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Generic Quasi-convergence for strongly order preserving semiflows: a new approach

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This paper is dedicated to Shui Nee Chow on the occasion of his 60th birthday

1 Introduction

It is now well-known that the generic pre-compact orbit of a strongly monotone dynamical system approaches the set of equilibria. Such a result is proved by Hirsch in [3] for ordinary differential equations where generic means that it holds for almost all initial data, relative to Lebesgue measure. This is extended to general strongly monotone systems in ordered spaces in [2, 6] where generic refers either to a residual set of initial data or all but a subset of Gaussian measure zero in suitable Banach spaces. Matano [10] announced similar results. Smith and Thieme [16, 15] and later Takáč [17] found mild conditions under which generic means the result holds for an open and dense set of initial data. Although these results hold for a quite general class of strongly ordered spaces, the proofs are not easy and, at least in the case that generic means open and dense, additional compactness assumptions on the semiflow are required. In the present paper, we give a short proof that convergence to the set of equilibria holds for an open and dense set of initial data. Of course, it is based on fundamental results of monotone systems theory, such as the Convergence Criterion and the Limit Set Dichotomy. However, instead of requiring additional compactness assumptions, we assume that compact invariant sets have a supremum and infimum in the state space. As the stronger property holds for the space of continuous functions on a compact set with the usual ordering (every compact subset has an infimum and supremum), our result covers the standard spaces used in applications to systems of ordinary and delay differential equations. For systems of reaction-diffusion equations, the choice of state space is more delicate but in

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many cases one can use continuous function spaces (see e.g. [13]). For an up-to-date survey of monotone systems theory, we refer the reader to our forthcoming paper [8].

2 Definitions and Basic Results

Let X be an ordered metric space with metric d and *partial order* relation $\leq$. Recall that a partial order relation satisfies:(i) reflexive: $x \leq x$ for all $x \in X$; (ii) transitive: $x \leq y$ and $y \leq z$ implies $x \leq z$;(iii) antisymmetric: $x \leq y$ and $y \leq x$ implies $x = y$. We write $x < y$ if $x \leq y$ and $x \neq y$. Given two subsets A and B of X, we write $A \leq B$ ($A < B$) when $x \leq y$ ($x < y$) holds for each choice of $x \in A$ and $y \in B$. We assume that the order relation and the topology on X are compatible in the sense that $x \leq y$ whenever $x_n \to x$ and $y_n \to y$ as $n \to \infty$ and $x_n \leq y_n$ for all n. This is just to say that the partial order relation is closed. For $A \subset X$ we write $\bar{A}$ for the closure of A and Int$A$ for the interior of A. A subset of an ordered space is *unordered* if it does not contain points x, y such that $x < y$. Let $A \subset X$ and let $L = \{x \in X : x \leq A\}$ be the (possibly empty) set of lower bounds for A in X. In the usual way, we define $\inf A := u$ if $u \in L$ and $L \leq u$; u is unique if it exists. Similarly, $\sup A$ is defined.

The notation $x \ll y$ means that there are open neighborhoods U, V of x, y respectively such that $U \leq V$. Equivalently, (x, y) belongs to the interior of the order relation. The relation $\ll$, sometimes referred to as the *strong ordering*, is transitive and vacuously antisymmetric, but not reflexive; in many cases it is empty. We write $x \geq y$ to mean $y \leq x$, and similarly for $>$ and $\gg$.

A *semiflow* on X is a continuous map $\Phi : \mathbb{R}^+ \times X \to X$, $(t, x) \mapsto \Phi_t(x)$ such that:

$$\Phi_0(x) = x, \quad (\Phi_t \circ \Phi_s)(x) = \Phi_{t+s}(x) \qquad (t, s \geq 0,\ x \in X)$$

The *orbit* of x is the set $O(x) = \{\Phi_t(x) : t \geq 0\}$. An *equilibrium* is a point x for which $O(x) = \{x\}$. The set of equilibria is denoted by E.

The *omega limit set* $\omega(x)$ of $x \in X$, defined in the usual way, is closed and positively invariant. When $\overline{O(x)}$ is compact, $\omega(x)$ is also nonempty, compact, invariant and connected; and it *attracts* x, that is, $\lim_{t\to\infty} \operatorname{dist}(\Phi_t(x), \omega(x)) = 0$. A point $x \in X$ is *quasiconvergent* if $\omega(x) \subset E$. The set of such points is denoted by Q. If $\omega(x)$ is single point, necessarily an equilibrium, then x is *convergent*. The set of convergent points is denoted by C.

Let Φ denote a semiflow in an ordered space X. We call Φ *monotone* provided

$$\Phi_t(x) \leq \Phi_t(y) \text{ whenever } x \leq y \text{ and } t \geq 0.$$

Φ is *strongly monotone* if $x < y$ implies that $\Phi_t(x) \ll \Phi_t(y)$ for all $t > 0$ and *eventually strongly monotone* if it is monotone and $x < y$ implies that $\Phi_t(x) \ll \Phi_t(y)$ for all large $t > 0$. Φ is *strongly order-preserving*, SOP for short, if it is monotone and whenever $x < y$ there exist open subsets U, V of X with $x \in U$ and $y \in V$ and $t_0 \geq 0$ such that

$$\Phi_{t_0}(U) \leq \Phi_{t_0}(V).$$

Monotonicity of Φ then implies that $\Phi_t(U) \le \Phi_t(V)$ for all $t \ge t_0$. It is easy to see that if Φ is SOP and K, L are compact subsets satisfying $K < L$, then we can choose neighborhoods U, V of K, L and $t_0 \ge 0$ such that the previous inequality holds for $t \ge t_0$. Strong monotonicity implies eventual strong monotonicity which implies SOP. See e.g. [16, 13].

In the remainder of this paper we assume Φ is a monotone semiflow in an ordered metric space X, such that every orbit has compact closure. The fundamental building blocks of the theory of monotone systems are the following results. Proofs can be found in the works of Hirsch [1, 6], in the monograph [13], or in the forthcoming survey [8].

Theorem 2.1 (Convergence Criterion) *Let Φ be monotone and $\Phi_T(x) \ge x$ for some $T > 0$. Then $\omega(x)$ is a T-periodic orbit. If $\Phi_t(x) \ge x$ for t belonging to some nonempty open subset of $(0, \infty)$ then $\Phi_t(x) \to p \in E$ as $t \to \infty$. If Φ is SOP and $\Phi_T(x) > x$ for some $T > 0$ then $\Phi_t(x) \to p \in E$ as $t \to \infty$.*

Theorem 2.2 (Nonordering of Limit Sets) *Let $\omega(z)$ be an omega limit set for Φ,*

(i) *No points of $\omega(z)$ are related by $\ll$.*

(ii) *If Φ is SOP, no points of $\omega(z)$ are related by $<$.*

Theorem 2.3 (Limit Set Dichotomy) *Let Φ be SOP. If $x < y$ then either*

(a) $\omega(x) < \omega(y)$, or

(b) $\omega(x) = \omega(y) \subset E$.

If case (b) holds and $t_k \to \infty$ then $\Phi_{t_k}(x) \to p$ if and only if $\Phi_{t_k}(y) \to p$.

Smith and Thieme [15] improve part (b) of the Limit Set Dichotomy to read $\omega(x) = \omega(y) = \{e\}$ for some $e \in E$ under additional smoothness and strong monotonicity conditions. For example, this strengthened Limit Set Dichotomy holds if $X \subset Y$ where Y is an ordered Banach space with cone Y_+ having non-empty interior, $\Phi_t(x)$ is C^1 in x and its derivative is a compact, strongly positive operator.

3 A New Approach to Generic Quasiconvergence

A point x is *doubly accessible from below* (respectively, above) if in every neighborhood of x there exist f, g with $f < g < x$ (respectively, $x < f < g$).

Consider the following condition on a semiflow having compact orbit closures:

(L) Either every omega limit set has an infimum in X and the set of points that are doubly accessible from below has dense interior, or every omega limit set has a supremum in X and the set of points that are doubly accessible from above has dense interior.

Axiom (L) is a restriction on both the space X (order and topology) and the semiflow (limit sets). It holds when $X = C(A, \mathbb{R})$, the Banach space of continuous functions on a compact set with the usual ordering, because every compact subset of $C(A, \mathbb{R})$ has a supremum and infimum (see Schaefer [11], Chapt. II, Prop. 7.6). In particular, it holds for $\mathbb{R}^n = C(N, \mathbb{R})$ and $C(A, \mathbb{R}^n) = C(A \times N, \mathbb{R})$ with the usual component-wise ordering where $N = \{1, 2, \cdots, n\}$ with discrete topology. Even if X is not a lattice, e.g., the Banach space of C^1 functions on a compact manifold, (L) may still be valid for certain dynamical systems.

Theorem 3.1 *Let $\Phi : X \to X$ be an SOP semiflow on the ordered metric space X, having compact orbit closures, and satisfying axiom* (L). *Then $X \setminus Q \subset \overline{\operatorname{Int} C}$, and $\operatorname{Int} Q$ is dense.*

The proof is based on the following result. For $p \in E$ define $C(p) := \{z \in X : \omega(z) = \{p\}\}$. Note that $C = \bigcup_{p \in E} C(p)$.

Lemma 3.2 *Suppose $x \in X \setminus Q$ and $a = \inf \omega(x)$. Then $\omega(a) = \{p\}$ with $p < \omega(x)$, and $x \in \overline{\operatorname{Int} C(p)}$ provided x is doubly accessible from below.*

Proof: Fix an arbitrary neighborhood M of x. Note that $a < \omega(x)$ because $\omega(x)$ is unordered (Theorem 2.2). By invariance of $\omega(x)$ we have $\Phi_t a \leq \omega(x)$, hence $\Phi_t a \leq a$. Therefore the Convergence Criterion Theorem 2.1 implies $\omega(a)$ is an equilibrium $p \leq a$. Because $p < \omega(x)$, SOP yields a neighborhood N of $\omega(x)$ and $s \geq 0$ such that $p \leq \Phi_t N$ for all $t \geq s$. Choose $r \geq 0$ with $\Phi_t x \in N$ for $t \geq r$. Then $p \leq \Phi_t x$ if $t \geq r + s$. The set $V := (\Phi_{r+s})^{-1}(N) \cap M$ is a neighborhood of x in M with the property that $p \leq \Phi_t V$ for all $t \geq r + 2s$. Hence:

$$u \in V \implies p \leq \omega(u) \tag{3.1}$$

Now assume x doubly accessible from below and fix $y_1, y \in V$ with $y_1 < y < x$. By the Limit Set Dichotomy $\omega(y) < \omega(x)$, because $\omega(x) \not\subset E$. By SOP we fix a neighborhood $U \subset V$ of y_1 and $t_0 > 0$ such that $\Phi_{t_0} u \leq \Phi_{t_0} y$ for all $u \in U$. The Limit Set Dichotomy implies $\omega(u) = \omega(y)$ or $\omega(u) < \omega(y)$; as $\omega(y) < \omega(x)$, we therefore have:

$$u \in U \implies \omega(u) < \omega(x) \tag{3.2}$$

For all $u \in U$, (3.2) implies $\omega(u) \leq \omega(a) = \{p\}$, while (3.1) entails $p \leq \omega(u)$. Hence $U \subset C(p) \cap M$, and the conclusion follows. ∎

Proof of Theorem 3.1 To fix ideas we assume the first alternative in (L), the other case being similar. Let X_0 denote a dense open set of points doubly accessible from below. Lemma 3.2 implies $X_0 \subset Q \cup \overline{\operatorname{Int} C} \subset Q \cup \overline{\operatorname{Int} Q}$, hence the open set $X_0 \setminus \overline{\operatorname{Int} Q}$ lies in Q. This prove $X_0 \setminus \overline{\operatorname{Int} Q} \subset \operatorname{Int} Q$, so $X_0 \setminus \overline{\operatorname{Int} Q} = \emptyset$. Therefore $\overline{\operatorname{Int} Q} \supset X_0$, hence $\overline{\operatorname{Int} Q} \supset \overline{X_0} = X$. ∎

4 An Example

We give a simple example illustrating that the application of Theorem 3.1 can lead to improvements in earlier results based on the approach of Smith and Thieme [16, 15]. The key message is that strong compactness assumptions for the semiflow can be avoided when the state space is a space of continuous functions on a compact set.

Consider the delay differential equation with positive delayed feedback

$$x'(t) = f(x(t), x(t-\tau)) \tag{4.1}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1 and satisfies

$$f_y(x, y) > 0. \tag{4.2}$$

The state space for (4.1) is $X := C([-\tau, 0], \mathbb{R})$ with the usual order. Given $\phi \in X$, there is a unique local solution of (4.1) satisfying

$$x(s) = \phi(s),\ \ s \in [-\tau, 0].$$

Proposition 4.1 *Suppose f satisfies (4.2) and suppose that the solution of every initial value problem is bounded on $t \geq 0$. Then the set of initial data ϕ corresponding to a convergent orbit contains a dense open set in $C([-\tau, 0], \mathbb{R})$.*

Proof: Eventual strong monotonicity, hence SOP, is implied by (4.2) Theorem 2.5 in Smith [12]. Orbits have compact closure by virtue of the boundedness of forward orbits. Hypothesis (L) holds on $C([-\tau, 0], \mathbb{R})$. Therefore, Theorem 3.1 implies that Q contains an open and dense set. But $Q \subset C$ since the set E of equilibria is totally ordered and limit sets are unordered.∎

Proposition 4.1 extends readily to cooperative and irreducible systems of functional differential equations as in chapter 5, Smith [13]. In particular, Theorem 4.1 [13] holds if in hypothesis (T) of that result, we drop the assumption that for every compact set $A \subset C([-\tau, 0], \mathbb{R}^n)$, there exists a closed and bounded set B such that for each $\phi \in A$, $x_t(\phi) \in B$ for all large t.

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