Title
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Journal
Transactions of the American Mathematical Society, 93(2)

ISSN
0002-9947

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Publication Date
1959

DOI
10.1090/s0002-9947-1959-0119214-4

Peer reviewed

IMMERSIONS OF MANIFOLDS(1)

BY

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INTRODUCTION

Let M and N be differentiable manifolds of dimensions k and n respectively, $k<n$. A differentiable map f: $M \rightarrow N$ is called an *immersion* if f is of class C^1 and the Jacobian matrix of f has rank k at each point of M. Such a map is also called ***regular***. Until recently, very little was known about the existence and classification of immersions of one manifold in another. The present work addresses itself to this problem and reduces it to the problem of constructing and classifying cross-sections of fibre bundles.

In 1944, Whitney [15] proved that every k-dimensional manifold can be immersed in Euclidean space of $2k-1$ dimensions, E^{2k-1}. The Whitney-Graustein theorem [13] classifies immersions of the circle S^1 in the plane E^2 up to ***regular homotopy***, which is a homotopy f_t with the property that for each t, f_t is an immersion, and the induced homotopy f_{t*} of the tangent bundle of M into the tangent bundle of N is continuous. In his thesis [8], Smale generalizes the Whitney-Graustein theorem to the case of immersions of S^1 in an arbitrary manifold. In [9] Smale classifies immersions of S^k in E^n for arbitrary $k<n$; the present work is based on this paper, in roughly the same way that obstruction theory is based on the theory of homotopy groups.

The paper is divided into two parts. The first part is devoted to building up machinery that will construct immersions over successive skeletons of a triangulated differentiable manifold, if certain conditions are satisfied. We start from the results of [9] and construct two invariants, Ω and τ. Given an immersion f: $S^{k-1} \rightarrow E^n$, $k<n$, and a field f' of vectors transversal to $f(S^{k-1})$, $\tau(f, f')$ is an element of a certain homotopy group with the following properties: (1) $\tau(f, f')=0$ if and only if f can be extended to an immersion g of the k-disk D^k whose normal derivative on the boundary of D^k is f'; (2) $\tau(f, f') = \tau(g, g')$ if (f, f') and (g, g') are "regularly homotopic" (in a sense to be defined later). Given two immersions f, g: $D^k \rightarrow E^n$ that agree on S^{k-1} and have the same first derivatives at points of S^{k-1}, $\Omega(f, g)$ is an element of a certain homotopy group, and has the following properties: (1) $\Omega(f, g)=0$ if and only if f and g are regularly homotopic "rel S^{k-1}," i.e., the homotopy agrees with f and g on S^{k-1} at each stage, up to the first derivative; (2) $\Omega(f, g)$ enjoys the usual algebraic properties of a difference cochain. At this point we should like to be able to make the following statement: "If f is an immersion of the

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Received by the editors September 29, 1958.

(1) The material in this paper is essentially a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the University of Chicago, 1958.

i-skeleton of M and $\tau(f)=0$, then f is extendible to the $i+1$-skeleton of M". Unfortunately, this makes no sense at all, because a skeleton of a manifold is not a manifold, and immersions are only defined for manifolds. A second difficulty is that τ is defined for immersions of boundaries of *disks*, while a triangulated manifold is composed of *simplices*. These difficulties are overcome by the notion of M-immersions. If B is a subset of M, an M-immersion of B is essentially an immersion of a neighborhood of B; we identify two M-immersions of B if their differentials agree at points of B. M-regular homotopies are similarly defined. If $B=M$, an M-immersion of B is simply an immersion of M. Ω and τ have natural generalizations to M-immersions and M-regular homotopies, in fact, we define τ only for M-immersions; M-immersions of skeletons make good sense, and Ω and τ can be defined for simplices as well as disks by "approximating" a simplex by a diffeomorphically embedded disk. The quoted statement above now makes sense, and is true, if *immersion* is replaced by *M-immersion*. An M-immersion has a well-defined differential, which is a map of the tangent bundle of M over B into the tangent bundle of A, taking fibers into fibers through vector space monomorphisms. The differential induces a map of the bundle of k-frames of M over B into the bundle of k-frames of A, and this map is equivariant with respect to the action of $GL(k)$. It turns out that an M-immersion of the i-skeleton of M can be extended to the $i+1$ skeleton if and only if the differential can be extended to an equivariant map of the bundle of k-frames of M over the $i+1$-skeleton. A similar statement holds for M-regular homotopies. These statements are made precise in Theorems 5.7 and 5.9, which are the main theorems of Part I. These theorems can be given the following interpretation: Given M of dimension k, A of dimension n, $k<n$, the regular homotopy classes of immersions of M in A are in one-one correspondence with homotopy classes of cross-sections of the bundle associated to the bundle of k-frames of M, whose fiber is the bundle of k-frames of A. (If $A=E^n$, the fiber may be taken to be the Stiefel manifold $V_{n,k}$.) For another interpretation, see the introduction to §5.

It should be remarked that the restriction $k<n$ is essential; the problem of immersing a manifold in another of the same dimension is much harder than the one we consider.

Part II consists of applications of the main theorems of Part I, chiefly to the problem of the existence of immersions in Euclidean spaces. "Best possible" results are obtained for projective spaces of dimensions less than 9; it is shown that a parallelizable manifold can be immersed in Euclidean space of one dimension higher with normal degree 0; certain sufficient conditions that two immersions be regularly homotopic are given. The problem of immersing a k-dimensional manifold in E^{2k-2} is solved in the compact case when $k\leqq 5$.

I wish to express my gratitude to my adviser, E. H. Spanier, for his generous help, and to thank S. Smale for many conversations, and for writing [9].

DEFINITIONS AND NOTATION. By *manifold* we shall always mean a C^∞ differentiable manifold, endowed with a fixed Riemannian metric d, with or without boundary. *Submanifold* means "C^∞-imbedded" submanifold in the sense of Whitney [16]. If M is a submanifold of N and X is a vector tangent to N at a point of M, X is called *transversal to* M if X is not tangent to M. $T(M)$ denotes the space of tangent vectors of M. A *k-frame* of M is a set of k linearly independent tangent vectors of M (with the same base point). $T_k(M)$ is the space of all k-frames of M. $\tilde{T}(M)$ is the space of unit tangent vectors of M. If B is any subset of M, $T(M/B)$ is the space of tangent vectors of M whose origin is in B; $T_k(M/B)$ and $\tilde{T}(M/B)$ are defined similarly. If $X \in T(M)$, $|X|$ is the length of X in the Riemannian metric of M.

E^n is n-dimensional Euclidean space, with an orthonormal coordinate system $x_1, \cdots, x_n$. If $m<n$, E^m is considered to be the subset of E^n defined by setting $x_i=0$, $i=m+1, \cdots, n$. The *k-disk* D^k is defined as $\{x \in E^k: |x| \leq 1\}$, where for any $x \in E^n$, $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$. The *boundary of* D^k is the unit sphere S^{k-1} of E^k, and is sometimes denoted by $\dot{D}^k$. The common origin of all the E^k is 0. The Stiefel manifold of p-frames in E^q is denoted by $V_{q,p}$ and is defined as $T_p(E^q/0)$. We shall frequently identify $V_{q,p}$ with $T_q(E^p)$, of which it is a deformation retract by parallel translation.

Immersion means a C^1 map of one manifold into another of not lower dimension whose Jacobian matrix has highest possible rank at each point. A *regular homotopy of* M *in* N is a family f_t of immersions of M in A, $t \in I$ $=$ unit interval, such that f_t is a homotopy in the ordinary sense, and differentials f_{t*} define a (continuous) homotopy of $T(M)$ into $T(N)$. If B is a closed subset of M, a map $f: B \to N$ is said to be *differentiable of class* C^k if for each $x \in B$ there is a neighborhood U of x and a C^k map $g: U \to N$ such that $g | U \cap B$ $= f | U \cap B$. Milnor [5] shows that this implies that f can be extended to a C^k map of a neighborhood of B.

If $f: M \to N$ is differentiable, then $f_*: T(M) \to T(N)$ denotes the differential of f. If f_* is a vector space monomorphism on each fiber of $T(M)$ (e.g., if f is an immersion) then f_* preserves linear independence, and there is an induced map $T_k(M) \to T_k(N)$ given by $\{Y_1, \cdots, Y_k\} \to \{f_* Y_1, \cdots, f_* Y_k\}$. We shall use f_* to denote this map, also.

If B is a subset of M, and $f, g: M \to N$ are immersions, then (f, f_*) *and* (g, g_*) *are tangent on* B if $f|B = g|B$ and $f_*|T(M/B) = g_*|T(M/B)$. This is denoted by $(f, f_*)|B = (g, g_*)|B$. f and g are *regularly homotopic* (rel B) if there is a regular homotopy f_t of M in N such that $f_0 = f$, $f_1 = g$, and $(f_t, f_{t*})|B$ $= (f, f_*)|B$ for all t.

PART I. GENERAL THEORY

1. Known results. Let M and N be manifolds, and $\mathfrak{X}$ a set of C^1 maps from M into N. The C^1 *topology* on $\mathfrak{X}$ is that which is obtained by considering $\mathfrak{X}$ as embedded in the space of all continuous maps $T(M) \to T(N)$, in the compact open topology, under the embedding $f \to f_*$.

DEFINITION. A map $f: Q \to R$ has the *covering homotopy property* if the following condition is satisfied: given a homotopy g_t of a polyhedron P into R and a map $h: P \to Q$ such that $fh = g_0$, there exists a homotopy h_t of P into Q such that $h_0 = h$, and $fh_t = g_t$.

Let $\mathcal{E} = \mathcal{E}_{k,n}$ be the space of all C^∞ immersions of D^k in E^n in the C^1 topology. Let $\mathcal{B} = \mathcal{B}_{k,n}$ be the set of all pairs (g, g') where $g: S^{k-1} \to E^n$ is a C^∞ immersion and $g': S^{k-1} \to T(E^n)$ is a C^∞ transversal field of g. $\mathcal{B}$ is topologized as a subspace of Cartesian product of the space of immersions $S^{k-1} \to E^n$, in the C^1 topology, with the space of continuous maps $S^{k-1} \to T(E^n)$, in the compact-open topology.

Let $\pi: \mathcal{E} \to \mathcal{B}$ as follows: if $h \in \mathcal{E}$, let $h': S^{k-1} \to T(E^n)$ be defined by $h'(x)$ $=$ derivative of h along the radius at $x \in S^{k-1}$, i.e., if $r(x)$ is the unit tangent vector of D that is normal to S^{k-1} at x and which points away from the origin, then $h'(x) = h_* r(x)$. $\pi(h)$ is defined to be $(h | S^{k-1}, h')$. It is clear that π is continuous.

THEOREM 1.1. *If $k < n$, then $\pi: \mathcal{E}_{k,n} \to \mathcal{B}_{k,n}$, has the covering homotopy property.*

The proof is found in [9].

The intuitive content of this theorem is as follows: If we are given an immersed disk in E^n and we deform the boundary of the disk and the normal derivatives along the boundary, then we can deform the whole disk at the same time so as to induce the given deformation on the boundary and normal derivatives. It is easily seen that this is false for $k = n$, as can be shown by very simple examples with $k = 1$ or 2.

Let $\mathcal{E}' = \mathcal{E}'_{k,n}$ be the space of all maps $D^k \to V_{n,k}$, in the compact-open topology. Let $\mathcal{B}' = \mathcal{B}'_{k,n}$ be the space of all maps $S^{k-1} \to V_{n,k}$, in the compact-open topology. Let $\pi': \mathcal{E}' \to \mathcal{B}'$ be the restriction map $\pi'(f) = f | S^{k-1}$. It is well known that π' has the covering homotopy property.

Let $e_i(x) \in T(E^k/x)$ be the ith coordinate vector based at x. For $f \in \mathcal{E}$, define $\Phi(f) \in \mathcal{E}'$ by $\Phi(f)(x) = f_*\{e_1(x), \cdots, e_k(x)\}$. Define $\tilde{\Phi}: \mathcal{B} \to \mathcal{B}'$ as follows: express $e_i(x)$ uniquely as $e_i'(x) + \lambda_i r(x)$, where $r(x)$ is the outward normal to S^{k-1} at $x \in S^{k-1}$ and $e_i'(x)$ is tangent to S^{k-1}. If $(g, g') \in \mathcal{B}$, then $\tilde{\Phi}(g, g')(x) = \{g_* e_i'(x) + \lambda_i g'(x), \cdots, g_* e_k'(x) + \lambda_k g'(x)\}$. These vectors are independent because $g_* e_i'(x)$ is tangent to $g(S^{k-1})$ and $g'(x)$ is transversal. The following diagram is commutative:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\Phi} & \mathcal{E}' \\ \downarrow \pi & & \downarrow \pi' \\ \mathcal{B} & \xrightarrow{\tilde{\Phi}} & \mathcal{B}' \end{array}$$

To see this, observe that if $g \in \mathcal{E}$ and $\pi(g) = (f, f')$, then $f = g | S^{k-1}$ and f' is by definition the derivative of g along $r(x)$, i.e., $f'(x)=g_*r(x)$. If X is any vector at $x\in S^{k-1}$, tangent to D, X can be expressed uniquely as $X'+\lambda r(x)$ with X' tangent to S^{k-1}. Then $g_*X=g_*X'+\lambda g_*r(x)=f_*X'+\lambda f'(x)$. Applying this to $X=e_i(x)$, $(\pi'\Phi)(g)(x)=g_*\{e_i(x),\cdots,e_k(x)\}=\{f_*e_i'(x)+\lambda_i f'(x),\cdots,f_*e_k'(x)+\lambda_k f'(x)\}=(\tilde{\Phi}\pi)(g)(x)$, for $\pi(g)(x)=g(x)$ if $x\in S^{k-1}$. Thus $\tilde{\Phi}\pi=\pi'\Phi$.

If $g\in\mathcal{E}_{k,n}$, put $\Gamma_{k,n}(g)=\{f\in\mathcal{E}_{k,n}:(f,\ f_*)\,|\,S^{k-1}=(g,\ g_*)\,|\,S^{k-1}\}$. This is equivalent to $\Gamma_{k,n}(g)=\pi^{-1}(\pi(g))$. Define

$$\Gamma'_{k,n}(g)=\{h:D^k\to V_{n,k}:h\,|\,S^{k-1}=\Phi(g)\,|\,S^{k-1}\}.$$

This is the same as $\Gamma'_{k,n}(g)=\pi'^{-1}(\pi'\Phi(g))$. Since the above diagram commutes, $\Phi(\Gamma_{k,n}(g))\subset\Gamma'_{k,n}(g)$.

Definition. If A and B are topological spaces, a map $f: A\to B$ is a *weak homotopy equivalence* if f induces a one-one correspondence between the arc-components of A and those of B, and $f:\pi_i(A)\to\pi_i(B)$ is an isomorphism for all $i=1, 2,\cdots$.

Theorem 1.2. *Let $i: D^k\to E^n$ be the inclusion map. If $k<n$, $\Phi:\Gamma_{k,n}(i)\to\Gamma'_{k,n}(i)$ is a weak homotopy equivalence.*

Proof. See [9].

In the next section we shall prove this for any $f\in\mathcal{E}_{k,n}$.

The following material is well known; see Steenrod's book [10] for details.

Let A be a topological space, simple in dimension k. Let $f, g: D^k\to A$, and assume $f(x)=g(x)$ if $x\in S^{k-1}$. There is an element $d(f, g)$ in $\pi_k(A)$ with the following properties:

Lemma 1.3.

(a) $d(f, g)=0$ *if and only if f and g are homotopic* (rel S^{k-1}).

(b) *If $h: D^k\to A$ is such that $h|S^{k-1}=f|S^{k-1}=g|S^{k-1}$, then $d(f, g)+d(g, h)=d(f, h)$.*

(c) $d(f, f)=0$.

(d) *Given f and $\alpha\in\pi_k(A)$, there exists $g: D^k\to A$ such that $d(f, g)=\alpha$.*

$d(f, g)$ is represented by mapping the "top" hemisphere of S^k by f and the "bottom" one by g, assuming that the orientation of S^k is given by the coordinate frame $\{e_1,\cdots,e_k\}$ at the "north" pole of S^k.

Now let $f, g\in\mathcal{E}_{k,n}$ with $g\in\Gamma(f)$, i.e., $\pi(g)=\pi(f)$. Then $\Phi(f)$ and $\Phi(g)$ are maps of D^k into $V_{n,k}$ that are tangent on S^{k-1}, and $V_{n,k}$ is simple in all dimensions. (Either $k=n$ or $n-1$, in which case $V_{n,k}$ is a group, or else $\pi_1(V_{n,k})=0$.) Therefore $d(\Phi(f), \Phi(g))$ is defined.

Definition. $d(\Phi(f), \Phi(g))\in\pi_k(V_{n,k})$ is called the *obstruction to a regular homotopy* (rel S^{k-1}) *between f and g* and is denoted by $\Omega(f, g)$.

Theorem 1.4. *Assume $k<n$, $f, g\in\mathcal{E}_{k,n}$, and $f, g\in\Gamma(i)$.*

(a) $\Omega(f, g)=0$ *if and only if f and g are regularly homotopic* (rel S^{k-1}).

(b) *If* $f, g, h\in\Gamma(i)$, $\Omega(f, g)+\Omega(g, h)=\Omega(f, h)$.

(c) $\Omega(f, f)=0$.

(d) *Given* $f\in\Gamma(i)$ *and* $\alpha\in\pi_k(V_{n,k})$, *there exists* $g\in\Gamma(i)$ *such that* $\Omega(f, g)=\alpha$.

(e) *If* $\Omega(f, g)=0$ *and* $H: D^k\times I\to V_{n,k}$ *is a homotopy* (rel S^{k-1}) *between* $\Phi(f)$ *and* $\Phi(g)$, *there exists a regular homotopy* (rel S^{k-1}) G_t *between* f *and* g *such that the map* $F: D^k\times I\to V_{n,k}$ *defined by* $F(x, t)=\Phi(G_t)(x)$ *is homotopic to* H (rel $S^{k-1}\times I\cup D^k\times\dot{I}$). ($\dot{I}=\{0, 1\}$.)

Proof. Follows from 1.2 and 1.3; for example to prove (a), assume $\Omega(f, g)=0$. By definition, this means that $d(\Phi(f), \Phi(g))$ vanishes. By 1.3, $\Phi(f)$ and $\Phi(g)$ are homotopic (rel S^{k-1}). This means there is an arc in $\Gamma'_{k,n}(f)$ joining $\Phi(f)$ to $\Phi(g)$. By 1.2, there must be an arc in $\Gamma_{k,n}(f)$ joining f to g. This arc provides us with a regular homotopy (rel S^{k-1}) between f and g. Conversely, if f and g are regularly homotopic (rel S^{k-1}) by a regular homotopy f_t, then $\Phi(f_t)$ is a homotopy (rel S^{k-1}) between $\Phi(f)$ and $\Phi(g)$, so $d(\Phi(f), \Phi(g))=0$ by 1.3a. The other statements of 1.4 follow similarly.

Let $f, g: S^k\to E^n$ be C^∞ immersions. Let U be an open set of S^k on which f and g agree, and let $p\in U$. Assume that S^k-U is diffeomorphic to D^k. Let $f_1=f|S^k-U$ and $g_1=g|S^k-U$, and consider f and g as C^∞ immersions of D^k in E^n. Since f and g agree on U, it follows that $\pi(f_1)=\pi(g_1)$, so that $\Omega(f_1, g_1)$ is defined. Put $\Omega'(f, g)=\Omega(f_1, g_1)\in\pi_k(V_{n,k})$.

THEOREM 1.5. *If* $\Omega'(f, g)=0$, *f and g are regularly homotopic* (rel p).

Proof. See [9] for details. The idea is to use 1.4a to obtain a regular homotopy (rel S^{k-1}) between f_1 and g_1, which is equivalent to a regular homotopy (rel boundary of U) between $f|S^k-U$ and $g|S^k-U$. It will be shown in 2.5 that this regular homotopy can be combined with the constant regular homotopy between $f|U$ and $g|U$ to produce the desired regular homotopy (rel a neighborhood of p, hence rel p) between f and g.

2. **Extension of 1.2 and 1.4.** The object of this section is to remove the restriction in 1.4 and 1.2 that f and g belong to $\Gamma(i)$, and to prove 2.5, which is used several times throughout Part I.

LEMMA 2.1. *Let* $f, g: M\to E^n$ *be* C^k *immersions that satisfy the following condition: for any* $X\in\tilde{T}(M)$, $|f_*X-g_*X|<|f_*X|$. *Then for any* $t\in I$, *the map* $h: M\to E$ *defined by* $h(x)=tf(x)+(1-t)g(x)$ *is a* C^k *immersion.*

Proof. By hypothesis, $|f_*X-g_*X|<|f_*X|$ if $X\neq 0$. This implies that g_*X cannot vanish, nor can g_*X be a negative multiple of f_*X, because in either of these cases, $|f_*X-g_*X|\geqq|f_*X|$. Therefore either f_*X and g_*X are independent, or they are positive multiples of each other; in either case, no positive combination of them can vanish. Therefore for every nonzero $X\in T(M)$, $h_*X\neq 0$, which is equivalent to saying that h is an immersion. It is obvious that h is C^k.

LEMMA 2.2. *Let Q and R be topological spaces, with R compact. Let A be a subset of Q; let U be an open subset of $Q\times R$ containing $A\times R$. Then there exists an open set V in Q containing A such that $A\times R\subset V\times R\subset U$.*

Proof. U can be written $U=\bigcup_i V_i\times W_i$, where i runs over an indexing set Λ and V_i is open in Q, W_i is open in R. For each $a\in A$, $a\times R$ is compact; it follows that there is a finite subset Λ_a of Λ such that $a\times R\subset\bigcup_{i\in\Lambda_a} V_i\times W_i$ and $a\in V_i$ if $i\in\Lambda_a$. Since Λ_a is finite, $V_a=\bigcap_{i\in\Lambda_a} V_i$ is an open set of Q containing a, and $a\times R\subset\bigcup_{i\in\Lambda_a} V_a\times W$. Define $V=\bigcup_a V_a$. It is easily checked that V has the required properties.

LEMMA 2.3. *Let M be a manifold and C a compact space. Let $p\in M$. Let F, G be continuous maps of C into the space of immersions of M in E^n, with the C^1 topology, such that for every $c\in C$, $(G(c), G(c)_*)\,|\,p=(F(c), F(c)_*)\,|\,p$.*

CONCLUSION. *There is a neighborhood W of p and positive real numbers α, β such that for any $X\in\tilde{T}(M/W)$, $c\in C$, $t\in I$, it is true that $\alpha<|tF(c)_*X+(1-t)G(c)_*X|<\beta$.*

Proof. Define $\eta: C\times T(M)\times I\to E^1$ by $\eta(c,x,t)=|tF(c)_*X+(1-t)G(c)_*X|$. η is continuous and assumes a minimum α_0 and a maximum β_0 on the compact set $C\times\tilde{T}(M/p)\times I$. Moreover, $\alpha_0>0$, for if $X\in\tilde{T}(M/p)$, $F(c)_*X=G(c)_*X$, hence $\eta(c, x, t)=|F(c)_*X|>0$, because $F(c)$ is regular. Next choose α and β such that $0<\alpha<\alpha_0<\beta_0<\beta$; then $\eta^{-1}(\alpha, \beta)$ is an open set U in $C\times\tilde{T}(M)\times I$ that contains $C\times\tilde{T}(M/p)\times I$. By 2.2 there is an open set V in $\tilde{T}(M)$ such that $C\times\tilde{T}(M/p)\times I\subset C\times V\times I\subset U$, and $\tilde{T}(M/p)\subset V$. Since $\tilde{T}(M)$ is locally a product, we may apply 2.2 again and obtain an open set W of M containing p such that $\tilde{T}(M/W)\subset V$. Then W, α, β, have the required properties.

LEMMA 2.4. (*The hypothesis and notation are as in* 2.3.) *Given $\epsilon>0$, there is a neighborhood U of p such that if $x\in U-p$ and $c\in C$,*

$$|F(c)(x)-G(c)(x)|/d(x,p)<\epsilon.$$

Proof. Choose a fixed $c\in C$. Then $F(c)(p)=G(c)(p)$, and we have the equality:

$$|F(c)(x)-G(c)(x)|/d(x,p) = |[F(c)(x)-F(c)(p)]-[G(c)(x)-G(c)(p)]|/d(x,p).$$

By the mean value theorem, there are points y, z on the geodesic between x and p (which we may assume to be unique if x is close to p) such that the expression on the right is equal to $|F(c)_*Y-G(c)_*Z|$ where Y and Z are unit tangents to the geodesic at y and z respectively. Let X be the unit tangent at p. Choose a convex neighborhood $V=V(c)$ of p such that if y and z lie on a geodesic in V through p, and X, Y, and Z are respectively the unit tangents to this geodesic at p, y, and z, then $|F(c)_*Y-G(c)_*Z|$ lies within ϵ of $|F(c)_*X-G(c)_*X|$. This last is 0, since by hypothesis $F(c)_*$ and $G(c)_*$ agree on vectors based at p. Thus for this particular c, $|F(c)(x)-G(c)(x)|/d(x,p)$

$<\epsilon$. Since F and G are continuous, this inequality holds for all c' in a neighborhood $W(c)$ of c. Since C is compact, there are points $c_1, \cdots, c_s$ such that $C \subset \bigcup_i W(c_i)$. It follows that $U = \bigcap_i V(c_i)$ satisfies the conditions of the lemma.

The next lemma is needed in several places. The following example may help to motivate it: Suppose we are given two immersions f and g, defined on a manifold M, such that for a point p of M, $(f, f_*)|p = (g, g_*)|p$. Then the lemma says that we can deform g so that it becomes equal to f in some neighborhood of p, and the deformation is constant outside an arbitrarily small neighborhood of p. Intuitively, we pinch the images of f and g together near p. Actually, the lemma allows us to do this for two regular homotopies that agree at p at each stage, and p can be replaced by a subcomplex of M.

LEMMA 2.5. *Let L be a finite complex embedded in the manifold M in such a way that each simplex of L is diffeomorphically embedded. Let W be a neighborhood of L in M. Let C be a compact space and F, G maps of C into the space of immersions of M in E^n (C^1 topology) such that for each $c \in C$, $(F(c), F(c)_*)|L = (G(c), G(c)_*)|L$.*

CONCLUSION. *There is an open set V of M such that $L \subset V \subset \overline{V} \subset W$, and a homotopy G_t of C into the space of immersions of M in E^n, satisfying the following conditions:*

(1) $G_0 = G$.
(2) $(G_t(c), G_t(c)_*)|M - W = (G(c), G(c)_*)|M - W$.
(3) $(G_1(c), G_1(c)_*)|V = (F(c), F(c)_*)|V$.
(4) *$G_t(c)$ is C^k if $G(c)$ and $F(c)$ are C^k.*
(5) $(G_t(c), G_t(c)_*)|x = (G(c), G(c)_*)|x$ *if* $(G(c), G(c)_*)|x = (F(c), F(c)_*)|x$.

Proof. Let L_i be the i-skeleton of L. We shall deform $G(c)$ over successive skeletons of L.

Let $\lambda: I \to I$ be a C^∞ function such that $\lambda = 0$ in a neighborhood of 0, $\lambda = 1$ in a neighborhood of 1, and whose derivative is never negative. Let $K > 0$ be an upper bound for the derivative.

Let p be a vertex of L. We shall define a $\delta > 0$ so that the closed ball U of radius δ and center p is contained in W. Let $\mu: M \to I$ be defined by $\mu(x) = \lambda(d(x, p)/\delta)$ if $x \in U$; $\mu(x) = 1$ if $x \in M - U$. Because of the properties of λ, μ is C^∞. Define $h: M \to E^n$ by $h(x) = (1 - \mu(x))f(x) + \mu(x)g(x)$, where for a fixed $c \in C$, $f(x) = F(c)(x)$, $g(x) = G(c)(x)$. By the proper choice of δ, it will turn out that for each $t \in I$, the map $G_t(c) = (1-t)g + th$ is a C^∞ immersion of M in E^n satisfying (1), (2), (4), and (5); (3) holds if V is interpreted to be some neighborhood of the vertex p. δ is chosen as follows: Choose a neighborhood U_1 of p and real numbers α, β so that for any $t \in I$, $X \in \tilde{T}(M/U_1)$, $c \in C$,

$$\text{(a)} \qquad 0 < \alpha < \left| tF(c)_*X + (1-t)G(c)_*X \right| < \beta.$$

This can be done by 2.3. Next choose a neighborhood U_2 of p so that for any $c \in C$ and $x \in U_2 - p$,

(b) $$|F(c)(x) - G(c)(x)|/d(x, p) < \alpha/2K.$$

This is possible by 2.4. Now choose a neighborhood U_3 of p such that for any $X\in\tilde{T}(M/U_3)$, $c\in C$,

(c) $$|F(c)_*X - G(c)_*X| < \alpha/2.$$

This can be done because for $X\in\tilde{T}(M/p)$, $F(c)_*X=G(c)_*X$ for all c. Now choose $\delta>0$ so small that the closed ball U of center p and radius δ is contained in $U_1\cap U_2\cap U_3$, and also U is contained in some coordinate neighborhood of p. Assume also that no vertex of L other than p is contained in U.

Now let $c\in C$; put $f=F(c)$ and $g=G(c)$. Define $h: M\to E^n$ by $h(x) =(1-\mu(x))f(x)+\mu(x)g(x)$. It is clear that h is C^∞ and depends continuously on c. We shall show now that for each $t\in I$, the map $g_t=(1-t)g+th$ is regular. By 2.1, it is enough to show that $|h_*X-g_*X|<|g_*X|$ for each $X\in\tilde{T}(M)$; however, because $\mu=1$ in a neighborhood of $M-U$, it follows that $h_*=g_*$ on $T(M/M-U)$, and therefore we need prove only that $|h_*X-g_*X|<|g_*X|$ for $X\in\tilde{T}(M/U)$. By (a), with $t=0$, $\alpha<|g_*X|$; thus it suffices to show that $|h_*X - g_*X| < \alpha$ for $X \in \tilde{T}(M/U)$. A direct computation yields $h_*X = \mu_*X(g(x) - f(x)) + \mu(x)g_*X + (1 - \mu(x))f_*X$. Therefore $|h_*X - g_*X| \leqq |\mu_*X(g(x)-f(x))|+|(1-\mu(x))(f_*X-g_*X)|$. By (c) the second term on the right is less than $\alpha/2$. If we show that $|\mu_*X|<K/d(x, p)$, then the first term is less than $K|g(x)-f(x)|/d(x, p)$ which is less than $K\alpha/2K$ by (b), and we are done. It is enough to show that $|\mu_*X|<K/\delta$. X has a unique expression $aY + bZ$, where Y is a unit vector normal to the submanifold $A=\{u\in U: d(u, p)=d(x, p)\}$ (X based at $x\in U$), and Z is a unit vector tangent to A. Observe that $|a|\leqq 1$ because $a^2+b^2=1$. Since μ is constant on A, $\mu_*X=a\mu_*Y$. Y is tangent to the geodesic though x normal to A; it follows from the definitions of μ and K that $|\mu_*Y|<K/\delta$. This proves that g_t is regular.

We repeat the above construction for every vertex of L, and we observe that we have proved the lemma for the special case dimension of $L=0$ by putting $G_t(c)=g_t$, and taking for V the union of the interiors of the U's chosen for the various vertices p. We proceed by induction on dimension of L.

Assume the lemma is true in case L has dimension i. Let L have dimension $i+1$, and let L_i be the i-skeleton of L. Given W, C, F, G as in the hypothesis, we apply the inductive hypothesis to L_i to conclude that there is a neighborhood U of L and a deformation G_t of G such that $L_i\subset U\subset\overline{U}\subset W$, (1), (2), (4), and (5) hold, and also

(3′) $$(G_1(c), G_1(c)_*)|U = (F(c), F(c)_*)|U.$$

Let σ be an $i+1$ simplex of L. We can find real numbers α, β, δ, greater than 0 such that if $d(x, \sigma)<\delta$ and $X\in\tilde{T}(M/x)$,

(a′) $0<\alpha<|tF(c)_*X+(1-t)G_1(c)_*X|<\beta$ for any $t\in I$, $c\in C$;

(b′) $|F(c)(x)-G_1(c)(x)|/d(x, \sigma)<\alpha/2K$ if $x\notin\sigma$, $c\in C$;

(c′) $|F(c)_*X - G_1(c)_*X| < \alpha/2$ for any $c \in C$.

This can be done by 2.3, 2.4, the compactness of σ, and the fact that $(F(c), F(c)_*)|\sigma = (G_1(c), G_1(c)_*)|\sigma$ by (5).

Let $B(\epsilon) = \{x \in M: d(x, \sigma) = \epsilon$, and x lies on a geodesic normal to $\sigma\}$. We can choose δ small enough so that if $0 < \epsilon \leqq \delta$, $B(\epsilon)$ is a C^∞ submanifold of M contained in W. Let B be the union of the $B(\epsilon)$ for $0 \leqq \epsilon \leqq \delta$. Let B' be the subset of B consisting of points of B that lie on geodesics normal to σ and passing through $\dot{\sigma}$. Make δ still smaller so that $B' \subset U$.

Define $\mu: B \to I$ by $\mu(x) = \lambda(d(x, \sigma)/\delta)$; it is clear that μ is C^∞ on the interior of B. Let $c \in C$ and put $f = F(c)$, $g = G_1(c)$, and define $h: B \to E^n$ by $h(x) = (1 - \mu(x))f(x) + \mu(x)g(x)$. It is easy to see that h is C^∞ on the interior of B. It turns out that $h = g$ on a neighborhood of the boundary of B: the boundary of B is $B' \cup B(\delta)$; $B' \subset U$, and $f(x) = g(x)$ in U by the assumption (3′), while $\mu = 1$ in a neighborhood of $B(\delta)$. It follows that we may extend h to all of M by setting $h(x) = g(x)$ if $x \in M - B$, and h: $M \to E^n$ is now C^∞.

We shall show that for each $t \in I$, the map G_{1t}: $M \to E^n$ defined by $G_{1t} = (1-t)g + th$ is regular. To do this, it suffices by 2.1 and (a′) to show that for any $X \in \tilde{T}(M/B)$, $|h_*X - g_*X| < \alpha$. The proof of this follows from (a′), (b′), and (c′) in the same way as in the case $\sigma = p$, done earlier.

We do this for each $i+1$ simplex of L in succession; (5) insures that we never undo our work, i.e., once $G_1(c)$ has been deformed to agree with $F(c)$ locally, any remaining deformations preserve this property. Let V be the union of U with the interiors of the B's constructed for each $i+1$ simplex σ of L. Then V, and the deformation obtained by first deforming G by G_t and then deforming G_1 by G_{1t}, satisfy (1) through (5). This completes the induction, and 2.5 is proved.

LEMMA 2.6. *The space $\mathcal{E}_{k,n}$ of C^∞ immersions of D^k in E^n is arcwise connected, provided $k < n$.*

Proof. Let f and g be any two elements of $\mathcal{E}_{k,n}$. We shall prove that both f and g can be joined to the inclusion i: $D^k \subset E^n$ by arcs in $\mathcal{E}_{k,n}$. It suffices to do this for g.

Since $k < n$, there is an arc T_t in the group of proper affine transformations of E^n such that T_0 is the identity and such that $(T_1g, (T_1g)_*)|0 = (i, i_*)|0$, where 0 is the center of D^k. Since T_tg is an arc in $\mathcal{E}_{k,n}$ from g to T_1g, we may assume that g itself has the property $(g, g_*)|0 = (i, i_*)|0$. By 2.5 (with $L = 0$, $C = c$, $F(c) = i$, $G(c) = g$) we may assume that there is an $r > 0$ such that if $|x| \leqq r$, $g(x) = i(x) = x$. For $0 \leqq t \leqq 1/2$ define g_t by $g_t(x) = g((1 - 2t + 2tr)x)$. Then $g_0 = g$ and $g_{1/2}(x) = g(rx) = i(rx) = rx$. For $1/2 \leqq t \leqq 1$ define $g_t(x)$ by $g_t(x) = (2t - 1 + 2r - 2tr)x$. Then under this definition $g_{1/2}(x) = rx$ and $g_1(x) = x$. Thus g_t, for $t \in I$, is an arc in $\mathcal{E}_{k,n}$ from g to i. This completes the proof.

If $k = n$, a similar argument shows there are two arc components, the elements of which respectively preserve and reverse orientation. We shall not use this result.

THEOREM 2.7. *The map* $\Phi\colon \Gamma_{k,n}(f)\to\Gamma'_{k,n}(\Phi(f))$ (*defined in* §1) *is a weak homotopy equivalence for every* $f\in\mathcal{E}_{k,n}$ *provided* $k<n$.

Proof. By 2.6, $\mathcal{E}_{k,n}$ is arcwise connected. Let a path in $\mathcal{E}_{k,n}$ running from f to i be chosen, and let $C\colon I\to\mathcal{B}_{k,n}$ be its image under π, so that $C(0)=\pi(i)$, $C(1)=\pi(f)$. Put $C'=\tilde{\Phi}C\colon I\to\mathcal{B}'_{k,n}$. C and C' induce isomorphisms $C_\#$ and $C'_\#$ such that the following diagram is commutative for all $j=0, 1, \cdots$, where π_0 is the set of arc-components:

$$\begin{array}{ccc} \pi_j(\Gamma(i), i) & \xrightarrow{(\Phi|\Gamma(i))_\#} & \pi_j(\Gamma'(i), \tilde{\Phi}(i)) \\ \downarrow C_\# & & \downarrow C'_\# \\ \pi_j(\Gamma(f), f) & \xrightarrow{(\Phi|\Gamma(f))_\#} & \pi_j(\Gamma'(f), \tilde{\Phi}(f)) \end{array}$$

By 1.2, $(\Phi|\Gamma(i))$ is an isomorphism. Since $C_\#$ and $C'_\#$ are also isomorphisms, $(\Phi|\Gamma(f))$ is an isomorphism.

THEOREM 2.8. *Let* f, $g\in\mathcal{E}_{k,n}$, $k<n$; *assume* $\pi(f)=\pi(g)$ *so that* $\Omega(f, g)$ *is defined. Then*

(a) $\Omega(f, g)=0$ *if and only if* f *and* g *are regularly homotopic* (rel S^{k-1}).

(b) $\Omega(f, g)+\Omega(g, h)=\Omega(f, h)$ *if* $h\in\pi(g)$.

(c) $\Omega(f, f)=0$.

(d) *Given* $\alpha\in\pi_k(V_{n,k})$, *there exists* $g\in\Gamma(f)$ *such that* $\Omega(f, g)=\alpha$.

(e) *If* $\Omega(f, g)=0$ *and* $H\colon D\times I\to V_{n,k}$ *is a homotopy* (rel S^{k-1}) *between* $\Phi(f)$ *and* $\Phi(g)$, *there is a regular homotopy* (rel S^{k-1}) $G\colon D^k\times I\to E^n$ *such that* $G_0=f$, $G_1=g$, *and the map* $F\colon D^k\times I\to V_{n,k}$ *defined by* $F(x, t)=\Phi(G_t)(x)$ *is homotopic to* H (rel $S^{k-1}\times I\cup D^k\times \dot{I}$).

Proof. Follows from 2.9 and 1.3 (compare 1.4).

3. M**-immersions.** Let A be an arbitrary subset of the manifold M. Let $f\colon A\to N$ and $f'\colon T(M/A)\to T(N)$ (N is a manifold) be continuous maps such that the following diagram commutes:

$$\begin{array}{ccc} T(M/A) & \xrightarrow{f'} & T(N) \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & N \end{array}$$

DEFINITION. The pair (f, f') is called an *M-regular map*, or *M-immersion*, *of* A *in* N if the following condition is satisfied: *there is a neighborhood* U *of* A *in* M *and an immersion* $g\colon U\to N$ *such that* $g_*|T(M/A)=f'$. It follows that $g|A=f$. g is called an *admissible extension of* f. Observe that f is completely determined by f'; nevertheless we shall use the redundant notation (f, f') for M-immersions. We say that (f, f') *is* C^k if g can be chosen to be C^k. We shall use the notation: $(f, f')\colon A\to N$ *is an M-immersion.* The following properties of M-immersions $(f, f')\colon A\to N$ are immediate:

(1) f' is a vector space monomorphism on each fiber of $T(M/A)$.

(2) If A is a submanifold of M, f is an immersion of A.

(3) If (g, g') is an N-immersion of B in V, and $f(A)\subset B$, then $(gf, g'f')$ is an M-immersion of A in V.

(4) If U is an open subset of M containing A, then an M-immersion of A is a U-immersion of A, and conversely.

(5) If $B\subset A$, then $(f|B, f'|T(M/B))$ is an M-immersion of B.

(6) *If $A=M$, f is an immersion of M and $f'=f_*$.*

(7) If V is a submanifold of M containing A, then $(f, f'|T(V/A))$ is a V-immersion of A.

(8) If W is an open set of N such that $f(A)\subset W$, then (f, f') is an M-immersion of A in W; conversely, any M-immersion of A in W is also an M-immersion of A in N.

DEFINITION. Let $B\subset A\subset M$ be subsets. If (f, f'), $(g, g'): A\to N$ are M-immersions such that $f|B=g|B$ and $f'|T(M/B)=g'|T(M/B)$, we say that (f, f') *and* (g, g') *are tangent on* B, and write this as $(f, f')|B=(g, g')|B$.

By (1) above, f' induces a map $T_k(M/A)\to T_k(N)$, given by $\{X_1, \cdots, X_k\}\to\{f'X_1, \cdots, f'X_k\}$. *We shall use f' to denote this map also.*

We topologize the set of M-immersions of A in N as a subspace of the space of continuous maps (compact-open topology) of $T(M/A)\to T(N)$, using the embedding $(f, f')\to f'$. It follows that the various compositions, restrictions, and inclusions discussed in (1) through (8) above are continuous maps.

Next we prove a lemma that gives certain sufficient conditions that (f, f') be an M-immersion.

LEMMA 3.1. *Let A be a submanifold of M; assume $\partial M=\partial A=$ empty set. Let $f: A\to N$ be a C^k immersion. Let $f': T(M/A)\to T(N)$ be a C^k map covering f with the following properties: f' is a vector space monomorphism on each fiber of $T(M/A)$, and $f'|T(A)=f_*$. Then (f, f') is a C^k M-immersion of A.*

Proof. Let U be a tubular neighborhood of A, and $\theta: U\to T(M/A)$ the identification of U with a neighborhood of the zero cross-section of the normal bundle of A in M. The map $g: U\to N$ given by

$$U \xrightarrow{\theta} T(M/A) \xrightarrow{f'} T(N) \to N,$$

where the last map is the exponential, is C^k, and $g_*|T(M/A)=f'$. Since f' preserves linear independence by the hypothesis, the Jacobian matrix of g has maximal rank at points of A; by continuity, it has maximal rank in a neighborhood of A. This means g is an admissible extension of f, and so (f, f') is an M-immersion. It is obvious that (f, f') is C^k.

DEFINITION. Such a g is called a *canonical extension* of f.

Let f, f', and A be as in 3.1. Suppose further that A has a family ϕ of normal frames of highest dimension, i.e., for each $x\in A$, $\phi(x)$ is an element of $T_r(M/A)$ which spans the fiber of the normal bundle to A, where $r = \dim M - \dim A$. Then $f'\phi(x)$ is a family of transversal r-frames on $f(A)$. Conversely, given a family of transversal r-frames $\psi(x)$, f' is completely determined by the conditions $f'X = f_*X$ if $X \in T(A)$, $f'\phi(x) = \psi(x)$. Keeping ϕ fixed, it is clear that $(f, f') \to (f, \psi)$ is a one-one correspondence between M-immersions (f, f') and pairs (g, ψ) where g is an immersion of A and ψ is a transversal r-field on $g(A)$; for given f and ψ, f' constructed as above satisfies the hypothesis of 3.1, and therefore (f, f') is an M-immersion. Specializing to the case $M = E^{k+r}$, $A = D^k$, $\phi(x) = \{e_{k+1}(x), \cdots, e_{k+r}(x)\}$ (where as usual $e_i(x)$ is the ith coordinate vector at x), we have:

THEOREM 3.2. *There is a homeomorphism between the space of E^{k+r}-immersions $(f, f'): D^k \to E^n$ and the space of pairs (g, ψ), $g: D^k \to E^n$ an immersion and ψ a transversal r-field. (This space is a subspace of (space of immersions, C^1 topology) $\times$ (space of maps $D^k \to T_r(E^n)$, compact-open topology).) The homeomorphism is given by $(f, f') \to (f, \psi)$ where $\psi(x) = f'\{e_{k+1}(x), \cdots, e_{k+r}(x)\}$. Moreover, (f, f') is C^k if and only if f and ψ are C^k.*

Proof. It suffices to check the continuity of the map and its inverse; this follows immediately from the definitions of the topologies involved.

An analogous result holds for S^{k-1}:

THEOREM 3.3. *There is a homeomorphism between the space of E^{k+r}-immersions $(f, f'): S^{k-1} \to E^n$ and the space of pairs (g, ψ), where $g: S^{k-1} \to E^n$ is an immersion and ψ is a transversal $(r+1)$-field. The homeomorphism is given as follows: Let ϕ be the normal $(r+1)$-field on S^{k-1} given by $\phi(x) = \{r(x), e_{k+1}(x), \cdots, e_{k+r}(x)\}$, where $r(x)$ is the outward unit normal to S^{k-1} in E^k; the homeomorphism is defined by $(f, f') \to (f, \psi)$ where $\psi(x) = f'\phi(x)$.*

These theorems will enable us to generalize the results of the earlier sections to E^{k+r}-immersions of D^k and S^{k-1}, by combining them with simple properties of transversal fields.

DEFINITION. Let (f, f') and (g, g') be M-immersions of A in N such that for a certain (possibly empty) subset B of A, (f, f') and (g, g') are tangent on B. We say that (f, f') and (g, g') are *M-regularly homotopic* (rel B) if there is a path (h_t, h_t') in the space of all M-immersions of A in N joining (f, f') to (g, g'), such that for each t, $(h_t, h_t') | B = (f, f') | B$. Such a path is called an *M-regular homotopy* (rel B), and it is C^k if every (h_t, h_t') is C^k.

DEFINITION. The space of all C^∞ E^q-immersions of D^k in E^n is denoted by $\mathcal{J}(k, n; q)$; the space of all C^∞ E^q-immersions of S^{k-1} in E^n is denoted by $\mathcal{J}'(k, n; q)$.

It is clear that $\mathcal{J}(k, n; k)$ is the same as the space $\mathcal{E}_{k,n}$ defined in §1; $\mathcal{J}'(k, n; k)$ is identified with $\mathcal{B}_{k,n}$ by means of 3.3.

In order to prove the next theorem we need the following lemma; it must be well known, but I do not know a reference. Let M be a manifold.

LEMMA 3.4. *Let $\mathfrak{X}$ be the space of C^∞ maps $D^k \to M$ and $\mathfrak{X}'$ the space of C^∞ maps $S^{k-1} \to M$, both spaces having the compact-open topology. Let $\pi: \mathfrak{X} \to \mathfrak{X}'$ be the restriction map. Conclusion: π has the covering homotopy property.*

Proof. Let P be a polyhedron and $G: P \times I \to \mathfrak{X}'$, $f: P \times 0 \to \mathfrak{X}$, such that $G(p, 0) = \pi f(p, 0)$. We must extend f to $F: P \times I \to \mathfrak{X}$ so that $\pi F = G$. By a compactness argument, I can be subdivided into subintervals I_j so small that if $t, t' \in I_j$, $p \in P$, $x \in S^{k-1}$, then $G(p, t)(x)$ and $G(p, t')(x)$ lie in the same convex open set of M. It suffices to define F successively on $P \times I_1$, $P \times I_2$, etc. Therefore we assume that I has the property that for any $t, t' \in I$, $p \in P$, $x \in S^{k-1}$, $G(p, t)(x)$ and $G(p, t')(x)$ lie in the same convex open set of M.

Let $u: I \times I - \{(1, 0)\} \to I$ be a function with the following properties:
$u(x, y)$ is C^∞ in x when y is held fixed;
$u(x, y) = 1$ if $x = 1$;
$u(x, y) = 0$ if $y = 0$ or $0 \leq x \leq 1/2$.

If a, b are in a convex neighborhood of M, $t \in I$, let $[a, b; t]$ be the point c on the geodesic joining a to b in the convex neighborhood such that $d(a, c)/d(a, b) = 1 - t$, $d(c, b)/d(a, b) = t$. If $t \in I$, $p \in P$, $x \in D^k$, define

$$F(p, t)(x) = \begin{cases} [f(p, 0)(x), G(p, t)(x/|x|); u(|x|, t)] & \text{if } x \neq 0, \\ f(p, 0)(x) & \text{if } x = 0. \end{cases}$$

This defines a map $F: P \times I \to \mathfrak{X}$ with the required properties.

Now let $\pi_q: \mathcal{J}(k, n; q) \to \mathcal{J}'(k, n; q)$ be defined by

$$\pi_q(f, f') = (f | S^{k-1}, f') | T(E^q/S^{k-1}).$$

If $q = k$, this is the map $\pi: \mathcal{E}_{k,n} \to \mathcal{B}_{k,n}$ defined in §1.

THEOREM 3.5. *π_q has the covering homotopy property if $k < n$.*

Proof. The case $q = k$ is covered by 1.1; assume $q > k$.

Let $\zeta: \mathcal{J}'(k, n; q) \to \mathcal{J}'(k, n; k)$ be defined by $\zeta(f, f') = (f, f') | T(E^k/S^{k-1})$. Let $\xi: \mathcal{J}'(k, n; k) \to \mathcal{B}_{k,n}$ be the homeomorphism of 3.3; explicitly, $\xi(f, f') = (f, \psi)$ with $\psi(x) = f' r(x)$. Define $\gamma: \mathcal{J}(k, n; q) \to \mathcal{E}_{k,n}$ by $\gamma(f, f') = f$. It is easily seen that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{J}(k, n; q) & \xrightarrow{\gamma} & \mathcal{E}_{k,n} \\ \downarrow \pi_q & & \downarrow \pi \\ \mathcal{J}'(k, n; q) & \xrightarrow{\xi\zeta} & \mathcal{B}_{k,n} \end{array}$$

Let P be a polyhedron, which we may assume to be a product of intervals, and suppose we are given maps $h: P \to \mathcal{J}$, $H: P \times I \to \mathcal{J}'$ such that $H(x, 0) = \pi_q h(x)$. We are to find $G: P \times I \to \mathcal{J}$ such that $\pi_q G = H$ and $G(x, 0) = h(x)$.

By 1.1, there exists $G': P \times I \to \mathcal{E}_{k,n}$ such that $\pi G' = \xi\zeta H$, and $G'(x, 0) = \gamma h(x)$. Let $A \to D^k \times P \times I$ be the bundle whose fiber over the point (d, p, t) is the space of $q - k$ frames transversal to $G'(p, t)(D^k)$ at the point $G'(p, t)(d)$.

Via 3.2 and 3.3, we interpret h and H as defining a cross-section of A over $D^k \times P \times 0 \cup S^{k-1} \times P \times I$, and it suffices to prove that this cross-section can be extended over $D^k \times P \times I$ so that it is C^∞ when restricted to $D^k \times p \times t$. Since $D^k \times P \times I$ admits $D^k \times P \times 0 \cup S^{k-1} \times P \times I$ as a deformation retract, the cross-section can be extended. Since P is a product of intervals, A is a trivial bundle, and a cross-section is merely a map into the fiber, which is a manifold. 3.4 implies that the cross-section can be chosen with the required differentiability properties.

DEFINITION. For $(f, f') \in \mathcal{J}(k, n; q)$ put $\Gamma_q(f, f') = \pi_q^{-1}(\pi_q(f, f'))$.

LEMMA 3.6. *Let* $\gamma: \mathcal{J}(k, n; q) \to \mathcal{E}_{k,n}$ *be the map described above. Then* $\gamma | \Gamma_q(f, f'): \Gamma_q(f, f') \to \Gamma(f)$ *has the covering homotopy property.*

Proof. The proof is similar in principle to that of 3.5 and the details are left to the reader. The idea is that by 3.2, $\Gamma_q(f, f')$ is the space of those immersions and transversal fields that agree with (f, f') on S^{k-1}, and γ projects this space onto the space of immersions that agree with f on S^{k-1}. 3.6 says that if the immersion is varied continuously, the transversal field can be varied along with it.

Let Φ_q: $\mathcal{J}(k, n; q) \to$(space of continuous maps $D^k \to V_{n,q}$), the latter space with the compact open topology, as follows: $\Phi_q(f, f')(x) = f'\{e_1(x), \cdots, e_q(x)\}$. If $q = k$, Φ_q is the map Φ defined in §1. Define

$$\Gamma_q'(f, f') = \{g;\ D^k \to V_{n,k};\ g \mid S^{k-1} = \Phi_q(f, f')\}$$

with the compact-open topology; then $\Phi_q(\Gamma_q(f, f')) \subset \Gamma_q'(f, f')$.

THEOREM 3.7. $\Phi_q: \Gamma_q(f, f') \to \Gamma_q'(f, f')$ *is a weak homotopy equivalence for every* (f, f') *in* $\mathcal{J}(k, n; q)$, *if* $k < n$.

Proof. Let $p: V_{n,q} \to V_{n,k}$ be the bundle projection $p\{X_1, \cdots, X_q\} = \{X_1, \cdots, X_k\}$. Let $\gamma': \Gamma_q'(f, f') \to \Gamma'(f)$ be defined by $\gamma'(g) = pg: D^k \to V_{n,k}$. p, and hence γ', has the covering homotopy property. It is easily seen that the following diagram commutes:

$$\begin{array}{ccc} \Gamma_q(f,f') & \xrightarrow{\Phi_q} & \Gamma_q'(f,f') \\ \downarrow \gamma & & \downarrow \gamma' \\ \Gamma(f) & \xrightarrow{\Phi} & \Gamma'(f) \end{array}$$

By 2.10, Φ is a weak homotopy equivalence. By 3.6 γ has the covering homotopy property. Let $Y = \gamma^{-1}(\gamma(f, f'))$ and $Y' = \gamma'^{-1}(\gamma'\Phi_q(f, f'))$. If we can show that $\Phi_q | Y: Y \to Y'$ is a weak homotopy equivalence we are done, for then Φ_q must be a weak homotopy equivalence, by applying the five lemma to the following commutative diagram:

$$\begin{array}{ccccccccc} \pi_{i+1}(\Gamma(f)) & \to & \pi_i(Y) & \to & \pi_i(\Gamma_q(f,f')) & \to & \pi_i(\Gamma(f)) & \to & \pi_{i-1}(Y) \\ \downarrow (\Phi|\Gamma)_\# & & \downarrow (\Phi_q|Y)_\# & & \downarrow \Phi_{q\#} & & \downarrow (\Phi|\Gamma)_\# & & \downarrow (\Phi_q|Y)_\# \\ \pi_{i+1}(\Gamma'(f)) & \to & \pi_i(Y') & \to & \pi_i(\Gamma_q'(f,f')) & \to & \pi_i(\Gamma'(f)) & \to & \pi_{i-1}'(Y') \end{array}$$

By 3.2 Y is the space of transversal $q-k$ fields of the immersion f, and Y' is the space of continuous maps $g: D^k \to V_{n,q}$ such that $pg = \Phi\gamma(f, f')$. Given a $q-k$ field ξ transversal to f we obtain, by Φ_q, a map $g: D^k \to V_{n,q}$ such that $\gamma'(g) = \Phi\gamma(f, f')$. Conversely, given $g: D^k \to V_{n,q}$, $g(x) = \{g_1(x), \cdots, g_q(x)\}$ such that $pg(x) = f_*\{e_1(x), \cdots, e_k(x)\}$, it follows that $\{g_{k+1}(x), \cdots, g_q(x)\}$ must be transversal to $f(D^k)$ at $f(x)$, and thus g defines a transversal $q-k$ field $\xi(g)$. Since $\Phi_q\xi$ and $\xi\Phi_q$ are identity maps on Y and Y' respectively, $\Phi_q | Y$ is a homeomorphism, and we are done.

Let (f, f'), $(g, g') \in \mathcal{G}(k, n; q)$, with $\pi_q(f, f') = \pi_q(g, g')$, so that $(f, f') \in \Gamma_q(g, g')$. Then $\Phi_q(f, f')$ and $\Phi_q(g, g')$ are maps $D^k \to V_{n,q}$ which are tangent on S^{k-1}.

Definition. $\Omega(f', g') = d(\Phi_q(f, f'), \Phi_q(g, g')) \in \pi_k(V_{n,q})$ is called the *obstruction to an E^q-regular homotopy* (rel S^{k-1}) between (f, f') and (g, g').

Theorem 3.8. *$\Omega(f', g')$ has the following properties, if $k < n$:*

(a) *$\Omega(f', g') = 0$ if and only if (f, f') and (g, g') are C^∞ E^q-regularly homotopic* (rel S^{k-1}),

(b) *Given $(f, f') \in \mathcal{G}(k, n; q)$ and $\alpha \in \pi_k(V_{n,q})$ there exists $(g, g') \in \Gamma_q(f, f')$ such that $\Omega(f', g') = \alpha$,*

(c) *If (g, g') and (h, h') are in $\Gamma_q(f, f')$, then $\Omega(f', g') + \Omega(g', h') = \Omega(f', h')$,*

(d) $\Omega(f', f') = 0$,

(e) *If $\Omega(f', g') = 0$ and $H: D^k \times I \to V_{n,q}$ is a homotopy* (rel S^{k-1}) *between $\Phi_q(f, f')$ and $\Phi_q(g, g')$, there is a C^∞ E^q-regular homotopy (f_t, f_t')* (rel S^{k-1}) *between (f, f') and (g, g') such that the map $D^k \times I \to V_{n,q}$ given by $(x, t) \to \Phi_q(f_t, f_t')(x)$ is homotopic to H* (rel $D^k \times \dot{I} \cup S^{k-1} \times I$).

Proof. Follows immediately from 3.7 and 1.3.

An explicit definition of $\Omega(f', g')$ is as follows: identify the upper and lower hemispheres of S^k with D^k. Let $\omega: S^k \to V_{n,q}$ be the map $\omega(x) = f'\{e_1(x), \cdots, e_q(x)\}$ if x is in the upper hemisphere, $\omega(x) = g'\{e_1(x), \cdots, e_q(x)\}$ if x is in the lower hemisphere; $\omega(x)$ is well defined on the equator because (f, f') and (g, g') agree on S^{k-1}. Then $\Omega(f', g')$ is the homotopy class of ω.

Next we define an invariant $\tau(g')$ defined for $(g, g') \in \mathcal{G}'(k, n; q)$ whose vanishing implies that (g, g') comes from $\mathcal{G}(k, n; q)$.

$(f, f') \in \mathcal{G}'(k, n; q)$ is said to be *extendible* if there is a $(g, g') \in \mathcal{G}(k, n; q)$ such that $\pi_q(g, g') = (f, f')$. Theorem 3.5 has the following important consequence: *if (h, h') and (f, f') are in $\mathcal{G}'(k, n; q)$ and are C^∞ E^q-regularly homotopic, then (h, h') is extendible if and only if (f, f') is extendible.*

Proof. To say that (f, f') and (h, h') are C^∞ E^q-regularly homotopic is the same as saying there is an arc joining them in $\mathcal{G}'(k, n; q)$. If (f, f') is extendible, then the map of a one-point polyhedron into it can be lifted to $\mathcal{G}(k, n; q)$, and by 3.5, the path connecting (f, f') and (h, h') can be lifted to $\mathcal{G}$ also; therefore (h, h') is in the image of π_q, i.e., is extendible.

DEFINITION. Let $(f, f'): S^k \to E^n$ be a C^∞ E^q-immersion, i.e., $(f, f') \in \mathcal{J}'(k, n; q)$. The *obstruction to extending* (f, f'), denoted by $\tau(f') \in \pi_{k-1}(V_{n,q})$, is the homotopy class of the map $S^{k-1} \to V_{n,q}$ defined by

$$x \to f'\{e_1(x), \cdots, e_q(x)\}.$$

THEOREM 3.9. *If $k < n$ and $\tau(f') = 0$, (f, f') is extendible.*

Proof. As is shown above, it suffices to show that (f, f') is C^∞ E^q-regularly homotopic to an extendible E^q-immersion, (i, i'). We shall do this as follows: first we make (f, f') agree with (i, i') in a neighborhood of a point p. Then we embed a $k-1$ disk diffeomorphically in the complement of p so that its boundary is in the neighborhood. We can consider f and i as E^q immersions of this disk that are tangent on the boundary; it turns out that the assumption $\tau = 0$ implies that these immersions are E^q-regularly homotopic (rel boundary of the disk). This will imply that (f, f') and (i, i') are C^∞ E^q-regularly homotopic. We can take for (i, i') either the identity or the reflection in a hyperplane; the latter is needed only if $q = n$ and f' reverses orientation. There is a proper affine transformation T of E^n such that $(Tf, Tf') | p = (i, i') | p$ where p is a point of S^{k-1}. Applying 2.5, (with $L = p$) we can deform (Tf, T_*f') so that it agrees with (i, i') in a neighborhood of p, and it suffices to show that this new E^q-immersion is extendible. We therefore assume: there is a point $p \in S^{k-1}$ and a neighborhood W of p such that $(f, f') | W = (i, i') | W$.

Let $\zeta: D^{k-1} \to S^{k-1} - p$ be a C^∞ regular homeomorphism such that $\zeta(S^{k-2}) \subset W$, and such that the orientations of the frames $\{e_1(y), \cdots, e_k(y)\}$ and $\{\zeta_* e_1(\zeta^{-1}(y)), \cdots, \zeta_* e_{k-1}(\zeta^{-1}(y)), r(y)\}$ agree for each $y \in \zeta(D^{k-1})$. It follows that the field of frames $\{e_i(y)\}$, $i = 1, \cdots, k$, defined on S^{k-1}, can be deformed in $T_k(E^k)$ to a family $\{e_i'(y)\}$ such that for $x \in D^{k-1}$, $e_i'(\zeta(x)) = \zeta_* e_i'(x)$ for $i = 1, \cdots, k-1$, and $e_k'(\zeta(x)) = r(\zeta(x))$. To see this, let $A(x)$ be the linear transformation, defined for each $x \in D^{k-1}$, that takes the frame $e_i(\zeta(x))$ into $e_i'(\zeta(x))$. Since these frames have the same orientation, $A(x)$ is in the component of the identity of $GL(k)$. Since D^{k-1} is contractible, the map $A: D^{k-1} \to GL(k)$ is homotopic to the constant map taking D^k into the identity. Such a homotopy yields the desired deformation of $\{e_i(y)\}$ for $y \in \zeta(D^{k-1})$; the deformation can be extended over S^{k-1} by the usual homotopy extension theorem. By 3.4 we may assume that the field $\{e_i'(y)\}$ is C^∞. For $i = k+1, \cdots, q$ put $e_i'(y) = e_i(y)$.

Define two C^∞ E^q-immersions (k_0, k_0'), $(k_1, k_1'): D^{k-1} \to E^n$ as follows:

$$k_0 = f\zeta;\ k_0' e_i(x) = \begin{cases} (f\zeta)_* e_i(x), & i = 1, \cdots, k-1, \\ f' e_i'(\zeta(x)), & i = k, \cdots, q, \end{cases}$$

$$k_1 = i\zeta;\ k_1' e_i(x) = \begin{cases} (i\zeta)_* e_i(x), & i = 1, \cdots, k-1, \\ i' e_i'(\zeta(x)), & i = k, \cdots, q. \end{cases}$$

Observe that (k_0, k_0') and (k_1, k_1') agree on S^{k-2} because (f, f') and (i, i')

agree on W. *Assertion*: $\Omega(k_0', k_1')=\tau(f')$. To see this, let $\phi, \phi':\zeta(D^{k-1})\to T_q(E^q/\zeta(D^{k-1}))$ be the cross-sections $\phi(y)=\{e_i(y)\}$, $\phi'(y)=\{e_i'(y)\}$, $i=1, \cdots, q$. By the definition of e_i', ϕ and ϕ' agree $\zeta(S^{k-2})$. Since i' and f' agree on $T(E^q/\zeta(S^{k-2}))$, the four maps $i'\phi$, $i'\phi'$, $f'\phi$, $f'\phi':\zeta(D^{k-1})\to V_{n,q}$ all agree on $\zeta(S^{k-2})$. We shall prove:

(1) $d(f'\phi, i'\phi)=\tau(f')$,

(2) $d(f'\phi', i'\phi')=\Omega(k_0', k_1')$,

(3) $d(f'\phi, i'\phi)=d(f'\phi', i'\phi')$,

where d is the obstruction cochain (see 1.3). This will prove $\tau(f')=\Omega(k_0', k_1')$. (3) is proved by observing that ϕ and ϕ' are homotopic, since $\{e_i'\}$ is a deformation of $\{e_i\}$. (2) follows from the definition of Ω. (1) holds because $f'\phi$ and $i'\phi$ have extensions λ, $\mu: S^{k-1}\to V_{n,q}$ defined by $\lambda(x)=f'\{e_i(x)\}$, $\mu(x)=i'\{e_i(x)\}$; λ and μ agree on W. $[\lambda]-[\mu]$ is the homotopy class of the map obtained by joining two spheres at a point, mapping one by λ, the other by μ, and composing this map with the map sending a third sphere onto the join, the equator going into the common point, and one hemisphere going onto each sphere, with degrees 1 and -1 respectively. Since λ and μ agree on W, this is the same as identifying each hemisphere with $\zeta(D^{k-1})$ and mapping one by $\lambda|\zeta(D^{k-1})$ and the other by $\mu|\zeta(D^{k-1})$, which is exactly the definition of $d(f'\phi, i'\phi)$. Since i' and $\{e_i(x)\}$ can be extended to the interior of D^k, μ is null homotopic. Finally, $[\lambda]=\tau(f')$ by definition. Thus $d(f'\phi, i'\phi)=[\lambda]-[\mu]=\tau(f')$. This proves (1) and so $\tau=\Omega$.

If $\tau(f')=0$, then by 3.8 there is a C^∞ E^q-regular homotopy (rel S^{k-2}), $(k_t, k_t'):D^{k-1}\to E^n$. $k_t\zeta^{-1}$ is a C^∞ regular homotopy of $\zeta(D^{k-1})$ and it is tangent to i on $\zeta(S^{k-2})$ at each stage t. By 2.5 (with $L=\zeta(S^{k-2})$, $C=I$, $F(t)=i$, $G(t)=\zeta^{-1}k_t$) we may deform $\zeta^{-1}k_t$ so that it agrees with i in a neighborhood of $\zeta(S^{k-2})$. We can cover the deformation by one of k_t' (see 3.2) so that (g_t, g_t') defined below is a C^∞ E^q-regular homotopy of S^{k-1} in E^n:

$$g_t(y) = \begin{cases} f(y) & \text{if } y\in S^{k-1}-\zeta(D^{k-1}),\\ k_t\zeta^{-1}(y) & \text{if } y\in\zeta(D^{k-1}),\end{cases}$$

$$g_t'e_i'(y) = \begin{cases} k_t'e_i(\zeta^{-1}(y)) & \text{if } y\in\zeta(D^{k-1}), & i=1,\cdots,q,\\ f'e_i'(y) & \text{if } y\in S^{k-1}-\zeta(D^{k-1}), & i=1,\cdots,q.\end{cases}$$

Then

$$g_0(y) = \begin{cases} f(y) & \text{if } y\in S^{k-1}-\zeta(D^{k-1}),\\ k_0\zeta^{-1}(y)=f\zeta\zeta^{-1}(y)=f(y) & \text{if } y\in\zeta(D^{k-1}).\end{cases}$$

Similarly, $g_1(y)=i(y)$; and

$$g_0'e_i'(y) = \begin{cases} f'e_i'(y) \quad \text{if } y\in S^{k-1}-\zeta(D^{k-1}),\\ k_0'e_i(\zeta^{-1}(y)) = \begin{cases} (f\zeta)_*e_i(\zeta^{-1}(y)) = f_*\zeta_*e_i(\zeta^{-1}(y)) = f'e_i'(y) \\ \qquad\qquad i=1,\cdots,k-1,\\ f'e_i'(\zeta\zeta^{-1}(y)) = f'e_i'(y), \qquad i=k,\cdots,q.\end{cases}\end{cases}$$

Similarly, $g_t' e_i'(y) = i' e_i'(y)$. This shows that (g_t, g_t') is a C^∞ E^q-regular homotopy between (f, f') and (i, i'). This proves 3.9.

4. Immersions of simplices. In this section we extend the definitions and properties of τ and Ω to E^q-immersions of simplices.

Δ^k denotes a fixed k-simplex in E^k whose barycenter is 0; $\dot{\Delta}^k$ is the boundary of Δ^k. θ: $S^{k-1} \to \dot{\Delta}^k$ is the radial projection from 0: if $x \in S^{k-1}$, $\theta(x)$ is the intersection with $\dot{\Delta}^k$ of the half line radiating from 0 through x; θ is a homeomorphism. $\dot{\Delta}^k$ is oriented so that θ preserves orientation.

Let (f, f'): $\dot{\Delta}^k \to E^n$ be a C^∞ E^q-immersion. Let ν: $S^{k-1} \to V_{n,q}$ be defined by $\nu(x) = f'\{e_1(\theta(x)), \cdots, e_q(\theta(x))\}$. We define $\tau(f') \in \pi_{k-1}(V_{n,q})$ to be the homotopy class of ν.

THEOREM 4.1. *(f, f') is extendible to a C^∞ E^q-immersion of Δ^k if $\tau(f') = 0$ and $k < n$.*

Proof. Let h: $U \to E^n$ be an admissible C^∞ extension of f, where U is an E^q-neighborhood of $\dot{\Delta}^k$. Choose $\epsilon > 0$ small enough so that $\{x: d(x, \dot{\Delta}^k) < \epsilon\} \subset U$. Let α: $D^k \to E^k$ be a regular C^∞ orientation preserving homeomorphism with the properties that $\alpha(S^{k-1}) \subset$ (interior of Δ^k), and if $x \in S^{k-1}$, $d(\alpha(x), \theta(x)) < \epsilon$; thus $\alpha(S^{k-1}) \subset U$.

Let κ: $S^{k-1} \to E^n$ be the C^∞ immersion $h(\alpha | S^{k-1})$. Define

$$\kappa': T(E^q | S^{k-1}) \to T(E^n) \text{ by } \kappa' e_i(x) = \begin{cases} h_* \alpha_* e_i(x) & i = 1, \cdots, k, \\ h_* e_i(\alpha(x)) & i = k+1, \cdots, q. \end{cases}$$

(κ, κ') is a C^∞ E^q-immersion of S^{k-1} in E^n by 3.3. We now show that $\tau(\kappa') = \tau(f')$: let α_t: $S^{k-1} \to U$ be defined by $\alpha_t(x) = (1-t)\alpha(x) + t(\theta(x) - \alpha(x))$. Then $\alpha_0 = \alpha | S^{k-1}$ and $\alpha_1 = \theta$. The two maps $S^{k-1} \to V_{k,k}$ defined respectively by $x \to \alpha_* e_i(x)$ and $x \to e_i \alpha(x)$, $i = 1, \cdots, k$, are homotopic, since each is null homotopic, being extendible to D^k. This implies that the map $S^{k-1} \to V_{n,q}$ given by $x \to h_*\{\alpha_* e_1(x), \cdots, \alpha_* e_k(x), e_{k+1}(\alpha(x)), \cdots, e_q(\alpha(x))\}$ whose homotopy class is $\tau(\kappa, \kappa')$, is homotopic to the map

$$x \to h_*\{e_1(\alpha(x)), \cdots, e_q(\alpha(x))\},$$

which in turn is homotopic to the map $x \to h_*\{e_1(\theta(x)), \cdots, e_q(\theta(x))\}$, using α_t. The homotopy class of this last map is $\tau(f')$, because $h_* X = f' X$ if $X \in T(E^q / \dot{\Delta}^k)$. Therefore $\tau(\kappa') = \tau(f')$.

Now assume $\tau(f') = 0$. By 3.9, (κ, κ') is extendible to a C^∞ E^q-immersion of D^k in E^n, which we shall also denote by (κ, κ'). Let (g, g'): $\alpha(D^k) \to E^n$ be the C^∞ E^q-immersion defined by $g(\alpha(x)) = \kappa(x)$, $g' e_i(\alpha(x)) = \kappa' e_i(x)$, if $i = k+1, \cdots, q$, and $g' \alpha_* e_i(x) = \kappa' e_i(x)$, $i = 1, \cdots, k$. Then $(g, g') | \alpha(S^{k-1}) = (h, h_*) | \alpha(S^{k-1})$, for if $x \in S^{k-1}$, we have by definition $g(\alpha(x)) = \kappa(x) = h(\alpha(x))$; if $i = k+1, \cdots, q$, $g' e_i(\alpha(x)) = \kappa' e_i(x) = h_* e_i(\alpha(x))$, while if $i = 1, \cdots, k$, $g' \alpha_* e_i(x) = \kappa' e_i(x) = h_* \alpha_* e_i(x)$. Let g_1 be an admissible extension of g; applying 2.5 we may assume that g_1 and h agree on a neighborhood V of $\alpha(S^{k-1})$ in E^q. Now define (f_0, f_0') as follows: f_0: $\Delta^k \to E^n$ is given by

$$f_0(x) = \begin{cases} h(x) & \text{if } x \in U \cap \Delta^k, \\ g(x) & \text{if } x \in \alpha(D^k). \end{cases}$$

Since g and h agree on V, f_0 is a well defined C^∞ immersion of Δ^k in E^n, and $f_0 | \dot{\Delta}^k = f$. $f_0' : T(E^q/\Delta^k) \to T(E^n)$ is defined as follows: if X is tangent to

$$\Delta^k \text{ at } x,\ f_0'X = \begin{cases} h_* X & \text{if } x \in U \cap \Delta^k, \\ g_{1*} X & \text{if } x \in \alpha(D^k). \end{cases}$$

f_0' is well defined because g_* and h_* agree on $T(E^q/V)$. It follows from 3.1 that (f_0, f_0') is a C^∞ E^q-immersion of Δ^k in E^n; although 3.1 is stated for submanifolds, we can extend f to a neighborhood of $\dot{\Delta}^k$ in E^k, (e.g., $U \cap E^k$) and cover it with an extension of f_0' and apply 3.1, taking the neighborhood as a submanifold of E^q. It is clear that (f_0, f_0') is an extension of (f, f'). This completes the proof.

REMARK. f has an admissible extension that agrees with h in a neighborhood of $\dot{\Delta}^k$. (Compare 5.8.)

Let $\theta' : D^k \to \Delta^k$ be an extension of $\theta : S^{k-1} \to \dot{\Delta}^k$. Let $\rho : S^k \to D^k$ be the vertical projection $\rho(x_1, \cdots, x_{k+1}) = (x_1, \cdots, x_k)$. Put $\rho' = \theta' \rho$, thus ρ' is a map of S^k onto Δ^k.

Let (f, f') and (g, g') be C^∞ E^q-immersion of $\dot{\Delta}^k$ in E^n that are tangent on $\dot{\Delta}^k$. Let $\eta : S^k \to V_{n,q}$ be defined by

$$\eta(x) = \begin{cases} f'\{e_i(\rho'(x))\}, & i = 1, \cdots, q, \text{ if } x_{k+1} \geqq 0, \\ g'\{e_i(\rho'(x))\}, & i = 1, \cdots, q, \text{ if } x_{k+1} \leqq 0. \end{cases}$$

DEFINITION. $\Omega(f', g') \in \pi_k(V_{n,q})$ is the homotopy class of η.

THEOREM 4.2. *If $k < n$ and $\Omega(f', g') = 0$, then (f, f') and (g, g') are C^∞ E^q-regularly homotopic.*

Proof. We leave the details to the reader, the proof being similar in principle to that of 4.1. Let f_1, g_1 be admissible extensions of f and g respectively; f_1 and g_1 are C^∞ immersions of an E^q-neighborhood of $\dot{\Delta}^k$; (f_1, f_{1*}) and (g_1, g_{1*}) are tangent to (f, f') and (g, g') respectively on $\dot{\Delta}^k$. We apply 2.5 and assume that f_1 and g_1 agree on a neighborhood U of $\dot{\Delta}^k$ in E^q. Let $\alpha : D^k \to E^k$ be a C^∞ regular orientation preserving homeomorphism such that $\alpha(S^{k-1}) \subset U \cap$(interior of Δ^k). $\alpha | S^{k-1}$ is homotopic in $U \cap \Delta^k$ to $\theta : S^{k-1} \to \dot{\Delta}^k$. Let $h_0 = f_1 \alpha$, $h_1 = g_1 \alpha$; then h_0 and h_1 are C^∞ immersions of D^k in E^n. Define

$$h_0', h_1' : T(E^q/D^k) \to T(E^n) \text{ by } h_0' e_i(x) = \begin{cases} f_{1*}\alpha_* e_i(x), & i = 1, \cdots, k, \\ f_{1*} e_i(\alpha(x)), & i = k+1, \cdots, q, \end{cases}$$

and h_1' is similarly defined in terms of g. Then for $j = 0, 1$, (h_j, h_j') are C^∞ E^q-immersions of D^k in E^n that are tangent on S^{k-1}. It is easy to see that $\Omega(h_0', h_1') = \Omega(f', g')$. If this last is 0, then by 3.8a there is a C^∞ E^q-regular homotopy (rel S^{k-1}) (h_t, h_t') joining (h_0, h_0') and (h_1, h_1'). This can be used to define an E^q-regular homotopy on $\alpha(D^k)$, which can be combined with the constant E^q-regular homotopy on $\Delta^k-\alpha(D^k)$ to produce an E^q-regular homotopy (rel $\dot{\Delta}^k$) between (f,f') and (g,g'). 2.5 is used to make this C^∞.

REMARK. If f and g agree on a neighborhood of $\dot{\Delta}^k$, we can choose the homotopy to be constant on a (possibly smaller) neighborhood of $\dot{\Delta}^k$.

THEOREM 4.3. *Let (f,f'), (g,g'), (h,h') be C^∞ E^q-immersions of Δ^k in E^n which are all tangent on $\dot{\Delta}^k$.*

(a) $\Omega(f',g')+\Omega(g',h')=\Omega(f',h')$.

(b) $\Omega(f',f')=0$.

(c) *Given $\alpha\in\pi_k(V_{n,q})$ there exists (g,g') such that $\Omega(f',g')=\alpha$.*

(d) *Suppose $\Omega(f',g')=0$, and $k<n$. Let $F:\Delta^k\times I\rightarrow V_{n,q}$ be a homotopy* (rel $\dot{\Delta}^k$) *between the maps F_0, $F_1:\Delta^k\rightarrow V_{n,q}$ defined respectively by $x\rightarrow f'\{e_i(x)\}$ and $x\rightarrow g'\{e_i(x)\}$, $i=1,\cdots,q$. Then there is a C^∞ E^q-regular homotopy (f_t,f_t') between (f,f') and (g,g') such that the map $\Delta^k\times I\rightarrow V_{n,q}$ defined by $(x,t)\rightarrow f_t'e_i(x)$ is homotopic* (rel $\dot{\Delta}^k\times I\cup\Delta^k\times\dot{I}$) *to F.*

Proof. (a) and (b) follow directly from the definition of Ω. (c) and (d) follow from (b) and (e) of 3.8.

5. **Immersions of manifolds.** In this section we consider the problem of extending an M-immersion, or an M-regular homotopy, defined on a skeleton of M, to the next skeleton. For convenience we shall assume that we are immersing M in a manifold N without boundary. This is not a strong restriction; an immersion into a manifold with a boundary can always be moved away from the boundary by means of a vector field normal to the boundary. The results of this section can be put in the following form: *The correspondence $f\rightarrow f_*$ induces a one-one correspondence between the regular homotopy classes of immersions $f:M\rightarrow N$ and the equivariant-homotopy classes of equivariant maps $T_k(M)\rightarrow T_k(N)$, where $k=\dim M<\dim N$.* First we discuss equivariant maps.

Let a group G act on topological spaces A and B on the right and left respectively. A map $f:A\rightarrow B$ is *equivariant* (with respect to the given actions of G) if for every $x\in A$ and $g\in G$, we have $f(xg)=g^{-1}f(x)$. A homotopy is equivariant if it is an equivariant map at each stage. If f and g are equivariantly homotopic, we shall write $f\simeq_e g$. If G acts on A, we let G act on $A\times I$ by $(x,t)g=(xg,t)$; similarly for $A\times I\times I$; by an equivariant map of $A\times I$, or $A\times I\times I$, we mean with respect to these actions.

THEOREM 5.1. *Let $p:E\rightarrow B$ be a principal G-bundle and let $p':E'\rightarrow B$ be an associated bundle with fiber Y. Then the space of cross-sections of E' is homeomorphic to the space of equivariant maps $E\rightarrow Y$ (both spaces having the compact-open topology).*

Proof. E' can be defined as the set of equivalence classes $\langle e,y\rangle$ of the following relation on $E\times Y$: $(e,y)\sim(eg,g^{-1}y)$, with $p'\langle e,y\rangle=p(e)$. If $f:E\rightarrow Y$

is equivariant, define the cross-section $f': B\rightarrow E'$ by $f'(x)=\langle e, f(e)\rangle$ where e is any element of $p^{-1}(x)$. If $g: B\rightarrow E'$ is a cross-section, define $g': E\rightarrow Y$ by $g'(e)=y$ where $gp(e)=\langle e, y\rangle$. It is easy to check that f' and g' are well defined, g' is equivariant, f' is a cross-section, $(f')'=f$, $(g')'=g$, and $f\rightarrow f'$ is a homeomorphism.

Now assume B is a simplicial complex. Let B_i denote the i-skeleton of B, and let $f: L\cup B_i\rightarrow E'$ be a cross-section. Assume also that the fiber Y is simple in dimension i. We recall the definition of the obstruction cochain of f: For each $x\in B$, let π_x be the group $\pi_i(p'^{-1}(x))$. The groups π_x form a *local system*; see [10] for the definition and properties of local systems. For each simplex σ of B let $b(\sigma)$ be its barycenter, and put $\pi_\sigma=\pi_{b(\sigma)}$. Let $C^{i+1}(B, L; \{\pi_x\})$ be the group of relative $i+1$-cochains of B mod L with values in the local system $\{\pi_x\}$; an element of this group assigns to each $i+1$ simplex σ of B an element of π_0. The obstruction cochain $C(f)$ is defined as follows: assume σ has an orientation; let $g_t: \dot{\sigma}\rightarrow\sigma$ be a homotopy such that $g_0(x)=x$, $g_1(x)=b(\sigma)$. Since f covers g_0, we can construct a homotopy f_t covering g_t such that $f_0=f$. Then $C(f)(\sigma)$ is the homotopy class of the map $f_1: \dot{\sigma}\rightarrow p'^{-1}(b(\sigma))$. For the properties of $C(f)$ see [10].

If we choose a local product representation $\phi: p'^{-1}(\sigma)\rightarrow\sigma\times Y$, then the homotopy class of

$$\dot{\sigma}\xrightarrow{f}p'^{-1}(\sigma)\xrightarrow{\phi}\sigma\times Y\rightarrow Y$$

defines $C(f)(\sigma)$ up to the action of $\pi_0(G)$ on $\pi_i(Y)$, where $\pi_0(G)$ is the quotient of G by the arc-component of the identity element, because two local product representations differ by an element of G.

A local product representation for $p'^{-1}(\sigma)$ is obtained by picking such a representation, $\psi: p^{-1}(\sigma)\rightarrow\sigma\times G$, for $p^{-1}(\sigma)$, and defining $\phi: p'^{-1}(\sigma)\rightarrow\sigma\times Y$ as follows: if $e\in p^{-1}(x)$, let $\psi(e)=(x, g)$. Then $\phi\langle e, y\rangle=(x, gy)$. It is easily seen that this is independent of the choice of e.

Now let $f: p^{-1}(\dot{\sigma})\rightarrow Y$ be equivariant and let $g: \dot{\sigma}\rightarrow p'^{-1}(\sigma)$ be the corresponding cross-section (5.1). Chasing through the definitions involved proves

LEMMA 5.2. *Up to the action of $\pi_0(G)$ on $\pi_i(y)$, $C(g)(\sigma)$ is obtained as follows: Let $\psi: p^{-1}(\sigma)\rightarrow\sigma\times G$ be a local product representation. Let $h\in G$ be arbitrary. Then $C(g)(\sigma)$ is the homotopy class of the map $\dot{\sigma}\rightarrow Y$ defined by $x\rightarrow f(\psi^{-1}(x, h))$.*

It is natural to define the *obstruction to extending an equivariant map* to be the obstruction cochain of the corresponding cross-section. If f is equivariant we denote its obstruction cochain by $C(f)$. If f is defined on $p^{-1}(L\cup B_i)$, f is extendible to an equivariant map of $p^{-1}(L\cup B_{i+1})$ if and only if $C(f)=0$.

If M is a manifold, we describe the action of $G=GL(r)$ on $T_q(M)$, $r\leqq q$. Let $\{X_i\}\in T_q(M)$ and $g\in G$, and suppose g is represented by the matrix

(a_{jk}). Then $\{X_i\}g=Y_i$ where Y_i is defined by $Y_i=X_i$ if $i=r+1, \cdots, q$, and $Y_i=\sum_{j=1}^{r}a_{ji}X_j$, if $i=1, \cdots, r$. This defines an action of G on the right; if G is to act on the left, we put $g\{X_i\}=\{X_i\}g^{-1}$. In particular, $GL(k)$ acts on $V_{n,k}$ on the left.

Now let $(f, f'): \dot{\Delta}^k \to E^n$ be an E^q-immersion. Put $T=T_q(E^q/\Delta^k)$ and let $p: T\to\Delta^k$ be the projection. Then $f': p^{-1}(\dot{\Delta}^k)\to V_{n,q}$ is equivariant. It follows from 5.2 that the obstruction to extending f' over T is the cochain that assigns to Δ^k the homotopy class of the map $\dot{\Delta}^k\to V_{n,q}$ defined by $x\to f'\{e_i(x)\}$, $i=1, \cdots, q$. Since this class is $\tau(f')$, we have proved

THEOREM 5.3. $\tau(f')=C(f')(\Delta^k)$.

If g_0, g_1 are equivariant maps agreeing on $p^{-1}(L\cup B_i)$, we define the *obstruction to the equivariant homotopy* (rel $L\cup B_i$) *between* g_0 *and* g_1 to be the obstruction to a homotopy (rel $L\cup B_i$) between the corresponding cross-sections. We denote the obstruction by $d(g_0, g_1)\in C^{i+1}(B, L; \{\pi_{i+1}(p'^{-1}(x))\})$. An analysis of definitions leads to

THEOREM 5.4. *Let $(f, f'), (g, g'): \Delta^k\to E^n$ be E^q-immersions tangent on $\dot{\Delta}^k$. Then $\Omega(f', g')=d(f', g')(\Delta^k)$.*

When τ and Ω vanish we have more precise results:

THEOREM 5.5. *Let $(f, f'): \dot{\Delta}^k\to E^n$ be a C^∞ E^q-immersion, $k<n$. Suppose $h': T_q(E^q/\Delta^k)\to V_{n,q}$ is an equivariant extension of f'. Then (f, f') can be extended to a C^∞ E^q-immersion $(g, g'): \Delta^k\to E^n$ such that $h'\simeq_e g'$* (rel $\dot{\Delta}^k$).

Proof. Since f' is extendible, $C(f')=0$, and therefore $\tau(f')=0$ by 5.3. By 4.1, (f, f') has a C^∞ extension (k, k'), and k' agrees with h' on $T_q(E^q/\dot{\Delta}^k)$. By 4.3c there is a C^∞ E^q-immersion (g, g') of Δ^k in E^n such that $\Omega(k', g')=d(k', h')$. Then $d(g', h')=d(g', k')+d(k', h')=\Omega(g', k')+d(k', h')$ by 5.4. Since $d(k', h')=\Omega(k', g')=-\Omega(g', k')$, $d(g', h')=0$, and so $g'\simeq_e h'$ (rel $\dot{\Delta}^k$).

THEOREM 5.6. *Let $(f, f'), (g, g'): \Delta^k\to E^n$ be C^∞ E^q-immersions tangent on $\dot{\Delta}^k$. Let $h'_t: T_q(E^q/\Delta^k)\to V_{n,q}$ be an equivariant homotopy (rel $\dot{\Delta}^k$) between f' and g'. There is a C^∞ E^q-regular homotopy (rel $\dot{\Delta}^k$) (f_t, f'_t) between (f, f') and (g, g') such that the two maps $T_q(E^q/\Delta^q)\times I\to V_{n,q}$ defined respectively by $(X, t)\to h'_t X$ and $(X, t)\to f'_t X$ are equivariantly homotopic* (rel $T_q(E^q/\dot{\Delta}^k)\times I\cup T_q(E^q/\Delta^k)\times \dot{I}$).

Proof. Follows from 4.3d.

In the rest of this section we make the following assumptions: M is a manifold of dimension k, N is a manifold without boundary of dimension $n>k$. $GL(k)$ acts on $T_k(M)$ on the right and on $T_k(N)$ on the left, in the manner described earlier in this section. *We assume that M has a C^∞ triangulation* in the sense of Milnor [5] or (equivalently) Whitehead [11]. We shall use only the property that each simplex of the triangulation is embedded in M by a C^∞ regular homeomorphism.

THEOREM 5.7. *Let K and L be subcomplexes of M with $K\subset L$. Let (f, f'): $K\rightarrow N$ be a C^∞ M-immersion and assume that f' is extendible to an equivariant map ϕ': $T_k(M/L)\rightarrow T_k(N)$. Let ϵ be a positive number.*

CONCLUSION. *There is a C^∞ M-immersion (g, g'): $L\rightarrow N$ such that*

(1) $(g, g')|K=(f, f')$;

(2) $g'\simeq_\epsilon\phi'$ (rel K);

(3) *If ϕ: $L\rightarrow N$ is the unique map covered by ϕ', then $d(g(x), \phi(x))<\epsilon$.*

Proof. The idea is to reduce everything to the case of extending an immersion defined on the boundary of a simplex (of a subdivision of M) into Euclidean space (a coordinate neighborhood of N) and then use the existence of an extension of f' to conclude that $\tau(f')=0$.

Let $\mathfrak{U}=\{U_i\}$ be a family of open sets of N with the following properties:

(a) $\phi(L)\subset\bigcup_i U_i$;

(b) Each U_i is a convex coordinate neighborhood of diameter less than ϵ.

Let L' be a subdivision of L so fine that each simplex of L' is contained in a coordinate neighborhood, and the image of each simplex under ϕ is contained in some U_i. For each simplex σ of L' let $U(\sigma)$ be a particular U_i containing $\phi(\sigma)$. Put $V(\sigma)=$intersection of all $U(\rho)$, $\sigma\subset\rho$. Then $V(\sigma)$ is a convex coordinate neighborhood, since it is a finite intersection of such, and it contains $\phi(\sigma)$; if σ is a face of ρ, $V(\sigma)\subset V(\rho)$. Let $\overline{L}_i$ be the union of K with the i-skeleton of L'. We shall define successively C^∞ M-immersions (h_i, h_i'): $\overline{L}_i\rightarrow N$ such that:

(i) $h_i(\sigma)\subset V(\sigma)$,

(ii) $h_i'\simeq_\epsilon\phi'|T_k(M/\overline{L}_i)$ (rel K) through an equivariant homotopy ψ_t with the property that if $\sigma\subset\overline{L}_i$, $X\in T_k(M/\sigma)$, then $\psi_t(X)\in T_k(N/V(\sigma))$.

(iii) $(h_i, h_i')|K=(f, f')$.

If L has dimension m, then (h_m, h_m') is the desired M-immersion of L.

Define an M-immersion (h_0, h_0'): $\overline{L}_0\rightarrow N$ by $(h_0, h_0')=(\phi, \phi')|\overline{L}_0$. This is trivially C^∞ and is an M-immersion because an admissible extension can easily be constructed by using the exponential map at points of $\overline{L}_0-K$, while at points of K, (ϕ, ϕ') coincides with the M-immersion (f, f'). It is clear that (i), (ii), and (iii) hold for $i=0$.

Now assume that (h_i, h_i'): $\overline{L}_i\rightarrow N$ is a C^∞ M-immersion satisfying (i), (ii) and (iii). Let $\sigma\subset\overline{L}_{i+1}$ have dimension $i+1$, $\sigma\not\subset K$. Let W be a coordinate neighborhood of σ in M; if we identify W with E^k, and $V(\sigma)$ with E^n, then $(h_i, h_i')|W$ is a C^∞ E^k-immersion of σ in E^n. By 5.5 we can extend (h_i, h_i') to σ so as to satisfy (ii). If $\sigma\subset K$, then (h_{i+1}, h_{i+1}') is defined to be (f, f') on σ. Thus (h_i, h_i') can be extended over each $i+1$ simplex of L. Moreover, we can assume that the extensions all agree in some neighborhood of L, by the remark following 4.1. Thus we can put all the extensions together to obtain (h_{i+1}, h_{i+1}'), a C^∞ M-immersion of L satisfying (i), (ii) and (iii); this completes the induction.

Before proving the next theorem we must prove a lemma motivated by the following considerations: Let A be a subset of M and (f_t, f_t') an M-regular homotopy of A in N. This means that f_t is a homotopy of A, and for each t, f_t can be extended to an immersion g_t of a neighborhood U_t, such that g_{t*} and f_t' agree on $T(M/A)$; but we cannot say that g_t is a homotopy of some neighborhood of A in M; even if the g_t are defined on some common neighborhood, they may not be continuous in t.

LEMMA 5.8. *Let $(f_t, f_t'): \Delta^k \to N$ be a C^∞ E^q-regular homotopy. Assume there is a neighborhood U of $\dot{\Delta}^k$ in E^q and a C^∞ regular homotopy g_t: $U \to N$ such that for each t, g_t is an admissible extension of $f_t | \dot{\Delta}^k$.*

CONCLUSION. *There exists a neighborhood W of Δ^k in E^q and a C^∞ regular homotopy h_t: $W \to N$ such that for each t, h_t is an admissible extension of f_t, and $h_t = g_t$ in an E^q neighborhood of $\dot{\Delta}^k$.*

Proof. Let $B \subset \Delta^k$ be obtained by shrinking Δ^k slightly toward 0; we assume that B is disjoint from $\dot{\Delta}^k$ and U contains the boundary $\dot{B}$ of B. For each t we can find an E^q neighborhood V_t of B and a *canonical extension* (see §3) k_t: $V_t \to N$ of f_t. By a compactness argument (compare 2.4) we can choose $V_t = V$ to be independent of t; as t varies, k_t is a regular homotopy of V in N. Since both g_t and k_t are admissible extensions of f_t on $\dot{B}$, it follows that $(g_t, g_{t*}) | \dot{B} = (k_t, k_{t*}) | \dot{B}$. By 2.5 (with $L = \dot{B}$, $C = I$, F and G given respectively by k_t and g_t, W of $2.5 = U \cap V$) we can deform g_t so that it agrees with k_t in an E^q neighborhood Y of $\dot{B}$, with $Y \subset U \cap V$. Let V' be an E^q neighborhood such that $B \subset Y \cup V' \subset V$ and $\mathrm{cl}(\Delta^k - B) \cap V'$ is empty. Let U' be an E^q neighborhood such that $\mathrm{cl}(\Delta^k - B) \subset U' \subset U$, and $\overline{U}' \cap V'$ is empty. Then $\Delta^k \subset U' \cup Y \cup V'$; call this last set W. Define h_t: $W \to N$,

$$h_t(x) = \begin{cases} g_t(x), & x \in U', \\ g_t(x) = k_t(x), & x \in Y, \\ k_t(x), & x = V'. \end{cases}$$

It is easily seen that h_t and W have the required properties.

THEOREM 5.9. *Let $K \subset L \subset M$ be subcomplexes of M. Let (f, f'), (g, g'): $L \to N$ be C^∞ M-immersions. Let $(f_t, f_t'): K \to N$ be a C^∞ M-regular homotopy satisfying the following conditions: $(f_0, f_0') = (f, f') | K$, $(f_1, f_1') = (g, g') | K$; there is a C^∞ regular homotopy of a neighborhood of K in N which is an admissible extension of f_t at each stage t. Assume also that f_t' can be extended to an equivariant homotopy $\phi_t' : T_k(M/L) \to T_k(N)$, such that $\phi_0' = f'$, $\phi_1' = g'$. Choose $\epsilon > 0$.*

CONCLUSION. *(f_t, f_t') can be extended to a C^∞ M-regular homotopy (h_t, h_t'): $L \to N$ such that:*

(1) *$(h_0, h_0') = (f, f')$ and $(h_1, h_1') = (g, g')$.*

(2) *The maps h' and ϕ': $T_k(M/L) \times I \to T_k(A)$ are equivariantly homotopic* (rel $T_k(M/L) \times \dot{I} \cup T_k(M/K) \times I$).

(3) *There is a C^∞ regular homotopy of a neighborhood of L which is an admissible extension of h_t for each t.*

(4) *If ϕ_t: $L \rightarrow N$ is the unique map covered by ϕ_t', $d(h_t(x), \phi_t(x)) < \epsilon$.*

Proof. Let $\mathfrak{U} = \{U_i\}$ be a family of open sets in N with the following properties:

(a) $\phi_t(L) \subset \bigcup_i U_i$ for each t.

(b) Each U_i is a convex coordinate neighborhood of diameter $< \epsilon$.

Choose a subdivision $0 = t_0 < \cdots < t_s = 1$ of I and a subdivision L' of L so fine that for any $\sigma \subset L'$ and $I_j = [t_{j-1}, t_j]$, $\phi(\sigma \times I_j)$ is contained in some U_i and each σ is contained in a coordinate neighborhood of M. Let $\phi(\sigma \times I_j) \subset U(\sigma \times I_j) \in \mathfrak{U}$; put $V(\sigma \times I_j) = \bigcap_{\sigma \subset \rho} U(\rho \times I_j)$.

Let W be a neighborhood of K in M and k_t: $W \rightarrow N$ a C^∞ regular homotopy such that for each t, k_t is an admissible extension of f_t, as in the hypothesis.

Put $\overline{L}_i = L_i' \cup K$. Define $(\theta_{0t}, \theta_{0t}')$: $\overline{L}_0 \rightarrow N$ by $(\theta_{0t}, \theta_{0t}') = (\phi_t, \phi_t') | \overline{L}_0$. By the definition of ϕ and ϕ', this is a C^∞ M-regular homotopy, when restricted to K; using the exponential map, it can be made into a C^∞ M-regular homotopy of $\overline{L}_0$. Moreover, there is a C^∞ M-regular homotopy $W_0 \rightarrow N$ of some neighborhood W_0 of $\overline{L}_0$ which is an admissible extension of θ_{0t} for each t; this follows from the assumptions made about f_t and properties of the exponential map. Observe that $(\theta_{01}, \theta_{01}') = (g, g') | \overline{L}_0$.

Now assume inductively that there is a C^∞ extension $(\theta_{it}, \theta_{it}')$: $\overline{L}_i \rightarrow N$ of f_t such that $\theta_{i1}, (\theta_{i1}') = (g, g') | \overline{L}_i$, and a C^∞ regular homotopy ψ_{it}: $W_i \rightarrow N$, where W_i is a neighborhood of $\overline{L}_i$, such that

(i) $\theta_i(\sigma \times I_j) \subset V(\sigma \times I_j)$ for each $\sigma \subset \overline{L}_i$, $j = 1, \cdots, s$.

(ii) $\theta_i' \simeq_e \phi' | T_k(M/\overline{L}_i) \times I$ (rel $T_k(M/\overline{L}_i) \times \dot{I} \cup T_k(M/K) \times I$).

(iii) ψ_{it} is an admissible extension of θ_{it} for each t. These conditions hold for $i = 0$. Let σ be an $i+1$ simplex of $\overline{L}_{i+1}$ not contained in K. We want to extend (θ_i, θ_i') to a C^∞ M-regular homotopy defined on σ. To do this, let α: $D^{i+1} \rightarrow$(interior of σ) be a C^∞ regular homeomorphism such that $\alpha(S^i) \subset W_i \cap \sigma$. There is a C^∞ map α': $T_k(E^k/D^{i+1}) \rightarrow T_k(M/\sigma)$ such that (α, α') is a C^∞ E^k-immersion. Then $(\psi_{it}\alpha, \psi_{it*}\alpha')$ is a C^∞ E^k-regular homotopy of S^i in N. If we restrict our attention to values of t in I_j, the homotopy takes place in the coordinate neighborhood $V(\sigma \times I_j)$ which may be identified with E^n. Since σ is contained in a coordinate neighborhood of M, we can apply 3.4 and conclude that $(\psi_{it}\alpha, \psi_{it*}\alpha')$ can be extended a C^∞ E^k-regular homotopy of D^{i+1}. Doing this for successive values of j we obtain a C^∞ extension (β_t, β_t') of $(\psi_{it}\alpha, \psi_{it*}\alpha')$. It follows that $(\beta_t\alpha^{-1}, \beta_t'\alpha'^{-1})$ is an extension of $(\psi_{it}, \psi_{it*}) | \alpha(S^i)$. By 2.5 we can make this extension agree with (ψ_{it}, ψ_{it*}) in a neighborhood of $\alpha(S^i)$, and together they form a C^∞ extension of $(\theta_{it}, \theta_{it}')$, say (q_t, q_t'). By 5.8 there is a C^∞ regular homotopy of a neighborhood of σ into N which is an admissible extension of q_t, for each t, and which agrees with ψ_{it} in a neighborhood of $\dot{\sigma}$. Therefore, after doing this for each σ, we can put all the (q_t, q_t') together, since they have admissible extensions agreeing with ψ_{it}, and we obtain a C^∞ M-regular homotopy $(\gamma_t, \gamma_t'): \bar{L}_{i+1} \to N$ which is an extension of $(\theta_{it}, \theta_{it}')$. By the inductive hypothesis, $(\gamma_1, \gamma_1') | \bar{L}_i = (g, g') | \bar{L}_i$. We must show that for each $i+1$ simplex $\sigma \subset \bar{L}_{i+1}$, the obstruction to an M-regular homotopy (rel $\dot{\sigma}$) between $(\gamma_1, \gamma_1') | \sigma$ and $(g, g') | \sigma$ vanishes. Since everything takes place in the coordinate neighborhood $V(\sigma \times I_s)$, we shall use results on E^q-immersions in Euclidean spaces already proved. If this obstruction vanishes for each σ, then the induction will be completed by applying 4.2 and obtaining a C^∞ M-regular homotopy between (γ_1, γ_1') and $(g, g') | \bar{L}_{i+1}$ (with the help of 5.8 to piece the homotopies on each σ together).

We shall use 5.4 and show that the equivariant maps γ_1' and g', considered as maps $T_k(M/\sigma) \to T_k(N/V(\sigma \times I_s))$, are equivariantly homotopic (rel $\dot{\sigma}$), and that the equivariant homotopy ξ_t can be chosen so that $\xi_t \otimes \gamma_t'$, where $\otimes$ means path multiplication, satisfies (ii). Since we are dealing with equivariant maps, by 5.1 we may replace them by cross-sections of a certain bundle over $\sigma \times I \times I$. Since such a bundle is trivial, we can replace the cross-sections by maps into the fiber $Y = T_k(A/V(\sigma \times I_s))$. Let $G: \dot{\sigma} \times I \times I \to Y$ be an equivariant homotopy as in (ii). Extend G to $\sigma \times 0 \times I \cup \sigma \times 1 \times I$ by $G(x, 0, t) = f''(x)$, $G(x, 1, t) = g''(x)$, where f'' and g'' are the maps $\sigma \to Y$ corresponding respectively to f' and g'. Then extend G to $\sigma \times I \times 1$ by $G(x, t, 1) = \phi_t''(x)$, where ϕ'' corresponds to ϕ'. Thus G maps all but the one face $\sigma \times I \times 0$ of the boundary of $\sigma \times I \times I$; by a well known theorem of homotopy theory, we can extend G to all of $\sigma \times I \times I$; assume this has been done. Now let $J = [-1, 0]$. Define $H: \sigma \times I \times \{-1\} \to Y$ by $H(x, t, -1) = \gamma_t''(x)$, where γ'' corresponds to γ'. Extend H to $\sigma \times 0 \times J \cup \dot{\sigma} \times I \times J \cup \sigma \times I \times 0$ by $H(x, 0, t) = \gamma_0''(x) = f''(x)$, $H(x, t, t') = G(x, t, 0)$, $H(x, t, 0) = G(x, t, 0)$. H is defined on all but one face of the boundary of $\sigma \times I \times J$. An extension to the last face provides an equivariant homotopy (rel $\dot{\sigma}$) between γ_1' and g', while an extension to the whole of $\sigma \times I \times J$ provides an equivariant homotopy as in (ii) for the case $i+1$. Since this extension is possible, the induction is complete and 5.9 is proven.

THEOREM 5.10. *Let f: $M \to N$ be a continuous map. If f is homotopic to an immersion g: $M \to N$, then f can be approximated by an immersion.* (*Recall that* dim $M <$ dim N.)

Proof. Since f is homotopic to g, and g can be covered by the equivariant map g_*: $T_k(M) \to T_k(N)$, f can be covered by such a map. The theorem follows upon applying 5.7 with K empty and $L = M$.

PART II. APPLICATIONS

6. Immersions in Euclidean spaces. In this section we give some conditions that are sufficient (and trivially necessary) for the inmersibility of a manifold in Euclidean space of given dimension. All the results have generalizations to M-immersions of subcomplexes, with practically identical proofs. As usual we identify $T_k(E^n)$ with $V_{n,k}$ by parallel translation. All immersions and cross-sections are to be understood to be C^∞. *M is a manifold of dimension* $k<n$.

THEOREM 6.1. *M is immersible in* E^n *with a transversal r-field if and only if there is an equivariant map* ψ: $T_k(M)\to V_{n,k+r}$. *Moreover, given such a* ψ, *there is an immersion* f: $M\to E^n$ *and a transversal r-field* ϕ *such that the map* $T_k(M)\to V_{n,k+r}$ *defined by*

$$\{X_1, \cdots, X_k\} \to \{f_*X_1, \cdots, f_*X_k, \phi_1(x), \cdots, \phi_r(x)\},$$

where the X_i *are based at* x *and* $\phi(x)=\{\phi_1(x), \cdots, \phi_r(x)\}$, *is equivariantly homotopic to* ψ.

Proof. Let p: $V_{n,k+r}\to V_{n,k}$ be the bundle projection that deletes the last r vectors of each $k+r$-frame. The map $p\psi$: $T_k(M)\to V_{n,k}$ is equivariant; by 5.7 (with $L=M$, $N=E^n$, K empty) there exists an immersion $f:M\to E^n$ such that $f_*:T_k(M)\to V_{n,k}$ is equivariantly homotopic to $p\psi$. It follows from the fact that p has the covering homotopy property that f_* can be lifted to an equivariant map g: $T_k(M)\to V_{n,k+r}$, so that $f_*=pg$. The following map $\phi:M\to V_{n,r}$ is transversal to $f:\phi(x)=\{Y_{k+1}, \cdots, Y_{k+r}\}$ where if $\{X_1, \cdots, X_k\}$ is any frame based at x, $g\{X_1, \cdots, X_k\}=\{Y_1, \cdots, Y_{k+r}\}$. By the definition of the action of $GL(k)$ on $V_{n,k+r}$, $\{Y_{k+1}, \cdots, Y_{k+r}\}$ depends only on x. ϕ defines a transversal field because $Y_1, \cdots, Y_k$ span the tangent plane to $f(M)$ at $f(x)$.

Using the correspondence between equivariant maps and cross-sections, described in 5.1, the last result can be restated as follows:

THEOREM 6.2. *M is immersible in* E^n *with a transversal r-field if and only if the bundle associated to* $T_k(M)$ *with fiber* $V_{n,k+r}$ *has a cross-section.*

It is interesting to observe that the immersibility of M in E^n (or any other manifold) depends only on the equivalence class of the principal bundle $T_k(M)$, and not on the complete differentiable structure of M. In other words, if M and M' are homeomorphic manifolds with equivalent frame bundles, they are immersible in exactly the same manifolds, in fact their regular homotopy classes of immersions correspond. For example, the manifolds constructed by Milnor [6] which are homeomorphic, but not diffeomorphic, to S^7, have trivial frame bundles, since $\pi_6(O(7))=0$, and hence they can be immersed in E^8. It is not known whether there exists a pair of homeomorphic manifolds with inequivalent frame bundles.

THEOREM 6.3. *If M is parallelizable it is immersible in* E^{k+1}.

Proof. If $T_k(M)$ is trivial, any associated bundle has a cross-section, and 6.2 can be applied with $r=0$, $n=k+1$.

THEOREM 6.4. *If M is immersible in* E^{n+r} *with a transversal r-field, it is immersible in* E^n(2).

(2) The proof shows that such an immersion in E^{n+r} regularly homotopic to one in E^n.

Proof. It suffices to show that M is immersible in E^{n+r-1} with a transversal $r-1$ field, and apply this result r times.

Let S be the unit $n+r-1$ sphere in E^{n+r}. Let $f: M \to E^{n+r}$ be an immersion with a transversal r-field. It is well known that this field can be normalized, so we assume that f has an orthonormal r-field $\chi(x) = \{\chi_1(x), \cdots, \chi_r(x)\}$. Consider $r(x)$ as having origin 0; then $\chi_r(x)$ defines an element of S. The map $\chi_r: M \to S$ can be covered by a map $\phi: T_k(M) \to T_{k+r-1}(S)$ by sending each frame $\{X_1, \cdots, X_k\}$ based at x into $\{f_* X_1, \cdots, f_* X_k, \chi_1(x), \cdots, \chi_{r-1}(x)\}$; and it is clear that ϕ is equivariant. Since $k < n$ and $r \geqq 1$, dim $M <$ dim S and therefore χ_r is homotopic to a constant map $M \to a \in S$. This homotopy can be covered by an equivariant homotopy ϕ_t, with $\phi_0 = \phi$. Then $\phi_1: T_k(M) \to T_{k+r-1}(S/a) = V_{n+r-1,k+r-1}$, and ϕ_1 is equivariant. The theorem is proved by applying 6.1.

THEOREM 6.5. *If M is immersible in E^n (any n) with a trivial normal bundle, it is immersible in E^{k+1}.*

Proof. Apply 6.4.

THEOREM 6.6 (WHITNEY). *M is immersible in E^{2k-1}.*

Proof. By 6.4 it suffices to immerse M^k in E^{2k} with a transversal field. M can always be immersed in E^{2k} because $\pi_i(V_{k+r,k}) = 0$ for $i = 0, \cdots, r-1$ [10] and therefore the bundle associated to $T_k(M)$ with fiber $V_{2k,k}$ has a cross-section, and we apply 6.2. We distinguish three cases: (1) M not compact, or with boundary. In this case $H^k(M) = 0$, and so the obstruction to a cross-section in the bundle associated to $T_k(M)$ with fiber $V_{2k-1,k}$ vanishes and again 6.2 can be applied. (2) M compact, without boundary, k odd. In this case the Stiefel-Whitney class $\overline{W}^k$ of M, which is the obstruction to a normal field, vanishes. (See [12] for the properties of $\overline{W}^k$.) This follows from the fact that if k is odd, $\overline{W}^k$ has order 2 [10, 38.11] while it lies in an infinite cyclic group [10, last sentence of 39.5]. (Observe that the local systems defined by the tangent sphere bundle of M and the normal sphere bundle of an immersion of M in E^{2k} are the same.) (3) M compact, k even. It will be shown in 8.2 that M can be immersed in E^{2k} in such a way that the normal class of the immersion vanishes. This implies that the immersion has a normal field.

Next we examine the problem of immersing the k-dimensional manifold M in E^{2k-2}. We are able to give complete results for compact M if $k = 3, 4, 5$, and $k \equiv 1 \pmod 4$. In these cases the immersibility of M in E^{2k-2} is a topological invariant of M.

THEOREM 6.7. *Every compact 3-manifold M is immersible in E^4.*

Proof. By 6.4 it suffices to immerse M in E^6 with a normal 2-field. By 6.6 M can be immersed in E^6. $\overline{W}^2(M)$ is the obstruction cohomology class to constructing a normal 2-field on M in E^6; if M is compact, $\overline{W}^2=0$ [14]. Therefore there is a normal 2-field on the 2-skeleton; the obstruction cochain to extending this field over M vanishes, since it has values in $\pi_2(V_{3,2})=0$ [10].

THEOREM 6.8. *Let M have dimension $k\equiv 1$ (mod 4). Then M is immersible in E^{2k-2} if and only if $\overline{W}^{k-1}(M)=0$.*

Proof. One implication is obvious. If on the other hand $\overline{W}^{k-1}=0$, then M is immersible in E^{2k} with a normal 2-field on the $k-1$ skeleton. This field can be extended to M, for the obstruction cochain to doing so has values in $\pi_{k-1}(V_{k,2})=0$ if $k\equiv 1$ (mod 4) [7].

Now suppose M has dimension 5. Using the formulas of Wu [17] for Stiefel-Whitney classes, it is easy to show that $\overline{W}^4(M)=0$. Applying 6.8 proves

THEOREM 6.9. *Every compact 5-manifold is immersible in E^8.*

THEOREM 6.10. *Let $\overline{W}^2$ and $\overline{P}^4$ be the normal Stiefel-Whitney and Pontryagin classes, respectively, of the compact 4-manifold M. M can be immersed in E^6 if and only if there exists an integral cohomology class $\alpha\in H^2(M)$ such that $\alpha\equiv\overline{W}^2$ (mod 2) and $\alpha^2=\overline{P}^4$.*

Proof. Let M be immersed in E^7 (6.6); $\overline{W}^2$ and $\overline{P}^4$ are determined by the normal 2-sphere bundle on the immersion. By [14] $\overline{W}^3=0$, i.e., the characteristic class of the normal sphere bundle vanishes. By a result of Massey [3], the existence of α is equivalent to the existence of a cross-section of the normal sphere bundle, i.e., a normal field. By 6.4, this is equivalent to the existence of an immersion of M in E^6.

The complex projective plane is an example of a 4-manifold that is not immersible in E^6. This is because $\overline{P}^4$ is three times the fundamental class, and the pairing $H^2\times H^2\rightarrow H^4$ given by the cup product is the natural pairing $Z\times Z\rightarrow Z$, while 3 is not the square of an integer.

7. **Immersions of projective spaces.** Let P^k denote the real projective space of dimension k.

THEOREM 7.1. *The following immersions are possible:*
(a) *P^3 in E^4,*
(b) *P^5 in E^7,*
(c) *P^6 in E^7,*
(d) *P^7 in E^8,*
(e) *P^9 in E^{16}.*

Proof. P^3 and P^7 are known to be parallelizable, so (a) and (d) follow from 6.3. Since P^7 is orientable, any immersion of it in E^8 has a transversal field, and since $P^5\subset P^6\subset P^7$, there exist immersions of P^5 and P^6 in E^8 with transversal fields, by d; (b) and (c) now follows from 6.4. Finally, (e) follows from 6.8 because $\overline{W}^8(P^9)=0$.

(a) was first proved by Milnor [4] and Lashof-Smale [2].

A consideration of normal Stiefel-Whitney classes shows that (a) through (d) are best possible, but the problem of immersing P^9 in E^{15} is open.

THEOREM 7.2. *Let k be even. P^k is immersible in E^{k+1} if and only if P^{k+1} is parallelizable*(3).

Proof. If P^{k+1} is parallelizable it is immersible in E^{k+2}, and any such immersion has a normal field (6.3). Therefore $P^k \subset P^{k+1}$ is immersible in E^{k+2} with a normal field, and by 6.4 P^k is immersible in E^{k+1}.

Conversely, if P^k is immersed in E^{k+1}, and k is even, its normal line bundle is the unique nontrivial line bundle over P^k, and is therefore equivalent to the normal line bundle of P^k in P^{k+1}. Therefore we can immerse a tubular neighborhood of P^k (in P^{k+1}) in $E^{k+1} \subset E^{k+2}$. The obstruction to extending this P^{k+1}-immersion of P^k in E^{k+2} takes values in $\pi_k(V_{k+2,k+1})$. The assumption that P^k is immersible in E^{k+1} means that all the normal Stiefel-Whitney classes must vanish, and this is only possible if k has the form 2^i-2, if k is even. For these values of k, $\pi_k(V_{k+2,k+1})=0$; see [7] for the cases $k=2, 6$; for higher values of k (of the form 2^i-2) see [1]. Therefore P^{k+1} is immersible in E^{k+2}. The composite immersion $S^{k+1} \to P^{k+1} \to E^{k+2}$, where the first map is the double covering, is an immersion of S^{k+1} in E^{k+2} with even *normal degree*. (The normal degree of an immersion $M^n \to E^{n+1}$ is the homological degree of the induced map $M^n \to S^n$, each point x of M^n going into the oriented line normal to the image of $T(M^n/x)$.) Milnor [4] shows that in this case S^{k+1} is parallelizable, and since P^{k+1} is immersed in E^{k+2}, P^{k+1} must also be parallelizable [4].

8. **Regular homotopy.** In [9] Smale proves that two immersions of S^{2k} in E^{4k} are regularly homotopic if and only if they have the same normal class, and that any even $2k$-dimensional cohomology class is the normal class of some such immersion. We generalize this result in two ways: replacing S^{2k} by M^{2k}, and replacing E^{4k} by N^{4k} (but not both!). Thus, for these special cases the normal class and the homotopy class are the only invariants of the regular homotopy class of the immersion.

LEMMA 8.1. *Let D be a k-disk, and $f, g: D \to E^{2k}$ immersions such that $(f, f_*) | \dot{D} = (g, g_*) | \dot{D}$. Let ϕ be a field transversal to f (and g) on $\dot{D}$. Let C be the value of the obstruction cochain to extending ϕ over $f(D)$ on the generator of $C_k(D, \dot{D})$, and C' the corresponding value of the obstruction to extending ϕ over $g(D)$; thus $C, C' \in \pi_{k-1}(Y)$, where Y is the fiber of the bundle of vectors transversal to f (or g). Let $\partial: \pi_k(V_{2k,k}) \to \pi_{k-1}(Y)$ be the boundary operator of the homotopy sequence of the bundle $V_{2k,k+1} \to V_{2k,k}$, whose fiber is also Y.*

CONCLUSION. $\partial\Omega(f, g) = C - C'$.

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(3) According to recent results of Kervaire, Bott, and Milnor, S^d, and hence P^d, is parallelizable only if $d=1, 3, 7$.

Proof. Let S be the k-sphere obtained by identifying the boundaries of two copies of D. Let $B \to S$ be the bundle obtained by identifying the transversal bundles of f and g over $\dot{D}$. Let $\phi: S \to V_{2k,k}$ be the map $\phi(x) = f_*\{e_i(x)\}$ if x is in the upper hemisphere of S, $\phi(x) = g_*\{e_i(x)\}$ if x is in the lower hemisphere, $i=1, \cdots, k$, each hemisphere being identified with D. Then by definition, $\Omega(f, g)$ is the homotopy class of ϕ, which is the same as $\phi_\#(s)$ where s is the generator of $\pi_k(S)$ that corresponds to the orientation of S given by orienting its upper hemisphere by the orientation of D. ϕ can be covered by the map $\bar{\Phi}: B \to V_{2k,k+1}$ defined by $\bar{\Phi}(X) = \{f_*e_1(x), \cdots, f_*e_k(x), X\}$ or $\bar{\Phi}(X) = \{g_*e_1(x), \cdots, g_*e_k(x), X\}$ according as X is a transversal vector to $f(D)$ at $f(x)$ or to $g(D)$ at $g(x)$; this map identifies the fiber of B over x with the fiber of $V_{2k,k+1}$ over $\bar{\Phi}(x)$. Therefore the following diagram is commutative:

$$\begin{array}{ccccc} \pi_k(B) & \rightarrow & \pi_k(S) & \xrightarrow{\partial} & \pi_{k-1}(Y) \\ \downarrow \bar{\Phi}_\# & & \downarrow \phi_\# & & \downarrow \text{identity} \\ \pi_k(V_{2k,k-1}) & \rightarrow & \pi_k(V_{2k,k}) & \xrightarrow{\Delta} & \pi_{k-1}(Y) \end{array}$$

Therefore $\partial\Omega = \partial\phi_\#(s) = \partial(s)$. By [10, 35.12] $\partial(s)$ is the Kronecker index of the characteristic class of B with the generator of $H_k(S)$. This characteristic class is the cohomology class of the obstruction cochain to a cross-section. Choosing the cross-section ϕ, we see that the obstruction cochain to extending ϕ takes the values C and $-C'$ on the respective upper and lower hemispheres of S, when they are coherently oriented. Therefore the Kronecker index is $C - C' = \partial(s) = \partial\phi_\#(s) = \partial\Omega$.

THEOREM 8.2. *Let M be a manifold of even dimension k. Two immersions $f, g: M \to E^{2k}$ are regularly homotopic if and only if they have the same normal class, and any even class of $H^k(M)$ is the normal class of some immersion of M in E^{2k}.*

(By $H^k(M)$ we mean cohomology with coefficients in the local system determined by the unit tangent sphere bundle of M; this is the same system as that determined by any immersion of M in E^{2k}; if M is compact and without boundary, $H^k(M)$ is infinite cyclic [10], otherwise $H^k(M) = 0$.)

Proof. If $H^k(M) = 0$, the obstruction cohomology class to making f and g regularly homotopic vanishes, since it lies in $H^k(M)$. (The first nonvanishing homotopy group of $V_{2k,k}$ is in dimension k.) This disposes of the cases M noncompact or with boundary.

Assume M is compact without boundary. If f and g are regularly homotopic, their normal bundles are equivalent, and so they have the same normal class. To prove the converse, we can deform f by a regular homotopy so that it agrees with g (and f_* with g_*) on the $k-1$ skeleton of M. This can be done by 5.9, because the obstruction to making f_* and g_* equivariantly homotopic on the $k-1$ skeleton of M has values in $\pi_{k-1}(V_{2k,k}) = 0$. We can choose the equivariant homotopy so that the obstruction cochain to extending it vanishes on the complement of a k-disk D embedded in M. Therefore we assume that f and g agree on $M-D$. Let ϕ be a transversal field on $f(M-D)=g(M-D)$; we assume ϕ is defined on the boundary of D also. It is clear that f and g are regularly homotopic if and only if $\Omega(f|D, g|D)=0$. Let $m \in H^k(M)$ be the generator corresponding to an orientation of D. It follows from definitions that $\overline{W}^k(f)=Cm$, $\overline{W}^k(g)=C'm$, where C and C' are defined as in 8.1. Now assume k is even. In this case the exact homotopy sequence of the bundle $V_{2k,k+1}\rightarrow V_{2k,k}$ with fiber Y reduces to

$$0\rightarrow\pi_k(V_{2k,k})\xrightarrow{\partial}\pi_{k-1}(Y)\rightarrow\pi_{k-1}(V_{2k,k+1})\rightarrow 0$$

[10] which is the sequence $0\rightarrow Z\rightarrow Z\rightarrow Z_2\rightarrow 0$. As in [10] there is an exact sequence

$$H^{k-1}(M;\ Z_2)\xrightarrow{\delta^*}H^k(M)\rightarrow H^k(M;\ Z_2)$$

induced from the short sequence above. Since $H^k(M)=Z$ and every element of

$$H^k(M;\ Z_2)$$

obviously has order 2, $\delta^*=0$ and by exactness ∂_* is one-one. It follows from 8.1 that $\partial(\Omega(f|D, g|D)m)=(C-C')m=\overline{W}^k(f)-\overline{W}^k(g)$, which is 0 if f and g have the same normal classes. Since ∂ is one-one, $\Omega=0$, and so f and g are regularly homotopic.

To prove the rest of the theorem, let $f: M\rightarrow E^{2k}$ be a fixed immersion, which exists by virtue of 6.2. By [17], $\overline{W}^k(f)$ is even. Let $\alpha=2\beta\in H^k(M)$. Letting D be a diffeomorphically embedded k-disk of M, let $g: D\rightarrow E^{2k}$ be an immersion such that $(g, g_*)|\dot{D}=(f, f_*)|\dot{D}$ and such that $\Omega((g, f)|D)=\beta$; this is possible by 2.8d. g can be extended to an immersion of M by defining $g(x)=f(x)$ for $x\in M-D$, and by 2.5 we can deform g slightly so that it is a C^∞ immersion of M. Then as above, $\partial\Omega((g, f)|D)=\overline{W}^k(g)-\overline{W}^k(f)$. Since ∂ is multiplication by 2, we have shown that $\overline{W}^k(g)=\alpha+\overline{W}^k(f)$. Since β was arbitrary, this completes the proof.

THEOREM 8.3. *Let S be a sphere of even dimension k and N a manifold of dimension $2k$. A necessary and sufficient condition that two immersions f, g: $S\rightarrow N$ be regularly homotopic is that they be homotopic and have the same normal class. Given a homotopy class of S in N and an even element of $H^k(S)$, there is an immersion in the homotopy class with the element as normal class.*

Proof. It suffices to prove the sufficiency of the condition. Choose a homotopy (not regular) $f_t: S\rightarrow N$ with $f_0=f$, $f_1=g$. Cover f_t by ϕ_t with $\phi_0=f: T_k(S)\rightarrow T_k(N)$. We can alter f_t slightly so that f_1 agrees with g on some neighborhood of S_0 (S_i is the i-skeleton), by 2.5. Thus $(f, f_*)|S_0$ and $(g, g_*)|S_0$ are regularly homotopic. The obstruction cohomology class to extending this regular homotopy to one between f and g lies in $H^1(S; \pi_1(T_k(N)))=0$. This means that there is a different regular homotopy which is extendible to S_1. We can choose this regular homotopy to approximate the first as a ***homotopy***, because by 5.9, choosing a different regular homotopy is equivalent to choosing a different equivariant map covering f_t, and then altering it slightly to obtain as its projection a regular homotopy, which it covers. This process can be continued inductively over successive skeletons of S until the following situation is reached: There is a regular homotopy f_t of f such that $(f_1, f_{1*}) | S_{k-1} = (g, g_*) | S_{k-1}$, and f_1 and g are homotopic (rel S_{k-1}) as continuous maps. Moreover, we can assume that S has a simplicial subdivision so fine that for any simplex σ, $f(\sigma)$ and $g(\sigma)$ are contained in the same coordinate neighborhood. Using the correspondence between regular homotopies and homotopies of cross-sections, we can alter f_t so that the obstruction cochain is any given element of its cohomology class. Thus we can assume that there is a k-disk D diffeomorphically embedded in S such that f_1 and g agree on $S-D$. Picking a transversal field ϕ on the closure of $f_1(S-D)=g(S-D)$, and observing that $f_1(D)$ and $g(D)$ are contained in the same coordinate neighborhood, we have reduced the first part of the theorem to 8.1, which has been proved. The second part follows from the fact that $\pi_{k-1}(V_{2k,k})=0$. This means that the map $\theta: H^k(S; \pi_k(T_k(N))\rightarrow H^k(S; \pi_k(N)))$, induced by the map $T_k(N)\rightarrow N$, is onto. Therefore given a map $f: S\rightarrow N$, to produce an immersion homotopic to f, take any immersion $g: S\rightarrow N$ (e.g., an immersion of S in a coordinate neighborhood of N) and choose an immersion $h: S\rightarrow N$ that agrees with g on S_{k-1} and such that $\theta(\Omega(g, h))=d(f, h)$, the difference cochain. This can be done by 5.9 and the fact that equivariant maps $T_k(S)\rightarrow T_k(N)$ are classified by $H^k(S; \pi_k(T_k(N)))$. It is clear that $d(g, h)=\theta(\Omega(g, h))$ and therefore $d(f, g)=0$, so that g is in the homotopy class of f. The rest of the theorem now follows from the analogous part of 8.2.

THEOREM 8.4. *Two immersions of M in E^{2k+1} are regularly homotopic.*

Proof. The obstruction to a regular homotopy has values in $\pi_k(V_{2k+1,k})=0$.

9. **The normal degree.** Let f be an immersion of the orientable k-manifold M in E^{k+1}. The *normal degree* of f, as defined in §7, is denoted by $\mathfrak{D}_f$.

THEOREM 9.1. *If M is parallelizable, there is an immersion f: $M\rightarrow E^{k+1}$ with $\mathfrak{D}_f=0$.*

Proof. Let ϕ: $T_k(M)\rightarrow V_{k,k}$ be an equivariant map, which exists because M is parallelizable. Let f: $M\rightarrow E^{k+1}$ be an immersion such that f_*: $T_k(M)\rightarrow V_{k+1,k}$ is equivariantly homotopic to the composite

$$T_k(M) \overset{\phi}{\rightarrow} V_{k,k} \subset V_{k+1,k}$$

(see 6.1). Let ψ: $M\rightarrow T_k(M)$ be the cross-section corresponding to ϕ (see 5.1).

Let p: $V_{k+1,k} \to S^k$ be the map assigning to $\{X_1, \cdots, X_k\}$ the vector Y of unit length such that $\{X_1, \cdots, X_k, Y\}$ is a $k+1$ frame inducing the standard orientation on E^{k+1}. It is easy to see that $\mathfrak{D}_f$ is the degree of the map $pf_*\psi$: $M \to S^k$, which is homotopic to $p\phi\psi$. Since $p | V_{k,k}$ is constant, $pf_*\psi$ is homotopic to a constant and therefore $\mathfrak{D}_f = 0$.

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