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Arc lengths of rational Pythagorean–hodograph curves

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Abstract

In a recent paper (*Comput. Aided Geom. Design* **31** (2014) 689–700) a family of rational Pythagorean–hodograph (PH) curves is introduced, characterized by constraints on the coefficients of a truncated Laurent series, and used to solve the first–order Hermite interpolation problem. Contrary to a claim made in this paper, it is shown that these rational PH curves have rational arc length functions only in degenerate cases, where the center of the Laurent series is a real value.

Keywords: Pythagorean–hodograph curve; parametric speed; arc length; truncated Laurent series; poles; residues; rational function integration.

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A Pythagorean–hodograph (PH) curve[1] $\mathbf{r}(t) = (x(t), y(t))$ has the distinctive property that its derivative $\mathbf{r}'(t) = (x'(t), y'(t))$ satisfies

$$x'^2(t) + y'^2(t) \;=\; \sigma^2(t)\,, \tag{1}$$

with $\sigma(t)$ lying in the same space of functions as $x(t), y(t)$. Thus, $\mathbf{r}(t)$ is said to be a *polynomial* or *rational* PH curve when (1) is satisfied by polynomial or rational functions, respectively. Both have rational unit tangents.

[1] For brevity, only planar PH curves are discussed here: the results also hold for spatial PH curves. A comprehensive treatment of PH curves may be found in [4].

Since the function $\sigma(t)$ represents the *parametric speed* of the curve $\mathbf{r}(t)$, i.e., the derivative $\mathrm{d}s/\mathrm{d}t$ of arc length s with respect to the parameter t, the cumulative arc length function $s(t)$ is simply the indefinite integral of $\sigma(t)$. Consequently, polynomial PH curves necessarily have polynomial arc length functions, but in general rational PH curves do *not* have rational arc length functions, since the integral of a rational function may involve transcendental (logarithmic or arctangent) terms. Since "simple" (polynomial/rational) arc length functions are important for real–time motion control [5, 6, 7, 12] the polynomial PH curves possess a clear advantage in this context.

The intent of this short communication is to emphasize the fundamental difference between the arc lengths of polynomial/rational PH curves — which is often glossed over, or totally misrepresented. For example, a recent paper [10] begins with "The Pythagorean–hodograph (PH) curves ... are a special class of polynomial/rational curves with polynomial/rational speed functions. They have polynomial/rational arc lengths ..." In fact, as shown below, the rational PH curves in [10] do *not* in general have rational arc lengths.

A geometrical approach to the construction of rational PH curves is based [8, 11] on the dual representation, in which a plane curve is interpreted as the envelope of a family of tangent lines, rather than a point locus. In [10], on the other hand, the complex model [1] is adopted, with the Cartesian components of a curve being regarded as real and imaginary parts of a complex–valued function $\mathbf{r}(t) = x(t) + \mathrm{i}\, y(t)$ of a real parameter t. In particular, the authors consider rational curves defined by truncated Laurent series of the form

$$\mathbf{r}(t) \;=\; \sum_{k=-m}^{n} \mathbf{a}_k (t - \mathbf{c})^k \,,$$

and investigate the conditions on the complex values $\mathbf{a}_{-m}, \ldots, \mathbf{a}_0, \ldots, \mathbf{a}_n$ and $\mathbf{c}$ such that $\sigma(t) = |\mathbf{r}'(t)|$ is a rational function. They focus, in particular, on the case $(m, n) = (1, 3)$ and show that for a non–polynomial curve ($\mathbf{a}_{-1} \neq 0$) these conditions amount to

$$\mathbf{a}_2 \;=\; 0 \qquad \text{and} \qquad \mathbf{a}_1^2 + 12\, \mathbf{a}_3 \mathbf{a}_{-1} \;=\; 0\,. \tag{2}$$

The case $\mathbf{a}_1 = 0$ is discounted, since the conditions (2) then imply that $\mathbf{a}_3 = 0$ if $\mathbf{a}_{-1} \neq 0$, and the locus $\mathbf{r}(t) = \mathbf{a}_{-1}/(t - \mathbf{c}) + \mathbf{a}_0$ simply defines [13] a circular arc with center $\mathbf{a}_0 + \mathrm{i}\, \mathbf{a}_{-1}/2\, \mathrm{Im}(\mathbf{c})$ and radius $|\mathbf{a}_{-1}/2\, \mathrm{Im}(\mathbf{c})|$. The length of a circular arc is obviously determined by its angular extent, which involves an arctangent dependence upon the parameter t.

Subject to the above conditions, differentiation of the rational curve

$$\mathbf{r}(t) \;=\; \frac{\mathbf{a}_{-1}}{t-\mathbf{c}} + \mathbf{a}_0 + \mathbf{a}_1(t-\mathbf{c}) + \mathbf{a}_3(t-\mathbf{c})^3 \tag{3}$$

yields the rational parametric speed function

$$\sigma(t) \;=\; |\mathbf{r}'(t)| \;=\; \frac{|\,6\,\mathbf{a}_3(t-\mathbf{c})^2 + \mathbf{a}_1\,|^2}{12\,|\mathbf{a}_3|\,|t-\mathbf{c}|^2}\,. \tag{4}$$

Note that $\mathrm{Im}(\mathbf{c}) \neq 0$ must be assumed to exclude real points at infinity.

The fact that integrating $\sigma(t)$ does not yield a rational arc length function can be verified as follows. Consider first the case $\mathrm{Im}(\mathbf{c}) \neq 0$. By expanding (4) and performing a partial fraction decomposition, we obtain

$$\begin{aligned}
\sigma(t) \;=&\; 3\,|\mathbf{a}_3|\,(t^2 - 2\,\mathrm{Re}(\mathbf{c})\,t + |\mathbf{c}|^2) \\
&+\; \frac{1}{|\mathbf{a}_3|}\left[\mathrm{Re}(\mathbf{a}_3\overline{\mathbf{a}}_1) \;+\; \mathrm{i}\,\mathrm{Im}(\mathbf{c})\left(\frac{\overline{\mathbf{a}}_3\mathbf{a}_1}{t-\mathbf{c}} \;-\; \frac{\mathbf{a}_3\overline{\mathbf{a}}_1}{t-\overline{\mathbf{c}}}\right)\right] \\
&+\; \frac{|\mathbf{a}_1|^2}{24\,|\mathbf{a}_3|\,\mathrm{i}\,\mathrm{Im}(\mathbf{c})}\left[\frac{1}{t-\mathbf{c}} \;-\; \frac{1}{t-\overline{\mathbf{c}}}\right].
\end{aligned}$$

Forming the indefinite integral then yields the arc length function

$$\begin{aligned}
s(t) \;=&\; |\mathbf{a}_3|\,(t^3 - 3\,\mathrm{Re}(\mathbf{c})\,t^2 + 3\,|\mathbf{c}|^2 t) \\
&+\; \frac{1}{|\mathbf{a}_3|}\;[\,\mathrm{Re}(\mathbf{a}_3\overline{\mathbf{a}}_1)\,t \;+\; \mathrm{i}\,\mathrm{Im}(\mathbf{c})\,(\overline{\mathbf{a}}_3\mathbf{a}_1\ln(t-\mathbf{c}) - \mathbf{a}_3\overline{\mathbf{a}}_1\ln(t-\overline{\mathbf{c}}))\,] \\
&+\; \frac{|\mathbf{a}_1|^2}{24\,|\mathbf{a}_3|\,\mathrm{i}\,\mathrm{Im}(\mathbf{c})}\;[\,\ln(t-\mathbf{c}) - \ln(t-\overline{\mathbf{c}})\,]\,.
\end{aligned}$$

This can be further reduced by using the complex logarithm expansion $\ln\mathbf{z} = \ln|\mathbf{z}| + \mathrm{i}\arg(\mathbf{z})$ and noting that $\ln|t-\overline{\mathbf{c}}| = \ln|t-\mathbf{c}|$, $\arg(t-\overline{\mathbf{c}}) = -\arg(t-\mathbf{c})$ to obtain

$$s(t) \;=\; \alpha_3 t^3 + \alpha_2 t^2 + \alpha_1 t + \alpha_0 + \beta\,\ln|t-\mathbf{c}| + \gamma\,\arg(t-\mathbf{c})\,, \tag{5}$$

where

$$\alpha_3 \;=\; |\mathbf{a}_3|\,, \quad \alpha_2 \;=\; -\,3\,|\mathbf{a}_3|\,\mathrm{Re}(\mathbf{c})\,, \quad \alpha_1 \;=\; 3\,|\mathbf{a}_3|\,|\mathbf{c}|^2 + \frac{\mathrm{Re}(\mathbf{a}_3\overline{\mathbf{a}}_1)}{|\mathbf{a}_3|}\,,$$

$$\beta \;=\; \frac{2\,\mathrm{Im}(\mathbf{a}_3\overline{\mathbf{a}}_1)\,\mathrm{Im}(\mathbf{c})}{|\mathbf{a}_3|}\,, \quad \gamma \;=\; \frac{|\mathbf{a}_1|^2}{12\,|\mathbf{a}_3|\,\mathrm{Im}(\mathbf{c})} \;-\; \frac{2\,\mathrm{Re}(\mathbf{a}_3\overline{\mathbf{a}}_1)\,\mathrm{Im}(\mathbf{c})}{|\mathbf{a}_3|}\,,$$

and the integration constant $\alpha_0 = \gamma \arg(\mathbf{c}) - \beta \ln|\mathbf{c}|$ yields $s(0) = 0$. Clearly, one must have $\beta = \gamma = 0$ if the arc length (5) is to be a rational (actually, polynomial) function. As noted above, for a true rational curve with $\mathbf{a}_{-1} \neq 0$ we have $\mathbf{a}_1 = 0 \iff \mathbf{a}_3 = 0$ from (2), and this special case identifies a circle. Otherwise, when $\mathbf{a}_1, \mathbf{a}_3$ are non–zero, β cannot vanish if $\mathrm{Im}(\mathbf{c}) \neq 0$. Thus, none of the rational curves defined by (2) and (3) with $\mathbf{a}_{-1} \neq 0$ and $\mathrm{Im}(\mathbf{c}) \neq 0$ has a rational arc length. The type of rational function integration performed above is well–known in the context of PH curves, e.g., in computing the elastic bending energy [2], and rotation–minimizing frames [3] on space curves.

For the degenerate case with $\mathrm{Im}(\mathbf{c}) = 0$, however, the situation is different. When $\mathbf{c}$ has the real value c, the parametric speed (4) becomes

$$\sigma(t) \,=\, |\mathbf{r}'(t)| \,=\, \frac{|\, 6\,\mathbf{a}_3(t-c)^2 + \mathbf{a}_1\,|^2}{12\,|\mathbf{a}_3|\,(t-c)^2}\,, \tag{6}$$

and the arc length is a rational function only in this special case, namely

$$s(t) \,=\, |\mathbf{a}_3|(t-c)^3 + \frac{\mathrm{Re}(\mathbf{a}_3\overline{\mathbf{a}}_1)}{|\mathbf{a}_3|}\,t - \frac{|\mathbf{a}_1|^2}{12|\mathbf{a}_3|(t-c)}\,. \tag{7}$$

A real value c for the center of the Laurent series generates a point at infinity on the curve $\mathbf{r}(t)$ — an undesirable feature in most practical applications — although for a finite curve segment one can always choose c to lie outside the prescribed curve parameter domain $t \in [\,a, b\,]$.

The *residue* of a rational function at a pole $t = \mathbf{c}$ is the coefficient of the term $(t-\mathbf{c})^{-1}$ in its partial fraction expansion, and the general condition for a rational function to have a rational integral is that the residues at each of its poles must vanish [9, §7.2]. If $\mathrm{Im}(\mathbf{c}) \neq 0$, the function (4) has the distinct simple poles $t = \mathbf{c}$ and $t = \overline{\mathbf{c}}$, with corresponding residues

$$(t-\mathbf{c})\sigma(t)\,|_{t=\mathbf{c}} = \frac{|\mathbf{a}_1|^2}{24|\mathbf{a}_3|\,\mathrm{i}\,\mathrm{Im}(\mathbf{c})}, \quad (t-\overline{\mathbf{c}})\sigma(t)\,|_{t=\overline{\mathbf{c}}} = -\,\frac{|\mathbf{a}_1 - 12\mathbf{a}_3\,\mathrm{i}\,\mathrm{Im}(\mathbf{c})|^2}{24|\mathbf{a}_3|\,\mathrm{i}\,\mathrm{Im}(\mathbf{c})}.$$

Clearly, these values cannot both vanish when $\mathbf{a}_1, \mathbf{a}_3$ are non–zero. On the other hand, when $\mathrm{Im}(\mathbf{c}) = 0$, and $\mathbf{c}$ has the real value c, the function (6) has only the double pole $t = c$. The residue of $\sigma(t)$ at this pole is

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(t-c)^2\sigma(t)\,\bigg|_{t=c} \,=\, 0\,,$$

and hence $s(t)$, the integral of $\sigma(t)$, is a rational function.

Example 1. Consider the rational curve (3) defined by the values

$$\mathbf{a}_{-1} = -\,2\,\mathrm{i}\,, \;\; \mathbf{a}_0 = 2+\mathrm{i}\,, \;\; \mathbf{a}_1 = 6+6\,\mathrm{i}\,, \;\; \mathbf{a}_3 = 3\,, \;\; \mathbf{c} \,=\, 1-\mathrm{i}\,.$$

Since these values satisfy the conditions (2), they specify a rational PH curve, for which the parametric speed (4) is

$$\sigma(t) \;=\; \frac{9\,t^4 - 36\,t^3 + 78\,t^2 - 72\,t + 26}{t^2 - 2\,t + 2}\,.$$

Omitting the integration constant α_0, the corresponding arc length function (5), satisfying $s'(t) = \sigma(t)$, is

$$s(t) \;=\; 3\,t^3 - 9\,t^2 + 24\,t \;+\; 12\,\ln\sqrt{t^2-2t+2} \;+\; 10\,\tan^{-1}\frac{1}{t-1}\,.$$

In this case, the arc length is clearly not a rational function of the parameter.

Example 2. We use the same values as in the previous example, except that the real value $c = 2$ is substituted for $\mathbf{c}$. The parametric speed (6) is then

$$\sigma(t) \;=\; \frac{9\,t^4 - 72\,t^3 + 222\,t^2 - 312\,t + 170}{(t-2)^2}\,.$$

Again omitting the integration constant α_0, the corresponding arc length (7) is defined by the rational function

$$s(t) \;=\; \frac{3\,t^4 - 24\,t^3 + 78\,t^2 - 108\,t + 46}{t-2}\,.$$

The vanishing–residue criterion might, in principle, be used as the point of departure for identifying more general classes of rational PH curves with rational arc lengths. However, it is not obvious how to impose this condition in a geometrically meaningful manner, and in practice it may prove quite restrictive. Clearly, the polynomial arc lengths of polynomial PH curves offer a much simpler and more robust framework for real–time motion control.

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