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# REGGE TRAJECTORIES AND THE POMERON SLOPE IN THE ABFST MULTIPERIPHERAL MODEL*

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Abstract: The forward and non-forward Amati-Bertocchi-Fubini-Stanghellini-Tonin (ABFST) multiperipheral integral equations with pion exchange and resonance production are solved in a factorizable approximation to the kernel. The solution, which determines the Regge trajectories, is continuous at $t = 0$ and is analytically continued and studied for all $t$. In this pion exchange model, the slope of the pomeron trajectory at $t = 0$ is calculated and is found to have a crucial dependence on the pion mass. The $\pi\pi$ cross section is also calculated in this approximation. The behavior of the trajectories near threshold is shown to be the same as in potential theory. The $t \to \pm \infty$ limits of the solution are also analyzed.

## 1. INTRODUCTION

The original multiperipheral model of ABFST [1, 2] although it does not incorporate Regge exchange, is still of great interest today. It provides insight into the dynamics of multiparticle contributions to unitarity. It also is used in the Schizophrenic pomeron model [3] to generate the input P′-like trajectory which is then coupled with pomeron Regge exchanges to bootstrap pomerons.

The ABFST integral equation has not been solved analytically, and approximate analytic studies of its solutions have usually been carried out with the 1st-Fredholm or trace approximation [4, 5]. In this paper we solve the forward and non-forward ABFST integral equations in the factorizable kernel approximation and investigate the properties of this solution. We choose a factorizable kernel which still contains much of the singularity structure in angular momentum $l$, and is continuous at $t = 0$, so that we can simultaneously solve both the forward and non-forward ABFST equations.

The solution gives a simple implicit equation for the Regge trajectories. With analytic methods we can readily find the dependence of the trajectories on the

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coupling constant and the pion mass. We can also qualitatively outline their motion in the complex $l$ plane as a function of $t$. The factorizable kernel solution may also prove useful in numerical calculations to compare with the results of the trace approximation.

In the original ABFST model, the pomeron is generated by a unitarity sum over multiperipheral pion exchange amplitudes. We calculate the slope of the pomeron trajectory in this model for $\alpha_P(0) = 1$ and show that it has a crucial logarithmic dependence on the pion mass. We also argue that even in the more completely unitarized pomeron bootstrap models which include pomeron exchange amplitudes, the slope will also depend on the pion mass. The $\pi\pi$ cross section has been calculated in the ABFST model with the trace approximation and shown to be independent of the pion mass for small pion mass [4, 1]. We include a calculation of the $\pi\pi$ cross section in the factorizable kernel approximation and obtain a similar result.

Although the trajectory equation for the ABFST model is derived for $t \leqslant 0$, we can analytically continue it to investigate the trajectories for any $t$. In particular, we investigate the motion of the poles asymptotically as $t \to \pm \infty$ and also near threshold $t = 4\mu^2$. Near threshold we find that the non-leading or complex poles converge to the point $l = -\frac{1}{2}$ at $t = 4\mu^2$ and their motion is the same as that found in potential models [6]. When $t$ approaches positive infinity, one Regge pole approaches each negative integer, with the leading trajectory outgoing to $-1$. The approach is from the positive imaginary direction. This behavior of the trajectories in the ABFST model is qualitatively the same as in potential theory.

In sect. 2 we begin by presenting the factorizable kernel approximation to the forward ABFST equation. We then solve it and present the implicit equation for the trajectories. Also, we calculate the $\pi\pi$ cross section assuming that the pomeron has intercept one. In sect. 3, we formulate the factorizable kernel approximation for the non-forward ABFST equation. The approximate equation, which is diagonalized in $l$, approaches smoothly the forward equation as $t \to 0$. We solve the non-forward equation and present the equation for the Regge trajectories. In sect. 4 we calculate the slope of the pomeron at $t = 0$ and show its dependence on the pion mass. We also prove that in a pomeron bootstrap model, the slope of the pomeron will also depend on the pion mass. By analytic continuation of the equation for the trajectories we investigate the behavior of the trajectories near threshold in sect. 5. The behavior of the trajectories as $t \to \pm \infty$ is analyzed in sect. 6. In sect. 7 we summarize our results.

## 2. FACTORIZABLE KERNEL FOR FORWARD ABFST EQUATION

We will begin by studying the forward ABFST equation in order to introduce the factorizable kernel method in the simplest case. It will also be used later to show its continuity with the non-forward approximation.

The absorptive part $A(s, u, v)$ of the forward $\pi\pi$ elastic scattering amplitude in

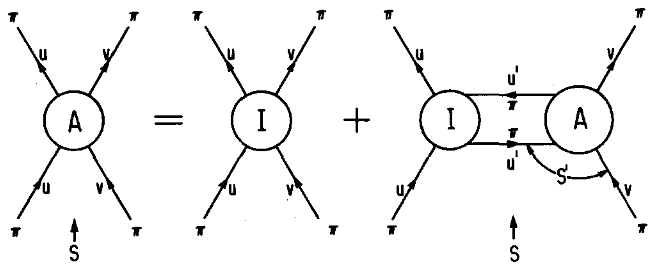


Fig. 1. The forward ABFST integral equation.

the ABFST model satisfies the integral equation [1, 2, 7] (see fig. 1)

$$A(s,u,v) = I(s,u,v) + \frac{\theta(s-4\mu^2)}{16\pi^3 \Delta^{\frac{1}{2}}(s,u,v)} \int_{4\mu^2}^{(s^{\frac{1}{2}}-\mu)^2} \mathrm{d}s_0 \int_{4\mu^2}^{(s^{\frac{1}{2}}-s_0^{\frac{1}{2}})^2} \mathrm{d}s'$$

$$\times \int_{u'_-}^{u'_+} \mathrm{d}u' \frac{I(s_0,u,u')A(s',u',v)}{(\mu^2-u')^2}\,, \tag{2.1}$$

where the inhomogeneous term $I$ corresponds to the two pion unitarity contribution, and $\mu$ is the pion mass. The limits on the $u'$ integration are

$$u'_\pm = s_0 + u - \frac{(s+u-v)(s+s_0-s')}{2s} \pm \frac{\Delta^{\frac{1}{2}}(s,u,v)\,\Delta^{\frac{1}{2}}(s,s_0,s')}{2s}\,,$$

$$\Delta(a,b,c) = a^2+b^2+c^2-2(ab+bc+ca)$$

In this paper we hold $u$ and $v$ negative to obtain this simple integration region. $v$ will be considered as very small. The isospin of the pion is ignored here, but the results can be directly generalized to include it. It has been shown that this equation can be exactly diagonalized in $l$ and we define the partial wave amplitude $A_l(u,v)$ by [8, 9, 10]

$$A_l(u,v) = \int_{4\mu^2}^{\infty} \mathrm{d}s \left[\frac{s-u-v+\Delta^{\frac{1}{2}}(s,u,v)}{2}\right]^{-l-1} A(s,u,v)\,,$$

$$\xrightarrow[s \gg u,v]{} \int_{4\mu^2}^{\infty} \mathrm{d}s\, s^{-l-1} A(s,u,v)\,. \tag{2.2}$$

The absorptive part is constructed from the inversion

$$A(s, u, v) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathrm{d}l \left[\frac{s-u-v+\Delta^{\frac{1}{2}}(s, u, v)}{2}\right]^{l+1} \frac{A_l(u, v)}{\Delta^{\frac{1}{2}}(s, u, v)},$$

$$\xrightarrow[s \gg u,v]{} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathrm{d}l\, s^l A_l(u, v). \tag{2.3}$$

The result of the partial wave projection of eq. (2.1) is [8, 9, 10]

$$A_l(u, v) = I_l(u, v) + \frac{1}{16\pi^3(l+1)} \int_{-\infty}^{0} \frac{\mathrm{d}u'(-u')^{l+1}}{(\mu^2-u')^2} I_l(u, u') A_l(u', v), \tag{2.4}$$

where $I_l(u, v)$ is defined as the partial wave of $I(s, u, v)$ by eq. (2.2). In general, $I(s_0, u, u')$ contains many $\pi\pi$ resonances as well as a background contribution from pomeron exchange. For reasons of simplicity we will study here only the case where $I$ is represented by one resonance with $M^2 = 0$ (1 GeV$^2$):

$$I(s_0, u, u') = \pi g^2 \delta(s_0 - M^2). \tag{2.5}$$

Our methods and results can be directly generalized to the case of any number of resonances. With eq. (2.5) as the kernel, the forward ABFST equation is: [8, 9, 10, 11]

$$A_l(u, v) = \pi g^2 \left[\frac{M^2-u-v+\Delta^{\frac{1}{2}}(M^2, u, v)}{2}\right]^{-l-1} + \frac{g^2}{16\pi^2(l+1)} \int_{-\infty}^{0} \mathrm{d}u'$$

$$\times \left[\frac{M^2-u-u'+\Delta^{\frac{1}{2}}(M^2, u, u')}{-2u'}\right]^{-l-1} \frac{A_l(u', v)}{(\mu^2-u')^2}. \tag{2.6}$$

Up to this point the equation is exact, but to solve it analytically we must make some approximation. The common one has been the trace approximation. In this paper we will apply the familiar factorizable kernel method in which the kernel is expanded in terms which are factorized in $u$ and $u'$. The approximation we use is to keep only the first factorized term. In order to approximate well the $l$-plane properties we must retain the source of the singularity structure. That means we must retain the correct behavior of the kernel at the physical boundary, i.e. at the limits of integration. The ABFST kernel has the following behavior at the boundaries:

$$\left(\frac{M^2 - u - u' + \Delta^{\frac{1}{2}}(M^2, u, u')}{-2u'}\right)^{-l-1} \xrightarrow[u' \to -\infty]{} 1\,,$$

$$\xrightarrow[u' \to 0]{} \left(\frac{-u'}{M^2 - u}\right)^{l+1},$$

$$\xrightarrow[u \to 0]{} \left(\frac{-u'}{M^2 - u'}\right)^{l+1}.$$

We have the following two ways to approximate the kernel and preserve much of the singularity structure of the full kernel:

(1) We take for the factorizable kernel the kernel for $u$ near zero,

$$K_0 = \frac{1}{l+1}\left(\frac{-u'}{M^2 - u'}\right)^{-l-1} \frac{1}{(\mu^2 - u')^2}\,,$$

since this has the correct behavior as $u' \to \infty$ and the correct divergence as $u' \to 0$. If we also consider $|v| \ll M^2$, then eq. (2.6) will be reduced to

$$A_l(u, v) = \pi g^2 (M^2)^{-l-1} + \frac{g^2}{16\pi^2(l+1)} \int_{-\infty}^{0} \mathrm{d}u' \left(\frac{-u'}{M^2 - u'}\right)^{l+1} \frac{A_l(u', v)}{(\mu^2 - u')^2}\,. \tag{2.7}$$

Since the factorized kernel is independent of $u$, $A_l(u, v)$ must be independent of $u$ and the solution to eq. (2.7) is obtained algebraically:

$$A_l(u, v) = \pi g^2 (M^2)^{-l-1}/D(l)\,, \tag{2.8}$$

where

$$D(l) = 1 - \frac{g^2}{16\pi^2(l+1)} \int_{-\infty}^{0} \mathrm{d}u' \left(\frac{-u'}{M^2 - u'}\right)^{l+1} \frac{1}{(\mu^2 - u')^2}\,. \tag{2.9}$$

(2) We may alternatively take the convergent kernel for $u'$ near zero,

$$K_0 = \frac{1}{l+1}\left(\frac{-u'}{M^2 - u}\right)^{-l-1} \frac{1}{(\mu^2 - u')^2}$$

By taking $|v| \ll M^2$, eq. (2.6) will be reduced to

$$A_l(u, v) = \pi g^2 \left(\frac{1}{M^2 - u}\right)^{l+1} + \frac{g^2}{16\pi^2(l+1)} \int_{-\infty}^{0} du' \times \left(\frac{-u'}{M^2 - u}\right)^{l+1} \frac{A_l(u', v)}{(\mu^2 - u')^2} . \tag{2.10}$$

By defining $\widetilde{A}_l(u, v) = (M^2 - u)^{l+1} A_l(u, v)$ we find the equation for $\widetilde{A}_l(u, v)$ to be the same as eq. (2.7). The solution is then

$$A_l(u, v) = \pi g^2 (M^2 - u)^{-l-1}/D(l) , \tag{2.11}$$

with the same $D(l)$ given by eq. (2.9). Both factorizable kernel approximations give the same denominator function $D(l)$ and will have the same Regge singularities. They are approximately equal for small $u$, $v$ and directly continuable to the mass shell $u = v = \mu^2 \ll M^2$.

We may evaluate $D(l)$ eq. (2.9) as a hypergeometric function

$$D(l) = 1 - \frac{g^2}{16\pi^2 M^2} \left\{ \frac{1}{l(l+1)} F(2, 1; -l+1; \mu^2/M^2) - \frac{\pi}{\sin \pi l} \left(\frac{\mu^2}{M^2}\right)^l \left(1 - \frac{\mu^2}{M^2}\right)^{-l-2} \right\} . \tag{2.12}$$

For $-1 < l \leqslant +1$ and $\mu^2 \ll M^2$ this is approximately

$$D(l) \simeq 1 - \frac{g^2}{16\pi^2 M^2} \left\{ \frac{1}{l(l+1)} \left(1 + \frac{2}{1-l} \frac{\mu^2}{M^2}\right) - \left(\frac{\mu^2}{M^2}\right)^l \frac{\pi}{\sin \pi l} \right\} . \tag{2.13}$$

There are no poles in $D(l)$ at $l = 0$ or at positive integers as is apparent from its definition eq. (2.9).

The Regge poles in $A_l(u, v)$ are given by $D(l) = 0$. For Re $l < 0$, the term proportional to $(\mu^2/M^2)^l$ is dominant and gives rise to complex poles [5, 11]. For Re $l > 0$ the leading pole is given by

$$D(l) = 0 \simeq 1 - \frac{g^2}{16\pi^2 M^2} \frac{1}{l(l+1)} . \tag{2.14}$$

The ABFST equation may be used for estimating the total $\pi\pi$ cross section [1, 4]. One requires that the coupling be arbitrarily adjusted to give a pomeron of intercept, $\alpha_p = 1$, which from (2.14) requires

$$\frac{g^2}{16\pi^2 M^2} = 2 \,. \tag{2.15}$$

For the residue of the absorptive part at $l = 1$ we need

$$\frac{\mathrm{d}D(l)}{\mathrm{d}l} = \frac{g^2}{16\pi^2 M^2} \frac{2l+1}{l^2(l+1)^2} \,.$$

This gives

$$A_l(u, v) \underset{l \approx 1}{=} \frac{1}{l-1} \frac{\pi g^2 (M^2)^{-2}}{\frac{\mathrm{d}}{\mathrm{d}l} D(l)|_{l=1}} = \frac{1}{l-1} \frac{16\pi^3}{M^2} \frac{4}{3} \,. \tag{2.16}$$

Using the inverse transform eq. (2.3) we find for the elastic forward absorptive part at large $s$

$$A(s, u, v) = \frac{64\pi^3}{3M^2} s \,. \tag{2.17}$$

If we assume, following Abarbanel et al., that the pomeron is an SU(3) singlet in the $t$-channel, then the projection to the $s$-channel $\pi\pi$ cross section at high energy will be

$$\sigma_{\pi\pi} = \frac{1}{8} \frac{A(s, u, v)}{s} = \frac{8}{3} \frac{\pi^3}{M^2} = 33 \text{ mb} \tag{2.18}$$

for $M^2 = (1 \text{ GeV})^2$. This is similar to the results of ref. [1] and ref. [4] and shows the independence of the cross section on the pion mass. In the ABFST model, one expects the pion propagators $(\mu^2 - u')^{-2}$ in the kernel to enhance the region between $u' = 0$ and $u' \approx 0\,(-\mu^2)$, which indeed occurs for $\mathrm{Re}\, l < 0$ and for complex poles. But for $\mathrm{Re}\, l > 0$, and $u'$ small, the kernel becomes

$$\left(\frac{-u'}{M^2 - u'}\right)^{l+1} \frac{1}{(\mu^2 - u')^2} \,,$$

whose integral is large only for $u' \lesssim 0\,(-M^2)$, and the cross section is largely independent of $\mu^2$.

We can see from this calculation that our factorizable kernel approximation is in fairly good agreement with the trace approximation which gives [4] $\sigma_{\pi\pi} = 36\pi^3/(11M^2)$.

## 3. FACTORIZABLE KERNEL APPROXIMATION TO NON-FORWARD ABFST EQUATION

We proceed to formulate and solve a factorizable kernel approximation to the non-forward ABFST equation. Much progress has been made recently in the exact partial wave decomposition and diagonalization of this equation [9, 10, 12]. We will use the results of their partial wave projection and refer the reader to these papers for the details.

The ABFST equation and its kinematics are given in fig. 2. With off mass shell momenta $u_\pm = (P + \frac{1}{2}Q)^2$, $u'_\pm = (P' \pm \frac{1}{2}Q)^2$, $v_\pm = (k \pm \frac{1}{2}Q)^2$, we define the useful combinations

$$u = P^2 = \tfrac{1}{2}(u_+ + u_-) - \tfrac{1}{4}t\,, \qquad \sin\psi = \frac{P\cdot Q}{(tu)^{\frac{1}{2}}} = \frac{u_- - u_+}{2(tu)^{\frac{1}{2}}}\,,$$

$$u' = P'^2 = \tfrac{1}{2}(u'_+ + u'_-) - \tfrac{1}{4}t\,, \qquad \sin\psi' = \frac{P'\cdot Q}{(tu')^{\frac{1}{2}}} = \frac{u'_- - u'_+}{2(tu')^{\frac{1}{2}}}\,,$$

$$v = k^2 = \tfrac{1}{2}(v_+ + v_-) - \tfrac{1}{4}t\,, \qquad \sin\phi = \frac{k\cdot Q}{(tv)^{\frac{1}{2}}} = \frac{v_- - v_+}{2(tv)^{\frac{1}{2}}}\,.$$

We hold the off mass shell momenta negative and treat $|v_+|$, $|v_-| \ll M^2$. The non-forward ABFST equation can then be written as [9, 10] (see fig. 2);

$$A(s, t; u, \psi; v, \phi) = I(s, t; u, \psi; v, \phi) + \frac{\theta(s-4\mu^2)}{32\pi^4} \int_{4\mu^2}^{(s^{\frac{1}{2}}-\mu)^2} \mathrm{d}s_0 \int_{4\mu^2}^{(s^{\frac{1}{2}}-s_0^{\frac{1}{2}})^2} \mathrm{d}s' \int_{u'_-}^{u'_+} \mathrm{d}u'$$

$$\times \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \mathrm{d}\sin\psi' (ut)^{\frac{1}{2}} \frac{\theta(-D)}{(-D)^{\frac{1}{2}}}$$

$$\times \frac{I(s_0, t; u, \psi; u'\psi')A(s', t; u', \psi'; v, \phi)}{[(\mu^2 - \frac{1}{4}t - u')^2 - u't\sin^2\psi']}\,. \tag{3.1}$$

The physical region is obtained from

$$D = \begin{vmatrix} t & (tu')^{\frac{1}{2}}\sin\psi' & (tu)^{\frac{1}{2}}\sin\psi & (tv)^{\frac{1}{2}}\sin\phi \\ (tu')^{\frac{1}{2}}\sin\psi' & u' & \frac{1}{2}(s_0 - u - u') & \frac{1}{2}(s' - u' - v) \\ (tu)^{\frac{1}{2}}\sin\psi & \frac{1}{2}(s_0 - u - u') & u & \frac{1}{2}(s - u - v) \\ (tv)^{\frac{1}{2}}\sin\phi & \frac{1}{2}(s' - u' - v) & \frac{1}{2}(s - u - v) & v \end{vmatrix}\,,$$

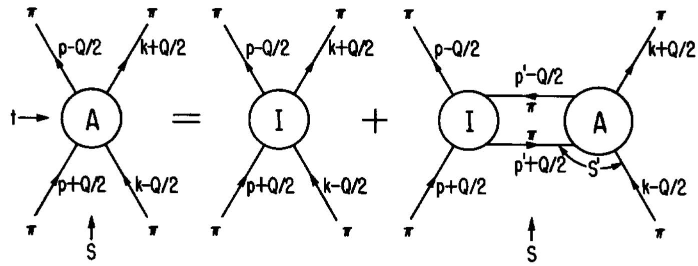


Fig. 2. The non-forward ABFST integral equation.

and

$$u'_{\pm} = s_0 + u - \frac{(s+u-v)(s+s_0-s')}{2s} \pm \frac{\Delta^{\frac{1}{2}}(s,u,v)\,\Delta^{\frac{1}{2}}(s,s_0,s')}{2s} .$$

Let us define the partial wave amplitude $A_l$ by [9, 10]

$$A_l(t;u,\psi;v,\phi) = \frac{1}{B(l+1,\frac{1}{2})} \int_0^{\infty} \frac{ds}{((uv)^{\frac{1}{2}}\cos\psi\cos\phi)^{l+1}}$$

$$\times\, Q_l\left(\frac{s-u-v-2(uv)^{\frac{1}{2}}\sin\psi\sin\phi}{2(uv)^{\frac{1}{2}}\cos\psi\cos\phi}\right) A(s,t;u,\psi;v,\phi)$$

$$\times \xrightarrow[s \gg |u|,|v|]{} \int_0^{\infty} ds\, s^{-l-1} A(s,t;u,\psi;v,\phi) . \tag{3.2}$$

The inverse gives the absorptive part

$$A(s,t;u,\psi,v,\phi) = \frac{1}{2\pi i}\,\frac{B(l+1,\frac{1}{2})}{2} \int_{c-i\infty}^{c+i\infty} dl\,(2l+1)\,((uv)^{\frac{1}{2}}\cos\psi\cos\phi)^l$$

$$\times\, P_l\left(\frac{s-u-v-2(uv)^{\frac{1}{2}}\sin\psi\cos\phi}{2(uv)^{\frac{1}{2}}\cos\psi\cos\phi}\right) A_l(t;u,\psi;v,\phi) , \tag{3.3}$$

$$A(s,t;u,\psi;v,\phi) \xrightarrow[s \gg |u|,|v|]{} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dl\, s^l A_l(t;u,\psi;v,\phi) .$$

As in the forward equation we choose a single resonance model $I = \pi g^2 \delta(s_0 - M^2)$. The partial wave amplitude will then satisfy the following integral equation [9, 10]

$$A_l(t;u,\psi;v,\phi) = \frac{\pi g^2}{B(l+1,\frac{1}{2})((uv)^{\frac{1}{2}}\cos\psi\cos\phi)^{l+1}}$$

$$\times Q_l\left(\frac{M^2-u-v-2(uv)^{\frac{1}{2}}\sin\psi\cos\phi}{2(uv)^{\frac{1}{2}}\cos\psi\cos\phi}\right) + \frac{g^2}{16\pi^3}\int_{-\infty}^{0}\mathrm{d}u'\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\mathrm{d}\psi'$$

$$\times\left[\left(\frac{u'}{u}\right)^{\frac{1}{2}}\frac{\cos\psi'}{\cos\psi}\right]^{l+1} Q_l\left(\frac{M^2-u-u'-2(uu')^{\frac{1}{2}}\sin\psi\sin\psi'}{2(uu')^{\frac{1}{2}}\cos\psi\cos\psi'}\right)$$

$$\times\frac{A_l(t;u',\psi';v,\phi)}{(\mu^2-\frac{1}{4}t-u')^2-u't\sin^2\psi'}\,. \tag{3.4}$$

To make a factorizable kernel approximation we again examine the behavior at the limits of integration $u' \to 0$, $u' \to -\infty$, and $\cos\psi' \to 0$. As $\psi' \to \pm\frac{1}{2}\pi$, the argument of the $Q_l$ function becomes infinite. Since the singularity structure of $A_l$ will be governed by the behavior at the limits of integration, our first approximation is to use the asymptotic form of $Q_l$. This is also a good approximation for small $u$ and $v$ since the minimum value of the argument of $Q_l$ in the integral is $\sim (M^2/-u)^{\frac{1}{2}}$. Then eq. (3.4) will become

$$A_l(t;u,\psi;v,\phi) = \frac{\pi g^2}{(M^2-u-v-2(uv)^{\frac{1}{2}}\sin\psi\sin\phi)^{l+1}} + \frac{g^2}{16\pi^3}B(l+1,\tfrac{1}{2})\int_{-\infty}^{0}\mathrm{d}u'$$

$$\times\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\mathrm{d}\psi'\left[\frac{-u'\cos^2\psi'}{M^2-u-u'-2(uu')^{\frac{1}{2}}\sin\psi\sin\psi'}\right]^{l+1}$$

$$\times\frac{A_l(t;u',\psi';v,\phi)}{(\mu^2-\frac{1}{4}t-u')^2-u't\sin^2\psi'}\,. \tag{3.5}$$

Though these approximations may seem crude, we note that the only dependence of the kernel on $t$ and on the pion mass is contained in the pion "propagators", and we treat these exactly. Also, the asymptotic form of $Q_l$ still contains the poles in $l$ at the negative integers.

Near the boundaries of $u'$ we have

$$\frac{-u'}{M^2-u-u'-2(uu')^{\frac{1}{2}}\sin\psi\sin\psi'} \xrightarrow[u'\to-\infty]{} 1\,,$$

$$\xrightarrow[u'\to 0]{} \frac{-u'}{M^2-u}\,,$$

$$\xrightarrow[u\to 0]{} \frac{-u'}{M^2-u'}\,. \tag{3.6}$$

In order to obtain a factorizable kernel we must drop the dependence of this factor on $\sin\psi\sin\psi'$. This is an allowable approximation since it does not affect the above boundary limits. In the inhomogeneous term we again treat $|v| \ll M^2$. We then have, as in sect. 2, the two following factorizable kernel equations depending on whether we take the $u\to 0$ or $u'\to 0$ forms for the above factor:

(1) $u$ near zero kernel

$$A_l(t;u,\psi;v,\phi) = \pi g^2(M^2)^{-l-1} + \frac{g^2}{16\pi^3}B(l+1,\tfrac{1}{2}) \int_{-\infty}^{0} \mathrm{d}u' \left(\frac{-u'}{M^2-u'}\right)^{l+1} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}$$

$$\times\, \mathrm{d}\psi' \frac{(\cos^2\psi')^{l+1}}{(\mu^2-t/4-u')^2-u't\sin^2\psi'} A_l(t;u',\psi';v,\phi)\,, \tag{3.7}$$

with the solution independent of $u,\psi,v,\phi$:

$$A_l = \frac{\pi g^2 (M^2)^{-l-1}}{D(l,t)}\,. \tag{3.8}$$

(2) $u'$ near zero kernel

$$A_l(t;u,\psi;v,\phi) = \frac{\pi g^2}{(M^2-u)^{l+1}} + \frac{g^2}{16\pi^3}B(l+1,\tfrac{1}{2}) \int_{-\infty}^{0} \mathrm{d}u' \left(\frac{-u'}{M^2-u}\right)^{l+1} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}$$

$$\times\, \mathrm{d}\psi' \frac{(\cos^2\psi')^{l+1}}{(\mu^2-\frac{1}{4}t-u')^2-u't\sin^2\psi'} A_l(t;u',\psi';v,\phi)\,, \tag{3.9}$$

with the solution

$$A_l = \frac{\pi g^2}{(M^2-u)^{l+1}D(l,t)}\,, \tag{3.11}$$

and the $D$ function is given by

$$D(l,t) = 1 - \frac{g^2}{16\pi^3} B(l+1, \tfrac{1}{2}) \int_{-\infty}^{0} du' \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\psi' \times \frac{(\cos^2\psi')^{l+1}}{(\mu^2 - \frac{1}{4}t - u')^2 - u't\sin^2\psi'} , \tag{3.11}$$

We may perform the $\psi'$ integration, and letting $y = -u'$ we have for $\mathrm{Re}\, l > -2$

$$D(l,t) = 1 - \frac{g^2}{16\pi^2(l+1)} \int_0^\infty dy \left(\frac{y}{M^2+y}\right)^{l+1} \frac{1}{(\mu^2 - \frac{1}{4}t + y)^2} \times F\left(1, \tfrac{1}{2}; l+2; \frac{-ty}{(\mu^2 - \frac{1}{4}t + y)^2}\right) . \tag{3.12}$$

This is the principal result of the factorizable kernel approximation and we will analyze it in the rest of the paper. We just note that our non-forward results, eqs. (3.8), (3.10), and (3.12), when continued to $t = 0$ give the same results as the forward approximation eqs. (2.8), (2.11), and (2.9).

We will use the result for the $D$ function to study the Regge trajectories $\alpha(t)$ in the rest of the paper. The Regge poles are determined by the implicit equation

$$D(\alpha(t), t) = 0 . \tag{3.13}$$

Although $D(l, t)$ has been derived from the ABFST model for $t \leqslant 0$, it may be analytically continued to $t > 0$ since it is continuous at $t = 0$. We can also analytically continue it to any $l$ and thereby find the Regge trajectories for all values of $t$ from eq. (3.13).

Eq. (3.12) is a suitable representation as long as the argument of the hypergeometric function has an absolute value less than or equal to one for all positive values of $y$. This is the case for $t$ in the range $-\infty < t \leqslant 2\mu^2$. To continue to positive ranges of $t$ we will use the transformation formula $F(\alpha, \beta; \gamma; z) = (1-z)^{-\beta} \times F(\beta, \gamma - \alpha; \gamma; z/(z-1))$. The representation for positive $t$ and $\mathrm{Re}\, l > -2$ is then

$$D(l,t) = 1 - \frac{g^2}{16\pi^2} \frac{1}{l+1} \int_0^\infty dy \left(\frac{y}{M^2+y}\right)^{l+1} \frac{1}{\mu^2 - \frac{1}{4}t + y} \times \frac{1}{[(\mu^2 - \frac{1}{4}t + y)^2 + ty]^{\frac{1}{2}}} F\left(\tfrac{1}{2}, l+1; l+2; \frac{ty}{(\mu^2 - \frac{1}{4}t + y)^2 + ty}\right), \tag{3.14}$$

where the hypergeometric series now converges for $-4\mu^2 < t < \infty$.

## 4. SLOPE OF THE POMERON TRAJECTORY AT $t = 0$

We compute the slope of the pomeron at $t = 0$ by using the equation $D(\alpha(t), t) = 0$ and differentiating with respect to $t$.

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} D(\alpha(t), t)|_{t=0} = \frac{\partial D}{\partial l}(\alpha(0), 0)\,\alpha'(0) + \frac{\partial D}{\partial t}(\alpha(0), 0)\,,$$

$$\alpha'(0) = -\left(\frac{\partial D}{\partial l}\bigg/\frac{\partial D}{\partial t}\right)_{t=0}. \tag{4.1}$$

To compute this we expand $D(l, t)$ about $t = 0$ by expanding the integral representation eq. (3.12)

$$D(l, t) = 1 - \frac{g^2}{16\pi^2(l+1)} \int_0^\infty \mathrm{d}y \left(\frac{y}{M^2+y}\right)^{l+1} \left[\frac{1}{(\mu^2+y)^2} + \frac{t}{2(\mu^2+y)^3} - \frac{ty}{2(l+2)(\mu^2+y)^4}\right] + \mathrm{O}(t^2)\,,$$

$$D(l, t) = 1 - \frac{g^2}{16\pi^2 M^2(l+1)} \left(\frac{\mu^2}{M^2}\right)^l \left\{B(l+2, 1) F\left(l+1; l+2; l+3; 1-\frac{\mu^2}{M^2}\right) + \frac{t}{2\mu^2} \times \left[B(l+2, 2) F\left(l+1, l+2; l+4; 1-\frac{\mu^2}{M^2}\right) - \frac{B(l+3, 2)}{l+2} \times F\left(l+1, l+3; l+5; 1-\frac{\mu^2}{M^2}\right)\right]\right\}. \tag{4.2}$$

For $l > 0$ we may approximate this for small $\mu^2/M^2$

$$D(l, t) = 1 - \frac{g^2}{16\pi^2 M^2}\left\{\frac{1}{l(l+1)} + \frac{t}{2M^2} \times \left[\frac{1}{l(l-1)(l+2)} - \frac{l\pi}{3\sin\pi(l-1)}\left(\frac{\mu^2}{M^2}\right)^{l-1}\right]\right\}. \tag{4.3}$$

The intercept at $t = 0$ is the solution to $D(\alpha_0, 0) = 0$ and is given by the first term

$$\alpha_0(\alpha_0 + 1) = \frac{g^2}{16\pi^2 M^2}. \tag{4.4}$$

The slope may be computed from (4.3) by using (4.1)

$$\alpha' = \frac{1}{2M^2}\,\frac{\alpha_0^2(\alpha_0+1)^2}{2\alpha_0+1}\left[\tfrac{1}{3}\alpha_0\,\frac{\pi}{\sin\pi\alpha_0}\left(\frac{M^2}{\mu^2}\right)^{1-\alpha_0} - \frac{1}{\alpha_0(1-\alpha_0)(2+\alpha_0)}\right]. \tag{4.5}$$

We see that the slope of the leading trajectory is very dependent on the pion mass. If the leading trajectory is taken to be the pomeron with intercept $\alpha_0 = 1$, and $g^2/(16\pi^2M^2) = 2$ we find its slope by expanding in $(1-\alpha_0)$;

$$\alpha'_P = \frac{1}{M^2}\,\frac{2}{9}\left[\ln\left(\frac{M^2}{\mu^2}\right) - \frac{7}{3}\right]. \tag{4.6}$$

The slope of the pomeron depends logarithmically on the pion mass. If we take $M^2 = 1\ \mathrm{GeV}^2$ then $\alpha'_P = 0.35\ \mathrm{GeV}^{-2}$.

In more complete multiperipheral models which contain pomeron exchange as well as AFS ladders the $D$ function will still contain an AFS kernel and the slope of the pomeron will still be dependent on the pion mass.

We can check the accuracy of our approximation by comparing it to the slope of the pomeron calculated from the BFT approximation [2] to the ABFST equation. The BFT $D$ function is

$$D^{\mathrm{BFT}}(l,t) = 1 - \frac{g^2}{16\pi^2M^2}\int_0^1 \mathrm{d}x\,\frac{x^l}{1-x}\int_0^1 \frac{\mathrm{d}z}{\dfrac{\mu^2}{M^2} + \dfrac{x}{(1-x)^2} - \dfrac{t}{4M^2}(1-z^2)}. \tag{4.7}$$

For small $\mu^2/M^2$ and $l > 0$ this gives a $D$ function near $t = 0$ very similar to (4.3);

$$D^{\mathrm{BFT}}(l,t) = 1 - \frac{g^2}{16\pi^2M^2}\left\{\frac{1}{l(l+1)} + \frac{t}{2M^2}\left[\frac{2}{l+1}\,\frac{1}{l(l-1)(l+2)} - \tfrac{1}{3}l \times \frac{\pi}{\sin\pi(l-1)}\left(\frac{\mu^2}{M^2}\right)^{l-1}\right]\right\}. \tag{4.8}$$

For a pomeron of intercept one, this gives a slope

$$\alpha'_P = \frac{2}{9M^2}\left[\ln\left(\frac{M^2}{\mu^2}\right) - \frac{17}{6}\right]. \tag{4.9}$$

This is in good agreement with the factorized kernel result (4.6) and gives a slope $\alpha'_P = 0.24\ \mathrm{GeV}^{-2}$.

## 5. BEHAVIOR OF TRAJECTORIES AT THRESHOLD

We can analytically continue the trajectory equation to $t > 0$ and investigate the behavior of the trajectories about the $t$ channel threshold $t = 4\mu^2$. In potential models it is known [6] that an infinite number of complex Regge poles accumulate at $l = -\frac{1}{2}$ as $t$ approaches $4\mu^2$. We will show in this section that this property is also obeyed by the trajectories in the ABFST model with the factorizable kernel approximation.

We begin with the representation (3.14) valid for $-4\mu^2 < t < \infty$. We approach from below threshold by defining

$$\epsilon = \mu^2 - \tfrac{1}{4}t > 0$$

and letting $\epsilon \to 0$.

$$D(l, t) = 1 - \frac{g^2}{16\pi^2(l+1)} \int_0^\infty \mathrm{d}y \left(\frac{y}{M^2+y}\right)^{l+1} \frac{1}{\epsilon + y} \frac{1}{[(\epsilon - y)^2 + 4\mu^2 y]^{\frac{1}{2}}}$$

$$\times F\left(\tfrac{1}{2}, l+1; l+2; \frac{(4\mu^2 - 4\epsilon)y}{(\epsilon - y)^2 + 4\mu^2 y}\right). \tag{5.1}$$

We also restrict our investigation near $l = -\frac{1}{2}$ and attempt to isolate the nature of the behavior near threshold rather than its exact numerical values. The hypergeometric function at $l = -\frac{1}{2}$ takes values between 1 and $\frac{1}{2}\pi$ and we replace it by a mean value $C_1 \lesssim \frac{1}{2}\pi$. We then have

$$D(l, t) = 1 - \frac{g^2 C_1}{16\pi^2(l+1)} \int_0^\infty \mathrm{d}y \left(\frac{y}{M^2+y}\right)^{l+1} \frac{1}{\epsilon + y} \frac{1}{[(\epsilon - y)^2 + 4\mu^2 y]^{\frac{1}{2}}}. \tag{5.2}$$

To simplify the integral we first approximate the square root by $(y^2 + 4\mu^2 y)^{\frac{1}{2}}$. This is good except in the region $y < \epsilon^2/4\mu^2$, but we have evaluated the difference in this region and found it to be of order $\epsilon^{2l+2} \sim \epsilon$. Since the crucial behavior at $l = -\frac{1}{2}$ comes from $y \lesssim 4\mu^2$, we rewrite the integral as

$$D(l, t) = 1 - \frac{g^2}{16\pi^2} \frac{C_1}{l+1} \left\{ \int_0^\infty \mathrm{d}y \frac{y^{l+1}}{(M^2)^{l+1}} \frac{1}{(\epsilon + y)(y^2 + 4\mu^2 y)^{\frac{1}{2}}} \right.$$

$$\left. + \int_0^\infty \mathrm{d}y \left[\left(\frac{y}{M^2+y}\right)^{l+1} - \left(\frac{y}{M^2}\right)^{l+1}\right] \frac{1}{(\epsilon + y)(y^2 + 4\mu^2 y)^{\frac{1}{2}}} \right\}. \tag{5.3}$$

The first integral becomes a hypergeometric function which for small $\epsilon$ and $l$ near $-\frac{1}{2}$ becomes

$$\frac{1}{l+\frac{1}{2}}\,\frac{1}{2\mu M}\left[\left(\frac{4\mu^2}{M^2}\right)^{l+\frac{1}{2}}-\left(\frac{\epsilon}{M^2}\right)^{l+\frac{1}{2}}\right].$$

In the second integral we encounter no difficulty in taking $\epsilon \to 0$ and setting $l = -\frac{1}{2}$. To leading order in $4\mu^2/M^2$ it is $-2/M^2$. The final result is then

$$D(l,t)\underset{\substack{\epsilon\to 0\\ l\to -\frac{1}{2}}}{\longrightarrow} 1-\frac{g^2}{16\pi^2}\,2C_1\left\{\frac{1}{l+\frac{1}{2}}\,\frac{1}{2\mu M}\left[\left(\frac{4\mu^2}{M^2}\right)^{l+\frac{1}{2}}-\left(\frac{\epsilon}{M^2}\right)^{l+\frac{1}{2}}\right]-\frac{2}{M^2}\right\}. \tag{5.5}$$

The equation for the trajectories near threshold and $l = -\frac{1}{2}$ becomes

$$\frac{4\mu}{M}\left(1+\frac{4\pi^2M^2}{g^2C_1}\right)=\frac{\left[1-\left(\frac{\epsilon}{4\mu^2}\right)^{l+\frac{1}{2}}\right]}{l+\frac{1}{2}}. \tag{5.6}$$

A similar equation has been derived for potential models by Desai and Newton [6]. The solutions they found with $l = l_{\mathrm{r}} + il_{\mathrm{i}}$ give an accumulation of an infinite number of trajectories at $l = -\frac{1}{2}$ as $t \to 4\mu^2$:

(A) $t-4\mu^2 \to 0^-$,

$$l_{\mathrm{r}} \simeq -\tfrac{1}{2}-\frac{2n^2\pi^2A^2}{\left[\ln\left(\dfrac{4\mu^2}{4\mu^2-t}\right)\right]^3}\,;\qquad l_{\mathrm{i}}\simeq\frac{2n\pi}{\ln\left(\dfrac{4\mu^2}{4\mu^2-t}\right)}\,; \tag{5.7}$$

(B) $t-4\mu^2 \to 0^+$,

$$l_{\mathrm{r}} = -\tfrac{1}{2}+\frac{2n\pi^2}{\left[\ln\left(\dfrac{4\mu^2}{t-4\mu^2}\right)\right]^2}\,;\qquad l_{\mathrm{i}}\simeq\frac{2n\pi}{\ln\left(\dfrac{4\mu^2}{t-4\mu^2}\right)}\,; \tag{5.8}$$

where $n = \pm 1, \pm 2, \pm 3, \ldots$, and

$$A=\frac{4\mu}{M}\left(1+\frac{4\pi^2M^2}{g^2C_1}\right). \tag{5.9}$$

## 6. BEHAVIOR AT $t \to \pm\infty$

In potential models, the leading trajectory approaches $l = -1$ as $t \to \pm\infty$. We will show that this is also true in the factorizable kernel approximation to the ABFST equation. In addition, we will investigate the asymptotic behavior of the non-leading trajectories.

(A) $t \to -\infty$

We investigate the behavior for $t \to -\infty$ from the representation (3.12) which shows that $D(l, t)$ has a pole at $l = -1$. To get the residue of the pole we set $l = -1$ and use $F(1, \frac{1}{2}, 1; x) = (1-x)^{-\frac{1}{2}}$:

$$D(l, t) \xrightarrow[l \to -1]{} 1 - \frac{g^2}{16\pi^2} \frac{1}{l+1} \int_0^\infty \mathrm{d}y \frac{1}{(\mu^2 - \frac{1}{4}t + y)\,[(\mu^2 - \frac{1}{4}t + y)^2 + ty]^{\frac{1}{2}}}, \tag{6.1}$$

$$D(l, t) \xrightarrow[l \to -1]{} 1 - \frac{g^2}{16\pi^2} \frac{1}{l+1} \frac{2}{(-t)} \left(1 - \frac{4\mu^2}{t}\right)^{-\frac{1}{2}} \ln\left(\frac{(1 - 4\mu^2/t)^{\frac{1}{2}} + 1}{(1 - 4\mu^2/t)^{\frac{1}{2}} - 1}\right). \tag{6.2}$$

As $t \to -\infty$ this gives a trajectory approaching $-1$ from above;

$$l \simeq -1 + \frac{g^2}{8\pi^2(-t)} \ln(-t/\mu^2). \tag{6.3}$$

For the non-leading trajectories we must use a representation of $D(l, t)$ obtained from (3.12) which is valid for $\mathrm{Re}\, l < -1$;

$$D(l, t) = 1 - \frac{g^2}{16\pi^2} \int_0^\infty \mathrm{d}y \left\{ \left[ \frac{1}{l+\frac{1}{2}} \left(\frac{y}{M^2 + y}\right)^{l+1} \frac{1}{(\mu^2 - \frac{1}{4}t + y)^2} \right.\right.$$
$$\left. \times F\left(1, \tfrac{1}{2}, -l + \tfrac{1}{2}; 1 + \frac{ty}{(\mu^2 - \frac{1}{4}t + y)^2}\right)\right] + \frac{\Gamma(l+1)\Gamma(-l-\frac{1}{2})}{\Gamma(\frac{1}{2})}$$
$$\left. \times \frac{1}{(M^2 + y)^{l+1}} \left[ \frac{[(\mu^2 - \frac{1}{4}t + y)^2 + ty]^{l+\frac{1}{2}}}{(-t)^{l+1}} \frac{1}{(\mu^2 - \frac{1}{4}t + y)} \right] \right\}. \tag{6.4}$$

The contribution of the integral of the first bracket is found to be $\propto t^{-1}$ as $t \to -\infty$ and will be neglected. In the second bracket, there is a great enhancement of the factor $[(y + \frac{1}{4}t + \mu^2)^2 - t\mu^2]^{l+\frac{1}{2}}$ for $\mathrm{Re}\, l < -1$ near $y = -\frac{1}{4}t - \mu^2$. In fact, for $y \ll -\frac{1}{4}t$ or $y \gg -\frac{1}{4}t$ the integral of the second bracket is found to be $\propto 1/t$ and will be neglected. Thus to get the leading asymptotic behavior we concentrate on the integral of the second bracket in a region about $y = -\frac{1}{4}t$ as $t \to -\infty$: $-\frac{1}{4}t - \delta(-t) <$

$y < -\frac{1}{4}t + \delta(-t)$. By taking $\delta \ll \frac{1}{4}$ and $t \to -\infty$ we can approximate the integral of the second bracket to be

$$I = \int_0^\infty \mathrm{d}y \frac{1}{(-\frac{1}{4}t)^{l+1}} \frac{1}{(-t)^{l+1}} \frac{1}{2(-\frac{1}{4}t)} \int_{-\frac{1}{4}t+\delta t-\mu^2}^{-\frac{1}{4}t-\delta t-\mu^2} \mathrm{d}y [(y + \tfrac{1}{4}t + \mu^2)^2 - t\mu^2]^{l+\frac{1}{2}} .$$

Changing the variable to $z = (y + \frac{1}{4}t + \mu^2)/(-t\mu^2)^{\frac{1}{2}}$ gives

$$I = \frac{1}{2\mu^2} \left(-\frac{t}{4\mu^2}\right)^{-l-2} 2 \int_0^{\delta(-t/\mu^2)^{\frac{1}{2}}} \mathrm{d}z [z^2+1]^{l+\frac{1}{2}} .$$

By holding $\delta$ fixed and letting $t \to -\infty$ this becomes

$$I = \frac{1}{2\mu^2} (-t/4\mu^2)^{-l-2} \frac{\Gamma(\frac{1}{2})\Gamma(-l-1)}{\Gamma(-l-\frac{1}{2})} ,$$

which is independent of $\delta$. Finally we have

$$D(l,t) \xrightarrow[t\to-\infty]{} 1 - \frac{g^2}{32\pi^2\mu^2} \frac{1}{l+1} \frac{\pi}{\sin \pi l} \left(-\frac{t}{4\mu^2}\right)^{-l-2} . \tag{6.5}$$

This is almost the same equation as derived by Gatto and Menotti [13] for the $t \to -\infty$ limit of the Wick-Cutkosky model of the Bethe-Salpeter equation. They showed that this implied an accumulation of Regge poles at $l = -2$ as $t \to -\infty$, and no poles in any bounded region of $\mathrm{Re}\, l < -2$ for $(-t)$ sufficiently large. This is also true of our result (6.5). To analyze the accumulation of poles at $l = -2$ let

$$l = -2 + \epsilon , \qquad \lambda = \frac{g^2}{32\pi^2\mu^2} , \qquad \tau = -t/4\mu^2 \to \infty .$$

and take $|\pi\epsilon| \ll 1$. Eq. (6.5) gives for the trajectories

$$\ln \tau = \frac{i\pi}{\epsilon} + \frac{1}{\epsilon} \ln \left(\frac{\lambda}{\epsilon}\right) . \tag{6.6}$$

Letting $\epsilon = \rho e^{i\theta}$ with $\rho$ positive, eq. (6.6) has real and imaginary parts

$$\ln \tau = \frac{1}{\rho} \ln \left(\frac{\lambda}{\rho}\right) \cos\theta \left[1 + \frac{(\pi-\theta)\tan\theta}{\ln\left(\frac{\lambda}{\rho}\right)}\right] , \tag{6.7}$$

$$\frac{\tan\theta}{\pi-\theta}=\frac{1}{\ln\left(\frac{\lambda}{\rho}\right)} \tag{6.8}$$

From (6.7) we see that $\tau \to +\infty$ can be obtained by $\rho \to 0$. Then as $\ln(\lambda/\rho) \to \infty$, eq. (6.8) has an infinity of solutions for $\theta$ near an integer multiple of $\pi$; $\theta_m = m\pi + \delta$ and for $|\delta| \ll 1$

$$\theta_m = m\pi - \frac{\pi(m-1)}{\ln\left(\frac{\lambda}{\rho}\right)} \qquad m = 0, \pm 1, \pm 2, \pm 3, \pm 4, \dots .$$

However, eq. (6.7) then becomes

$$\ln\tau = \frac{1}{\rho}\ln\left(\frac{\lambda}{\rho}\right)\cos\theta_m[1+\delta^2]\ , \tag{6.9}$$

and this has solutions only for $\cos\theta > 0$, or $m$ even;

$$\theta_n = 2n\pi - \frac{\pi(2n-1)}{\ln\left(\frac{\lambda}{\rho}\right)}\ . \tag{6.10}$$

This gives an accumulation of trajectories at $l = -2$ which approach from the side $\mathrm{Re}\,\alpha_n > -2$;

$$\alpha_n = -2 + \rho e^{i\theta_n}\ , \tag{6.11}$$

and $\rho$ is given by

$$\ln\left(\frac{-t}{4\mu^2}\right) = \frac{1}{\rho}\ln\frac{g^2}{32\pi^2\mu^2\rho}\ . \tag{6.12}$$

(B) $t \to +\infty$

For this limit we start from the representation (3.14) for $t + i\epsilon$ and scale $y$ to $x = 4y/t$. For $t \to \infty$ we drop terms of order $4M^2/t$ and $4\mu^2/t$ to obtain

$$D(l, t + i\epsilon) \xrightarrow[t\to+\infty]{} 1 - \frac{g^2}{4\pi^2(l+1)t} \int_0^\infty \mathrm{d}x \frac{F\left(\frac{1}{2}, l+1; l+2; \frac{4x}{(x+1)^2}\right)}{(x-1-i\epsilon)(x+1)},$$

$$= 1 - \frac{g^2}{4\pi^2(l+1)t}\left[\tfrac{1}{2}i\pi F(\tfrac{1}{2}, l+1; l+2; 1) + \mathrm{P}\int_0^\infty \frac{\mathrm{d}x}{(x-1)(x+1)}\right.$$

$$\left. \times F\left(\tfrac{1}{2}, l+1; l+2; \frac{4x}{(x+1)^2}\right)\right] . \qquad (6.13)$$

The principal value integral may be shown to vanish by the following steps. The hypergeometric function may be considered as a function of

$$1 - \frac{4x}{(1+x)^2} = \left(\frac{1-x}{1+x}\right)^2 .$$

For the integral from 0 to $1-\epsilon$ we substitute $z = (1-x)/(1+x)$, and for the integral from $1+\epsilon$ to $\infty$ we substitute $z' = (x-1)/(1+x)$. These integrals then will be seen to cancel as $\epsilon \to 0$.

Then we have to leading order in $1/t$;

$$D(l, t + i\epsilon) \to \frac{-ig^2}{8\pi t} \frac{\Gamma(\frac{1}{2})\Gamma(l+1)}{\Gamma(l+\frac{3}{2})} . \qquad (6.14)$$

The equation $D(l, t) = 0$ can only be solved for $t \to +\infty$ for a point near a pole of $D(l, t)$. These occur at $l = -n$, $n = 1, 2, 3, ...$ and give the trajectories the asymptotic behavior.

$$\alpha_n(t) \xrightarrow[t\to+\infty]{} -n + \frac{ig^2}{8\pi^2 t} \frac{\Gamma(\frac{1}{2})\Gamma(n-\frac{1}{2})}{\Gamma(n)} . \qquad (6.15)$$

The trajectories approach to the negative integers from the positive imaginary direction as in potential models.

## 7. CONCLUSIONS

In this paper we have analytically solved the forward and non-forward ABFST equations in the simple factorizable kernel approximation. This approximation is reasonable as it retains the sources of $l$-plane singularities, leaves the $\mu^2$ and $t$ dependence of the pion propagators intact, and is continuous at $t = 0$. The approximation gives an explicit denominator function $D(l, t)$ from which the Regge trajectories of real and complex poles have been studied. We have calculated a reasonable slope for the pomeron trajectory at $t = 0$ and shown it to depend logarithmically on the pion mass $\mu^2$, in contrast to the total cross section which is independent of $\mu^2$ for small $\mu^2$.

Although the ABFST equation is derived and solved in the $s$-channel physical region with $t \leqslant 0$, the explicit equation for the trajectories, $D(\alpha(t), t) = 0$, can be analytically continued to $t > 0$. We have examined the region near threshold and found an accumulation of poles at $l = -\frac{1}{2}$ which have the same behavior as in potential theory. Another similarity with potential theory is that the Regge poles approach to negative integers as $t \to +\infty$.

It is encouraging that so many reasonable properties follow from such a simple approximation to the ABFST equation. This approximation may be of use as an input in more thoroughly unitarized bootstrap models where the exchange of the pomeron is included in the kernel. In addition, similar factorizable approximations may be useful in more complex kernels which have higher $t$-channel thresholds and could lead to trajectories which rise to higher $l$ values for $t$ positive.

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