap \mathcal{M}, \tag{C.14}$$

$$V(G(x, \kappa_D(x))) - V(x) \leq -L_D(x, \kappa_D(x)) \qquad \forall x \in D_\kappa \cap \mathcal{M}. \tag{C.15}$$

Moreover, if

a) Item 1, item 4, or item 5 above hold, define

$$\rho(x,\kappa(x)) := \begin{cases} L_C(x,\kappa_C(x)) & \text{if } x \in C_\kappa \setminus D_\kappa \\ \min\{L_C(x,\kappa_C(x)), L_D(x,\kappa_D(x))\} & \text{if } x \in C_\kappa \cap D_\kappa \\ L_D(x,\kappa_D(x)) & \text{if } x \in D_\kappa \setminus C_\kappa \end{cases}$$

b) Item 2 above holds, define

$$\rho(x,\kappa(x)) := \begin{cases} \eta(|x|_{\mathcal{A}}) & \text{if } x \in C_\kappa \setminus D_\kappa \\ \min\{\eta(|x|_{\mathcal{A}}), L_D(x,\kappa_D(x))\} & \text{if } x \in C_\kappa \cap D_\kappa \\ L_D(x,\kappa_D(x)) & \text{if } x \in D_\kappa \setminus C_\kappa \end{cases}$$

c) Item 3 above holds, define

$$\rho(x,\kappa(x)) := \begin{cases} L_C(x,\kappa_C(x)) & \text{if } x \in C_\kappa \setminus D_\kappa \\ \min\{L_C(x,\kappa_C(x)), \eta(|x|_{\mathcal{A}})\} & \text{if } x \in C_\kappa \cap D_\kappa \\ \eta(|x|_{\mathcal{A}}) & \text{if } x \in D_\kappa \setminus C_\kappa \end{cases}$$

d) Item 6 above holds, define

$$\rho(x,\kappa(x)) := \begin{cases} \lambda_C V(x) & \text{if } x \in C_\kappa \setminus D_\kappa \\ \min\{\lambda_C V(x), e^{\lambda_D} V(x)\} & \text{if } x \in C_\kappa \cap D_\kappa \\ e^{\lambda_D} V(x) & \text{if } x \in D_\kappa \setminus C_\kappa \end{cases}$$

Thus, given that from (C.14) and (C.15), for each case above the function ρ satisfies

$$\langle \nabla V(x), F(x,\kappa_C(x)) \rangle \le -\rho(x,\kappa(x)) \quad \forall x \in C_\kappa \cap \mathcal{M}, \tag{C.16}$$

$$V(G(x,\kappa_D(x))) - V(x) \le -\rho(x,\kappa(x)) \quad \forall x \in D_\kappa \cap \mathcal{M}. \tag{C.17}$$

Thanks to [21, Theorem 3.19], the set $\mathcal{A}$ is pAS for $\mathcal{H}_\kappa$. □

Appendix D

Proofs of Chapter 6

D.1 Proof of Proposition 6.2.1

Pick any solution ϕ to $\mathcal{H}_s$ as in (2.1) with no inputs from $\xi \in (\overline{C} \cup D)$, where $\{t_j\}_{j=0}^{\sup_j \operatorname{dom}\phi}$ is a nondecreasing sequence associated to the definition of the hybrid time domain of ϕ as in Definition 2.2.2. Observe that, for each $(T, J) \in \operatorname{dom}\phi$,

$$V(\phi(T,J)) - V(\xi) = \sum_{j=0}^{J} \int_{t_j}^{t_{j+1}} \frac{d}{dt} V(\phi(t,j))dt + \sum_{j=0}^{J-1} \Big(V(\phi(t_{j+1}, j+1)) - V(\phi(t_{j+1}, j)) \Big) \tag{D.1}$$

where $t_{J+1} := T$, and $V \circ \phi$ is locally Lipschitz on every $I_\phi^j = [t_j, t_{j+1}]$ with $j \in \mathbb{N}$ and nonempty interior. In particular, for each $j \in \mathbb{N}$ and for almost all $t \in I_\phi^j$,

$$\frac{d}{dt} V(\phi(t,j)) \leq \sup_{f \in F(\phi(t,j))} \langle \nabla V(\phi(t,j)), f \rangle \tag{D.2}$$

Moreover, (6.8a) implies that for each $j \in \mathbb{N}$ and for almost all $t \in I_\phi^j$,

$$\frac{d}{dt} V(\phi(t,j)) \leq -L_C(\phi(t,j)). \tag{D.3}$$

Similarly, for every $(t_{j+1}, j) \in \operatorname{dom}\phi$ such that $(t_{j+1}, j+1) \in \operatorname{dom}\phi$, (6.8b) implies that

$$V(\phi(t_{j+1}, j+1)) - V(\phi(t_{j+1}, j)) \leq -L_D(\phi(t_{j+1}, j)). \tag{D.4}$$

Then, by combining (D.1), (D.3), and (D.4), we obtain

$$V(\phi(T,J)) - V(\xi) \leq -\sum_{j=0}^{J} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j))dt - \sum_{j=0}^{J-1} L_D(\phi(t_{j+1}, j)) \tag{D.5}$$

By taking the limit when $(T, J) \to \sup \operatorname{dom} \phi$, thanks to (6.9), (D.5) implies

$$\begin{aligned}\widetilde{\mathcal{J}}(\phi) = & \sum_{j=0}^{\sup_j \operatorname{dom} \phi} \int_{t_j}^{t_{j+1}} L_C(\phi(t,j))dt + \sum_{j=0}^{\sup_j \operatorname{dom} \phi - 1} L_D(\phi(t_{j+1}, j)) + \limsup_{\substack{(t,j) \to \sup \operatorname{dom} \phi \\ (t,j) \in \operatorname{dom} \phi}} q(\phi(t,j)) \\ & \leq V(\xi)\end{aligned}$$

which gives (6.10), concluding the proof. □

D.2 Proof of Corollary 6.2.3

Pick any solution ϕ^* to (6.14) and observe that since for each $x \in C$

$$\arg\max_{f \in F(x)} \langle \nabla V(x), f \rangle \subset F(x)$$

and for each $x \in D$

$$\arg\max_{g \in G(x)} V(g) \subset G(x)$$

ϕ^* is a solution to (2.1) as well. Let $\{t_j\}_{j=0}^{\sup_j \operatorname{dom} \phi^*}$ be a nondecreasing sequence associated to the definition of the hybrid time domain of ϕ^* as in Definition 2.2.2. Moreover, for each $j \in \mathbb{N}$ such that $I_{\phi^*}^j = [t_j, t_{j+1}]$ has a nonempty interior $\operatorname{int} I_{\phi^*}^j$, we have, for almost all $t \in I_{\phi^*}^j$,

$$\frac{d}{dt} V(\phi^*(t,j)) = \max_{f \in F(\phi^*(t,j))} \langle \nabla V(\phi^*(t,j)), f \rangle \tag{D.6a}$$

and thanks to (6.13a)

$$\frac{d}{dt} V(\phi^*(t,j)) + L_C(\phi^*(t,j)) = 0. \tag{D.6b}$$

Likewise, for each $(t_{j+1}, j) \in \operatorname{dom} \phi^*$ such that $(t_{j+1}, j+1) \in \operatorname{dom} \phi^*$

$$V(\phi^*(t_{j+1}, j+1)) = \max_{g \in G(\phi^*(t_{j+1}, j))} V(g). \tag{D.7a}$$

and thanks to (6.13b)

$$V(\phi^*(t_{j+1}, j+1)) - V(\phi^*(t_{j+1}, j)) + L_D(\phi^*(t_{j+1}, j)) = 0. \tag{D.7b}$$

Following the same arguments as in the proof of Proposition 6.2.1, for each $\phi \in \mathcal{S}_{\mathcal{H}}(\xi)$, (D.6) and (D.7) yield

$$\widetilde{\mathcal{J}}(\phi) \leq \widetilde{\mathcal{J}}(\phi^*) = V(\xi) \tag{D.8}$$

Notice that in the light of the inequality in (D.8), since $\phi^* \in \mathcal{S}_{\mathcal{H}_s}(\xi)$ and $\widetilde{\mathcal{J}}(\phi^*) = V(\xi)$, one has that $\phi^* \in \arg\max_{\psi \in \mathcal{S}_{\mathcal{H}_s}(\xi)} \mathcal{J}(\psi)$. On the other hand, by definition one has that

$$\mathcal{J}(\xi) = \sup_{\phi \in \mathcal{S}_{\mathcal{H}_s}(\xi)} \mathcal{J}(\phi)$$

That is, $\mathcal{J}(\xi) = \mathcal{J}(\phi^*)$, and this concludes the proof.

□

D.3 Proof of Proposition 6.3.1

From (6.16), and given a solution (ϕ, u) to $\mathcal{H}_s$ from $\xi \in \Pi(\overline{C}) \cup \Pi(D)$, for each $j \in \mathbb{N}$ such that $I^j_\phi = [t_j, t_{j+1}]$ has a nonempty interior $\mathrm{int} I^j_\phi$, we have, for all $t \in \mathrm{int} I^j_\phi$,

$$\begin{aligned} &L_C(\phi(t,j), u_C(t,j)) + \frac{dV}{dt}(\phi(t,j)) \\ &\quad \leq L_C(\phi(t,j), u_C(t,j)) + \sup_{f \in F(\phi(t,j), u_C(t,j))} \langle \nabla V(\phi(t,j)), f \rangle \leq 0 \end{aligned} \tag{D.9}$$

In addition, from (6.17), for every $(t,j) \in \operatorname{dom} \phi$ such that $(t, j+1) \in \operatorname{dom} \phi$, we have

$$\begin{aligned} &L_D(\phi(t,j), u_D(t,j)) + V(\phi(t,j+1)) - V(\phi(t,j)) \\ &\quad \leq L_D(\phi(t,j), u_D(t,j)) + \sup_{g \in G(\phi(t,j), u_D(t,j))} V(g) - V(\phi(t,j)) \leq 0 \end{aligned} \tag{D.10}$$

Then, thanks to (D.9) and (D.10), by applying a version of Proposition 3.2.2 where (ϕ, u) is a solution to $\mathcal{H}_s$ as in (2.1), we have that $\widetilde{\mathcal{J}}(\phi, u) \leq V(\xi)$, with $\widetilde{\mathcal{J}}$ defined as in (6.2). □

D.4 Proof of Proposition 6.3.2

Following [47, 78], given $\xi \in \Pi(\overline{C}) \cup \Pi(D)$ and a solution (ϕ^*, u^*) to (6.19), (that thanks to Lemma 6.3.4 is also a solution to $\mathcal{H}_s$ as in (2.1)), given that V satisfies (6.20), for each $j \in \mathbb{N}$ such that $I^j_{\phi^*} = [t_j, t_{j+1}]$ has a nonempty interior $\mathrm{int} I^j_{\phi^*}$, we have:

1. for all $t \in \mathrm{int} I^j_{\phi^*}$,

$$0 = L_C(\phi^*(t,j), u_C(t,j)) + \max_{f \in F(\phi^*(t,j), u_C(t,j))} \langle \nabla V(\phi^*(t,j)), f \rangle \tag{D.11}$$

   and $(\phi^*(t,j), u(t,j)) \in C$.

Given that (ϕ^*, u^*) is a solution to $\mathcal{H}_{\max}$, and V is continuously differentiable on a neighborhood of $\Pi(C)$, we can express its total derivative as

$$\frac{dV}{dt}(\phi^*(t,j)) = \max_{f \in F(\phi^*(t,j), u_C(t,j))} \langle \nabla V(\phi^*(t,j)), f \rangle$$

for every $(t,j) \in \operatorname{int}(I^j_{\phi^*}) \times \{j\}$ with $\operatorname{int}(I^j_{\phi^*})$ nonempty. Given that V satisfies (6.21), we have:

1. for every $(t,j) \in \operatorname{dom} \phi^*$ such that $(t, j+1) \in \operatorname{dom} \phi^*$,

$$\begin{aligned} V(\phi^*(t,j)) &= L_D(\phi^*(t,j), u_D(t,j)) + \max_{g \in G(\phi^*(t,j), u_D(t,j))} V(g) \\ &= L_D(\phi^*(t,j), u_D(t,j)) + V(\phi^*(t, j+1)) \end{aligned} \tag{D.12}$$

   where $(\phi^*(t,j), u(t,j)) \in D$.

Now, thanks to (D.11) and (D.12), by applying a version of Proposition 3.2.2 where (ϕ, u) is a solution to $\mathcal{H}_s$ as in (2.1) and Corollary 3.2.3, we have that

$$V(\xi) = \widetilde{\mathcal{J}}(\phi^*, u). \tag{D.13}$$

Using (6.20), for all x such that $(x, u_C) \in C$, it holds

$$0 \geq L_C(x, u_C) + \langle \nabla V(x), f \rangle \quad \forall f \in F(x, u_C) \tag{D.14}$$

and using (6.21), for all x such that $(x, u_D) \in D$, we have

$$V(x) \geq L_D(x, u_D) + V(g) \quad \forall g \in G(x, u_D). \tag{D.15}$$

Thus, for any arbitrary $(\phi, u) \in \mathcal{S}_{\mathcal{H}_s}(\xi)$, we have from Proposition 6.3.1 and (D.13) that

$$\widetilde{\mathcal{J}}(\phi, u) \leq \widetilde{\mathcal{J}}(\phi^*, u) = V(\xi) \tag{D.16}$$

which also implies that for the control action u, the largest cost of solutions from ξ satisfies

$$\mathcal{J}(\xi, u) = V(\xi). \tag{D.17}$$

□

D.5 Proof of Lemma 6.3.4

Pick any solution (ϕ, u) to $\mathcal{H}_{\max}$ from $\xi \in \Pi(C \cup D)$. Then, $\phi \in \mathcal{X}$, $u = (u_C, u_D) \in \mathcal{U}$, $\text{dom}\phi = \text{dom}u$, and

- $(\phi(0,0), u_C(0,0)) \in \overline{C}$ or $(\phi(0,0), u_D(0,0)) \in D$,
- For each $j \in \mathbb{N}$ such that I_ϕ^j has a nonempty interior $\text{int}I_\phi^j$, we have, for all $t \in \text{int}I_\phi^j$,

  $$(\phi(t,j), u_C(t,j)) \in C$$

  and, for almost all $t \in I_\phi^j$,

  $$\frac{d}{dt}\phi(t,j) = \underset{f \in F(\phi(t,j), u_C(t,j))}{\text{argmax}} \langle \nabla V(x), f \rangle \in F(\phi(t,j), u_C(t,j))$$

- For all $(t,j) \in \text{dom}\,\phi$ such that $(t, j+1) \in \text{dom}\,\phi$,

  $$(\phi(t,j), u_D(t,j)) \in D$$
  $$\phi(t, j+1) = \underset{g \in G(\phi(t,j), u_D(t,j))}{\text{argmax}} V(g) \in G(\phi(t,j), u_D(t,j))$$

which, according to Definition 2.2.4, defines a solution pair to (2.1). □

D.6 Proof of Lemma 6.3.5

Given that V is continuously differentiable on a neighborhood of $\Pi(C)$, the gradient $\nabla V(x)$ exists for any $x \in \Pi(C)$. Since the map $F(x, u_C)$ is compact for each $(x, u_C) \in C$ and the function $f \mapsto \langle \nabla V(x), f \rangle$ is continuous for any selection f of $F(x, u_C)$, then it attains its maximum in $F(x, u_C)$.

Given that V is continuous in $\Pi(\overline{C}) \cup \Pi(D) \cup G(D)$, and the map $G(x, u_D)$ is compact for each $(x, u_D) \in D$, then $g \mapsto V(g)$ attains its maximum in $G(x, u_D)$. □

D.7 Proof of Theorem 6.3.7

To show the claim we proceed as follows:

1. Pick any initial condition ξ and evaluate the cost associated to the solutions yield by $\kappa = (\kappa_C, \kappa_D)$ from ξ. Find an upper bound for this cost.

2. Lower bound the cost associated to the worst-case solutions from ξ when P_2 plays $\kappa_2 := (\kappa_{C2}, \kappa_{D2})$ by the value of the function V evaluated at ξ.

3. Upper bound the cost associated to the solutions from ξ when P_1 plays $\kappa_1 := (\kappa_{C1}, \kappa_{D1})$ by the value of the function V evaluated at ξ.

4. By showing that the cost of the solutions from ξ when P_1 plays κ_1 are not greater than the worst-case cost of the solutions yielded by $\kappa = (\kappa_1, \kappa_2)$ from ξ, and by showing that the worst-case cost of the solutions from ξ when P_2 plays κ_2 is not smaller than the worst-case cost of the solutions yielded by κ from ξ, we establish (6.35).

Proceeding as in item a above, pick any $\xi \in \Pi(\overline{C}) \cup \Pi(D)$ and any solution (ϕ^*, u^*) to (6.19) from ξ with $\operatorname{dom} \phi^* \ni (t,j) \mapsto u^*(t,j) = \kappa(\phi^*(t,j))$. Thanks to Proposition 6.3.2, we have that

$$V(\xi) = \widetilde{\mathcal{J}}(\phi^*, u^*). \tag{D.18}$$

and for any arbitrary $(\phi, u^*) \in \mathcal{S}_{\mathcal{H}_s}(\xi)$, we have that

$$\widetilde{\mathcal{J}}(\phi, u^*) \leq \widetilde{\mathcal{J}}(\phi^*, u^*) = V(\xi) \tag{D.19}$$

which also implies that for the control action u^*, defined by the feedback law κ, the largest cost of solutions from ξ satisfies

$$\mathcal{J}(\xi, u^*) = V(\xi). \tag{D.20}$$

Proceeding with item b as above, pick any solution (ϕ_s, u^s) to $\mathcal{H}_{\max}$ as in (6.19) from the initial condition ξ, with $u^s = (u^s_1, u^s_2)$, $\operatorname{dom} \phi_s \ni (t,j) \mapsto u^s_1(t,j) = \bar{\kappa}_1(\phi_s(t,j))$ for some $\bar{\kappa}_1 := (\bar{\kappa}_{C1}, \bar{\kappa}_{D1}) \in \mathcal{K}_1$, and $\operatorname{dom} \phi_s \ni (t,j) \mapsto u^s_2(t,j) = \kappa_2(\phi_s(t,j))$ for $\kappa_2 := (\kappa_{C2}, \kappa_{D2})$. Thanks to Proposition 6.3.2 and Corollary 6.3.3, we have that

$$V(\xi) \leq \widetilde{\mathcal{J}}(\phi_s, u^s), \tag{D.21}$$

and for any arbitrary $(\phi, u^s) \in \mathcal{S}_{\mathcal{H}_s}(\xi)$, we have that

$$\widetilde{\mathcal{J}}(\phi, u^s) \leq \widetilde{\mathcal{J}}(\phi_s, u^s) \tag{D.22}$$

and

$$V(\xi) \leq \mathcal{J}(\xi, u^s). \tag{D.23}$$

Notice that it is not possible to guarantee that $V(\xi)$ is a lower bound for any other solution rendered by u^s.

Proceeding with item c as above, pick any $(\phi_w, u^w) \in \mathcal{S}^w_{\mathcal{H}_s}(\xi)$ with $u^w := (u^w_C, u^w_D)$ where $\mathcal{S}^w_{\mathcal{H}_s}(\xi) := \{(\phi, (u_1, u_2)) \in \mathcal{S}_{\mathcal{H}_s}(\xi) : \operatorname{dom}\phi \ni (t,j) \mapsto u_1(t,j) = \kappa_1(\phi(t,j))$, and $\operatorname{dom}\phi \ni (t,j) \mapsto u_2(t,j) = \bar{\kappa}_2(\phi(t,j))$, $\bar{\kappa}_2 \in \mathcal{K}_2\}$. Since $\bar{\kappa}_2$ does not necessarily attain the upper bound in (6.30), then, for each $j \in \mathbb{N}$ such that $I^j_{\phi_w} = [t_j, t_{j+1}]$ has a nonempty interior $\operatorname{int} I^j_{\phi_w}$, we have that for every $t \in \operatorname{int} I^j_{\phi_w}$,

$$0 \geq L_C(\phi_w(t,j), u^w_C(t,j)) + \langle \nabla V(\phi_w(t,j)), f \rangle \qquad \forall f \in F(\phi_w(t,j), u^w_C(t,j))$$

which implies

$$0 \geq L_C(\phi_w(t,j), u^w_C(t,j)) + \frac{dV}{dt}(\phi_w(t,j)). \tag{D.24}$$

and by integrating over the interval $[t_j, t_{j+1}]$, we obtain

$$0 \geq \int_{t_j}^{t_{j+1}} \left(L_C(\phi_w(t,j), u^w_C(t,j)) + \frac{dV}{dt}(\phi_w(t,j)) \right) dt$$

from which we have

$$V(\phi_w(t_j,j)) \geq \int_{t_j}^{t_{j+1}} L_C(\phi_w(t,j), u^w_C(t,j)) dt + V(\phi_w(t_{j+1},j))$$

Summing both sides from $j = 0$ to $j = J_{\phi_w}$, we obtain

$$\sum_{j=0}^{J_{\phi_w}} V(\phi_w(t_j,j)) \geq \sum_{j=0}^{J_{\phi_w}} \int_{t_j}^{t_{j+1}} L_C(\phi_w(t,j), u^w_C(t,j)) dt + \sum_{j=0}^{J_{\phi_w}} V(\phi_w(t_{j+1},j))$$

Then, solving for V at the initial condition $\phi_w(0,0)$, we obtain

$$\begin{aligned} V(\phi_w(0,0)) \geq & \sum_{j=0}^{J_{\phi_w}} \int_{t_j}^{t_{j+1}} L_C(\phi_w(t,j), u^w_C(t,j)) dt \\ & + V(\phi_w(t_1,0)) + \sum_{j=1}^{J_{\phi_w}} \left(V(\phi_w(t_{j+1},j)) - V(\phi_w(t_j,j)) \right) \end{aligned} \tag{D.25}$$

In addition, since $\bar{\kappa}_2$ does not necessarily attain the upper bound in (6.33), then, for every $(t,j) \in \operatorname{dom}\phi_w$ such that $(t,j+1) \in \operatorname{dom}\phi_w$, we have

$$V(\phi_w(t,j)) \geq L_D(\phi_w(t,j), u^w_D(t,j)) + V(g) \qquad \forall g \in G(\phi_w(t,j), u^w_D(t,j))$$

and

$$V(\phi_w(t,j)) \geq L_D(\phi_w(t,j), u_D^w(t,j)) + V(\phi_w(t,j+1)) \tag{D.26}$$

Summing both sides from $j = 0$ to $j = J_{\phi_w} - 1$, we obtain

$$\sum_{j=0}^{J_{\phi_w}-1} V(\phi_w(t,j)) \geq \sum_{j=0}^{J_{\phi_w}-1} L_D(\phi_w(t,j), u_D^w(t,j)) + \sum_{j=0}^{J_{\phi_w}-1} V(\phi_w(t,j+1))$$

Then, solving for V at the first jump time, we obtain

$$\begin{aligned} V(\phi_w(t_1,0)) \geq V(\phi_w(t_1,1)) \ &+ \sum_{j=0}^{J_{\phi_w}-1} L_D(\phi_w(t,j), u_D^w(t,j)) \\ &+ \sum_{j=1}^{J_{\phi_w}-1} (V(\phi_w(t,j+1)) - V(\phi_w(t,j))) \end{aligned} \tag{D.27}$$

In addition, given that $\phi_w(0,0) = \xi$, lower bounding $V(\phi_w(t_1,0))$ in (D.25) by the right-hand side of (D.27), we obtain

$$\begin{aligned} V(\xi) \geq& \sum_{j=0}^{J_{\phi_w}} \int_{t_j}^{t} L_C(\phi_w(t,j), u_C^w(t,j))dt + V(\phi_w(t_1,0)) + \sum_{j=1}^{J_{\phi_w}} (V(\phi_w(t,j)) - V(\phi_w(t_j,j))) \\ \geq& \sum_{j=0}^{J_{\phi_w}} \int_{t_j}^{t} L_C(\phi_w(t,j), u_C^w(t,j))dt + \sum_{j=0}^{J_{\phi_w}-1} L_D(\phi_w(t,j), u_D^w(t,j)) \\ &+ \sum_{j=1}^{J_{\phi_w}-1} (V(\phi_w(t,j+1)) - V(\phi_w(t,j))) + \sum_{j=1}^{J_{\phi_w}} (V(\phi_w(t,j)) - V(\phi_w(t_j,j))) \\ &+ V(\phi_w(t_1,1)) \end{aligned}$$

Since

$$\begin{aligned} &V(\phi_w(t_1,1)) + \sum_{j=1}^{J_{\phi_w}-1} (V(\phi_w(t,j+1)) - V(\phi_w(t,j))) + \sum_{j=1}^{J_{\phi_w}} (V(\phi_w(t,j)) - V(\phi_w(t_j,j))) \\ &= V(\phi_w(t_{J_{\phi_w}+1}, J_{\phi_w})) + V(\phi_w(t_1,1)) + \sum_{j=1}^{J_{\phi_w}-1} (V(\phi_w(t,j+1))) - \sum_{j=1}^{J_{\phi_w}} (V(\phi_w(t_j,j))) \\ &= V(\phi_w(t_{J_{\phi_w}+1}, J_{\phi_w})) \end{aligned}$$

then we have

$$\begin{aligned} V(\xi) \geq& \sum_{j=0}^{J_{\phi_w}} \int_{t_j}^{t} L_C(\phi_w(t,j), u_C^w(t,j))dt + \sum_{j=0}^{J_{\phi_w}-1} L_D(\phi_w(t,j), u_D^w(t,j)) \\ &+ V(\phi_w(t_{J_{\phi_w}+1}, J_{\phi_w})) \end{aligned}$$

By taking the limit when $(t_{J_{\phi_w}+1}, J_{\phi_w}) \to \sup \operatorname{dom} \phi_w$, and given that (6.34) holds, we have

$$\begin{aligned} V(\xi) \geq & \sum_{j=0}^{\sup_j \operatorname{dom} \phi_w} \int_{t_j}^{t_{j+1}} L_C(\phi_w(t,j), u_C^w(t,j))dt + \sum_{j=0}^{\sup_j \operatorname{dom} \phi_w - 1} L_D(\phi_w(t_{j+1},j), u_D^w(t_{j+1},j)) \\ & + \limsup_{\substack{t+j\to\infty \\ (t,j)\in \operatorname{dom}\phi_w}} V(\phi_w(t,j)) \\ = & J(\phi_w, u^w) \end{aligned} \tag{D.28}$$

Finally, by proceeding as in item d above, by applying the infimum on each side of (D.23) over the set $\mathcal{S}_{\mathcal{H}_{\max}}(\xi)$, we obtain

$$V(\xi) \leq \inf_{u_1:(\phi_s,(u_1,\kappa_2(\phi_s)))\in\mathcal{S}_{\mathcal{H}_{\max}}(\xi)} \mathcal{J}(\xi,(u_1,\kappa_2(\phi_s)) := \overline{V}(\xi).$$

By applying the supremum on each side of (D.28) over the set $\mathcal{S}^w_{\mathcal{H}_s}(\xi)$, we obtain

$$V(\xi) \geq \sup_{(\phi_w,u_2):(\phi_w,(\kappa_1(\phi_w),u_2))\in\mathcal{S}^w_{\mathcal{H}_s}(\xi)} \widetilde{\mathcal{J}}(\phi_w,(\kappa_1(\phi_w),u_2)) =: \underline{V}(\xi).$$

Given that $V(\xi) = \mathcal{J}(\xi, u^*)$ from (D.20), we have that for any $\xi \in \Pi(\overline{C}) \cup \Pi(D)$, each $(\phi^*, u^*) \in \mathcal{S}_{\mathcal{H}_{\max}}(\xi)$ with $u^* = (\kappa_1(\phi^*), \kappa_2(\phi^*))$ satisfies

$$\underline{V}(\xi) \leq \mathcal{J}(\xi, u^*) \leq \overline{V}(\xi) \tag{D.29}$$

Since $(\phi^*, u^*) \in \mathcal{S}^w_{\mathcal{H}_s}(\xi) \cap \mathcal{S}_{\mathcal{H}_{\max}}(\xi)$, we have

$$\operatorname*{arginf}_{u_1:(\phi_s,(u_1,\kappa_2(\phi_s)))\in\mathcal{S}_{\mathcal{H}_{\max}}(\xi)} \mathcal{J}(\xi,(u_1,\kappa_2(\phi_s))) = \kappa_1(\phi^*)$$

and

$$\operatorname*{argsup}_{(\phi_w,u_2):(\phi_w,(\kappa_1(\phi_w),u_2))\in\mathcal{S}^w_{\mathcal{H}_s}(\xi)} \widetilde{\mathcal{J}}(\phi_w,(\kappa_1(\phi_w),u_w)) = (\phi^*, \kappa_2(\phi^*))$$

Thus, this implies that $\underline{V}(\xi) = \mathcal{J}(\xi,(\kappa_1(\phi^*),\kappa_2(\phi^*))) = \overline{V}(\xi)$, which together with (D.29) leads to

$$\mathcal{J}(\xi, u^*) = \min_{u_1} \max_{\substack{u_2 \\ (u_1,u_2)\in\mathcal{U}_{\mathcal{H}_s}(\xi)}} \mathcal{J}(\xi,(u_1,u_2)) \tag{D.30}$$

Thus, from (D.20) and (D.30), $V(\xi)$ is the value function for $\mathcal{H}_s$ and the worst-case solution that the strategy $\kappa = (\kappa_1, \kappa_2)$ renders attains it. □

D.8 Proof of Theorem 6.4.2

Since, by assumption, we have that $C_\kappa = \Pi(C), D_\kappa = \Pi(D)$, and $V, \kappa := (\kappa_C, \kappa_D) = ((\kappa_{C1}, \kappa_{C2}), (\kappa_{D1}, \kappa_{D2}))$ are such that (6.28)-(6.33) hold, and for each $\xi \in \overline{C_\kappa} \cup D_\kappa$, each $\phi \in \mathcal{S}_{\mathcal{H}_\kappa}(\xi)$ satisfies (6.34), we have from Theorem 6.3.7 that V is the value function as in (6.4) for $\mathcal{H}_\kappa$ at $\overline{C_\kappa} \cup D_\kappa$ and the feedback law κ is the saddle-point equilibrium for this game. Then, V is a Lyapunov candidate for $\mathcal{H}_\kappa$ [69, Definition 3.16] since $\overline{C_\kappa} \cup D_\kappa \cup G(D_\kappa) \subset \operatorname{dom} V = \mathbb{R}^n$ and V is continuously differentiable on an open set containing $\overline{C_\kappa}$. From (6.28) and (6.31), we have

$$\langle \nabla V(x), f \rangle \leq -L_C(x, \kappa_C(x)) \quad \forall x \in C_\kappa, f \in F(x, \kappa_C(x)) \tag{D.31}$$

$$V(g) - V(x) \leq -L_D(x, \kappa_D(x)) \quad \forall x \in D_\kappa, g \in G(x, \kappa_D(x)). \tag{D.32}$$

Moreover, if

1. Item 1 above holds, define

$$\rho(x, \kappa(x)) := \min\{L_C(x, \kappa_C(x)), L_D(x, \kappa_D(x))\}$$

2. Item 2 above holds, define

$$\rho(x, \kappa(x)) := \min\{\eta(|x|_{\mathcal{A}}), L_D(x, \kappa_D(x))\}$$

3. Item 3 above holds, define

$$\rho(x, \kappa(x)) := \min\{L_C(x, \kappa_C(x)), \eta(|x|_{\mathcal{A}})\}$$

Thus, given the functions α_1, α_2 satisfying (6.36), and given that from (D.31) and (D.32), for each case above the continuous function ρ satisfies

$$\langle \nabla V(x), f \rangle \leq -\rho(x, \kappa(x)) \quad \forall x \in C_\kappa, f \in F(x, \kappa_C(x)) \tag{D.33}$$

$$V(g) - V(x) \leq -\rho(x, \kappa(x)) \quad \forall x \in D_\kappa, g \in G(x, \kappa_D(x)) \tag{D.34}$$

thanks to [69, Theorem 3.18], the set $\mathcal{A}$ is uniformly globally pre-asymptotically stable for $\mathcal{H}_\kappa$. Furthermore, when maximal solutions to $\mathcal{H}_\kappa$ are complete, we can argue uniform global asymptotic stability of $\mathcal{A}$ as in [21]. □

D.9 Proof of Corollary 6.5.1

We show that when conditions (6.39)-(6.41) hold, by using the result in Theorem 6.3.7, the value function is equal to the function V and with the feedback law $\kappa := (\kappa_C, \kappa_D)$ with values as in (6.42) and (6.43), such a cost is attained. A sufficient condition for (6.28)-(6.30) in Theorem 6.3.7 for the single-valued flow map F as in (6.38) is

$$\begin{aligned} 0 &= \min_{u_{C1}} \max_{u_{C2}}_{u_C=(u_{C1},u_{C2})\in\Pi_u(x,C)} \mathcal{L}_C(x,u_C), \\ \mathcal{L}_C(x,u_C) &= x_p^\top Q_C x_p + u_{C1}^\top R_{C1} u_{C1} + u_{C2}^\top R_{C2} u_{C2} \\ &\quad + 2x_p^\top P(\tau)(A_C x_p + B_C u_C) + x_p^\top \frac{\mathrm{d}P(\tau)}{\mathrm{d}\tau} x_p \end{aligned} \tag{D.35}$$

First, given that (6.39) holds, and $x_p^\top(P(\tau)A_C + A_C^\top P(\tau))x_p = 2x_p^\top P(\tau)A_C x_p$ for every $x \in \Pi(C)$, one has

$$\begin{aligned} \mathcal{L}_C(x,u_C) &= x_p^\top P(\tau)(B_{C2}R_{C2}^{-1}B_{C2}^\top + B_{C1}R_{C1}^{-1}B_{C1}^\top)P(\tau)x_p \\ &\quad + u_{C1}^\top R_{C1} u_{C1} + u_{C2}^\top R_{C2} u_{C2} + 2x_p^\top P(\tau) B_C u_C \end{aligned}$$

The first-order necessary conditions for optimality

$$\frac{\partial}{\partial u_{C1}} \mathcal{L}_C(x,u_C)\Big|_{u_C^*} = 0, \quad \frac{\partial}{\partial u_{C2}} \mathcal{L}_C(x,u_C)\Big|_{u_C^*} = 0$$

for all $(x,u_C) \in C$ are satisfied by the point $u_C^* = (u_{C1}^*, u_{C2}^*)$, with values

$$u_{C1}^* = -R_{C1}^{-1}B_{C1}^\top P(\tau)x_p, \quad u_{C2}^* = -R_{C2}^{-1}B_{C2}^\top P(\tau)x_p \tag{D.36}$$

for each $x = (x_p, \tau) \in \Pi(C)$. Given that $R_{C1}, -R_{C2} \in \mathbb{S}_+^{m_D}$, the second-order sufficient conditions for optimality

$$\frac{\partial^2}{\partial u_{C1}^2} \mathcal{L}_C(x,u_C)\Big|_{u_C^*} \succeq 0, \quad \frac{\partial^2}{\partial u_{C2}^2} \mathcal{L}_C(x,u_C)\Big|_{u_C^*} \preceq 0,$$

for all $(x,u_C) \in C$ hold, rendering u_C^* as in (D.36) as an optimizer of the min-max problem in (D.35). In addition, it satisfies $\mathcal{L}_C(x,u_C^*) = 0$, making $V(x) = x_p^\top P(\tau)x_p$ a solution to (6.28)-(6.30) in Theorem 6.3.7.

On the other hand, given that P is nonincreasing, then

$$\sup_{t\in[0,\bar{T}]} (A_D x_p + B_D u_D)^\top P(t)(A_D x_p + B_D u_D) = (A_D x_p + B_D u_D)^\top P(0)(A_D x_p + B_D u_D)$$

and we can write (6.31)-(6.33) in Theorem 6.3.7 as

$$x_p^\top P(\bar{T})x_p = \mathcal{L}_D(x, \kappa_D(x)) \qquad \forall x : (x, \kappa_D(x)) \in D, \tag{D.37}$$

$$x_p^\top P(\bar{T})x_p \leq \mathcal{L}_D(x, (u_{D1}, \kappa_{D2}(x))) \quad \forall (x, u_{D1}) : (x, (u_{D1}, \kappa_{D2}(x)))) \in D, \tag{D.38}$$

$$x_p^\top P(\bar{T})x_p \geq \mathcal{L}_D(x, (\kappa_{D1}(x), u_{D2})) \quad \forall (x, u_{D2}) : (x, (\kappa_{D1}(x), u_{D2})) \in D, \tag{D.39}$$

$$\begin{aligned}\mathcal{L}_D(x, u_D) = x_p^\top Q_D x_p + u_{D1}^\top R_{D1} u_{D1} + u_{D2}^\top R_{D2} u_{D2} \\ +(A_D x_p + B_D u_D)^\top P(0)(A_D x_p + B_D u_D)\end{aligned} \tag{D.40}$$

Similar to the case along flows, a sufficient condition for (D.37)-(D.39) is

$$x_p^\top P(\bar{T})x_p = \min_{\substack{u_{D1} \\ u_D = (u_{D1}, u_{D2}) \in \Pi_u(x, D)}} \max_{u_{D2}} \mathcal{L}_D(x, u_D), \tag{D.41}$$

and the first-order necessary conditions for optimality are satisfied by the point $u_D^* = (u_{D1}^*, u_{D2}^*)$, such that, for each $x_p \in \Pi(D)$,

$$u_D^* = -\begin{bmatrix} R_{D1} + B_{D1}^\top P(0) B_{D1} & B_{D1}^\top P(0) B_{D2} \\ B_{D2}^\top P(0) B_{D1} & R_{D2} + B_{D2}^\top P(0) B_{D2} \end{bmatrix}^{-1} \begin{bmatrix} B_{D1}^\top P(0) A_D \\ B_{D2}^\top P(0) A_D \end{bmatrix} x_p \tag{D.42}$$

Given that (6.40) holds, the second-order sufficient conditions for optimality are satisfied, rendering u_D^* as in (D.42) as an optimizer of the min-max problem in (D.41). In addition, u_D^* satisfies $\mathcal{L}_D(x, u_D^*) = x_p^\top P(\bar{T})x_p$, with $P(\bar{T})$ as in (6.41), making $V(x) = x_p^\top P(\tau)x_p$ satisfy (6.31)-(6.33) in Theorem 6.3.7.

Then, given that V is continuously differentiable on a neighborhood of $\Pi(C)$ and that $q(x) = V(x)$ for all $x \in \operatorname{dom} V$, satisfying (6.34), by applying Theorem 6.3.7, in particular from (6.35), for every $\xi = (\xi_p, \xi_\tau) \in \Pi(\overline{C} \cup D)$ the value function is $\mathcal{J}^*(\xi) = \mathcal{J}(\xi, ((u_{C1}^*, u_{D1}^*), (u_{C2}^*, u_{D2}^*)) = \xi_p^\top P(\xi_\tau)\xi_p$. When P_1 plays u_1^* defined by $\kappa_1 = (\kappa_{C1}, \kappa_{D1})$ with values as in (6.42) and (6.43), and P_2 plays any disturbance u_2 such that solutions to $\mathcal{H}_s$ with data as in (6.38) are complete, then the cost is upper bounded by $\mathcal{J}(\xi, u^*)$, satisfying (3.2). ☐

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