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Authors
Hirsch, Morris W
Smith, Hal

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ASYMPTOTICALLY STABLE EQUILIBRIA FOR MONOTONE SEMIFLOWS

M.W. Hirsch

Department of Mathematics
University of California
Berkeley, CA 94720-3840, USA

H.L. Smith

Department of Mathematics and Statistics
Arizona State University
Tempe, AZ 85287-1804, USA

Abstract. Conditions for the existence of a stable equilibrium and for the existence of an asymptotically stable equilibrium for a strongly order preserving semiflow are presented. Analyticity of the semiflow and the compactness of certain subsets of the set of equilibria are required for the latter and yield finiteness of the equilibrium set. Our results are applied to semilinear parabolic partial differential equations and to the classical Kolmogorov competition system with diffusion.

1. Introduction and Preview of Results.

It is well known that the generic orbit of a strongly order preserving semiflow, all of whose orbits are precompact, converges to the set E of equilibria and that, under mild additional hypotheses, it converges to a single equilibrium. See [11, 29, 12, 27]. Furthermore, an omega limit set not consisting of equilibria, is necessarily unstable in a strong sense. It is therefore reasonable to expect that there exist equilibria with some stability properties. The purpose of the present paper is to establish conditions for the existence of a stable equilibrium for a strongly order preserving semiflow and for the existence of an asymptotically stable equilibrium under certain additional assumptions. These additional assumptions require that E, or some closed subset of it, be compact and that a maximal totally ordered subset of E be finite. Such a seemingly strong assumption may appear to be difficult to verify. However, if the semiflow is dissipative then E will be compact and, with the additional assumption that the semiflow is analytic, a result of Jiang and Yu [15] allows one to conclude that an equilibrium is either isolated as a member of E or belongs to an analytic monotone arc of equilibria. This arc may be continued to a global analytic monotone arc but the assumed compactness of E may conflict with the continuability of the arc, producing a contradiction, enabling us to conclude that our finiteness assumption holds and therefore, the existence of an asymptotically stable equilibrium. The

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analyticity hypothesis was previously used by Smith and Thieme [29] to rule out the existence of totally ordered connected sets of equilibria and thereby to conclude that the set of asymptotically stable points is dense for strongly order preserving semiflows on Banach lattices.

Our results apply to the wide variety of differential equations that generate monotone semiflows (see e.g. [12, 27]) but they are most useful when the usual linearization technique becomes unfeasible — that is, for infinite dimensional problems such as delay-differential equations and systems of quasimonotone reaction diffusion systems. Many such systems arising in applications are analytic so this assumption is not so restrictive. See Cosner and Cantrell [1] for examples of reaction diffusion systems in ecology. For a single reaction diffusion equation with nonlinearity not depending on the gradient of the unknown function, a great deal is known about the existence of asymptotically stable equilibria and the behavior of solutions by monotonicity techniques as well as by variational methods. See for example Lions [16], Martin [17], Matano [18, 19], Simon [25], Poláčik [24], Du [5], Zhao [31]. Even more is known in the special case of one spatial variable; see Hale [8] for a recent review.

In the remainder of this section we preview some of our results, deferring proofs to a later section. We begin with some examples which are merely intended to illustrate the kinds of results possible with no intent at generality.

Consider the reaction-diffusion equation for describing the density of a population occupying a region $\Omega \subset \mathbb{R}^m$ with smooth lethal boundary (other boundary conditions and more general elliptic operators can be treated):

$$\begin{aligned} u_t &= d\nabla^2 u + uf(x,u),\ x \in \Omega \\ u &= 0,\ x \in \partial\Omega \end{aligned} \tag{1}$$

Obviously, $u = 0$ is an equilibrium. Denote by λ the principal eigenvalue of

$$\begin{aligned} \lambda u &= d\nabla^2 u + uf(x,0),\ x \in \Omega \\ u &= 0,\ x \in \partial\Omega \end{aligned} \tag{2}$$

Let $X = L^p(\Omega)$. The Laplace operator with Dirichlet boundary conditions induces a closed linear operator A on X which generates an analytic, strongly positive, compact semigroup e^{At}. Equation (1) then defines a local semiflow on the fractional power space $Y := X^\alpha$ where we choose $\alpha \in (1/2, 1)$ and $p > m$ such that $2\alpha - m/p > 1$ [9]. X^α continuously imbeds in $C^1(\overline{\Omega})$. Y is strongly ordered relative to the cone $Y_+ = \{u \in X^\alpha : u \geq 0\}$ in the sense that Y_+ has non-empty interior in Y. We write $u \leq v$ when $v - u \in Y_+$, $u < v$ when $v - u \in Y_+ \setminus \{0\}$, and $u \ll v$ when $v - u \in \mathrm{Int}Y_+$. Let E denote the set of (non-negative) equilibria of (1).

Theorem 1. *Let f be analytic and assume that there is $u_0 > 0$ such that*

$$f(x,u) < 0,\ u > u_0, x \in \Omega.$$

Then E is finite and one of its points is asymptotically stable relative to Y_+. In fact, this same point is asymptotically stable point relative to the positive cone in $C^1(\overline{\Omega})$.

Remark 1. Under our assumptions, a result of Simon [25] shows that every non-negative solution converges to an equilibrium.

Theorem 1, while certainly not surprising in light of results of Lions [16], nevertheless adds significant stability information to our understanding of single-population dynamics. It and our next result were motivated by the recent book of Cantrell and Cosner [1] in which we note a lack of results on stability of equilibria for simple population models.

Consider the equation

$$\begin{aligned} u_t &= d\nabla^2 u + F(x,u),\ x \in \Omega \\ u &= 0,\ x \in \partial\Omega \end{aligned} \tag{3}$$

Theorem 2. *Let F be analytic and suppose there are distinct real numbers $a \leq 0 \leq b$ such that*

$$F(x,u) > 0,\ u < a, \quad F(x,u) < 0,\ u > b,\ x \in \overline{\Omega}.$$

Then the set E of equilibria in $I := \{u \in Y : a \leq u(x) \leq b,\ x \in \Omega\}$ is finite and there exists an asymptotically stable equilibrium relative to I. It is also asymptotically stable relative to the $C^1(\overline{\Omega})$ norm.

The following example of the nonexistence of an asymptotically stable equilibrium for an analytic dissipative system is due to C. Cosner [2]. Suppose the largest eigenvalue of

$$\begin{aligned} \lambda u &= \nabla^2 u + au,\ x \in \Omega \\ u &= 0,\ x \in \partial\Omega \end{aligned} \tag{4}$$

is positive for some $a > 0$. Then, the logistic equation

$$\begin{aligned} u_t &= \nabla^2 u + u(a-u),\ x \in \Omega \\ u &= 0,\ x \in \partial\Omega \end{aligned} \tag{5}$$

has a unique positive equilibrium w (see Cantrell & Cosner [1]) which is asymptotically stable relative to Y_+ by Theorem 1. Now consider the degenerate Lotka-Volterra competition system:

$$\begin{aligned} u_t &= \nabla^2 u + u(a-u-v),\ x \in \Omega \\ v_t &= \nabla^2 v + v(a-u-v),\ x \in \Omega \\ u &= v = 0,\ x \in \partial\Omega \end{aligned} \tag{6}$$

It has a continuum of solutions $u = sw,\ v = (1-s)w,\ 0 \leq s \leq 1$. (These correspond to the line of equilibria $(as, a(1-s))$ for the ODE system.) Note that this system is analytic, dissipative and strongly monotone (in the usual competitive sense). It clearly doesn't have an asymptotically stable equilibrium; instead, it has a continuum of stable equilibria. Note that $w = u+v$ satisfies (5) hence the manifold of equilibria attracts all non-trivial initial data. In Theorem 3 below we show that an attracting continua of equilibria is the only alternative to finiteness of E and the existence of an asymptotically stable equilibrium for an analytic competitive system.

We consider the general Kolmogorov competition system

$$\begin{aligned} u_t &= d_1\nabla^2 u + uf_1(x,u,v),\ x \in \Omega \\ v_t &= d_2\nabla^2 v + vf_2(x,u,v),\ x \in \Omega \\ u &= v = 0,\ x \in \partial\Omega \end{aligned} \tag{7}$$

where f_i is analytic, $d_i > 0$, and

$$\frac{\partial f_i}{\partial v}(x,u,v), \frac{\partial f_i}{\partial u}(x,u,v) < 0, \quad i = 1,2. \tag{8}$$

We assume that the trivial equilibrium $E_0 = (0,0)$ is repelling and that there exists unique single population equilibria $E_1 \equiv (\bar{u}(x), 0)$ and $E_2 \equiv (0, \bar{v}(x))$, each of which attracts all non-trivial solutions starting on their respective coordinate axes. See [27, 1] for such conditions.

Consider the linearization of (7) about E_1. The eigenvalue problem associated with the variational system at E_1 is given by

$$\begin{aligned} \lambda u &= d_1 \nabla^2 u + u[f_1(x,\bar{u}(x),0) + \bar{u}(x)\frac{\partial f_1}{\partial u}(x,\bar{u}(x),0)] + \bar{u}(x)\frac{\partial f_1}{\partial v}(x,\bar{u}(x),0)v, \\ \lambda v &= d_2 \nabla^2 v + v f_2(x,\bar{u}(x),0), \ x \in \Omega \\ u &= v = 0, \ x \in \partial\Omega \end{aligned} \tag{9}$$

The second equation decouples from the first and we denote by σ_{12} the principal eigenvalue of

$$\begin{aligned} \lambda v &= d_2 \nabla^2 v + v f_2(x,\bar{u}(x),0), \ x \in \Omega \\ v &= 0, \ x \in \partial\Omega \end{aligned} \tag{10}$$

The following is well known consequence of (8) (see e.g. Thm 8.3.2 [27]) :

(a) $\sigma_{12} < 0$ implies that all eigenvalues of (9) have negative real part and E_1 is asymptotically stable.

(b) $\sigma_{12} \geq 0$ implies that it is a simple eigenvalue of (9), strictly larger than any other, and the corresponding eigenspace is spanned by an eigenvector $z := (-U, V)$ where $U, V > 0$ in Ω.

(c) $\sigma_{12} > 0$ implies that E_1 is unstable.

(d) E_1 is an isolated equilibrium in cases (a) and (c).

Similar considerations apply to the eigenvalue problem associated with E_2 and σ_{21} is the corresponding principal eigenvalue. E denotes the set of equilibria of (7).

The system (6) generates a strictly monotone semiflow Φ on $Y_+ \times Y_+$ with respect to the order relation $\leq_K$ generated by the cone $K = Y_+ \times (-Y_+)$ where, $Y = X^\alpha$ and Y_+ are as in Theorem 1.

Theorem 3. *Assume the hypotheses as described above. Namely, the f_i are analytic, E_0 is a repeller and there are unique single-population equilibria $E_i, \ i = 1,2$. Then either*

(i) *there exists a strongly $\ll_K$-monotone analytic arc C of equilibria connecting E_1 to E_2 and $E = \{E_0\} \cup C$, or*

(ii) *E is finite and there exists an asymptotically stable equilibrium.*

If (i) holds, then every orbit except E_0 converges to a single equilibrium in C. If either $\sigma_{ij} \neq 0$, then (ii) holds. If $\sigma_{ij} > 0, i \neq j$, then there exists an asymptotically stable equilibrium $(u^*, v^*) \in Y_+ \times Y_+$ with $u^*(x) > 0, v^*(x) > 0, \ x \in \Omega$.

By a strongly monotone analytic arc C of equilibria, we mean there is an analytic mapping $g : [0,1] \to Y_+ \times Y_+$, satisfying $g(0) = E_2,\ g(1) = E_1,\ s < t \Rightarrow g(s) \ll_K g(t)$, and $C = \{g(t) : 0 \le t \le 1\}$ is contained in E.

The result should extend to abstract two-species competition dynamics as treated by Hsu et al. [13] and Smith & Thieme [30].

For related results on the existence, uniqueness and stability of positive solutions of competitive systems with diffusion see [4, 1, 6, 5].

The flavor of our abstract results can be seen from the following result (Corollary 2 in the next section) providing conditions for the existence of a stable (asymptotically stable) equilibrium of a monotone semiflow.

Theorem 4. *Let Y be an ordered Banach space and $X \subset Y$ either a nonempty open set, a closed order interval, or a closed subcone of Y^+. Let Φ be a strongly order preserving semiflow on X with every orbit closure compact and let E denote the set of equilibria. Assume that some maximal totally ordered subset $R \subset E$ is nonempty and compact, and that one of the following two statements holds:*

(a) *Y is normally ordered*

(b) *Y is strongly ordered, X is open in Y, Φ is strongly monotone, and every equilibrium has a neighborhood attracted to a compact set.*

Then there exists a stable equilibrium, and an asymptotically stable equilibrium when R is finite.

See the following section for definitions. We note that the existence of a maximal totally ordered subset R of E follows from Zorn's Lemma and the compactness assumption on R follows if E is compact, which in turn will follow from dissipativity of Φ.

2. Main Results.

2.1. Existence of asymptotically stable equilibria.

We assume, unless explicitly mentioned, that X is an ordered metric space with closed order relation $\le$. We write $x < y$ if $x \le y$ and $x \ne y$ and $x \ll y$ if $u \le v$ for all u in some neighborhood of x and all v in some neighborhood of y. Order intervals are defined by $[u,v] = \{x \in X : u \le x \le v\}$ and $[[u,v]] = \{x \in X : u \ll x \ll v\}$.

The metric space X is *normally ordered* if there exists a *normality constant* $\kappa > 0$ such that $d(x,y) \le \kappa d(u,v)$ whenever $u, v \in X$ and $x, y \in [u,v]$. In a normally ordered space order intervals are bounded and the diameter of $[u,v]$ goes to zero with $d(u,v)$.

Let $\Phi : [0,\infty) \times X \to X$ be a semiflow on X. The orbit of z, denoted by $O(z)$, consists of $\{\Phi_t(z) : t \ge 0\}$ and its omega limit set is denoted by $\omega(z)$. Let $E = \{x \in X : \Phi_t(x) = x,\ t \ge 0\}$ be the set of equilibria. Φ is monotone if $x \le y$ implies $\Phi_t(x) \le \Phi_t(y),\ t \ge 0$. It is strictly (strongly) monotone if $x < y$ implies $\Phi_t(x) < \Phi_t(y)$ ($\Phi_t(x) \ll \Phi_t(y)$) for $t > 0$. Φ is strongly order-preserving (SOP) if Φ is monotone and whenever $x < y$ there exist neighborhoods U, V of x, y respectively, and $t_0 \ge 0$, such that $\Phi_{t_0}(U) \le \Phi_{t_0}(V)$.

We assume, unless explicitly mentioned to the contrary, that Φ is an SOP semiflow on the ordered metric space X with the property that all orbits have compact closure.

The *diameter* of a set Z is $\mathrm{diam}\, Z := \sup_{x,y \in Z} d(x,y)$.

We now introduce some familiar stability notions. A point $x \in X$ is *stable* (relative to $R \subset X$) if for every $\epsilon > 0$ there exists a neighborhood U of x such that

$\operatorname{diam}\Phi_t(U\cap R) < \epsilon$ for all $t \geq 0$. In case $R = X$, the set of stable points is denoted by S. x is *stable from above* (respectively, *from below*) if x is stable relative to the set R of points $\geq x$ (resp., $\leq x$). The set of points stable from above (resp., below is denoted by S_+ (resp., S_-).

The *basin of* x in R is the union of all subsets of R of the form $V \cap R$ where $V \subset X$ is an open neighborhood of x such that

$$\lim_{t\to\infty} \operatorname{diam}\Phi_t(V\cap R) = 0$$

Notice that $\omega(x) = \omega(y)$ for all y in the basin. If the basin of x in R is nonempty, we say x is *asymptotically stable* relative to R. This implies x is stable relative to R. If x is asymptotically stable relative to X we say x is *asymptotically stable*. The set of asymptotically stable points is an open set denoted by A. x is *asymptotically stable from above* (respectively, *below)* if it is asymptotically stable relative to the set of points $\geq x$ (resp., $\leq x$). The basin of x relative to this set is called the *upper* (resp., *lower*) basin of x. The set of such x is denoted by A_+ (resp., A_-).

Continuity of Φ implies that asymptotic stability relative to R implies stability relative to R. In particular, $A \subset S$, $A_+ \subset S_+$ and $A_- \subset S_-$.

These stability notions for x depend only on the topology of X, and not on the metric, provided the orbit of x has compact closure.

We state without proof several results proved in [12]. The next two results record useful stability properties of SOP dynamics in normally ordered spaces. A point x is accessible from below (above) in X if there is a sequence $x_n \to x$ with $x_n < x$ ($x_n > x$).

Proposition 1. *Assume X is normally ordered.*

(a) $x \in S_+$ (respectively, S_-) *provided there exists a sequence* $y_n \to x$ *such that* $y_n > x$ (resp., $y_n < x$) *and* $\lim_{n\to\infty} \sup_{t>0} d(\Phi_t(x), \Phi_t(y_n)) = 0$.
(b) $x \in S$ *provided* $x \in S_+ \cap S_-$ *and x is accessible from above and below*
(c) $x \in A$ *provided* $x \in A_+ \cap A_-$ *and x is accessible from above and below.*
(d) *Suppose* $a < b$ *and* $\omega(a) = \omega(b)$. *Then* $a \in A_+$ *and* $b \in A_-$. *If* $a < x < b$ *then* $x \in A$ *and the basin of x includes* $[a, b] \setminus \{a, b\}$.

In particular, (d) shows that an equilibrium e is in A_+ if $x > e$ and $\Phi_t(x) \to e$ (provided X is normally ordered); and dually for A_-.

Proposition 2. *Assume X is normally ordered, $p \in E$, and $\{K_n\}$ is a sequence of nonempty compact invariant sets such that $K_n < p$ and* $\operatorname{dist}(K_n, p) \to 0$. *Then:*

(a) *p is stable from below*
(b) *If z is such that $\omega(z) = p$, then z is stable from below.*

In particular, if p is the limit of a sequence of equilibria $< p$ then p is stable from below.

Many Banach spaces arising in applications are not normally ordered. Function spaces with norms involving derivatives of the functions (such as X^α) are notable examples. We would like to extend our stability results for normal spaces to the strongly ordered Banach space Y. The *order topology* on Y is the topology generated by open order intervals. An *order norm* on the topological vector space $\hat{Y}$ is defined by fixing $u \gg 0$ and assigning to x the smallest ϵ such that $x \in [-\epsilon u, \epsilon u]$. It is easy to see that $\hat{Y}$ is normally ordered by the order norm, with normality constant 1.

Every order neighborhood of p in $\hat{Y}$ contains $[p-\epsilon u, p+\epsilon u]$ for all sufficiently small numbers $\epsilon > 0$.

The induced topology on any subset $Z \subset Y$ is also referred to as the order topology, and the resulting topological space is denoted by $\hat{Z}$. A neighborhood in $\hat{Z}$ is an *order neighborhood.* Every open subset of $\hat{Z}$ is open in Z, i.e., the identity map of Z is continuous from Z to $\hat{Z}$ since the latter topology is generated by the open (in Y) order intervals $[[u,v]] \cap Z$. Therefore $\hat{Z} = Z$ as topological spaces when Z is compact. As shown below, if Ψ is a monotone local semiflow in Z, it is also a local semiflow in $\hat{Z}$, denoted by $\hat{\Psi}$. Evidently Ψ and $\hat{\Psi}$ have the same orbits and the same invariant sets.

Lemma 1. *Let Ψ be a monotone local semiflow in a subset X of a strongly ordered Banach space Y that extends to a monotone local semiflow in an open subset of Y. Then:*

(a) *$\hat{\Psi}$ is a monotone local semiflow.*
(b) *If Ψ is a strongly monotone, then $\hat{\Psi}$ is SOP.*

Lemma 1 and the following results are proved in [12].

A set K is said to *attract* a set S if for every neighborhood U of K there exists $t_0 \geq 0$ such that $t > t_0 \implies \Psi_t(S) \subset U$. An equilibrium p for semiflow $\Psi : \mathbb{R}^+ \times X \to X$ is *order stable* (respectively, *asymptotically order stable*) if p is stable (respectively, asymptotically stable) for $\hat{\Psi}$.

Proposition 3. *Let Ψ be a monotone local semiflow in a subset X of a strongly ordered Banach space Y that extends to a monotone local semiflow in some open subset of Y. Assume p is an equilibrium having a neighborhood W that is attracted to a compact set $K \subset X$. If p is order stable (respectively, asymptotically order stable), it is stable (respectively, asymptotically stable).*

The following corollary of the above results will help in finding asymptotically stable equilibria.

Corollary 1. *$e \in E$ is asymptotically stable if there exists $a, b \in X$ satisfying $\omega(a) = \omega(b) = \{e\}$ and*

(a) *X is normally ordered and $a < e < b$ or $a = e = \inf X < b$, or*

(b) *If X is an open subset of a strongly ordered Banach space, e has a neighborhood attracted to a compact set, and $a \ll e \ll b$ or $a = e = \inf X \ll b$.*

Proof. Part (a) follows from Proposition 1; part (b) is proved by first using part (a) and Lemma 1 to conclude that e is asymptotically order stable. Proposition 3 implies e is asymptotically stable. □

Obviously, there is a parallel result to part (b) with $b = e = \sup X$ replacing $a = e = \inf X$.

A point x is quasiconvergent if $\omega(x) \subset E$ and the set of all quasiconvergent points is denoted by Q. It is well known that Q is dense (even open and dense) for an SOP semiflow under suitable hypotheses. See e.g. Theorem 1.22 of [12]. Below are some simple consequences.

Proposition 4. *Assume Q is dense. Let $p, q \in E$ be such that $p < q$, p is accessible from above, and q is accessible from below. Then there exists $z \in X$ satisfying one of the following conditions:*

(a) $p < z < q$, *and* $\Phi_t(z) \to p$ *or* $\Phi_t(z) \to q$
(b) $p < z < q$ *and* $z \in E$
(c) $z > p$ *and* $p \in O(z)$, *or* $z < q$ *and* $q \in O(z)$

We can now state one of our main results.

Theorem 5. *Suppose* X *is normally ordered and the following three conditions hold:*

(a) Q *is dense*
(b) *if* $e \in E$ *and* e *is not accessible from above (respectively, below) then* $e = \sup X$ *(resp.,* $e = \inf X$*)*
(c) *there is a maximal totally ordered subset* $R \subset E$ *that is nonempty and compact*

Then R *contains a stable equilibrium, and an an asymptotically stable equilibrium if* R *is finite.*

Proof. It is easy to see that $\sup R$ ($\inf R$) exists and is a maximal (minimal) element of E (see e.g., Lemma 1.1 of [12]). We first prove that every maximal equilibrium q is in A_+. This holds vacuously when $q = \sup X$. Suppose $q \neq \sup X$. If q is in the orbit of some point $> q$ then $q \in A_+$ by Proposition 1. Hence we can assume:

$$t \geq 0, \quad y > q \implies \Phi_t(y) > q$$

By hypothesis we can choose $y > q$. By SOP there is an open neighborhood U of q and $s > 0$ such that $\Phi_s(y) \geq \Phi_s(U)$. By hypothesis we can choose $z \in U$ such that $\Phi_s(y) \neq \Phi_s(z)$ and $z > q$. Set $x_2 = \Phi_s(y), x_1 = \Phi_s z$. Then $x_2 > x_1 > q$ and by SOP and the assumption above there is a neighborhood V_2 of x_2 and $t_0 \geq 0$ such that

$$t > t_0 \implies q < \Phi_t(x_1) \leq \Phi_t(V_2)$$

Choose $v \in V_2 \cap Q$. Then $q < \Phi_t(v)$ for $t \geq t_0$, hence $q \leq \omega(v) = \omega(\Phi_{t_0}(v)) \subset E$. Therefore $\Phi_t(v) \to q$ by maximality of q, so Proposition 1(d) implies $q \in A_+$, as required. The dual argument shows that every minimal equilibria is in A_-.

Assumption (c) and previous arguments establish that $q = \sup R$ and $p = \inf R$ satisfy $p \leq q$ and $q \in A_+$, $p \in A_-$.

Suppose $p = q$; in this case we prove $q \in A$. As q is both maximal and minimal in E, we have $q \in A_+ \cap A_-$. If q is accessible from above and below then $q \in A$ by Proposition 1(c). If q is not accessible from above then by hypothesis $q = \sup X$, in which case the fact that $q \in A_-$ implies $q \in A$. Similarly, $q \in A$ if q is not accessible from below.

Henceforth we assume $p < q$. As R is compact and $R \cap S_- \neq \emptyset$ because $p \in R$, R contains the equilibrium $r := \sup(R \cap S_-)$. Note that $r \in S_-$, because this holds by definition of r if r is isolated in $\{r' \in R : r' \leq r\}$, and otherwise $r \in S_-$ by Proposition 2(a). If $r = q$ a modification of the preceding paragraph proves $q \in S$.

Henceforth we assume $r < q$; therefore r is accessible from above.

If r is not accessible from below then $r = p = \inf X$ so $r \in S$ and we are done; so we may as well assume r is accessible from below as well as from above. If r is the limit of a sequence of equilibria $> r$ then $r \in S_+$ by the dual of Proposition 2, hence $r \in S$ by Proposition 1(b). Therefore we can assume R contains a smallest equilibrium $r_1 > r$. Note that $r_1 \notin S_-$ by maximality of r. We apply Proposition 4 to r, r_1: among its conclusions, the only one possible here is that $z > r$ and $\Phi_t(z) \to r$ (and perhaps $r \in O(z)$). Therefore $r \in S_+$ by Proposition 1(a), whence $r \in S$ by Proposition 1(b). When R is finite, a modification of the preceding arguments proves $\max(R \cap A_-) \subset A$. □

Assumption (b) in Theorem 5 holds for many subsets X of an ordered Banach space Y, including open sets, subcones of Y_+, closed order intervals, and so forth. This result is similar to Theorem 10.2 of Hirsch [11], which establishes equilibria that are merely order stable, but does not require normality.

Assumption (c) holds when E is compact, and also in the following situation: $X \subset Y$ where Y is an L^p space, $1 \leq p < \infty$, and E is a nonempty, closed, and order bounded subset of X; then every order bounded increasing or decreasing sequence converges.

We now extend Theorem 5 to spaces that are not normally ordered.

Theorem 6. *Assume X is an open subset of a strongly ordered Banach space Y such that each equilibrium has a neighborhood attracted to some compact set. Let Φ be a strongly monotone semiflow in X such that hypotheses* (a), (b), (c) *of Theorem 5 hold; let $R \subset E$ be as in part (c). Then R contains a stable equilibrium, and an asymptotically stable equilibrium when R is finite.*

Proof. Apply Theorem 5 to the semiflow $\hat{\Phi}$ on $\hat{X}$ with the metric coming from an order norm on $\hat{Y}$; this makes $\hat{X}$ normally ordered and $\hat{\Phi}$ is SOP by Lemma 1. Consequently, by Theorem 5 R contains a stable equilibrium p for $\hat{\Phi}$. This means p is order stable for Φ, whence Proposition 3 shows that p is stable for Φ. The final assertion follows similarly. □

Putting together Theorem 5 and Theorem 6, we have the following result (Theorem 4 in the introduction).

Corollary 2. *Let Y be an ordered Banach space and $X \subset Y$ either a nonempty open set, a closed order interval, or a closed subcone of Y^+. Assume that some maximal totally ordered subset $R \subset E$ is nonempty and compact, and that one of the following two statements holds:*

(a) *Y is normally ordered*
(b) *Y is strongly ordered, X is open in Y, Φ is strongly monotone, and every equilibrium has a neighborhood attracted to a compact set.*

Then there exists a stable equilibrium, and an asymptotically stable equilibrium when R is finite.

Proof. The assumptions on X imply the hypothesis of Theorem 1.22 of [12], so Q is residual and thus dense in X. They also imply assumption (b) of Theorem 5, whence the conclusion follows by Theorems 5 and 6. □

2.2. **Analyticity and finiteness of E.** The following result of Jiang and Yu [15] will be useful in showing that R is finite. A totally ordered arc in an ordered Banach space Y is the image of a strictly increasing mapping of an open interval of $\mathbb{R}$. For the remainder of this section we suspend our default assumptions mentioned at the beginning of the section.

Proposition 5. *[Jiang & Yu] Let Y be a real ordered Banach space such that Y^+ has nonempty interior and $X \subset Y$ is order open. Suppose that $T : X \to X$ is analytic and order compact. If p is a fixed point of T and $DT(p)$ is strongly positive with $\rho(DT(p)) = 1$, then there exists a neighborhood U of p and a totally ordered arc $C \subset U$ containing p such that either C consists of fixed points of T or p is a unique fixed point of T in U. In the latter case, if p is order-stable, then it is asymptotically order stable.*

Remark 2. The proof of Proposition 5 uses the hypothesis that $DT(p)$ is strongly positive only to conclude (1) the simplicity of the eigenvalue $\rho(DT(p)) = 1$, (2) that all other eigenvalues have strictly smaller modulus, and (3) that the associated eigenvector is positive, i.e., $DT(p)v = v \gg 0$. Therefore, the result holds if we replace this hypothesis with these three consequences. In addition, the proof establishes that the set of fixed points in some neighborhood of p are contained in the arc C and that C is the image of an analytic mapping.

Corollary 3. *Let X be an order-open subset of a strongly ordered Banach space Y, Φ is an order-compact, analytic semiflow on X, and let R be a maximal totally ordered subset of E that is non-empty and compact in X. Suppose that $D\Phi_t(x)$ is strongly positive at each $x \in X$. Then R is finite.*

Proof. Let $a = \sup R$ and $b = \inf R$. If R is not finite, then it has a limit point $p = \lim x_n$, $x_n \in R$, $x_n \neq p$. We may assume that $x_n > p$ for all n. Using

$$\Phi_t(x_n) = \Phi_t(p) + D\Phi_t(p)(x_n - p) + o(\|x_n - p\|)$$

and the compactness of $D\Phi_t(p)$, we find that a subsequence of $\{(x_n - p)/\|x_n - p\|\}$ converges to $v > 0$ such that $D\Phi_t(p)v = v$. It follows from strong positivity that $\rho(D\Phi_t(p)) = 1$ for $t > 0$.

Fix $t > 0$. By Proposition 5 applied to Φ_t, there is a neighborhood U of p and an analytic arc $C \subset U$, containing p and consisting of fixed points of Φ_t. Furthermore, every fixed point of Φ_t in U belongs to C. For each $s \in (0, t)$, $\Phi_s C$ also consists of fixed points of Φ_t and so $\Phi_s C \cap U \subset C$. Therefore, for all $x \in C$ sufficiently near p, $\Phi_s x \in C$ for all $s \in (0, t)$. But C is totally ordered so either $\Phi_s x = x$ or $\Phi_s x < x$ or $\Phi_s x > x$. The latter two alternatives are impossible since a periodic orbit of a strongly monotone semiflow cannot have two related points. We conclude that, by shrinking the arc C and neighborhood U if necessary, we may assume that $C \subset E$ and all fixed points of Φ_t in U belong to C. We claim that $C \subset R$. We show that if $x \in R$ and $y \in C$ then x and y are related, which proves the claim since R is maximal. If $x \in U$, then $x \in C$ and obviously it is related to y. If $x \notin U$ then $x < p$ or $p < x$. If $x < p$, observe that $\{z \in C : x < z\} = \{z \in C : x \ll z\}$ is nonempty and relatively open in C. It is also relatively closed in C since $x \notin C$. Therefore it coincides with C and we are done. A similar argument applies if $p < x$.

Maximality of R implies $p \neq a, p \neq b$. Now R is compact so the endpoints of C, $\inf C$ and $\sup C$, exist in R (see Lemma 1.1 of [12]) and are also limit points of R and consequently Proposition 5 applies to these points as well, extending C to a larger analytic arc contained in R. Let $S \subset R$ be the maximal (with respect to set inclusion) analytic arc in R, which exists by Zorn's Lemma. But R being compact implies the endpoints of S (its infimum and supremum) are limit points of R and thus S can be extended, contradicting maximality of S. This contradiction shows that R is finite. □

It is useful to define $[a, \infty] = \{x \in Y : x \geq a\}$, $[[a, \infty]] = \{x \in Y : x \gg a\}$, similarly for $[-\infty, a]$ and $[[-\infty, a]]$ and $[-\infty, +\infty] = [[-\infty, +\infty]] = Y$.

Corollary 4. *Let Y be a strongly ordered Banach space and let $X = [a, b]$ for some a, b with $a \ll b$, where we allow $a = -\infty$ and/or $b = +\infty$. Let Φ be an order-compact semiflow on X that is analytic on $[[a, b]]$, $a \in Y \Rightarrow a \in E$ and similarly for b. Let R be a maximal totally ordered subset of equilibria that is non-empty and compact in X and $R \setminus \{a, b\} \subset [[a, b]]$. Suppose that $D\Phi_t(x)$ is strongly positive at each $x \in [[a, b]]$.*

If $a, b \in E$ and either a or b is an isolated equilibrium, then R is countable with no accumulation point except possibly a or b, whichever is nonisolated. If both a, b are isolated equilibria, then R is finite.

R is finite if $a \in E$ is an isolated equilibrium and $b = +\infty$ or if $a = -\infty$ and $b \in E$ is an isolated equilibrium or if $a = -\infty$ and $b = +\infty$.

Proof. Assume a is an isolated equilibrium. We can assume that $R' := R \cap [[a, b]]$ is non-empty. We claim that no point of R' is a point of accumulation of R'. If false, there is an accumulation point $p \in R'$. Since $p \in [[a, b]]$ we may argue as in Corollary 3 to obtain an totally ordered arc C containing p, itself contained in R. Let $C_- := C \cap [a, p]$. Observe that $\inf C_-$ belongs to R and it is a point of accumulation of R distinct from a by the isolatedness of a. The argument now proceeds exactly as in the proof of Corollary 3 to obtain a maximal totally ordered arc $S \subset R \cap [a, p]$. Since R is compact, $\inf S$ exists and belongs to R but cannot be a. Hence we can extend S using Proposition 5 to get a contradiction to the maximality of S. This contradiction proves that no point of R' is a point of accumulation of R' and, of course, a is not a point of accumulation of R'. This proves the result. □

For related results on stable equilibria see Jiang [14], Mierczyński [20, 21], and Hirsch [10].

3. **Proofs.**

Proof of Theorem 1. Our assumptions imply that (1) defines a strongly monotone semiflow Φ on the positive cone Y_+ (see [23]) which has a compact attractor K in Y_+ (see [7, 24]). Indeed, if $D := \{u \in Y : 0 \leq u(x) \leq u_0,\ x \in \Omega\}$, then $\Phi_t(D)$ is precompact in Y [24]. Thus the set E of equilibria is compact. Furthermore, strong monotonicity implies that if $w \in E,\ w \neq 0$, then $w \gg 0$.

The map $\Phi_t|_{\mathrm{Int} Y_+}$ is analytic by our assumption on f and Corollary 3.4.5 of Henry.

According to Corollary 4, if 0 is an isolated equilibrium and R is a maximal totally ordered subset of E, then R is finite. In that case, if $R = \{0\}$ then 0 is asymptotically stable by Lemma 1 since it is the omega limit set of some nontrivial orbit. If R is not trivial then it contains a smallest nonzero element u. An application of the Dancer-Hess trichotomy [3] to Φ on $[0, u]$ implies that either all orbits in $[0, u]$ converge to 0 except u, or all orbits converge to u except 0. The former case implies that 0 is asymptotically stable by Lemma 1. In the latter case, Theorem 6, with $X = \mathrm{Int} Y_+$, implies the existence of an asymptotically stable equilibrium, which of course is nontrivial. Therefore, the existence of an asymptotically stable equilibrium and the finiteness of every maximal totally ordered subset of E follows from the isolatedness of 0.

Suppose that 0 is not isolated. Equilibria of (1) satisfy

$$0 = Au + F(u),$$

where $F(u)(x) := u(x) f(x, u(x))$, and can be viewed as fixed points

$$u = T_\lambda(u) := (\lambda I - A)^{-1}[F(u) + \lambda u]$$

of an analytic map of some open order interval $[[a, b]] \subset Y$ containing 0. The resolvent operator is compact and strongly positive for $\lambda > 0$ and $F + \lambda I$ is strictly increasing on $[[a, b]]$ for large enough λ. Furthermore, as we assume 0 is a nonisolated fixed point of T_λ relative to Y_+, it follows that $\rho(DT_\lambda(0)) = 1$. Thus, T_λ satisfies the hypotheses of Proposition 5 and we conclude that there is a totally ordered arc C of fixed points of T_λ containing 0. $C_+ := C \cap Y_+$ is a totally ordered arc-with-end-point 0 consisting of equilibria for Φ. But because Φ is dissipative, this leads to a contradiction just as in the proof of Corollary 3. Therefore, 0 is isolated as asserted.

As every maximal totally ordered subset of E is finite, E itself must be finite. If not it has a nonzero point w of accumulation since it is compact. But then the hypotheses of Proposition 5 hold at $p = w$ and there must be a totally ordered arc containing w, contradicting that every totally ordered subset of E is finite.

The finiteness of E and the fact that (1) generates a strongly monotone, completely continuous semiflow on the positive cone in $C^1(\overline{\Omega})$ (see [22]) and that $Y_+ \subset C^1_+(\overline{\Omega})$ is dense implies that we may use the same arguments (Lemma 1 and Dancer-Hess trichotomy) as above to prove that the asymptotically stable equilibrium relative to Y_+ is also asymptotically stable relative to the positive cone in $C^1(\overline{\Omega})$. $\square$

Proof of Theorem 3. We restrict Φ to the order interval $I := [E_2, E_1]_K$ containing the equilibria $E_0 \equiv (0,0)$ and the E_i. I contains the set E of equilibria [13]. Restricted to I, Φ has a global attractor so E is compact. The restriction of Φ to $[[E_2, E_1]]_K$ is analytic and strongly monotone. Furthermore, if $e \in E$ is distinct from E_0, E_1, E_2, then $e \in [[E_2, E_1]]_K$. See e.g. [13].

If $E = \{E_0, E_1, E_2\}$, then one of the E_i is asymptotically stable since one of them attracts all initial data w satisfying $E_2 <_K w <_K E_1$. See Theorem B of [13]. Lemma 1 implies asymptotic stability of the attracting E_i. We are done in this case so we assume the existence of an equilibrium w distinct from E_0, E_1, E_2. It follows that $E_2 <<_K w <<_K E_1$.

We claim that E_1 is either an isolated equilibrium or there is a unique analytic arc $C_1 \subset E$ containing E_1 (part of which extends outside of $Y_+ \times Y_+$) tangent at E_1 to the eigenvector $z \ll_K 0$ identified in (b) above, and monotone with respect to the strong ordering $\ll_K$. We may as well assume that $\sigma_{12} = 0$ since otherwise E_1 is isolated by (d). The claim follows from Remark 2 following Proposition 5 and (b) applied to the mapping

$$T(u,v) := ((\lambda I - A_1)^{-1} u[f_1(\cdot, u, v) + \lambda], (\lambda I - A_2)^{-1} v[f_2(\cdot, u, v) + \lambda]),$$

defined in some open order interval $[[y, z]]_K \subset Y \times Y$ containing E_1, where $\lambda > 0$ and A_i is a realization of $d_i \nabla^2$ on X^α as in Theorem 1. Fixed points of T are equilibria of (6). Indeed, it follows from (b) and $\sigma_{12} = 0$ that $\rho(DT(E_1)) = 1$, that one is a simple eigenvalue strictly larger in modulus than any other and with eigenspace spanned by z. Similarly, E_2 is either an isolated equilibrium or there is a unique analytic arc $C_2 \subset E$ containing E_2, tangent to a vector $\bar{z} \gg_K 0$ and monotone with respect to the strong order $\ll_K$.

Now, clearly the hypotheses of Proposition 5 are satisfied at $p = \inf C_1$, where now the relevant mapping is the time t map Φ_t having p as a fixed point, implying that the analytic arc C_1 can be extended in a unique way to an analytic, strictly monotone arc C_1'. There exists a maximal extension C of C_1 as a strictly monotone analytic arc of equilibria contained in I, which is unique due to the local uniqueness assertion mentioned in Remark 2. As $E_2 \leq_K C$, $\inf C$ exists and belongs to E. If $\inf C = p \gg_K E_2$, then as it is an accumulation point of E, we may again apply Proposition 5 to obtain a contradiction to the maximality of C. Therefore,

$\inf C = E_2$ which is not an isolated equilibrium. Hence $C_2 \cap I \subset C$ and C is an analytic, strictly monotone arc connecting E_1 to E_2.

We conclude that E_1 is a nonisolated equilibrium if and only if E_2 is, in which case, there exists an analytic strictly monotone arc C connecting E_1 and E_2. When they are both nonisolated, then $E = C \cup \{E_0\}$. Suppose $w \in E$ but $w \notin C \cup \{E_0\}$. As $E_2 \ll_K w \ll_K E_1$, we may find $e_i \in C$ such that $e_2 \leq_K w \leq_K e_1$ with the property that e_2 is maximal and e_1 is minimal. As both inequalities are strict, strong monotonicity implies $e_2 \ll_K w \ll_K e_1$, a contradiction to the maximality of e_2 and minimality of e_1. A similar argument shows that every nontrivial orbit converges to C. This is obvious if its omega limit set ω is contained in E. If ω is not contained in E, then it must satisfy $E_2 \leq_K \omega \leq_K E_1$ and, by strong monotonicity, it must contain some point (u, v) with $E_2 \ll_K (u, v) \ll_K E_1$. Therefore, by the Non-ordering of omega limit sets, $E_1 <_K \omega <_K E_2$, and then by strong monotonicity, $E_2 \ll_K \omega \ll_K E_1$. Choose $e_1, e_2 \in C$ such that $e_2 \leq_K \omega \leq_K e_1$ and such that e_2 is maximal and e_1 is minimal with this property. Equality cannot hold by the Non-ordering of omega limit sets so strong monotonicity again produces the contradiction $e_2 \ll_K \omega \ll_K e_1$. We conclude there is no such omega limit set, proving our claim. The omega limit set of any orbit other than E_0 not only belongs to C but consists of a single point of C, by the Non-ordering of omega limit sets.

Hereafter, we may assume that both E_i are isolated equilibria but not the only nontrivial equilibria. The finiteness of E follows from Corollary 4. There are equilibria e_i satisfying $E_2 \ll_K e_2$ and $e_1 \ll_K E_1$ where e_2 is minimal and e_1 is maximal. Either $\Phi_t(x) \to E_1$ for all $x \in [[e_1, E_1]]_K$ or $\Phi_t(x) \to e_1$ for all $x \in [[e_1, E_1]]_K$ by the Dancer-Hess trichotomy (also see Prop. 2.2 [13]) since there are no other equilibria in $[e_1, E_1]_K$. If E_1 is the attractor, then it is asymptotically stable by Lemma 1 and we are done. An analogous dichotomy holds on $[E_2, e_2]$; if E_2 is the attractor, then it is asymptotically stable by Lemma 1. If, on the other hand, e_1 and e_2 are the attractors on their respective order intervals, then the existence of an asymptotically stable equilibrium follows from Theorem 6 applied to the restriction of Φ to $[[E_2, E_1]]_K$. □

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E-mail address: mwhirsch@chorus.net
E-mail address: halsmith@asu.edu