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The Design of Mechanisms to Draw Plane Curves

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The Design of Mechanisms to Draw Plane Curves

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mechanical and Aerospace Engineering

by

Yang Liu

Thesis Committee:
Professor J. Michael McCarthy, Chair
Professor Lorenzo Valdevit
Professor Haithem Taha

2018

DEDICATION

To my parents,
thank you for your love and support.

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ABSTRACT OF THE DISSERTATION

The Design of Mechanisms to Draw Plane Curves

By

Yang Liu

Doctor of Philosophy in Mechanical and Aerospace Engineering

University of California, Irvine, 2018

Professor J. Michael McCarthy, Chair

This dissertation develops a mechanism design procedures to draw algebraic plane curves. In 1876, Alfred B. Kempe published a proof that showed how to construct a linkage system obtained from the equation of an algebraic curve that draws the curve. He admitted that his approach lead to complex devices and recommended further study to achieve practical designs. Kempe's result, now called the Kempe Universality Theorem, was proven with modern mathematical precision by John Milson and Michael Kapovich in 2000. The resulting designs remain complex due to the generality of the proof. In this dissertation the focus is on the design of practical linkage systems that draw algebraic curves, trigonometric curves and Bezier curves. We also explore the realization of these linkage systems using solid models and additive manufacturing.

Chapter 1

Mechanisms to Draw Curves

1.1 Overview

This dissertation studies the design of mechanisms to draw plane curves. Mathematicians have shown that for every algebraic curve there is a mechanism that draws the curve, however the existence proof yields complex devices with hundreds of links. Here we obtain simpler drawing linkages by using computing mechanism elements. One result is that for general algebraic curves the number of parts can be reduced by at least one-half, while simplifying the structure of the device. A second result is that trigonometric curves can be drawn either by assemblies of Scotch yoke mechanisms or by coupled serial chains, both of which can result in a reduction in part count by a factor of ten. Finally, because Bezier curves are trigonometric curves, we show that any curve that can be approximated by a sequence of n-degree Bezier curves can be drawn by a sequence of 2n-coupled serial chains. Application of this research shows that in practice complex curves associated with cursive hand-writing can be drawn by a sequence four-link coupled serial chains. Prototypes of these devices were constructed using additive manufacturing to demonstrate the results.

1.2 Algebraic Curves

In 1887 Kempe [28] presented a procedure that begins with the equation of a plane algebraic curve and yields a planar linkage that draws this curve. To do this he introduced four calculating linkages, the reversor, the multiplicator, the additor and the translator, that he used to construct mechanical constraints on the two angles of a planar RR serial chain so that it draws the curve—R denotes a revolute, or hinged, joint.

Kempe recognized that his existence proof yielded complex linkages. He stated, "It is hardly necessary to add, that this method would not be practically useful on account of the complexity of the linkwork...". However, he posed an interesting question, asking the "mathematical artist to discover the simplest linkworks that will describe particular curves". This search for the simplest linkage is more easily achieved on a case by case basis, as is found in Artobolevskii[2], who provides a large number of specialized linkages adapted to the geometry of specific classes of algebraic plane curves.

Recent work by mathematicians, Jordan and Steiner [25] and Kapovich and Millson [27] provide a modern proof to Kempe's result, which they call *Kempe's Universality Theorem*. These results expose a direct mathematical connection between algebraic curves and linkages. This connection may obscured by the elementary way the mechanical calculations are implemented in Kempe's method, which results in complex linkage systems.

The goal is to maintain Kempe's general procedure but simplify the resulting linkages by replacing his four computing linkages with differentials and cable drives. The differential performs addition and the cable drives are configured to reverse, multiply and translate angular values. The result is a set of constraints on the angles of an RR serial chain that cause it draws a given algebraic curve. The resulting linkages are simpler and more clearly illustrate Kempe's approach. However, they remain complex devices that deserve more attention to simplify their construction.

Practical applications for a mechanical system that replaces what is currently done by electrical systems arise in circumstances where size is an issue such as MEMS or nano applications, and where electric power is unavailable or undesirable. Our current focus is on reducing the complexity of this important theoretical result, we look forward to working on applications in the near future.

The early work on linkages that draw curves is described in Nolle [48, 49] and Koetsier [32, 33]. In 1876 Kempe showed that a drawing linkage for an arbitrary algebraic curve could be constructed using computing linkage to calculate the angular values of a planar RR serial chain, which he presented as a parallelogram. Saxena [57] presents a step-by-step construction of a drawing linkage, using Kempe's method, for a quadratic curve defined by the product of two lines, and obtains the linkage shown in Figure 1.1 that has 48 links and 70 joints. Our synthesis methodology results in a simpler design showing in Figure 1.2.

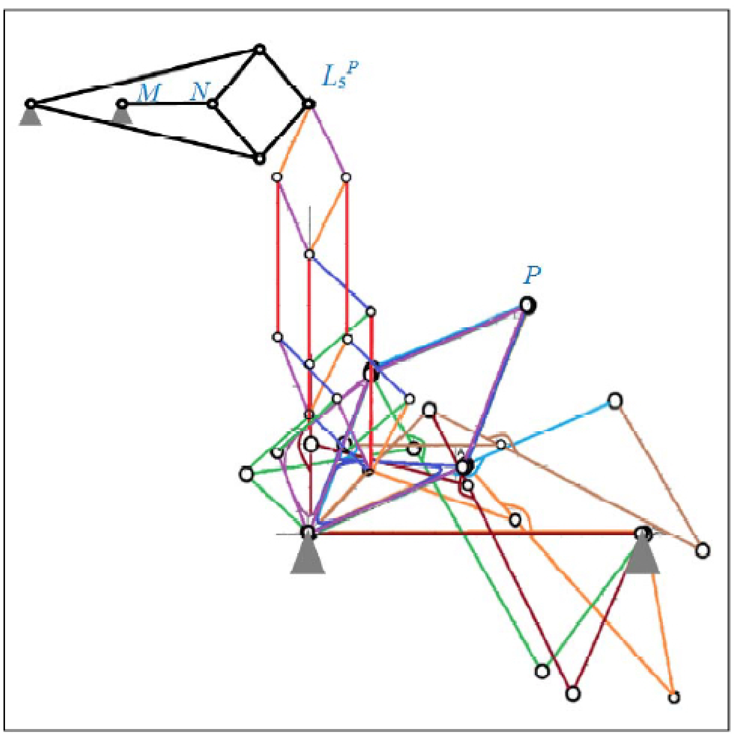


Figure 1.1: The linkage obtained by Saxena [57] follows Kempe's procedure to obtain a linkage that moves the point P to draw a curve defined by the product of two straight lines. It consists of 48 links and 70 joints.

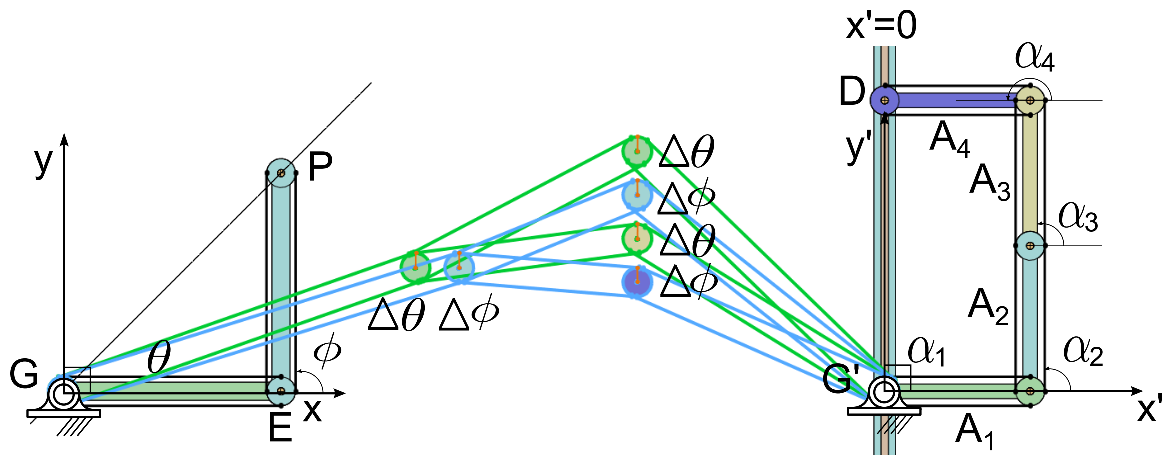


Figure 1.2: The linkage constructed bevel gears differential and configuration of cable drives as the function of Kempe's additor, reversor, multuplicator and translator to draw the same line as Saxena's linkage constructing using exactly Kempe's approach.

The difference between Kempe's and Artobolevskii's approach can be seen by comparing Saxena's linkage for a quadratic curve to Artobolevskii's eight-bar conograph linkage which can be adapted to draw any quadratic curve, see Figure 1.3.

Recent work by Gao et al. [61] and Abbott [1] generalized Kempe's methodlogy and showed that a curve of degree n requires at least $O(n^2)$ bars. Kobel [31] used the dynamic geometry system *Cinderella* to construct a number of drawing linkages. See Figure 1.4, which is the drawing linkage for a elliptic cubic curve. By applying our synthesis methodology, we obtain a drawing linkage for this case showing in Figure 1.5.

A different approach to curve-drawing linkages was introduced by Roth and Freudenstein [55], who formulated equations for the dimensions of a four-bar linkage that interpolated a set of nine accuracy points along the designed curve. Also see Ramakrishna and Sen [53], and Bai and Angeles [3].

Wampler et al.[60] obtained the complete solution for four-bar path generation problem and showed that there are as many as 4326 distinct solutions for a prescribed set of nine points on a curve. This point path generation technique was extend by Kim, et al. [30] to six-bar

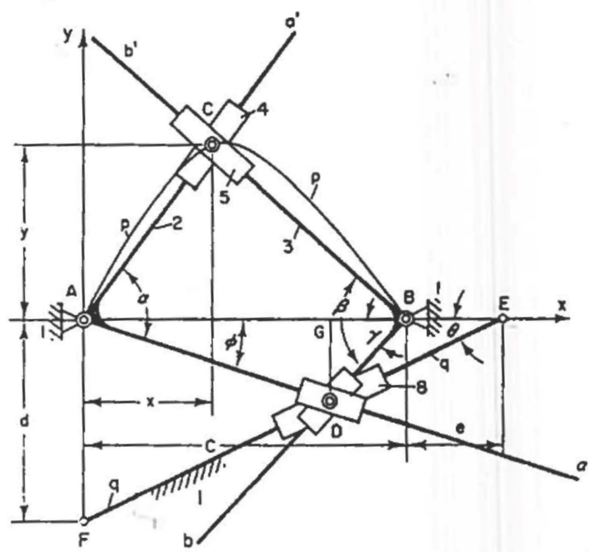


FIG. 136

Figure 1.3: Artobolevskii shows that a linkage to draw any planar quadratic curve can be obtained by adjusting the dimensions of this linkage consisting of eight bar and 10 joint, known as a *conograph*. This is simpler than the linkage obtained by Kempe's general method.

mechanism design to guide the coupler going though 15 accurate points. Recent research by Plecnik [51, 52] shows that the equations for 15 precision points six-bar linkage path generation has a Bezout bound over 10^{46}, which means finding all the six-bar linkages for a given set 15 points is beyond our current computation capabilities.

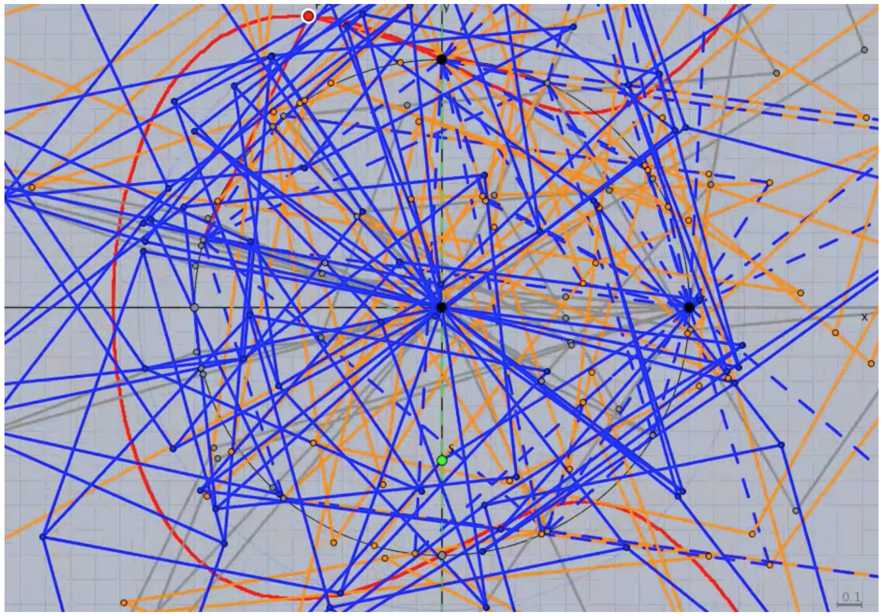


Figure 1.4: This linkage that draws an elliptic cubic curve was obtained by Kobel [31] using the dynamic geometry system *Cinderella*. The cubic curve is in the background covered by many linkage elements that guide a point seen near the top of the figure along the curve.

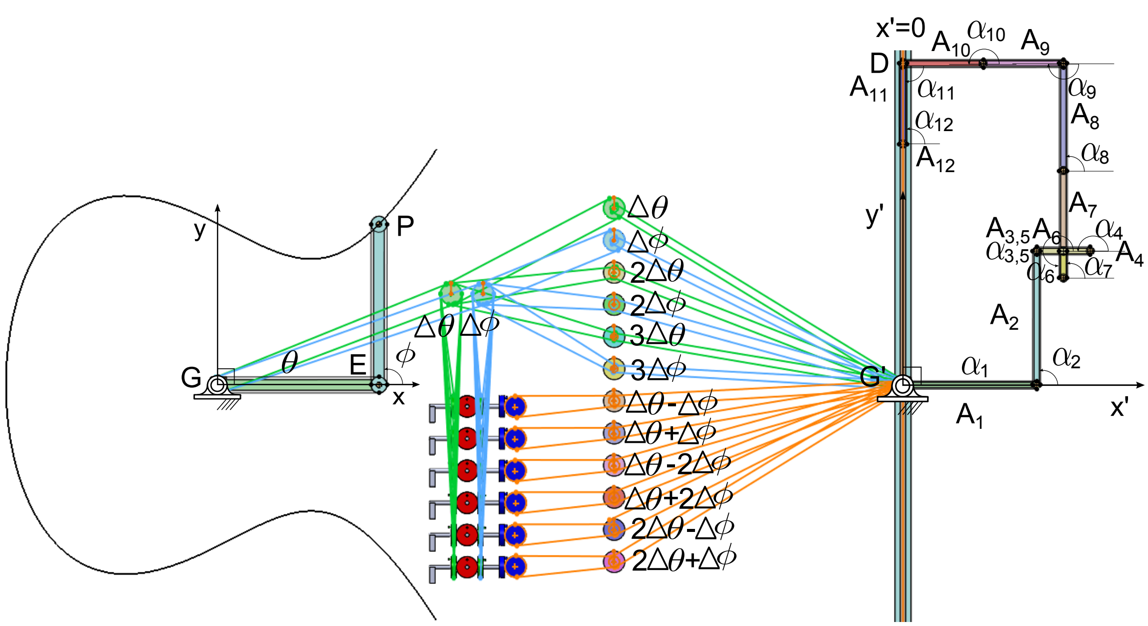


Figure 1.5: The drawing linkage for the elliptic cubic curve presented by Kobel consists of the RR serial chain constrained by the linkage consisting 12 hinged links that end in a prismatic joint that moves along line $x' = 0$.

1.3 Trigonometric Curves

The design of mechanisms to draw plane curves have found recent application in the construction of mechanical characters, [9, 59]. Nolle [49, 48] and Koetsier [32, 33] describe the history of mechanism design to draw curves. Also see Liu and McCarthy [39].

We show how to obtain a mechanical system that draws curves that have x and y coordinates defined by finite Fourier series, known as trigonometric curves. The class of trigonometric curves include well-known curves such as the limacon of Pascal, the Cardioid, Trifolium, Hypocycloid and Lissajous figures, as well as many other examples [22]. Artobolevskii [2] presents specialized linkages for drawing curves including the linkage in Figure 1.6 to draw the Trifolium. Another comparison of the mechanism obtained by Kempe's existence proof and the work by "mathematicians and artist" in the words of Kempe can be seen in the linkage obtained by Kobel [31] to draw the quartic trifolium, Fig. 1.7. He reports that his algorithm generates too many bars to be able to count. Our results can be viewed as a version of Kempe's Universality Theorem [28, 25, 27], which shows that linkages exist to draw a particular class of plane curves. As comparison, our mechanical system to draw the same Trifolium curve is shown in Figure 1.8 1.9 1.10.

Nie and Krovi [46] present the method of designing single degree-of-freedom coupled serial chain to draw plane curves. The coupled serial chain is driven by pulleys and belts. They use discrete Fourier transform approach to obtain serial chain that can go through the sample points on a specified curve. The number of links can be reduced using optimization method.

Lord Kelvin invent the first harmonic analyzer in 1872. Miller [44] developed a 32-element harmonic synthesizer in 1916 based on Fourier's Theorem. This device was built according to the calculation of the amplitude and phase of each Fourier component in the curve equation. The continue work was done by Brown [4], a thirty terms harmonic analyzer is constructed to drive a pencil to draw a specified curve on a board.

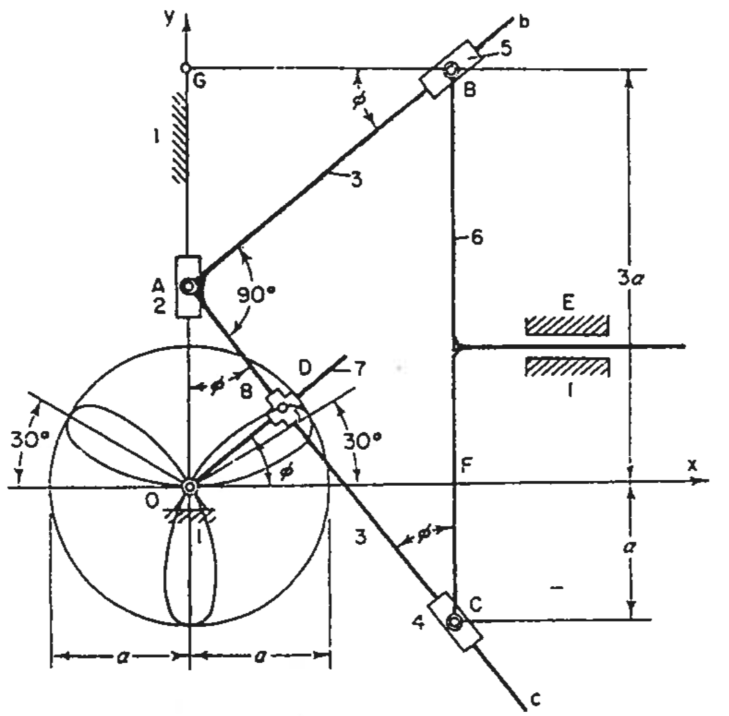


Figure 1.6: Artobolevskii [2] designed this mechanism to draw the Trifolium (three petal) curve.

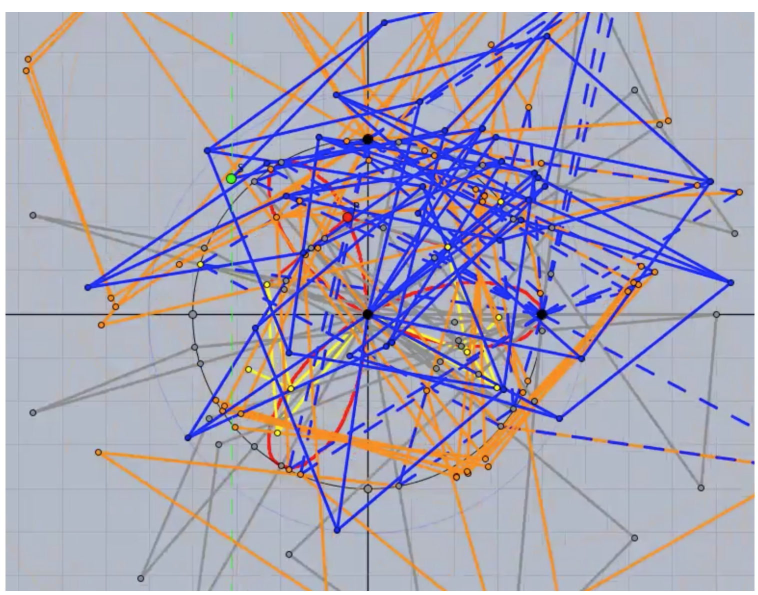


Figure 1.7: Kobel use software Cinderella to generate this linkage to draw trifolium curve.

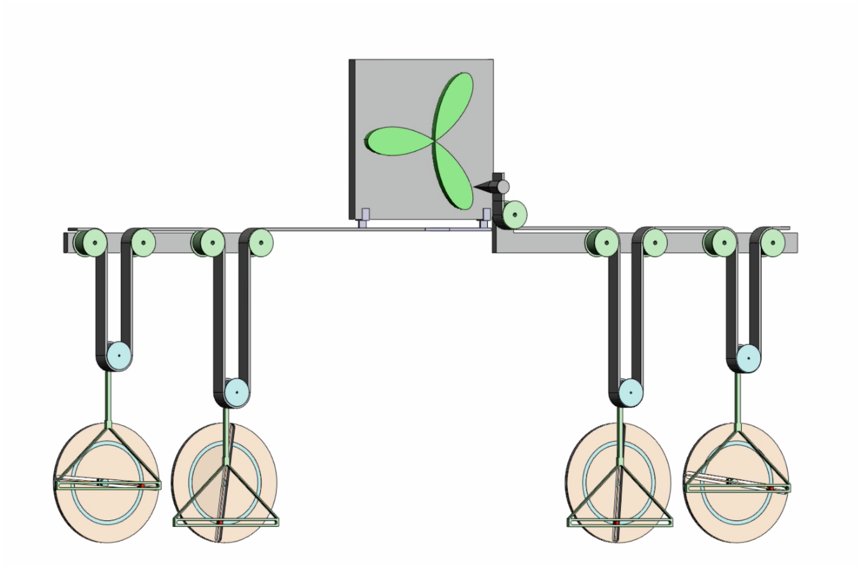


Figure 1.8: A system of Scotch yoke mechanisms driven by a single input that draws the Trifolium.

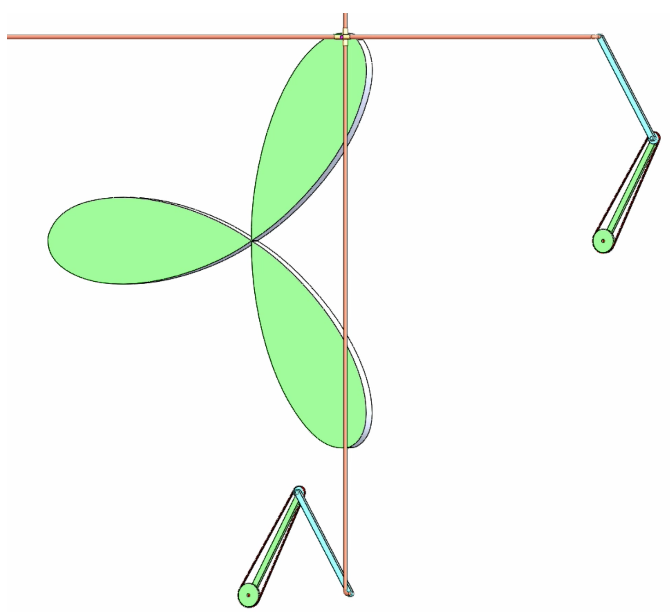


Figure 1.9: A system of two coupled serial chains driven by a single input that draws the Trifolium.

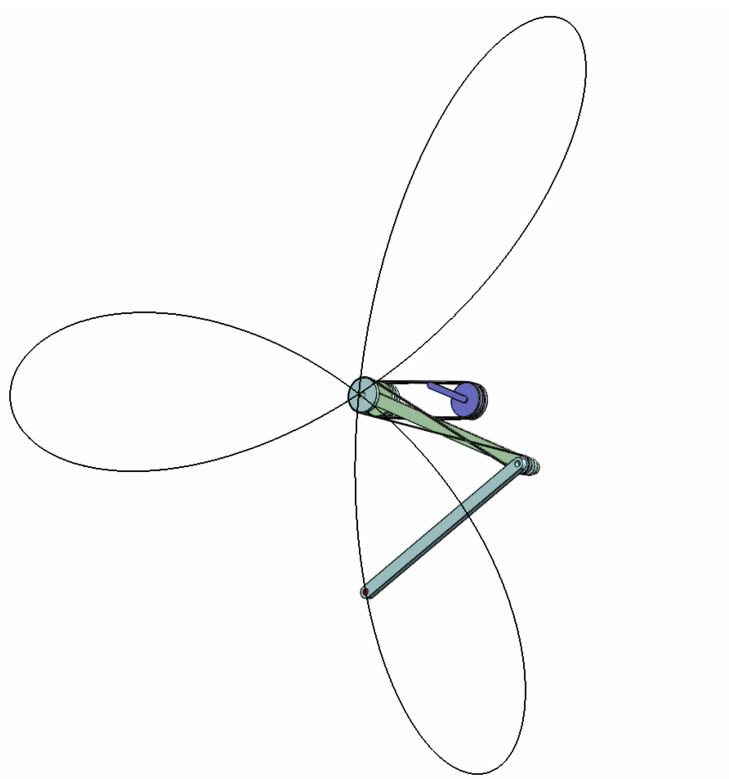


Figure 1.10: A single constrained coupled serial chain with one input that draws the Trifolium.

While it is known that cams can be cut to obtain the desired coordinate functions for a plane curve, or shaped to drive linkages to achieve draw a desired curve [45], our focus is on a simpler realizable way to construct mechanisms to draw plane curves. We increase the set of standard linkages used in synthesis of curve-tracing mechanism to include belts and pulleys, but avoid the requirement to cut a cam to achieve a specified function. We use belts and pulleys to add and translate the projection of Scotch yoke mechanisms and to constrain the joints in coupled-serial chain mechanisms to achieve the summation of Fourier coefficients that define a plane curve. The result is a wide range of devices that draw complex plane curves.

We present three ways to draw these curves. First, we use the mechanical Fourier synthesis system described by D. C. Miller [44]. Next, we follow Nie and Krovi [46] and obtain coupled serial chains defined by the coordinate Fourier series. Finally, we present a way to define a single coupled serial chain that draws a given trigonometric plane curve. The methodology is illustrated with several examples, which include the the use of boundary points and the discrete Fourier transform [50] to define the trigonometric curve.

1.4 Bezier Curves

Recent research on Kempe's universaility theorem has yielded drawing linkages for rational plane and space curves, Hegedus et al (2015) [21], Gallet et al (2017) [14], and Li et al (2017) [37], by factoring a suitable motion polynomial in the Clifford algebra of dual quaternions [43]. Bezier curves formed from dual quaternions with trigonometric weighting functions [15, 16, 11] have been used for robot path planning.

Ge et al (1997) [17] modified the polynomial formulation of Bezier curves using trigonometric weighting functions, which they called harmonic rational Bezier curves, in order to eliminate

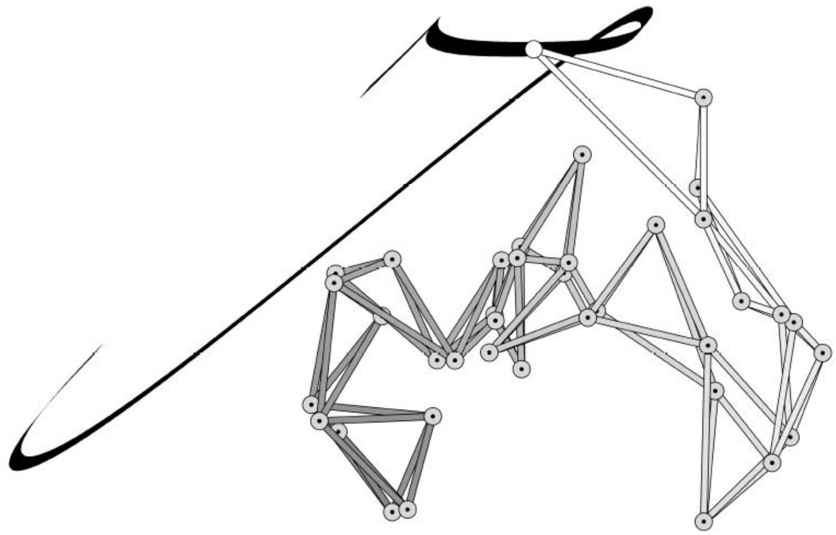


Figure 1.11: A linkage curve approximating 'J' and linkage to draw it.

higher harmonics. Sanchez-Reyes (1998) [56] and Han (2003) [18] reformulated harmonic Bezier curves as trigonometric Bezier curves to draw plane curves. More recently, Juhasz and Roth (2014) [26] show how to use trigonometric spline curves for interpolation.

Our use of cubic trigonometric Bezier curves follows the formulation of Han (2004) [19], who uses four control points to define a curve that has the same properties as cubic polynomial Bezier curves [12]. Therefore, in this paper we define a system of plane curves using cubic Bezier curve segments, approximate these curves using cubic trigonometric Bezier curve segments, and then use the results of Liu and McCarthy (2017) [40] to design a mechanical system that draws the curves.

Our examples consist of cursive alphabet letters [5] and script Chinese characters [64] that are drawn with a set of continuous strokes forming plane curves. We design planar systems of coupled serial chains that draw these curves using a single input.

An example in Gallet et al (2017) [14] showed using rational curve to approximate the 'J' in John Hancock's signature is in Figure 1.11. As comparison, our linkage draw a Chinese character using Bezier curve approach is shown in Figure 1.12.

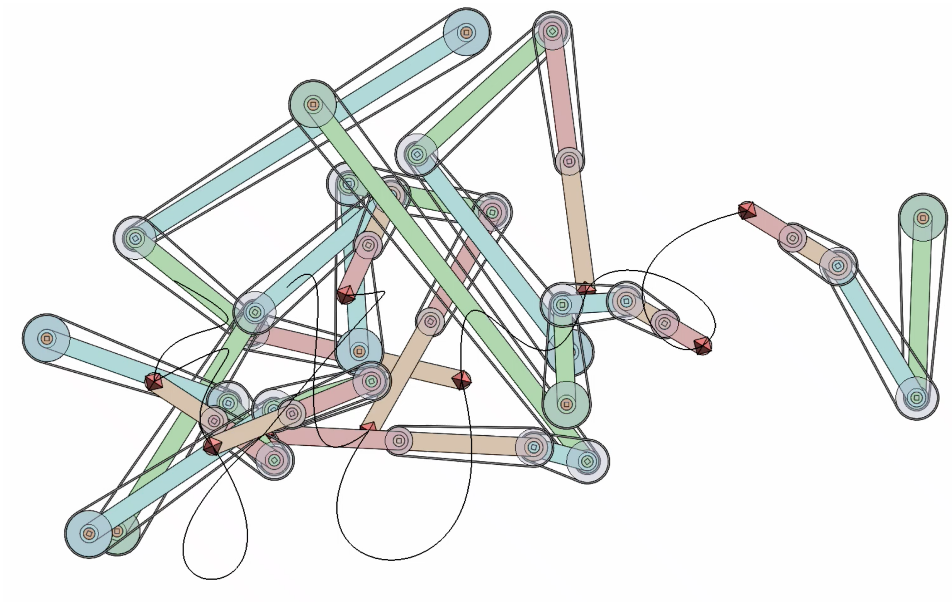


Figure 1.12: The linkage system that draws the script form of the Chinese character "long."

Krovi et al. [34, 47] provide a design methodology for single degree of freedom linkage systems formed as serial chains with joints coupled by belt and cable drives. Recent work by Liu and McCarthy [41] shows the usage of belt and cable drives to design mechanisms to draw plane algebraic curves. The use of belt and cable drives to actuate the joints of robotic hands can be found in [36, 23, 62]. Collins [8] uses a combination of cables and gears to actuate finger joints, while Lin [38] uses gears. Our approach follows Krovi, but uses gearing to provide the joint coupling.

1.5 Additive Manufacture

The use of additive manufacturing for the fabrication of mechanical systems can be drawn to Mavroidis [42] and Laliberte [35]. The manufacturing options include selective laser melting [63], stereolithography and selective laser sintering [6], and layer building technology [7]. Research in additive manufacturing of mechanisms generally focuses on techniques to avoid assembly [10]. In this paper, we use fused deposition modeling by the Stratasys Fortus 450mc, which requires manufacture of individual components and assembly to maintain tolerances, but is fast and inexpensive for prototype development [54].

1.6 Contributions

The contribution of this research consists of new methods for the design of mechanisms that draw plane curves that are less complex, measured in terms of the number of parts, than the state of the art. In particular, this dissertation shows that

1. The use of mechanical computational elements such as the bevel-gear differential for adding and pulleys for multiplication and translation reduces the number of parts in Kempe's linkage for an algebraic curve by more than half,

2. The use of finite Fourier series to approximate the coordinate functions of a plane curve yields a trigonometric curve that can be drawn by Scotch yoke linkages defined by each term,

3. A trigonometric plane curve can also be drawn by a serial chain in which each of the joints are coupled by specific gear ratios, resulting in significant simplification of the linkage system,

4. The trigonometric representation of the component functions of Bezier curves yields $2n$-link coupled serial chains for degree n Bezier functions,

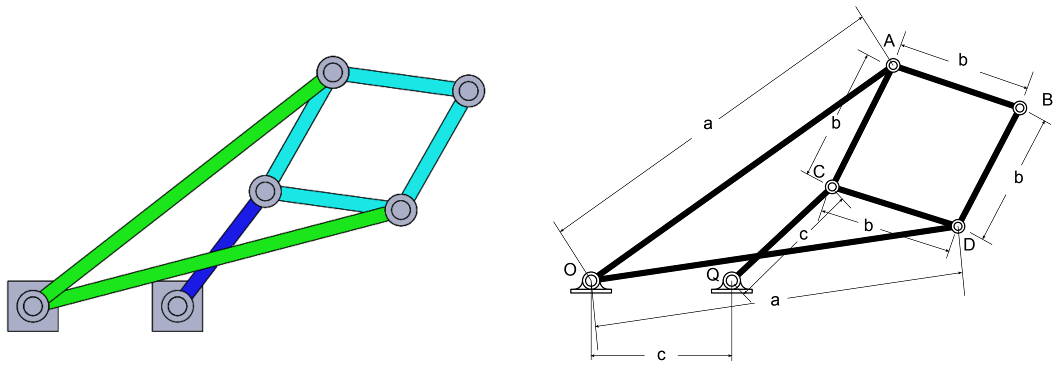
5. Example designs for cursive writing approximated by cubic Bezier curves can be simplified to systems four-link coupled serial chains.

Chapter 2

Computing Linkages

2.1 Introduction

In this chapter, we present the Peaucellier-Lipkin eight-bar linkage that draws a straight line and then show that there are three four-bar linkages that draw approximate straight lines. Similarly, we then present Kempe's approach to the design of linkages that draw algebraic curves, and then show that we can obtain simpler designs using computing linkages such as the bevel-gear differential as an adder and a gear train as multipliers. We then show that Fourier series approximations to an algebraic curve yields less complicated drawing mechanisms. In particular, we obtain serial chains that draw Bezier curves, which means we obtain a drawing mechanism for any curve that is approximated by a set of Bezier curves.



(a) (b)

Figure 2.1: (a) Solidworks model of Peaucellier's linkage (b) The dimensions of Peaucellier's linkage.

2.2 Straight-Line Mechanisms

A mechanism that can achieve draw a straight line was obtained by Peaucellier[20] in 1864, Fig.2.1. It is an eight-bar linkage that is known as a inversor, because it mechanically computes the inversion of points in a circle. The crank draws a circle that passes through the origin of the inversion, which transformed into a straight line.

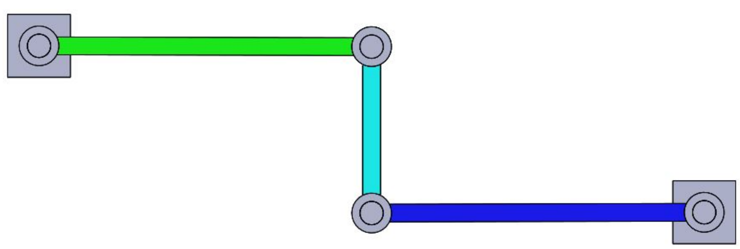


Figure 2.2: SolidWorks Model of Watt's Linkage.

An approximation to a straight line can be generated by the simpler four-bar mechanism. The best known of the approximate straight-line mechanisms is the Watt Parallel Motion Generator that enabled double acting expansion for steam engines [20]. It can be found today in automotive suspensions to resist side swaying movement. The SolidWorks model of Watt's linkage can be seen in Fig2.2.

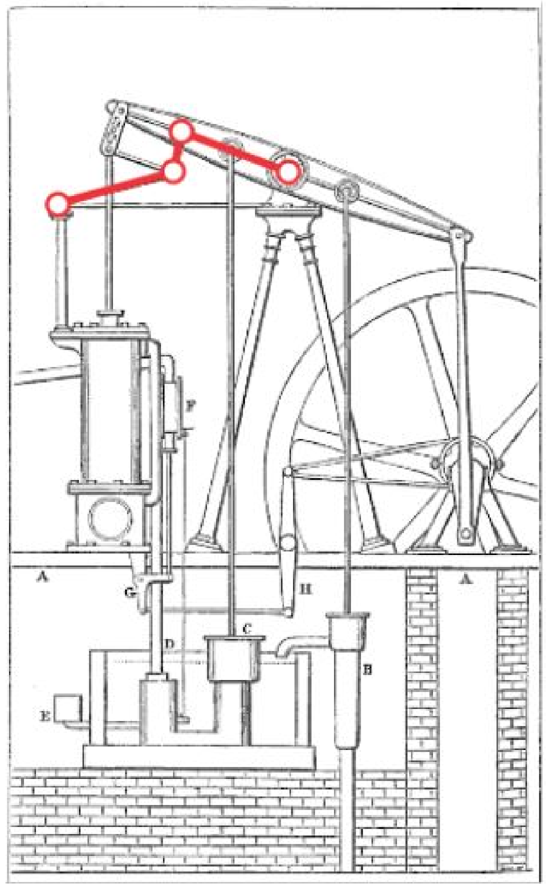
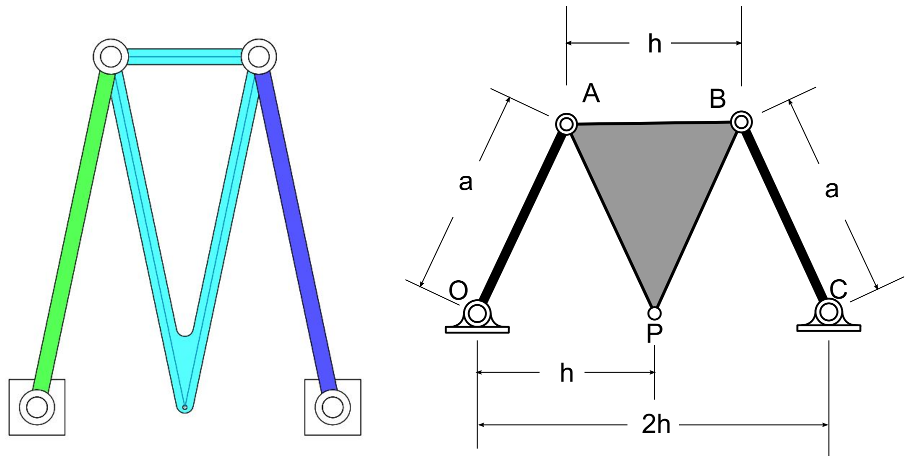


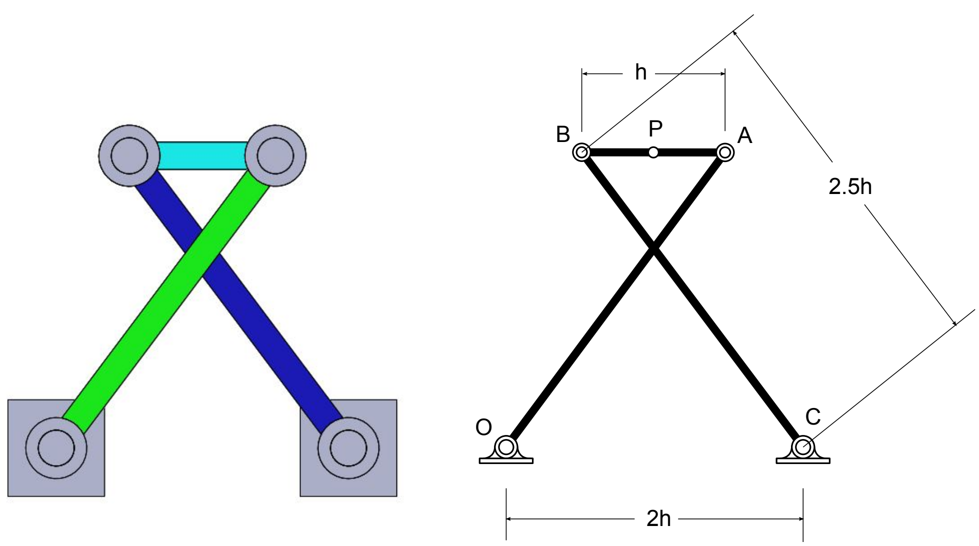
Figure 2.3: A schematic of Watt-Boulton steam engine with the Watt linkage highlighted.

Other approximate straight line mechanisms were obtained by Roberts, Fig.2.4 and Chebyshev, Fig.2.5. The website DMG-LIB.org provides an example of the walking machine developed by Chebyshev using his approximate straight line linkage and its cognate.



(a) (b)

Figure 2.4: (a) SolidWorks model of Robert's Linkage (b) The dimension of Robert's linkage.



(a) (b)

Figure 2.5: (a) Solidworks model of Chybyshev's linkage (b) The dimensions of Chebysheve's linkage.

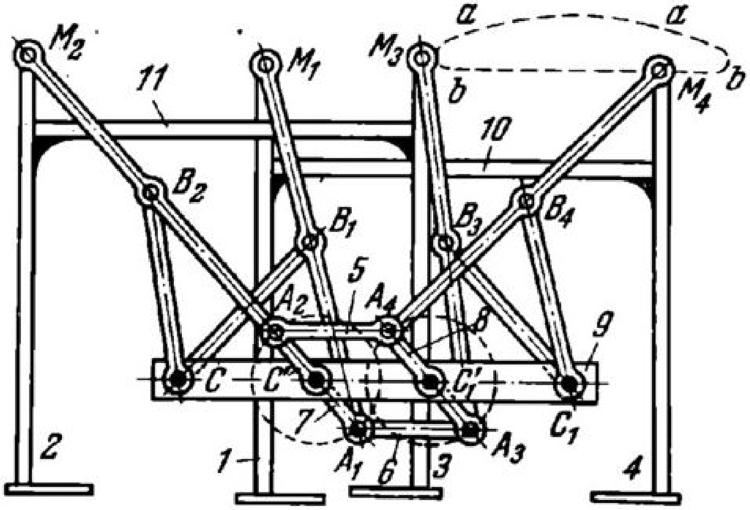


Figure 2.6: A schematic of Chebyshev's walking machine based on his approximate straight line linkage.

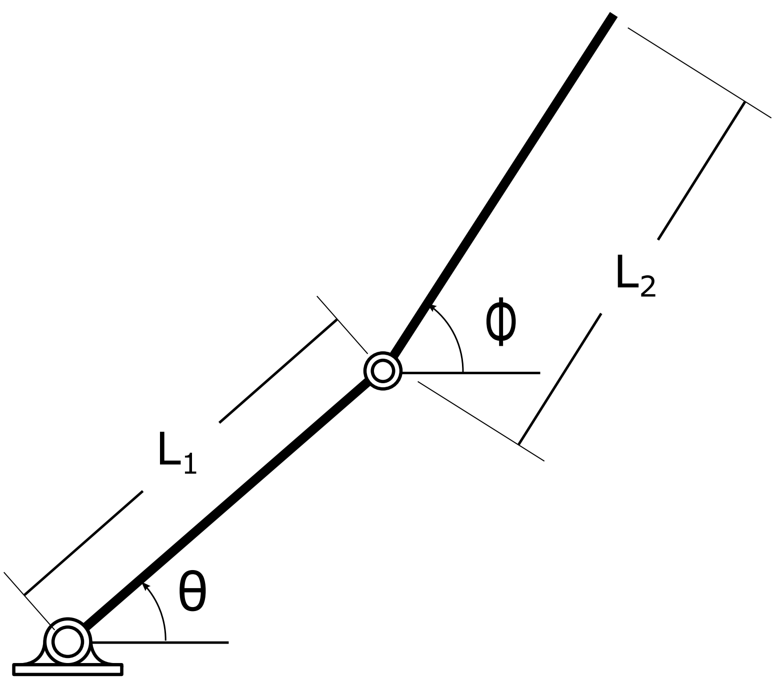


Figure 2.7: The RR robot.

2.3 Curve Drawing Mechanisms

In 1887 A. B. Kempe showed that an arbitrary algebraic equation for a plane curve can be used to define a linkage that draws the curve. To do this, he considered the curve to be drawn by an RR planar chain, where R denotes a revolute or hinged joint, Fig 2.7. The coordinates of the end-point of the RR chain are defined by the equations,

$$\mathbf{P} = \begin{Bmatrix} x(\theta,\phi) \\ y(\theta,\phi) \end{Bmatrix} = \begin{Bmatrix} L_1\cos\theta + L_2\cos\phi \\ L_1\sin\theta + L_2\sin\phi \end{Bmatrix}. \tag{2.1}$$

If the joint angles of this serial chain are coordinated correctly, then it can draw any curve within its workspace.

2.3.1 Kempe's computing linkages

In order to obtain the mechanism that coordinates these joint angles, Kempe considered curves defined by the polynomial equation [28],

$$f(x,y) = \sum b_{jk}x^j y^k = 0, \tag{2.2}$$

where the summation is over non-negative integers j and k with $j+k \leq n$, where n is the degree of the polynomial. Substitute Eq. 2.1 into this polynomial

$$f(x(\theta,\phi), y(\theta,\phi)) = \sum b_{jk}(L_1\cos\theta + L_2\cos\phi)^j(L_1\sin\theta + L_2\sin\phi)^k = 0. \tag{2.3}$$

and then expand using trigonometric identities to obtain an equation of the form,

$$f(\theta,\phi) = \sum_{i=1}^{n} A_i\cos(r_i\phi + s_i\theta + \beta_i) - K = 0, \tag{2.4}$$

where $\beta_i = 0$ or $\pi/2$, and the A_i and K are real constants.

Equation 2.4 is the x-projection of an n-bar serial chain formed from links of length A_i. The angles between each of the links of this chain are defined as linear combinations of the joint angles θ and ϕ of the RR chain. This equation coordinates the values of θ and ϕ so that this RR chain draws the desired curve, and can be written as

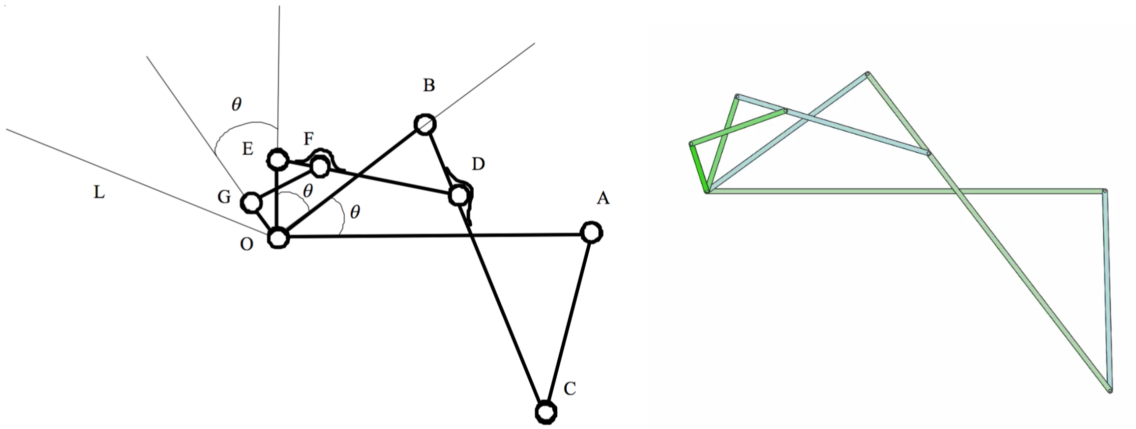
$$x' = \sum_{i=1}^{n} A_i \cos \alpha_i = K, \tag{2.5}$$

where

$$\alpha_i = r_i\phi + s_i\theta + \beta_i. \tag{2.6}$$

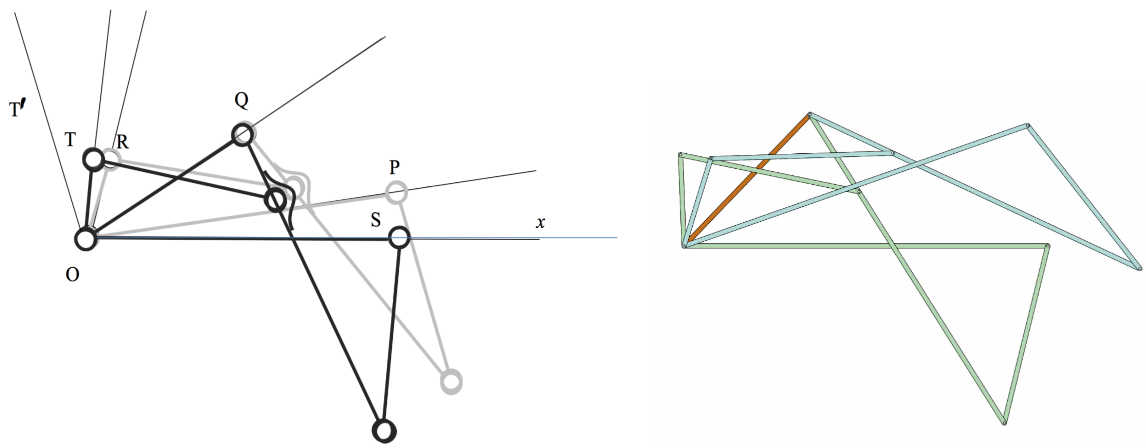
The primary tool of his approach is the introduction of computing mechanisms that yield the angles α_i for the n-bar coordinating serial chain as the required linear combinations defined in Eq 2.4. These are the multiplicator and additor linkages shown in Fig 2.8 and Fig 2.9. He also introduced a reversor that would perform subtraction.

The resulting curve drawing linkage consists of a set of multiplicator and additor linkages that performed each of the n joint angle calculations in Eq 2.4. These computing linkages are connected by a series of parallelogram linkages, that Kempe called Translators), to the angles of n joint angles of the coordinating serial chain and to the joint angles of the RR serial chain. Kempe states "It is hardly necessary to add, that this method would not be practically useful on account of the complexity of the linkwork employed, a necessary consequence of the perfect generality of the demonstration," [28].



(a) (b)

Figure 2.8: (a) Kempe's construction of Multiplicator, see [28] (b) Construction of SolidWorks model of Multiplicator.



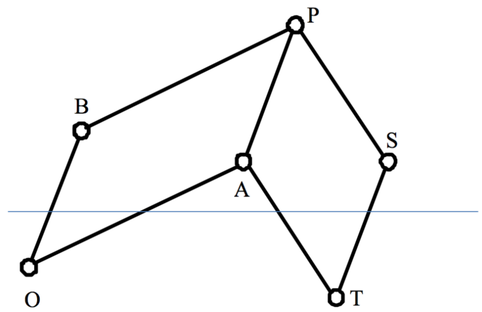
(a) (b)

Figure 2.9: (a) Kempe's construction of Additor, see [28] (b) Construction of SolidWorks model of Additor.

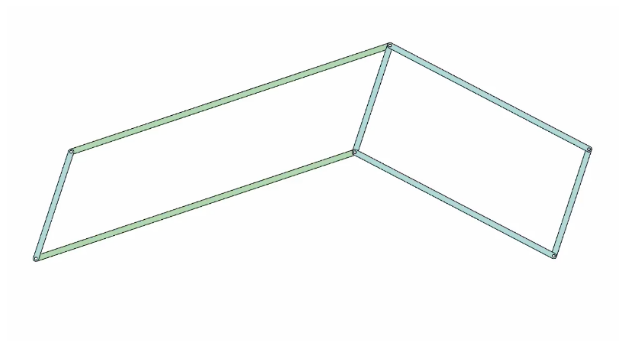
2.3.2 Alternative computing mechanisms

In this section, we present Kempe's method for the construction of a drawing mechanism in more detail, but rather than his additor, reversor, multiplicator and translator linkages, we use the simpler mechanisms formed by bevel gear differentials and cable drives, Figure 2.11.

These examples show that Kempe constructs a constraining linkage and intervening computing linkages in order to coordinate angles of an RR chain so that it draws the desired



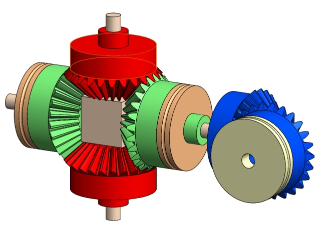
(a)



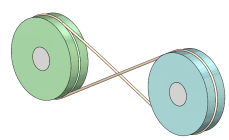
(b)

Figure 2.10: (a) Kempe's construction of Translator, see [28] (b) Construction of SolidWorks model of Translator.

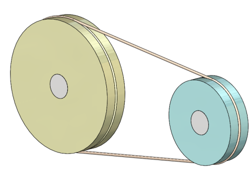
curve. In our approach, we use differentials and cable drives as the intervening computing linkages to coordinate the angles of the constraining linkage and RR chain.



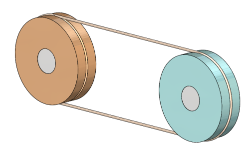
(a) Addition



(b) Negation



(c) Multiplication



(d) Translation

Figure 2.11: A bevel gear differential and configurations of cable drives perform the same functions as Kempe's additor, reversor, multiplicator and translator linkages, and simplify the resulting device.

Kempe recognized that (2.4) can be viewed as the x'-projection of a serial chain constructed from n links of length $A_i, i = 1, \ldots n$ with n joints at angles $\alpha_i = r_i\phi + s_i\theta + \beta_i, i = 1, \ldots, n$ relative to the x'-axis. We call this Kempe's constraining linkage, because the values of the angles α_i are mechanically calculated from θ and ϕ using differentials and cable drives, Figure 2.11, so that the RR chain draws the curve. The movement of the end-point of this serial chain along the line $x' = K$ can be viewed as the input to the system. Kempe achieved this constraint by using a Peaucilier linkage, which replace with a prismatic, or sliding, joint.

For each term in Equation (2.4), we can use the alternative computing elements to construct

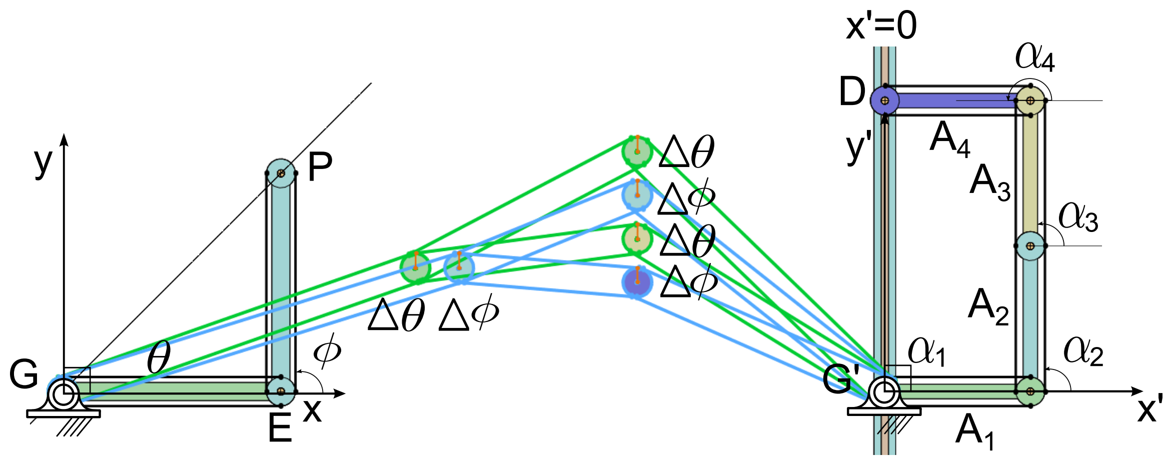


Figure 2.12: The linkage constructed bevel gears differential and configuration of cable drives as the function of Kempe's additor, reversor, multuplicator and translator to draw a line.

the linkage shown in Figure 2.12 as following steps

1. A_i is the link length of the constraint linkage, the number of links in the constraint linkage is determined by the number of terms in Equation 2.4,
2. The multiplication of $r_i\phi$ and $s_i\theta$ is achieved by gear trains with gear ratio r_i and s_i,
3. The addition of $r_i\phi$ and $s_i\theta$ is achieved by bevel gear differentials,
4. The phase angel β_i is the angel of Kempe's constraining linkage with the x-axis in the setup potion,
5. As we clarified in Equation (2.5), the cosine operation gives us the projection of each link in Kempe's constraining linkage on the x-axis, the summation of this projection is a constant number K,
6. We constraint the end-point of Kempe's constraint linkage move on the line $x' = K$,
7. The cable drive set up for the RR chain to draw a specified algebraic curve is shown in Figure 2.13.

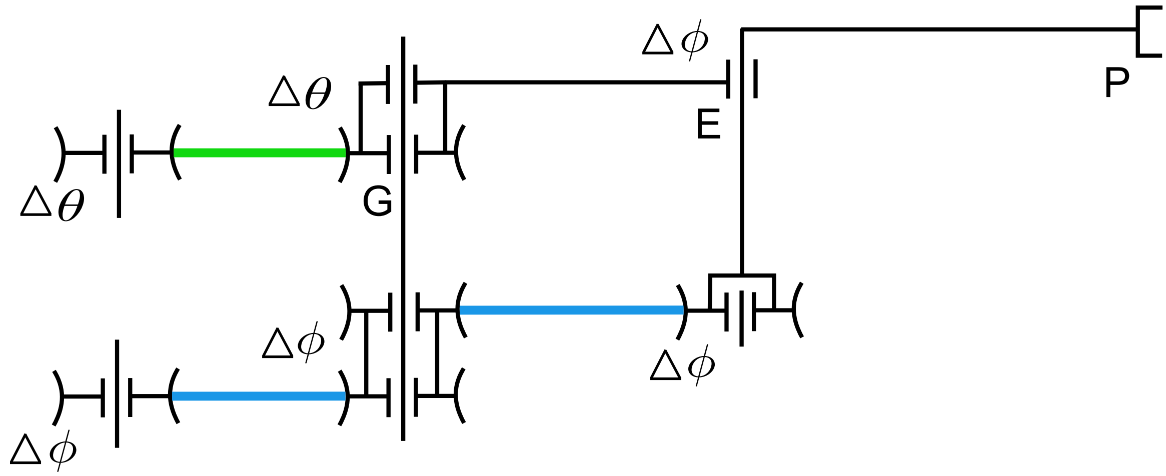


Figure 2.13: Cable drive set up on the RR chain to draw an algebraic curve. The green cable translates angle $\Delta\theta$ to link GE and the two blue cables translate angle $\Delta\phi$ to link EP.

2.3.3 Counting parts

The part counts of links, differentials and cable drives using our approach is as follows. There are two links in the RR linkage that draws the curve, and n links in Kempe's serial chain, corresponding to the n terms in (2.4), so $c_l = n + 2$.

Let $a_i, i = 1, \ldots, n$ be the number of addition operations in the i^{th} term, which is equal to 0, 1 or 2, therefore,

$$c_a = \sum_{i}^{n} a_i. \tag{2.7}$$

Similarly, let v_i denote the number of multiplications in the i^{th} term, which again will be 0, 1 or 2, that is,

$$c_m = \sum_{i}^{n} v_i. \tag{2.8}$$

Recall that we perform the actual multiplications $r_i\phi(r_i \geq 2)$ and $s_i\theta(s_i \geq 2)$ using the relative sizes of pulleys in cable drives.

Three cable drives are needed to position the RR chain that draws the curve. The i^{th} joint of Kempe's serial chain requires i cable drives to constrain it. Thus, the number of cables drives for Kempe's serial chain is,

$$c_t = n(n+1)/2 + 3. \tag{2.9}$$

The total number of parts for our drawing linkage for n terms is given by,

$$p = c_l + c_t + c_a + c_m = (n^2 + 3n + 10)/2 + \sum_i^n a_i + \sum_i^n v_i. \tag{2.10}$$

2.4 Fourier Series Approximation

While Kapovich and Millson proved that Kempe was correct that every algebraic curve has a mechanism that draws the linkage, our goal is a more practical mechanical system to draw the curve. In this section we introduce computing linkages that assist in the Fourier approximation to parameterized curves.

The existence of a local parameterization to an algebraic curve is provided by the implicit function theorem [24]. For the algebraic equation $f(x, y) = 0$ which defines y as a function of x implicitly, there exists an paramerization of the curve $y = g(x)$ if

$$\frac{\partial f}{\partial y} \neq 0, \tag{2.11}$$

in which case,

$$\frac{dg}{dx} = -\frac{\partial f}{\partial y}^{-1} \frac{\partial f}{\partial x}. \tag{2.12}$$

In what follows, we assume that an implicit curve has a parameterization that we can approximate with a finite Fourier series. Curves of this type are called trigonometric curves [65]. It is possible to generate a curve with finite Fourier series components in three different ways. The first method used Scotch Yoke mechanisms for each of the sine and cosine terms in the coordinate functions. The second method uses coupled serial chains to compute the coordinate functions. The third method calculates a single coupled serial chain that generates the Fourier coordinate functions.

2.4.1 Component Scotch Yoke Mechanisms

Given a finite Fourier series representation of the coordinate functions of a curve, we have

$$P = \begin{Bmatrix} x(\theta) \\ y(\theta) \end{Bmatrix} = \begin{Bmatrix} \sum_{k=0}^{m} a_k \cos k\theta + b_k \sin k\theta \\ \sum_{k=0}^{m} c_k \cos k\theta + d_k \sin k\theta \end{Bmatrix}, \tag{2.13}$$

where a_k, b_k, c_k and d_k, $k = 0, \dots, m$, are real coefficients and $\theta \in [0, 2\pi]$.

Consider the x component of (2.13) and rewrite the equation in the form,

$$x(\theta) = \sum_{k=0}^{m} L_k \cos(k\theta - \psi_k), \tag{2.14}$$

where

$$L_k = \sqrt{a_k^2 + b_k^2}, \quad \psi_k = \arctan \frac{b_k}{a_k}. \tag{2.15}$$

This equation defines $x(\theta)$ as the sum of m Scotch yoke mechanisms each with an input crank length of L_k and initial angle ψ_k. Figure 2.14 shows the construction of one unit of Scotch yoke mechanism. The system is driven such that the angle of the crank k is given by,

$$\phi_k = k\theta - \psi_k. \tag{2.16}$$

The output of the set of m Scotch yoke linkages are added by acting on a belt or cable drive to generate the $x(\theta)$ component of the curve P. The initial configuration of the system is defined by the phase angles ψ_k.

The y component of the curve P yields a similar relationship,

$$y(\theta) = \sum_{k=0}^{m} M_k \cos(k\theta - \eta_k), \tag{2.17}$$

where

$$M_k = \sqrt{c_k^2 + d_k^2}, \quad \eta_k = \arctan \frac{d_k}{c_k}. \tag{2.18}$$

This equation defines a set of Scotch yoke mechanisms that generate the y component of the curve P.

The Scotch yoke mechanisms for both components are driven by the same input θ, thus the mechanical system has one degree of freedom. Let the output of the two sets of Scotch yoke mechanisms drive the x and y components of a cursor to draw the desired curve. Figure 2.15 shows the system of a set of Scotch yoke mechanisms that can draw a Hypocycloid curve.

To construct a Scotch Yoke mechanical system to draw trigonometric curve, the steps are a following

1. We need two sets of Scotch Yoke mechanisms to approximate the x component and y component of the curve respectively,

2. For the set operating in the x direction, the number of Scotch Yoke mechanism is m,

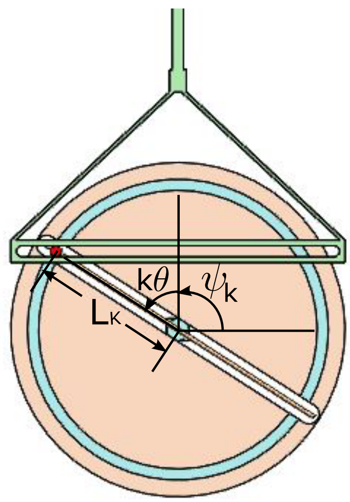


Figure 2.14: A Scotch yoke mechanism that transforms the rotation of a crank into a cosine curve.

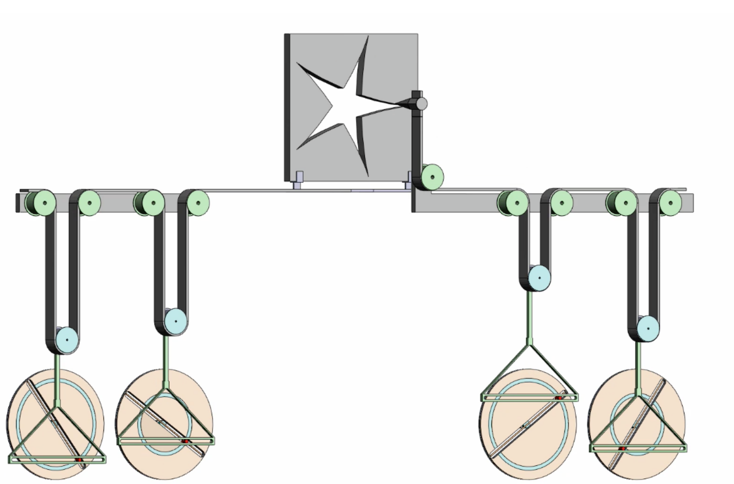


Figure 2.15: A set of Scotch yoke mechanisms drive the x-component and a another set drives the y-component of a cursor to draw a plane curve.

the radius of each Scotch yoke mechanism is L_k, the speed frequency is $k\theta$ and the initial setup phase angle of each element is given by ψ_k,

3. For the set operating in the y direction, the number of Scotch Yoke mechanism is m, the radius of each Scotch yoke mechanism is M_k, the speed frequency is $k\theta$ and the initial setup phase angle of each element is given by η_k,

4. Finally, one belt add x Scotch yoke mechanisms outputs together and drive the board horizontally; one belt add y Scotch yoke mechanisms outputs together and drive the end-effector to draw the specified trigonometric curve.

2.4.2 Component Coupled Serial Chains

The Scotch yoke mechanisms of the previous section can be replaced by the projection of serial chains that have their joints driven by belt or cable drives coupled to a single input joint angle.

Let the x component equation (2.14) be the x projection of the end-point of a serial chain formed from m links of length L_k that are each positioned at an angle α_k relative to the x axis, given by

$$\alpha_k = k\theta - \psi_k. \tag{2.19}$$

These angles are driven by a single input angle θ through a cable drive that has a decreasing pulley diameters at each link in order to increase the rotation angle required by $k\theta$. The input angles of the k_{th} link and $(k-1)_{th}$ link are related by the ratio of the pulley diameters $D_k/D_{k-1} = (k-1)/k$.

In the same way, the y component of P can be generated by a serial chain constructed from

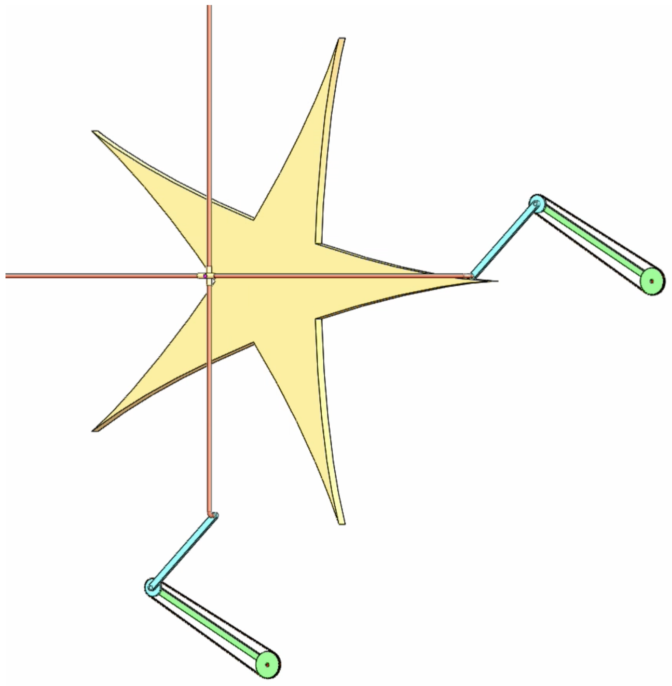


Figure 2.16: A coupled serial chain drives the x-component and a separate coupled serial chain drives the y-component of a cursor to draw a plane curve.

m links of length M_k that are each positioned at an angle β_k relative to the y axis, given by

$$\beta_k = k\theta - \eta_k. \tag{2.20}$$

The coupled serial chains for the x and y components have the same inputs, thus this mechanical system has degree of freedom one. The two serial chains can be coupled so they move a cursor along the x and y coordinates to draw the curve P. The system of coupled serial chains that draws a Hypocycloid is shown in Figure 2.16.

To construct a component coupled serial chain mechanical system to draw trigonometric curve, the steps are a following

1. We need two sets of coupled serial chain mechanisms to approximate the x component and y component of the curve respectively,

2. For the set operating in the x direction, the number of links is m, the length of each

link is L_k, the speed frequency is $k\theta$ and the initial setup phase angle of each link is given by ψ_k,

3. For the set operating in the y direction, the number of links is m, the length of each link is M_k, the speed frequency is $k\theta$ and the initial setup phase angle of each link is given by η_k,

4. Finally, the end-points of the x coupled serial chain and the y coupled serial chain are connected by horizontal and vertical sliders that intersect at the tracing point, and both chains are driven by the same input rotation.

2.4.3 Single Coupled Serial Chain

In this section, we show how to obtain a single coupled serial chain that draws trigonometric curves (2.21). Starting with the equation of the curve with component finite Fourier series, we have

$$P = \begin{Bmatrix} x(\theta) \\ y(\theta) \end{Bmatrix} = \begin{Bmatrix} \sum_{k=0}^{m} a_k \cos k\theta + b_k \sin k\theta \\ \sum_{k=0}^{m} c_k \cos k\theta + d_k \sin k\theta \end{Bmatrix}, \tag{2.21}$$

and introduce the complex form of curve 2.21 as,

$$P(\theta) = x(\theta) + iy(\theta) = \sum_{k=0}^{m}(a_k \cos k\theta + b_k \sin k\theta) + \sum_{k=0}^{m} i(c_k \cos k\theta + d_k \sin k\theta). \tag{2.22}$$

Introduce the identities,

$$\cos\phi = \frac{1}{2}(e^{i\phi} + e^{-i\phi}), \sin\phi = \frac{1}{2i}(e^{i\phi} - e^{-i\phi}), \tag{2.23}$$

to obtain

$$P(\theta) = \frac{1}{2}\sum_{k=0}^{m}((a_k + d_k) + i(c_k - b_k))e^{ik\theta} + ((a_k - d_k) + i(c_k + b_k))e^{-ik\theta}. \tag{2.24}$$

Introduce the magnitudes L_k and M_k given by

$$L_k = \frac{1}{2}\sqrt{(a_k + d_k)^2 + (c_k - b_k)^2}, \quad M_k = \frac{1}{2}\sqrt{(a_k - d_k)^2 + (c_k + b_k)^2}, \tag{2.25}$$

and the associated phase angles

$$\psi_k = \arctan\frac{c_k - b_k}{a_k + d_k}, \quad \eta_k = \arctan\frac{c_k + b_k}{a_k - d_k}. \tag{2.26}$$

so (2.24) becomes,

$$P(\theta) = \frac{1}{2}\sum_{k=0}^{m} L_k e^{i(k\theta - \psi_k)} + M_k e^{-i(k\theta + \eta_k)}. \tag{2.27}$$

The components of the coupled serial chain that draws the curve can now be obtained as

$$P(\theta) = \begin{Bmatrix} \sum_{k=0}^{m} L_k \cos(k\theta - \psi_k) + M_k \cos(-k\theta - \eta_k) \\ \sum_{k=0}^{m} L_k \sin(k\theta - \psi_k) + M_k \sin(-k\theta - \eta_k) \end{Bmatrix}. \tag{2.28}$$

The links L_k have $k\theta$ as input and therefore rotate counter-clockwise and the links M_k have $-k\theta$ as input and therefor rotate clock-wise. The phase angles ψ_k and η_k define the initial configuration of the system.

Figure 2.17 shows a single coupled two link serial chain that draws a Hypocycloid curve.

Steps to construct a single coupled serial chain mechanical system that draws a trigonometric curve:

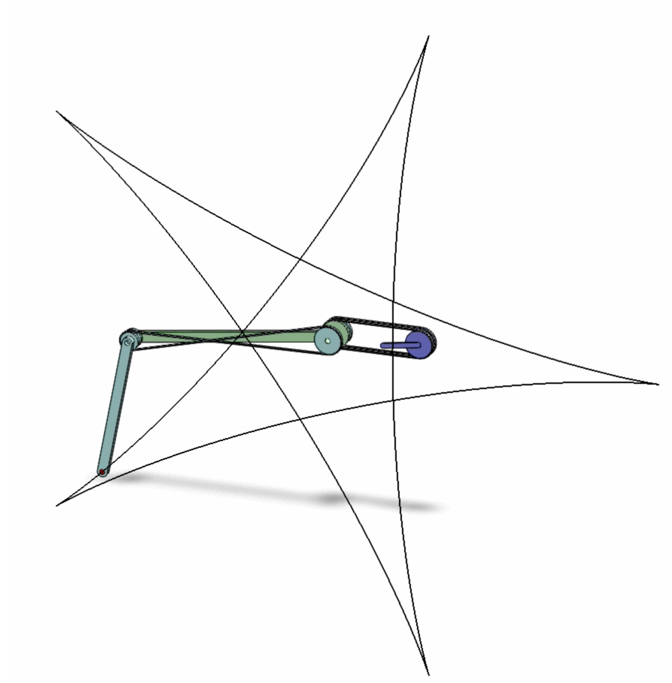


Figure 2.17: The end-point of a single coupled serial chain draws a plane curve.

1. The single coupled serial chain contains m links rotating counter-clockwise and m links rotating clockwise, each counter-clockwise rotating link followed by a clockwise rotating link,
2. For the links rotating counter-clockwise, each link length is L_k, the speed frequency is $k\theta$ and the initial setup phase angle of each link is given by ψ_k,
3. For the links rotating clockwise, each link length is M_k, the speed frequency is $k\theta$ and the initial setup phase angle of each link is given by η_k,
4. Finally, the end-point of the single coupled serial chain can draw a specified trigonometric curve.

2.5 Drawing Cubic Bezier Curves

In this section, we determine a cubic trigonometric Bezier curve that approximates a given cubic Bezier curve. A cubic Bezier curve is the parameterized curve $r(t)$ defined as the weighted sum of four control points P_0, P_1, P_2 and P_3, given by,

$$r(t) = (1-t)^3 P_0 + 3(1-t)^2 t P_1 + 3(1-t)t^2 P_2 + t^3 P_3, \quad 0 \le t \le 1. \tag{2.29}$$

A cubic trigonometric Bezier curve has weighting functions that are cubic polynomials of sine and cosine functions [19]. For the same four control points, it is the parameterized curve $s(t, \lambda)$, given by,

$$\begin{aligned} s(t,\lambda) &= (1-\sin\tfrac{\pi t}{2})^2(1-\lambda\sin\tfrac{\pi t}{2})P_0 + \sin\tfrac{\pi t}{2}(1-\sin\tfrac{\pi t}{2})(2+\lambda(1-\sin\tfrac{\pi t}{2}))P_1 \\ &+ \cos\tfrac{\pi t}{2}(1-\cos\tfrac{\pi t}{2})(2+\lambda(1-\cos\tfrac{\pi t}{2}))P_2 + (1-\cos\tfrac{\pi t}{2})^2(1-\lambda\cos\tfrac{\pi t}{2})P_3, \\ &\qquad 0 \le t \le 1. \end{aligned} \tag{2.30}$$

Once each of the cubic Bezier curves is replaced by cubic trigonometric Bezier curve, we can use the results of Liu and McCarthy [40] to design the associated coupled serial chain. First, we must convert the cubic trigonometric Bezier curve $s(t, \lambda)$ into the standard form of the trigonometric plane curve, $p(\theta)$, given by

$$p(\theta) = \sum_{k=0}^{n} a_k \cos k\theta + \sum_{k=0}^{n} b_k \sin k\theta, \tag{2.31}$$

where $a_k = (a_{xk}, a_{yk})$ and $b_k = (b_{xk}, b_{yk})$, $k = 0, \ldots, n$, are real coefficients and $\theta \in [0, 2\pi]$.

The conversion consists of a reparameterization defined by $t = 2\theta/\pi$ and an expansion of the powers of sine and cosine to obtain,

$$s(t,\lambda) = a_0 + a_1\cos\theta + a_2\cos 2\theta + a_3\cos 3\theta \\ +b_0 + b_1\sin\theta + b_2\sin 2\theta + b_3\sin 3\theta, \tag{2.32}$$

where the coefficient vectors a_k and b_k, $i = 0, 1, 2, 3$, are given by,

$$\begin{aligned}
a_0 &= (\frac{3}{2}+\lambda)P_0 - (1+\lambda)P_1 - (1+\lambda)P_2 + (\frac{3}{2}+\lambda)P_3,\\
a_1 &= (2+\frac{7}{4}\lambda)P_2 - (2+\frac{7}{4}\lambda)P_3,\\
a_2 &= (-\frac{1}{2}-\lambda)P_0 + (1+\lambda)P_1 - (1+\lambda)P_2 + (\frac{1}{2}+\lambda)P_3,\\
a_3 &= \frac{1}{4}\lambda P_2 - \frac{1}{4}\lambda P_3, \quad \text{and}\\
b_0 &= 0,\\
b_1 &= (-2-\frac{7}{4}\lambda)P_0 + (2+\frac{7}{4}\lambda)P_1,\\
b_2 &= 0,\\
b_3 &= \frac{1}{4}\lambda P_0 - \frac{1}{4}\lambda P_1.
\end{aligned} \tag{2.33}$$

Liu and McCarthy [40] show how for each value k the coefficients a_k and b_k are used to compute the magnitudes, L_k and M_k, and phase angles ψ_k and η_k of vectors defining the links of the coupled serial chain,

$$\begin{aligned}
L_k &= \tfrac{1}{2}\sqrt{(a_{xk}+b_{yk})^2 + (a_{yk}-b_{xk})^2},\\
M_k &= \tfrac{1}{2}\sqrt{(a_{xk}-b_{yk})^2 + (a_{yk}+b_{xk})^2}, \quad k = 0, 1, 2, 3,
\end{aligned} \tag{2.34}$$

and

$$\psi_k = \arctan\frac{a_{yk}-b_{xk}}{a_{xk}+b_{yk}}, \quad \eta_k = \arctan\frac{a_{yk}+b_{xk}}{a_{xk}-b_{yk}}, \quad k = 0, 1, 2, 3. \tag{2.35}$$

The result is the equation of the cubic trigonometric curve takes the form

$$s(\theta, \lambda) = \sum_{k=0}^{3} L_k u_k + \sum_{k=0}^{3} M_k v_k, \tag{2.36}$$

where u_k and v_k are unit vectors given by

$$u_k = \begin{bmatrix} \cos(k\theta - \psi_k) \\ \sin(k\theta - \psi_k) \end{bmatrix}, \; v_k = \begin{bmatrix} \cos(-k\theta - \eta_k) \\ \sin(-k\theta - \eta_k) \end{bmatrix}, \quad k = 0, 1, 2, 3. \tag{2.37}$$

The ground pivot, G, for the serial chain is given by

$$G = s(0, \lambda) = L_0 u_0 + M_0 v_0. \tag{2.38}$$

This shows that the cubic trigonometric Bezier curve is drawn by a six link serial chain with three links rotating counter-clockwise and three links rotating clockwise.

2.6 Drawing Spherical Curves

In this section, we introduce the steps constructing the needed format of rotation matrix that was used to formulate the equation of the spherical mechanism.

The configuration of the single coupled serial chain to draw the spherical curve is similar to the planar one. For planar trigonometric curve, we have

$$P(\theta) = \frac{1}{2} \sum_{k=0}^{m} ((a_k + d_k) + i(c_k - b_k))e^{ik\theta} + ((a_k - d_k) + i(c_k + b_k))e^{-ik\theta}. \tag{2.39}$$

The magnitudes L_k and M_k given by

$$L_k = \frac{1}{2}\sqrt{(a_k + d_k)^2 + (c_k - b_k)^2}, \quad M_k = \frac{1}{2}\sqrt{(a_k - d_k)^2 + (c_k + b_k)^2}, \tag{2.40}$$

and the associated phase angles

$$\psi_k = \arctan\frac{c_k - b_k}{a_k + d_k}, \quad \eta_k = \arctan\frac{c_k + b_k}{a_k - d_k}. \tag{2.41}$$

The components of the coupled serial chain that draws the curve can now be obtained as

$$P(\theta) = \begin{Bmatrix} \sum_{k=0}^{m} L_k \cos(k\theta - \psi_k) + M_k \cos(-k\theta - \eta_k) \\ \sum_{k=0}^{m} L_k \sin(k\theta - \psi_k) + M_k \sin(-k\theta - \eta_k) \end{Bmatrix}. \tag{2.42}$$

The equation of this curve in space can be obtained as

$$\mathbf{P}(\theta) = \begin{Bmatrix} \sum_{k=0}^{m} L_k \cos(k\theta - \psi_k) + M_k \cos(-k\theta - \eta_k) \\ \sum_{k=0}^{m} L_k \sin(k\theta - \psi_k) + M_k \sin(-k\theta - \eta_k) \\ r \end{Bmatrix}. \tag{2.43}$$

where r is the radius of the sphere.

Now project the curve onto the sphere by normalizing the vector curve $\mathbf{P}(\theta)$,

$$\mathbf{S}_P(\theta) = r\mathbf{P}(\theta)/|\mathbf{P}|. \tag{2.44}$$

We now introduce a procedure to define a spherical coupled serial chain obtained from the dimensions of the planar coupled serial chain that approximates the projected spherical

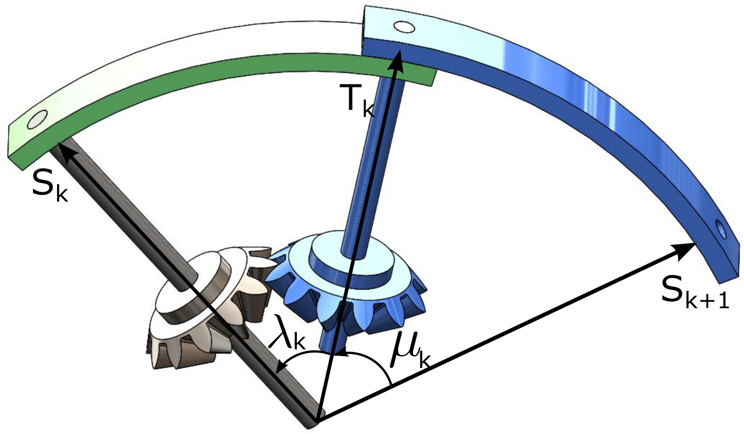


Figure 2.18: Spherical chain.

curve. For each of the links L_k and M_k, $k = 1, \ldots, m$, we compute the projected angles,

$$\lambda_k = \arctan L_k/r, \quad \mu_k = \arctan M_k/r. \tag{2.45}$$

If R is the length of the planar serial chain, then we select $r > 2R/\pi$, so the spherical coupled serial chain and the curve it generates lies in the upper hemisphere.

The coordinate vectors that define the axes of the spherical chain are denoted $\mathbf{S}_k$ and $\mathbf{T}_k$, $k = 1, \ldots, m$. The end-point of the spherical chain is $\mathbf{S}_{m+1}$. See Figure 2.18. We select the first axis of the serial chain to be $\mathbf{S}_1$, that is

$$\mathbf{S}_1 = \mathbf{k} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}. \tag{2.46}$$

In order to define $\mathbf{T}_1$, we rotate the x-axis, which we denote as $\mathbf{V}_1 = \mathbf{i}$, around $\mathbf{S}_1$ by the phase angle ψ_1 to define the vector $\mathbf{W}_1$, and then rotate about $\mathbf{W}_1$ by the angle λ_1, that is

$$\mathbf{W}_1 = [R(\psi_1, \mathbf{S}_1)]\mathbf{V}_1, \quad \text{and} \quad \mathbf{T}_1 = [R(\lambda_1, \mathbf{W}_1)]\mathbf{S}_1. \tag{2.47}$$

The axis $\mathbf{S}_2$ is now obtained by rotating $\mathbf{W}_1$ around $\mathbf{T}_1$ by the phase η_1 to obtain $\mathbf{V}_2$, that is

$$\mathbf{V}_2 = [R(\eta_1, \mathbf{T}_1)]\mathbf{W}_1, \quad \text{and} \quad \mathbf{S}_2 = [R(\mu_1, \mathbf{V}_2)]\mathbf{T}_1. \tag{2.48}$$

This process continues so that for given the axis $\mathbf{S}_i$ with phase angle ψ_i, we have

$$\mathbf{W}_i = [R(\psi_i, \mathbf{S}_i)]\mathbf{V}_i, \quad \text{and} \quad \mathbf{T}_i = [R(\lambda_i, \mathbf{W}_i)]\mathbf{S}_i. \tag{2.49}$$

Similarly, for $\mathbf{T}_i$ with phase angle η_i, we have

$$\mathbf{V}_{i+1} = [R(\eta_i, \mathbf{T}_i)]\mathbf{W}_i, \quad \text{and} \quad \mathbf{S}_{i+1} = [R(\mu_i, \mathbf{V}_{i+1})]\mathbf{T}_i. \tag{2.50}$$

Table 2.1: Configurations for projected curve $\mathbf{S}_P(\theta)$.

k	$\mathbf{S}_k$	λ_k	ψ_k	$\mathbf{T}_k$	μ_k	η_k
1	$\mathbf{S}_1$	λ_1	ψ_1	$\mathbf{T}_1$	μ_1	η_1
2	$\mathbf{S}_2$	λ_2	ψ_2	$\mathbf{T}_2$	μ_2	η_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	$\mathbf{S}_m$	λ_m	ψ_m	$\mathbf{T}_m$	μ_m	η_m

We now obtain the equation for the mechanical generated spherical curve that approximates the spherical projection as,

$$\begin{aligned}\mathbf{S}_P =& [R(\theta, \mathbf{S}_1)][R(-\theta, \mathbf{T}_1][R(2\theta, \mathbf{S}_2)][R(-2\theta, \mathbf{T}_2)] \ldots [R(k\theta, \mathbf{S}_k)][R(-k\theta, \mathbf{T}_k)] \\ & \ldots [R(m\theta, \mathbf{S}_m)][R(-m\theta, \mathbf{T}_m)]\mathbf{S}_{m+1},\end{aligned} \tag{2.51}$$

where k defines the speed ratio for the rotating links. If the linkage angle λ_k or μ_k equals zero then the matrix associated with this link is not included in this equation.

2.7 Summary

In this chapter, we presented the traditional mechanism that can draw straight lines. We then showed the kinematic of serial chain mechanisms. The algorithm of Kempe's Universality Theorem to draw an arbitrary algebraic curve was presented. We demonstrated our alternative computing mechanism for Kempe's theorem. Finally, we illustrated three methods to apply Fourier series to approximate an algebraic curve.

Chapter 3

Mechanisms to Draw Algebraic Curves

3.1 Introduction

In this chapter, we present our design methodology for a mechanical system that draws an algebraic plane curve and consider its complexity. Then we foucus on mechanical system to draw trigonometric plane curve. In particular we show that we can obtain the boundary of a image and computer the Fourier transform to draw the curve. In what follows, we provide a detailed construction of the linkage systems that generate each of the two lines of Saxena's example, shown in Figure 1.1, so the two versions of the linkage can be compared. We then construct the linkage that draws the elliptic cubic of Kobel's example as shown in Figure 1.4.

3.2 Saxena's Example

In this section, we use Saxena's [57] example to illustrate our version of Kempe's procedure for constructing curve-drawing linkages. The linkage shown in Figure 1.1 draws the quadratic curve generated by the product of two lines given by,

$$f(x, y) = (x - y)(x + y + \sqrt{2}/2) = 0. \tag{3.1}$$

In what follows, we design a separate linkage for each of these two lines.

3.2.1 Line: $x - y = 0$

Here we show the process of designing one degree of freedom linkage to draw the line defined by

$$g(x, y) = x - y = 0. \tag{3.2}$$

Select the lengths of the RR chain to be $L_1 = L_2 = 1$. In this case, the trajectory of the end-point P is given by,

$$P = \begin{Bmatrix} x(\theta, \phi) \\ y(\theta, \phi) \end{Bmatrix} = \begin{Bmatrix} \cos\theta + \cos\phi \\ \sin\theta + \sin\phi \end{Bmatrix}. \tag{3.3}$$

Substitute (3.3) into (3.2) to obtain,

$$g(\theta, \phi) = \cos\theta + \cos\phi + \cos(\theta + \pi/2) + \cos(\phi + \pi/2) = 0. \tag{3.4}$$

Kempe interpreted this equation as the x' projection of a simple closed chain consisting of four revolute joints with the end-link constrained by a prismatic joint that slides along the line $x' = K$, that is

$$x' = A_1 \cos\alpha_1 + A_2 \cos\alpha_2 + A_3 \cos\alpha_3 + A_4 \cos\alpha_4 = K, \tag{3.5}$$

where A_i are the lengths of the individual links, α_i are the angles of these links relative to the x'-axis. Each of the angles α_i is given by

$$\alpha_i = \Delta\alpha_i + \alpha_{i0}, \tag{3.6}$$

where α_{i0} define the initial configuration of the constraining linkage.

In order to determine the angles α_{i0}, select a point P on the line $g(x, y)$ that is to be the initial configuration of the system. Compute the initial values θ_0 and ϕ_0 for the joint angles of the RR chain, so that

$$\theta = \Delta\theta + \theta_0, \quad \phi = \Delta\phi + \phi_0. \tag{3.7}$$

Then the constraint equation (3.5) takes the form,

$$g(\theta, \phi) = \cos(\Delta\theta + \theta_0) + \cos(\Delta\phi + \phi_0) + \cos(\Delta\theta + \theta_0 + \pi/2) + \cos(\Delta\phi + \phi_0 + \pi/2) = 0. \tag{3.8}$$

For this example, we choose,

$$P = \begin{Bmatrix} \cos\theta_0 + \cos\phi_0 \\ \sin\theta_0 + \sin\phi_0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \tag{3.9}$$

and solve to obtain two configurations,

$$\begin{Bmatrix} \theta_0 \\ \phi_0 \end{Bmatrix} = \begin{Bmatrix} \pi/2 \\ 0 \end{Bmatrix}, \text{or} \begin{Bmatrix} \theta_0 \\ \phi_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \pi/2 \end{Bmatrix}. \quad (3.10)$$

These solutions define the "elbow-up" and "elbow-down" configurations of the RR chain.

Select the elbow down solution and set $\Delta\theta = \Delta\phi = 0$, to obtain the initial angles of the constraining linkage,

$$\alpha_{10} = \theta_0, \alpha_{20} = \phi_0, \alpha_{30} = \theta_0 + \frac{\pi}{2}, \text{and } \alpha_{40} = \phi_0 + \frac{\pi}{2}. \quad (3.11)$$

Thus, the dimensions of Kempe's constraining linkage are obtained by equating (3.5) and (3.8). The results are listed in Table 3.1.

Table 3.1: Dimensions of Kempe's constraining linkage for the line $x - y = 0$

Link, i	Link Length, A_i	Angle, $\Delta\alpha_i$	Initial config., α_{i0}
1	1	$\Delta\theta$	0
2	1	$\Delta\phi$	$\pi/2$
3	1	$\Delta\theta$	$\pi/2$
4	1	$\Delta\phi$	π

Cable drives are used to connect the angles $\Delta\theta$ and $\Delta\phi$ to drive the joint angles θ and ϕ of the RR chain, see Figure 3.1. The cables connecting $\Delta\alpha_i$ to the joint angles α_i of the constraining linkage are routed in the same way.

The movement of the end-link of the constraining linkage along the line $x' = 0$ generates the curve, see Figure 3.2. A construction of a linkage that draws this straight line using Kempe's approach can be found in [31].

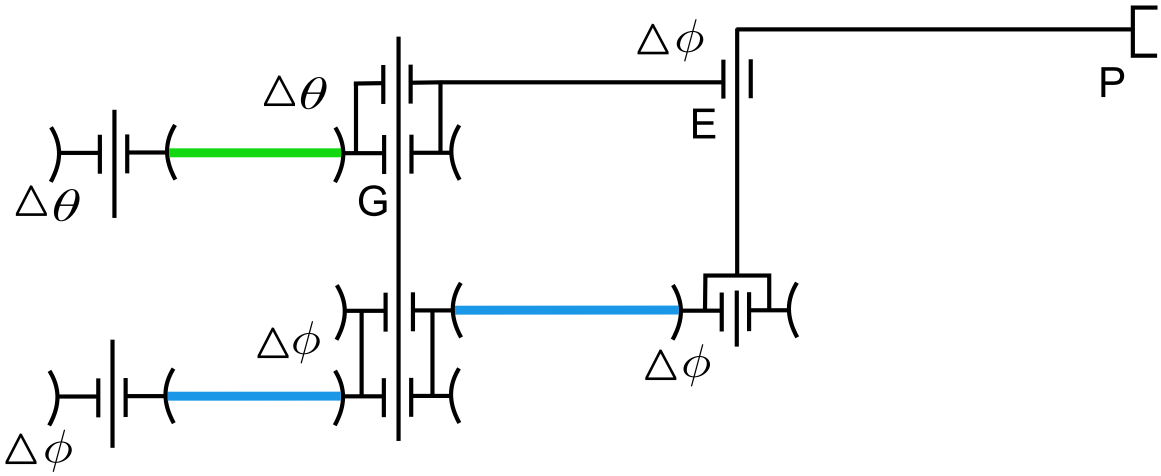


Figure 3.1: The green cable translates angle $\Delta\theta$ to link GE and the two blue cables translate angle $\Delta\phi$ to link EP.

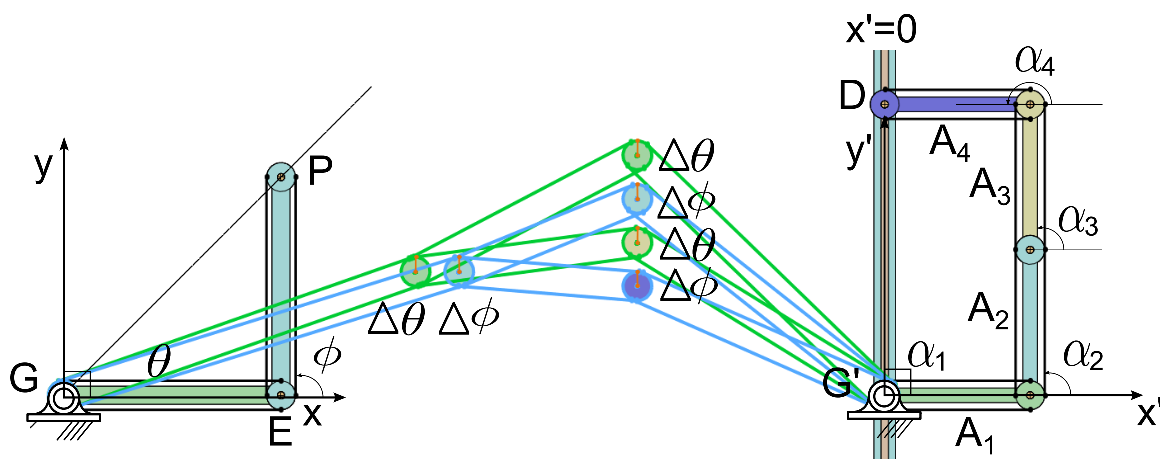


Figure 3.2: The end-point of the linkage $A_1A_2A_3A_4$ slides along the vertical guide to impose constraints, through cable drives, on the angles of the RR chain so that the point P follows the line $x - y = 0$.

3.2.2 Line: $x + y + \sqrt{2}/2 = 0$

In order to define a one degree of freedom drawing linkage for the line,

$$h(x, y) = x + y + \sqrt{2}/2 = 0, \tag{3.12}$$

substitute (3.3) into this equation to obtain the constraint equation,

$$h(\theta,\phi) = \cos\theta + \cos\phi + \cos(\theta - \pi/2) + \cos(\phi - \pi/2) = -\sqrt{2}/2. \tag{3.13}$$

This defines the constraining linkage with dimensions listed in Table 3.2 and illustrated in Figure 3.3.

Table 3.2: Dimensions of Kempe's constraining linkage for the line $x + y + \sqrt{2}/2 = 0$

Link, i	Link length, A_i	Link angle, $\Delta\alpha_i$	Initial config., α_i^0
1	1	$\Delta\theta$	$-5\pi/12$
2	1	$\Delta\phi$	$11\pi/12$
3	1	$\Delta\theta$	$-11\pi/12$
4	1	$\Delta\phi$	$5\pi/12$

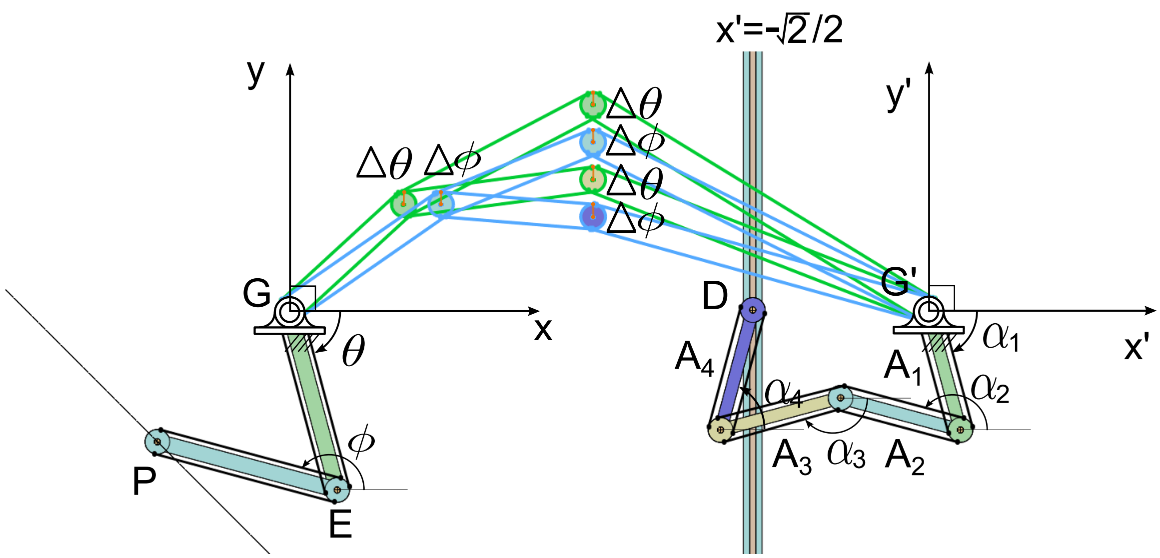


Figure 3.3: This mechanical system uses Kempe's theory to design a mechanical system to draw a straight line. Cable drives is used to provide the transmission to constrain the RR chain to draw this curve.

An initial configuration of the constraining linkage is obtained by setting (3.3) equal to the point $P = (-\sqrt{2}/4, -\sqrt{2}/4)$ on the line $h(x,y)$ to determine the angles, θ_0 and ϕ_0. Solve these equations to obtain values for angles α_{i0} for Kempe's constraining linkage listed in Table 3.2.

The differences between the drawing linkages for the two different lines in Saxena's example consists of the calculation of the angles α_i from θ and ϕ, the value $x' = K$, and the initial configuration.

3.3 Cubic Curve

In this section, we introduce Kempe's method to design planar linkages to draw an algebraic curve. Then we modify his approach to simplify the resulting design.

Let $f(x, y) = 0$ be an algebraic curve. Kempe introduced planar serial chain formed from two revolute joints with link lengths L_1 and L_2 to draw this curve. Thus, the goal is to coordinate the angles θ and ϕ fo this RR chain, so that x and y are given by,

$$P = \begin{Bmatrix} x(\theta, \phi) \\ y(\theta, \phi) \end{Bmatrix} = \begin{Bmatrix} L_1 \cos\theta + L_2 \cos\phi \\ L_1 \sin\theta + L_2 \sin\phi \end{Bmatrix}, \tag{3.14}$$

such that

$$f(x(\theta, \phi), y(\theta, \phi)) = 0. \tag{3.15}$$

Kempe shows that this equation can always be reduced to the form,

$$f(\theta, \phi) = \Sigma_i^n A_i \cos(r_i\phi + s_i\theta + \alpha) - C = 0, \tag{3.16}$$

where $\alpha = 0$ or $\pi/2$, where the A_i and C are constants.

Rather than follow Kempe and introduce his multiplicator, additor, and translator linkages, we use gears, differentials and pulleys to perform these operations. For each r_i and s_i we perform the multiplication using a set of meshing gears, which means for n terms there are at most $g = 2n$ gear pairs. The addition of the terms $r_i\phi + s_i\theta$ are each performed by a gear differential, thus for n terms, we have at most $d = n$ differentials. Finally, we assemble Kempe's serial chain consisting of bars of lengths A_i. We constrain this serial chain to move along the line $x = C$ by a prismatic joint.

In order to obtain the constraint on θ and ϕ to draw the curve f, we connect the gears, differentials and joints of Kempe's serial chain using belts and pulleys. Each pair of gears requires one belt, differential requires two belts, and the n joints of the serial chain requires $n(n+1)/2$ belts. Finally, three belts are required to drive the RR chain. Thus, the number of belts can be estimated to be,

$$b = g + 2d + n(n+1)/2 + 3. \tag{3.17}$$

In order to demonstrate this procedure, we obtain the mechanism that draws the cubic curve,

$$f(x, y) = x^3 - y - 1 = 0. \tag{3.18}$$

Let the lengths of the links of the RR chain that is to draw this curve be $L_1 = L_2 = 1$ and substitute the resulting $x(\theta, \phi)$ and $y(\theta, \phi)$ into to $f(x, y)$ to obtain,

$$f(\theta, \phi) = \cos^3\theta + \cos^3\phi + 3\cos^2\theta\cos\phi + 3\cos^2\phi\cos\theta - \sin\theta - \sin\phi - 1 = 0. \tag{3.19}$$

The powers of cosine are reduced to first degree using the identities,

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}, \quad \cos^3\theta = \frac{3\cos\theta + \cos(3\theta)}{4}. \tag{3.20}$$

Similarly, the trigonometric sum and difference identities can be used to obtain

$$\begin{aligned} f(\theta,\phi) = \frac{9}{4}\cos\theta + \frac{9}{4}\cos\phi + \frac{1}{4}\cos 3\theta + \frac{1}{4}\cos 3\phi + \frac{3}{4}\cos(2\theta-\phi) + \frac{3}{4}\cos(2\theta+\phi) \\ +\frac{3}{4}\cos(2\phi-\theta) + \frac{3}{4}\cos(2\phi+\theta) + \cos(\frac{\pi}{2}+\theta) + \cos(\frac{\pi}{2}+\phi) = 1, \end{aligned} \tag{3.21}$$

which has $n = 10$ terms.

In order to determine the initial configuration of the system, set $P = (1, 1)$ and solve the equations,

$$P = \begin{Bmatrix} \cos\theta_0 + \cos\phi_0 \\ \sin\theta_0 + \sin\phi_0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}. \tag{3.22}$$

Select the elbow down solution to obtain

$$\theta_0 = -\pi/3, \quad \phi_0 = \pi/3. \tag{3.23}$$

Substitute these initial angles into the constraint equation (3.21) to obtain,

$$\begin{aligned} f(\theta,\phi) =& 2.25\cos(\Delta\theta - \pi/3) + 2.25\cos(\Delta\phi + \pi/3) + 0.25\cos(3\Delta\theta - \pi) + 0.25\cos(3\Delta\phi + \pi) \\ &+ 0.75\cos(2\Delta\theta - \Delta\phi - \pi) + 0.75\cos(2\Delta\theta + \Delta\phi - \pi/3) + 0.75\cos(2\Delta\phi - \Delta\theta + \pi) \\ &+ 0.75\cos(2\Delta\phi + \Delta\theta + \pi/3) + \cos(\Delta\theta + \pi/6) + \cos(\Delta\phi + 5\pi/6) = 1. \end{aligned} \tag{3.24}$$

In order to compare our linkage to Kempe's construction, we consider the simplest cases of Kempe's additor, multiplicator and translator linkages. The additor has six bars and is required for each addition including the constants. A multiplication by k requires a multiplicator with at least $m(k) = 2(k-2)+6$ bars. We model the translator as a parallelogram linkage that requires three bars for each belt used in our design, which means t=3b. There-

Table 3.3: Serial Chain Configuration.

Link, i	Link length, A_i	Link angle, $\Delta\alpha_i$	Initial config., α_{i0}
1	2.25	$\Delta\theta$	$-\pi/3$
2	2.25	$\Delta\phi$	$\pi/3$
3	0.25	$3\Delta\theta$	$-\pi$
4	0.25	$3\Delta\phi$	π
5	0.75	$2\Delta\theta - \Delta\phi$	$-\pi$
6	0.75	$2\Delta\theta + \Delta\phi$	$-\pi/3$
7	0.75	$2\Delta\phi - \Delta\theta$	π
8	0.75	$2\Delta\phi + \Delta\theta$	$\pi/3$
9	1	$\Delta\theta$	$\pi/6$
10	1	$\Delta\phi$	$5\pi/6$

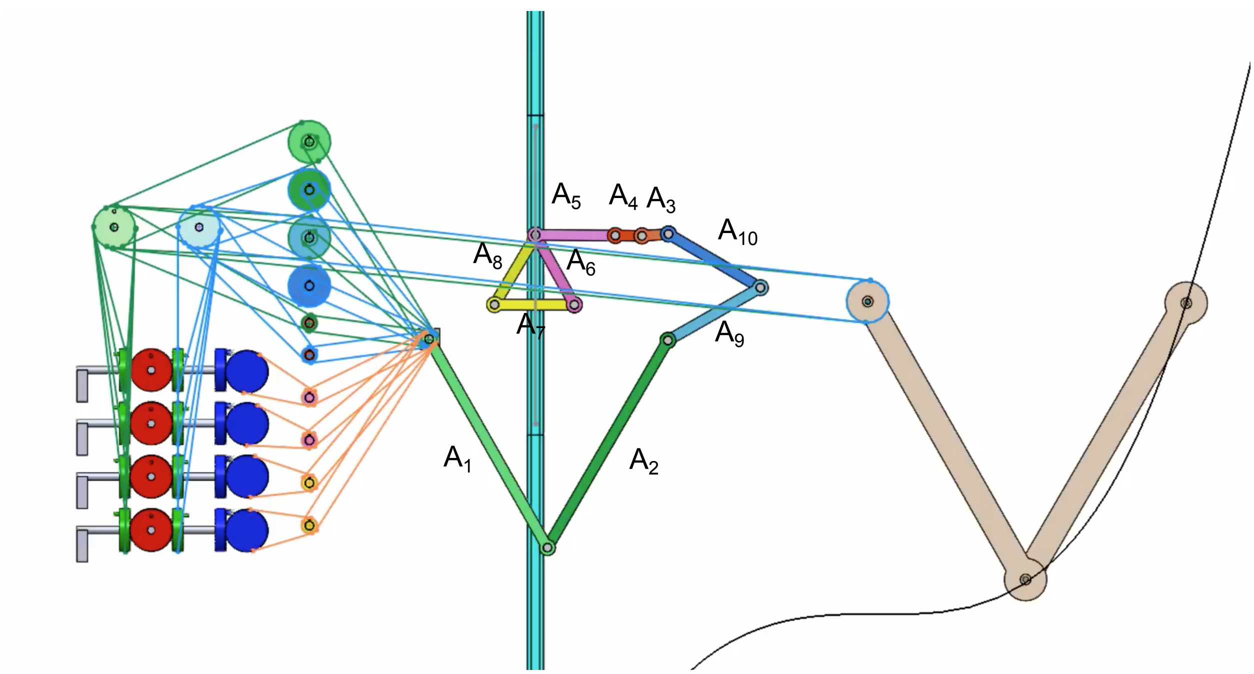


Figure 3.4: This mechanical system uses Kempe's theory to design a mechanical system to draw a cubic curve. Gear pairs, differentials and belt drives are used to provide the multiplications, additions and transmission necessary to constrain the RR chain to draw this curve.

fore, in order to estimate Kempe's linkage, we note that (3.21), requires $a = 6$ additors, $m(2) = 4$ multiplicators with $k = 2$ and $m(3) = 2$ with $k = 3$, thus

$$p = 6a + m(2)6 + m(3)8 + 3b = 36 + 24 + 16 + 216 = 292. \tag{3.25}$$

Thus, we can estimate Kempe's construction to require at least 292 parts for this example.

If we count the individual parts for our method, we have two gears per multiplication and four gears per addition, and two pulleys for each belt. Thus, the part count is

$$p = 2g + 4d + 2b + b = 12 + 15 + 144 + 72 = 244. \tag{3.26}$$

This comparison shows that the primary difference arises from the complexity of the multiplicator linkage. Our method simplifies this further by using the sizes of pulleys to perform the multiplication. This also shows the dominant role that the translator linkages play in the part count of Kempe's designs. It is our expectation that effective use of gears, differentials, belts and pulleys can simplify the application of Kempe's results to a wide range of algebraic curves.

3.4 Elliptic Cubic Curve

In this section, we follow the same procedure presented above to obtain the one degree of freedom drawing linkage system for the elliptic cubic curve given by

$$f(x, y) = x^3 - y^2 - x + 1 = 0. \tag{3.27}$$

This is the curve drawn by the Kempe's linkage shown in Figure 1.4 obtained by Kobel.

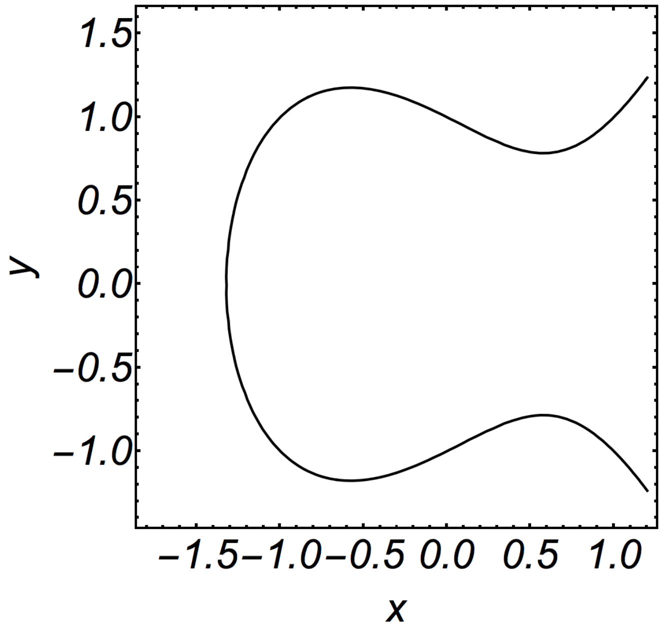


Figure 3.5: A plot of the elliptic cubic curve defined by $f(x, y) = x^3 - y^2 - x + 1 = 0$.

Introduce the RR chain with link lengths, $L_1 = L_2 = 1$ and substitute the equation for the trajectory of its end-point $P(\theta, \phi)$ into (3.27) to obtain the constraint equation,

$$\begin{aligned} f(\theta,\phi) =& \cos^3\theta + \cos^3\phi + 3\cos^2\theta\cos\phi + 3\cos^2\phi\cos\theta + \cos^2\theta + \cos^2\phi \\ & - 2\sin\theta\sin\phi - \cos\theta - \cos\phi - 1 = 0. \end{aligned} \tag{3.28}$$

Eliminate the powers of cosine using the identities,

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}, \quad \cos^3\theta = \frac{3\cos\theta + \cos(3\theta)}{4}. \tag{3.29}$$

And use the trigonometric sum and difference identities to obtain the constraint equation,

$$\begin{aligned} f(\theta,\phi) =& 1.25\cos\theta + 1.25\cos\phi + 0.50\cos 2\theta + 0.50\cos 2\phi + 0.25\cos 3\theta \\ & + 0.25\cos 3\phi + \cos(\theta - \phi + \pi) + \cos(\theta+\phi) + 0.75\cos(\theta - 2\phi) \\ & + 0.75\cos(\theta + 2\phi) + 0.75\cos(2\theta - \phi) + 0.75\cos(2\theta+\phi) = 0. \end{aligned} \tag{3.30}$$

This equation is used to define Kempe's constraining linkage which has the x component defined by,

$$x' = \sum_{i=1}^{n} A_i \cos\alpha_i = K, \tag{3.31}$$

where A_i are the link lengths, the angles $\alpha_i = \Delta\alpha_i + \alpha_{i0}$ are measured relative to the x'-axis, and $x' = K$ locates the slider that guides the end-point.

In order to determine the initial configuration of the system, set $P = (1, 1)$ and solve the equations,

$$P = \begin{Bmatrix} \cos\theta_0 + \cos\phi_0 \\ \sin\theta_0 + \sin\phi_0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}. \tag{3.32}$$

Select the elbow down solution to obtain

$$\theta_0 = 0, \quad \phi_0 = \pi/2. \tag{3.33}$$

Substitute these initial angles into the constraint equation (3.30) to obtain,

$$\begin{aligned} f(\theta,\phi) =& 1.25\cos\Delta\theta + 1.25\cos(\Delta\phi + \pi/2) + 0.50\cos 2\Delta\theta + 0.50\cos(2\Delta\phi + \pi) \\ &+ 0.25\cos 3\Delta\theta + 0.25\cos(3\Delta\phi + 3\pi/2) + \cos(\Delta\theta - \Delta\phi + \pi/2) \\ &+ \cos(\Delta\theta + \Delta\phi + \pi/2) + 0.75\cos(\Delta\theta - 2\Delta\phi - \pi) \\ &+ 0.75\cos(\Delta\theta + 2\Delta\phi + \pi) + 0.75\cos(2\Delta\theta - \Delta\phi - \pi/2) \\ &+ 0.75\cos(2\Delta\theta + \Delta\phi + \pi/2) = 0. \end{aligned} \tag{3.34}$$

From this equation we determine the angles $\alpha_i = \Delta\alpha_i + \alpha_{i0}$ that define the constraining linkage. See Table 3.4. Figure 3.6 shows the linkage system that draws this elliptic cubic curve.

Table 3.4: Dimensions for Kempe's constraining linkage that defines the elliptic cubic curve $x^3 - y^2 - x + 1 = 0$

Link, i	Link length, A_i	Link angle, $\Delta\alpha_i$	Initial config., α_{i0}
1	1.25	$\Delta\theta$	0
2	1.25	$\Delta\phi$	$\pi/2$
3	0.50	$2\Delta\theta$	0
4	0.50	$2\Delta\phi$	π
5	0.25	$3\Delta\theta$	0
6	0.25	$3\Delta\phi$	$3\pi/2$
7	1	$\Delta\theta - \Delta\phi$	$\pi/2$
8	1	$\Delta\theta + \Delta\phi$	$\pi/2$
9	0.75	$\Delta\theta - 2\Delta\phi$	$-\pi$
10	0.75	$\Delta\theta + 2\Delta\phi$	π
11	0.75	$2\Delta\theta - \Delta\phi$	$-\pi/2$
12	0.75	$2\Delta\theta + \Delta\phi$	$\pi/2$

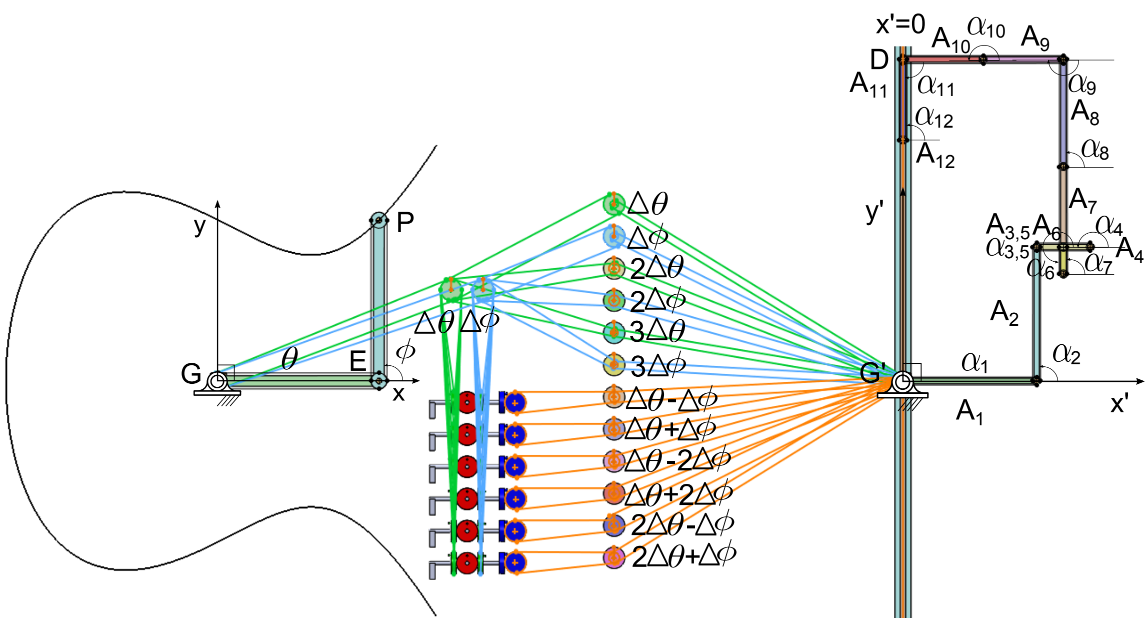


Figure 3.6: The drawing linkage for the elliptic cubic curve presented by Kobel consists of the RR serial chain constrained by the linkage consisting 12 hinged links that end in a prismatic joint that moves along line $x' = 0$.

3.5 Part Counts

In this section, we compute the number of parts in our drawing and compare it to the number of links using Kempe's approach.

3.5.1 The Number of Links Needed in Kempe's Method

In order to estimate the number of links for the drawing linkage obtained using Kempe's method, notice that as before Kempe's serial has n links, one for each terms in (2.4) together with two links for the RR chain that draws the curve, thus $c_l = n + 2$.

Kempe's multiplicator is a series of scaled contra-parallelograms, which performs a multiplication by q using $2q + 2$ links. This linkage is needed only if there is a multiplication by 2 or more, that is $q \geq 2$. Recall that i^{th} has the multiplication $r_i\phi$ and $s_i\theta$. Thus, the number of links for multiplication is given by,

$$c_m = \sum_{i}^{n}(2r_i + 2) + \sum_{i}^{n}(2s_i + 2), \tag{3.35}$$

where each term is counted only if the multiplication is by factor of 2 or more.

Kempe's additor has 11 links. Let $a_i, i = 1, \ldots n$ denote the number of additions in the i^{th} term. Notice that in Kempe's construction, additors are required for additions between θ or ϕ with a constant angle. This means the number of links used for additions is,

$$c_a = \sum_{i}^{n} 11a_i. \tag{3.36}$$

Kempe's translator is constructed from a series of parallelograms linkage that are positioned along Kempe's serial chain to constrain its angular values. The i^{th} link requires $i - 1$ parallelograms to connect it to the calculating linkages. This can be assembled by adding two additional links for parallelogram, which yields,

$$c_t = n(n-1). \tag{3.37}$$

Thus, the number of links required for Kempe's drawing linkage can be estimated to be,

$$p = c_l + c_t + c_a + c_m = n^2 + 2 + \sum_i^n 11a_i + \sum_i^n (2r_i + 2) + \sum_i^n (2s_i + 2), \tag{3.38}$$

where n is the number of links in Kempe's serial chain.

3.5.2 Part Count for Straight Line

Links, differentials and cable drives The number parts for our mechanism to draw a straight line can be estimated as follows. Here we use the straight line defined in (3.2) as an example. Examine (3.8) to see that there are $n = 4$ terms, the number of additions c_a and multiplications c_m are both zero. Thus, the part count for the straight line drawing linkage is computed to be,

$$p_{1a} = (n^2 + 3n + 10)/2 + \sum_i^n a_i + \sum_i^n v_i = 19, \tag{3.39}$$

where the subscript $1a$ denotes our drawing linkage for the first example.

Links in Kempe's method Kempe's drawing linkage for the straight line requires two additors that $a_i = 1, i = 3, 4$ for the addition of θ or ϕ with a constant angle . There is no mutiplicators needed in this case. Thus, the number of parts used in Kempe's construction is computed to be,

$$p_{1b} = n^2 + 2 + \sum_i^n 11a_i + \sum_i^n (2r_i + 2) + \sum_i^n (2s_i + 2) = 18 + 22 = 40. \tag{3.40}$$

Thus, we can estimate Kempe's construction to draw the straight line requires at least 40 links.

3.5.3 Part Count for Elliptic Cubic Curve

Links, differentials and cable drives The number parts for the drawing linkage for the elliptic cubic curve is obtained from (3.30). We see that there are $n = 12$ terms, the number of additions is $c_a = 6$ and the number of multiplications is $c_m = 8$. Thus, the part count for the elliptic cubic curve drawing linkage is,

$$p_{2a} = (n^2 + 3n + 10)/2 + \sum_i^n a_i + \sum_i^n v_i = 95 + 6 + 8 = 109, \tag{3.41}$$

where the subscript $2a$ denotes our drawing linkage for our second example.

Links in Kempe's method Finally, the number of links in Kempe's linkage for the elliptic cubic curve is obtained from equation (3.34). It has $n = 12$ terms, which requires six additors needed are $a_i = 1, i = 7, \ldots 12$, and eight mutiplicators, which are $r_4 = 2$, $r_6 = 3$, $r_9 = 2$, $r_{10} = 2$ and $s_3 = 2$, $s_5 = 3$, $s_{11} = 2$, $s_{12} = 2$. Thus, the number of links for the elliptic cubic curve drawing linkage is estimated to be,

$$p_{2b} = n^2 + 2 + \sum_i^n 11a_i + \sum_i^n (2r_i + 2) + \sum_i^n (2s_i + 2) = 146 + 66 + 26 + 26 = 264. \tag{3.42}$$

Thus, we can estimate Kempe's construction to draw the elliptic cubic curve requires at least 264 links.

This part count comparison shows that our use of differentials and cable drives in the drawing linkage reduces the part count by more than one-half.

3.6 Workspace Analysis

The plots of a number of algebraic curves are in the region $-\infty < x < \infty$ or $-\infty < y < \infty$. But the mechanism drawing the algebraic curve is designed to work in a specific range. In this section, the moving range of the drawing linkage and the relationship between θ and ϕ are analyzed.

The extreme points are hit when the RR chain is fully stretched out. This constraint the moving range of the curve drawing mechanism. Thus, angles θ and ϕ for the extreme position can be obtained by solving,

$$\theta = \phi, \quad \text{and} \quad f(\theta, \phi) = \sum_{i}^{n} A_i \cos(r_i\phi + s_i\theta + \alpha_i) - K = 0. \tag{3.43}$$

Here we use the straight line defined by (3.2) as an example. In order to obtain the moving range of the mechanical system constructing from our approach, we solve for Eq (3.4) by setting $\theta = \phi$ which yields,

$$\theta = \phi = 0.785 + 2k\pi, \text{ and } 3.927 + 2k\pi, k = 0, \pm 1, \pm 2, \ldots. \tag{3.44}$$

The first solution gives the configuration of the RR chain when it reach the maximum y value while the second solution is for the situation it reaches the minimum y value. Substituting Eq (3.44) into Eq (3.3) to compute the two extreme points that the RR chain can reach are,

$$P_1 = (1.414, 1.414), \text{ and } P_2 = (-1.414, -1.414). \tag{3.45}$$

Equation (3.45) provides us the RR chain's workspace. The part of the straight line been drawn is within the boundary that $-1.414 < x < 1.414$, $-1.414 < y < 1.414$.

We follow the same procedure to get the mechanical system's workspace for the elliptic cubic curve. Solving Eq (3.30) by setting $\theta = \phi$ to obtain,

$$\theta = \phi = 0.824 + 2k\pi, \text{ and } 5.459 + 2k\pi, k = 0, \pm 1, \pm 2, \ldots. \tag{3.46}$$

Substituting Eq (3.46) into Eq (3.3) to compute the two extreme points the RR chain can reach on the elliptic cubic curve are,

$$P_1 = (1.359, 1.467), \text{ and } P_2 = (1.359, -1.467). \tag{3.47}$$

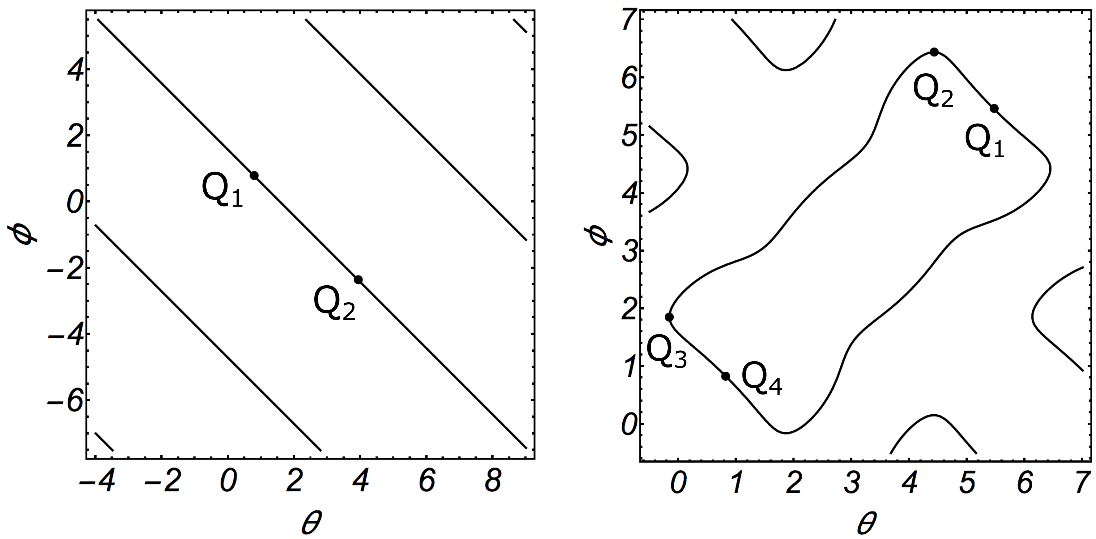
We set $y = 0$ in Eq (3.27) to obtain the minimum x value to be -1.325. The elliptic cubic curve is symmetrical with respect to $y = 0$. Thus the part of elliptic cubic curve been drawn is withing the boundary that $-1.325 < x < 1.359, -1.467 < y < 1.467$.

Note that the region of the curve been drawn is dependent on the lengths of the RR chain. The above workspace calculation for the straight line and elliptic cubic curve are both based on our construction that $L_1 = L_2 = 1$.

Since we have constraint angle θ and ϕ through Kempe's serial chain, the degree of freedom of the resulting mechanical system is one. We now analyze the relationship between θ and ϕ.

In order to see how θ or ϕ change as the RR chain draws the algebraic curve, we plot the relationship between θ and ϕ from Eq (3.4) and Eq (3.30). We focus on the plot for θ and ϕ when $k = 0$ in Eq (3.44) and Eq (3.46), see Figure 3.7.

In Figure 3.7(a), all the lines defined by equation $\phi = -\theta + \pi/2 + 2k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$



(a) Straight line: The relationship between θ and ϕ for $k = 0$ in Eq (3.44).

(b) Elliptic: The relationship between θ and ϕ for $k = 0$ in Eq (3.46).

Figure 3.7: In Eq (3.44) and Eq (3.46), setting $k = 0$ to obtain two specific plots. In (a), angle θ and ϕ change linearly from Q_1 to Q_2. In (b), points Q_1 and Q_4 are two boundary points where the RR chain changes elbow directions; points Q_2 and Q_3 are places θ and ϕ reach its maximum or minimum values.

represent the relationship of θ and ϕ to constraint the RR chain to draw the straight line. From Q_1 to Q_2, the RR chain moves from one fully stretched-out position with maximum y value to the other fully stretched-out position with minimum y value. During this movement, angle θ increases linearly from $\pi/4$ to $5\pi/4$ while ϕ decreases linearly from $\pi/4$ to $-3\pi/4$.

In Figure 3.7(b), the upper left part from Q_1 counter-clockwise to Q_4 shows the constraint between θ and ϕ for the RR chain starts from "elbow-down" configuration while the lower right part from Q_1 clockwise to Q_4 is for the RR chain starts from "elbow-up" configuration. Now we focus on the upper left part resulting from our construction for elliptic cubic curve in Figure 3.6. It shows θ keeps decreasing from point Q_1 to Q_3 and starts increasing from Q_3 to Q_4. Thus we can obtain that θ decreases from 5.459 to -0.156 and then increases to 0.824 . While ϕ starts increasing from point Q_1 to Q_2 then keeps decreasing from Q_2 to Q_4. We can compute that ϕ increases from 5.459 to 6.440 and then decreases to 0.824.

3.7 Summary

In this chapter, we present a simplified version of Kempe's method for the design of a drawing linkage for a general plane algebraic curve. We replace his additor, reversor, multiplicator and translator computing linkages, with differential and cable drives to perform the equivalent mechanical calculations. In order to compare these two approaches, we estimate the part count and find that our approach uses less than one-half the number of parts. Kempe's universality theorem guarantees the existence of a drawing linkage for any algebraic curve, therefore our simplified linkage exists as well. We use Saxena's and Kobel's examples to illustrate our method, which has the added benefit of illustrating the central role played by Kempe's constraining linkage in controlling the movement of a RR chain that draws the curve.

Chapter 4

Mechanisms to Draw Trigonometric Curves

4.1 Introduction

This chapter describes a mechanism design methodology that draws plane curves that have finite Fourier series parameterizations, known as trigonometric curves. We present three ways to use the coefficients of this parameterization to construct a mechanical system that draws the curve. One uses Scotch yoke mechanisms for each of the terms in the coordinate trigonometric functions, which are then added using a belt or cable drive. The second approach uses two coupled serial chains obtained from the coordinate trigonometric functions. The third approach combines the coordinate trigonometric functions to define a single coupled serial chain that draws the plane curve. This work is a version of Kempe's Universality Theorem that demonstrates that every plane trigonometric curve has a linkage that draws the curve. Several examples illustrate the method including the use of boundary points and the discrete Fourier transform to define the trigonometric curve.

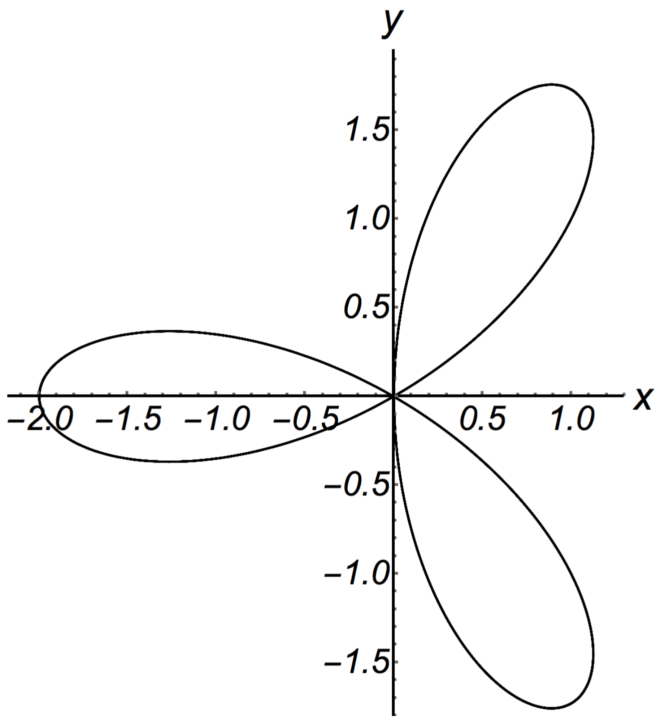


Figure 4.1: Trifolium with the size of the petal set to $a = 2$.

4.2 Trifolium

In this section, we demonstrate the use of the equations in the previous section to design mechanisms to draw the Trifolium curve. Figure 4.1 shows the Trifolium curve defined by Eq. (4.1) with α equals to 2.

4.2.1 Trifolium Using Scotch Yoke Mechanisms

The Trifolium is often defined in radial coordinates by the formula [29, 58],

$$P: \ \rho = -a \cos 3\theta, \tag{4.1}$$

where a point P is defined by the length $\rho(\theta)$ of a radius vector at the angle θ to the x axis. The constant a defines the size of the petals of the Trifolium.

The coordinate functions for this curve are given by,

$$P_T = \begin{Bmatrix} \rho\cos\theta \\ \rho\sin\theta \end{Bmatrix} = \begin{Bmatrix} x(\theta) \\ y(\theta) \end{Bmatrix} = \begin{Bmatrix} -a\cos 3\theta\cos\theta \\ -a\cos 3\theta\sin\theta \end{Bmatrix}. \tag{4.2}$$

Set the size of the petals to $a = 2$, and expand the products of sine and cosine functions to obtain the trigonometric form of this curve,

$$P_T = \begin{Bmatrix} -\cos 2\theta - \cos 4\theta \\ \sin 2\theta - \sin 4\theta \end{Bmatrix}. \tag{4.3}$$

$$L_2 = \sqrt{a_2^2 + b_2^2} = \sqrt{(-1)^2} = 1 \quad \psi_2 = \arctan(b_2/a_2) = \arctan(0/-1) = \pi \tag{4.4}$$

$$L_4 = \sqrt{a_4^2 + b_4^2} = \sqrt{(-1)^2} = 1 \quad \psi_4 = \arctan(b_4/a_4) = \arctan(0/-1) = \pi \tag{4.5}$$

$$M_2 = \sqrt{c_2^2 + d_2^2} = \sqrt{(-1)^2} = 1 \quad \eta_2 = \arctan(d_2/c_2) = \arctan(1/0) = \pi/2 \tag{4.6}$$

$$M_4 = \sqrt{c_4^2 + d_4^2} = \sqrt{(-1)^2} = 1 \quad \eta_4 = \arctan(d_4/c_4) = \arctan(-1/0) = -\pi/2 \tag{4.7}$$

The coefficients of the component trigonometric equations listed in Table 4.1 can be used to

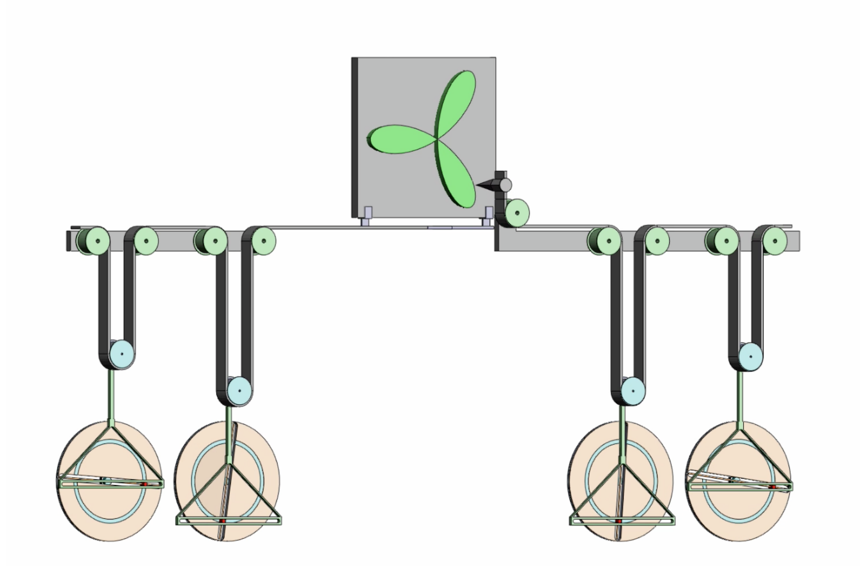


Figure 4.2: A system of Scotch yoke mechanisms driven by a single input that draws the Trifolium.

Table 4.1: Coefficients of the component trigonometric functions for the Trifolium curve.

k	a_k	b_k	c_k	d_k
0	0	0	0	0
1	0	0	0	0
2	-1	0	0	1
3	0	0	0	0
4	-1	0	0	-1

determine the dimensions of a set of component Scotch yoke mechanisms, using Eqs (2.15) and (2.18). These dimensions are listed in Table 4.2. The mechanical system that draws this Trifolium curve is shown in Figure 4.2.

4.2.2 Trifolium Using Component Coupled Serial Chains

The analysis provided in the previous section shows that the same dimensions used to construct the Scotch yoke mechanisms can be used to define a pair of component serial chains that position a cursor to draw a trigonometric curve like the Trifolium. Figure 4.3 shows the mechanical system of coupled serial chains that draws the Trifolium curve.

Table 4.2: Component dimensions for component Scotch yoke mechanisms and coupled serial chains to draw the Trifolium.

k	L_k	ψ_k	M_k	η_k
1	0	0	0	0
2	1	π	1	$\pi/2$
3	0	0	0	0
4	1	π	1	$-\pi/2$

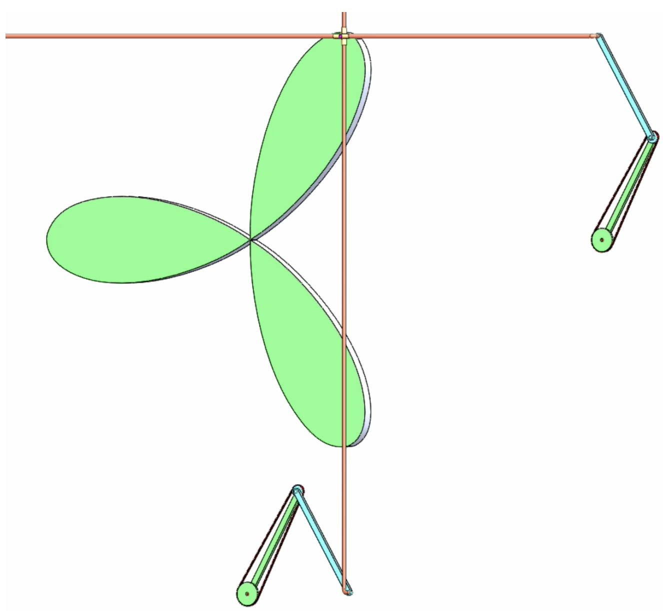


Figure 4.3: A system of two coupled serial chains driven by a single input that draws the Trifolium.

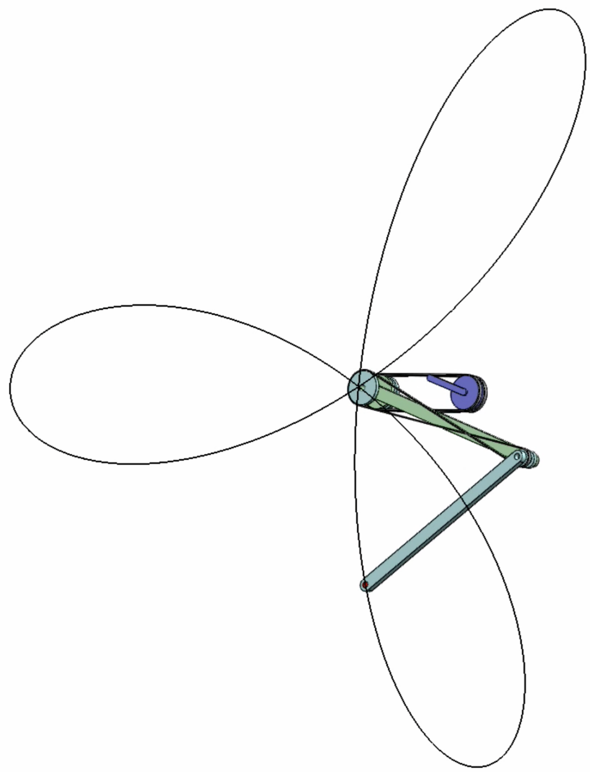


Figure 4.4: A single constrained coupled serial chain with one input that draws the Trifolium.

4.2.3 Trifolium Using a Single Coupled Serial Chain

In order to define the single coupled serial chain that draws the Trifolium curve, we use the coefficients of the trigonometric form of the equation, Table 4.1, and compute the dimensions L_k, M_k and initial angles ψ_k, η_k, using (7.3) and (7.4). These dimensions are listed in Table 4.3. Recall that for a single coupled serial chain, those links denoted L_k rotate counter clockwise, while those links denoted M_k rotate clockwise and the angles ψ_k and η_k denote the initial configurations of the system. Figure 4.4 shows the single coupled serial chain that draws the Trifolium curve.

$$P_T = \begin{Bmatrix} \cos(-2\theta - \pi) + \cos(4\theta - \pi) \\ \sin(-2\theta - \pi) + \sin(4\theta - \pi) \end{Bmatrix} \tag{4.8}$$

Table 4.3: Dimensions of the single coupled serial chain to draw the Trifolium.

k	L_k	ψ_k	M_k	η_k
1	0	0	0	0
2	0	0	1	π
3	0	0	0	0
4	1	π	0	0

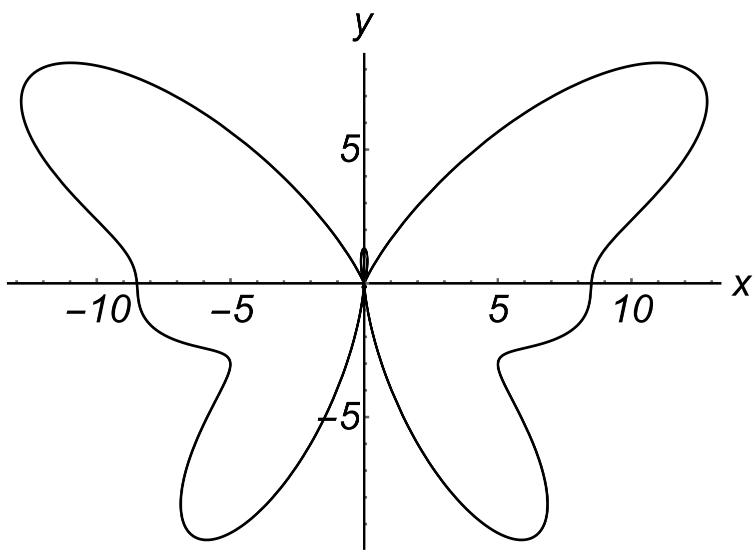


Figure 4.5: The plot of a Butterfly curve from its polar equation.

This drawing linkage for the Trifolium curve consists of two links, as compared to the exceedingly complex drawing linkage obtained by Kobel [31] using Kempe's method. This is also simpler than Artobolevskii's eight-bar linkage [2] showing in Figure 1.6.

4.3 Butterfly Curve

A version of the Butterfly curve [13] can be generated by the formula,

$$P_B: \ \rho(\theta) = 7 - \sin\theta + 2.3\sin 3\theta + 2.5\sin 5\theta - 2\sin 7\theta - 0.4\sin 9\theta + 4\cos 2\theta - 2.5\cos 4\theta, \tag{4.9}$$

where radius vector ρ lies at the angle θ relative to the x-axis. Figure 4.5 shows this Butterfly curve.

The trigonometric form of the Butterfly curve is obtained by expanding the coordinates, $P_B = (\rho\cos\theta, \rho\sin\theta) = (p_{Bx}, p_{By})$, to obtain,

$$P_B = \left\{\begin{matrix} 9\cos\theta + 0.65\sin 2\theta + 0.75\cos 3\theta + 2.4\sin 4\theta - 1.25\cos 5\theta + 0.25\sin 6\theta - 1.2\sin 8\theta - 0.2\sin 10\theta \\ -0.5 + 5\sin\theta + 1.65\cos 2\theta + 3.25\sin 3\theta + 0.1\cos 4\theta - 1.25\sin 5\theta - 2.25\cos 6\theta + 0.8\cos 8\theta + 0.2\cos 10\theta \end{matrix}\right\} \quad (4.10)$$

The coefficients of these equations are listed in Table 4.4.

Table 4.4: Coefficients of the component trigonometric functions for the Butterfly curve.

k	a_k	b_k	c_k	d_k
0	0	0	-0.5	0
1	9	0	0	5
2	0	0.65	1.65	0
3	0.75	0	0	3.25
4	0	2.4	0.1	0
5	-1.25	0	0	-1.25
6	0	0.25	-2.25	0
7	0	0	0	0
8	0	-1.2	0.8	0
9	0	0	0	0
10	0	-0.2	0.2	0

4.3.1 Butterfly Using Scotch Yoke Mechanisms

The coefficients of the component trigonometric equations listed in Table 4.4 are used to calculate the dimensions of the component Scotch yoke mechanisms, using Eqs (2.15) and (2.18). These dimensions are listed in Table 4.5. These dimensions can also be used to construct the component coupled serial chains that draw this curve.

Table 4.5: Dimensions for the component Scotch yoke mechanisms to draw the Butterfly curve.

k	L_k	ψ_k	M_k	η_k
0	0	0	0.5	π
1	9	0	5	$\pi/2$
2	0.65	$\pi/2$	1.65	0
3	0.75	0	3.25	$\pi/2$
4	2.4	$\pi/2$	0.1	0
5	1.25	π	1.25	$-\pi/2$
6	0.25	$\pi/2$	2.25	π
7	0	0	0	0
8	1.2	$-\pi/2$	0.8	0
9	0	0	0	0
10	0.2	$-\pi/2$	0.2	0

4.3.2 Butterfly Using a Single Coupled Serial Chain

In order to define the single coupled serial chain that draws the Butterfly curve, we use the coefficients of the trigonometric form of the equation, Table 4.4, and compute the dimensions L_k, M_k and initial angles ψ_k, η_k, using (2.25) and (2.26). These dimensions are listed in Table 4.6. The constant terms $k = 0$ define the location of the first ground pivot $(-0.5, 0)$. There are 14 links in this coupled serial chain driven by a single input. The single coupled serial chain that draws the Butterfly curve is shown in Figure 4.6.

Table 4.6: Dimensions of the single coupled serial chain to draw the Butterfly curve.

k	L_k	ψ_k	M_k	η_k
0	0.25	$-\pi/2$	0.25	$-\pi/2$
1	7	0	2	0
2	0.5	$\pi/2$	1.15	$\pi/2$
3	2	0	1.25	π
4	1.15	$-\pi/2$	1.25	$\pi/2$
5	1.25	π	0	0
6	1.25	$-\pi/2$	1	$-\pi/2$
7	0	0	0	0
8	1	$\pi/2$	0.2	$-\pi/2$
9	0	0	0	0
10	0.2	$\pi/2$	0	0

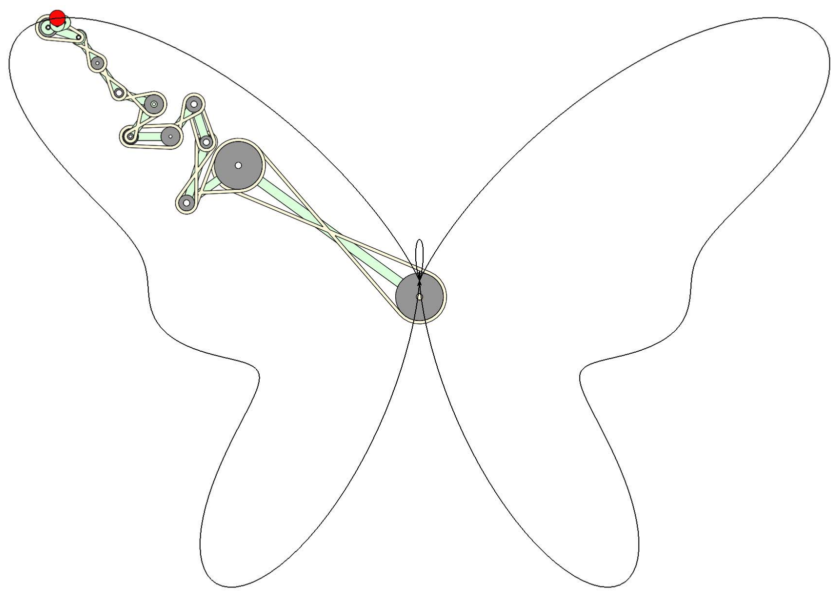


Figure 4.6: The constraint coupled serial chain to draw this Butterfly curve consists of 14 terms.

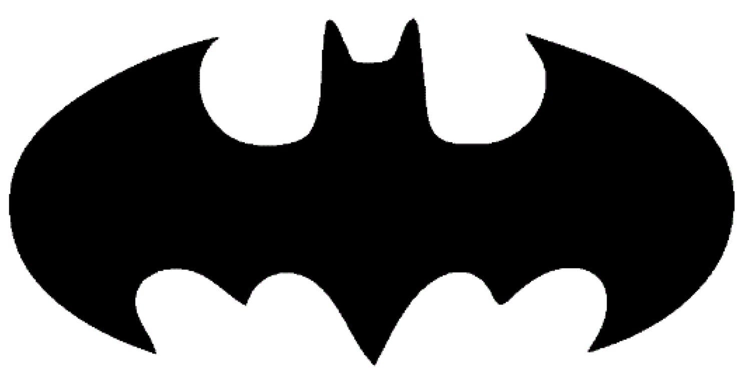


Figure 4.7: The shape of the Batman logo.

4.4 Batman Logo

In this example, we use the software *Mathematica* to extract 3235 points to define the boundary curve of the Batman logo showing in Figure 4.7. Twenty terms of the discrete Fourier transform of these points yields the curve $P = (x(\theta), y(\theta))$ in the trigonometric form that approximates this Batman logo curve, Table 4.7. The formulas obtained above yield mechanisms that draw our approximation to the Batman logo. Figure 4.8 shows the Batman logo curve achieved by 20 terms of Fourier series.

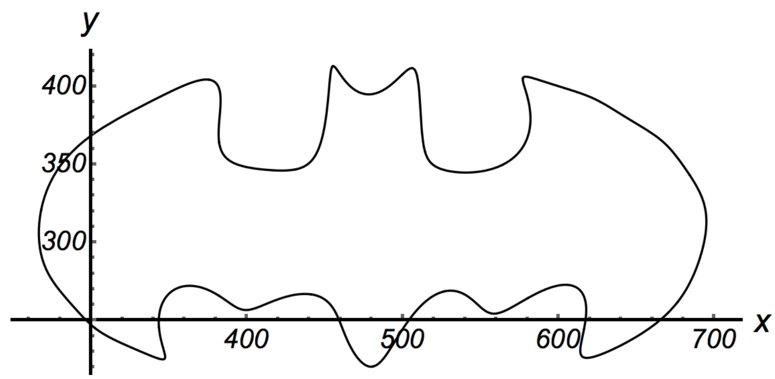


Figure 4.8: Batman logo curve obtained using 20 terms of the discrete Fourier transform of boundary points.

4.4.1 Batman Logo Using Scotch Yoke Mechanisms

The coefficients of the component trigonometric equations listed in Table 4.7 can be used to determine the dimensions of a set of component Scotch yoke mechanisms, using Eqs (2.15) and (2.18). These dimensions are listed in Table 4.8. Figure 4.9 shows the system of Scotch yoke mechanisms that draws this curve.

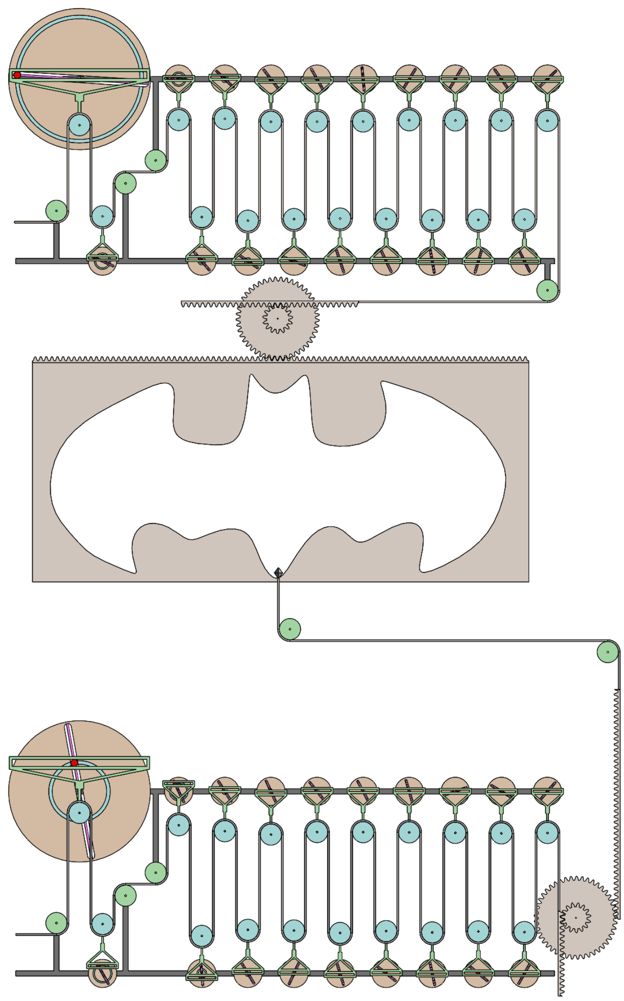


Figure 4.9: The system of component Scotch yoke mechanisms that draws the Batman logo consists of two sets of 19 mechanisms.

Table 4.7: Coefficients of the component trigonometric functions for the Batman logo.

k	a_k	b_k	c_k	d_k
0	480.22	0	317.57	0
1	11.36	-175.47	-80.64	-14.59
2	2.08	-18.25	-2.05	-0.60
3	-8.80	19.26	21.64	12.16
4	-2.80	2.14	5.38	3.28
5	5.49	-5.70	-22.26	-10.47
6	1.48	0.92	3.54	-1.62
7	-6.68	5.58	-11.81	-1.03
8	-2.53	0.84	-0.64	1.85
9	1.00	0.59	4.17	1.90
10	-1.04	2.64	-3.47	-1.05
11	3.59	-5.21	0.99	1.19
12	-1.65	0.79	-5.35	-3.42
13	-0.19	-0.20	-2.64	-1.41
14	-1.93	0.18	-3.03	-2.42
15	-2.44	1.36	-0.41	0.35
16	0.35	-0.17	-1.95	-0.42
17	0.44	-0.30	1.77	2.25
18	1.03	0.30	-1.05	0.59
19	1.58	-0.28	0.75	1.25

4.4.2 Batman Logo Using a Single Coupled Serial Chain

By using the coefficients of the component trigonometric equations listed in Table 4.7, Eqs (7.3) and (7.4) yield the dimensions, L_k, M_k and the initial angles ψ_k and η_k of the serial chain that draws the Batman logo. These dimensions are listed in Table 4.9. The $k = 0$ terms identify the ground pivot coordinates $(480.22, 317.57)$. This system is a serial chain consisting of 38 links with one input that draws our approximation of the Batman logo. Figure 4.10 shows the single coupled serial chain that can draw this Batman logo.

Table 4.8: Dimensions for the component Scotch yoke mechanisms to draw the Batman logo.

k	L_k	ψ_k	M_k	η_k
0	480.22	0	317.57	0
1	175.84	-1.50	81.95	-2.96
2	18.37	-1.45	2.14	-2.85
3	21.18	1.99	24.83	0.51
4	3.52	2.48	6.30	0.54
5	7.92	-0.80	24.60	-2.70
6	1.75	0.55	3.90	-0.42
7	8.71	2.44	11.85	-3.05
8	2.66	2.82	1.95	1.90
9	1.16	0.53	4.58	0.42
10	2.84	1.94	3.63	-2.84
11	6.33	-0.96	1.55	0.87
12	1.83	2.69	6.35	-2.57
13	0.28	-2.34	3.00	-2.65
14	1.94	3.04	3.88	-2.46
15	2.80	2.63	0.54	2.42
16	0.40	-0.46	2.00	-2.92
17	0.53	-0.61	2.87	0.90
18	1.08	0.28	1.21	2.62
19	1.61	-0.17	1.46	1.03

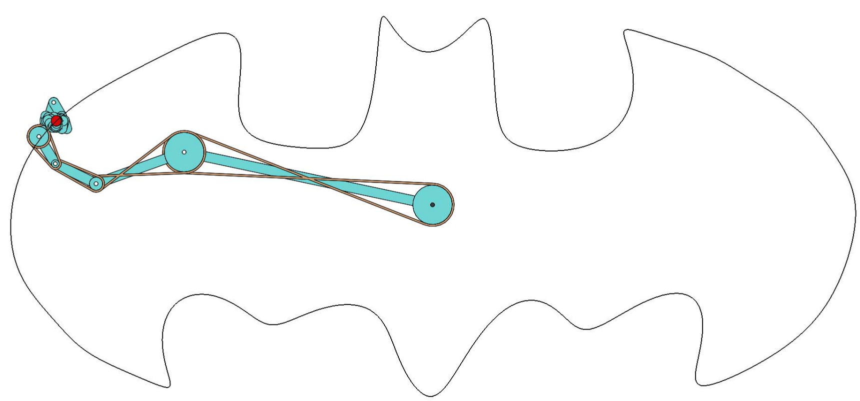


Figure 4.10: The single coupled serial chain that draws the Batman logo consists of 38 links.

Table 4.9: Dimensions of the single coupled serial chain that draws the Batman logo.

k	L_k	ψ_k	M_k	η_k
0	287.86	0.58	287.86	0.58
1	47.44	1.60	128.71	-1.46
2	8.13	1.47	10.24	-1.43
3	2.06	0.61	22.98	2.04
4	1.63	1.42	4.84	2.25
5	8.64	-1.86	16.10	-1.05
6	1.31	1.62	2.72	0.96
7	9.51	-1.98	4.20	-2.30
8	0.81	-2.00	2.19	3.09
9	2.30	0.88	2.42	1.75
10	3.23	-1.90	0.41	-1.54
11	3.92	0.91	2.42	-1.05
12	3.98	-2.26	2.44	-1.20
13	1.46	-2.15	1.54	-1.16
14	2.71	-2.50	1.44	-1.40
15	1.37	-2.43	1.47	2.81
16	0.88	-1.60	1.13	-1.22
17	1.70	0.65	1.16	2.46
18	1.06	-0.69	0.43	-1.04
19	1.51	0.35	0.28	0.95

Chapter 5

Mechanisms to Draw Bezier Curves

This chapter presents a design methodology for a coupled serial chain that draws a cubic Bezier curve. The four control points of the cubic Bezier curve are used to define an associated cubic trigonometric Bezier curve, which can be adjusted to minimize the maximum distance between the two Bezier curves. A coupled serial chain is then defined that draws the cubic trigonometric Bezier curve. For multiple Bezier curves, the input of each of the coupled serial chains can be connected to a single drive. The result is a methodology for the design of a mechanical system that draws complex plane curves, which is demonstrated by drawing cursive alphabet letters and a script Chinese character.

5.1 Cursive Letters

In this section, we design the system of eight coupled serial chains that draw the eight cubic Bezier curves shown in Figure 5.1, which spell the name "Yang" in cursive letters. The control points for each of these eight curves are listed in Table 5.1. The link lengths of the m^{th} coupled serial chain are obtained using equation (7.3) with $\lambda = 0.01$ for each of the eight

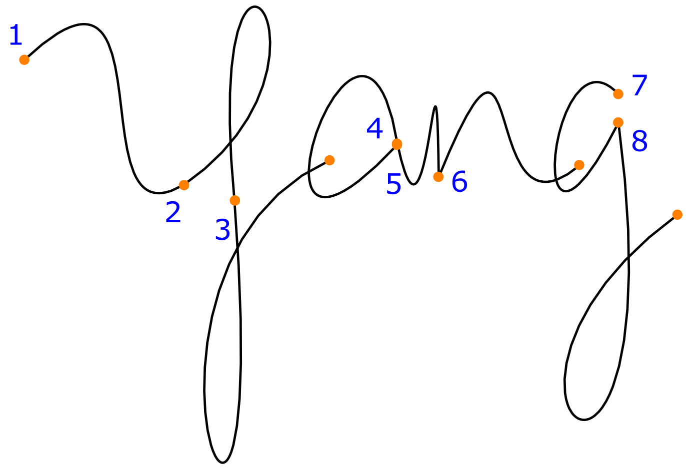


Figure 5.1: The cursive letters that define "Yang" are defined by eight cubic Bezier curves listed in Table 5.1.

Table 5.1: Control points for the eight Bezier curves that define "Yang," (units mm).

Curve j	P_0^j	P_1^j	P_2^j	P_3^j
1	(85.37, 839.79)	(180.60, 929.26)	(116.57, 719.11)	(186.35, 760.16)
2	(186.35, 760.16)	(300.52, 838.94)	(197.06, 974.37)	(218.36, 750.31)
3	(218.36, 750.31)	(245.45, 394.84)	(125.64, 707.59)	(279.11, 775.76)
4	(321.80, 786.43)	(298.81, 927.62)	(207.71, 663.29)	(321.80, 785.61)
5	(321.80, 786.43)	(341.53, 691.99)	(346.43, 894.79)	(348.06, 765.09)
6	(348.06, 765.09)	(405.41, 906.64)	(371.58, 716.91)	(437.54, 772.47)
7	(462.17, 817.62)	(419.19, 865.06)	(398.30, 673.90)	(462.17, 799.56)
8	(462.17, 799.56)	(502.42, 493.34)	(339.86, 624.71)	(499.93, 741.28)

curve segments. These lengths are listed in Table 5.2.

In Table 5.2, we see for this example that the last two links are less than 1% of the length of the other links in the serial chain. Therefore, we evaluate the effect of using only the first four-links to draw the Bezier curves using the same measure $\epsilon = \Delta/d$ presented above.

Consider the first curve in collection that has the control points,

$$P_0 = \begin{bmatrix} 85.37 \\ 839.78 \end{bmatrix}, P_1 = \begin{bmatrix} 180.60 \\ 929.26 \end{bmatrix}, P_2 = \begin{bmatrix} 116.57 \\ 719.11 \end{bmatrix}, P_3 = \begin{bmatrix} 186.34 \\ 760.16 \end{bmatrix}. \tag{5.1}$$

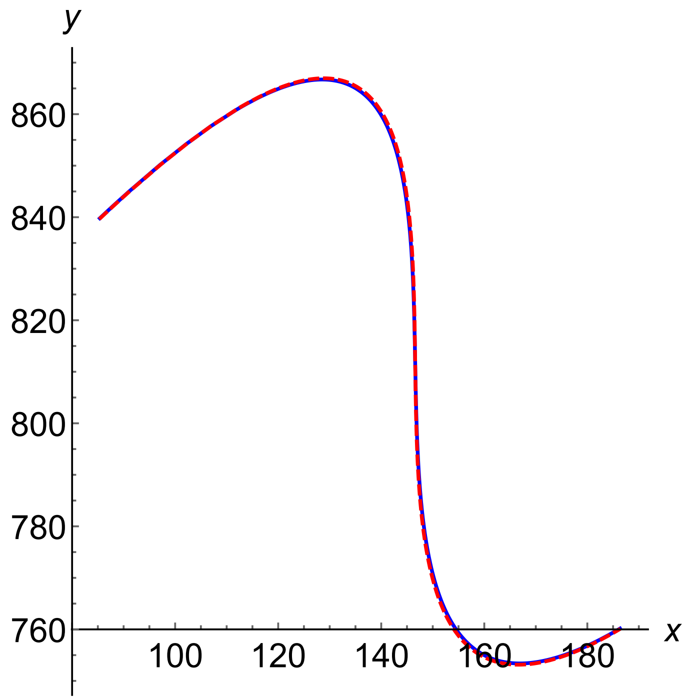


Figure 5.2: Comparison of the curve 1 in "Yang" drawn by a six-link chain (solid blue) and by a four-link chain (dashed red).

Table 5.2: Serial chain ground pivot positions and link lengths for "Yang," (units mm).

Curve j	G_j	L_1^j	M_1^j	L_2^j	M_2^j	L_3^j	M_3^j
1	(110.16, 751.07)	138.90	169.68	103.63	103.63	0.21	0.17
2	(108.55, 449.37)	125.10	355.82	93.15	93.15	0.44	0.16
3	(376.38, 1190.91)	522.31	207.94	169.45	169.45	0.26	0.64
4	(460.24, 766.96)	103.86	296.30	140.99	140.99	0.37	0.13
5	(316.65, 740.14)	147.30	177.45	107.94	107.94	0.22	0.18
6	(401.51, 681.93)	137.06	209.34	105.53	105.53	0.26	0.17
7	(570.09, 887.60)	85.04	203.83	92.53	92.53	0.25	0.10
8	(602.07, 1197.44)	496.26	166.32	122.50	122.50	0.21	0.61

This yields a coupled serial chain with the six link lengths,

$$L_1^1 = 138.90, M_1^1 = 169.68, L_2^1 = 103.63, M_2^1 = 103.63, L_3^1 = 0.21, M_3^1 = 0.17, \qquad (5.2)$$

As discussed above we compare the curves generated by the six-link chain and those generated by the four-link chain, by selecting $N = 1000$ points on the two curves and computing the maximum difference between corresponding points to obtain,

$$\epsilon = \frac{\Delta}{d} = 0.002. \qquad (5.3)$$

Figure 5.2 shows a comparison of these two curves.

Because the last two links of the six-link drawing linkage are small, we neglect them and obtain a simplified drawing four link linkage. However, this introduces a gap between the curves drawn by each of the four-link serial chains that is approximately the size of the last two links. In the next section, we close this gap and show the affect on the resulting curves.

Table 5.3: Gap between consecutive curves using four links serial chain, (units mm).

Gap	d_1^2	d_2^3	d_4^5	d_5^6	d_7^8
Value	0.45	0.76	0.85	0.24	0.50

5.1.1 Correction for Four-link Serial Chains

Neglecting the last two links of the six-link serial chains used to draw "Yang" introduces a gap of length d_m^{m+1} between the end point of curve m and the start point of curve $m+1$. The size of this gap is given bounded by the size of the neglected links,

$$|d_m^{m+1}| \le |L_3^m| + |M_3^m|, \qquad (5.4)$$

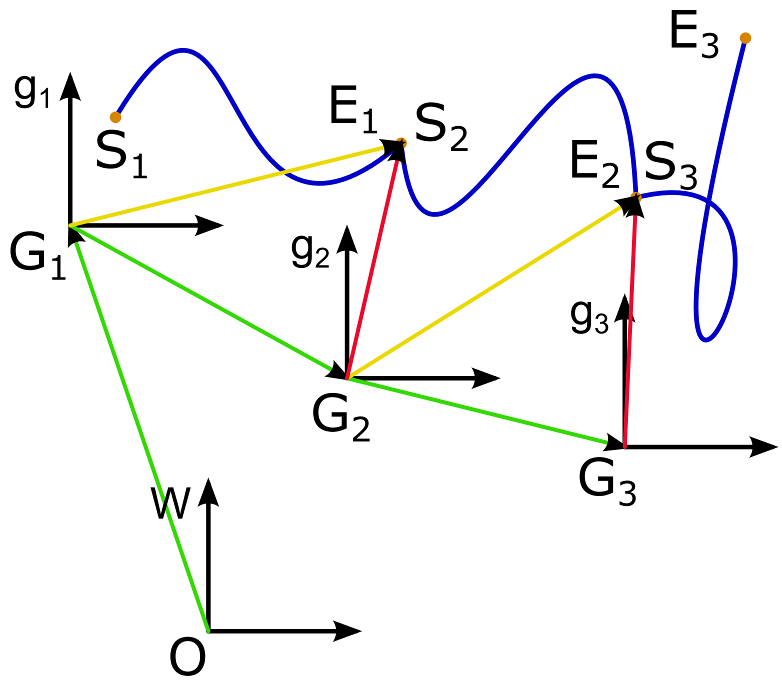


Figure 5.3: An example showing the vectors used to calculate the ground pivots G.

and are listed in Table 5.3. We eliminate this gap by adjusting the ground pivot location of each of the four-link drawing linkages.

For convenience, we introduce local coordinates for each of the m serial chains so that $G_j = (0,0), j = 1,\ldots,m$. This yields the local coordinates of these curves

$$h_j(\theta,\lambda) = L_1^j u_1^j + M_1^j v_1^j + L_2^j u_2^j + M_2^j v_2^j, \quad j = 1,\ldots,m. \tag{5.5}$$

The start and end points, S_j and E_j of these curves are given by

$$S_j = h_j(0,\lambda), \text{ and } E_j = h_j(\pi/2,\lambda), \quad j = 1,\ldots,m. \tag{5.6}$$

Compute the ground pivots for each of the serial chains relative to the ground pivot of the first chain, G_{1j}, by requiring that the end point E_j of curve j be the start point S_{j+1} of the curve $j+1$, see Figure 5.3. This yields

$$G_{1j} = \sum_{i=1}^{j-1} (E_i - S_{i+1}), \quad j = 2, \ldots, m. \tag{5.7}$$

Assemble the new system of curves by positioning the ground pivot of the first serial chain at the original ground pivot location to obtain,

$$\bar{s}_j(\theta, \lambda) = G_1 + G_{1j} + h_j(\theta, \lambda), \quad j = 1, \ldots, m, \tag{5.8}$$

where we consider $G_{11} = 0$,

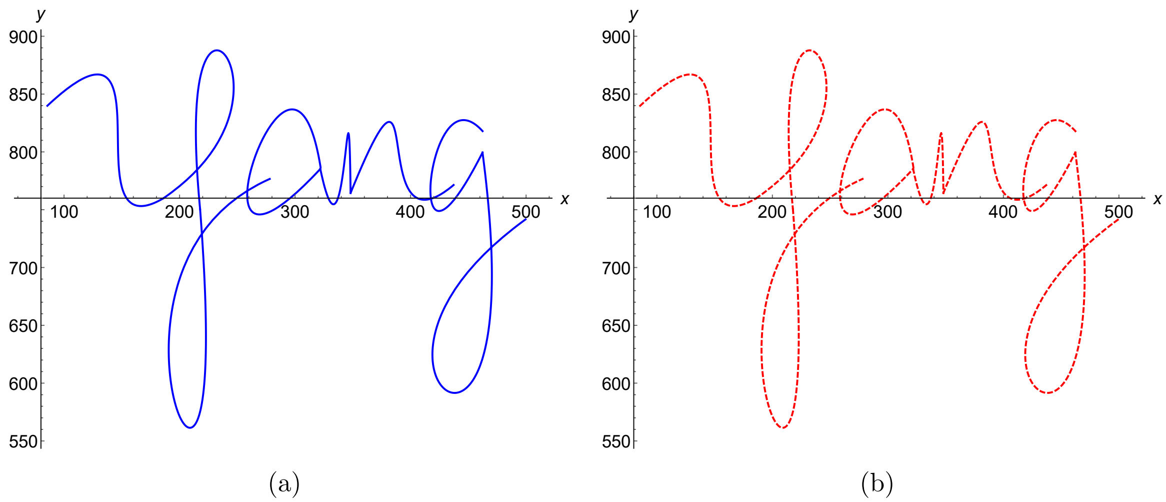


Figure 5.4: (a) The curves generated by six links serial chains (b) The curves generated by corrected four links serial chains.

Figure 5.4 shows the curves $s_j(\theta, \lambda)$ drawn by the six-link serial chains and the curves $\bar{s}_j(\theta, \lambda)$ drawn by four-link serial chains. And Figure 5.5 provides a comparison of the two curves.

Thus, the drawing linkage for the name "Yang" consists of eight four-link coupled serial chains that are shown separately in Figure 5.6 and together in Figure 5.7.

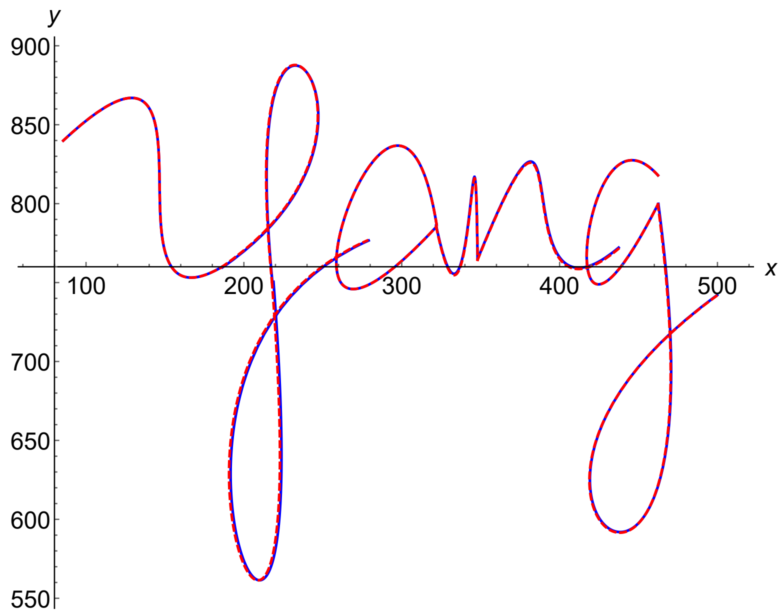


Figure 5.5: The comparison of curves generated by six links and corrected four links serial chains.

Table 5.4: Link lengths and phase angles to sign a name.

Curve j	G_j	L_1^j	ψ_1^j	M_1^j	η_1^j	L_2^j	ψ_2^j	M_2^j	η_2^j
1	(110.16, 751.07)	138.90	-1.43	169.68	2.81	103.63	0.98	103.63	0.98
2	(108.26, 449.70)	125.10	1.09	355.82	1.86	93.15	-0.87	93.15	-0.87
3	(375.42, 1190.88)	522.31	-2.96	207.94	-0.20	169.45	-1.11	169.45	-1.11
4	(460.24, 766.96)	103.87	-1.30	296.30	-2.62	140.99	1.24	140.99	1.24
5	(316.70, 739.29)	147.30	2.29	177.45	1.02	107.94	-1.53	107.94	-1.53
6	(401.35, 681.18)	137.07	-0.98	209.34	3.13	105.53	1.18	105.53	1.18
7	(570.09, 887.60)	85.04	-1.77	203.83	-2.15	92.53	1.46	92.53	1.46
8	(601.78, 1197.03)	496.27	-2.82	166.32	-0.48	122.50	-0.72	122.50	-0.72

Table 5.5: Control points for the nine Bezier curves that define the Chinese character "long," (units mm).

Curve j	P_0^j	P_1^j	P_2^j	P_3^j
1	(175.31, 724.27)	(224.14, 698.20)	(168.03, 709.63)	(166.37, 680.51)
2	(166.37, 680.51)	(226.98, 713.13)	(171.78, 671.48)	(191.97, 652.84)
3	(191.97, 652.84)	(239.31, 579.43)	(135.07, 587.58)	(214.57, 657.92)
4	(214.57, 657.92)	(273.38, 729.78)	(271.95, 717.82)	(250.07, 718.32)
5	(223.72, 721.20)	(252.27, 750.96)	(217.29, 626.42)	(259.36, 659.51)
6	(259.36, 659.51)	(213.07, 595.59)	(323.64, 585.64)	(299.35, 681.46)
7	(299.35, 681.46)	(295.66, 750.18)	(337.36, 658.35)	(353.31, 720.09)
8	(353.31, 720.09)	(366.15, 742.17)	(424.29, 711.36)	(403.75, 696.70)
9	(403.75, 696.70)	(385.52, 686.45)	(356.55, 739.57)	(423.29, 755.08)

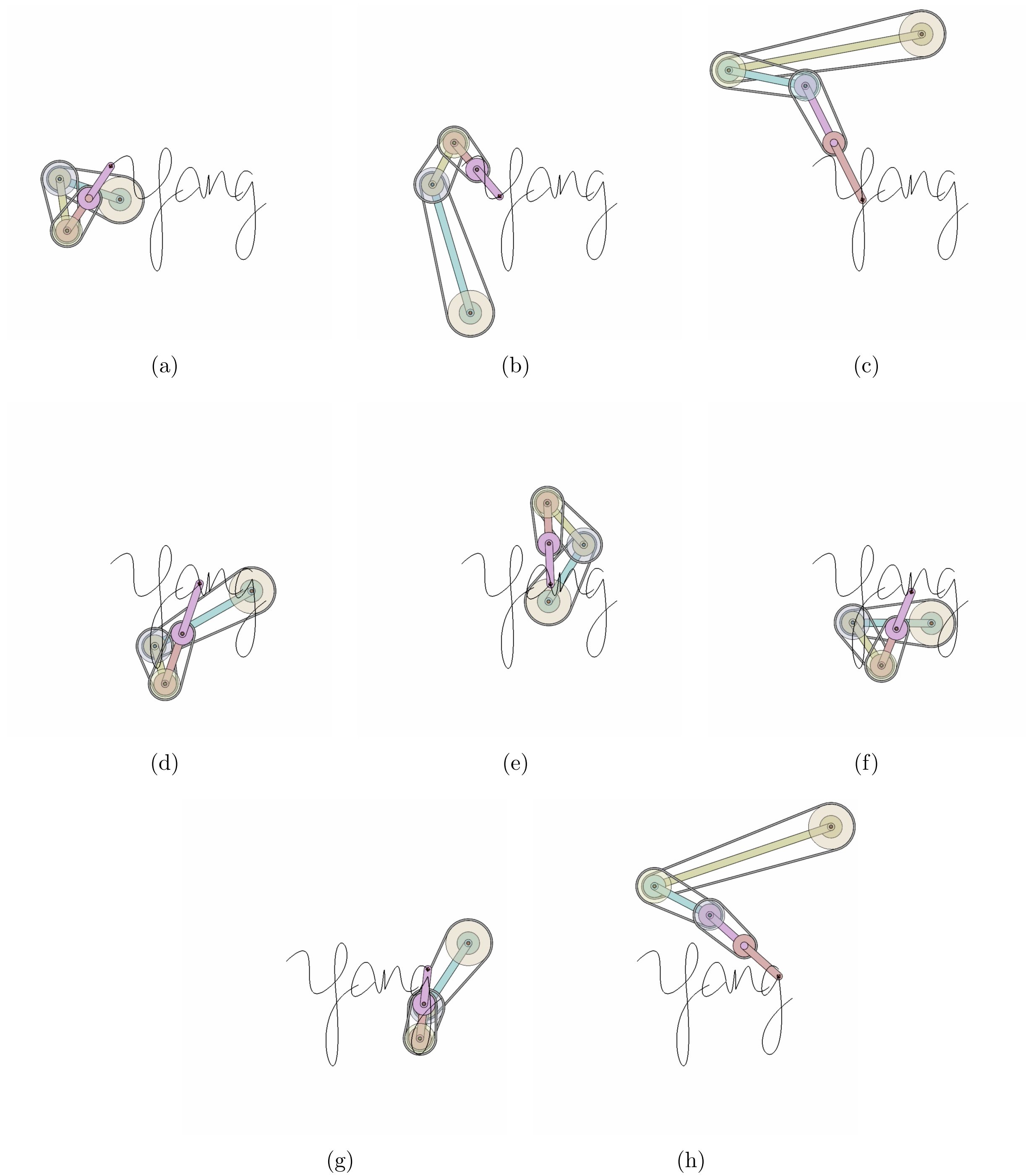


Figure 5.6: The eight four-link coupled serial chains that draw the Bezier curves defining the cursive "Yang."

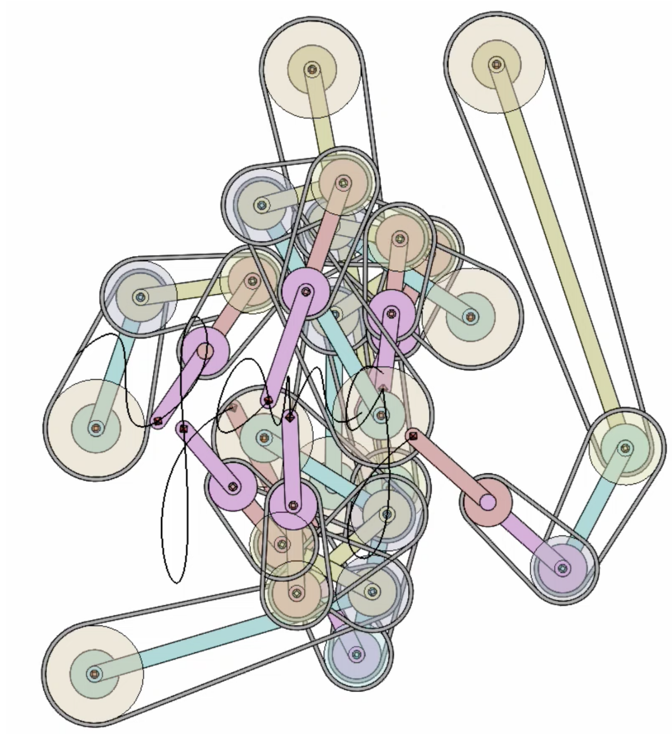


Figure 5.7: The linkage system that can draw "Yang" in cursive. These chains are driven by the same input.

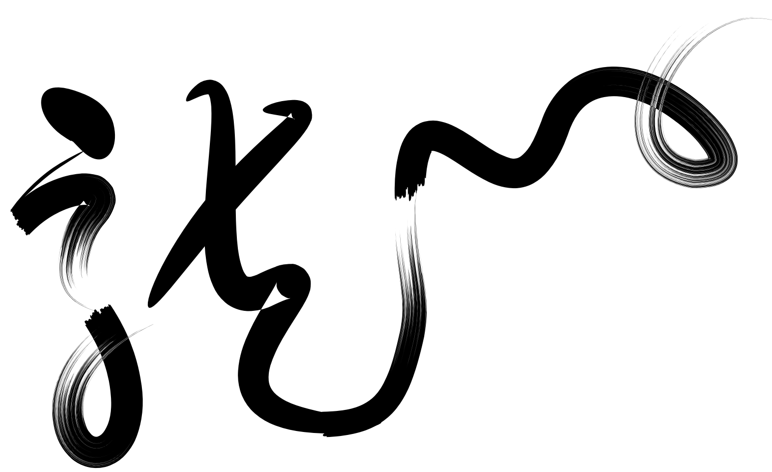


Figure 5.8: Script form of the Chinese character "long," or dragon.

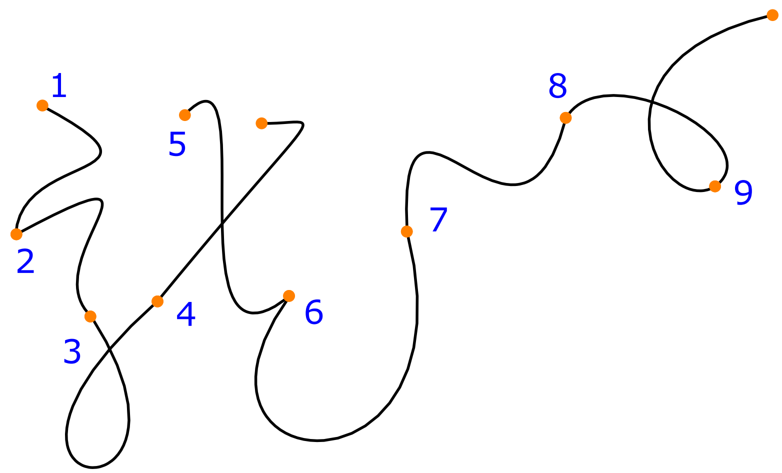
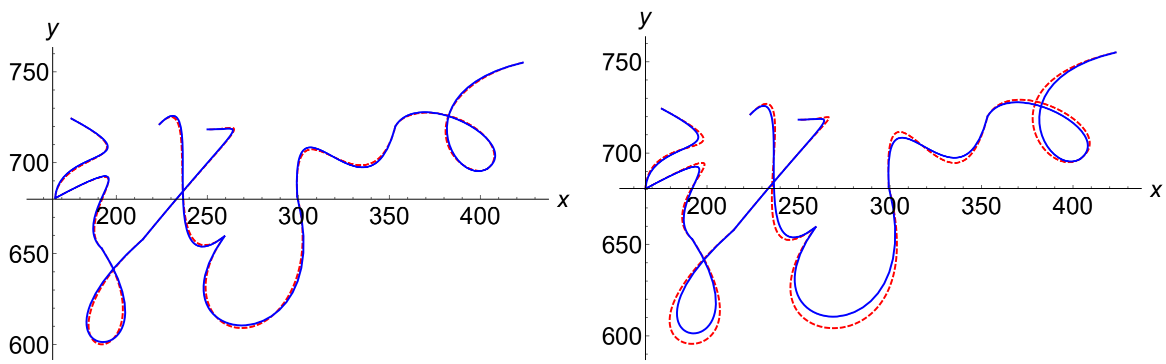


Figure 5.9: Nine Bezier curves used to define the script form of the Chinese character "long." The control points are listed in Table 5.5.

5.2 Script Chinese Characters

In this section, we design the drawing linkage for the script form of the Chinese character "long," which means dragon. Figure 5.8 is a copy of the Chinese script for "long" drawn using the drawing software Inkscape. From this drawing, we obtain nine cubic Bezier curves that form the character, shown in Figure 5.9. The control points for these nine cubic Bezier curves are listed in Table 5.5.



(a) (b)

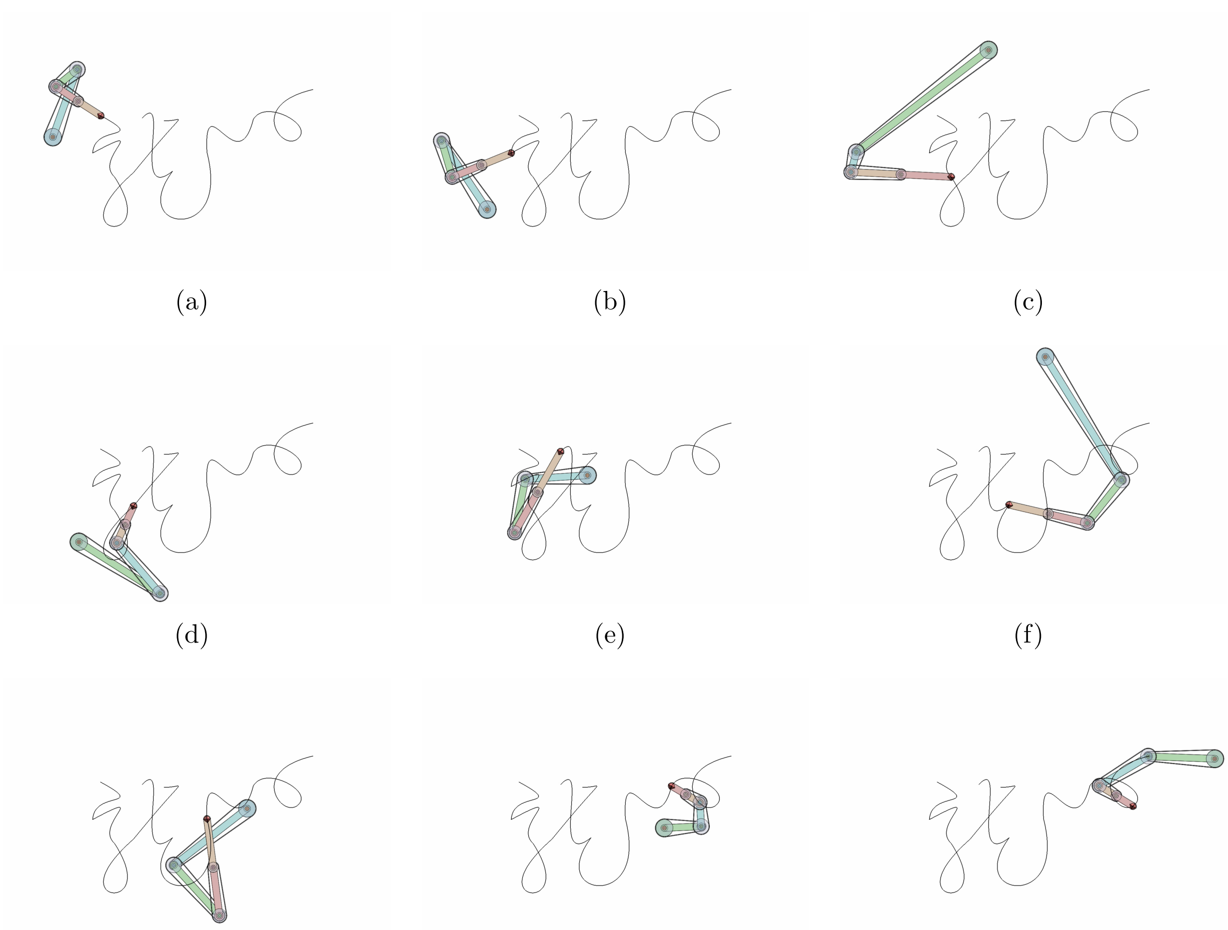
Figure 5.10: (a) The cubic trigonometric Bezier curves with $\lambda = -0.5$ (dashed red) fits the cubic Bezier curves (solid blue) (b) The cubic trigonometric Bezier curves with $\lambda = 0.01$ (dashed red) varies from the cubic Bezier curves (solid blue).

For this example, we found that the shape parameter can be used either to match the cubic Bezier curves or to reduce the size of the end-links to simplify the coupled serial chain. Figure 5.10 shows that by specifying $\lambda = -0.5$, we can make the maximum deviation $\epsilon \leq 0.01$ for all the curves. On the other hand, if we adjust the desired shape of this Chinese character so that $\lambda = 0.01$, we find that the last two links are of lengths less than 1% of the other four links. This provides a simpler drawing linkage.

Therefore, we select $\lambda = 0.01$ and, as was done in the previous example, we drop these last two links of the resulting serial chain. The character "long" is drawn using nine four-link coupled serial chains. The dimension of these chains are listed in Table 5.6, and illustrated in Figure 5.11, 5.12.

Table 5.6: Link lengths and phase angles for the script Chinese character.

Curve j	G_j	L_1^j	ψ_1^j	M_1^j	η_1^j	L_2^j	ψ_2^j	M_2^j	η_2^j
1	(119.85, 699.30)	31.64	-2.46	83.46	1.23	31.07	-0.58	31.07	-0.58
2	(138.17, 615.00)	44.15	-1.28	96.07	2.16	37.13	0.39	37.13	0.39
3	(235.22, 800.41)	194.63	-2.49	23.99	-1.83	58.47	-0.05	58.47	-0.05
4	(150.23, 616.07)	111.90	-0.56	77.48	2.28	23.56	1.14	23.56	1.14
5	(255.19, 693.70)	63.41	-1.77	72.61	-3.08	54.22	1.055	54.22	1.05
6	(301.55, 831.49)	63.98	-2.25	168.72	-1.02	46.86	2.91	46.86	2.91
7	(346.22, 693.40)	79.14	-0.83	107.94	-2.48	56.70	1.70	56.70	1.70
8	(344.94, 670.83)	43.03	0.04	27.79	1.63	19.09	2.61	19.09	2.61
9	(499.26, 751.36)	77.71	3.11	66.37	-2.60	22.96	-0.55	22.96	-0.55



(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

Figure 5.11: The nine four-link coupled serial chains that draw the Bezier curves defining Chinese character "long."

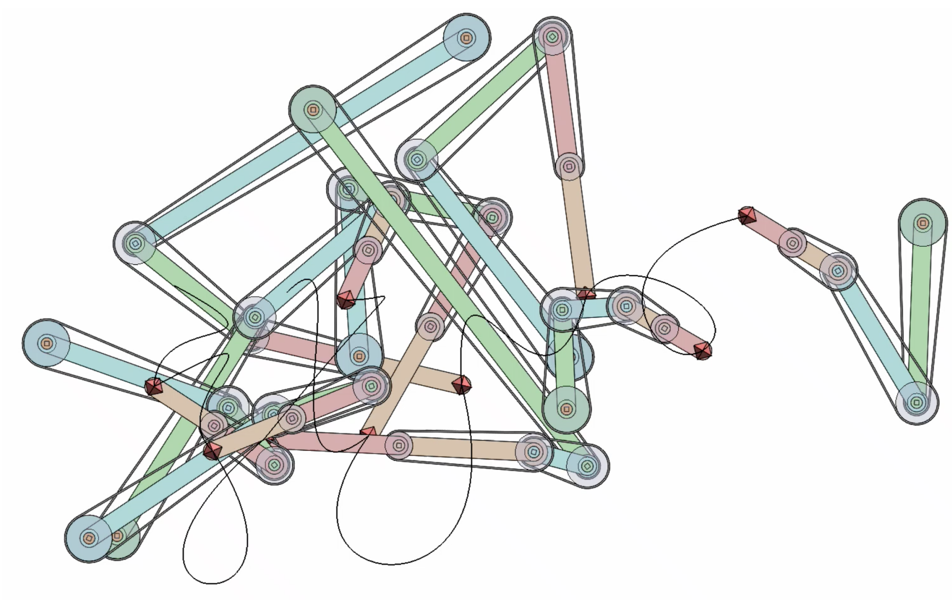


Figure 5.12: The linkage system that draws the script form of the Chinese character “long.”

5.3 Practical Considerations

This chapter shows that a cubic Bezier curve can be drawn by a six-link serial chain. The designer can use the shape parameter of cubic trigonometric Bezier curves to fit the specified cubic Bezier curves, or to adjust the curve so that the two end-links of the six-link serial chain are small enough to be neglected resulting in a four-link drawing linkage.

This means that an assembly of m cubic Bezier curves can be drawn mechanically by m coupled six-link serial chains connected to a single actuator, and in some cases by m four-link serial chains. This is a significant reduction in complexity of drawing linkages relative to current implementations of Kempe's universality theorem [41, 21, 14, 37].

However, the manufacture of such an assembly of coupled serial chains shown in Figures 5.7 and 5.12 requires further study to evaluate accuracy and interference.

5.4 Summary

This chapter presents a methodology for the design one degree-of-freedom coupled serial chains to draw Bezier curves. The control points of cubic Bezier curves are used to define cubic trigonometric Bezier curves that can be drawn by six-link coupled serial chains. We show that the shape parameter of the cubic trigonometric Bezier curves can be used to approximate the cubic Bezier curve or to adjust the curve to eliminate the last two links of the serial chain.

Two examples demonstrate this methodology. The first yields a linkage system that draws the cursive letters spelling "Yang" using eight four-link coupled serial chains driven by one actuator. The second draws a script version of the Chinese character "long," or dragon, using eight inerconnected four-link coupled serial chains.

Chapter 6

Mechanisms to Draw Spherical Curves

This chapter presents the theory of design a spherical linkage to draw a spherical curve. The purpose is to design a linkage that can draw the projection of the planar trigonometric curve onto sphere. Two examples are provided to show the steps to construct the desired spherical linkage. The spherical Trifolium linkage can trace the projection of the planar Trifolium exactly, while for the spherical butterfly linkage, there exists little offset which is caused by the relative scale of the specified spherical curve and the sphere.

6.1 Spherical Trifolium Curve

Liu and McCarthy [40] have developed the method of synthesis of coupled serial planar linkage to draw trigonometric curves. Here we would like to develop the method of synthesis a equivalent spherical linkage.

We start with defining a sphere with radius r. The Trifolium curve $\mathbf{P}_T(\theta)$ in space is defined

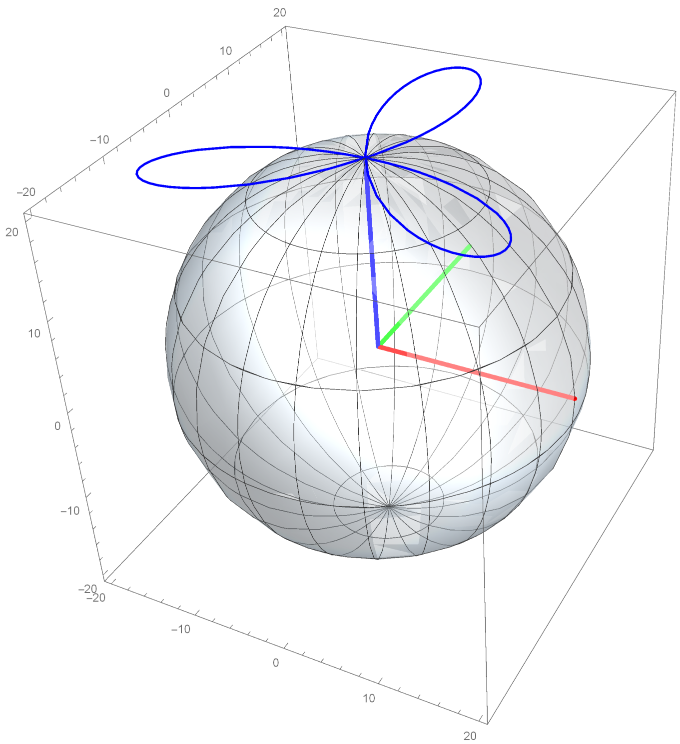


Figure 6.1: Plotting Trifolium $\mathbf{P}_T(\theta)$ in the plane tangent to sphere.

as

$$\mathbf{P}_T(\theta) = \begin{Bmatrix} -\cos(2\theta) - \cos(4\theta) \\ \sin(2\theta) - \sin(4\theta) \\ r \end{Bmatrix} \tag{6.1}$$

in which $r = 2$. Figure 6.1 shows a plot of $\mathbf{P}_T(\theta)$.

Table 6.1 reproduces the dimensions that define the coupled serial chain that draw planar Trifolium curve $\mathbf{P}_T(\theta)$.

We can project the Trifolium onto the sphere by normalizing the vector curve $\mathbf{P}_T(\theta)$ to obtain

$$\mathbf{S}_T(\theta) = r\mathbf{P}_T(\theta)/|\mathbf{P}_T|. \tag{6.2}$$

Table 6.1: Dimensions of the single coupled serial chain to draw planar Trifolium curve $\mathbf{P}_T(\theta)$.

k	L_k	ψ_k	M_k	η_k
1	0	0	0	0
2	0	0	1	π
3	0	0	0	0
4	1	π	0	0

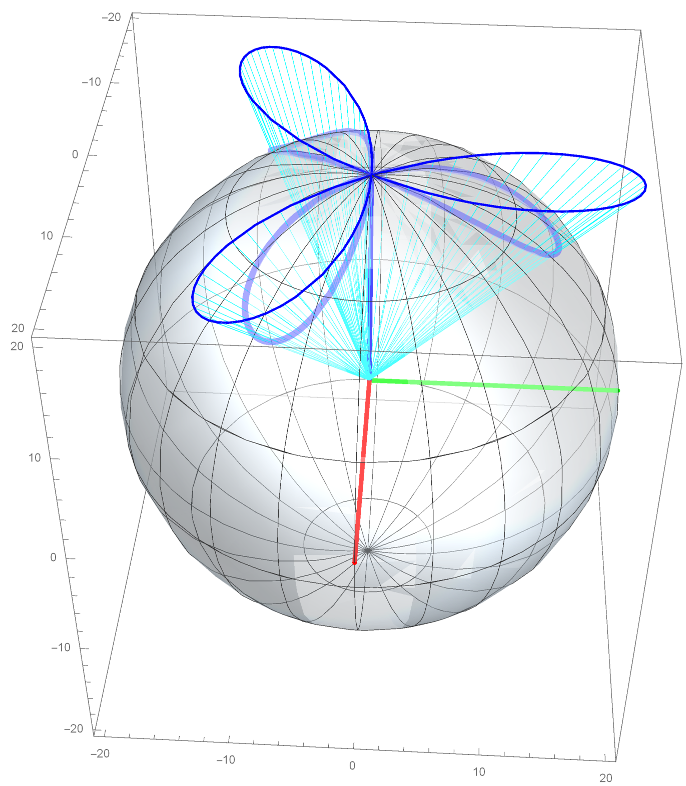


Figure 6.2: The projected curve $\mathbf{S}_T(\theta)$.

The projected curve $\mathbf{S}_T(\theta)$ is shown in Figure 6.2.

In order to define a spherical coupled serial chain that defines a curve approximating the projected Trifolium, we follow the procedure presented in Chapter 2.

The first step is to compute the angular link dimensions, λ_k and μ_k, of the spherical coupled serial chain from the planar linkage dimensions L_k and M_k, $k = 1, \ldots, 4$, using the equations,

$$\lambda_k = \arctan L_k/r, \quad \mu_k = \arctan M_k/r. \tag{6.3}$$

Thus, we have

$$\lambda_1 = 0, \quad \mu_1 = 0, \quad \lambda_2 = 0, \quad \mu_2 = 0.46, \quad \lambda_3 = 0, \quad \mu_3 = 0, \quad \lambda_4 = 0.46, \quad \mu_4 = 0. \tag{6.4}$$

The coordinate vectors $\mathbf{S}_k$ and $\mathbf{T}_k$, $k = 1, \ldots, 4$, which define each of the joints of the spherical coupled serial, are obtained following the procedure defined in Chapter 2:

1. The first axis of the serial chain is $\mathbf{S}_1$, given by

$$\mathbf{S}_1 = \mathbf{k} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}. \tag{6.5}$$

2. The $\mathbf{T}_1$, is obtained by rotating the x-axis, $\mathbf{V}_1 = \mathbf{i}$, around $\mathbf{S}_1$ by the phase angle ψ_1 to define the vector $\mathbf{W}_1$. We then rotate about $\mathbf{W}_1$ by the angle λ_1, that is

$$\mathbf{W}_1 = [R(\psi_1, \mathbf{S}_1)]\mathbf{V}_1 = \mathbf{i}, \quad \text{and} \quad \mathbf{T}_1 = [R(\lambda_1, \mathbf{W}_1)]\mathbf{S}_1 = \mathbf{k}. \tag{6.6}$$

3. The axis $\mathbf{S}_2$ is obtained by rotating $\mathbf{W}_1$ around $\mathbf{T}_1$ by the phase η_1 to obtain $\mathbf{V}_2$, that is

$$\mathbf{V}_2 = [R(\eta_1, \mathbf{T}_1)]\mathbf{W}_1 = \mathbf{i}, \quad \text{and} \quad \mathbf{S}_2 = [R(\mu_1, \mathbf{V}_2)]\mathbf{T}_1 = \mathbf{k}. \tag{6.7}$$

4. The axis $\mathbf{T}_2$ is obtained as

$$\mathbf{W}_2 = [R(\psi_2, \mathbf{S}_2)]\mathbf{V}_2 = \mathbf{i}, \quad \text{and} \quad \mathbf{T}_2 = [R(\lambda_2, \mathbf{W}_2)]\mathbf{S}_2 = \mathbf{k}. \tag{6.8}$$

5. The axis $\mathbf{S}_3$ is obtained as

$$\mathbf{V}_3 = [R(\eta_2, \mathbf{T}_2)]\mathbf{W}_2 = -\mathbf{i}, \quad \text{and} \quad \mathbf{S}_3 = [R(\mu_2, \mathbf{V}_3)]\mathbf{T}_2 = \begin{Bmatrix} 0 \\ 0.44 \\ 0.90 \end{Bmatrix}. \tag{6.9}$$

6. The axis $\mathbf{T}_3$ is obtained as

$$\mathbf{W}_3 = [R(\psi_3, \mathbf{S}_3)]\mathbf{V}_3 = -\mathbf{i}, \quad \text{and} \quad \mathbf{T}_3 = [R(\lambda_3, \mathbf{W}_3)]\mathbf{S}_3 = \mathbf{S}_3 = \begin{Bmatrix} 0 \\ 0.44 \\ 0.90 \end{Bmatrix}. \tag{6.10}$$

7. The axis $\mathbf{S}_4$ is obtained as

$$\mathbf{V}_4 = [R(\eta_3, \mathbf{T}_3)]\mathbf{W}_3 = -\mathbf{i}, \quad \text{and} \quad \mathbf{S}_4 = [R(\mu_3, \mathbf{V}_4)]\mathbf{T}_3 = \begin{Bmatrix} 0 \\ 0.44 \\ 0.90 \end{Bmatrix}. \tag{6.11}$$

8. The axis $\mathbf{T}_4$ ise obtained as

$$\mathbf{W}_4 = [R(\psi_4, \mathbf{S}_4)]\mathbf{V}_4 = -\mathbf{i}, \quad \text{and} \quad \mathbf{T}_4 = [R(\lambda_4, \mathbf{W}_4)]\mathbf{S}_4 = \mathbf{S}_4 = \begin{Bmatrix} 0 \\ 0.79 \\ 0.61 \end{Bmatrix}. \tag{6.12}$$

9. The axis $\mathbf{S}_5$ is obtained as

$$\mathbf{V}_5 = [R(\eta_4, \mathbf{T}_4)]\mathbf{W}_4 = -\mathbf{i}, \quad \text{and} \quad \mathbf{S}_5 = [R(\mu_4, \mathbf{V}_5)]\mathbf{T}_4 = \begin{Bmatrix} 0 \\ 0.79 \\ 0.61 \end{Bmatrix}. \tag{6.13}$$

This data is assembled into Table 6.2. We compute the rotation matrices for each of the non-zero link angles, and obtain the equation for the spherical coupled serial chain,

$$\mathbf{L}_T(\theta) = [R(-2\theta, \mathbf{T}_2)][R(4\theta, \mathbf{S}_4)]\mathbf{S}_5. \tag{6.14}$$

This is our mechanically generated spherical Trifolium curve.

Table 6.2: Configurations for projected Trifolium curve $\mathbf{L}_T(\theta)$.

k	$\mathbf{S}_k$	λ_k	ψ_k	$\mathbf{T}_k$	μ_k	η_k
1	$(0,0,1)^T$	0	0	$(0,0,1)^T$	0	0
2	$(0,0,1)^T$	0	0	$(0,0,1)^T$	0.46	π
3	$(0,0.44,0.90)^T$	0	0	$(0,0.44,0.90)^T$	0	0
4	$(0,0.44,0.90)^T$	0.46	π	$(0,0.79,0.61)^T$	0	0

The mechanically generated spherical Trifolium $\mathbf{L}_T(\theta)$ is shown in Figure 6.3 and compared to the projected Trifolium, $\mathbf{S}_T(\theta)$, Figure 6.4 and Figure 6.5.

A SolidWorks model, Figure 6.6 and Figure 6.7, shows the curve generated by the spherical coupled serial chain.

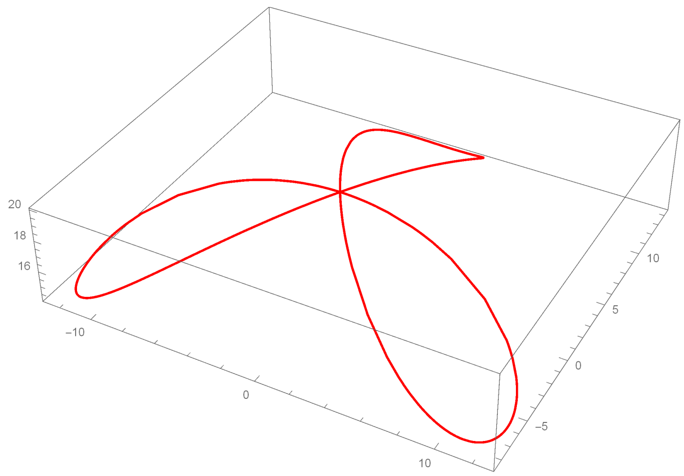


Figure 6.3: Spherical Trifolium curve $\mathbf{L}_T(\theta)$ generated by spherical linkage.

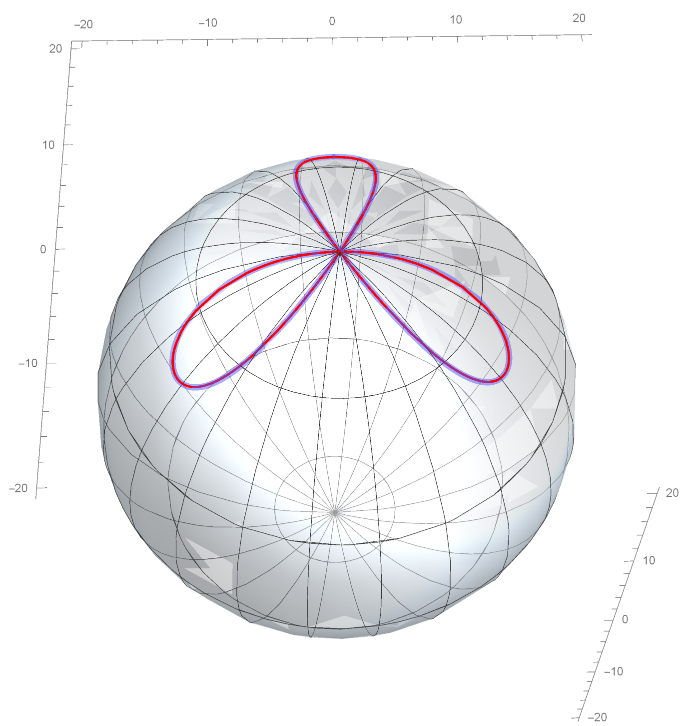


Figure 6.4: The generated spherical Trifolium curve $\mathbf{L}_T(\theta)$ approximates the projected curve $\mathbf{S}_T(\theta)$.

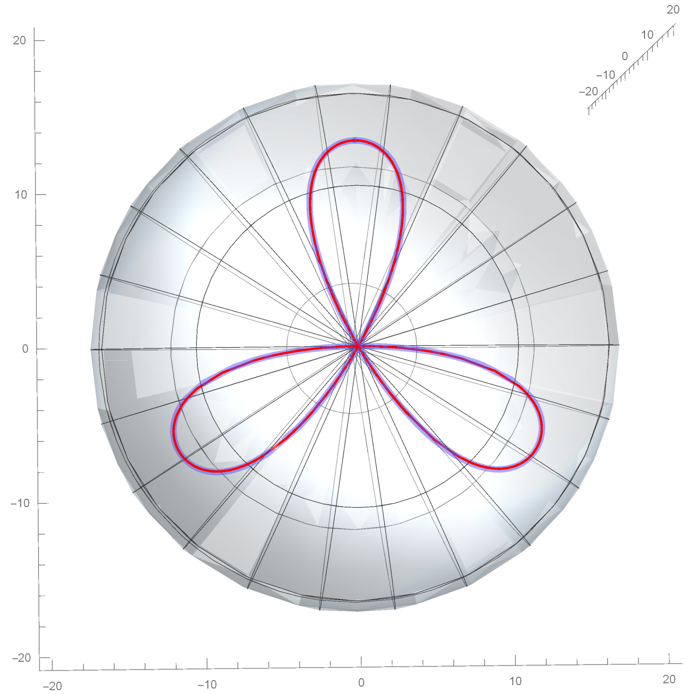


Figure 6.5: Top view of the generated spherical Trifolium curve $\mathbf{L}_T(\theta)$ and the projected curve $\mathbf{S}_T(\theta)$.

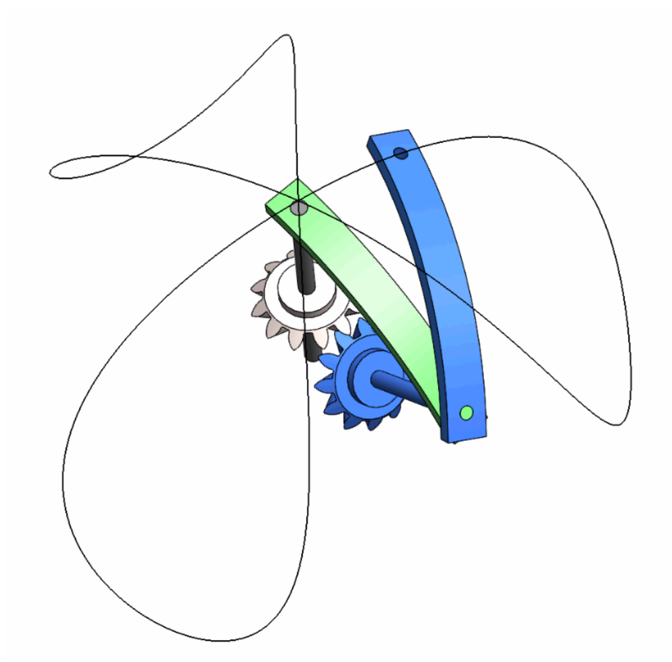


Figure 6.6: Spherical Trifolium curve $\mathbf{L}_T(\theta)$ simulation.

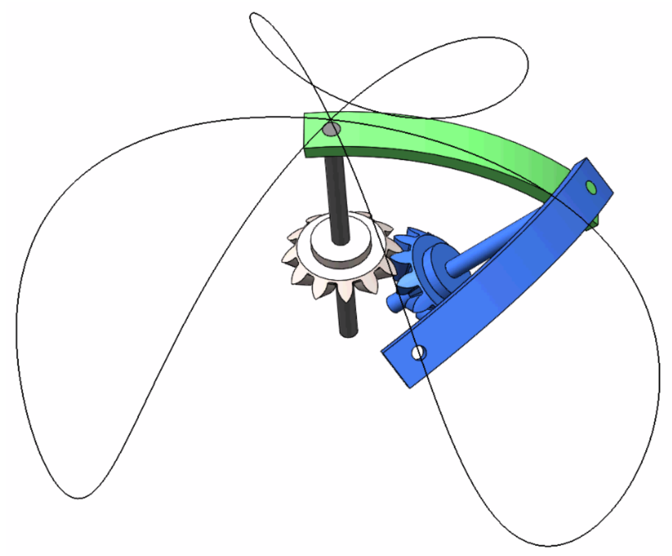


Figure 6.7: Spherical Trifolium curve $\mathbf{L}_T(\theta)$ simulation.

6.2 Spherical Butterfly Curve

In this section, we determine the spherical coupled serial chain that mechanically generates an approximation to the projection of the butterfly curve. The planar butterfly curve $\mathbf{P}_B(\theta)$ shown in Figure 6.8 is given by,

$$\mathbf{P}_B(\theta) = \begin{Bmatrix} 9\cos(\theta) + 0.75\cos(3\theta) - 1.25\cos(5\theta) + 1.25\sin(2\theta) + 2.4\sin(4\theta) + 0.25\sin(6\theta) - \sin(8\theta) \\ 1.15\cos(2\theta) + 0.1\cos(4\theta) - 2.25\cos(6\theta) + \cos(8\theta) + 5\sin(\theta) + 3.25\sin(3\theta) - 1.25\sin(5\theta) \\ r \end{Bmatrix} \tag{6.15}$$

where r is the radius of the sphere and been set to be $r = 20$. This radius is chosen so that the generated spherical butterfly curve is located above latitude 60 degrees.

The projection of the butterfly curve onto the sphere is obtained by normalizing the curve $\mathbf{P}_B(\theta)$,

$$\mathbf{S}_B(\theta) = r\mathbf{P}_B(\theta)/|\mathbf{P}_B|. \tag{6.16}$$

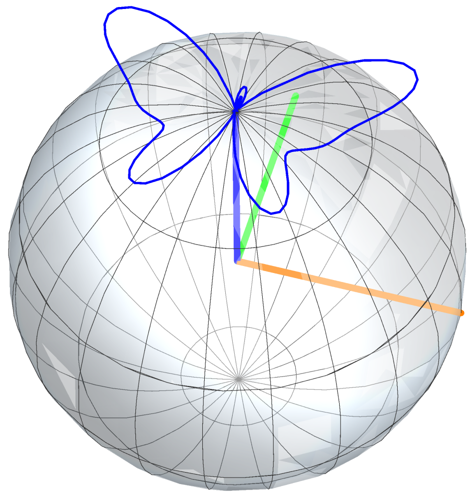


Figure 6.8: Planar butterfly curve $\mathbf{P}_B(\theta)$.

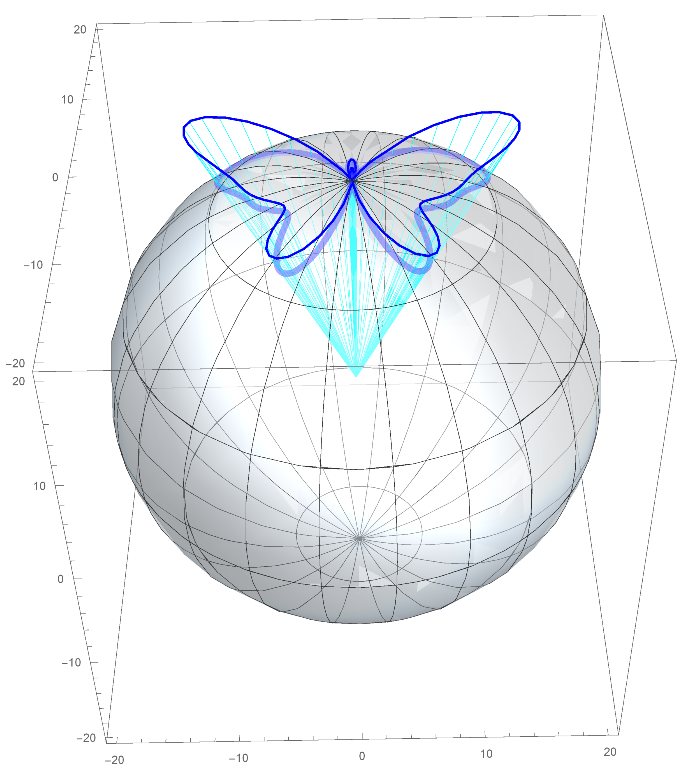


Figure 6.9: The projected butterfly curve $\mathbf{S}_B(\theta)$.

The dimensions of the links of the planar coupled serial chain that draws the butterfly curve obtained above is reproduced in Table Table 6.3.

Table 6.3: Speed ratios, link length ratios and phase angles for the planar butterfly curve $\mathbf{P}_B(\theta)$

k	L_k	ψ_k	M_k	η_k
1	7	0	2	0
2	0	0	1.15	$\pi/2$
3	2	0	1.25	π
4	1.15	$-\pi/2$	1.25	$\pi/2$
5	1.25	π	0	0
6	1.25	$-\pi/2$	1	$-\pi/2$
7	0	0	0	0
8	1	$\pi/2$	0	0

The spherical coupled serial chain that draws this curve is is constructed the procedure in Chapter 2 and presented in detail above for the Trifolium linkage. The result is a set of eight sets of link angles λ_k and μ_k, and points $\mathbf{S}_k$ and $\mathbf{T}_k$, $i = 1, \ldots, 8$, listed in Table 6.4.

Table 6.4: Configurations for projected Butterfly curve $\mathbf{L}_B(\theta)$.

k	$\mathbf{S}_k$	λ_k	ψ_k	$\mathbf{T}_k$	μ_k	η_k
1	$(0, 0, 1)^T$	0.34	0	$(0.32, 0, 0.95)^T$	0.1	0
2	$(0.41, 0, 0.91)^T$	0	0	$(0.41, 0, 0.91)^T$	0.06	$\pi/2$
3	$(0.5, 0, 0.87)^T$	0.1	0	$(0.44, 0, 0.90)^T$	0.06	π
4	$(0.39, 0, 0.92)^T$	0.06	$-\pi/2$	$(0.39, 0.05, 0.92)^T$	0.06	$\pi/2$
5	$(0.39, 0, 0.92)^T$	0.06	π	$(0.39, 0.05, 0.92)^T$	0	0
6	$(0.39, 0.50, 0.92)^T$	0.06	$-\pi/2$	$(0.39, 0, 0.92)^T$	0.05	$-\pi/2$
7	$(0.39, -0.05, 0.92)^T$	0	0	$(0.39, -0.05, 0.92)^T$	0	0
8	$(0.39, -0.05, 0.92)^T$	0.05	$\pi/2$	$(0.39, 0, 0.92)^T$	0	0

Ignoring the matrices associated with zero links angles, we obtain the equation of the mechanically generated spherical butterfly curve as,

$$\begin{aligned}\mathbf{L}_B(\theta) =& [R(\theta, \mathbf{S}_1)][R(-\theta, \mathbf{T}_1][R(-2\theta, \mathbf{T}_2)][R(3\theta, \mathbf{S}_3)][R(-3\theta, \mathbf{T}_3)] \\ & [R(4\theta, \mathbf{S}_4)][R(-4\theta, \mathbf{T}_4)][R(5\theta, \mathbf{S}_5)][R(6\theta, \mathbf{S}_6)][R(-6\theta, \mathbf{T}_6)][R(8\theta, \mathbf{S}_8)]\mathbf{S}_9,\end{aligned} \tag{6.17}$$

where $\mathbf{S}_9 = (0.39, 0, 0.92)^T$.

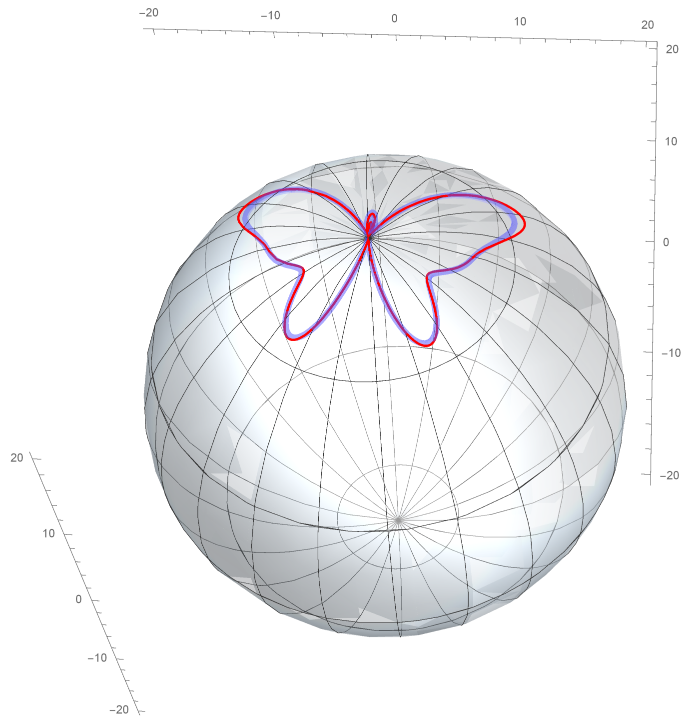


Figure 6.10: Comparison of the spherical butterfly curve $\mathbf{L}_B(\theta)$ and the planar one's projection $\mathbf{S}_B(\theta)$.

The comparison of the projected butterfly curve $\mathbf{S}_B(\theta)$ and our mechanically generated spherical butterfly curve, $\mathbf{L}_B(\theta)$ is shown in Figure 6.10 and Figure 6.11.

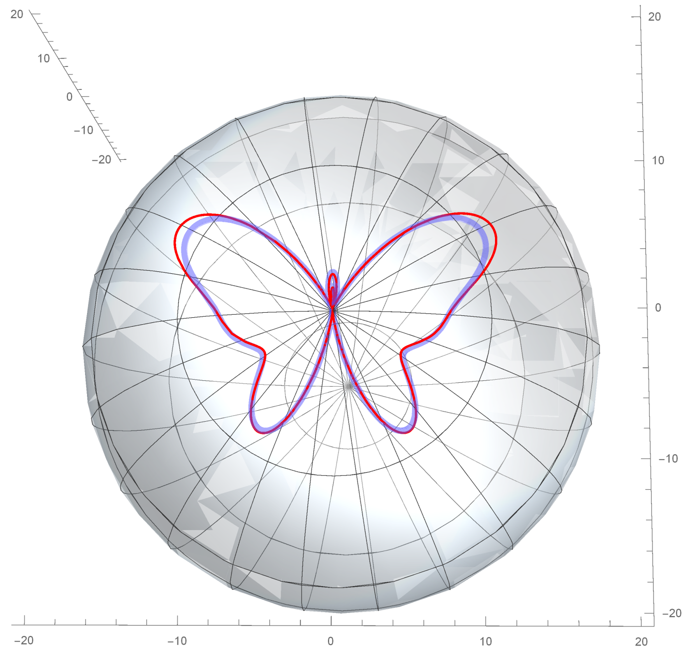


Figure 6.11: Comparison of the spherical butterfly curve $\mathbf{L}_B(\theta)$ and the planar one's projection $\mathbf{S}_B(\theta)$.

6.3 Summary

This chapter presents the methodology of synthesis a spherical linkage to draw a specified spherical curve. This method is demonstrated by generating a spherical Trifolium curve and a spherical butterfly curve. We show how to obtain the configurations of the spherical linkage. Future work will be researching the synthesis coupled serial spherical chain to draw Bezier curve.

Chapter 7

Manufacture of Coupled Serial Chains

7.1 Introduction

In this section, we presents a design and manufacturing methodology for a mechanical system that draws trigonometric curves assembled from a series of links connected by gears. We demonstrate this technique using 11 gear-coupled links that draw a butterfly curve. The equation of a butterfly curve is converted to the relative rotations of the links of a coupled serial chain assembled so it operates with one input. We present a procedure to determine the adjustments to the gear ratios and link dimensions necessary for practical manufacture of the mechanism. The results are demonstrated by a working prototype.

7.2 Preliminaries

Liu and McCarthy [40] have shown that a trigonometric plane curve, $\mathbf{P} = (x(\theta), y(\theta))$,

$$P = \begin{Bmatrix} x(\theta) \\ y(\theta) \end{Bmatrix} = \begin{Bmatrix} \sum_{k=0}^{m} a_k \cos k\theta + b_k \sin k\theta \\ \sum_{k=0}^{m} c_k \cos k\theta + d_k \sin k\theta \end{Bmatrix}, \tag{7.1}$$

where a_k, b_k, c_k and d_k, $k = 0, \ldots, m$, are real coefficients and $\theta \in [0, 2\pi]$, can be converted to the form,

$$P(\theta) = \begin{Bmatrix} \sum_{k=0}^{m} L_k \cos(k\theta - \psi_k) + M_k \cos(-k\theta - \eta_k) \\ \sum_{k=0}^{m} L_k \sin(k\theta - \psi_k) + M_k \sin(-k\theta - \eta_k) \end{Bmatrix}, \tag{7.2}$$

in which,

$$\begin{aligned} L_k &= \frac{1}{2}\sqrt{(a_k + d_k)^2 + (c_k - b_k)^2}, \\ M_k &= \frac{1}{2}\sqrt{(a_k - d_k)^2 + (c_k + b_k)^2}, \end{aligned} \tag{7.3}$$

and

$$\psi_k = \arctan \frac{c_k - b_k}{a_k + d_k}, \quad \eta_k = \arctan \frac{c_k + b_k}{a_k - d_k}. \tag{7.4}$$

The result is an equation that can be interpreted as a sequence of links driven at a prescribed set of speed ratios $k = 1, \ldots, m$ to draw the curve. Equation 7.3 defines the lengths of the links L_k and M_k and (7.4) defines the phase angles ψ_k and η_k of their movement.

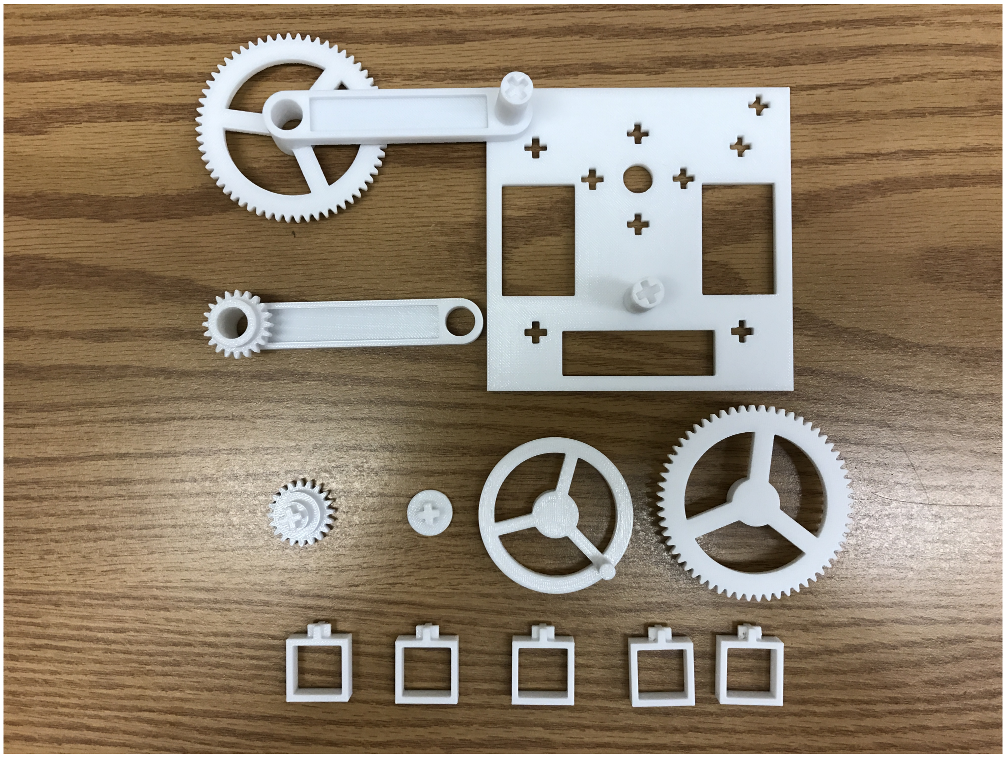


Figure 7.1: A photograph of the component parts of the Trifolium drawing mechanism that were manufactured from ABS using additive manufacture.

7.3 Physical Prototype for the Trifolium

We used the Stratasys Fortus additive manufacturing system to build components for the Trifolium linkage using ABS plastic. Figure 7.1 shows the individual components. Figure 7.2 is the assembled Trifolium drawing mechanism.

In order to show the curve generated by the mechanism, we attached a blue LED to the drawing tip and used a DC motor to drive the base at 100 RPM. A photograph of the draw of the LED is shown in Figure 7.3.

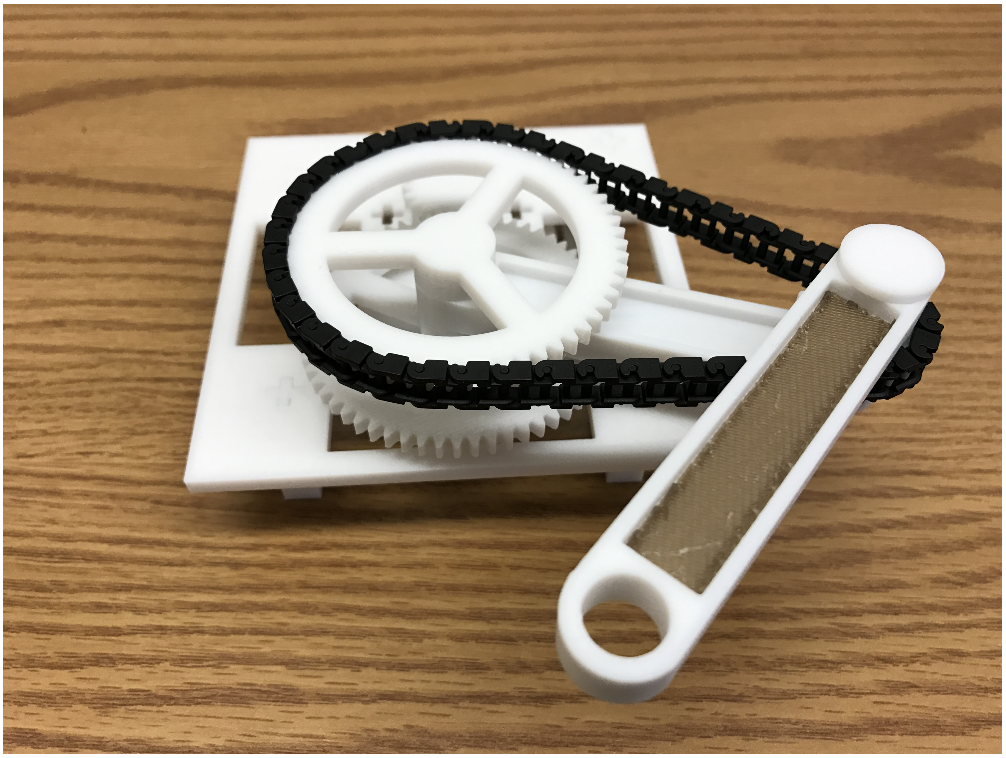


Figure 7.2: A photograph of the assembled mechanism that draws the Trifolium.

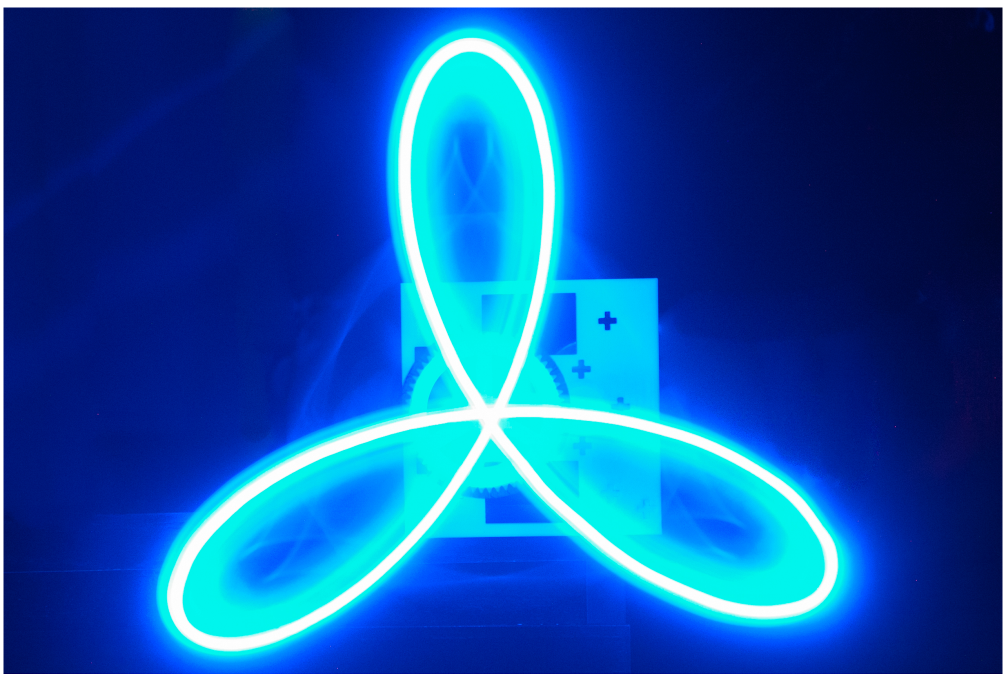


Figure 7.3: This is a photograph showing the Trfolium drawn by the trajectoy of a blue LED positioned at the drawing point of the mechanism.

7.4 Butterfly Curve

When the equation of the Butterly curve [13],

$$P_B: \ \rho(\theta) = 7 - \sin\theta + 2.3\sin 3\theta + 2.5\sin 5\theta - 2\sin 7\theta \\ - 0.4\sin 9\theta + 4\cos 2\theta - 2.5\cos 4\theta, \tag{7.5}$$

is written in the form of (2), we obtain the link lengths and phase angles listed in Table 7.1. The $k = 0$ row defines the ground pivot position. There are totally 14 links consisting the coupled serial chain to generate the butterfly curve shown in Figure 7.4.

Table 7.1: Speed ratios, link length ratios and phase angles for the butterfly curve

k	L_k	ψ_k	M_k	η_k
0	0.25	$-\pi/2$	0.25	$-\pi/2$
1	7	0	2	0
2	0.5	$\pi/2$	1.15	$\pi/2$
3	2	0	1.25	π
4	1.15	$-\pi/2$	1.25	$\pi/2$
5	1.25	π	0	0
6	1.25	$-\pi/2$	1	$-\pi/2$
7	0	0	0	0
8	1	$\pi/2$	0.2	$-\pi/2$
9	0	0	0	0
10	0.2	$\pi/2$	0	0

This is the starting point for the design of our mechanism. In what follows, we adjust these dimensions to facilitate manufacture using Additive Manufacturing.

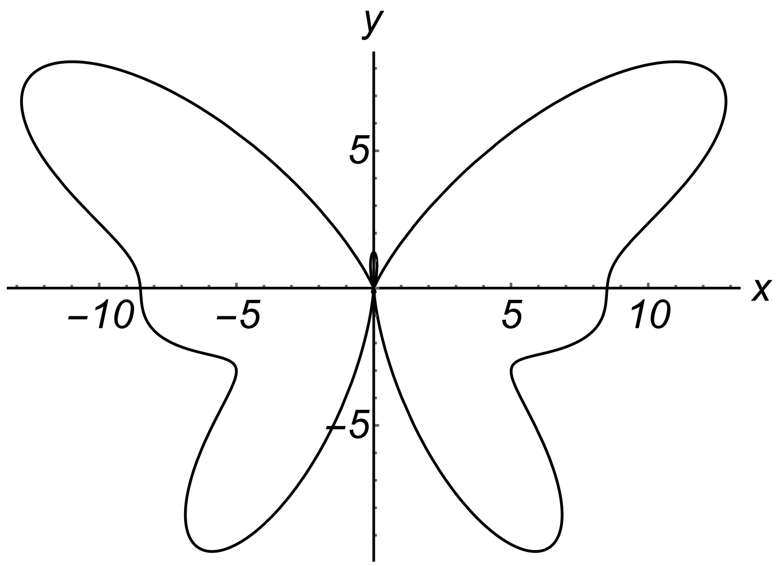


Figure 7.4: A plot of a butterfly curve of dimension ±12 cm wide

7.5 Design Modification

The build envelope of the Stratasys Fortus 450mc additive manufacturing system is 406 × 355 × 406 mm, which sets a limit to the size of our mechanism. Our experiments showed that the smallest feature size that we could reliably generate was a 10mm hole for the link joints. This restricted the size of our gears to a minimum diameter of 15mm, which in turn sets our minimum link dimension to be 20mm.

If the dimension of the smallest link in Table 7.1 to be 20mm, the longest link must be 700mm, which is beyond the build envelop of the Stratasys Fortus printer. If we eliminate the three shortest links, we can reduce the size of the largest link to 210mm. Specifically, we delete the links denoted as L_2, L_{10} and M_8. The resulting Butterfly curve differs from the original curve by a maximum of 21.6mm, see Figure 7.5. The dimensions for manufacturing prototype are listed in Table 7.2. This removes the high frequency terms, which reduces the curvature and softens the curve.

Table 7.2: Speed ratios, link length ratios and phase angles for the manufacturing prototype

k	L_k	ψ_k	M_k	η_k
0	0.25	$-\pi/2$	0.25	$-\pi/2$
1	7	0	2	0
2	0	0	1.15	$\pi/2$
3	2	0	1.25	π
4	1.15	$-\pi/2$	1.25	$\pi/2$
5	1.25	π	0	0
6	1.25	$-\pi/2$	1	$-\pi/2$
7	0	0	0	0
8	1	$\pi/2$	0	0
9	0	0	0	0
10	0	0	0	0

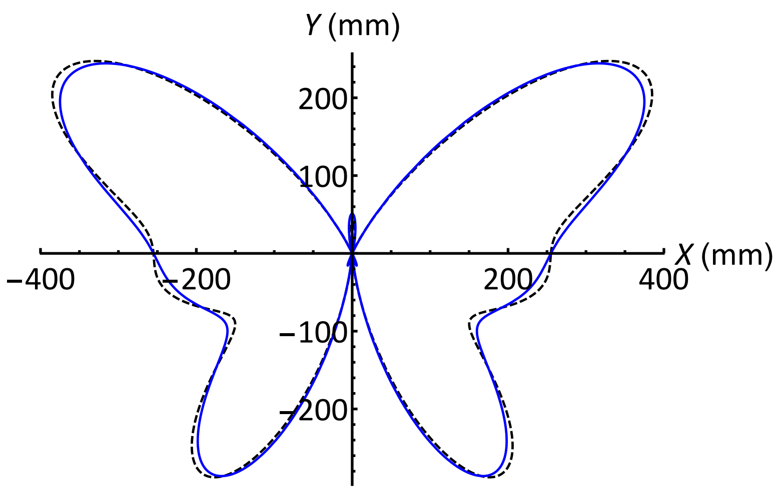


Figure 7.5: The Butterfly mechanism is to draw a curve ±40 cm wide (solid line). Using 11 links in the coupled serial chain introduces variations from the original 14 link version (dashed line)

7.6 Drive Train

The drawing mechanism for trigonometric curves uses pairs of links that rotate in opposite directions [40]. The speed ratio for each link in the chain is given by $+k$ for links L_k and $-k$ for links M_k. This speed ratio is measured relative to the world frame. By introducing relative speed ratios at each link, we can simplify the construction of the mechanism using gear trains.

The drawing mechanism consists of a sequence of joints G_j and links B_j, $j = 1, \ldots, 11$ connected in series and attached by joint G_1 to a base, Figure 7.6. Each link B_j has a gear fixed to the link and centered on its joint G_j, which for convenience we call G_j as well. This joint includes a bearing that engages the axle mounted on the previous link B_{j-1}. The link B_j has an axle at the end opposite to joint G_j that engages the bearing of G_{j+1} on link B_{j+1}.

Each joint G_j includes a second gear D_j that is rigidly mounted to the axle of previous link B_{j-1}. Similarly, the axle at the other end of link B_j engages and is fixed to the gear D_{j+1}. The drive system consists of a sequence of connections between gear D_j and the gear G_{j+1}. To start, D_1 is rigidly mounted to an axle attached to the base, and the mechanism is actuated by a drive gear that engages G_1.

This configuration allows us to define the relative speed ratios between the gears D_j and G_{j+1} when the link B_j is held fixed,

$$\frac{\omega_{j+1,j}}{\omega_{j-1,j}} = \frac{T_{D,j}}{T_{G,j+1}}, \tag{7.6}$$

where $T_{D,j}$ and $T_{G,j+1}$ are the teeth number of the gear D_j and G_{j+1}.

The relative speed ratios are related to world frame speed ratios listed in Table 7.2, by

$$\omega_{j+1,j} = \omega_{j+1} - \omega_j, \tag{7.7}$$

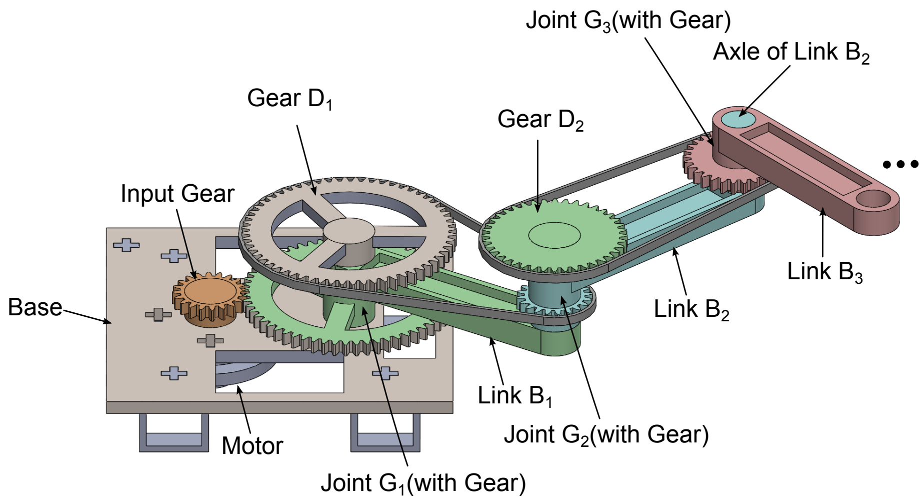


Figure 7.6: The structure of a coupled serial chain consisting of three links

therefore the relative speed ratio are given by,

$$\frac{\omega_{j+1,j}}{\omega_{j-1,j}} = \frac{\omega_{j+1} - \omega_j}{\omega_{j-1} - \omega_j}. \tag{7.8}$$

From these equations, we calculate the gears that provide the world frame speed ratios that we need.

Figure 7.6 shows the structure of a serial chain consisting of three links. The gear D_1 is rigidly attached to the base. Link B_1, B_2 and B_3 have gear G_1, G_2 and G_3 fixed to them respectively. The relative rotation is achieved by adding chain driven between gear D_1 and G_2, and between gear D_2 and G_3. The resulting mechanical system has a single degree of freedom.

In the chain driving configuration, we assume the links are lined up with one counter-clockwise rotating link followed by one clockwise rotating link. The world frame is built

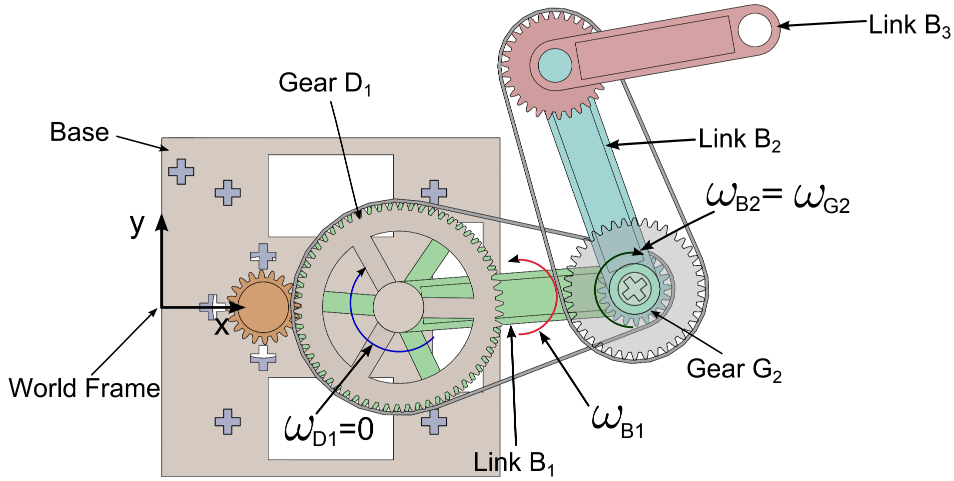


Figure 7.7: The angular velocities of the components for the first two links in the world frame

along the base. In the world frame, we denote the angular velocities of link B_j and gear G_j as ω_{Bj} and ω_{Gj}, respectively. It is obvious that $\omega_{Bj} = \omega_{Gj}$. We use the denotation ω_{Dj} for the angular velocity of gear D_j. From Table 7.2, we have the speed ratios for all the links.

Figure 7.7 shows the angular velocities of the components that drive the first two links in the world frame. The gear D_1 is static relative to the world frame thus $\omega_{D1} = 0$. Reference frame 1 is built along link B_1. Figure 7.8 shows the angular velocities of the components in reference frame 1. We denote the teeth number of the gear D_j and G_j as $T_{D,j}$ and $T_{G,j}$, respectively. Thus we can calculate the ratio of the teeth number between gear D_1 and G_2 as

$$\frac{T_{D,1}}{T_{G,2}} = \frac{\omega'_{G2}}{\omega'_{D1}} = \frac{\omega_{B2} - \omega_{B1}}{-\omega_{B1}} \tag{7.9}$$

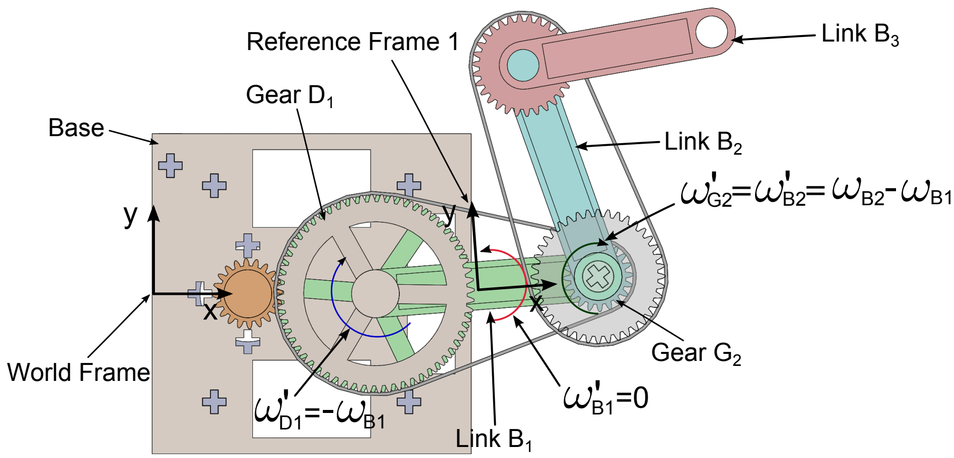


Figure 7.8: The angular velocities of the components for the first two links in the reference frame 1

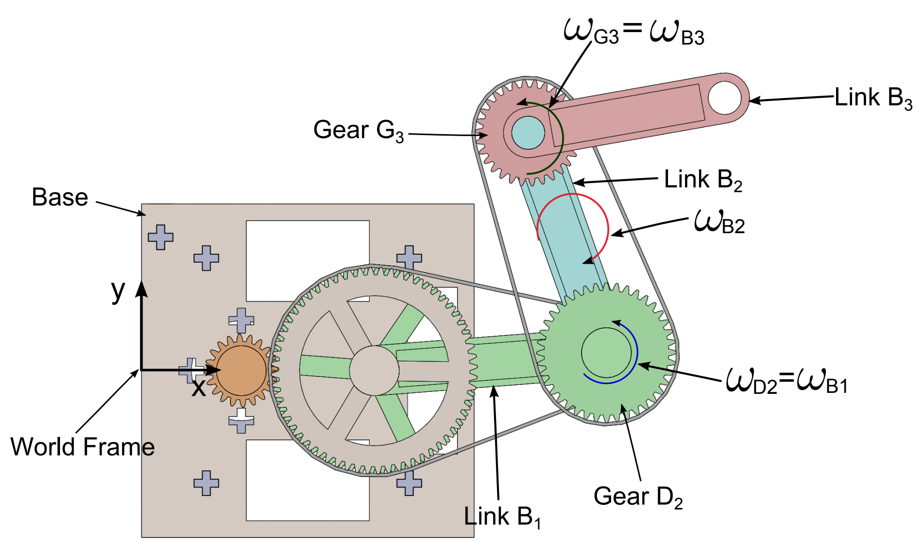


Figure 7.9: The angular velocities of the driving components from link B_2 to link B_3 in the world frame

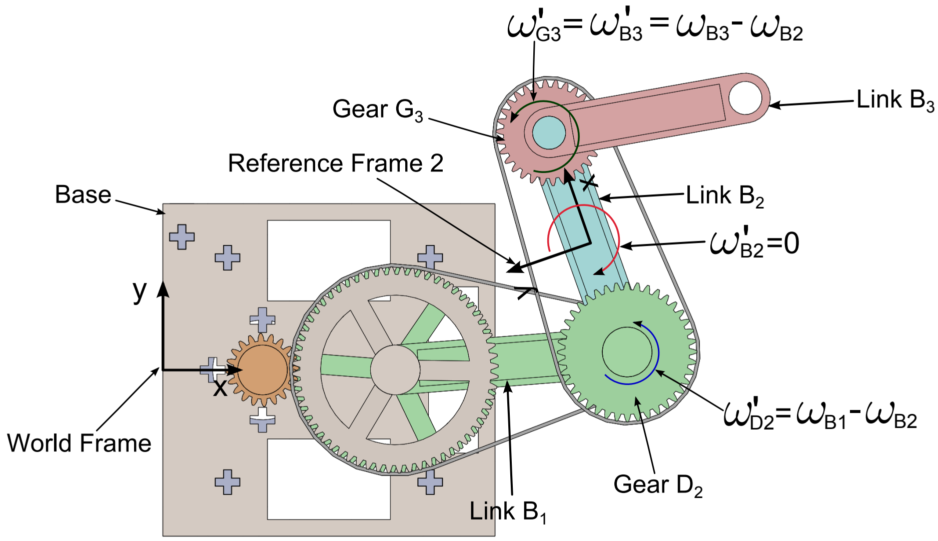


Figure 7.10: The angular velocities of the driving components from link B_2 to link B_3 in the reference frame 2

Figure 7.9 and Figure 7.10 show the angular velocities of the driving components from link B_2 to link B_3 in the world frame and reference frame 2, respectively. Similarly, we can calculate the relative size of gear D_2 and G_3 as

$$\frac{T_{D,2}}{T_{G,3}} = \frac{\omega'_{G3}}{\omega'_{D2}} = \frac{\omega_{B3} - \omega_{B2}}{\omega_{B1} - \omega_{B2}} \tag{7.10}$$

Therefor the equation to calculate the relative size of gear D_j with respect to gear G_{j+1} can be obtained as

$$\frac{T_{D,j}}{T_{G,j+1}} = \frac{\omega'_{G(j+1)}}{\omega'_{Dj}} = \frac{\omega_{B(j+1)} - \omega_{B(j)}}{\omega_{B(j-1)} - \omega_{Bj}} \tag{7.11}$$

7.7 The Coupled Serial Chain Mechanism

In this section, we present the methodology of using relative rotation to build a coupled serial chain mechanism that can draw the butterfly curve showing in solid line in Figure 7.5. The resulting mechanism has a single degree of freedom. Recall that we have eleven links in our manufacturing version of coupled serial chain. The column k in Table 7.2 defines the speed ratios for all the links. The ground pivot position is given by the $k = 0$ row in Table 7.2. We have six links that rotate counter-clockwise are L_1, L_3, L_4, L_5, L_6 and L_8, and another set of five links rotate clockwise are M_1, M_2, M_3, M_4 and M_6. In our configuration, we line up all the links from the longest to the shortest according to their length S_j. Additionally, the links are assembled in a way such that the counter-clockwise rotating links are followed by a clockwise rotating link and vice versa. Therefore, the sequence of the eleven links in the serial chain is L_1, M_1, L_3, M_3, L_5, M_4, L_6, M_2, L_4, M_6 and L_8. The links in this sequence are denoted as B_j ($j = 1 \cdots 11$) and the associated phase angle for each link is denoted as δ_j.

Rather than a chain driven, we introduce an alternative method of driving the mechanism to include a middle gear M_j between gear D_j and G_{j+1}. We replace the links with plates and locate the middle gear M_j a position such that its pitch circle is tangent to the pitch circles of both the gear D_j and G_{j+1}. Figure 7.11 shows the configuration the components from link B_j to link B_{j+2}. Note that in order to keep the relative rotation speed, the middle gear M_j has to be the same size as gear D_j or the G_{j+1}. The teeth number of the gear M_j is denoted as $T_{M,j}$.

By applying (7.11), we can obtain the relative size of the gear D_j and G_{j+1}. We make the module of all the gears to be m. The addendum of the gear is denoted as a. In order to make the three gears assembled on each plate mesh correctly, the following two conditions

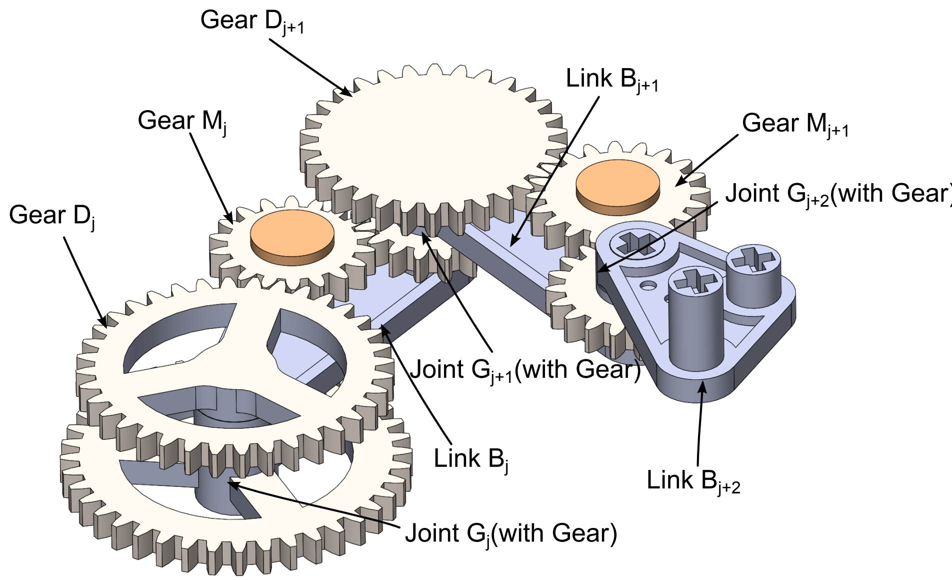


Figure 7.11: The configuration of gear driving from Link B_j to Link B_{j+2}

must be satisfied, such that

$$\frac{m}{2}(T_{D,j} + T_{G,j+1}) + 2a < S_k \tag{7.12}$$

$$\begin{aligned} &\frac{m}{2}(3T_{D,j} + T_{G,j+1}) \geq S_k \quad or \\ &\quad \frac{m}{2}(T_{D,j} + 3T_{G,j+1}) \geq S_k \end{aligned} \tag{7.13}$$

Equation 7.12 makes sure the gear D_j and G_{j+1} do not conflict with each other. Equation 7.13 guarantees that the gear D_j, M_j and G_{j+1} are close enough to mesh with each other correctly. In our model, we choose the gear module $m = 1.5$ and addendum $a = 1.5$mm. We make the length of the smallest link to be 30mm. The dimensions for all the components are listed in Table 7.3. The manufacturing model of our linkage is shown in Figure 7.12. The simulation of the motion of our mechanism is shown in Figure 7.13.

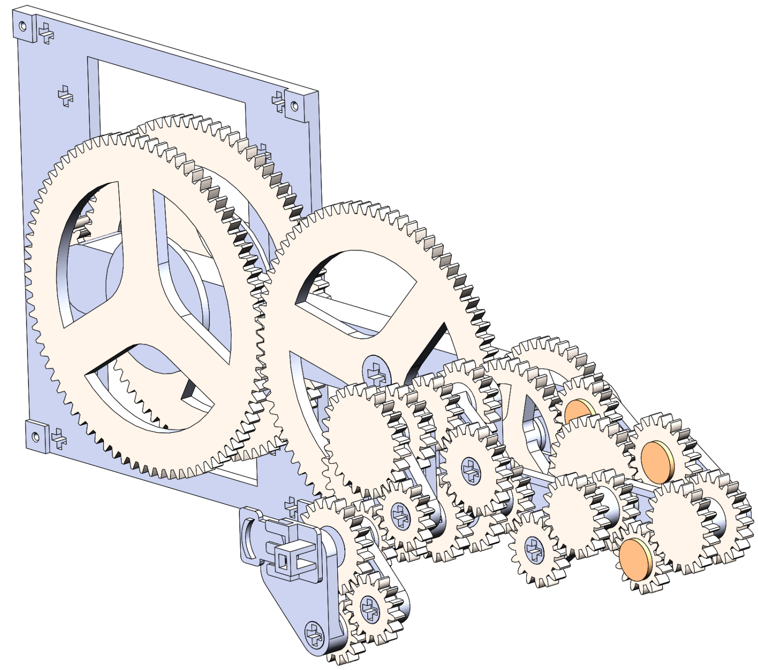


Figure 7.12: The solid model of the Butterfly mechanism to be built using additive manufacturing

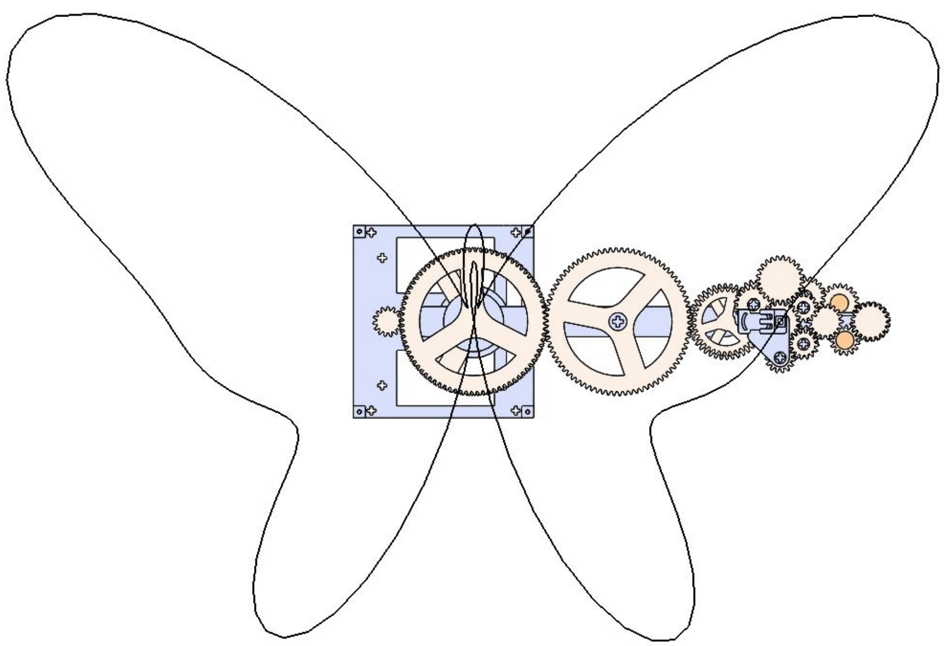


Figure 7.13: Simulation of the end-effector movement of the Butterfly mechanism using *SolidWorks Motion Analysis*

Table 7.3: Manufacturing dimensions of the single coupled serial chain to draw the butterfly curve

j	k	B_j	S_j	δ_j	$T_{D,j}$	$T_{M,j}$	$T_{G,j+1}$
1	1	L_1	210	0	80	80	40
2	-1	M_1	60	0	36	18	18
3	3	L_3	60	0	30	20	20
4	-3	M_3	37.5	π	20	15	15
5	5	L_5	37.5	π	18	16	16
6	-4	M_4	37.5	$\pi/2$	20	18	18
7	6	L_6	37.5	$-\pi/2$	16	20	20
8	-2	M_2	34.5	$\pi/2$	15	20	20
9	4	L_4	34.5	$-\pi/2$	25	15	15
10	-6	M_6	30	$-\pi/2$	21	15	15
11	8	L_8	30	$\pi/2$	0	0	0

7.8 Physical Prototype for the Butterfly Mechanism

We used the Stratsys Fortus 450mc additive manufacturing system to build all the components for our linkage. These include 11 links, 32 gears, ground pivot and the input crank. The resolution of the Stratsys printer is 0.127 mm. Through repeated experimentation, we sized the clearance between the revolute joint and the pivot to be 0.1 mm. An acetone and silicon combination solution was used to simultaneously wear the size of joint pin and hole to a tight tolerance while lubricating the contact surface to minimize friction. In order to ensure the cap gear tightly assembled to the link joint, the clearance fitting was tested and determined to be 0.05 mm. The clearance between components is shown in Figure 7.14 and Figure 7.15.

We manufactured all the components individually and assembled them together to obtain our serial coupled serial chain that can draw the butterfly curve. Part of the components are shown in Figure 7.16. The final manufactured prototype is shown in Figure 7.17.

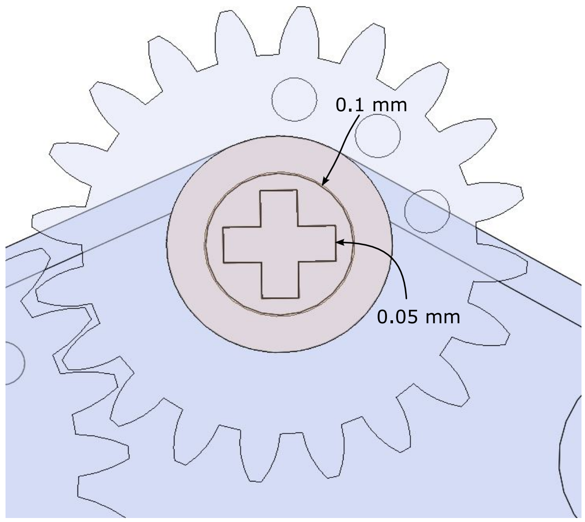


Figure 7.14: Dimensional differences for mounting features for gear and link axle are specified to be less than the resolution of the Fortus 450mc system. The parts are manually fit to ensure performance

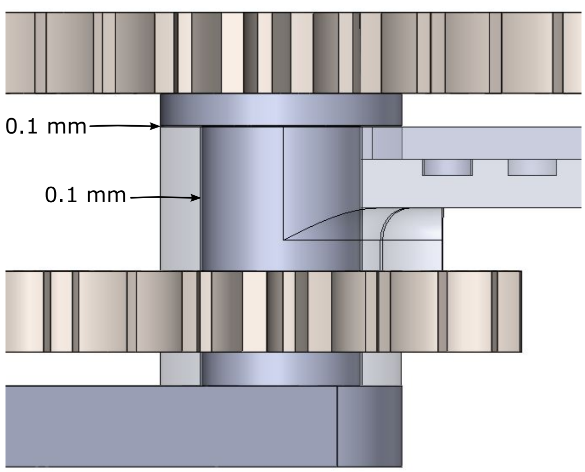


Figure 7.15: Dimensional differences between assembly features is less than the resolution of the Fortus 450mc system, therefore manual fit is required

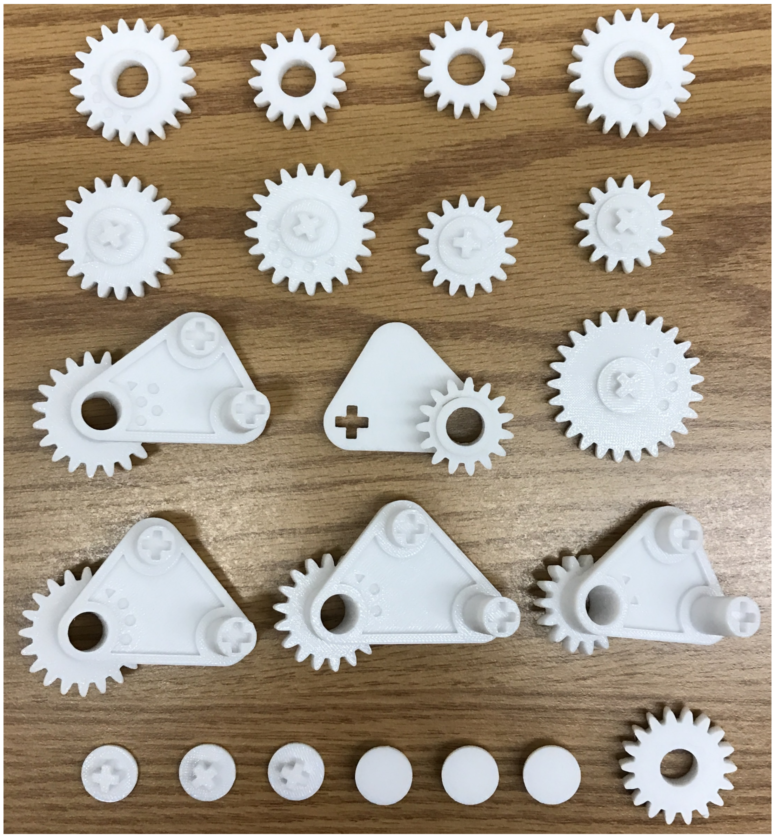


Figure 7.16: Example of the Butterfly mechanism drive components manufactured by the Fortus 450mc system.

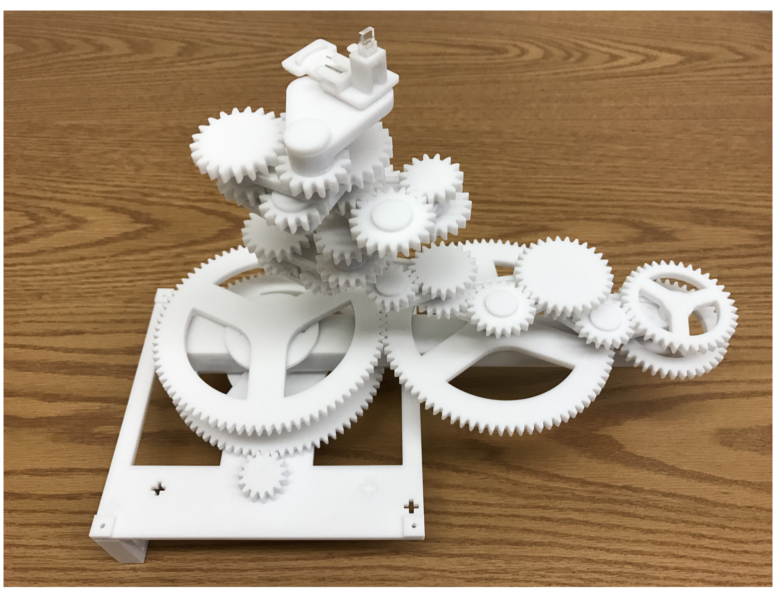


Figure 7.17: The fully assembled Butterfly mechanism

7.9 Summary

This chapter presents the manufacturing details for the construction of complex systems to draw trigonometric curves using additive manufacturing. The method is demonstrated for the 14 link Butterfly curve, which must simplified to an 11 link system to fit the requirements of the Stratasys Fortus system. We show how to adjust the dimensions of the gears to accomplish the desired joint movement within the tolerances available to this manufacturing process. The result is a functional prototype butterfly linkage drawing mechanism. Future work will simplify design and improve the manufacturing quality of these systems.

Chapter 8

Conclusion

This dissertation has presented a design methodology for linkage systems to draw complex plane curves. The additor, multiplicator, reversor and translator linkages of Kempe's proof are replaced by bevel gear differentials, pulleys, Scotch yoke linkages and coupled serial chains to obtain simple devices to draw plane curves.

We see that using standard mechanical computing elements such as the bevel gear differential to add and pulleys or gears to multiply, allows us to reduced the number of parts in Kempe's projection of a serial chain by at least half.

Further simplification is obtained by representing the coordinate functions of a curve by a finite Fourier series. This can also obtained by computing the Discrete Fourier transform of a set of points. The resulting curve is drawn by either the sum of Scotch yoke mechanisms or by a coupled serial chain, in which the joints are interconnected. Examples demonstrate a significant reduction in part count.

Bezier curves can be defined by coordinate function that are finite Fourier series, and we find that an n degree Bezier curve is drawn by a $2n$-link coupled serial chain.

Finally, examples demonstrate that curves obtained from cursive writing in English and Chinese using cubic Bezier curves, can be simplified to be drawn by a system of four-link coupled serial chains.

We also showed the method and process of manufacturing a complex mechanical system to draw trigonometric curves. This method is demonstrated by showing a working prototype of coupled serial chain drawing a Trifolium curve and a butterfly curve.

Future work would be applying this curve drawing technique to some useful applications. This could be some bio-inspired robots, for example, mimic the leg movement of a running animal or the wing trajectory of a bird. Another application could be medical device used for minimally invasive surgery. The challenge will be reduce the size of the mechanical system and high precision manufacturing.

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Appendices

A Fourier Transform Mathematica Code

Below is the Mathematica code for the implementation of Fourier Transform on trigonometric curves.

Single Circle

```
a = R1 * Cos[w1 * t];
```

```
b = R1 * Sin[w1 * t];
```

```
FTa = FourierTransform[a, t, w]
```

$\sqrt{\frac{\pi}{2}}$ R1 DiracDelta[w - w1] + $\sqrt{\frac{\pi}{2}}$ R1 DiracDelta[w + w1]

```
FTb = Expand[I * FourierTransform[b, t, w]]
```

$-\sqrt{\frac{\pi}{2}}$ R1 DiracDelta[w - w1] + $\sqrt{\frac{\pi}{2}}$ R1 DiracDelta[w + w1]

```
FTa + FTb
```

$\sqrt{2\pi}$ R1 DiracDelta[w + w1]

```
(FTa + FTb) * (1 / Sqrt[2 Pi])
```

R1 DiracDelta[w + w1]

Note: The -w1 means conterclock rotating

Heart

```
x = 12 Sin[t] - 4 Sin[3 t];
```

```
y = 13 Cos[t] - 5 Cos[2 t] - 2 Cos[3 t] - Cos[4 t];
```

```
FTy = N[Chop[Simplify[FourierTransform[Expand[N[y], Trig -> True], t, w]]], 10]
```

```
- 6.26657 DiracDelta[2. - 1.000000000 w] - 2.50663 DiracDelta[3. - 1.000000000 w] -
 1.25331 DiracDelta[4. - 1.000000000 w] + 16.2931 DiracDelta[-1.000000000 + w] +
 16.2931 DiracDelta[1.000000000 + w] - 6.26657 DiracDelta[2. + w] -
 2.50663 DiracDelta[3. + w] - 1.25331 DiracDelta[4. + w]
```

```
FTx = N[Chop[Expand[N[I * FourierTransform[Expand[N[x], Trig -> True], t, w]]]], 10]
```

```
5.01326 DiracDelta[3. - 1. w] - 15.0398 DiracDelta[-1. + w] +
 15.0398 DiracDelta[1. + w] - 5.01326 DiracDelta[3. + w]
```

```
Expand[(FTy + FTx) * (1 / Sqrt[2 Pi])]
```

```
1. DiracDelta[3. - 1. w] - 2.5 DiracDelta[2. - 1.000000000 w] -
 0.5 DiracDelta[4. - 1.000000000 w] + 0.5 DiracDelta[-1. + w] + 12.5 DiracDelta[1. + w] -
 2.5 DiracDelta[2. + w] - 3. DiracDelta[3. + w] - 0.5 DiracDelta[4. + w]
```

Batman

```
param[x_, m_, t_] :=
 Module[{f, n = Length[x], nf}, f = Chop[Fourier[x]][[ ;; Ceiling[Length[x] / 2]]];
  nf = Length[f];
  Total[
   Rationalize[2 Abs[f] / Sqrt[n] Sin[Pi / 2 - Arg[f] + 2. Pi Range[0, nf - 1] t], .01][[
     ;; Min[m, nf]]]]]
```

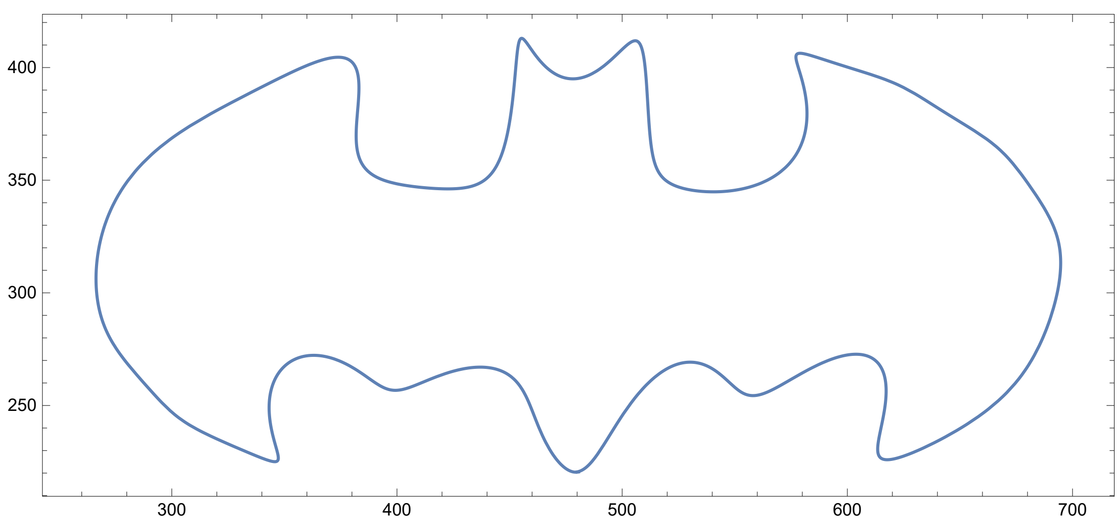
```
tocurve[Line[data_], m_, t_] := param[#, m, t] & /@ Transpose[data]
```

```
img = Import["C:\\Users\\Onion\\Dropbox\\1_Fourier Transform\\Batman.GIF"];
```

```
img = Binarize[img~ColorConvert~"Grayscale"~ImageResize~500~Blur~3]~Blur~3;
```

```
lines = Cases[
   Normal@ListContourPlot[Reverse@ImageData[img], Contours → {0.5}], _Line, -1];
```

```
ParametricPlot[Evaluate[tocurve[#, 20, t] & /@ lines],
 {t, 0, 1}, Frame → True, Axes → False, ImageSize → Large]
```



```
f = Evaluate[tocurve[#, 20, t] & /@ lines]
```

$$\Big\{\Big\{\frac{4322}{9}-\frac{14}{5}\,\text{Sin}\Big[\frac{17}{16}-\frac{377\,t}{4}\Big]-\frac{37}{19}\,\text{Sin}\Big[\frac{28}{19}-\frac{2199\,t}{25}\Big]-\frac{11}{6}\,\text{Sin}\Big[\frac{9}{8}-\frac{377\,t}{5}\Big]-$$
$$\frac{37}{13}\,\text{Sin}\Big[\frac{3}{8}-\frac{377\,t}{6}\Big]-\frac{8}{3}\,\text{Sin}\Big[\frac{5}{4}-\frac{553\,t}{11}\Big]-\frac{61}{7}\,\text{Sin}\Big[\frac{7}{8}-\frac{1891\,t}{43}\Big]-\frac{60}{17}\,\text{Sin}\Big[\frac{11}{12}-\frac{201\,t}{8}\Big]-$$
$$\frac{233}{11}\,\text{Sin}\Big[\frac{3}{7}-\frac{132\,t}{7}\Big]+\frac{2286}{13}\,\text{Sin}\Big[\frac{40}{13}+\frac{44\,t}{7}\Big]+\frac{147}{8}\,\text{Sin}\Big[\frac{109}{36}+\frac{88\,t}{7}\Big]+\frac{103}{13}\,\text{Sin}\Big[\frac{19}{8}+\frac{377\,t}{12}\Big]+$$
$$\frac{7}{4}\,\text{Sin}\Big[\frac{61}{60}+\frac{377\,t}{10}\Big]+\frac{7}{6}\,\text{Sin}\Big[\frac{31}{30}+\frac{509\,t}{9}\Big]+\frac{19}{3}\,\text{Sin}\Big[\frac{33}{13}+\frac{553\,t}{8}\Big]+\frac{2}{7}\,\text{Sin}\Big[\frac{47}{12}+\frac{1307\,t}{16}\Big]+$$
$$\frac{2}{5}\,\text{Sin}\Big[\frac{61}{30}+\frac{1307\,t}{13}\Big]+\frac{7}{13}\,\text{Sin}\Big[\frac{24}{11}+\frac{1175\,t}{11}\Big]+\frac{13}{12}\,\text{Sin}\Big[\frac{9}{7}+\frac{1131\,t}{10}\Big]+\frac{21}{13}\,\text{Sin}\Big[\frac{7}{4}+\frac{955\,t}{8}\Big],$$
$$\frac{2223}{7}-\frac{17}{14}\,\text{Sin}\Big[\frac{19}{18}-\frac{1131\,t}{10}\Big]-\frac{6}{11}\,\text{Sin}\Big[\frac{6}{7}-\frac{377\,t}{4}\Big]-\frac{47}{24}\,\text{Sin}\Big[\frac{1}{3}-\frac{553\,t}{11}\Big]+$$
$$\frac{1885}{23}\,\text{Sin}\Big[\frac{68}{15}+\frac{44\,t}{7}\Big]+\frac{15}{7}\,\text{Sin}\Big[\frac{31}{7}+\frac{88\,t}{7}\Big]+\frac{149}{6}\,\text{Sin}\Big[\frac{18}{17}+\frac{132\,t}{7}\Big]+\frac{82}{13}\,\text{Sin}\Big[\frac{46}{45}+\frac{201\,t}{8}\Big]+$$
$$\frac{123}{5}\,\text{Sin}\Big[\frac{47}{11}+\frac{377\,t}{12}\Big]+\frac{39}{10}\,\text{Sin}\Big[2+\frac{377\,t}{10}\Big]+\frac{83}{7}\,\text{Sin}\Big[\frac{37}{8}+\frac{1627\,t}{37}\Big]+\frac{55}{12}\,\text{Sin}\Big[\frac{8}{7}+\frac{509\,t}{9}\Big]+$$
$$\frac{40}{11}\,\text{Sin}\Big[\frac{53}{12}+\frac{377\,t}{6}\Big]+\frac{14}{9}\,\text{Sin}\Big[\frac{9}{13}+\frac{553\,t}{8}\Big]+\frac{108}{17}\,\text{Sin}\Big[\frac{29}{7}+\frac{377\,t}{5}\Big]+3\,\text{Sin}\Big[\frac{38}{9}+\frac{1307\,t}{16}\Big]+$$
$$\frac{35}{9}\,\text{Sin}\Big[\frac{105}{26}+\frac{2023\,t}{23}\Big]+2\,\text{Sin}\Big[\frac{9}{2}+\frac{1307\,t}{13}\Big]+\frac{23}{8}\,\text{Sin}\Big[\frac{2}{3}+\frac{1175\,t}{11}\Big]+\frac{22}{15}\,\text{Sin}\Big[\frac{7}{13}+\frac{955\,t}{8}\Big]\Big\}\Big\}$$

```
f1 = f[[1, 1]]
```

$$\frac{4322}{9}-\frac{14}{5}\,\text{Sin}\Big[\frac{17}{16}-\frac{377\,t}{4}\Big]-\frac{37}{19}\,\text{Sin}\Big[\frac{28}{19}-\frac{2199\,t}{25}\Big]-\frac{11}{6}\,\text{Sin}\Big[\frac{9}{8}-\frac{377\,t}{5}\Big]-$$
$$\frac{37}{13}\,\text{Sin}\Big[\frac{3}{8}-\frac{377\,t}{6}\Big]-\frac{8}{3}\,\text{Sin}\Big[\frac{5}{4}-\frac{553\,t}{11}\Big]-\frac{61}{7}\,\text{Sin}\Big[\frac{7}{8}-\frac{1891\,t}{43}\Big]-\frac{60}{17}\,\text{Sin}\Big[\frac{11}{12}-\frac{201\,t}{8}\Big]-$$
$$\frac{233}{11}\,\text{Sin}\Big[\frac{3}{7}-\frac{132\,t}{7}\Big]+\frac{2286}{13}\,\text{Sin}\Big[\frac{40}{13}+\frac{44\,t}{7}\Big]+\frac{147}{8}\,\text{Sin}\Big[\frac{109}{36}+\frac{88\,t}{7}\Big]+\frac{103}{13}\,\text{Sin}\Big[\frac{19}{8}+\frac{377\,t}{12}\Big]+$$
$$\frac{7}{4}\,\text{Sin}\Big[\frac{61}{60}+\frac{377\,t}{10}\Big]+\frac{7}{6}\,\text{Sin}\Big[\frac{31}{30}+\frac{509\,t}{9}\Big]+\frac{19}{3}\,\text{Sin}\Big[\frac{33}{13}+\frac{553\,t}{8}\Big]+\frac{2}{7}\,\text{Sin}\Big[\frac{47}{12}+\frac{1307\,t}{16}\Big]+$$
$$\frac{2}{5}\,\text{Sin}\Big[\frac{61}{30}+\frac{1307\,t}{13}\Big]+\frac{7}{13}\,\text{Sin}\Big[\frac{24}{11}+\frac{1175\,t}{11}\Big]+\frac{13}{12}\,\text{Sin}\Big[\frac{9}{7}+\frac{1131\,t}{10}\Big]+\frac{21}{13}\,\text{Sin}\Big[\frac{7}{4}+\frac{955\,t}{8}\Big]$$

```
N[f1]
```

```
480.222 - 2.8 Sin[1.0625 - 94.25 t] -
 1.94737 Sin[1.47368 - 87.96 t] - 1.83333 Sin[1.125 - 75.4 t] -
 2.84615 Sin[0.375 - 62.8333 t] - 2.66667 Sin[1.25 - 50.2727 t] -
 8.71429 Sin[0.875 - 43.9767 t] - 3.52941 Sin[0.916667 - 25.125 t] -
 21.1818 Sin[0.428571 - 18.8571 t] + 175.846 Sin[3.07692 + 6.28571 t] +
 18.375 Sin[3.02778 + 12.5714 t] + 7.92308 Sin[2.375 + 31.4167 t] +
 1.75 Sin[1.01667 + 37.7 t] + 1.16667 Sin[1.03333 + 56.5556 t] +
 6.33333 Sin[2.53846 + 69.125 t] + 0.285714 Sin[3.91667 + 81.6875 t] +
 0.4 Sin[2.03333 + 100.538 t] + 0.538462 Sin[2.18182 + 106.818 t] +
 1.08333 Sin[1.28571 + 113.1 t] + 1.61538 Sin[1.75 + 119.375 t]
```

f2 = f[[1, 2]]

$$\frac{2223}{7}-\frac{17}{14}\,\text{Sin}\left[\frac{19}{18}-\frac{1131\,t}{10}\right]-\frac{6}{11}\,\text{Sin}\left[\frac{6}{7}-\frac{377\,t}{4}\right]-\frac{47}{24}\,\text{Sin}\left[\frac{1}{3}-\frac{553\,t}{11}\right]+\frac{1885}{23}\,\text{Sin}\left[\frac{68}{15}+\frac{44\,t}{7}\right]+\frac{15}{7}\,\text{Sin}\left[\frac{31}{7}+\frac{88\,t}{7}\right]+\frac{149}{6}\,\text{Sin}\left[\frac{18}{17}+\frac{132\,t}{7}\right]+\frac{82}{13}\,\text{Sin}\left[\frac{46}{45}+\frac{201\,t}{8}\right]+\frac{123}{5}\,\text{Sin}\left[\frac{47}{11}+\frac{377\,t}{12}\right]+\frac{39}{10}\,\text{Sin}\left[2+\frac{377\,t}{10}\right]+\frac{83}{7}\,\text{Sin}\left[\frac{37}{8}+\frac{1627\,t}{37}\right]+\frac{55}{12}\,\text{Sin}\left[\frac{8}{7}+\frac{509\,t}{9}\right]+\frac{40}{11}\,\text{Sin}\left[\frac{53}{12}+\frac{377\,t}{6}\right]+\frac{14}{9}\,\text{Sin}\left[\frac{9}{13}+\frac{553\,t}{8}\right]+\frac{108}{17}\,\text{Sin}\left[\frac{29}{7}+\frac{377\,t}{5}\right]+3\,\text{Sin}\left[\frac{38}{9}+\frac{1307\,t}{16}\right]+\frac{35}{9}\,\text{Sin}\left[\frac{105}{26}+\frac{2023\,t}{23}\right]+2\,\text{Sin}\left[\frac{9}{2}+\frac{1307\,t}{13}\right]+\frac{23}{8}\,\text{Sin}\left[\frac{2}{3}+\frac{1175\,t}{11}\right]+\frac{22}{15}\,\text{Sin}\left[\frac{7}{13}+\frac{955\,t}{8}\right]$$

N[f2]

```
317.571 - 1.21429 Sin[1.05556 - 113.1 t] - 0.545455 Sin[0.857143 - 94.25 t] -
 1.95833 Sin[0.333333 - 50.2727 t] + 81.9565 Sin[4.53333 + 6.28571 t] +
 2.14286 Sin[4.42857 + 12.5714 t] + 24.8333 Sin[1.05882 + 18.8571 t] +
 6.30769 Sin[1.02222 + 25.125 t] + 24.6 Sin[4.27273 + 31.4167 t] + 3.9 Sin[2. + 37.7 t] +
 11.8571 Sin[4.625 + 43.973 t] + 4.58333 Sin[1.14286 + 56.5556 t] +
 3.63636 Sin[4.41667 + 62.8333 t] + 1.55556 Sin[0.692308 + 69.125 t] +
 6.35294 Sin[4.14286 + 75.4 t] + 3. Sin[4.22222 + 81.6875 t] +
 3.88889 Sin[4.03846 + 87.9565 t] + 2. Sin[4.5 + 100.538 t] +
 2.875 Sin[0.666667 + 106.818 t] + 1.46667 Sin[0.538462 + 119.375 t]
```

```
FTf1 = N[Chop[Simplify[FourierTransform[Expand[N[f1], Trig -> True], t, w]]], 10]
(14.2426 - 219.93 i) DiracDelta[6.28571 - 1.000000000 w] +
 (2.61546 - 22.8806 i) DiracDelta[12.5714 - 1.000000000 w] -
 (11.0324 - 24.1465 i) DiracDelta[18.8571 - 1.000000000 w] -
 (3.51036 - 2.69154 i) DiracDelta[25.125 - 1.000000000 w] +
 (6.88836 - 7.15244 i) DiracDelta[31.4167 - 1.000000000 w] +
 (1.86509 + 1.15412 i) DiracDelta[37.7 - 1.000000000 w] -
 (8.38291 - 7.0008 i) DiracDelta[43.9767 - 1.000000000 w] -
 (3.17167 - 1.05386 i) DiracDelta[50.2727 - 1.000000000 w] +
 (1.25604 + 0.748585 i) DiracDelta[56.5556 - 1.000000000 w] -
 (1.30654 - 3.31924 i) DiracDelta[62.8333 - 1.000000000 w] +
 (4.50243 - 6.53716 i) DiracDelta[69.125 - 1.000000000 w] -
 (2.07318 - 0.990733 i) DiracDelta[75.4 - 1.000000000 w] -
 (0.25058 + 0.255808 i) DiracDelta[81.6875 - 1.000000000 w] -
 (2.42916 - 0.236646 i) DiracDelta[87.96 - 1.000000000 w] -
 (3.06562 - 1.70793 i) DiracDelta[94.25 - 1.000000000 w] +
 (0.448648 - 0.223702 i) DiracDelta[100.538 - 1.000000000 w] +
 (0.552754 - 0.387171 i) DiracDelta[106.818 - 1.000000000 w] +
 (1.30296 + 0.38185 i) DiracDelta[113.1 - 1.000000000 w] +
 (1.99216 - 0.360874 i) DiracDelta[119.375 - 1.000000000 w] +
 1203.74 DiracDelta[w] + (14.2426 + 219.93 i) DiracDelta[6.28571 + w] +
 (2.61546 + 22.8806 i) DiracDelta[12.5714 + w] -
 (11.0324 + 24.1465 i) DiracDelta[18.8571 + w] -
 (3.51036 + 2.69154 i) DiracDelta[25.125 + w] +
 (6.88836 + 7.15244 i) DiracDelta[31.4167 + w] +
 (1.86509 - 1.15412 i) DiracDelta[37.7 + w] -
 (8.38291 + 7.0008 i) DiracDelta[43.9767 + w] -
 (3.17167 + 1.05386 i) DiracDelta[50.2727 + w] +
 (1.25604 - 0.748585 i) DiracDelta[56.5556 + w] -
 (1.30654 + 3.31924 i) DiracDelta[62.8333 + w] +
 (4.50243 + 6.53716 i) DiracDelta[69.125 + w] -
 (2.07318 + 0.990733 i) DiracDelta[75.4 + w] -
 (0.25058 - 0.255808 i) DiracDelta[81.6875 + w] -
 (2.42916 + 0.236646 i) DiracDelta[87.96 + w] -
 (3.06562 + 1.70793 i) DiracDelta[94.25 + w] +
 (0.448648 + 0.223702 i) DiracDelta[100.538 + w] +
 (0.552754 + 0.387171 i) DiracDelta[106.818 + w] +
 (1.30296 - 0.38185 i) DiracDelta[113.1 + w] +
 (1.99216 + 0.360874 i) DiracDelta[119.375 + w]
```

```
FTf2 = N[Chop[Expand[N[I * FourierTransform[Expand[N[f2], Trig -> True], t, w]]]], 10]
(18.294 - 101.075 i) DiracDelta[6.28571 - 1. w] +
 (0.752049 - 2.57823 i) DiracDelta[12.5714 - 1. w] -
 (15.2476 - 27.1332 i) DiracDelta[18.8571 - 1. w] -
 (4.1225 - 6.74553 i) DiracDelta[25.125 - 1. w] +
 (13.1229 - 27.8993 i) DiracDelta[31.4167 - 1. w] +
 (2.03409 + 4.44458 i) DiracDelta[37.7 - 1. w] +
 (1.29701 - 14.804 i) DiracDelta[43.973 - 1. w] -
 (2.31931 + 0.803069 i) DiracDelta[50.2727 - 1. w] -
 (2.38389 - 5.22635 i) DiracDelta[56.5556 - 1. w] +
 (1.3282 - 4.35967 i) DiracDelta[62.8333 - 1. w] -
 (1.50075 - 1.24446 i) DiracDelta[69.125 - 1. w] +
 (4.29354 - 6.70542 i) DiracDelta[75.4 - 1. w] +
 (1.77008 - 3.31723 i) DiracDelta[81.6875 - 1. w] +
 (3.04167 - 3.80843 i) DiracDelta[87.9565 - 1. w] -
 (0.447502 + 0.516804 i) DiracDelta[94.25 - 1. w] +
 (0.528387 - 2.4503 i) DiracDelta[100.538 - 1. w] -
 (2.83177 - 2.22816 i) DiracDelta[106.818 - 1. w] -
 (0.749899 + 1.3243 i) DiracDelta[113.1 - 1. w] -
 (1.57809 - 0.942655 i) DiracDelta[119.375 - 1. w] +
 (0. + 796.034 i) DiracDelta[w] - (18.294 + 101.075 i) DiracDelta[6.28571 + w] -
 (0.752049 + 2.57823 i) DiracDelta[12.5714 + w] +
 (15.2476 + 27.1332 i) DiracDelta[18.8571 + w] +
 (4.1225 + 6.74553 i) DiracDelta[25.125 + w] -
 (13.1229 + 27.8993 i) DiracDelta[31.4167 + w] -
 (2.03409 - 4.44458 i) DiracDelta[37.7 + w] - (1.29701 + 14.804 i) DiracDelta[43.973 + w] +
 (2.31931 - 0.803069 i) DiracDelta[50.2727 + w] +
 (2.38389 + 5.22635 i) DiracDelta[56.5556 + w] -
 (1.3282 + 4.35967 i) DiracDelta[62.8333 + w] +
 (1.50075 + 1.24446 i) DiracDelta[69.125 + w] -
 (4.29354 + 6.70542 i) DiracDelta[75.4 + w] -
 (1.77008 + 3.31723 i) DiracDelta[81.6875 + w] -
 (3.04167 + 3.80843 i) DiracDelta[87.9565 + w] +
 (0.447502 - 0.516804 i) DiracDelta[94.25 + w] -
 (0.528387 + 2.4503 i) DiracDelta[100.538 + w] +
 (2.83177 + 2.22816 i) DiracDelta[106.818 + w] +
 (0.749899 - 1.3243 i) DiracDelta[113.1 + w] +
 (1.57809 + 0.942655 i) DiracDelta[119.375 + w]

Solution = Expand[(FTf1 + FTf2) * (1 / Sqrt[2 Pi])]
```

```
(12.9802 - 128.062 i) DiracDelta[6.28571 - 1. w] +
 (1.34344 - 10.1566 i) DiracDelta[12.5714 - 1. w] -
 (10.4842 - 20.4577 i) DiracDelta[18.8571 - 1. w] -
 (3.04507 - 3.76485 i) DiracDelta[25.125 - 1. w] +
 (7.98335 - 13.9836 i) DiracDelta[31.4167 - 1. w] +
 (1.55555 + 2.23356 i) DiracDelta[37.7 - 1. w] +
 (0.517433 - 5.90595 i) DiracDelta[43.973 - 1. w] -
 (2.19058 - 0.100052 i) DiracDelta[50.2727 - 1. w] -
 (0.449945 - 2.38365 i) DiracDelta[56.5556 - 1. w] +
 (0.00864035 - 0.415074 i) DiracDelta[62.8333 - 1. w] +
 (1.1975 - 2.11148 i) DiracDelta[69.125 - 1. w] +
 (0.885795 - 2.27983 i) DiracDelta[75.4 - 1. w] +
 (0.606193 - 1.42543 i) DiracDelta[81.6875 - 1. w] +
 (1.21345 - 1.51934 i) DiracDelta[87.9565 - 1. w] -
 (1.40153 - 0.47519 i) DiracDelta[94.25 - 1. w] +
 (0.38978 - 1.06677 i) DiracDelta[100.538 - 1. w] -
 (0.909196 - 0.734448 i) DiracDelta[106.818 - 1. w] +
 (0.220638 - 0.375984 i) DiracDelta[113.1 - 1. w] +
 (0.165192 + 0.232097 i) DiracDelta[119.375 - 1. w] -
 (3.3443 - 2.79291 i) DiracDelta[43.9767 - 1.000000000 w] -
 (0.969097 - 0.094408 i) DiracDelta[87.96 - 1.000000000 w] +
 (480.222 + 317.571 i) DiracDelta[w] - (1.61626 - 47.4162 i) DiracDelta[6.28571 + w] +
 (0.743394 + 8.09949 i) DiracDelta[12.5714 + w] +
 (1.68162 + 1.19153 i) DiracDelta[18.8571 + w] +
 (0.244208 + 1.61731 i) DiracDelta[25.125 + w] -
 (2.48723 + 8.27681 i) DiracDelta[31.4167 + w] -
 (0.0674224 - 1.3127 i) DiracDelta[37.7 + w] -
 (0.517433 + 5.90595 i) DiracDelta[43.973 + w] -
 (3.3443 + 2.79291 i) DiracDelta[43.9767 + w] -
 (0.340042 + 0.740808 i) DiracDelta[50.2727 + w] +
 (1.45212 + 1.78637 i) DiracDelta[56.5556 + w] -
 (1.05111 + 3.06344 i) DiracDelta[62.8333 + w] +
 (2.39492 + 3.10442 i) DiracDelta[69.125 + w] -
 (2.53995 + 3.07032 i) DiracDelta[75.4 + w] -
 (0.806127 + 1.22133 i) DiracDelta[81.6875 + w] -
 (1.21345 + 1.51934 i) DiracDelta[87.9565 + w] -
 (0.969097 + 0.094408 i) DiracDelta[87.96 + w] -
 (1.04448 + 0.887541 i) DiracDelta[94.25 + w] -
 (0.0318111 + 0.888286 i) DiracDelta[100.538 + w] +
 (1.35023 + 1.04337 i) DiracDelta[106.818 + w] +
 (0.81897 - 0.680656 i) DiracDelta[113.1 + w] +
 (1.42432 + 0.520033 i) DiracDelta[119.375 + w]
```

Remove the solution ω=0

```
ModSolution =
 Cases[Solution, Except[(480.2222222222222` + 317.5714285714285` ⅈ) DiracDelta[w]]]
```

```
{(12.9802 - 128.062 i) DiracDelta[6.28571 - 1. w],
 (1.34344 - 10.1566 i) DiracDelta[12.5714 - 1. w],
 (-10.4842 + 20.4577 i) DiracDelta[18.8571 - 1. w],
 (-3.04507 + 3.76485 i) DiracDelta[25.125 - 1. w],
 (7.98335 - 13.9836 i) DiracDelta[31.4167 - 1. w],
 (1.55555 + 2.23356 i) DiracDelta[37.7 - 1. w],
 (0.517433 - 5.90595 i) DiracDelta[43.973 - 1. w],
 (-2.19058 + 0.100052 i) DiracDelta[50.2727 - 1. w],
 (-0.449945 + 2.38365 i) DiracDelta[56.5556 - 1. w],
 (0.00864035 - 0.415074 i) DiracDelta[62.8333 - 1. w],
 (1.1975 - 2.11148 i) DiracDelta[69.125 - 1. w],
 (0.885795 - 2.27983 i) DiracDelta[75.4 - 1. w],
 (0.606193 - 1.42543 i) DiracDelta[81.6875 - 1. w],
 (1.21345 - 1.51934 i) DiracDelta[87.9565 - 1. w],
 (-1.40153 + 0.47519 i) DiracDelta[94.25 - 1. w],
 (0.38978 - 1.06677 i) DiracDelta[100.538 - 1. w],
 (-0.909196 + 0.734448 i) DiracDelta[106.818 - 1. w],
 (0.220638 - 0.375984 i) DiracDelta[113.1 - 1. w],
 (0.165192 + 0.232097 i) DiracDelta[119.375 - 1. w],
 (-3.3443 + 2.79291 i) DiracDelta[43.9767 - 1.000000000 w],
 (-0.969097 + 0.094408 i) DiracDelta[87.96 - 1.000000000 w],
 (-1.61626 + 47.4162 i) DiracDelta[6.28571 + w],
 (0.743394 + 8.09949 i) DiracDelta[12.5714 + w],
 (1.68162 + 1.19153 i) DiracDelta[18.8571 + w],
 (0.244208 + 1.61731 i) DiracDelta[25.125 + w],
 (-2.48723 - 8.27681 i) DiracDelta[31.4167 + w],
 (-0.0674224 + 1.3127 i) DiracDelta[37.7 + w],
 (-0.517433 - 5.90595 i) DiracDelta[43.973 + w],
 (-3.3443 - 2.79291 i) DiracDelta[43.9767 + w],
 (-0.340042 - 0.740808 i) DiracDelta[50.2727 + w],
 (1.45212 + 1.78637 i) DiracDelta[56.5556 + w],
 (-1.05111 - 3.06344 i) DiracDelta[62.8333 + w],
 (2.39492 + 3.10442 i) DiracDelta[69.125 + w],
 (-2.53995 - 3.07032 i) DiracDelta[75.4 + w],
 (-0.806127 - 1.22133 i) DiracDelta[81.6875 + w],
 (-1.21345 - 1.51934 i) DiracDelta[87.9565 + w],
 (-0.969097 - 0.094408 i) DiracDelta[87.96 + w],
 (-1.04448 - 0.887541 i) DiracDelta[94.25 + w],
 (-0.0318111 - 0.888286 i) DiracDelta[100.538 + w],
 (1.35023 + 1.04337 i) DiracDelta[106.818 + w],
 (0.81897 - 0.680656 i) DiracDelta[113.1 + w],
 (1.42432 + 0.520033 i) DiracDelta[119.375 + w]}
```

```
num = Length[ModSolution]
```

42

Test

```
S = ModSolution[[1, 1]]
```

12.9802 - 128.062 i

```
Re[S]
```

12.9802

```
Im[S]
```

-128.062

```
NSolve[
 {R1 * Cos[α] == 12.980230312510045`, R1 * Sin[α] == -128.0624006550793`}, {R1, α}]
```

NSolve::ifun : Inverse functions are being used by NSolve,
so some solutions may not be found; use Reduce for complete solution information. >>

{{R1 → -128.719, α → 1.67181}, {R1 → 128.719, α → -1.46978}}

```
128.71854893723696` * Cos[-1.4697826714263014`]
```

12.9802

```
128.71854893723696` * Sin[-1.4697826714263014`]
```

-128.062

Get the coefficient of Delta function

```
CoeSolution = Table[ModSolution[[i, 1]], {i, num}]
```

{12.9802 - 128.062 i, 1.34344 - 10.1566 i, -10.4842 + 20.4577 i, -3.04507 + 3.76485 i,
7.98335 - 13.9836 i, 1.55555 + 2.23356 i, 0.517433 - 5.90595 i, -2.19058 + 0.100052 i,
-0.449945 + 2.38365 i, 0.00864035 - 0.415074 i, 1.1975 - 2.11148 i, 0.885795 - 2.27983 i,
0.606193 - 1.42543 i, 1.21345 - 1.51934 i, -1.40153 + 0.47519 i, 0.38978 - 1.06677 i,
-0.909196 + 0.734448 i, 0.220638 - 0.375984 i, 0.165192 + 0.232097 i,
-3.3443 + 2.79291 i, -0.969097 + 0.094408 i, -1.61626 + 47.4162 i, 0.743394 + 8.09949 i,
1.68162 + 1.19153 i, 0.244208 + 1.61731 i, -2.48723 - 8.27681 i, -0.0674224 + 1.3127 i,
-0.517433 - 5.90595 i, -3.3443 - 2.79291 i, -0.340042 - 0.740808 i, 1.45212 + 1.78637 i,
-1.05111 - 3.06344 i, 2.39492 + 3.10442 i, -2.53995 - 3.07032 i, -0.806127 - 1.22133 i,
-1.21345 - 1.51934 i, -0.969097 - 0.094408 i, -1.04448 - 0.887541 i,
-0.0318111 - 0.888286 i, 1.35023 + 1.04337 i, 0.81897 - 0.680656 i, 1.42432 + 0.520033 i}

Solve for Link lengths and Phases

```
LpSolution = Table[NSolve[{R * Cos[α] == Re[CoeSolution[[i]]],
    R * Sin[α] == Im[CoeSolution[[i]]]}, {R, α}], {i, num}]
```

NSolve::ifun : Inverse functions are being used by NSolve,
so some solutions may not be found; use Reduce for complete solution information. ≫

NSolve::ifun : Inverse functions are being used by NSolve,
so some solutions may not be found; use Reduce for complete solution information. ≫

NSolve::ifun : Inverse functions are being used by NSolve,
so some solutions may not be found; use Reduce for complete solution information. ≫

General::stop : Further output of NSolve::ifun will be suppressed during this calculation. ≫

```
{{{R → -128.719, α → 1.67181}, {R → 128.719, α → -1.46978}},
 {{R → -10.2451, α → 1.70231}, {R → 10.2451, α → -1.43929}},
 {{R → -22.9877, α → -1.09721}, {R → 22.9877, α → 2.04438}},
 {{R → -4.84216, α → -0.890703}, {R → 4.84216, α → 2.25089}},
 {{R → -16.102, α → 2.08955}, {R → 16.102, α → -1.05204}},
 {{R → -2.72186, α → -2.17913}, {R → 2.72186, α → 0.96246}},
 {{R → -5.92857, α → 1.65819}, {R → 5.92857, α → -1.48341}},
 {{R → -2.19287, α → -0.0456418}, {R → 2.19287, α → 3.09595}},
 {{R → -2.42575, α → -1.38423}, {R → 2.42575, α → 1.75736}},
 {{R → -0.415164, α → 1.59161}, {R → 0.415164, α → -1.54998}},
 {{R → -2.42742, α → 2.0867}, {R → 2.42742, α → -1.05489}},
 {{R → -2.44587, α → 1.94138}, {R → 2.44587, α → -1.20021}},
 {{R → -1.54898, α → 1.97289}, {R → 1.54898, α → -1.1687}},
 {{R → -1.94444, α → 2.24472}, {R → 1.94444, α → -0.896869}},
 {{R → -1.4799, α → -0.326887}, {R → 1.4799, α → 2.81471}},
 {{R → -1.13575, α → 1.92111}, {R → 1.13575, α → -1.22048}},
 {{R → -1.16878, α → -0.679478}, {R → 1.16878, α → 2.46211}},
 {{R → -0.435941, α → 2.10147}, {R → 0.435941, α → -1.04012}},
 {{R → -0.284882, α → -2.18936}, {R → 0.284882, α → 0.952236}},
 {{R → -4.35714, α → -0.695796}, {R → 4.35714, α → 2.4458}},
 {{R → -0.973684, α → -0.0971121}, {R → 0.973684, α → 3.04448}},
 {{R → -47.4437, α → -1.53672}, {R → 47.4437, α → 1.60487}},
 {{R → -8.13354, α → -1.66232}, {R → 8.13354, α → 1.47927}},
 {{R → -2.06096, α → -2.52514}, {R → 2.06096, α → 0.616448}},
 {{R → -1.63564, α → -1.72066}, {R → 1.63564, α → 1.42093}},
 {{R → -8.64244, α → 1.27888}, {R → 8.64244, α → -1.86272}},
 {{R → -1.31443, α → -1.51948}, {R → 1.31443, α → 1.62211}},
 {{R → -5.92857, α → 1.48341}, {R → 5.92857, α → -1.65819}},
 {{R → -4.35714, α → 0.695796}, {R → 4.35714, α → -2.4458}},
 {{R → -0.815123, α → 1.14047}, {R → 0.815123, α → -2.00112}},
 {{R → -2.30212, α → -2.25335}, {R → 2.30212, α → 0.888244}},
 {{R → -3.23875, α → 1.24027}, {R → 3.23875, α → -1.90132}},
 {{R → -3.92085, α → -2.22789}, {R → 3.92085, α → 0.913704}},
 {{R → -3.98475, α → 0.879653}, {R → 3.98475, α → -2.26194}},
 {{R → -1.46338, α → 0.987395}, {R → 1.46338, α → -2.1542}},
 {{R → -1.94444, α → 0.896869}, {R → 1.94444, α → -2.24472}},
 {{R → -0.973684, α → 0.0971121}, {R → 0.973684, α → -3.04448}},
 {{R → -1.37064, α → 0.704347}, {R → 1.37064, α → -2.43725}},
 {{R → -0.888856, α → 1.535}, {R → 0.888856, α → -1.60659}},
 {{R → -1.70638, α → -2.4837}, {R → 1.70638, α → 0.657892}},
 {{R → -1.0649, α → 2.44817}, {R → 1.0649, α → -0.693426}},
 {{R → -1.51629, α → -2.79152}, {R → 1.51629, α → 0.350071}}}
```

Pick up solutions that R>0

```
LpSolution[[1, 2]]
```

```
{R → 128.719, α → -1.46978}
```

```
ModLpSolution = Table[LpSolution[[i, 2]], {i, num}]
```

```
{{R → 128.719, α → -1.46978}, {R → 10.2451, α → -1.43929},
 {R → 22.9877, α → 2.04438}, {R → 4.84216, α → 2.25089}, {R → 16.102, α → -1.05204},
 {R → 2.72186, α → 0.96246}, {R → 5.92857, α → -1.48341}, {R → 2.19287, α → 3.09595},
 {R → 2.42575, α → 1.75736}, {R → 0.415164, α → -1.54998}, {R → 2.42742, α → -1.05489},
 {R → 2.44587, α → -1.20021}, {R → 1.54898, α → -1.1687}, {R → 1.94444, α → -0.896869},
 {R → 1.4799, α → 2.81471}, {R → 1.13575, α → -1.22048}, {R → 1.16878, α → 2.46211},
 {R → 0.435941, α → -1.04012}, {R → 0.284882, α → 0.952236},
 {R → 4.35714, α → 2.4458}, {R → 0.973684, α → 3.04448}, {R → 47.4437, α → 1.60487},
 {R → 8.13354, α → 1.47927}, {R → 2.06096, α → 0.616448}, {R → 1.63564, α → 1.42093},
 {R → 8.64244, α → -1.86272}, {R → 1.31443, α → 1.62211}, {R → 5.92857, α → -1.65819},
 {R → 4.35714, α → -2.4458}, {R → 0.815123, α → -2.00112}, {R → 2.30212, α → 0.888244},
 {R → 3.23875, α → -1.90132}, {R → 3.92085, α → 0.913704}, {R → 3.98475, α → -2.26194},
 {R → 1.46338, α → -2.1542}, {R → 1.94444, α → -2.24472}, {R → 0.973684, α → -3.04448},
 {R → 1.37064, α → -2.43725}, {R → 0.888856, α → -1.60659},
 {R → 1.70638, α → 0.657892}, {R → 1.0649, α → -0.693426}, {R → 1.51629, α → 0.350071}}
```

Get Delta functions

```
DeltaSolution = Table[ModSolution[[i, 2]], {i, num}]
```

```
{DiracDelta[6.28571 - 1. w], DiracDelta[12.5714 - 1. w],
 DiracDelta[18.8571 - 1. w], DiracDelta[25.125 - 1. w], DiracDelta[31.4167 - 1. w],
 DiracDelta[37.7 - 1. w], DiracDelta[43.973 - 1. w], DiracDelta[50.2727 - 1. w],
 DiracDelta[56.5556 - 1. w], DiracDelta[62.8333 - 1. w],
 DiracDelta[69.125 - 1. w], DiracDelta[75.4 - 1. w], DiracDelta[81.6875 - 1. w],
 DiracDelta[87.9565 - 1. w], DiracDelta[94.25 - 1. w], DiracDelta[100.538 - 1. w],
 DiracDelta[106.818 - 1. w], DiracDelta[113.1 - 1. w], DiracDelta[119.375 - 1. w],
 DiracDelta[43.9767 - 1.000000000 w], DiracDelta[87.96 - 1.000000000 w],
 DiracDelta[6.28571 + w], DiracDelta[12.5714 + w], DiracDelta[18.8571 + w],
 DiracDelta[25.125 + w], DiracDelta[31.4167 + w], DiracDelta[37.7 + w],
 DiracDelta[43.973 + w], DiracDelta[43.9767 + w], DiracDelta[50.2727 + w],
 DiracDelta[56.5556 + w], DiracDelta[62.8333 + w], DiracDelta[69.125 + w],
 DiracDelta[75.4 + w], DiracDelta[81.6875 + w], DiracDelta[87.9565 + w],
 DiracDelta[87.96 + w], DiracDelta[94.25 + w], DiracDelta[100.538 + w],
 DiracDelta[106.818 + w], DiracDelta[113.1 + w], DiracDelta[119.375 + w]}
```

```
Solve[DeltaSolution[[1]] == DiracDelta[0], w]
```

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

$\{\{w \to \frac{44}{7}\}\}$

```
DiracDelta[2]
```

0

Solve for Frequencies

```
FreqSolution = Table[Solve[DeltaSolution[[i]] == DiracDelta[0], w], {i, num}]
```

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

General::stop : Further output of Solve::ifun will be suppressed during this calculation. ≫

$$\{\{\{w \to \frac{44}{7}\}\}, \{\{w \to \frac{88}{7}\}\}, \{\{w \to \frac{132}{7}\}\}, \{\{w \to \frac{201}{8}\}\}, \{\{w \to \frac{377}{12}\}\}, \{\{w \to \frac{377}{10}\}\},$$
$$\{\{w \to \frac{1627}{37}\}\}, \{\{w \to \frac{553}{11}\}\}, \{\{w \to \frac{509}{9}\}\}, \{\{w \to \frac{377}{6}\}\}, \{\{w \to \frac{553}{8}\}\}, \{\{w \to \frac{377}{5}\}\},$$
$$\{\{w \to \frac{1307}{16}\}\}, \{\{w \to \frac{2023}{23}\}\}, \{\{w \to \frac{377}{4}\}\}, \{\{w \to \frac{1307}{13}\}\}, \{\{w \to \frac{1175}{11}\}\},$$
$$\{\{w \to \frac{1131}{10}\}\}, \{\{w \to \frac{955}{8}\}\}, \{\{w \to \frac{1891}{43}\}\}, \{\{w \to \frac{2199}{25}\}\}, \{\{w \to -\frac{44}{7}\}\},$$
$$\{\{w \to -\frac{88}{7}\}\}, \{\{w \to -\frac{132}{7}\}\}, \{\{w \to -\frac{201}{8}\}\}, \{\{w \to -\frac{377}{12}\}\}, \{\{w \to -\frac{377}{10}\}\},$$
$$\{\{w \to -\frac{1627}{37}\}\}, \{\{w \to -\frac{1891}{43}\}\}, \{\{w \to -\frac{553}{11}\}\}, \{\{w \to -\frac{509}{9}\}\}, \{\{w \to -\frac{377}{6}\}\},$$
$$\{\{w \to -\frac{553}{8}\}\}, \{\{w \to -\frac{377}{5}\}\}, \{\{w \to -\frac{1307}{16}\}\}, \{\{w \to -\frac{2023}{23}\}\}, \{\{w \to -\frac{2199}{25}\}\},$$
$$\{\{w \to -\frac{377}{4}\}\}, \{\{w \to -\frac{1307}{13}\}\}, \{\{w \to -\frac{1175}{11}\}\}, \{\{w \to -\frac{1131}{10}\}\}, \{\{w \to -\frac{955}{8}\}\}\}$$

```
N[FreqSolution]
```

```
{{{w → 6.28571}}, {{w → 12.5714}}, {{w → 18.8571}}, {{w → 25.125}}, {{w → 31.4167}},
 {{w → 37.7}}, {{w → 43.973}}, {{w → 50.2727}}, {{w → 56.5556}}, {{w → 62.8333}},
 {{w → 69.125}}, {{w → 75.4}}, {{w → 81.6875}}, {{w → 87.9565}}, {{w → 94.25}},
 {{w → 100.538}}, {{w → 106.818}}, {{w → 113.1}}, {{w → 119.375}}, {{w → 43.9767}},
 {{w → 87.96}}, {{w → -6.28571}}, {{w → -12.5714}}, {{w → -18.8571}},
 {{w → -25.125}}, {{w → -31.4167}}, {{w → -37.7}}, {{w → -43.973}}, {{w → -43.9767}},
 {{w → -50.2727}}, {{w → -56.5556}}, {{w → -62.8333}}, {{w → -69.125}},
 {{w → -75.4}}, {{w → -81.6875}}, {{w → -87.9565}}, {{w → -87.96}}, {{w → -94.25}},
 {{w → -100.538}}, {{w → -106.818}}, {{w → -113.1}}, {{w → -119.375}}}
```

Get the solution set for each link

```
Table[Flatten[Append[ModLpSolution[[i]], FreqSolution[[i]]]], {i, num}]
```

$\{\{R\to 128.719, \alpha\to -1.46978, w\to \frac{44}{7}\}, \{R\to 10.2451, \alpha\to -1.43929, w\to \frac{88}{7}\},$
$\{R\to 22.9877, \alpha\to 2.04438, w\to \frac{132}{7}\}, \{R\to 4.84216, \alpha\to 2.25089, w\to \frac{201}{8}\},$
$\{R\to 16.102, \alpha\to -1.05204, w\to \frac{377}{12}\}, \{R\to 2.72186, \alpha\to 0.96246, w\to \frac{377}{10}\},$
$\{R\to 5.92857, \alpha\to -1.48341, w\to \frac{1627}{37}\}, \{R\to 2.19287, \alpha\to 3.09595, w\to \frac{553}{11}\},$
$\{R\to 2.42575, \alpha\to 1.75736, w\to \frac{509}{9}\}, \{R\to 0.415164, \alpha\to -1.54998, w\to \frac{377}{6}\},$
$\{R\to 2.42742, \alpha\to -1.05489, w\to \frac{553}{8}\}, \{R\to 2.44587, \alpha\to -1.20021, w\to \frac{377}{5}\},$
$\{R\to 1.54898, \alpha\to -1.1687, w\to \frac{1307}{16}\}, \{R\to 1.94444, \alpha\to -0.896869, w\to \frac{2023}{23}\},$
$\{R\to 1.4799, \alpha\to 2.81471, w\to \frac{377}{4}\}, \{R\to 1.13575, \alpha\to -1.22048, w\to \frac{1307}{13}\},$
$\{R\to 1.16878, \alpha\to 2.46211, w\to \frac{1175}{11}\}, \{R\to 0.435941, \alpha\to -1.04012, w\to \frac{1131}{10}\},$
$\{R\to 0.284882, \alpha\to 0.952236, w\to \frac{955}{8}\}, \{R\to 4.35714, \alpha\to 2.4458, w\to \frac{1891}{43}\},$
$\{R\to 0.973684, \alpha\to 3.04448, w\to \frac{2199}{25}\}, \{R\to 47.4437, \alpha\to 1.60487, w\to -\frac{44}{7}\},$
$\{R\to 8.13354, \alpha\to 1.47927, w\to -\frac{88}{7}\}, \{R\to 2.06096, \alpha\to 0.616448, w\to -\frac{132}{7}\},$
$\{R\to 1.63564, \alpha\to 1.42093, w\to -\frac{201}{8}\}, \{R\to 8.64244, \alpha\to -1.86272, w\to -\frac{377}{12}\},$
$\{R\to 1.31443, \alpha\to 1.62211, w\to -\frac{377}{10}\}, \{R\to 5.92857, \alpha\to -1.65819, w\to -\frac{1627}{37}\},$
$\{R\to 4.35714, \alpha\to -2.4458, w\to -\frac{1891}{43}\}, \{R\to 0.815123, \alpha\to -2.00112, w\to -\frac{553}{11}\},$
$\{R\to 2.30212, \alpha\to 0.888244, w\to -\frac{509}{9}\}, \{R\to 3.23875, \alpha\to -1.90132, w\to -\frac{377}{6}\},$
$\{R\to 3.92085, \alpha\to 0.913704, w\to -\frac{553}{8}\}, \{R\to 3.98475, \alpha\to -2.26194, w\to -\frac{377}{5}\},$
$\{R\to 1.46338, \alpha\to -2.1542, w\to -\frac{1307}{16}\}, \{R\to 1.94444, \alpha\to -2.24472, w\to -\frac{2023}{23}\},$
$\{R\to 0.973684, \alpha\to -3.04448, w\to -\frac{2199}{25}\}, \{R\to 1.37064, \alpha\to -2.43725, w\to -\frac{377}{4}\},$
$\{R\to 0.888856, \alpha\to -1.60659, w\to -\frac{1307}{13}\}, \{R\to 1.70638, \alpha\to 0.657892, w\to -\frac{1175}{11}\},$
$\{R\to 1.0649, \alpha\to -0.693426, w\to -\frac{1131}{10}\}, \{R\to 1.51629, \alpha\to 0.350071, w\to -\frac{955}{8}\}\}$

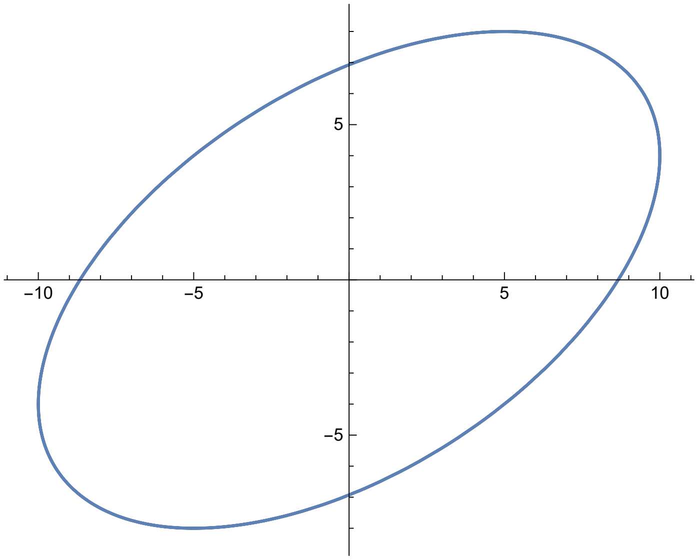
```
Table[Flatten[Append[ModLpSolution[[i]], N[FreqSolution[[i]]]]], {i, num}]
```

```
{{R → 128.719, α → -1.46978, w → 6.28571}, {R → 10.2451, α → -1.43929, w → 12.5714},
 {R → 22.9877, α → 2.04438, w → 18.8571}, {R → 4.84216, α → 2.25089, w → 25.125},
 {R → 16.102, α → -1.05204, w → 31.4167}, {R → 2.72186, α → 0.96246, w → 37.7},
 {R → 5.92857, α → -1.48341, w → 43.973}, {R → 2.19287, α → 3.09595, w → 50.2727},
 {R → 2.42575, α → 1.75736, w → 56.5556}, {R → 0.415164, α → -1.54998, w → 62.8333},
 {R → 2.42742, α → -1.05489, w → 69.125}, {R → 2.44587, α → -1.20021, w → 75.4},
 {R → 1.54898, α → -1.1687, w → 81.6875}, {R → 1.94444, α → -0.896869, w → 87.9565},
 {R → 1.4799, α → 2.81471, w → 94.25}, {R → 1.13575, α → -1.22048, w → 100.538},
 {R → 1.16878, α → 2.46211, w → 106.818}, {R → 0.435941, α → -1.04012, w → 113.1},
 {R → 0.284882, α → 0.952236, w → 119.375}, {R → 4.35714, α → 2.4458, w → 43.9767},
 {R → 0.973684, α → 3.04448, w → 87.96}, {R → 47.4437, α → 1.60487, w → -6.28571},
 {R → 8.13354, α → 1.47927, w → -12.5714}, {R → 2.06096, α → 0.616448, w → -18.8571},
 {R → 1.63564, α → 1.42093, w → -25.125}, {R → 8.64244, α → -1.86272, w → -31.4167},
 {R → 1.31443, α → 1.62211, w → -37.7}, {R → 5.92857, α → -1.65819, w → -43.973},
 {R → 4.35714, α → -2.4458, w → -43.9767}, {R → 0.815123, α → -2.00112, w → -50.2727},
 {R → 2.30212, α → 0.888244, w → -56.5556}, {R → 3.23875, α → -1.90132, w → -62.8333},
 {R → 3.92085, α → 0.913704, w → -69.125}, {R → 3.98475, α → -2.26194, w → -75.4},
 {R → 1.46338, α → -2.1542, w → -81.6875}, {R → 1.94444, α → -2.24472, w → -87.9565},
 {R → 0.973684, α → -3.04448, w → -87.96}, {R → 1.37064, α → -2.43725, w → -94.25},
 {R → 0.888856, α → -1.60659, w → -100.538}, {R → 1.70638, α → 0.657892, w → -106.818},
 {R → 1.0649, α → -0.693426, w → -113.1}, {R → 1.51629, α → 0.350071, w → -119.375}}
```

Test Complex Number

```
c = Expand[10 * Cos[Pi / 6 + 2 t]];
d = 8 * Sin[Pi / 3 + 2 t];
```

```
ParametricPlot[{c, d}, {t, 0, 2 Pi}]
```




```
FTc = N[Chop[Simplify[FourierTransform[Expand[N[c], Trig -> True], t, w]]], 10]
```

```
(10.854 - 6.26657 i) DiracDelta[2. - 1.000000000 w] +
 (10.854 + 6.26657 i) DiracDelta[2. + w]
```

```
FTd = N[Chop[Expand[N[I * FourierTransform[Expand[N[d], Trig -> True], t, w]]]], 10]
```

```
(-5.01326 + 8.68322 i) DiracDelta[2. - 1. w] + (5.01326 + 8.68322 i) DiracDelta[2. + w]
```

```
Expand[(FTc + FTd) * (1 / Sqrt[2 Pi])]
```

```
(2.33013 + 0.964102 i) DiracDelta[2. - 1. w] + (6.33013 + 5.9641 i) DiracDelta[2. + w]
```

```
NSolve[
 R1 * Cos[α1] == 2.330127018922194` && R1 * Sin[α1] == 0.964101615137755`, {R1, α1}]
```

NSolve::ifun : Inverse functions are being used by NSolve,
so some solutions may not be found; use Reduce for complete solution information. >>

```
{{R1 → -2.5217, α1 → -2.74929}, {R1 → 2.5217, α1 → 0.392308}}
```

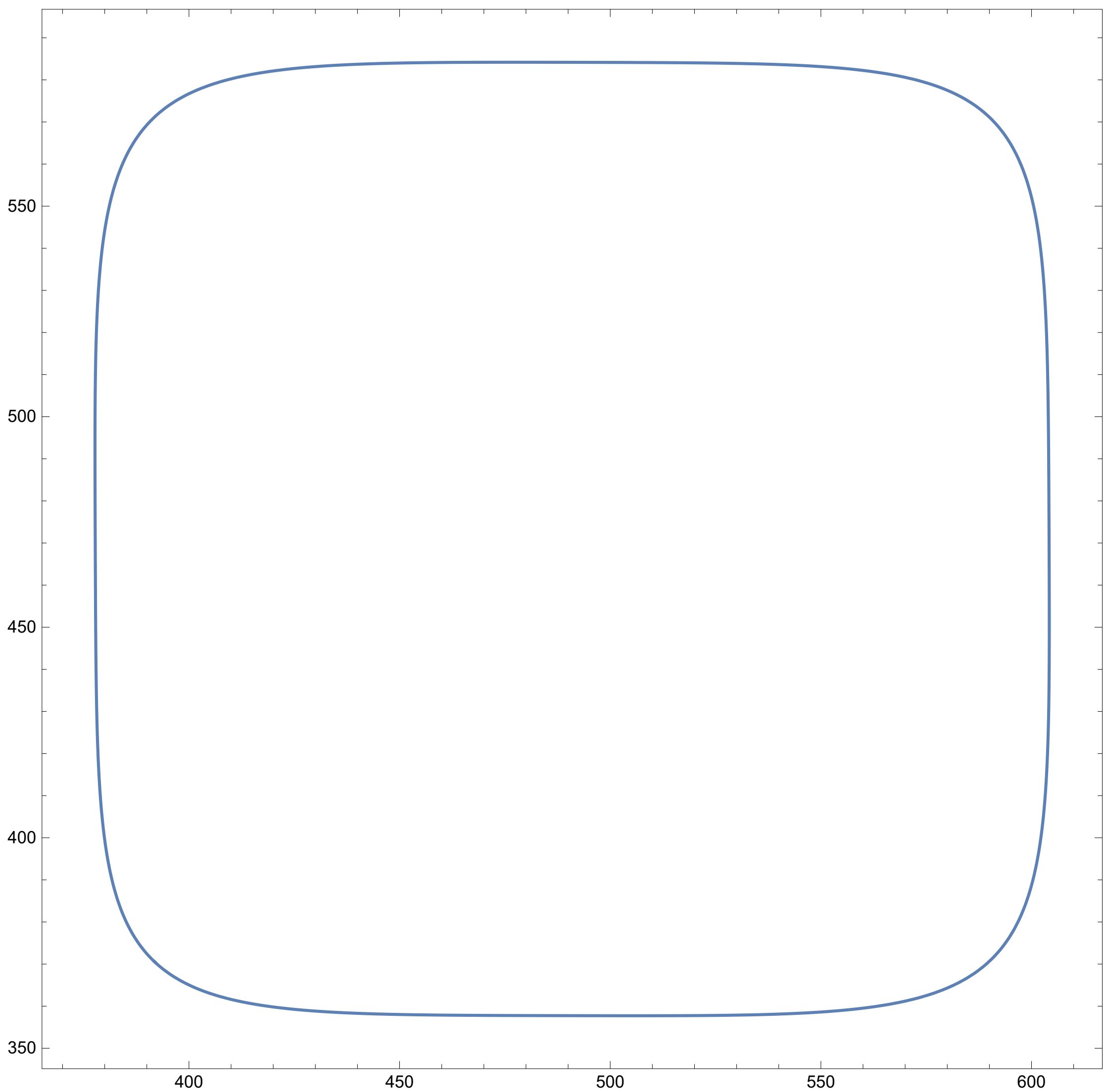
```
NSolve[
 R1 * Cos[α1] == 6.330127018922194` && R1 * Sin[α1] == 5.9641016151377535`, {R1, α1}]
```

NSolve::ifun : Inverse functions are being used by NSolve,
so some solutions may not be found; use Reduce for complete solution information. >>

```
{{R1 → -8.69718, α1 → -2.38596}, {R1 → 8.69718, α1 → 0.755635}}
```

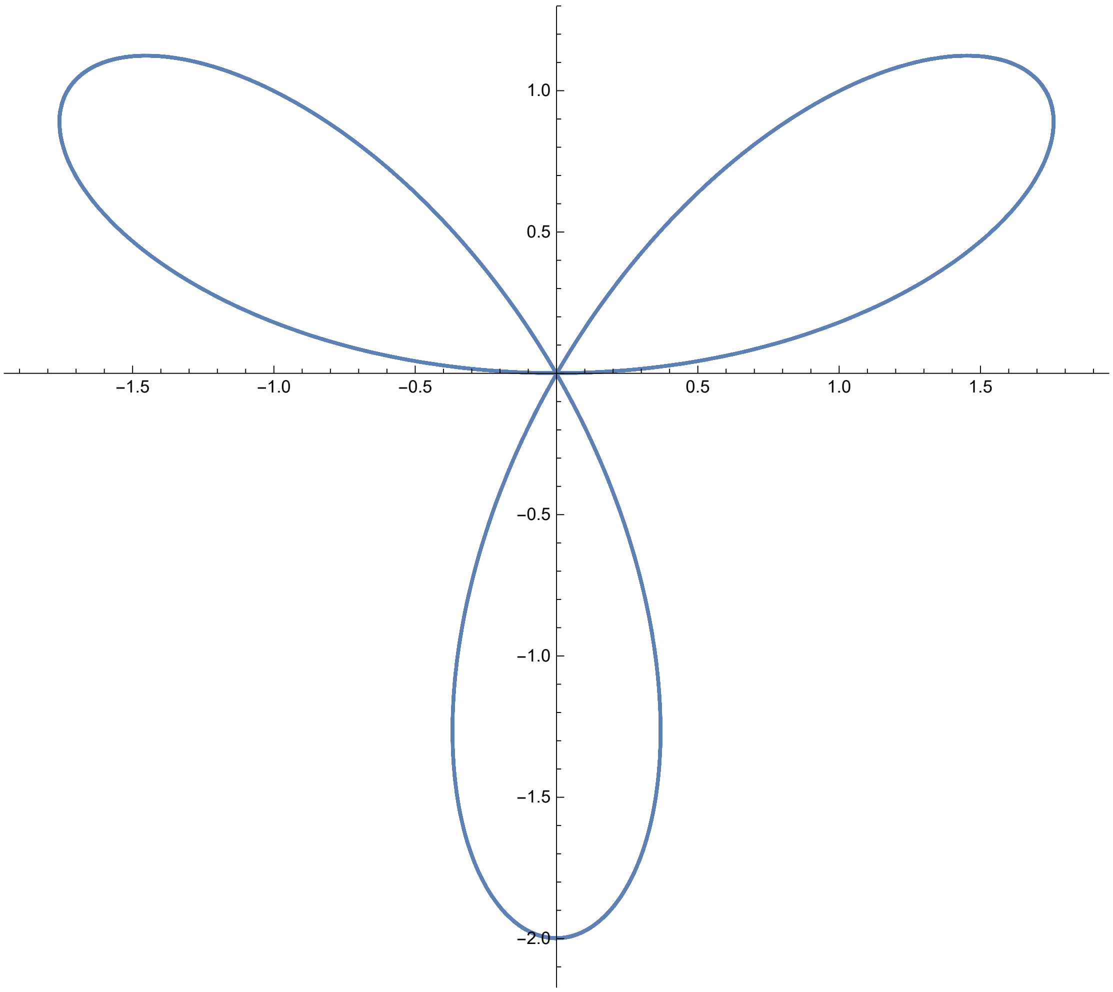
Test Square

```
img = Import["C:\\Users\\Onion\\Dropbox\\1_Fourier Transform\\Square.GIF"];

img = Binarize[img~ColorConvert~"Grayscale"~ImageResize~500~Blur~3]~Blur~3;

lines = Cases[
   Normal@ListContourPlot[Reverse@ImageData[img], Contours → {0.5}], _Line, -1];

ParametricPlot[Evaluate[tocurve[#, 4, t] & /@ lines],
 {t, 0, 1}, Frame → True, Axes → False, ImageSize → Large]
```



Three Pedalled Rose

```
x = Sin[4 t] + Sin[2 t];

y = Cos[2 t] - Cos[4 t];

ParametricPlot[{x, y}, {t, 0, 100}, ImageSize → Large]
```



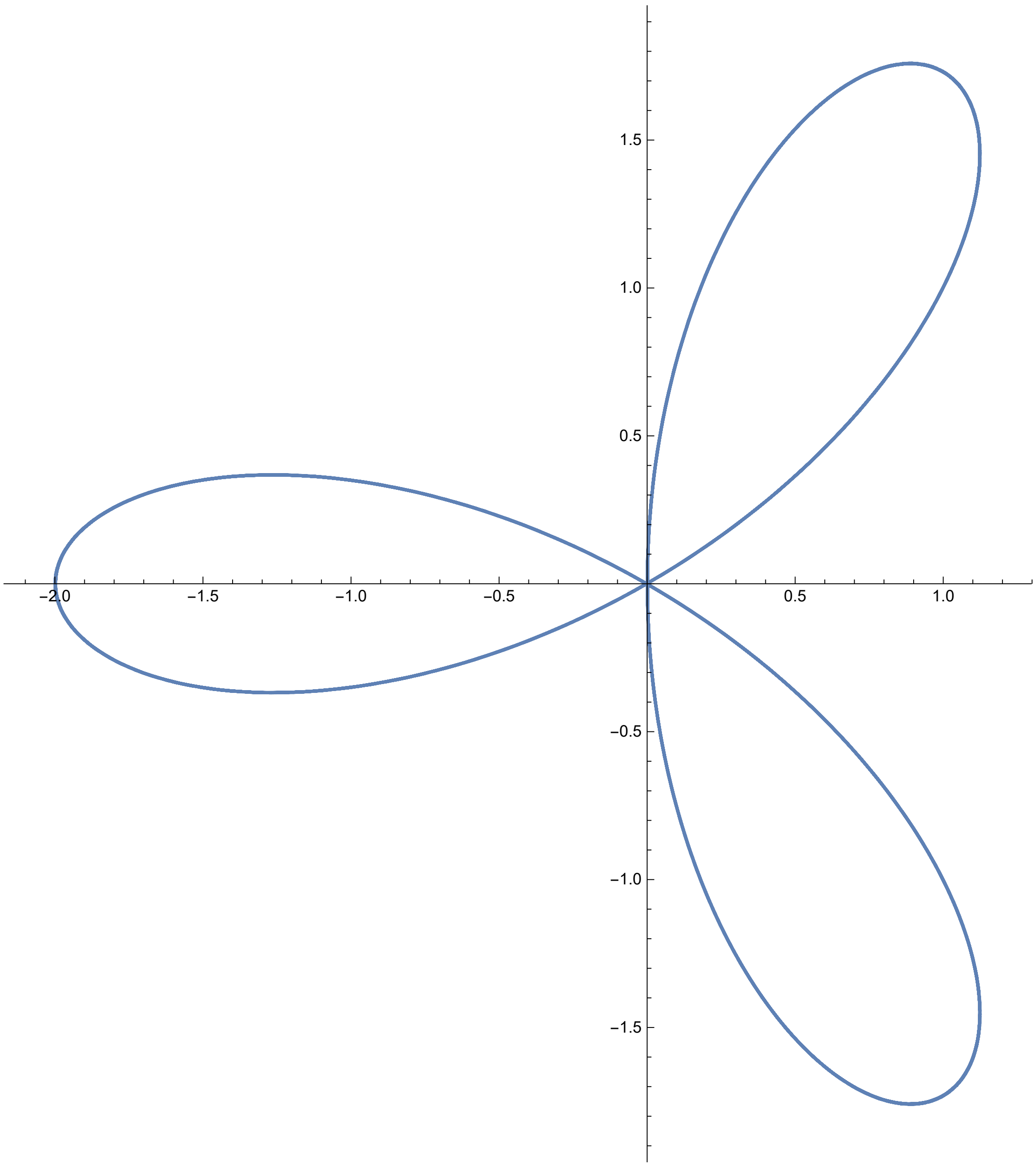

```
FTy = N[Chop[Simplify[FourierTransform[Expand[N[y], Trig -> True], t, w]]], 10]
```

1.253314137 DiracDelta[2. - 1.000000000 w] - 1.25331 DiracDelta[4. - 1.000000000 w] +
 1.253314137 DiracDelta[2. + w] - 1.25331 DiracDelta[4. + w]

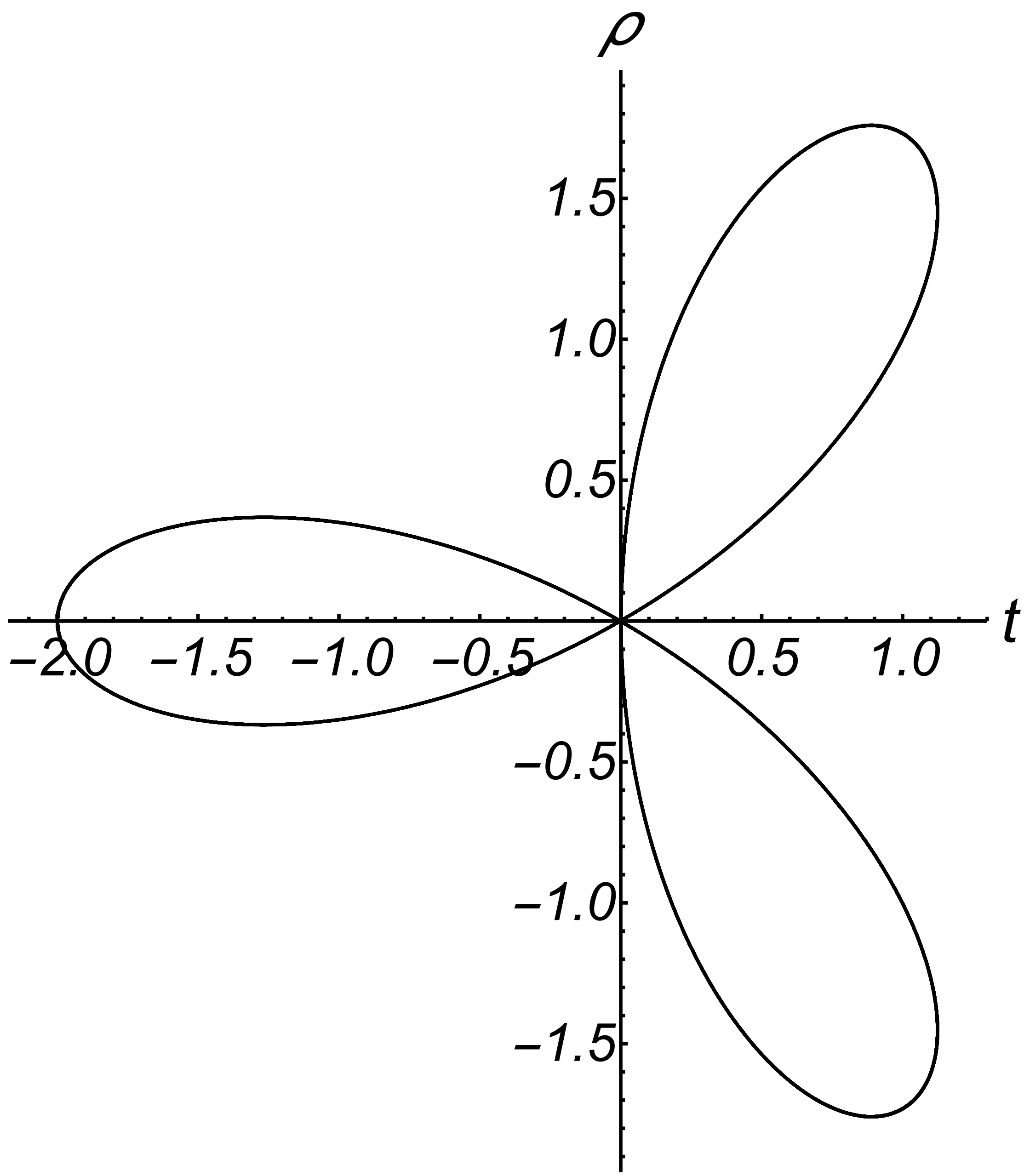
```
FTx = N[Chop[Expand[N[I * FourierTransform[Expand[N[x], Trig -> True], t, w]]]], 10]
```

-1.25331 DiracDelta[2. - 1. w] - 1.25331 DiracDelta[4. - 1. w] +
 1.25331 DiracDelta[2. + w] + 1.25331 DiracDelta[4. + w]

```
Chop[Expand[(FTy + FTx) * (1 / Sqrt[2 Pi])]]
-1. DiracDelta[4. - 1. w] + 1. DiracDelta[2. + w]

x = -(Cos[2 t] + Cos[4 t]);

y = -(Sin[4 t] - Sin[2 t]);

ParametricPlot[{x, y}, {t, 0, 100}, ImageSize → Large]
```



```
ParametricPlot[{x, y}, {t, 0, 100}, PlotStyle → Directive[Black, Thickness[.003]],
 AxesLabel → {Style["x", 35, Black, Italic], Style["y", 35, Black, Italic]},
 AxesStyle → Directive[Black, 30, Thick],
 LabelStyle → Directive[30, Black, Italic], ImageSize → Large]
```




```
FTx = N[Chop[Simplify[FourierTransform[Expand[x, Trig -> True], t, w]]], 10]
```

```
-1.253314137 DiracDelta[-4.000000000 + w] - 1.253314137 DiracDelta[-2.000000000 + w] -
 1.253314137 DiracDelta[2.000000000 + w] - 1.253314137 DiracDelta[4.000000000 + w]
```

```
FTy = N[Chop[Expand[N[-I * FourierTransform[Expand[y, Trig -> True], t, w]]]], 10]
```

-1.25331 DiracDelta[-4. + w] + 1.25331 DiracDelta[-2. + w] -
 1.25331 DiracDelta[2. + w] + 1.25331 DiracDelta[4. + w]

```
Chop[Expand[(FTy + FTx) * (1 / Sqrt[2 Pi])]]
```

-1. DiracDelta[-4. + w] - 1. DiracDelta[2. + w]

Test

```
FourierTransform[Cos[2 t], t, w]
```

$\sqrt{\frac{\pi}{2}}$ DiracDelta[-2 + w] + $\sqrt{\frac{\pi}{2}}$ DiracDelta[2 + w]

```
FourierTransform[Sin[2 t], t, w]
```

$i\sqrt{\frac{\pi}{2}}$ DiracDelta[-2 + w] - $i\sqrt{\frac{\pi}{2}}$ DiracDelta[2 + w]

Oblique Bifolium

```
a = 0; b = 150 ;
```

Get x component

```
xIni = Simplify[(a Cos[t]^3 + b Sin[t] Cos[t]^2) Cos[t]]
```

150 Cos[t]^3 Sin[t]

```
xSim = Expand[xIni /. Cos[t]^3 → (3 Cos[t] + Cos[3 t]) / 4]
```

$\frac{225}{2}$ Cos[t] Sin[t] + $\frac{75}{2}$ Cos[3 t] Sin[t]

```
xSim2 = Expand[xSim /. Cos[t]^2 → (1 + Cos[2 t]) / 2]
```

$\frac{225}{2}$ Cos[t] Sin[t] + $\frac{75}{2}$ Cos[3 t] Sin[t]

```
x = Expand[xSim2 /. {Cos[t] Cos[3 t] → 0.5 (Cos[2 t] + Cos[4 t]),
    Cos[t] Sin[t] → 0.5 (Sin[2 t]), Cos[3 t] Sin[t] → 0.5 (Sin[4 t] - Sin[2 t])}]
```

37.5 Sin[2 t] + 18.75 Sin[4 t]

Get y component

```
yIni = Simplify[(a Cos[t]^3 + b Sin[t] Cos[t]^2) Sin[t]]
75/2 Sin[2 t]^2

ySim = Expand[yIni /. Cos[t]^2 → (1 + Cos[2 t]) / 2]
75/2 Sin[2 t]^2

ySim2 = Expand[ySim /. Sin[t]^2 → (1 - Cos[2 t]) / 2]
75/2 Sin[2 t]^2

ySim3 = Expand[ySim2 /. {Cos[2 t]^2 → (1 + Cos[4 t]) / 2, Cos[t] Sin[t] → 0.5 (Sin[2 t])}]
75/2 Sin[2 t]^2

y = ySim3 /. Cos[2 t] Sin[2 t] → 0.5 (Sin[4 t])
75/2 Sin[2 t]^2
```

Get Chain

Fourier Transform here don't need to make x and y expression the format of summation of sin and cos

```
FTx = N[Chop[Simplify[FourierTransform[Expand[N[x], Trig -> True], t, w]]], 10]
(0. + 46.9993 i) DiracDelta[2. - 1.000000000 w] +
 (0. + 23.4996 i) DiracDelta[4. - 1.000000000 w] -
 (0. + 46.9993 i) DiracDelta[2. + w] - (0. + 23.4996 i) DiracDelta[4. + w]

FTy = N[Chop[Expand[N[I * FourierTransform[Expand[N[y], Trig -> True], t, w]]]], 10]
(0. - 23.4996 i) DiracDelta[4. - 1. w] +
 (0. + 46.9993 i) DiracDelta[w] - (0. + 23.4996 i) DiracDelta[4. + w]

Solution = Chop[Expand[(FTy + FTx) * (1 / Sqrt[2 Pi])]]
(0. + 18.75 i) DiracDelta[2. - 1.000000000 w] + (0. + 18.75 i) DiracDelta[w] -
 (0. + 18.75 i) DiracDelta[2. + w] - (0. + 18.75 i) DiracDelta[4. + w]
```

Remove the solution ω=0

```
ModSolution = Cases[Solution, Except[(0.1875` - 0.0675` i) DiracDelta[w]]]
{(0. + 18.75 i) DiracDelta[2. - 1.000000000 w], (0. + 18.75 i) DiracDelta[w],
 (0. - 18.75 i) DiracDelta[2. + w], (0. - 18.75 i) DiracDelta[4. + w]}
```

```
num = Length[ModSolution]
```

4

Get the coefficient of Delta function

```
CoeSolution = Table[ModSolution[[i, 1]], {i, num}]
```

{0. + 18.75 i, 0. + 18.75 i, 0. - 18.75 i, 0. - 18.75 i}

Solve for Link lengths and Phases

```
LpSolution = Table[NSolve[{R * Cos[α] == Re[CoeSolution[[i]]],
    R * Sin[α] == Im[CoeSolution[[i]]]}, {R, α}], {i, num}]
```

NSolve::ifun : Inverse functions are being used by NSolve,
so some solutions may not be found; use Reduce for complete solution information. ≫

NSolve::ifun : Inverse functions are being used by NSolve,
so some solutions may not be found; use Reduce for complete solution information. ≫

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General::stop : Further output of NSolve::ifun will be suppressed during this calculation. ≫

{{{R → -18.75, α → -1.5708}, {R → 18.75, α → 1.5708}},
{{R → -18.75, α → -1.5708}, {R → 18.75, α → 1.5708}},
{{R → -18.75, α → 1.5708}, {R → 18.75, α → -1.5708}},
{{R → -18.75, α → 1.5708}, {R → 18.75, α → -1.5708}}}

Pick up solutions that R>0

```
ModLpSolution = Table[LpSolution[[i, 2]], {i, num}]
```

{{R → 18.75, α → 1.5708}, {R → 18.75, α → 1.5708},
{R → 18.75, α → -1.5708}, {R → 18.75, α → -1.5708}}

Get Delta functions

```
DeltaSolution = Table[ModSolution[[i, 2]], {i, num}]
```

{DiracDelta[2. - 1.000000000 w], DiracDelta[w], DiracDelta[2. + w], DiracDelta[4. + w]}

Solve for Frequencies

```
FreqSolution = Table[Solve[DeltaSolution[[i]] == DiracDelta[0], w], {i, num}]
```

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

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Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

General::stop : Further output of Solve::ifun will be suppressed during this calculation. >>

```
{{{w → 2}}, {{w → 0}}, {{w → -2}}, {{w → -4}}}
```

```
N[FreqSolution]
```

```
{{{w → 2.}}, {{w → 0.}}, {{w → -2.}}, {{w → -4.}}}
```

Get the solution set for each link

```
Table[Flatten[Append[ModLpSolution[[i]], FreqSolution[[i]]]], {i, num}]
```

```
{{R → 18.75, α → 1.5708, w → 2}, {R → 18.75, α → 1.5708, w → 0},
 {R → 18.75, α → -1.5708, w → -2}, {R → 18.75, α → -1.5708, w → -4}}
```

```
Table[Flatten[Append[ModLpSolution[[i]], N[FreqSolution[[i]]]]], {i, num}]
```

```
{{R → 18.75, α → 1.5708, w → 2.}, {R → 18.75, α → 1.5708, w → 0.},
 {R → 18.75, α → -1.5708, w → -2.}, {R → 18.75, α → -1.5708, w → -4.}}
```

Geometric Petal

```
a =.; b =.; n =.;
```

```
ρ = a + b Cos[2 n * t]
```

```
a + b Cos[2 n t]
```

Get x component

```
xIni = Expand[Simplify[ρ Cos[t]]]
```

```
a Cos[t] + b Cos[t] Cos[2 n t]
```

```
xSim = Expand[xIni /. Cos[t]^3 → (3 Cos[t] + Cos[3 t]) / 4]
a Cos[t] + b Cos[t] Cos[2 n t]

xSim2 = Expand[xSim /. Cos[t]^2 → (1 + Cos[2 t]) / 2]
a Cos[t] + b Cos[t] Cos[2 n t]

x = Expand[xSim2 /. {Cos[t] Cos[2 n t] → 0.5 (Cos[2 n t - t] + Cos[2 n t + t]),
    Cos[t] Sin[t] → 0.5 (Sin[2 t]), Cos[3 t] Sin[t] → 0.5 (Sin[4 t] - Sin[2 t])}]
a Cos[t] + 0.5 b Cos[t - 2 n t] + 0.5 b Cos[t + 2 n t]
```

Get y component

```
yIni = Expand[Simplify[ρ Sin[t]]]
a Sin[t] + b Cos[2 n t] Sin[t]

ySim = Expand[yIni /. Cos[t]^2 → (1 + Cos[2 t]) / 2]
a Sin[t] + b Cos[2 n t] Sin[t]

ySim2 = Expand[ySim /. Sin[t]^2 → (1 - Cos[2 t]) / 2]
a Sin[t] + b Cos[2 n t] Sin[t]

ySim3 = Expand[ySim2 /. {Cos[2 t]^2 → (1 + Cos[4 t]) / 2,
    Cos[2 n t] Sin[t] → 0.5 (Sin[2 n t + t] - Sin[2 n t - t])}]
a Sin[t] + 0.5 b Sin[t - 2 n t] + 0.5 b Sin[t + 2 n t]

y = ySim3 /. Cos[2 t] Sin[2 t] → 0.5 (Sin[4 t])
a Sin[t] + 0.5 b Sin[t - 2 n t] + 0.5 b Sin[t + 2 n t]

a = 20; b = 70; n = 3;
```

Get Chain

Fourier Transform here don't need to make x and y expression the format of summation of sin and cos

```
FTx = N[Chop[Simplify[FourierTransform[Expand[N[x], Trig -> True], t, w]]], 10]
43.866 DiracDelta[5. - 1.000000000 w] + 43.866 DiracDelta[7. - 1.000000000 w] +
 25.0663 DiracDelta[-1.000000000 + w] + 25.0663 DiracDelta[1.000000000 + w] +
 43.866 DiracDelta[5. + w] + 43.866 DiracDelta[7. + w]

FTy = N[Chop[Expand[N[I * FourierTransform[Expand[N[y], Trig -> True], t, w]]]], 10]
43.866 DiracDelta[5. - 1. w] - 43.866 DiracDelta[7. - 1. w] - 25.0663 DiracDelta[-1. + w] +
 25.0663 DiracDelta[1. + w] - 43.866 DiracDelta[5. + w] + 43.866 DiracDelta[7. + w]
```

```
Solution = Chop[Expand[(FTy + FTx) * (1 / Sqrt[2 Pi])]]
35. DiracDelta[5. - 1. w] + 20. DiracDelta[1. + w] + 35. DiracDelta[7. + w]
```

Remove the solution ω=0

```
ModSolution = Cases[Solution, Except[(0.1875` - 0.0675` i) DiracDelta[w]]]
{35. DiracDelta[5. - 1. w], 20. DiracDelta[1. + w], 35. DiracDelta[7. + w]}

num = Length[ModSolution]
3
```

Get the coefficient of Delta function

```
CoeSolution = Table[ModSolution[[i, 1]], {i, num}]
{35., 20., 35.}
```

Solve for Link lengths and Phases

```
LpSolution = Table[NSolve[{R * Cos[α] == Re[CoeSolution[[i]]],
    R * Sin[α] == Im[CoeSolution[[i]]]}, {R, α}], {i, num}]
{{{R → -35., α → -3.14159}, {R → -35., α → 3.14159}, {R → 35., α → 0.}},
 {{R → -20., α → -3.14159}, {R → -20., α → 3.14159}, {R → 20., α → 0.}},
 {{R → -35., α → -3.14159}, {R → -35., α → 3.14159}, {R → 35., α → 0.}}}
```

Pick up solutions that R>0

```
ModLpSolution = Table[LpSolution[[i, 3]], {i, num}]
{{R → 35., α → 0.}, {R → 20., α → 0.}, {R → 35., α → 0.}}
```

Get Delta functions

```
DeltaSolution = Table[ModSolution[[i, 2]], {i, num}]
{DiracDelta[5. - 1. w], DiracDelta[1. + w], DiracDelta[7. + w]}
```

Solve for Frequencies

```
FreqSolution = Table[Solve[DeltaSolution[[i]] == DiracDelta[0], w], {i, num}]
{{{w → 5}}, {{w → -1}}, {{w → -7}}}
```

```
N[FreqSolution]
{{{w → 5.}}, {{w → -1.}}, {{w → -7.}}}
```

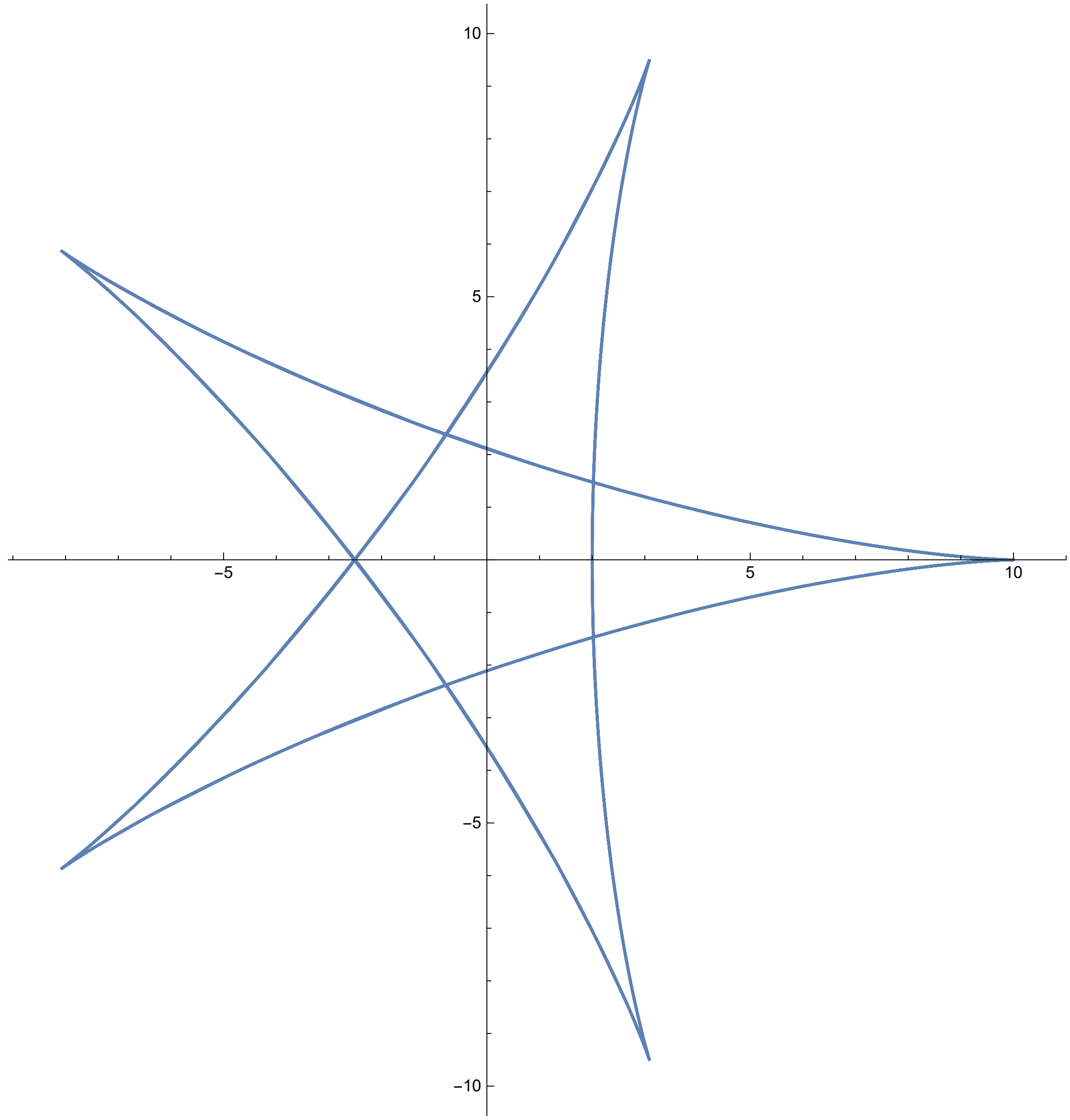
Get the solution set for each link

```
Table[Flatten[Append[ModLpSolution[[i]], FreqSolution[[i]]]], {i, num}]
{{R → 35., α → 0., w → 5}, {R → 20., α → 0., w → -1}, {R → 35., α → 0., w → -7}}

Table[Flatten[Append[ModLpSolution[[i]], N[FreqSolution[[i]]]]], {i, num}]
{{R → 35., α → 0., w → 5.}, {R → 20., α → 0., w → -1.}, {R → 35., α → 0., w → -7.}}
```

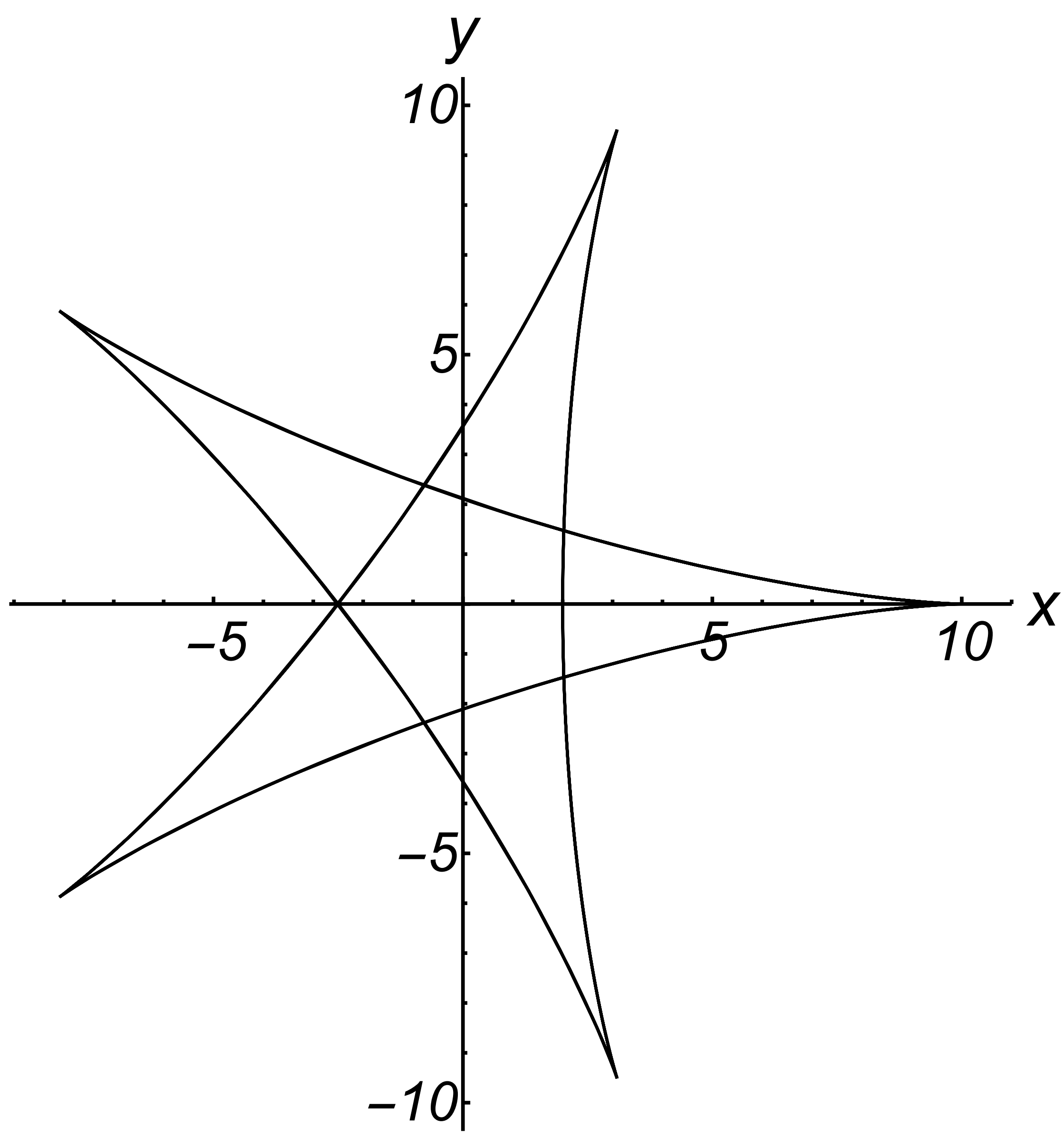
Hypercloid

```
LR =.; m =.;

x = (LR - m LR) Cos[m t] + m LR Cos[t - m t]
(LR - LR m) Cos[m t] + LR m Cos[t - m t]

y = (LR - m LR) Sin[m t] - m LR Sin[t - m t]
(LR - LR m) Sin[m t] - LR m Sin[t - m t]

LR = 10; m = .4;
x
6. Cos[0.4 t] + 4. Cos[0.6 t]

y
6. Sin[0.4 t] - 4. Sin[0.6 t]
```

```
ParametricPlot[{x, y}, {t, 0, 100}, ImageSize → Large]
```



```
ParametricPlot[{x, y}, {t, 0, 100}, PlotStyle → Directive[Black, Thickness[.003]],
 AxesLabel → {Style["x", 35, Black, Italic], Style["y", 35, Black, Italic]},
 AxesStyle → Directive[Black, 30, Thick],
 LabelStyle → Directive[30, Black, Italic], ImageSize → Large]
```



Get Chain

Fourier Transform here don't need to make x and y expression the format of summation of sin and cos

```
FTx = N[Chop[Simplify[FourierTransform[Expand[N[x], Trig -> True], t, w]]], 10]
```

```
7.51988 DiracDelta[0.4 - 1.000000000 w] + 5.01326 DiracDelta[0.6 - 1.000000000 w] +
 7.51988 DiracDelta[0.4 + w] + 5.01326 DiracDelta[0.6 + w]
```

```
FTy = N[Chop[Expand[N[I * FourierTransform[Expand[N[y], Trig -> True], t, w]]]], 10]
```

```
-7.51988 DiracDelta[0.4 - 1. w] + 5.01326 DiracDelta[0.6 - 1. w] +
 7.51988 DiracDelta[0.4 + w] - 5.01326 DiracDelta[0.6 + w]
```

```
Solution = Chop[Expand[(FTy + FTx) * (1 / Sqrt[2 Pi])]]
```

```
4. DiracDelta[0.6 - 1. w] + 6. DiracDelta[0.4 + w]
```

Remove the solution ω=0

```
ModSolution = Cases[Solution, Except[(0.1875` - 0.0675` i) DiracDelta[w]]]
```

```
{4. DiracDelta[0.6 - 1. w], 6. DiracDelta[0.4 + w]}
```

```
num = Length[ModSolution]
```

```
2
```

Get the coefficient of Delta function

```
CoeSolution = Table[ModSolution[[i, 1]], {i, num}]
```

```
{4., 6.}
```

Solve for Link lengths and Phases

```
LpSolution = Table[NSolve[{R * Cos[α] == Re[CoeSolution[[i]]],
     R * Sin[α] == Im[CoeSolution[[i]]]}, {R, α}], {i, num}]
```

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NSolve::ifun : Inverse functions are being used by NSolve,
so some solutions may not be found; use Reduce for complete solution information. >>

```
{{{R → -4., α → -3.14159}, {R → -4., α → 3.14159}, {R → 4., α → 0.}},
 {{R → -6., α → -3.14159}, {R → -6., α → 3.14159}, {R → 6., α → 0.}}}
```

Pick up solutions that R>0

```
ModLpSolution = Table[LpSolution[[i, 3]], {i, num}]
```

```
{{R → 4., α → 0.}, {R → 6., α → 0.}}
```

Get Delta functions

```
DeltaSolution = Table[ModSolution[[i, 2]], {i, num}]
```

```
{DiracDelta[0.6 - 1. w], DiracDelta[0.4 + w]}
```

Solve for Frequencies

```
FreqSolution = Table[Solve[DeltaSolution[[i]] == DiracDelta[0], w], {i, num}]
```

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

$$\left\{\left\{\left\{w \to \frac{3}{5}\right\}\right\}, \left\{\left\{w \to -\frac{2}{5}\right\}\right\}\right\}$$

```
N[FreqSolution]
```

```
{{{w → 0.6}}, {{w → -0.4}}}
```

Get the solution set for each link

```
Table[Flatten[Append[ModLpSolution[[i]], FreqSolution[[i]]]], {i, num}]
```

$$\left\{\left\{R \to 4., \alpha \to 0., w \to \frac{3}{5}\right\}, \left\{R \to 6., \alpha \to 0., w \to -\frac{2}{5}\right\}\right\}$$

```
Table[Flatten[Append[ModLpSolution[[i]], N[FreqSolution[[i]]]]], {i, num}]
```

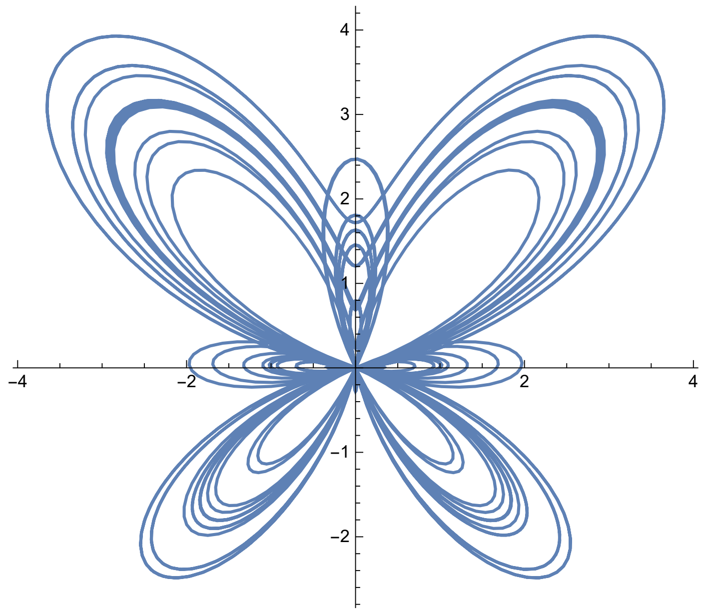
```
{{R → 4., α → 0., w → 0.6}, {R → 6., α → 0., w → -0.4}}
```

\[RawEscape]

\[RawEscape]

Butterfly

```
ParametricPlot[{(e^Sin[t] - 2 Cos[4 t] + Sin[(2 t - Pi) / 24]^5) Cos[t],
  (e^Sin[t] - 2 Cos[4 t] + Sin[(2 t - Pi) / 24]^5) Sin[t]}, {t, 0, 100}]
```




```
x = (e^Sin[t] - 2 Cos[4 t] + Sin[(2 t - Pi) / 24]^5) Cos[t];

y = (e^Sin[t] - 2 Cos[4 t] + Sin[(2 t - Pi) / 24]^5) Sin[t];
```

Get Chain

Fourier Transform here don't need to make x and y expression the format of summation of sin and cos

```
FTx = N[Chop[Simplify[FourierTransform[Expand[N[x], Trig -> True], t, w]]], 10]

0.751988 DiracDelta[0.4 - 1.000000000 w] + 0.501326 DiracDelta[0.6 - 1.000000000 w] +
 0.751988 DiracDelta[0.4 + w] + 0.501326 DiracDelta[0.6 + w]

FTy = N[Chop[Expand[N[I * FourierTransform[Expand[N[y], Trig -> True], t, w]]]], 10]

-0.751988 DiracDelta[0.4 - 1. w] - 0.501326 DiracDelta[0.6 - 1. w] +
 0.751988 DiracDelta[0.4 + w] + 0.501326 DiracDelta[0.6 + w]

Solution = Chop[Expand[(FTy + FTx) * (1 / Sqrt[2 Pi])]]

0.6 DiracDelta[0.4 + w] + 0.4 DiracDelta[0.6 + w]
```

Remove the solution ω=0

```
ModSolution = Cases[Solution, Except[(0.1875` - 0.0675` i) DiracDelta[w]]]
{0.6 DiracDelta[0.4 + w], 0.4 DiracDelta[0.6 + w]}
```

```
num = Length[ModSolution]
2
```

Get the coefficient of Delta function

```
CoeSolution = Table[ModSolution[[i, 1]], {i, num}]
{0.6, 0.4}
```

Solve for Link lengths and Phases

```
LpSolution = Table[NSolve[{R * Cos[α] == Re[CoeSolution[[i]]],
    R * Sin[α] == Im[CoeSolution[[i]]]}, {R, α}], {i, num}]
```

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NSolve::ifun : Inverse functions are being used by NSolve,
so some solutions may not be found; use Reduce for complete solution information. >>

```
{{{R → -0.6, α → -3.14159}, {R → -0.6, α → 3.14159}, {R → 0.6, α → 0.}},
 {{R → -0.4, α → -3.14159}, {R → -0.4, α → 3.14159}, {R → 0.4, α → 0.}}}
```

Pick up solutions that R>0

```
ModLpSolution = Table[LpSolution[[i, 3]], {i, num}]
{{R → 0.6, α → 0.}, {R → 0.4, α → 0.}}
```

Get Delta functions

```
DeltaSolution = Table[ModSolution[[i, 2]], {i, num}]
{DiracDelta[0.4 + w], DiracDelta[0.6 + w]}
```

Solve for Frequencies

```
FreqSolution = Table[Solve[DeltaSolution[[i]] == DiracDelta[0], w], {i, num}]
```

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

$$\left\{\left\{\left\{w \to -\frac{2}{5}\right\}\right\}, \left\{\left\{w \to -\frac{3}{5}\right\}\right\}\right\}$$

```
N[FreqSolution]
```

{{{w → -0.4}}, {{w → -0.6}}}

Get the solution set for each link

```
Table[Flatten[Append[ModLpSolution[[i]], FreqSolution[[i]]]], {i, num}]
```

$$\left\{\left\{R \to 0.6, \alpha \to 0., w \to -\frac{2}{5}\right\}, \left\{R \to 0.4, \alpha \to 0., w \to -\frac{3}{5}\right\}\right\}$$

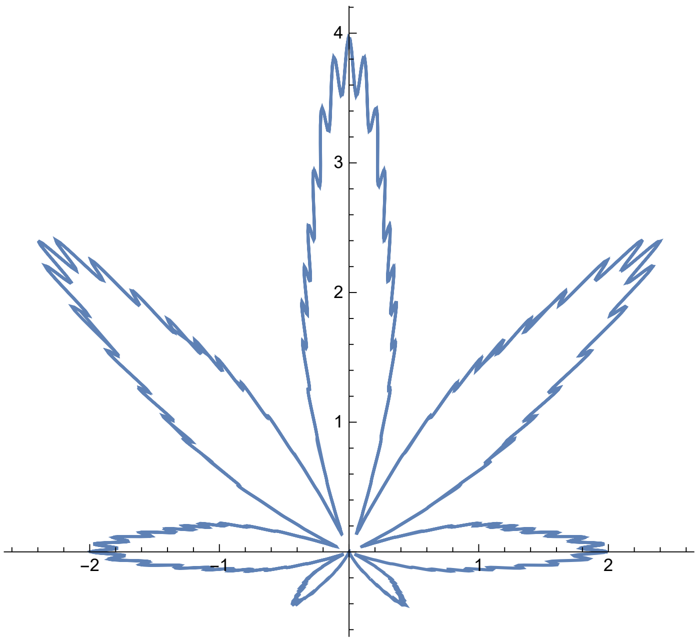
```
Table[Flatten[Append[ModLpSolution[[i]], N[FreqSolution[[i]]]]], {i, num}]
```

{{R → 0.6, α → 0., w → -0.4}, {R → 0.4, α → 0., w → -0.6}}

Leaf

```
x = (1 + 0.9 Cos[8 t]) (1 + 0.1 Cos[24 t]) (0.9 + 0.05 Cos[200 t]) (1 + Sin[t]) Cos[t];
y = (1 + 0.9 Cos[8 t]) (1 + 0.1 Cos[24 t]) (0.9 + 0.05 Cos[200 t]) (1 + Sin[t]) Sin[t];
```

```
ParametricPlot[{x, y}, {t, 0, 10}]
```

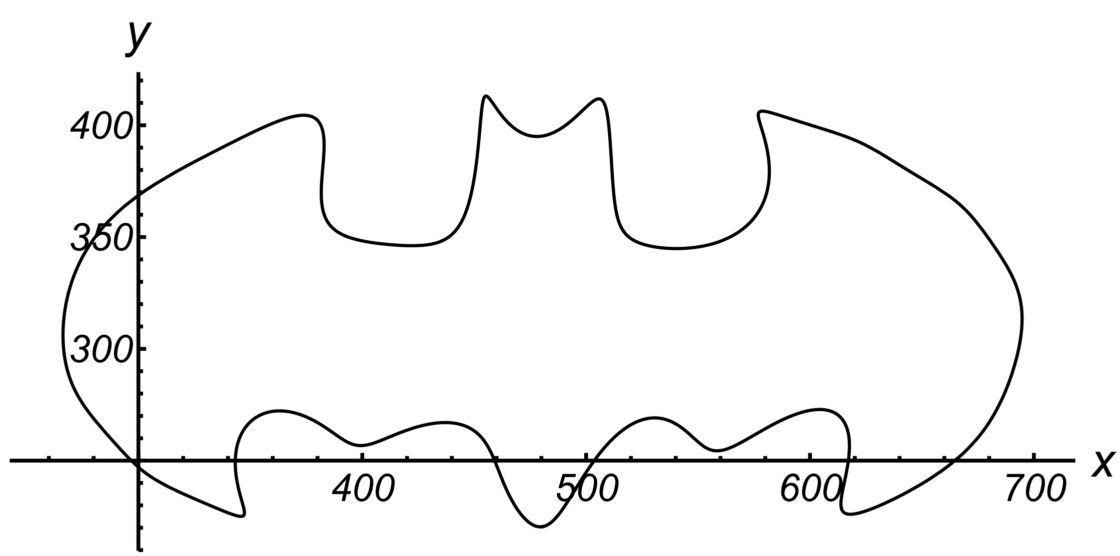


B Batman Logo Synthesis Mathematica Code

Below is the Mathematica code for the computation of linkage system to draw Batman logo.

Batman

```
param[x_, m_, t_] :=
 Module[{f, n = Length[x], nf}, f = Chop[Fourier[x]][[ ;; Ceiling[Length[x] / 2]]];
  nf = Length[f];
  Total[Rationalize[
     2 Abs[f] / Sqrt[n] Sin[Pi / 2 - Arg[f] + Range[0, nf - 1] t], .01][[ ;; Min[m, nf]]]]]

tocurve[Line[data_], m_, t_] := param[#, m, t] & /@ Transpose[data]

img = Import["C:\\Users\\Yang\\Dropbox\\1_Fourier Transform\\Batman.GIF"];

img = Binarize[img ~ ColorConvert ~ "Grayscale" ~ ImageResize ~ 500 ~ Blur ~ 3] ~ Blur ~ 3;

lines = Cases[
   Normal@ListContourPlot[Reverse@ImageData[img], Contours → {0.5}], _Line, -1];

ParametricPlot[Evaluate[tocurve[#, 20, t] & /@ lines],
 {t, 0, 2 Pi}, PlotStyle → Directive[Black, Thickness[.003]],
 AxesLabel → {Style["x", 25, Black, Italic], Style["y", 25, Black, Italic]},
 AxesStyle → Directive[Black, 20, Thick],
 LabelStyle → Directive[10, Black, Italic], ImageSize → Large]
```



f = Evaluate[tocurve[#, 20, t] & /@ lines]

$$\left\{\left\{\frac{4322}{9}-\frac{14}{5}\,\text{Sin}\left[\frac{17}{16}-15\,t\right]-\frac{37}{19}\,\text{Sin}\left[\frac{28}{19}-14\,t\right]-\frac{11}{6}\,\text{Sin}\left[\frac{9}{8}-12\,t\right]-\right.\right.$$
$$\frac{37}{13}\,\text{Sin}\left[\frac{3}{8}-10\,t\right]-\frac{8}{3}\,\text{Sin}\left[\frac{5}{4}-8\,t\right]-\frac{61}{7}\,\text{Sin}\left[\frac{7}{8}-7\,t\right]-\frac{60}{17}\,\text{Sin}\left[\frac{11}{12}-4\,t\right]-$$
$$\frac{233}{11}\,\text{Sin}\left[\frac{3}{7}-3\,t\right]+\frac{2286}{13}\,\text{Sin}\left[\frac{40}{13}+t\right]+\frac{147}{8}\,\text{Sin}\left[\frac{109}{36}+2\,t\right]+\frac{103}{13}\,\text{Sin}\left[\frac{19}{8}+5\,t\right]+$$
$$\frac{7}{4}\,\text{Sin}\left[\frac{61}{60}+6\,t\right]+\frac{7}{6}\,\text{Sin}\left[\frac{31}{30}+9\,t\right]+\frac{19}{3}\,\text{Sin}\left[\frac{33}{13}+11\,t\right]+\frac{2}{7}\,\text{Sin}\left[\frac{47}{12}+13\,t\right]+$$
$$\frac{2}{5}\,\text{Sin}\left[\frac{61}{30}+16\,t\right]+\frac{7}{13}\,\text{Sin}\left[\frac{24}{11}+17\,t\right]+\frac{13}{12}\,\text{Sin}\left[\frac{9}{7}+18\,t\right]+\frac{21}{13}\,\text{Sin}\left[\frac{7}{4}+19\,t\right],$$
$$\frac{2223}{7}-\frac{17}{14}\,\text{Sin}\left[\frac{19}{18}-18\,t\right]-\frac{6}{11}\,\text{Sin}\left[\frac{6}{7}-15\,t\right]-\frac{47}{24}\,\text{Sin}\left[\frac{1}{3}-8\,t\right]+$$
$$\frac{1885}{23}\,\text{Sin}\left[\frac{68}{15}+t\right]+\frac{15}{7}\,\text{Sin}\left[\frac{31}{7}+2\,t\right]+\frac{149}{6}\,\text{Sin}\left[\frac{18}{17}+3\,t\right]+\frac{82}{13}\,\text{Sin}\left[\frac{46}{45}+4\,t\right]+$$
$$\frac{123}{5}\,\text{Sin}\left[\frac{47}{11}+5\,t\right]+\frac{39}{10}\,\text{Sin}[2+6\,t]+\frac{83}{7}\,\text{Sin}\left[\frac{37}{8}+7\,t\right]+\frac{55}{12}\,\text{Sin}\left[\frac{8}{7}+9\,t\right]+$$
$$\frac{40}{11}\,\text{Sin}\left[\frac{53}{12}+10\,t\right]+\frac{14}{9}\,\text{Sin}\left[\frac{9}{13}+11\,t\right]+\frac{108}{17}\,\text{Sin}\left[\frac{29}{7}+12\,t\right]+3\,\text{Sin}\left[\frac{38}{9}+13\,t\right]+$$
$$\left.\left.\frac{35}{9}\,\text{Sin}\left[\frac{105}{26}+14\,t\right]+2\,\text{Sin}\left[\frac{9}{2}+16\,t\right]+\frac{23}{8}\,\text{Sin}\left[\frac{2}{3}+17\,t\right]+\frac{22}{15}\,\text{Sin}\left[\frac{7}{13}+19\,t\right]\right\}\right\}$$

f1 = f[[1, 1]]

$$\frac{4322}{9}-\frac{14}{5}\,\text{Sin}\left[\frac{17}{16}-15\,t\right]-\frac{37}{19}\,\text{Sin}\left[\frac{28}{19}-14\,t\right]-\frac{11}{6}\,\text{Sin}\left[\frac{9}{8}-12\,t\right]-$$
$$\frac{37}{13}\,\text{Sin}\left[\frac{3}{8}-10\,t\right]-\frac{8}{3}\,\text{Sin}\left[\frac{5}{4}-8\,t\right]-\frac{61}{7}\,\text{Sin}\left[\frac{7}{8}-7\,t\right]-\frac{60}{17}\,\text{Sin}\left[\frac{11}{12}-4\,t\right]-$$
$$\frac{233}{11}\,\text{Sin}\left[\frac{3}{7}-3\,t\right]+\frac{2286}{13}\,\text{Sin}\left[\frac{40}{13}+t\right]+\frac{147}{8}\,\text{Sin}\left[\frac{109}{36}+2\,t\right]+\frac{103}{13}\,\text{Sin}\left[\frac{19}{8}+5\,t\right]+$$
$$\frac{7}{4}\,\text{Sin}\left[\frac{61}{60}+6\,t\right]+\frac{7}{6}\,\text{Sin}\left[\frac{31}{30}+9\,t\right]+\frac{19}{3}\,\text{Sin}\left[\frac{33}{13}+11\,t\right]+\frac{2}{7}\,\text{Sin}\left[\frac{47}{12}+13\,t\right]+$$
$$\frac{2}{5}\,\text{Sin}\left[\frac{61}{30}+16\,t\right]+\frac{7}{13}\,\text{Sin}\left[\frac{24}{11}+17\,t\right]+\frac{13}{12}\,\text{Sin}\left[\frac{9}{7}+18\,t\right]+\frac{21}{13}\,\text{Sin}\left[\frac{7}{4}+19\,t\right]$$

Length[f1]

20

Compute the x coefficient ak and bk

```
nf1 = N[f1]
```

```
480.222 - 2.8 Sin[1.0625 - 15. t] - 1.94737 Sin[1.47368 - 14. t] -
 1.83333 Sin[1.125 - 12. t] - 2.84615 Sin[0.375 - 10. t] - 2.66667 Sin[1.25 - 8. t] -
 8.71429 Sin[0.875 - 7. t] - 3.52941 Sin[0.916667 - 4. t] -
 21.1818 Sin[0.428571 - 3. t] + 175.846 Sin[3.07692 + t] + 18.375 Sin[3.02778 + 2. t] +
 7.92308 Sin[2.375 + 5. t] + 1.75 Sin[1.01667 + 6. t] + 1.16667 Sin[1.03333 + 9. t] +
 6.33333 Sin[2.53846 + 11. t] + 0.285714 Sin[3.91667 + 13. t] + 0.4 Sin[2.03333 + 16. t] +
 0.538462 Sin[2.18182 + 17. t] + 1.08333 Sin[1.28571 + 18. t] + 1.61538 Sin[1.75 + 19. t]
```

```
expandf1 = Simplify[TrigExpand[nf1]]
```

```
480.222 + 11.364 Cos[t] + 2.08684 Cos[2. t] - 8.80257 Cos[3. t] -
 2.80086 Cos[4. t] + 5.49612 Cos[5. t] + 1.48813 Cos[6. t] - 6.68859 Cos[7. t] -
 2.53063 Cos[8. t] + 1.00218 Cos[9. t] - 1.04247 Cos[10. t] + 3.59242 Cos[11. t] -
 1.65416 Cos[12. t] - 0.199934 Cos[13. t] - 1.93819 Cos[14. t] - 2.44601 Cos[15. t] +
 0.357969 Cos[16. t] + 0.441033 Cos[17. t] + 1.03961 Cos[18. t] +
 1.58952 Cos[19. t] - 175.479 Sin[t] - 18.2561 Sin[2. t] + 19.2661 Sin[3. t] +
 2.14754 Sin[4. t] - 5.70682 Sin[5. t] + 0.920856 Sin[6. t] + 5.58583 Sin[7. t] +
 0.84086 Sin[8. t] + 0.597285 Sin[9. t] + 2.64837 Sin[10. t] - 5.2159 Sin[11. t] +
 0.79049 Sin[12. t] - 0.204105 Sin[13. t] + 0.188816 Sin[14. t] + 1.36273 Sin[15. t] -
 0.178488 Sin[16. t] - 0.308918 Sin[17. t] + 0.304673 Sin[18. t] - 0.287936 Sin[19. t]
```

```
Length[expandf1]
```

```
39
```

```
a0 =  expandf1[[1]]; a1 = Coefficient[expandf1, Cos[t]];
a2 = Coefficient[expandf1, Cos[2. t]]; a3 = Coefficient[expandf1, Cos[3. t]];
a4 = Coefficient[expandf1, Cos[4. t]]; a5 = Coefficient[expandf1, Cos[5. t]];
a6 = Coefficient[expandf1, Cos[6. t]]; a7 = Coefficient[expandf1, Cos[7. t]];
a8 = Coefficient[expandf1, Cos[8. t]]; a9 = Coefficient[expandf1, Cos[9. t]];
ax = Coefficient[expandf1, Cos[10. t]]; ax1 = Coefficient[expandf1, Cos[11. t]];
ax2 = Coefficient[expandf1, Cos[12. t]]; ax3 = Coefficient[expandf1, Cos[13. t]];
ax4 = Coefficient[expandf1, Cos[14. t]]; ax5 = Coefficient[expandf1, Cos[15. t]];
ax6 = Coefficient[expandf1, Cos[16. t]]; ax7 = Coefficient[expandf1, Cos[17. t]];
ax8 = Coefficient[expandf1, Cos[18. t]]; ax9 = Coefficient[expandf1, Cos[19. t]];
```

```
ak = {a0, a1, a2, a3, a4, a5, a6, a7,
  a8, a9, ax, ax1, ax2, ax3, ax4, ax5, ax6, ax7, ax8, ax9}
```

```
{480.222, 11.364, 2.08684, -8.80257, -2.80086, 5.49612,
 1.48813, -6.68859, -2.53063, 1.00218, -1.04247, 3.59242, -1.65416,
 -0.199934, -1.93819, -2.44601, 0.357969, 0.441033, 1.03961, 1.58952}
```

```
b0 = 0; b1 = Coefficient[expandf1, Sin[t]];
b2 = Coefficient[expandf1, Sin[2. t]]; b3 = Coefficient[expandf1, Sin[3. t]];
b4 = Coefficient[expandf1, Sin[4. t]]; b5 = Coefficient[expandf1, Sin[5. t]];
b6 = Coefficient[expandf1, Sin[6. t]]; b7 = Coefficient[expandf1, Sin[7. t]];
b8 = Coefficient[expandf1, Sin[8. t]]; b9 = Coefficient[expandf1, Sin[9. t]];
bx = Coefficient[expandf1, Sin[10. t]]; bx1 = Coefficient[expandf1, Sin[11. t]];
bx2 = Coefficient[expandf1, Sin[12. t]]; bx3 = Coefficient[expandf1, Sin[13. t]];
bx4 = Coefficient[expandf1, Sin[14. t]]; bx5 = Coefficient[expandf1, Sin[15. t]];
bx6 = Coefficient[expandf1, Sin[16. t]]; bx7 = Coefficient[expandf1, Sin[17. t]];
bx8 = Coefficient[expandf1, Sin[18. t]]; bx9 = Coefficient[expandf1, Sin[19. t]];
```

```
bk = {b0, b1, b2, b3, b4, b5, b6, b7,
  b8, b9, bx, bx1, bx2, bx3, bx4, bx5, bx6, bx7, bx8, bx9}
```

```
{0, -175.479, -18.2561, 19.2661, 2.14754, -5.70682, 0.920856,
 5.58583, 0.84086, 0.597285, 2.64837, -5.2159, 0.79049, -0.204105,
 0.188816, 1.36273, -0.178488, -0.308918, 0.304673, -0.287936}
```

Compute the y coefficient ck and dk

```
f2 = f[[1, 2]]
```

$$\frac{2223}{7} - \frac{17}{14}\,\mathrm{Sin}\left[\frac{19}{18} - 18\,t\right] - \frac{6}{11}\,\mathrm{Sin}\left[\frac{6}{7} - 15\,t\right] - \frac{47}{24}\,\mathrm{Sin}\left[\frac{1}{3} - 8\,t\right] + \frac{1885}{23}\,\mathrm{Sin}\left[\frac{68}{15} + t\right] + \frac{15}{7}\,\mathrm{Sin}\left[\frac{31}{7} + 2\,t\right] + \frac{149}{6}\,\mathrm{Sin}\left[\frac{18}{17} + 3\,t\right] + \frac{82}{13}\,\mathrm{Sin}\left[\frac{46}{45} + 4\,t\right] + \frac{123}{5}\,\mathrm{Sin}\left[\frac{47}{11} + 5\,t\right] + \frac{39}{10}\,\mathrm{Sin}[2 + 6\,t] + \frac{83}{7}\,\mathrm{Sin}\left[\frac{37}{8} + 7\,t\right] + \frac{55}{12}\,\mathrm{Sin}\left[\frac{8}{7} + 9\,t\right] + \frac{40}{11}\,\mathrm{Sin}\left[\frac{53}{12} + 10\,t\right] + \frac{14}{9}\,\mathrm{Sin}\left[\frac{9}{13} + 11\,t\right] + \frac{108}{17}\,\mathrm{Sin}\left[\frac{29}{7} + 12\,t\right] + 3\,\mathrm{Sin}\left[\frac{38}{9} + 13\,t\right] + \frac{35}{9}\,\mathrm{Sin}\left[\frac{105}{26} + 14\,t\right] + 2\,\mathrm{Sin}\left[\frac{9}{2} + 16\,t\right] + \frac{23}{8}\,\mathrm{Sin}\left[\frac{2}{3} + 17\,t\right] + \frac{22}{15}\,\mathrm{Sin}\left[\frac{7}{13} + 19\,t\right]$$

```
Length[f2]
```

```
20
```

```
nf2 = N[f2]
```

```
317.571 - 1.21429 Sin[1.05556 - 18. t] - 0.545455 Sin[0.857143 - 15. t] -
 1.95833 Sin[0.333333 - 8. t] + 81.9565 Sin[4.53333 + t] + 2.14286 Sin[4.42857 + 2. t] +
 24.8333 Sin[1.05882 + 3. t] + 6.30769 Sin[1.02222 + 4. t] + 24.6 Sin[4.27273 + 5. t] +
 3.9 Sin[2. + 6. t] + 11.8571 Sin[4.625 + 7. t] + 4.58333 Sin[1.14286 + 9. t] +
 3.63636 Sin[4.41667 + 10. t] + 1.55556 Sin[0.692308 + 11. t] +
 6.35294 Sin[4.14286 + 12. t] + 3. Sin[4.22222 + 13. t] + 3.88889 Sin[4.03846 + 14. t] +
 2. Sin[4.5 + 16. t] + 2.875 Sin[0.666667 + 17. t] + 1.46667 Sin[0.538462 + 19. t]
```

```
expandf2 = Simplify[TrigExpand[nf2]]

317.571 - 80.6462 Cos[t] - 2.05713 Cos[2. t] + 21.6492 Cos[3. t] +
 5.38216 Cos[4. t] - 22.2604 Cos[5. t] + 3.54626 Cos[6. t] - 11.8119 Cos[7. t] -
 0.640756 Cos[8. t] + 4.17002 Cos[9. t] - 3.47852 Cos[10. t] + 0.992935 Cos[11. t] -
 5.35015 Cos[12. t] - 2.64676 Cos[13. t] - 3.03869 Cos[14. t] -
 0.41235 Cos[15. t] - 1.95506 Cos[16. t] + 1.77781 Cos[17. t] - 1.05664 Cos[18. t] +
 0.75213 Cos[19. t] - 14.5965 Sin[t] - 0.600048 Sin[2. t] + 12.1658 Sin[3. t] +
 3.28928 Sin[4. t] - 10.4706 Sin[5. t] - 1.62297 Sin[6. t] - 1.03487 Sin[7. t] +
 1.85054 Sin[8. t] + 1.90207 Sin[9. t] - 1.05975 Sin[10. t] + 1.19743 Sin[11. t] -
 3.42575 Sin[12. t] - 1.41232 Sin[13. t] - 2.4269 Sin[14. t] + 0.357055 Sin[15. t] -
 0.421592 Sin[16. t] + 2.25943 Sin[17. t] + 0.598332 Sin[18. t] + 1.25913 Sin[19. t]

Length[expandf2]

39

c0 = expandf2[[1]]; c1 = Coefficient[expandf2, Cos[t]];
c2 = Coefficient[expandf2, Cos[2. t]]; c3 = Coefficient[expandf2, Cos[3. t]];
c4 = Coefficient[expandf2, Cos[4. t]]; c5 = Coefficient[expandf2, Cos[5. t]];
c6 = Coefficient[expandf2, Cos[6. t]]; c7 = Coefficient[expandf2, Cos[7. t]];
c8 = Coefficient[expandf2, Cos[8. t]]; c9 = Coefficient[expandf2, Cos[9. t]];
cx = Coefficient[expandf2, Cos[10. t]]; cx1 = Coefficient[expandf2, Cos[11. t]];
cx2 = Coefficient[expandf2, Cos[12. t]]; cx3 = Coefficient[expandf2, Cos[13. t]];
cx4 = Coefficient[expandf2, Cos[14. t]]; cx5 = Coefficient[expandf2, Cos[15. t]];
cx6 = Coefficient[expandf2, Cos[16. t]]; cx7 = Coefficient[expandf2, Cos[17. t]];
cx8 = Coefficient[expandf2, Cos[18. t]]; cx9 = Coefficient[expandf2, Cos[19. t]];

ck = {c0, c1, c2, c3, c4, c5, c6, c7,
  c8, c9, cx, cx1, cx2, cx3, cx4, cx5, cx6, cx7, cx8, cx9}

{317.571, -80.6462, -2.05713, 21.6492, 5.38216, -22.2604,
 3.54626, -11.8119, -0.640756, 4.17002, -3.47852, 0.992935, -5.35015,
 -2.64676, -3.03869, -0.41235, -1.95506, 1.77781, -1.05664, 0.75213}

d0 = 0; d1 = Coefficient[expandf2, Sin[t]];
d2 = Coefficient[expandf2, Sin[2. t]]; d3 = Coefficient[expandf2, Sin[3. t]];
d4 = Coefficient[expandf2, Sin[4. t]]; d5 = Coefficient[expandf2, Sin[5. t]];
d6 = Coefficient[expandf2, Sin[6. t]]; d7 = Coefficient[expandf2, Sin[7. t]];
d8 = Coefficient[expandf2, Sin[8. t]]; d9 = Coefficient[expandf2, Sin[9. t]];
dx = Coefficient[expandf2, Sin[10. t]]; dx1 = Coefficient[expandf2, Sin[11. t]];
dx2 = Coefficient[expandf2, Sin[12. t]]; dx3 = Coefficient[expandf2, Sin[13. t]];
dx4 = Coefficient[expandf2, Sin[14. t]]; dx5 = Coefficient[expandf2, Sin[15. t]];
dx6 = Coefficient[expandf2, Sin[16. t]]; dx7 = Coefficient[expandf2, Sin[17. t]];
dx8 = Coefficient[expandf2, Sin[18. t]]; dx9 = Coefficient[expandf2, Sin[19. t]];
```

```
dk = {d0, d1, d2, d3, d4, d5, d6, d7,
  d8, d9, dx, dx1, dx2, dx3, dx4, dx5, dx6, dx7, dx8, dx9}
{0, -14.5965, -0.600048, 12.1658, 3.28928, -10.4706,
 -1.62297, -1.03487, 1.85054, 1.90207, -1.05975, 1.19743, -3.42575,
 -1.41232, -2.4269, 0.357055, -0.421592, 2.25943, 0.598332, 1.25913}
```

Compute the configuration for the Scotch yoke Mechanisms

```
Lk = Sqrt[ak^2 + bk^2]
{480.222, 175.846, 18.375, 21.1818, 3.52941, 7.92308, 1.75, 8.71429, 2.66667, 1.16667,
 2.84615, 6.33333, 1.83333, 0.285714, 1.94737, 2.8, 0.4, 0.538462, 1.08333, 1.61538}
```

```
psi = ArcTan[ak, bk]
{0., -1.50613, -1.45698, 1.99937, 2.48746, -0.804204, 0.55413,
 2.4458, 2.8208, 0.537463, 1.9458, -0.967665, 2.6958, -2.34587,
 3.04448, 2.6333, -0.462537, -0.611022, 0.285082, -0.179204}
```

```
Mk = Sqrt[ck^2 + dk^2]
{317.571, 81.9565, 2.14286, 24.8333, 6.30769, 24.6, 3.9, 11.8571, 1.95833, 4.58333,
 3.63636, 1.55556, 6.35294, 3., 3.88889, 0.545455, 2., 2.875, 1.21429, 1.46667}
```

```
eta = ArcTan[ck, dk]
{0., -2.96254, -2.85778, 0.511973, 0.548574, -2.70193,
 -0.429204, -3.0542, 1.90413, 0.427939, -2.84587, 0.878489, -2.57206,
 -2.65143, -2.46767, 2.42794, -2.9292, 0.90413, 2.62635, 1.03233}
```

Compute the configuration for the single coupled serial chain

```
Lk = 0.5 * Sqrt[(ak + dk) ^2 + (ck - bk) ^2]
{287.865, 47.4437, 8.13354, 2.06096, 1.63564, 8.64244,
 1.31443, 9.51752, 0.815123, 2.30212, 3.23875, 3.92085, 3.98475,
 1.46338, 2.71435, 1.37064, 0.888856, 1.70638, 1.0649, 1.51629}
```

```
psik = ArcTan[ak + dk, ck - bk]
```

```
{0.584279, 1.60487, 1.47927, 0.616448, 1.42093, -1.86272,
 1.62211, -1.9886, -2.00112, 0.888244, -1.90132, 0.913704, -2.26194,
 -2.1542, -2.50492, -2.43725, -1.60659, 0.657892, -0.693426, 0.350071}
```

```
Mk = 0.5 * Sqrt[(ak - dk) ^2 + (ck + bk) ^2]
```

```
{287.865, 128.719, 10.2451, 22.9877, 4.84216, 16.102,
 2.72186, 4.20501, 2.19287, 2.42575, 0.415164, 2.42742, 2.44587,
 1.54898, 1.44574, 1.4799, 1.13575, 1.16878, 0.435941, 0.284882}
```

```
etak = ArcTan[ak - dk, ck + bk]
```

```
{0.584279, -1.46978, -1.43929, 2.04438, 2.25089, -1.05204,
 0.96246, -2.30805, 3.09595, 1.75736, -1.54998, -1.05489, -1.20021,
 -1.1687, -1.40096, 2.81471, -1.22048, 2.46211, -1.04012, 0.952236}
```

Fourier Transform

```
FTf1 = N[Chop[Simplify[FourierTransform[Expand[N[f1], Trig -> True], t, w]]], 10]
(2.61546 - 22.8806 i) DiracDelta[2. - 1.000000000 w] -
 (11.0324 - 24.1465 i) DiracDelta[3. - 1.000000000 w] -
 (3.51036 - 2.69154 i) DiracDelta[4. - 1.000000000 w] +
 (6.88836 - 7.15244 i) DiracDelta[5. - 1.000000000 w] +
 (1.86509 + 1.15412 i) DiracDelta[6. - 1.000000000 w] -
 (8.38291 - 7.0008 i) DiracDelta[7. - 1.000000000 w] -
 (3.17167 - 1.05386 i) DiracDelta[8. - 1.000000000 w] +
 (1.25604 + 0.748585 i) DiracDelta[9. - 1.000000000 w] -
 (1.30654 - 3.31924 i) DiracDelta[10. - 1.000000000 w] +
 (4.50243 - 6.53716 i) DiracDelta[11. - 1.000000000 w] -
 (2.07318 - 0.990733 i) DiracDelta[12. - 1.000000000 w] -
 (0.25058 + 0.255808 i) DiracDelta[13. - 1.000000000 w] -
 (2.42916 - 0.236646 i) DiracDelta[14. - 1.000000000 w] -
 (3.06562 - 1.70793 i) DiracDelta[15. - 1.000000000 w] +
 (0.448648 - 0.223702 i) DiracDelta[16. - 1.000000000 w] +
 (0.552754 - 0.387171 i) DiracDelta[17. - 1.000000000 w] +
 (1.30296 + 0.38185 i) DiracDelta[18. - 1.000000000 w] +
 (1.99216 - 0.360874 i) DiracDelta[19. - 1.000000000 w] +
 (14.2426 - 219.93 i) DiracDelta[-1.000000000 + w] +
 1203.74 DiracDelta[w] + (14.2426 + 219.93 i) DiracDelta[1.000000000 + w] +
 (2.61546 + 22.8806 i) DiracDelta[2. + w] - (11.0324 + 24.1465 i) DiracDelta[3. + w] -
 (3.51036 + 2.69154 i) DiracDelta[4. + w] + (6.88836 + 7.15244 i) DiracDelta[5. + w] +
 (1.86509 - 1.15412 i) DiracDelta[6. + w] - (8.38291 + 7.0008 i) DiracDelta[7. + w] -
 (3.17167 + 1.05386 i) DiracDelta[8. + w] + (1.25604 - 0.748585 i) DiracDelta[9. + w] -
 (1.30654 + 3.31924 i) DiracDelta[10. + w] + (4.50243 + 6.53716 i) DiracDelta[11. + w] -
 (2.07318 + 0.990733 i) DiracDelta[12. + w] - (0.25058 - 0.255808 i) DiracDelta[13. + w] -
 (2.42916 + 0.236646 i) DiracDelta[14. + w] - (3.06562 + 1.70793 i) DiracDelta[15. + w] +
 (0.448648 + 0.223702 i) DiracDelta[16. + w] +
 (0.552754 + 0.387171 i) DiracDelta[17. + w] +
 (1.30296 - 0.38185 i) DiracDelta[18. + w] + (1.99216 + 0.360874 i) DiracDelta[19. + w]
```

```
FTf2 = N[Chop[Expand[N[I * FourierTransform[Expand[N[f2], Trig -> True], t, w]]]], 10]
(0.752049 - 2.57823 i) DiracDelta[2. - 1. w] -
 (15.2476 - 27.1332 i) DiracDelta[3. - 1. w] - (4.1225 - 6.74553 i) DiracDelta[4. - 1. w] +
 (13.1229 - 27.8993 i) DiracDelta[5. - 1. w] + (2.03409 + 4.44458 i) DiracDelta[6. - 1. w] +
 (1.29701 - 14.804 i) DiracDelta[7. - 1. w] - (2.31931 + 0.803069 i) DiracDelta[8. - 1. w] -
 (2.38389 - 5.22635 i) DiracDelta[9. - 1. w] + (1.3282 - 4.35967 i) DiracDelta[10. - 1. w] -
 (1.50075 - 1.24446 i) DiracDelta[11. - 1. w] +
 (4.29354 - 6.70542 i) DiracDelta[12. - 1. w] +
 (1.77008 - 3.31723 i) DiracDelta[13. - 1. w] +
 (3.04167 - 3.80843 i) DiracDelta[14. - 1. w] -
 (0.447502 + 0.516804 i) DiracDelta[15. - 1. w] +
 (0.528387 - 2.4503 i) DiracDelta[16. - 1. w] -
 (2.83177 - 2.22816 i) DiracDelta[17. - 1. w] -
 (0.749899 + 1.3243 i) DiracDelta[18. - 1. w] -
 (1.57809 - 0.942655 i) DiracDelta[19. - 1. w] + (18.294 - 101.075 i) DiracDelta[-1. + w] +
 (0. + 796.034 i) DiracDelta[w] - (18.294 + 101.075 i) DiracDelta[1. + w] -
 (0.752049 + 2.57823 i) DiracDelta[2. + w] + (15.2476 + 27.1332 i) DiracDelta[3. + w] +
 (4.1225 + 6.74553 i) DiracDelta[4. + w] - (13.1229 + 27.8993 i) DiracDelta[5. + w] -
 (2.03409 - 4.44458 i) DiracDelta[6. + w] - (1.29701 + 14.804 i) DiracDelta[7. + w] +
 (2.31931 - 0.803069 i) DiracDelta[8. + w] + (2.38389 + 5.22635 i) DiracDelta[9. + w] -
 (1.3282 + 4.35967 i) DiracDelta[10. + w] + (1.50075 + 1.24446 i) DiracDelta[11. + w] -
 (4.29354 + 6.70542 i) DiracDelta[12. + w] - (1.77008 + 3.31723 i) DiracDelta[13. + w] -
 (3.04167 + 3.80843 i) DiracDelta[14. + w] + (0.447502 - 0.516804 i) DiracDelta[15. + w] -
 (0.528387 + 2.4503 i) DiracDelta[16. + w] + (2.83177 + 2.22816 i) DiracDelta[17. + w] +
 (0.749899 - 1.3243 i) DiracDelta[18. + w] + (1.57809 + 0.942655 i) DiracDelta[19. + w]
```

```
Solution = Expand[(FTf1 + FTf2) * (1 / Sqrt[2 Pi])]
(1.34344 - 10.1566 i) DiracDelta[2. - 1. w] - (10.4842 - 20.4577 i) DiracDelta[3. - 1. w] -
 (3.04507 - 3.76485 i) DiracDelta[4. - 1. w] + (7.98335 - 13.9836 i) DiracDelta[5. - 1. w] +
 (1.55555 + 2.23356 i) DiracDelta[6. - 1. w] - (2.82686 + 3.11303 i) DiracDelta[7. - 1. w] -
 (2.19058 - 0.100052 i) DiracDelta[8. - 1. w] -
 (0.449945 - 2.38365 i) DiracDelta[9. - 1. w] +
 (0.00864035 - 0.415074 i) DiracDelta[10. - 1. w] +
 (1.1975 - 2.11148 i) DiracDelta[11. - 1. w] +
 (0.885795 - 2.27983 i) DiracDelta[12. - 1. w] +
 (0.606193 - 1.42543 i) DiracDelta[13. - 1. w] +
 (0.244353 - 1.42494 i) DiracDelta[14. - 1. w] -
 (1.40153 - 0.47519 i) DiracDelta[15. - 1. w] +
 (0.38978 - 1.06677 i) DiracDelta[16. - 1. w] -
 (0.909196 - 0.734448 i) DiracDelta[17. - 1. w] +
 (0.220638 - 0.375984 i) DiracDelta[18. - 1. w] +
 (0.165192 + 0.232097 i) DiracDelta[19. - 1. w] +
 (12.9802 - 128.062 i) DiracDelta[-1. + w] + (480.222 + 317.571 i) DiracDelta[w] -
 (1.61626 - 47.4162 i) DiracDelta[1. + w] + (0.743394 + 8.09949 i) DiracDelta[2. + w] +
 (1.68162 + 1.19153 i) DiracDelta[3. + w] + (0.244208 + 1.61731 i) DiracDelta[4. + w] -
 (2.48723 + 8.27681 i) DiracDelta[5. + w] - (0.0674224 - 1.3127 i) DiracDelta[6. + w] -
 (3.86173 + 8.69886 i) DiracDelta[7. + w] - (0.340042 + 0.740808 i) DiracDelta[8. + w] +
 (1.45212 + 1.78637 i) DiracDelta[9. + w] - (1.05111 + 3.06344 i) DiracDelta[10. + w] +
 (2.39492 + 3.10442 i) DiracDelta[11. + w] - (2.53995 + 3.07032 i) DiracDelta[12. + w] -
 (0.806127 + 1.22133 i) DiracDelta[13. + w] -
 (2.18255 + 1.61375 i) DiracDelta[14. + w] - (1.04448 + 0.887541 i) DiracDelta[15. + w] -
 (0.0318111 + 0.888286 i) DiracDelta[16. + w] + (1.35023 + 1.04337 i) DiracDelta[17. + w] +
 (0.81897 - 0.680656 i) DiracDelta[18. + w] + (1.42432 + 0.520033 i) DiracDelta[19. + w]
```

Remove the solution ω=0

```
ModSolution =
 Cases[Solution, Except[(480.2222222222222` + 317.5714285714285` i) DiracDelta[w]]]
```

```
{(1.34344 - 10.1566 i) DiracDelta[2. - 1. w],
 (-10.4842 + 20.4577 i) DiracDelta[3. - 1. w],
 (-3.04507 + 3.76485 i) DiracDelta[4. - 1. w],
 (7.98335 - 13.9836 i) DiracDelta[5. - 1. w], (1.55555 + 2.23356 i) DiracDelta[6. - 1. w],
 (-2.82686 - 3.11303 i) DiracDelta[7. - 1. w],
 (-2.19058 + 0.100052 i) DiracDelta[8. - 1. w],
 (-0.449945 + 2.38365 i) DiracDelta[9. - 1. w],
 (0.00864035 - 0.415074 i) DiracDelta[10. - 1. w],
 (1.1975 - 2.11148 i) DiracDelta[11. - 1. w],
 (0.885795 - 2.27983 i) DiracDelta[12. - 1. w],
 (0.606193 - 1.42543 i) DiracDelta[13. - 1. w],
 (0.244353 - 1.42494 i) DiracDelta[14. - 1. w],
 (-1.40153 + 0.47519 i) DiracDelta[15. - 1. w],
 (0.38978 - 1.06677 i) DiracDelta[16. - 1. w],
 (-0.909196 + 0.734448 i) DiracDelta[17. - 1. w],
 (0.220638 - 0.375984 i) DiracDelta[18. - 1. w],
 (0.165192 + 0.232097 i) DiracDelta[19. - 1. w],
 (12.9802 - 128.062 i) DiracDelta[-1. + w], (-1.61626 + 47.4162 i) DiracDelta[1. + w],
 (0.743394 + 8.09949 i) DiracDelta[2. + w], (1.68162 + 1.19153 i) DiracDelta[3. + w],
 (0.244208 + 1.61731 i) DiracDelta[4. + w], (-2.48723 - 8.27681 i) DiracDelta[5. + w],
 (-0.0674224 + 1.3127 i) DiracDelta[6. + w], (-3.86173 - 8.69886 i) DiracDelta[7. + w],
 (-0.340042 - 0.740808 i) DiracDelta[8. + w], (1.45212 + 1.78637 i) DiracDelta[9. + w],
 (-1.05111 - 3.06344 i) DiracDelta[10. + w], (2.39492 + 3.10442 i) DiracDelta[11. + w],
 (-2.53995 - 3.07032 i) DiracDelta[12. + w], (-0.806127 - 1.22133 i) DiracDelta[13. + w],
 (-2.18255 - 1.61375 i) DiracDelta[14. + w], (-1.04448 - 0.887541 i) DiracDelta[15. + w],
 (-0.0318111 - 0.888286 i) DiracDelta[16. + w],
 (1.35023 + 1.04337 i) DiracDelta[17. + w], (0.81897 - 0.680656 i) DiracDelta[18. + w],
 (1.42432 + 0.520033 i) DiracDelta[19. + w]}
```

```
num = Length[ModSolution]
```

```
38
```

Test

```
S = ModSolution[[1, 1]]
```

```
1.34344 - 10.1566 i
```

```
Re[S]
```

```
1.34344
```

```
Im[S]
```

```
-10.1566
```

```
NSolve[
 {R1 * Cos[α] == 12.980230312510045`, R1 * Sin[α] == -128.0624006550793`}, {R1, α}]
```

NSolve::ifun : Inverse functions are being used by NSolve,
so some solutions may not be found; use Reduce for complete solution information. >>

```
{{R1 → -128.719, α → 1.67181}, {R1 → 128.719, α → -1.46978}}
```

```
128.71854893723696` * Cos[-1.4697826714263014`]
```

```
12.9802
```

```
128.71854893723696` * Sin[-1.4697826714263014`]
```

```
-128.062
```

Get the coefficient of Delta function

```
CoeSolution = Table[ModSolution[[i, 1]], {i, num}]
```

```
{1.34344 - 10.1566 i, -10.4842 + 20.4577 i, -3.04507 + 3.76485 i, 7.98335 - 13.9836 i,
 1.55555 + 2.23356 i, -2.82686 - 3.11303 i, -2.19058 + 0.100052 i, -0.449945 + 2.38365 i,
 0.00864035 - 0.415074 i, 1.1975 - 2.11148 i, 0.885795 - 2.27983 i, 0.606193 - 1.42543 i,
 0.244353 - 1.42494 i, -1.40153 + 0.47519 i, 0.38978 - 1.06677 i, -0.909196 + 0.734448 i,
 0.220638 - 0.375984 i, 0.165192 + 0.232097 i, 12.9802 - 128.062 i, -1.61626 + 47.4162 i,
 0.743394 + 8.09949 i, 1.68162 + 1.19153 i, 0.244208 + 1.61731 i, -2.48723 - 8.27681 i,
 -0.0674224 + 1.3127 i, -3.86173 - 8.69886 i, -0.340042 - 0.740808 i,
 1.45212 + 1.78637 i, -1.05111 - 3.06344 i, 2.39492 + 3.10442 i, -2.53995 - 3.07032 i,
 -0.806127 - 1.22133 i, -2.18255 - 1.61375 i, -1.04448 - 0.887541 i,
 -0.0318111 - 0.888286 i, 1.35023 + 1.04337 i, 0.81897 - 0.680656 i, 1.42432 + 0.520033 i}
```

Solve for Link lengths and Phases

```
LpSolution = Table[NSolve[{R * Cos[α] == Re[CoeSolution[[i]]],
    R * Sin[α] == Im[CoeSolution[[i]]]}, {R, α}], {i, num}]
```

NSolve::ifun : Inverse functions are being used by NSolve,
so some solutions may not be found; use Reduce for complete solution information. >>

NSolve::ifun : Inverse functions are being used by NSolve,
so some solutions may not be found; use Reduce for complete solution information. >>

NSolve::ifun : Inverse functions are being used by NSolve,
so some solutions may not be found; use Reduce for complete solution information. >>

General::stop : Further output of NSolve::ifun will be suppressed during this calculation. ≫

```
{{{R → -10.2451, α → 1.70231}, {R → 10.2451, α → -1.43929}},
 {{R → -22.9877, α → -1.09721}, {R → 22.9877, α → 2.04438}},
 {{R → -4.84216, α → -0.890703}, {R → 4.84216, α → 2.25089}},
 {{R → -16.102, α → 2.08955}, {R → 16.102, α → -1.05204}},
 {{R → -2.72186, α → -2.17913}, {R → 2.72186, α → 0.96246}},
 {{R → -4.20501, α → 0.833538}, {R → 4.20501, α → -2.30805}},
 {{R → -2.19287, α → -0.0456418}, {R → 2.19287, α → 3.09595}},
 {{R → -2.42575, α → -1.38423}, {R → 2.42575, α → 1.75736}},
 {{R → -0.415164, α → 1.59161}, {R → 0.415164, α → -1.54998}},
 {{R → -2.42742, α → 2.0867}, {R → 2.42742, α → -1.05489}},
 {{R → -2.44587, α → 1.94138}, {R → 2.44587, α → -1.20021}},
 {{R → -1.54898, α → 1.97289}, {R → 1.54898, α → -1.1687}},
 {{R → -1.44574, α → 1.74063}, {R → 1.44574, α → -1.40096}},
 {{R → -1.4799, α → -0.326887}, {R → 1.4799, α → 2.81471}},
 {{R → -1.13575, α → 1.92111}, {R → 1.13575, α → -1.22048}},
 {{R → -1.16878, α → -0.679478}, {R → 1.16878, α → 2.46211}},
 {{R → -0.435941, α → 2.10147}, {R → 0.435941, α → -1.04012}},
 {{R → -0.284882, α → -2.18936}, {R → 0.284882, α → 0.952236}},
 {{R → -128.719, α → 1.67181}, {R → 128.719, α → -1.46978}},
 {{R → -47.4437, α → -1.53672}, {R → 47.4437, α → 1.60487}},
 {{R → -8.13354, α → -1.66232}, {R → 8.13354, α → 1.47927}},
 {{R → -2.06096, α → -2.52514}, {R → 2.06096, α → 0.616448}},
 {{R → -1.63564, α → -1.72066}, {R → 1.63564, α → 1.42093}},
 {{R → -8.64244, α → 1.27888}, {R → 8.64244, α → -1.86272}},
 {{R → -1.31443, α → -1.51948}, {R → 1.31443, α → 1.62211}},
 {{R → -9.51752, α → 1.153}, {R → 9.51752, α → -1.9886}},
 {{R → -0.815123, α → 1.14047}, {R → 0.815123, α → -2.00112}},
 {{R → -2.30212, α → -2.25335}, {R → 2.30212, α → 0.888244}},
 {{R → -3.23875, α → 1.24027}, {R → 3.23875, α → -1.90132}},
 {{R → -3.92085, α → -2.22789}, {R → 3.92085, α → 0.913704}},
 {{R → -3.98475, α → 0.879653}, {R → 3.98475, α → -2.26194}},
 {{R → -1.46338, α → 0.987395}, {R → 1.46338, α → -2.1542}},
 {{R → -2.71435, α → 0.636676}, {R → 2.71435, α → -2.50492}},
 {{R → -1.37064, α → 0.704347}, {R → 1.37064, α → -2.43725}},
 {{R → -0.888856, α → 1.535}, {R → 0.888856, α → -1.60659}},
 {{R → -1.70638, α → -2.4837}, {R → 1.70638, α → 0.657892}},
 {{R → -1.0649, α → 2.44817}, {R → 1.0649, α → -0.693426}},
 {{R → -1.51629, α → -2.79152}, {R → 1.51629, α → 0.350071}}}
```

Pick up solutions that R>0

```
LpSolution[[1, 2]]
```

```
{R → 10.2451, α → -1.43929}
```

```
ModLpSolution = Table[LpSolution[[i, 2]], {i, num}]
```

```
{{R → 10.2451, α → -1.43929}, {R → 22.9877, α → 2.04438}, {R → 4.84216, α → 2.25089},
 {R → 16.102, α → -1.05204}, {R → 2.72186, α → 0.96246}, {R → 4.20501, α → -2.30805},
 {R → 2.19287, α → 3.09595}, {R → 2.42575, α → 1.75736}, {R → 0.415164, α → -1.54998},
 {R → 2.42742, α → -1.05489}, {R → 2.44587, α → -1.20021}, {R → 1.54898, α → -1.1687},
 {R → 1.44574, α → -1.40096}, {R → 1.4799, α → 2.81471}, {R → 1.13575, α → -1.22048},
 {R → 1.16878, α → 2.46211}, {R → 0.435941, α → -1.04012}, {R → 0.284882, α → 0.952236},
 {R → 128.719, α → -1.46978}, {R → 47.4437, α → 1.60487}, {R → 8.13354, α → 1.47927},
 {R → 2.06096, α → 0.616448}, {R → 1.63564, α → 1.42093}, {R → 8.64244, α → -1.86272},
 {R → 1.31443, α → 1.62211}, {R → 9.51752, α → -1.9886}, {R → 0.815123, α → -2.00112},
 {R → 2.30212, α → 0.888244}, {R → 3.23875, α → -1.90132}, {R → 3.92085, α → 0.913704},
 {R → 3.98475, α → -2.26194}, {R → 1.46338, α → -2.1542}, {R → 2.71435, α → -2.50492},
 {R → 1.37064, α → -2.43725}, {R → 0.888856, α → -1.60659},
 {R → 1.70638, α → 0.657892}, {R → 1.0649, α → -0.693426}, {R → 1.51629, α → 0.350071}}
```

Get Delta functions

```
DeltaSolution = Table[ModSolution[[i, 2]], {i, num}]
```

```
{DiracDelta[2. - 1. w], DiracDelta[3. - 1. w], DiracDelta[4. - 1. w],
 DiracDelta[5. - 1. w], DiracDelta[6. - 1. w], DiracDelta[7. - 1. w],
 DiracDelta[8. - 1. w], DiracDelta[9. - 1. w], DiracDelta[10. - 1. w],
 DiracDelta[11. - 1. w], DiracDelta[12. - 1. w], DiracDelta[13. - 1. w],
 DiracDelta[14. - 1. w], DiracDelta[15. - 1. w], DiracDelta[16. - 1. w],
 DiracDelta[17. - 1. w], DiracDelta[18. - 1. w], DiracDelta[19. - 1. w],
 DiracDelta[-1. + w], DiracDelta[1. + w], DiracDelta[2. + w], DiracDelta[3. + w],
 DiracDelta[4. + w], DiracDelta[5. + w], DiracDelta[6. + w], DiracDelta[7. + w],
 DiracDelta[8. + w], DiracDelta[9. + w], DiracDelta[10. + w], DiracDelta[11. + w],
 DiracDelta[12. + w], DiracDelta[13. + w], DiracDelta[14. + w], DiracDelta[15. + w],
 DiracDelta[16. + w], DiracDelta[17. + w], DiracDelta[18. + w], DiracDelta[19. + w]}
```

```
Solve[DeltaSolution[[1]] == DiracDelta[0], w]
```

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

```
{{w → 2}}
```

```
DiracDelta[2]
```

```
0
```

Solve for Frequencies

```
FreqSolution = Table[Solve[DeltaSolution[[i]] == DiracDelta[0], w], {i, num}]
```

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

General::stop : Further output of Solve::ifun will be suppressed during this calculation. ≫

```
{{{w → 2}}, {{w → 3}}, {{w → 4}}, {{w → 5}}, {{w → 6}}, {{w → 7}}, {{w → 8}},
 {{w → 9}}, {{w → 10}}, {{w → 11}}, {{w → 12}}, {{w → 13}}, {{w → 14}}, {{w → 15}},
 {{w → 16}}, {{w → 17}}, {{w → 18}}, {{w → 19}}, {{w → 1}}, {{w → -1}},
 {{w → -2}}, {{w → -3}}, {{w → -4}}, {{w → -5}}, {{w → -6}}, {{w → -7}},
 {{w → -8}}, {{w → -9}}, {{w → -10}}, {{w → -11}}, {{w → -12}}, {{w → -13}},
 {{w → -14}}, {{w → -15}}, {{w → -16}}, {{w → -17}}, {{w → -18}}, {{w → -19}}}
```

```
N[FreqSolution]
```

```
{{{w → 2.}}, {{w → 3.}}, {{w → 4.}}, {{w → 5.}}, {{w → 6.}}, {{w → 7.}}, {{w → 8.}},
 {{w → 9.}}, {{w → 10.}}, {{w → 11.}}, {{w → 12.}}, {{w → 13.}}, {{w → 14.}}, {{w → 15.}},
 {{w → 16.}}, {{w → 17.}}, {{w → 18.}}, {{w → 19.}}, {{w → 1.}}, {{w → -1.}},
 {{w → -2.}}, {{w → -3.}}, {{w → -4.}}, {{w → -5.}}, {{w → -6.}}, {{w → -7.}},
 {{w → -8.}}, {{w → -9.}}, {{w → -10.}}, {{w → -11.}}, {{w → -12.}}, {{w → -13.}},
 {{w → -14.}}, {{w → -15.}}, {{w → -16.}}, {{w → -17.}}, {{w → -18.}}, {{w → -19.}}}
```

Get the solution set for each link

```
Table[Flatten[Append[ModLpSolution[[i]], FreqSolution[[i]]]], {i, num}]
```

```
{{R → 10.2451, α → -1.43929, w → 2}, {R → 22.9877, α → 2.04438, w → 3},
 {R → 4.84216, α → 2.25089, w → 4}, {R → 16.102, α → -1.05204, w → 5},
 {R → 2.72186, α → 0.96246, w → 6}, {R → 4.20501, α → -2.30805, w → 7},
 {R → 2.19287, α → 3.09595, w → 8}, {R → 2.42575, α → 1.75736, w → 9},
 {R → 0.415164, α → -1.54998, w → 10}, {R → 2.42742, α → -1.05489, w → 11},
 {R → 2.44587, α → -1.20021, w → 12}, {R → 1.54898, α → -1.1687, w → 13},
 {R → 1.44574, α → -1.40096, w → 14}, {R → 1.4799, α → 2.81471, w → 15},
 {R → 1.13575, α → -1.22048, w → 16}, {R → 1.16878, α → 2.46211, w → 17},
 {R → 0.435941, α → -1.04012, w → 18}, {R → 0.284882, α → 0.952236, w → 19},
 {R → 128.719, α → -1.46978, w → 1}, {R → 47.4437, α → 1.60487, w → -1},
 {R → 8.13354, α → 1.47927, w → -2}, {R → 2.06096, α → 0.616448, w → -3},
 {R → 1.63564, α → 1.42093, w → -4}, {R → 8.64244, α → -1.86272, w → -5},
 {R → 1.31443, α → 1.62211, w → -6}, {R → 9.51752, α → -1.9886, w → -7},
 {R → 0.815123, α → -2.00112, w → -8}, {R → 2.30212, α → 0.888244, w → -9},
 {R → 3.23875, α → -1.90132, w → -10}, {R → 3.92085, α → 0.913704, w → -11},
 {R → 3.98475, α → -2.26194, w → -12}, {R → 1.46338, α → -2.1542, w → -13},
 {R → 2.71435, α → -2.50492, w → -14}, {R → 1.37064, α → -2.43725, w → -15},
 {R → 0.888856, α → -1.60659, w → -16}, {R → 1.70638, α → 0.657892, w → -17},
 {R → 1.0649, α → -0.693426, w → -18}, {R → 1.51629, α → 0.350071, w → -19}}
```

```
SL = Table[Flatten[Append[ModLpSolution[[i]], N[FreqSolution[[i]]]]], {i, num}]
```

```
{{R → 10.2451, α → -1.43929, w → 2.}, {R → 22.9877, α → 2.04438, w → 3.},
 {R → 4.84216, α → 2.25089, w → 4.}, {R → 16.102, α → -1.05204, w → 5.},
 {R → 2.72186, α → 0.96246, w → 6.}, {R → 4.20501, α → -2.30805, w → 7.},
 {R → 2.19287, α → 3.09595, w → 8.}, {R → 2.42575, α → 1.75736, w → 9.},
 {R → 0.415164, α → -1.54998, w → 10.}, {R → 2.42742, α → -1.05489, w → 11.},
 {R → 2.44587, α → -1.20021, w → 12.}, {R → 1.54898, α → -1.1687, w → 13.},
 {R → 1.44574, α → -1.40096, w → 14.}, {R → 1.4799, α → 2.81471, w → 15.},
 {R → 1.13575, α → -1.22048, w → 16.}, {R → 1.16878, α → 2.46211, w → 17.},
 {R → 0.435941, α → -1.04012, w → 18.}, {R → 0.284882, α → 0.952236, w → 19.},
 {R → 128.719, α → -1.46978, w → 1.}, {R → 47.4437, α → 1.60487, w → -1.},
 {R → 8.13354, α → 1.47927, w → -2.}, {R → 2.06096, α → 0.616448, w → -3.},
 {R → 1.63564, α → 1.42093, w → -4.}, {R → 8.64244, α → -1.86272, w → -5.},
 {R → 1.31443, α → 1.62211, w → -6.}, {R → 9.51752, α → -1.9886, w → -7.},
 {R → 0.815123, α → -2.00112, w → -8.}, {R → 2.30212, α → 0.888244, w → -9.},
 {R → 3.23875, α → -1.90132, w → -10.}, {R → 3.92085, α → 0.913704, w → -11.},
 {R → 3.98475, α → -2.26194, w → -12.}, {R → 1.46338, α → -2.1542, w → -13.},
 {R → 2.71435, α → -2.50492, w → -14.}, {R → 1.37064, α → -2.43725, w → -15.},
 {R → 0.888856, α → -1.60659, w → -16.}, {R → 1.70638, α → 0.657892, w → -17.},
 {R → 1.0649, α → -0.693426, w → -18.}, {R → 1.51629, α → 0.350071, w → -19.}}
```

```
Length[SL]
```

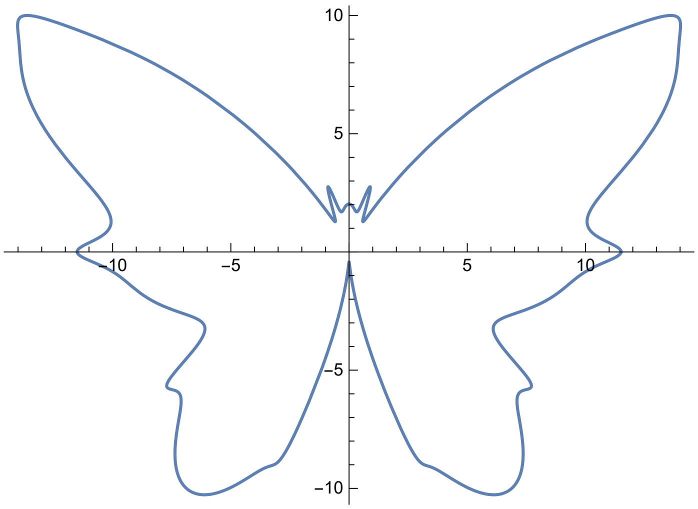
```
38
```

C Butterfly Curve Synthesis Mathematica Code

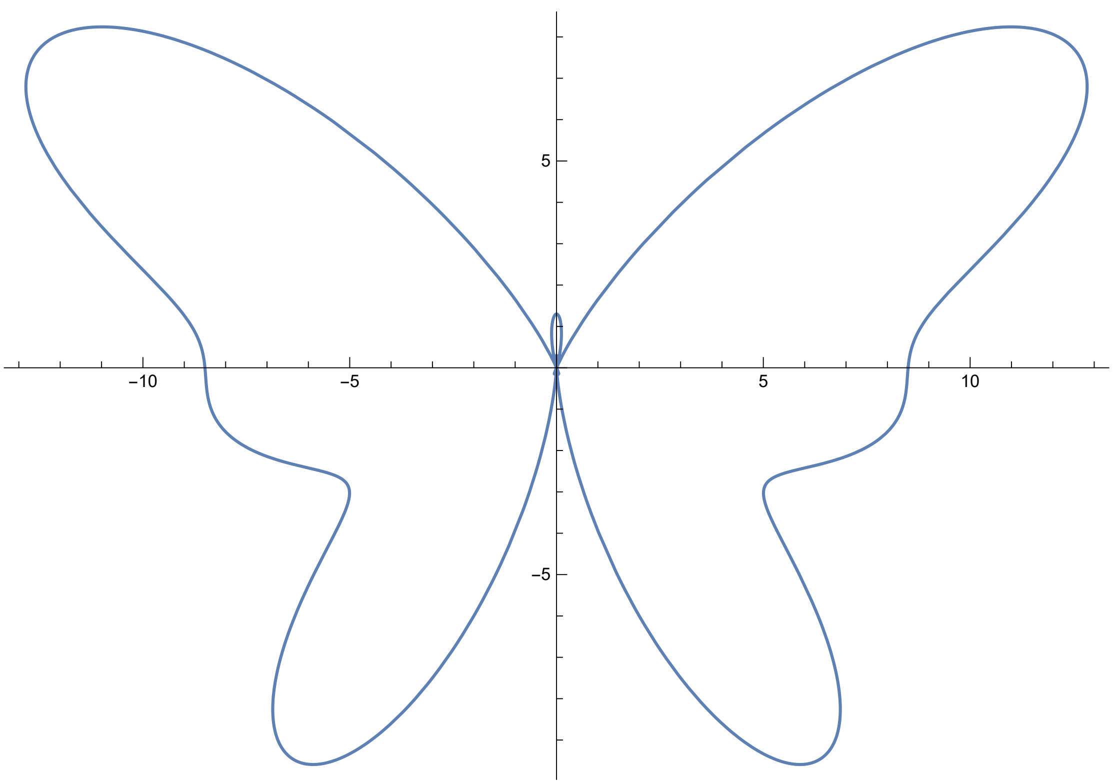
Below is the Mathematica code for the computation of linkage system to draw Butterfly Curve.

Butterfly Single Coupled Serial Chain

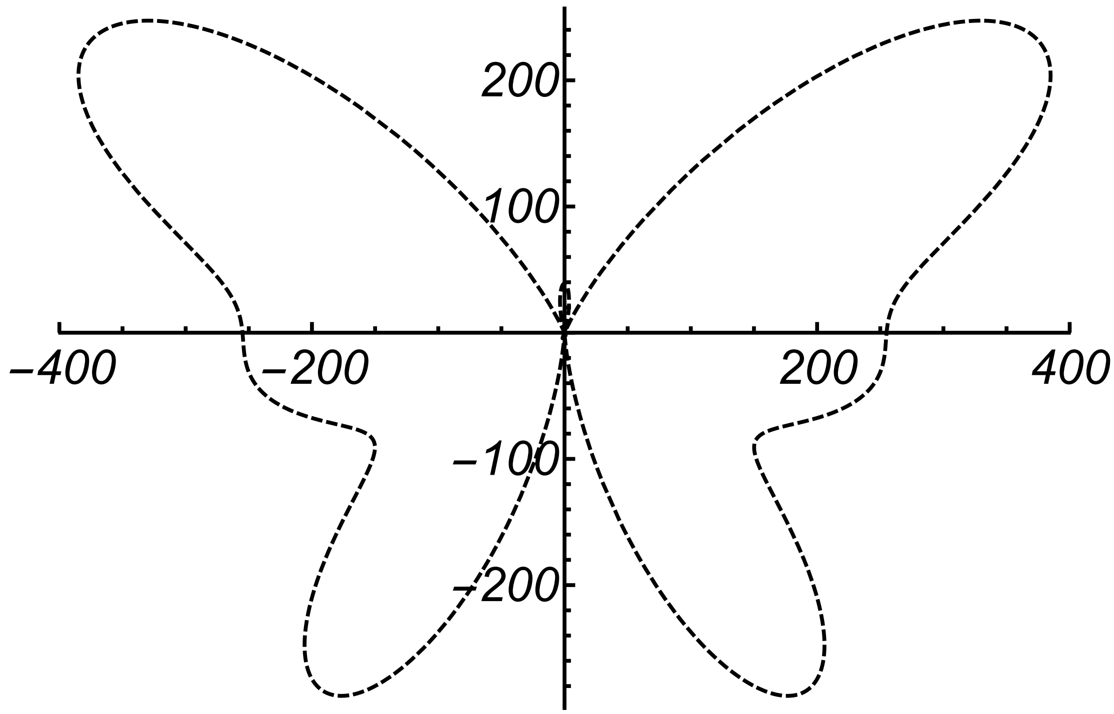
```
PolarPlot[{7 - Sin[θ] + 2.3 Sin[3 θ] + 2.5 Sin[5 θ] - 2 Sin[7 θ] - 0.4 Sin[9 θ] +
   4 Cos[2 θ] - 2.5 Cos[4 θ] + 3 / Sqrt[(4 Sin[5 θ]) ^2 + 1]}, {θ, 0, 2 Pi}]
```



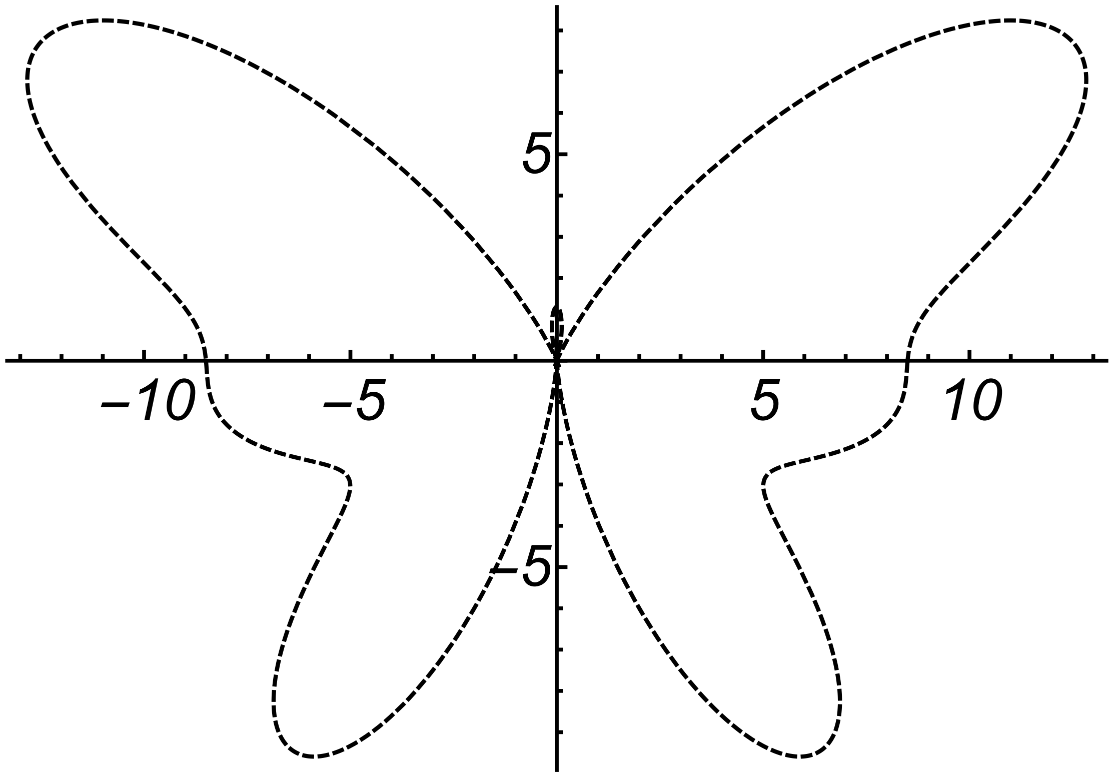

```
PolarPlot[{7 - Sin[θ] + 2.3 Sin[3 θ] + 2.5 Sin[5 θ] - 2 Sin[7 θ] -
   0.4 Sin[9 θ] + 4 Cos[2 θ] - 2.5 Cos[4 θ]}, {θ, 0, 2 Pi}, ImageSize → Large]
```



```
InitialPlot =
 PolarPlot[{30 * (7 - Sin[θ] + 2.3 Sin[3 θ] + 2.5 Sin[5 θ] - 2 Sin[7 θ] - 0.4 Sin[9 θ] +
      4 Cos[2 θ] - 2.5 Cos[4 θ])}, {θ, 0, 2 Pi},
  PlotStyle → Directive[Black, Dashed, Thickness[.0038]],
  AxesStyle → Directive[25, Thick, Black],
  LabelStyle → Directive[30, Black, Italic], ImageSize → Large]
```



```
PolarPlot[{7 - Sin[θ] + 2.3 Sin[3 θ] +
    2.5 Sin[5 θ] - 2 Sin[7 θ] - 0.4 Sin[9 θ] + 4 Cos[2 θ] - 2.5 Cos[4 θ]},
 {θ, 0, 2 Pi}, PlotStyle → Directive[Black, Dashed, Thickness[.0038]],
 AxesStyle → Directive[30, Thick, Black],
 LabelStyle → Directive[30, Black, Italic], ImageSize → Large]
```




```
roi = 7 - Sin[θ] + 2.3 Sin[3 θ] + 2.5 Sin[5 θ] -
   2 Sin[7 θ] - 0.4 Sin[9 θ] + 4 Cos[2 θ] - 2.5 Cos[4 θ]
```

```
7 + 4 Cos[2 θ] - 2.5 Cos[4 θ] - Sin[θ] + 2.3 Sin[3 θ] + 2.5 Sin[5 θ] - 2 Sin[7 θ] - 0.4 Sin[9 θ]
```

```
x = Expand[roi * Cos[θ]]
```

```
7 Cos[θ] + 4 Cos[θ] Cos[2 θ] - 2.5 Cos[θ] Cos[4 θ] - Cos[θ] Sin[θ] +
 2.3 Cos[θ] Sin[3 θ] + 2.5 Cos[θ] Sin[5 θ] - 2 Cos[θ] Sin[7 θ] - 0.4 Cos[θ] Sin[9 θ]
```

```
xReduce = TrigReduce[x]
```

```
9 Cos[θ] + 0.75 Cos[3 θ] - 1.25 Cos[5 θ] + 0.65 Sin[2 θ] +
 2.4 Sin[4 θ] + 0.25 Sin[6 θ] - 1.2 Sin[8 θ] - 0.2 Sin[10 θ]
```

```
y = Expand[roi * Sin[θ]]
```

```
7 Sin[θ] + 4 Cos[2 θ] Sin[θ] - 2.5 Cos[4 θ] Sin[θ] - Sin[θ]^2 +
 2.3 Sin[θ] Sin[3 θ] + 2.5 Sin[θ] Sin[5 θ] - 2 Sin[θ] Sin[7 θ] - 0.4 Sin[θ] Sin[9 θ]
```

```
yReduce = TrigReduce[y]
```

```
-0.5 + 1.65 Cos[2 θ] + 0.1 Cos[4 θ] - 2.25 Cos[6 θ] +
 0.8 Cos[8 θ] + 0.2 Cos[10 θ] + 5 Sin[θ] + 3.25 Sin[3 θ] - 1.25 Sin[5 θ]
```

Coefficients

```
ak = Table[Coefficient[xReduce, Cos[i θ]], {i, 1, 10}]
```

```
{9, 0, 0.75, 0, -1.25, 0, 0, 0, 0, 0}
```

```
bk = Table[Coefficient[xReduce, Sin[i θ]], {i, 1, 10}]
```

```
{0, 0.65, 0, 2.4, 0, 0.25, 0, -1.2, 0, -0.2}
```

```
ck = Table[Coefficient[yReduce, Cos[i θ]], {i, 1, 10}]
```

```
{0, 1.65, 0, 0.1, 0, -2.25, 0, 0.8, 0, 0.2}
```

```
dk = Table[Coefficient[yReduce, Sin[i θ]], {i, 1, 10}]
```

```
{5, 0, 3.25, 0, -1.25, 0, 0, 0, 0, 0}
```

Ground pivot coefficients

```
a0 = xReduce - Sum[ak[[i]] Cos[i θ] + bk[[i]] Sin[i θ], {i, 1, 10}]
```

```
0.
```

```
b0 = 0
```

```
0
```

```
c0 = yReduce - Sum[ck[[i]] Cos[i θ] + dk[[i]] Sin[i θ], {i, 1, 10}]
```

```
-0.5
```

```
d0 = 0
```

```
0
```

Single coupled serial chain Config.

```
Lk = 0.5 * Sqrt[(ak + dk)^2 + (ck - bk)^2]
```

```
{7., 0.5, 2., 1.15, 1.25, 1.25, 0., 1., 0., 0.2}
```

```
psik = ArcTan[Chop[ak + dk], Chop[ck - bk]]
```

```
MessageTemplate[ArcTan, indet,
 Indeterminate expression ArcTan[0, 0] encountered., 2, 18, 1, 29 507 351 937 250 181 510, Local]
```

```
MessageTemplate[ArcTan, indet,
 Indeterminate expression ArcTan[0, 0] encountered., 2, 18, 2, 29 507 351 937 250 181 510, Local]
```

```
{0, 1.5708, 0., -1.5708, 3.14159, -1.5708,
 Indeterminate, 1.5708, Indeterminate, 1.5708}
```

```
psik = psik /. {Indeterminate → 0}
```

```
{0, 1.5708, 0., -1.5708, 3.14159, -1.5708, 0, 1.5708, 0, 1.5708}
```

```
Mk = 0.5 * Sqrt[(ak - dk) ^2 + (ck + bk) ^2]
```

```
{2., 1.15, 1.25, 1.25, 0., 1., 0., 0.2, 0., 0.}
```

```
etak = ArcTan[Chop[ak - dk], Chop[ck + bk]]
```

```
MessageTemplate[ArcTan, indet,
 Indeterminate expression ArcTan[0, 0] encountered., 2, 21, 3, 29 507 351 937 250 181 510, Local]
```

```
MessageTemplate[ArcTan, indet,
 Indeterminate expression ArcTan[0, 0] encountered., 2, 21, 4, 29 507 351 937 250 181 510, Local]
```

```
MessageTemplate[ArcTan, indet,
 Indeterminate expression ArcTan[0, 0] encountered., 2, 21, 5, 29 507 351 937 250 181 510, Local]
```

```
MessageTemplate[General, stop, Further output of ArcTan::indet will be suppressed during this calculation.,
 2, 21, 6, 29 507 351 937 250 181 510, Local]
```

```
{0, 1.5708, 3.14159, 1.5708, Indeterminate, -1.5708,
 Indeterminate, -1.5708, Indeterminate, Indeterminate}
```

```
etak = etak /. {Indeterminate → 0}
```

```
{0, 1.5708, 3.14159, 1.5708, 0, -1.5708, 0, -1.5708, 0, 0}
```

Ground pivot position Config.

```
L0 = 0.5 * Sqrt[(a0 + d0) ^2 + (c0 - b0) ^2]
```

```
0.25
```

```
psi0 = ArcTan[a0 + d0, c0 - b0] / Pi
```

-0.5

```
M0 = 0.5 * Sqrt[(a0 - d0)^2 + (c0 + b0)^2]
```

0.25

```
eta0 = ArcTan[a0 - d0, c0 + b0] / Pi
```

-0.5

Plot the curve Starting from Ground pivot position

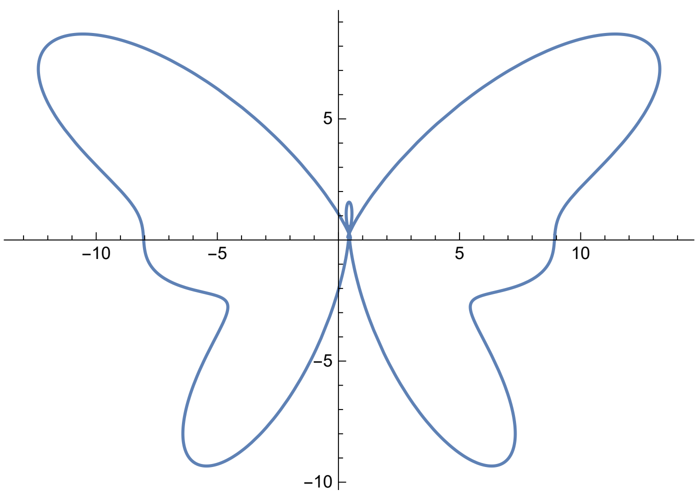
```
Newxθ = Sum[Lk[[i]] Cos[psik[[i]] + i θ] + Mk[[i]] Cos[etak[[i]] - i θ], {i, 1, 10}] +
  L0 Cos[psi0] + M0 Cos[eta0]
```

```
0.438791 + 1.25 Cos[1.5708 - 6 θ] + 2.4 Cos[1.5708 - 4 θ] +
 1.25 Cos[3.14159 - 3 θ] + 1.15 Cos[1.5708 - 2 θ] + 9. Cos[θ] +
 0.5 Cos[1.5708 + 2 θ] + 2. Cos[0. + 3 θ] + 1.25 Cos[3.14159 + 5 θ] +
 1. Cos[1.5708 + 6 θ] + 1.2 Cos[1.5708 + 8 θ] + 0.2 Cos[1.5708 + 10 θ]
```

```
Newyθ = Sum[Lk[[i]] Sin[psik[[i]] + i θ] + Mk[[i]] Sin[etak[[i]] - i θ], {i, 1, 10}] +
  L0 Sin[psi0] + M0 Sin[eta0]
```

```
-0.239713 - 1.25 Sin[1.5708 - 6 θ] + 0.1 Sin[1.5708 - 4 θ] +
 1.25 Sin[3.14159 - 3 θ] + 1.15 Sin[1.5708 - 2 θ] + 5. Sin[θ] +
 0.5 Sin[1.5708 + 2 θ] + 2. Sin[0. + 3 θ] + 1.25 Sin[3.14159 + 5 θ] -
 1. Sin[1.5708 + 6 θ] + 0.8 Sin[1.5708 + 8 θ] + 0.2 Sin[1.5708 + 10 θ]
```

```
ParametricPlot[{Newxθ, Newyθ}, {θ, 0, 2 Pi}]
```



Plot the curve NOT Starting from Ground pivot position

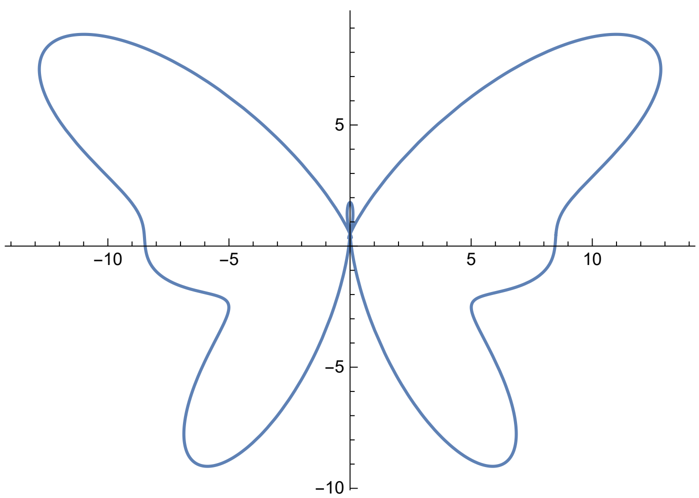
```
Newxθ1 = Sum[Lk[[i]] Cos[psik[[i]] + i θ] + Mk[[i]] Cos[etak[[i]] - i θ], {i, 1, 10}]
```

```
0. + 1.25 Cos[1.5708 - 6 θ] + 2.4 Cos[1.5708 - 4 θ] +
 1.25 Cos[3.14159 - 3 θ] + 1.15 Cos[1.5708 - 2 θ] + 9. Cos[θ] +
 0.5 Cos[1.5708 + 2 θ] + 2. Cos[0. + 3 θ] + 1.25 Cos[3.14159 + 5 θ] +
 1. Cos[1.5708 + 6 θ] + 1.2 Cos[1.5708 + 8 θ] + 0.2 Cos[1.5708 + 10 θ]
```

```
Newyθ1 = Sum[Lk[[i]] Sin[psik[[i]] + i θ] + Mk[[i]] Sin[etak[[i]] - i θ], {i, 1, 10}]
```

```
0. - 1.25 Sin[1.5708 - 6 θ] + 0.1 Sin[1.5708 - 4 θ] +
 1.25 Sin[3.14159 - 3 θ] + 1.15 Sin[1.5708 - 2 θ] + 5. Sin[θ] +
 0.5 Sin[1.5708 + 2 θ] + 2. Sin[0. + 3 θ] + 1.25 Sin[3.14159 + 5 θ] -
 1. Sin[1.5708 + 6 θ] + 0.8 Sin[1.5708 + 8 θ] + 0.2 Sin[1.5708 + 10 θ]
```

```
ParametricPlot[{Newxθ1, Newyθ1}, {θ, 0, 2 Pi}]
```



Simplification of the Plot

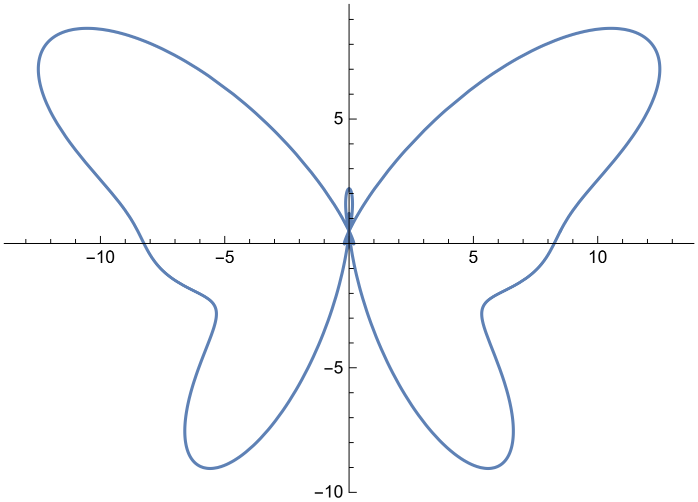
```
Newxθ2 = Sum[Lk[[i]] Cos[psik[[i]] + i θ], {i, 1, 8}] +
  Sum[Mk[[j]] Cos[etak[[j]] - j θ], {j, 1, 6}]
```

```
0. + 1.25 Cos[1.5708 - 6 θ] + 2.4 Cos[1.5708 - 4 θ] + 1.25 Cos[3.14159 - 3 θ] +
 1.15 Cos[1.5708 - 2 θ] + 9. Cos[θ] + 0.5 Cos[1.5708 + 2 θ] + 2. Cos[0. + 3 θ] +
 1.25 Cos[3.14159 + 5 θ] + 1. Cos[1.5708 + 6 θ] + 1. Cos[1.5708 + 8 θ]
```

```
Newyθ2 = Sum[Lk[[i]] Sin[psik[[i]] + i θ], {i, 1, 8}] +
  Sum[Mk[[j]] Sin[etak[[j]] - j θ], {j, 1, 6}]
```

```
0. - 1.25 Sin[1.5708 - 6 θ] + 0.1 Sin[1.5708 - 4 θ] + 1.25 Sin[3.14159 - 3 θ] +
 1.15 Sin[1.5708 - 2 θ] + 5. Sin[θ] + 0.5 Sin[1.5708 + 2 θ] + 2. Sin[0. + 3 θ] +
 1.25 Sin[3.14159 + 5 θ] - 1. Sin[1.5708 + 6 θ] + 1. Sin[1.5708 + 8 θ]
```

```
ParametricPlot[{Newxθ2, Newyθ2}, {θ, 0, 2 Pi}]
```

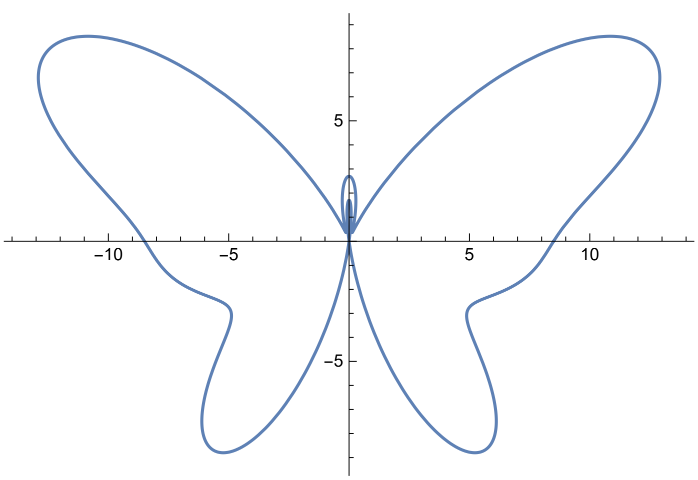


Further Simplification of the Plot

```
Newxθ3 = Sum[Lk[[i]] Cos[psik[[i]] + i θ], {i, 1, 8}] +
  Sum[Mk[[j]] Cos[etak[[j]] - j θ], {j, 1, 6}] - Lk[[2]] Cos[psik[[2]] + 2 θ]
0. + 1.25 Cos[1.5708 - 6 θ] + 2.4 Cos[1.5708 - 4 θ] +
 1.25 Cos[3.14159 - 3 θ] + 1.15 Cos[1.5708 - 2 θ] + 9. Cos[θ] + 2. Cos[0. + 3 θ] +
 1.25 Cos[3.14159 + 5 θ] + 1. Cos[1.5708 + 6 θ] + 1. Cos[1.5708 + 8 θ]
```

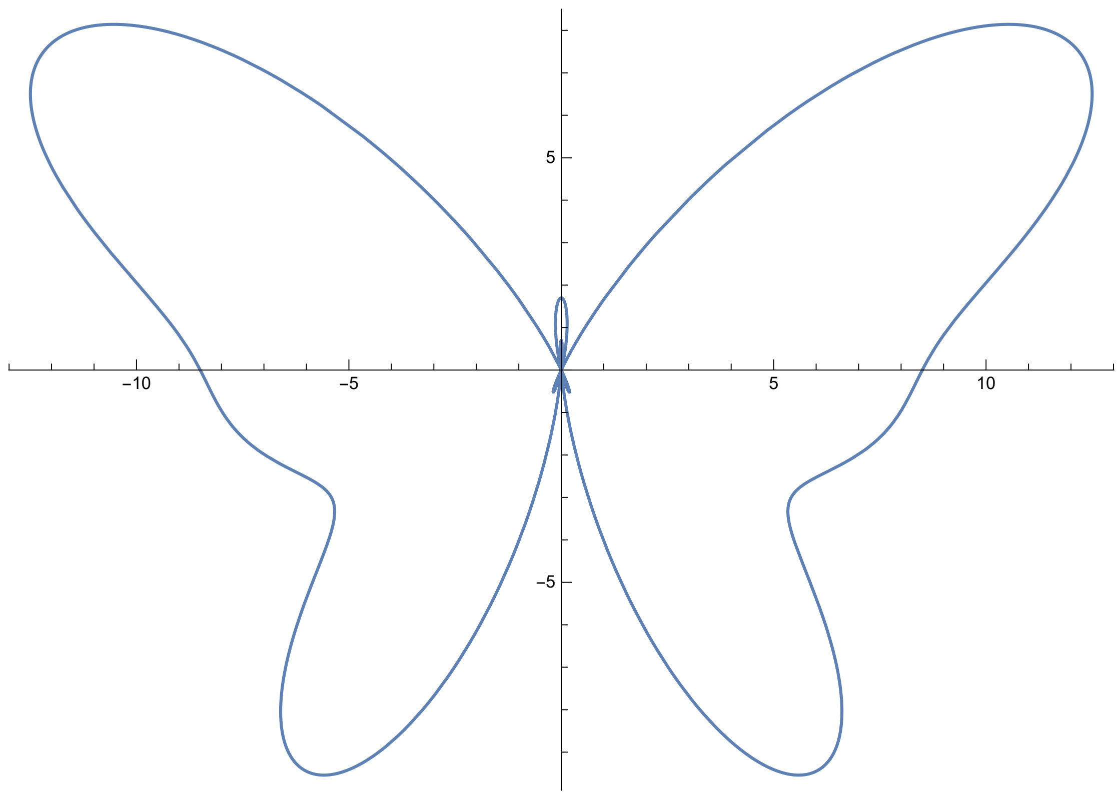
```
Newyθ3 = Sum[Lk[[i]] Sin[psik[[i]] + i θ], {i, 1, 8}] +
  Sum[Mk[[j]] Sin[etak[[j]] - j θ], {j, 1, 6}] - Lk[[2]] Sin[psik[[2]] + 2 θ]
0. - 1.25 Sin[1.5708 - 6 θ] + 0.1 Sin[1.5708 - 4 θ] +
 1.25 Sin[3.14159 - 3 θ] + 1.15 Sin[1.5708 - 2 θ] + 5. Sin[θ] + 2. Sin[0. + 3 θ] +
 1.25 Sin[3.14159 + 5 θ] - 1. Sin[1.5708 + 6 θ] + 1. Sin[1.5708 + 8 θ]
```

```
ParametricPlot[{Newxθ3, Newyθ3}, {θ, 0, 2 Pi}]
```

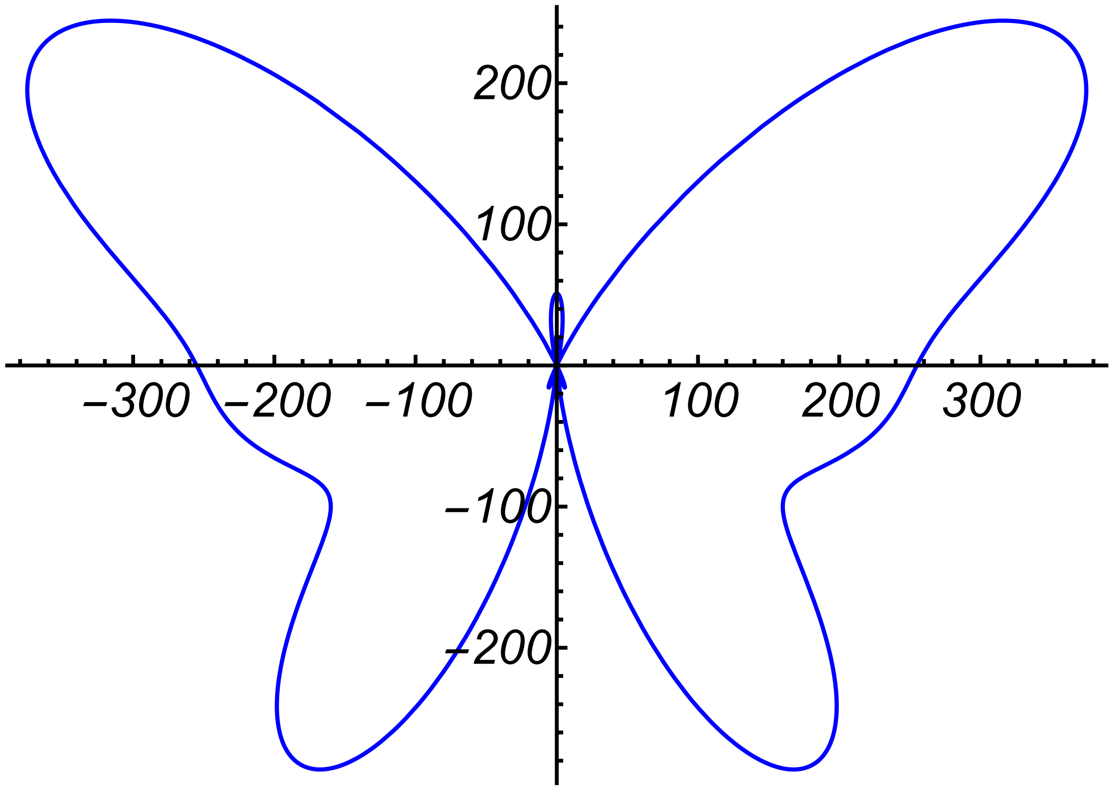


Truncate one term in the original curve

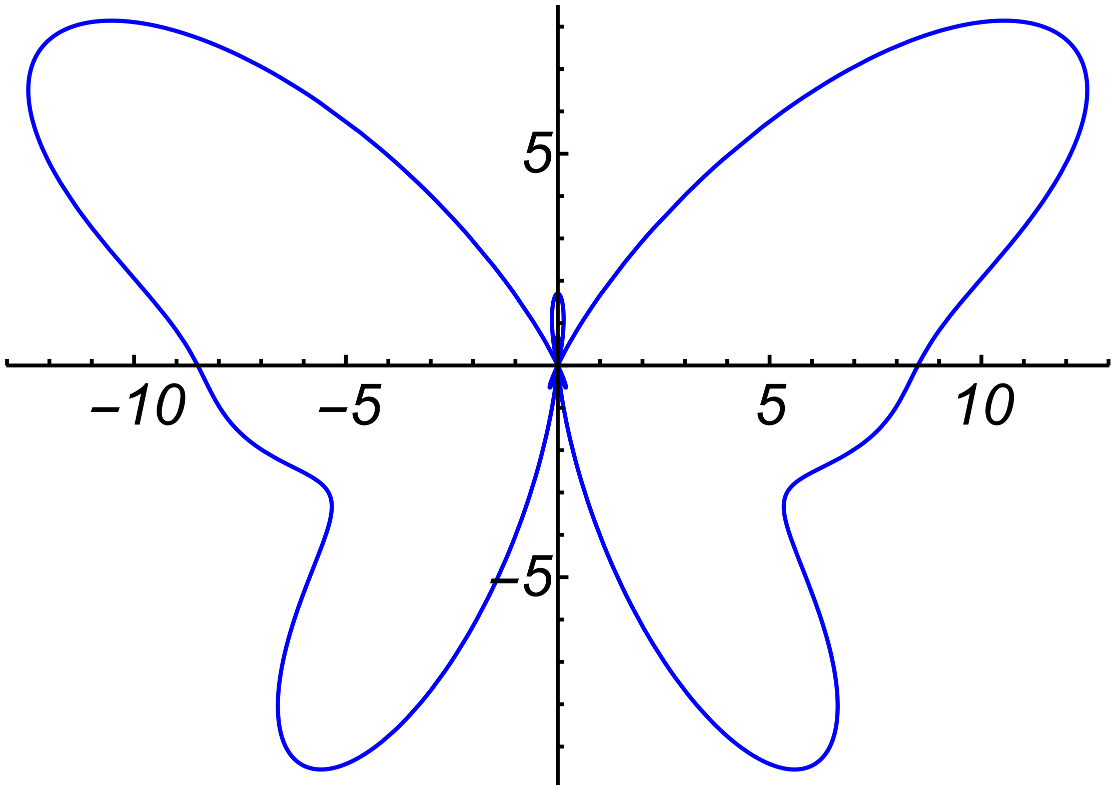
```
PolarPlot[{7 - Sin[θ] + 2.3 Sin[3 θ] + 2.5 Sin[5 θ] - 2 Sin[7 θ] + 4 Cos[2 θ] - 2.5 Cos[4 θ]},
 {θ, 0, 2 Pi}, ImageSize → Large]
```



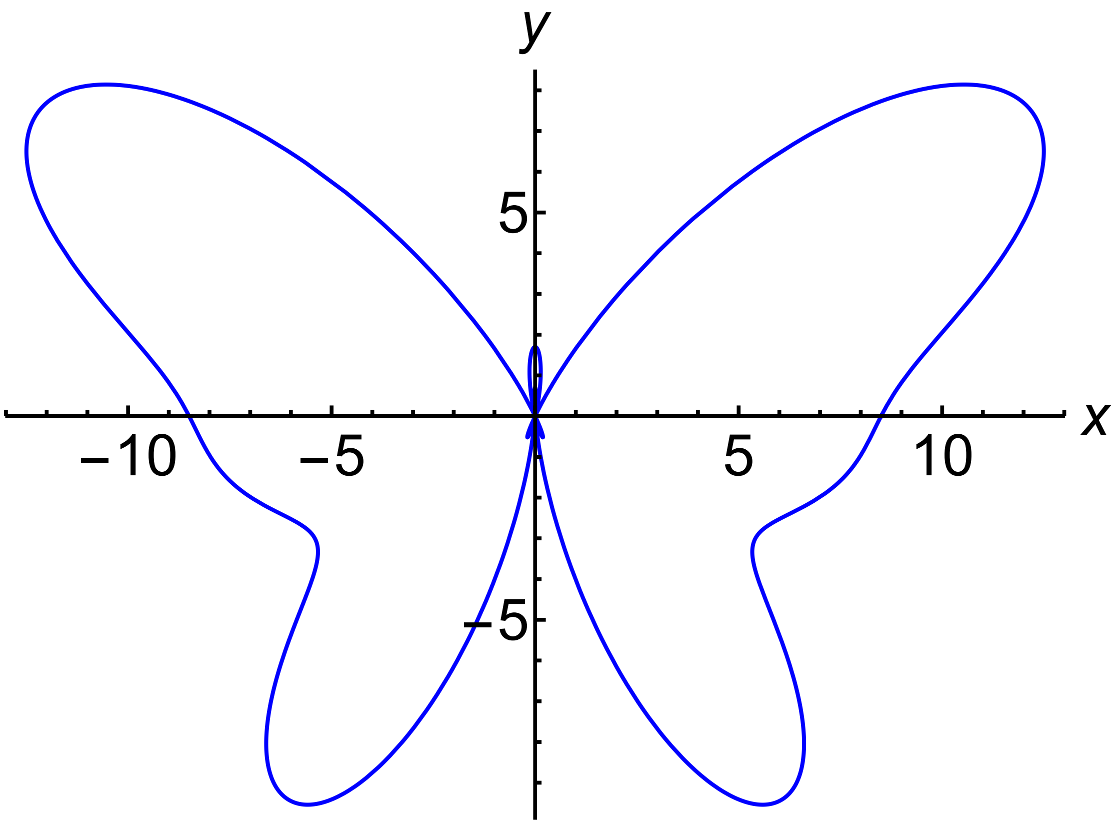
```
TruncatePlot2 = PolarPlot[
  {30 * (7 - Sin[θ] + 2.3 Sin[3 θ] + 2.5 Sin[5 θ] - 2 Sin[7 θ] + 4 Cos[2 θ] - 2.5 Cos[4 θ])},
  {θ, 0, 2 Pi}, PlotStyle → Directive[Blue, Thickness[.0038]],
  AxesStyle → Directive[25, Thick, Black],
  LabelStyle → Directive[30, Black, Italic], ImageSize → Large]
```



```
TruncatePlot =
 PolarPlot[{7 - Sin[θ] + 2.3 Sin[3 θ] + 2.5 Sin[5 θ] - 2 Sin[7 θ] + 4 Cos[2 θ] - 2.5 Cos[4 θ]},
  {θ, 0, 2 Pi}, PlotStyle → Directive[Blue, Thickness[.0038]],
  AxesStyle → Directive[30, Thick, Black],
  LabelStyle → Directive[30, Black, Italic], ImageSize → Large]
```



```
Show[TruncatePlot, AxesLabel → {HoldForm[x], HoldForm[y]},
 PlotLabel → None, LabelStyle → {GrayLevel[0]}]
```




```
newroi = 7 - Sin[θ] + 2.3 Sin[3 θ] + 2.5 Sin[5 θ] - 2 Sin[7 θ] + 4 Cos[2 θ] - 2.5 Cos[4 θ]
```

```
7 + 4 Cos[2 θ] - 2.5 Cos[4 θ] - Sin[θ] + 2.3 Sin[3 θ] + 2.5 Sin[5 θ] - 2 Sin[7 θ]
```

```
x = Expand[newroi * Cos[θ]]
```

```
7 Cos[θ] + 4 Cos[θ] Cos[2 θ] - 2.5 Cos[θ] Cos[4 θ] - Cos[θ] Sin[θ] +
 2.3 Cos[θ] Sin[3 θ] + 2.5 Cos[θ] Sin[5 θ] - 2 Cos[θ] Sin[7 θ]
```

```
Reduce = TrigReduce[x]
```

```
MessageTemplate[ Set , wrsym , Symbol Reduce is Protected. , 2 , 44 , 7 , 29 507 351 937 250 181 510 , Local ]
```

```
9 Cos[θ] + 0.75 Cos[3 θ] - 1.25 Cos[5 θ] +
 0.65 Sin[2 θ] + 2.4 Sin[4 θ] + 0.25 Sin[6 θ] - Sin[8 θ]
```

```
y = Expand[newroi * Sin[θ]]
```

```
7 Sin[θ] + 4 Cos[2 θ] Sin[θ] - 2.5 Cos[4 θ] Sin[θ] - Sin[θ]^2 +
 2.3 Sin[θ] Sin[3 θ] + 2.5 Sin[θ] Sin[5 θ] - 2 Sin[θ] Sin[7 θ]
```

```
yReduce = TrigReduce[y]
-0.5 + 1.65 Cos[2 θ] + 0.1 Cos[4 θ] - 2.25 Cos[6 θ] +
 Cos[8 θ] + 5 Sin[θ] + 3.25 Sin[3 θ] - 1.25 Sin[5 θ]
```

Coefficients

```
ak = Table[Coefficient[xReduce, Cos[i θ]], {i, 1, 10}]
{9, 0, 0.75, 0, -1.25, 0, 0, 0, 0, 0}
```

```
bk = Table[Coefficient[xReduce, Sin[i θ]], {i, 1, 10}]
{0, 0.65, 0, 2.4, 0, 0.25, 0, -1.2, 0, -0.2}
```

```
ck = Table[Coefficient[yReduce, Cos[i θ]], {i, 1, 10}]
{0, 1.65, 0, 0.1, 0, -2.25, 0, 1, 0, 0}
```

```
dk = Table[Coefficient[yReduce, Sin[i θ]], {i, 1, 10}]
{5, 0, 3.25, 0, -1.25, 0, 0, 0, 0, 0}
```

Ground pivot coefficients

```
a0 = xReduce - Sum[ak[[i]] Cos[i θ] + bk[[i]] Sin[i θ], {i, 1, 10}]
0.
```

```
b0 = 0
0
```

```
c0 = yReduce - Sum[ck[[i]] Cos[i θ] + dk[[i]] Sin[i θ], {i, 1, 10}]
-0.5
```

```
d0 = 0
0
```

Single coupled serial chain Config.

```
Lk = 0.5 * Sqrt[(ak + dk) ^2 + (ck - bk) ^2]
{7., 0.5, 2., 1.15, 1.25, 1.25, 0., 1.1, 0., 0.1}
```

```
psik = ArcTan[Chop[ak + dk], Chop[ck - bk]]
```

MessageTemplate[ArcTan, indet, Indeterminate expression ArcTan[0, 0] encountered., 2, 56, 8, 29 507 351 937 250 181 510, Local]

MessageTemplate[ArcTan, indet, Indeterminate expression ArcTan[0, 0] encountered., 2, 56, 9, 29 507 351 937 250 181 510, Local]

```
{0, 1.5708, 0., -1.5708, 3.14159, -1.5708,
 Indeterminate, 1.5708, Indeterminate, 1.5708}
```

```
psik = psik /. {Indeterminate → 0}
```

```
{0, 1.5708, 0., -1.5708, 3.14159, -1.5708, 0, 1.5708, 0, 1.5708}
```

```
Mk = 0.5 * Sqrt[(ak - dk)^2 + (ck + bk)^2]
```

```
{2., 1.15, 1.25, 1.25, 0., 1., 0., 0.1, 0., 0.1}
```

```
etak = ArcTan[Chop[ak - dk], Chop[ck + bk]]
```

MessageTemplate[ArcTan, indet, Indeterminate expression ArcTan[0, 0] encountered., 2, 59, 10, 29 507 351 937 250 181 510, Local]

MessageTemplate[ArcTan, indet, Indeterminate expression ArcTan[0, 0] encountered., 2, 59, 11, 29 507 351 937 250 181 510, Local]

MessageTemplate[ArcTan, indet, Indeterminate expression ArcTan[0, 0] encountered., 2, 59, 12, 29 507 351 937 250 181 510, Local]

MessageTemplate[General, stop, Further output of ArcTan::indet will be suppressed during this calculation., 2, 59, 13, 29 507 351 937 250 181 510, Local]

```
{0, 1.5708, 3.14159, 1.5708, Indeterminate,
 -1.5708, Indeterminate, -1.5708, Indeterminate, -1.5708}
```

```
etak = etak /. {Indeterminate → 0}
```

```
{0, 1.5708, 3.14159, 1.5708, 0, -1.5708, 0, -1.5708, 0, -1.5708}
```

Plot the curve NOT Starting from Ground pivot position

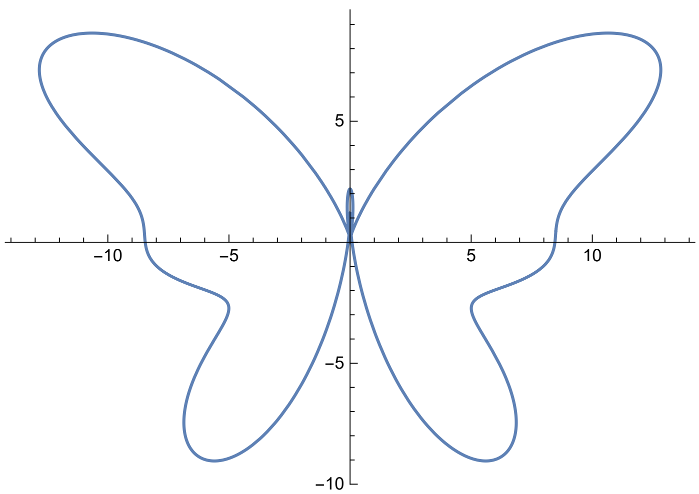
```
Newxθ = Sum[Lk[[i]] Cos[psik[[i]] + i θ] + Mk[[i]] Cos[etak[[i]] - i θ], {i, 1, 10}]
```

```
0. + 1.25 Cos[1.5708 - 6 θ] + 2.4 Cos[1.5708 - 4 θ] +
 1.25 Cos[3.14159 - 3 θ] + 1.15 Cos[1.5708 - 2 θ] + 9. Cos[θ] +
 0.5 Cos[1.5708 + 2 θ] + 2. Cos[0. + 3 θ] + 1.25 Cos[3.14159 + 5 θ] +
 1. Cos[1.5708 + 6 θ] + 1.2 Cos[1.5708 + 8 θ] + 0.2 Cos[1.5708 + 10 θ]
```

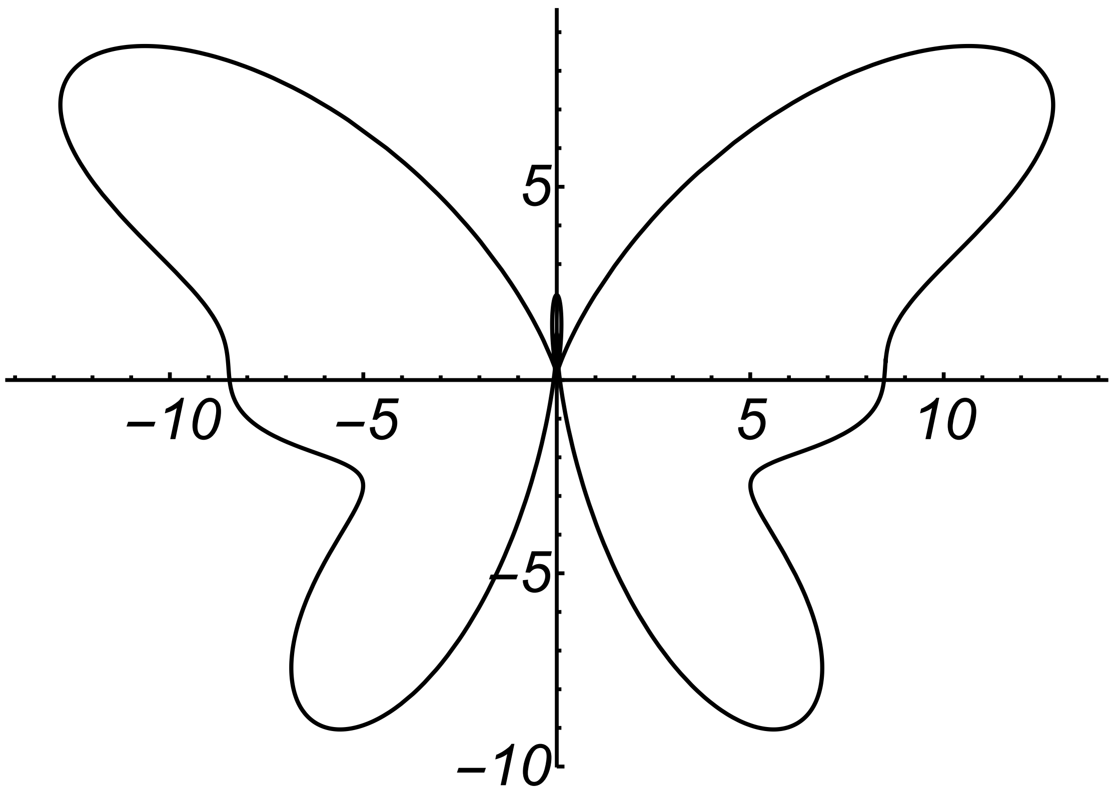
```
Newyθ = Sum[Lk[[i]] Sin[psik[[i]] + i θ] + Mk[[i]] Sin[etak[[i]] - i θ], {i, 1, 10}]
```

```
0. - 1.25 Sin[1.5708 - 6 θ] + 0.1 Sin[1.5708 - 4 θ] + 1.25 Sin[3.14159 - 3 θ] +
 1.15 Sin[1.5708 - 2 θ] + 5. Sin[θ] + 0.5 Sin[1.5708 + 2 θ] + 2. Sin[0. + 3 θ] +
 1.25 Sin[3.14159 + 5 θ] - 1. Sin[1.5708 + 6 θ] + 1. Sin[1.5708 + 8 θ]
```

```
ParametricPlot[{Newxθ, Newyθ}, {θ, 0, 2 Pi}]
```



```
ParametricPlot[{Newxθ, Newyθ}, {θ, 0, 2 Pi},
 PlotStyle → Directive[Black, Thickness[.0038]],
 AxesStyle → Directive[30, Thick, Black],
 LabelStyle → Directive[30, Black, Italic], ImageSize → Large]
```



■ Simplification of the Plot

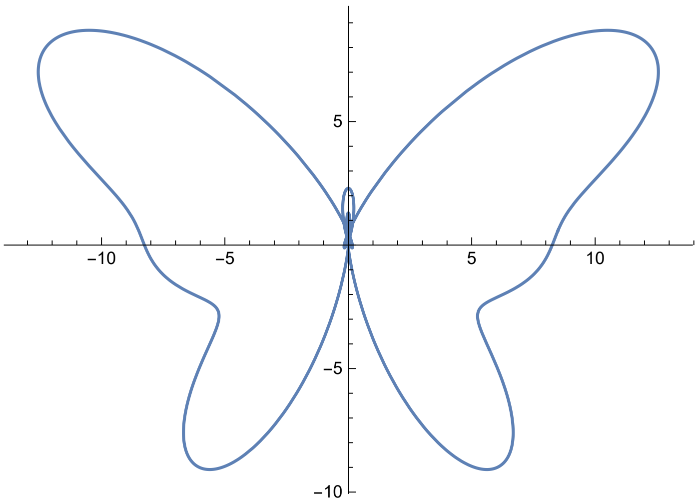
```
Newxθ = Sum[Lk[[i]] Cos[psik[[i]] + i θ], {i, 1, 8}] +
  Sum[Mk[[j]] Cos[etak[[j]] - j θ], {j, 1, 6}]
```

```
0. + 1.25 Cos[1.5708 - 6 θ] + 2.4 Cos[1.5708 - 4 θ] + 1.25 Cos[3.14159 - 3 θ] +
 1.15 Cos[1.5708 - 2 θ] + 9. Cos[θ] + 0.5 Cos[1.5708 + 2 θ] + 2. Cos[0. + 3 θ] +
 1.25 Cos[3.14159 + 5 θ] + 1. Cos[1.5708 + 6 θ] + 1.1 Cos[1.5708 + 8 θ]
```

```
Newyθ = Sum[Lk[[i]] Sin[psik[[i]] + i θ], {i, 1, 8}] +
  Sum[Mk[[j]] Sin[etak[[j]] - j θ], {j, 1, 6}]
```

```
0. - 1.25 Sin[1.5708 - 6 θ] + 0.1 Sin[1.5708 - 4 θ] + 1.25 Sin[3.14159 - 3 θ] +
 1.15 Sin[1.5708 - 2 θ] + 5. Sin[θ] + 0.5 Sin[1.5708 + 2 θ] + 2. Sin[0. + 3 θ] +
 1.25 Sin[3.14159 + 5 θ] - 1. Sin[1.5708 + 6 θ] + 1.1 Sin[1.5708 + 8 θ]
```

```
ParametricPlot[{Newxθ, Newyθ}, {θ, 0, 2 Pi}]
```



Further Simplification of the Plot

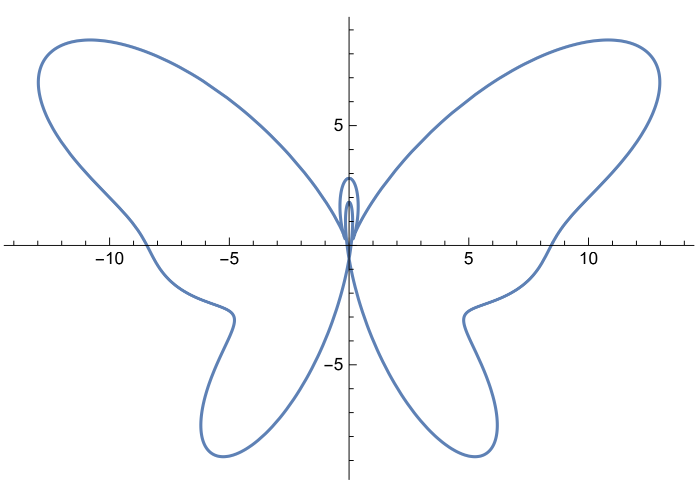
```
Newxθ = Sum[Lk[[i]] Cos[psik[[i]] + i θ], {i, 1, 8}] +
  Sum[Mk[[j]] Cos[etak[[j]] - j θ], {j, 1, 6}] - Lk[[2]] Cos[psik[[2]] + 2 θ]
```

```
0. + 1.25 Cos[1.5708 - 6 θ] + 2.4 Cos[1.5708 - 4 θ] +
 1.25 Cos[3.14159 - 3 θ] + 1.15 Cos[1.5708 - 2 θ] + 9. Cos[θ] + 2. Cos[0. + 3 θ] +
 1.25 Cos[3.14159 + 5 θ] + 1. Cos[1.5708 + 6 θ] + 1.1 Cos[1.5708 + 8 θ]
```

```
Newyθ = Sum[Lk[[i]] Sin[psik[[i]] + i θ], {i, 1, 8}] +
  Sum[Mk[[j]] Sin[etak[[j]] - j θ], {j, 1, 6}] - Lk[[2]] Sin[psik[[2]] + 2 θ]
```

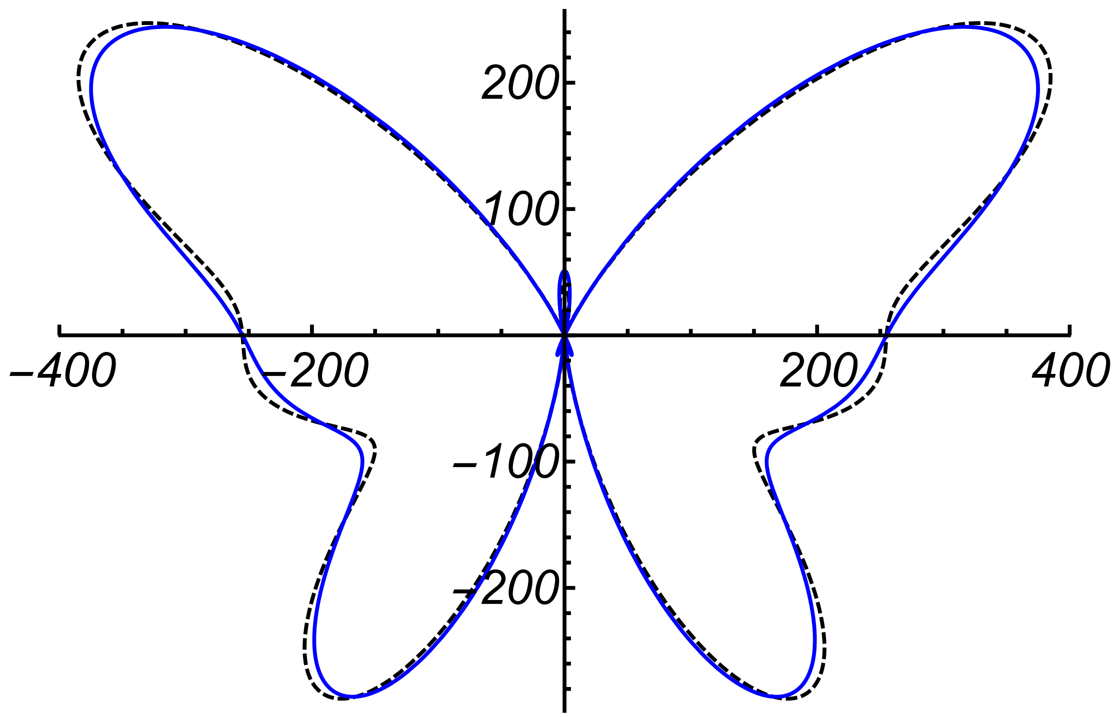
```
0. - 1.25 Sin[1.5708 - 6 θ] + 0.1 Sin[1.5708 - 4 θ] +
 1.25 Sin[3.14159 - 3 θ] + 1.15 Sin[1.5708 - 2 θ] + 5. Sin[θ] + 2. Sin[0. + 3 θ] +
 1.25 Sin[3.14159 + 5 θ] - 1. Sin[1.5708 + 6 θ] + 1.1 Sin[1.5708 + 8 θ]
```

```
ParametricPlot[{Newxθ, Newyθ}, {θ, 0, 2 Pi}]
```

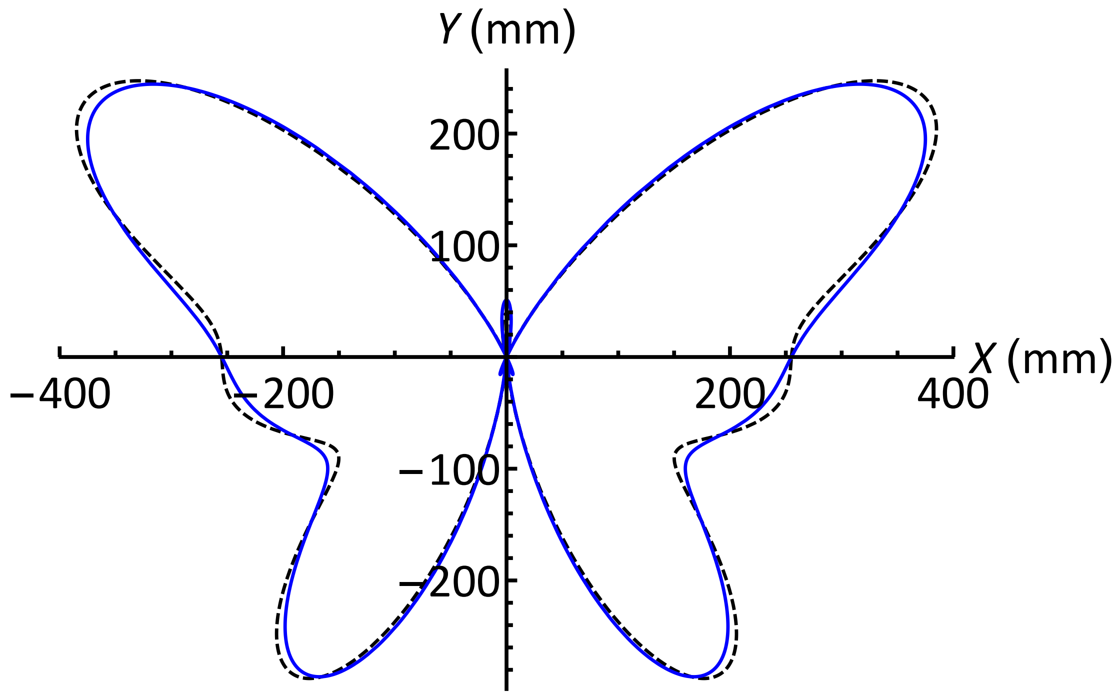


Compare Initial plot and Truncated plot

```
Show[InitialPlot, TruncatePlot2]
```



```
Show[%5, AxesLabel → {HoldForm[X " (mm) "], HoldForm[Y " (mm) "]},
 PlotLabel → None, LabelStyle → {FontFamily → "Calibri", 14, GrayLevel[0]}]
```




```
OriginalSize = Sqrt[Newxθ1^2 + Newyθ1^2]
```

```
√((0. + 1.25 Cos[1.5708 - 6 θ] + 2.4 Cos[1.5708 - 4 θ] +
      1.25 Cos[3.14159 - 3 θ] + 1.15 Cos[1.5708 - 2 θ] + 9. Cos[θ] +
      0.5 Cos[1.5708 + 2 θ] + 2. Cos[0. + 3 θ] + 1.25 Cos[3.14159 + 5 θ] +
      1. Cos[1.5708 + 6 θ] + 1.2 Cos[1.5708 + 8 θ] + 0.2 Cos[1.5708 + 10 θ])^2 +
   (0. - 1.25 Sin[1.5708 - 6 θ] + 0.1 Sin[1.5708 - 4 θ] + 1.25 Sin[3.14159 - 3 θ] +
      1.15 Sin[1.5708 - 2 θ] + 5. Sin[θ] + 0.5 Sin[1.5708 + 2 θ] +
      2. Sin[0. + 3 θ] + 1.25 Sin[3.14159 + 5 θ] - 1. Sin[1.5708 + 6 θ] +
      0.8 Sin[1.5708 + 8 θ] + 0.2 Sin[1.5708 + 10 θ])^2)
```

```
MaxValue[OriginalSize, θ]
```

```
14.9768
```

```
TruncateTermX = Newxθ1 - Newxθ3
```

```
0. + 0.5 Cos[1.5708 + 2 θ] + 0.2 Cos[1.5708 + 8 θ] + 0.2 Cos[1.5708 + 10 θ]
```

```
TruncateTermY = Newyθ1 - Newyθ3
```

```
0. + 0.5 Sin[1.5708 + 2 θ] - 0.2 Sin[1.5708 + 8 θ] + 0.2 Sin[1.5708 + 10 θ]
```

```
Distance = Sqrt[TruncateTermX^2 + TruncateTermY^2]
```

```
√((0. + 0.5 Cos[1.5708 + 2 θ] + 0.2 Cos[1.5708 + 8 θ] + 0.2 Cos[1.5708 + 10 θ])^2 +
   (0. + 0.5 Sin[1.5708 + 2 θ] - 0.2 Sin[1.5708 + 8 θ] + 0.2 Sin[1.5708 + 10 θ])^2)
```

```
MaxValue[Distance, θ]
0.9
```

D Whale Bezier Curve Synthesis Mathematica Code

Below is the Mathematica code for the computation of linkage system to draw whale Bezier Curve.

Cubic Trigonometric Bezier Curve_V5_Whale

Cubic Trigonometric Bezier Basis Functions

```
b0t = (1 - λ Sin[Pi t / 2]) (1 -  Sin[Pi t / 2]) ^2
```

$$\left(1 - \text{Sin}\left[\frac{\pi\, t}{2}\right]\right)^2 \left(1 - \lambda\, \text{Sin}\left[\frac{\pi\, t}{2}\right]\right)$$

```
b1t = Sin[Pi t / 2] (1 - Sin[Pi t / 2]) (2 + λ (1 - Sin[Pi t / 2]))
```

$$\left(2 + \lambda \left(1 - \text{Sin}\left[\frac{\pi\, t}{2}\right]\right)\right) \left(1 - \text{Sin}\left[\frac{\pi\, t}{2}\right]\right) \text{Sin}\left[\frac{\pi\, t}{2}\right]$$

```
b2t = Cos[Pi t / 2] (1 - Cos[Pi t / 2]) (2 + λ (1 - Cos[Pi t / 2]))
```

$$\left(2 + \lambda \left(1 - \text{Cos}\left[\frac{\pi\, t}{2}\right]\right)\right) \left(1 - \text{Cos}\left[\frac{\pi\, t}{2}\right]\right) \text{Cos}\left[\frac{\pi\, t}{2}\right]$$

```
b3t = (1 - λ Cos[Pi t / 2]) (1 -  Cos[Pi t / 2]) ^2
```

$$\left(1 - \text{Cos}\left[\frac{\pi\, t}{2}\right]\right)^2 \left(1 - \lambda\, \text{Cos}\left[\frac{\pi\, t}{2}\right]\right)$$

Define the number of segments

```
n = 4;
```

Define Control Points

```
P0 = {{575.09, 328.26}, {575.09, 328.26}, {578.89, 365.72}, {578.89, 365.72}};
P1 = {{375.59, 85.09}, {570.70, 226.05}, {524.49, 397.62}, {640.27, 405.68}};
P2 = {{199.34, 607.75}, {496.00, 47.25}, {524.04, 359.18}, {614.44, 357.36}};
P3 = {{153.85, 193.39}, {153.85, 193.39}, {575.09, 328.26}, {575.09, 328.26}};
```

Cubic Triginometric Bezier Curve

```
rt = Table[b0t P0[[k]] + b1t P1[[k]]  + b2t P2[[k]] + b3t P3[[k]], {k, 1, n}]
```

$$\Big\{\Big\{199.34\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+$$

$$153.85\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+375.59\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+575.09\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right),$$

$$607.75\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+$$

$$193.39\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+85.09\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+328.26\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)\Big\},$$

$$\Big\{496.\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+$$

$$153.85\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+570.7\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+575.09\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right),$$

$$47.25\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+$$

$$193.39\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+226.05\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+328.26\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)\Big\},$$

$$\Big\{524.04\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+$$

$$575.09\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+524.49\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+578.89\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right),$$

$$359.18\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+$$

$$328.26\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+397.62\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+365.72\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)\Big\},$$

$$\Big\{614.44\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+$$

$$575.09\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+640.27\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+578.89\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right),$$

$$357.36\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+$$

$$328.26\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+405.68\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+365.72\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)\Big\}\Big\}$$

```
rtx = Table[TrigReduce[rt[[k, 1]]], {k, 1, n}]
```

$$\{518.48 + 154.01\lambda + 90.98\,\text{Cos}[\tfrac{\pi t}{2}] + 79.6075\lambda\,\text{Cos}[\tfrac{\pi t}{2}] - 34.37\,\text{Cos}[\pi t] - 244.99\lambda\,\text{Cos}[\pi t] + 11.3725\lambda\,\text{Cos}[\tfrac{3\pi t}{2}] - 399.\,\text{Sin}[\tfrac{\pi t}{2}] - 349.125\lambda\,\text{Sin}[\tfrac{\pi t}{2}] + 49.875\lambda\,\text{Sin}[\tfrac{3\pi t}{2}],\ 26.71 - 337.76\lambda + 684.3\,\text{Cos}[\tfrac{\pi t}{2}] + 598.763\lambda\,\text{Cos}[\tfrac{\pi t}{2}] - 135.92\,\text{Cos}[\pi t] - 346.54\lambda\,\text{Cos}[\pi t] + 85.5375\lambda\,\text{Cos}[\tfrac{3\pi t}{2}] - 8.78\,\text{Sin}[\tfrac{\pi t}{2}] - 7.6825\lambda\,\text{Sin}[\tfrac{\pi t}{2}] + 1.0975\lambda\,\text{Sin}[\tfrac{3\pi t}{2}],\ 682.44 + 105.45\lambda - 102.1\,\text{Cos}[\tfrac{\pi t}{2}] - 89.3375\lambda\,\text{Cos}[\tfrac{\pi t}{2}] - 1.45\,\text{Cos}[\pi t] - 3.35\lambda\,\text{Cos}[\pi t] - 12.7625\lambda\,\text{Cos}[\tfrac{3\pi t}{2}] - 108.8\,\text{Sin}[\tfrac{\pi t}{2}] - 95.2\lambda\,\text{Sin}[\tfrac{\pi t}{2}] + 13.6\lambda\,\text{Sin}[\tfrac{3\pi t}{2}],\ 476.26 - 100.73\lambda + 78.7\,\text{Cos}[\tfrac{\pi t}{2}] + 68.8625\lambda\,\text{Cos}[\tfrac{\pi t}{2}] + 23.93\,\text{Cos}[\pi t] + 22.03\lambda\,\text{Cos}[\pi t] + 9.8375\lambda\,\text{Cos}[\tfrac{3\pi t}{2}] + 122.76\,\text{Sin}[\tfrac{\pi t}{2}] + 107.415\lambda\,\text{Sin}[\tfrac{\pi t}{2}] - 15.345\lambda\,\text{Sin}[\tfrac{3\pi t}{2}]\}$$

```
rty = Table[TrigReduce[rt[[k, 2]]], {k, 1, n}]
```

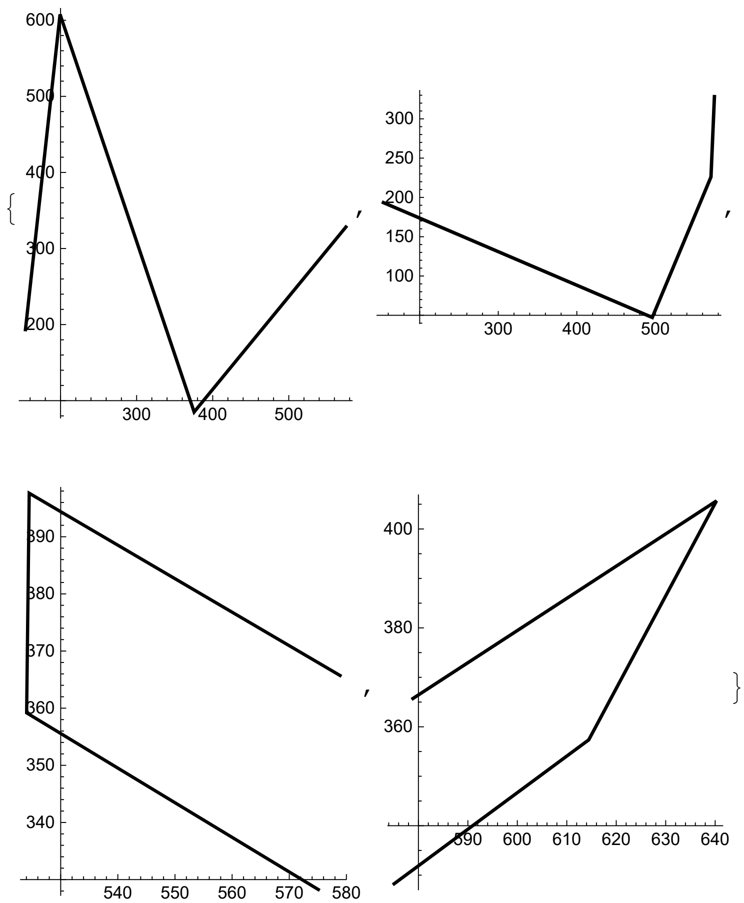
$$\{89.635 - 171.19\lambda + 828.72\,\text{Cos}[\tfrac{\pi t}{2}] + 725.13\lambda\,\text{Cos}[\tfrac{\pi t}{2}] - 590.095\,\text{Cos}[\pi t] - 657.53\lambda\,\text{Cos}[\pi t] + 103.59\lambda\,\text{Cos}[\tfrac{3\pi t}{2}] - 486.34\,\text{Sin}[\tfrac{\pi t}{2}] - 425.547\lambda\,\text{Sin}[\tfrac{\pi t}{2}] + 60.7925\lambda\,\text{Sin}[\tfrac{3\pi t}{2}],\ 509.175 + 248.35\lambda - 292.28\,\text{Cos}[\tfrac{\pi t}{2}] - 255.745\lambda\,\text{Cos}[\tfrac{\pi t}{2}] + 111.365\,\text{Cos}[\pi t] + 43.93\lambda\,\text{Cos}[\pi t] - 36.535\lambda\,\text{Cos}[\tfrac{3\pi t}{2}] - 204.42\,\text{Sin}[\tfrac{\pi t}{2}] - 178.867\lambda\,\text{Sin}[\tfrac{\pi t}{2}] + 25.5525\lambda\,\text{Sin}[\tfrac{3\pi t}{2}],\ 284.17 - 62.82\lambda + 61.84\,\text{Cos}[\tfrac{\pi t}{2}] + 54.11\lambda\,\text{Cos}[\tfrac{\pi t}{2}] + 19.71\,\text{Cos}[\pi t] + 0.98\lambda\,\text{Cos}[\pi t] + 7.73\lambda\,\text{Cos}[\tfrac{3\pi t}{2}] + 63.8\,\text{Sin}[\tfrac{\pi t}{2}] + 55.825\lambda\,\text{Sin}[\tfrac{\pi t}{2}] - 7.975\lambda\,\text{Sin}[\tfrac{3\pi t}{2}],\ 277.93 - 69.06\lambda + 58.2\,\text{Cos}[\tfrac{\pi t}{2}] + 50.925\lambda\,\text{Cos}[\tfrac{\pi t}{2}] + 29.59\,\text{Cos}[\pi t] + 10.86\lambda\,\text{Cos}[\pi t] + 7.275\lambda\,\text{Cos}[\tfrac{3\pi t}{2}] + 79.92\,\text{Sin}[\tfrac{\pi t}{2}] + 69.93\lambda\,\text{Sin}[\tfrac{\pi t}{2}] - 9.99\lambda\,\text{Sin}[\tfrac{3\pi t}{2}]\}$$

Set λ value

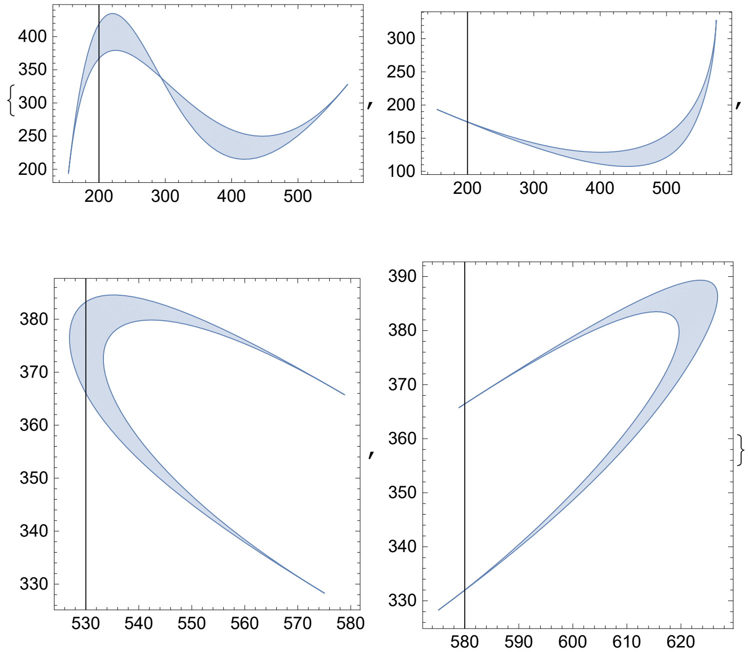
```
λ =.;
```

Curve Plot

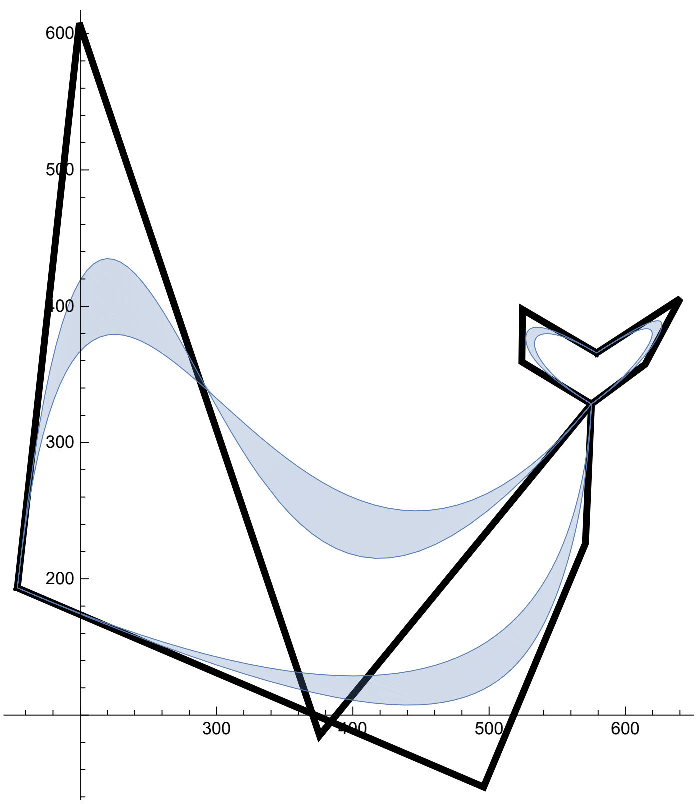
```
PolyG = Table[Graphics[{Thickness[0.01],
    Line[{P0[[k]], P1[[k]], P2[[k]], P3[[k]]}]}, Axes → True], {k, 1, n}]
```



```
Curve = Table[ParametricPlot[{rtx[[k]], rty[[k]]}, {t, 0, 1}, {λ, 0, 1}], {k, 1, n}]
```




```
Show[PolyG, Curve]
```



Set λ value

```
λ = 0.01;
```

Trigonometric Functions for x and y componnets

```
rtx
```

$$\left\{520.02 + 91.7761\,\text{Cos}\left[\frac{\pi t}{2}\right] - 36.8199\,\text{Cos}[\pi t] + 0.113725\,\text{Cos}\left[\frac{3\pi t}{2}\right] - 402.491\,\text{Sin}\left[\frac{\pi t}{2}\right] + 0.49875\,\text{Sin}\left[\frac{3\pi t}{2}\right],\ 23.3324 + 690.288\,\text{Cos}\left[\frac{\pi t}{2}\right] - 139.385\,\text{Cos}[\pi t] + 0.855375\,\text{Cos}\left[\frac{3\pi t}{2}\right] - 8.85682\,\text{Sin}\left[\frac{\pi t}{2}\right] + 0.010975\,\text{Sin}\left[\frac{3\pi t}{2}\right],\ 683.495 - 102.993\,\text{Cos}\left[\frac{\pi t}{2}\right] - 1.4835\,\text{Cos}[\pi t] - 0.127625\,\text{Cos}\left[\frac{3\pi t}{2}\right] - 109.752\,\text{Sin}\left[\frac{\pi t}{2}\right] + 0.136\,\text{Sin}\left[\frac{3\pi t}{2}\right],\ 475.253 + 79.3886\,\text{Cos}\left[\frac{\pi t}{2}\right] + 24.1503\,\text{Cos}[\pi t] + 0.098375\,\text{Cos}\left[\frac{3\pi t}{2}\right] + 123.834\,\text{Sin}\left[\frac{\pi t}{2}\right] - 0.15345\,\text{Sin}\left[\frac{3\pi t}{2}\right]\right\}$$

```
rty
```

$$\left\{87.9231 + 835.971\,\text{Cos}\left[\frac{\pi t}{2}\right] - 596.67\,\text{Cos}[\pi t] + 1.0359\,\text{Cos}\left[\frac{3\pi t}{2}\right] - 490.595\,\text{Sin}\left[\frac{\pi t}{2}\right] + 0.607925\,\text{Sin}\left[\frac{3\pi t}{2}\right],\ 511.658 - 294.837\,\text{Cos}\left[\frac{\pi t}{2}\right] + 111.804\,\text{Cos}[\pi t] - 0.36535\,\text{Cos}\left[\frac{3\pi t}{2}\right] - 206.209\,\text{Sin}\left[\frac{\pi t}{2}\right] + 0.255525\,\text{Sin}\left[\frac{3\pi t}{2}\right],\ 283.542 + 62.3811\,\text{Cos}\left[\frac{\pi t}{2}\right] + 19.7198\,\text{Cos}[\pi t] + 0.0773\,\text{Cos}\left[\frac{3\pi t}{2}\right] + 64.3582\,\text{Sin}\left[\frac{\pi t}{2}\right] - 0.07975\,\text{Sin}\left[\frac{3\pi t}{2}\right],\ 277.239 + 58.7093\,\text{Cos}\left[\frac{\pi t}{2}\right] + 29.6986\,\text{Cos}[\pi t] + 0.07275\,\text{Cos}\left[\frac{3\pi t}{2}\right] + 80.6193\,\text{Sin}\left[\frac{\pi t}{2}\right] - 0.0999\,\text{Sin}\left[\frac{3\pi t}{2}\right]\right\}$$

```
Expand[rtx]
```

$$\left\{520.02 + 91.7761\,\text{Cos}\left[\frac{\pi t}{2}\right] - 36.8199\,\text{Cos}[\pi t] + 0.113725\,\text{Cos}\left[\frac{3\pi t}{2}\right] - 402.491\,\text{Sin}\left[\frac{\pi t}{2}\right] + 0.49875\,\text{Sin}\left[\frac{3\pi t}{2}\right],\ 23.3324 + 690.288\,\text{Cos}\left[\frac{\pi t}{2}\right] - 139.385\,\text{Cos}[\pi t] + 0.855375\,\text{Cos}\left[\frac{3\pi t}{2}\right] - 8.85682\,\text{Sin}\left[\frac{\pi t}{2}\right] + 0.010975\,\text{Sin}\left[\frac{3\pi t}{2}\right],\ 683.495 - 102.993\,\text{Cos}\left[\frac{\pi t}{2}\right] - 1.4835\,\text{Cos}[\pi t] - 0.127625\,\text{Cos}\left[\frac{3\pi t}{2}\right] - 109.752\,\text{Sin}\left[\frac{\pi t}{2}\right] + 0.136\,\text{Sin}\left[\frac{3\pi t}{2}\right],\ 475.253 + 79.3886\,\text{Cos}\left[\frac{\pi t}{2}\right] + 24.1503\,\text{Cos}[\pi t] + 0.098375\,\text{Cos}\left[\frac{3\pi t}{2}\right] + 123.834\,\text{Sin}\left[\frac{\pi t}{2}\right] - 0.15345\,\text{Sin}\left[\frac{3\pi t}{2}\right]\right\}$$

```
Expand[rty]
```

$\{87.9231 + 835.971\,\text{Cos}[\frac{\pi t}{2}] - 596.67\,\text{Cos}[\pi t] + 1.0359\,\text{Cos}[\frac{3\pi t}{2}] - 490.595\,\text{Sin}[\frac{\pi t}{2}] + 0.607925\,\text{Sin}[\frac{3\pi t}{2}],\ 511.658 - 294.837\,\text{Cos}[\frac{\pi t}{2}] + 111.804\,\text{Cos}[\pi t] - 0.36535\,\text{Cos}[\frac{3\pi t}{2}] - 206.209\,\text{Sin}[\frac{\pi t}{2}] + 0.255525\,\text{Sin}[\frac{3\pi t}{2}],\ 283.542 + 62.3811\,\text{Cos}[\frac{\pi t}{2}] + 19.7198\,\text{Cos}[\pi t] + 0.0773\,\text{Cos}[\frac{3\pi t}{2}] + 64.3582\,\text{Sin}[\frac{\pi t}{2}] - 0.07975\,\text{Sin}[\frac{3\pi t}{2}],\ 277.239 + 58.7093\,\text{Cos}[\frac{\pi t}{2}] + 29.6986\,\text{Cos}[\pi t] + 0.07275\,\text{Cos}[\frac{3\pi t}{2}] + 80.6193\,\text{Sin}[\frac{\pi t}{2}] - 0.0999\,\text{Sin}[\frac{3\pi t}{2}]\}$

Note that because t∈[0,1], thus $\theta \in [0, \pi/2]$

```
xθ = Table[rtx[[k]] /. { Pi t → 2 θ}, {k, 1, n}]
```

```
{520.02 + 91.7761 Cos[θ] - 36.8199 Cos[2 θ] + 0.113725 Cos[3 θ] -
  402.491 Sin[θ] + 0.49875 Sin[3 θ], 23.3324 + 690.288 Cos[θ] -
  139.385 Cos[2 θ] + 0.855375 Cos[3 θ] - 8.85682 Sin[θ] + 0.010975 Sin[3 θ],
 683.495 - 102.993 Cos[θ] - 1.4835 Cos[2 θ] - 0.127625 Cos[3 θ] -
  109.752 Sin[θ] + 0.136 Sin[3 θ], 475.253 + 79.3886 Cos[θ] +
  24.1503 Cos[2 θ] + 0.098375 Cos[3 θ] + 123.834 Sin[θ] - 0.15345 Sin[3 θ]}
```

```
yθ = Table[rty[[k]] /. {Pi t → 2 θ}, {k, 1, n}]
```

```
{87.9231 + 835.971 Cos[θ] - 596.67 Cos[2 θ] + 1.0359 Cos[3 θ] -
  490.595 Sin[θ] + 0.607925 Sin[3 θ], 511.658 - 294.837 Cos[θ] +
  111.804 Cos[2 θ] - 0.36535 Cos[3 θ] - 206.209 Sin[θ] + 0.255525 Sin[3 θ],
 283.542 + 62.3811 Cos[θ] + 19.7198 Cos[2 θ] + 0.0773 Cos[3 θ] +
  64.3582 Sin[θ] - 0.07975 Sin[3 θ], 277.239 + 58.7093 Cos[θ] +
  29.6986 Cos[2 θ] + 0.07275 Cos[3 θ] + 80.6193 Sin[θ] - 0.0999 Sin[3 θ]}
```

```
Expand[xθ]
```

```
{520.02 + 91.7761 Cos[θ] - 36.8199 Cos[2 θ] + 0.113725 Cos[3 θ] -
  402.491 Sin[θ] + 0.49875 Sin[3 θ], 23.3324 + 690.288 Cos[θ] -
  139.385 Cos[2 θ] + 0.855375 Cos[3 θ] - 8.85682 Sin[θ] + 0.010975 Sin[3 θ],
 683.495 - 102.993 Cos[θ] - 1.4835 Cos[2 θ] - 0.127625 Cos[3 θ] -
  109.752 Sin[θ] + 0.136 Sin[3 θ], 475.253 + 79.3886 Cos[θ] +
  24.1503 Cos[2 θ] + 0.098375 Cos[3 θ] + 123.834 Sin[θ] - 0.15345 Sin[3 θ]}
```

```
Expand[yθ]
```

```
{87.9231 + 835.971 Cos[θ] - 596.67 Cos[2 θ] + 1.0359 Cos[3 θ] -
   490.595 Sin[θ] + 0.607925 Sin[3 θ], 511.658 - 294.837 Cos[θ] +
   111.804 Cos[2 θ] - 0.36535 Cos[3 θ] - 206.209 Sin[θ] + 0.255525 Sin[3 θ],
  283.542 + 62.3811 Cos[θ] + 19.7198 Cos[2 θ] + 0.0773 Cos[3 θ] +
   64.3582 Sin[θ] - 0.07975 Sin[3 θ], 277.239 + 58.7093 Cos[θ] +
   29.6986 Cos[2 θ] + 0.07275 Cos[3 θ] + 80.6193 Sin[θ] - 0.0999 Sin[3 θ]}
```

Single coupled serial chain coefficients

```
ak = Table[Table[Coefficient[xθ[[k]], Cos[i θ]], {i, 1, 3}], {k, 1, n}]
```

```
{{91.7761, -36.8199, 0.113725}, {690.288, -139.385, 0.855375},
 {-102.993, -1.4835, -0.127625}, {79.3886, 24.1503, 0.098375}}
```

```
bk = Table[Coefficient[xθ[[k]], Sin[i θ]], {k, 1, n}, {i, 1, 3}]
```

```
{{-402.491, 0, 0.49875}, {-8.85682, 0, 0.010975},
 {-109.752, 0, 0.136}, {123.834, 0, -0.15345}}
```

```
ck = Table[Coefficient[yθ[[k]], Cos[i θ]], {k, 1, n}, {i, 1, 3}]
```

```
{{835.971, -596.67, 1.0359}, {-294.837, 111.804, -0.36535},
 {62.3811, 19.7198, 0.0773}, {58.7093, 29.6986, 0.07275}}
```

```
dk = Table[Coefficient[yθ[[k]], Sin[i θ]], {k, 1, n}, {i, 1, 3}]
```

```
{{-490.595, 0, 0.607925}, {-206.209, 0, 0.255525},
 {64.3582, 0, -0.07975}, {80.6193, 0, -0.0999}}
```

Link Lengths and Phase angles

```
Lk = Table[0.5 * Sqrt[(ak[[k]] + dk[[k]])^2 + (ck[[k]] - bk[[k]])^2], {k, 1, n}]
```

```
{{650.547, 298.903, 0.449808}, {281.122, 89.3427, 0.586455},
 {88.2078, 9.88776, 0.107761}, {86.3768, 19.1393, 0.113103}}
```

```
psik = Table[ArcTan[ak[[k]] + dk[[k]], ck[[k]] - bk[[k]]], {k, 1, n}]
```

```
{{1.88234, -1.63243, 0.639866}, {-0.533607, 2.46556, -0.326624},
 {1.79159, 1.64588, -2.86575}, {-0.386535, 0.88807, 1.57754}}
```

```
Mk = Table[0.5 * Sqrt[(ak[[k]] - dk[[k]])^2 + (ck[[k]] + bk[[k]])^2], {k, 1, n}]
```

```
{{362.995, 298.903, 0.80613}, {473.269, 89.3427, 0.348354},
 {86.9635, 9.88776, 0.109303}, {91.2738, 19.1393, 0.107034}}
```

```
etak = Table[ArcTan[ak[[k]] - dk[[k]], ck[[k]] + bk[[k]]], {k, 1, n}]
```

```
{{0.639866, -1.63243, 1.88234}, {-0.326624, 2.46556, -0.533607},
 {-2.86575, 1.64588, 1.79159}, {1.57754, 0.88807, -0.386535}}
```

Ground pivot coefficients

```
a0 = Table[xθ[[k]] - Sum[ak[[k, i]] Cos[i θ] + bk[[k, i]] Sin[i θ], {i, 1, 3}], {k, 1, n}]
```

```
{520.02, 23.3324, 683.495, 475.253}
```

```
b0 = Table[0 * k, {k, 1, n}]
```

```
{0, 0, 0, 0}
```

```
c0 = Table[yθ[[k]] - Sum[ck[[k, i]] Cos[i θ] + dk[[k, i]] Sin[i θ], {i, 1, 3}], {k, 1, n}]
```

```
{87.9231, 511.658, 283.542, 277.239}
```

```
d0 = Table[0 * k, {k, 1, n}]
```

```
{0, 0, 0, 0}
```

Ground pivot position Config.

```
L0 = Table[0.5 * Sqrt[(a0[[k]] + d0[[k]])^2 + (c0[[k]] - b0[[k]])^2], {k, 1, n}]
```

```
{263.7, 256.095, 369.987, 275.103}
```

```
psi0 = Table[ArcTan[a0[[k]] + d0[[k]], c0[[k]] - b0[[k]]], {k, 1, n}]
```

```
{0.167492, 1.52523, 0.393235, 0.528088}
```

```
M0 = Table[0.5 * Sqrt[(a0[[k]] - d0[[k]])^2 + (c0[[k]] + b0[[k]])^2], {k, 1, n}]
```

```
{263.7, 256.095, 369.987, 275.103}
```

```
eta0 = Table[ArcTan[a0[[k]] - d0[[k]], c0[[k]] + b0[[k]]], {k, 1, n}]
```

```
{0.167492, 1.52523, 0.393235, 0.528088}
```

Plot the curve

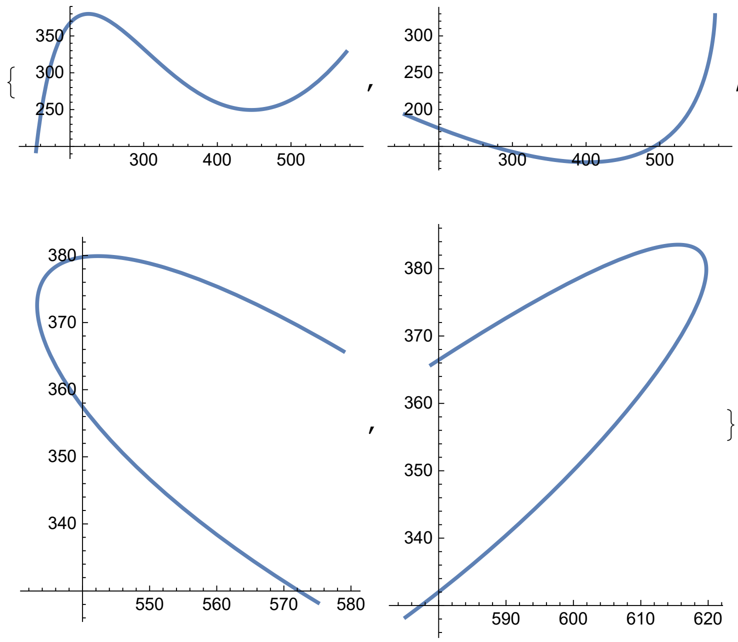
```
Newxθ = Table[
  Sum[Lk[[k, i]] Cos[psik[[k, i]] + i θ] + Mk[[k, i]] Cos[etak[[k, i]] - i θ], {i, 1, 3}] +
   L0 [[k]] Cos[psi0[[k]]] + M0[[k]] Cos[eta0[[k]]], {k, 1, n}]
```

```
{520.02 + 0.80613 Cos[1.88234 - 3 θ] + 298.903 Cos[1.63243 - 2 θ] +
  362.995 Cos[0.639866 - θ] + 650.547 Cos[1.88234 + θ] +
  298.903 Cos[1.63243 + 2 θ] + 0.449808 Cos[0.639866 + 3 θ],
 23.3324 + 0.586455 Cos[0.326624 - 3 θ] + 89.3427 Cos[2.46556 - 2 θ] +
  281.122 Cos[0.533607 - θ] + 473.269 Cos[0.326624 + θ] +
  89.3427 Cos[2.46556 + 2 θ] + 0.348354 Cos[0.533607 + 3 θ],
 683.495 + 0.109303 Cos[1.79159 - 3 θ] + 0.107761 Cos[2.86575 - 3 θ] +
  9.88776 Cos[1.64588 - 2 θ] + 88.2078 Cos[1.79159 + θ] +
  86.9635 Cos[2.86575 + θ] + 9.88776 Cos[1.64588 + 2 θ],
 475.253 + 19.1393 Cos[0.88807 - 2 θ] + 86.3768 Cos[0.386535 - θ] +
  91.2738 Cos[1.57754 - θ] + 19.1393 Cos[0.88807 + 2 θ] +
  0.107034 Cos[0.386535 + 3 θ] + 0.113103 Cos[1.57754 + 3 θ]}
```

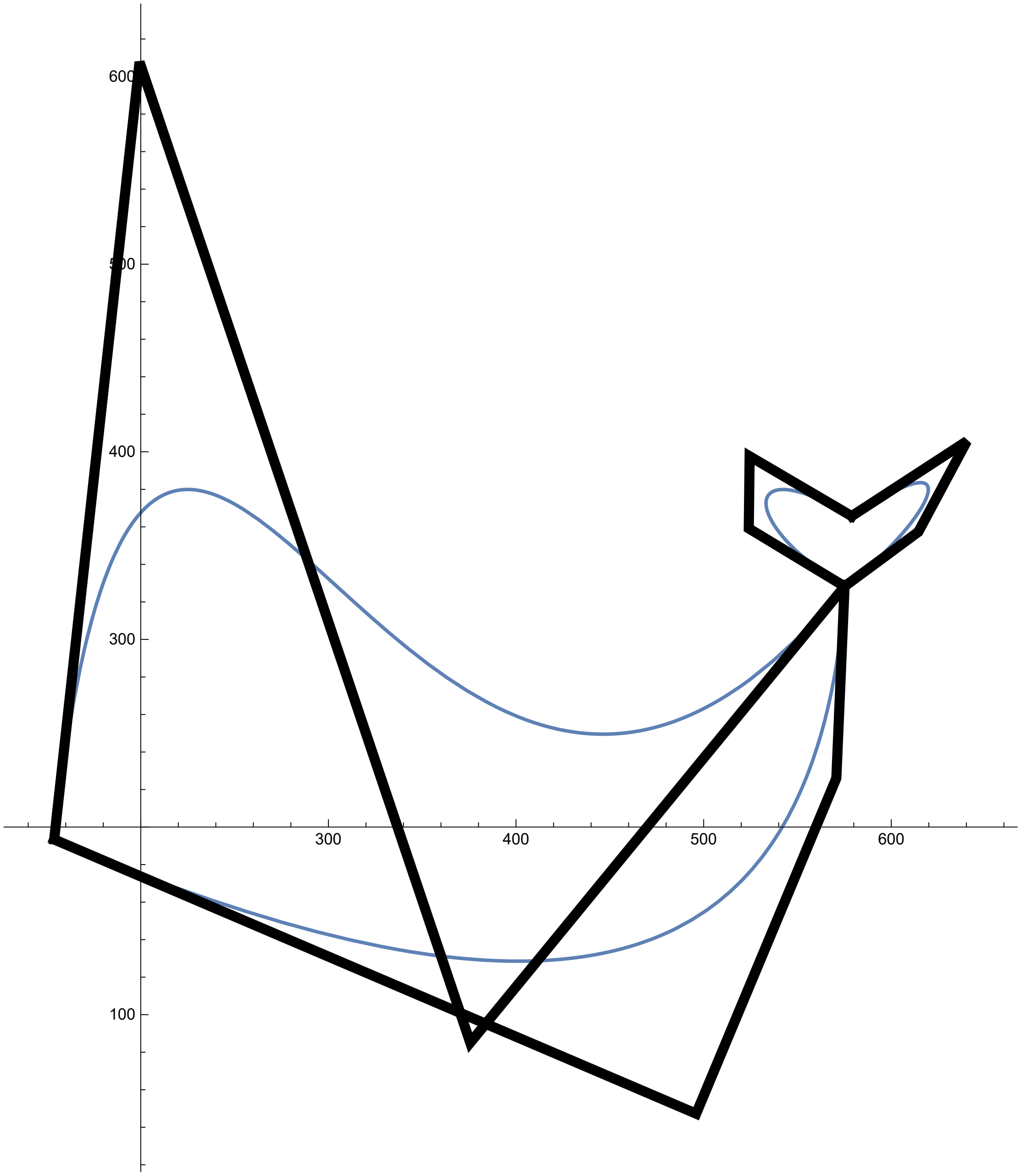
```
Newyθ = Table[
  Sum[Lk[[k, i]] Sin[psik[[k, i]] + i θ] + Mk[[k, i]] Sin[etak[[k, i]] - i θ], {i, 1, 3}] +
   L0[[k]] Sin[psi0[[k]]] + M0[[k]] Sin[eta0[[k]]], {k, 1, n}]
```

```
{87.9231 + 0.80613 Sin[1.88234 - 3 θ] - 298.903 Sin[1.63243 - 2 θ] +
  362.995 Sin[0.639866 - θ] + 650.547 Sin[1.88234 + θ] -
  298.903 Sin[1.63243 + 2 θ] + 0.449808 Sin[0.639866 + 3 θ],
 511.658 - 0.586455 Sin[0.326624 - 3 θ] + 89.3427 Sin[2.46556 - 2 θ] -
  281.122 Sin[0.533607 - θ] - 473.269 Sin[0.326624 + θ] +
  89.3427 Sin[2.46556 + 2 θ] - 0.348354 Sin[0.533607 + 3 θ],
 283.542 + 0.109303 Sin[1.79159 - 3 θ] - 0.107761 Sin[2.86575 - 3 θ] +
  9.88776 Sin[1.64588 - 2 θ] + 88.2078 Sin[1.79159 + θ] -
  86.9635 Sin[2.86575 + θ] + 9.88776 Sin[1.64588 + 2 θ],
 277.239 + 19.1393 Sin[0.88807 - 2 θ] - 86.3768 Sin[0.386535 - θ] +
  91.2738 Sin[1.57754 - θ] + 19.1393 Sin[0.88807 + 2 θ] -
  0.107034 Sin[0.386535 + 3 θ] + 0.113103 Sin[1.57754 + 3 θ]}
```

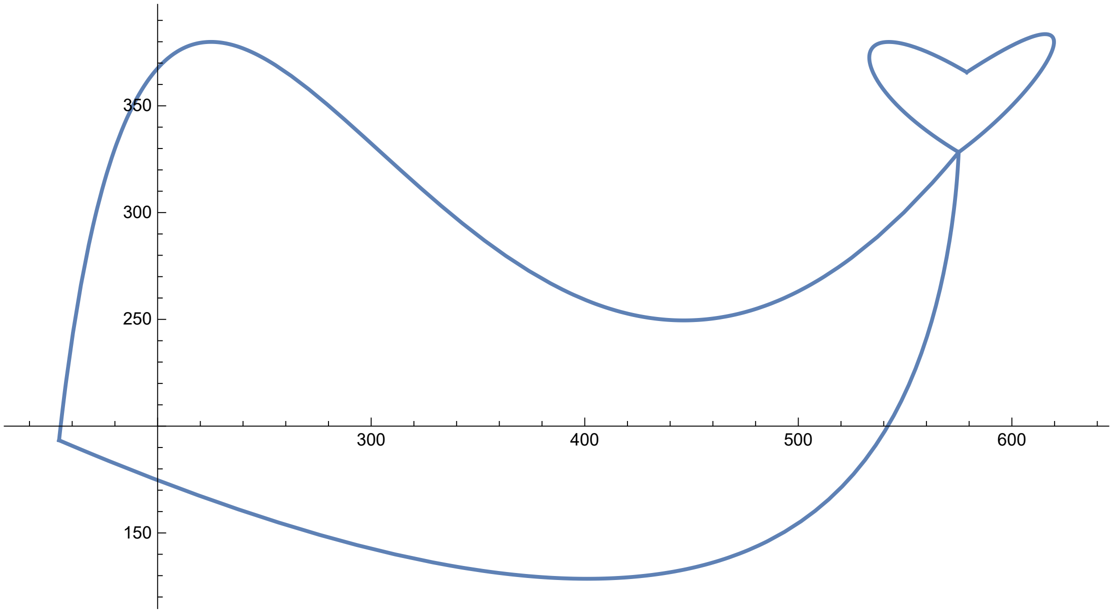
```
Segm = Table[ParametricPlot[
   {Newxθ[[k]], Newyθ[[k]]}, {θ, 0, Pi / 2}, PlotStyle → Thick], {k, 1, n}]
```



```
Show[Segm, PolyG, PlotRange → All, ImageSize → Large]
```



```
Show[Segm, PlotRange → All, ImageSize → Large]
```



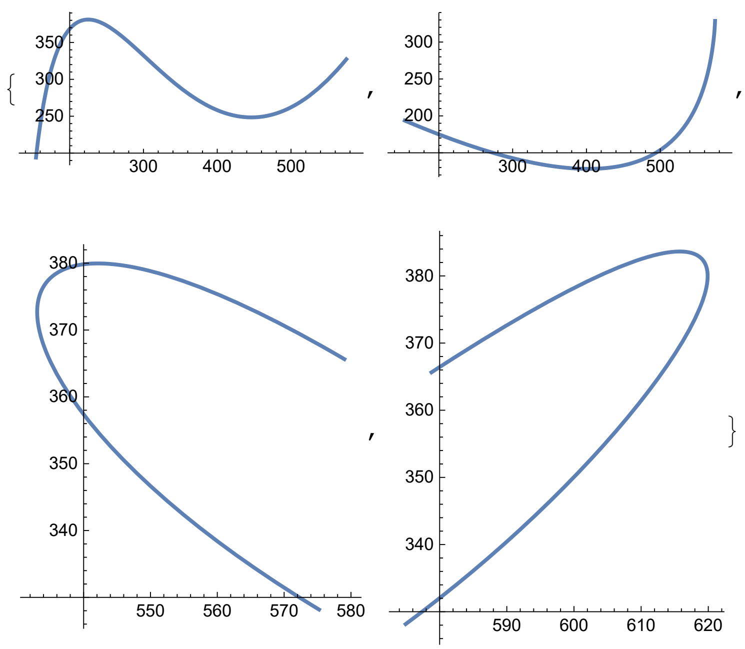
Simplification by truncating the 3rd Term

```
Newxθ = Table[
  Sum[Lk[[k, i]] Cos[psik[[k, i]] + i θ] + Mk[[k, i]] Cos[etak[[k, i]] - i θ], {i, 1, 2}] +
   L0 [[k]] Cos[psi0[[k]]] + M0[[k]] Cos[eta0[[k]]], {k, 1, n}]
```

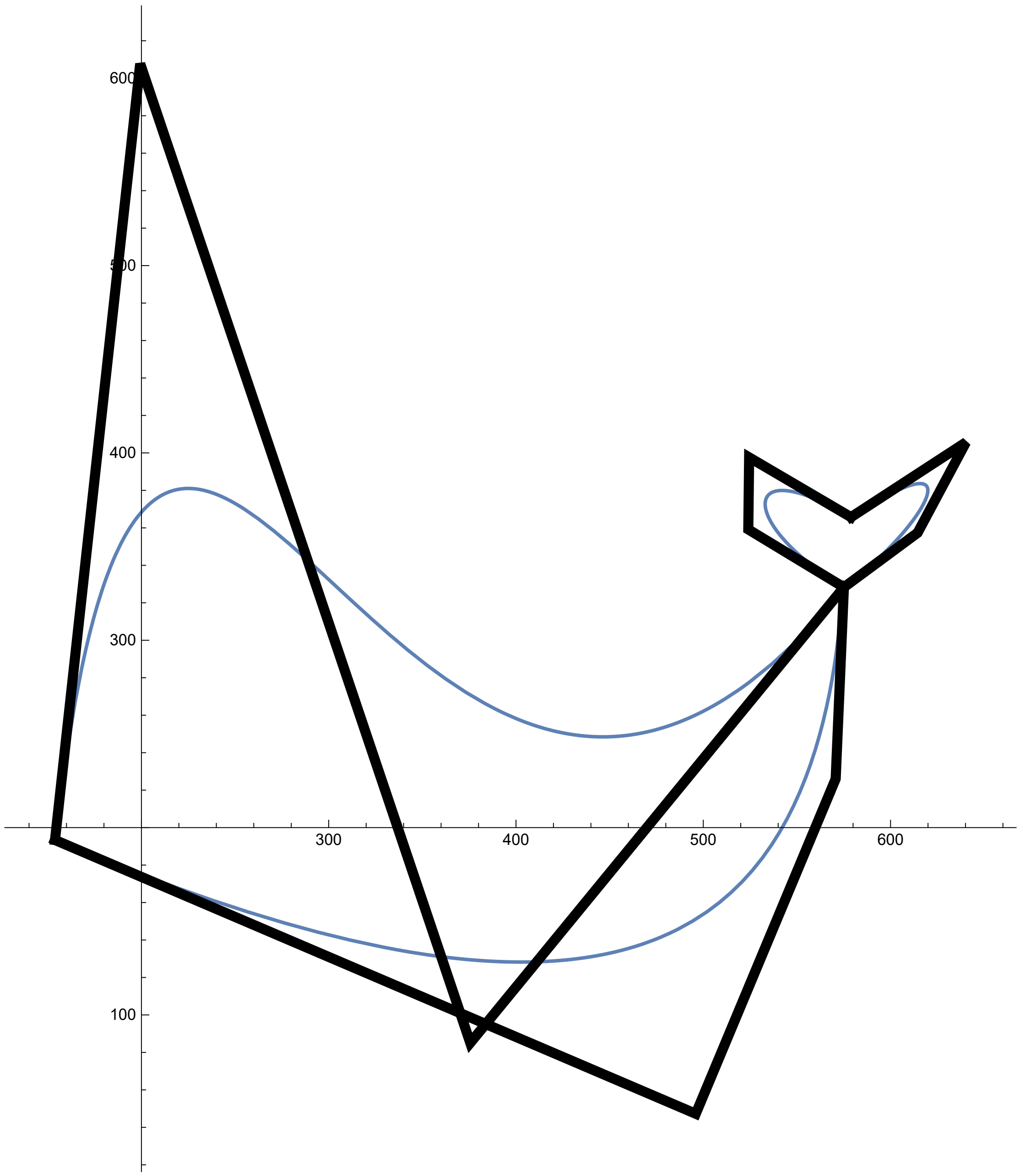
```
{520.02 + 298.903 Cos[1.63243 - 2 θ] + 362.995 Cos[0.639866 - θ] +
  650.547 Cos[1.88234 + θ] + 298.903 Cos[1.63243 + 2 θ],
 23.3324 + 89.3427 Cos[2.46556 - 2 θ] + 281.122 Cos[0.533607 - θ] +
  473.269 Cos[0.326624 + θ] + 89.3427 Cos[2.46556 + 2 θ],
 683.495 + 9.88776 Cos[1.64588 - 2 θ] + 88.2078 Cos[1.79159 + θ] +
  86.9635 Cos[2.86575 + θ] + 9.88776 Cos[1.64588 + 2 θ],
 475.253 + 19.1393 Cos[0.88807 - 2 θ] + 86.3768 Cos[0.386535 - θ] +
  91.2738 Cos[1.57754 - θ] + 19.1393 Cos[0.88807 + 2 θ]}
```

```
Newyθ = Table[
  Sum[Lk[[k, i]] Sin[psik[[k, i]] + i θ] + Mk[[k, i]] Sin[etak[[k, i]] - i θ], {i, 1, 2}] +
   L0[[k]] Sin[psi0[[k]]] + M0[[k]] Sin[eta0[[k]]], {k, 1, n}]
{87.9231 - 298.903 Sin[1.63243 - 2 θ] + 362.995 Sin[0.639866 - θ] +
  650.547 Sin[1.88234 + θ] - 298.903 Sin[1.63243 + 2 θ],
 511.658 + 89.3427 Sin[2.46556 - 2 θ] - 281.122 Sin[0.533607 - θ] -
  473.269 Sin[0.326624 + θ] + 89.3427 Sin[2.46556 + 2 θ],
 283.542 + 9.88776 Sin[1.64588 - 2 θ] + 88.2078 Sin[1.79159 + θ] -
  86.9635 Sin[2.86575 + θ] + 9.88776 Sin[1.64588 + 2 θ],
 277.239 + 19.1393 Sin[0.88807 - 2 θ] - 86.3768 Sin[0.386535 - θ] +
  91.2738 Sin[1.57754 - θ] + 19.1393 Sin[0.88807 + 2 θ]}
```

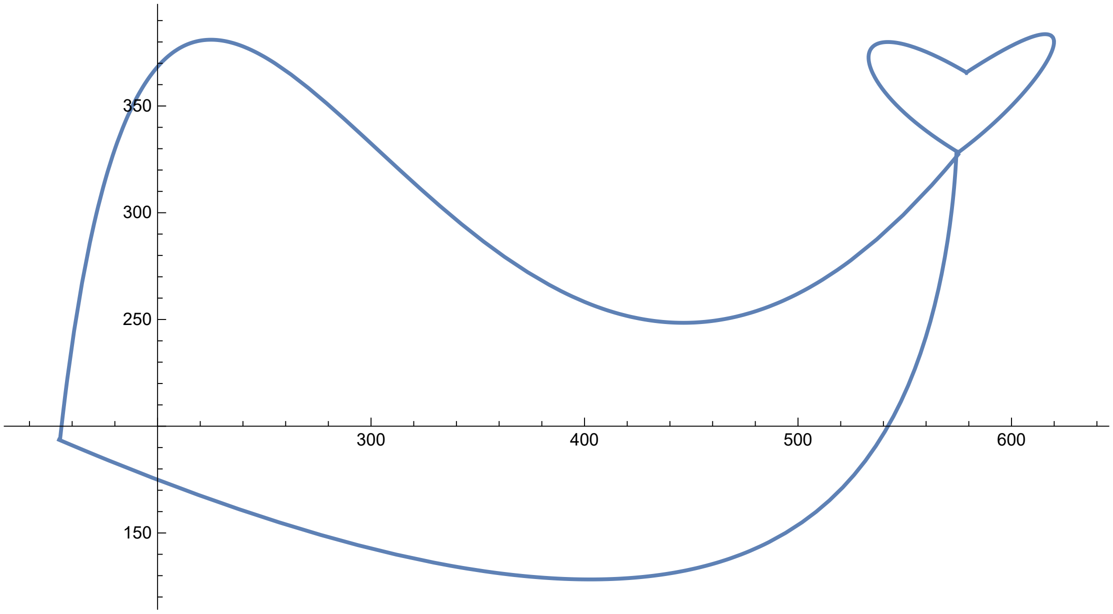
```
Segm = Table[ParametricPlot[
   {Newxθ[[k]], Newyθ[[k]]}, {θ, 0, Pi / 2}, PlotStyle → Thick], {k, 1, n}]
```



```
Show[Segm, PolyG, PlotRange → All, ImageSize → Large]
```



```
Show[Segm, PlotRange → All, ImageSize → Large]
```



E Name Bezier Curve Synthesis Mathematica Code

Below is the Mathematica code for the computation of linkage system to sign name.

Sign Your Name using Cubic Trigonometric Bezier

Cubic Trigonometric Bezier Basis Functions

```
b0t = (1 - λ Sin[Pi t / 2]) (1 -  Sin[Pi t / 2]) ^2
```

$$\left(1 - \text{Sin}\left[\frac{\pi t}{2}\right]\right)^2 \left(1 - \lambda\, \text{Sin}\left[\frac{\pi t}{2}\right]\right)$$

```
b1t = Sin[Pi t / 2] (1 - Sin[Pi t / 2]) (2 + λ (1 - Sin[Pi t / 2]))
```

$$\left(2 + \lambda \left(1 - \text{Sin}\left[\frac{\pi t}{2}\right]\right)\right) \left(1 - \text{Sin}\left[\frac{\pi t}{2}\right]\right) \text{Sin}\left[\frac{\pi t}{2}\right]$$

```
b2t = Cos[Pi t / 2] (1 - Cos[Pi t / 2]) (2 + λ (1 - Cos[Pi t / 2]))
```

$$\left(2 + \lambda \left(1 - \text{Cos}\left[\frac{\pi t}{2}\right]\right)\right) \left(1 - \text{Cos}\left[\frac{\pi t}{2}\right]\right) \text{Cos}\left[\frac{\pi t}{2}\right]$$

```
b3t = (1 - λ Cos[Pi t / 2]) (1 -  Cos[Pi t / 2]) ^2
```

$$\left(1 - \text{Cos}\left[\frac{\pi t}{2}\right]\right)^2 \left(1 - \lambda\, \text{Cos}\left[\frac{\pi t}{2}\right]\right)$$

```
rt = b0t P0 + b1t P1 + b2t P2 + b3t P3
```

$$\text{P2} \left(2 + \lambda \left(1 - \text{Cos}\left[\frac{\pi t}{2}\right]\right)\right) \left(1 - \text{Cos}\left[\frac{\pi t}{2}\right]\right) \text{Cos}\left[\frac{\pi t}{2}\right] + \text{P3} \left(1 - \text{Cos}\left[\frac{\pi t}{2}\right]\right)^2 \left(1 - \lambda\, \text{Cos}\left[\frac{\pi t}{2}\right]\right) + \text{P1} \left(2 + \lambda \left(1 - \text{Sin}\left[\frac{\pi t}{2}\right]\right)\right) \left(1 - \text{Sin}\left[\frac{\pi t}{2}\right]\right) \text{Sin}\left[\frac{\pi t}{2}\right] + \text{P0} \left(1 - \text{Sin}\left[\frac{\pi t}{2}\right]\right)^2 \left(1 - \lambda\, \text{Sin}\left[\frac{\pi t}{2}\right]\right)$$

Define Control Points

```
P0 = {{85.374, 839.788}, {186.346, 760.160}, {218.361, 750.309}, {321.796, 786.429},
    {321.796, 786.429}, {348.065, 765.086}, {462.171, 817.624}, {462.171, 799.564}};
P1 = {{180.60, 929.26}, {300.52, 838.94}, {245.45, 394.84}, {298.81, 927.62},
   {341.53, 691.99}, {405.41, 906.64}, {419.19, 865.06}, {502.42, 493.34}};
P2 = {{116.57, 719.11}, {197.06, 974.37}, {125.64, 707.59}, {207.71, 663.29},
   {346.43, 894.79}, {371.58, 716.91}, {398.30, 673.90}, {339.86, 624.71}};
P3 = {{186.346, 760.160}, {218.361, 750.309}, {279.109, 775.758}, {321.796, 785.608},
   {348.065, 765.086}, {437.544, 772.474}, {462.171, 799.564}, {499.933, 741.279}};
```

■ Define the number of segments

```
n = Length[P0]
```

```
8
```

■ Cubic Trigonometric Bezier Curve

```
rt = Table[b0t P0[[k]] + b1t P1[[k]] + b2t P2[[k]] + b3t P3[[k]], {k, 1, n}]
```

$$\Big\{\Big\{116.57 \left(2+\lambda \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right) \mathrm{Cos}\left[\frac{\pi t}{2}\right] +$$

$$186.346 \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right) + 180.6 \left(2+\lambda \left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right) \mathrm{Sin}\left[\frac{\pi t}{2}\right] + 85.374 \left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right),$$

$$719.11 \left(2+\lambda \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right) \mathrm{Cos}\left[\frac{\pi t}{2}\right] +$$

$$760.16 \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right) + 929.26 \left(2+\lambda \left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right) \mathrm{Sin}\left[\frac{\pi t}{2}\right] + 839.788 \left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)\Big\},$$

$$\Big\{197.06 \left(2+\lambda \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right) \mathrm{Cos}\left[\frac{\pi t}{2}\right] +$$

$$218.361 \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right) + 300.52 \left(2+\lambda \left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right) \mathrm{Sin}\left[\frac{\pi t}{2}\right] + 186.346 \left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right),$$

$$974.37 \left(2+\lambda \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right) \mathrm{Cos}\left[\frac{\pi t}{2}\right] +$$

$$750.309 \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right) + 838.94 \left(2+\lambda \left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right) \mathrm{Sin}\left[\frac{\pi t}{2}\right] + 760.16 \left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)\Big\},$$

$$\Big\{125.64 \left(2+\lambda \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right) \mathrm{Cos}\left[\frac{\pi t}{2}\right] +$$

$$279.109 \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right) + 245.45 \left(2+\lambda \left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right) \mathrm{Sin}\left[\frac{\pi t}{2}\right] + 218.361 \left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right),$$

$$707.59 \left(2+\lambda \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right) \mathrm{Cos}\left[\frac{\pi t}{2}\right] +$$

$$775.758 \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right) + 394.84 \left(2+\lambda \left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right) \mathrm{Sin}\left[\frac{\pi t}{2}\right] + 750.309 \left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)\Big\},$$

$$\Big\{207.71 \left(2+\lambda \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right) \mathrm{Cos}\left[\frac{\pi t}{2}\right] +$$

$$321.796 \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+298.81 \left(2+\lambda \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right) \text{Sin}\left[\frac{\pi t}{2}\right]+321.796 \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right),$$

$$663.29 \left(2+\lambda \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right) \text{Cos}\left[\frac{\pi t}{2}\right]+$$

$$785.608 \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+927.62 \left(2+\lambda \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right) \text{Sin}\left[\frac{\pi t}{2}\right]+786.429 \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)\Big\},$$

$$\Big\{346.43 \left(2+\lambda \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right) \text{Cos}\left[\frac{\pi t}{2}\right]+$$

$$348.065 \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+341.53 \left(2+\lambda \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right) \text{Sin}\left[\frac{\pi t}{2}\right]+321.796 \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right),$$

$$894.79 \left(2+\lambda \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right) \text{Cos}\left[\frac{\pi t}{2}\right]+$$

$$765.086 \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+691.99 \left(2+\lambda \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right) \text{Sin}\left[\frac{\pi t}{2}\right]+786.429 \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)\Big\},$$

$$\Big\{371.58 \left(2+\lambda \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right) \text{Cos}\left[\frac{\pi t}{2}\right]+$$

$$437.544 \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+405.41 \left(2+\lambda \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right) \text{Sin}\left[\frac{\pi t}{2}\right]+348.065 \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right),$$

$$716.91 \left(2+\lambda \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right) \text{Cos}\left[\frac{\pi t}{2}\right]+$$

$$772.474 \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+906.64 \left(2+\lambda \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right) \text{Sin}\left[\frac{\pi t}{2}\right]+765.086 \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)\Big\},$$

$$\Big\{398.3 \left(2+\lambda \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right) \text{Cos}\left[\frac{\pi t}{2}\right]+$$

$$462.171 \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+419.19 \left(2+\lambda \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right) \text{Sin}\left[\frac{\pi t}{2}\right]+462.171 \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right),$$

$$673.9 \left(2+\lambda \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right) \text{Cos}\left[\frac{\pi t}{2}\right]+$$

$$799.564 \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+865.06 \left(2+\lambda \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$

$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right) \text{Sin}\left[\frac{\pi t}{2}\right]+817.624 \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)\Big\},$$

$$\Big\{339.86 \left(2+\lambda \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right) \text{Cos}\left[\frac{\pi t}{2}\right]+$$

```
499.933 (1 - Cos[π t/2])^2 (1 - λ Cos[π t/2]) + 502.42 (2 + λ (1 - Sin[π t/2]))
 (1 - Sin[π t/2]) Sin[π t/2] + 462.171 (1 - Sin[π t/2])^2 (1 - λ Sin[π t/2]),
624.71 (2 + λ (1 - Cos[π t/2])) (1 - Cos[π t/2]) Cos[π t/2] +
 741.279 (1 - Cos[π t/2])^2 (1 - λ Cos[π t/2]) + 493.34 (2 + λ (1 - Sin[π t/2]))
 (1 - Sin[π t/2]) Sin[π t/2] + 799.564 (1 - Sin[π t/2])^2 (1 - λ Sin[π t/2])}}
```

```
rtx = Table[TrigReduce[rt[[k, 1]]], {k, 1, n}]
```

$$
\begin{aligned}
\{&110.41 - 25.45\,\lambda - 139.552\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] - 122.108\,\lambda\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + \\
&114.516\,\mathrm{Cos}[\pi t] + 165.002\,\lambda\,\mathrm{Cos}[\pi t] - 17.444\,\lambda\,\mathrm{Cos}\left[\frac{3\pi t}{2}\right] + \\
&190.452\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] + 166.645\,\lambda\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - 23.8065\,\lambda\,\mathrm{Sin}\left[\frac{3\pi t}{2}\right], \\
&109.481 - 92.873\,\lambda - 42.602\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] - 37.2767\,\lambda\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + \\
&119.467\,\mathrm{Cos}[\pi t] + 135.475\,\lambda\,\mathrm{Cos}[\pi t] - 5.32525\,\lambda\,\mathrm{Cos}\left[\frac{3\pi t}{2}\right] + \\
&228.348\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] + 199.804\,\lambda\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - 28.5435\,\lambda\,\mathrm{Sin}\left[\frac{3\pi t}{2}\right], \\
&375.115 + 126.38\,\lambda - 306.938\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] - 268.571\,\lambda\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + \\
&150.184\,\mathrm{Cos}[\pi t] + 180.558\,\lambda\,\mathrm{Cos}[\pi t] - 38.3673\,\lambda\,\mathrm{Cos}\left[\frac{3\pi t}{2}\right] + \\
&54.178\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] + 47.4057\,\lambda\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - 6.77225\,\lambda\,\mathrm{Sin}\left[\frac{3\pi t}{2}\right], \\
&458.868 + 137.072\,\lambda - 228.172\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] - 199.65\,\lambda\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + 91.1\,\mathrm{Cos}[\pi t] + \\
&91.1\,\lambda\,\mathrm{Cos}[\pi t] - 28.5215\,\lambda\,\mathrm{Cos}\left[\frac{3\pi t}{2}\right] - 45.972\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - 40.2255\,\lambda\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] + \\
&5.7465\,\lambda\,\mathrm{Sin}\left[\frac{3\pi t}{2}\right],\ 316.831 - 18.099\,\lambda - 3.27\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] - 2.86125\,\lambda\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + \\
&8.2345\,\mathrm{Cos}[\pi t] + 21.369\,\lambda\,\mathrm{Cos}[\pi t] - 0.40875\,\lambda\,\mathrm{Cos}\left[\frac{3\pi t}{2}\right] + \\
&39.468\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] + 34.5345\,\lambda\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - 4.9335\,\lambda\,\mathrm{Sin}\left[\frac{3\pi t}{2}\right], \\
&401.424 + 8.619\,\lambda - 131.928\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] - 115.437\,\lambda\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + 78.5695\,\mathrm{Cos}[\pi t] + \\
&123.309\,\lambda\,\mathrm{Cos}[\pi t] - 16.491\,\lambda\,\mathrm{Cos}\left[\frac{3\pi t}{2}\right] + 114.69\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] + 100.354\,\lambda\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - \\
&14.3363\,\lambda\,\mathrm{Sin}\left[\frac{3\pi t}{2}\right],\ 569.023 + 106.852\,\lambda - 127.742\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] - \\
&111.774\,\lambda\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + 20.89\,\mathrm{Cos}[\pi t] + 20.89\,\lambda\,\mathrm{Cos}[\pi t] - 15.9677\,\lambda\,\mathrm{Cos}\left[\frac{3\pi t}{2}\right] - \\
&85.962\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - 75.2167\,\lambda\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] + 10.7452\,\lambda\,\mathrm{Sin}\left[\frac{3\pi t}{2}\right], \\
&600.876 + 119.824\,\lambda - 320.146\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] - 280.128\,\lambda\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + \\
&181.441\,\mathrm{Cos}[\pi t] + 200.322\,\lambda\,\mathrm{Cos}[\pi t] - 40.0182\,\lambda\,\mathrm{Cos}\left[\frac{3\pi t}{2}\right] + \\
&80.498\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] + 70.4358\,\lambda\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - 10.0623\,\lambda\,\mathrm{Sin}\left[\frac{3\pi t}{2}\right]\}
\end{aligned}
$$

rty = Table[TrigReduce[rt[[k, 2]]], {k, 1, n}]

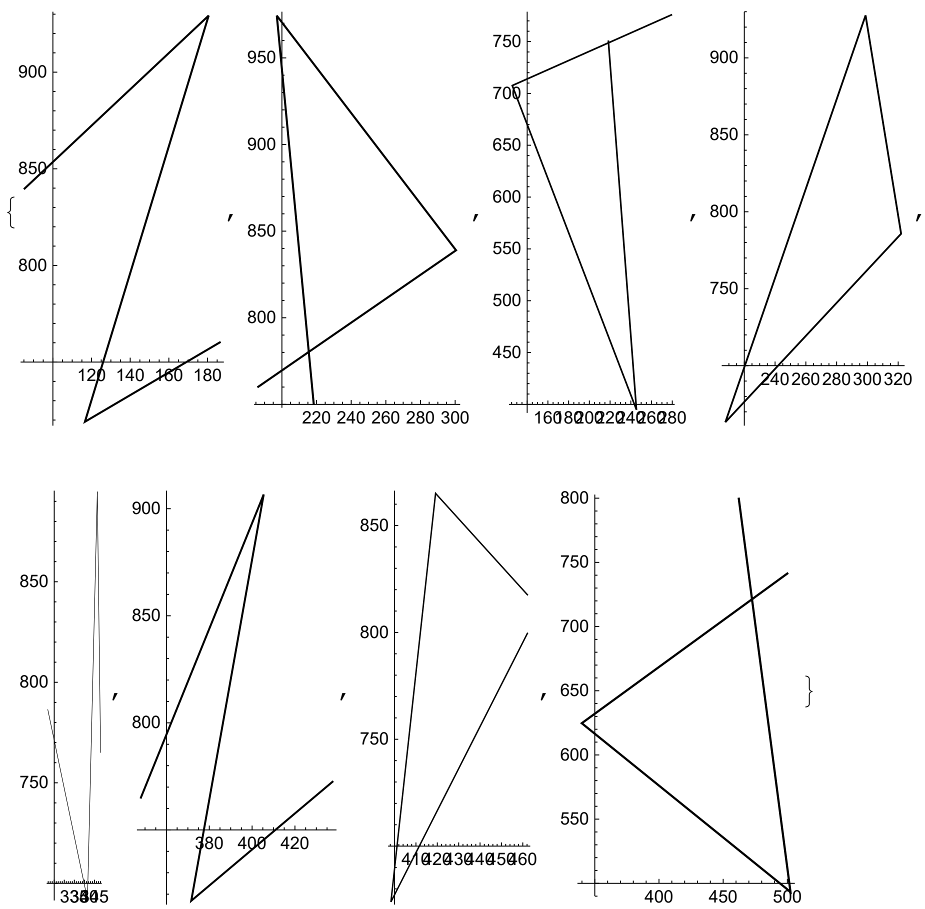
$$\{751.552 - 48.422\,\lambda - 82.1\,\text{Cos}\left[\frac{\pi t}{2}\right] - 71.8375\,\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right] +$$
$$170.336\,\text{Cos}[\pi t] + 130.522\,\lambda\,\text{Cos}[\pi t] - 10.2625\,\lambda\,\text{Cos}\left[\frac{3\pi t}{2}\right] +$$
$$178.944\,\text{Sin}\left[\frac{\pi t}{2}\right] + 156.576\,\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right] - 22.368\,\lambda\,\text{Sin}\left[\frac{3\pi t}{2}\right],$$
$$452.393 - 302.841\,\lambda + 448.122\,\text{Cos}\left[\frac{\pi t}{2}\right] + 392.107\,\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right] -$$
$$140.355\,\text{Cos}[\pi t] - 145.281\,\lambda\,\text{Cos}[\pi t] + 56.0153\,\lambda\,\text{Cos}\left[\frac{3\pi t}{2}\right] +$$
$$157.56\,\text{Sin}\left[\frac{\pi t}{2}\right] + 137.865\,\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right] - 19.695\,\lambda\,\text{Sin}\left[\frac{3\pi t}{2}\right],$$
$$1186.67 + 423.637\,\lambda - 136.336\,\text{Cos}\left[\frac{\pi t}{2}\right] - 119.294\,\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right] -$$
$$300.026\,\text{Cos}[\pi t] - 287.301\,\lambda\,\text{Cos}[\pi t] - 17.042\,\lambda\,\text{Cos}\left[\frac{3\pi t}{2}\right] -$$
$$710.938\,\text{Sin}\left[\frac{\pi t}{2}\right] - 622.071\,\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right] + 88.8673\,\lambda\,\text{Sin}\left[\frac{3\pi t}{2}\right],$$
$$767.145 - 18.873\,\lambda - 244.636\,\text{Cos}\left[\frac{\pi t}{2}\right] - 214.056\,\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right] +$$
$$263.92\,\text{Cos}[\pi t] + 263.509\,\lambda\,\text{Cos}[\pi t] - 30.5795\,\lambda\,\text{Cos}\left[\frac{3\pi t}{2}\right] +$$
$$282.382\,\text{Sin}\left[\frac{\pi t}{2}\right] + 247.084\,\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right] - 35.2978\,\lambda\,\text{Sin}\left[\frac{3\pi t}{2}\right],$$
$$740.493 - 35.265\,\lambda + 259.408\,\text{Cos}\left[\frac{\pi t}{2}\right] + 226.982\,\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right] -$$
$$213.471\,\text{Cos}[\pi t] - 224.143\,\lambda\,\text{Cos}[\pi t] + 32.426\,\lambda\,\text{Cos}\left[\frac{3\pi t}{2}\right] -$$
$$188.878\,\text{Sin}\left[\frac{\pi t}{2}\right] - 165.268\,\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right] + 23.6097\,\lambda\,\text{Sin}\left[\frac{3\pi t}{2}\right],$$
$$682.79 - 85.99\,\lambda - 111.128\,\text{Cos}\left[\frac{\pi t}{2}\right] - 97.237\,\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right] + 193.424\,\text{Cos}[\pi t] +$$
$$197.118\,\lambda\,\text{Cos}[\pi t] - 13.891\,\lambda\,\text{Cos}\left[\frac{3\pi t}{2}\right] + 283.108\,\text{Sin}\left[\frac{\pi t}{2}\right] +$$
$$247.719\,\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right] - 35.3885\,\lambda\,\text{Sin}\left[\frac{3\pi t}{2}\right],\ 886.822 + 78.228\,\lambda -$$
$$251.328\,\text{Cos}\left[\frac{\pi t}{2}\right] - 219.912\,\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right] + 182.13\,\text{Cos}[\pi t] + 173.1\,\lambda\,\text{Cos}[\pi t] -$$
$$31.416\,\lambda\,\text{Cos}\left[\frac{3\pi t}{2}\right] + 94.872\,\text{Sin}\left[\frac{\pi t}{2}\right] + 83.013\,\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right] - 11.859\,\lambda\,\text{Sin}\left[\frac{3\pi t}{2}\right],$$
$$1193.21 + 422.793\,\lambda - 233.138\,\text{Cos}\left[\frac{\pi t}{2}\right] - 203.996\,\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right] -$$
$$160.513\,\text{Cos}[\pi t] - 189.655\,\lambda\,\text{Cos}[\pi t] - 29.1422\,\lambda\,\text{Cos}\left[\frac{3\pi t}{2}\right] -$$
$$612.448\,\text{Sin}\left[\frac{\pi t}{2}\right] - 535.892\,\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right] + 76.556\,\lambda\,\text{Sin}\left[\frac{3\pi t}{2}\right]\}$$

■ Set λ value

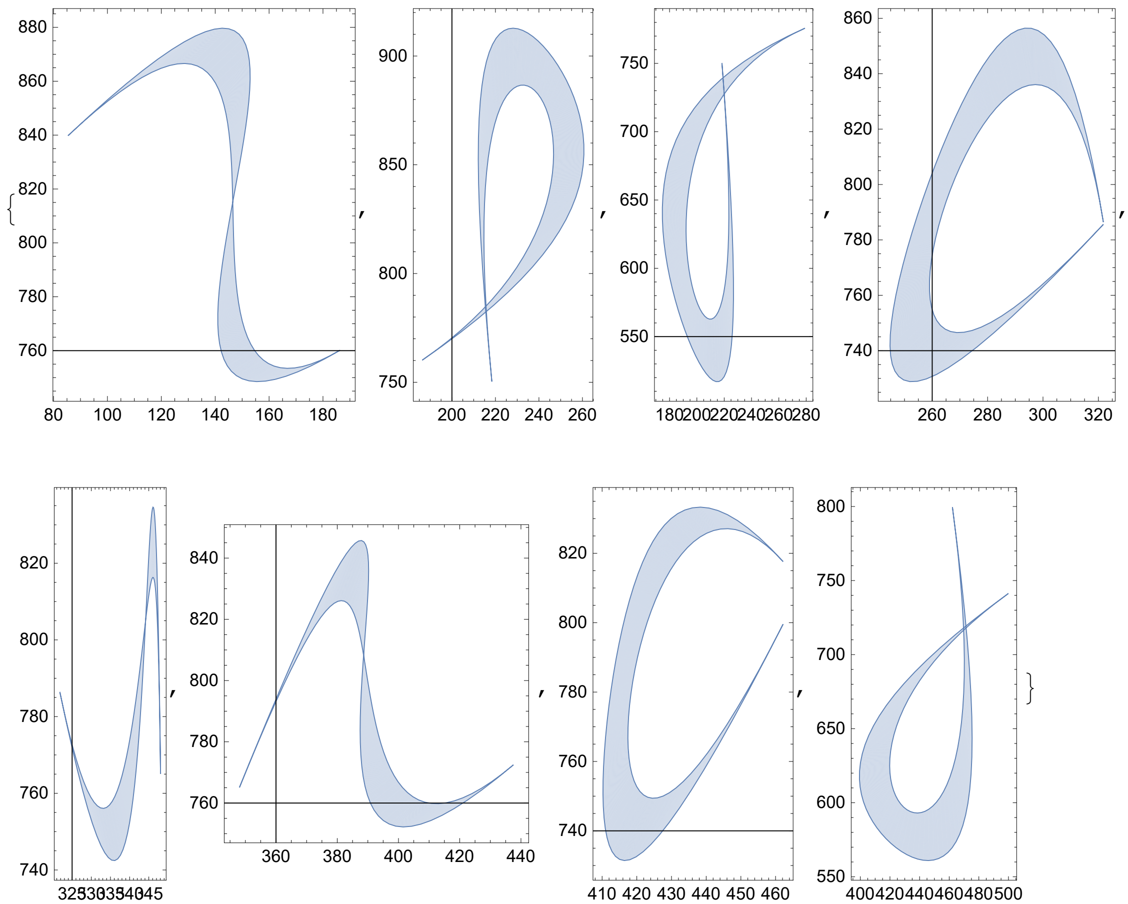
```
λ = .;
```

■ Curve Plot

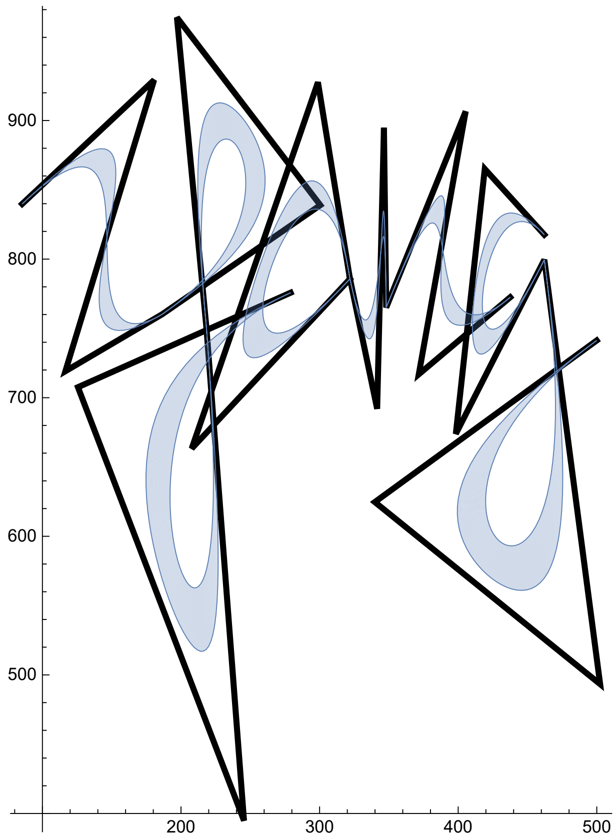
```
PolyG = Table[Graphics[{Thickness[0.01],
    Line[{P0[[k]], P1[[k]], P2[[k]], P3[[k]]}]}, Axes → True], {k, 1, n}]
```



```
Curve = Table[ParametricPlot[{rtx[[k]], rty[[k]]}, {t, 0, 1}, {λ, 0, 1}], {k, 1, n}]
```



```
Show[PolyG, Curve]
```



Set λ value

```
λ = 0.01;
```

Trigonometric Functions for x and y components

rtx

$$\left\{110.156 - 140.773 \operatorname{Cos}\left[\frac{\pi t}{2}\right] + 116.166 \operatorname{Cos}[\pi t] - 0.17444 \operatorname{Cos}\left[\frac{3 \pi t}{2}\right] + 192.118 \operatorname{Sin}\left[\frac{\pi t}{2}\right] - 0.238065 \operatorname{Sin}\left[\frac{3 \pi t}{2}\right],\ 108.552 - 42.9748 \operatorname{Cos}\left[\frac{\pi t}{2}\right] + 120.822 \operatorname{Cos}[\pi t] - 0.0532525 \operatorname{Cos}\left[\frac{3 \pi t}{2}\right] + 230.346 \operatorname{Sin}\left[\frac{\pi t}{2}\right] - 0.285435 \operatorname{Sin}\left[\frac{3 \pi t}{2}\right],\right.$$

$$376.379 - 309.624 \operatorname{Cos}\left[\frac{\pi t}{2}\right] + 151.99 \operatorname{Cos}[\pi t] - 0.383672 \operatorname{Cos}\left[\frac{3 \pi t}{2}\right] + 54.6521 \operatorname{Sin}\left[\frac{\pi t}{2}\right] - 0.0677225 \operatorname{Sin}\left[\frac{3 \pi t}{2}\right],\ 460.239 - 230.169 \operatorname{Cos}\left[\frac{\pi t}{2}\right] + 92.011 \operatorname{Cos}[\pi t] - 0.285215 \operatorname{Cos}\left[\frac{3 \pi t}{2}\right] - 46.3743 \operatorname{Sin}\left[\frac{\pi t}{2}\right] + 0.057465 \operatorname{Sin}\left[\frac{3 \pi t}{2}\right],$$

$$316.651 - 3.29861 \operatorname{Cos}\left[\frac{\pi t}{2}\right] + 8.44819 \operatorname{Cos}[\pi t] - 0.0040875 \operatorname{Cos}\left[\frac{3 \pi t}{2}\right] + 39.8133 \operatorname{Sin}\left[\frac{\pi t}{2}\right] - 0.049335 \operatorname{Sin}\left[\frac{3 \pi t}{2}\right],\ 401.51 - 133.082 \operatorname{Cos}\left[\frac{\pi t}{2}\right] + 79.8026 \operatorname{Cos}[\pi t] - 0.16491 \operatorname{Cos}\left[\frac{3 \pi t}{2}\right] + 115.694 \operatorname{Sin}\left[\frac{\pi t}{2}\right] - 0.143363 \operatorname{Sin}\left[\frac{3 \pi t}{2}\right],$$

$$\left.570.092 - 128.86 \operatorname{Cos}\left[\frac{\pi t}{2}\right] + 21.0989 \operatorname{Cos}[\pi t] - 0.159677 \operatorname{Cos}\left[\frac{3 \pi t}{2}\right] - 86.7142 \operatorname{Sin}\left[\frac{\pi t}{2}\right] + 0.107452 \operatorname{Sin}\left[\frac{3 \pi t}{2}\right],\ 602.074 - 322.947 \operatorname{Cos}\left[\frac{\pi t}{2}\right] + 183.444 \operatorname{Cos}[\pi t] - 0.400182 \operatorname{Cos}\left[\frac{3 \pi t}{2}\right] + 81.2024 \operatorname{Sin}\left[\frac{\pi t}{2}\right] - 0.100623 \operatorname{Sin}\left[\frac{3 \pi t}{2}\right]\right\}$$

rty

$$\{751.068 - 82.8184\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + 171.641\,\mathrm{Cos}[\pi t] - 0.102625\,\mathrm{Cos}\left[\frac{3\pi t}{2}\right] + 180.51\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - 0.22368\,\mathrm{Sin}\left[\frac{3\pi t}{2}\right],\ 449.365 + 452.043\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] - 141.808\,\mathrm{Cos}[\pi t] + 0.560153\,\mathrm{Cos}\left[\frac{3\pi t}{2}\right] + 158.939\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - 0.19695\,\mathrm{Sin}\left[\frac{3\pi t}{2}\right],$$

$$1190.91 - 137.529\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] - 302.899\,\mathrm{Cos}[\pi t] - 0.17042\,\mathrm{Cos}\left[\frac{3\pi t}{2}\right] - 717.159\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] + 0.888673\,\mathrm{Sin}\left[\frac{3\pi t}{2}\right],\ 766.957 - 246.777\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + 266.555\,\mathrm{Cos}[\pi t] - 0.305795\,\mathrm{Cos}\left[\frac{3\pi t}{2}\right] + 284.853\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - 0.352978\,\mathrm{Sin}\left[\frac{3\pi t}{2}\right],$$

$$740.14 + 261.678\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] - 215.713\,\mathrm{Cos}[\pi t] + 0.32426\,\mathrm{Cos}\left[\frac{3\pi t}{2}\right] - 190.531\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] + 0.236097\,\mathrm{Sin}\left[\frac{3\pi t}{2}\right],\ 681.93 - 112.1\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + 195.395\,\mathrm{Cos}[\pi t] - 0.13891\,\mathrm{Cos}\left[\frac{3\pi t}{2}\right] + 285.585\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - 0.353885\,\mathrm{Sin}\left[\frac{3\pi t}{2}\right],$$

$$887.604 - 253.527\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + 183.861\,\mathrm{Cos}[\pi t] - 0.31416\,\mathrm{Cos}\left[\frac{3\pi t}{2}\right] + 95.7021\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - 0.11859\,\mathrm{Sin}\left[\frac{3\pi t}{2}\right],\ 1197.44 - 235.178\,\mathrm{Cos}\left[\frac{\pi t}{2}\right] - 162.409\,\mathrm{Cos}[\pi t] - 0.291422\,\mathrm{Cos}\left[\frac{3\pi t}{2}\right] - 617.807\,\mathrm{Sin}\left[\frac{\pi t}{2}\right] + 0.76556\,\mathrm{Sin}\left[\frac{3\pi t}{2}\right]\}$$

Expand[rtx]

$$\{110.156 - 140.773\,\text{Cos}\left[\frac{\pi t}{2}\right] + 116.166\,\text{Cos}[\pi t] - 0.17444\,\text{Cos}\left[\frac{3\pi t}{2}\right] + 192.118\,\text{Sin}\left[\frac{\pi t}{2}\right] - 0.238065\,\text{Sin}\left[\frac{3\pi t}{2}\right],\ 108.552 - 42.9748\,\text{Cos}\left[\frac{\pi t}{2}\right] + 120.822\,\text{Cos}[\pi t] - 0.0532525\,\text{Cos}\left[\frac{3\pi t}{2}\right] + 230.346\,\text{Sin}\left[\frac{\pi t}{2}\right] - 0.285435\,\text{Sin}\left[\frac{3\pi t}{2}\right],$$

$$376.379 - 309.624\,\text{Cos}\left[\frac{\pi t}{2}\right] + 151.99\,\text{Cos}[\pi t] - 0.383672\,\text{Cos}\left[\frac{3\pi t}{2}\right] + 54.6521\,\text{Sin}\left[\frac{\pi t}{2}\right] - 0.0677225\,\text{Sin}\left[\frac{3\pi t}{2}\right],\ 460.239 - 230.169\,\text{Cos}\left[\frac{\pi t}{2}\right] + 92.011\,\text{Cos}[\pi t] - 0.285215\,\text{Cos}\left[\frac{3\pi t}{2}\right] - 46.3743\,\text{Sin}\left[\frac{\pi t}{2}\right] + 0.057465\,\text{Sin}\left[\frac{3\pi t}{2}\right],$$

$$316.651 - 3.29861\,\text{Cos}\left[\frac{\pi t}{2}\right] + 8.44819\,\text{Cos}[\pi t] - 0.0040875\,\text{Cos}\left[\frac{3\pi t}{2}\right] + 39.8133\,\text{Sin}\left[\frac{\pi t}{2}\right] - 0.049335\,\text{Sin}\left[\frac{3\pi t}{2}\right],\ 401.51 - 133.082\,\text{Cos}\left[\frac{\pi t}{2}\right] + 79.8026\,\text{Cos}[\pi t] - 0.16491\,\text{Cos}\left[\frac{3\pi t}{2}\right] + 115.694\,\text{Sin}\left[\frac{\pi t}{2}\right] - 0.143363\,\text{Sin}\left[\frac{3\pi t}{2}\right],$$

$$570.092 - 128.86\,\text{Cos}\left[\frac{\pi t}{2}\right] + 21.0989\,\text{Cos}[\pi t] - 0.159677\,\text{Cos}\left[\frac{3\pi t}{2}\right] - 86.7142\,\text{Sin}\left[\frac{\pi t}{2}\right] + 0.107452\,\text{Sin}\left[\frac{3\pi t}{2}\right],\ 602.074 - 322.947\,\text{Cos}\left[\frac{\pi t}{2}\right] + 183.444\,\text{Cos}[\pi t] - 0.400182\,\text{Cos}\left[\frac{3\pi t}{2}\right] + 81.2024\,\text{Sin}\left[\frac{\pi t}{2}\right] - 0.100623\,\text{Sin}\left[\frac{3\pi t}{2}\right]\}$$

```
Expand[rty]
```

$\{751.068 - 82.8184\,\mathrm{Cos}[\frac{\pi t}{2}] + 171.641\,\mathrm{Cos}[\pi t] - 0.102625\,\mathrm{Cos}[\frac{3\pi t}{2}] + 180.51\,\mathrm{Sin}[\frac{\pi t}{2}] - 0.22368\,\mathrm{Sin}[\frac{3\pi t}{2}],\ 449.365 + 452.043\,\mathrm{Cos}[\frac{\pi t}{2}] - 141.808\,\mathrm{Cos}[\pi t] + 0.560153\,\mathrm{Cos}[\frac{3\pi t}{2}] + 158.939\,\mathrm{Sin}[\frac{\pi t}{2}] - 0.19695\,\mathrm{Sin}[\frac{3\pi t}{2}],$

$1190.91 - 137.529\,\mathrm{Cos}[\frac{\pi t}{2}] - 302.899\,\mathrm{Cos}[\pi t] - 0.17042\,\mathrm{Cos}[\frac{3\pi t}{2}] - 717.159\,\mathrm{Sin}[\frac{\pi t}{2}] + 0.888673\,\mathrm{Sin}[\frac{3\pi t}{2}],\ 766.957 - 246.777\,\mathrm{Cos}[\frac{\pi t}{2}] + 266.555\,\mathrm{Cos}[\pi t] - 0.305795\,\mathrm{Cos}[\frac{3\pi t}{2}] + 284.853\,\mathrm{Sin}[\frac{\pi t}{2}] - 0.352978\,\mathrm{Sin}[\frac{3\pi t}{2}],$

$740.14 + 261.678\,\mathrm{Cos}[\frac{\pi t}{2}] - 215.713\,\mathrm{Cos}[\pi t] + 0.32426\,\mathrm{Cos}[\frac{3\pi t}{2}] - 190.531\,\mathrm{Sin}[\frac{\pi t}{2}] + 0.236097\,\mathrm{Sin}[\frac{3\pi t}{2}],\ 681.93 - 112.1\,\mathrm{Cos}[\frac{\pi t}{2}] + 195.395\,\mathrm{Cos}[\pi t] - 0.13891\,\mathrm{Cos}[\frac{3\pi t}{2}] + 285.585\,\mathrm{Sin}[\frac{\pi t}{2}] - 0.353885\,\mathrm{Sin}[\frac{3\pi t}{2}],$

$887.604 - 253.527\,\mathrm{Cos}[\frac{\pi t}{2}] + 183.861\,\mathrm{Cos}[\pi t] - 0.31416\,\mathrm{Cos}[\frac{3\pi t}{2}] + 95.7021\,\mathrm{Sin}[\frac{\pi t}{2}] - 0.11859\,\mathrm{Sin}[\frac{3\pi t}{2}],\ 1197.44 - 235.178\,\mathrm{Cos}[\frac{\pi t}{2}] - 162.409\,\mathrm{Cos}[\pi t] - 0.291422\,\mathrm{Cos}[\frac{3\pi t}{2}] - 617.807\,\mathrm{Sin}[\frac{\pi t}{2}] + 0.76556\,\mathrm{Sin}[\frac{3\pi t}{2}]\}$

Note that because t∈[0,1], thus $\theta \in [0, \pi/2]$

```
xθ = Table[rtx[[k]] /. { Pi t → 2 θ}, {k, 1, n}]
```

```
{110.156 - 140.773 Cos[θ] + 116.166 Cos[2 θ] - 0.17444 Cos[3 θ] +
  192.118 Sin[θ] - 0.238065 Sin[3 θ], 108.552 - 42.9748 Cos[θ] +
  120.822 Cos[2 θ] - 0.0532525 Cos[3 θ] + 230.346 Sin[θ] - 0.285435 Sin[3 θ],
 376.379 - 309.624 Cos[θ] + 151.99 Cos[2 θ] - 0.383672 Cos[3 θ] +
  54.6521 Sin[θ] - 0.0677225 Sin[3 θ], 460.239 - 230.169 Cos[θ] +
  92.011 Cos[2 θ] - 0.285215 Cos[3 θ] - 46.3743 Sin[θ] + 0.057465 Sin[3 θ],
 316.651 - 3.29861 Cos[θ] + 8.44819 Cos[2 θ] - 0.0040875 Cos[3 θ] +
  39.8133 Sin[θ] - 0.049335 Sin[3 θ], 401.51 - 133.082 Cos[θ] +
  79.8026 Cos[2 θ] - 0.16491 Cos[3 θ] + 115.694 Sin[θ] - 0.143363 Sin[3 θ],
 570.092 - 128.86 Cos[θ] + 21.0989 Cos[2 θ] - 0.159677 Cos[3 θ] -
  86.7142 Sin[θ] + 0.107452 Sin[3 θ], 602.074 - 322.947 Cos[θ] +
  183.444 Cos[2 θ] - 0.400182 Cos[3 θ] + 81.2024 Sin[θ] - 0.100623 Sin[3 θ]}
```

```
yθ = Table[rty[[k]] /. {Pi t → 2 θ}, {k, 1, n}]
```

```
{751.068 - 82.8184 Cos[θ] + 171.641 Cos[2 θ] - 0.102625 Cos[3 θ] +
   180.51 Sin[θ] - 0.22368 Sin[3 θ], 449.365 + 452.043 Cos[θ] -
   141.808 Cos[2 θ] + 0.560153 Cos[3 θ] + 158.939 Sin[θ] - 0.19695 Sin[3 θ],
  1190.91 - 137.529 Cos[θ] - 302.899 Cos[2 θ] - 0.17042 Cos[3 θ] -
   717.159 Sin[θ] + 0.888673 Sin[3 θ], 766.957 - 246.777 Cos[θ] +
   266.555 Cos[2 θ] - 0.305795 Cos[3 θ] + 284.853 Sin[θ] - 0.352978 Sin[3 θ],
  740.14 + 261.678 Cos[θ] - 215.713 Cos[2 θ] + 0.32426 Cos[3 θ] -
   190.531 Sin[θ] + 0.236097 Sin[3 θ], 681.93 - 112.1 Cos[θ] +
   195.395 Cos[2 θ] - 0.13891 Cos[3 θ] + 285.585 Sin[θ] - 0.353885 Sin[3 θ],
  887.604 - 253.527 Cos[θ] + 183.861 Cos[2 θ] - 0.31416 Cos[3 θ] +
   95.7021 Sin[θ] - 0.11859 Sin[3 θ], 1197.44 - 235.178 Cos[θ] -
   162.409 Cos[2 θ] - 0.291422 Cos[3 θ] - 617.807 Sin[θ] + 0.76556 Sin[3 θ]}
```

```
Expand[xθ]
```

```
{110.156 - 140.773 Cos[θ] + 116.166 Cos[2 θ] - 0.17444 Cos[3 θ] +
   192.118 Sin[θ] - 0.238065 Sin[3 θ], 108.552 - 42.9748 Cos[θ] +
   120.822 Cos[2 θ] - 0.0532525 Cos[3 θ] + 230.346 Sin[θ] - 0.285435 Sin[3 θ],
  376.379 - 309.624 Cos[θ] + 151.99 Cos[2 θ] - 0.383672 Cos[3 θ] +
   54.6521 Sin[θ] - 0.0677225 Sin[3 θ], 460.239 - 230.169 Cos[θ] +
   92.011 Cos[2 θ] - 0.285215 Cos[3 θ] - 46.3743 Sin[θ] + 0.057465 Sin[3 θ],
  316.651 - 3.29861 Cos[θ] + 8.44819 Cos[2 θ] - 0.0040875 Cos[3 θ] +
   39.8133 Sin[θ] - 0.049335 Sin[3 θ], 401.51 - 133.082 Cos[θ] +
   79.8026 Cos[2 θ] - 0.16491 Cos[3 θ] + 115.694 Sin[θ] - 0.143363 Sin[3 θ],
  570.092 - 128.86 Cos[θ] + 21.0989 Cos[2 θ] - 0.159677 Cos[3 θ] -
   86.7142 Sin[θ] + 0.107452 Sin[3 θ], 602.074 - 322.947 Cos[θ] +
   183.444 Cos[2 θ] - 0.400182 Cos[3 θ] + 81.2024 Sin[θ] - 0.100623 Sin[3 θ]}
```

```
Expand[yθ]
```

```
{751.068 - 82.8184 Cos[θ] + 171.641 Cos[2 θ] - 0.102625 Cos[3 θ] +
   180.51 Sin[θ] - 0.22368 Sin[3 θ], 449.365 + 452.043 Cos[θ] -
   141.808 Cos[2 θ] + 0.560153 Cos[3 θ] + 158.939 Sin[θ] - 0.19695 Sin[3 θ],
  1190.91 - 137.529 Cos[θ] - 302.899 Cos[2 θ] - 0.17042 Cos[3 θ] -
   717.159 Sin[θ] + 0.888673 Sin[3 θ], 766.957 - 246.777 Cos[θ] +
   266.555 Cos[2 θ] - 0.305795 Cos[3 θ] + 284.853 Sin[θ] - 0.352978 Sin[3 θ],
  740.14 + 261.678 Cos[θ] - 215.713 Cos[2 θ] + 0.32426 Cos[3 θ] -
   190.531 Sin[θ] + 0.236097 Sin[3 θ], 681.93 - 112.1 Cos[θ] +
   195.395 Cos[2 θ] - 0.13891 Cos[3 θ] + 285.585 Sin[θ] - 0.353885 Sin[3 θ],
  887.604 - 253.527 Cos[θ] + 183.861 Cos[2 θ] - 0.31416 Cos[3 θ] +
   95.7021 Sin[θ] - 0.11859 Sin[3 θ], 1197.44 - 235.178 Cos[θ] -
   162.409 Cos[2 θ] - 0.291422 Cos[3 θ] - 617.807 Sin[θ] + 0.76556 Sin[3 θ]}
```

Single coupled serial chain coefficients

```
ak = Table[Table[Coefficient[xθ[[k]], Cos[i θ]], {i, 1, 3}], {k, 1, n}]
{{-140.773, 116.166, -0.17444}, {-42.9748, 120.822, -0.0532525},
 {-309.624, 151.99, -0.383672}, {-230.169, 92.011, -0.285215},
 {-3.29861, 8.44819, -0.0040875}, {-133.082, 79.8026, -0.16491},
 {-128.86, 21.0989, -0.159677}, {-322.947, 183.444, -0.400182}}

bk = Table[Coefficient[xθ[[k]], Sin[i θ]], {k, 1, n}, {i, 1, 3}]
{{192.118, 0, -0.238065}, {230.346, 0, -0.285435},
 {54.6521, 0, -0.0677225}, {-46.3743, 0, 0.057465}, {39.8133, 0, -0.049335},
 {115.694, 0, -0.143363}, {-86.7142, 0, 0.107452}, {81.2024, 0, -0.100623}}

ck = Table[Coefficient[yθ[[k]], Cos[i θ]], {k, 1, n}, {i, 1, 3}]
{{-82.8184, 171.641, -0.102625}, {452.043, -141.808, 0.560153},
 {-137.529, -302.899, -0.17042}, {-246.777, 266.555, -0.305795},
 {261.678, -215.713, 0.32426}, {-112.1, 195.395, -0.13891},
 {-253.527, 183.861, -0.31416}, {-235.178, -162.409, -0.291422}}

dk = Table[Coefficient[yθ[[k]], Sin[i θ]], {k, 1, n}, {i, 1, 3}]
{{180.51, 0, -0.22368}, {158.939, 0, -0.19695},
 {-717.159, 0, 0.888673}, {284.853, 0, -0.352978}, {-190.531, 0, 0.236097},
 {285.585, 0, -0.353885}, {95.7021, 0, -0.11859}, {-617.807, 0, 0.76556}}
```

Link Lengths and Phase angles

```
Lk = Table[0.5 * Sqrt[(ak[[k]] + dk[[k]])^2 + (ck[[k]] - bk[[k]])^2], {k, 1, n}]
{{138.897, 103.628, 0.210264}, {125.097, 93.1499, 0.440914},
 {522.306, 169.446, 0.257668}, {103.865, 140.994, 0.367167},
 {147.304, 107.939, 0.219887}, {137.065, 105.532, 0.259407},
 {85.0382, 92.5338, 0.252582}, {496.265, 122.504, 0.206098}}

psik = Table[ArcTan[ak[[k]] + dk[[k]], ck[[k]] - bk[[k]]], {k, 1, n}]
{{-1.42726, 0.975814, 2.81368}, {1.08886, -0.865136, 1.85848},
 {-2.95657, -1.10572, -0.200626}, {-1.30441, 1.23842, -2.62413},
 {2.28885, -1.53165, 1.01506}, {-0.98085, 1.18306, 3.13301},
 {-1.76701, 1.45654, -2.15417}, {-2.81717, -0.724652, -0.481249}}
```

```
Mk = Table[0.5 * Sqrt[(ak[[k]] - dk[[k]])^2 + (ck[[k]] + bk[[k]])^2], {k, 1, n}]
```

```
{{169.683, 103.628, 0.172115}, {355.817, 93.1499, 0.155015},
 {207.938, 169.446, 0.64722}, {296.304, 140.994, 0.128705},
 {177.449, 107.939, 0.182533}, {209.341, 105.532, 0.169845},
 {203.833, 92.5338, 0.105376}, {166.321, 122.504, 0.61495}}
```

```
etak = Table[ArcTan[ak[[k]] - dk[[k]], ck[[k]] + bk[[k]]], {k, 1, n}]
```

```
{{2.81368, 0.975814, -1.42726}, {1.85848, -0.865136, 1.08886},
 {-0.200626, -1.10572, -2.95657}, {-2.62413, 1.23842, -1.30441},
 {1.01506, -1.53165, 2.28885}, {3.13301, 1.18306, -0.98085},
 {-2.15417, 1.45654, -1.76701}, {-0.481249, -0.724652, -2.81717}}
```

Ground pivot coefficients

```
a0 = Table[xθ[[k]] - Sum[ak[[k, i]] Cos[i θ] + bk[[k, i]] Sin[i θ], {i, 1, 3}], {k, 1, n}]
```

```
{110.156, 108.552, 376.379, 460.239, 316.651, 401.51, 570.092, 602.074}
```

```
b0 = Table[0 * k, {k, 1, n}]
```

```
{0, 0, 0, 0, 0, 0, 0, 0}
```

```
c0 = Table[yθ[[k]] - Sum[ck[[k, i]] Cos[i θ] + dk[[k, i]] Sin[i θ], {i, 1, 3}], {k, 1, n}]
```

```
{751.068, 449.365, 1190.91, 766.957, 740.14, 681.93, 887.604, 1197.44}
```

```
d0 = Table[0 * k, {k, 1, n}]
```

```
{0, 0, 0, 0, 0, 0, 0, 0}
```

Ground pivot position Config.

```
L0 = Table[0.5 * Sqrt[(a0[[k]] + d0[[k]])^2 + (c0[[k]] - b0[[k]])^2], {k, 1, n}]
```

```
{379.551, 231.145, 624.484, 447.225, 402.515, 395.676, 527.458, 670.142}
```

```
psi0 = Table[ArcTan[a0[[k]] + d0[[k]], c0[[k]] - b0[[k]]], {k, 1, n}]
```

```
{1.42517, 1.33377, 1.26469, 1.03031, 1.16654, 1.03866, 0.999866, 1.10491}
```

```
M0 = Table[0.5 * Sqrt[(a0[[k]] - d0[[k]])^2 + (c0[[k]] + b0[[k]])^2], {k, 1, n}]
```

```
{379.551, 231.145, 624.484, 447.225, 402.515, 395.676, 527.458, 670.142}
```

```
eta0 = Table[ArcTan[a0[[k]] - d0[[k]], c0[[k]] + b0[[k]]], {k, 1, n}]
```

```
{1.42517, 1.33377, 1.26469, 1.03031, 1.16654, 1.03866, 0.999866, 1.10491}
```

Ground Pivot Coordinates

```
gx = Table[L0 [[k]] Cos[psi0[[k]]] + M0[[k]] Cos[eta0[[k]]], {k, 1, n}]
```

```
{110.156, 108.552, 376.379, 460.239, 316.651, 401.51, 570.092, 602.074}
```

```
gy = Table[L0[[k]] Sin[psi0[[k]]] + M0[[k]] Sin[eta0[[k]]], {k, 1, n}]
```

```
{751.068, 449.365, 1190.91, 766.957, 740.14, 681.93, 887.604, 1197.44}
```

```
gCoordinates = Table[{gx[[k]], gy[[k]]}, {k, 1, n}]
```

```
{{110.156, 751.068}, {108.552, 449.365}, {376.379, 1190.91}, {460.239, 766.957},
 {316.651, 740.14}, {401.51, 681.93}, {570.092, 887.604}, {602.074, 1197.44}}
```

Plot the curve

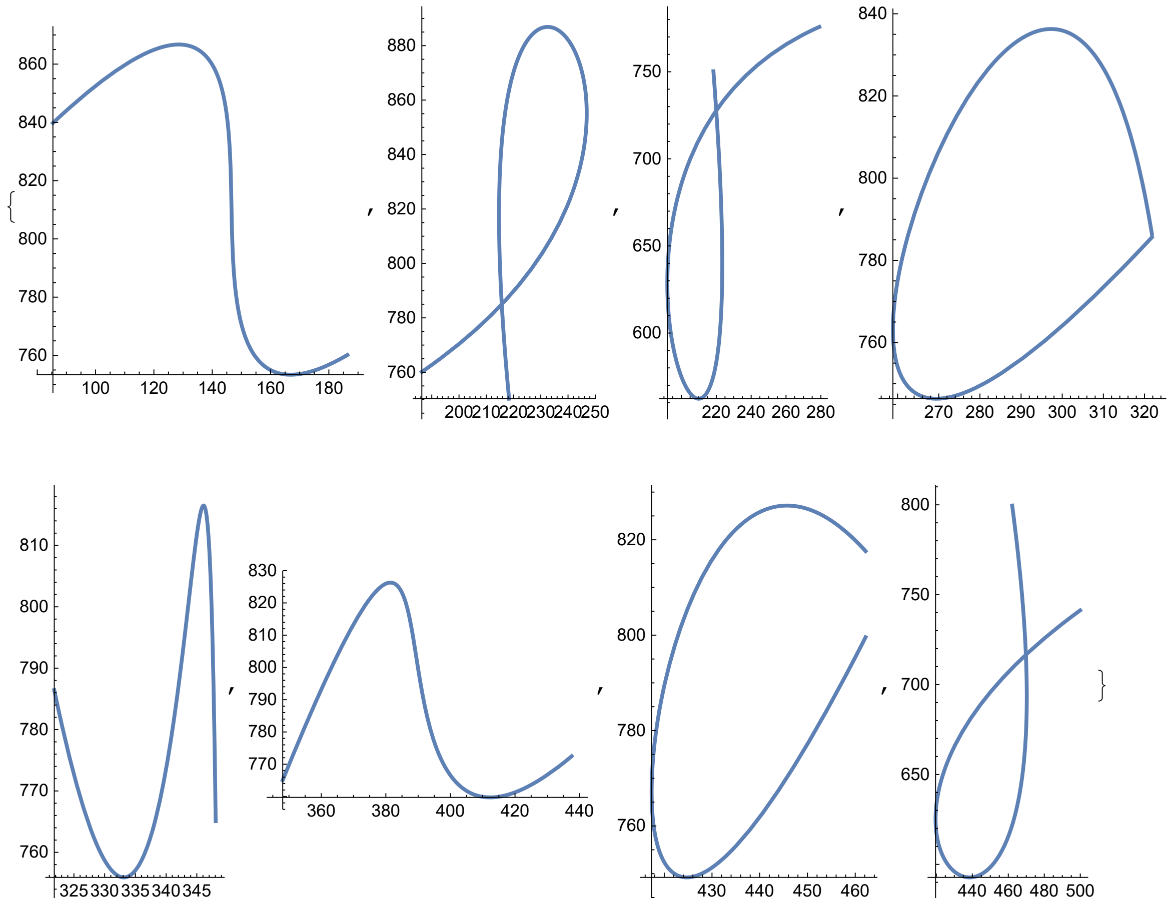
```
Newxθ = Table[
  Sum[Lk[[k, i]] Cos[psik[[k, i]] + i θ] + Mk[[k, i]] Cos[etak[[k, i]] - i θ], {i, 1, 3}] +
   L0 [[k]] Cos[psi0[[k]]] + M0[[k]] Cos[eta0[[k]]], {k, 1, n}]
```

```
{110.156 + 103.628 Cos[0.975814 - 2 θ] + 138.897 Cos[1.42726 - θ] +
  169.683 Cos[2.81368 - θ] + 103.628 Cos[0.975814 + 2 θ] +
  0.172115 Cos[1.42726 + 3 θ] + 0.210264 Cos[2.81368 + 3 θ],
 108.552 + 0.155015 Cos[1.08886 - 3 θ] + 93.1499 Cos[0.865136 - 2 θ] +
  355.817 Cos[1.85848 - θ] + 125.097 Cos[1.08886 + θ] +
  93.1499 Cos[0.865136 + 2 θ] + 0.440914 Cos[1.85848 + 3 θ],
 376.379 + 0.257668 Cos[0.200626 - 3 θ] + 169.446 Cos[1.10572 - 2 θ] +
  522.306 Cos[2.95657 - θ] + 207.938 Cos[0.200626 + θ] +
  169.446 Cos[1.10572 + 2 θ] + 0.64722 Cos[2.95657 + 3 θ],
 460.239 + 0.367167 Cos[2.62413 - 3 θ] + 140.994 Cos[1.23842 - 2 θ] +
  103.865 Cos[1.30441 - θ] + 296.304 Cos[2.62413 + θ] +
  140.994 Cos[1.23842 + 2 θ] + 0.128705 Cos[1.30441 + 3 θ],
 316.651 + 0.182533 Cos[2.28885 - 3 θ] + 107.939 Cos[1.53165 - 2 θ] +
  177.449 Cos[1.01506 - θ] + 147.304 Cos[2.28885 + θ] +
  107.939 Cos[1.53165 + 2 θ] + 0.219887 Cos[1.01506 + 3 θ],
 401.51 + 105.532 Cos[1.18306 - 2 θ] + 137.065 Cos[0.98085 - θ] + 209.341 Cos[3.13301 - θ] +
  105.532 Cos[1.18306 + 2 θ] + 0.169845 Cos[0.98085 + 3 θ] + 0.259407 Cos[3.13301 + 3 θ],
 570.092 + 0.252582 Cos[2.15417 - 3 θ] + 92.5338 Cos[1.45654 - 2 θ] +
  85.0382 Cos[1.76701 - θ] + 203.833 Cos[2.15417 + θ] +
  92.5338 Cos[1.45654 + 2 θ] + 0.105376 Cos[1.76701 + 3 θ],
 602.074 + 0.206098 Cos[0.481249 - 3 θ] + 122.504 Cos[0.724652 - 2 θ] +
  496.265 Cos[2.81717 - θ] + 166.321 Cos[0.481249 + θ] +
  122.504 Cos[0.724652 + 2 θ] + 0.61495 Cos[2.81717 + 3 θ]}
```

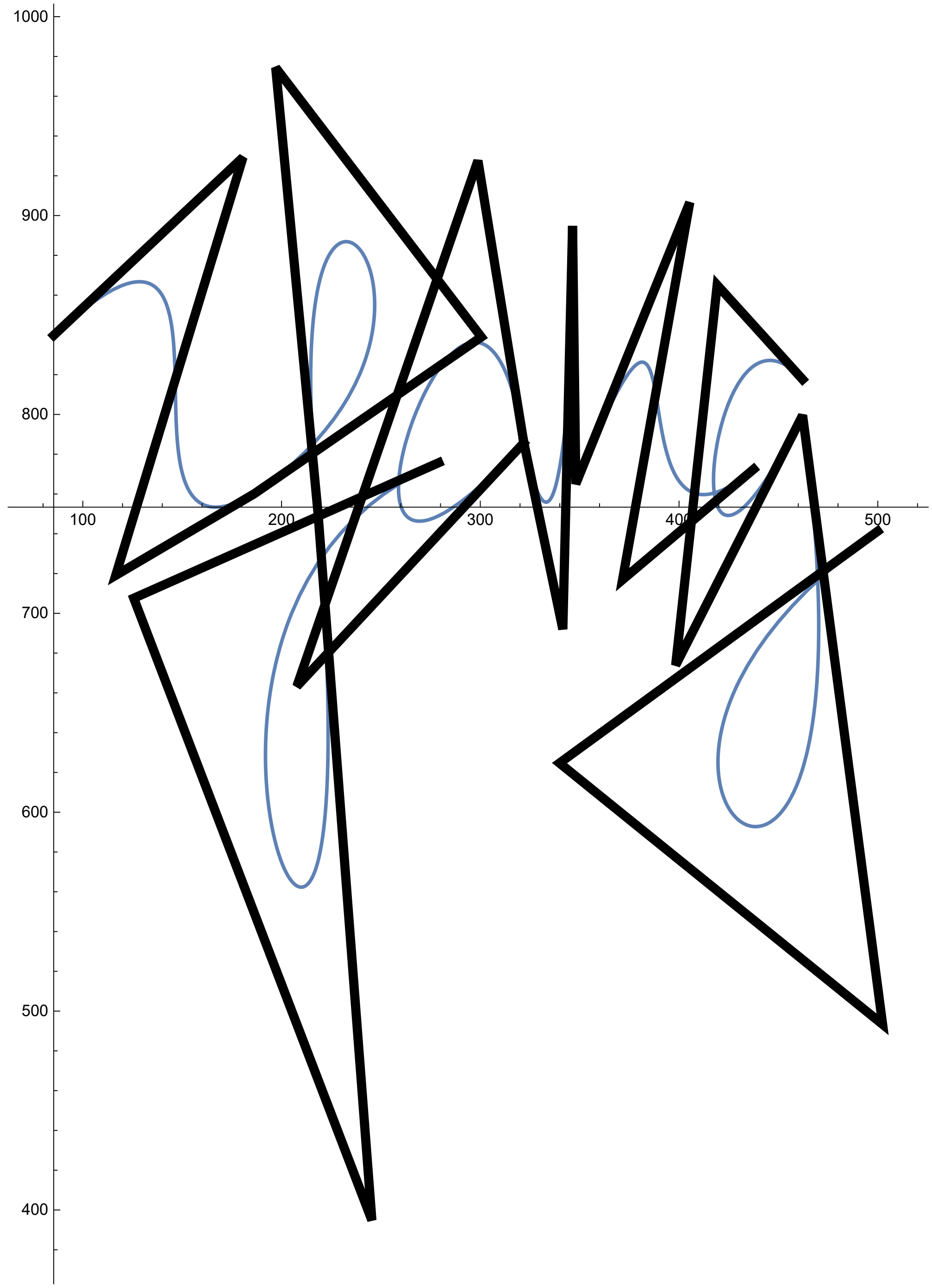
```
NewyΘ = Table[
  Sum[Lk[[k, i]] Sin[psik[[k, i]] + i Θ] + Mk[[k, i]] Sin[etak[[k, i]] - i Θ], {i, 1, 3}] +
   L0[[k]] Sin[psi0[[k]]] + M0[[k]] Sin[eta0[[k]]], {k, 1, n}]
```

```
{751.068 + 103.628 Sin[0.975814 - 2 Θ] - 138.897 Sin[1.42726 - Θ] +
  169.683 Sin[2.81368 - Θ] + 103.628 Sin[0.975814 + 2 Θ] -
  0.172115 Sin[1.42726 + 3 Θ] + 0.210264 Sin[2.81368 + 3 Θ],
 449.365 + 0.155015 Sin[1.08886 - 3 Θ] - 93.1499 Sin[0.865136 - 2 Θ] +
  355.817 Sin[1.85848 - Θ] + 125.097 Sin[1.08886 + Θ] -
  93.1499 Sin[0.865136 + 2 Θ] + 0.440914 Sin[1.85848 + 3 Θ],
 1190.91 - 0.257668 Sin[0.200626 - 3 Θ] - 169.446 Sin[1.10572 - 2 Θ] -
  522.306 Sin[2.95657 - Θ] - 207.938 Sin[0.200626 + Θ] -
  169.446 Sin[1.10572 + 2 Θ] - 0.64722 Sin[2.95657 + 3 Θ],
 766.957 - 0.367167 Sin[2.62413 - 3 Θ] + 140.994 Sin[1.23842 - 2 Θ] -
  103.865 Sin[1.30441 - Θ] - 296.304 Sin[2.62413 + Θ] +
  140.994 Sin[1.23842 + 2 Θ] - 0.128705 Sin[1.30441 + 3 Θ],
 740.14 + 0.182533 Sin[2.28885 - 3 Θ] - 107.939 Sin[1.53165 - 2 Θ] +
  177.449 Sin[1.01506 - Θ] + 147.304 Sin[2.28885 + Θ] -
  107.939 Sin[1.53165 + 2 Θ] + 0.219887 Sin[1.01506 + 3 Θ],
 681.93 + 105.532 Sin[1.18306 - 2 Θ] - 137.065 Sin[0.98085 - Θ] + 209.341 Sin[3.13301 - Θ] +
  105.532 Sin[1.18306 + 2 Θ] - 0.169845 Sin[0.98085 + 3 Θ] + 0.259407 Sin[3.13301 + 3 Θ],
 887.604 - 0.252582 Sin[2.15417 - 3 Θ] + 92.5338 Sin[1.45654 - 2 Θ] -
  85.0382 Sin[1.76701 - Θ] - 203.833 Sin[2.15417 + Θ] +
  92.5338 Sin[1.45654 + 2 Θ] - 0.105376 Sin[1.76701 + 3 Θ],
 1197.44 - 0.206098 Sin[0.481249 - 3 Θ] - 122.504 Sin[0.724652 - 2 Θ] -
  496.265 Sin[2.81717 - Θ] - 166.321 Sin[0.481249 + Θ] -
  122.504 Sin[0.724652 + 2 Θ] - 0.61495 Sin[2.81717 + 3 Θ]}
```

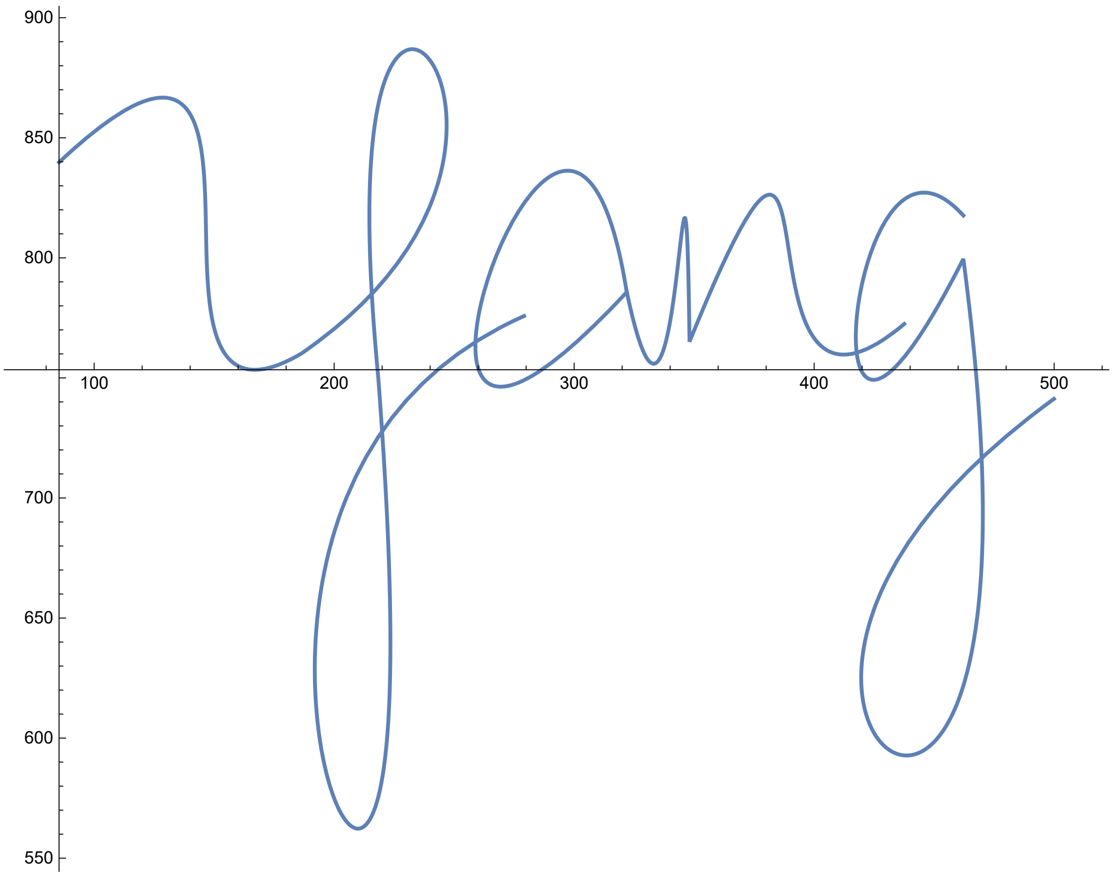
```
Segm = Table[ParametricPlot[
    {Newxθ[[k]], Newyθ[[k]]}, {θ, 0, Pi/2}, PlotStyle → Thick], {k, 1, n}]
```



```
Show[Segm, PolyG, PlotRange → All, ImageSize → Large]
```



```
Show[Segm, PlotRange → All, ImageSize → Large]
```



Simplification by truncating the 3rd Term

```
Newxθ = Table[
  Sum[Lk[[k, i]] Cos[psik[[k, i]] + i θ] + Mk[[k, i]] Cos[etak[[k, i]] - i θ], {i, 1, 2}] +
   L0 [[k]] Cos[psi0[[k]]] + M0[[k]] Cos[eta0[[k]]], {k, 1, n}]
```

```
{110.156 + 103.628 Cos[0.975814 - 2 θ] + 138.897 Cos[1.42726 - θ] +
  169.683 Cos[2.81368 - θ] + 103.628 Cos[0.975814 + 2 θ],
 108.552 + 93.1499 Cos[0.865136 - 2 θ] + 355.817 Cos[1.85848 - θ] +
  125.097 Cos[1.08886 + θ] + 93.1499 Cos[0.865136 + 2 θ],
 376.379 + 169.446 Cos[1.10572 - 2 θ] + 522.306 Cos[2.95657 - θ] +
  207.938 Cos[0.200626 + θ] + 169.446 Cos[1.10572 + 2 θ],
 460.239 + 140.994 Cos[1.23842 - 2 θ] + 103.865 Cos[1.30441 - θ] +
  296.304 Cos[2.62413 + θ] + 140.994 Cos[1.23842 + 2 θ],
 316.651 + 107.939 Cos[1.53165 - 2 θ] + 177.449 Cos[1.01506 - θ] +
  147.304 Cos[2.28885 + θ] + 107.939 Cos[1.53165 + 2 θ],
 401.51 + 105.532 Cos[1.18306 - 2 θ] + 137.065 Cos[0.98085 - θ] +
  209.341 Cos[3.13301 - θ] + 105.532 Cos[1.18306 + 2 θ],
 570.092 + 92.5338 Cos[1.45654 - 2 θ] + 85.0382 Cos[1.76701 - θ] +
  203.833 Cos[2.15417 + θ] + 92.5338 Cos[1.45654 + 2 θ],
 602.074 + 122.504 Cos[0.724652 - 2 θ] + 496.265 Cos[2.81717 - θ] +
  166.321 Cos[0.481249 + θ] + 122.504 Cos[0.724652 + 2 θ]}
```

```
θ =.;
```

```
x0 = Newxθ /. {θ → 0}
```

```
{85.5484, 186.399, 218.745, 322.081, 321.8, 348.23, 462.331, 462.571}
```

```
θ =.;
```

```
xe = Newxθ /. {θ → π/2}
```

```
{186.108, 218.076, 279.041, 321.853, 348.016, 437.401, 462.278, 499.832}
```

```
Newyθ = Table[
  Sum[Lk[[k, i]] Sin[psik[[k, i]] + i θ] + Mk[[k, i]] Sin[etak[[k, i]] - i θ], {i, 1, 2}] +
   L0[[k]] Sin[psi0[[k]]] + M0[[k]] Sin[eta0[[k]]], {k, 1, n}]
```

```
{751.068 + 103.628 Sin[0.975814 - 2 θ] - 138.897 Sin[1.42726 - θ] +
  169.683 Sin[2.81368 - θ] + 103.628 Sin[0.975814 + 2 θ],
 449.365 - 93.1499 Sin[0.865136 - 2 θ] + 355.817 Sin[1.85848 - θ] +
  125.097 Sin[1.08886 + θ] - 93.1499 Sin[0.865136 + 2 θ],
 1190.91 - 169.446 Sin[1.10572 - 2 θ] - 522.306 Sin[2.95657 - θ] -
  207.938 Sin[0.200626 + θ] - 169.446 Sin[1.10572 + 2 θ],
 766.957 + 140.994 Sin[1.23842 - 2 θ] - 103.865 Sin[1.30441 - θ] -
  296.304 Sin[2.62413 + θ] + 140.994 Sin[1.23842 + 2 θ],
 740.14 - 107.939 Sin[1.53165 - 2 θ] + 177.449 Sin[1.01506 - θ] +
  147.304 Sin[2.28885 + θ] - 107.939 Sin[1.53165 + 2 θ],
 681.93 + 105.532 Sin[1.18306 - 2 θ] - 137.065 Sin[0.98085 - θ] +
  209.341 Sin[3.13301 - θ] + 105.532 Sin[1.18306 + 2 θ],
 887.604 + 92.5338 Sin[1.45654 - 2 θ] - 85.0382 Sin[1.76701 - θ] -
  203.833 Sin[2.15417 + θ] + 92.5338 Sin[1.45654 + 2 θ],
 1197.44 - 122.504 Sin[0.724652 - 2 θ] - 496.265 Sin[2.81717 - θ] -
  166.321 Sin[0.481249 + θ] - 122.504 Sin[0.724652 + 2 θ]}
```

```
θ =.;
```

```
y0 = Newyθ /. {θ → 0}
```

```
{839.891, 759.6, 750.479, 786.735, 786.105, 765.225, 817.938, 799.855}
```

```
θ =.;
```

```
ye = Newyθ /. {θ → π/2}
```

```
{759.936, 750.112, 776.647, 785.255, 765.322, 772.12, 799.445, 742.045}
```

Calculate the gap

```
StartPoints = Table[{x0[[i]], y0[[i]]}, {i, 1, 8}]
```

```
{{85.5484, 839.891}, {186.399, 759.6}, {218.745, 750.479}, {322.081, 786.735},
 {321.8, 786.105}, {348.23, 765.225}, {462.331, 817.938}, {462.571, 799.855}}
```

```
EndPoints = Table[{xe[[i]], ye[[i]]}, {i, 1, 8}]
```

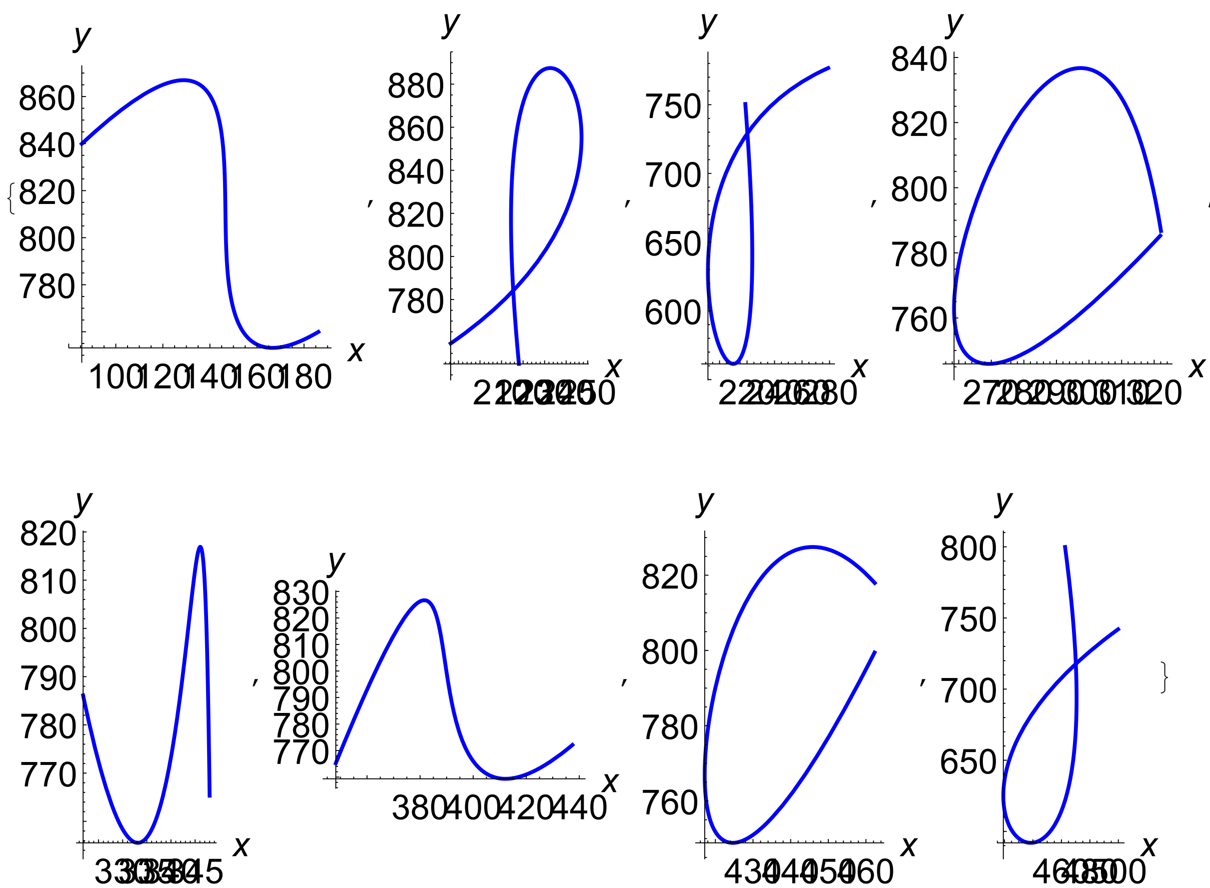
```
{{186.108, 759.936}, {218.076, 750.112}, {279.041, 776.647}, {321.853, 785.255},
 {348.016, 765.322}, {437.401, 772.12}, {462.278, 799.445}, {499.832, 742.045}}
```

```
Table[Norm[EndPoints[[i]] - StartPoints[[i + 1]]], {i, 1, 7}]
```

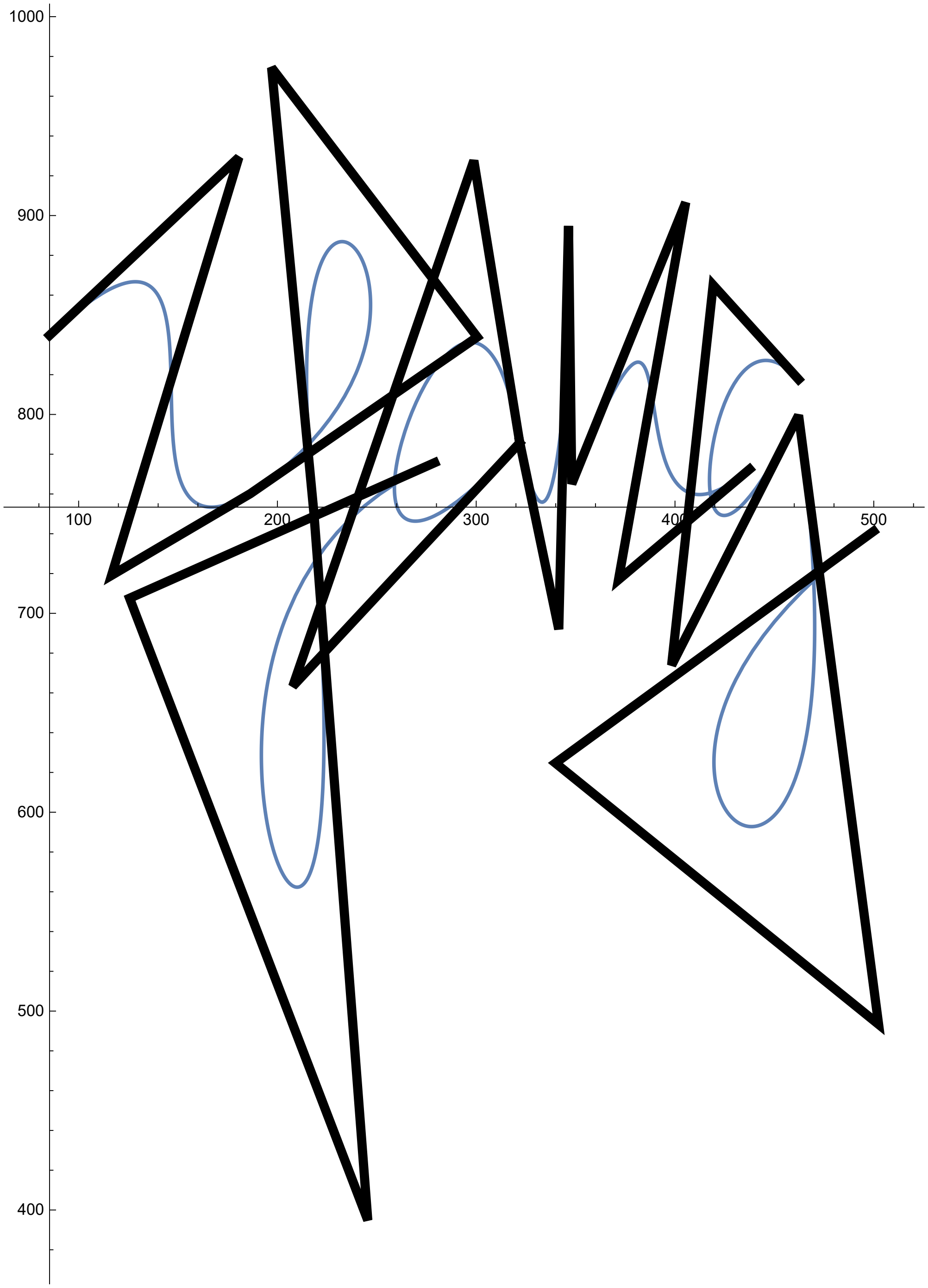
```
{0.445061, 0.763325, 44.2064, 0.851392, 0.235258, 52.1613, 0.503787}
```

Plot the segments

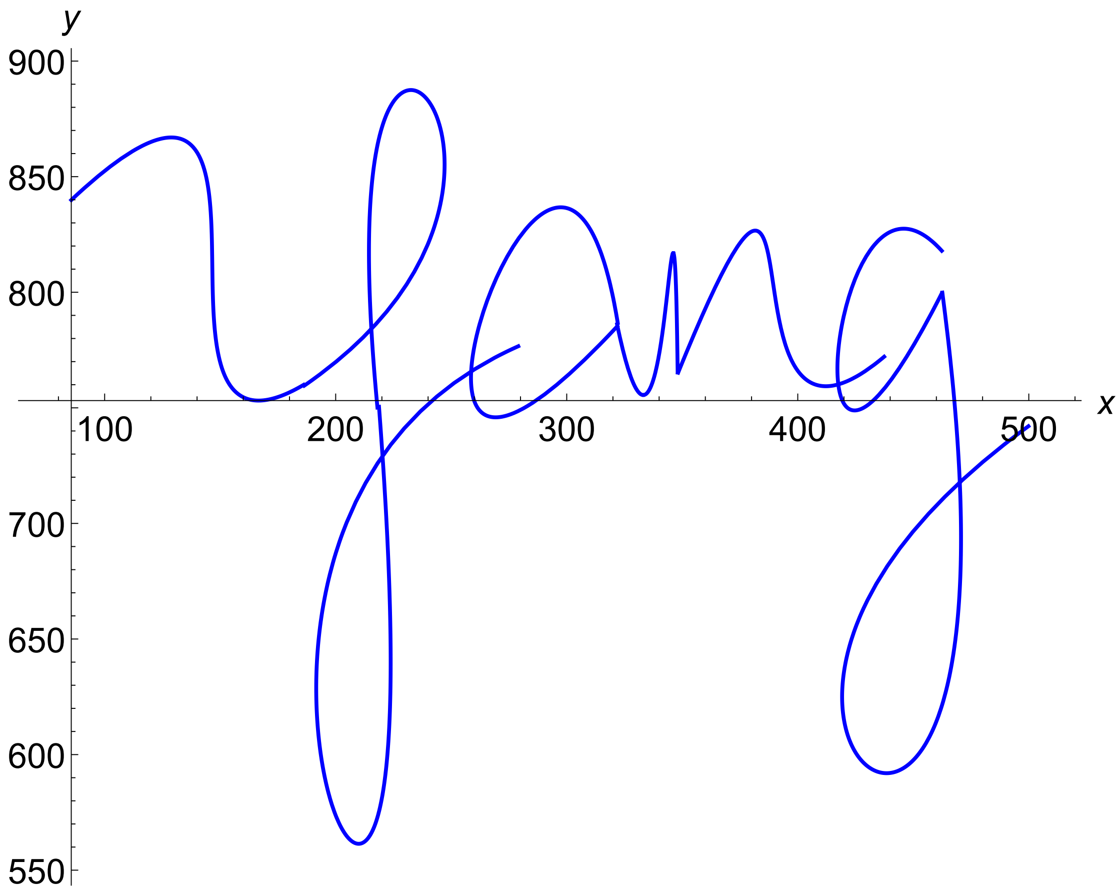
```
Segm6 = Table[
  ParametricPlot[{Newxθ[[k]], Newyθ[[k]]}, {θ, 0, Pi/2}, PlotStyle → {Blue, Thick},
   AxesLabel → {x, y}, AxesStyle → Directive[18, Black]], {k, 1, n}]
```



```
Show[Segm, PolyG, PlotRange → All, ImageSize → Large]
```



```
Show[Segm6, PlotRange → All, ImageSize → Large]
```



Find Revised Ground Pivot Position

```
Link1GroundPosion = {{L0 [[1]] Cos[psi0[[1]]] + M0[[1]] Cos[eta0[[1]]],
   L0[[1]] Sin[psi0[[1]]] + M0[[1]] Sin[eta0[[1]]]}}
```

```
{{110.156, 751.068}}
```

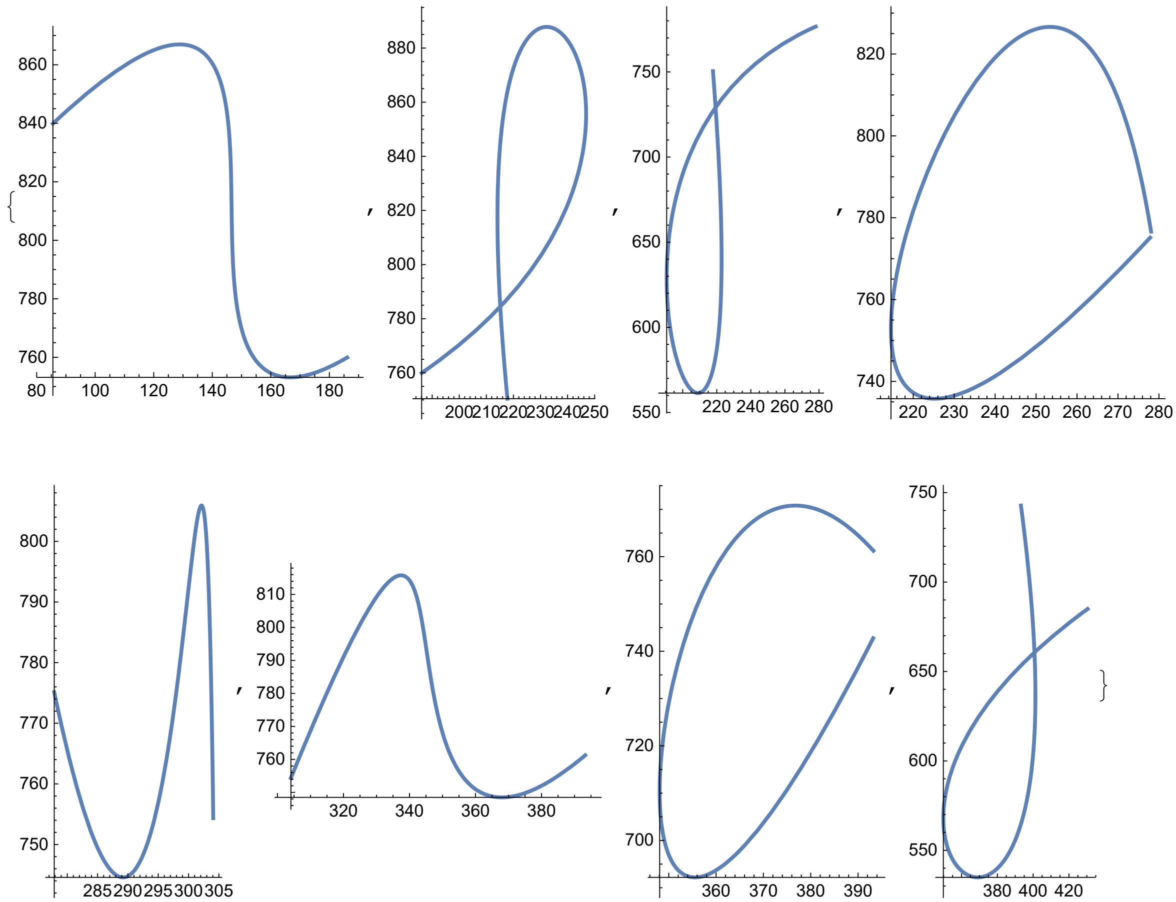
```
SegStartPoint = Table[{Lk[[i, 1]] Cos[psik[[i, 1]]] + Mk[[i, 1]] Cos[etak[[i, 1]]] +
    Lk[[i, 2]] Cos[psik[[i, 2]]] + Mk[[i, 2]] Cos[etak[[i, 2]]],
   Lk[[i, 1]] Sin[psik[[i, 1]]] + Mk[[i, 1]] Sin[etak[[i, 1]]] +
    Lk[[i, 2]] Sin[psik[[i, 2]]] + Mk[[i, 2]] Sin[etak[[i, 2]]]}, {i, Length[Lk]}]
```

```
{{-24.6071, 88.8228}, {77.8475, 310.235},
 {-157.634, -440.427}, {-138.158, 19.778}, {5.14958, 45.9649},
 {-53.2798, 83.2948}, {-107.761, -69.6661}, {-139.503, -397.587}}
```

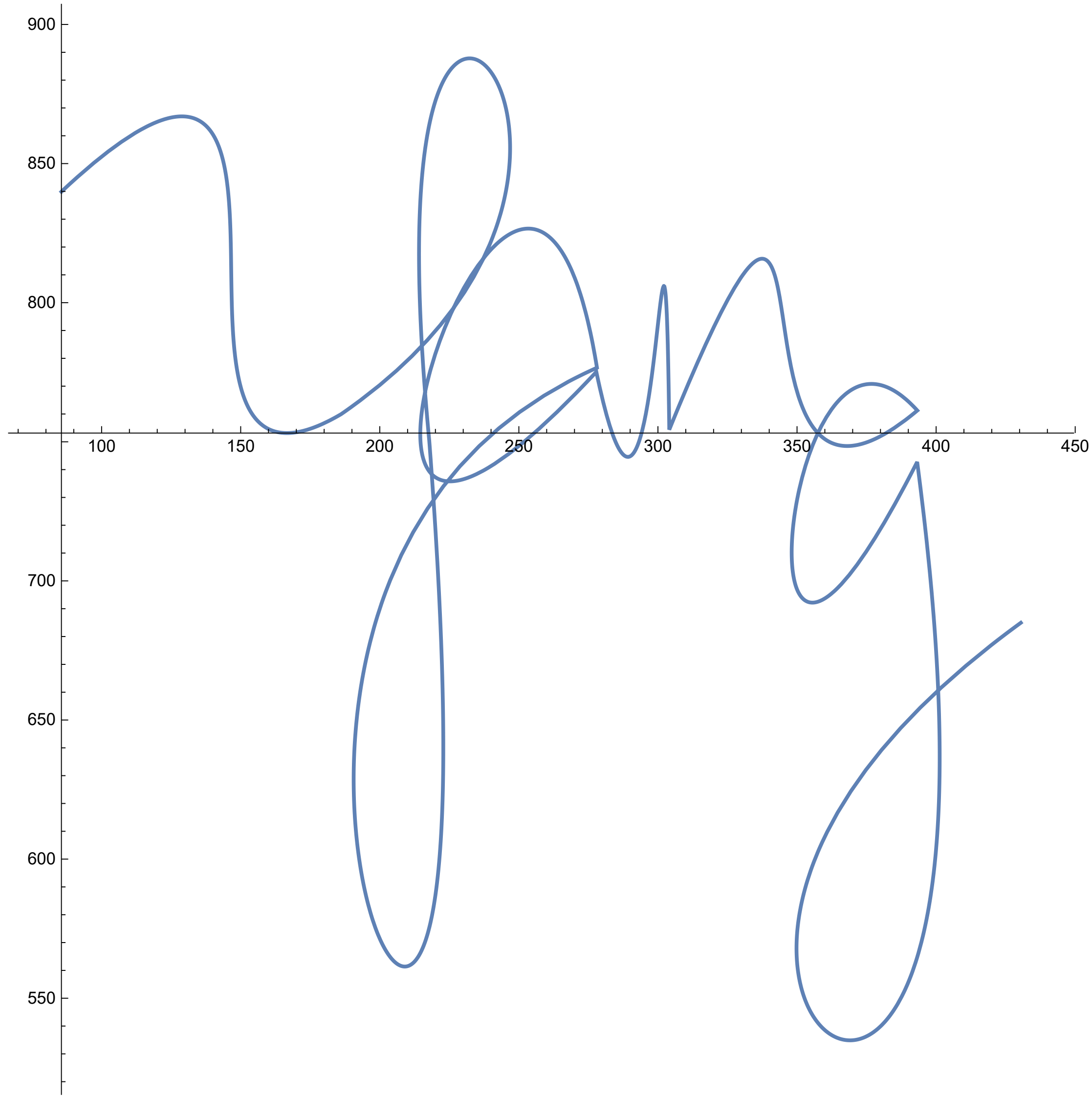
```
SegEndPoint =
 Table[{Lk[[i, 1]] Cos[psik[[i, 1]] + Pi/2] + Mk[[i, 1]] Cos[etak[[i, 1]] - Pi/2] +
     Lk[[i, 2]] Cos[psik[[i, 2]] + Pi] + Mk[[i, 2]] Cos[etak[[i, 2]] - Pi],
    Lk[[i, 1]] Sin[psik[[i, 1]] + Pi/2] + Mk[[i, 1]] Sin[etak[[i, 1]] - Pi/2] +
     Lk[[i, 2]] Sin[psik[[i, 2]] + Pi] +
     Mk[[i, 2]] Sin[etak[[i, 2]] - Pi]}, {i, Length[Lk]}]
{{75.9524, 8.86854}, {109.524, 300.747}, {-97.3375, -414.26}, {-138.385, 18.2983},
 {31.3652, 25.1822}, {35.8909, 90.19}, {-107.813, -88.1589}, {-102.242, -455.398}}

RelativePosition = Join[Link1GroundPosion,
  Table[SegEndPoint[[i]] - SegStartPoint[[i + 1]], {i, Length[Lk] - 1}]]
{{110.156, 751.068}, {-1.89505, -301.366}, {267.158, 741.174}, {40.82, -434.038},
 {-143.535, -27.6666}, {84.6449, -58.1126}, {143.652, 159.856}, {31.69, 309.428}}

AllNewGroundPosition =
 Map[Fold[Plus, Take[RelativePosition, #]] &, Range[Length[Lk]]]
{{110.156, 751.068}, {108.26, 449.702}, {375.418, 1190.88}, {416.238, 756.838},
 {272.704, 729.171}, {357.348, 671.059}, {501., 830.915}, {532.69, 1140.34}}

Newxθ = Table[
  Sum[Lk[[k, i]] Cos[psik[[k, i]] + i θ] + Mk[[k, i]] Cos[etak[[k, i]] - i θ], {i, 1, 2}] +
   AllNewGroundPosition[[k, 1]], {k, 1, n}]
{110.156 + 103.628 Cos[0.975814 - 2 θ] + 138.897 Cos[1.42726 - θ] +
  169.683 Cos[2.81368 - θ] + 103.628 Cos[0.975814 + 2 θ],
 108.26 + 93.1499 Cos[0.865136 - 2 θ] + 355.817 Cos[1.85848 - θ] +
  125.097 Cos[1.08886 + θ] + 93.1499 Cos[0.865136 + 2 θ],
 375.418 + 169.446 Cos[1.10572 - 2 θ] + 522.306 Cos[2.95657 - θ] +
  207.938 Cos[0.200626 + θ] + 169.446 Cos[1.10572 + 2 θ],
 416.238 + 140.994 Cos[1.23842 - 2 θ] + 103.865 Cos[1.30441 - θ] +
  296.304 Cos[2.62413 + θ] + 140.994 Cos[1.23842 + 2 θ],
 272.704 + 107.939 Cos[1.53165 - 2 θ] + 177.449 Cos[1.01506 - θ] +
  147.304 Cos[2.28885 + θ] + 107.939 Cos[1.53165 + 2 θ],
 357.348 + 105.532 Cos[1.18306 - 2 θ] + 137.065 Cos[0.98085 - θ] +
  209.341 Cos[3.13301 - θ] + 105.532 Cos[1.18306 + 2 θ],
 501. + 92.5338 Cos[1.45654 - 2 θ] + 85.0382 Cos[1.76701 - θ] +
  203.833 Cos[2.15417 + θ] + 92.5338 Cos[1.45654 + 2 θ],
 532.69 + 122.504 Cos[0.724652 - 2 θ] + 496.265 Cos[2.81717 - θ] +
  166.321 Cos[0.481249 + θ] + 122.504 Cos[0.724652 + 2 θ]}
```

```
NewyΘ = Table[
  Sum[Lk[[k, i]] Sin[psik[[k, i]] + i Θ] + Mk[[k, i]] Sin[etak[[k, i]] - i Θ], {i, 1, 2}] +
   AllNewGroundPosition[[k, 2]], {k, 1, n}]

{751.068 + 103.628 Sin[0.975814 - 2 Θ] - 138.897 Sin[1.42726 - Θ] +
  169.683 Sin[2.81368 - Θ] + 103.628 Sin[0.975814 + 2 Θ],
 449.702 - 93.1499 Sin[0.865136 - 2 Θ] + 355.817 Sin[1.85848 - Θ] +
  125.097 Sin[1.08886 + Θ] - 93.1499 Sin[0.865136 + 2 Θ],
 1190.88 - 169.446 Sin[1.10572 - 2 Θ] - 522.306 Sin[2.95657 - Θ] -
  207.938 Sin[0.200626 + Θ] - 169.446 Sin[1.10572 + 2 Θ],
 756.838 + 140.994 Sin[1.23842 - 2 Θ] - 103.865 Sin[1.30441 - Θ] -
  296.304 Sin[2.62413 + Θ] + 140.994 Sin[1.23842 + 2 Θ],
 729.171 - 107.939 Sin[1.53165 - 2 Θ] + 177.449 Sin[1.01506 - Θ] +
  147.304 Sin[2.28885 + Θ] - 107.939 Sin[1.53165 + 2 Θ],
 671.059 + 105.532 Sin[1.18306 - 2 Θ] - 137.065 Sin[0.98085 - Θ] +
  209.341 Sin[3.13301 - Θ] + 105.532 Sin[1.18306 + 2 Θ],
 830.915 + 92.5338 Sin[1.45654 - 2 Θ] - 85.0382 Sin[1.76701 - Θ] -
  203.833 Sin[2.15417 + Θ] + 92.5338 Sin[1.45654 + 2 Θ],
 1140.34 - 122.504 Sin[0.724652 - 2 Θ] - 496.265 Sin[2.81717 - Θ] -
  166.321 Sin[0.481249 + Θ] - 122.504 Sin[0.724652 + 2 Θ]}
```

```
Segm = Table[ParametricPlot[
    {Newxθ[[k]], Newyθ[[k]]}, {θ, 0, Pi/2}, PlotStyle → Thick], {k, 1, n}]
```



```
Show[Segm, PlotRange → All, ImageSize → Large]
```



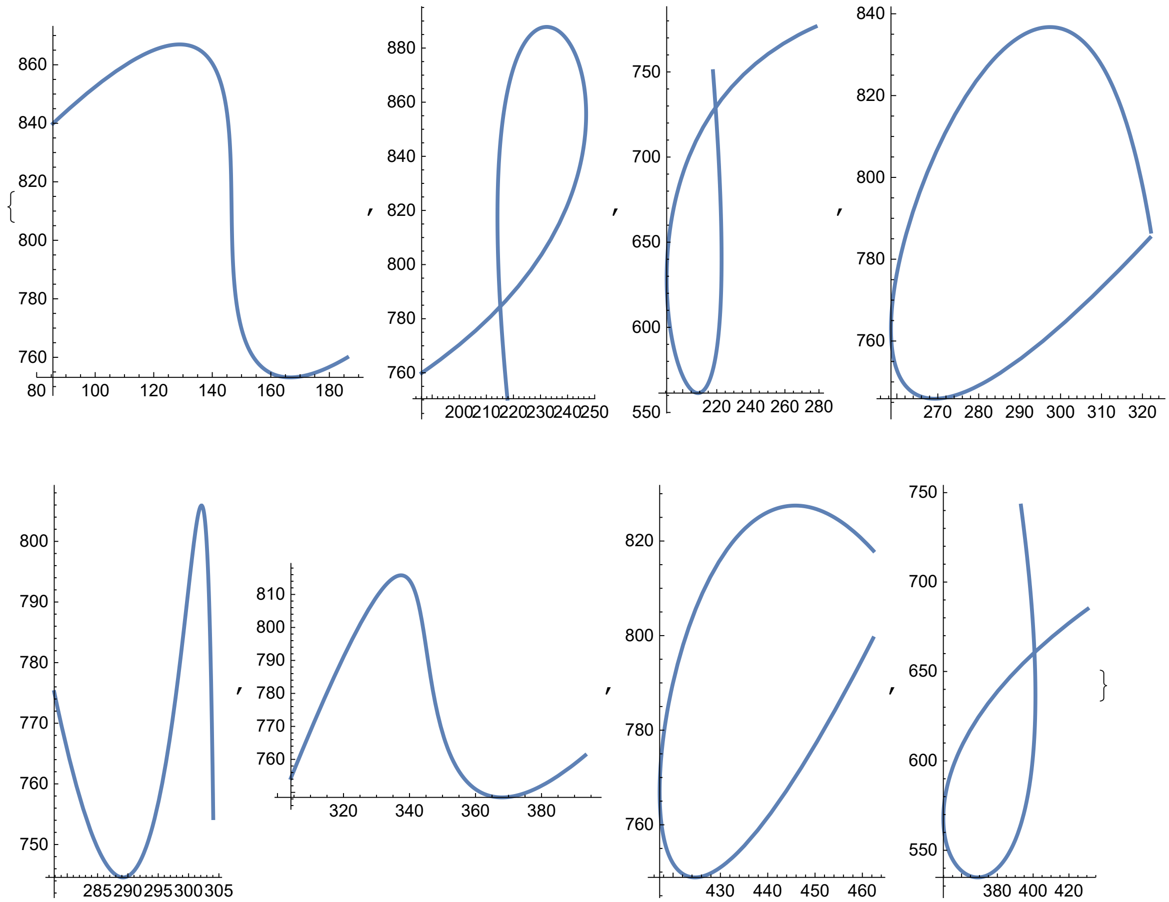
- Add letter break(The break has to be recalculate at every break places)

```
Newxθ1to3 = Table[
   Sum[Lk[[k, i]] Cos[psik[[k, i]] + i θ] + Mk[[k, i]] Cos[etak[[k, i]] - i θ], {i, 1, 2}] +
    AllNewGroundPosition[[k, 1]], {k, 1, 3}];

Newyθ1to3 = Table[
   Sum[Lk[[k, i]] Sin[psik[[k, i]] + i θ] + Mk[[k, i]] Sin[etak[[k, i]] - i θ], {i, 1, 2}] +
    AllNewGroundPosition[[k, 2]], {k, 1, 3}];
```

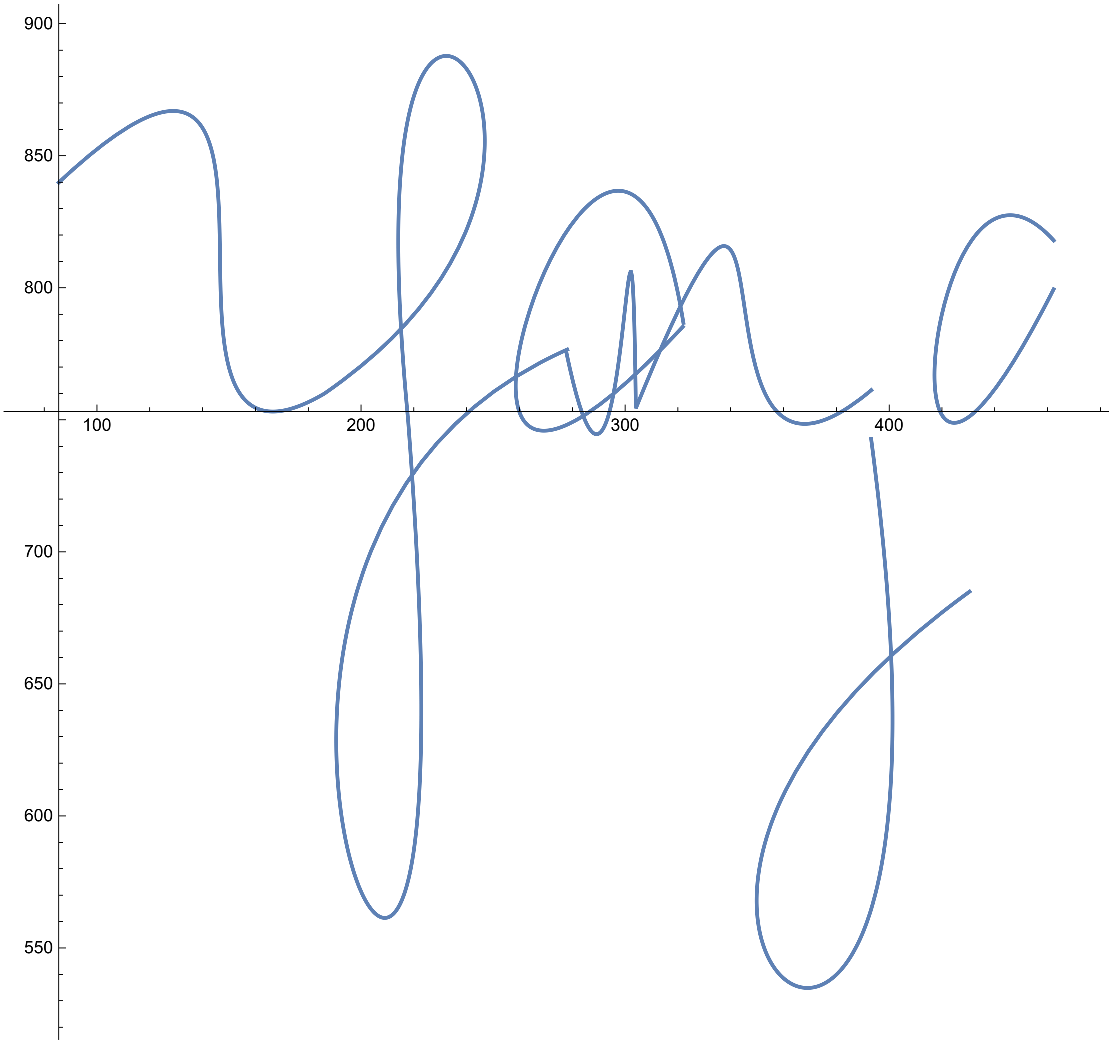
```
Newxθ4 = Table[
    Sum[Lk[[k, i]] Cos[psik[[k, i]] + i θ] + Mk[[k, i]] Cos[etak[[k, i]] - i θ], {i, 1, 2}] +
     L0 [[4]] Cos[psi0[[4]]] + M0[[4]] Cos[eta0[[4]]], {k, 4, 4}];

Newyθ4 = Table[
    Sum[Lk[[k, i]] Sin[psik[[k, i]] + i θ] + Mk[[k, i]] Sin[etak[[k, i]] - i θ], {i, 1, 2}] +
     L0[[4]] Sin[psi0[[4]]] + M0[[4]] Sin[eta0[[4]]], {k, 4, 4}];

Newxθ5to6 = Table[
    Sum[Lk[[k, i]] Cos[psik[[k, i]] + i θ] + Mk[[k, i]] Cos[etak[[k, i]] - i θ], {i, 1, 2}] +
     AllNewGroundPosition[[k, 1]], {k, 5, 6}];

Newyθ5to6 = Table[
    Sum[Lk[[k, i]] Sin[psik[[k, i]] + i θ] + Mk[[k, i]] Sin[etak[[k, i]] - i θ], {i, 1, 2}] +
     AllNewGroundPosition[[k, 2]], {k, 5, 6}];

Newxθ7 = Table[
    Sum[Lk[[k, i]] Cos[psik[[k, i]] + i θ] + Mk[[k, i]] Cos[etak[[k, i]] - i θ], {i, 1, 2}] +
     L0 [[7]] Cos[psi0[[7]]] + M0[[7]] Cos[eta0[[7]]], {k, 7, 7}];

Newyθ7 = Table[
    Sum[Lk[[k, i]] Sin[psik[[k, i]] + i θ] + Mk[[k, i]] Sin[etak[[k, i]] - i θ], {i, 1, 2}] +
     L0[[7]] Sin[psi0[[7]]] + M0[[7]] Sin[eta0[[7]]], {k, 7, 7}];

Newxθ8 = Table[
    Sum[Lk[[k, i]] Cos[psik[[k, i]] + i θ] + Mk[[k, i]] Cos[etak[[k, i]] - i θ], {i, 1, 2}] +
     AllNewGroundPosition[[k, 1]], {k, 8, 8}];

Newyθ8 = Table[
    Sum[Lk[[k, i]] Sin[psik[[k, i]] + i θ] + Mk[[k, i]] Sin[etak[[k, i]] - i θ], {i, 1, 2}] +
     AllNewGroundPosition[[k, 2]], {k, 8, 8}];

(*Newxθ9=
  Table[Sum[Lk[[k,i]]Cos[psik[[k,i]]+i θ]+Mk[[k,i]]Cos[etak[[k,i]]-i θ],{i,1,2}]+
    L0 [[9]]Cos[psi0[[9]]]+M0[[9]] Cos[eta0[[9]]],{k,9,9}];*)

(*Newyθ9=
  Table[Sum[Lk[[k,i]]Sin[psik[[k,i]]+i θ]+Mk[[k,i]]Sin[etak[[k,i]]-i θ],{i,1,2}]+
    L0[[9]] Sin[psi0[[9]]]+M0[[9]] Sin[eta0[[9]]],{k,9,9}];*)

(*Newxθ10to14=
  Table[Sum[Lk[[k,i]]Cos[psik[[k,i]]+i θ]+Mk[[k,i]]Cos[etak[[k,i]]-i θ],{i,1,2}]+
    AllNewGroundPosition[[k,1]],{k,10,14}];*)

(*Newyθ10to14=
  Table[Sum[Lk[[k,i]]Sin[psik[[k,i]]+i θ]+Mk[[k,i]]Sin[etak[[k,i]]-i θ],{i,1,2}]+
    AllNewGroundPosition[[k,2]],{k,10,14}];*)
```

```
Newxθ = Join[Newxθ1to3, Newxθ4, Newxθ5to6, Newxθ7, Newxθ8(*,Newxθ9,Newxθ10to14*)]
{110.156 + 103.628 Cos[0.975814 - 2 θ] + 138.897 Cos[1.42726 - θ] +
   169.683 Cos[2.81368 - θ] + 103.628 Cos[0.975814 + 2 θ],
  108.26 + 93.1499 Cos[0.865136 - 2 θ] + 355.817 Cos[1.85848 - θ] +
   125.097 Cos[1.08886 + θ] + 93.1499 Cos[0.865136 + 2 θ],
  375.418 + 169.446 Cos[1.10572 - 2 θ] + 522.306 Cos[2.95657 - θ] +
   207.938 Cos[0.200626 + θ] + 169.446 Cos[1.10572 + 2 θ],
  460.239 + 140.994 Cos[1.23842 - 2 θ] + 103.865 Cos[1.30441 - θ] +
   296.304 Cos[2.62413 + θ] + 140.994 Cos[1.23842 + 2 θ],
  272.704 + 107.939 Cos[1.53165 - 2 θ] + 177.449 Cos[1.01506 - θ] +
   147.304 Cos[2.28885 + θ] + 107.939 Cos[1.53165 + 2 θ],
  357.348 + 105.532 Cos[1.18306 - 2 θ] + 137.065 Cos[0.98085 - θ] +
   209.341 Cos[3.13301 - θ] + 105.532 Cos[1.18306 + 2 θ],
  570.092 + 92.5338 Cos[1.45654 - 2 θ] + 85.0382 Cos[1.76701 - θ] +
   203.833 Cos[2.15417 + θ] + 92.5338 Cos[1.45654 + 2 θ],
  532.69 + 122.504 Cos[0.724652 - 2 θ] + 496.265 Cos[2.81717 - θ] +
   166.321 Cos[0.481249 + θ] + 122.504 Cos[0.724652 + 2 θ]}

Newyθ = Join[Newyθ1to3, Newyθ4, Newyθ5to6, Newyθ7, Newyθ8(*,Newyθ9,Newyθ10to14*)]
{751.068 + 103.628 Sin[0.975814 - 2 θ] - 138.897 Sin[1.42726 - θ] +
   169.683 Sin[2.81368 - θ] + 103.628 Sin[0.975814 + 2 θ],
  449.702 - 93.1499 Sin[0.865136 - 2 θ] + 355.817 Sin[1.85848 - θ] +
   125.097 Sin[1.08886 + θ] - 93.1499 Sin[0.865136 + 2 θ],
  1190.88 - 169.446 Sin[1.10572 - 2 θ] - 522.306 Sin[2.95657 - θ] -
   207.938 Sin[0.200626 + θ] - 169.446 Sin[1.10572 + 2 θ],
  766.957 + 140.994 Sin[1.23842 - 2 θ] - 103.865 Sin[1.30441 - θ] -
   296.304 Sin[2.62413 + θ] + 140.994 Sin[1.23842 + 2 θ],
  729.171 - 107.939 Sin[1.53165 - 2 θ] + 177.449 Sin[1.01506 - θ] +
   147.304 Sin[2.28885 + θ] - 107.939 Sin[1.53165 + 2 θ],
  671.059 + 105.532 Sin[1.18306 - 2 θ] - 137.065 Sin[0.98085 - θ] +
   209.341 Sin[3.13301 - θ] + 105.532 Sin[1.18306 + 2 θ],
  887.604 + 92.5338 Sin[1.45654 - 2 θ] - 85.0382 Sin[1.76701 - θ] -
   203.833 Sin[2.15417 + θ] + 92.5338 Sin[1.45654 + 2 θ],
  1140.34 - 122.504 Sin[0.724652 - 2 θ] - 496.265 Sin[2.81717 - θ] -
   166.321 Sin[0.481249 + θ] - 122.504 Sin[0.724652 + 2 θ]}
```

```
Segm = Table[ParametricPlot[
   {Newxθ[[k]], Newyθ[[k]]}, {θ, 0, Pi/2}, PlotStyle → Thick], {k, 1, n}]
```



```
Show[Segm, PlotRange → All, ImageSize → Large]
```



- Recalculate Revised Ground Pivot Position (Breaks at 1st, 4th, 7th and 9th segments)

```
Link1GroundPosion = {{L0 [[1]] Cos[psi0[[1]]] + M0[[1]] Cos[eta0[[1]]],
   L0[[1]] Sin[psi0[[1]]] + M0[[1]] Sin[eta0[[1]]]}}
```

```
{{110.156, 751.068}}
```

```
Link4GroundPosion = {{L0 [[4]] Cos[psi0[[4]]] + M0[[4]] Cos[eta0[[4]]],
   L0[[4]] Sin[psi0[[4]]] + M0[[4]] Sin[eta0[[4]]]}}
```

```
{{460.239, 766.957}}
```

```
Link7GroundPosion = {{L0 [[7]] Cos[psi0[[7]]] + M0[[7]] Cos[eta0[[7]]],
   L0[[7]] Sin[psi0[[7]]] + M0[[7]] Sin[eta0[[7]]]}}
{{570.092, 887.604}}

(*Link9GroundPosion={{L0 [[9]]Cos[psi0[[9]]]+M0[[9]] Cos[eta0[[9]]],
   L0[[9]] Sin[psi0[[9]]]+M0[[9]] Sin[eta0[[9]]]}}*)

SegStartPoint = Table[{Lk[[i, 1]] Cos[psik[[i, 1]]] + Mk[[i, 1]] Cos[etak[[i, 1]]] +
    Lk[[i, 2]] Cos[psik[[i, 2]]] + Mk[[i, 2]] Cos[etak[[i, 2]]],
   Lk[[i, 1]] Sin[psik[[i, 1]]] + Mk[[i, 1]] Sin[etak[[i, 1]]] +
    Lk[[i, 2]] Sin[psik[[i, 2]]] + Mk[[i, 2]] Sin[etak[[i, 2]]]}, {i, Length[Lk]}]
{{-24.6071, 88.8228}, {77.8475, 310.235},
 {-157.634, -440.427}, {-138.158, 19.778}, {5.14958, 45.9649},
 {-53.2798, 83.2948}, {-107.761, -69.6661}, {-139.503, -397.587}}

SegEndPoint =
 Table[{Lk[[i, 1]] Cos[psik[[i, 1]] + Pi/2] + Mk[[i, 1]] Cos[etak[[i, 1]] - Pi/2] +
    Lk[[i, 2]] Cos[psik[[i, 2]] + Pi] + Mk[[i, 2]] Cos[etak[[i, 2]] - Pi],
   Lk[[i, 1]] Sin[psik[[i, 1]] + Pi/2] + Mk[[i, 1]] Sin[etak[[i, 1]] - Pi/2] +
    Lk[[i, 2]] Sin[psik[[i, 2]] + Pi] +
    Mk[[i, 2]] Sin[etak[[i, 2]] - Pi]}, {i, Length[Lk]}]
{{75.9524, 8.86854}, {109.524, 300.747}, {-97.3375, -414.26}, {-138.385, 18.2983},
 {31.3652, 25.1822}, {35.8909, 90.19}, {-107.813, -88.1589}, {-102.242, -455.398}}

RelativePosition1to3 =
 Join[Link1GroundPosion, Table[SegEndPoint[[i]] - SegStartPoint[[i + 1]], {i, 2}]]
{{110.156, 751.068}, {-1.89505, -301.366}, {267.158, 741.174}}

AllNewGroundPosition1to3 = Map[Fold[Plus, Take[RelativePosition1to3, #]] &, Range[3]]
{{110.156, 751.068}, {108.26, 449.702}, {375.418, 1190.88}}

RelativePosition4to6 =
 Join[Link4GroundPosion, Table[SegEndPoint[[i]] - SegStartPoint[[i + 1]], {i, 4, 5}]]
{{460.239, 766.957}, {-143.535, -27.6666}, {84.6449, -58.1126}}

AllNewGroundPosition4to6 = Map[Fold[Plus, Take[RelativePosition4to6, #]] &, Range[3]]
{{460.239, 766.957}, {316.704, 739.29}, {401.349, 681.178}}

RelativePosition7to8 =
 Join[Link7GroundPosion, Table[SegEndPoint[[i]] - SegStartPoint[[i + 1]], {i, 7, 7}]]
{{570.092, 887.604}, {31.69, 309.428}}

AllNewGroundPosition7to8 = Map[Fold[Plus, Take[RelativePosition7to8, #]] &, Range[2]]
{{570.092, 887.604}, {601.782, 1197.03}}
```

```
(*RelativePosition9to14=
 Join[Link9GroundPosion,Table[SegEndPoint[[i]]-SegStartPoint[[i+1]],{i,9,13}]]*)

(*AllNewGroundPosition9to14=
 Map[Fold[Plus,Take[RelativePosition9to14,#]]&,Range[6]]*)
```

Plot the letters

```
Newxθ1to3 = Table[
    Sum[Lk[[k, i]] Cos[psik[[k, i]] + i θ] + Mk[[k, i]] Cos[etak[[k, i]] - i θ], {i, 1, 2}] +
     AllNewGroundPosition1to3[[k, 1]], {k, 1, 3}];

Newyθ1to3 = Table[
    Sum[Lk[[k, i]] Sin[psik[[k, i]] + i θ] + Mk[[k, i]] Sin[etak[[k, i]] - i θ], {i, 1, 2}] +
     AllNewGroundPosition1to3[[k, 2]], {k, 1, 3}];

Newxθ4to6 = Table[
    Sum[Lk[[k, i]] Cos[psik[[k, i]] + i θ] + Mk[[k, i]] Cos[etak[[k, i]] - i θ], {i, 1, 2}] +
     AllNewGroundPosition4to6[[k - 3, 1]], {k, 4, 6}];

Newyθ4to6 = Table[
    Sum[Lk[[k, i]] Sin[psik[[k, i]] + i θ] + Mk[[k, i]] Sin[etak[[k, i]] - i θ], {i, 1, 2}] +
     AllNewGroundPosition4to6[[k - 3, 2]], {k, 4, 6}];

Newxθ7to8 = Table[
    Sum[Lk[[k, i]] Cos[psik[[k, i]] + i θ] + Mk[[k, i]] Cos[etak[[k, i]] - i θ], {i, 1, 2}] +
     AllNewGroundPosition7to8[[k - 6, 1]], {k, 7, 8}];

Newyθ7to8 = Table[
    Sum[Lk[[k, i]] Sin[psik[[k, i]] + i θ] + Mk[[k, i]] Sin[etak[[k, i]] - i θ], {i, 1, 2}] +
     AllNewGroundPosition7to8[[k - 6, 2]], {k, 7, 8}];

(*Newxθ9to14=
  Table[Sum[Lk[[k,i]]Cos[psik[[k,i]]+i θ]+Mk[[k,i]]Cos[etak[[k,i]]-i θ],{i,1,2}]+
    AllNewGroundPosition9to14[[k-8,1]],{k,9,14}];*)

(*Newyθ9to14=
  Table[Sum[Lk[[k,i]]Sin[psik[[k,i]]+i θ]+Mk[[k,i]]Sin[etak[[k,i]]-i θ],{i,1,2}]+
    AllNewGroundPosition9to14[[k-8,2]],{k,9,14}];*)
```

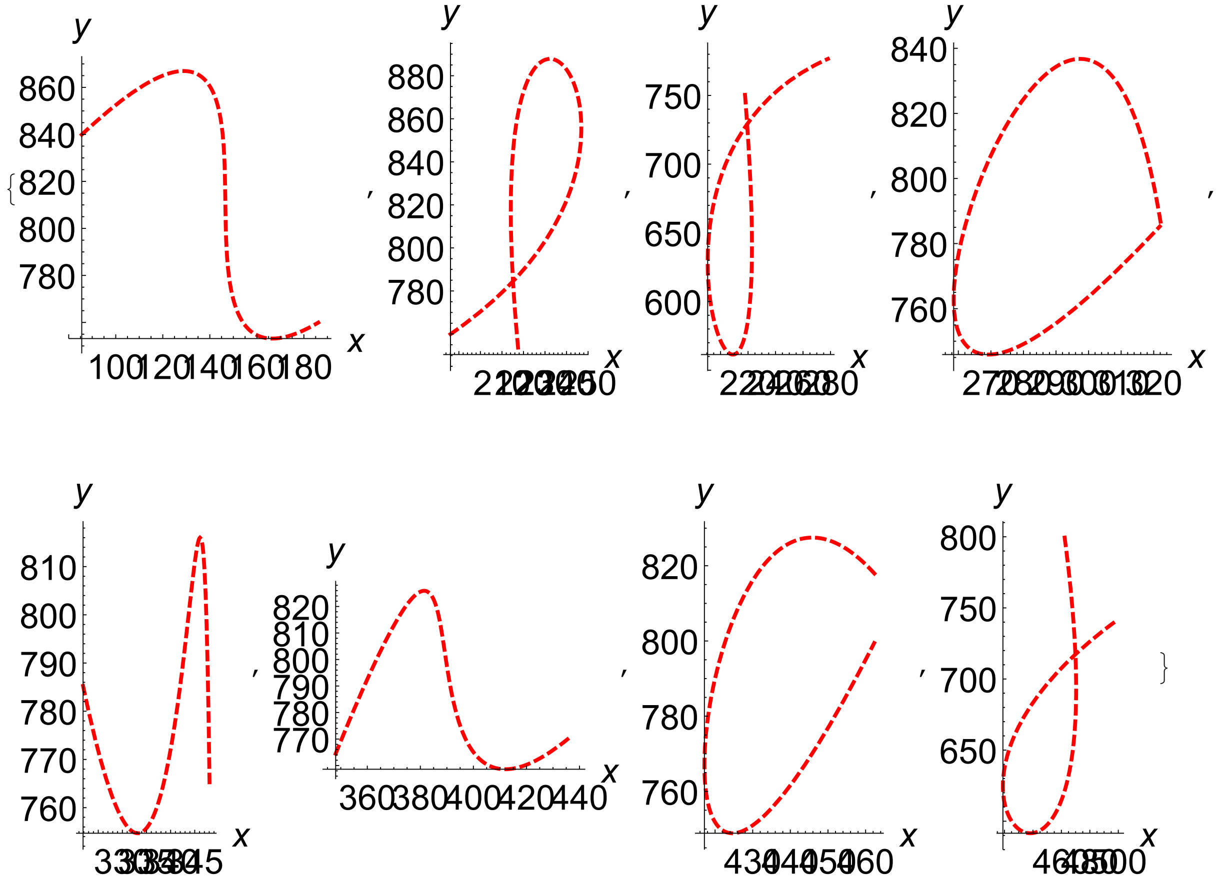
```
Newxθ = Join[Newxθ1to3, Newxθ4to6, Newxθ7to8(*,Newxθ9to14*)]
```

```
{110.156 + 103.628 Cos[0.975814 - 2 θ] + 138.897 Cos[1.42726 - θ] +
   169.683 Cos[2.81368 - θ] + 103.628 Cos[0.975814 + 2 θ],
  108.26 + 93.1499 Cos[0.865136 - 2 θ] + 355.817 Cos[1.85848 - θ] +
   125.097 Cos[1.08886 + θ] + 93.1499 Cos[0.865136 + 2 θ],
  375.418 + 169.446 Cos[1.10572 - 2 θ] + 522.306 Cos[2.95657 - θ] +
   207.938 Cos[0.200626 + θ] + 169.446 Cos[1.10572 + 2 θ],
  460.239 + 140.994 Cos[1.23842 - 2 θ] + 103.865 Cos[1.30441 - θ] +
   296.304 Cos[2.62413 + θ] + 140.994 Cos[1.23842 + 2 θ],
  316.704 + 107.939 Cos[1.53165 - 2 θ] + 177.449 Cos[1.01506 - θ] +
   147.304 Cos[2.28885 + θ] + 107.939 Cos[1.53165 + 2 θ],
  401.349 + 105.532 Cos[1.18306 - 2 θ] + 137.065 Cos[0.98085 - θ] +
   209.341 Cos[3.13301 - θ] + 105.532 Cos[1.18306 + 2 θ],
  570.092 + 92.5338 Cos[1.45654 - 2 θ] + 85.0382 Cos[1.76701 - θ] +
   203.833 Cos[2.15417 + θ] + 92.5338 Cos[1.45654 + 2 θ],
  601.782 + 122.504 Cos[0.724652 - 2 θ] + 496.265 Cos[2.81717 - θ] +
   166.321 Cos[0.481249 + θ] + 122.504 Cos[0.724652 + 2 θ]}
```

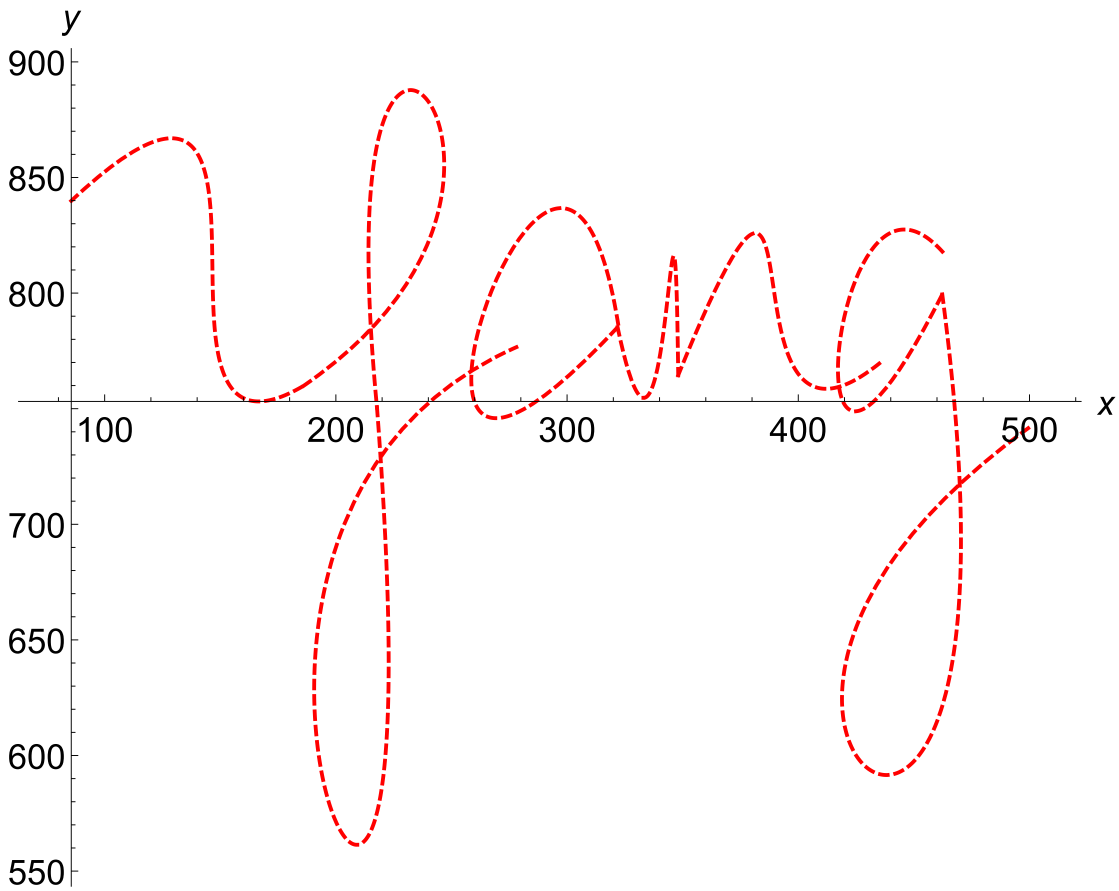
```
Newyθ = Join[Newyθ1to3, Newyθ4to6, Newyθ7to8(*,Newyθ9to14*)]
```

```
{751.068 + 103.628 Sin[0.975814 - 2 θ] - 138.897 Sin[1.42726 - θ] +
   169.683 Sin[2.81368 - θ] + 103.628 Sin[0.975814 + 2 θ],
  449.702 - 93.1499 Sin[0.865136 - 2 θ] + 355.817 Sin[1.85848 - θ] +
   125.097 Sin[1.08886 + θ] - 93.1499 Sin[0.865136 + 2 θ],
  1190.88 - 169.446 Sin[1.10572 - 2 θ] - 522.306 Sin[2.95657 - θ] -
   207.938 Sin[0.200626 + θ] - 169.446 Sin[1.10572 + 2 θ],
  766.957 + 140.994 Sin[1.23842 - 2 θ] - 103.865 Sin[1.30441 - θ] -
   296.304 Sin[2.62413 + θ] + 140.994 Sin[1.23842 + 2 θ],
  739.29 - 107.939 Sin[1.53165 - 2 θ] + 177.449 Sin[1.01506 - θ] +
   147.304 Sin[2.28885 + θ] - 107.939 Sin[1.53165 + 2 θ],
  681.178 + 105.532 Sin[1.18306 - 2 θ] - 137.065 Sin[0.98085 - θ] +
   209.341 Sin[3.13301 - θ] + 105.532 Sin[1.18306 + 2 θ],
  887.604 + 92.5338 Sin[1.45654 - 2 θ] - 85.0382 Sin[1.76701 - θ] -
   203.833 Sin[2.15417 + θ] + 92.5338 Sin[1.45654 + 2 θ],
  1197.03 - 122.504 Sin[0.724652 - 2 θ] - 496.265 Sin[2.81717 - θ] -
   166.321 Sin[0.481249 + θ] - 122.504 Sin[0.724652 + 2 θ]}
```

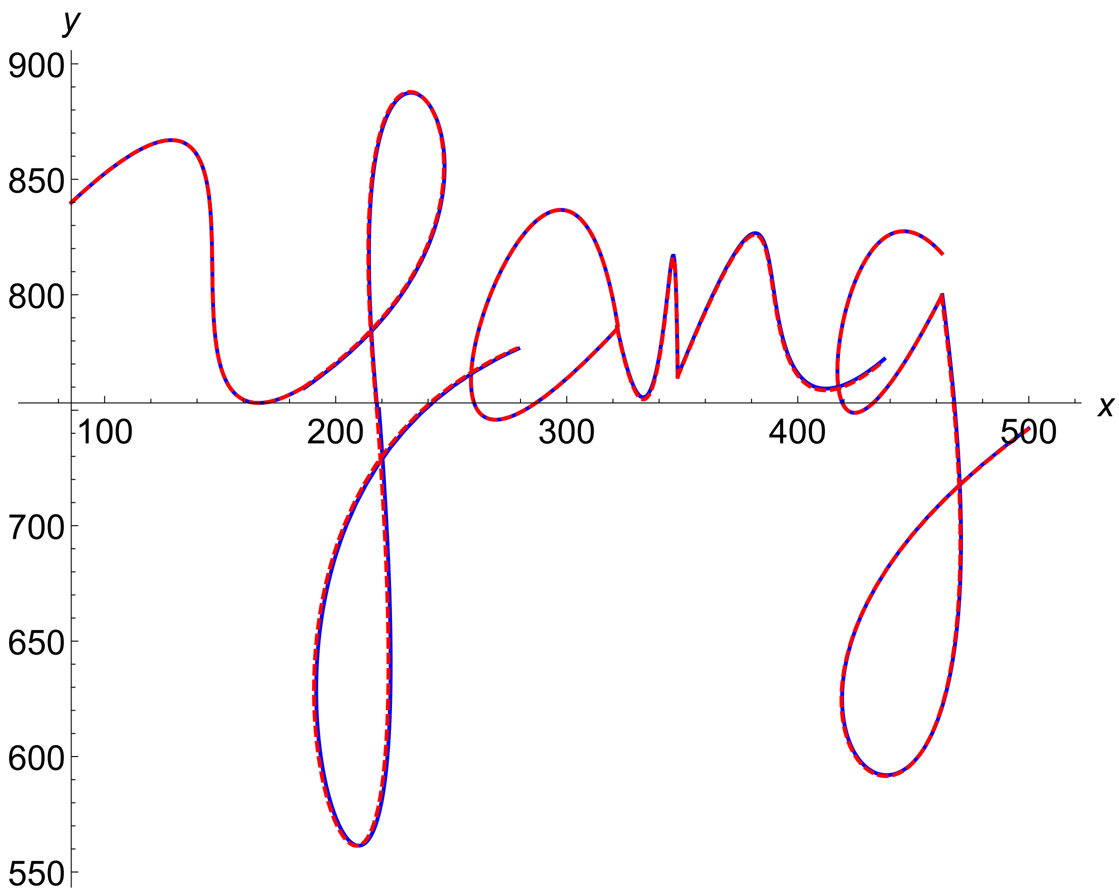
```
Segm4 = Table[ParametricPlot[{Newxθ[[k]], Newyθ[[k]]},
   {θ, 0, Pi/2}, PlotStyle → {Red, Dashed, Thick},
   AxesLabel → {x, y}, AxesStyle → Directive[18, Black]], {k, 1, n}]
```



```
Show[Segm4, PlotRange → All, ImageSize → Large]
```



```
Show[Segm6, Segm4, PlotRange → All, ImageSize → Large]
```



■ Compare link length range

```
LinksL = Join[Table[Lk[[i, 1]], {i, 1, n}], Table[Lk[[i, 2]], {i, 1, n}]]
```

```
{138.897, 125.097, 522.306, 103.865, 147.304, 137.065, 85.0382, 496.265,
 103.628, 93.1499, 169.446, 140.994, 107.939, 105.532, 92.5338, 122.504}
```

```
Max[LinksL]
```

```
522.306
```

```
Min[LinksL]
```

```
85.0382
```

```
LinksM = Join[Table[Mk[[i, 1]], {i, 1, n}], Table[Mk[[i, 2]], {i, 1, n}]]
```

```
{169.683, 355.817, 207.938, 296.304, 177.449, 209.341, 203.833, 166.321,
 103.628, 93.1499, 169.446, 140.994, 107.939, 105.532, 92.5338, 122.504}
```

```
Max[LinksM]
355.817
Min[LinksM]
92.5338
```

F Chinese Character Bezier Curve Synthesis Mathematica Code

Below is the Mathematica code for the computation of linkage system to draw Chinese character.

Chinese Dragon using Cubic Trigonometric Bezier

Cubic Trigonometric Bezier Basis Functions

```
b0t = (1 - λ Sin[Pi t/2]) (1 -  Sin[Pi t/2])^2
```

$$\left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)$$

```
b1t = Sin[Pi t/2] (1 - Sin[Pi t/2]) (2 + λ (1 - Sin[Pi t/2]))
```

$$\left(2+\lambda\left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)\mathrm{Sin}\left[\frac{\pi t}{2}\right]$$

```
b2t = Cos[Pi t/2] (1 - Cos[Pi t/2]) (2 + λ (1 - Cos[Pi t/2]))
```

$$\left(2+\lambda\left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)\mathrm{Cos}\left[\frac{\pi t}{2}\right]$$

```
b3t = (1 - λ Cos[Pi t/2]) (1 -  Cos[Pi t/2])^2
```

$$\left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)$$

Define Control Points

```
P0 = {{175.313, 724.266}, {166.372, 680.506}, {191.966, 652.835},
    {214.568, 657.916}, {223.715, 721.195}, {259.358, 659.505},
    {299.345, 681.455}, {353.309, 720.086}, {403.752, 696.697}};
P1 = {{224.14, 698.20}, {226.98, 713.13}, {239.31, 579.43},
   {273.38, 729.78}, {252.27, 750.96}, {213.07, 595.59},
   {295.66, 750.18}, {366.15, 742.17}, {385.52, 686.45}};
P2 = {{168.03, 709.63}, {171.78, 671.48}, {135.07, 587.58}, {271.95, 717.82}, {217.29,
    626.42}, {323.64, 585.64}, {337.36, 658.35}, {424.29, 711.36}, {356.55, 739.57}};
P3 = {{166.372, 680.506}, {191.966, 652.835}, {214.568, 657.916},
   {250.070, 718.316}, {259.358, 659.505}, {299.345, 681.455},
   {353.309, 720.086}, {403.752, 696.697}, {423.286, 755.077}};
```

Define the number of segments

```
n = Length[P0]
```

9

Cubic Trigonometric Bezier Curve

```
rt = Table[b0t P0[[k]] + b1t P1[[k]] + b2t P2[[k]] + b3t P3[[k]], {k, 1, n}]
```

$$\{\{168.03\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+ 166.372\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+224.14\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+175.313\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right),$$

$$709.63\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+ 680.506\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+698.2\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+724.266\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)\},$$

$$\{171.78\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+ 191.966\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+226.98\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+166.372\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right),$$

$$671.48\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+ 652.835\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+713.13\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+680.506\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)\},$$

$$\{135.07\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+ 214.568\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+239.31\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+191.966\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right),$$

$$587.58\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+ 657.916\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+579.43\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right) \left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+652.835\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)\},$$

$$\{271.95\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+$$

$$250.07\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+273.38\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$
$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+214.568\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right),$$
$$717.82\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+$$
$$718.316\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+729.78\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$
$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+657.916\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)\Big\},$$
$$\Big\{217.29\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+$$
$$259.358\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+252.27\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$
$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+223.715\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right),$$
$$626.42\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+$$
$$659.505\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+750.96\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$
$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+721.195\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)\Big\},$$
$$\Big\{323.64\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+$$
$$299.345\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+213.07\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$
$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+259.358\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right),$$
$$585.64\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+$$
$$681.455\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+595.59\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$
$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+659.505\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)\Big\},$$
$$\Big\{337.36\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+$$
$$353.309\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+295.66\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$
$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+299.345\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right),$$
$$658.35\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+$$
$$720.086\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)+750.18\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)$$
$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]+681.455\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)\Big\},$$
$$\Big\{424.29\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]+$$

```
  403.752 (1 - Cos[π t/2])^2 (1 - λ Cos[π t/2]) + 366.15 (2 + λ (1 - Sin[π t/2]))
    (1 - Sin[π t/2]) Sin[π t/2] + 353.309 (1 - Sin[π t/2])^2 (1 - λ Sin[π t/2]),
 711.36 (2 + λ (1 - Cos[π t/2])) (1 - Cos[π t/2]) Cos[π t/2] +
  696.697 (1 - Cos[π t/2])^2 (1 - λ Cos[π t/2]) + 742.17 (2 + λ (1 - Sin[π t/2]))
    (1 - Sin[π t/2]) Sin[π t/2] + 720.086 (1 - Sin[π t/2])^2 (1 - λ Sin[π t/2])},
{356.55 (2 + λ (1 - Cos[π t/2])) (1 - Cos[π t/2]) Cos[π t/2] +
  423.286 (1 - Cos[π t/2])^2 (1 - λ Cos[π t/2]) + 385.52 (2 + λ (1 - Sin[π t/2]))
    (1 - Sin[π t/2]) Sin[π t/2] + 403.752 (1 - Sin[π t/2])^2 (1 - λ Sin[π t/2]),
 739.57 (2 + λ (1 - Cos[π t/2])) (1 - Cos[π t/2]) Cos[π t/2] +
  755.077 (1 - Cos[π t/2])^2 (1 - λ Cos[π t/2]) + 686.45 (2 + λ (1 - Sin[π t/2]))
    (1 - Sin[π t/2]) Sin[π t/2] + 696.697 (1 - Sin[π t/2])^2 (1 - λ Sin[π t/2])}}
```

rtx = Table[TrigReduce[rt[[k, 1]]], {k, 1, n}]

$$
\begin{array}{l}
\{120.358 - 50.485\,\lambda + 3.316\,\text{Cos}\left[\frac{\pi\,t}{2}\right] + 2.9015\,\lambda\,\text{Cos}\left[\frac{\pi\,t}{2}\right] + \\
\quad 51.6395\,\text{Cos}[\pi\,t] + 47.169\,\lambda\,\text{Cos}[\pi\,t] + 0.4145\,\lambda\,\text{Cos}\left[\frac{3\,\pi\,t}{2}\right] + \\
\quad 97.654\,\text{Sin}\left[\frac{\pi\,t}{2}\right] + 85.4473\,\lambda\,\text{Sin}\left[\frac{\pi\,t}{2}\right] - 12.2068\,\lambda\,\text{Sin}\left[\frac{3\,\pi\,t}{2}\right], \\
138.747 - 40.422\,\lambda - 40.372\,\text{Cos}\left[\frac{\pi\,t}{2}\right] - 35.3255\,\lambda\,\text{Cos}\left[\frac{\pi\,t}{2}\right] + \\
\quad 67.997\,\text{Cos}[\pi\,t] + 80.794\,\lambda\,\text{Cos}[\pi\,t] - 5.0465\,\lambda\,\text{Cos}\left[\frac{3\,\pi\,t}{2}\right] + \\
\quad 121.216\,\text{Sin}\left[\frac{\pi\,t}{2}\right] + 106.064\,\lambda\,\text{Sin}\left[\frac{\pi\,t}{2}\right] - 15.152\,\lambda\,\text{Sin}\left[\frac{3\,\pi\,t}{2}\right], \\
235.421 + 32.154\,\lambda - 158.996\,\text{Cos}\left[\frac{\pi\,t}{2}\right] - 139.122\,\lambda\,\text{Cos}\left[\frac{\pi\,t}{2}\right] + 115.541\,\text{Cos}[\pi\,t] + \\
\quad 126.842\,\lambda\,\text{Cos}[\pi\,t] - 19.8745\,\lambda\,\text{Cos}\left[\frac{3\,\pi\,t}{2}\right] + 94.688\,\text{Sin}\left[\frac{\pi\,t}{2}\right] + \\
\quad 82.852\,\lambda\,\text{Sin}\left[\frac{\pi\,t}{2}\right] - 11.836\,\lambda\,\text{Sin}\left[\frac{3\,\pi\,t}{2}\right], 151.627 - 80.692\,\lambda + 43.76\,\text{Cos}\left[\frac{\pi\,t}{2}\right] + \\
\quad 38.29\,\lambda\,\text{Cos}\left[\frac{\pi\,t}{2}\right] + 19.181\,\text{Cos}[\pi\,t] + 36.932\,\lambda\,\text{Cos}[\pi\,t] + 5.47\,\lambda\,\text{Cos}\left[\frac{3\,\pi\,t}{2}\right] + \\
\quad 117.624\,\text{Sin}\left[\frac{\pi\,t}{2}\right] + 102.921\,\lambda\,\text{Sin}\left[\frac{\pi\,t}{2}\right] - 14.703\,\lambda\,\text{Sin}\left[\frac{3\,\pi\,t}{2}\right], \\
255.05 + 13.513\,\lambda - 84.136\,\text{Cos}\left[\frac{\pi\,t}{2}\right] - 73.619\,\lambda\,\text{Cos}\left[\frac{\pi\,t}{2}\right] + 52.8015\,\text{Cos}[\pi\,t] + \\
\quad 70.623\,\lambda\,\text{Cos}[\pi\,t] - 10.517\,\lambda\,\text{Cos}\left[\frac{3\,\pi\,t}{2}\right] + 57.11\,\text{Sin}\left[\frac{\pi\,t}{2}\right] + \\
\quad 49.9713\,\lambda\,\text{Sin}\left[\frac{\pi\,t}{2}\right] - 7.13875\,\lambda\,\text{Sin}\left[\frac{3\,\pi\,t}{2}\right], 301.345 + 21.993\,\lambda + \\
\quad 48.59\,\text{Cos}\left[\frac{\pi\,t}{2}\right] + 42.5163\,\lambda\,\text{Cos}\left[\frac{\pi\,t}{2}\right] - 90.5765\,\text{Cos}[\pi\,t] - 70.583\,\lambda\,\text{Cos}[\pi\,t] + \\
\quad 6.07375\,\lambda\,\text{Cos}\left[\frac{3\,\pi\,t}{2}\right] - 92.576\,\text{Sin}\left[\frac{\pi\,t}{2}\right] - 81.004\,\lambda\,\text{Sin}\left[\frac{\pi\,t}{2}\right] + 11.572\,\lambda\,\text{Sin}\left[\frac{3\,\pi\,t}{2}\right], \\
345.961 + 19.634\,\lambda - 31.898\,\text{Cos}\left[\frac{\pi\,t}{2}\right] - 27.9108\,\lambda\,\text{Cos}\left[\frac{\pi\,t}{2}\right] - 14.718\,\text{Cos}[\pi\,t] + \\
\quad 12.264\,\lambda\,\text{Cos}[\pi\,t] - 3.98725\,\lambda\,\text{Cos}\left[\frac{3\,\pi\,t}{2}\right] - 7.37\,\text{Sin}\left[\frac{\pi\,t}{2}\right] - 6.44875\,\lambda\,\text{Sin}\left[\frac{\pi\,t}{2}\right] + \\
\quad 0.92125\,\lambda\,\text{Sin}\left[\frac{3\,\pi\,t}{2}\right], 345.151 - 33.379\,\lambda + 41.076\,\text{Cos}\left[\frac{\pi\,t}{2}\right] + \\
\quad 35.9415\,\lambda\,\text{Cos}\left[\frac{\pi\,t}{2}\right] - 32.9185\,\text{Cos}[\pi\,t] - 7.697\,\lambda\,\text{Cos}[\pi\,t] + 5.1345\,\lambda\,\text{Cos}\left[\frac{3\,\pi\,t}{2}\right] + \\
\quad 25.682\,\text{Sin}\left[\frac{\pi\,t}{2}\right] + 22.4717\,\lambda\,\text{Sin}\left[\frac{\pi\,t}{2}\right] - 3.21025\,\lambda\,\text{Sin}\left[\frac{3\,\pi\,t}{2}\right], \\
498.487 + 84.968\,\lambda - 133.472\,\text{Cos}\left[\frac{\pi\,t}{2}\right] - 116.788\,\lambda\,\text{Cos}\left[\frac{\pi\,t}{2}\right] + \\
\quad 38.737\,\text{Cos}[\pi\,t] + 48.504\,\lambda\,\text{Cos}[\pi\,t] - 16.684\,\lambda\,\text{Cos}\left[\frac{3\,\pi\,t}{2}\right] - \\
\quad 36.464\,\text{Sin}\left[\frac{\pi\,t}{2}\right] - 31.906\,\lambda\,\text{Sin}\left[\frac{\pi\,t}{2}\right] + 4.558\,\lambda\,\text{Sin}\left[\frac{3\,\pi\,t}{2}\right]\}
\end{array}
$$

```
rty = Table[TrigReduce[rt[[k, 2]]], {k, 1, n}]
```

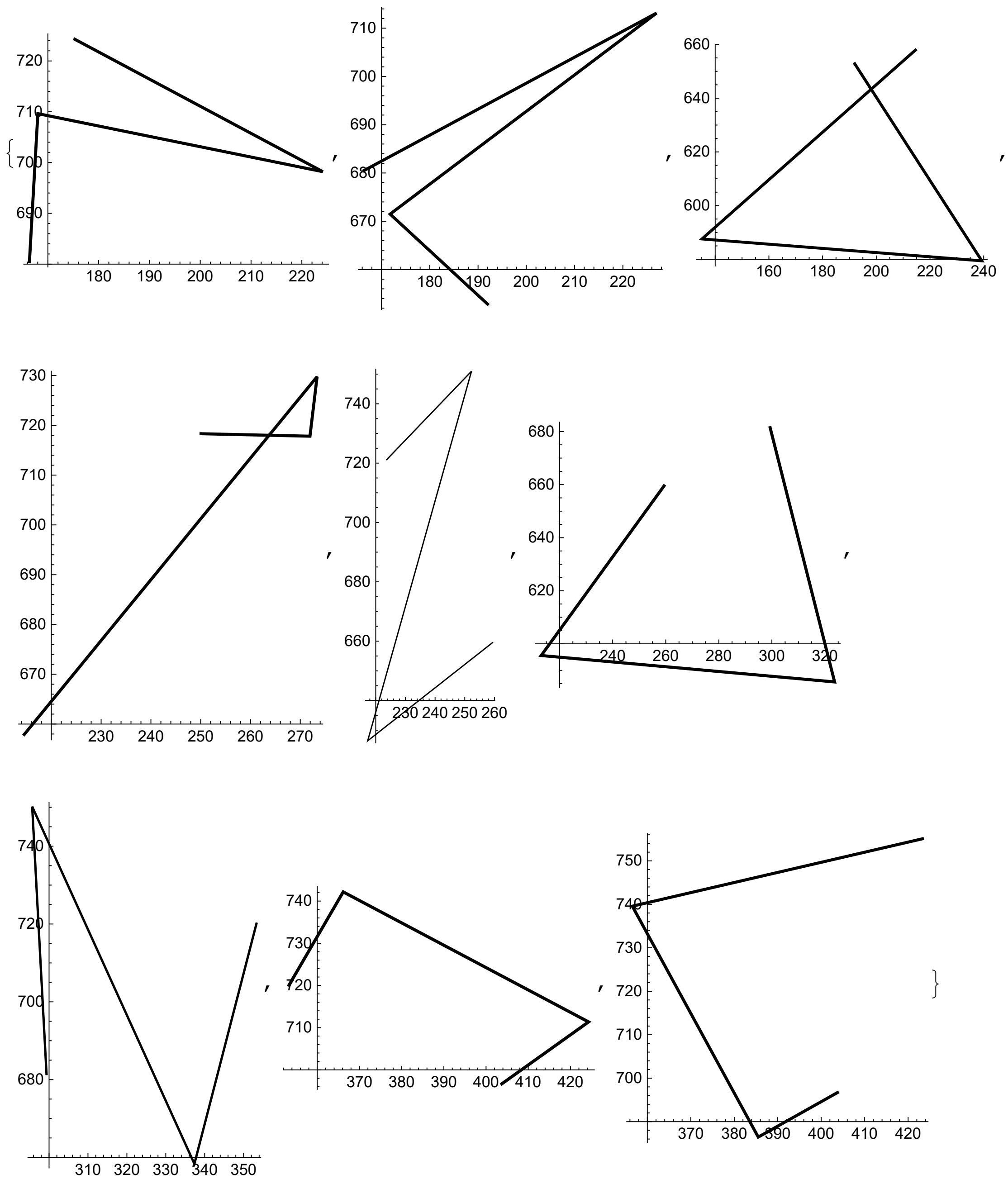
```
{699.328 - 3.058 λ + 58.248 Cos[(π t)/2] + 50.967 λ Cos[(π t)/2] - 33.31 Cos[π t] - 55.19 λ Cos[π t] +
   7.281 λ Cos[(3 π t)/2] - 52.132 Sin[(π t)/2] - 45.6155 λ Sin[(π t)/2] + 6.5165 λ Sin[(3 π t)/2],
 615.401 - 51.269 λ + 37.29 Cos[(π t)/2] + 32.6288 λ Cos[(π t)/2] + 27.8145 Cos[π t] +
   13.979 λ Cos[π t] + 4.66125 λ Cos[(3 π t)/2] + 65.248 Sin[(π t)/2] + 57.092 λ Sin[(π t)/2] -
   8.156 λ Sin[(3 π t)/2], 799.117 + 143.741 λ - 140.672 Cos[(π t)/2] -
   123.088 λ Cos[(π t)/2] - 5.6095 Cos[π t] - 3.069 λ Cos[π t] - 17.584 λ Cos[(3 π t)/2] -
   146.81 Sin[(π t)/2] - 128.459 λ Sin[(π t)/2] + 18.3513 λ Sin[(3 π t)/2],
 616.748 - 71.368 λ - 0.992 Cos[(π t)/2] - 0.868 λ Cos[(π t)/2] + 42.16 Cos[π t] + 72.36 λ Cos[π t] -
   0.124 λ Cos[(3 π t)/2] + 143.728 Sin[(π t)/2] + 125.762 λ Sin[(π t)/2] - 17.966 λ Sin[(3 π t)/2],
 693.67 + 3.32 λ - 66.17 Cos[(π t)/2] - 57.8988 λ Cos[(π t)/2] + 93.695 Cos[π t] + 62.85 λ Cos[π t] -
   8.27125 λ Cos[(3 π t)/2] + 59.53 Sin[(π t)/2] + 52.0888 λ Sin[(π t)/2] - 7.44125 λ Sin[(3 π t)/2],
 830.21 + 159.73 λ - 191.63 Cos[(π t)/2] - 167.676 λ Cos[(π t)/2] + 20.925 Cos[π t] +
   31.9 λ Cos[π t] - 23.9538 λ Cos[(3 π t)/2] - 127.83 Sin[(π t)/2] - 111.851 λ Sin[(π t)/2] +
   15.9787 λ Sin[(3 π t)/2], 693.781 - 6.989 λ - 123.472 Cos[(π t)/2] -
   108.038 λ Cos[(π t)/2] + 111.145 Cos[π t] + 130.461 λ Cos[π t] - 15.434 λ Cos[(3 π t)/2] +
   137.45 Sin[(π t)/2] + 120.269 λ Sin[(π t)/2] - 17.1812 λ Sin[(3 π t)/2],
 671.645 - 36.747 λ + 29.326 Cos[(π t)/2] + 25.6603 λ Cos[(π t)/2] + 19.1155 Cos[π t] +
   7.421 λ Cos[π t] + 3.66575 λ Cos[(3 π t)/2] + 44.168 Sin[(π t)/2] +
   38.647 λ Sin[(π t)/2] - 5.521 λ Sin[(3 π t)/2], 751.641 + 25.754 λ - 31.014 Cos[(π t)/2] -
   27.1372 λ Cos[(π t)/2] - 23.93 Cos[π t] + 5.26 λ Cos[π t] - 3.87675 λ Cos[(3 π t)/2] -
   20.494 Sin[(π t)/2] - 17.9322 λ Sin[(π t)/2] + 2.56175 λ Sin[(3 π t)/2]}
```

Set λ value

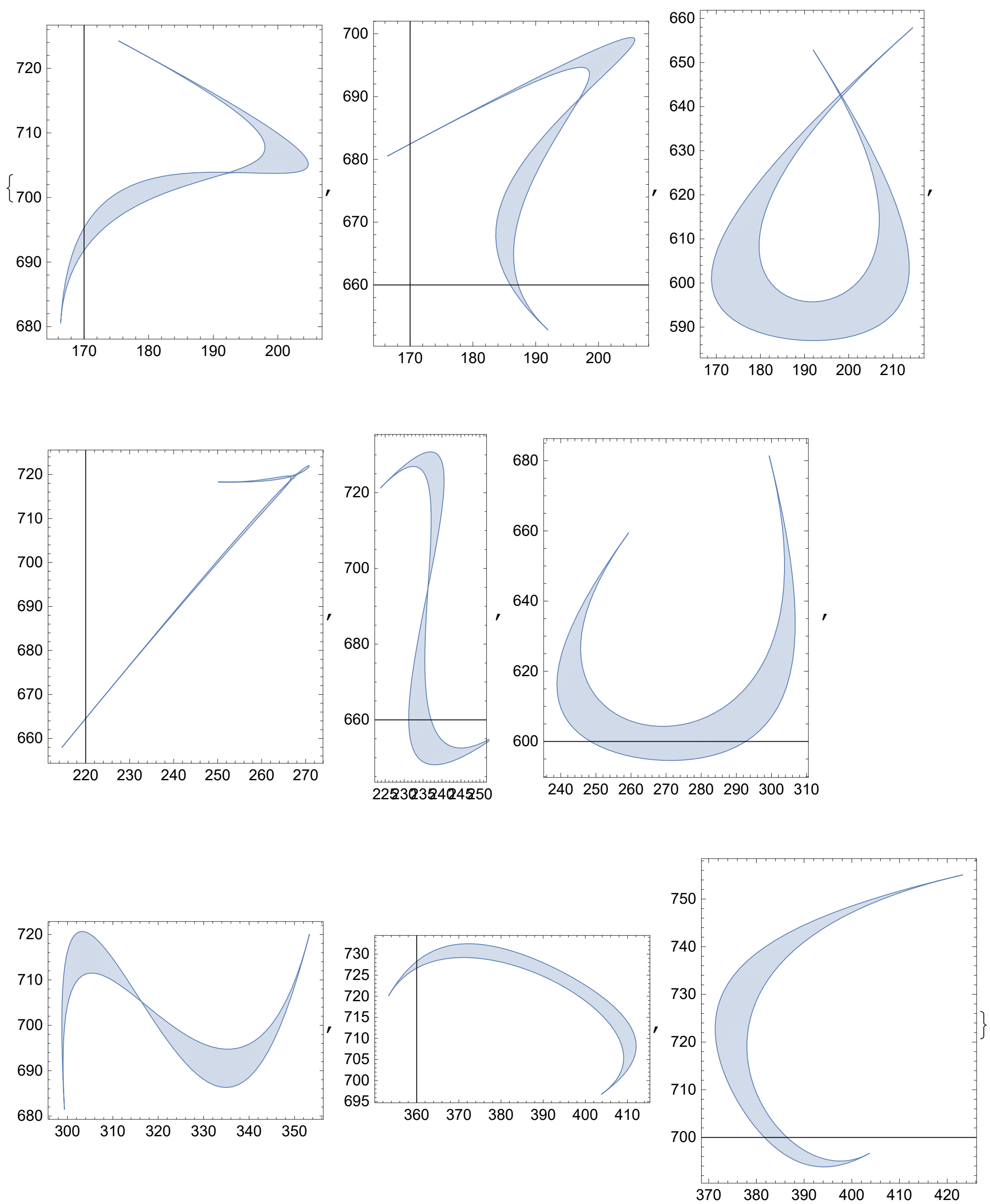
```
λ =.;
```

Curve Plot

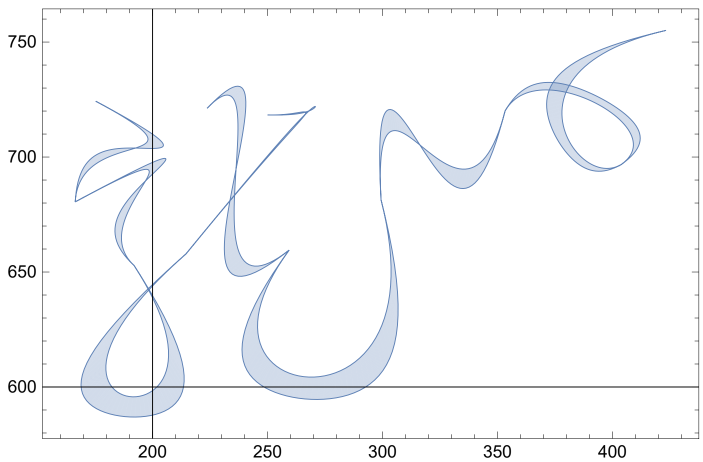
```
PolyG = Table[Graphics[{Thickness[0.01],
    Line[{P0[[k]], P1[[k]], P2[[k]], P3[[k]]}]}, Axes → True], {k, 1, n}]
```



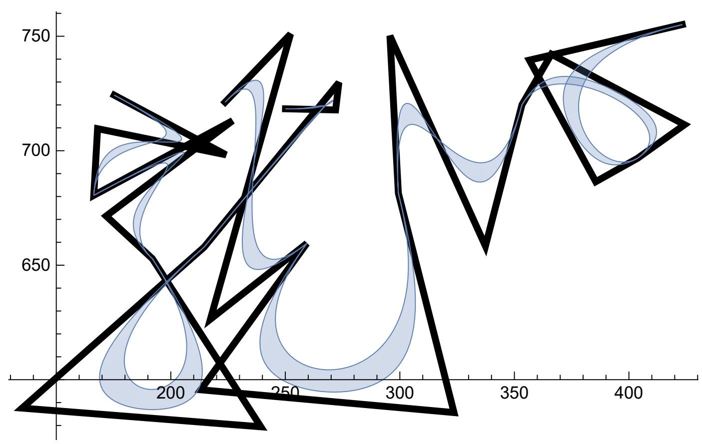
```
Curve = Table[ParametricPlot[{rtx[[k]], rty[[k]]}, {t, 0, 1}, {λ, 0, 1}], {k, 1, n}]
```



```
Show[Curve, PlotRange → All]
```




```
Show[PolyG, Curve]
```



■ Set λ value

```
λ = 0.01;
```

Trigonometric Functions for x and y components

```
rtx
```

$$\{119.853 + 3.34501 \mathrm{Cos}[\frac{\pi t}{2}] + 52.1112 \mathrm{Cos}[\pi t] + 0.004145 \mathrm{Cos}[\frac{3 \pi t}{2}] + 98.5085 \mathrm{Sin}[\frac{\pi t}{2}] - 0.122068 \mathrm{Sin}[\frac{3 \pi t}{2}],$$
$$138.343 - 40.7253 \mathrm{Cos}[\frac{\pi t}{2}] + 68.8049 \mathrm{Cos}[\pi t] - 0.050465 \mathrm{Cos}[\frac{3 \pi t}{2}] + 122.277 \mathrm{Sin}[\frac{\pi t}{2}] - 0.15152 \mathrm{Sin}[\frac{3 \pi t}{2}],$$
$$235.743 - 160.387 \mathrm{Cos}[\frac{\pi t}{2}] + 116.809 \mathrm{Cos}[\pi t] - 0.198745 \mathrm{Cos}[\frac{3 \pi t}{2}] + 95.5165 \mathrm{Sin}[\frac{\pi t}{2}] - 0.11836 \mathrm{Sin}[\frac{3 \pi t}{2}],$$
$$150.82 + 44.1429 \mathrm{Cos}[\frac{\pi t}{2}] + 19.5503 \mathrm{Cos}[\pi t] + 0.0547 \mathrm{Cos}[\frac{3 \pi t}{2}] + 118.653 \mathrm{Sin}[\frac{\pi t}{2}] - 0.14703 \mathrm{Sin}[\frac{3 \pi t}{2}],$$
$$255.185 - 84.8722 \mathrm{Cos}[\frac{\pi t}{2}] + 53.5077 \mathrm{Cos}[\pi t] - 0.10517 \mathrm{Cos}[\frac{3 \pi t}{2}] + 57.6097 \mathrm{Sin}[\frac{\pi t}{2}] - 0.0713875 \mathrm{Sin}[\frac{3 \pi t}{2}],$$
$$301.564 + 49.0152 \mathrm{Cos}[\frac{\pi t}{2}] - 91.2823 \mathrm{Cos}[\pi t] + 0.0607375 \mathrm{Cos}[\frac{3 \pi t}{2}] - 93.386 \mathrm{Sin}[\frac{\pi t}{2}] + 0.11572 \mathrm{Sin}[\frac{3 \pi t}{2}],$$
$$346.157 - 32.1771 \mathrm{Cos}[\frac{\pi t}{2}] - 14.5954 \mathrm{Cos}[\pi t] - 0.0398725 \mathrm{Cos}[\frac{3 \pi t}{2}] - 7.43449 \mathrm{Sin}[\frac{\pi t}{2}] + 0.0092125 \mathrm{Sin}[\frac{3 \pi t}{2}],$$
$$344.818 + 41.4354 \mathrm{Cos}[\frac{\pi t}{2}] - 32.9955 \mathrm{Cos}[\pi t] + 0.051345 \mathrm{Cos}[\frac{3 \pi t}{2}] + 25.9067 \mathrm{Sin}[\frac{\pi t}{2}] - 0.0321025 \mathrm{Sin}[\frac{3 \pi t}{2}],$$
$$499.337 - 134.64 \mathrm{Cos}[\frac{\pi t}{2}] + 39.222 \mathrm{Cos}[\pi t] - 0.16684 \mathrm{Cos}[\frac{3 \pi t}{2}] - 36.7831 \mathrm{Sin}[\frac{\pi t}{2}] + 0.04558 \mathrm{Sin}[\frac{3 \pi t}{2}]\}$$

rty

$$\left\{699.297 + 58.7577 \,\mathrm{Cos}\left[\frac{\pi t}{2}\right] - 33.8619 \,\mathrm{Cos}[\pi t] + 0.07281 \,\mathrm{Cos}\left[\frac{3 \pi t}{2}\right] - 52.5882 \,\mathrm{Sin}\left[\frac{\pi t}{2}\right] + 0.065165 \,\mathrm{Sin}\left[\frac{3 \pi t}{2}\right],\right.$$

$$614.889 + 37.6163 \,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + 27.9543 \,\mathrm{Cos}[\pi t] + 0.0466125 \,\mathrm{Cos}\left[\frac{3 \pi t}{2}\right] + 65.8189 \,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - 0.08156 \,\mathrm{Sin}\left[\frac{3 \pi t}{2}\right],$$

$$800.554 - 141.903 \,\mathrm{Cos}\left[\frac{\pi t}{2}\right] - 5.64019 \,\mathrm{Cos}[\pi t] - 0.17584 \,\mathrm{Cos}\left[\frac{3 \pi t}{2}\right] - 148.095 \,\mathrm{Sin}\left[\frac{\pi t}{2}\right] + 0.183513 \,\mathrm{Sin}\left[\frac{3 \pi t}{2}\right],$$

$$616.034 - 1.00068 \,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + 42.8836 \,\mathrm{Cos}[\pi t] - 0.00124 \,\mathrm{Cos}\left[\frac{3 \pi t}{2}\right] + 144.986 \,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - 0.17966 \,\mathrm{Sin}\left[\frac{3 \pi t}{2}\right],$$

$$693.703 - 66.749 \,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + 94.3235 \,\mathrm{Cos}[\pi t] - 0.0827125 \,\mathrm{Cos}\left[\frac{3 \pi t}{2}\right] + 60.0509 \,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - 0.0744125 \,\mathrm{Sin}\left[\frac{3 \pi t}{2}\right],$$

$$831.807 - 193.307 \,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + 21.244 \,\mathrm{Cos}[\pi t] - 0.239538 \,\mathrm{Cos}\left[\frac{3 \pi t}{2}\right] - 128.949 \,\mathrm{Sin}\left[\frac{\pi t}{2}\right] + 0.159787 \,\mathrm{Sin}\left[\frac{3 \pi t}{2}\right],$$

$$693.712 - 124.552 \,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + 112.45 \,\mathrm{Cos}[\pi t] - 0.15434 \,\mathrm{Cos}\left[\frac{3 \pi t}{2}\right] + 138.653 \,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - 0.171812 \,\mathrm{Sin}\left[\frac{3 \pi t}{2}\right],$$

$$671.277 + 29.5826 \,\mathrm{Cos}\left[\frac{\pi t}{2}\right] + 19.1897 \,\mathrm{Cos}[\pi t] + 0.0366575 \,\mathrm{Cos}\left[\frac{3 \pi t}{2}\right] + 44.5545 \,\mathrm{Sin}\left[\frac{\pi t}{2}\right] - 0.05521 \,\mathrm{Sin}\left[\frac{3 \pi t}{2}\right],$$

$$\left.751.899 - 31.2854 \,\mathrm{Cos}\left[\frac{\pi t}{2}\right] - 23.8774 \,\mathrm{Cos}[\pi t] - 0.0387675 \,\mathrm{Cos}\left[\frac{3 \pi t}{2}\right] - 20.6733 \,\mathrm{Sin}\left[\frac{\pi t}{2}\right] + 0.0256175 \,\mathrm{Sin}\left[\frac{3 \pi t}{2}\right]\right\}$$

Expand[rtx]

$$\left\{119.853 + 3.34501 \cos\left[\frac{\pi t}{2}\right] + 52.1112 \cos[\pi t] + 0.004145 \cos\left[\frac{3 \pi t}{2}\right] + 98.5085 \sin\left[\frac{\pi t}{2}\right] - 0.122068 \sin\left[\frac{3 \pi t}{2}\right],\right.$$

$$138.343 - 40.7253 \cos\left[\frac{\pi t}{2}\right] + 68.8049 \cos[\pi t] - 0.050465 \cos\left[\frac{3 \pi t}{2}\right] + 122.277 \sin\left[\frac{\pi t}{2}\right] - 0.15152 \sin\left[\frac{3 \pi t}{2}\right],$$

$$235.743 - 160.387 \cos\left[\frac{\pi t}{2}\right] + 116.809 \cos[\pi t] - 0.198745 \cos\left[\frac{3 \pi t}{2}\right] + 95.5165 \sin\left[\frac{\pi t}{2}\right] - 0.11836 \sin\left[\frac{3 \pi t}{2}\right],$$

$$150.82 + 44.1429 \cos\left[\frac{\pi t}{2}\right] + 19.5503 \cos[\pi t] + 0.0547 \cos\left[\frac{3 \pi t}{2}\right] + 118.653 \sin\left[\frac{\pi t}{2}\right] - 0.14703 \sin\left[\frac{3 \pi t}{2}\right],$$

$$255.185 - 84.8722 \cos\left[\frac{\pi t}{2}\right] + 53.5077 \cos[\pi t] - 0.10517 \cos\left[\frac{3 \pi t}{2}\right] + 57.6097 \sin\left[\frac{\pi t}{2}\right] - 0.0713875 \sin\left[\frac{3 \pi t}{2}\right],$$

$$301.564 + 49.0152 \cos\left[\frac{\pi t}{2}\right] - 91.2823 \cos[\pi t] + 0.0607375 \cos\left[\frac{3 \pi t}{2}\right] - 93.386 \sin\left[\frac{\pi t}{2}\right] + 0.11572 \sin\left[\frac{3 \pi t}{2}\right],$$

$$346.157 - 32.1771 \cos\left[\frac{\pi t}{2}\right] - 14.5954 \cos[\pi t] - 0.0398725 \cos\left[\frac{3 \pi t}{2}\right] - 7.43449 \sin\left[\frac{\pi t}{2}\right] + 0.0092125 \sin\left[\frac{3 \pi t}{2}\right],$$

$$344.818 + 41.4354 \cos\left[\frac{\pi t}{2}\right] - 32.9955 \cos[\pi t] + 0.051345 \cos\left[\frac{3 \pi t}{2}\right] + 25.9067 \sin\left[\frac{\pi t}{2}\right] - 0.0321025 \sin\left[\frac{3 \pi t}{2}\right],$$

$$\left.499.337 - 134.64 \cos\left[\frac{\pi t}{2}\right] + 39.222 \cos[\pi t] - 0.16684 \cos\left[\frac{3 \pi t}{2}\right] - 36.7831 \sin\left[\frac{\pi t}{2}\right] + 0.04558 \sin\left[\frac{3 \pi t}{2}\right]\right\}$$

```
Expand[rty]
```

$$\{699.297 + 58.7577 \cos\left[\frac{\pi t}{2}\right] - 33.8619 \cos[\pi t] + 0.07281 \cos\left[\frac{3\pi t}{2}\right] - 52.5882 \sin\left[\frac{\pi t}{2}\right] + 0.065165 \sin\left[\frac{3\pi t}{2}\right],$$

$$614.889 + 37.6163 \cos\left[\frac{\pi t}{2}\right] + 27.9543 \cos[\pi t] + 0.0466125 \cos\left[\frac{3\pi t}{2}\right] + 65.8189 \sin\left[\frac{\pi t}{2}\right] - 0.08156 \sin\left[\frac{3\pi t}{2}\right],$$

$$800.554 - 141.903 \cos\left[\frac{\pi t}{2}\right] - 5.64019 \cos[\pi t] - 0.17584 \cos\left[\frac{3\pi t}{2}\right] - 148.095 \sin\left[\frac{\pi t}{2}\right] + 0.183513 \sin\left[\frac{3\pi t}{2}\right],$$

$$616.034 - 1.00068 \cos\left[\frac{\pi t}{2}\right] + 42.8836 \cos[\pi t] - 0.00124 \cos\left[\frac{3\pi t}{2}\right] + 144.986 \sin\left[\frac{\pi t}{2}\right] - 0.17966 \sin\left[\frac{3\pi t}{2}\right],$$

$$693.703 - 66.749 \cos\left[\frac{\pi t}{2}\right] + 94.3235 \cos[\pi t] - 0.0827125 \cos\left[\frac{3\pi t}{2}\right] + 60.0509 \sin\left[\frac{\pi t}{2}\right] - 0.0744125 \sin\left[\frac{3\pi t}{2}\right],$$

$$831.807 - 193.307 \cos\left[\frac{\pi t}{2}\right] + 21.244 \cos[\pi t] - 0.239538 \cos\left[\frac{3\pi t}{2}\right] - 128.949 \sin\left[\frac{\pi t}{2}\right] + 0.159787 \sin\left[\frac{3\pi t}{2}\right],$$

$$693.712 - 124.552 \cos\left[\frac{\pi t}{2}\right] + 112.45 \cos[\pi t] - 0.15434 \cos\left[\frac{3\pi t}{2}\right] + 138.653 \sin\left[\frac{\pi t}{2}\right] - 0.171812 \sin\left[\frac{3\pi t}{2}\right],$$

$$671.277 + 29.5826 \cos\left[\frac{\pi t}{2}\right] + 19.1897 \cos[\pi t] + 0.0366575 \cos\left[\frac{3\pi t}{2}\right] + 44.5545 \sin\left[\frac{\pi t}{2}\right] - 0.05521 \sin\left[\frac{3\pi t}{2}\right],$$

$$751.899 - 31.2854 \cos\left[\frac{\pi t}{2}\right] - 23.8774 \cos[\pi t] - 0.0387675 \cos\left[\frac{3\pi t}{2}\right] - 20.6733 \sin\left[\frac{\pi t}{2}\right] + 0.0256175 \sin\left[\frac{3\pi t}{2}\right]\}$$

Note that because t∈[0,1], thus $\theta \in [0, \pi/2]$

```
xθ = Table[rtx[[k]] /. { Pi t → 2 θ}, {k, 1, n}]
```

```
{119.853 + 3.34501 Cos[θ] + 52.1112 Cos[2 θ] + 0.004145 Cos[3 θ] +
   98.5085 Sin[θ] - 0.122068 Sin[3 θ], 138.343 - 40.7253 Cos[θ] +
   68.8049 Cos[2 θ] - 0.050465 Cos[3 θ] + 122.277 Sin[θ] - 0.15152 Sin[3 θ],
 235.743 - 160.387 Cos[θ] + 116.809 Cos[2 θ] - 0.198745 Cos[3 θ] +
   95.5165 Sin[θ] - 0.11836 Sin[3 θ], 150.82 + 44.1429 Cos[θ] +
   19.5503 Cos[2 θ] + 0.0547 Cos[3 θ] + 118.653 Sin[θ] - 0.14703 Sin[3 θ],
 255.185 - 84.8722 Cos[θ] + 53.5077 Cos[2 θ] - 0.10517 Cos[3 θ] + 57.6097 Sin[θ] -
   0.0713875 Sin[3 θ], 301.564 + 49.0152 Cos[θ] - 91.2823 Cos[2 θ] +
   0.0607375 Cos[3 θ] - 93.386 Sin[θ] + 0.11572 Sin[3 θ], 346.157 - 32.1771 Cos[θ] -
   14.5954 Cos[2 θ] - 0.0398725 Cos[3 θ] - 7.43449 Sin[θ] + 0.0092125 Sin[3 θ],
 344.818 + 41.4354 Cos[θ] - 32.9955 Cos[2 θ] + 0.051345 Cos[3 θ] +
   25.9067 Sin[θ] - 0.0321025 Sin[3 θ], 499.337 - 134.64 Cos[θ] +
   39.222 Cos[2 θ] - 0.16684 Cos[3 θ] - 36.7831 Sin[θ] + 0.04558 Sin[3 θ]}
```

```
yθ = Table[rty[[k]] /. {Pi t → 2 θ}, {k, 1, n}]
```

```
{699.297 + 58.7577 Cos[θ] - 33.8619 Cos[2 θ] + 0.07281 Cos[3 θ] -
   52.5882 Sin[θ] + 0.065165 Sin[3 θ], 614.889 + 37.6163 Cos[θ] +
   27.9543 Cos[2 θ] + 0.0466125 Cos[3 θ] + 65.8189 Sin[θ] - 0.08156 Sin[3 θ],
 800.554 - 141.903 Cos[θ] - 5.64019 Cos[2 θ] - 0.17584 Cos[3 θ] - 148.095 Sin[θ] +
   0.183513 Sin[3 θ], 616.034 - 1.00068 Cos[θ] + 42.8836 Cos[2 θ] -
   0.00124 Cos[3 θ] + 144.986 Sin[θ] - 0.17966 Sin[3 θ], 693.703 - 66.749 Cos[θ] +
   94.3235 Cos[2 θ] - 0.0827125 Cos[3 θ] + 60.0509 Sin[θ] - 0.0744125 Sin[3 θ],
 831.807 - 193.307 Cos[θ] + 21.244 Cos[2 θ] - 0.239538 Cos[3 θ] -
   128.949 Sin[θ] + 0.159787 Sin[3 θ], 693.712 - 124.552 Cos[θ] +
   112.45 Cos[2 θ] - 0.15434 Cos[3 θ] + 138.653 Sin[θ] - 0.171812 Sin[3 θ],
 671.277 + 29.5826 Cos[θ] + 19.1897 Cos[2 θ] + 0.0366575 Cos[3 θ] +
   44.5545 Sin[θ] - 0.05521 Sin[3 θ], 751.899 - 31.2854 Cos[θ] -
   23.8774 Cos[2 θ] - 0.0387675 Cos[3 θ] - 20.6733 Sin[θ] + 0.0256175 Sin[3 θ]}
```

```
Expand[xθ]
```

```
{119.853 + 3.34501 Cos[θ] + 52.1112 Cos[2 θ] + 0.004145 Cos[3 θ] +
   98.5085 Sin[θ] - 0.122068 Sin[3 θ], 138.343 - 40.7253 Cos[θ] +
   68.8049 Cos[2 θ] - 0.050465 Cos[3 θ] + 122.277 Sin[θ] - 0.15152 Sin[3 θ],
  235.743 - 160.387 Cos[θ] + 116.809 Cos[2 θ] - 0.198745 Cos[3 θ] +
   95.5165 Sin[θ] - 0.11836 Sin[3 θ], 150.82 + 44.1429 Cos[θ] +
   19.5503 Cos[2 θ] + 0.0547 Cos[3 θ] + 118.653 Sin[θ] - 0.14703 Sin[3 θ],
  255.185 - 84.8722 Cos[θ] + 53.5077 Cos[2 θ] - 0.10517 Cos[3 θ] + 57.6097 Sin[θ] -
   0.0713875 Sin[3 θ], 301.564 + 49.0152 Cos[θ] - 91.2823 Cos[2 θ] +
   0.0607375 Cos[3 θ] - 93.386 Sin[θ] + 0.11572 Sin[3 θ], 346.157 - 32.1771 Cos[θ] -
   14.5954 Cos[2 θ] - 0.0398725 Cos[3 θ] - 7.43449 Sin[θ] + 0.0092125 Sin[3 θ],
  344.818 + 41.4354 Cos[θ] - 32.9955 Cos[2 θ] + 0.051345 Cos[3 θ] +
   25.9067 Sin[θ] - 0.0321025 Sin[3 θ], 499.337 - 134.64 Cos[θ] +
   39.222 Cos[2 θ] - 0.16684 Cos[3 θ] - 36.7831 Sin[θ] + 0.04558 Sin[3 θ]}
```

```
Expand[yθ]
```

```
{699.297 + 58.7577 Cos[θ] - 33.8619 Cos[2 θ] + 0.07281 Cos[3 θ] -
   52.5882 Sin[θ] + 0.065165 Sin[3 θ], 614.889 + 37.6163 Cos[θ] +
   27.9543 Cos[2 θ] + 0.0466125 Cos[3 θ] + 65.8189 Sin[θ] - 0.08156 Sin[3 θ],
  800.554 - 141.903 Cos[θ] - 5.64019 Cos[2 θ] - 0.17584 Cos[3 θ] - 148.095 Sin[θ] +
   0.183513 Sin[3 θ], 616.034 - 1.00068 Cos[θ] + 42.8836 Cos[2 θ] -
   0.00124 Cos[3 θ] + 144.986 Sin[θ] - 0.17966 Sin[3 θ], 693.703 - 66.749 Cos[θ] +
   94.3235 Cos[2 θ] - 0.0827125 Cos[3 θ] + 60.0509 Sin[θ] - 0.0744125 Sin[3 θ],
  831.807 - 193.307 Cos[θ] + 21.244 Cos[2 θ] - 0.239538 Cos[3 θ] -
   128.949 Sin[θ] + 0.159787 Sin[3 θ], 693.712 - 124.552 Cos[θ] +
   112.45 Cos[2 θ] - 0.15434 Cos[3 θ] + 138.653 Sin[θ] - 0.171812 Sin[3 θ],
  671.277 + 29.5826 Cos[θ] + 19.1897 Cos[2 θ] + 0.0366575 Cos[3 θ] +
   44.5545 Sin[θ] - 0.05521 Sin[3 θ], 751.899 - 31.2854 Cos[θ] -
   23.8774 Cos[2 θ] - 0.0387675 Cos[3 θ] - 20.6733 Sin[θ] + 0.0256175 Sin[3 θ]}
```

Single coupled serial chain coefficients

```
ak = Table[Table[Coefficient[xθ[[k]], Cos[i θ]], {i, 1, 3}], {k, 1, n}]
```

```
{{3.34501, 52.1112, 0.004145},
 {-40.7253, 68.8049, -0.050465}, {-160.387, 116.809, -0.198745},
 {44.1429, 19.5503, 0.0547}, {-84.8722, 53.5077, -0.10517},
 {49.0152, -91.2823, 0.0607375}, {-32.1771, -14.5954, -0.0398725},
 {41.4354, -32.9955, 0.051345}, {-134.64, 39.222, -0.16684}}
```

```
bk = Table[Coefficient[xθ[[k]], Sin[i θ]], {k, 1, n}, {i, 1, 3}]
{{98.5085, 0, -0.122068}, {122.277, 0, -0.15152}, {95.5165, 0, -0.11836},
 {118.653, 0, -0.14703}, {57.6097, 0, -0.0713875}, {-93.386, 0, 0.11572},
 {-7.43449, 0, 0.0092125}, {25.9067, 0, -0.0321025}, {-36.7831, 0, 0.04558}}

ck = Table[Coefficient[yθ[[k]], Cos[i θ]], {k, 1, n}, {i, 1, 3}]
{{58.7577, -33.8619, 0.07281},
 {37.6163, 27.9543, 0.0466125}, {-141.903, -5.64019, -0.17584},
 {-1.00068, 42.8836, -0.00124}, {-66.749, 94.3235, -0.0827125},
 {-193.307, 21.244, -0.239538}, {-124.552, 112.45, -0.15434},
 {29.5826, 19.1897, 0.0366575}, {-31.2854, -23.8774, -0.0387675}}

dk = Table[Coefficient[yθ[[k]], Sin[i θ]], {k, 1, n}, {i, 1, 3}]
{{-52.5882, 0, 0.065165}, {65.8189, 0, -0.08156}, {-148.095, 0, 0.183513},
 {144.986, 0, -0.17966}, {60.0509, 0, -0.0744125}, {-128.949, 0, 0.159787},
 {138.653, 0, -0.171812}, {44.5545, 0, -0.05521}, {-20.6733, 0, 0.0256175}}
```

Link Lengths and Phase angles

```
LkOrig = Table[0.5 * Sqrt[(ak[[k]] + dk[[k]])^2 + (ck[[k]] - bk[[k]])^2], {k, 1, n}]
{{31.6426, 31.0733, 0.103418},
 {44.1505, 37.1334, 0.119045}, {194.634, 58.4728, 0.0297321},
 {111.9, 23.5649, 0.0960075}, {63.4058, 54.2218, 0.0899696},
 {63.9795, 46.8609, 0.209069}, {79.1417, 56.6967, 0.133753},
 {43.0342, 19.085, 0.0344343}, {77.7052, 22.9592, 0.082247}}

Lk = Table[Take[LkOrig[[i]], 2], {i, 9}]
{{31.6426, 31.0733}, {44.1505, 37.1334}, {194.634, 58.4728},
 {111.9, 23.5649}, {63.4058, 54.2218}, {63.9795, 46.8609},
 {79.1417, 56.6967}, {43.0342, 19.085}, {77.7052, 22.9592}}

psikOrig = Table[ArcTan[ak[[k]] + dk[[k]], ck[[k]] - bk[[k]]], {k, 1, n}]
{{-2.46246, -0.576235, 1.22909},
 {-1.28264, 0.385911, 2.15858}, {-2.48564, -0.0482479, -1.82985},
 {-0.564088, 1.14305, 2.27941}, {-1.7678, 1.05478, -3.07861},
 {-2.24552, 2.91293, -1.01526}, {-0.832959, 1.69987, -2.48377},
 {0.0427219, 2.61482, 1.62695}, {3.10621, -0.546847, -2.60319}}

psik = Table[Take[psikOrig[[i]], 2], {i, 9}]
{{-2.46246, -0.576235}, {-1.28264, 0.385911}, {-2.48564, -0.0482479},
 {-0.564088, 1.14305}, {-1.7678, 1.05478}, {-2.24552, 2.91293},
 {-0.832959, 1.69987}, {0.0427219, 2.61482}, {3.10621, -0.546847}}
```

```
MkOrig = Table[0.5 * Sqrt[(ak[[k]] - dk[[k]])^2 + (ck[[k]] + bk[[k]])^2], {k, 1, n}]
```

```
{{83.4583, 31.0733, 0.0392101},
 {96.0695, 37.1334, 0.0547094}, {23.9938, 58.4728, 0.241182},
 {77.478, 23.5649, 0.138662}, {72.6055, 54.2218, 0.0785698},
 {168.719, 46.8609, 0.0792806}, {107.939, 56.6967, 0.0980691},
 {27.7885, 19.085, 0.0533262}, {66.3734, 22.9592, 0.096289}}
```

```
Mk = Table[Take[MkOrig[[i]], 2], {i, 9}]
```

```
{{83.4583, 31.0733}, {96.0695, 37.1334}, {23.9938, 58.4728},
 {77.478, 23.5649}, {72.6055, 54.2218}, {168.719, 46.8609},
 {107.939, 56.6967}, {27.7885, 19.085}, {66.3734, 22.9592}}
```

```
etakOrig = Table[ArcTan[ak[[k]] - dk[[k]], ck[[k]] + bk[[k]]], {k, 1, n}]
```

```
{{1.22909, -0.576235, -2.46246},
 {2.15858, 0.385911, -1.28264}, {-1.82985, -0.0482479, -2.48564},
 {2.27941, 1.14305, -0.564088}, {-3.07861, 1.05478, -1.7678},
 {-1.01526, 2.91293, -2.24552}, {-2.48377, 1.69987, -0.832959},
 {1.62695, 2.61482, 0.0427219}, {-2.60319, -0.546847, 3.10621}}
```

```
etak = Table[Take[etakOrig[[i]], 2], {i, 9}]
```

```
{{1.22909, -0.576235}, {2.15858, 0.385911}, {-1.82985, -0.0482479},
 {2.27941, 1.14305}, {-3.07861, 1.05478}, {-1.01526, 2.91293},
 {-2.48377, 1.69987}, {1.62695, 2.61482}, {-2.60319, -0.546847}}
```

Ground pivot coefficients

```
a0 = Table[xθ[[k]] - Sum[ak[[k, i]] Cos[i θ] + bk[[k, i]] Sin[i θ], {i, 1, 3}], {k, 1, n}]
```

```
{119.853, 138.343, 235.743, 150.82, 255.185, 301.564, 346.157, 344.818, 499.337}
```

```
b0 = Table[0 * k, {k, 1, n}]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
c0 = Table[yθ[[k]] - Sum[ck[[k, i]] Cos[i θ] + dk[[k, i]] Sin[i θ], {i, 1, 3}], {k, 1, n}]
```

```
{699.297, 614.889, 800.554, 616.034, 693.703, 831.807, 693.712, 671.277, 751.899}
```

```
d0 = Table[0 * k, {k, 1, n}]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Ground pivot position Config.

```
L0 = Table[0.5 * Sqrt[(a0[[k]] + d0[[k]])^2 + (c0[[k]] - b0[[k]])^2], {k, 1, n}]
{354.747, 315.13, 417.271, 317.114, 369.575, 442.392, 387.641, 377.33, 451.3}

psi0 = Table[ArcTan[a0[[k]] + d0[[k]], c0[[k]] - b0[[k]]], {k, 1, n}]
{1.40106, 1.34949, 1.28442, 1.33069, 1.2183, 1.22299, 1.10795, 1.09627, 0.984572}

M0 = Table[0.5 * Sqrt[(a0[[k]] - d0[[k]])^2 + (c0[[k]] + b0[[k]])^2], {k, 1, n}]
{354.747, 315.13, 417.271, 317.114, 369.575, 442.392, 387.641, 377.33, 451.3}

eta0 = Table[ArcTan[a0[[k]] - d0[[k]], c0[[k]] + b0[[k]]], {k, 1, n}]
{1.40106, 1.34949, 1.28442, 1.33069, 1.2183, 1.22299, 1.10795, 1.09627, 0.984572}
```

■ Plot the curve

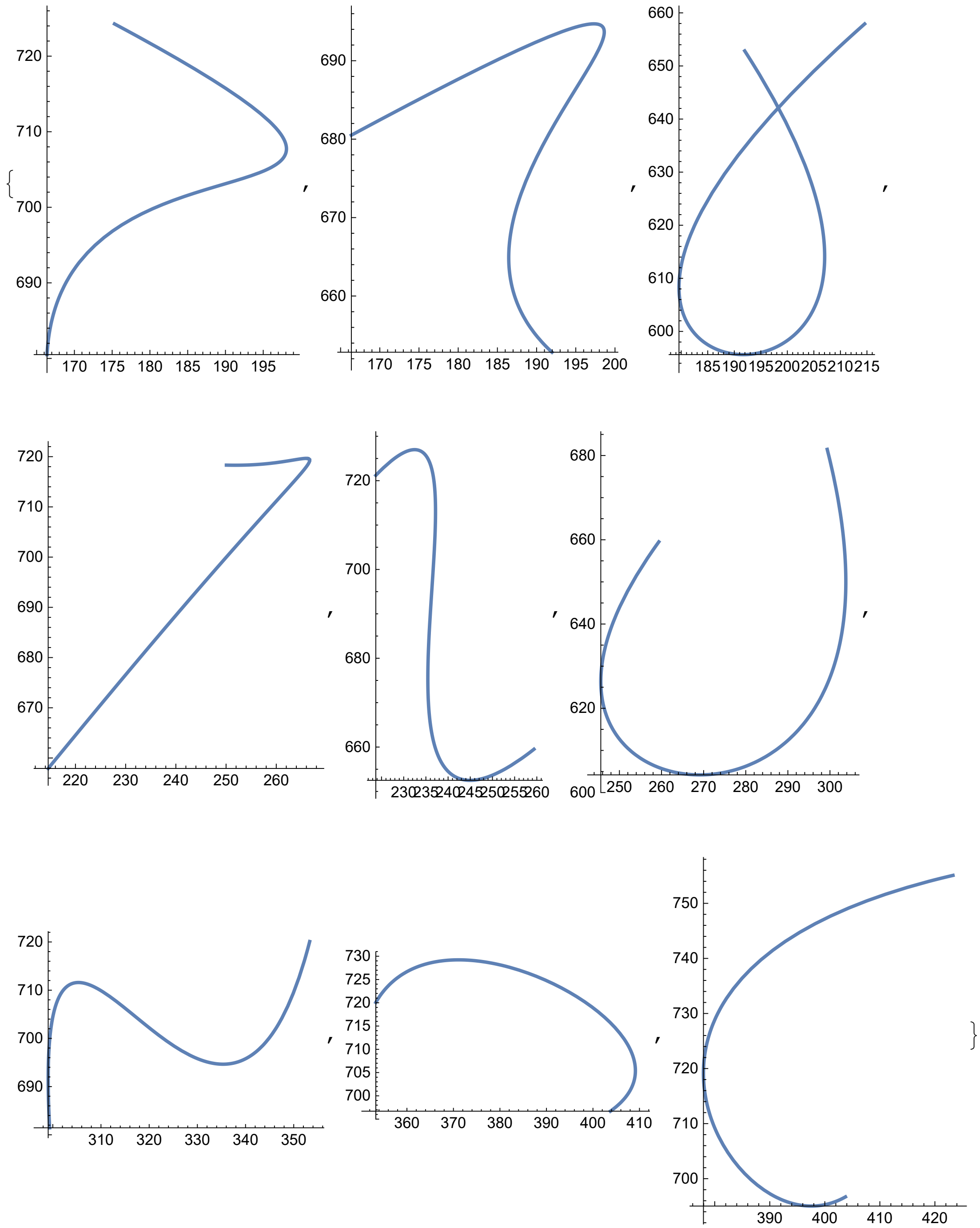
```
Newxθ = Table[Sum[LkOrig[[k, i]] Cos[psikOrig[[k, i]] + i θ] +
      MkOrig[[k, i]] Cos[etakOrig[[k, i]] - i θ], {i, 1, 3}] +
    L0 [[k]] Cos[psi0[[k]]] + M0[[k]] Cos[eta0[[k]]], {k, 1, n}]
```

```
{119.853 + 31.0733 Cos[0.576235 - 2 θ] + 83.4583 Cos[1.22909 - θ] +
   31.6426 Cos[2.46246 - θ] + 31.0733 Cos[0.576235 + 2 θ] +
   0.103418 Cos[1.22909 + 3 θ] + 0.0392101 Cos[2.46246 + 3 θ],
 138.343 + 37.1334 Cos[0.385911 - 2 θ] + 44.1505 Cos[1.28264 - θ] +
   96.0695 Cos[2.15858 - θ] + 37.1334 Cos[0.385911 + 2 θ] +
   0.0547094 Cos[1.28264 + 3 θ] + 0.119045 Cos[2.15858 + 3 θ],
 235.743 + 0.0297321 Cos[1.82985 - 3 θ] + 58.4728 Cos[0.0482479 - 2 θ] +
   194.634 Cos[2.48564 - θ] + 23.9938 Cos[1.82985 + θ] +
   58.4728 Cos[0.0482479 + 2 θ] + 0.241182 Cos[2.48564 + 3 θ],
 150.82 + 23.5649 Cos[1.14305 - 2 θ] + 111.9 Cos[0.564088 - θ] + 77.478 Cos[2.27941 - θ] +
   23.5649 Cos[1.14305 + 2 θ] + 0.138662 Cos[0.564088 + 3 θ] + 0.0960075 Cos[2.27941 + 3 θ],
 255.185 + 0.0899696 Cos[3.07861 - 3 θ] + 54.2218 Cos[1.05478 - 2 θ] +
   63.4058 Cos[1.7678 - θ] + 72.6055 Cos[3.07861 + θ] + 54.2218 Cos[1.05478 + 2 θ] +
   0.0785698 Cos[1.7678 + 3 θ], 301.564 + 0.209069 Cos[1.01526 - 3 θ] +
   46.8609 Cos[2.91293 - 2 θ] + 63.9795 Cos[2.24552 - θ] + 168.719 Cos[1.01526 + θ] +
   46.8609 Cos[2.91293 + 2 θ] + 0.0792806 Cos[2.24552 + 3 θ],
 346.157 + 0.133753 Cos[2.48377 - 3 θ] + 56.6967 Cos[1.69987 - 2 θ] +
   79.1417 Cos[0.832959 - θ] + 107.939 Cos[2.48377 + θ] +
   56.6967 Cos[1.69987 + 2 θ] + 0.0980691 Cos[0.832959 + 3 θ],
 344.818 + 0.0533262 Cos[0.0427219 - 3 θ] + 19.085 Cos[2.61482 - 2 θ] +
   27.7885 Cos[1.62695 - θ] + 43.0342 Cos[0.0427219 + θ] +
   19.085 Cos[2.61482 + 2 θ] + 0.0344343 Cos[1.62695 + 3 θ],
 499.337 + 0.082247 Cos[2.60319 - 3 θ] + 0.096289 Cos[3.10621 - 3 θ] +
   22.9592 Cos[0.546847 - 2 θ] + 66.3734 Cos[2.60319 + θ] +
   77.7052 Cos[3.10621 + θ] + 22.9592 Cos[0.546847 + 2 θ]}
```

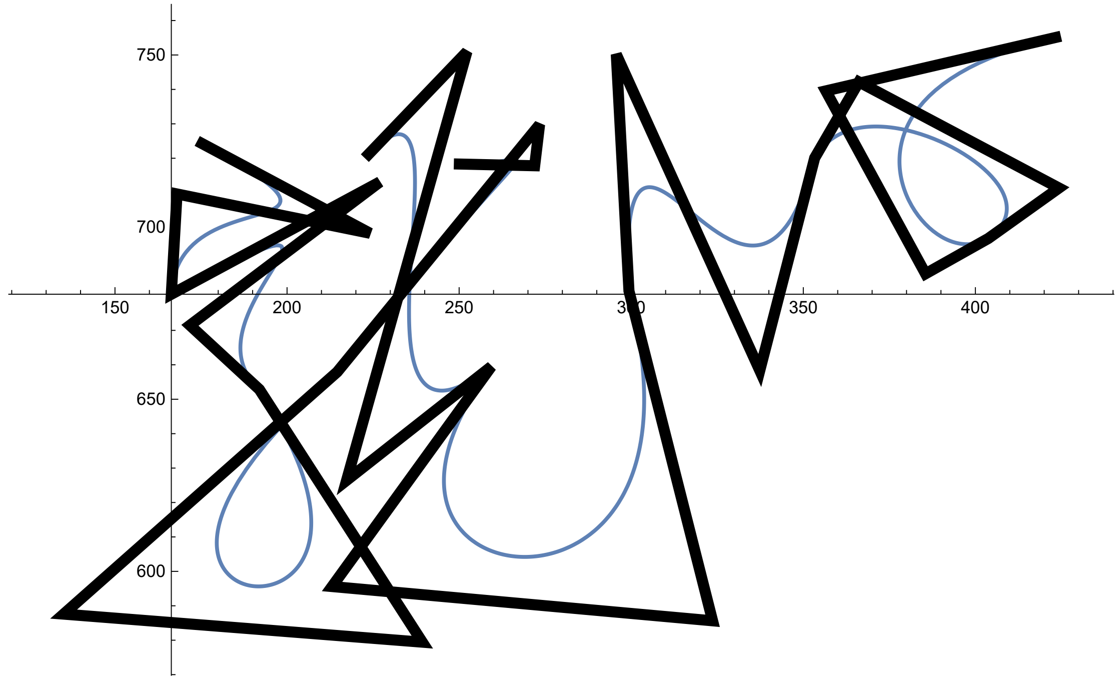
```
Newyθ = Table[Sum[LkOrig[[k, i]] Sin[psikOrig[[k, i]] + i θ] +
      MkOrig[[k, i]] Sin[etakOrig[[k, i]] - i θ], {i, 1, 3}] +
    L0[[k]] Sin[psi0[[k]]] + M0[[k]] Sin[eta0[[k]]], {k, 1, n}]
```

```
{699.297 - 31.0733 Sin[0.576235 - 2 θ] + 83.4583 Sin[1.22909 - θ] -
  31.6426 Sin[2.46246 - θ] - 31.0733 Sin[0.576235 + 2 θ] +
  0.103418 Sin[1.22909 + 3 θ] - 0.0392101 Sin[2.46246 + 3 θ],
 614.889 + 37.1334 Sin[0.385911 - 2 θ] - 44.1505 Sin[1.28264 - θ] +
  96.0695 Sin[2.15858 - θ] + 37.1334 Sin[0.385911 + 2 θ] -
  0.0547094 Sin[1.28264 + 3 θ] + 0.119045 Sin[2.15858 + 3 θ],
 800.554 - 0.0297321 Sin[1.82985 - 3 θ] - 58.4728 Sin[0.0482479 - 2 θ] -
  194.634 Sin[2.48564 - θ] - 23.9938 Sin[1.82985 + θ] -
  58.4728 Sin[0.0482479 + 2 θ] - 0.241182 Sin[2.48564 + 3 θ],
 616.034 + 23.5649 Sin[1.14305 - 2 θ] - 111.9 Sin[0.564088 - θ] + 77.478 Sin[2.27941 - θ] +
  23.5649 Sin[1.14305 + 2 θ] - 0.138662 Sin[0.564088 + 3 θ] + 0.0960075 Sin[2.27941 + 3 θ],
 693.703 - 0.0899696 Sin[3.07861 - 3 θ] + 54.2218 Sin[1.05478 - 2 θ] -
  63.4058 Sin[1.7678 - θ] - 72.6055 Sin[3.07861 + θ] + 54.2218 Sin[1.05478 + 2 θ] -
  0.0785698 Sin[1.7678 + 3 θ], 831.807 - 0.209069 Sin[1.01526 - 3 θ] +
  46.8609 Sin[2.91293 - 2 θ] - 63.9795 Sin[2.24552 - θ] - 168.719 Sin[1.01526 + θ] +
  46.8609 Sin[2.91293 + 2 θ] - 0.0792806 Sin[2.24552 + 3 θ],
 693.712 - 0.133753 Sin[2.48377 - 3 θ] + 56.6967 Sin[1.69987 - 2 θ] -
  79.1417 Sin[0.832959 - θ] - 107.939 Sin[2.48377 + θ] +
  56.6967 Sin[1.69987 + 2 θ] - 0.0980691 Sin[0.832959 + 3 θ],
 671.277 + 0.0533262 Sin[0.0427219 - 3 θ] + 19.085 Sin[2.61482 - 2 θ] +
  27.7885 Sin[1.62695 - θ] + 43.0342 Sin[0.0427219 + θ] +
  19.085 Sin[2.61482 + 2 θ] + 0.0344343 Sin[1.62695 + 3 θ],
 751.899 - 0.082247 Sin[2.60319 - 3 θ] + 0.096289 Sin[3.10621 - 3 θ] -
  22.9592 Sin[0.546847 - 2 θ] - 66.3734 Sin[2.60319 + θ] +
  77.7052 Sin[3.10621 + θ] - 22.9592 Sin[0.546847 + 2 θ]}
```

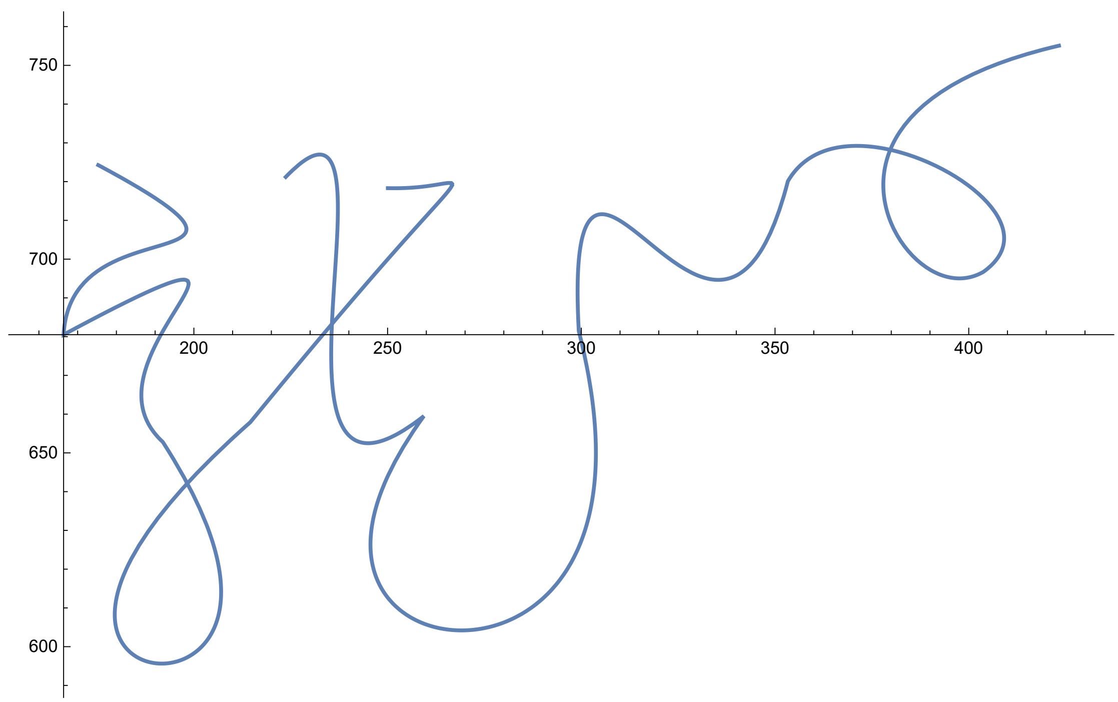
```
Segm = Table[ParametricPlot[
    {Newxθ[[k]], Newyθ[[k]]}, {θ, 0, Pi/2}, PlotStyle → Thick], {k, 1, n}]
```



```
Show[Segm, PolyG, PlotRange → All, ImageSize → Large]
```




```
Show[Segm, PlotRange → All, ImageSize → Large]
```



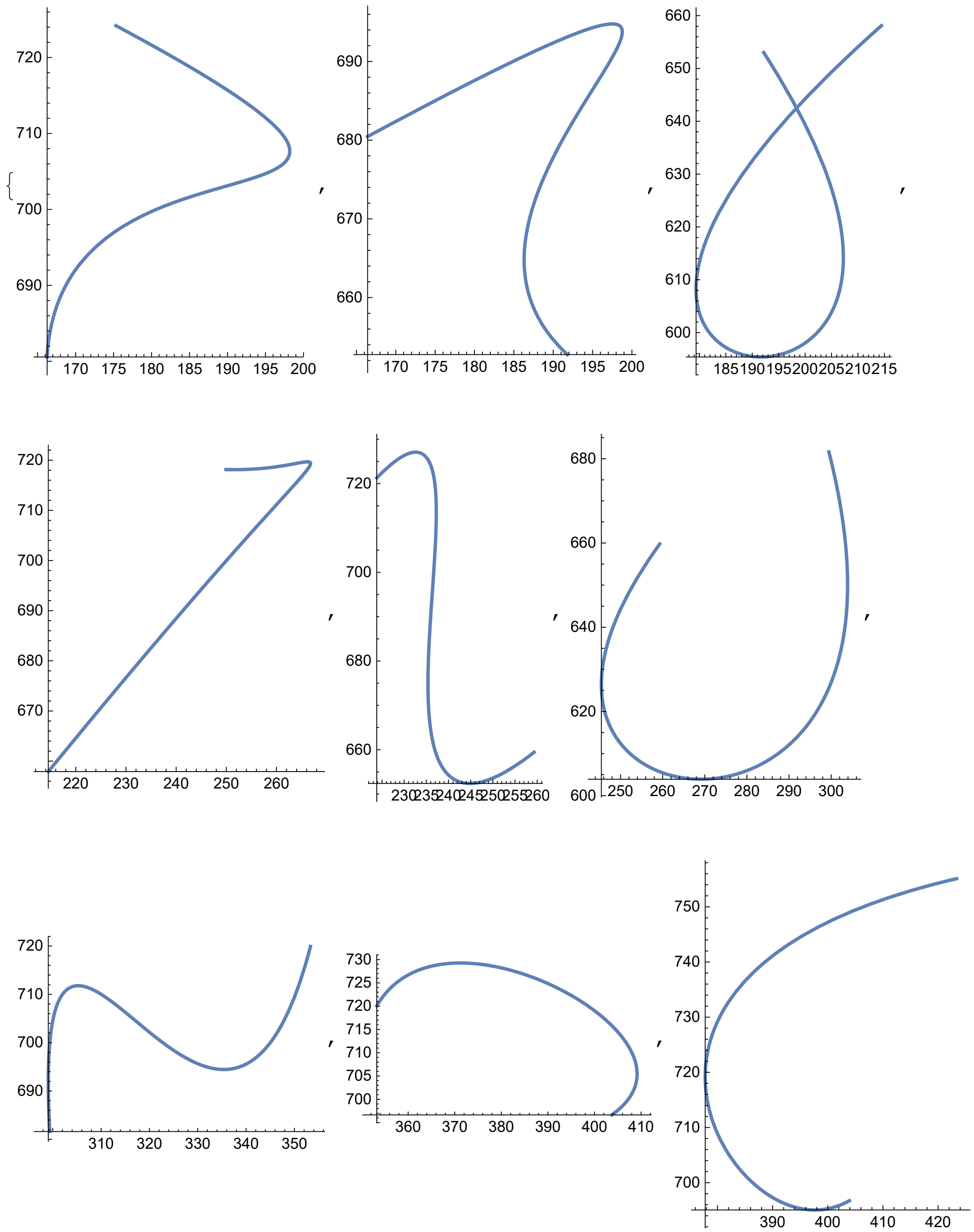
Simplification by truncating the 3rd Term

```
Newxθ = Table[
  Sum[Lk[[k, i]] Cos[psik[[k, i]] + i θ] + Mk[[k, i]] Cos[etak[[k, i]] - i θ], {i, 1, 2}] +
   L0 [[k]] Cos[psi0[[k]]] + M0[[k]] Cos[eta0[[k]]], {k, 1, n}]
```

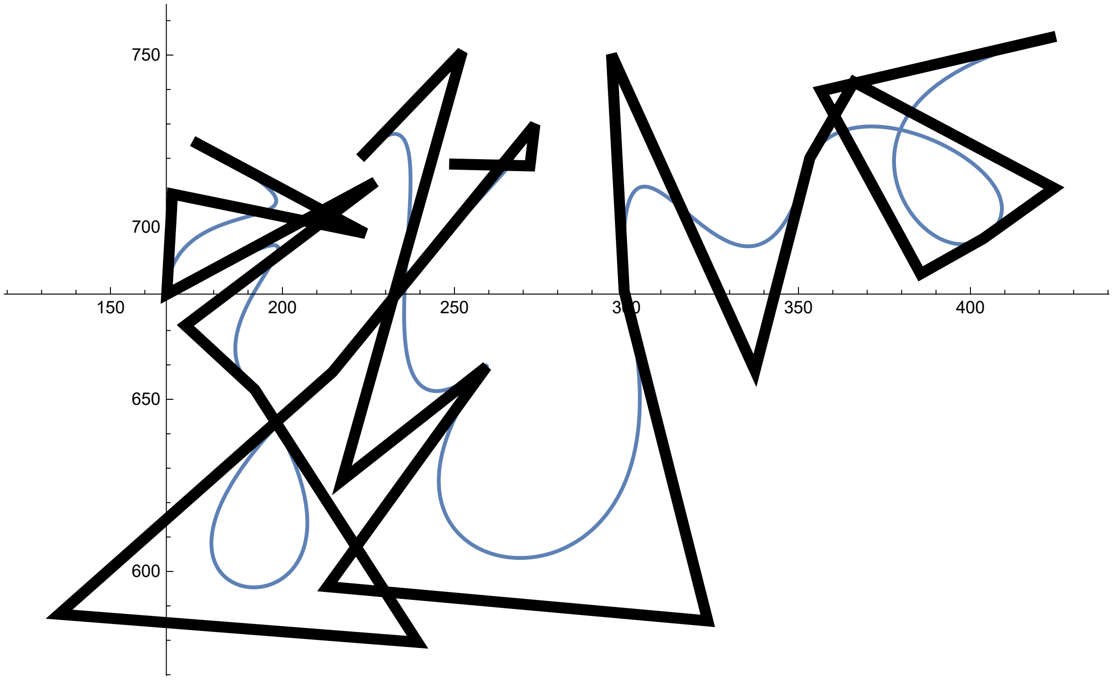
```
{119.853 + 31.0733 Cos[0.576235 - 2 θ] + 83.4583 Cos[1.22909 - θ] +
  31.6426 Cos[2.46246 - θ] + 31.0733 Cos[0.576235 + 2 θ],
 138.343 + 37.1334 Cos[0.385911 - 2 θ] + 44.1505 Cos[1.28264 - θ] +
  96.0695 Cos[2.15858 - θ] + 37.1334 Cos[0.385911 + 2 θ],
 235.743 + 58.4728 Cos[0.0482479 - 2 θ] + 194.634 Cos[2.48564 - θ] +
  23.9938 Cos[1.82985 + θ] + 58.4728 Cos[0.0482479 + 2 θ],
 150.82 + 23.5649 Cos[1.14305 - 2 θ] + 111.9 Cos[0.564088 - θ] + 77.478 Cos[2.27941 - θ] +
  23.5649 Cos[1.14305 + 2 θ], 255.185 + 54.2218 Cos[1.05478 - 2 θ] +
  63.4058 Cos[1.7678 - θ] + 72.6055 Cos[3.07861 + θ] + 54.2218 Cos[1.05478 + 2 θ],
 301.564 + 46.8609 Cos[2.91293 - 2 θ] + 63.9795 Cos[2.24552 - θ] +
  168.719 Cos[1.01526 + θ] + 46.8609 Cos[2.91293 + 2 θ],
 346.157 + 56.6967 Cos[1.69987 - 2 θ] + 79.1417 Cos[0.832959 - θ] +
  107.939 Cos[2.48377 + θ] + 56.6967 Cos[1.69987 + 2 θ],
 344.818 + 19.085 Cos[2.61482 - 2 θ] + 27.7885 Cos[1.62695 - θ] +
  43.0342 Cos[0.0427219 + θ] + 19.085 Cos[2.61482 + 2 θ],
 499.337 + 22.9592 Cos[0.546847 - 2 θ] + 66.3734 Cos[2.60319 + θ] +
  77.7052 Cos[3.10621 + θ] + 22.9592 Cos[0.546847 + 2 θ]}
```

```
NewyΘ = Table[
  Sum[Lk[[k, i]] Sin[psik[[k, i]] + i Θ] + Mk[[k, i]] Sin[etak[[k, i]] - i Θ], {i, 1, 2}] +
   L0[[k]] Sin[psi0[[k]]] + M0[[k]] Sin[eta0[[k]]], {k, 1, n}]
{699.297 - 31.0733 Sin[0.576235 - 2 Θ] + 83.4583 Sin[1.22909 - Θ] -
  31.6426 Sin[2.46246 - Θ] - 31.0733 Sin[0.576235 + 2 Θ],
 614.889 + 37.1334 Sin[0.385911 - 2 Θ] - 44.1505 Sin[1.28264 - Θ] +
  96.0695 Sin[2.15858 - Θ] + 37.1334 Sin[0.385911 + 2 Θ],
 800.554 - 58.4728 Sin[0.0482479 - 2 Θ] - 194.634 Sin[2.48564 - Θ] -
  23.9938 Sin[1.82985 + Θ] - 58.4728 Sin[0.0482479 + 2 Θ],
 616.034 + 23.5649 Sin[1.14305 - 2 Θ] - 111.9 Sin[0.564088 - Θ] +
  77.478 Sin[2.27941 - Θ] + 23.5649 Sin[1.14305 + 2 Θ],
 693.703 + 54.2218 Sin[1.05478 - 2 Θ] - 63.4058 Sin[1.7678 - Θ] -
  72.6055 Sin[3.07861 + Θ] + 54.2218 Sin[1.05478 + 2 Θ],
 831.807 + 46.8609 Sin[2.91293 - 2 Θ] - 63.9795 Sin[2.24552 - Θ] -
  168.719 Sin[1.01526 + Θ] + 46.8609 Sin[2.91293 + 2 Θ],
 693.712 + 56.6967 Sin[1.69987 - 2 Θ] - 79.1417 Sin[0.832959 - Θ] -
  107.939 Sin[2.48377 + Θ] + 56.6967 Sin[1.69987 + 2 Θ],
 671.277 + 19.085 Sin[2.61482 - 2 Θ] + 27.7885 Sin[1.62695 - Θ] +
  43.0342 Sin[0.0427219 + Θ] + 19.085 Sin[2.61482 + 2 Θ],
 751.899 - 22.9592 Sin[0.546847 - 2 Θ] - 66.3734 Sin[2.60319 + Θ] +
  77.7052 Sin[3.10621 + Θ] - 22.9592 Sin[0.546847 + 2 Θ]}
```

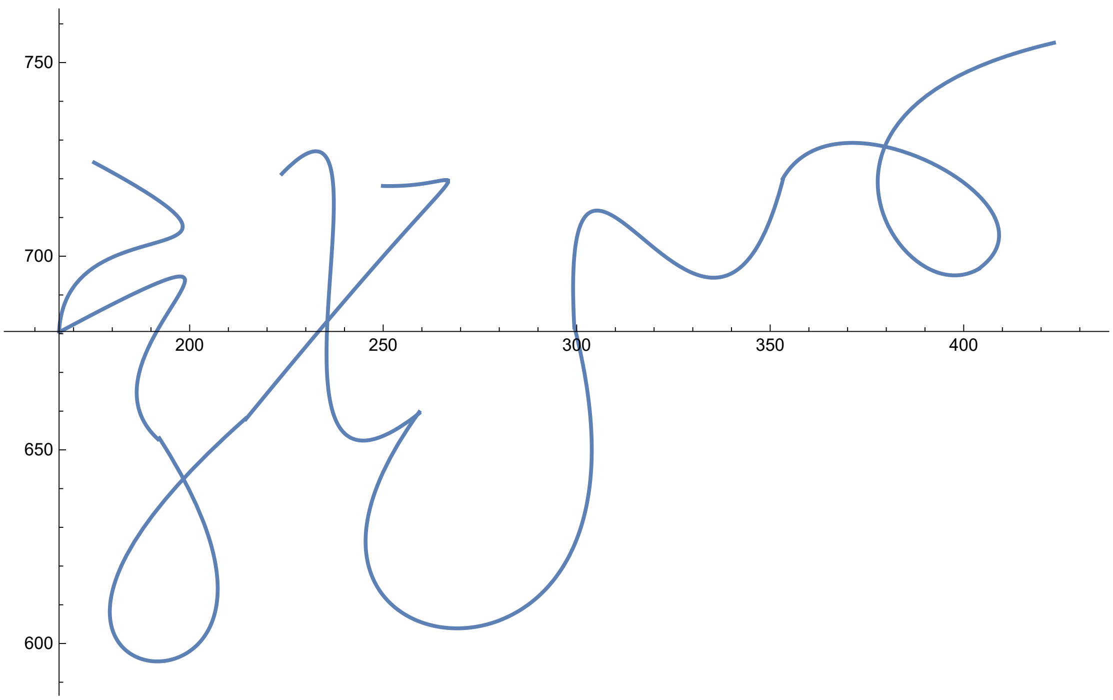
```
Segm = Table[ParametricPlot[
   {Newxθ[[k]], Newyθ[[k]]}, {θ, 0, Pi / 2}, PlotStyle → Thick], {k, 1, n}]
```



```
Show[Segm, PolyG, PlotRange → All, ImageSize → Large]
```




```
Show[Segm, PlotRange → All, ImageSize → Large]
```



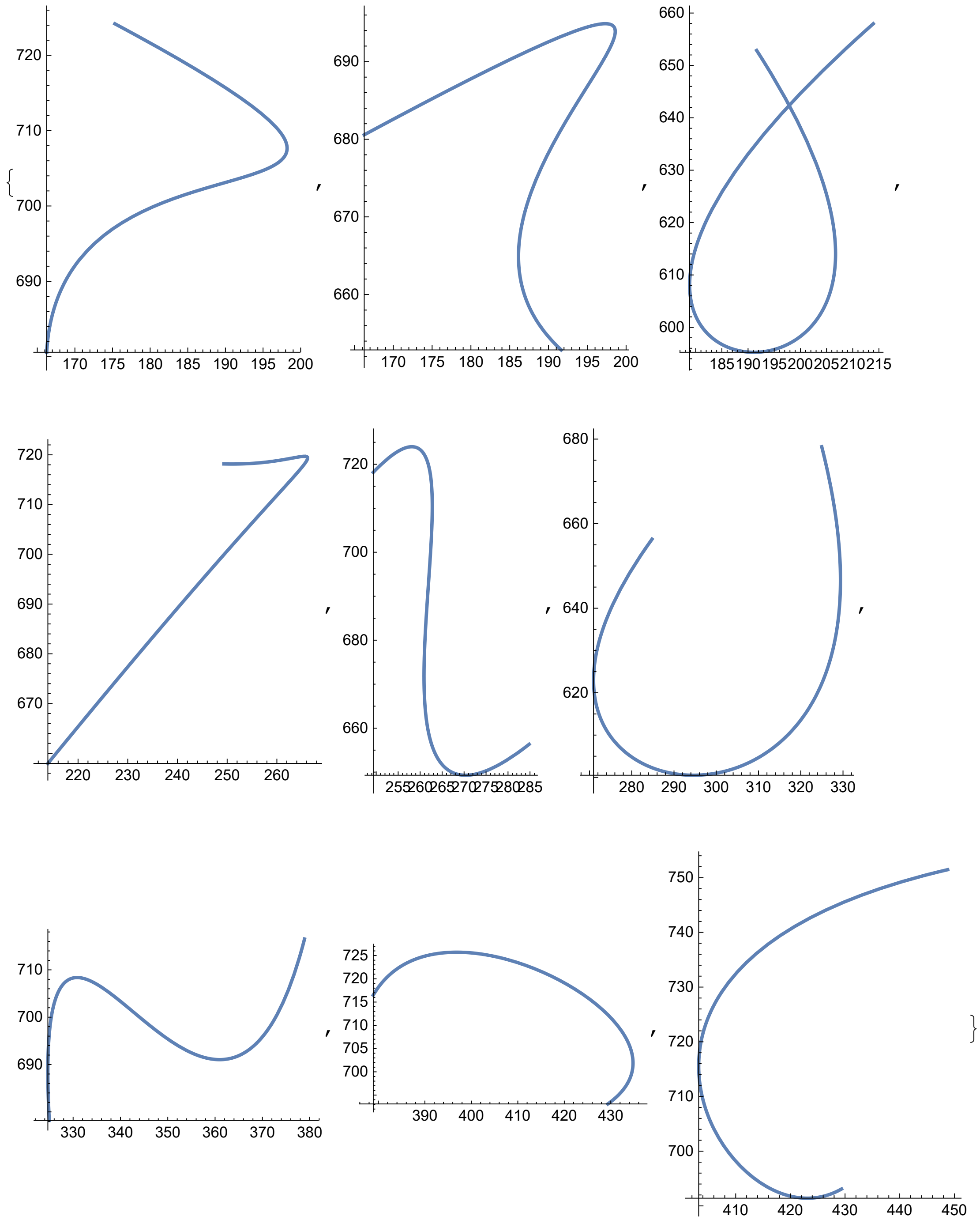
Find Revised Ground Pivot Position

```
Link1GroundPosion = {{L0 [[1]] Cos[psi0[[1]]] + M0[[1]] Cos[eta0[[1]]],
   L0[[1]] Sin[psi0[[1]]] + M0[[1]] Sin[eta0[[1]]]}}
{{119.853, 699.297}}

SegStartPoint = Table[{Lk[[i, 1]] Cos[psik[[i, 1]]] + Mk[[i, 1]] Cos[etak[[i, 1]]] +
     Lk[[i, 2]] Cos[psik[[i, 2]]] + Mk[[i, 2]] Cos[etak[[i, 2]]],
    Lk[[i, 1]] Sin[psik[[i, 1]]] + Mk[[i, 1]] Sin[etak[[i, 1]]] +
     Lk[[i, 2]] Sin[psik[[i, 2]]] + Mk[[i, 2]] Sin[etak[[i, 2]]]}, {i, Length[Lk]}]
{{55.4562, 24.8958}, {28.0797, 65.5706}, {-43.5778, -147.543},
 {63.6932, 41.8829}, {-31.3645, 27.5745}, {-42.2672, -172.063},
 {-46.7725, -12.1023}, {8.43994, 48.7723}, {-95.4178, -55.1628}}

SegEndPoint =
 Table[{Lk[[i, 1]] Cos[psik[[i, 1]] + Pi / 2] + Mk[[i, 1]] Cos[etak[[i, 1]] - Pi / 2] +
     Lk[[i, 2]] Cos[psik[[i, 2]] + Pi] + Mk[[i, 2]] Cos[etak[[i, 2]] - Pi],
    Lk[[i, 1]] Sin[psik[[i, 1]] + Pi / 2] + Mk[[i, 1]] Sin[etak[[i, 1]] - Pi / 2] +
     Lk[[i, 2]] Sin[psik[[i, 2]] + Pi] +
     Mk[[i, 2]] Sin[etak[[i, 2]] - Pi]}, {i, Length[Lk]}]
{{46.3973, -18.7263}, {53.4717, 37.8646}, {-21.2929, -142.454},
 {99.1029, 102.102}, {4.10198, -34.2726}, {-2.10371, -150.193},
 {7.16087, 26.2026}, {58.9022, 25.3648}, {-76.0051, 3.20408}}

RelativePosition = Join[Link1GroundPosion,
  Table[SegEndPoint[[i]] - SegStartPoint[[i + 1]], {i, Length[Lk] - 1}]]
{{119.853, 699.297}, {18.3176, -84.2968}, {97.0495, 185.408},
 {-84.9861, -184.337}, {130.467, 74.5275}, {46.3692, 137.79},
 {44.6688, -138.09}, {-1.27907, -22.5697}, {154.32, 80.5275}}

AllNewGroundPosition =
 Map[Fold[Plus, Take[RelativePosition, #]] &, Range[Length[Lk]]]
{{119.853, 699.297}, {138.17, 615.001}, {235.22, 800.408},
 {150.234, 616.071}, {280.701, 690.598}, {327.07, 828.389},
 {371.739, 690.298}, {370.46, 667.729}, {524.78, 748.256}}
```

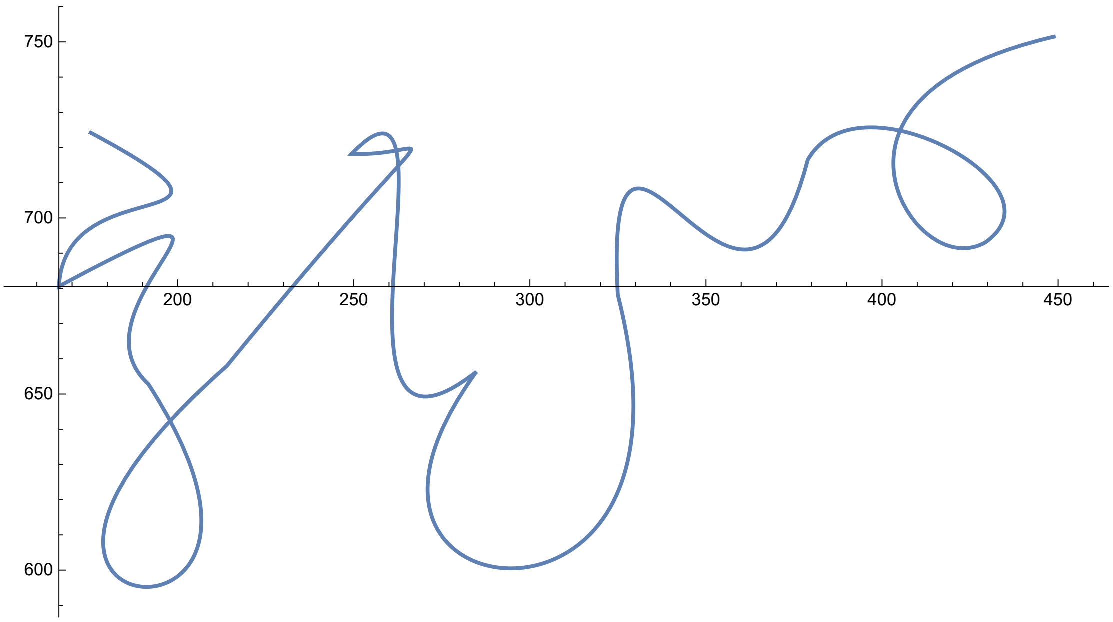
```
NewxΘ = Table[
  Sum[Lk[[k, i]] Cos[psik[[k, i]] + i Θ] + Mk[[k, i]] Cos[etak[[k, i]] - i Θ], {i, 1, 2}] +
   AllNewGroundPosition[[k, 1]], {k, 1, n}]

{119.853 + 31.0733 Cos[0.576235 - 2 Θ] + 83.4583 Cos[1.22909 - Θ] +
  31.6426 Cos[2.46246 - Θ] + 31.0733 Cos[0.576235 + 2 Θ],
 138.17 + 37.1334 Cos[0.385911 - 2 Θ] + 44.1505 Cos[1.28264 - Θ] +
  96.0695 Cos[2.15858 - Θ] + 37.1334 Cos[0.385911 + 2 Θ],
 235.22 + 58.4728 Cos[0.0482479 - 2 Θ] + 194.634 Cos[2.48564 - Θ] +
  23.9938 Cos[1.82985 + Θ] + 58.4728 Cos[0.0482479 + 2 Θ],
 150.234 + 23.5649 Cos[1.14305 - 2 Θ] + 111.9 Cos[0.564088 - Θ] +
  77.478 Cos[2.27941 - Θ] + 23.5649 Cos[1.14305 + 2 Θ],
 280.701 + 54.2218 Cos[1.05478 - 2 Θ] + 63.4058 Cos[1.7678 - Θ] +
  72.6055 Cos[3.07861 + Θ] + 54.2218 Cos[1.05478 + 2 Θ],
 327.07 + 46.8609 Cos[2.91293 - 2 Θ] + 63.9795 Cos[2.24552 - Θ] +
  168.719 Cos[1.01526 + Θ] + 46.8609 Cos[2.91293 + 2 Θ],
 371.739 + 56.6967 Cos[1.69987 - 2 Θ] + 79.1417 Cos[0.832959 - Θ] +
  107.939 Cos[2.48377 + Θ] + 56.6967 Cos[1.69987 + 2 Θ],
 370.46 + 19.085 Cos[2.61482 - 2 Θ] + 27.7885 Cos[1.62695 - Θ] +
  43.0342 Cos[0.0427219 + Θ] + 19.085 Cos[2.61482 + 2 Θ],
 524.78 + 22.9592 Cos[0.546847 - 2 Θ] + 66.3734 Cos[2.60319 + Θ] +
  77.7052 Cos[3.10621 + Θ] + 22.9592 Cos[0.546847 + 2 Θ]}
```

```
Newyθ = Table[
  Sum[Lk[[k, i]] Sin[psik[[k, i]] + i θ] + Mk[[k, i]] Sin[etak[[k, i]] - i θ], {i, 1, 2}] +
   AllNewGroundPosition[[k, 2]], {k, 1, n}]

{699.297 - 31.0733 Sin[0.576235 - 2 θ] + 83.4583 Sin[1.22909 - θ] -
  31.6426 Sin[2.46246 - θ] - 31.0733 Sin[0.576235 + 2 θ],
 615.001 + 37.1334 Sin[0.385911 - 2 θ] - 44.1505 Sin[1.28264 - θ] +
  96.0695 Sin[2.15858 - θ] + 37.1334 Sin[0.385911 + 2 θ],
 800.408 - 58.4728 Sin[0.0482479 - 2 θ] - 194.634 Sin[2.48564 - θ] -
  23.9938 Sin[1.82985 + θ] - 58.4728 Sin[0.0482479 + 2 θ],
 616.071 + 23.5649 Sin[1.14305 - 2 θ] - 111.9 Sin[0.564088 - θ] +
  77.478 Sin[2.27941 - θ] + 23.5649 Sin[1.14305 + 2 θ],
 690.598 + 54.2218 Sin[1.05478 - 2 θ] - 63.4058 Sin[1.7678 - θ] -
  72.6055 Sin[3.07861 + θ] + 54.2218 Sin[1.05478 + 2 θ],
 828.389 + 46.8609 Sin[2.91293 - 2 θ] - 63.9795 Sin[2.24552 - θ] -
  168.719 Sin[1.01526 + θ] + 46.8609 Sin[2.91293 + 2 θ],
 690.298 + 56.6967 Sin[1.69987 - 2 θ] - 79.1417 Sin[0.832959 - θ] -
  107.939 Sin[2.48377 + θ] + 56.6967 Sin[1.69987 + 2 θ],
 667.729 + 19.085 Sin[2.61482 - 2 θ] + 27.7885 Sin[1.62695 - θ] +
  43.0342 Sin[0.0427219 + θ] + 19.085 Sin[2.61482 + 2 θ],
 748.256 - 22.9592 Sin[0.546847 - 2 θ] - 66.3734 Sin[2.60319 + θ] +
  77.7052 Sin[3.10621 + θ] - 22.9592 Sin[0.546847 + 2 θ]}
```

```
Segm = Table[ParametricPlot[
   {Newxθ[[k]], Newyθ[[k]]}, {θ, 0, Pi / 2}, PlotStyle → Thick], {k, 1, n}]
```



```
Show[Segm, PlotRange → All, ImageSize → Large]
```



- Add letter break(The break has to be recalculate at every break places)

```
(*Newxθ1to4=
  Table[Sum[Lk[[k,i]]Cos[psik[[k,i]]+i θ]+Mk[[k,i]]Cos[etak[[k,i]]-i θ],{i,1,2}]+
    AllNewGroundPosition[[k,1]],{k,1,4}];*)

(*Newyθ1to4=
  Table[Sum[Lk[[k,i]]Sin[psik[[k,i]]+i θ]+Mk[[k,i]]Sin[etak[[k,i]]-i θ],{i,1,2}]+
    AllNewGroundPosition[[k,2]],{k,1,4}];*)

(*Newxθ5to9=
  Table[Sum[Lk[[k,i]]Cos[psik[[k,i]]+i θ]+Mk[[k,i]]Cos[etak[[k,i]]-i θ],{i,1,2}]+
    AllNewGroundPosition[[k,1]],{k,5,9}];*)

(*Newyθ10to14=
  Table[Sum[Lk[[k,i]]Sin[psik[[k,i]]+i θ]+Mk[[k,i]]Sin[etak[[k,i]]-i θ],{i,1,2}]+
    AllNewGroundPosition[[k,2]],{k,5,9}];*)

(*Newxθ=Join[Newxθ1to3,Newxθ4,Newxθ5to6,Newxθ7,Newxθ8,Newxθ9,Newxθ10to14]*)

(*Newyθ=Join[Newyθ1to3,Newyθ4,Newyθ5to6,Newyθ7,Newyθ8,Newyθ9,Newyθ10to14]*)

(*Segm=Table[
  ParametricPlot[{Newxθ[[k]],Newyθ[[k]]},{θ,0,Pi/2},PlotStyle→Thick],{k,1,n}]*)

(*Show[Segm,PlotRange→All,ImageSize→Large]*)
```

Recalculate Revised Ground Pivot Position (Breaks at 5th segments)

```
Link1GroundPosion = {{L0 [[1]] Cos[psi0[[1]]] + M0[[1]] Cos[eta0[[1]]],
   L0[[1]] Sin[psi0[[1]]] + M0[[1]] Sin[eta0[[1]]]}}
{{119.853, 699.297}}

Link5GroundPosion = {{L0 [[5]] Cos[psi0[[5]]] + M0[[5]] Cos[eta0[[5]]],
   L0[[5]] Sin[psi0[[5]]] + M0[[5]] Sin[eta0[[5]]]}}
{{255.185, 693.703}}

(*Link7GroundPosion={{L0 [[7]]Cos[psi0[[7]]]+M0[[7]] Cos[eta0[[7]]],
   L0[[7]] Sin[psi0[[7]]]+M0[[7]] Sin[eta0[[7]]]}}*)

(*Link9GroundPosion={{L0 [[9]]Cos[psi0[[9]]]+M0[[9]] Cos[eta0[[9]]],
   L0[[9]] Sin[psi0[[9]]]+M0[[9]] Sin[eta0[[9]]]}}*)

SegStartPoint = Table[{Lk[[i, 1]] Cos[psik[[i, 1]]] + Mk[[i, 1]] Cos[etak[[i, 1]]] +
    Lk[[i, 2]] Cos[psik[[i, 2]]] + Mk[[i, 2]] Cos[etak[[i, 2]]],
   Lk[[i, 1]] Sin[psik[[i, 1]]] + Mk[[i, 1]] Sin[etak[[i, 1]]] +
    Lk[[i, 2]] Sin[psik[[i, 2]]] + Mk[[i, 2]] Sin[etak[[i, 2]]]}, {i, Length[Lk]}]
{{55.4562, 24.8958}, {28.0797, 65.5706}, {-43.5778, -147.543},
 {63.6932, 41.8829}, {-31.3645, 27.5745}, {-42.2672, -172.063},
 {-46.7725, -12.1023}, {8.43994, 48.7723}, {-95.4178, -55.1628}}

SegEndPoint =
 Table[{Lk[[i, 1]] Cos[psik[[i, 1]] + Pi / 2] + Mk[[i, 1]] Cos[etak[[i, 1]] - Pi / 2] +
    Lk[[i, 2]] Cos[psik[[i, 2]] + Pi] + Mk[[i, 2]] Cos[etak[[i, 2]] - Pi],
   Lk[[i, 1]] Sin[psik[[i, 1]] + Pi / 2] + Mk[[i, 1]] Sin[etak[[i, 1]] - Pi / 2] +
    Lk[[i, 2]] Sin[psik[[i, 2]] + Pi] +
    Mk[[i, 2]] Sin[etak[[i, 2]] - Pi]}, {i, Length[Lk]}]
{{46.3973, -18.7263}, {53.4717, 37.8646}, {-21.2929, -142.454},
 {99.1029, 102.102}, {4.10198, -34.2726}, {-2.10371, -150.193},
 {7.16087, 26.2026}, {58.9022, 25.3648}, {-76.0051, 3.20408}}

RelativePosition1to4 =
 Join[Link1GroundPosion, Table[SegEndPoint[[i]] - SegStartPoint[[i + 1]], {i, 3}]]
{{119.853, 699.297}, {18.3176, -84.2968}, {97.0495, 185.408}, {-84.9861, -184.337}}

AllNewGroundPosition1to4 = Map[Fold[Plus, Take[RelativePosition1to4, #]] &, Range[4]]
{{119.853, 699.297}, {138.17, 615.001}, {235.22, 800.408}, {150.234, 616.071}}
```

```
RelativePosition5to9 =
 Join[Link5GroundPosion, Table[SegEndPoint[[i]] - SegStartPoint[[i + 1]], {i, 5, 8}]]
{{255.185, 693.703}, {46.3692, 137.79},
 {44.6688, -138.09}, {-1.27907, -22.5697}, {154.32, 80.5275}}

AllNewGroundPosition5to9 = Map[Fold[Plus, Take[RelativePosition5to9, #]] &, Range[5]]
{{255.185, 693.703}, {301.554, 831.493},
 {346.223, 693.403}, {344.943, 670.833}, {499.263, 751.361}}

(*RelativePosition7to8=
 Join[Link7GroundPosion,Table[SegEndPoint[[i]]-SegStartPoint[[i+1]],{i,7,7}]]*)

(*AllNewGroundPosition7to8=
 Map[Fold[Plus,Take[RelativePosition7to8,#]]&,Range[2]]*)

(*RelativePosition9to14=
 Join[Link9GroundPosion,Table[SegEndPoint[[i]]-SegStartPoint[[i+1]],{i,9,13}]]*)

(*AllNewGroundPosition9to14=
 Map[Fold[Plus,Take[RelativePosition9to14,#]]&,Range[6]]*)
```

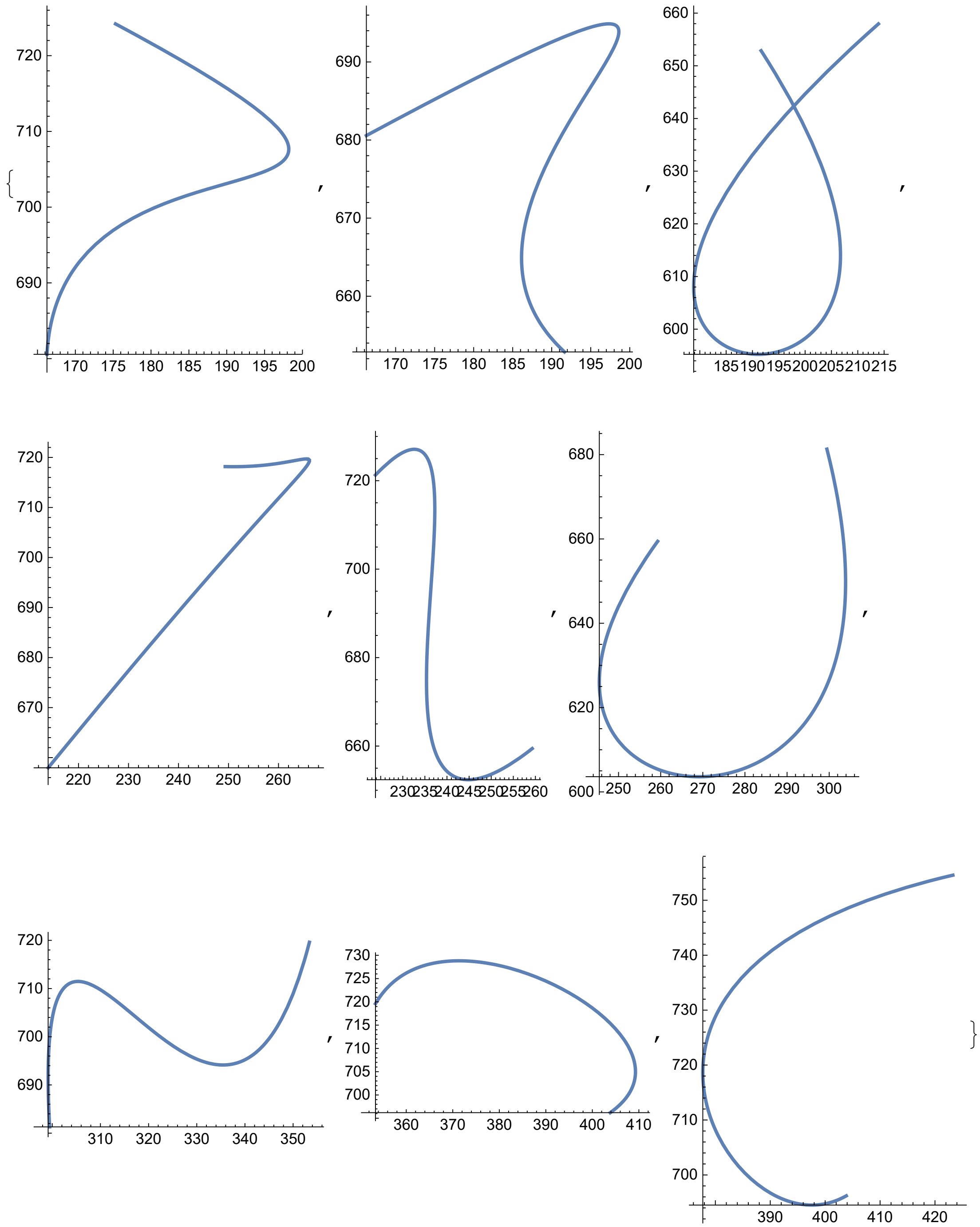
Plot the letters

```
Newxθ1to4 = Table[
   Sum[Lk[[k, i]] Cos[psik[[k, i]] + i θ] + Mk[[k, i]] Cos[etak[[k, i]] - i θ], {i, 1, 2}] +
    AllNewGroundPosition1to4[[k, 1]], {k, 1, 4}];

Newyθ1to4 = Table[
   Sum[Lk[[k, i]] Sin[psik[[k, i]] + i θ] + Mk[[k, i]] Sin[etak[[k, i]] - i θ], {i, 1, 2}] +
    AllNewGroundPosition1to4[[k, 2]], {k, 1, 4}];

Newxθ5to9 = Table[
   Sum[Lk[[k, i]] Cos[psik[[k, i]] + i θ] + Mk[[k, i]] Cos[etak[[k, i]] - i θ], {i, 1, 2}] +
    AllNewGroundPosition5to9[[k - 4, 1]], {k, 5, 9}];

Newyθ5to9 = Table[
   Sum[Lk[[k, i]] Sin[psik[[k, i]] + i θ] + Mk[[k, i]] Sin[etak[[k, i]] - i θ], {i, 1, 2}] +
    AllNewGroundPosition5to9[[k - 4, 2]], {k, 5, 9}];

(*Newxθ7to8=
  Table[Sum[Lk[[k,i]]Cos[psik[[k,i]]+i θ]+Mk[[k,i]]Cos[etak[[k,i]]-i θ],{i,1,2}]+
    AllNewGroundPosition7to8[[k-6,1]],{k,7,8}];*)

(*Newyθ7to8=
  Table[Sum[Lk[[k,i]]Sin[psik[[k,i]]+i θ]+Mk[[k,i]]Sin[etak[[k,i]]-i θ],{i,1,2}]+
    AllNewGroundPosition7to8[[k-6,2]],{k,7,8}];*)
```

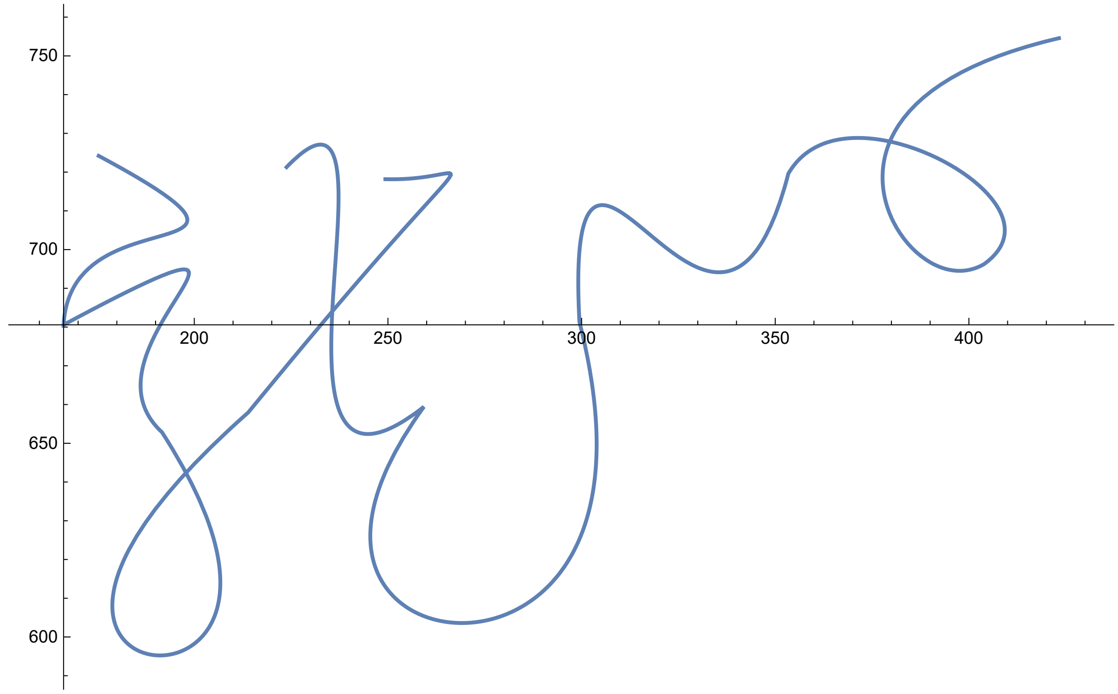
```
(*Newxθ9to14=
  Table[Sum[Lk[[k,i]]Cos[psik[[k,i]]+i θ]+Mk[[k,i]]Cos[etak[[k,i]]-i θ],{i,1,2}]+
    AllNewGroundPosition9to14[[k-8,1]],{k,9,14}];*)

(*Newyθ9to14=
  Table[Sum[Lk[[k,i]]Sin[psik[[k,i]]+i θ]+Mk[[k,i]]Sin[etak[[k,i]]-i θ],{i,1,2}]+
    AllNewGroundPosition9to14[[k-8,2]],{k,9,14}];*)

Newxθ = Join[Newxθ1to4, Newxθ5to9]

{119.853 + 31.0733 Cos[0.576235 - 2 θ] + 83.4583 Cos[1.22909 - θ] +
   31.6426 Cos[2.46246 - θ] + 31.0733 Cos[0.576235 + 2 θ],
 138.17 + 37.1334 Cos[0.385911 - 2 θ] + 44.1505 Cos[1.28264 - θ] +
   96.0695 Cos[2.15858 - θ] + 37.1334 Cos[0.385911 + 2 θ],
 235.22 + 58.4728 Cos[0.0482479 - 2 θ] + 194.634 Cos[2.48564 - θ] +
   23.9938 Cos[1.82985 + θ] + 58.4728 Cos[0.0482479 + 2 θ],
 150.234 + 23.5649 Cos[1.14305 - 2 θ] + 111.9 Cos[0.564088 - θ] +
   77.478 Cos[2.27941 - θ] + 23.5649 Cos[1.14305 + 2 θ],
 255.185 + 54.2218 Cos[1.05478 - 2 θ] + 63.4058 Cos[1.7678 - θ] +
   72.6055 Cos[3.07861 + θ] + 54.2218 Cos[1.05478 + 2 θ],
 301.554 + 46.8609 Cos[2.91293 - 2 θ] + 63.9795 Cos[2.24552 - θ] +
   168.719 Cos[1.01526 + θ] + 46.8609 Cos[2.91293 + 2 θ],
 346.223 + 56.6967 Cos[1.69987 - 2 θ] + 79.1417 Cos[0.832959 - θ] +
   107.939 Cos[2.48377 + θ] + 56.6967 Cos[1.69987 + 2 θ],
 344.943 + 19.085 Cos[2.61482 - 2 θ] + 27.7885 Cos[1.62695 - θ] +
   43.0342 Cos[0.0427219 + θ] + 19.085 Cos[2.61482 + 2 θ],
 499.263 + 22.9592 Cos[0.546847 - 2 θ] + 66.3734 Cos[2.60319 + θ] +
   77.7052 Cos[3.10621 + θ] + 22.9592 Cos[0.546847 + 2 θ]}
```

```
Newyθ = Join[Newyθ1to4, Newyθ5to9]

{699.297 - 31.0733 Sin[0.576235 - 2 θ] + 83.4583 Sin[1.22909 - θ] -
   31.6426 Sin[2.46246 - θ] - 31.0733 Sin[0.576235 + 2 θ],
  615.001 + 37.1334 Sin[0.385911 - 2 θ] - 44.1505 Sin[1.28264 - θ] +
   96.0695 Sin[2.15858 - θ] + 37.1334 Sin[0.385911 + 2 θ],
  800.408 - 58.4728 Sin[0.0482479 - 2 θ] - 194.634 Sin[2.48564 - θ] -
   23.9938 Sin[1.82985 + θ] - 58.4728 Sin[0.0482479 + 2 θ],
  616.071 + 23.5649 Sin[1.14305 - 2 θ] - 111.9 Sin[0.564088 - θ] +
   77.478 Sin[2.27941 - θ] + 23.5649 Sin[1.14305 + 2 θ],
  693.703 + 54.2218 Sin[1.05478 - 2 θ] - 63.4058 Sin[1.7678 - θ] -
   72.6055 Sin[3.07861 + θ] + 54.2218 Sin[1.05478 + 2 θ],
  831.493 + 46.8609 Sin[2.91293 - 2 θ] - 63.9795 Sin[2.24552 - θ] -
   168.719 Sin[1.01526 + θ] + 46.8609 Sin[2.91293 + 2 θ],
  693.403 + 56.6967 Sin[1.69987 - 2 θ] - 79.1417 Sin[0.832959 - θ] -
   107.939 Sin[2.48377 + θ] + 56.6967 Sin[1.69987 + 2 θ],
  670.833 + 19.085 Sin[2.61482 - 2 θ] + 27.7885 Sin[1.62695 - θ] +
   43.0342 Sin[0.0427219 + θ] + 19.085 Sin[2.61482 + 2 θ],
  751.361 - 22.9592 Sin[0.546847 - 2 θ] - 66.3734 Sin[2.60319 + θ] +
   77.7052 Sin[3.10621 + θ] - 22.9592 Sin[0.546847 + 2 θ]}
```

```
Segm = Table[ParametricPlot[
   {Newxθ[[k]], Newyθ[[k]]}, {θ, 0, Pi / 2}, PlotStyle → Thick], {k, 1, n}]
```



```
Show[Segm, PlotRange → All, ImageSize → Large]
```



Compare link length range

```
LinksL = Join[Table[Lk[[i, 1]], {i, 1, n}], Table[Lk[[i, 2]], {i, 1, n}]]
```

```
{31.6426, 44.1505, 194.634, 111.9, 63.4058, 63.9795, 79.1417, 43.0342, 77.7052,
 31.0733, 37.1334, 58.4728, 23.5649, 54.2218, 46.8609, 56.6967, 19.085, 22.9592}
```

```
Max[LinksL]
```

```
194.634
```

```
Min[LinksL]
```

```
19.085
```

```
LinksM = Join[Table[Mk[[i, 1]], {i, 1, n}], Table[Mk[[i, 2]], {i, 1, n}]]
```

```
{83.4583, 96.0695, 23.9938, 77.478, 72.6055, 168.719, 107.939, 27.7885, 66.3734,
 31.0733, 37.1334, 58.4728, 23.5649, 54.2218, 46.8609, 56.6967, 19.085, 22.9592}
```

```
Max[LinksM]
```

```
168.719
```

```
Min[LinksM]
```

```
19.085
```

```
Min[Table[Lk[[i, 1]], {i, n}]]
```

31.6426

```
Min[Table[Mk[[i, 1]], {i, n}]]
```

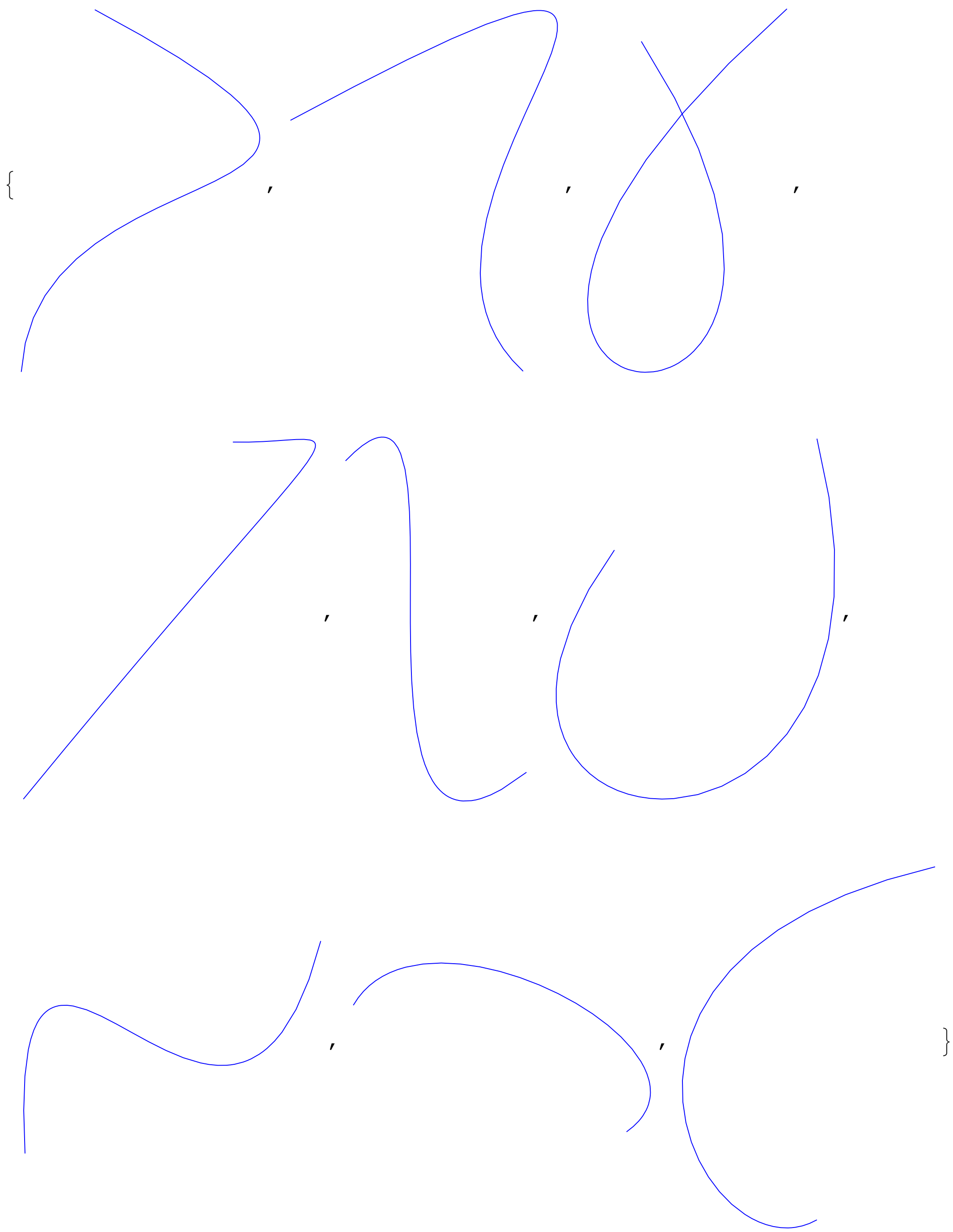
23.9938

Compare the Original Bezier curve and the Trigonometric Bezier curve

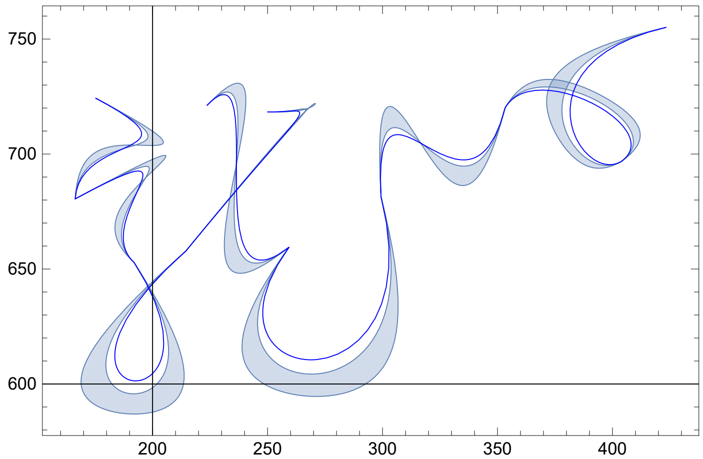
■ Original Bezier Curve Using Mathematica's Bezier Function

```
Needs["Splines`"]
```

```
OrigBezier = Map[
  Graphics[{Blue, Spline[{P0[[#]], P1[[#]], P2[[#]], P3[[#]]}, Bezier]}] &, Range[n]]
```



```
Show[Curve, OrigBezier, PlotRange → All]
```

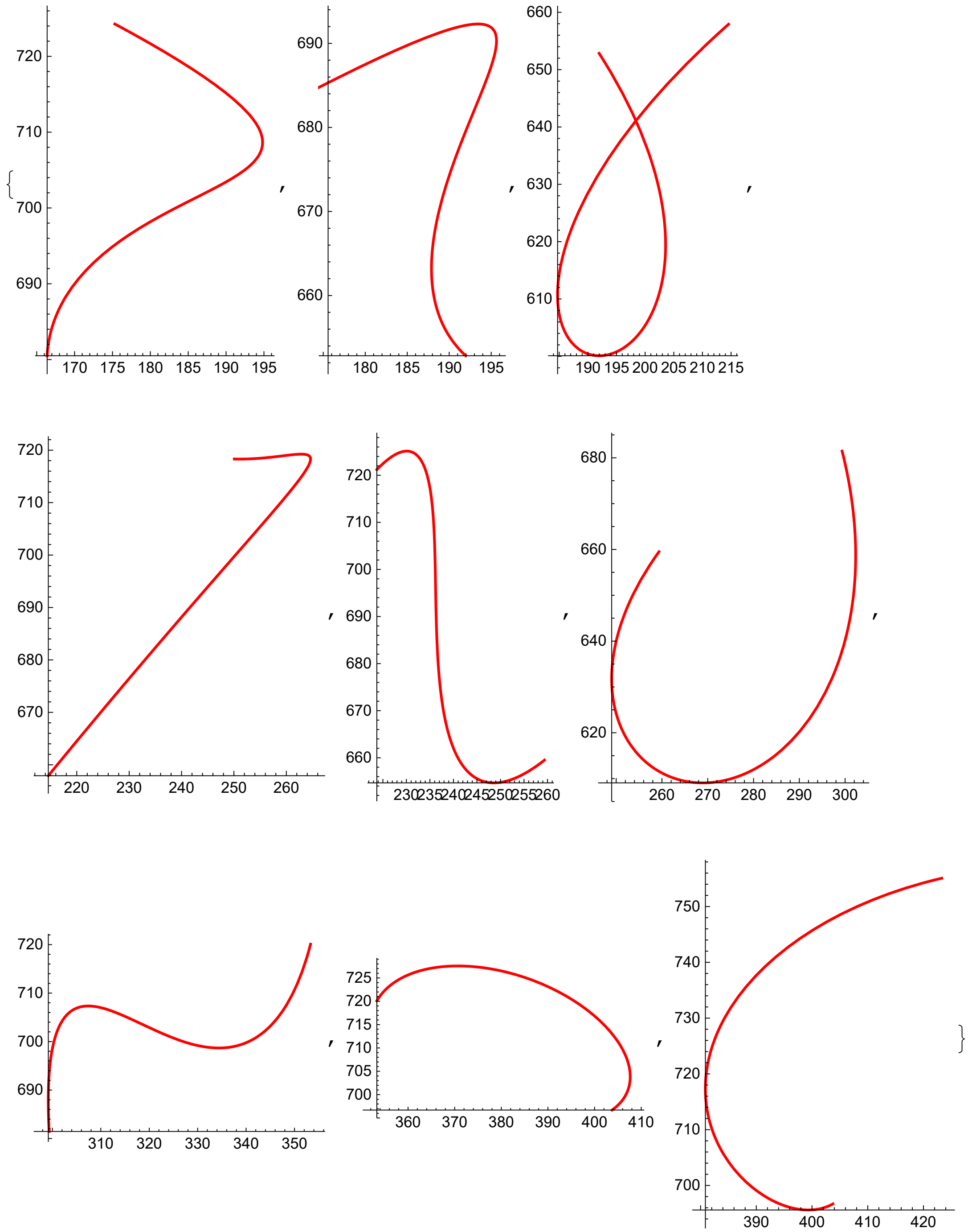



```
λ = .;
```

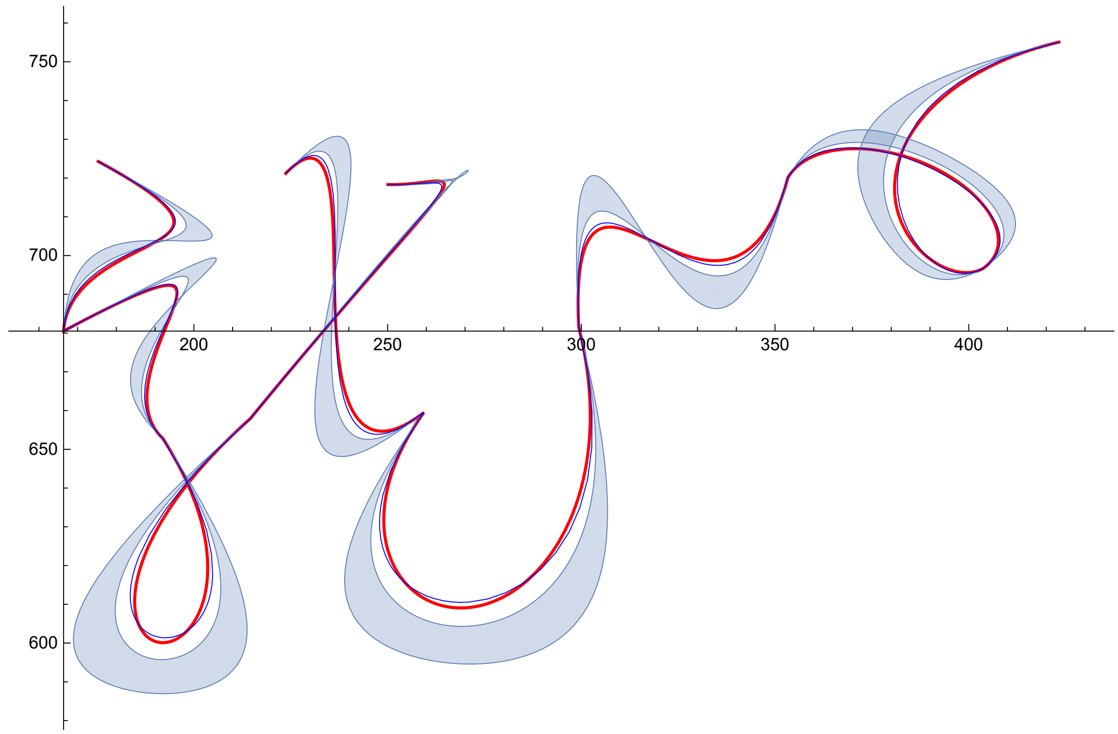
```
rtx; rty;
```

```
λ = -0.5;
```

```
Curve0 =
 Table[ParametricPlot[{rtx[[k]], rty[[k]]}, {t, 0, 1}, PlotStyle → Red], {k, 1, n}]
```

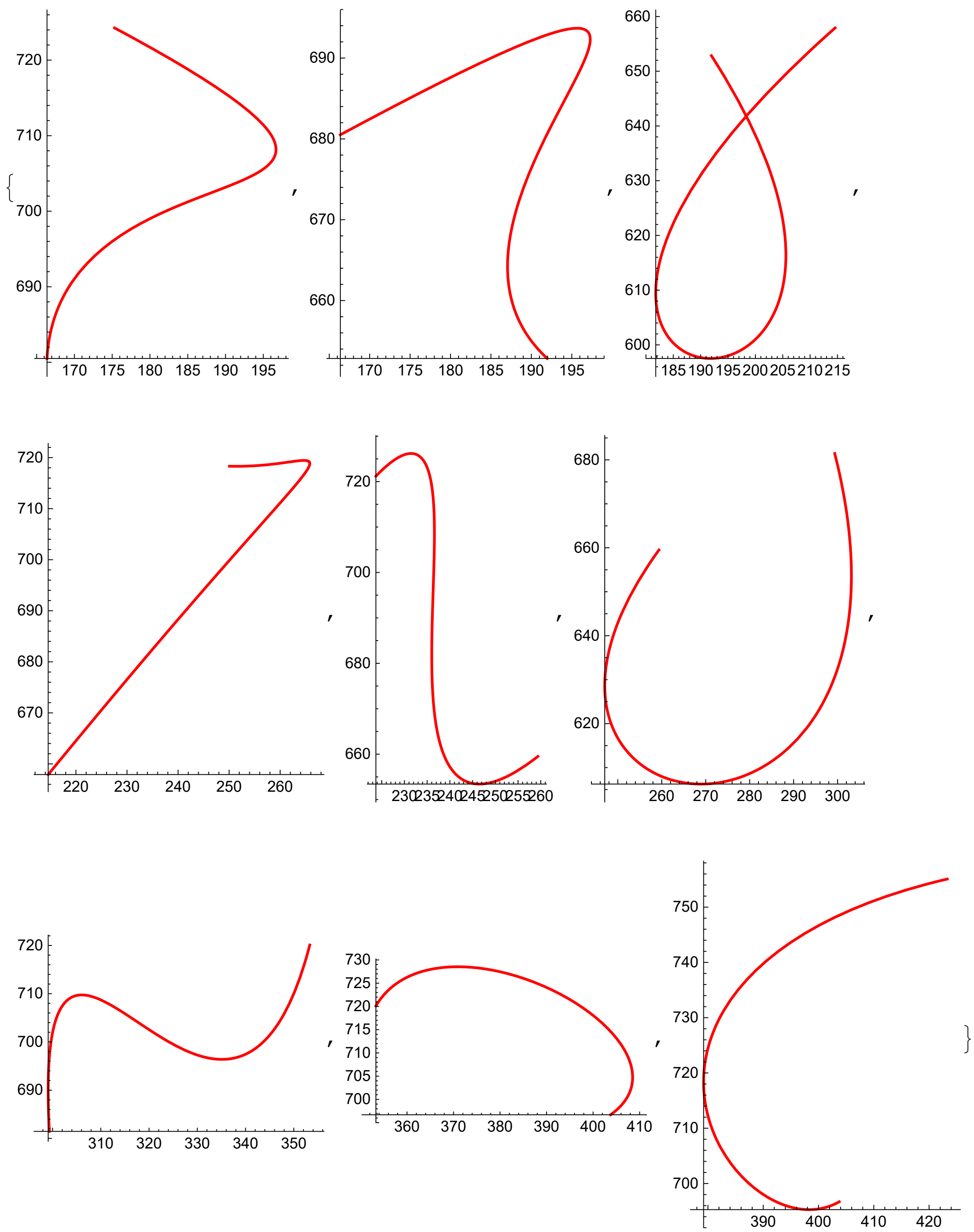


```
Show[Curve0, Curve, OrigBezier, PlotRange → All, ImageSize → Large]
```

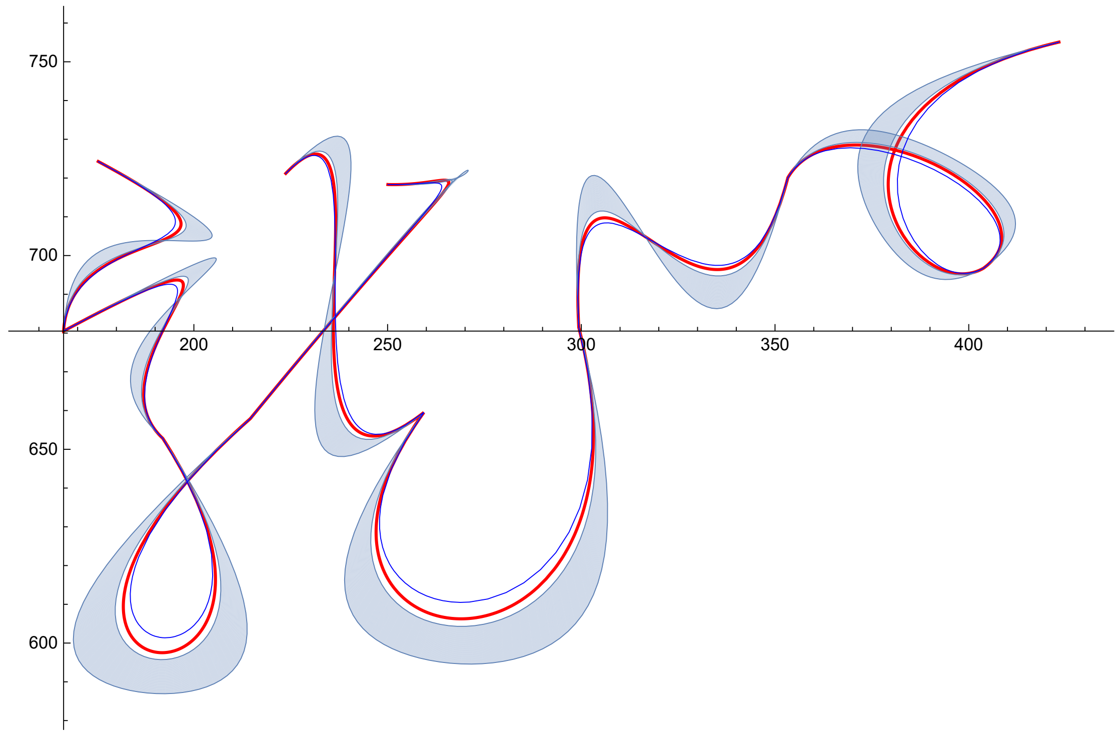



```
λ = -0.205;
```

```
Curve01 =
 Table[ParametricPlot[{rtx[[k]], rty[[k]]}, {t, 0, 1}, PlotStyle → Red], {k, 1, n}]
```



```
Show[Curve01, Curve, OrigBezier, PlotRange → All, ImageSize → Large]
```



Construct the Bezier curve original format

```
λ =.;
```

Compare with the Trigonometric Bezier curve

```
b0t = (1 - λ Sin[Pi t / 2]) (1 - Sin[Pi t / 2]) ^2
```

$$\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)$$

```
b1t = Sin[Pi t / 2] (1 - Sin[Pi t / 2]) (2 + λ (1 - Sin[Pi t / 2]))
```

$$\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right]$$

```
b2t = Cos[Pi t / 2] (1 - Cos[Pi t / 2]) (2 + λ (1 - Cos[Pi t / 2]))
```

$$\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right]$$

```
b3t = (1 - λ Cos[Pi t / 2]) (1 - Cos[Pi t / 2]) ^2
```

$$\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2 \left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right)$$

Original Bezier curve

```
ob0t = (1 - t)^3;
ob1t = 3 (1 - t)^2 * t;
ob2t = 3 (1 - t) * t^2;
ob3t = t^3;
```

"ob" means original Bezier

```
ort = ob0t * P0[[1]] + ob1t * P1[[1]] + ob2t * P2[[1]] + ob3t * P3[[1]]
```

$$\{175.313\,(1-t)^3 + 672.42\,(1-t)^2\,t + 504.09\,(1-t)\,t^2 + 166.372\,t^3,\ 724.266\,(1-t)^3 + 2094.6\,(1-t)^2\,t + 2128.89\,(1-t)\,t^2 + 680.506\,t^3\}$$

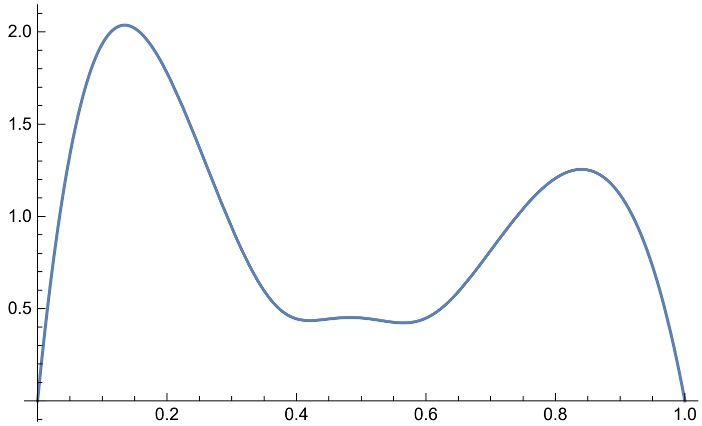
```
rt = b0t * P0[[1]] + b1t * P1[[1]] + b2t * P2[[1]] + b3t * P3[[1]]
```

$$\Big\{168.03\left(2+\lambda\left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)\mathrm{Cos}\left[\frac{\pi t}{2}\right] + 166.372\left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right) + 224.14\left(2+\lambda\left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)\mathrm{Sin}\left[\frac{\pi t}{2}\right] + 175.313\left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right),$$
$$709.63\left(2+\lambda\left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)\mathrm{Cos}\left[\frac{\pi t}{2}\right] + 680.506\left(1-\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\mathrm{Cos}\left[\frac{\pi t}{2}\right]\right) + 698.2\left(2+\lambda\left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)\mathrm{Sin}\left[\frac{\pi t}{2}\right] + 724.266\left(1-\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\mathrm{Sin}\left[\frac{\pi t}{2}\right]\right)\Big\}$$

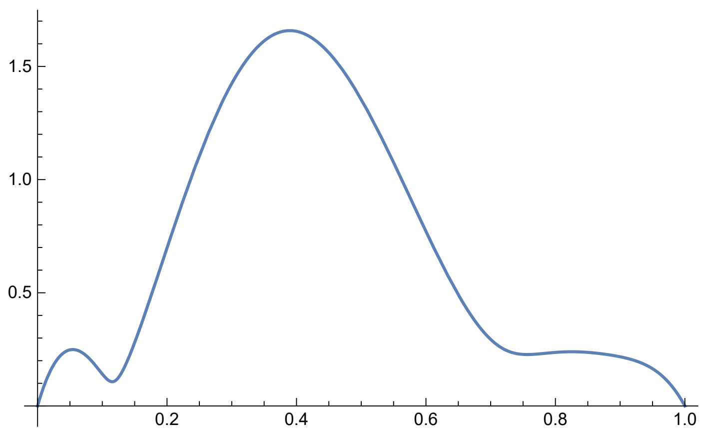
```
sub = Sqrt[(ort[[1]] - rt[[1]])^2 + (ort[[2]] - rt[[2]])^2]
```

$$\sqrt{\Big(\Big(724.266\,(1-t)^3 + 2094.6\,(1-t)^2\,t + 2128.89\,(1-t)\,t^2 + 680.506\,t^3 - 709.63\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right] - 680.506\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right) - 698.2\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right] - 724.266\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)\Big)^2 + \Big(175.313\,(1-t)^3 + 672.42\,(1-t)^2\,t + 504.09\,(1-t)\,t^2 + 166.372\,t^3 - 168.03\left(2+\lambda\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)\text{Cos}\left[\frac{\pi t}{2}\right] - 166.372\left(1-\text{Cos}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Cos}\left[\frac{\pi t}{2}\right]\right) - 224.14\left(2+\lambda\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\right)\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)\text{Sin}\left[\frac{\pi t}{2}\right] - 175.313\left(1-\text{Sin}\left[\frac{\pi t}{2}\right]\right)^2\left(1-\lambda\,\text{Sin}\left[\frac{\pi t}{2}\right]\right)\Big)^2\Big)}$$

```
Plot[sub /. λ → -0.5, {t, 0, 1}]
```



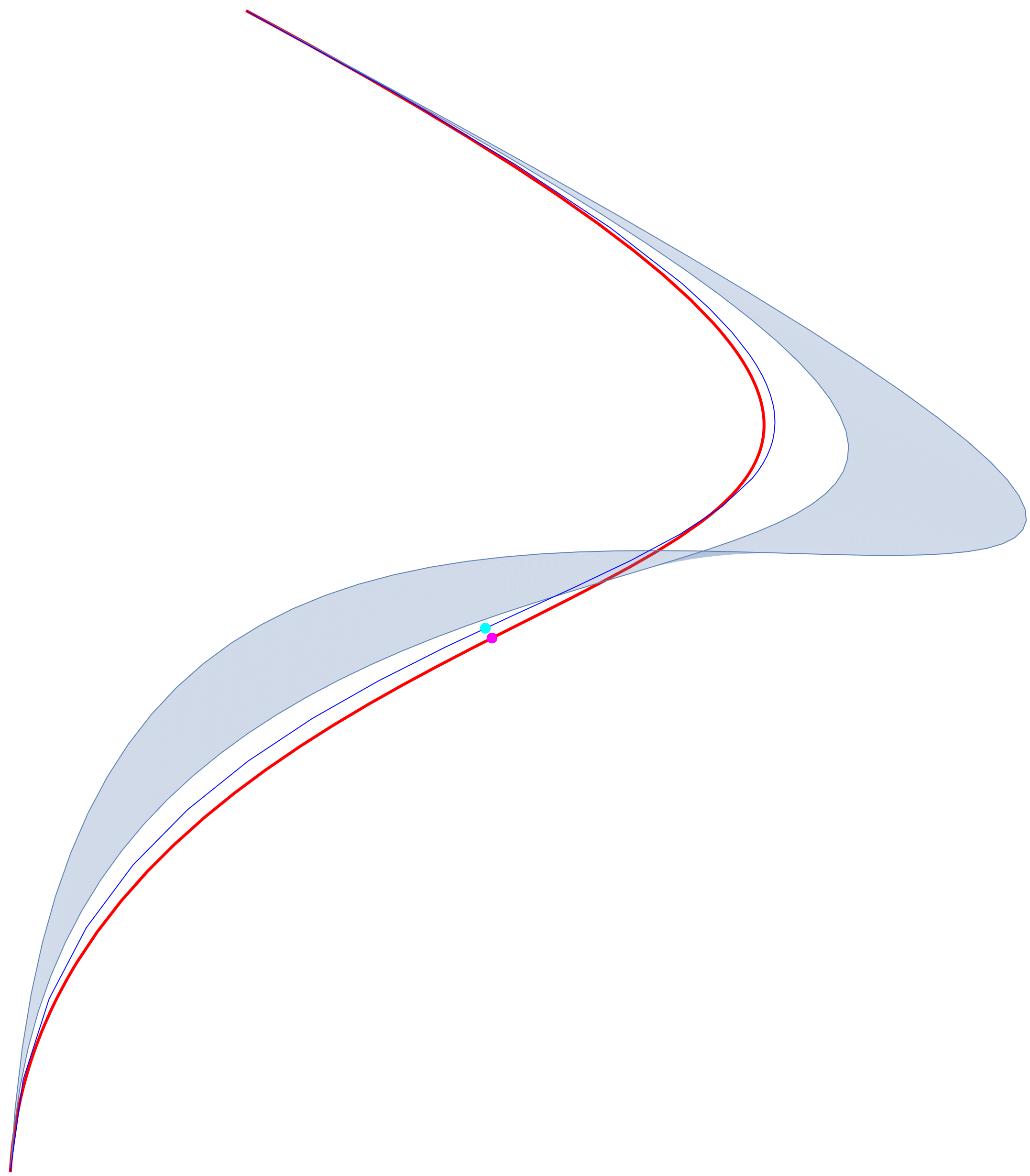

```
Plot[sub /. λ → -0.205, {t, 0, 1}]
```



```
Maximize[{sub /. λ → -0.5, 0 ≤ t ≤ 1}, t]
{2.03627, {t → 0.134579}}

Maximize[{sub /. λ → -0.205, 0 ≤ t ≤ 1}, t]
{1.65808, {t → 0.389663}}

FindMaximum[{sub, λ == -0.205, 0 < t < 1}, {λ, t}]
{0.239483, {λ → -0.205, t → 0.825204}}

λ =.; t =.;

FindMaximum[{sub, -0.3 ≤ λ ≤ -0.1, 0 ≤ t ≤ 1}, {{λ, -1}, {t, 0}}]
{0.714069, {λ → -0.3, t → 0.0888849}}

Sqrt[(P0[[1, 1]] - P3[[1, 1]])^2 + (P0[[1, 2]] - P3[[1, 2]])^2]
44.6641

t = 0.6;

ort
{184.298, 700.984}

λ = -0.5;

rt
{184.55, 700.613}

Norm[ort - rt]
0.44841
```

```
Point2 = Graphics[{PointSize[0.01], Cyan, Point[ort], Magenta, Point[rt]}]
```

```
Show[Point2, Curve0[[1]], Curve[[1]],
 OrigBezier[[1]], Point2, PlotRange → All, ImageSize → Large]
```



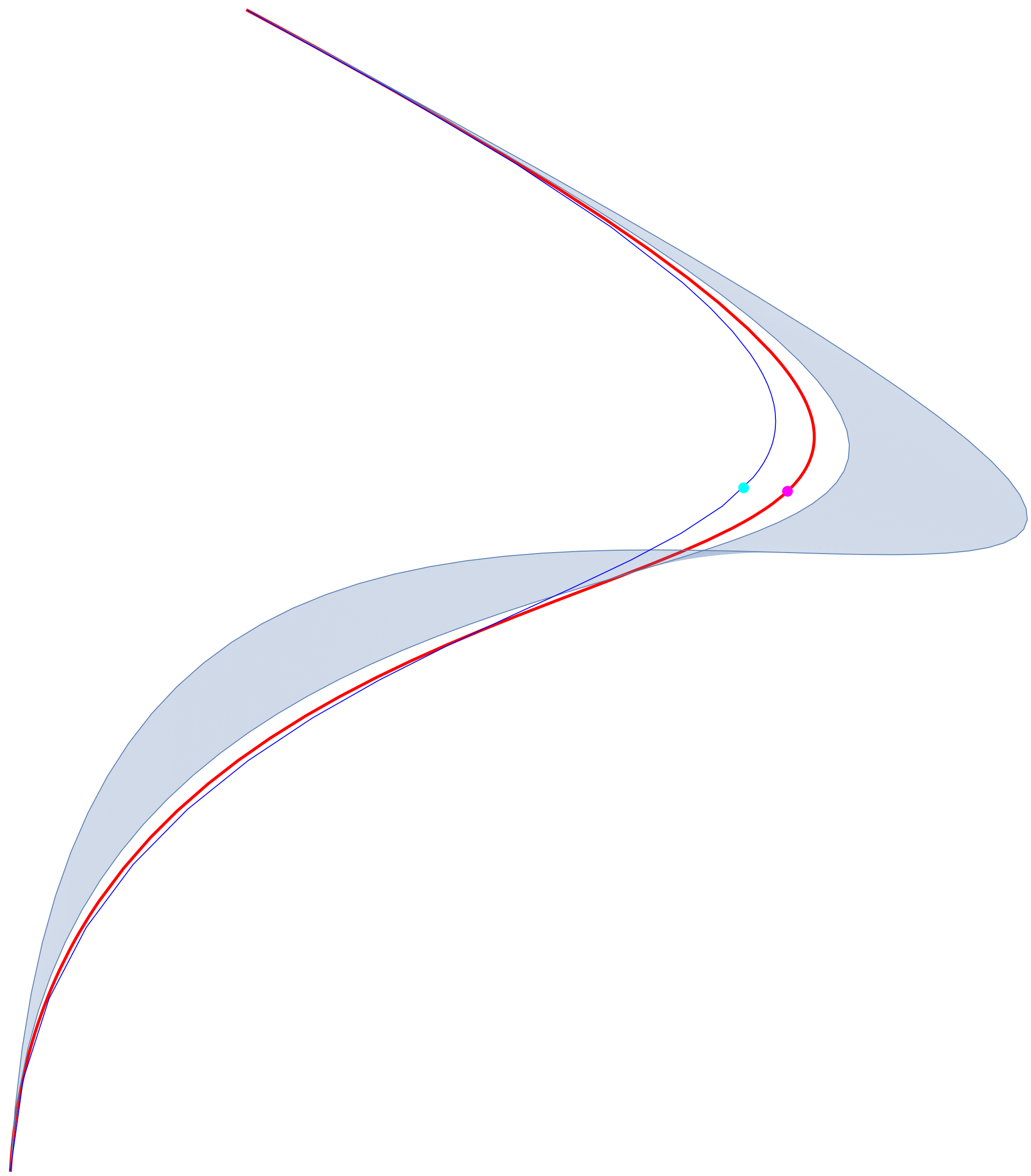
```
t = 0.3896628382725411`;

λ = -0.205;

ort

{194.022, 706.257}

rt

{195.674, 706.126}
```

```
dis = Sqrt[(ort[[1]] - rt[[1]])^2 + (ort[[2]] - rt[[2]])^2]
```

```
1.65808
```

```
Point2 = Graphics[{PointSize[0.01], Cyan, Point[ort], Magenta, Point[rt]}]
```

```
Show[Point2, Curve01[[1]], Curve[[1]],
 OrigBezier[[1]], Point2, PlotRange → All, ImageSize → Large]
```



G Spherical Bezier Curve Mathematica Code

Below is the Mathematica code for the computation of spherical Bezier Curve.

Projection of Trifolium on a Sphere

Define the link length of the planar links

```
L1 = 10;
L2 = 10;
```

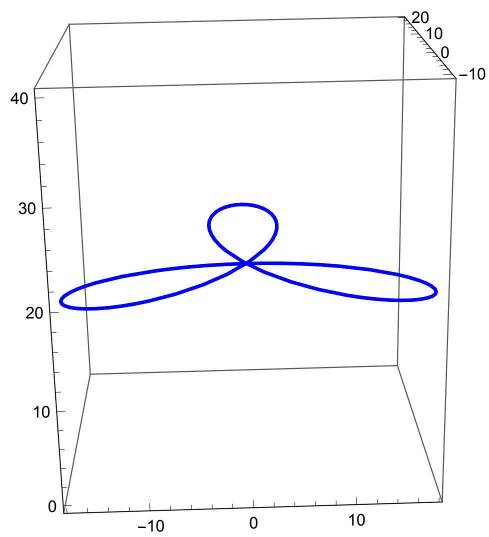
Define the radius of the sphere

```
r = 20;
```

The coordinates is build in the center of the sphere
Represent the coordinates of the Trifolium on the plane

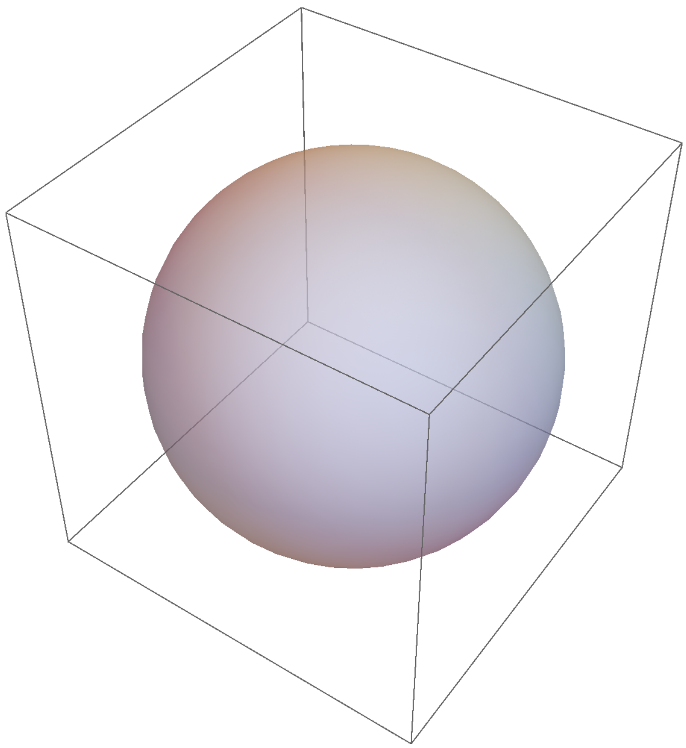
```
Pt = { L1 Sin[θ] - L2 Sin[2 θ], L1 Cos[θ] + L2 Cos[2 θ], r}
{10 Sin[θ] - 10 Sin[2 θ], 10 Cos[θ] + 10 Cos[2 θ], 20}
```

```
TriPlanar =
 ParametricPlot3D[{Pt[[1]], Pt[[2]], Pt[[3]]}, {θ, 0, 2 Pi}, PlotStyle → {Blue}]
```

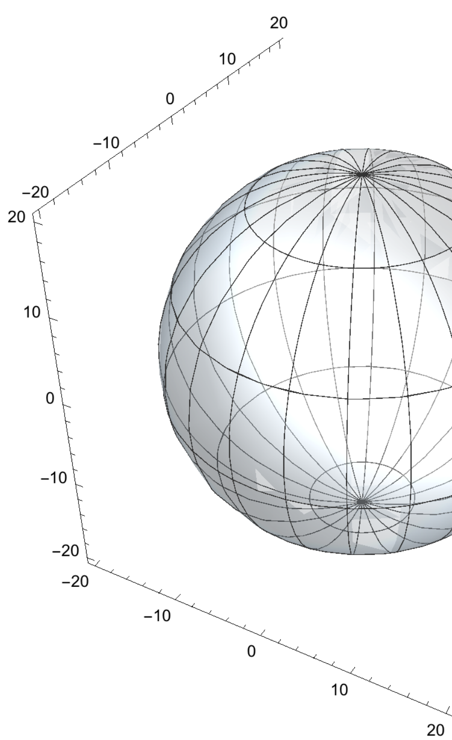



```
TriSteps =
  Table [{ L1 Sin[θ] - L2 Sin[2 θ], L1 Cos[θ] + L2 Cos[2 θ], r}, {θ, 0, 2 π, 2 π / 100}];
```

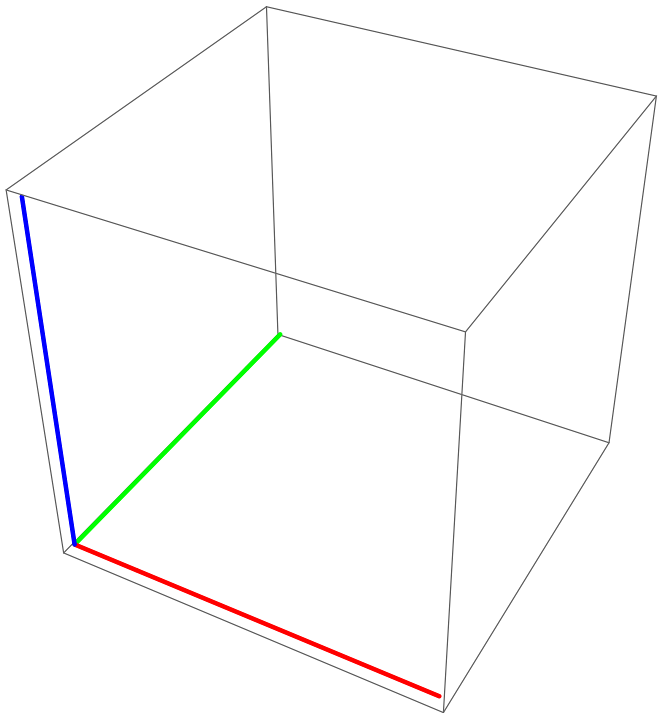
```
Sph = Graphics3D[{Opacity[0.5], LightBlue, Sphere[{0, 0, 0}, r]}]
```



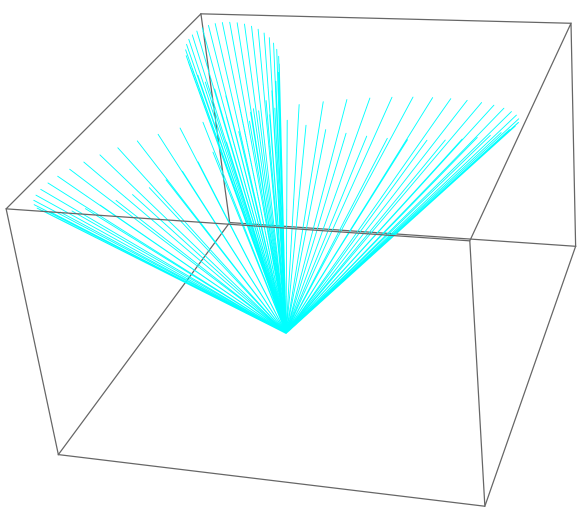
```
Sph = SphericalPlot3D[{r}, {α, 0, 2 Pi},
  {β, 0, 2 Pi}, PlotStyle → {{FaceForm[None], EdgeForm[None]},
    {Opacity[0.3], FaceForm[LightBlue], EdgeForm[None]}},
  Mesh → 10, Boxed -> False, Lighting → "Neutral"]
```



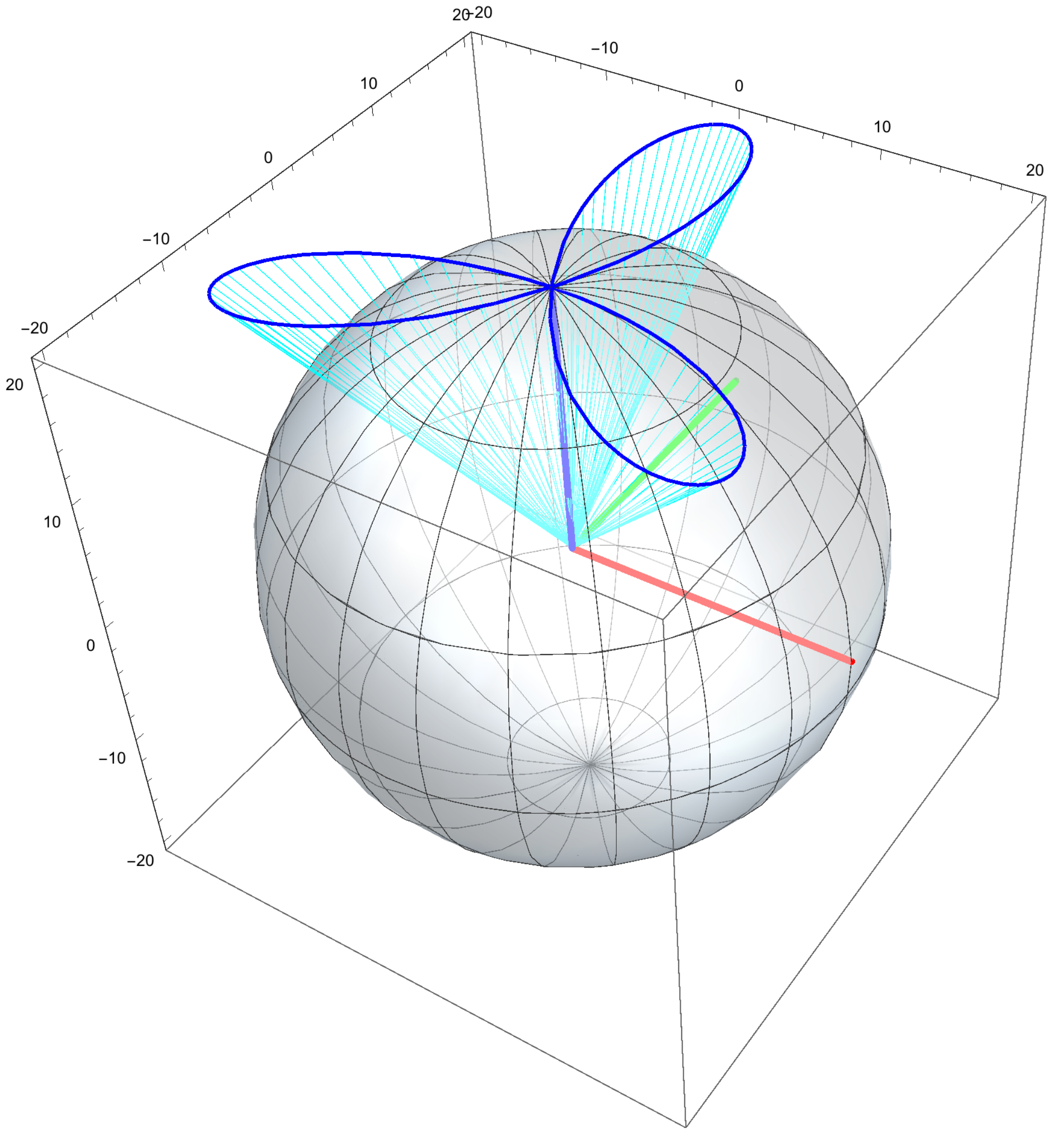
```
axis = Graphics3D[{Red, Thickness[0.007], Line[{{0, 0, 0}, {r, 0, 0}}],
   Green, Line[{{0, 0, 0}, {0, r, 0}}],
   Blue, Line[{{0, 0, 0}, {0, 0, r}}]} ]
```



```
ProjectionLine = Graphics3D[
  {Cyan, Table[Line[{{0, 0, 0}, TriSteps[[i]]}], {i, 1, Length[TriSteps]}]}]
```



```
Show[TriPlanar, ProjectionLine, Sph, axis, PlotRange → All, ImageSize → Large]
```

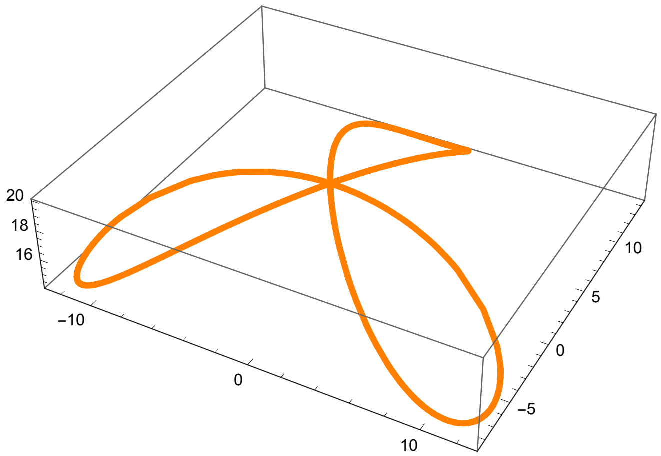


Projection on the Sphere

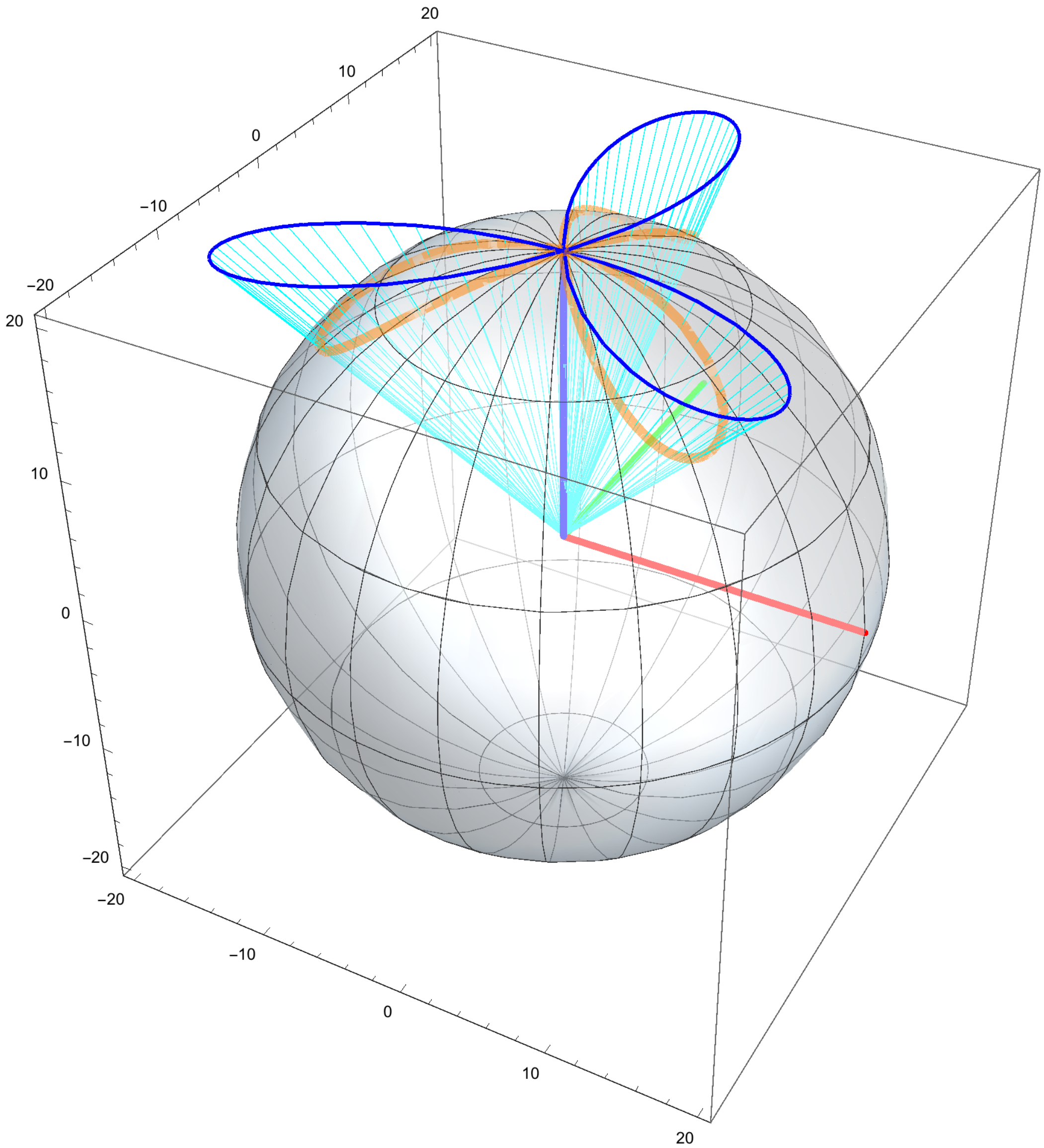
```
PtProj = r * Normalize[Pt]
```

$$\left\{(20\ (10\ \mathrm{Sin}[\theta]-10\ \mathrm{Sin}[2\ \theta]))\Big/\left(\sqrt{(400+\mathrm{Abs}[10\ \mathrm{Cos}[\theta]+10\ \mathrm{Cos}[2\ \theta]]^2+\mathrm{Abs}[10\ \mathrm{Sin}[\theta]-10\ \mathrm{Sin}[2\ \theta]]^2)}\right),\right.$$
$$(20\ (10\ \mathrm{Cos}[\theta]+10\ \mathrm{Cos}[2\ \theta]))\Big/\left(\sqrt{(400+\mathrm{Abs}[10\ \mathrm{Cos}[\theta]+10\ \mathrm{Cos}[2\ \theta]]^2+\mathrm{Abs}[10\ \mathrm{Sin}[\theta]-10\ \mathrm{Sin}[2\ \theta]]^2)}\right),$$
$$\left.400\Big/\left(\sqrt{(400+\mathrm{Abs}[10\ \mathrm{Cos}[\theta]+10\ \mathrm{Cos}[2\ \theta]]^2+\mathrm{Abs}[10\ \mathrm{Sin}[\theta]-10\ \mathrm{Sin}[2\ \theta]]^2)}\right)\right\}$$

```
TriProj = ParametricPlot3D[{PtProj[[1]], PtProj[[2]], PtProj[[3]]},
  {θ, 0, 2 Pi}, PlotStyle → {Orange, Thickness[0.01], Opacity[0.5]}]
```



```
Show[TriPlanar, ProjectionLine, Sph,
 axis, TriProj, PlotRange → All, ImageSize → Large]
```



Define Quaternion

Define the start point p0 on the sphere

```
p0 = {1, 0, 0, 0};
```

```
MatrixForm[p0]
```

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Define Skew axis

```
s[sx_, sy_, sz_] := {sx, sy, sy}
```

Define the Quaternion [Q]

```
Q = {sx * Sin[ϕ / 2], sy * Sin[ϕ / 2], sz * Sin[ϕ / 2], Cos[ϕ / 2]};
MatrixForm[Q]
```

$$\begin{pmatrix} sx\,Sin\left[\frac{\phi}{2}\right] \\ sy\,Sin\left[\frac{\phi}{2}\right] \\ sz\,Sin\left[\frac{\phi}{2}\right] \\ Cos\left[\frac{\phi}{2}\right] \end{pmatrix}$$

Define Quaternion conjugate [Qc]

```
Qc = {-sx * Sin[ϕ / 2], -sy * Sin[ϕ / 2], -sz * Sin[ϕ / 2], Cos[ϕ / 2]};
MatrixForm[Qc]
```

$$\begin{pmatrix} -sx\,Sin\left[\frac{\phi}{2}\right] \\ -sy\,Sin\left[\frac{\phi}{2}\right] \\ -sz\,Sin\left[\frac{\phi}{2}\right] \\ Cos\left[\frac{\phi}{2}\right] \end{pmatrix}$$

Define [Q+] (Qplus)

```
Qplus = {{Q[[4]], -Q[[3]], Q[[2]], Q[[1]]}, {Q[[3]], Q[[4]], -Q[[1]], Q[[2]]},
   {-Q[[2]], Q[[1]], Q[[4]], Q[[3]]}, {-Q[[1]], -Q[[2]], -Q[[3]], Q[[4]]}};
MatrixForm[
 Qplus]
```

$$\begin{pmatrix} Cos\left[\frac{\phi}{2}\right] & -sz\,Sin\left[\frac{\phi}{2}\right] & sy\,Sin\left[\frac{\phi}{2}\right] & sx\,Sin\left[\frac{\phi}{2}\right] \\ sz\,Sin\left[\frac{\phi}{2}\right] & Cos\left[\frac{\phi}{2}\right] & -sx\,Sin\left[\frac{\phi}{2}\right] & sy\,Sin\left[\frac{\phi}{2}\right] \\ -sy\,Sin\left[\frac{\phi}{2}\right] & sx\,Sin\left[\frac{\phi}{2}\right] & Cos\left[\frac{\phi}{2}\right] & sz\,Sin\left[\frac{\phi}{2}\right] \\ -sx\,Sin\left[\frac{\phi}{2}\right] & -sy\,Sin\left[\frac{\phi}{2}\right] & -sz\,Sin\left[\frac{\phi}{2}\right] & Cos\left[\frac{\phi}{2}\right] \end{pmatrix}$$

Define [Qc-] (QcMinu)

```
QcMinu = {{Qc[[4]], Qc[[3]], -Qc[[2]], Qc[[1]]},
   {-Qc[[3]], Qc[[4]], Qc[[1]], Qc[[2]]}, {Qc[[2]], -Qc[[1]], Qc[[4]], Qc[[3]]},
   {-Qc[[1]], -Qc[[2]], -Qc[[3]], Qc[[4]]}};
MatrixForm[
 QcMinu]
```

$$\begin{pmatrix} \mathrm{Cos}\left[\frac{\phi}{2}\right] & -\mathrm{sz}\ \mathrm{Sin}\left[\frac{\phi}{2}\right] & \mathrm{sy}\ \mathrm{Sin}\left[\frac{\phi}{2}\right] & -\mathrm{sx}\ \mathrm{Sin}\left[\frac{\phi}{2}\right] \\ \mathrm{sz}\ \mathrm{Sin}\left[\frac{\phi}{2}\right] & \mathrm{Cos}\left[\frac{\phi}{2}\right] & -\mathrm{sx}\ \mathrm{Sin}\left[\frac{\phi}{2}\right] & -\mathrm{sy}\ \mathrm{Sin}\left[\frac{\phi}{2}\right] \\ -\mathrm{sy}\ \mathrm{Sin}\left[\frac{\phi}{2}\right] & \mathrm{sx}\ \mathrm{Sin}\left[\frac{\phi}{2}\right] & \mathrm{Cos}\left[\frac{\phi}{2}\right] & -\mathrm{sz}\ \mathrm{Sin}\left[\frac{\phi}{2}\right] \\ \mathrm{sx}\ \mathrm{Sin}\left[\frac{\phi}{2}\right] & \mathrm{sy}\ \mathrm{Sin}\left[\frac{\phi}{2}\right] & \mathrm{sz}\ \mathrm{Sin}\left[\frac{\phi}{2}\right] & \mathrm{Cos}\left[\frac{\phi}{2}\right] \end{pmatrix}$$

Define [Q+][Qc-]

```
Qt = MatrixForm[FullSimplify[Qplus.QcMinu, (sx)^2 + (sy)^2 + (sz)^2 == 1]]
```

$$\begin{pmatrix} 1+\mathrm{sy}^2\ (-1+\mathrm{Cos}[\phi])+\mathrm{sz}^2\ (-1+\mathrm{Cos}[\phi]) & \mathrm{sx\ sy}-\mathrm{sx\ sy\ Cos}[\phi]-\mathrm{sz\ Sin}[\phi] & \mathrm{sx\ sz}-\mathrm{sx\ sz\ Cos} \\ \mathrm{sx\ sy}-\mathrm{sx\ sy\ Cos}[\phi]+\mathrm{sz\ Sin}[\phi] & \mathrm{sy}^2\ (1-\mathrm{Cos}[\phi])+\mathrm{Cos}[\phi] & \mathrm{sy\ sz}-\mathrm{sy\ sz\ Cos} \\ \mathrm{sx\ sz}-\mathrm{sx\ sz\ Cos}[\phi]-\mathrm{sy\ Sin}[\phi] & \mathrm{sy\ sz}-\mathrm{sy\ sz\ Cos}[\phi]+\mathrm{sx\ Sin}[\phi] & \mathrm{sz}^2\ (1-\mathrm{Cos}[\phi \\ 0 & 0 & 0 \end{pmatrix}$$

```
Qtn[φ_, sx_, sy_, sz_] :=
 {{1 + sy^2 (-1 + Cos[φ]) + sz^2 (-1 + Cos[φ]),
   sx sy - sx sy Cos[φ] - sz Sin[φ], sx sz - sx sz Cos[φ] + sy Sin[φ], 0},
  {sx sy - sx sy Cos[φ] + sz Sin[φ], sy^2 (1 - Cos[φ]) + Cos[φ],
   sy sz - sy sz Cos[φ] - sx Sin[φ], 0},
  {sx sz - sx sz Cos[φ] - sy Sin[φ], sy sz - sy sz Cos[φ] + sx Sin[φ],
   sz^2 (1 - Cos[φ]) + Cos[φ], 0},
  {0,
   0,
   0,
   1}}
```

Verification

```
MatrixForm[Qtn[0, 0, 0, 0]]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```
Qtn[Pi / 2, 0, 0, 1].p0
```

{0, 1, 0, 0}

```
Qtn[2 Pi / 3, 1 / Sqrt[3], 1 / Sqrt[3], 1 / Sqrt[3]].{1, 0, 0, 0}
```

{0, 1, 0, 0}

Serial chain Contruction

Projection of the link L1 and L2 on the sphere

```
SL1 = r * Normalize[{0, L1, r}]
```

$\{0, 4\sqrt{5}, 8\sqrt{5}\}$

```
SL2 = r * Normalize[{0, L1 + L2, r}]
```

$\{0, 10\sqrt{2}, 10\sqrt{2}\}$

Verification

```
N[VectorAngle[{0, 0, r}, SL1]]
```

0.463648

```
N[VectorAngle[SL1, SL2]]
```

0.321751

Convert SL1 and SL2 to Quaternion

```
SL1q = {SL1[[1]], SL1[[2]], SL1[[3]], 0}
```

$\{0, 4\sqrt{5}, 8\sqrt{5}, 0\}$

```
SL2q = {SL2[[1]], SL2[[2]], SL2[[3]], 0}
```

$\{0, 10\sqrt{2}, 10\sqrt{2}, 0\}$

Rotation of SL1 and SL2

```
Skew1 = Normalize[{0, 0, r}]
```

{0, 0, 1}

```
Curve2 = Qtn[ψ, Skew1[[1]], Skew1[[2]], Skew1[[3]]].SL1q
```

$\{-4\sqrt{5}\,\mathrm{Sin}[\psi], 4\sqrt{5}\,\mathrm{Cos}[\psi], 8\sqrt{5}, 0\}$

```
Skew2 = Normalize[SL1]
```

$\left\{0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\}$

```
Curve3 = Qtn[ψ, Skew1[[1]], Skew1[[2]], Skew1[[3]]].
  Qtn[-3 ψ, Skew2[[1]], Skew2[[2]], Skew2[[3]]].SL2q
```

$$\left\{10\sqrt{2}\left(-\left(\frac{2}{5}-\frac{2}{5}\text{Cos}[3\psi]\right)\text{Sin}[\psi]-\frac{\text{Cos}[\psi]\,\text{Sin}[3\psi]}{\sqrt{5}}\right)+\right.$$

$$10\sqrt{2}\left(-\left(\frac{1}{5}(1-\text{Cos}[3\psi])+\text{Cos}[3\psi]\right)\text{Sin}[\psi]+\frac{2\,\text{Cos}[\psi]\,\text{Sin}[3\psi]}{\sqrt{5}}\right),$$

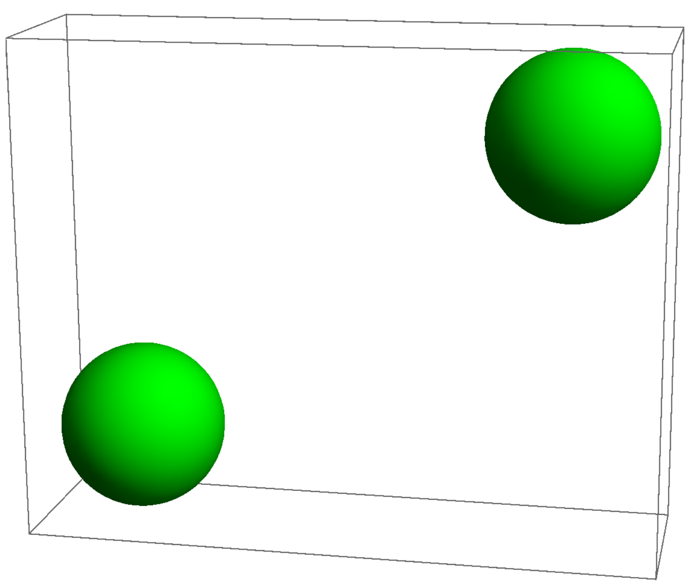
$$10\sqrt{2}\left(\text{Cos}[\psi]\left(\frac{2}{5}-\frac{2}{5}\text{Cos}[3\psi]\right)-\frac{\text{Sin}[\psi]\,\text{Sin}[3\psi]}{\sqrt{5}}\right)+$$

$$10\sqrt{2}\left(\text{Cos}[\psi]\left(\frac{1}{5}(1-\text{Cos}[3\psi])+\text{Cos}[3\psi]\right)+\frac{2\,\text{Sin}[\psi]\,\text{Sin}[3\psi]}{\sqrt{5}}\right),$$

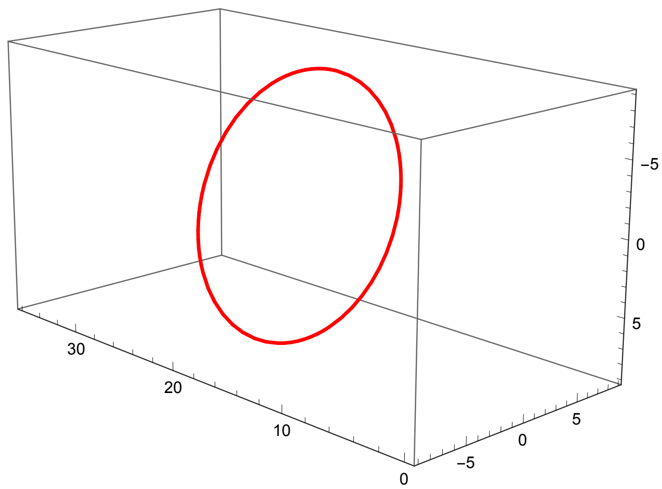
$$\left.10\sqrt{2}\left(\frac{2}{5}-\frac{2}{5}\text{Cos}[3\psi]\right)+10\sqrt{2}\left(\frac{4}{5}(1-\text{Cos}[3\psi])+\text{Cos}[3\psi]\right),\ 0\right\}$$

Plot the curve generated by the spherical serial chain

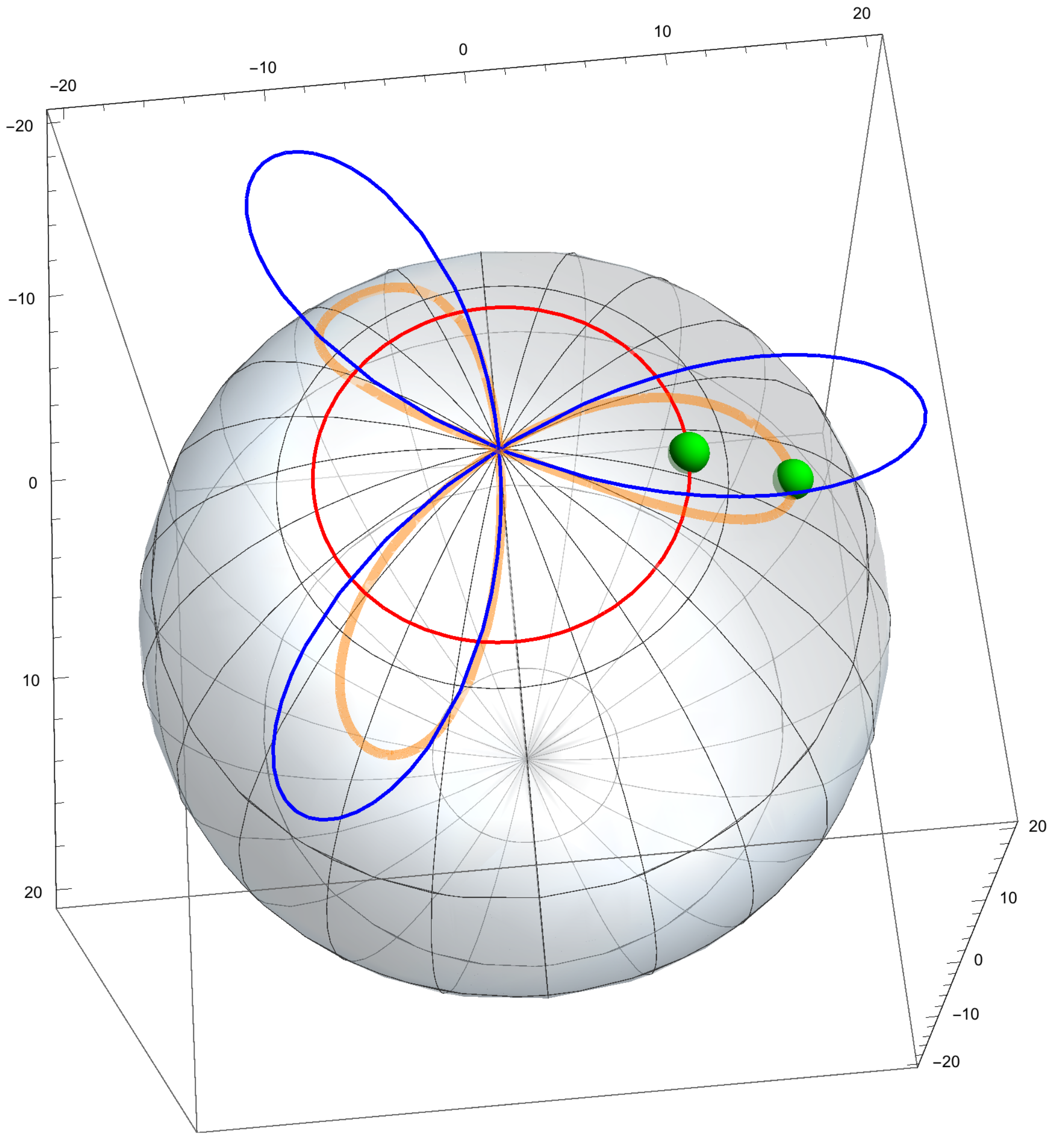
```
TracePoints = Graphics3D[{Green, Sphere[SL1, 1], Sphere[SL2, 1]}]
```



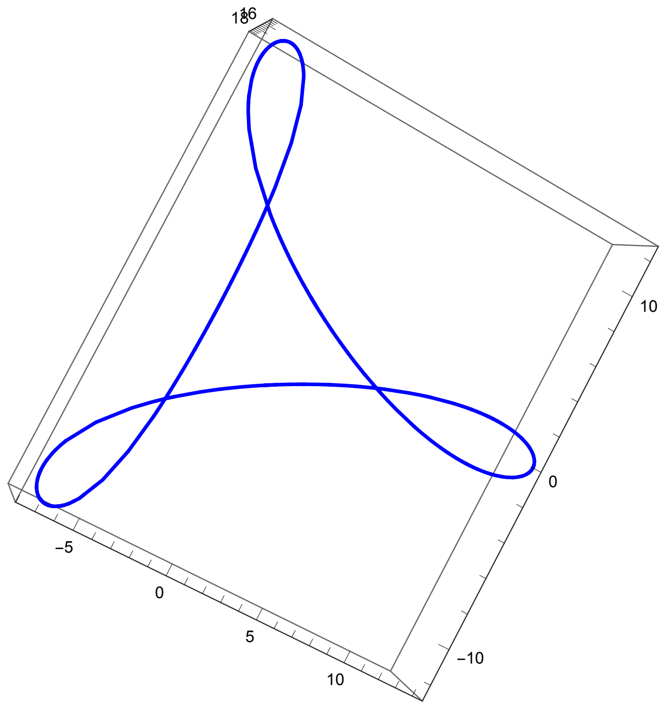
```
TraceSL1q = ParametricPlot3D[{Curve2[[1]], Curve2[[2]], Curve2[[3]]},
  {ψ, 0, 2 Pi}, PlotStyle → {Red}, ImageSize → Medium]
```



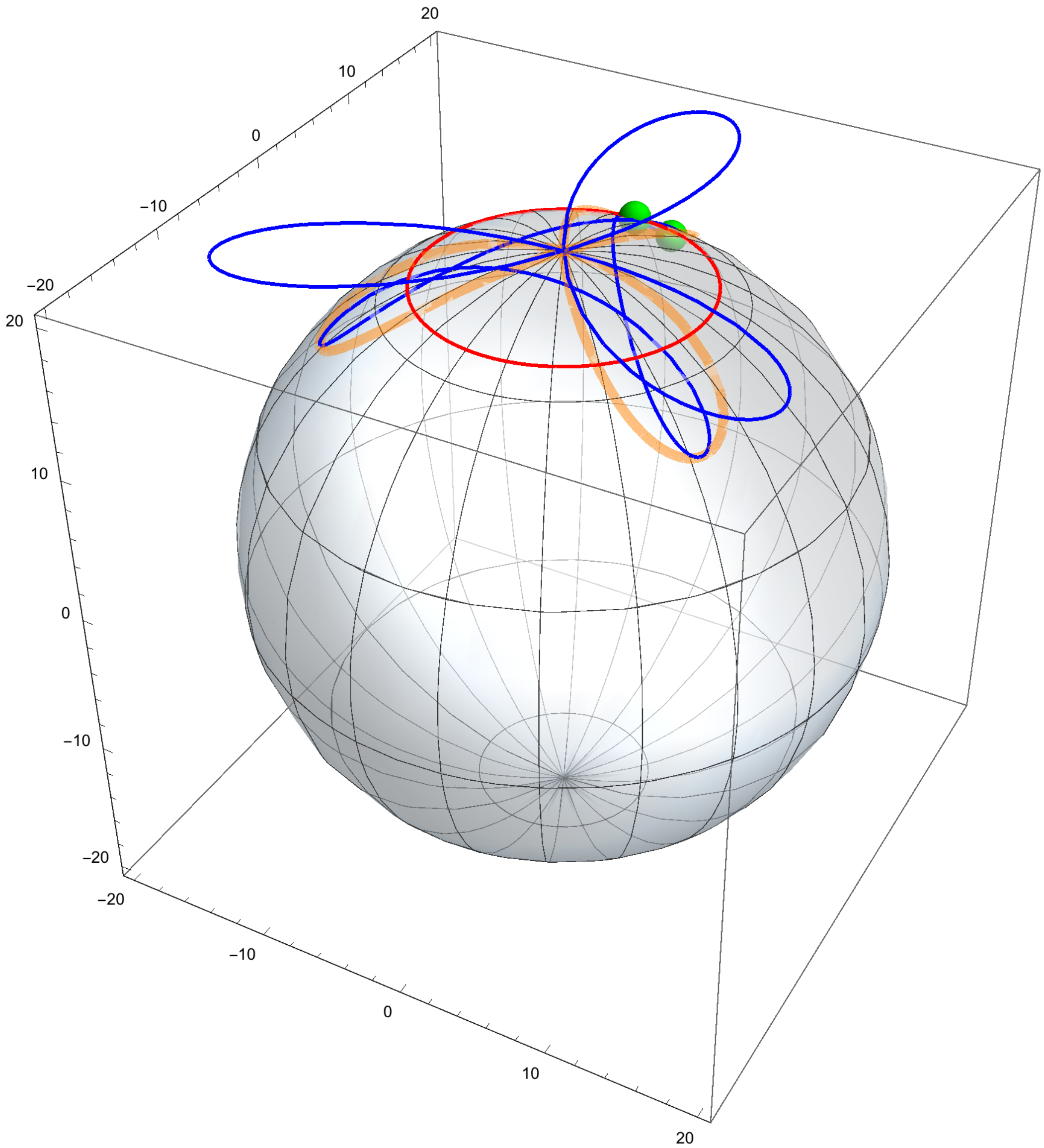
```
Show[TriPlanar, Sph, TriProj, TraceSL1q,
 TracePoints, PlotRange → All, ImageSize → Large]
```



```
TraceSL2q = ParametricPlot3D[{Curve3[[1]], Curve3[[2]], Curve3[[3]]},
  {ψ, 0, 2 Pi}, PlotStyle → {Blue}, ImageSize → Medium]
```



```
Show[TriPlanar, Sph, TriProj, TraceSL1q,
 TraceSL2q, TracePoints, PlotRange → All, ImageSize → Large]
```



Revised Spherical Serial Chain

Projection L2 on the sphere

```
SL2 = r * Normalize[{0, L1 + L2, r}]
```

$\{0, 10\sqrt{2}, 10\sqrt{2}\}$

```
angle = N[VectorAngle[{0, 0, r}, SL2]]
```

0.785398

Convert SL2 to Quaternion

```
SL2q = {SL2[[1]], SL2[[2]], SL2[[3]], 0}
```

$\{0, 10\sqrt{2}, 10\sqrt{2}, 0\}$

Projection L1 on the sphere

```
SL1q = Qtn[angle / 2, 1, 0, 0].SL2q
{0., 7.65367, 18.4776, 0.}

SL1 = {SL1q[[1]], SL1q[[2]], SL1q[[3]]}
{0., 7.65367, 18.4776}
```

Verification

```
N[VectorAngle[{0, 0, r}, SL1]]
0.392699

N[VectorAngle[SL1, SL2]]
0.392699
```

Rotation of SL1 and SL2

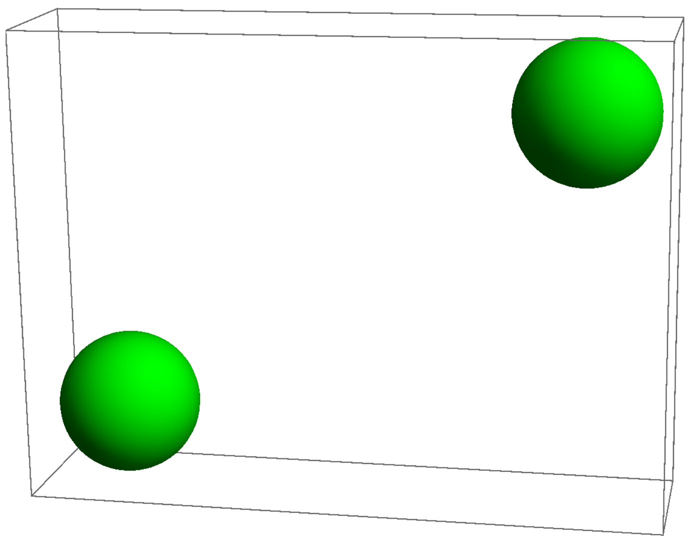
```
Skew1 = Normalize[{0, 0, r}]
{0, 0, 1}

Curve2 = Qtn[ψ, Skew1[[1]], Skew1[[2]], Skew1[[3]]].SL1q
{0. - 7.65367 Sin[ψ], 0. + 7.65367 Cos[ψ], 18.4776, 0.}

Skew2 = Normalize[SL1]
{0., 0.382683, 0.92388}

Curve3 = Qtn[ψ, Skew1[[1]], Skew1[[2]], Skew1[[3]]].
  Qtn[-3 ψ, Skew2[[1]], Skew2[[2]], Skew2[[3]]].SL2q
```

$\{10\sqrt{2}\ (-(0.353553 - 0.353553\ \mathrm{Cos}[3\psi])\ \mathrm{Sin}[\psi] + \mathrm{Cos}[\psi]\ (0. - 0.382683\ \mathrm{Sin}[3\psi])) + 10$
$\sqrt{2}\ (-(0.146447\ (1 - \mathrm{Cos}[3\psi]) + \mathrm{Cos}[3\psi])\ \mathrm{Sin}[\psi] + \mathrm{Cos}[\psi]\ (0. + 0.92388\ \mathrm{Sin}[3\psi])),$
$10\sqrt{2}\ (\mathrm{Cos}[\psi]\ (0.353553 - 0.353553\ \mathrm{Cos}[3\psi]) + \mathrm{Sin}[\psi]\ (0. - 0.382683\ \mathrm{Sin}[3\psi])) +$
$10\sqrt{2}\ (\mathrm{Cos}[\psi]\ (0.146447\ (1 - \mathrm{Cos}[3\psi]) + \mathrm{Cos}[3\psi]) + \mathrm{Sin}[\psi]\ (0. + 0.92388\ \mathrm{Sin}[3\psi])),$
$10\sqrt{2}\ (0.353553 - 0.353553\ \mathrm{Cos}[3\psi]) + 10\sqrt{2}\ (0.853553\ (1 - \mathrm{Cos}[3\psi]) + \mathrm{Cos}[3\psi]), 0\}$

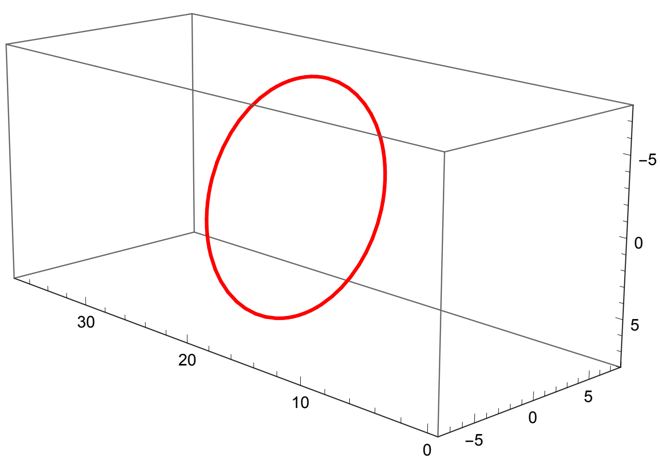
```
Sim_curve3 = Simplify[Curve3]
{(-7.07107 - 7.07107 Cos[3 ψ]) Sin[ψ] + 7.65367 Cos[ψ] Sin[3 ψ],
 Cos[ψ] (7.07107 + 7.07107 Cos[3 ψ]) + 7.65367 Sin[ψ] Sin[3 ψ],
 17.0711 - 2.92893 Cos[3 ψ], 0}
```

Plot the curve generated by the spherical serial chain

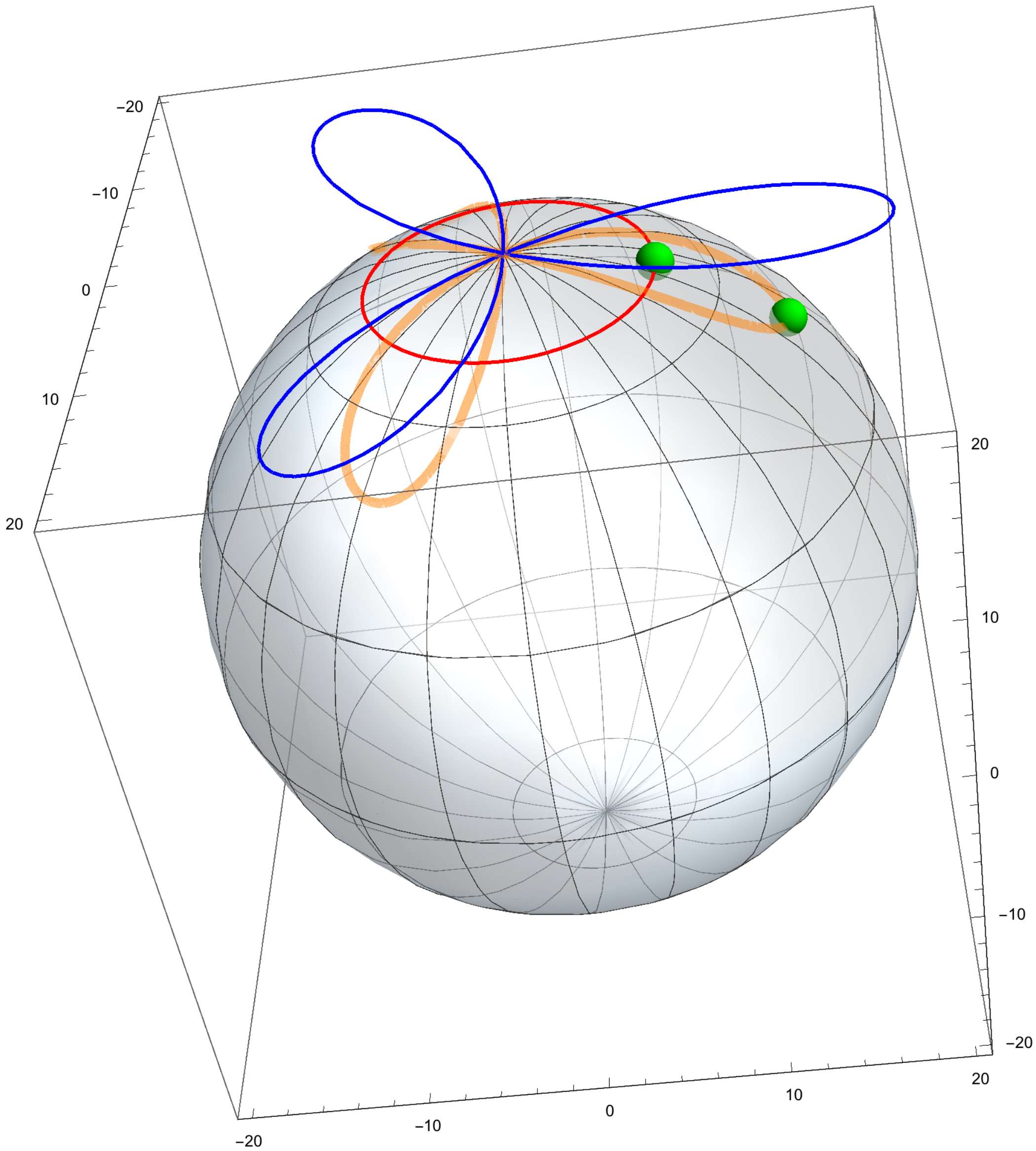
```
TracePoints = Graphics3D[{Green, Sphere[SL1, 1], Sphere[SL2, 1]}]
```



```
ψ =.;

TraceSL1q = ParametricPlot3D[{Curve2[[1]], Curve2[[2]], Curve2[[3]]},
  {ψ, 0, 2 Pi}, PlotStyle → {Red}, ImageSize → Medium]
```

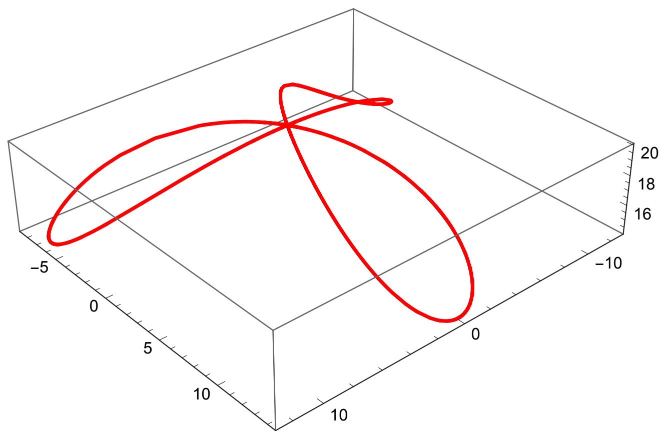


```
Show[TriPlanar, Sph, TriProj, TraceSL1q,
 TracePoints, PlotRange → All, ImageSize → Large]
```

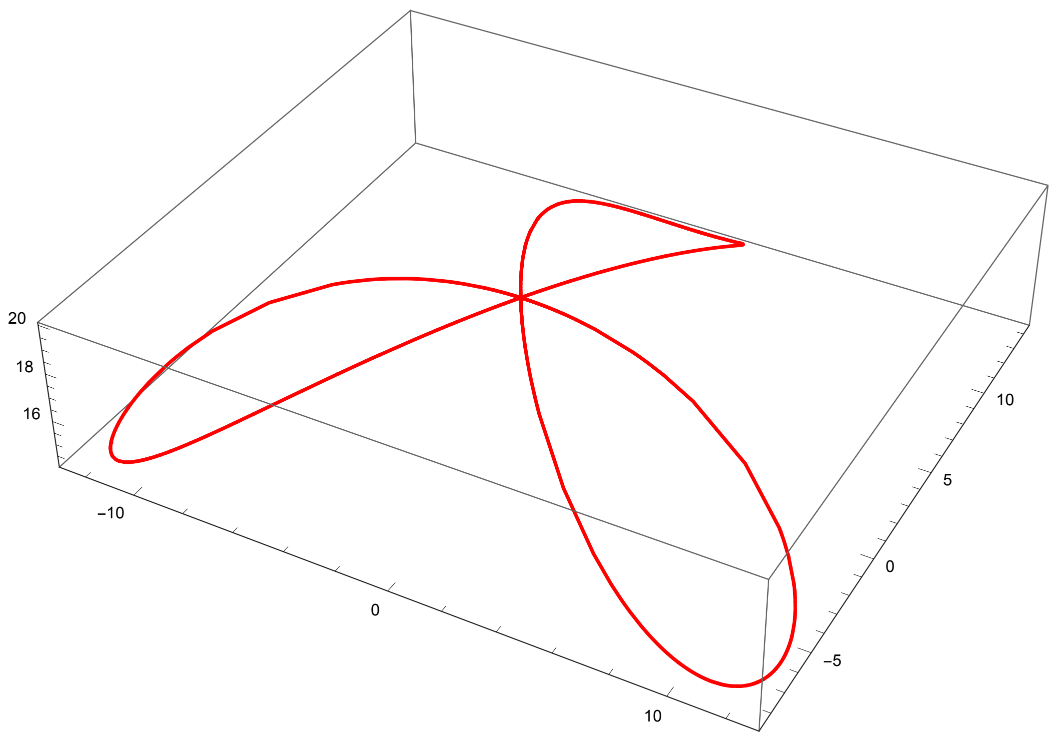



```
ψ =.;
```

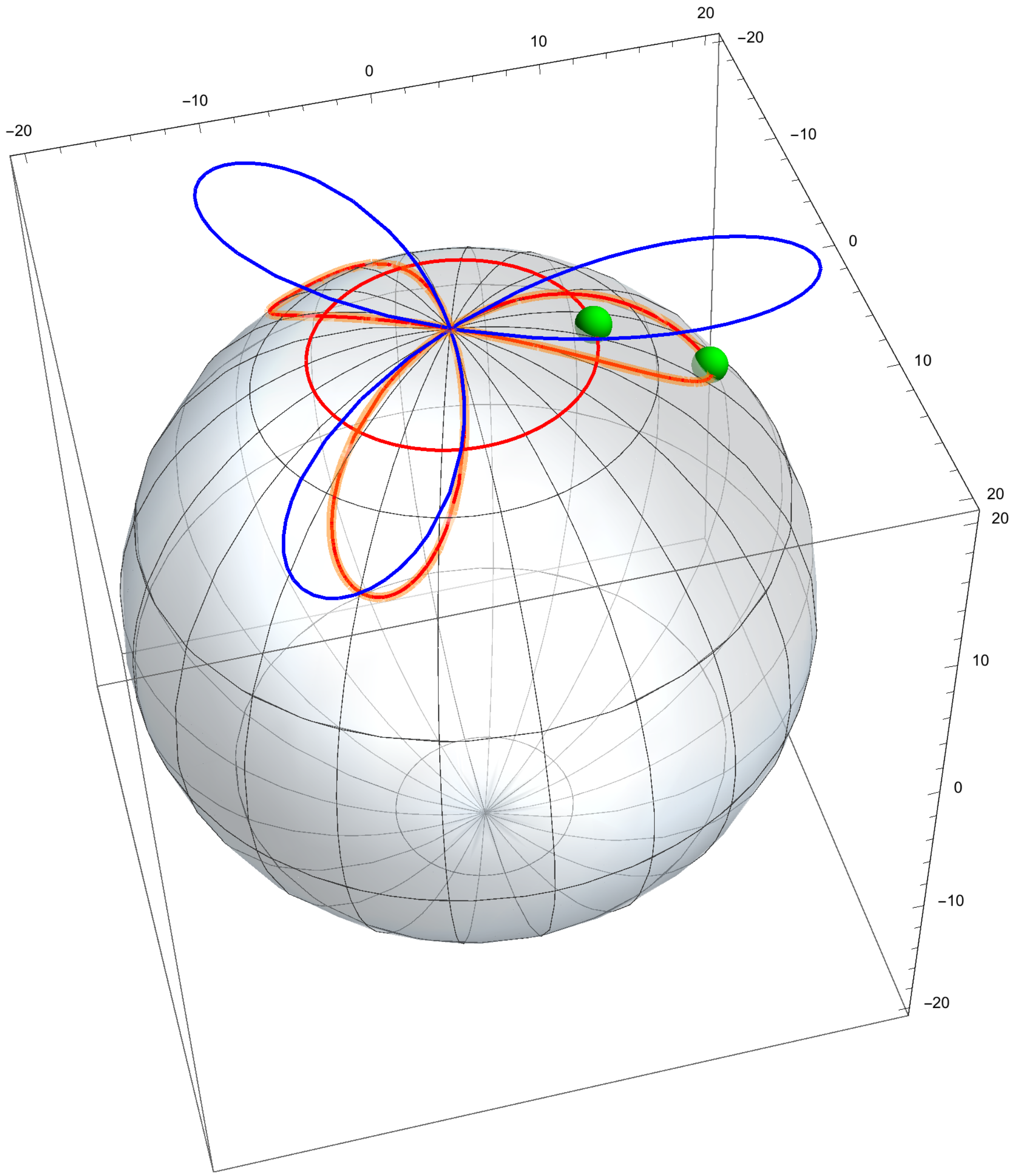
```
TraceSL2q = ParametricPlot3D[{Curve3[[1]], Curve3[[2]], Curve3[[3]]},
  {ψ, 0, 2 Pi}, PlotStyle → {Red}, ImageSize → Medium]
```



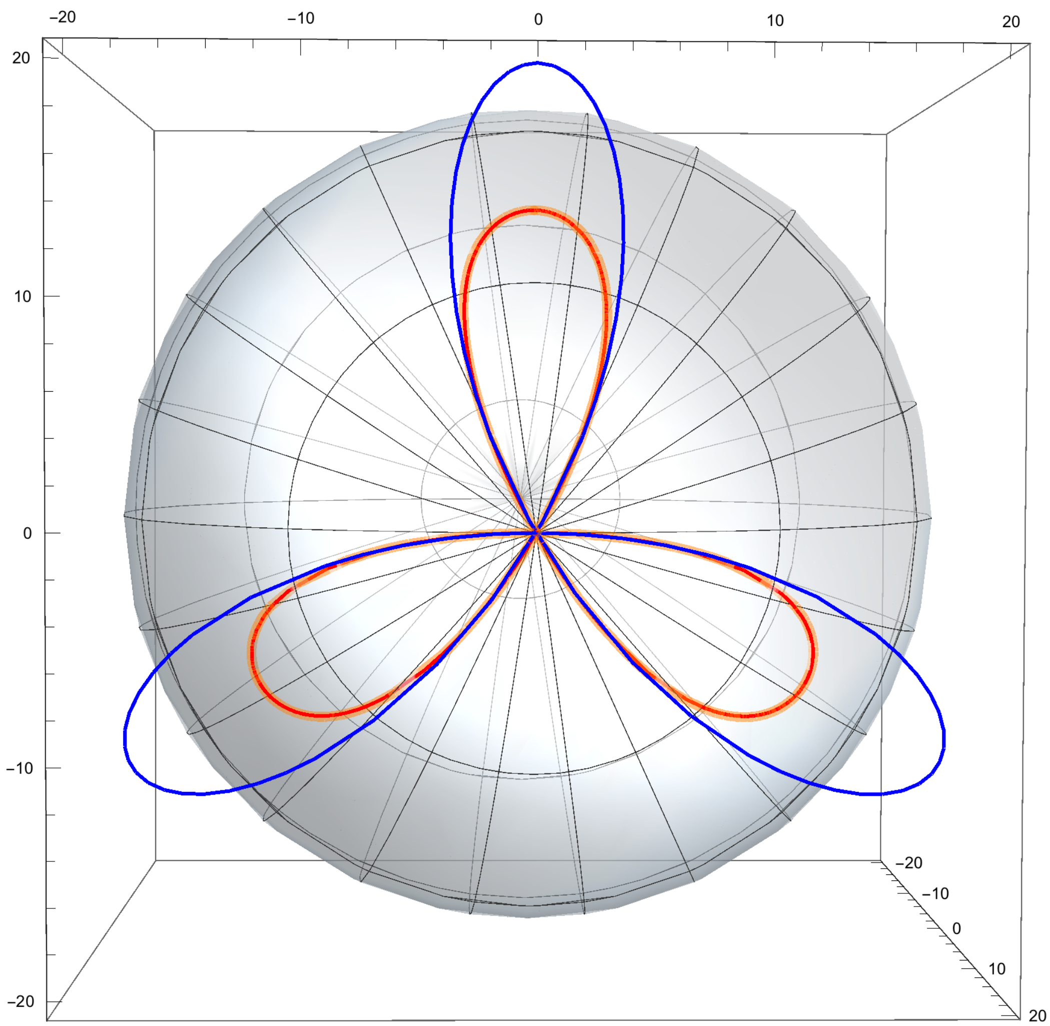

```
Show[TraceSL2q, ImageSize → Large]
```



```
Show[TriPlanar, Sph, TriProj, TraceSL1q,
 TraceSL2q, TracePoints, PlotRange → All, ImageSize → Large]
```



```
Show[TriPlanar, Sph, TriProj, TraceSL2q, PlotRange → All, ImageSize → Large]
```



H Spherical Butterfly Curve Mathematica Code

Below is the Mathematica code for the computation of mechanical system to draw a spherical butterfly curve.

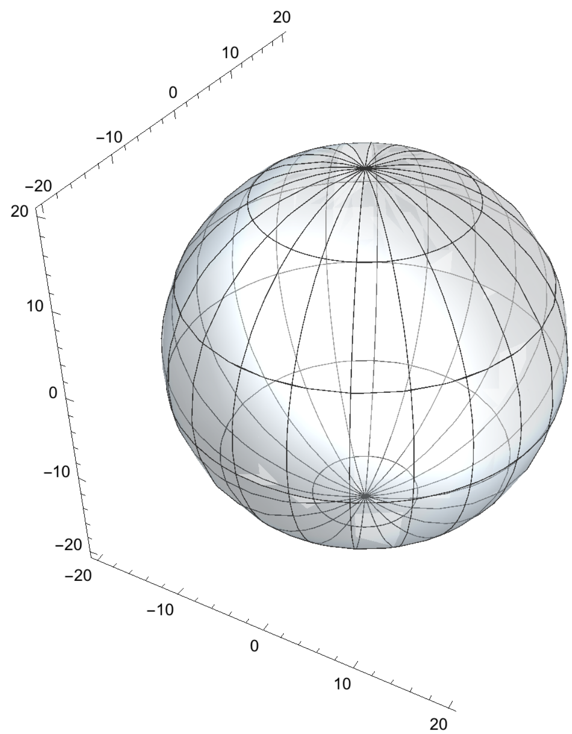
Spherical Serial chain for Butterfly

Specify the radius of the sphere

```
r = 20;
```

Drawing the sphere

```
Sph = SphericalPlot3D[{r}, {α, 0, 2 Pi},
  {β, 0, 2 Pi}, PlotStyle → {{FaceForm[None], EdgeForm[None]},
    {Opacity[0.3], FaceForm[LightBlue], EdgeForm[None]}},
  Mesh → 10, Boxed -> False, Lighting → "Neutral"]
```



Planar Butterfly

```
Bx = 7 Cos[θ] + 2 Cos[-θ] + 2 Cos[3 θ] + 1.25 Cos[Pi - 3 θ] + 1.25 Cos[Pi + 5 θ] +
  1.25 Cos[Pi / 2 - 4 θ] + 1.25 Cos[-Pi / 2 + 6 θ] + 1.15 Cos[Pi / 2 - 2 θ] +
  1.15 Cos[-Pi / 2 + 4 θ] + 1 Cos[-Pi / 2 - 6 θ] + 1 Cos[Pi / 2 + 8 θ]
```

```
9 Cos[θ] + 0.75 Cos[3 θ] - 1.25 Cos[5 θ] +
 1.15 Sin[2 θ] + 2.4 Sin[4 θ] + 0.25 Sin[6 θ] - Sin[8 θ]
```

```
By = 7 Sin[θ] + 2 Sin[-θ] + 2 Sin[3 θ] + 1.25 Sin[Pi - 3 θ] + 1.25 Sin[Pi + 5 θ] +
  1.25 Sin[Pi / 2 - 4 θ] + 1.25 Sin[-Pi / 2 + 6 θ] + 1.15 Sin[Pi / 2 - 2 θ] +
  1.15 Sin[-Pi / 2 + 4 θ] + 1 Sin[-Pi / 2 - 6 θ] + 1 Sin[Pi / 2 + 8 θ]

1.15 Cos[2 θ] + 0.1 Cos[4 θ] - 2.25 Cos[6 θ] +
 Cos[8 θ] + 5 Sin[θ] + 3.25 Sin[3 θ] - 1.25 Sin[5 θ]

Pt = {Bx, By, r}

{9 Cos[θ] + 0.75 Cos[3 θ] - 1.25 Cos[5 θ] + 1.15 Sin[2 θ] +
  2.4 Sin[4 θ] + 0.25 Sin[6 θ] - Sin[8 θ], 1.15 Cos[2 θ] + 0.1 Cos[4 θ] -
  2.25 Cos[6 θ] + Cos[8 θ] + 5 Sin[θ] + 3.25 Sin[3 θ] - 1.25 Sin[5 θ], 20}
```

******************New Section*****************

Define planar linkage end position function

```
Bxlist = {7 Cos[θ], 2 Cos[-θ] , 2 Cos[3 θ] , 1.25 Cos[Pi - 3 θ] , 1.25 Cos[Pi + 5 θ] ,
  1.25 Cos[Pi / 2 - 4 θ], 1.25 Cos[-Pi / 2 + 6 θ] , 1.15 Cos[Pi / 2 - 2 θ] ,
  1.15 Cos[-Pi / 2 + 4 θ] , 1 Cos[-Pi / 2 - 6 θ], 1 Cos[Pi / 2 + 8 θ]}

{7 Cos[θ], 2 Cos[θ], 2 Cos[3 θ], -1.25 Cos[3 θ], -1.25 Cos[5 θ], 1.25 Sin[4 θ],
 1.25 Sin[6 θ], 1.15 Sin[2 θ], 1.15 Sin[4 θ], -Sin[6 θ], -Sin[8 θ]}

Bylist = {7 Sin[θ], 2 Sin[-θ] , 2 Sin[3 θ] , 1.25 Sin[Pi - 3 θ] , 1.25 Sin[Pi + 5 θ] ,
  1.25 Sin[Pi / 2 - 4 θ], 1.25 Sin[-Pi / 2 + 6 θ] , 1.15 Sin[Pi / 2 - 2 θ] ,
  1.15 Sin[-Pi / 2 + 4 θ] , 1 Sin[-Pi / 2 - 6 θ], 1 Sin[Pi / 2 + 8 θ]}

{7 Sin[θ], -2 Sin[θ], 2 Sin[3 θ], 1.25 Sin[3 θ], -1.25 Sin[5 θ], 1.25 Cos[4 θ],
 -1.25 Cos[6 θ], 1.15 Cos[2 θ], -1.15 Cos[4 θ], -Cos[6 θ], Cos[8 θ]}

PLEx[i_] := Fold[Plus, Take[Bxlist, i]] /. {θ → 0}

PLEy[i_] := Fold[Plus, Take[Bylist, i]] /. {θ → 0}
```

PLE refers to Planar Linkage End

test

```
PLEx[1]

7

PLEx[2]

9

PLEx[3]

11

PLEy[1]

0

PLEy[6]

1.25
```

Linkage end-effector point

```
StartPoint = Chop[Pt /. {θ → 0}]
```

```
{8.5, 0, 20}
```

```
IniAngle = N[VectorAngle[{0, 0, r}, StartPoint]]
```

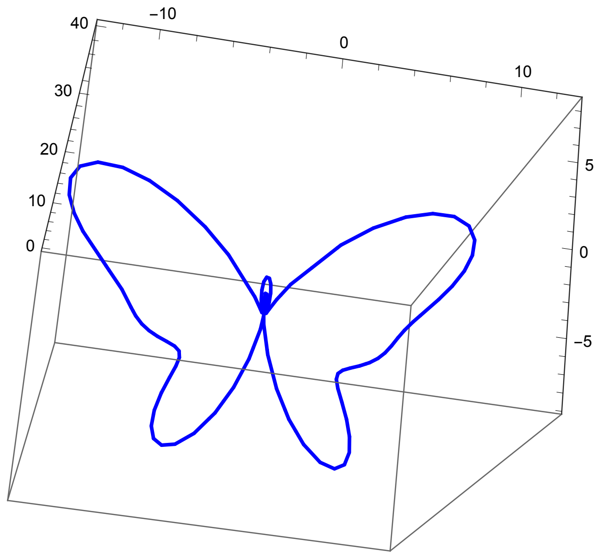
```
0.401871
```

Consider the initial position, 7θ+2θ+2θ-1.25θ-1.25θ=Iniangle, thus we can calculate the first spherical link angle as

```
S1angle = IniAngle * 7 / 8.5
```

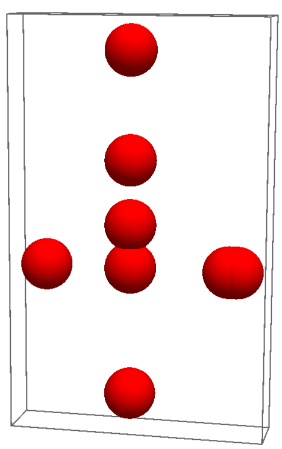
```
0.330952
```

```
BPlanar =
 ParametricPlot3D[{Pt[[1]], Pt[[2]], Pt[[3]]}, {θ, 0, 2 Pi}, PlotStyle → {Blue}]
```

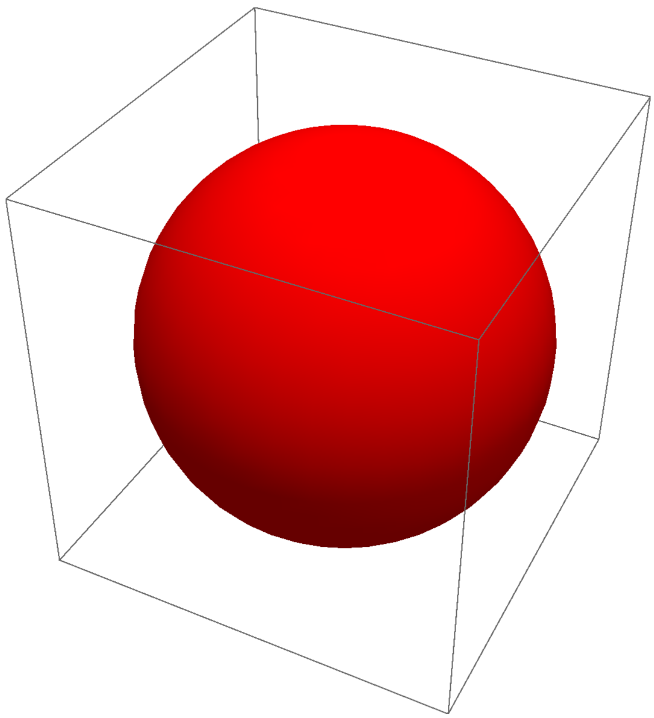



```
ButterflySteps = Table [{ Pt[[1]], Pt[[2]], Pt[[3]]}, {θ, 0, 2 π, 2 π / 100}];
```

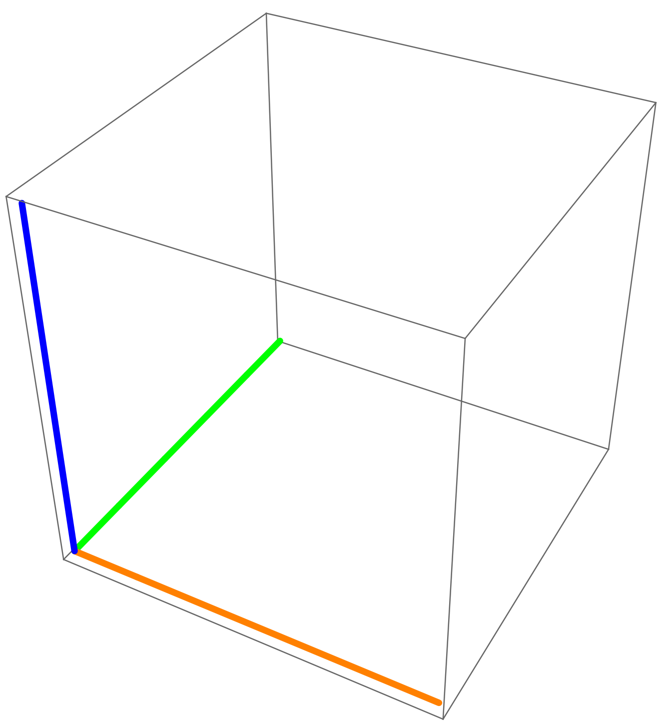
```
PlnarLinkage =
 Graphics3D[{Red, Table[Sphere[{PLEx[i], PLEy[i], r}, 0.3], {i, Length[Bxlist]}]}]
```



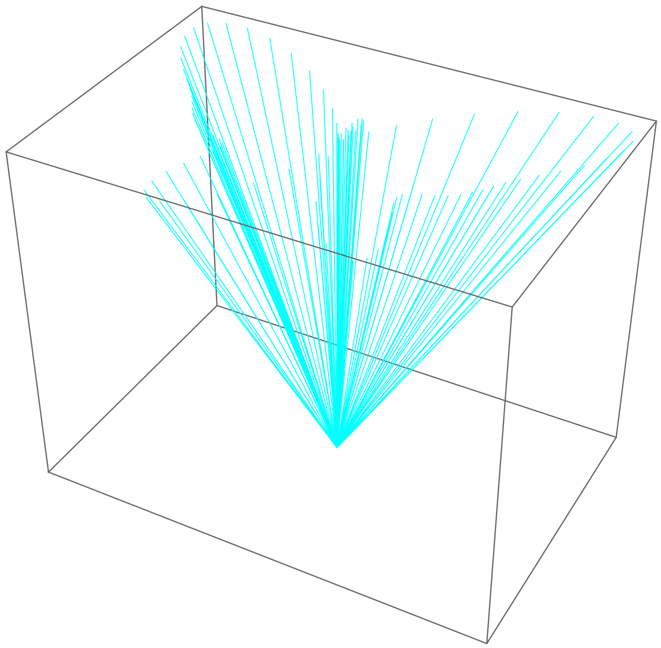
```
SP = Graphics3D[{Red, Sphere[StartPoint, 0.3]}]
```



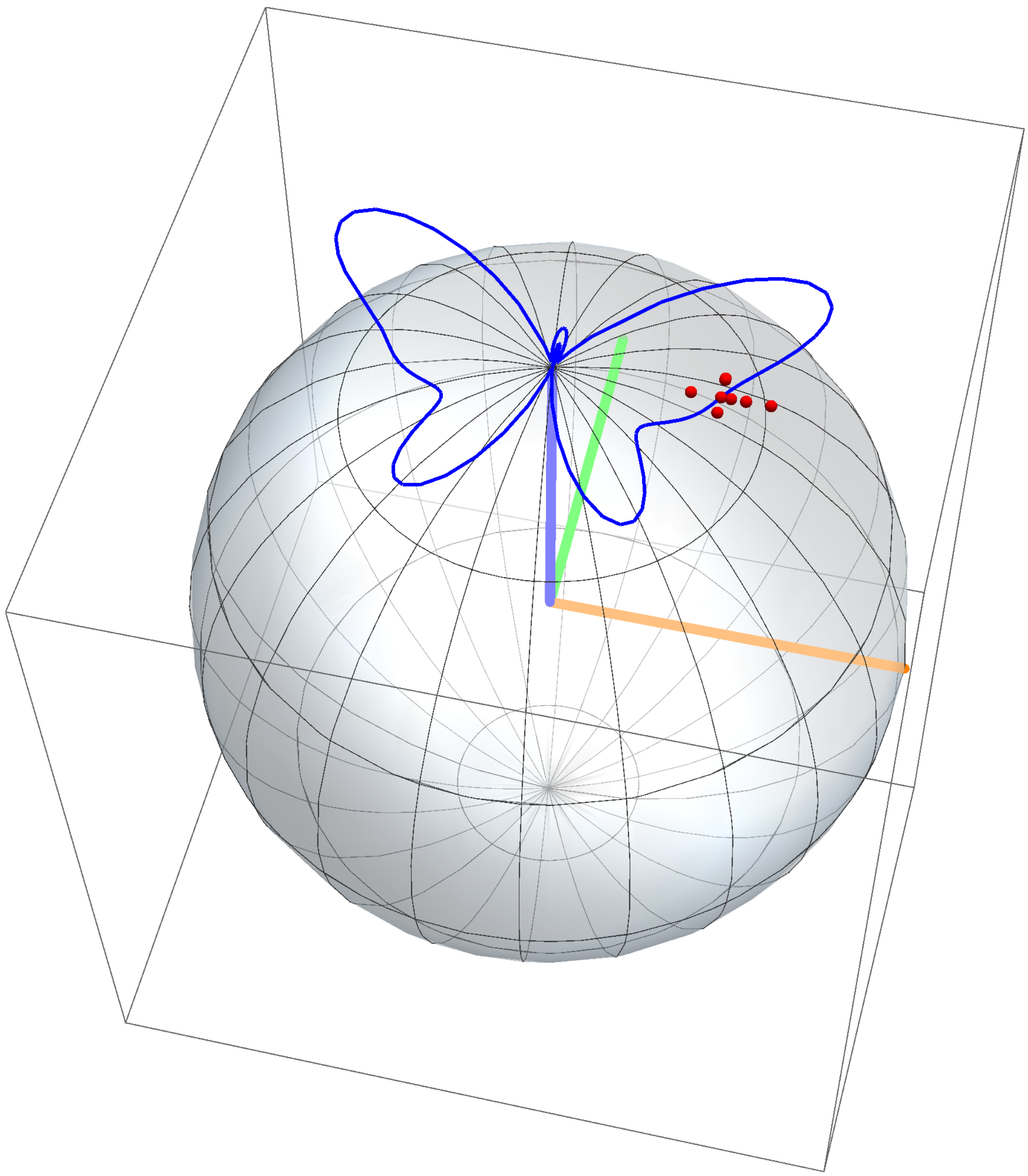
```
xyz = Graphics3D[{Orange, Thickness[0.01], Line[{{0, 0, 0}, {r, 0, 0}}],
   Green, Line[{{0, 0, 0}, {0, r, 0}}], Blue, Line[{{0, 0, 0}, {0, 0, r}}]}]
```



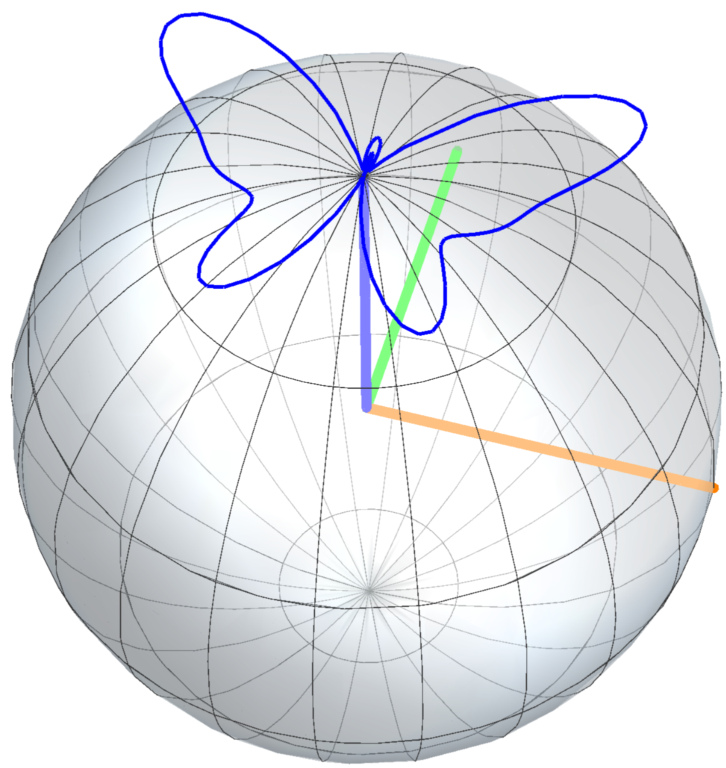
```
ProjectionLine = Graphics3D[{Cyan, Table[
    Line[{{0, 0, 0}, ButterflySteps[[i]]}], {i, 1, Length[ButterflySteps]}]}]
```



```
Show[xyz, BPlanar, Sph, SP, PlnarLinkage, PlotRange → All, ImageSize → Large]
```



```
Show[xyz, BPlanar, Sph, PlotRange → All, ImageSize → Large, Boxed → False]
```



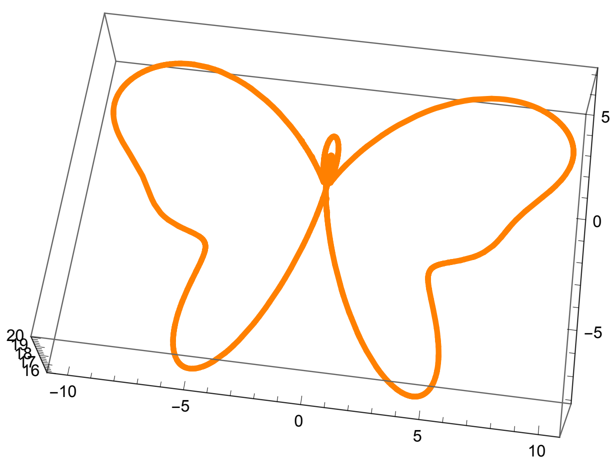
```
PtProj = r * Normalize[Pt]
```

```
{(20 (9 Cos[θ] + 0.75 Cos[3 θ] - 1.25 Cos[5 θ] +
        1.15 Sin[2 θ] + 2.4 Sin[4 θ] + 0.25 Sin[6 θ] - Sin[8 θ])) /
   (√(400 + Abs[1.15 Cos[2 θ] + 0.1 Cos[4 θ] - 2.25 Cos[6 θ] + Cos[8 θ] + 5 Sin[θ] +
            3.25 Sin[3 θ] - 1.25 Sin[5 θ]]^2 + Abs[9 Cos[θ] + 0.75 Cos[3 θ] -
            1.25 Cos[5 θ] + 1.15 Sin[2 θ] + 2.4 Sin[4 θ] + 0.25 Sin[6 θ] - Sin[8 θ]]^2)),
 (20 (1.15 Cos[2 θ] + 0.1 Cos[4 θ] - 2.25 Cos[6 θ] + Cos[8 θ] + 5 Sin[θ] +
        3.25 Sin[3 θ] - 1.25 Sin[5 θ])) /
   (√(400 + Abs[1.15 Cos[2 θ] + 0.1 Cos[4 θ] - 2.25 Cos[6 θ] + Cos[8 θ] + 5 Sin[θ] +
            3.25 Sin[3 θ] - 1.25 Sin[5 θ]]^2 + Abs[9 Cos[θ] + 0.75 Cos[3 θ] -
            1.25 Cos[5 θ] + 1.15 Sin[2 θ] + 2.4 Sin[4 θ] + 0.25 Sin[6 θ] - Sin[8 θ]]^2)),
 400 / (√(400 + Abs[1.15 Cos[2 θ] + 0.1 Cos[4 θ] - 2.25 Cos[6 θ] + Cos[8 θ] +
            5 Sin[θ] + 3.25 Sin[3 θ] - 1.25 Sin[5 θ]]^2 + Abs[9 Cos[θ] + 0.75 Cos[3 θ] -
            1.25 Cos[5 θ] + 1.15 Sin[2 θ] + 2.4 Sin[4 θ] + 0.25 Sin[6 θ] - Sin[8 θ]]^2))}
```

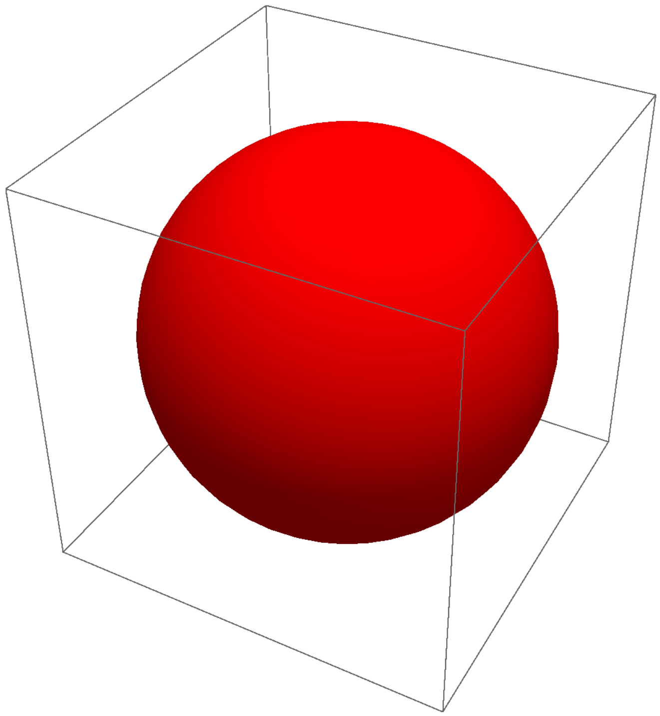
```
SphStartPoint = Chop[PtProj /. {θ → 0}]
```

```
{7.82281, 0, 18.4066}
```

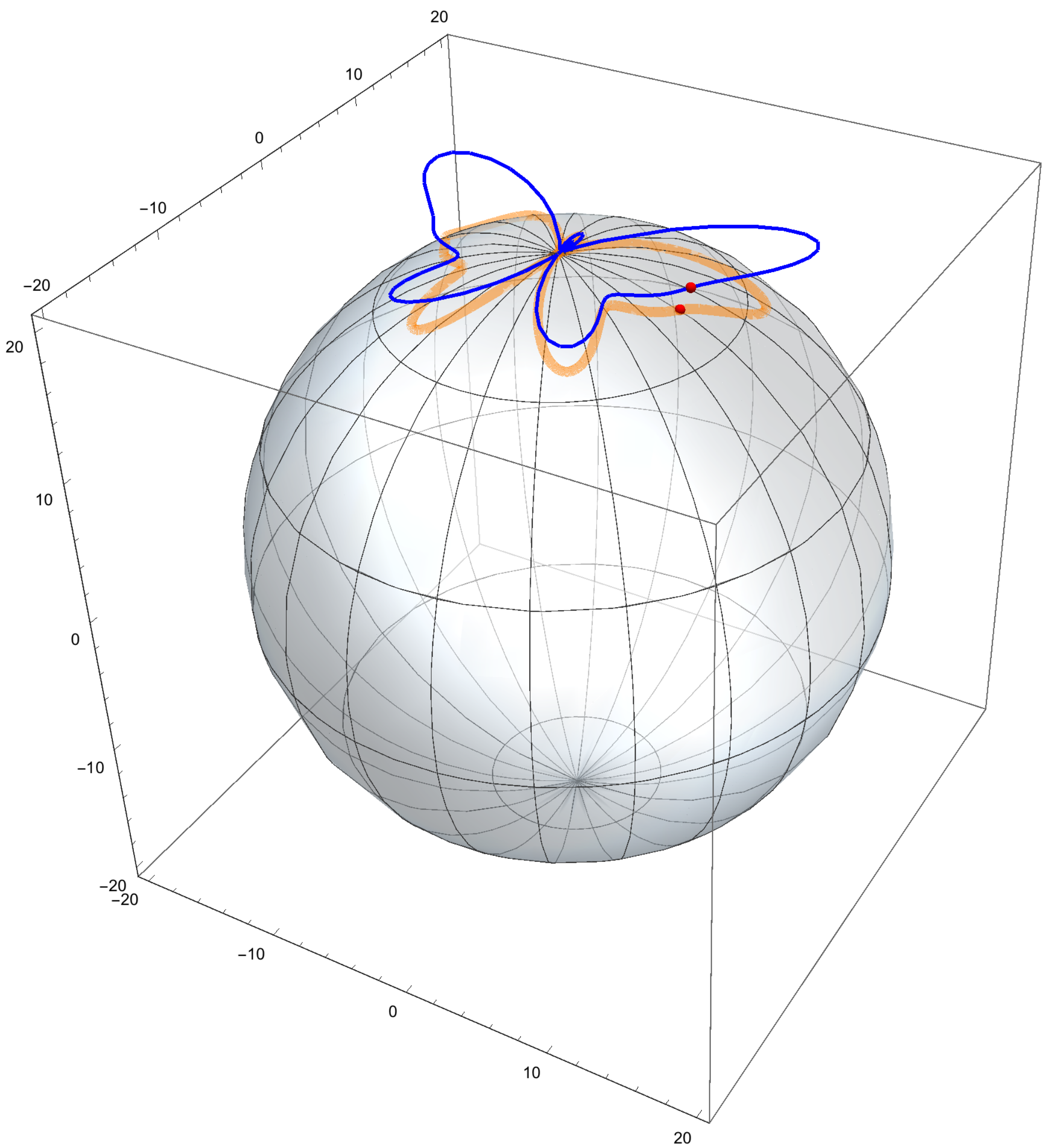
```
BProj = ParametricPlot3D[{PtProj[[1]], PtProj[[2]], PtProj[[3]]},
  {θ, 0, 2 Pi}, PlotStyle → {Orange, Thickness[0.01], Opacity[0.5]}]
```



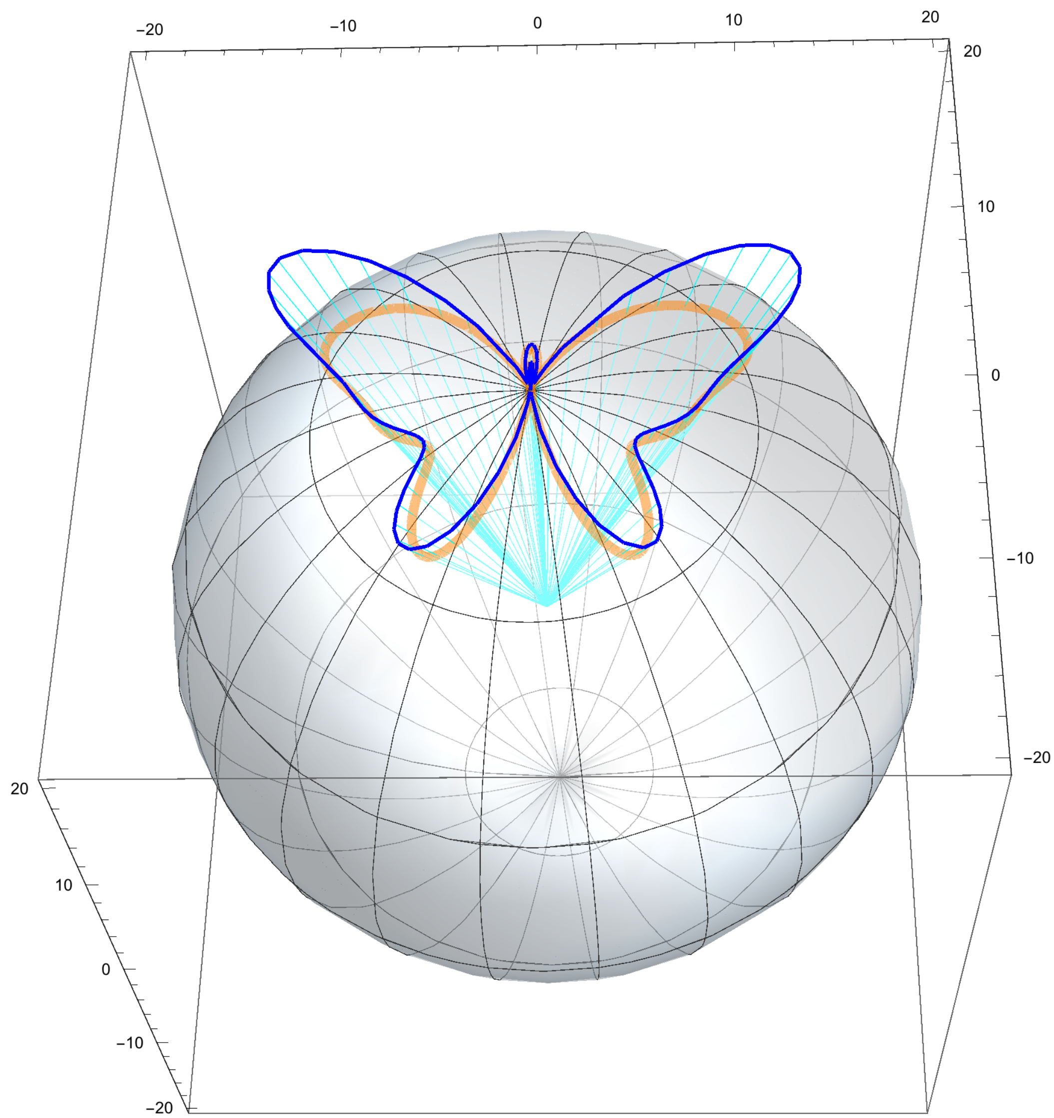
```
SphSP = Graphics3D[{Red, Sphere[SphStartPoint, 0.3]}]
```



```
Show[BPlanar, Sph, SP, SphSP, BProj, PlotRange → All, ImageSize → Large]
```



```
Show[BPlanar, ProjectionLine, Sph, BProj, PlotRange → All, ImageSize → Large]
```



Define Quaternion

Define the start point p0 on the sphere

```
(*p0={1,0,0,0};*)

(*MatrixForm[p0]*)
```

Define Skew axis

```
s[sx_, sy_, sz_] := {sx, sy, sy}
```

Define the Quaternion [Q]

```
Q = {sx * Sin[ϕ / 2], sy * Sin[ϕ / 2], sz * Sin[ϕ / 2], Cos[ϕ / 2]};
MatrixForm[Q]
```

$$\begin{pmatrix} sx\,Sin\left[\frac{\phi}{2}\right] \\ sy\,Sin\left[\frac{\phi}{2}\right] \\ sz\,Sin\left[\frac{\phi}{2}\right] \\ Cos\left[\frac{\phi}{2}\right] \end{pmatrix}$$

Define Quaternion conjugate [Qc]

```
Qc = {-sx * Sin[ϕ / 2], -sy * Sin[ϕ / 2], -sz * Sin[ϕ / 2], Cos[ϕ / 2]};
MatrixForm[Qc]
```

$$\begin{pmatrix} -sx\,Sin\left[\frac{\phi}{2}\right] \\ -sy\,Sin\left[\frac{\phi}{2}\right] \\ -sz\,Sin\left[\frac{\phi}{2}\right] \\ Cos\left[\frac{\phi}{2}\right] \end{pmatrix}$$

Define [Q+] (Qplus)

```
Qplus = {{Q[[4]], -Q[[3]], Q[[2]], Q[[1]]}, {Q[[3]], Q[[4]], -Q[[1]], Q[[2]]},
    {-Q[[2]], Q[[1]], Q[[4]], Q[[3]]}, {-Q[[1]], -Q[[2]], -Q[[3]], Q[[4]]}};
MatrixForm[
 Qplus]
```

$$\begin{pmatrix} Cos\left[\frac{\phi}{2}\right] & -sz\,Sin\left[\frac{\phi}{2}\right] & sy\,Sin\left[\frac{\phi}{2}\right] & sx\,Sin\left[\frac{\phi}{2}\right] \\ sz\,Sin\left[\frac{\phi}{2}\right] & Cos\left[\frac{\phi}{2}\right] & -sx\,Sin\left[\frac{\phi}{2}\right] & sy\,Sin\left[\frac{\phi}{2}\right] \\ -sy\,Sin\left[\frac{\phi}{2}\right] & sx\,Sin\left[\frac{\phi}{2}\right] & Cos\left[\frac{\phi}{2}\right] & sz\,Sin\left[\frac{\phi}{2}\right] \\ -sx\,Sin\left[\frac{\phi}{2}\right] & -sy\,Sin\left[\frac{\phi}{2}\right] & -sz\,Sin\left[\frac{\phi}{2}\right] & Cos\left[\frac{\phi}{2}\right] \end{pmatrix}$$

Define [Qc-] (QcMinu)

```
QcMinu = {{Qc[[4]], Qc[[3]], -Qc[[2]], Qc[[1]]},
    {-Qc[[3]], Qc[[4]], Qc[[1]], Qc[[2]]}, {Qc[[2]], -Qc[[1]], Qc[[4]], Qc[[3]]},
    {-Qc[[1]], -Qc[[2]], -Qc[[3]], Qc[[4]]}};
MatrixForm[
 QcMinu]
```

$$\begin{pmatrix} Cos\left[\frac{\phi}{2}\right] & -sz\,Sin\left[\frac{\phi}{2}\right] & sy\,Sin\left[\frac{\phi}{2}\right] & -sx\,Sin\left[\frac{\phi}{2}\right] \\ sz\,Sin\left[\frac{\phi}{2}\right] & Cos\left[\frac{\phi}{2}\right] & -sx\,Sin\left[\frac{\phi}{2}\right] & -sy\,Sin\left[\frac{\phi}{2}\right] \\ -sy\,Sin\left[\frac{\phi}{2}\right] & sx\,Sin\left[\frac{\phi}{2}\right] & Cos\left[\frac{\phi}{2}\right] & -sz\,Sin\left[\frac{\phi}{2}\right] \\ sx\,Sin\left[\frac{\phi}{2}\right] & sy\,Sin\left[\frac{\phi}{2}\right] & sz\,Sin\left[\frac{\phi}{2}\right] & Cos\left[\frac{\phi}{2}\right] \end{pmatrix}$$

Define [Q+][Qc-]

```
Qt = MatrixForm[FullSimplify[Qplus.QcMinu, (sx)^2 + (sy)^2 + (sz)^2 == 1]]
```

$$\begin{pmatrix} 1 + sy^2 (-1 + \text{Cos}[\phi]) + sz^2 (-1 + \text{Cos}[\phi]) & sx\,sy - sx\,sy\,\text{Cos}[\phi] - sz\,\text{Sin}[\phi] & sx\,sz - sx\,sz\,\text{Cos} \\ sx\,sy - sx\,sy\,\text{Cos}[\phi] + sz\,\text{Sin}[\phi] & sy^2 (1 - \text{Cos}[\phi]) + \text{Cos}[\phi] & sy\,sz - sy\,sz\,\text{Cos} \\ sx\,sz - sx\,sz\,\text{Cos}[\phi] - sy\,\text{Sin}[\phi] & sy\,sz - sy\,sz\,\text{Cos}[\phi] + sx\,\text{Sin}[\phi] & sz^2 (1 - \text{Cos}[\phi \\ 0 & 0 & 0 \end{pmatrix}$$

```
Qtn[ϕ_, sx_, sy_, sz_] :=
 {{1 + sy^2 (-1 + Cos[ϕ]) + sz^2 (-1 + Cos[ϕ]),
   sx sy - sx sy Cos[ϕ] - sz Sin[ϕ], sx sz - sx sz Cos[ϕ] + sy Sin[ϕ], 0},
  {sx sy - sx sy Cos[ϕ] + sz Sin[ϕ], sy^2 (1 - Cos[ϕ]) + Cos[ϕ],
   sy sz - sy sz Cos[ϕ] - sx Sin[ϕ], 0},
  {sx sz - sx sz Cos[ϕ] - sy Sin[ϕ], sy sz - sy sz Cos[ϕ] + sx Sin[ϕ],
   sz^2 (1 - Cos[ϕ]) + Cos[ϕ], 0},
  {0,
   0,
   0,
   1}}
```

Verification

```
MatrixForm[Qtn[0, 0, 0, 0]]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```
(*Qtn[Pi/2,0,0,1].p0*)
```

```
Qtn[2 Pi / 3, 1 / Sqrt[3], 1 / Sqrt[3], 1 / Sqrt[3]].{1, 0, 0, 0}
```

```
{0, 1, 0, 0}
```

Generate the Spherical Linkage

The link length and phase angle of the planar Butterfly linkage

Number of links

```
n = 11;
```

```
(*L0=0
 L1=7
 L2=2
 L3=2
 L4=1.25
 L5=1.25
 L6=1.25
 L7=1.25
 L8=1.15
 L9=1.15
 L10=1
 L11=1*)
```

```
L[i_] := Take[{0, 7, 2, 2, 1.25, 1.25, 1.25, 1.25, 1.15, 1.15, 1, 1}, i + 1][[i + 1]]
```

Test

```
L[0]
```

0

```
L[1]
```

7

```
L[11]
```

1

```
(*θo0=0
 θo1=0
 θo2=0
 θo3=0
 θo4=Pi
 θo5=Pi
 θo6=Pi/2
 θo7=-Pi/2
 θo8=Pi/2
 θo9=-Pi/2
 θo10=-Pi/2
 θo11=Pi/2*)
```

```
θo[i_] :=
 Take[{0, 0, 0, 0, Pi, Pi, Pi / 2, -Pi / 2, Pi / 2, -Pi / 2, -Pi / 2, Pi / 2}, i + 1][[i + 1]]
```

Test

```
θo[0]
```

0

```
θo[7]
```

$-\frac{\pi}{2}$

Get the coordinates on the plane in 3D space

```
QS[i_] := {L[i] Cos[θo[i]], L[i] Sin[θo[i]], r}
```

Test

```
QS[0]
{0, 0, 20}

QS[1]
{7, 0, 20}

QS[5]
{-1.25, 0., 20}
```

***********New Section********** Calculate the scaler

```
Angle1 = N[VectorAngle[{0, 0, r}, QS[1]]]
0.336675

(*Scaler=S1angle/Angle1*)

Scaler = 1
1
```

Get the coordinates on the Sphere in 3D space

```
(*IniP[i_]:=r*Normalize[{L[i] Cos[θo[i]],L[i] Sin[θo[i]],r}]*)
```

test

```
(*IniP[0]*)

(*IniP[1]*)

(*IniP[2]*)
```

Get the angles between P[i] (i>0) with P[0]

```
(*angle[i_]:=N[VectorAngle[IniP[0],IniP[i]]]*)

(*axis[i_]:=Normalize[Cross[IniP[0],IniP[i]]]*)

(*Rvaxis[i_]:=Normalize[Cross[IniP[i],IniP[0]]]*)
```

Test

```
(*angle[1]*)

(*axis[0]*)

(*axis[1]*)
```

Convert Pi to Quaternion

```
(*QP[i_]:=Join[IniP[i],{0}]*)
```

Test

```
(*QP[0]*)
```

```
(*QP[1]*)
```

```
(*Norm[QP[1]]/r*)
```

*******New Section***********

```
(*P[i_]:=Take[Qtn[N[angle[i]]-Scaler*angle[i],Normalize[Rvaxis[i]][[1]],
    Normalize[Rvaxis[i]][[2]],Normalize[Rvaxis[i]][[3]]].QP[i],3]*)
```

```
(*P[0]*)
```

```
(*P[1]*)
```

Test

```
(*VectorAngle[P[0],IniP[1]]*)
```

```
(*VectorAngle[P[0],P[1]]*)
```

```
(*VectorAngle[P[0],P[1]]/VectorAngle[P[0],IniP[1]]*)
```

*****

```
(*VectorAngle[P[0],IniP[2]]*)
```

```
(*VectorAngle[P[0],P[2]]*)
```

```
(*VectorAngle[P[0],P[2]]/VectorAngle[P[0],IniP[2]]*)
```

******

```
(*VectorAngle[P[0],IniP[4]]*)
```

```
(*VectorAngle[P[0],P[4]]*)
```

```
(*VectorAngle[P[0],P[4]]/VectorAngle[P[0],IniP[4]]*)
```

*****

```
(*VectorAngle[P[0],IniP[6]]*)
```

```
(*VectorAngle[P[0],P[6]]*)
```

```
(*VectorAngle[P[0],P[6]]/VectorAngle[P[0],IniP[6]]*)
```

```
(*VectorAngle[P[0],IniP[9]]*)
```

*****

```
(*VectorAngle[P[0],P[9]]*)
```

```
(*VectorAngle[P[0],P[9]]/VectorAngle[P[0],IniP[9]]*)
```

Define the function

```
(*sPr[i_]:=
 Take[Qtn[angle[i-1],axis[i-1][[1]],axis[i-1][[2]],axis[i-1][[3]]].QP[i],3]*)
```

Test

```
(*sPr[1]*)
```

```
(*sPr[2]*)
```

```
(*sPr[3]=Take[Qtn[N[VectorAngle[P[0],sPr[2]]],
    Normalize[Cross[P[0],sPr[2]]][[1]],Normalize[Cross[P[0],sPr[2]]][[2]],
    Normalize[Cross[P[0],sPr[2]]][[3]]].QP[3],3]*)
```

```
(*sPr[4]=Qtn[N[VectorAngle[P[0],sPr[3]]],
   Normalize[Cross[P[0],sPr[3]]][[1]],Normalize[Cross[P[0],sPr[3]]][[2]],
   Normalize[Cross[P[0],sPr[3]]][[3]]].QP[4]*)
```

New Definition

```
(*QPlist=Table[QP[i],{i,0,n}]*)
```

```
(*Flatten[Take[QPlist,1]]*)
```

Use irritation to define Pr[i]

```
(*Pr[0]=P[0]*)
```

```
(*Pr[1]=P[1]*)
```

```
(*Pr[2]=Take[Qtn[N[VectorAngle[P[0],Pr[1]]],
    Normalize[Cross[P[0],Pr[1]]][[1]],Normalize[Cross[P[0],Pr[1]]][[2]],
    Normalize[Cross[P[0],Pr[1]]][[3]]].QP[2],3]*)
```

```
(*Table[Pr[n]=Take[Qtn[N[VectorAngle[P[0],Pr[n-1]]],
     Normalize[Cross[P[0],Pr[n-1]]][[1]],Normalize[Cross[P[0],Pr[n-1]]][[2]],
     Normalize[Cross[P[0],P[1]]][[3]]].QP[n],3],{n,3,11}]*)
```

```
(*Pr[3]*)
```

```
(*Pr[4]*)
```

```
(*Pr[i_]:=Nest[Take[Qtn[N[VectorAngle[P[0],#]],
      Normalize[Cross[P[0],#]][[1]],Normalize[Cross[P[0],#]][[2]],
      Normalize[Cross[P[0],#]][[3]]].QP[Norm[#]/r+1],3]&,P[1],i-1]*)
```

```
(*QPr[i_]:=Join[Pr[i],{0}]*)
```

```
(*QPr[1]*)
```

Define the skews

```
(*Skew[i_]:=Normalize[Pr[i]]*)
```

Test

```
(*Skew[0]*)

(*Skew[1]*)

(*Skew[2]*)

(*Skew[3]*)

(*Skew[4]*)

(*Skew[5]*)

(*Skew[6]*)

(*Skew[7]*)

(*Skew[8]*)

(*Skew[9]*)

(*Skew[10]*)

(*Skew[11]*)
```

**

****************Redefine Skews******************

```
P[0] = {0, 0, r, 0}
{0, 0, 20, 0}

arc1 = 0.33;

P[1] = Qtn[arc1, 0, 1, 0].P[0]
{6.48086, 0., 18.9208, 0.}

P[2] = Qtn[arc1 * (7 + 2) / 7, 0, 1, 0].P[0]
{8.2334, 0., 18.2267, 0.}

P[3] = Qtn[arc1 * (7 + 2 + 2) / 7, 0, 1, 0].P[0]
{9.9128, 0., 17.3706, 0.}

P[4] = Qtn[arc1 * (7 + 2 + 2 - 1.25) / 7, 0, 1, 0].P[0]
{8.87256, 0., 17.9242, 0.}

P[5] = Qtn[arc1 * (7 + 2 + 2 - 1.25 - 1.25) / 7, 0, 1, 0].P[0]
{7.80152, 0., 18.4157, 0.}
```

```
Raxis = Normalize[
  Cross[Take[Qtn[arc1 * (7 + 2 + 2 - 1.25 - 1.25) / 7, 0, 1, 0].P[0], 3], {0, r, 0}]]
{-0.920783, 0., 0.390076}

testP[6] = Qtn[arc1 * (1.25) / 7, Raxis[[1]], Raxis[[2]], Raxis[[3]]].P[5]
{7.78798, 1.17789, 18.3837, 0.}

P[6] = Qtn[arc1 * (1.25) / 7,
   Normalize[Cross[Take[Qtn[arc1 * (7 + 2 + 2 - 1.25 - 1.25) / 7, 0, 1, 0].P[0], 3],
      {0, r, 0}]][[1]], Normalize[Cross[
      Take[Qtn[arc1 * (7 + 2 + 2 - 1.25 - 1.25) / 7, 0, 1, 0].P[0], 3], {0, r, 0}]][[2]],
   Normalize[Cross[Take[Qtn[arc1 * (7 + 2 + 2 - 1.25 - 1.25) / 7, 0, 1, 0].P[0], 3],
      {0, r, 0}]][[3]]].Qtn[arc1 * (7 + 2 + 2 - 1.25 - 1.25) / 7, 0, 1, 0].P[0]
{7.78798, 1.17789, 18.3837, 0.}

P[7] = P[5]
{7.80152, 0., 18.4157, 0.}

P[8] = Qtn[arc1 * (1.15) / 7, Raxis[[1]], Raxis[[2]], Raxis[[3]]].P[5]
{7.79006, 1.08375, 18.3886, 0.}

P[9] = P[5]
{7.80152, 0., 18.4157, 0.}

P[10] = Qtn[arc1 * (-1) / 7, Raxis[[1]], Raxis[[2]], Raxis[[3]]].P[5]
{7.79286, -0.942508, 18.3952, 0.}

P[11] = P[5]
{7.80152, 0., 18.4157, 0.}
```

************************New Skews****************************

```
Skew[0] = Take[Normalize[P[0]], 3]
{0, 0, 1}

Skew[1] = Take[Normalize[P[1]], 3]
{0.324043, 0., 0.946042}

Skew[2] = Take[Normalize[P[2]], 3]
{0.41167, 0., 0.911333}

Skew[3] = Take[Normalize[P[3]], 3]
{0.49564, 0., 0.868528}

Skew[4] = Take[Normalize[P[4]], 3]
{0.443628, 0., 0.896211}

Skew[5] = Take[Normalize[P[5]], 3]
{0.390076, 0., 0.920783}
```

```
Skew[6] = Take[Normalize[P[6]], 3]
{0.389399, 0.0588945, 0.919184}

Skew[7] = Take[Normalize[P[7]], 3]
{0.390076, 0., 0.920783}

Skew[8] = Take[Normalize[P[8]], 3]
{0.389503, 0.0541877, 0.91943}

Skew[9] = Take[Normalize[P[9]], 3]
{0.390076, 0., 0.920783}

Skew[10] = Take[Normalize[P[10]], 3]
{0.389643, -0.0471254, 0.91976}

Skew[11] = Take[Normalize[P[11]], 3]
{0.390076, 0., 0.920783}
```

Specify the link Speed

```
θ[i_] := Take[{0, 1, 1, 3, 3, 5, 4, 6, 2, 4, 6, 8}, i + 1][[i + 1]]
```

Test

```
θ[0]
0

θ[11]
8
```

Define the Ratio of the speed

```
v[i_] :=  (-1)^(i + 1) (θ[i] + θ[i - 1])
```

Test

```
v[1]
1

v[2]
-2

v[3]
4

v[4]
-6
```

```
v[5]
8
v[6]
-9
v[7]
10
v[8]
-8
v[9]
6
v[10]
-10
v[11]
14
```

Generating the curves

```
Curve1 = Qtn[v[1] θ, Skew[0][[1]], Skew[0][[2]], Skew[0][[3]]].P[1]
{0. + 6.48086 Cos[θ], 0. + 6.48086 Sin[θ], 18.9208, 0.}
```

Define the relation of θ and ϕ

```
Curve2 = TrigReduce[Qtn[θ, Skew[0][[1]], Skew[0][[2]], Skew[0][[3]]].
    Qtn[v[2] θ, Skew[1][[1]], Skew[1][[2]], Skew[1][[3]]].P[2]]
{8.2842 Cos[θ] - 0.050799 Cos[3 θ],
 4.61995 Sin[θ] - 0.050799 Sin[3 θ], 18.8368 - 0.610148 Cos[2 θ], 0.}
```

Generate the list of Qtn

```
list =
 Table[Qtn[v[i] θ, Skew[i - 1][[1]], Skew[i - 1][[2]], Skew[i - 1][[3]]], {i, 1, n}]
{{{Cos[θ], -Sin[θ], 0, 0}, {Sin[θ], Cos[θ], 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}},
 {{1. + 0.894996 (-1 + Cos[2 θ]), 0. + 0.946042 Sin[2 θ], 0.306558 - 0.306558 Cos[2 θ],
   0}, {0. - 0.946042 Sin[2 θ], 0. + Cos[2 θ], 0. + 0.324043 Sin[2 θ], 0},
  {0.306558 - 0.306558 Cos[2 θ], 0. - 0.324043 Sin[2 θ],
   0.894996 (1 - Cos[2 θ]) + Cos[2 θ], 0}, {0, 0, 0, 1}},
 {{1. + 0.830528 (-1 + Cos[4 θ]), 0. - 0.911333 Sin[4 θ], 0.375168 - 0.375168 Cos[4 θ],
   0}, {0. + 0.911333 Sin[4 θ], 0. + Cos[4 θ], 0. - 0.41167 Sin[4 θ], 0},
  {0.375168 - 0.375168 Cos[4 θ], 0. + 0.41167 Sin[4 θ],
   0.830528 (1 - Cos[4 θ]) + Cos[4 θ], 0}, {0, 0, 0, 1}},
 {{1. + 0.754341 (-1 + Cos[6 θ]), 0. + 0.868528 Sin[6 θ], 0.430477 - 0.430477 Cos[6 θ],
   0}, {0. - 0.868528 Sin[6 θ], 0. + Cos[6 θ], 0. + 0.49564 Sin[6 θ], 0},
  {0.430477 - 0.430477 Cos[6 θ], 0. - 0.49564 Sin[6 θ],
   0.754341 (1 - Cos[6 θ]) + Cos[6 θ], 0}, {0, 0, 0, 1}},
 {{1. + 0.803194 (-1 + Cos[8 θ]), 0. - 0.896211 Sin[8 θ], 0.397584 - 0.397584 Cos[8 θ],
   0}, {0. + 0.896211 Sin[8 θ], 0. + Cos[8 θ], 0. - 0.443628 Sin[8 θ], 0},
  {0.397584 - 0.397584 Cos[8 θ], 0. + 0.443628 Sin[8 θ],
   0.803194 (1 - Cos[8 θ]) + Cos[8 θ], 0}, {0, 0, 0, 1}},
 {{1. + 0.847841 (-1 + Cos[9 θ]), 0. + 0.920783 Sin[9 θ], 0.359175 - 0.359175 Cos[9 θ],
   0}, {0. - 0.920783 Sin[9 θ], 0. + Cos[9 θ], 0. + 0.390076 Sin[9 θ], 0},
  {0.359175 - 0.359175 Cos[9 θ], 0. - 0.390076 Sin[9 θ],
   0.847841 (1 - Cos[9 θ]) + Cos[9 θ], 0}, {0, 0, 0, 1}},
 {{1 + 0.848368 (-1 + Cos[10 θ]), 0.0229335 - 0.0229335 Cos[10 θ] - 0.919184 Sin[10 θ],
   0.35793 - 0.35793 Cos[10 θ] + 0.0588945 Sin[10 θ], 0},
  {0.0229335 - 0.0229335 Cos[10 θ] + 0.919184 Sin[10 θ], 0.00346856 (1 - Cos[10 θ]) +
    Cos[10 θ], 0.0541349 - 0.0541349 Cos[10 θ] - 0.389399 Sin[10 θ], 0},
  {0.35793 - 0.35793 Cos[10 θ] - 0.0588945 Sin[10 θ], 0.0541349 - 0.0541349 Cos[10 θ] +
    0.389399 Sin[10 θ], 0.8449 (1 - Cos[10 θ]) + Cos[10 θ], 0}, {0, 0, 0, 1}},
 {{1. + 0.847841 (-1 + Cos[8 θ]), 0. + 0.920783 Sin[8 θ], 0.359175 - 0.359175 Cos[8 θ],
   0}, {0. - 0.920783 Sin[8 θ], 0. + Cos[8 θ], 0. + 0.390076 Sin[8 θ], 0},
  {0.359175 - 0.359175 Cos[8 θ], 0. - 0.390076 Sin[8 θ],
   0.847841 (1 - Cos[8 θ]) + Cos[8 θ], 0}, {0, 0, 0, 1}},
 {{1 + 0.848287 (-1 + Cos[6 θ]), 0.0211063 - 0.0211063 Cos[6 θ] - 0.91943 Sin[6 θ],
   0.358121 - 0.358121 Cos[6 θ] + 0.0541877 Sin[6 θ], 0},
  {0.0211063 - 0.0211063 Cos[6 θ] + 0.91943 Sin[6 θ], 0.00293631 (1 - Cos[6 θ]) +
    Cos[6 θ], 0.0498218 - 0.0498218 Cos[6 θ] - 0.389503 Sin[6 θ], 0},
  {0.358121 - 0.358121 Cos[6 θ] - 0.0541877 Sin[6 θ], 0.0498218 - 0.0498218 Cos[6 θ] +
    0.389503 Sin[6 θ], 0.845351 (1 - Cos[6 θ]) + Cos[6 θ], 0}, {0, 0, 0, 1}},
 {{1. + 0.847841 (-1 + Cos[10 θ]), 0. + 0.920783 Sin[10 θ],
   0.359175 - 0.359175 Cos[10 θ], 0},
  {0. - 0.920783 Sin[10 θ], 0. + Cos[10 θ], 0. + 0.390076 Sin[10 θ], 0},
  {0.359175 - 0.359175 Cos[10 θ], 0. - 0.390076 Sin[10 θ],
   0.847841 (1 - Cos[10 θ]) + Cos[10 θ], 0}, {0, 0, 0, 1}},
 {{1 + 0.848179 (-1 + Cos[14 θ]), -0.0183621 + 0.0183621 Cos[14 θ] - 0.91976 Sin[14 θ],
   0.358378 - 0.358378 Cos[14 θ] - 0.0471254 Sin[14 θ], 0},
  {-0.0183621 + 0.0183621 Cos[14 θ] + 0.91976 Sin[14 θ], 0.0022208 (1 - Cos[14 θ]) +
    Cos[14 θ], -0.043344 + 0.043344 Cos[14 θ] - 0.389643 Sin[14 θ], 0},
  {0.358378 - 0.358378 Cos[14 θ] + 0.0471254 Sin[14 θ],
   -0.043344 + 0.043344 Cos[14 θ] + 0.389643 Sin[14 θ],
   0.845958 (1 - Cos[14 θ]) + Cos[14 θ], 0}, {0, 0, 0, 1}}}
```

Test

```
list[[1]]
{{Cos[θ], -Sin[θ], 0, 0}, {Sin[θ], Cos[θ], 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
```

Define Curve[j]

```
Curve[i_] := Fold[Dot, Take[list, i]].P[i]

Curve[1]

{0. + 6.48086 Cos[θ], 0. + 6.48086 Sin[θ], 18.9208, 0.}

Curve[2]

{0. + 8.2334 (Cos[θ] (1. + 0.894996 (-1 + Cos[2 θ])) - Sin[θ] (0. - 0.946042 Sin[2 θ])) +
   18.2267 (Cos[θ] (0.306558 - 0.306558 Cos[2 θ]) - Sin[θ] (0. + 0.324043 Sin[2 θ])),
 0. + 8.2334 ((1. + 0.894996 (-1 + Cos[2 θ])) Sin[θ] + Cos[θ] (0. - 0.946042 Sin[2 θ])) +
   18.2267 ((0.306558 - 0.306558 Cos[2 θ]) Sin[θ] + Cos[θ] (0. + 0.324043 Sin[2 θ])),
 0. + 8.2334 (0.306558 - 0.306558 Cos[2 θ]) +
   18.2267 (0.894996 (1 - Cos[2 θ]) + Cos[2 θ]), 0.}

Curve[3]

{0. + 17.3706 ((0.375168 - 0.375168 Cos[4 θ])
        (Cos[θ] (1. + 0.894996 (-1 + Cos[2 θ])) - Sin[θ] (0. - 0.946042 Sin[2 θ])) +
       (0.830528 (1 - Cos[4 θ]) + Cos[4 θ])
        (Cos[θ] (0.306558 - 0.306558 Cos[2 θ]) - Sin[θ] (0. + 0.324043 Sin[2 θ])) +
       (-(0. + Cos[2 θ]) Sin[θ] + Cos[θ] (0. + 0.946042 Sin[2 θ]))
        (0. - 0.41167 Sin[4 θ])) +
   9.9128 ((1. + 0.830528 (-1 + Cos[4 θ])) (Cos[θ] (1. + 0.894996 (-1 + Cos[2 θ])) -
          Sin[θ] (0. - 0.946042 Sin[2 θ])) + (0.375168 - 0.375168 Cos[4 θ])
        (Cos[θ] (0.306558 - 0.306558 Cos[2 θ]) - Sin[θ] (0. + 0.324043 Sin[2 θ])) +
       (-(0. + Cos[2 θ]) Sin[θ] + Cos[θ] (0. + 0.946042 Sin[2 θ]))
        (0. + 0.911333 Sin[4 θ])),
 0. + 17.3706 ((0.375168 - 0.375168 Cos[4 θ]) ((1. + 0.894996 (-1 + Cos[2 θ])) Sin[θ] +
          Cos[θ] (0. - 0.946042 Sin[2 θ])) + (0.830528 (1 - Cos[4 θ]) + Cos[4 θ])
        ((0.306558 - 0.306558 Cos[2 θ]) Sin[θ] + Cos[θ] (0. + 0.324043 Sin[2 θ])) +
       (Cos[θ] (0. + Cos[2 θ]) + Sin[θ] (0. + 0.946042 Sin[2 θ]))
        (0. - 0.41167 Sin[4 θ])) +
   9.9128 ((1. + 0.830528 (-1 + Cos[4 θ])) ((1. + 0.894996 (-1 + Cos[2 θ])) Sin[θ] +
          Cos[θ] (0. - 0.946042 Sin[2 θ])) + (0.375168 - 0.375168 Cos[4 θ])
        ((0.306558 - 0.306558 Cos[2 θ]) Sin[θ] + Cos[θ] (0. + 0.324043 Sin[2 θ])) +
       (Cos[θ] (0. + Cos[2 θ]) + Sin[θ] (0. + 0.946042 Sin[2 θ]))
        (0. + 0.911333 Sin[4 θ])),
 0. + 17.3706 ((0.306558 - 0.306558 Cos[2 θ]) (0.375168 - 0.375168 Cos[4 θ]) +
       (0.894996 (1 - Cos[2 θ]) + Cos[2 θ]) (0.830528 (1 - Cos[4 θ]) + Cos[4 θ]) +
       (0. - 0.324043 Sin[2 θ]) (0. - 0.41167 Sin[4 θ])) +
   9.9128 ((0.306558 - 0.306558 Cos[2 θ]) (1. + 0.830528 (-1 + Cos[4 θ])) +
       (0.894996 (1 - Cos[2 θ]) + Cos[2 θ]) (0.375168 - 0.375168 Cos[4 θ]) +
       (0. - 0.324043 Sin[2 θ]) (0. + 0.911333 Sin[4 θ])), 0.}

TrigReduce[Curve[3]]

{8.19672 Cos[θ] + 1.74876 Cos[3 θ] - 0.0327903 Cos[5 θ] + 0.000112815 Cos[7 θ],
 4.65012 Sin[θ] + 1.8062 Sin[3 θ] - 0.0246527 Sin[5 θ] + 0.000112815 Sin[7 θ],
 18.7531 - 1.21623 Cos[2 θ] - 0.167705 Cos[4 θ] + 0.00135502 Cos[6 θ], 0.}
```

```
Chop[TrigReduce[Curve[11]]]
```

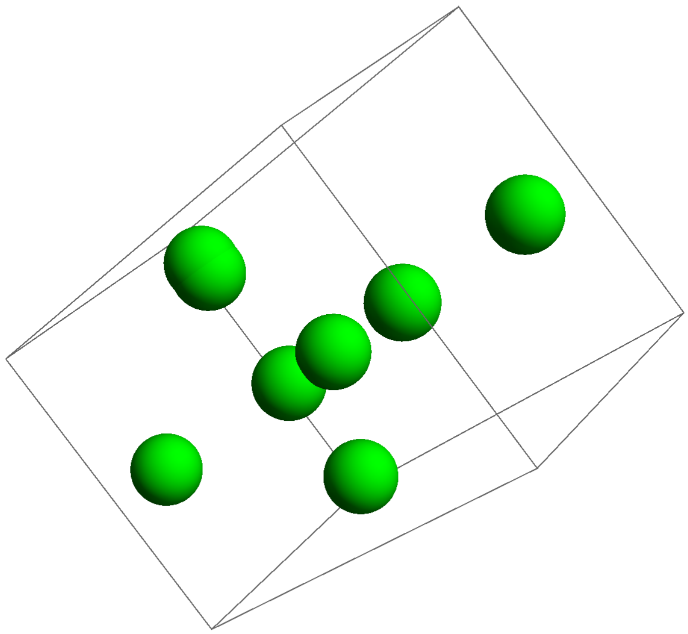
```
{8.13688 Cos[θ] + 0.685661 Cos[3 θ] - 1.05839 Cos[5 θ] + 0.0718358 Cos[7 θ] +
  0.00268695 Cos[9 θ] - 0.0301305 Cos[11 θ] - 0.0176531 Cos[13 θ] +
  0.00817962 Cos[15 θ] + 0.00370497 Cos[17 θ] - 0.00134114 Cos[19 θ] +
  0.0000649934 Cos[21 θ] + 0.0000214065 Cos[23 θ] + 5.91767 × 10^-6 Cos[25 θ] -
  8.07344 × 10^-6 Cos[27 θ] - 1.75266 × 10^-6 Cos[29 θ] + 8.78683 × 10^-7 Cos[31 θ] -
  5.2034 × 10^-8 Cos[33 θ] - 1.1505 × 10^-8 Cos[35 θ] - 4.47566 × 10^-9 Cos[37 θ] +
  8.33593 × 10^-10 Cos[39 θ] + 6.9641 × 10^-10 Cos[41 θ] + 0.916184 Sin[2 θ] +
  2.05574 Sin[4 θ] + 0.201361 Sin[6 θ] - 0.84026 Sin[8 θ] + 0.0427638 Sin[10 θ] -
  0.0173188 Sin[12 θ] - 0.00493218 Sin[14 θ] - 0.0031796 Sin[16 θ] +
  0.00159398 Sin[18 θ] + 0.000639815 Sin[20 θ] - 8.77074 × 10^-6 Sin[22 θ] +
  0.0000322871 Sin[24 θ] + 2.56154 × 10^-6 Sin[26 θ] - 7.54771 × 10^-7 Sin[28 θ] -
  1.15492 × 10^-6 Sin[30 θ] - 3.85509 × 10^-7 Sin[32 θ] + 8.53685 × 10^-9 Sin[34 θ] -
  1.76081 × 10^-8 Sin[36 θ] - 3.14417 × 10^-9 Sin[38 θ] + 3.91838 × 10^-10 Sin[40 θ],
 -0.0336882 + 1.00301 Cos[2 θ] + 0.133986 Cos[4 θ] - 1.98661 Cos[6 θ] +
  0.932747 Cos[8 θ] - 0.0585539 Cos[10 θ] + 0.00298683 Cos[12 θ] +
  0.00629618 Cos[14 θ] + 0.000733702 Cos[16 θ] + 0.000806828 Cos[18 θ] -
  0.00175034 Cos[20 θ] + 0.0000573663 Cos[22 θ] - 0.0000219992 Cos[24 θ] -
  9.10853 × 10^-6 Cos[26 θ] + 3.25644 × 10^-6 Cos[28 θ] - 5.35043 × 10^-7 Cos[30 θ] +
  4.13274 × 10^-7 Cos[32 θ] - 2.76369 × 10^-8 Cos[34 θ] + 8.74276 × 10^-9 Cos[36 θ] +
  5.68036 × 10^-9 Cos[38 θ] - 1.22759 × 10^-9 Cos[40 θ] + 4.56031 Sin[θ] + 2.89915 Sin[3 θ] -
  1.15718 Sin[5 θ] + 0.0352672 Sin[7 θ] + 0.0235306 Sin[9 θ] + 0.00934776 Sin[11 θ] -
  0.0238775 Sin[13 θ] - 0.00197889 Sin[15 θ] + 0.00642771 Sin[17 θ] -
  0.00140151 Sin[19 θ] + 0.0000269857 Sin[21 θ] + 6.64991 × 10^-6 Sin[23 θ] +
  0.0000145 Sin[25 θ] - 2.72411 × 10^-6 Sin[27 θ] - 3.58317 × 10^-6 Sin[29 θ] +
  9.48259 × 10^-7 Sin[31 θ] - 3.09139 × 10^-8 Sin[33 θ] - 6.32873 × 10^-9 Sin[35 θ] -
  3.19463 × 10^-9 Sin[37 θ] - 5.83817 × 10^-10 Sin[39 θ] + 7.30696 × 10^-10 Sin[41 θ],
 18.5258 - 1.01101 Cos[2 θ] + 0.679487 Cos[4 θ] + 0.258074 Cos[6 θ] +
  0.0504979 Cos[8 θ] - 0.0270017 Cos[10 θ] - 0.101696 Cos[12 θ] + 0.0442221 Cos[14 θ] -
  0.00317007 Cos[16 θ] + 0.0000587 Cos[18 θ] + 0.000318024 Cos[20 θ] +
  5.99047 × 10^-6 Cos[22 θ] + 0.0000439916 Cos[24 θ] - 0.000028876 Cos[26 θ] +
  2.05574 × 10^-6 Cos[28 θ] - 2.72961 × 10^-7 Cos[30 θ] - 2.49511 × 10^-7 Cos[32 θ] +
  1.12961 × 10^-8 Cos[34 θ] - 2.49724 × 10^-8 Cos[36 θ] - 2.99208 × 10^-10 Cos[38 θ] -
  5.97907 × 10^-10 Cos[40 θ] - 0.176687 Sin[θ] - 1.00609 Sin[3 θ] -
  0.77841 Sin[5 θ] + 0.510695 Sin[7 θ] + 0.297369 Sin[9 θ] - 0.110119 Sin[11 θ] +
  0.00416625 Sin[13 θ] + 0.00151557 Sin[15 θ] + 0.000161198 Sin[17 θ] -
  0.000941697 Sin[19 θ] - 0.000109549 Sin[21 θ] + 0.000195797 Sin[23 θ] -
  0.0000358711 Sin[25 θ] + 3.39198 × 10^-7 Sin[27 θ] + 3.20875 × 10^-7 Sin[29 θ] +
  4.17621 × 10^-7 Sin[31 θ] + 1.17493 × 10^-7 Sin[33 θ] - 7.59475 × 10^-8 Sin[35 θ] +
  3.07876 × 10^-9 Sin[37 θ] - 1.73674 × 10^-10 Sin[39 θ] - 1.50819 × 10^-10 Sin[41 θ], 0}
```

```
(*Curve[11]*)
```

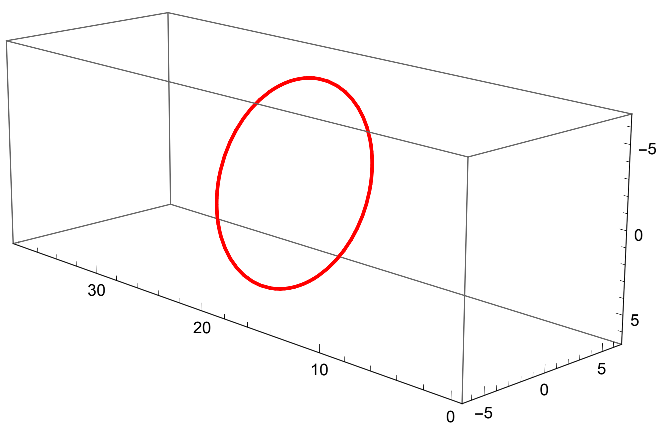
Alt+ . is used to kill one step large calculation.

- Plot the curve generated by the spherical serial chain

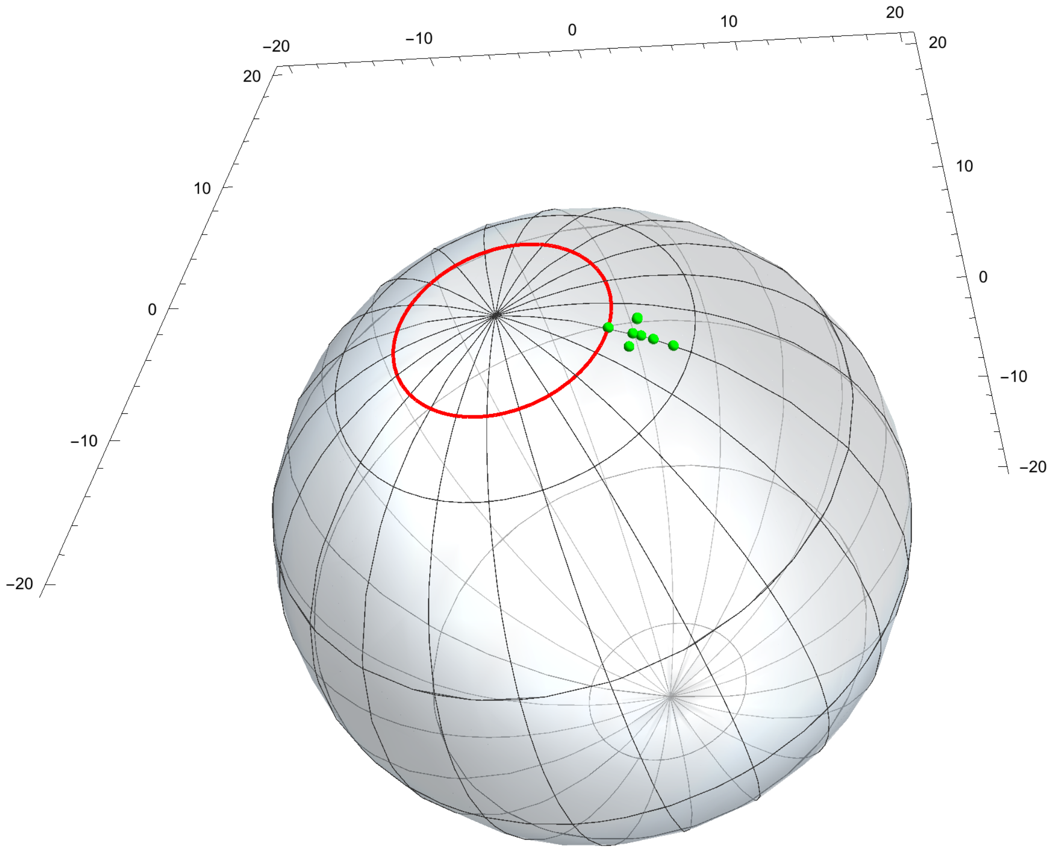
```
TracePoints = Graphics3D[{Green, Table[Sphere[Take[P[i], 3], 0.3], {i, 1, 11}]}]
```



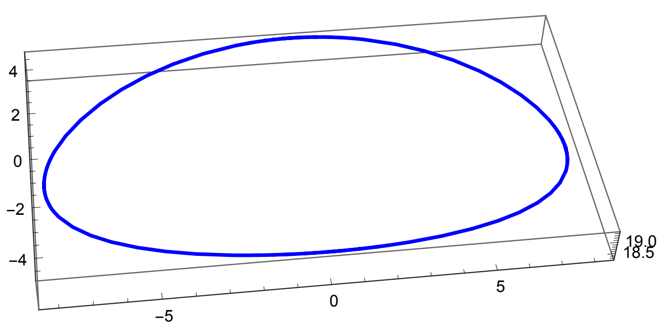
```
TraceP1 = ParametricPlot3D[{Curve1[[1]], Curve1[[2]], Curve1[[3]]},
  {θ, 0, 2 Pi}, PlotStyle → {Red}, ImageSize → Medium]
```



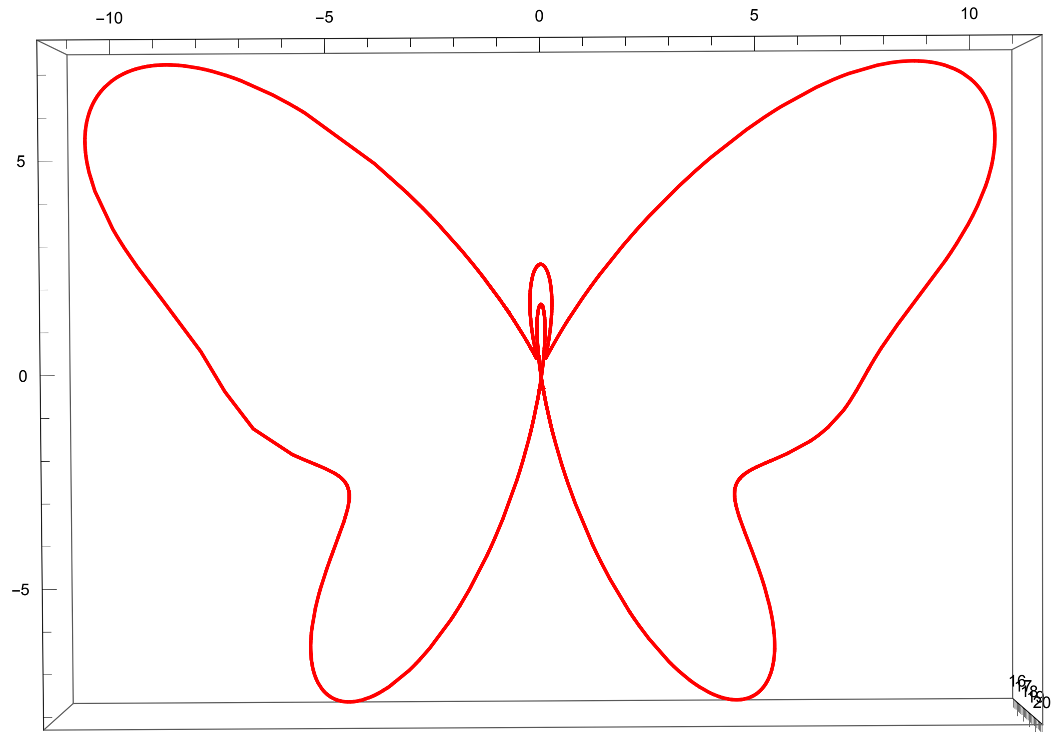
```
Show[Sph, TracePoints, TraceP1, PlotRange → All, ImageSize → Large]
```



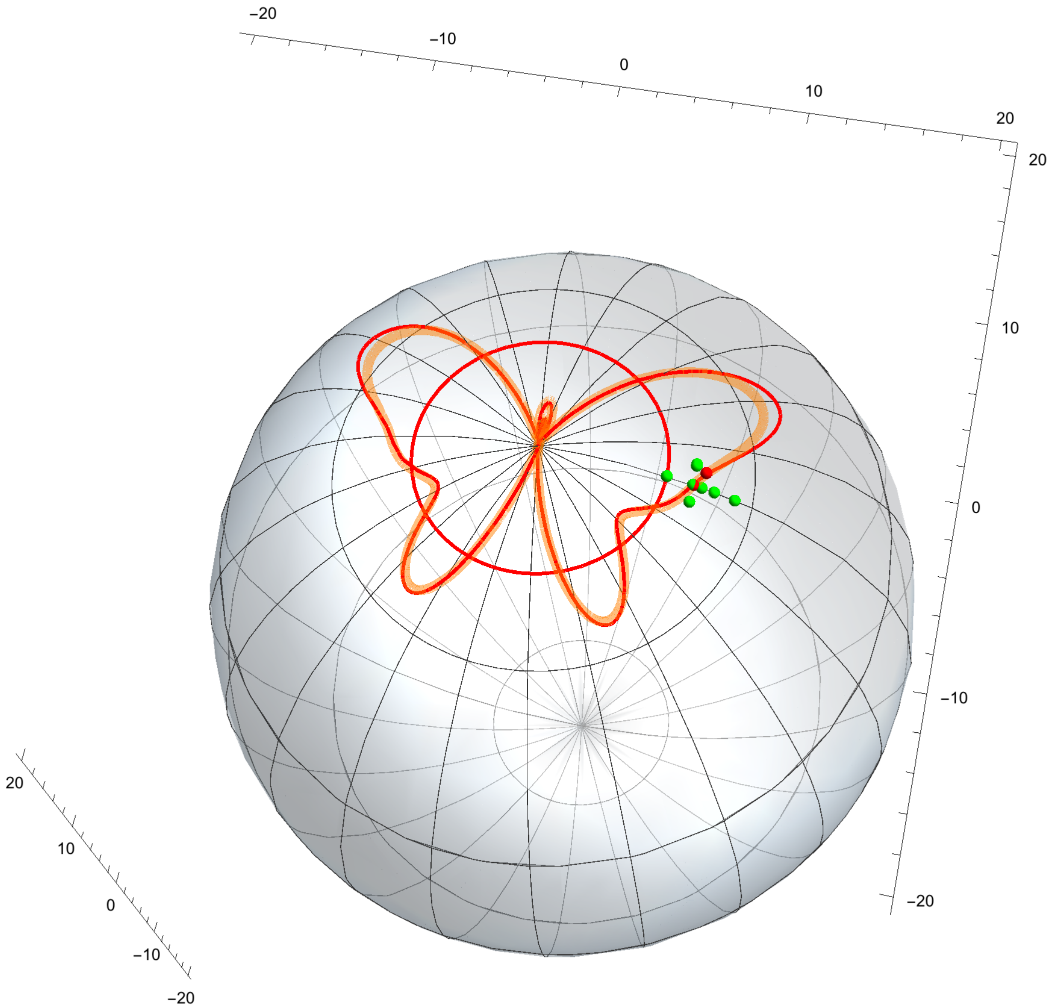

```
TraceP2 = ParametricPlot3D[{Curve2[[1]], Curve2[[2]], Curve2[[3]]},
   {θ, 0, 2 Pi}, PlotStyle → {Blue}, ImageSize → Medium]
```



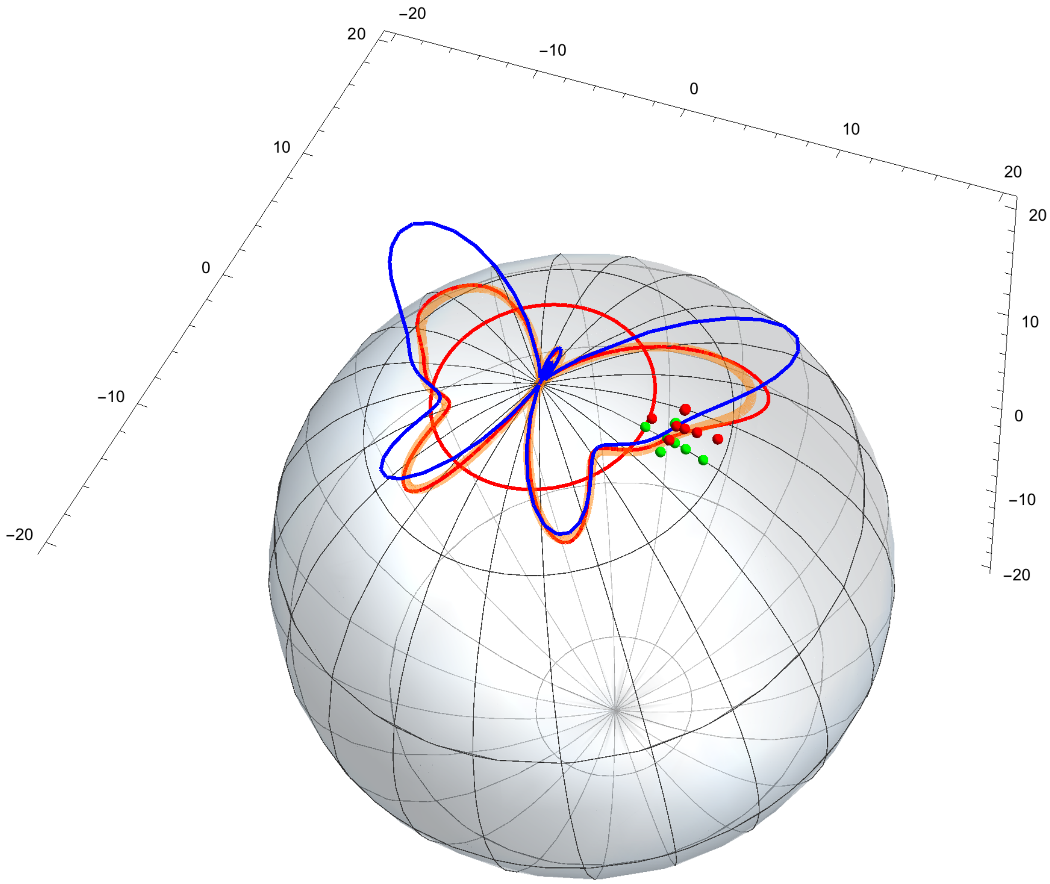
```
TraceP11 = ParametricPlot3D[{Curve[11][[1]], Curve[11][[2]], Curve[11][[3]]},
  {θ, 0, 2 Pi}, PlotStyle → {Red}, ImageSize → Large]
```



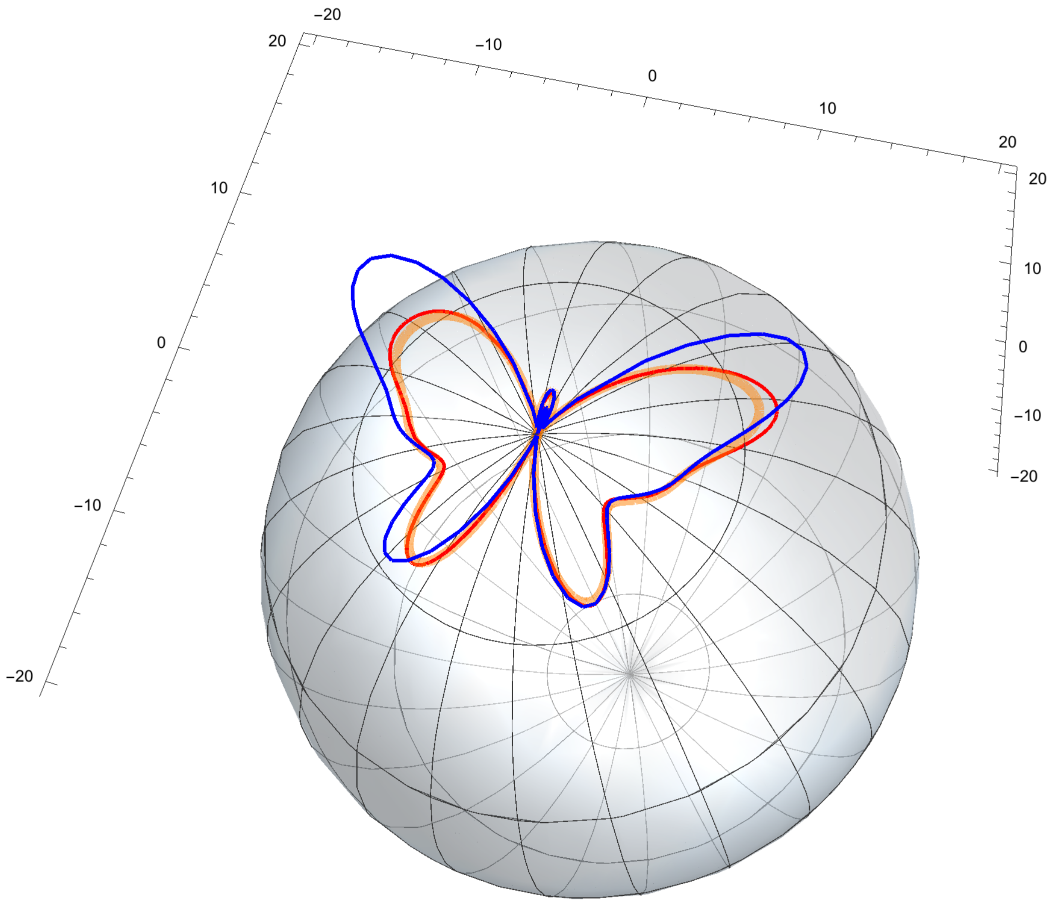
```
Show[Sph, BProj, SP, SphSP, TracePoints,
 TraceP1, TraceP11, PlotRange → All, ImageSize → Large]
```



```
Show[Sph, BPlanar, BProj, PlnarLinkage, SP, SphSP,
 TracePoints, TraceP1, TraceP11, PlotRange → All, ImageSize → Large]
```



```
Show[Sph, BPlanar, BProj, TraceP11, PlotRange → All, ImageSize → Large]
```



```
Show[Sph, BProj, TraceP11, PlotRange → All, ImageSize → Large]
```