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Authors

Gilton, Thomas
Neeman, Itay

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ABRAHAM-RUBIN-SHELAH OPEN COLORINGS AND A LARGE CONTINUUM

THOMAS GILTON AND ITAY NEEMAN

Abstract. We show that the Abraham-Rubin-Shelah Open Coloring Axiom is consistent with a large continuum, in particular, consistent with $2^{\aleph_0} = \aleph_3$. This answers one of the main open questions from [2]. As in [2], we need to construct names for so-called *preassignments of colors* in order to add the necessary homogeneous sets. However, these names are constructed over models satisfying the CH. In order to address this difficulty, we show how to construct such names with very strong symmetry conditions. This symmetry allows us to combine them in many different ways, using a new type of poset called a *partition product*, and thereby obtain a model of this axiom in which $2^{\aleph_0} = \aleph_3$.

1. Introduction

Ramsey's Theorem, regarding colorings of tuples of ω, is a fundamental result in combinatorics. Naturally, set theorists have studied generalizations of this theorem which concern colorings of pairs of countable ordinals, that is to say, colorings on ω_1. The most straightforward generalization of this theorem is the assertion that any coloring of pairs of countable ordinals has an uncountable homogeneous set. However, this naive generalization is provably false, at least in ZFC. In order to obtain consistent generalizations of Ramsey's Theorem to ω_1, various topological restrictions are placed on the colorings, resulting in so-called Coloring Axioms. Let us now discuss the two most prominent of these which have appeared in the literature; we will use the notation $[A]^2$ to denote all two-element subsets of A.

Definition 1.1. *A function $\chi : [\omega_1]^2 \longrightarrow \{0, 1\}$ is said to be an* open coloring *if it is continuous with respect to some second countable, Hausdorff topology on ω_1. $A \subseteq \omega_1$ is said to be* χ-homogeneous *if χ is constant on $[A]^2$.*

The Abraham-Rubin-Shelah Open Coloring Axiom*, abbreviated OCA_{ARS}, states that for any open coloring χ on ω_1, there exists a partition $\omega_1 = \bigcup_{n<\omega} A_n$ such that each A_n is χ-homogeneous.*

Abraham and Shelah ([1]) first studied a weaker version of this axiom during the course of their investigation into the relationship between Martin's Axiom and Baumgartner's Axiom ([3]). This weaker version is concerned just with monotonic subfunctions of injective, real-valued functions. The full version made its debut in [2], where the authors studied it alongside a number of other axioms about $\aleph_1$-sized sets of reals. In particular, they showed that OCA_{ARS} is consistent with ZFC.

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A little later, Todorčević isolated the following axiom ([5]):

Definition 1.2. *The* Todorčević Open Coloring Axiom*, abbreviated* OCA_T*, states the following: let* A *be a set of reals, and suppose that* $[A]^2 = K_0 \cup K_1$*, where* K_0 *is open in* $[A]^2$*. Then either there is an uncountable* $A_0 \subseteq A$ *such that* $[A_0]^2 \subseteq K_0$*, or there is a partition* $A = \bigcup_{n<\omega} A_n$ *such that* $[A_n]^2 \subseteq K_1$ *for each* $n < \omega$*.*

If we restrict our attention to sets of reals A *with size* $\aleph_1$*, we denote this axiom by* $\mathsf{OCA}_T(\aleph_1)$*.*

Both of these axioms imply that the CH is false. Indeed, OCA_T implies that the bounding number $\mathfrak{b}$ is $\aleph_2$ (see [5]); see [1] and the citations therein for the proof that OCA_{ARS} implies that the CH is false. Moreover, the conjunction of these axioms is consistent with $2^{\aleph_0} = \aleph_2$; for instance, they are both consequences of PFA, though their conjunction can be shown to be consistent directly. Thus each of these coloring axioms has some effect on the size of the continuum.

It is therefore of interest whether or not these axioms, individually or jointly, actually decide the value of the continuum. In the case of OCA_T, Farah has shown in an unpublished note that $\mathsf{OCA}_T(\aleph_1)$ is consistent with an arbitrarily large value of the continuum, though it is not known whether the full OCA_T is consistent with larger values of the continuum than $\aleph_2$. On the other hand, Moore has shown ([4]) that $\mathsf{OCA}_T + \mathsf{OCA}_{ARS}$ does decide that the continuum is exactly $\aleph_2$.

However, the question of whether OCA_{ARS} is powerful enough to decide the value of the continuum on its own, first asked in [2], has remained open. There are a number of difficulties in obtaining a model of OCA_{ARS} with a "large continuum," i.e., with $2^{\aleph_0} > \aleph_2$. Chief among these difficulties is to construct so-called *preassignments of colors*. The authors of [1] first discovered the technique of preassigning colors and used this technique to prove the consistency of the weaker version of OCA_{ARS} mentioned above. As used in [2], a preassignment of colors is a function which decides, in the ground model, whether the forcing will place a countable ordinal α inside *some* 0-homogeneous or *some* 1-homogeneous set, with respect to a fixed coloring. The key to the consistency of OCA_{ARS} is to construct preassignments in such a way that the posets which add the requisite homogeneous sets, as guided by the preassignments, are c.c.c.

However, the known constructions of "good" preassignments only work under the CH. Since forcing iterations whose strict initial segments satisfy the CH can only lead to a model where the continuum is at most $\aleph_2$, this creates considerable difficulties for obtaining models of OCA_{ARS} in which the continuum is, say, $\aleph_3$.

In this paper, we prove the following theorem, thereby providing an answer to this question:

Theorem 1.3. *If* ZFC *is consistent, then so is* $\mathsf{ZFC} + \mathsf{OCA}_{ARS} + 2^{\aleph_0} = \aleph_3$*.*

The key to our solution is to construct names for preassignments with a substantial amount of symmetry. Roughly, suppose that $\mathbb{P}$ is a "nice" iteration of $\aleph_1$-sized, c.c.c. posets, where the length of $\mathbb{P}$ is less than ω_2; note that $\mathbb{P}$ preserves the CH. We show how to construct $\mathbb{P}$-names $\dot{f}$ for preassignments so that the name $\dot{f}$ can be interpreted by a host of different V-generics for $\mathbb{P}$ and still give rise to a c.c.c. product of posets. We then combine such shorter iterations, which function as a type of alphabet, into much longer ones which we call *Partition Products*. Finally, we force with a large partition product to construct a model of OCA_{ARS} wherein $2^{\aleph_0} = \aleph_3$. The general theme of the paper, then, is the following: short iterations are necessary to preserve the CH and thereby construct effective preassignments; longer iterations, built out of these smaller ones in specific ways, can be used to obtain models with a large continuum.

The above method is general enough that it can be adapted to strengthen Theorem 1.3 to obtain the forcing axiom $\mathsf{FA}(\aleph_2, \text{Knaster}(\aleph_1))$; this forcing axiom asserts that for any Knaster poset $\mathbb{P}$ of size $\leq \aleph_1$ and any sequence $\langle D_i : i < \omega_2 \rangle$ of $\aleph_2$-many dense subsets of $\mathbb{P}$, there is a filter for $\mathbb{P}$ which meets each of the D_i. Thus we may obtain the following theorem:

Theorem 1.4. *If* ZFC *is consistent, then so is* $\mathsf{ZFC} + \mathsf{OCA}_{ARS} + 2^{\aleph_0} = \aleph_3 + \mathsf{FA}(\aleph_2, \text{Knaster}(\aleph_1))$.

The outline of the paper is as follows: in Section 2, we introduce the definition of a Partition Product and prove a number of general facts about this type of poset. In Section 3, we develop the machinery to combine partition products in a variety of ways. In Section 4, we show how to construct "partition product names" for highly symmetric preassignments; this section forms the technical heart of the paper. And finally, in Section 5, we show how to recursively construct partition products in L, using the results from Section 4 to push the construction through successor stages. In particular, we construct the partition product which witnesses Theorem 1.3. It would be helpful, though not necessary, for the reader to be familiar with the first few sections of the paper [2], in particular, their construction of preassignments and the role which preassignments play in showing the consistency of OCA_{ARS}.

A few remarks about notation are in order: first, if f is a function and $A \subseteq \text{dom}(f)$, then we use $f[A]$ to denote $\{f(x) : x \in A\}$. Second, we will often be working in the context of a poset $\mathbb{R}$ as well as various other posets related to it; these other posets will have notational decorations, for example, $\mathbb{R}^*$. If $\dot{G}$ is the canonical $\mathbb{R}$-name for a generic filter, we use the corresponding decorations, such as $\dot{G}^*$, to denote the related names.

2. Partition Products

2.1. **Definition and Basic Facts.** Our first goal in this section is to define the notion of a *partition product* and prove a few basic lemmas. Roughly speaking, this is a class of finite support iterations which are built in very specific ways, but which is rich enough to be closed under products, closed under products of iterations taken over a common initial segment, and closed under more general "partitioned products" of segments of the iterations taken over common earlier segments. After the definition, we will provide comments which motivate it, as well as what is to come.

We begin by fixing some unbounded set $C \subseteq \omega_2$; this set will be specified in Section 5. We define by recursion the notion of a *partition product* based upon a sequence $\underline{\mathbb{P}} \restriction \kappa = \langle \mathbb{P}_\delta : \delta \in C \cap \kappa \rangle$ of posets and a sequence $\underline{\dot{\mathbb{Q}}} \restriction \kappa = \langle \dot{\mathbb{Q}}_\delta : \delta \in C \cap \kappa \rangle$ of names, where $\kappa \in C \cup \{\omega_2\}$. Every such object will be a poset $\mathbb{R}$ consisting of various finite partial functions on some set X of ordinals. This set X will be definable from $\mathbb{R}$ and will be called the *domain* of $\mathbb{R}$. The definition is by recursion on κ, and we make the following recursive assumptions about the objects $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$:

(i) for each $\delta \in C \cap \kappa$, $\mathbb{P}_\delta$ (the so-called *canonical δ-partition product*) is a partition product based upon $\underline{\mathbb{P}} \restriction \delta$ and $\underline{\dot{\mathbb{Q}}} \restriction \delta$, and $\dot{\mathbb{Q}}_\delta$ is a $\mathbb{P}_\delta$-name for a poset. The domain of $\mathbb{P}_\delta$ is an ordinal, which we call ρ_δ, and $\rho_\delta \leq \delta^+$.

The definition of partition products below is such that for each $\delta \leq \kappa$, every partition product $\mathbb{R}$, with domain X, say, based upon $\underline{\mathbb{P}} \restriction \delta$ and $\underline{\dot{\mathbb{Q}}} \restriction \delta$ comes equipped with two additional functions $\text{base}_{\mathbb{R}}$ and $\text{index}_{\mathbb{R}}$ defined on X. These functions for $\mathbb{R}$ satisfy (among other properties to be specified later) the following:

(ii) for each $\xi \in X$, $\text{index}_{\mathbb{R}}(\xi) \in C \cap \delta$, and $\text{base}_{\mathbb{R}}(\xi)$ is a pair
$$\text{base}_{\mathbb{R}}(\xi) = (b_{\mathbb{R}}(\xi), \pi^{\mathbb{R}}_\xi),$$
where $b_{\mathbb{R}}(\xi) \subseteq X \cap \xi$ and $\pi^{\mathbb{R}}_\xi$ is a bijection from $\rho_{\text{index}_{\mathbb{R}}(\xi)}$ onto $b_{\mathbb{R}}(\xi)$.

For each $\delta \in C \cap \kappa$, we abbreviate $\text{index}_{\mathbb{P}_\delta}$ by index_δ, and we abbreviate $\text{base}_{\mathbb{P}_\delta}(\xi)$ by $\text{base}_\delta(\xi) = (b_\delta(\xi), \pi^\delta_\xi)$, for each $\xi < \rho_\delta$. While (ii) above holds for all partition products, the next recursive assumption specifically concerns the canonical partition products $\mathbb{P}_\delta$:

(iii) for each $\xi < \rho_\delta$, $b_\delta(\xi)$ has ordertype $\rho_{\text{index}_\delta(\xi)}$, and $\pi^\delta_\xi : \rho_{\text{index}_\delta(\xi)} \longrightarrow b_\delta(\xi)$ is the order isomorphism.

Given a partition product $\mathbb{R}$ with domain X based upon $\underline{\mathbb{P}} \restriction \delta$ and $\underline{\dot{\mathbb{Q}}} \restriction \delta$, for some $\delta \in C \cap (\kappa+1)$, we say that a bijection $\sigma : X \longrightarrow X^*$ is an *acceptable rearrangement of* $\mathbb{R}$ if for all $\zeta, \xi \in X$, if $\zeta \in b_{\mathbb{R}}(\xi)$, then $\sigma(\zeta) < \sigma(\xi)$. The definition of partition products below is such that the following holds:

(iv) let $\delta \in C \cap \kappa$, let $\mathbb{R}$ be a partition product with domain X based upon $\underline{\mathbb{P}} \restriction \delta$ and $\underline{\dot{\mathbb{Q}}} \restriction \delta$, and suppose that $\sigma : X \longrightarrow X^*$ is an acceptable rearrangement of $\mathbb{R}$. Then σ lifts uniquely to an isomorphism (also denoted σ) from $\mathbb{R}$ to a partition product $\mathbb{R}^*$ on X^* based upon $\underline{\mathbb{P}} \restriction \delta$ and $\underline{\dot{\mathbb{Q}}} \restriction \delta$. We also have that any $\mathbb{R}$-name $\dot{\tau}$ lifts to a name in $\mathbb{R}^*$, which we denote by $\sigma(\dot{\tau})$, such that if G is generic for $\mathbb{R}$ and if G^* is the isomorphic generic induced by σ, then $\dot{\tau}[G] = \sigma(\dot{\tau})[G^*]$.

We call the partition product $\mathbb{R}^*$ in (iv) the *σ-rearrangement of* $\mathbb{R}$ and denote it by $\sigma[\mathbb{R}]$; we also refer to the $\mathbb{R}^*$-name $\sigma(\dot{\tau})$ as the *σ-rearrangement of* $\dot{\tau}$. Our definition of $\mathbb{R}^*$ and the lifted embedding, which we give later, are such that the next item holds:

(v) let δ, X, $\mathbb{R}$, and σ be as in (iv). Then for each $\xi \in X$, $\text{base}_{\sigma[\mathbb{R}]}(\sigma(\xi)) = (\sigma[b_{\mathbb{R}}(\xi)], \sigma \circ \pi^{\mathbb{R}}_\xi)$ and $\text{index}_{\sigma[\mathbb{R}]}(\sigma(\xi)) = \text{index}_{\mathbb{R}}(\xi)$.

In light of the requirement from (i) that $\rho_\delta \leq \delta^+$, for each $\delta \in C \cap \kappa$, let us fix surjections $\varphi_{\delta,\mu} : \delta \longrightarrow \mu$ for each $\mu < \rho_\delta$. We refer to this sequence of surjections as $\vec{\varphi}$, and we fix this notation until specified in Section 5.

Suppose that $\bar{\delta} \leq \delta$ are both in $C \cap \kappa$, $\bar{\mu} < \rho_{\bar{\delta}}$, and $\mu < \rho_\delta$. We say that a subset A of μ *matches* $\langle \delta, \mu \rangle$ to $\langle \bar{\delta}, \bar{\mu} \rangle$ if the following three conditions are satisfied:

(a) A is of the form $\varphi_{\delta,\mu}[\bar{\delta}]$;
(b) A is a countably closed subset of μ, i.e., closed under limit points less than μ of cofinality ω;
(c) if $\mu > \bar{\delta}$, then $\delta \in A$, $A \cap \delta = \bar{\delta}$, and, letting j denote the transitive collapse of A, we have that
$$j \circ \varphi_{\delta,\mu} \restriction \bar{\delta} = \varphi_{\bar{\delta},\bar{\mu}}.$$

We will now define what it means for two functions base and index on a set X to support a partition product, and after doing so, we will finally define a partition product.

Definition 2.1. *Let X be a set of ordinals, and let* base *and* index *be two functions with domain X. We say that* base *and* index *are functions which* support a partition product on X based upon $\underline{\mathbb{P}} \upharpoonright \kappa$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa$ *if the following conditions are satisfied:*

1. *for each $\xi \in X$, $\operatorname{index}(\xi) \in C \cap \kappa$ and $\operatorname{base}(\xi)$ is a pair $(b(\xi), \pi_\xi)$, where $b(\xi) \subseteq X \cap \xi$ and $\pi_\xi : \rho_{\operatorname{index}(\xi)} \longrightarrow b(\xi)$ is an acceptable rearrangement of $\mathbb{P}_{\operatorname{index}(\xi)}$;*
2. *let $\xi \in X$, and set $\delta := \operatorname{index}(\xi)$. Then for all $\zeta \in b(\xi)$, setting $\zeta_0 := \pi_\xi^{-1}(\zeta)$, we have $\operatorname{base}(\zeta) = (\pi_\xi[b_\delta(\zeta_0)], \pi_\xi \circ \pi^\delta_{\zeta_0})$ and $\operatorname{index}(\zeta) = \operatorname{index}_\delta(\zeta_0)$;*
3. *let $\xi_1, \xi_2 \in X$. Suppose that $\operatorname{index}(\xi_1) \leq \operatorname{index}(\xi_2)$ and that there is some $\zeta \in b(\xi_1) \cap b(\xi_2)$. Set $\mu_1 := \pi_{\xi_1}^{-1}(\zeta)$ and $\mu_2 := \pi_{\xi_2}^{-1}(\zeta)$. Then $\pi_{\xi_1}[\mu_1] \subseteq \pi_{\xi_2}[\mu_2]$, and $\pi_{\xi_2}^{-1}[\pi_{\xi_1}[\mu_1]]$ matches $\langle \operatorname{index}(\xi_2), \mu_2 \rangle$ to $\langle \operatorname{index}(\xi_1), \mu_1 \rangle$.*

Definition 2.2. *We say that $\mathbb{R}$ is a* partition product *with domain X, based upon $\underline{\mathbb{P}} \upharpoonright \kappa$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa$, with base and index functions* $\operatorname{base}_{\mathbb{R}}$ *and* $\operatorname{index}_{\mathbb{R}}$ *if*

1. $\operatorname{base}_{\mathbb{R}}$ *and* $\operatorname{index}_{\mathbb{R}}$ *support a partition product on X based upon $\underline{\mathbb{P}} \upharpoonright \kappa$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa$ as in Definition 2.1;*
2. *$\mathbb{R}$ consists of all finite partial functions p with $\operatorname{dom}(p) \subseteq X$ so that for all $\xi \in \operatorname{dom}(p)$, $p(\xi)$ is a canonical $\pi^{\mathbb{R}}_\xi[\mathbb{P}_{\operatorname{index}_{\mathbb{R}}(\xi)}]$-name for an element of $\dot{\mathbb{U}}_\xi := \pi^{\mathbb{R}}_\xi(\dot{\mathbb{Q}}_{\operatorname{index}_{\mathbb{R}}(\xi)})$, i.e., the $\pi^{\mathbb{R}}_\xi$-rearrangement of the $\mathbb{P}_{\operatorname{index}_{\mathbb{R}}(\xi)}$-name $\dot{\mathbb{Q}}_{\operatorname{index}_{\mathbb{R}}(\xi)}$, as in (iv).*

$\mathbb{R}$ is ordered as follows: $q \leq_{\mathbb{R}} p$ iff $\operatorname{dom}(p) \subseteq \operatorname{dom}(q)$, and for all $\xi \in \operatorname{dom}(p)$,

$$q \upharpoonright b_{\mathbb{R}}(\xi) \Vdash_{\pi^{\mathbb{R}}_\xi[\mathbb{P}_{\operatorname{index}_{\mathbb{R}}(\xi)}]} q(\xi) \leq_{\dot{\mathbb{U}}_\xi} p(\xi).$$

The definition of a partition product refers not only to the sequences $\underline{\mathbb{P}} \upharpoonright \kappa$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa$, but additionally to the ordinal κ, and to the sequence of functions $\operatorname{index}_\delta$, $\operatorname{base}_\delta$, and $\varphi_{\delta,\mu}$ for $\delta \in C \cap \kappa$ and $\mu < \rho_\delta$. We suppress this dependence in the notation, viewing these additional objects as implicit in $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa$.

We have one final bit of notation before making a number of additional remarks about the definition: given a partition product $\mathbb{R}$ with domain X, say, and given $X_0 \subseteq X$, we define $\mathbb{R} \upharpoonright X_0$ to be the set $\{p \in \mathbb{R} : \operatorname{dom}(p) \subseteq X_0\}$, with the restriction of $\leq_{\mathbb{R}}$, which may or may not itself be a partition product.

Remark 2.3. Note that the definition of the ordering $\leq_{\mathbb{R}}$ in Definition 2.2 presupposes that for each $q \in \mathbb{R}$ and $\xi \in \operatorname{dom}(q)$, $q \upharpoonright b_{\mathbb{R}}(\xi)$ is a condition in $\pi^{\mathbb{R}}_\xi[\mathbb{P}_{\operatorname{index}_{\mathbb{R}}(\xi)}]$. This holds as follows: fix $q \in \mathbb{R}$, $\xi \in \operatorname{dom}(q)$, and set $\delta := \operatorname{index}_{\mathbb{R}}(\xi)$. Let $\mathbb{S}$ abbreviate the poset $\pi^{\mathbb{R}}_\xi[\mathbb{P}_\delta]$. As $\delta < \kappa$, we know by recursion that $\mathbb{S}$ consists of all finite partial functions u on $b_{\mathbb{R}}(\xi)$ such that for each $\zeta \in \operatorname{dom}(u)$, $u(\zeta)$ is a canonical $\pi^{\mathbb{S}}_\zeta[\mathbb{P}_{\operatorname{index}_{\mathbb{S}}(\zeta)}]$-name for an element of $\pi^{\mathbb{S}}_\zeta(\dot{\mathbb{Q}}_{\operatorname{index}_{\mathbb{S}}(\zeta)})$. Now fixing $\zeta \in b_{\mathbb{R}}(\xi) \cap \operatorname{dom}(q)$, by (2) of Definition 2.1 and item (v), $\operatorname{base}_{\mathbb{R}}(\zeta) = \operatorname{base}_{\mathbb{S}}(\zeta)$ and $\operatorname{index}_{\mathbb{R}}(\zeta) = \operatorname{index}_{\mathbb{S}}(\zeta)$, and therefore, $q(\zeta)$ is indeed a canonical $\pi^{\mathbb{S}}_\zeta[\mathbb{P}_{\operatorname{index}_{\mathbb{S}}(\zeta)}]$-name for a condition in $\pi^{\mathbb{S}}_\zeta(\dot{\mathbb{Q}}_{\operatorname{index}_{\mathbb{S}}(\zeta)})$. Thus $q \upharpoonright b_{\mathbb{R}}(\xi)$ is a condition in $\mathbb{S}$.

Note also that by similar reasoning, every condition in $\pi^{\mathbb{R}}_\xi[\mathbb{P}_{\operatorname{index}_{\mathbb{R}}(\xi)}]$ is a condition in $\mathbb{R}$, and in fact, $\mathbb{R} \upharpoonright b_{\mathbb{R}}(\xi)$ equals $\pi^{\mathbb{R}}_\xi[\mathbb{P}_{\operatorname{index}_{\mathbb{R}}(\xi)}]$.

Remark 2.4. A partition product based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$ should be viewed (roughly) as an iteration into which we can fit many copies of the shorter posets $\mathbb{P}_\delta$ and $\mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta$, for $\delta \in C \cap \kappa$. In this way, the canonical partition products function as a kind of "alphabet" with which we build other partition products. In most of our intended applications, each name $\dot{\mathbb{Q}}_\delta$ will either be Cohen forcing for adding a single real or will be a $\mathbb{P}_\delta$-name for a poset to decompose ω_1 into countably-many homogeneous sets with respect to some open coloring $\dot{\chi}$.

Remark 2.5. A partition product is a somewhat flexible object in that we have a limited, but non-trivial, ability to rearrange coordinates. The reason we need these rearrangements to be *acceptable*, as defined above, is that if $\mathbb{R}$ is a partition product and $\zeta \in b_{\mathbb{R}}(\xi)$, then what happens at coordinate ξ depends on what happens at the earlier coordinate ζ, and therefore the image of ζ under a rearrangement must remain below the image of ξ. The ability to rearrange coordinates will be useful later on when we need to check (roughly) that there are not too many isomorphism types of sufficiently simple partition products (see Lemma 4.17).

Remark 2.6. We will prove Theorem 1.3 by forcing over L with a partition product $\mathbb{P}_{\omega_2}$ with domain ω_3. When we construct these objects in L, the set C in the definition will consist, roughly, of all uncountable $\kappa < \omega_2$ which look locally like ω_2. More specifically, we will show how to construct the sequences $\underline{\mathbb{P}} = \langle \mathbb{P}_\delta : \delta \in C \cup \{\omega_2\} \rangle$ and $\underline{\dot{\mathbb{Q}}} = \langle \dot{\mathbb{Q}}_\delta : \delta \in C \rangle$ in such a way that for each $\kappa \in C \cup \{\omega_2\}$, every partition product based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$ is c.c.c. In particular, our final partition product $\mathbb{P}_{\omega_2}$ will be c.c.c., which is the result that we need.

Every partition product is a dense subset of an iteration, as the next lemma shows.

Lemma 2.7. *Let $\mathbb{R}$ be a partition product with domain X. Then $\mathbb{R}$ is a dense subset of a finite support iteration on X.*

Proof. Let $\mathbb{R}^*$ be the finite support iteration based upon the sequence of names $\langle \dot{\mathbb{U}}_\xi : \xi \in X \rangle$, where the names are defined as in Definition 2.2 (2). Then $\mathbb{R}$ is a dense subset of $\mathbb{R}^*$; the proof is straightforward, using the fact that for each $\xi \in X$, $\dot{\mathbb{U}}_\xi$ is an $\mathbb{R} \restriction b_{\mathbb{R}}(\xi)$-name for a poset. □

Remark 2.8. In studying partition products, we choose to work with this dense subset, rather than the iteration itself, to avoid various technicalities, especially with regards to restricting conditions.

We now want to understand further circumstances wherein we may restrict a partition product with domain X to various subsets of X and still obtain a partition product. This motivates the following key definition.

Definition 2.9. *Let $\mathbb{R}$ be a partition product, say with domain X, and let $B \subseteq X$. We say that B is* base-closed with respect to $\mathbb{R}$ *if for all $\xi \in B$, $b_{\mathbb{R}}(\xi) \subseteq B$.*

If the partition product $\mathbb{R}$ is clear from context, we will often drop the phrase "with respect to $\mathbb{R}$" in the above definition and simply say that $B \subseteq X$ is *base-closed*. We will also drop the "$\mathbb{R}$" from expressions such as $\text{index}_{\mathbb{R}}$, $b_{\mathbb{R}}(\xi)$, and $\pi^{\mathbb{R}}_\xi$ if the context is clear.

Lemma 2.10. *Let $\mathbb{R}$ be a partition product with domain X, and let $\xi \in X$. Then $b(\xi)$ is base-closed. Also, for each $\zeta \in b(\xi)$, $\text{index}(\zeta) < \text{index}(\xi)$.*

Proof. Set $\delta := \mathrm{index}(\xi)$, let $\zeta \in b(\xi)$, and set $\zeta_0 := \pi_\xi^{-1}(\zeta)$. Then since π_ξ is an acceptable rearrangement of $\mathbb{P}_\delta$, condition (2) in Definition 2.1 and item (v) imply that $b(\zeta) = b_{\pi_\xi[\mathbb{P}_\delta]}(\zeta)$ and also that $b_{\pi_\xi[\mathbb{P}_\delta]}(\zeta)$ equals $\pi_\xi[b_\delta(\zeta_0)] \subseteq b(\xi)$. Thus $b(\zeta) \subseteq b(\xi)$.

To see that $\mathrm{index}(\zeta) < \delta$, we recall that $\mathrm{index}(\zeta) = \mathrm{index}_{\pi_\xi[\mathbb{P}_\delta]}(\zeta)$ which in turn equals $\mathrm{index}_\delta(\zeta_0)$. Since $\mathbb{P}_\delta$ is a partition product based upon $\underline{\mathbb{P}} \restriction \delta$ and $\underline{\dot{\mathbb{Q}}} \restriction \delta$, we must have $\mathrm{index}_\delta(\zeta_0) \in C \cap \delta$, and therefore $\mathrm{index}(\zeta) = \mathrm{index}_\delta(\zeta_0)$ is below δ. $\square$

The following lemma tells us that we may restrict the functions in a partition product to a base-closed subset and obtain a partition product.

Lemma 2.11. *Suppose that $\mathbb{R}$ is a partition product with domain X and that $B \subseteq X$ is base-closed. Then* $\mathrm{base} \restriction B$ *and* $\mathrm{index} \restriction B$ *support a partition product on B, and this partition product is exactly $\mathbb{R} \restriction B$. Moreover, if there is a $\beta \in C$ such that $\{\mathrm{index}(\xi) : \xi \in B\} \subseteq \beta$, then $\mathbb{R} \restriction B$ is a partition product based upon $\underline{\mathbb{P}} \restriction \beta$ and $\underline{\dot{\mathbb{Q}}} \restriction \beta$. Finally, $\mathbb{R} \restriction B$ is a complete subposet of $\mathbb{R}$.*

Proof. It is straightforward to check that $\mathrm{base} \restriction B$ and $\mathrm{index} \restriction B$ support a partition product on B, using the fact that B is base-closed and also to check that $\mathbb{R} \restriction B$ is the partition product supported by these functions. It is also straightforward to see that $\mathbb{R} \restriction B$ is based upon $\underline{\mathbb{P}} \restriction \beta$ and $\underline{\dot{\mathbb{Q}}} \restriction \beta$ if $\mathrm{index}(\xi) < \beta$, for all $\xi \in B$.

We now verify that the inclusion is a complete embedding of $\mathbb{R} \restriction B$ into $\mathbb{R}$. The only non-trivial property which we must check is the following: if $p \in \mathbb{R}$, $q \in \mathbb{R} \restriction B$, and $q \leq_{\mathbb{R} \restriction B} p \restriction B$, then q and p are compatible in $\mathbb{R}$. To see this, fix such p and q. We claim that $r := q \cup p \restriction (X \setminus \mathrm{dom}(q))$ is a condition in $\mathbb{R}$ which is below p and q. As it is clear that r is a condition, by (2) of Definition 2.2, we check that it is below both p and q. Fix $\xi \in \mathrm{dom}(r)$, and suppose that $r \restriction \xi$ is below both $p \restriction \xi$ and $q \restriction \xi$. If ξ is not in $\mathrm{dom}(q) \cap \mathrm{dom}(p)$, then it is clear that $r \restriction (\xi + 1)$ is a condition below both $p \restriction (\xi+1)$ and $q \restriction (\xi+1)$. So suppose that $\xi \in \mathrm{dom}(q) \cap \mathrm{dom}(p)$, and in particular, that $\xi \in B$. Since q extends $p \restriction B$ in $\mathbb{R} \restriction B$ and since the base and index functions for $\mathbb{R} \restriction B$ are the restrictions of those for $\mathbb{R}$, we have that $q \restriction b(\xi)$ forces in $\pi_\xi[\mathbb{P}_{\mathrm{index}(\xi)}]$ that $q(\xi) \leq_{\dot{\mathbb{U}}_\xi} p(\xi)$, where $\dot{\mathbb{U}}_\xi = \pi_\xi(\dot{\mathbb{Q}}_{\mathrm{index}(\xi)})$. Since $r \restriction \xi$ extends $q \restriction \xi$ and since $b(\xi) \subseteq \xi$, we know that $r \restriction b(\xi)$ extends $q \restriction b(\xi)$ in $\pi_\xi[\mathbb{P}_{\mathrm{index}(\xi)}]$. Therefore $r \restriction b(\xi)$ also forces that $q(\xi)$ is below $p(\xi)$ in $\dot{\mathbb{U}}_\xi$. Since $r(\xi) = q(\xi)$, this finishes the proof. $\square$

If $\mathbb{R}$, X, and B are as in the previous lemma, and if G is generic for $\mathbb{R}$, we use $G \restriction B$ to denote $\{p \restriction B : p \in G\}$, which is generic for $\mathbb{R} \restriction B$.

2.2. **Rearranging Partition Products.** Our next main goal is to prove the Rearrangement Lemma, which, as the name suggests, allows us to use an acceptable rearrangement to shift around the coordinates of a partition product and still obtain a partition product. More specifically, if we have an acceptable rearrangement of a partition product $\mathbb{R}$, then we can "compose" it with the base and index functions from $\mathbb{R}$, as stated in the next definition.

Definition 2.12. *Suppose that $\sigma : X \longrightarrow X^*$ is an acceptable rearrangement of $\mathbb{R}$, a partition product with domain X. We define the functions $\sigma[\mathrm{base}_{\mathbb{R}}]$ and $\sigma[\mathrm{index}_{\mathbb{R}}]$ on X^* as follows: fix $\xi \in X$. Then set $\sigma[\mathrm{index}_{\mathbb{R}}](\sigma(\xi)) = \mathrm{index}_{\mathbb{R}}(\xi)$, and set $\sigma[\mathrm{base}_{\mathbb{R}}](\sigma(\xi))$ to be the pair*

$$(b^*(\sigma(\xi)), \pi^*_{\sigma(\xi)}),$$

where $b^*(\sigma(\xi)) = \sigma[b_{\mathbb{R}}(\xi)]$*, and where* $\pi^*_{\sigma(\xi)} = \sigma \circ \pi^{\mathbb{R}}_{\xi}$.

The following item, known as the Rearrangement Lemma, shows that the objects as in Definition 2.12 support a partition product isomorphic to the original one. The Rearrangement Lemma yields condition (iv) above. It is proved for partition products based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$ by induction on κ, assuming it is already known for $\delta < \kappa$.

Lemma 2.13. (Rearrangement Lemma) *Suppose that* $\mathbb{R}$ *is a partition product with domain* X *and that* $\sigma : X \longrightarrow X^*$ *is an acceptable rearrangement of* $\mathbb{R}$*. Then* $\sigma[\mathrm{base}_{\mathbb{R}}]$ *and* $\sigma[\mathrm{index}_{\mathbb{R}}]$ *support a partition product on* X^**. Moreover, letting* $\sigma[\mathbb{R}]$ *be this partition product, we have that there is a unique lift of* σ *to an isomorphism from* $\mathbb{R}$ *to* $\sigma[\mathbb{R}]$*.*

Proof. It is straightforward to check that the functions $\sigma[\mathrm{base}_{\mathbb{R}}]$ and $\sigma[\mathrm{index}_{\mathbb{R}}]$ satisfy all three conditions of Definition 2.1, since σ is an acceptable rearrangement. Thus we show that σ lifts to an isomorphism, also denoted σ, from $\mathbb{R}$ onto $\sigma[\mathbb{R}]$. Let $p \in \mathbb{R}$. Then we set $\sigma(p)$ to be the function with domain $\sigma[\mathrm{dom}(p)]$ such that for each $\xi \in \mathrm{dom}(p)$, $\sigma(p)(\sigma(\xi))$ equals the $\sigma \restriction b_{\mathbb{R}}(\xi)$-rearrangement of the name $p(\xi)$, as in (iv). This is well-defined by an inductive application of the Rearrangement Lemma to the acceptable rearrangement $\sigma \restriction b_{\mathbb{R}}(\xi)$ of the partition product $\mathbb{R} \restriction b_{\mathbb{R}}(\xi)$, which is based upon the sequence up to $\mathrm{index}_{\mathbb{R}}(\xi) < \kappa$. It is straightforward to see that $\sigma(p)$ is a condition in $\sigma[\mathbb{R}]$ and that this defines an isomorphism. □

Remark 2.14. Given $\mathbb{R}$ and σ as in Lemma 2.13 and setting $\mathbb{R}^* := \sigma[\mathbb{R}]$, if G is generic for $\mathbb{R}$, then we use $\sigma(G)$ to denote the generic $\{\sigma(p) : p \in \mathbb{R}\}$ for $\mathbb{R}^*$. Furthermore, given an $\mathbb{R}$-name $\dot{\tau}$, we recursively define $\sigma(\dot{\tau})$ to be the $\sigma[\mathbb{R}]$-name $\{\langle \sigma(p), \sigma(\dot{x})\rangle : \langle p, \dot{x}\rangle \in \dot{\tau}\}$. It is straightforward to check that $\dot{\tau}[G] = \sigma(\dot{\tau})[\sigma(G)]$ for any generic G for $\mathbb{R}$. This name $\sigma(\dot{\tau})$ is the σ-rearrangement of $\dot{\tau}$ as in (iv) above.

Remark 2.15. Suppose that M and M^* are transitive, satisfy enough of ZFC, and that $\sigma : M \longrightarrow M^*$ is sufficiently elementary. Also, suppose that $\mathbb{R} \in M$ is a partition product, say with domain X, and that $\mathbb{R}$ is based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$. It is straightforward to check that $\pi := \sigma \restriction X$ provides an acceptable rearrangement of $\mathbb{R}$. There is now a potential conflict between the π-rearrangements of conditions in $\mathbb{R}$ and the images of these conditions under the embedding σ. However, these are the same if σ doesn't move any members of the "alphabet" $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$. The next lemma summarizes what we need about this situation and will be used crucially in the final proof of Theorem 1.3 in Section 5. For the next lemma, we will continue to use π to denote rearrangements, and we will keep σ as the elementary map.

Lemma 2.16. *Let* $\sigma : M \longrightarrow M^*$*,* $\mathbb{R}$*,* X*,* κ*, and* π *be as in Remark 2.15. Further suppose that for each* $\delta \in C \cap \kappa$*,* σ *is the identity on every element of* $\mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta \cup \left\{\mathbb{P}_\delta, \dot{\mathbb{Q}}_\delta\right\}$*. Then for each* $p \in \mathbb{R}$*,* $\pi(p) = \sigma(p)$*.*

Furthermore, setting $\mathbb{R}^* := \sigma(\mathbb{R})$*,* $\sigma[X]$ *is a base-closed subset of* $\mathbb{R}^*$*, and* $\mathbb{R}^* \restriction \sigma[X]$ *equals* $\pi[\mathbb{R}]$*, the* π*-rearrangement of* $\mathbb{R}$*.*

Additionally, suppose that G *is* V*-generic for* $\mathbb{R}$*,* G^* *is* V*-generic for* $\mathbb{R}^*$*, and* σ *extends to a sufficiently elementary embedding* $\sigma^* : M[G] \longrightarrow M^*[G^*]$*. Suppose also that* $\dot{\tau}$ *is an* $\mathbb{R}$*-name (not necessarily in* M*) and* $\pi(\dot{\tau})$ *is the* π*-rearrangement* *of $\dot{\tau}$. Then $\pi(\dot{\tau})$ is an $\mathbb{R}^*$-name, and $\dot{\tau}[G] = \pi(\dot{\tau})[G^*]$. Finally, if $\dot{\mathbb{Q}}$ is an $\mathbb{R}$-name in M of M-cardinality $< \mathrm{crit}(\sigma)$ and names a poset contained in $\mathrm{crit}(\sigma)$, then $\sigma(\dot{\mathbb{Q}}) = \pi(\dot{\mathbb{Q}})$.*

Proof. We only prove the second and third parts. For the second part, fix some $\xi \in X$. Then $b_{\mathbb{R}}(\xi)$ is in bijection, via a bijection in M, with some ρ_α, for $\alpha < \kappa$. However, ρ_α is below $\mathrm{crit}(\sigma)$, since σ is the identity on $\mathbb{P}_\alpha$. Therefore,

$$b_{\mathbb{R}^*}(\sigma(\xi)) = \sigma(b_{\mathbb{R}}(\xi)) = \sigma[b_{\mathbb{R}}(\xi)],$$

where the first equality holds by the elementarity of σ and the second since $\mathrm{crit}(\sigma) > |b_{\mathbb{R}}(\xi)|$. This implies that $\sigma[X]$ is base-closed, and therefore $\mathbb{R}^* \restriction \sigma[X]$ is a partition product by Lemma 2.11. By the first part of the current lemma, we see that every condition in $\mathbb{R}^* \restriction \sigma[X]$ is in the image of σ. However, $\pi(p) = \sigma(p)$ for each condition $p \in \mathbb{R}$, and consequently $\mathbb{R}^* \restriction \sigma[X]$ equals $\pi[\mathbb{R}]$, the π-rearrangement of $\mathbb{R}$.

For the third part, let G and G^* be as in the statement of the lemma. Also let $\pi(G)$ denote the π-rearrangement of the filter G, as defined in Remark 2.14. By same remark, we have that $\dot{\tau}[G] = \pi(\dot{\tau})[\pi(G)]$. We also see that $\pi(\dot{\tau})$ is an $\mathbb{R}^*$-name, since it is a $\pi[\mathbb{R}]$-name and since, by the second part of the lemma, $\pi[\mathbb{R}] = \mathbb{R}^* \restriction \sigma[X]$ and $\sigma[X]$ is base-closed. Furthermore, $\sigma[G]$ is a subset of G^*, by the elementarity of σ^*. However, by the first part of the current lemma, $\sigma[G] = \{\sigma(p) : p \in G\} = \{\pi(p) : p \in G\} = \pi(G)$, and therefore

$$\dot{\tau}[G] = \pi(\dot{\tau})[\pi(G)] = \pi(\dot{\tau})[G^*].$$

Finally, if $\dot{\mathbb{Q}} \in M$ and satisfies the assumptions in the statement of the lemma, then $\sigma(\dot{\mathbb{Q}}) = \sigma[\dot{\mathbb{Q}}]$, and $\sigma[\dot{\mathbb{Q}}] = \pi(\dot{\mathbb{Q}})$. This completes the proof of the lemma. $\square$

Before we give applications of the Rearrangement Lemma, we record our definition of an embedding.

Definition 2.17. *Suppose that $\mathbb{R}$ and $\mathbb{R}^*$ are partition products with respective domains X and X^*. We say that an injection $\sigma : X \longrightarrow X^*$* embeds $\mathbb{R}$ *into* $\mathbb{R}^*$ *if $\sigma : X \longrightarrow \mathrm{ran}(\sigma)$ is an acceptable rearrangement of $\mathbb{R}$, and if $\sigma[\mathrm{base}_{\mathbb{R}}] = \mathrm{base}_{\mathbb{R}^*} \restriction \mathrm{ran}(\sigma)$ and $\sigma[\mathrm{index}_{\mathbb{R}}] = \mathrm{index}_{\mathbb{R}^*} \restriction \mathrm{ran}(\sigma)$.*

It is straightforward to check that if σ is an embedding as in Definition 2.17, and if G^* is generic over $\mathbb{R}^*$, then the filter $\sigma^{-1}(G^*) := \{p \in \mathbb{R} : \sigma(p) \in G^*\}$ is generic over $\mathbb{R}$. We also remark that, in the context of the above definition, $\sigma[\mathbb{R}] = \mathbb{R}^* \restriction \mathrm{ran}(\sigma)$.

Lemma 2.18. *Suppose that $\mathbb{R}$ is a partition product with domain X and $B \subseteq X$ is base-closed. Then $\mathbb{R}$ is isomorphic to a partition product $\mathbb{R}^*$ with a domain X^* such that B is an initial segment of X^* and $\mathbb{R}^* \restriction B = \mathbb{R} \restriction B$.*

Proof. We define a map σ with domain X which will lift to give us $\mathbb{R}^*$. Let $\xi \in X$. If $\xi \in B$, then set $\sigma(\xi) = \xi$. On the other hand, if $\xi \in X \backslash B$, say that ξ is the γth element of $X \backslash B$, then we define $\sigma(\xi) = \sup(X) + 1 + \gamma$.

We show that σ is an acceptable rearrangement of $\mathbb{R}$, and then we may set $\mathbb{R}^* := \sigma[\mathbb{R}]$ by Lemma 2.13. So suppose that $\zeta, \xi \in X$ and $\zeta \in b(\xi)$; we check that $\sigma(\zeta) < \sigma(\xi)$. There are two cases. On the one hand, if $\xi \in B$, then $b(\xi) \subseteq B$, since B is base-closed, and therefore $\zeta \in B$. Then $\sigma(\zeta) = \zeta < \xi = \sigma(\xi)$. On the other hand, if $\xi \notin B$, then either $\zeta \in B$ or not. If $\zeta \in B$, then $\sigma(\zeta) = \zeta < \sup(X) + 1 \leq \sigma(\xi)$, and if $\zeta \notin B$, then $\sigma(\zeta) < \sigma(\xi)$ since σ is order-preserving on $X \backslash B$. $\square$

It will be helpful later on to know that we can apply Lemma 2.18 ω-many times, as in the following corollary.

Corollary 2.19. *Suppose that $\mathbb{R}$ is a partition product with domain X and that for each $n < \omega$, π_n is an acceptable rearrangement of $\mathbb{R}$. Suppose that $\langle B_n : n \in \omega\rangle$ is a $\subseteq$-increasing sequence of base-closed subsets of X where $B_0 = \varnothing$ and where $X = \bigcup_n B_n$. Then there is a partition product $\mathbb{R}^*$ which has domain an ordinal ρ^* and an acceptable rearrangement $\sigma : X \longrightarrow \rho^*$ of $\mathbb{R}$ which lifts to an isomorphism of $\mathbb{R}$ onto $\mathbb{R}^*$ and which also satisfies that for each $n < \omega$, $\sigma[B_n]$ is an ordinal and $\pi_n \circ \sigma^{-1}$ is order-preserving on $\sigma[B_{n+1} \backslash B_n]$.*

Proof. We aim to recursively construct a sequence $\langle \mathbb{R}_n : n < \omega\rangle$ of partition products, where $\mathbb{R}_n$ has domain X_n, and a sequence $\langle \sigma_n : n < \omega\rangle$ of bijections, where $\sigma_n : X \longrightarrow X_n$, so that

(1) σ_n is an acceptable rearrangement of $\mathbb{R}$;
(2) $\sigma_n[B_n]$ is an ordinal, and in particular, an initial segment of X_n;
(3) for each $k < m < \omega$, $\sigma_k[B_k] = \sigma_m[B_k]$;
(4) for each $n < \omega$, $\pi_n \circ \sigma_{n+1}^{-1}$ is order-preserving on $\sigma_{n+1}[B_{n+1} \backslash B_n]$.

Suppose that we can do this. Then we define a map σ on X, by taking $\sigma(\xi)$ to be the eventual value of the sequence $\langle \sigma_n(\xi) : n < \omega\rangle$; we see that this sequence is eventually constant by (3) and the assumption that $\bigcup_n B_n = X$. By (2) and (3), $\sigma[B_n]$ is an ordinal, for each $n < \omega$, and therefore the range of σ is an ordinal, which we call ρ^*. Furthermore, $\pi_n \circ \sigma^{-1}$ is order-preserving on $\sigma[B_{n+1} \backslash B_n]$ by (4), and since σ and σ_{n+1} agree on B_{n+1}. Finally, by (1) we see that σ is an acceptable rearrangement of $\mathbb{R}$, and we thus take $\mathbb{R}^*$ to be the partition product isomorphic to $\mathbb{R}$ via σ, by Lemma 2.13.

We now show how to create the above objects. Suppose that $\langle \mathbb{R}_m : m < n\rangle$ and $\langle \sigma_m : m < n\rangle$ have been constructed. If $n = 0$, we take $\mathbb{R}_0 = \mathbb{R}$ and σ_0 to be the identity; since $B_0 = \varnothing$, this completes the base case. So suppose $n > 0$. Apply Lemma 2.18 to the partition product $\mathbb{R}_{n-1}$ and the base-closed subset $\sigma_{n-1}[B_n]$ of X_{n-1} to create a partition product $\mathbb{R}_n$ on a set X_n which is isomorphic to $\mathbb{R}_{n-1}$ via the acceptable rearrangement $\tau_n : X_{n-1} \longrightarrow X_n$ and which satisfies that $\sigma_{n-1}[B_n]$ is an initial segment of X_n. Since $\sigma_{n-1}[B_{n-1}]$ is an ordinal, by (2) applied to $n-1$, and since $\sigma_{n-1}[B_n]$ is an initial segment of X_n, we see that τ_n is the identity on $\sigma_{n-1}[B_{n-1}]$. Also, by composing τ_n with a further function and relabelling if necessary, we may assume that $\pi_{n-1} \circ \tau_n^{-1}$ just shifts the ordinals in $\sigma_{n-1}[B_n \backslash B_{n-1}]$ in an order-preserving way and that $\tau_n \circ \sigma_{n-1}[B_n]$ is an ordinal. We now take σ_n to be $\tau_n \circ \sigma_{n-1}$, and we see that σ_n and $\mathbb{R}_n$ satisfy the recursive hypotheses. $\square$

Lemma 2.20. *Suppose that $\beta \in C \cap \kappa$ and that $\mathbb{R}$ is a partition product with domain X based upon $\underline{\mathbb{P}} \restriction (\beta+1)$ and $\underline{\dot{\mathbb{Q}}} \restriction (\beta+1)$. Then, letting $B := \{\xi \in X : \mathrm{index}(\xi) < \beta\}$ and $I := \{\xi \in X : \mathrm{index}(\xi) = \beta\}$, B is base-closed, and $\mathbb{R}$ is isomorphic to*

$$(\mathbb{R} \restriction B) * \prod_{\xi \in I} \dot{\mathbb{Q}}_\beta \left[\pi_\xi^{-1}\left(\dot{G}_B \restriction b(\xi)\right)\right],$$

where $\dot{G}_B$ is the canonical $\mathbb{R} \restriction B$-name for the generic filter.

Proof. To see that B is base-closed, fix $\xi \in B$. Then for all $\zeta \in b(\xi)$, $\mathrm{index}(\zeta) < \mathrm{index}(\xi) < \beta$ by Lemma 2.10, and so $\zeta \in B$. Thus by Lemma 2.18, we may assume that B is an initial segment of X, and hence I is a tail segment of X. Now let

G_B be generic for $\mathbb{R} \restriction B$, and for each $\xi \in I$, let $G_{B,\xi}$ denote $\pi_\xi^{-1}(G_B \restriction b(\xi))$, which is generic for $\mathbb{P}_\beta$. The sequence of posets $\langle \dot{\mathbb{Q}}_\beta[G_{B,\xi}] : \xi \in I \rangle$ is in $V[G_B]$, and consequently the finite support iteration of $\langle \dot{\mathbb{Q}}_\beta[G_{B,\xi}] : \xi \in I \rangle$ in $V[G_B]$ is isomorphic to the (finite support) product $\prod_{\xi \in I} \dot{\mathbb{Q}}_\beta[G_{B,\xi}]$. Therefore, in V, $\mathbb{R}$ is isomorphic to the poset in the statement of the lemma. □

Remark 2.21. The previous lemma shows that a partition product does indeed have product-like behavior, and it is part of the justification for our term "partition product."

2.3. **Further Remarks on Matching.** In this subsection we state and prove a few consequences of the matching conditions (a)-(c) above. These results will, in combination with the ability to rearrange a partition product, allow us to find isomorphism types of sufficiently simple partition products inside sufficiently elementary, countably-closed models (see Lemma 4.17).

Remark 2.22. As mentioned earlier, we will carry out the construction of partition products in L. The matching conditions (a)-(c), combined with Definition 2.1, are roughly meant to capture the idea that the base functions, up to some rearranging, behave like the ordinals in a countably-closed Skolem hull of some suitable level of L.

Lemma 2.23. *Let $\mathbb{R}$ be a partition product, say with domain X, based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$. Let $\xi_1, \xi_2 \in X$, set $\delta_i = \mathrm{index}(\xi_i)$, for $i = 1, 2$, and suppose that $\delta_1 \leq \delta_2$. Finally, let $A := \pi_{\xi_2}^{-1}[b(\xi_1) \cap b(\xi_2)]$. Then A is definable in any sufficiently elementary substructure from $\vec{\varphi}$, the ordinals δ_1 and δ_2, and any cofinal $Z \subseteq A$.*

Proof. Let $Z \subseteq A$ be cofinal. For each $\alpha \in Z$, we have from Definition 2.1 (3) and condition (a) in the definition of matching that $A \cap \alpha = \varphi_{\delta_2,\alpha}[\delta_1]$. Therefore $A = \bigcup_{\alpha \in Z} \varphi_{\delta_2,\alpha}[\delta_1]$, which is inside any such sufficiently elementary substructure containing the requisite parameters. □

Corollary 2.24. *Let $\mathbb{R}$, X, ξ_1, ξ_2, and A be as in Lemma 2.23. Assume that for all $\xi \in C$, $\rho_\xi < \omega_2$. Let M be a sufficiently elementary, countably-closed substructure containing the objects $\underline{\mathbb{P}} \restriction \kappa$, $\underline{\dot{\mathbb{Q}}} \restriction \kappa$, $\vec{\varphi}$, and δ_1, δ_2. Then A is a member of M.*

Proof. First observe that A is a subset of ρ_{δ_2}, which is a member of M. Since $\rho_{\delta_2} < \omega_2$ and M contains ω_1 as a subset, $\rho_{\delta_2} \subseteq M$. In particular, $\sup(A)$ is an element of M.

Next, consider the case that $\sup(A)$ has countable cofinality. Then by the countable closure of M, we can find a cofinal subset Z of A inside M. By Lemma 2.23, we then conclude that $A \in M$.

Now suppose that $\sup(A)$ has uncountable cofinality. Recall from condition (b) of the definition of matching that A is countably closed in $\sup(A)$. Moreover, since $A \cap \alpha = \varphi_{\delta_2,\alpha}[\delta_1]$ for each $\alpha \in A$, we know that the sequence of sets $\langle \varphi_{\delta_2,\alpha}[\delta_1] : \alpha \in A \rangle$ is $\subseteq$-increasing. By the elementarity of M, we may find an ω-closed, cofinal subset Z of $\sup(A)$ such that $Z \in M$ for which the sequence of sets $\langle \varphi_{\delta_2,\alpha}[\delta_1] : \alpha \in Z \rangle$ is $\subseteq$-increasing. Combining this with the fact that $Z \cap A$ is also ω-closed and cofinal in $\sup(A)$, we have that

$$A = \bigcup_{\alpha \in A \cap Z} \varphi_{\delta_2,\alpha}[\delta_1] = \bigcup_{\alpha \in Z} \varphi_{\delta_2,\alpha}[\delta_1],$$

and hence A is in M, as $\bigcup_{\alpha \in Z} \varphi_{\delta_2,\alpha}[\delta_1]$ is in M by elementarity. $\square$

3. Combining Partition Products

In this section, we develop the machinery necessary to combine partition products in various ways. This will be essential for later arguments where, in the context of working with a partition product $\mathbb{R}$, we will want to create another partition product $\mathbb{R}^*$ into which $\mathbb{R}$ embeds in a variety of ways. Forcing with $\mathbb{R}^*$ will then add plenty of generics for $\mathbb{R}$, with various amounts of agreement or mutual genericity.

The main result of this section is a so-called "grafting lemma" which gives conditions under which, given partition products $\mathbb{P}$ and $\mathbb{R}$, we may extend $\mathbb{R}$ to another partition product $\mathbb{R}^*$ in such a way that $\mathbb{R}^*$ subsumes an isomorphic image of $\mathbb{P}$; in this case $\mathbb{P}$ is, in some sense, "grafted onto" $\mathbb{R}$. One trivial way of doing this, we will show, is to take the partition product $\mathbb{P} \times \mathbb{R}$. However, the issue becomes somewhat delicate if we desire, as later on we often will, that $\mathbb{R}$ and the isomorphic copy of $\mathbb{P}$ in $\mathbb{R}^*$ have coordinates in common, and hence share some part of their generics. Doing so requires that we keep track of more information about the structure of a partition product, and we begin with the relevant definition in the first subsection.

3.1. **Shadow Bases.**

Definition 3.1. *A triple $\langle x, \pi_x, \alpha \rangle$ is said to be a* shadow base *if the following conditions are satisfied: $\alpha \in C$, π_x has domain γ_x for some $\gamma_x \leq \rho_\alpha$, and $\pi_x : \gamma_x \longrightarrow x$ is an acceptable rearrangement of $\mathbb{P}_\alpha \restriction \gamma_x$.*

Moreover, if $\mathbb{R}$ is a partition product, say with domain X, we say that a shadow base $\langle x, \pi_x, \alpha \rangle$ is an $\mathbb{R}$-shadow base *if $x \subseteq X$ is base-closed and if π_x embeds $\mathbb{P}_\alpha \restriction \gamma_x$ into $\mathbb{R} \restriction x$.*

For example, if $\mathbb{R}$ is a partition product with domain X, then for any $\xi \in X$ the triple $\langle b(\xi), \pi_\xi, \text{index}(\xi) \rangle$ is an $\mathbb{R}$- "shadow" base; this is part of the motivation for the term. In practice, a shadow base will be an initial segment, in a sense we will specify soon, of such a triple.

Definition 3.2. *Suppose that $\langle x, \pi_x, \alpha \rangle$ and $\langle y, \pi_y, \beta \rangle$ are two shadow bases. We say that they* cohere *if the following holds: suppose that $\alpha \leq \beta$ and that there is some $\zeta \in x \cap y$. Define $\mu_x := \pi_x^{-1}(\zeta)$ and $\mu_y := \pi_y^{-1}(\zeta)$. Then*

(1) *$\pi_x[\mu_x] \subseteq \pi_y[\mu_y]$; and*
(2) *$\pi_y^{-1}[\pi_x[\mu_x]]$ matches $\langle \beta, \mu_y \rangle$ to $\langle \alpha, \mu_x \rangle$.*

A collection $\mathcal{B}$ of shadow bases is said to cohere *if any two elements of $\mathcal{B}$ cohere.*

Note that with this definition, item (3) of Definition 2.1 could be rephrased as saying that the two shadow bases $\langle b(\xi_1), \pi_{\xi_1}, \text{index}(\xi_1) \rangle$ and $\langle b(\xi_2), \pi_{\xi_2}, \text{index}(\xi_2) \rangle$ cohere.

Remark 3.3. It is straightforward to check that Corollary 2.24 holds for shadow bases too, in the following sense. Suppose that $\langle x, \pi_x, \alpha \rangle$ and $\langle y, \pi_y, \beta \rangle$ are two coherent shadow bases, say with $\alpha \leq \beta$. Then $\pi_y^{-1}[x \cap y]$ is a member of any M as in the statement of Corollary 2.24, provided that α and β, as well as the additional parameters $\underline{\mathbb{P}} \restriction \beta$, $\underline{\dot{\mathbb{Q}}} \restriction \beta$, and $\vec{\varphi}$, are all in M.

Definition 3.4. *Given a shadow base $\langle x, \pi_x, \alpha \rangle$ and some $a \subseteq x$, we say that a is an* initial segment *of $\langle x, \pi_x, \alpha \rangle$ if a is of the form $\pi_x[\mu]$ for some $\mu \leq \text{dom}(\pi_x)$.*

Given two shadow bases $\langle x_0, \pi_{x_0}, \alpha_0\rangle$ and $\langle x, \pi_x, \alpha\rangle$, we say that $\langle x_0, \pi_{x_0}, \alpha_0\rangle$ is an initial segment *of $\langle x, \pi_x, \alpha\rangle$ if $\alpha_0 = \alpha$, x_0 is an initial segment of $\langle x, \pi_x, \alpha\rangle$, and $\pi_x \restriction \mathrm{dom}(\pi_{x_0}) = \pi_{x_0}$.*

Remark 3.5. A simple but useful observation is that if $\langle x_0, \pi_{x_0}, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ are two coherent shadow bases, $\langle x_0, \pi_{x_0}, \alpha\rangle$ is an initial segment of $\langle x, \pi_x, \alpha\rangle$, and $(x\backslash x_0) \cap y = \varnothing$, then $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ cohere.

Lemma 3.6. *Suppose that $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ are coherent shadow bases and $\alpha \leq \beta$. Then $\pi_x^{-1}[x \cap y]$ is an ordinal $\leq \mathrm{dom}(\pi_x)$, and hence $x \cap y$ is an initial segment of $\langle x, \pi_x, \alpha\rangle$.*

Proof. Fix $\xi \in x \cap y$. By the definition of coherence and the fact that $\alpha \leq \beta$, we see that $\pi_x^{-1}(\xi) + 1 \subseteq \pi_x^{-1}[x \cap y]$. Thus

$$\pi_x^{-1}[x \cap y] = \bigcup_{\xi \in x \cap y} (\pi_x^{-1}(\xi) + 1),$$

and therefore $\pi_x^{-1}[x \cap y]$ is an ordinal. $\square$

Lemma 3.7. *Suppose that $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ are two coherent shadow bases, where $\alpha \leq \beta$. Let $\zeta \in x \cap y$, and define $\mu_x := \pi_x^{-1}(\zeta)$ and $\mu_y := \pi_y^{-1}(\zeta)$. Then $\pi_y^{-1} \circ \pi_x$ is an order preserving map from μ_x into μ_y. In particular, $\mu_x \leq \mu_y$, and $\pi_x^{-1} \circ \pi_y$ is the transitive collapse of $\pi_y^{-1}[\pi_x[\mu_x]]$.*

Proof. By Definition 3.2 (1), we know that $\pi_x[\mu_x]$ is a subset of $\pi_y[\mu_y]$, and so $\pi_y^{-1} \circ \pi_x$ is indeed a map from μ_x into μ_y. Let us abbreviate $\pi_y^{-1} \circ \pi_x$ by j. Suppose that $\zeta < \eta < \mu_x$, and we show $j(\zeta) < j(\eta)$. Set $\zeta_y = j(\zeta)$ and $\eta_y = j(\eta)$. Since $\pi_x(\eta) = \pi_y(\eta_y) \in x \cap y$, Definition 3.2 (1) implies that $\pi_x[\eta] \subseteq \pi_y[\eta_y]$. Next, as $\zeta < \eta$, $\pi_x(\zeta) \in \pi_x[\eta]$, and so $\pi_y(\zeta_y) \in \pi_y[\eta_y]$. Finally, since π_y is a bijection we conclude that $\zeta_y \in \eta_y$, i.e., $j(\zeta) < j(\eta)$. $\square$

As a result of the previous lemma, if two coherent shadow bases have the same "index", then their intersection is an initial segment of both.

Corollary 3.8. *Suppose that $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \alpha\rangle$ are two coherent shadow bases and that $\zeta \in x \cap y$. Then $\zeta_0 := \pi_x^{-1}(\zeta) = \pi_y^{-1}(\zeta)$, and in fact, $\pi_x \restriction (\zeta_0 + 1) = \pi_y \restriction (\zeta_0 + 1)$.*

Proof. Fix $\eta \in x \cap y$. Since both shadow bases have index α, we know from Lemma 3.7 that $\pi_x^{-1}(\eta) = \pi_y^{-1}(\eta)$. Since this holds for any $\eta \in x \cap y$, the result follows. $\square$

Remark 3.9. In the context of Corollary 3.8, we note that $\pi_x^{-1}[x \cap y] = \pi_y^{-1}[x \cap y]$ is an ordinal $\leq \rho_\alpha$, and if $x \neq y$, then this ordinal is strictly less than ρ_α.

We conclude this subsection with a very useful lemma.

Lemma 3.10. *Suppose that $\langle x, \pi_x, \alpha\rangle$, $\langle y, \pi_y, \beta\rangle$, and $\langle z, \pi_z, \gamma\rangle$ are shadow bases such that $\alpha, \beta \leq \gamma$. Suppose further that $x \cap y \subseteq z$, that $\langle x, \pi_x, \alpha\rangle$ and $\langle z, \pi_z, \gamma\rangle$ cohere, and that $\langle y, \pi_y, \beta\rangle$ and $\langle z, \pi_z, \gamma\rangle$ cohere. Then $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ cohere.*

Proof. By relabeling if necessary, we assume that $\alpha \leq \beta$. Let $\zeta \in x \cap y$, and we will show that (1) and (2) of Definition 3.2 hold. Define $\mu_x := \pi_x^{-1}(\zeta)$ and $\mu_y := \pi_y^{-1}(\zeta)$. As $x \cap y \subseteq z$, $\zeta \in z$, and therefore we may also define $\mu_z := \pi_z^{-1}(\zeta)$. Applying the coherence assumptions in the statement of the lemma, we conclude that

$$\pi_z^{-1}\left[\pi_x[\mu_x]\right] = \varphi_{\gamma,\mu_z}[\alpha] \text{ and } \pi_z^{-1}[\pi_y[\mu_y]] = \varphi_{\gamma,\mu_z}[\beta].$$

Since $\alpha \leq \beta$, it then follows that $\pi_x[\mu_x] \subseteq \pi_y[\mu_y]$.

We next show that $\pi_y^{-1}[\pi_x[\mu_x]] = \varphi_{\beta,\mu_y}[\alpha]$. By Lemma 3.7 applied to the shadow bases $\langle y, \pi_y, \beta\rangle$ and $\langle z, \pi_z, \gamma\rangle$, we conclude that $\pi_y^{-1} \circ \pi_z$, which we abbreviate as $j_{z,y}$, is the transitive collapse of $\pi_z^{-1}[\pi_y[\mu_y]]$. Furthermore, the definition of coherence also implies that $j_{z,y} \circ \varphi_{\gamma,\mu_z} \restriction \beta = \varphi_{\beta,\mu_y}$. Since $\alpha \leq \beta$ and since $\pi_z^{-1}[\pi_x[\mu_x]] = \varphi_{\gamma,\mu_z}[\alpha]$, we apply $j_{z,y}$ to conclude that $\pi_y^{-1}[\pi_x[\mu_x]] = \varphi_{\beta,\mu_y}[\alpha]$.

Now let $j_{y,x}$ denote the transitive collapse of $\pi_y^{-1}[\pi_x[\mu_x]]$; we check that $j_{y,x} \circ \varphi_{\beta,\mu_y} \restriction \alpha = \varphi_{\alpha,\mu_x}$. We also let $j_{z,x}$ be the transitive collapse of $\pi_z^{-1}[\pi_x[\mu_x]]$. From Lemma 3.7, we know that $j_{y,x} = \pi_x^{-1} \circ \pi_y$ and $j_{z,x} = \pi_x^{-1} \circ \pi_z$. Thus $j_{z,x} = j_{y,x} \circ j_{z,y}$. Since $j_{z,y} \circ \varphi_{\gamma,\mu_z} \restriction \beta = \varphi_{\beta,\mu_y}$ and $\alpha \leq \beta$, we conclude that $\varphi_{\alpha,\mu_x} = j_{y,x} \circ \varphi_{\beta,\mu_y} \restriction \alpha$, completing the proof. □

3.2. **Enriched Partition Products.** In this subsection, we will consider in greater detail how shadow bases interact with partition products. We begin with the following definition.

Definition 3.11. *Let $\mathbb{R}$ be a partition product with domain X. A collection $\mathcal{B}$ of $\mathbb{R}$-shadow bases is said to be* $\mathbb{R}$-full *if for all $\xi \in X$, $\langle b(\xi), \pi_\xi, \text{index}(\xi)\rangle \in \mathcal{B}$. $\mathcal{B}$ is said to be an* $\mathbb{R}$-enrichment *if $\mathcal{B}$ is both coherent and $\mathbb{R}$-full.*

An enriched partition product *is a pair $(\mathbb{R}, \mathcal{B})$ where $\mathcal{B}$ is an enrichment of $\mathbb{R}$.*

The next definition is a strengthening of the notion of a base-closed subset which allows us to restrict an enrichment.

Definition 3.12. *Let $(\mathbb{R}, \mathcal{B})$ be an enriched partition product with domain X. A base-closed subset $B \subseteq X$ is said to* cohere *with $(\mathbb{R}, \mathcal{B})$ if for all triples $\langle x, \pi_x, \alpha\rangle$ in $\mathcal{B}$ and for every $\zeta \in B \cap x$, if $\zeta = \pi_x(\zeta_0)$, say, then $\pi_x[\zeta_0] \subseteq B$.*

Lemma 3.13. *Suppose that $(\mathbb{R}, \mathcal{B})$ is an enriched partition product with domain X and that $B \subseteq X$ coheres with $(\mathbb{R}, \mathcal{B})$. Let $\langle x, \pi_x, \alpha\rangle \in \mathcal{B}$, and define $\pi_{x\cap B}$ to be the restriction of π_x mapping onto $x \cap B$. Then $\langle x \cap B, \pi_{x\cap B}, \alpha\rangle$ is a shadow base.*

Additionally, if we define

$$\mathcal{B} \restriction B := \{\langle x \cap B, \pi_{x\cap B}, \alpha\rangle : \langle x, \pi_x, \alpha\rangle \in \mathcal{B}\},$$

then $(\mathbb{R} \restriction B, \mathcal{B} \restriction B)$ is an enriched partition product.

Proof. To see that $\langle x\cap B, \pi_{x\cap B}, \alpha\rangle$ is a shadow base, it suffices to show that $\pi_x^{-1}[x\cap B]$ is an ordinal. This holds since for each $\xi \in x \cap B$, by the coherence of B with $(\mathbb{R}, \mathcal{B})$, $\pi_x^{-1}(\xi) + 1 \subseteq \pi_x^{-1}[x \cap B]$.

Now we need to verify that $(\mathbb{R} \restriction B, \mathcal{B} \restriction B)$ is an enriched partition product. It is straightforward to see that $\mathcal{B} \restriction B$ is $(\mathbb{R} \restriction B)$-full, since B is base-closed and since the base and index functions for $\mathbb{R} \restriction B$ are exactly the restrictions of those for $\mathbb{R}$. Similarly, we see that each shadow base in $\mathcal{B} \restriction B$ is in fact an $(\mathbb{R} \restriction B)$-shadow base. Thus we need to check that any two elements of $\mathcal{B} \restriction B$ cohere. Fix $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ in $\mathcal{B}$, and suppose that there exists $\zeta \in (x \cap B) \cap (y \cap B)$. Let $\mu_x < \rho_\alpha$ be such that $\zeta = \pi_{x\cap B}(\mu_x)$, and let $\mu_y < \rho_\beta$ be such that $\zeta = \pi_{y\cap B}(\mu_y)$. Then since B coheres with $(\mathbb{R}, \mathcal{B})$, $\pi_x \restriction (\mu_x + 1) = \pi_{x\cap B} \restriction (\mu_x + 1)$, and similarly $\pi_y \restriction (\mu_y + 1) = \pi_{y\cap B} \restriction (\mu_y + 1)$. Therefore conditions (1) and (2) of Definition 3.2 at ζ follow from their applications to $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ at ζ. □

Definition 3.14. *Suppose that $\mathbb{P}$ and $\mathbb{R}$ are partition products and σ embeds $\mathbb{P}$ into $\mathbb{R}$. If $\langle x, \pi_x, \alpha\rangle$ is a $\mathbb{P}$-shadow base, we define $\sigma(\langle x, \pi_x, \alpha\rangle)$ to be the triple*

$$\langle \sigma[x], \sigma \circ \pi_x, \alpha\rangle.$$

If $\mathcal{B}$ is a collection of $\mathbb{P}$-shadow bases, we define $\sigma[\mathcal{B}] := \{\sigma(t) : t \in \mathcal{B}\}$.

The proof of the following lemma is routine.

Lemma 3.15. *Suppose that $\mathbb{P}$ and $\mathbb{R}$ are partition products, σ embeds $\mathbb{P}$ into $\mathbb{R}$, and $\mathcal{B}$ is a collection of $\mathbb{P}$-shadow bases. Then $\sigma[\mathcal{B}]$ is a collection of $\mathbb{R}$-shadow bases.*

The following technical lemma will be of some use later.

Lemma 3.16. *Suppose that $\mathbb{R}$ and $\mathbb{R}^*$ are partition products, σ_1, σ_2 are embeddings of $\mathbb{R}$ into $\mathbb{R}^*$, and $\langle x, \pi_x, \alpha\rangle$ and $\langle y, \pi_y, \beta\rangle$ are two coherent $\mathbb{R}$-shadow bases, with $\alpha \leq \beta$. Let a be an initial segment of x such that $a \subseteq y$, $\sigma_1 \restriction a = \sigma_2 \restriction a$, and $\sigma_1[x \backslash a]$ is disjoint from $\sigma_2[y \backslash a]$. Then $\sigma_1(\langle x, \pi_x, \alpha\rangle)$ and $\sigma_2(\langle y, \pi_y, \beta\rangle)$ are coherent $\mathbb{R}^*$-shadow bases.*

Proof. From Lemma 3.15, we see that $\sigma_1(\langle x, \pi_x, \alpha\rangle)$ and $\sigma_2(\langle y, \pi_y, \beta\rangle)$ are $\mathbb{R}^*$-shadow bases. Furthermore, if $\zeta^* \in \sigma_1[x] \cap \sigma_2[y]$, then ζ^* must be in $\sigma_1[a] \cap \sigma_2[a]$, since $\sigma_1[x \backslash a] \cap \sigma_2[y \backslash a] = \varnothing$ and since $\sigma_1 \restriction a = \sigma_2 \restriction a$. As the injections σ_1 and σ_2 are equal on a, we then have that $\sigma_1^{-1}(\zeta^*) = \sigma_2^{-1}(\zeta^*) =: \zeta$. Thus $\zeta \in x \cap y$, and the coherence of the original triples at ζ implies the coherence of their images at ζ^*. $\square$

We next define a notion of embedding for enriched partition products.

Definition 3.17. *Suppose that $(\mathbb{P}, \mathcal{B})$ is an enriched partition product with domain X, $(\mathbb{R}, \mathcal{D})$ is an enriched partition product with domain Y, and $\sigma : X \longrightarrow Y$ is a function. We say that σ* embeds $(\mathbb{P}, \mathcal{B})$ into $(\mathbb{R}, \mathcal{D})$ *if σ embeds $\mathbb{P}$ into $\mathbb{R}$, as in Definition 2.17, and if $\sigma[\mathcal{B}] \subseteq \mathcal{D}$.*

We may now state and prove the Grafting Lemma; enrichments play a crucial role in its proof.

Lemma 3.18. (Grafting Lemma) *Let $(\mathbb{P}, \mathcal{B})$ and $(\mathbb{R}, \mathcal{D})$ be enriched partition products with respective domains X and Y. Suppose that $\hat{X} \subseteq X$ coheres with $(\mathbb{P}, \mathcal{B})$ and that there is a map $\sigma : \hat{X} \longrightarrow Y$ which embeds $(\mathbb{P} \restriction \hat{X}, \mathcal{B} \restriction \hat{X})$ into $(\mathbb{R}, \mathcal{D})$.Then there is an enriched partition product $(\mathbb{R}^*, \mathcal{D}^*)$ with domain Y^* such that $Y \subseteq Y^*$, $\mathbb{R}^* \restriction Y = \mathbb{R}$, $\mathcal{D} \subseteq \mathcal{D}^*$, and such that there is an extension σ^* of σ which embeds $(\mathbb{P}, \mathcal{B})$ into $(\mathbb{R}^*, \mathcal{D}^*)$ and which satisfies $\sigma^* \left[X \backslash \hat{X}\right] = Y^* \backslash Y$.*

Proof. We first define the map σ^* extending σ: if $\xi \in \hat{X}$, then set $\sigma^*(\xi) := \sigma(\xi)$. If $\xi \in X \backslash \hat{X}$, say ξ is the γth such element, then we set $\sigma^*(\xi) := \sup(Y) + 1 + \gamma$. Then σ^* is an acceptable rearrangement, since $\hat{X}$ is base-closed. Let $Y^* := Y \cup \operatorname{ran}(\sigma^*)$. Recalling that σ embeds $\mathbb{P} \restriction \hat{X}$ into $\mathbb{R}$, we know that $\sigma^*[\mathrm{base}_{\mathbb{P}}] \restriction \operatorname{ran}(\sigma) = \mathrm{base}_{\mathbb{R}} \restriction \operatorname{ran}(\sigma)$ and that $\sigma^*[\mathrm{index}_{\mathbb{P}}] \restriction \operatorname{ran}(\sigma) = \mathrm{index}_{\mathbb{R}} \restriction \operatorname{ran}(\sigma)$. Thus if we define $\mathrm{base}^* := \mathrm{base}_{\mathbb{R}} \cup \sigma^*[\mathrm{base}_{\mathbb{P}}]$ and $\mathrm{index}^* := \mathrm{index}_{\mathbb{R}} \cup \sigma^*[\mathrm{index}_{\mathbb{P}}]$, then base^* and index^* are functions.

Before we check that base^* and index^* support a partition product on Y^*, we need to check that $\mathcal{D} \cup \sigma^*[\mathcal{B}]$ consists of a coherent collection of shadow bases. To facilitate the discussion, we set $\mathcal{B}^* := \sigma^*[\mathcal{B}]$ and $\mathcal{D}^* := \mathcal{D} \cup \mathcal{B}^*$. So fix $\langle x, \pi_x, \alpha\rangle \in \mathcal{B}$ and $\langle y, \pi_y, \beta\rangle$ in $\mathcal{D}$, and we check that $\langle y, \pi_y, \beta\rangle$ and $\langle x^*, \pi_{x^*}, \alpha\rangle$ cohere, where $x^* := \sigma^*[x]$ and $\pi_{x^*} := \sigma^* \circ \pi_x$. By our assumption that σ embeds $(\mathbb{P} \restriction \hat{X}, \mathcal{B} \restriction \hat{X})$ into $(\mathbb{R}, \mathcal{D})$, we know that $\langle y, \pi_y, \beta\rangle$ and $\langle \sigma[x \cap \hat{X}], \sigma \circ \pi_{x\cap\hat{X}}, \alpha\rangle$ cohere. However, $\langle \sigma[x \cap \hat{X}], \sigma \circ \pi_{x\cap\hat{X}}, \alpha\rangle$ is an initial segment of $\langle x^*, \pi_{x^*}, \alpha\rangle$, as in Definition 3.4. Therefore by Remark 3.5, since $\sigma^*[X\backslash\hat{X}]$ is disjoint from y, we have that $\langle y, \pi_y, \beta\rangle$ and $\langle x^*, \pi_{x^*}, \alpha\rangle$ cohere.

We now check that base* and index* support a partition product on Y^*. Conditions (1) and (2) of Definition 2.1 for base* and index* follow because they hold for base$_\mathbb{R}$ and index$_\mathbb{R}$, as well as $\sigma^*[\text{base}_\mathbb{P}]$ and $\sigma^*[\text{index}_\mathbb{P}]$ individually, and since base* and index* are functions. Thus we need to verify condition (3). For this it suffices to check that it holds for $\xi_1 \in Y$ and $\xi_2 \in Y^*\backslash Y$. Rephrasing, we need to show that the triples $\langle b^*(\xi_1), \pi^*_{\xi_1}, \text{index}^*(\xi_1)\rangle$ and $\langle b^*(\xi_2), \pi^*_{\xi_2}, \text{index}^*(\xi_2)\rangle$ cohere. The first triple equals $\langle b_\mathbb{R}(\xi_1), \pi^\mathbb{R}_{\xi_1}, \text{index}_\mathbb{R}(\xi_1)\rangle$ and so is in $\mathcal{D}$ since $\mathcal{D}$ is $\mathbb{R}$-full. The second triple is in $\mathcal{B}^*$, since it equals $\langle \sigma^*[b_\mathbb{P}(\hat{\xi}_2)], \sigma^* \circ \pi^\mathbb{P}_{\hat{\xi}_2}, \text{index}_\mathbb{P}(\hat{\xi}_2)\rangle$, where $\sigma^*(\hat{\xi}_2) = \xi_2$. Consequently, both shadow bases are in $\mathcal{D}^*$ and are therefore coherent, by the previous paragraph. Thus condition (3) of Definition 2.1 is satisfied.

Thus base* and index* support a partition product on Y^*, which we call $\mathbb{R}^*$. Since the restrictions of base* and index* to Y equal base$_\mathbb{R}$ and index$_\mathbb{R}$, respectively, we have that $\mathbb{R}^* \restriction Y = \mathbb{R}$. Additionally, σ^* embeds $\mathbb{P}$ into $\mathbb{R}^*$, since base* and index* restricted to $\text{ran}(\sigma^*)$ equal $\sigma^*[\text{base}_\mathbb{P}]$ and $\sigma^*[\text{index}_\mathbb{P}]$ respectively. Thus it remains to check that $\mathcal{D}^*$ is an enrichment of $\mathbb{R}^*$, and for this, it only remains to check that $\mathcal{D}^*$ is $\mathbb{R}^*$-full. However, $\mathcal{D}$ is $\mathbb{R}$-full, and since $\mathcal{B}$ is $\mathbb{P}$-full, $\mathcal{B}^*$ is full with respect to $\mathbb{R}^* \restriction \text{ran}(\sigma^*)$. Thus $\mathcal{D}^*$ is $\mathbb{R}^*$-full. □

Definition 3.19. *Let $(\mathbb{P}, \mathcal{B})$, $(\mathbb{R}, \mathcal{D})$, $(\mathbb{R}^*, \mathcal{D}^*)$, $\hat{X}$, σ, and σ^* be as in Lemma 3.18. We will say in this case that $(\mathbb{R}^*, \mathcal{D}^*)$ is* the extension of $(\mathbb{R}, \mathcal{D})$ by grafting $(\mathbb{P}, \mathcal{B})$ over σ, *and we will call σ^* the* grafting embedding.

Note that as a corollary, we get that the product of two partition products is isomorphic to a partition product; this fact could also be proven directly from the definitions.

Corollary 3.20. *Suppose that $\mathbb{P}$ and $\mathbb{R}$ are partition products with respective domains X and Y. Then $\mathbb{P} \times \mathbb{R}$ is isomorphic to a partition product $\mathbb{R}^*$.*

In fact, by Lemma 2.13 we may assume that $X \cap Y = \varnothing$, that $\mathbb{R}^*$ is a partition product on $X \cup Y$, and that $\mathbb{R}^* \restriction X = \mathbb{P}$ and $\mathbb{R}^* \restriction Y = \mathbb{R}$. Finally, in this case, if $\mathcal{B}$ and $\mathcal{D}$ are enrichments of $\mathbb{P}$ and $\mathbb{R}$ respectively, then $\mathcal{B} \cup \mathcal{D}$ is an enrichment of $\mathbb{R}^*$.

The following lemma gives a situation under which, after creating a single grafting embedding, we may extend a number of other embeddings without further grafting; it will be used later in constructing preassignments (see Lemma **??**).

Lemma 3.21. *Let $(\mathbb{P}, \mathcal{B})$ and $(\mathbb{R}, \mathcal{D})$ be enriched partition products with domains X and Y respectively. Suppose that X can be written as $X = X_0 \cup X_1$, where both X_0 and X_1 cohere with $(\mathbb{P}, \mathcal{B})$. Let $\mathcal{F}$ be a finite collection of maps which embed $(\mathbb{P} \restriction X_0, \mathcal{B} \restriction X_0)$ into $(\mathbb{R}, \mathcal{D})$, and suppose that for each $\sigma_0, \sigma_1 \in \mathcal{F}$,*

$$\sigma_0[X_0 \cap X_1] = \sigma_1[X_0 \cap X_1].$$

Finally, fix a particular $\sigma_0 \in \mathcal{F}$*, let* $(\mathbb{R}^*, \mathcal{D}^*)$ *be the extension of* $(\mathbb{R}, \mathcal{D})$ *by grafting* $(\mathbb{P}, \mathcal{B})$ *over* σ_0*, and let* σ_0^* *be the grafting embedding. Then for all* $\sigma \in \mathcal{F}$*, the map*

$$\sigma^* := \sigma \cup (\sigma_0^* \restriction (X_1 \backslash X_0))$$

embeds $(\mathbb{P}, \mathcal{B})$ *into* $(\mathbb{R}^*, \mathcal{D}^*)$*.*

Proof. Fix $\sigma \in \mathcal{F}$. Before we continue, we note that σ^* and σ_0^* agree on all of X_1, since they agree on $X_0 \cap X_1$ by assumption and on $X_1 \backslash X_0$ by definition.

We first verify that σ^* provides an acceptable rearrangement of $\mathbb{P}$. So let $\zeta, \xi \in X$ so that $\zeta \in b_{\mathbb{P}}(\xi)$. If $\xi \in X_0$, then ζ is too, since X_0 is base-closed. Then $\sigma^*(\zeta) = \sigma(\zeta) < \sigma(\xi) = \sigma^*(\xi)$, since σ is an acceptable rearrangement of $\mathbb{P} \restriction X_0$. On the other hand, if $\xi \in X_1$, then $\zeta \in X_1$. Since $\sigma^* \restriction X_1 = \sigma_0^* \restriction X_1$, and $\sigma_0^* \restriction X_1$ is an acceptable rearrangement of $\mathbb{P} \restriction X_1$, we get that $\sigma^*(\zeta) < \sigma^*(\xi)$.

We may now see that σ^* embeds $\mathbb{P}$ into $\mathbb{R}^*$, as follows: let base* and index* be the functions which support $\mathbb{R}^*$. Then $\sigma^*[\text{index}_{\mathbb{P}}]$ and $\sigma^*[\text{base}_{\mathbb{P}}]$ agree with index* and base* on $\text{ran}(\sigma)$, since σ embeds $\mathbb{P} \restriction X_0$ into $\mathbb{R}$. Furthermore, $\sigma^*[\text{index}_{\mathbb{P}}]$ and $\sigma^*[\text{base}_{\mathbb{P}}]$ agree with index* and base* on $\sigma^*[X_1]$, since they are equal, respectively, to $\sigma_0^*[\text{index}_{\mathbb{P}}]$ and $\sigma_0^*[\text{base}_{\mathbb{P}}]$ restricted to $\sigma_0^*[X_1]$. Thus $\sigma^*[\text{index}_{\mathbb{P}}]$ and $\sigma^*[\text{base}_{\mathbb{P}}]$ are equal to the restriction of index* and base* to $\text{ran}(\sigma^*)$, and consequently, σ^* embeds $\mathbb{P}$ into $\mathbb{R}^*$.

We finish the proof of the lemma by showing that $\sigma^*[\mathcal{B}] \subseteq \mathcal{D}^*$. To see this, fix some $\langle x, \pi_x, \alpha \rangle \in \mathcal{B}$. We first claim that either $x \subseteq X_0$ or $x \subseteq X_1$. If this is false, then there exist $\alpha \in x \backslash X_0$ and $\beta \in x \backslash X_1$. Since $X_0 \cup X_1 = X$, we then have $\alpha \in X_1$ and $\beta \in X_0$. We suppose, by relabeling if necessary, that $\alpha_0 := \pi_x^{-1}(\alpha) < \pi_x^{-1}(\beta) =: \beta_0$. By the coherence of X_0 with $(\mathbb{P}, \mathcal{B})$, we conclude that $\pi_x[\beta_0] \subseteq X_0$. However, $\alpha = \pi_x(\alpha_0) \in \pi_x[\beta_0]$, and therefore $\alpha \in X_0$, a contradiction.

We now show that the shadow base $\langle x^*, \pi_{x^*}, \alpha \rangle$ is in $\mathcal{D}^*$, where $x^* := \sigma^*[x]$ and $\pi_{x^*} = \sigma^* \circ \pi_x$. On the one hand, if $x \subseteq X_0$, then the shadow base $\langle x, \pi_x, \alpha \rangle$ is in $\mathcal{B} \restriction X_0$, and therefore $\langle \sigma[x], \sigma \circ \pi_x, \alpha \rangle$ is a member of $\mathcal{D} \subseteq \mathcal{D}^*$. Since $\sigma = \sigma^* \restriction X_0$, $\langle x^*, \pi_{x^*}, \alpha \rangle = \langle \sigma[x], \sigma \circ \pi_x, \alpha \rangle$, completing this subcase. On the other hand, if $x \subseteq X_1$, then we see that $\langle x^*, \pi_{x^*}, \alpha \rangle = \langle \sigma_0^*[x], \sigma_0^* \circ \pi_x, \alpha \rangle$, since $\sigma^* \restriction X_1 = \sigma_0^* \restriction X_1$. It is therefore a member of $\mathcal{D}^*$, which finishes the proof. □

4. Constructing Preassignments

In this section we show how to construct the particular names for preassignments of colors that we need. Throughout this section, we make the following assumptions.

Assumption 4.1. The CH holds. $\kappa < \omega_2$ is in C, and for each $\xi \in C \cap \kappa$, ρ_ξ is below ω_2. Additionally, the κ-canonical partition product $\mathbb{P}_\kappa$ is defined, and in particular, $\mathbb{P}_\kappa$ is a partition product based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$. We also assume that the $\mathbb{P}_\kappa$-names $\dot{S}_\kappa$ and $\dot{\chi}_\kappa$ are defined and satisfy that $\dot{S}_\kappa$ names a countable basis for a second countable, Hausdorff topology on ω_1 and $\dot{\chi}_\kappa$ names a coloring open with respect to the topology generated by $\dot{S}_\kappa$. And finally, we assume that any partition product based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$ is c.c.c.

Remark 4.2. Our goal is to show, under Assumption 4.1, how to construct a $\mathbb{P}_\kappa$-name $\dot{\mathbb{Q}}_\kappa$ for a poset which decomposes ω_1 into countably-many $\dot{\chi}_\kappa$-homogeneous sets, in such a way that any partition product based upon $\underline{\mathbb{P}} \restriction (\kappa{+}1)$ and $\underline{\dot{\mathbb{Q}}} \restriction (\kappa{+}1)$

is c.c.c. In Section 5 we will use this as part of an inductive construction of a sequence $\underline{\mathbb{P}}$ which provides the right building blocks for our main theorem.

4.1. κ-Suitable Collections.

We now consider how various copies of $\mathbb{P}_\kappa$ fit into a partition product $\mathbb{R}$, where $\mathbb{R}$ is based upon $\underline{\mathbb{P}} \upharpoonright \kappa$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa$. Even though we have yet to construct the name $\dot{\mathbb{Q}}_\kappa$, we would still like to isolate the relevant behavior of copies of $\mathbb{P}_\kappa$ inside such an $\mathbb{R}$ which these copies *would* have if $\mathbb{R}$ *were* of the form

$$\mathbb{R} = \mathbb{R}^* \upharpoonright \{\xi \in \operatorname{dom}(\mathbb{R}^*) : \operatorname{index}(\xi) < \kappa\},$$

for some partition product $\mathbb{R}^*$ based upon $\underline{\mathbb{P}} \upharpoonright (\kappa+1)$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright (\kappa+1)$. This leads to the following definition.

Definition 4.3. *Let $\mathbb{R}$ be a partition product with domain X based upon $\underline{\mathbb{P}} \upharpoonright \kappa$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa$. Let $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ be a set of pairs, where each $B_\iota \subseteq X$ is base-closed and where $\psi_\iota : \rho_\kappa \longrightarrow B_\iota$ is a bijection which embeds $\mathbb{P}_\kappa$ into $\mathbb{R}$. We say that the collection $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ is* κ-suitable with respect to $\mathbb{R}$ *if*

$$\{\langle B_\iota, \psi_\iota, \kappa\rangle : \iota \in I\} \cup \{\langle b(\xi), \pi_\xi, \operatorname{index}(\xi)\rangle : \xi \in X\}$$

is a coherent set of $\mathbb{R}$-shadow bases.

Moreover, if $(\mathbb{R}, \mathcal{B})$ is an enriched partition product, we say that $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ is κ-suitable with respect to $(\mathbb{R}, \mathcal{B})$ *if $\{\langle B_\iota, \psi_\iota, \kappa\rangle : \iota \in I\} \subseteq \mathcal{B}$ and if $\alpha \leq \kappa$ for all $\langle x, \pi_x, \alpha\rangle \in \mathcal{B}$.*

As the next lemma shows, κ-suitable collections give us subsets which cohere with the original partition product, since the indices of the triples in the enrichment do not exceed κ.

Lemma 4.4. *Suppose that $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ is κ-suitable with respect to an enriched partition product $(\mathbb{R}, \mathcal{B})$. Then for any $I_0 \subseteq I$, $\bigcup_{\iota \in I_0} B_\iota$ coheres with $(\mathbb{R}, \mathcal{B})$.*

Proof. Let $\langle x, \pi_x, \alpha\rangle \in \mathcal{B}$, and suppose that there exists $\zeta \in (\bigcup_{\iota \in I_0} B_\iota) \cap x$. Fix some $\iota \in I_0$ such that $\zeta \in B_\iota \cap x$. Then $\langle B_\iota, \psi_\iota, \kappa\rangle$ is in $\mathcal{B}$. Furthermore, $\alpha \leq \kappa$, by definition of κ-suitability with respect to $(\mathbb{R}, \mathcal{B})$. Since $\mathcal{B}$ is coherent, by definition of an enrichment, and since $\alpha \leq \kappa$, we have by Definition 3.2 that

$$\pi_x[\pi_x^{-1}(\zeta)] \subseteq \psi_\iota[\psi_\iota^{-1}(\zeta)].$$

Since $\operatorname{ran}(\psi_\iota) = B_\iota$, this finishes the proof. □

We will often be interested in the following strengthening of the notion of an embedding, one which preserves the κ-suitable structure.

Definition 4.5. *Let $\mathbb{R}$ and $\mathbb{R}^*$ be two partition products, and let $\mathcal{S} = \{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ and $\mathcal{S}^* = \left\{\langle B^*_\eta, \psi^*_\eta\rangle : \eta \in I^*\right\}$ be κ-suitable collections with respect to $\mathbb{R}$ and $\mathbb{R}^*$ respectively. An embedding σ of $\mathbb{R}$ into $\mathbb{R}^*$ is said to be* $(\mathcal{S}, \mathcal{S}^*)$-suitable *if for each $\iota \in I$, there is some $\eta \in I^*$ such that $\sigma \upharpoonright B_\iota$ isomorphs $\mathbb{R} \upharpoonright B_\iota$ onto $\mathbb{R}^* \upharpoonright B^*_\eta$ and $\psi^*_\eta = \sigma \circ \psi_\iota$. A collection $\mathcal{F}$ of embeddings is said to be* $(\mathcal{S}, \mathcal{S}^*)$-suitable *if each $\sigma \in \mathcal{F}$ is $(\mathcal{S}, \mathcal{S}^*)$-suitable.*

If σ is $(\mathcal{S}, \mathcal{S}^*)$-suitable, we let h_σ denote the injection from I into I^* such that σ maps B_ι onto $B^*_{h_\sigma(\iota)}$ for each $\iota \in I$.

The following technical lemmas will be used later in this section.

Lemma 4.6. *Suppose that $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ is κ-suitable with respect to an enriched partition product $(\mathbb{R}, \mathcal{B})$ and that the elements of $\{B_\iota : \iota \in I\}$ are pairwise disjoint. Then for any $\langle x, \pi_x, \alpha\rangle \in \mathcal{B}$, $x \cap (\bigcup_{\iota \in I} B_\iota) = x \cap B_{\iota_0}$ for a unique $\iota_0 \in I$.*

Proof. Suppose otherwise, and fix $\langle x, \pi_x, \alpha\rangle \in \mathcal{B}$ as well as distinct $\iota_0, \iota_1 \in I$ such that $x \cap B_{\iota_0} \neq \varnothing$ and $x \cap B_{\iota_1} \neq \varnothing$. Let $\zeta \in x \cap B_{\iota_0}$ and $\eta \in x \cap B_{\iota_1}$. Then $\zeta \neq \eta$, since $B_{\iota_0} \cap B_{\iota_1} = \varnothing$. Define $\zeta_0 := \pi_x^{-1}(\zeta)$ and $\eta_0 := \pi_x^{-1}(\eta)$. Since $\zeta_0 \neq \eta_0$, we suppose, by relabeling if necessary, that $\zeta_0 < \eta_0$. By definition of an enrichment, we know that $\langle B_{\iota_1}, \psi_{\iota_1}, \kappa\rangle$ and $\langle x, \pi_x, \alpha\rangle$ cohere, and since $\alpha \leq \kappa$ and $\eta \in B_{\iota_1} \cap x$, we conclude that $\pi_x[\eta_0] \subseteq B_{\iota_1}$. However, $\zeta_0 < \eta_0$, and so $\zeta = \pi_x(\zeta_0) \in \pi_x[\eta_0]$, which implies that $\zeta \in B_{\iota_1}$. This contradicts the fact that $B_{\iota_0} \cap B_{\iota_1} = \varnothing$. □

The next lemma gives a sufficient condition for creating an enrichment; it will be used in the construction of preassignments (see Lemma **??**).

Lemma 4.7. *Suppose that $\mathcal{S} = \{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ is κ-suitable with respect to an enriched partition product $(\mathbb{R}, \mathcal{B})$ and that the elements of $\{B_\iota : \iota \in I\}$ are pairwise disjoint. Suppose further that $\mathbb{R}^*$ is a partition product with domain X^* and that $\mathcal{S}^* = \left\{\langle B^*_\eta, \psi^*_\eta\rangle : \eta \in I^*\right\}$ is κ-suitable with respect to $\mathbb{R}^*$. Finally, set $\hat{X} := \bigcup_{\iota \in I} B_\iota$, and suppose that there exists a finite collection $\mathcal{F}$ of $(\mathcal{S}, \mathcal{S}^*)$-suitable embeddings of $\mathbb{R} \restriction \hat{X}$ into $\mathbb{R}^*$ such that for any distinct $\iota_0, \iota_1 \in I$ and any (not necessarily distinct) $\pi_0, \pi_1 \in \mathcal{F}$,*
$$B^*_{h_{\pi_0}(\iota_0)} \cap B^*_{h_{\pi_1}(\iota_1)} = \varnothing,$$
where for each $\pi \in \mathcal{F}$, h_π is the associated injection. Then
$$\mathcal{B}^* := \left\{\langle b^*(\xi), \pi^*_\xi, \mathrm{index}^*(\xi)\rangle : \xi \in X^*\right\} \cup \bigcup_{\pi \in \mathcal{F}} \pi\left[\mathcal{B} \restriction \hat{X}\right] \cup \left\{\langle B^*_\eta, \psi^*_\eta, \kappa\rangle : \eta \in I^*\right\}$$
is an enrichment of $\mathbb{R}^*$ and $\mathcal{S}^*$ is κ-suitable with respect to $(\mathbb{R}^*, \mathcal{B}^*)$.

Proof. We will first show that $\bigcup_{\pi \in \mathcal{F}} \pi\left[\mathcal{B} \restriction \hat{X}\right]$ is a coherent collection of $\mathbb{R}^*$-shadow bases. Since each $\pi \in \mathcal{F}$ is an embedding of $\mathbb{R} \restriction \hat{X}$ into $\mathbb{R}^*$, Lemma 3.15 implies that this is a set of $\mathbb{R}^*$-shadow bases. Thus we check coherence.

Fix $\pi_0, \pi_1 \in \mathcal{F}$ and $\langle x, \pi_x, \alpha\rangle, \langle y, \pi_y, \beta\rangle \in \mathcal{B} \restriction \hat{X}$, and assume, by relabeling if necessary, that $\alpha \leq \beta$. We show that $\langle x^*, \pi_{x^*}, \alpha\rangle$ and $\langle y^*, \pi_{y^*}, \beta\rangle$ cohere, where $x^* := \pi_0[x]$ and $\pi_{x^*} := \pi_0 \circ \pi_x$, and where $y^* := \pi_1[y]$, $\pi_{y^*} := \pi_1 \circ \pi_y$. By Lemma 4.6, and since x and y are subsets of $\hat{X}$, we may fix $\iota_0, \iota_1 \in I$ such that $x = x \cap \hat{X} = x \cap B_{\iota_0}$ and $y = y \cap \hat{X} = y \cap B_{\iota_1}$. There are two cases.

First suppose that $\iota_0 \neq \iota_1$. Then we must have that $x^* \cap y^* = \varnothing$. To see this, observe that
$$x^* = \pi_0[x] = \pi_0[x \cap B_{\iota_0}] \subseteq B^*_{h_{\pi_0}(\iota_0)},$$
and
$$y^* = \pi_1[y] = \pi_1[y \cap B_{\iota_1}] \subseteq B^*_{h_{\pi_1}(\iota_1)}.$$
Therefore $x^* \cap y^* = \varnothing$, as $B^*_{h_{\pi_0}(\iota_0)} \cap B^*_{h_{\pi_1}(\iota_1)} = \varnothing$, by assumption. We thus trivially have the coherence of $\langle x^*, \pi_{x^*}, \alpha\rangle$ and $\langle y^*, \pi_{y^*}, \beta\rangle$ in this case.

On the other hand, suppose that $\iota := \iota_0 = \iota_1$. Define $a \subseteq x$ to be the largest initial segment (see Definition 3.4) of $\langle x, \pi_x, \alpha\rangle$ on which π_0 and π_1 agree, and set $a^* := \pi_0[a] = \pi_1[a]$. In order to see that $\langle x^*, \pi_{x^*}, \alpha\rangle$ and $\langle y^*, \pi_{y^*}, \beta\rangle$ cohere, it suffices, in light of Lemma 3.16, to show that $\pi_0[x \backslash a]$ is disjoint from $\pi_1[y \backslash a]$.

Towards this end, fix some $\zeta^* \in x^* \cap y^*$, and suppose for a contradiction that $\zeta^* \notin a^*$. Define $\mu_x := \pi_{x^*}^{-1}(\zeta^*)$, and observe that μ_x is greater than the ordinal $\pi_x^{-1}[a]$, since $\zeta^* \notin a^*$. Using the abbreviation $\eta_i := h_{\pi_i}(\iota)$, for $i \in \{0,1\}$, we see that $\zeta^* \in B^*_{\eta_0} \cap B^*_{\eta_1}$, as $x^* = \pi_0[x \cap B_\iota] \subseteq B^*_{\eta_0}$, and as $y^* = \pi_1[y \cap B_\iota] \subseteq B^*_{\eta_1}$. Set $\zeta_0 := (\psi^*_{\eta_0})^{-1}(\zeta^*)$. Since the $\mathbb{R}^*$-shadow bases $\langle B^*_{\eta_0}, \psi^*_{\eta_0}, \kappa\rangle$ and $\langle B^*_{\eta_1}, \psi^*_{\eta_1}, \kappa\rangle$ cohere, Corollary 3.8 implies that $\psi^*_{\eta_0} \restriction (\zeta_0+1) = \psi^*_{\eta_1} \restriction (\zeta_0+1)$.

Now we observe that

$$\pi_{x^*}(\mu_x) = \pi_0(\pi_x(\mu_x)) = \zeta^* = \psi^*_{\eta_0}(\zeta_0) = \pi_0(\psi_\iota(\zeta_0)),$$

and therefore $\pi_x(\mu_x) = \psi_\iota(\zeta_0)$. Let us call this ordinal ζ. Since $\zeta \in B_\iota \cap x$, the coherence of $\langle x, \pi_x, \alpha\rangle$ with $\langle B_\iota, \psi_\iota, \kappa\rangle$ and the fact that $\alpha \leq \kappa$ imply that

$$\pi_x[\mu_x + 1] \subseteq \psi_\iota[\zeta_0 + 1].$$

As noted above, $\psi^*_{\eta_0} \restriction (\zeta_0+1) = \psi^*_{\eta_1} \restriction (\zeta_0+1)$, and therefore by the commutativity assumed in the statement of the lemma, π_0 and π_1 agree on $\psi_\iota[\zeta_0+1]$. In particular, they agree on $\pi_x[\mu_x+1]$. Thus $\pi_x[\mu_x+1]$ is an initial segment of $\langle x, \pi_x, \alpha\rangle$ on which π_0 and π_1 agree. Since $\zeta = \pi_x(\mu_x) \notin a$, this contradicts the maximality of a.

At this point, we have shown that $\bigcup_{\pi \in \mathcal{F}} \pi\left[\mathcal{B} \restriction \hat{X}\right]$ is a coherent collection of $\mathbb{R}^*$-shadow bases. We introduce the abbreviation

$$\mathcal{B}^*_0 := \left\{\langle b^*(\xi), \pi^*_\xi, \mathrm{index}^*(\xi)\rangle : \xi \in X^*\right\} \cup \left\{\langle B^*_\eta, \psi^*_\eta, \kappa\rangle : \eta \in I^*\right\}.$$

We know that $\mathcal{B}^*_0$ is a coherent set of $\mathbb{R}^*$-shadow bases, by the definition of κ-suitability. Therefore, to finish showing that $\mathcal{B}^*$ is an enrichment of $\mathbb{R}^*$, we now check that if $\langle y, \pi_y, \beta\rangle \in \mathcal{B}^*_0$, $\pi \in \mathcal{F}$, and $\langle x, \pi_x, \alpha\rangle \in \mathcal{B} \restriction \hat{X}$, then $\langle y, \pi_y, \beta\rangle$ and $\langle x^*, \pi_{x^*}, \alpha\rangle$ cohere, where $x^* := \pi[x]$ and $\pi_{x^*} = \pi \circ \pi_x$. By Lemma 4.6, let $\iota \in I$ be such that $x = x \cap \hat{X} = x \cap B_\iota$. Then $x^* = \pi[x] = \pi[x \cap B_\iota] \subseteq B^*_{h_\pi(\iota)}$. Now $\langle x, \pi_x, \alpha\rangle$ and $\langle B_\iota, \psi_\iota, \kappa\rangle$ cohere, and moreover, π isomorphs $\mathbb{R} \restriction B_\iota$ onto $\mathbb{R}^* \restriction B^*_{h_\pi(\iota)}$ and satisfies that $\psi^*_{h_\pi(\iota)} = \pi \circ \psi_\iota$. It is straightforward to see from this that $\langle x^*, \pi_{x^*}, \alpha\rangle$ and $\langle B^*_{h_\pi(\iota)}, \psi^*_{h_\pi(\iota)}, \kappa\rangle$ cohere. However, $\langle y, \pi_y, \beta\rangle$ and $\langle B^*_{h_\pi(\iota)}, \psi^*_{h_\pi(\iota)}, \kappa\rangle$ also cohere, by definition of κ-suitability. Since $\alpha, \beta \leq \kappa$, Lemma 3.10 therefore implies that $\langle y, \pi_y, \beta\rangle$ and $\langle x^*, \pi_{x^*}, \alpha\rangle$ cohere, which is what we wanted to show. $\square$

4.2. **What suffices.**

Given a (possibly partial) 2-coloring χ on ω_1 and a function f from ω_1 into $\{0,1\}$, we use $\mathbb{Q}(\chi, f)$ to denote the poset to decompose ω_1 into countably-many χ-homogeneous sets which respect the function f. More precisely, a condition is a finite partial function q with $\mathrm{dom}(q) \subseteq \omega$ such that for each $n \in \mathrm{dom}(q)$, $q(n)$ is a finite subset of ω_1 on which f is constant, say with value i, and $q(n)$ is χ-homogeneous with color i, meaning that if $x, y \in q(n)$ and $\langle x, y\rangle \in \mathrm{dom}(\chi)$, then $\chi(x,y) = i$. The ordering is $q_1 \leq q_0$ iff $\mathrm{dom}(q_0) \subseteq \mathrm{dom}(q_1)$, and for each $n \in \mathrm{dom}(q_0)$, $q_0(n) \subseteq q_1(n)$.

Remark 4.8. Forcing with $\mathbb{Q}(\chi, f)$ adds reals over V. Say that G is generic for $\mathbb{Q}(\chi, f)$, giving the partition $\omega_1 = \bigcup_n A_n$ into χ-homogeneous sets. Then by an easy density argument, the map which sends $m < \omega$ to the unique n s.t. $m \in A_n$ is a new real.

Following [2], we refer to any such f as a *preassignment of colors.* Our main goal in this section is to come up with a $\mathbb{P}_\kappa$-name $\dot{f}$ for a particularly nice preassignment of colors for $\dot{\chi}_\kappa$, in the following sense:

Proposition 4.9. *There is a $\mathbb{P}_\kappa$-name $\dot{f}$ for a preassignment of colors so that for any partition product $\mathbb{R}$ based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$, any generic G for $\mathbb{R}$, and any finite collection $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ which is κ-suitable with respect to $\mathbb{R}$, the poset $\prod_{\iota \in I} \mathbb{Q}\left(\dot{\chi}_\kappa\left[\psi_\iota^{-1}(G \restriction B_\iota)\right], \dot{f}\left[\psi_\iota^{-1}(G \restriction B_\iota)\right]\right)$ is c.c.c. in $V[G]$.*

Remark 4.10. Observe that in the previous proposition, the same name $\dot{f}$ is interpreted in a variety of ways, namely, by various generics for $\mathbb{P}_\kappa$ added by forcing with $\mathbb{R}$. Moreover, $\dot{f}$ is strong enough that the product of the induced homogeneous set posets is c.c.c. This is what we mean by referring to the name as "symmetric."

Corollary 4.11. *Let $\dot{f}_\kappa$ be a name witnessing Proposition 4.9, and set $\dot{\mathbb{Q}}_\kappa$ to be the $\mathbb{P}_\kappa$-name $\mathbb{Q}(\dot{\chi}_\kappa, \dot{f}_\kappa)$. Then any partition product based upon $\underline{\mathbb{P}} \restriction (\kappa+1)$ and $\underline{\dot{\mathbb{Q}}} \restriction (\kappa+1)$ is c.c.c.*

Proof of Corollary 4.11. Let $\mathbb{R}$ be a partition product based upon $\underline{\mathbb{P}} \restriction (\kappa+1)$ and $\underline{\dot{\mathbb{Q}}} \restriction (\kappa+1)$, and let X be the domain of $\mathbb{R}$. Set $\hat{X} := \{\xi \in X : \mathrm{index}(\xi) < \kappa\}$, and let $I := \{\xi \in X : \mathrm{index}(\xi) = \kappa\}$. By Lemma 2.20, $\mathbb{R}$ is isomorphic to $(\mathbb{R} \restriction \hat{X}) * \prod_{\xi \in I} \dot{\mathbb{Q}}_\kappa\left[\pi_\xi^{-1}\left(\dot{G} \restriction b(\xi)\right)\right]$, and $\mathbb{R} \restriction \hat{X}$ is a partition product based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$. By Assumption 4.1, $\mathbb{R} \restriction \hat{X}$ is c.c.c. It is also straightforward to check that $\{\langle b(\xi), \pi_\xi\rangle : \xi \in I\}$ is κ-suitable, by the definition of $\mathbb{R}$ as a partition product based upon $\underline{\mathbb{P}} \restriction (\kappa+1)$ and $\underline{\dot{\mathbb{Q}}} \restriction (\kappa+1)$. Finally, from Proposition 4.9, we know each finitely-supported subproduct of

$$\prod_{\xi \in I} \dot{\mathbb{Q}}_\kappa\left[\pi_\xi^{-1}\left(G \restriction b(\xi)\right)\right]$$

is c.c.c. in $V[G \restriction \hat{X}]$, and hence the entire product is c.c.c. Since $\mathbb{R} \restriction \hat{X}$ is c.c.c. in V, this finishes the proof. □

We will prove Proposition 4.9 by working backwards through a series of reductions; the final proof of Proposition 4.9 occurs in Subsection 4.4. We first want to see what happens if a finite product $\prod_{l<m} \mathbb{Q}(\chi_l, f_l)$ is *not* c.c.c., where each χ_l is an open coloring on ω_1 with respect to some second countable, Hausdorff topology τ_l on ω_1 and $f_l : \omega_1 \longrightarrow \{0,1\}$ is an arbitrary preassignment. In light of this discussion, we are able to simplify the sufficient conditions for proving Proposition 4.9 by reducing the scope of our investigation to so-called *finitely generated* partition products (see Definition 4.13). With this simplification in place, we continue in the third subsection to isolate a property of the name $\dot{f}$, which we call the *partition product preassignment property.* This will complete the final reduction, isolating exactly what we need to show in order to ensure that the desired posets are c.c.c. And finally, we show how to construct names with the partition product preassignment property.

Now consider a sequence $\langle \tau_l : l < m\rangle$ of second countable, Hausdorff topologies on ω_1 with respective open colorings $\langle \chi_l : l < m\rangle$ and preassignments $\langle f_l : l < m\rangle$. Let us define $\tau := \biguplus \tau_l$, a topology on $X := \biguplus_{l<m} \omega_1$, as well as $f := \biguplus f_l$ and $\chi := \biguplus \chi_l$. So, for example, if $x \in X$, then $f(x) = f_l(x)$, where l is unique s.t. x is

in the lth copy of ω_1, and if $x, y \in X$ then $\chi(x, y)$ is defined iff x and y are distinct and belong to the same copy of ω_1, say the lth, and in this case, $\chi(x, y) = \chi_l(x, y)$. With this notation, we may view a condition in the product $\prod_{l<m} \mathbb{Q}(\chi_l, f_l)$ as a condition in $\mathbb{Q}(\chi, f)$. Note that χ is partial, and this is the only reason we allowed partial colorings in the definition of $\mathbb{Q}(\chi, f)$.

Now suppose that $\prod_{l<m} \mathbb{Q}(\chi_l, f_l)$ has an uncountable antichain. Then we claim that there exists an $n < \omega$, an uncountable subset A of X^n and a closed (in X^n) set $F \supseteq A$ so that

(1) the function $\langle x(0), \ldots, x(n-1)\rangle \mapsto \langle f(x(0)), \ldots, f(x(n-1))\rangle$ is constant on A, say with value $d \in 2^n$. Abusing notion we also denote this function by f;
(2) no two tuples in A have any elements in common;
(3) for every distinct $x, y \in F$, there exists some $i < n$ so that $\chi(x(i), y(i))$ is defined and $\chi(x(i), y(i)) \neq d(i)$.

To see that this is true, take an antichain of size $\aleph_1$ in the product $\prod_{l<m} \mathbb{Q}(\chi_l, f_l)$, and first thin it to assume that for each l, k all conditions contribute the same number of elements to the kth homogeneous set for χ_l. Now viewing the elements in the antichain as sequences arranging the members according to the coloring and homogeneous set they contribute to, call the resulting set A. Let n be the length of each sequence in A. We further thin A to secure (1). Next, thin A to become a Δ-system, and note that by taking n to be minimal, we secure (2). Now observe that, for each $x \in A$, if $i < j < n$ and $x(i)$ and $x(j)$ are part of the same homogeneous set for the same coloring χ_l, say with color c, then as χ_l is an open coloring, there exists a pair of open sets $U_{i,j} \times V_{i,j}$ in $\tau_i \times \tau_j$ such that

$$\langle x(i), x(j)\rangle \in U_{i,j} \times V_{i,j} \subseteq \chi_l^{-1}(\{c\}).$$

With this x still fixed, by intersecting at most finitely-many open sets around each $x(i)$, we may remove the dependence on coordinates $j \neq i$, and thereby obtain for each i, an open set U_i around $x(i)$ witnessing the values of χ. In particular, for any $i < j < n$ such that $x(i)$ and $x(j)$ are in the same homogeneous set for the same coloring, say χ_l, we have

$$\langle x(i), x(j)\rangle \in U_i \times U_j \subseteq \chi_l^{-1}(\{c\}),$$

where $c = \chi_l(x(i), x(j))$. By using basic open sets, of which there are only countably-many, we may thin A to assume that the sequence of open sets $\langle U_i : i < n\rangle$ is the same for all $x \in A$. As a result of fixing these open sets, and since A is an antichain, we see that (3) holds for the elements of A. Since χ is an open coloring, (3) also hold for F, the closure of A in X^n.

Remark 4.12. The conditions in the previous paragraph are equivalent to the existence of $n < \omega$, $d \in 2^n$, and a closed set $F \subseteq X^n$ so that (i) for any distinct $x, y \in F$, $\chi(x(i), y(i))$ is defined for all $i < n$, and for some $i < n$, $\chi(x(i), y(i)) \neq d(i)$; and (ii) for every countable $z \subseteq F$, there exists $x \in F \backslash z$ so that $f \circ x = d$. Indeed, it is immediate that (1)-(3) give (i) and (ii), and for the other direction, iterate (ii) to obtain the uncountable set A.

Any F as in Remark 4.12 is a closed subset of a second countable space, and so F is coded by a real. Thus if $\mathbb{R}$ is a partition product as in the statement of Proposition 4.9, then any $\mathbb{R}$-name $\dot{F}$ for such a closed set will only involve conditions intersecting countably-many support coordinates, since $\mathbb{R}$, by Assumption 4.1, is c.c.c. This motivates the following definition and subsequent remark.

Definition 4.13. *A partition product $\mathbb{R}$ with domain X, say, based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$ is said to be* finitely generated *if there is a finite, κ-suitable collection $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$, and a countable $Z \subseteq X$, such that*

$$X = Z \cup \bigcup_{\xi \in Z} b(\xi) \cup \bigcup_{\iota \in I} B_\iota.$$

In this case, we will refer to Z as the auxiliary part.

Note that in the above definition, if there is some $\xi \in Z \cap B_\iota$, then $b(\xi) \subseteq B_\iota$, since B_ι is base-closed. Thus it poses no loss of generality to assume that Z is disjoint from $\bigcup_\iota B_\iota$, and we will do so.

Remark 4.14. As shown by the arguments preceding Definition 4.13, Proposition 4.9 follows from its restriction to finitely generated partition products $\mathbb{R}$.

We further remark that Definition 4.13 refers implicitly to the following objects: index_δ, base_δ, and $\varphi_{\delta,\mu}$ for $\delta < \kappa$, as well as $\mathbb{P}_\kappa$, index_κ, base_κ, and $\varphi_{\kappa,\mu}$, which are needed in order to define a suitable collection.

It is also straightforward to see that grafting a finitely generated partition product over another such results in a partition product which is still finitely generated, as stated in the following lemma.

Lemma 4.15. *Let $(\mathbb{P}, \mathcal{B})$, $(\mathbb{R}, \mathcal{D})$, and σ be as in Lemma 3.18. Suppose that both $(\mathbb{P}, \mathcal{B})$ and $(\mathbb{R}, \mathcal{D})$ are finitely generated and that $(\mathbb{R}^*, \mathcal{D}^*)$ is the extension of $(\mathbb{R}, \mathcal{D})$ by grafting $(\mathbb{P}, \mathcal{B})$ over σ. Then $(\mathbb{R}^*, \mathcal{D}^*)$ is also finitely generated.*

One of the main advantages of looking at finitely generated partition products is that there are not too many of them, as made precise by the following two items.

Lemma 4.16. *Let M be any sufficiently elementary, countably closed substructure with $\underline{\mathbb{P}} \restriction (\kappa+1)$, $\underline{\dot{\mathbb{Q}}} \restriction \kappa \in M$. Then if $\mathbb{R}$ is a finitely generated partition product based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$, then $\mathbb{R}$ is isomorphic to a partition product which has domain an ordinal ρ below $M \cap \omega_2$.*

Proof. Fix such an M, and let $\mathbb{R}$ be a finitely generated partition product based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$, say with domain X. Let $\{\langle B_m, \psi_m\rangle : m < n\}$ be the κ-suitable collection and Z the auxiliary part, where we assume that Z is disjoint from the union of the B_m. Let us enumerate Z as $\langle \xi_k : k < \omega\rangle$ and set $\delta_k := \text{index}(\xi_k)$ for each $k < \omega$. Furthermore, we let π_k be the rearrangement of $\mathbb{P}_{\delta_k}$ associated to $\text{base}(\xi_k)$.

We intend to apply Corollary 2.19, and so we define a sequence $\langle \tau_m : m < \omega\rangle$ of rearrangements of $\mathbb{R}$ and base-closed subsets $\langle D_m : m < \omega\rangle$ of X. For each $m < n$, set τ_m to be the rearrangement which first shifts the ordinals in $X \backslash B_m$ past $\sup(B_m)$ and then acts as ψ_m^{-1} on B_m. For each $m \geq n$, say $m = n + k$, we set τ_m to be the rearrangement which first shifts the ordinals in $X \backslash (b(\xi_k) \cup \{\xi_k\})$ past ξ_k and then acts as π_k^{-1} on $b(\xi_k)$ and sends ξ_k to ρ_{δ_k}. We set $D_0 := \varnothing$, $D_{m+1} := \bigcup_{k \leq m} B_k$ for $m < n$, and $D_{n+1+k} := D_n \cup \bigcup_{l \leq k}(b(\xi_l) \cup \{\xi_l\})$ for $k < \omega$.

By Corollary 2.19, let σ be a rearrangement of $\mathbb{R}$ so that $\text{ran}(\sigma)$ is an ordinal ρ and so that for each $m < \omega$, $\sigma[D_m]$ is an ordinal and $\tau_m \circ \sigma^{-1}$ is order-preserving on $\sigma[D_{m+1} \backslash D_m]$. We then see that ρ equals $\sum_{m<\omega} \text{ot}(\sigma[D_{m+1} \backslash D_m])$. However, if $m <$

n, then $\mathrm{ot}(\sigma[D_{m+1}\backslash D_m])$ is no larger than ρ_κ, and if $m \geq n$, then $\mathrm{ot}(\sigma[D_{m+1}\backslash D_m])$ is no larger than $\rho_{\delta_k} + 1$, where $m = n + k$. Therefore

$$\rho = \sum_{m<\omega} \mathrm{ot}(\sigma[D_{m+1}\backslash D_m]) \leq \rho_\kappa \cdot n + \sum_{k<\omega} (\rho_{\delta_k} + 1).$$

By the elementarity and countable closure of M, the ordinal on the right hand side is an element of $M \cap \omega_2$. Since $M \cap \omega_2$ is an ordinal, ρ is also a member of $M \cap \omega_2$. $\square$

Lemma 4.17. *Let M be any sufficiently elementary, countably closed substructure containing $\underline{\mathbb{P}} \upharpoonright (\kappa+1)$, $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa$ and $\vec{\varphi}$ as members. Then, if $\mathbb{R}$ is a finitely generated partition product based upon $\underline{\mathbb{P}} \upharpoonright \kappa$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright \kappa$, then $\mathbb{R}$ is isomorphic to a partition product which belongs to M.*

Proof. Let M be fixed as in the statement of the lemma, and let $\mathbb{R}$ be finitely generated. Let $\{\langle B_k, \psi_k\rangle : k < n\}$ be the κ-suitable collection and Z the auxiliary part associated to $\mathbb{R}$. By Lemma 4.16, we may assume that $\mathbb{R}$ is a partition product on some ordinal ρ and that $\rho \in M \cap \omega_2$. Since $M \cap \omega_2$ is an ordinal, $\rho \subseteq M$. Then $Z \subseteq M$, and so by the countable closure of M, Z is a member of M. Hence by the elementarity and countable closure of M, setting $\delta_\xi := \mathrm{index}(\xi)$ for each $\xi \in Z$, the sequence $\langle \delta_\xi : \xi \in Z\rangle$ is in M.

Now fix $k < n$ and $\xi \in Z$, and note that by Remark 3.3, since δ_ξ and κ are in M, $\psi_k^{-1}[B_k \cap b(\xi)]$ is in M. Next consider the relation in μ, ν which holds iff $\pi_\xi(\mu) = \psi_k(\nu)$, and observe that by Lemma 3.7, this holds iff ν is the μth element of $\psi_k^{-1}[B_k \cap b(\xi)]$. Therefore, this relation is a member of M. By the countable closure of M, the relation in ξ, k, μ, ν which holds iff $\pi_\xi(\mu) = \psi_k(\nu)$ is in M too. Similarly, the relation (in ξ, ζ, μ, ν) which holds iff $\pi_\xi(\mu) = \pi_\zeta(\nu)$ and the relation (in k, l, μ, ν) which holds iff $\psi_k(\mu) = \psi_l(\nu)$ are also in M.

We now apply the elementarity of M to find a finitely generated partition product $\mathbb{R}^*$ with domain ρ which has the following properties, where base* and index* denote the functions supporting $\mathbb{R}^*$:

(1) $\mathbb{R}^*$ has κ-suitable collection $\{\langle B_k^*, \psi_k^*\rangle : k < n\}$ and auxiliary part Z; moreover, for each $\xi \in Z$, $\mathrm{index}^*(\xi) = \delta_\xi$;
(2) for each $\mu, \nu < \rho$ and each $\xi, \zeta \in Z$, $\pi_\xi(\mu) = \pi_\zeta(\nu)$ iff $\pi_\xi^*(\mu) = \pi_\zeta^*(\nu)$, and similarly with one of the ψ_k (resp. ψ_k^*) replacing one or both of the π_i (resp. π_i^*).

We now define a bijection $\sigma : \rho \longrightarrow \rho$ which will be the rearrangement witnessing that $\mathbb{R}$ and $\mathbb{R}^*$ are isomorphic. Set $\sigma(\alpha) = \beta$ iff $\alpha = \beta$ are both in Z; or for some $\xi \in Z$, $\alpha = \pi_\xi(\mu)$ and $\beta = \pi_\xi^*(\mu)$; or for some $k < n$, $\alpha = \psi_k(\mu)$ and $\beta = \psi_k^*(\mu)$. By (2), we see that σ is well-defined, i.e., there is no conflict when some of these conditions overlap. It is also straightforward to see that σ is an acceptable rearrangement of $\mathbb{R}$ and in fact, $\sigma[\mathrm{base}] = \mathrm{base}^*$ and $\sigma[\mathrm{index}] = \mathrm{index}^*$, so that σ is an isomorphism from $\mathbb{R}$ onto $\mathbb{R}^*$. $\square$

Recall that we are assuming the CH holds (Assumption 4.1). Thus for the rest of Section 4, we fix a sufficiently elementary substructure M satisfying the conclusion of Lemma 4.17 such that $|M| = \aleph_1$. We write $M = \bigcup_{\gamma<\omega_1} M_\gamma$, for a continuous, increasing sequence of sufficiently elementary, countable M_γ, such that the relevant parameters are in M_0.

Remark 4.18. The crucial use of the CH in the paper is to fix the model M. We will use the decomposition $M = \bigcup_{\gamma<\omega_1} M_\gamma$ to partition a tail of ω_1 into the slices $[M_\gamma \cap \omega_1, M_{\gamma+1} \cap \omega_1)$. We will show that it suffices to define the preassignment one slice at a time, with the values of the preassignment on one slice independent of the others. As Lemma 4.20 below shows, the preassignment restricted to the slice $[M_\gamma \cap \omega_1, M_{\gamma+1} \cap \omega_1)$ only needs to anticipate "partition product names" which are members of M_γ. This idea that the preassignment need only work in the above slices goes back to Lemma 3.2 of [2]. Furthermore, the proof of our Lemma 4.19 is more or less the same as Lemma 3.2 of [2]; we are simply working in slightly greater generality in order to analyze products of posets rather than just a single poset.

We recall that $\dot{S}_\kappa$ names a countable basis for a second countable, Hausdorff topology on ω_1.

Lemma 4.19. *Suppose that $\dot{f}$ is a $\mathbb{P}_\kappa$-name for a function from ω_1 into $\{0,1\}$ which satisfies the following: for any finitely generated partition product $\mathbb{R}$, with κ-suitable collection $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ and auxiliary part Z, say, all of which are in M; for every γ sufficiently large so that $\mathbb{R}$, the κ-suitable collection, and Z are in M_γ; for any $\mathbb{R}$-name $\dot{F}$ in M_γ for a set of n-tuples in $X := \biguplus_{\iota\in I}\omega_1$, which is closed in $\left(\biguplus_\iota \dot{S}_\kappa[\psi_\iota^{-1}(\dot{G} \restriction B_\iota)]\right)^n$; for any generic G for $\mathbb{R}$; and for any x with*

$$x \in \dot{F}[G] \cap (M_{\gamma+1}[G] \backslash M_\gamma[G])^n,$$

there exist pairwise distinct tuples y, y' in $\dot{F}[G] \cap M_\gamma[G]$ so that for every $i < n$ and $\iota \in I$, if $x(i)$ is in the ι-th copy of ω_1, then so are $y(i)$ and $y'(i)$, and

$$\dot{\chi}_\kappa[\psi_\iota^{-1}(G \restriction B_\iota)](y(i), y'(i)) = \dot{f}[\psi_\iota^{-1}(G \restriction B_\iota)](x(i)).$$

Then $\dot{f}$ satisfies Proposition 4.9.

Proof. Let $\dot{f}$ be as in the statement of the lemma, and suppose that $\dot{f}$ failed to satisfy Proposition 4.9. By Remarks 4.12 and 4.14 there exist a finitely generated partition product $\mathbb{R}$, a condition $p \in \mathbb{R}$, an integer $n < \omega$, a sequence $d \in 2^n$, and an $\mathbb{R}$-name for a closed set $\dot{F}$ of n-tuples such that p forces that these objects satisfy Remark 4.12. We may assume that $\mathbb{R} \in M$ by Lemma 4.17. Since M is countably closed and contains $\mathbb{R}$, and since $\mathbb{R}$ is c.c.c. (by Assumption 4.1), we know that the name $\dot{F}$ belongs to M too. Thus we may find some $\gamma < \omega_1$ such that $\dot{F}$ and all other relevant objects are in M_γ.

Now let G be a generic for $\mathbb{R}$ containing the condition p. Let $S := \biguplus_\iota \dot{S}_\kappa[\psi_\iota^{-1}(G \restriction B_\iota)]$, let $f := \biguplus_\iota \dot{f}[\psi_\iota^{-1}(G \restriction B_\iota)]$, and let $\chi := \biguplus_\iota \dot{\chi}_\kappa[\psi_\iota^{-1}(G \restriction B_\iota)]$. By (ii) of Remark 4.12, we may find some $x \in F \cap (X \backslash M_\gamma[G])^n$, where $F := \dot{F}[G]$. We now want to consider how the models $\langle M_\beta : \gamma \leq \beta < \omega_1\rangle$ separate the elements of x, and then we will apply the assumptions of the lemma to each $\beta \in [\gamma, \omega_1)$ such that $M_{\beta+1}[G] \backslash M_\beta[G]$ contains an element of x. Indeed, consider the finite set a of $\beta \in [\gamma, \omega_1)$ such that x contains at least one element in $M_{\beta+1}[G] \backslash M_\beta[G]$, and let $\langle \gamma_k : k < l\rangle$ be the increasing enumeration of a. Further, let x_k, for each $k < l$, be the subsequence of x consisting of all the elements of x inside $M_{\gamma_k+1}[G] \backslash M_{\gamma_k}[G]$.

We now work downwards from l to define a sequence of formulas $\langle \varphi_k : k \leq l\rangle$. We will maintain as recursion hypotheses that if $0 < k < l$, then (i) $\varphi_{k+1}(x_0, \ldots, x_k)$ is satisfied, and that (ii) the parameters of φ_{k+1} are in $M_{\gamma_0}[G]$. Let $\varphi_l(u_0, \ldots, u_{l-1})$ state that $u_0{}^\frown \ldots {}^\frown u_{l-1} \in F$; then (i) and (ii) are satisfied. Now suppose that

$0 < k < l$ and that φ_{k+1} is defined. Let F_k be the closure of the set of all tuples u such that $\varphi_{k+1}(x_0, \ldots, x_{k-1}, u)$ is satisfied. By (ii) and the fact that $x_0{}^\frown \ldots {}^\frown x_{k-1} \in M_{\gamma_k}[G]$, we see that F_k is in $M_{\gamma_k}[G]$. Furthermore, $x_k \in F_k$. Therefore, by the assumptions of the lemma, we may find pairwise distinct tuples $v_{k,L}, v_{k,R}$ in $M_{\gamma_k}[G] \cap F_k$ such that for every $i < n$ and $\iota \in I$, if $x_k(i)$ is in the ι-th copy of ω_1, then so are $v_{k,L}(i)$ and $v_{k,R}(i)$, and

$$\dot{\chi}_\kappa[\psi_\iota^{-1}(G \restriction B_\iota)](v_{k,L}(i), v_{k,R}(i)) = \dot{f}[\psi_\iota^{-1}(G \restriction B_\iota)](x_k(i)).$$

For each such i, fix a pair of disjoint, basic open sets U_i, V_i from $\dot{S}_\kappa[\psi_\iota^{-1}(G \restriction B_\iota)]$ witnessing this coloring statement. By definition of F_k, we may find two further tuples $u_{k,L}, u_{k,R}$ such that for each $Z \in \{L, R\}$, $\varphi_{k+1}(x_0, \ldots, x_{k-1}, u_{k,Z})$ is satisfied, and such that the pair $\langle u_{k,L}(i), u_{k,R}(i)\rangle$ is in $U_i \times V_i$. Now define $\varphi_k(u_0, \ldots, u_{k-1})$ to be the following formula:

$$\exists w_{k,L}, w_{k,R} \left(\bigwedge_{Z \in \{L,R\}} \varphi_{k+1}(u_0, \ldots, u_{k-1}, w_{k,Z}) \wedge \bigwedge_i \Big(\langle w_{k,L}(i), w_{k,R}(i)\rangle \in U_i \times V_i \Big) \right).$$

Then (i) is satisfied, and since the only additional parameters are the basic open sets U_i and V_i, (ii) is also satisfied.

This completes the construction of the sequence $\langle \varphi_k : k \leq l\rangle$. Now using the fact that the sentence φ_0 is true and has only parameters in M_{γ_0}, we may work our way upwards through the sequence $\varphi_0, \varphi_1, \ldots, \varphi_l$ in order to find two tuples x_L, x_R of the same length as x such that $x_L, x_R \in F$, and such that for each $i < n$, $\langle x_L(i), x_R(i)\rangle \in U_i \times V_i$. In particular, for each $i < n$,

$$\dot{\chi}_\kappa[\psi_\iota^{-1}(G \restriction B_\iota)](x_L(i), x_R(i)) = \dot{f}[\psi_\iota^{-1}(G \restriction B_\iota)](x(i)),$$

where ι is such that $x(i)$ is in the ι-th copy of ω_1. However, recalling Remark 4.12 and the assumptions about the condition p, this contradicts the fact that $f \circ x = d$, and that there is some $i < n$ so that $\chi(x_L(i), x_R(i)) \neq d(i)$. $\square$

The following lemma gives a nice streamlining of the previous one and applies to any collection $\dot{U}$ of n-tuples in ω_1, not just collections $\dot{F}$ which are closed in the appropriate topology. The greater generality here is only apparent, as we can always take closures and obtain, because the colorings are open, the same result from its application to closed sets of tuples. However, it is technically convenient. Also, as a matter of notation, for each $\gamma < \omega_1$, we fix an enumeration $\langle \nu_{\gamma,n} : n < \omega\rangle$ of the slice $[M_\gamma \cap \omega_1, M_{\gamma+1} \cap \omega_1)$.

Lemma 4.20. *Suppose that $\dot{f}$ is a $\mathbb{P}_\kappa$-name for a function from ω_1 into $\{0,1\}$ satisfying the following: for any finitely generated partition product $\mathbb{R}$, say with κ-suitable collection $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ and auxiliary part Z, all of which are in M; for any γ sufficiently large such that M_γ contains $\mathbb{R}$, $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$, and Z; for any $l < \omega$; for any $\mathbb{R}$-name $\dot{U}$ in M_γ for a set of l-tuples in ω_1; and for any generic G for $\mathbb{R}$, if $\langle \nu_{\gamma,0}, \ldots, \nu_{\gamma,l-1}\rangle \in \dot{U}[G]$, then there exist pairwise distinct l-tuples $\vec{\mu}, \vec{\mu}'$ in $M_\gamma[G] \cap \dot{U}[G]$ so that for all $k < l$ and all $\iota \in I$,*

$$\dot{\chi}_\kappa[\psi_\iota^{-1}(G \restriction B_\iota)](\mu_k, \mu_k') = \dot{f}[\psi_\iota^{-1}(G \restriction B_\iota)](\nu_{\gamma,k}).$$

Then $\dot{f}$ satisfies Lemma 4.19.

Proof. We want to first observe that Lemma 4.19 follows from its restriction to sequences z which are bijections from some $n < \omega$ onto $\biguplus_\iota \{\nu_{\gamma,l} : l < m\}$, for some $m < \omega$. Towards this end, fix $\dot{F}$, G, and a tuple $x \in \dot{F}[G]$ as in the statement of Lemma 4.19. First, if x is not such a surjection, we may add additional coordinates to x to form a sequence x' which is a surjection onto $\biguplus_\iota \{\nu_{\gamma,l} : l < m\}$, for some $m < \omega$. Then we define the name $\dot{F}'$ as the product of $\dot{F}$ with the requisite, finite number of copies of ω_1, so that x' is a member of $\dot{F}'[G]$. Second, if x' contains repetitions, then we make the necessary shifts in x' to eliminate the repetitions and call the resulting sequence x''. We then consider the name $\dot{F}''$ of all tuples from $\dot{F}'$ which have the same corresponding shifts in their tuples as x''. $\dot{F}''$ still names a closed set and is still an element of M_γ. Thus $x'' \in \dot{F}''[G]$, and x'' is a bijection from some integer onto $\biguplus_\iota \{\nu_{\gamma,l} : l < m\}$, for some $m < \omega$. By applying the restricted version of Lemma 4.19 to x'' and $\dot{F}''$, we see that the desired result holds for x and $\dot{F}$.

To verify Lemma 4.19, fix $\dot{F}$, a generic G, and a sequence $x \in \dot{F}[G]$ as in the statement thereof, where we assume that x is a bijection from some n onto $\biguplus_\iota \{\nu_{\gamma,l} : l < m\}$, for some $m < \omega$. Define $\dot{U}$ to be the $\mathbb{R}$-name for the set of all tuples $\vec{\xi} = \langle \xi_0, \ldots, \xi_{m-1} \rangle$ in ω_1 such that $\vec{\xi}$ concatenated with itself $|I|$-many times is an element of $\dot{F}$, noting that $\dot{U}$ is still a member of M_γ. Since x is a bijection as described above, $\langle \nu_{\gamma,0}, \ldots, \nu_{\gamma,m-1} \rangle \in \dot{U}[G]$. Now apply the assumptions in the statement of the current lemma to find two pairwise distinct m-tuples $\vec{\mu}, \vec{\mu}'$ in $M_\gamma[G] \cap \dot{U}[G]$ so that for all $l < m$ and $\iota \in I$,

$$\dot{\chi}_\kappa[\psi_\iota^{-1}(G \upharpoonright B_\iota)](\mu_l, \mu_l') = \dot{f}[\psi_\iota^{-1}(G \upharpoonright B_\iota)](\nu_{\gamma,l}).$$

Let y be the $|I|$-fold concatenation of $\vec{\mu}$ with itself, and let y' be defined similarly with respect to $\vec{\mu}'$. Then as $\vec{\mu}, \vec{\mu}' \in \dot{U}[G]$, we have y, y' are in $\dot{F}[G]$. And since $\vec{\mu}, \vec{\mu}'$ satisfy the appropriate coloring requirements, we have that y and y' satisfy the conclusion of Lemma 4.19. $\square$

Lemma 4.20 gives a sufficient condition for Proposition 4.9, and it thus implies that any partition product based upon $\underline{\mathbb{P}} \upharpoonright (\kappa+1)$ and $\underline{\dot{\mathbb{Q}}} \upharpoonright (\kappa+1)$ is c.c.c. In the next subsection, we consider how to obtain a $\mathbb{P}_\kappa$-name $\dot{f}$ as in Lemma 4.20.

4.3. **How to get there.**

In this subsection, which forms the technical heart of the paper, we show how to obtain a $\mathbb{P}_\kappa$-name $\dot{f}$ as in Lemma 4.20. In light of Remark 4.18 and Lemma 4.20, it suffices to define the name $\dot{f}$ separately for each of its restrictions to the slices $[M_\gamma \cap \omega_1, M_{\gamma+1} \cap \omega_1)$, and so let $\gamma < \omega_1$ be fixed for the remainder of this subsection. To simplify notation, we drop the γ-subscript from the enumeration $\langle \nu_{\gamma,n} : n < \omega \rangle$ of $[M_\gamma \cap \omega_1, M_{\gamma+1} \cap \omega_1)$, preferring instead to simply write $\langle \nu_n : n < \omega \rangle$. We note that the values of $\dot{f}$ on the countable ordinal $M_0 \cap \omega_1$ are irrelevant, by Remark 4.12.

In order to define the name $\dot{f}$, we recursively specify the $\mathbb{P}_\kappa$-name equal to $\dot{f}(\nu_k)$, which we call $\dot{a}_k$. Each $\dot{a}_k$ will be a canonical name, which we view as a function from a maximal antichain in $\mathbb{P}_\kappa$ into $\{0, 1\}$. We refer to these more specifically as *canonical color names*. By a *partial canonical color name* we mean a function from an antichain in $\mathbb{P}_\kappa$, possibly not maximal, into $\{0, 1\}$. When viewing such functions as names $\dot{a}$, we say that $\dot{a}[G]$, where G is generic for $\mathbb{P}_\kappa$, is defined and equal to i if there is some $p \in G$ which belongs to the domain of the function $\dot{a}$ and gets mapped to i. The upcoming definition isolates exactly what we need.

Definition 4.21. *Suppose that $\dot{a}_0, \dots, \dot{a}_{l-1}$ are partial canonical color names. We say that they have the* partition product preassignment property at γ *if for every finitely generated partition product $\mathbb{R}$ with κ-suitable collection $\{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$, say, all of which are in M_γ; for every $\mathbb{R}$-name $\dot{U} \in M_\gamma$ for a collection of l-tuples in ω_1; and for every generic G for $\mathbb{R}$, the following holds: if $\langle \nu_0, \dots, \nu_{l-1}\rangle \in \dot{U}[G]$, then there exist two pairwise distinct tuples $\vec{\mu}, \vec{\mu}' \in \dot{U}[G] \cap M_\gamma[G]$ so that for every $\iota \in I$ and $k < l$, if $\dot{a}_k[\psi_\iota^{-1}(G \restriction B_\iota)]$ is defined, then*

$$\dot{\chi}_\kappa[\psi_\iota^{-1}(G \restriction B_\iota)](\mu_k, \mu_k') = \dot{a}_k[\psi_\iota^{-1}(G \restriction B_\iota)].$$

Remark 4.22. In the context of Definition 4.21, we say that two sequences $\vec{\mu}$ and $\vec{\mu}'$ of length l *match $\dot{a}_0, \dots, \dot{a}_{l-1}$ at ι with respect to G*, or *match at B_ι with respect to G* if for every $k < l$ such that $\dot{a}_k[\psi_\iota^{-1}(G \restriction B_\iota)]$ is defined,

$$\dot{\chi}_\kappa[\psi_\iota^{-1}(G \restriction B_\iota)](\mu_k, \mu_k') = \dot{a}_k[\psi_\iota^{-1}(G \restriction B_\iota)].$$

We say that two sequences $\vec{\mu}$ and $\vec{\mu}'$ *match $\dot{a}_0, \dots, \dot{a}_{l-1}$ on I with respect to G* if for every $\iota \in I$, $\vec{\mu}$ and $\vec{\mu}'$ match $\dot{a}_0, \dots, \dot{a}_{l-1}$ at ι with respect to G. If the filter G is clear from context, we drop the phrase "with respect to G." Furthermore, we will often want to avoid talking about the index set I explicitly, and so we will also say that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \dots, \dot{a}_{l-1}$ on $\mathcal{S} := \{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$, if for each $\langle B, \psi\rangle \in \mathcal{S}$, we have that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \dots, \dot{a}_{l-1}$ at B.

To prove Lemma 4.20, and in turn Proposition 4.9, we recursively construct the sequence $\langle \dot{a}_k : k < \omega\rangle$ in such a way that for each $l < \omega$, $\dot{a}_0, \dots, \dot{a}_{l-1}$ have the partition product preassignment property at γ. More precisely, we show that if $\dot{a}_0, \dots, \dot{a}_{l-1}$ are *total* canonical color names with the partition product preassignment property at γ, then there is a total name $\dot{a}_l$ so that $\dot{a}_0, \dots, \dot{a}_l$ have the partition product preassignment property at γ.

For this in turn it is enough to prove that if $\dot{a}_0, \dots, \dot{a}_{l-1}$ are total canonical color names, $\dot{a}_l$ is a partial canonical color name, $\dot{a}_0, \dots, \dot{a}_l$ have the partition product preassignment property at γ, and $p \in \mathbb{P}_\kappa$ is incompatible with all conditions in the domain of $\dot{a}_l$, then there exist $p^* \leq_{\mathbb{P}_\kappa} p$ and $c \in \{0, 1\}$ so that $\dot{a}_0, \dots, \dot{a}_l \cup \{p^* \mapsto c\}$ have the partition product preassignment property at γ. By a transfinite iteration of this process we can construct a sequence of names $\dot{a}_l^\xi$ with increasing domains, continuing until we reach a name whose domain is a maximal antichain. This final name is then total.

To prove the "one condition" extension above, we assume that it fails with $c = 0$ and prove that it then holds with $c = 1$. Our assumption is the following:

Assumption 4.23. $\dot{a}_0, \dots, \dot{a}_{l-1}$ are total canonical color names, $\dot{a}_l$ is partial, $\dot{a}_0, \dots, \dot{a}_l$ have the partition product preassignment property at γ, $p \in \mathbb{P}_\kappa$ is incompatible with all conditions in $\mathrm{dom}(\dot{a}_l)$, but for every $p^* \leq_{\mathbb{P}_\kappa} p$, $\dot{a}_0, \dots, \dot{a}_l \cup \{p^* \mapsto 0\}$ do *not* have the partition product preassignment property at γ.

Our goal is to show that $\dot{a}_0, \dots, \dot{a}_l \cup \{p \mapsto 1\}$ do have the partition product preassignment property at γ. The following lemma is the key technical result which allows us to prove that $p \mapsto 1$ works in this sense and thereby continue the construction of the name $\dot{a}_l$. We note that the lemma is stated in terms of enriched partition products; the enrichments are used to propagate the induction hypothesis needed for its proof.

Lemma 4.24. *Let $(\mathbb{R}, \mathcal{B})$ be an enriched partition product with domain X which is finitely generated by a κ-suitable collection $\mathcal{S} = \{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ and auxiliary part Z, all of which belong to M_γ. Let $\bar{p}$ be a condition in $\mathbb{R}$, and let $\vec{\nu} := \langle \nu_0, \ldots, \nu_l\rangle$. Finally, let $\bar{\mathcal{S}} \subseteq \mathcal{S}$ be non-empty. Then there exist the following objects:*

- (a) *an enriched partition product $(\mathbb{R}^*, \mathcal{B}^*)$ with domain X^*, finitely generated by a κ-suitable collection $\mathcal{S}^*$ and an auxiliary part Z^*, all of which are in M_γ;*
- (b) *a condition $p^* \in \mathbb{R}^*$;*
- (c) *an $\mathbb{R}^*$-name $\dot{U}^*$ in M_γ for a collection of $l+1$-tuples in ω_1;*
- (d) *a non-empty, finite collection $\mathcal{F}$ in M_γ of embeddings from $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^*, \mathcal{B}^*)$;*

satisfying that for each $\pi \in \mathcal{F}$:

- (1) *$p^* \leq_{\mathbb{R}^*} \pi(\bar{p})$;*

and also satisfying that p^* forces the following statements in $\mathbb{R}^*$:

- (2) *$\vec{\nu} \in \dot{U}^*$;*
- (3) *for any pairwise distinct tuples $\vec{\mu}, \vec{\mu}'$ in $\dot{U}^* \cap M_\gamma[\dot{G}^*]$, if $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l$ on $\mathcal{S}^*$, then there is some $\pi \in \mathcal{F}$ such that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\pi[\bar{\mathcal{S}}]$.*

Proof. For the remainder of the proof, fix the objects $(\mathbb{R}, \mathcal{B})$, X, $\mathcal{S}$, Z, $\bar{p}$, and $\bar{\mathcal{S}}$ as in the statement of the lemma. We also set $J := \left\{\iota \in I : \langle B_\iota, \psi_\iota\rangle \in \bar{\mathcal{S}}\right\}$. Before we continue, let us introduce the following terminology: suppose that $p' \leq_{\mathbb{P}_\kappa} p$, $c \in \{0,1\}$, and $\tilde{p} \in \mathbb{P}_\kappa$. We say that $\tilde{p}$ is *decisive* about the sequence of names $\dot{a}_0, \ldots, \dot{a}_l \cup \{p' \mapsto c\}$ if for each $k < l$, $\tilde{p}$ extends a unique element of $\mathrm{dom}(\dot{a}_k)$, and if $\tilde{p}$ either extends a unique element of $\mathrm{dom}(\dot{a}_l) \cup \{p'\}$ or is incompatible with all conditions therein. Note that any $\tilde{p}$ may be extended to a decisive condition, as $\mathrm{dom}(\dot{a}_k)$ is a maximal antichain in $\mathbb{P}_\kappa$, for each $k < l$.

For each $\iota \in I$ we set p_ι to be the condition $\psi_\iota^{-1}(\bar{p} \restriction B_\iota)$ in $\mathbb{P}_\kappa$. By extending $\bar{p}$ if necessary, we may assume that for each $\iota \in I$, p_ι is decisive about $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$. Let us also define

$$J_p := \{\iota \in J : p_\iota \leq_{\mathbb{P}_\kappa} p\},$$

noting that for each $\iota \in J \backslash J_p$, p_ι is incompatible with p in $\mathbb{P}_\kappa$, since p_ι is decisive.

We will prove by induction that there exist objects as in (a)-(d) satisfying (1)-(3). The induction concerns properties of $\bar{\mathcal{S}}$, which we will refer to as the *matching core* of $\mathcal{S}$, in light of the requirement in (3) that the desired matching occurs on the image of $\bar{\mathcal{S}}$ under some $\pi \in \mathcal{F}$. By the definition of κ-suitability and Remark 3.9, for each distinct $\iota_0, \iota_1 \in I$, $\psi_{\iota_0}^{-1}[B_{\iota_0} \cap B_{\iota_1}] = \psi_{\iota_1}^{-1}[B_{\iota_0} \cap B_{\iota_1}]$ is an ordinal $< \rho_\kappa$. We will denote this ordinal by $\mathrm{ht}(B_{\iota_0}, B_{\iota_1})$; it is helpful to note that $\mathrm{ht}(B_{\iota_0}, B_{\iota_1}) = \max\{\alpha < \rho_\kappa : \psi_{\iota_0}[\alpha] = \psi_{\iota_1}[\alpha]\} = \sup\{\xi + 1 : \psi_{\iota_0}(\xi) = \psi_{\iota_1}(\xi)\}$. The induction will be first on the ordinal

$$\mathrm{ht}\left(\bar{\mathcal{S}}\right) := \max\{\mathrm{ht}(B_{\iota_0}, B_{\iota_1}) : \iota_0, \iota_1 \in J \wedge \iota_0 \neq \iota_1\}$$

and second on the finite size of $\bar{\mathcal{S}}$.

Case 1: $\operatorname{ht}\left(\bar{\mathcal{S}}\right)=0$ (note that this includes as a subcase $|J|=1$). For each $\iota \in J_p$, p_ι extends p in $\mathbb{P}_\kappa$, and so, by Assumption 4.23, $\dot{a}_0, \ldots, \dot{a}_l \cup \{p_\iota \mapsto 0\}$ do not have the partition product preassignment property at γ. For each $\iota \in J_p$, we fix the following objects as witnesses to this:

$(1)_\iota$ a partition product $\mathbb{R}^*_\iota$, say with domain X^*_ι, which is finitely generated by the κ-suitable collection $\mathcal{S}^*_\iota = \left\{\langle B^*_{\iota,\eta}, \psi^*_{\iota,\eta}\rangle : \eta \in I^*(\iota)\right\}$ and auxiliary part Z^*_ι, all of which are in M_γ;

$(2)_\iota$ a condition p^*_ι in $\mathbb{R}^*_\iota$;

$(3)_\iota$ an $\mathbb{R}^*_\iota$-name $\dot{U}^*_\iota$ in M_γ for a set of $l+1$-tuples in ω_1;

such that p^*_ι forces in $\mathbb{R}^*_\iota$ that

$(4)_\iota$ $\vec{\nu} \in \dot{U}^*_\iota$, and for any pairwise distinct tuples $\vec{\mu}, \vec{\mu}'$ in $\dot{U}^*_\iota \cap M_\gamma[\dot{G}^*_\iota]$, $\vec{\mu}$ and $\vec{\mu}'$ do *not* match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p_\iota \mapsto 0\}$ on $I^*(\iota)$.

For each $\eta \in I^*(\iota)$, let $p_{\iota,\eta}$ denote the $\mathbb{P}_\kappa$-condition $(\psi^*_{\iota,\eta})^{-1}\left(p^*_\iota \upharpoonright B^*_{\iota,\eta}\right)$, and note that by extending the condition p^*_ι, we may assume that each $p_{\iota,\eta}$ is decisive about $\dot{a}_0, \ldots, \dot{a}_l \cup \{p_\iota \mapsto 0\}$. It is straightforward to check that since each such $p_{\iota,\eta}$ is decisive and since, by Assumption 4.23, $\dot{a}_0, \ldots, \dot{a}_l$ do have the partition product preassignment property at γ, we must have that

$$J^*(\iota) := \{\eta \in I^*(\iota) : p_{\iota,\eta} \leq_{\mathbb{P}_\kappa} p_\iota\} \neq \varnothing,$$

as otherwise we contradict $(4)_\iota$.

Let us introduce some further notation which will facilitate the exposition. For $\iota \in J \backslash J_p$, define $\mathbb{R}^*_\iota$ to be some isomorphic copy of $\mathbb{P}_\kappa$ with domain X^*_ι, say with isomorphism $\psi^*_{\iota,\iota}$; we will denote X^*_ι additionally by $B^*_{\iota,\iota}$ in order to streamline the notation in later arguments. For $\iota \in J \backslash J_p$, we set $\mathcal{S}^*_\iota := \left\{\langle B^*_{\iota,\iota}, \psi^*_{\iota,\iota}\rangle\right\}$ with index set $I^*(\iota) = \{\iota\}$ which we also denote by $J^*(\iota)$. Next, we define p^*_ι to be the image of p_ι under the isomorphism $\psi^*_{\iota,\iota}$ from $\mathbb{P}_\kappa$ onto $\mathbb{R}^*_\iota$, and we set $\dot{U}^*_\iota$ to be the $\mathbb{R}^*_\iota$-name for all $l+1$-tuples in ω_1. We remark here for later use that for each $\iota \in J$ and $\eta \in J^*(\iota)$,

$$(\psi^*_{\iota,\eta})^{-1}\left(p^*_\iota \upharpoonright B^*_{\iota,\eta}\right) \leq_{\mathbb{P}_\kappa} (\psi_\iota)^{-1}(\bar{p} \upharpoonright B_\iota).$$

Our next step is to amalgamate all of the above into one much larger partition product. Without loss of generality, by shifting if necessary, we may assume that the domains X^*_ι, for $\iota \in J$, are pairwise disjoint. Then, by Corollary 3.20, the poset $\mathbb{R}^*(0) := \prod_{\iota \in J} \mathbb{R}^*_\iota$ is a partition product with domain $\bigcup_{\iota \in J} X^*_\iota$. It is also a member of M_γ. Additionally, $\mathbb{R}^*(0)$ is finitely generated by the κ-suitable collection $\mathcal{S}^* := \bigcup_{\iota \in J} \mathcal{S}^*_\iota$ and auxiliary part $\bigcup_{\iota \in J_p} Z^*_\iota$. Let us abbreviate $\bigcup_{\iota \in J} B_\iota$ by X_0 and $\bigcup_{\iota \in J} X^*_\iota$ by X^*_0. We also let $p^*(0)$ be the condition in $\mathbb{R}^*(0)$ whose restriction to X^*_ι equals p^*_ι, and we let $\dot{U}^*$ be the $\mathbb{R}^*(0)$-name for the intersection of all the $\dot{U}^*_\iota$, for $\iota \in J$.

Now consider the product of indices

$$\hat{J} := \prod_{\iota \in J} J^*(\iota);$$

$\hat{J}$ is non-empty, finite, and an element of M_γ, since J and each $J^*(\iota)$ are. Let $\langle h_k : k < n\rangle$ enumerate $\hat{J}$. Each h_k selects, for every $\iota \in J$, an image of the $\mathbb{P}_\kappa$-"branch" B_ι inside $\mathbb{R}^*_\iota$. For each $k < n$, we define the map $\pi_k : X_0 \longrightarrow X^*_0$ corresponding to h_k by taking $\pi_k \upharpoonright B_\iota$ to be equal to $\psi^*_{\iota,h_k(\iota)} \circ \psi_\iota^{-1}$, for each $\iota \in J$. This is well-defined since, by our assumption that $\operatorname{ht}\left(\bar{\mathcal{S}}\right)=0$, we know that the sets B_ι, for $\iota \in J$, are pairwise disjoint. We also see that each π_k embeds $\mathbb{R} \restriction X_0$ into $\mathbb{R}^*(0)$, since it isomorphs $\mathbb{R} \restriction B_\iota$ onto $\mathbb{R}^*(0) \restriction B^*_{\iota,h_k(\iota)}$, for each $\iota \in J$. In fact, each π_k is $(\bar{\mathcal{S}}, \mathcal{S}^*)$-suitable by construction, and h_k is the associated injection h_{π_k} (see Definition 4.5). Finally, we want to see that $p^*(0)$ extends $\pi_k(\bar{p} \restriction X_0)$ for each $k < n$; but this follows by definition of π_k and our above observation that for each $\iota \in J$ and $\eta \in J^*(\iota)$,

$$(\psi^*_{\iota,\eta})^{-1}\left(p^*_\iota \restriction B^*_{\iota,\eta}\right) \leq_{\mathbb{P}_\kappa} (\psi_\iota)^{-1}(\bar{p} \restriction B_\iota).$$

Using Lemma 4.7, fix an enrichment $\mathcal{B}^*_0$ of $\mathbb{R}^*(0)$ such that $\mathcal{B}^*_0$ contains the image of $\mathcal{B} \restriction X_0$ under each π_k and such that $\left\{\langle B^*_{\iota,\eta}, \psi^*_{\iota,\eta}\rangle : \iota \in J \wedge \eta \in J^*(\iota)\right\}$ is κ-suitable with respect to $(\mathbb{R}^*(0), \mathcal{B}^*_0)$. Note that the assumptions of Lemma 4.7 are satisfied because the sets X^*_ι, for $\iota \in J$, are pairwise disjoint and $\{\pi_k : k < n\}$ is a collection of $(\bar{\mathcal{S}}, \mathcal{S}^*)$-suitable maps.

Before continuing with the main argument, we want to consider an "illustrative case" in which we make the simplifying assumption that the domain of $\mathbb{R}$ is just X_0. The key ideas of the matching argument are present in this illustrative case, and after working through the details, we will show how to extend the argument to work in the more general setting wherein the domain of $\mathbb{R}$ has elements beyond X_0.

Proceeding, then, under the assumption that the domain of $\mathbb{R}_0$ is exactly X_0, we specify the objects from (a)-(d) satisfying (1)-(3). Namely, the finitely generated partition product $(\mathbb{R}^*(0), \mathcal{B}^*_0)$, generated by $\mathcal{S}^*$ and $\bigcup_{\iota \in J_p} Z^*_\iota$; the condition $p^*(0)$; the $\mathbb{R}^*(0)$-name $\dot{U}^*$; and the collection $\{\pi_k : k < n\}$ of embeddings are the requisite objects. From the fact that $\bar{p} = \bar{p} \restriction X_0$ we have that $p^*(0)$ is below $\pi_k(\bar{p})$ for each $k < n$. Since p^*_ι forces that $\vec{\nu} \in \dot{U}^*_\iota$ for each $\iota \in J$, we see that $p^*(0)$ forces that $\vec{\nu} \in \dot{U}^*$. Thus (3) remains to be checked.

Towards this end, fix a generic G^* for $\mathbb{R}^*(0)$ containing $p^*(0)$, and for each $\iota \in J$, set $G^*_\iota := G^* \restriction \mathbb{R}^*_\iota$. Also set $U^* := \dot{U}^*[G^*]$. Let us also fix two pairwise distinct tuples $\vec{\mu}$ and $\vec{\mu}'$ in $U^* \cap M_\gamma[G^*]$ which match $\dot{a}_0, \ldots, \dot{a}_l$ on $\mathcal{S}^*$. Our goal is to find some $k < n$ such that $\vec{\mu}$ and $\vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\pi_k[\bar{\mathcal{S}}]$. We will first show the following claim.

Claim 4.25. *For each $\iota \in J_p$, there is some $\eta \in J^*(\iota)$ such that*

$$\dot{\chi}_\kappa\left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta})\right](\mu_l, \mu'_l) = 1.$$

Proof of Claim 4.25. Recall that for each $\iota \in J_p$, by $(4)_\iota$ above, we know that the condition p^*_ι forces in $\mathbb{R}^*_\iota$ that for any two pairwise distinct tuples $\vec{\xi}, \vec{\xi}'$ in $\dot{U}^*_\iota \cap M_\gamma[\dot{G}^*_\iota]$, $\vec{\xi}$ and $\vec{\xi}'$ do *not* match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p_\iota \mapsto 0\}$ on $I^*(\iota)$. Fix some $\iota \in J_p$, and let $U^*_\iota := \dot{U}^*_\iota[G^*_\iota]$. Now observe that $\vec{\mu}$ and $\vec{\mu}'$ are in $U^*_\iota \cap M_\gamma[G^*_\iota]$: first, $U^* \subseteq U^*_\iota$; second, all of the posets under consideration are c.c.c. by Assumption 4.1, and therefore $M_\gamma[G^*]$ has the same ordinals as $M_\gamma[G^*_\iota]$. Since $\vec{\mu}, \vec{\mu}' \in U^*_\iota \cap M_\gamma[G^*_\iota]$, $\vec{\mu}, \vec{\mu}'$ fail to match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p_\iota \mapsto 0\}$ at some $\eta \in I^*(\iota)$. That is to say, one of the following holds:

(a) there is some $k \leq l$ such that

$$\dot{\chi}_\kappa\left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta})\right](\mu_k, \mu'_k) = 1 - \dot{a}_k\left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta})\right]$$

(and in case $k = l$, $\dot{a}_k\left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta})\right]$ is defined);

(b) or $(\{p_\iota \mapsto 0\})\left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta})\right]$ is defined and

$$\dot\chi_\kappa\left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta})\right](\mu_l,\mu'_l) = 1 - (\{p_\iota \mapsto 0\})\left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta})\right].$$

However, we assumed that $\vec\mu$ and $\vec\mu'$ match $\dot a_0,\dots,\dot a_l$ on $\mathcal{S}^*$. Therefore (a) is false and (b) holds. This implies in particular that $\psi^*_{\iota,\eta}(p_\iota) \in G^*_\iota \restriction B^*_{\iota,\eta}$ and $(\{p_\iota \mapsto 0\})\left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta})\right] = 0$. Thus

$$\dot\chi_\kappa\left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta})\right](\mu_l,\mu'_l) = 1 - (\dot a_l \cup \{p_\iota \mapsto 0\})\left[(\psi^*_{\iota,\eta})^{-1}(G^*_\iota \restriction B^*_{\iota,\eta})\right] = 1.$$

Since $p^*_\iota \in G^*_\iota$, p^*_ι and $\psi^*_{\iota,\eta}(p_\iota)$ are compatible, and therefore p^*_ι, being decisive, extends $\psi^*_{\iota,\eta}(p_\iota)$. Thus $\eta \in J^*(\iota)$. $\square$

This completes the proof of the above claim. As a result, we fix some function h on J_p such that for each $\iota \in J_p$, $h(\iota) \in J^*(\iota)$ provides a witness to the claim for ι. Let $k < n$ such that $h = h_k \restriction J_p$. We now check that $\vec\mu, \vec\mu'$ match $\dot a_0,\dots,\dot a_l \cup \{p \mapsto 1\}$ on $\pi_k[\bar{\mathcal{S}}]$.

Observe that since $\vec\mu$ and $\vec\mu'$ match $\dot a_0,\dots,\dot a_l$ on $\mathcal{S}^*$, we only need to check that for each $\iota \in J$, if $p \in (\psi^*_{\iota,h_k(\iota)})^{-1}\left(G^* \restriction B^*_{\iota,h_k(\iota)}\right)$, then

$$\dot\chi_\kappa\left[\left(\psi^*_{\iota,h_k(\iota)}\right)^{-1}\left(G^*_\iota \restriction B^*_{\iota,h_k(\iota)}\right)\right](\mu_l,\mu'_l) = 1.$$

But this is clear: for $\iota \in J_p$, the conclusion of the implication holds, by the last claim and the choice of h_k. For $\iota \notin J_p$ the hypothesis of the implication fails, since $(\psi^*_{\iota,h_k(\iota)})^{-1}(p^*(0))$ extends p_ι which, for $\iota \notin J_p$, is incompatible with p.

We have now completed our discussion of the illustrative case when the domain of $\mathbb{R}$ consists entirely of X_0. We next work in full generality to finish with this case; we will proceed by grafting multiple copies of the part of $\mathbb{R}$ outside X_0 onto $\mathbb{R}^*(0)$. In more detail, recall that the maps π_k each embed $(\mathbb{R} \restriction X_0, \mathcal{B} \restriction X_0)$ into $(\mathbb{R}^*(0), \mathcal{B}^*_0)$. Thus we may apply Lemma 3.18 in M_γ, once for each $k < n$, to construct a sequence of enriched partition products $\langle(\mathbb{R}^*(k+1), \mathcal{B}^*_{k+1}) : k < n\rangle$ such that for each $k < n$, letting X^*_k denote the domain of $\mathbb{R}^*(k)$, $X^*_k \subseteq X^*_{k+1}$, $\mathbb{R}^*(k+1) \restriction X^*_k = \mathbb{R}^*(k)$, $\mathcal{B}^*_k \subseteq \mathcal{B}^*_{k+1}$, and such that π_k extends to an embedding, which we call π^*_k, of $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^*(k+1), \mathcal{B}^*_{k+1})$. We remark that by the grafting construction, for each $k < n$,

$$\pi^*_k[X \backslash X_0] = X^*_{k+1} \backslash X^*_k.$$

Let us now use $\mathbb{R}^*$ to denote $\mathbb{R}^*(n)$, X^* to denote the domain of $\mathbb{R}^*$, and $\mathcal{B}^*$ to denote $\mathcal{B}^*_n$. Also, observe that π^*_k embeds $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^*, \mathcal{B}^*)$, since it embeds $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^*(k+1), \mathcal{B}_{k+1})$ and since $\mathcal{B}_{k+1} \subseteq \mathcal{B}^*$ and $\mathbb{R}^*(k+1) = \mathbb{R}^* \restriction X^*_{k+1}$. We claim that $(\mathbb{R}^*, \mathcal{B}^*)$ witnesses the lemma in this case.

We first address item (a). Since $(\mathbb{R}^*(0), \mathcal{B}^*_0)$ and $(\mathbb{R}, \mathcal{B})$ are both finitely generated and since $(\mathbb{R}^*, \mathcal{B}^*)$ was constructed from them by finitely-many applications of the Grafting Lemma, $(\mathbb{R}^*, \mathcal{B}^*)$ is itself finitely generated by Lemma 4.15. Moreover, as all of the partition products under consideration are in M_γ, the suitable collection and auxiliary part for $(\mathbb{R}^*, \mathcal{B}^*)$ are also in M_γ.

For (b), we define a sequence of conditions in $\mathbb{R}^*$ by recursion, beginning with $p^*(0)$. Suppose that we have constructed the condition $p^*(k)$ in $\mathbb{R}^*(k)$ such that if $k > 0$, then $p^*(k) \restriction \mathbb{R}^*(k-1) = p^*(k-1)$ and $p^*(k)$ extends $\pi^*_{k-1}(\bar p)$. To construct

$p^*(k+1)$, note that $p^*(k)$ extends $\pi_k(\bar{p} \restriction X_0)$, since $p^*(0)$ does, as observed before the illustrative case, and since $p^*(k) \restriction \mathbb{R}^*(0) = p^*(0)$. Moreover,

$$\pi_k^*[X \backslash X_0] \cap \operatorname{dom}(p^*(k)) = \varnothing,$$

as $\operatorname{dom}(p^*(k)) \subseteq X_k^*$, and as $\pi_k^*[X \backslash X_0] \cap X_k^* = \varnothing$. Thus we see that

$$p^*(k+1) := p^*(k) \cup \pi_k^* \left(\bar{p} \restriction (X \backslash X_0)\right)$$

is a condition in $\mathbb{R}^*(k+1)$ which extends $\pi_k^*(\bar{p})$. This completes the construction of the sequence of conditions, and so we now let p^* be the condition $p^*(n)$ in $\mathbb{R}^*$.

We take the same $\mathbb{R}^*(0)$-name $\dot{U}^*$ for (c). To address (d), we let $\mathcal{F} = \{\pi_k^* : k < n\}$, each of which, as noted above, is an embedding of $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^*, \mathcal{B}^*)$ and a member of M_γ.

This now defines the objects from (a)-(d), and so we check that conditions (1)-(3) hold. By the construction of p^* above, p^* extends $\pi_k^*(\bar{p})$ for each $k < n$, so (1) is satisfied. Moreover, we already know that $p^* \Vdash_{\mathbb{R}^*} \vec{\nu} \in \dot{U}^*$, since $p^*(0) \Vdash_{\mathbb{R}^*(0)} \vec{\nu} \in \dot{U}^*$ and since $\mathbb{R}^* \restriction X_0^* = \mathbb{R}^*(0)$. And finally, the proof of condition (3) is the same as in the illustrative case, using the fact that each π_k^* extends π_k. This completes the proof of the lemma in the case that $\operatorname{ht}(\bar{\mathcal{S}}) = 0$.

Case 2: $\operatorname{ht}\left(\bar{\mathcal{S}}\right) > 0$ (in particular, $\bar{\mathcal{S}}$ has at least 2 elements). We abbreviate $\operatorname{ht}\left(\bar{\mathcal{S}}\right)$ by δ in what follows. Fix $\iota_0, \iota_1 \in J$ which satisfy $\delta = \operatorname{ht}(B_{\iota_0}, B_{\iota_1})$, and set $\hat{J} := J \backslash \{\iota_0\}$.

By Lemma 4.4, $\hat{X}_0 := \bigcup_{\iota \in \hat{J}} B_\iota$ coheres with $(\mathbb{R}, \mathcal{B})$. Let $\hat{\mathbb{R}}$ be the partition product $\mathbb{R} \restriction \hat{X}_0$, and set $\hat{\mathcal{B}} := \mathcal{B} \restriction \hat{X}_0$, which, by Lemma 3.13, is an enrichment of $\hat{\mathbb{R}}$. Furthermore, $\hat{\mathbb{R}}$ is finitely generated with an empty auxiliary part and with $\hat{\mathcal{S}} := \left\{\langle B_\iota, \psi_\iota \rangle : \iota \in \hat{J}\right\}$ as κ-suitable with respect to $(\hat{\mathbb{R}}, \hat{\mathcal{B}})$. We also let $\hat{p}$ be the condition $\bar{p} \restriction \hat{X}_0 \in \hat{\mathbb{R}}$. Finally, we let $\bar{\mathbb{R}} := \mathbb{R} \restriction \bigcup_{\iota \in J} B_\iota$, and $\bar{\mathcal{B}} = \mathcal{B} \restriction \bigcup_{\iota \in J} B_\iota$, so that $(\bar{\mathbb{R}}, \bar{\mathcal{B}})$ is also an enriched partition product.

Since $|\hat{\mathcal{S}}| < |\bar{\mathcal{S}}|$ and $\operatorname{ht}(\hat{\mathcal{S}}) \leq \operatorname{ht}(\bar{\mathcal{S}})$, we may apply the induction hypothesis to $(\hat{\mathbb{R}}, \hat{\mathcal{B}})$, the condition $\hat{p}$, the $\hat{\mathbb{R}}$-name for all $l+1$-tuples in ω_1, and with $\hat{\mathcal{S}}$ as the matching core. This produces the following objects:

- $(a)^*$ an enriched partition product $(\mathbb{R}^*, \mathcal{B}^*)$ with domain X^*, say, finitely generated by a κ-suitable collection $\mathcal{S}^*$ and an auxiliary part Z^*, all of which are in M_γ;
- $(b)^*$ a condition $p^* \in \mathbb{R}^*$;
- $(c)^*$ an $\mathbb{R}^*$-name $\dot{W}^*$ in M_γ for a collection of $l+1$-tuples in ω_1;
- $(d)^*$ a nonempty, finite collection $\mathcal{F}$ in M_γ of embeddings of $(\hat{\mathbb{R}}, \hat{\mathcal{B}})$ into $(\mathbb{R}^*, \mathcal{B}^*)$;

satisfying that for each $\pi \in \mathcal{F}$:

- $(1)^*$ p^* extends $\pi(\hat{p})$ in $\mathbb{R}^*$;

and also satisfying that p^* forces the following statements in $\mathbb{R}^*$:

- $(2)^*$ $\vec{\nu} \in \dot{W}^*$;
- $(3)^*$ for any pairwise distinct $l+1$-tuples $\vec{\mu}$ and $\vec{\mu}'$ in $\dot{W}^* \cap M_\gamma[\dot{G}^*]$, if $\vec{\mu}$ and $\vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l$ on $\mathcal{S}^*$, then there is some $\pi \in \mathcal{F}$ such that $\vec{\mu}$ and $\vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\pi[\hat{\mathcal{S}}]$.

Our next step is to restore many copies of the segment $\psi_{\iota_0}[\rho_\kappa \backslash \delta]$ of the lost branch B_{ι_0} in such a way that the restored copies form a κ-suitable collection with smaller height than δ; this will allow another application of the induction hypothesis. Towards this end, define

$$\mathcal{R} := \{\pi \circ \psi_{\iota_1}[\delta] : \pi \in \mathcal{F}\},$$

and, recalling that $\mathcal{F}$ is finite, let $x_0, \ldots, x_{d-1}$ enumerate $\mathcal{R}$. We choose, for each $k < d$, a map $\pi_k \in \mathcal{F}$ so that $\pi_k \circ \psi_{\iota_1}[\delta] = x_k$.

We now work in M_γ to graft one copy of $\psi_{\iota_0}[\rho_\kappa \backslash \delta]$ onto $(\mathbb{R}^*, \mathcal{B}^*)$ over π_k, for each $k < n$. Indeed, since π_k embeds $(\hat{\mathbb{R}}, \hat{\mathcal{B}})$ into $(\mathbb{R}^*, \mathcal{B}^*)$, we may successively apply the Grafting Lemma to find an enriched partition product $(\mathbb{R}^{**}, \mathcal{B}^{**})$ on a domain X^{**} so that $\mathbb{R}^{**} \restriction X^* = \mathbb{R}^*$, $\mathcal{B}^* \subseteq \mathcal{B}^{**}$, and so that for each $k < d$, π_k extends to an embedding π_k^* of $(\bar{\mathbb{R}}, \bar{\mathcal{B}})$ into $(\mathbb{R}^{**}, \mathcal{B}^{**})$. Since $(\mathbb{R}^{**}, \mathcal{B}^{**})$ is finitely generated, by Lemma 4.15, we may let $\mathcal{S}^{**}$ denote the finite, κ-suitable collection for $(\mathbb{R}^{**}, \mathcal{B}^{**})$.

Let us make a number of observations about the above situation. First, we want to see that for each $\pi \in \mathcal{F}$, we may extend π to embed $(\bar{\mathbb{R}}, \bar{\mathcal{B}})$ into $(\mathbb{R}^{**}, \mathcal{B}^{**})$. Thus fix $\pi \in \mathcal{F}$, and let $k < d$ such that $\pi \circ \psi_{\iota_1}[\delta] = x_k$. We want to apply Lemma 3.21, and for this we need to see that π and π_k agree on $X_0 \cap B_{\iota_0}$. To verify this, we first claim that $X_0 \cap B_{\iota_0} = B_{\iota_1} \cap B_{\iota_0}$. Suppose that this is false, for a contradiction. Then there is some $\alpha \in X_0 \cap B_{\iota_0} \backslash B_{\iota_1}$. Fix $\iota \in J$ s.t. $\alpha \in B_\iota \cap B_{\iota_0}$. Then $\psi_{\iota_0}^{-1}[B_\iota \cap B_{\iota_0}] \leq \mathrm{ht}(\bar{\mathcal{S}}) = \delta$, and so $\alpha \in \psi_{\iota_0}[\delta]$. But $\psi_{\iota_0} \restriction \delta = \psi_{\iota_1} \restriction \delta$, and therefore $\alpha \in B_{\iota_1}$, a contradiction.

Thus $X_0 \cap B_{\iota_0} = B_{\iota_1} \cap B_{\iota_0}$. But $B_{\iota_1} \cap B_{\iota_0} = \psi_{\iota_1}[\delta]$, and therefore

$$\pi[B_{\iota_1} \cap B_{\iota_0}] = \pi \circ \psi_{\iota_1}[\delta] = x_k = \pi_k \circ \psi_{\iota_1}[\delta] = \pi_k[B_{\iota_1} \cap B_{\iota_0}].$$

Hence π and π_k agree on $X_0 \cap B_{\iota_0}$. By Lemma 3.21, the map

$$\pi^* := \pi \cup \pi_k^* \restriction (\psi_{\iota_0}[\rho_\kappa \backslash \delta])$$

is an extension of π which embeds $(\bar{\mathbb{R}}, \bar{\mathcal{B}})$ into $(\mathbb{R}^{**}, \mathcal{B}^{**})$. We make the observation that $\pi^*[B_{\iota_0}] = \pi_k^*[B_{\iota_0}]$, which will be useful later.

For each $k < d$, we use x_k^* to denote the image of B_{ι_0} under the map π_k^*. Let $\bar{\mathcal{S}}^{**} := \{\langle x_k^*, \pi_k^* \circ \psi_{\iota_0}, \kappa \rangle : k < d\}$. Then $\bar{\mathcal{S}}^{**} \subseteq \mathcal{S}^{**}$, and in particular, $\bar{\mathcal{S}}^{**}$ is κ-suitable. For $k \neq l$ we have

$$(\pi_k^* \circ \psi_{\iota_0})[\delta] = x_k \neq x_l = (\pi_l^* \circ \psi_{\iota_0})[\delta],$$

and hence $\mathrm{ht}(x_k^*, x_l^*) < \delta$. Therefore $\mathrm{ht}(\bar{\mathcal{S}}^{**}) < \delta$, since $\bar{\mathcal{S}}^{**}$ is finite.

We now have a collection $\mathcal{F}^* := \{\pi^* : \pi \in \mathcal{F}\}$ of embeddings of $(\bar{\mathbb{R}}, \bar{\mathcal{B}})$ into $(\mathbb{R}^{**}, \mathcal{B}^{**})$ and a finite, κ-suitable subcollection $\bar{\mathcal{S}}^{**}$ of $\mathcal{S}^{**}$ such that the height of $\bar{\mathcal{S}}^{**}$ is less than δ. But before we apply the induction hypothesis, we need to extend $(\mathbb{R}^{**}, \mathcal{B}^{**})$ to add generics for the full $\mathbb{R}$ and to also define a few more objects. Towards this end, we work in M_γ to successively apply the Grafting Lemma to each map π^* in $\mathcal{F}^*$ to graft $(\mathbb{R}, \mathcal{B})$ onto $(\mathbb{R}^{**}, \mathcal{B}^{**})$ over π^*. This results in a partition product $(\mathbb{R}^{***}, \mathcal{B}^{***})$ in M_γ with domain X^{***} so that $\mathbb{R}^{***} \restriction X^{**} = \mathbb{R}^{**}$, $\mathcal{B}^{**} \subseteq \mathcal{B}^{***}$, and so that each map $\pi^* \in \mathcal{F}^*$ extends to an embedding π^{***} of $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^{***}, \mathcal{B}^{***})$. By Lemma 4.15, $(\mathbb{R}^{***}, \mathcal{B}^{***})$ is still finitely generated, say with κ-suitable collection $\mathcal{S}^{***}$.

We now want to define a condition p^{***} in $\mathbb{R}^{***}$ by adding further coordinates to the condition $p^* \in \mathbb{R}^* \subseteq \mathbb{R}^{***}$ from $(a)^*$. By the grafting construction of $\mathbb{R}^{**}$, if $k < l < d$, then the images of $\psi_{\iota_0}[\rho_k \backslash \delta]$ under π_k^* and π_l^* are disjoint. Thus

$$p^{**} := p^* \cup \bigcup_{k<d} \pi_k^* \left(\bar{p} \restriction \psi_{\iota_0}[\rho_k \backslash \delta]\right)$$

is a condition in $\mathbb{R}^{**}$. Since by $(1)^*$, p^* extends $\pi(\hat{p})$ in $\mathbb{R}^*$ for each $\pi \in \mathcal{F}$, we conclude that p^{**} extends $\pi_k^*(\bar{p} \restriction \bigcup_{\iota \in J} B_\iota)$ for each $k < d$. Furthermore, if $\pi^* \in \mathcal{F}^*$, then for some $k < d$, π^* agrees with π_k^* on B_{ι_0}, as observed above. It is straightforward to see that this implies that p^{**} in fact extends $\pi^*(\bar{p} \restriction \bigcup_{\iota \in J} B_\iota)$ for each $\pi^* \in \mathcal{F}^*$. And finally, by the grafting construction of $\mathbb{R}^{***}$, we know that if π and σ are distinct embeddings in $\mathcal{F}$, then the images of $X \setminus \bigcup_{\iota \in J} B_\iota$ under π^{***} and σ^{***} are disjoint. Consequently,

$$p^{***} := p^{**} \cup \bigcup_{\pi \in \mathcal{F}} \pi^{***} \left(\bar{p} \restriction \left(X \setminus \bigcup_{\iota \in J} B_\iota \right) \right)$$

is a condition in $\mathbb{R}^{***}$ which extends $\pi^{***}(\bar{p})$ for each $\pi \in \mathcal{F}$.

We are now ready to apply the induction hypothesis to the partition product $(\mathbb{R}^{***}, \mathcal{B}^{***})$, the condition $p^{***} \in \mathbb{R}^{***}$, and the matching core $\bar{\mathcal{S}}^{**}$, which has height below δ. This results in the following objects:

- $(a)^{**}$ an enriched partition product $(\mathbb{R}^{****}, \mathcal{B}^{****})$ on a set X^{****} which is finitely generated, say with κ-suitable collection $\mathcal{S}^{****}$ and auxiliary part Z^{****}, all of which are in M_γ;
- $(b)^{**}$ a condition p^{****} in $\mathbb{R}^{****}$;
- $(c)^{**}$ an $\mathbb{R}^{****}$-name $\dot{U}^{****}$ in M_γ for a collection of $l+1$ tuples in ω_1;
- $(d)^{**}$ a nonempty, finite collection $\mathcal{G}$ in M_γ of embeddings of $(\mathbb{R}^{***}, \mathcal{B}^{***})$ into $(\mathbb{R}^{****}, \mathcal{B}^{****})$;

satisfying that for each $\sigma \in \mathcal{G}$

- $(1)^{**}$ p^{****} extends $\sigma(p^{***})$ in $\mathbb{R}^{****}$;

and such that p^{****} forces in $\mathbb{R}^{****}$ that

- $(2)^{**}$ $\vec{\nu} \in \dot{U}^{****}$;
- $(3)^{**}$ for any pairwise distinct tuples $\vec{\mu}, \vec{\mu}'$ in $M_\gamma[\dot{G}^{****}] \cap \dot{U}^{****}$ such that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l$ on $\mathcal{S}^{****}$, there is some $\sigma \in \mathcal{G}$ such that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\sigma[\bar{\mathcal{S}}^{**}]$.

This completes the construction of our final partition product. To finish the proof, we will need to define a number of embeddings from our original partition product $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^{****}, \mathcal{B}^{****})$ and check that the appropriate matching obtains. For $\sigma \in \mathcal{G}$ and $\pi \in \mathcal{F}$, we define the map $\tau(\pi, \sigma)$ to be the composition $\sigma \circ \pi^{***}$, which embeds $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^{****}, \mathcal{B}^{****})$. We also observe that $p^{****} \leq \tau(\pi, \sigma)(\bar{p})$ for each such π and σ since p^{****} extends $\sigma(p^{***})$ in $\mathbb{R}^{****}$, and since p^{***} extends $\pi^{***}(\bar{p})$ in $\mathbb{R}^{***}$. Now define the $\mathbb{R}^{****}$-name $\dot{V}^*$ to be

$$\dot{U}^{****} \cap \bigcap_{\sigma \in \mathcal{G}} \dot{W}^* \left[\sigma^{-1}(\dot{G}^{****}) \restriction X^* \right].$$

We observe that this is well-defined, since for each $\sigma \in \mathcal{G}$ and generic G^{****} for $\mathbb{R}^{****}$, $\sigma^{-1}(G^{****})$ is generic for $\mathbb{R}^{***}$, and hence its restriction to X^* is generic for $\mathbb{R}^*$. We also see that p^{****} forces that $\vec{\nu} \in \dot{V}^*$ because p^{****} forces $\vec{\nu} \in \dot{U}^{****}$, p^* is in $\sigma^{-1}(G^{****})$ for any generic G^{****} containing p^{****}, and p^* forces in $\mathbb{R}^*$ that $\vec{\nu} \in \dot{W}^*$.

We finish the proof of the lemma in this case by showing that the partition product $(\mathbb{R}^{****}, \mathcal{B}^{****})$, the condition $p^{****} \in \mathbb{R}^{****}$, the name $\dot{V}^*$, and the collection $\{\tau(\pi, \sigma) : \pi \in \mathcal{F} \wedge \sigma \in \mathcal{G}\}$ of embeddings satisfy (1)-(3). We already know that p^{****} extends $\tau(\pi, \sigma)(\bar{p})$ for each π and σ and that $p^{****} \Vdash \vec{\nu} \in \dot{V}^*$. So now we check the matching condition. Towards this end, fix a generic H for $\mathbb{R}^{****}$ and two pairwise distinct tuples $\vec{\mu}, \vec{\mu}'$ in $\dot{V}^*[H] \cap M_\gamma[H]$ which match $\dot{a}_0, \dots, \dot{a}_l$ on $\mathcal{S}^{****}$. We need to find some π and σ such that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \dots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\tau(\pi, \sigma)[\bar{\mathcal{S}}]$.

By $(3)^{**}$, we know that we can find some σ such that

(i) $\vec{\mu}$ and $\vec{\mu}'$ match $\dot{a}_0, \dots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\sigma[\bar{\mathcal{S}}^{**}]$.

Let t denote the triple $\langle B_{\iota_0}, \psi_{\iota_0}, \kappa\rangle$. By construction of the maps π^*, for each $\pi \in \mathcal{F}$, there is some k so that $\pi^{***}(t) = \pi^*(t) = \pi^*_k(t) \in \bar{\mathcal{S}}^{**}$. Using (i) it follows that:

(ii) for every $\pi \in \mathcal{F}$, $\vec{\mu}$ and $\vec{\mu}'$ match $\dot{a}_0, \dots, \dot{a}_l \cup \{p \mapsto 1\}$ at $\sigma \circ \pi^{***}(t) = \tau(\pi, \sigma)(t)$.

Now consider the filter $G^*_\sigma := \sigma^{-1}(H) \restriction X^*$, which is generic for $\mathbb{R}^*$ and contains p^*. By Assumption 4.1, we know that all the posets under consideration are c.c.c., and therefore the models $M_\gamma[H]$ and $M_\gamma[G^*_\sigma]$ have the same ordinals, namely those of M_γ. Thus $\vec{\mu}, \vec{\mu}' \in M_\gamma[G^*_\sigma]$. Furthermore, by definition of $\dot{V}^*[H]$, we have that $\vec{\mu}, \vec{\mu}' \in \dot{W}^*[G^*_\sigma]$, and as a result $\vec{\mu}, \vec{\mu}' \in M_\gamma[G^*_\sigma] \cap \dot{W}^*[G^*_\sigma]$. Thus by $(3)^*$, we can find some $\pi \in \mathcal{F}$ so that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \dots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\pi[\hat{\mathcal{S}}]$. Because π^{***} extends π, we may rephrase this to say that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \dots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\pi^{***}[\hat{\mathcal{S}}]$. Since σ embeds $(\mathbb{R}^{***}, \mathcal{B}^{***})$ into $(\mathbb{R}^{****}, \mathcal{B}^{****})$,

(iii) $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \dots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\tau(\pi, \sigma)[\hat{\mathcal{S}}]$.

Finally, (ii) and (iii) imply that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \dots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\tau(\pi, \sigma)[\bar{\mathcal{S}}]$, as $\bar{\mathcal{S}} = \hat{\mathcal{S}} \cup \{t\}$. This completes the proof of the lemma.

□

Corollary 4.26. *Under the assumptions of Lemma 4.24, suppose that $\dot{U}$ is an $\mathbb{R}$-name in M_γ for a set of $l{+}1$-tuples in ω_1 such that $\bar{p} \Vdash_{\mathbb{R}} \vec{\nu} \in \dot{U}$. Then the conclusion of Lemma 4.24 may be strengthened to say that $p^* \Vdash_{\mathbb{R}^*} \dot{U}^* \subseteq \bigcap_{\pi \in \mathcal{F}} \dot{U}[\pi^{-1}(\dot{G}^*)]$.*

Proof. Let $\dot{U}$ be fixed, and let $\dot{U}^*$ be as in the conclusion of Lemma 4.24. Define $\dot{U}^{**}$ to be the name $\dot{U}^* \cap \bigcap_{\pi \in \mathcal{F}} \dot{U}[\pi^{-1}(\dot{G}^*)]$, and observe that this name is still in M_γ. By condition (1) of Lemma 4.24, we know that p^* forces that $\bar{p}$ is in $\pi^{-1}(\dot{G}^*)$, for each $\pi \in \mathcal{F}$. Since each such $\pi^{-1}(\dot{G}^*)$ is forced to be V-generic for $\mathbb{R}$ and since $\bar{p} \Vdash_{\mathbb{R}} \vec{\nu} \in \dot{U}$, this implies that p^* forces that $\vec{\nu}$ is a member of $\dot{U}^{**}$. Finally, condition (3) of Lemma 4.24 still holds, since $\dot{U}^{**}$ is forced to be a subset of $\dot{U}^*$. □

Corollary 4.27. (Under Assumption 4.23) $\dot{a}_0, \dots, \dot{a}_l \cup \{p \mapsto 1\}$ *have the partition product preassignment property at γ.*

Proof. Suppose otherwise, for a contradiction. Then there exists a partition product $\mathbb{R}$, say with domain X, finitely generated by $\mathcal{S} = \{\langle B_\iota, \psi_\iota\rangle : \iota \in I\}$ and an auxiliary part Z, all of which are in M_γ; an $\mathbb{R}$-name $\dot{U}$ in M_γ; and a condition $\bar{p} \in \mathbb{R}$ (not necessarily in M_γ), such that $\bar{p}$ forces that $\vec{\nu} \in \dot{U}$, but also that for any pairwise distinct tuples $\vec{\mu}, \vec{\mu}'$ in $\dot{U} \cap M_\gamma[\dot{G}]$, there exists some $\iota_0 \in I$ such that $\vec{\mu}, \vec{\mu}'$ fail to match $\dot{a}_0, \dots, \dot{a}_l \cup \{p \mapsto 1\}$ at ι_0. Apply Lemma 4.24 and Corollary 4.26 to these objects, with $\bar{\mathcal{S}} := \mathcal{S}$ and with the enrichment

$$\mathcal{B} := \{\langle b(\xi), \pi_\xi, \mathrm{index}(\xi)\rangle : \xi \in X\} \cup \mathcal{S},$$

to construct the objects as in the conclusions of Lemma 4.24 and Corollary 4.26. Also, fix a generic G^* for $\mathbb{R}^*$ which contains the condition p^*.

We now apply the fact that $\dot{a}_0, \ldots, \dot{a}_l$ have the partition product preassignment property at γ to the objects in the conclusion of Lemma 4.24: since $\vec{\nu} \in U^* := \dot{U}^*[G^*]$, we can find two pairwise distinct tuples $\vec{\mu}, \vec{\mu}'$ in $U^* \cap M_\gamma[G^*]$ which match $\dot{a}_0, \ldots, \dot{a}_l$ on I^*. Thus by (3) of Lemma 4.24, there is some embedding π of $(\mathbb{R}, \mathcal{B})$ into $(\mathbb{R}^*, \mathcal{B}^*)$ so that $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\pi[\mathcal{S}]$. Now consider $G := \pi^{-1}(G^*)$, which is generic for $\mathbb{R}$ and contains the condition $\bar{p}$, since $p^* \leq_{\mathbb{R}^*} \pi(\bar{p})$. Since $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\pi[\mathcal{S}]$ and π is an embedding, $\vec{\mu}, \vec{\mu}'$ match $\dot{a}_0, \ldots, \dot{a}_l \cup \{p \mapsto 1\}$ on $\mathcal{S}$ with respect to the filter G. Finally, observe that $\vec{\mu}$ and $\vec{\mu}'$ are both in $\dot{U}[G] \cap M_\gamma[G]$: they are in $\dot{U}[G]$ by Corollary 4.26, since U^* is a subset of $\dot{U}[G]$. They are both in M_γ, hence in $M_\gamma[G]$, since by Assumption 4.1, $\mathbb{R}^*$ is c.c.c. However, this contradicts what we assumed about $\bar{p}$. $\square$

4.4. **Putting it together.** Let us now put together the results from the previous three subsections.

Lemma 4.28. *Suppose that $\dot{a}_0, \ldots, \dot{a}_{l-1}$ are total canonical color names which have the partition product preassignment property at γ. Then there is a total canonical color name $\dot{a}_l$ so that $\dot{a}_0, \ldots, \dot{a}_l$ have the partition product preassignment property at γ.*

Proof. We recursively construct a sequence $\dot{a}_l^\xi$ of names, taking unions at limit stages. If $\dot{a}_l^\xi$ has been constructed and $\mathrm{dom}(\dot{a}_l^\xi)$ is a maximal antichain in $\mathbb{P}_\kappa$, we set $\dot{a}_l = \dot{a}_l^\xi$. Otherwise, we pick some condition $p \in \mathbb{P}_\kappa$ incompatible with all conditions therein. If there is some extension $p^* \leq_{\mathbb{P}_\kappa} p$ so that $\dot{a}_0, \ldots, \dot{a}_{l-1}, \dot{a}_l^\xi \cup \{p^* \mapsto 0\}$ have the partition product preassignment property at γ, we pick some such p^* and set $\dot{a}_l^{\xi+1} := \dot{a}_l^\xi \cup \{p^* \mapsto 0\}$. Otherwise, Assumption 4.23 is satisfied, and hence by Corollary 4.27, $\dot{a}_0, \ldots, \dot{a}_{l-1}, \dot{a}_l^\xi \cup \{p \mapsto 1\}$ have the partition product preassignment property at γ. In this case we set $\dot{a}_l^{\xi+1} := \dot{a}_l^\xi \cup \{p \mapsto 1\}$. Note that the construction of the sequence $\dot{a}_l^\zeta$ halts at some countable stage, since $\mathbb{P}_\kappa$ is c.c.c., by Assumption 4.1. $\square$

We now prove Proposition 4.9:

Proof of Proposition 4.9. Recall that for each $\gamma < \omega_1$, $\langle \nu_{\gamma,l} : l < \omega \rangle$ enumerates the slice $[M_\gamma \cap \omega_1, M_{\gamma+1} \cap \omega_1)$. By Lemma 4.28, we may construct, for each $\gamma < \omega_1$, a sequence of $\mathbb{P}_\kappa$-names $\langle \dot{a}_{\gamma,l} : l < \omega \rangle$ such that for each $l < \omega$, $\dot{a}_{\gamma,0}, \ldots, \dot{a}_{\gamma,l}$ have the partition product preassignment property at γ. We now define a function $\dot{f}$ by taking $\dot{f}(\nu_{\gamma,l}) = \dot{a}_{\gamma,l}$, for each $\gamma < \omega_1$ and $l < \omega$. The values of $\dot{f}$ on ordinals $\nu < M_0 \cap \omega_1$ are irrelevant, so we simply set $\dot{f}(\nu)$ to name 0 for each such ν. Then $\dot{f}$ satisfies the assumptions of Lemma 4.20 and hence satisfies Proposition 4.9. $\square$

5. Constructing Partition Products in L

In this section, we show how to construct the desired partition products in L. In particular, we will construct the ω_2-canonical partition product $\mathbb{P}_{\omega_2}$, which will have domain ω_3. Forcing with $\mathbb{P}_{\omega_2}$ will provide the model which witness our theorem. We assume for this section that $V = L$, and before we introduce some more definitions, let us fix a sufficiently large, finite fragment F of ZFC which is satisfied in $H(\omega_3)$. F will be large enough, in particular, to prove all of the results of the previous sections. As a matter of notation, by the Gödel pairing function, we view each ordinal γ as coding a pair of ordinals, where $(\gamma)_k$, for $k \leq 1$, denotes the kth ordinal coded by γ; this will be useful for bookkeeping later.

5.1. Local ω_2's and Witnesses.

Definition 5.1. *Let $\omega_1 < \kappa \leq \omega_2$. We say that κ is a* local ω_2 *if there is some $\delta > \kappa$ such that L_δ is closed under ω-sequences and such that*

$$L_\delta \models \kappa = \aleph_2 \wedge F \wedge \kappa \text{ is the largest cardinal.}$$

If κ is a local ω_2, we will refer to any such δ as above as a witness that κ is a local ω_2, *or simply say that δ is a* witness for κ. *We let $C \subseteq \omega_2$ be the set of all local ω_2's below ω_2.*

For each $\kappa \in C$ and $\mu < \omega_2$, we let $\varphi_{\kappa,\mu}$ be the $<_L$-least surjection from κ onto μ, and we let $\vec{\varphi}$ be the sequence of these surjections.

We begin our discussion with the following straightforward lemma.

Lemma 5.2. *Suppose that L_δ is closed under ω-sequences, and let $\gamma < \delta$. Then $\mathrm{Hull}^{L_\delta}(\omega_1)$ is also closed under ω-sequences.*

The next lemma shows how a local ω_2 can project to another.

Lemma 5.3. *Suppose that δ is a witness for κ, let $\gamma < \delta$, and define $H := \mathrm{Hull}^{L_\delta}(\omega_1)$. Suppose that $H \cap \kappa = \bar{\kappa} < \kappa$. Then $\bar{\kappa}$ is a local ω_2, and $\mathrm{ot}(H \cap \delta)$ is a witness for $\bar{\kappa}$.*

Proof. Let $\pi : H \longrightarrow L_{\bar{\delta}}$ be the transitive collapse, so that $\pi(\kappa) = \bar{\kappa}$. Since H is closed under ω-sequences, by Lemma 5.2, $L_{\bar{\delta}}$ is too. Thus by the elementarity of π, $\bar{\kappa}$ is a local ω_2 and $\bar{\delta}$ is a witness. □

For each $\kappa \in C \cup \{\omega_2\}$, we define the *canonical sequence of witnesses for κ*, denoted $\langle \delta_i(\kappa) : i < \gamma(\kappa) \rangle$. We set $\delta_0(\kappa)$ to be the least witness for κ. Suppose that $\langle \delta_i(\kappa) : i < \gamma \rangle$ is defined, for some γ. If there exists a witness $\tilde{\delta}$ for κ such that $\tilde{\delta} > \sup_{i<\gamma} \delta_i(\kappa)$, then we set $\delta_\gamma(\kappa)$ to be the least such. Otherwise, we halt the construction and set $\gamma(\kappa) = \gamma$.

Remark 5.4. It is straightforward to check that if κ is a local ω_2 and $\gamma < \gamma(\kappa)$, then because $L_{\delta_\gamma(\kappa)}$ is countably closed, being a witness for κ is absolute between $L_{\delta_\gamma(\kappa)}$ and V. Thus the sequence $\langle \delta_i(\kappa) : i < \gamma \rangle$, and consequently the ordinal γ, is definable in $L_{\delta_\gamma(\kappa)}$ as the longest sequence of witnesses for κ. Furthermore, in the case that $\kappa = \omega_2$, we see that $\gamma(\omega_2) = \omega_3$.

For each $\kappa \in C \cup \{\omega_2\}$ and $\gamma < \gamma(\kappa)$, we define $H(\kappa, \gamma)$ to be $\mathrm{Hull}^{L_{\delta_\gamma(\kappa)}}(\omega_1)$. We also let $j_{\kappa,\gamma}$ be the transitive collapse embedding of $H(\kappa, \gamma)$ and let $\tau(\kappa, \gamma)$ be the level of L to which $H(\kappa, \gamma)$ collapses.

Suppose that $\kappa \in C$ is such that $\gamma(\kappa)$ is a successor, say $\gamma + 1$, and further suppose that $H(\kappa, \gamma)$ contains κ as a subset. Then we refer to $\delta_\gamma(\kappa)$, the final element on the canonical sequence of witnesses for κ, as the *stable witness for κ*. It is stable in the sense that we cannot condense the hull further.

Lemma 5.5. *Suppose that $\gamma + 1 < \gamma(\kappa)$. Then $H(\kappa, \gamma) \cap \kappa \in \kappa$.*

Proof. Suppose otherwise. Then $\kappa \subseteq H(\kappa, \gamma)$. Since $\gamma + 1 < \gamma(\kappa)$, we know that $\hat{\delta} := \delta_{\gamma+1}(\kappa)$ exists, and in particular, $\delta_\gamma(\kappa) < \hat{\delta}$. Observe that $H(\kappa, \gamma)$ is a member of $L_{\hat{\delta}}$, and therefore we may find a surjection from ω_1 onto $H(\kappa, \gamma)$ in $L_{\hat{\delta}}$. Since $\kappa \subseteq H(\kappa, \gamma)$, this contradicts our assumption that $L_{\hat{\delta}}$ satisfies that κ is $\aleph_2$. □

If $\gamma+1<\gamma(\kappa)$, then the collapse of $H(\kappa,\gamma)$ moves κ. The level to which $H(\kappa,\gamma)$ collapses is then the stable witness for the image of κ, as shown in the following lemma.

Lemma 5.6. *Suppose that $\gamma+1<\gamma(\kappa)$, let $\bar{\kappa}=H(\kappa,\gamma)\cap\kappa$, and let $\bar{\gamma}=j_{\kappa,\gamma}(\gamma)$. Then $\bar{\gamma}+1=\gamma(\bar{\kappa})$ and $\tau(\kappa,\gamma)=\delta_{\bar{\gamma}}(\bar{\kappa})$ is the stable witness for $\bar{\kappa}$.*

Proof. Let us abbreviate $H(\kappa,\gamma)$, $j_{\kappa,\gamma}$, and $\tau(\kappa,\gamma)$ by H, j, and τ respectively. By Remark 5.4, we have that $\langle\delta_i(\kappa):i<\gamma\rangle\in H(\kappa,\gamma)$. By the elementarity of j, j sends this sequence to $\langle\delta_i(\bar{\kappa}):i<\bar{\gamma}\rangle$. Now we check that $\tau=\delta_{\bar{\gamma}}(\bar{\kappa})$. By Lemma 5.3, we know that τ is a witness for $\bar{\kappa}$. Furthermore, τ is the least witness for $\bar{\kappa}$ above $\sup_{i<\bar{\gamma}}\delta_i(\bar{\kappa})$: suppose that there were a witness $\bar{\delta}$ for $\bar{\kappa}$ between $\sup_{i<\bar{\gamma}}\delta_i(\bar{\kappa})$ and τ. Then L_τ satisfies that $\bar{\delta}$ is a witness for $\bar{\kappa}$. By the elementarity of j^{-1}, setting $\delta:=j^{-1}(\bar{\delta})$, we see that $L_{\delta_\gamma(\kappa)}$ satisfies that δ is a witness for κ. Since $L_{\delta_\gamma(\kappa)}$ is closed under ω-sequences, δ is in fact a witness for κ. As δ is between $\sup_{i<\gamma}\delta_i(\kappa)$ and $\delta_\gamma(\kappa)$, this is a contradiction. Therefore τ is the least witness for $\bar{\kappa}$ above $\sup_{i<\bar{\gamma}}\delta_i(\bar{\kappa})$. However, because L_τ is the collapse of H, we see that $\mathrm{Hull}^{L_\tau}(\omega_1)$ is all of L_τ. Therefore τ is the stable witness for $\bar{\kappa}$. □

5.2. **Building Partition Products.** In this subsection, we show how to construct the desired sequence of partition products $\underline{\mathbb{P}}=\langle\mathbb{P}_\delta:\delta\in C\cup\{\omega_2\}\rangle$ and names $\underline{\dot{\mathbb{Q}}}=\langle\dot{\mathbb{Q}}_\delta:\delta\in C\rangle$. The ω_2-canonical partition product $\mathbb{P}_{\omega_2}$ will force OCA_{ARS} and $2^{\aleph_0}=\aleph_3$, which proves Theorem 1.3. We will also show how to adapt our construction so that our model additionally satisfies $\mathsf{FA}(\aleph_2,\mathrm{Knaster}(\aleph_1))$; recall that this axiom asserts that we can meet any $\aleph_2$-many dense subsets of an $\aleph_1$-sized, Knaster poset.

Suppose that $\kappa\in C\cup\{\omega_2\}$ and that we have defined $\underline{\mathbb{P}}\restriction\kappa$ and $\underline{\dot{\mathbb{Q}}}\restriction\kappa$ in such a way that the following recursive assumptions are satisfied:

(a) for each $\bar{\kappa}\in C\cap\kappa$, $\mathbb{P}_{\bar{\kappa}}$ is a partition product based upon $\underline{\mathbb{P}}\restriction\bar{\kappa}$ and $\underline{\dot{\mathbb{Q}}}\restriction\bar{\kappa}$, and $\dot{\mathbb{Q}}_{\bar{\kappa}}$ is a $\mathbb{P}_{\bar{\kappa}}$-name. In particular, conditions (i)-(v) from Section 2 are satisfied.

(b) every partition product based upon $\underline{\mathbb{P}}\restriction\kappa$ and $\underline{\dot{\mathbb{Q}}}\restriction\kappa$ is c.c.c.

We now aim to define the partition product $\mathbb{P}_\kappa$ and, in the case that $\kappa<\omega_2$, the $\mathbb{P}_\kappa$-name $\dot{\mathbb{Q}}_\kappa$. We assume that $\mathbb{P}_\kappa\restriction\gamma$ is defined, as well as the base and index functions $\mathrm{base}_\kappa\restriction\gamma$ and $\kappa\restriction\gamma$. We divide into two cases.

Case 1: $\gamma+1=\gamma(\kappa)$, or $\gamma=\gamma(\kappa)$ is a limit.

If Case 1 obtains, then we halt the construction, setting $\rho_\kappa=\gamma$ and $\mathbb{P}_\kappa=\mathbb{P}_\kappa\restriction\gamma$. If $\kappa<\omega_2$, then we need to define the name $\dot{\mathbb{Q}}_\kappa$. Suppose that the $(\gamma)_0$-th element under $<_L$ is a pair $\langle\dot{S}_\kappa,\dot{\chi}_\kappa\rangle$ of $\mathbb{P}_\kappa$-names, where $\dot{S}_\kappa$ names a countable basis for a second countable, Hausdorff topology on ω_1 and $\dot{\chi}_\kappa$ names a coloring on ω_1 which is open with respect to the topology generated by $\dot{S}_\kappa$. Then let $\dot{f}_\kappa$ be the $<_L$-least $\mathbb{P}_\kappa$-name satisfying Proposition 4.9, and set $\dot{\mathbb{Q}}_\kappa:=\mathbb{Q}(\dot{\chi}_\kappa,\dot{f}_\kappa)$, so that by Corollary 4.11, any partition product based upon $\underline{\mathbb{P}}\restriction(\kappa+1)$ and $\underline{\dot{\mathbb{Q}}}\restriction(\kappa+1)$ is c.c.c. If $(\gamma)_0$ does not code such a pair, then we simply let $\dot{\mathbb{Q}}_\kappa$ name Cohen forcing for adding a single real. It is clear in this case also, by Lemma 2.20, that any partition product based upon $\underline{\mathbb{P}}\restriction(\kappa+1)$ and $\underline{\dot{\mathbb{Q}}}\restriction(\kappa+1)$ is c.c.c.

On the other hand, if $\kappa = \omega_2$, then the partition product $\mathbb{P}_{\omega_2}$ is defined. After completing the rest of the construction, we show that forcing with $\mathbb{P}_{\omega_2}$ provides the desired model witnessing our theorem.

Case 2: $\gamma + 1 < \gamma(\kappa)$.

In this case, we desire to continue the construction another step. Let $\bar{\kappa} := H(\kappa, \gamma) \cap \kappa$, which is below κ by Lemma 5.5, and let $j := j_{\kappa,\gamma}$ as well as $\bar{\gamma} := j(\gamma)$. We halt the construction if either $\mathbb{P}_\kappa \restriction \gamma$ is not a member of $H(\kappa, \gamma)$, or if it is a member of $H(\kappa, \gamma)$ but is not mapped to $\mathbb{P}_{\bar{\kappa}} \restriction \bar{\gamma}$ by j (we will later show that this does not in fact occur).

Suppose, on the other hand, that $\mathbb{P}_\kappa \restriction \gamma$ is a member of $H(\kappa, \gamma)$ and is mapped by j to $\mathbb{P}_{\bar{\kappa}} \restriction \bar{\gamma}$. We shall specify the next name $\dot{\mathbb{U}}_\gamma$ as well as the values $\text{base}_\kappa(\gamma)$ and $\text{index}_\kappa(\gamma)$. By Lemma 5.6, we have that $\bar{\gamma}+1 = \gamma(\bar{\kappa})$. By recursion, this means that $\bar{\gamma} = \rho_{\bar{\kappa}}$, i.e., that $\mathbb{P}_{\bar{\kappa}} = \mathbb{P}_{\bar{\kappa}} \restriction \bar{\gamma}$. We now pull these objects back along j^{-1}.

In more detail, we observe that, setting $\pi_\gamma := j^{-1}$, $\pi_\gamma \restriction \rho_{\bar{\kappa}}$ provides an acceptable rearrangement of $\mathbb{P}_{\bar{\kappa}}$, since π_γ is order-preserving. In fact, the π_γ-rearrangement of $\mathbb{P}_{\bar{\kappa}}$ is exactly equal to $(\mathbb{P}_\kappa \restriction \gamma) \restriction \pi_\gamma[\rho_{\bar{\kappa}}]$, by Lemma 2.16; this Lemma applies since for each $\delta \in C \cap \bar{\kappa}$, π_γ is the identity on $\mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta \cup \left\{\mathbb{P}_\delta, \dot{\mathbb{Q}}_\delta\right\}$. By assumption (iv) of Section 2, we see that the π_γ-rearrangement of $\dot{\mathbb{Q}}_{\bar{\kappa}}$ is defined (though not necessarily an element of $L_{\delta_\gamma(\kappa)}$), and so we let $\dot{\mathbb{U}}_\gamma$ be the π_γ-rearrangement of $\dot{\mathbb{Q}}_{\bar{\kappa}}$. We now set $\text{base}_\kappa(\gamma) := (\pi_\gamma[\rho_{\bar{\kappa}}], \pi_\gamma \restriction \rho_{\bar{\kappa}})$ and set $\text{index}_\kappa(\gamma) := \bar{\kappa}$. In particular, we observe that

$$b_\kappa(\gamma) = H(\kappa, \gamma) \cap \gamma$$

is an initial segment of the ordinals of $H(\kappa, \gamma)$.

Claim 5.7. $\text{base}_\kappa \restriction (\gamma + 1)$ *and* $\text{index}_\kappa \restriction (\gamma + 1)$ *support a partition product based upon* $\underline{\mathbb{P}} \restriction \kappa$ *and* $\underline{\dot{\mathbb{Q}}} \restriction \kappa$.

Proof of Claim 5.7. Condition (1) of Definition 2.1 follows from the comments in the above paragraph. Condition (2) holds at γ by the elementarity of π_γ and at all smaller ordinals by recursion. So we need to check condition (3), where it suffices to verify the matching condition for γ and some $\beta < \gamma$. So suppose that there is some $\xi \in b_\kappa(\beta) \cap b_\kappa(\gamma)$. We define $\bar{\kappa}^*$ to be $H(\kappa, \beta) \cap \kappa$, so that $\bar{\kappa}^* = \text{index}_\kappa(\beta)$, and we let π_β denote $j^{-1}_{\kappa,\beta}$.

Now the models $H(\kappa, \beta)$ and $H(\kappa, \gamma)$ are both sufficiently elementary, in particular, with respect to the sequence of surjections $\vec{\varphi}$. Since κ is the largest cardinal in $H(\kappa, \gamma)$,

$$H(\kappa, \gamma) \cap \xi = \varphi_{\kappa,\xi}[H(\kappa, \gamma) \cap \kappa] = \varphi_{\kappa,\xi}[\bar{\kappa}],$$

and therefore $b_\kappa(\gamma) \cap \xi = \varphi_{\kappa,\xi}[\bar{\kappa}]$. Similarly, $b_\kappa(\beta) \cap \xi = \varphi_{\kappa,\xi}[\bar{\kappa}^*]$.

With this observation in mind, we now verify that (3) holds. Suppose that $\bar{\kappa}^* \leq \bar{\kappa}$, and let $\zeta_0 := \pi_\beta^{-1}(\xi)$ and $\zeta_1 := \pi_\gamma^{-1}(\xi)$. If $\bar{\kappa}^* = \bar{\kappa}$, then by the calculations in the previous paragraph, (3) holds trivially, since the models $H(\kappa, \beta)$ and $H(\kappa, \gamma)$ have the same intersection with $\xi + 1$. Thus we proceed under the assumption that $\bar{\kappa}^* < \bar{\kappa}$. Since the above paragraph shows that $\pi_\beta[\zeta_0] \subseteq \pi_\gamma[\zeta_1]$, we need to check that $A := \pi_\gamma^{-1}[\pi_\beta[\zeta_0]]$ matches $\langle \bar{\kappa}, \zeta_1 \rangle$ to $\langle \bar{\kappa}^*, \zeta_0 \rangle$.

Now $\pi_\beta[\zeta_0] = b_\kappa(\beta)\cap\xi$ has the form $\varphi_{\kappa,\xi}[\bar{\kappa}^*]$. Since $\bar{\kappa}^* < \bar{\kappa}$, we have that κ, ξ, and $\bar{\kappa}^*$ are all in $H(\kappa,\gamma)$. Thus so is $\pi_\beta[\zeta_0]$. Applying the elementarity of $j_{\kappa,\gamma} = \pi_\gamma^{-1}$, we see that $\pi_\gamma^{-1} \circ \varphi_{\kappa,\xi} \restriction \bar{\kappa}^* = \varphi_{\bar{\kappa},\zeta_1} \restriction \bar{\kappa}^*$, which shows that A has the form $\varphi_{\bar{\kappa},\zeta_1}[\bar{\kappa}^*]$. Therefore condition (a) in the definition of matching holds. Additionally, if we let σ denote the transitive collapse of A, then we see that $\sigma \circ \pi_\gamma^{-1}$ is the transitive collapse of $\pi_\beta[\zeta_0] = \varphi_{\kappa,\xi}[\bar{\kappa}^*]$, which is just $\pi_\beta^{-1} = j_{\kappa,\beta}$. However, the elementarity of π_β^{-1} implies that $\pi_\beta^{-1} \circ \varphi_{\kappa,\xi} \restriction \bar{\kappa}^* = \varphi_{\bar{\kappa}^*,\zeta_0}$, and therefore $\sigma \circ \varphi_{\bar{\kappa},\zeta_1} \restriction \bar{\kappa}^* = \varphi_{\bar{\kappa}^*,\zeta_0}$. And finally, to see that (b) holds, we first observe that $b_\kappa(\beta) \cap \xi$ is closed under limit points of cofinality ω below its supremum, because $H(\kappa,\beta)$ is closed under ω-sequences. Since $b_\kappa(\beta) \cap \xi$ is in $H(\kappa,\gamma)$, by applying $j_{\kappa,\gamma}$, we conclude that $L_{\tau(\kappa,\gamma)}$ satisfies that A is closed under limit points of cofinality ω below its supremum. However, $L_{\tau(\kappa,\gamma)}$ is closed under ω-sequences, and therefore A is in fact closed under limit points of cofinality ω below its supremum. Thus (b) is satisfied. Since the proof in the case that $\bar{\kappa} < \bar{\kappa}^*$ is entirely similar, this completes the proof of the claim. □

We have now completed the construction of the desired sequence of partition products. Before we prove our main theorem, we need to verify that for each $\kappa \in C \cup \{\omega_2\}$, we obtain a partition product of the appropriate length, i.e., that the construction does not halt prematurely, as described at the beginning of Case 2.

Lemma 5.8. *For each $\kappa \in C \cup \{\omega_2\}$, $\rho_\kappa = \gamma(\kappa)$ if $\gamma(\kappa)$ is a limit or equals $\gamma(\kappa) - 1$ if $\gamma(\kappa)$ is a successor.*

Proof. Suppose that $\kappa \in C \cup \{\omega_2\}$ and that $\gamma + 1 < \gamma(\kappa)$. We need to show that $\mathbb{P}_\kappa \restriction \gamma$ is a member of $H(\kappa,\gamma)$ and gets mapped by $j := j_{\kappa,\gamma}$ to $\mathbb{P}_{\bar{\kappa}} \restriction \bar{\gamma}$, where $\bar{\kappa} = j(\kappa)$ and $\bar{\gamma} = j(\gamma)$. However, it is clear that $\mathbb{P}_\kappa \restriction \gamma$ is a member of $L_{\delta_\gamma(\kappa)}$. Moreover, the above construction of partition products is uniform, so that $\mathbb{P}_\kappa \restriction \gamma$ is definable in $L_{\delta_\gamma(\kappa)}$ from κ and γ by the same definition which defines $\mathbb{P}_{\bar{\kappa}} \restriction \bar{\gamma}$ in $L_{\delta_{\bar{\gamma}}(\bar{\kappa})}$ from $\bar{\kappa}$ and $\bar{\gamma}$. Thus $\mathbb{P}_\kappa \restriction \gamma$ is a member of $H(\kappa,\gamma)$ and gets mapped to $\mathbb{P}_{\bar{\kappa}} \restriction \bar{\gamma}$ by j. □

We now prove Theorem 1.3.

Proof of Theorem 1.3. We force over L with $\mathbb{P}_{\omega_2}$. By Lemma 5.8, $\mathbb{P}_{\omega_2}$ is a partition product with domain $\gamma(\omega_2)$, and by Remark 5.4, $\gamma(\omega_2) = \omega_3$. Let us denote the sequence of names used to form $\mathbb{P}_{\omega_2}$ by $\langle \dot{\mathbb{U}}_\gamma : \gamma < \omega_3 \rangle$. Since $\mathbb{P}_{\omega_2}$ is a partition product based upon $\underline{\mathbb{P}} \restriction \omega_2$ and $\underline{\dot{\mathbb{Q}}}$, it is c.c.c. Hence all cardinals are preserved. Since $\mathbb{P}_{\omega_2}$ has size $\aleph_3$ and is c.c.c., it forces that the continuum has size no more than $\aleph_3$. However, $\mathbb{P}_{\omega_2}$ adds $\aleph_3$-many reals, and to see this, we first recall that by Lemma 2.7, $\mathbb{P}_{\omega_2}$ is a dense subset of the finite support iteration of the names $\langle \dot{\mathbb{U}}_\gamma : \gamma < \omega_3 \rangle$. Next, each $\dot{\mathbb{U}}_\gamma$ either names Cohen forcing or one of the homogeneous set posets, and each of the latter adds a real by Remark 4.8. Thus $\mathbb{P}_{\omega_2}$ forces that the continuum has size exactly $\aleph_3$. We now want to see that $\mathbb{P}_{\omega_2}$ forces that OCA_{ARS} holds.

Towards this end, let $\langle \dot{S}, \dot{\chi} \rangle$ be a pair of $\mathbb{P}_{\omega_2}$-names, where $\dot{S}$ names a countable basis for a second countable, Hausdorff topology on ω_1 and $\dot{\chi}$ names a coloring which is open with respect to the topology generated by $\dot{S}$. Let $\gamma < \omega_3$ so that $\langle \dot{S}, \dot{\chi} \rangle$ is the $(\gamma)_0$-th element under $<_L$ and so that $\langle \dot{S}, \dot{\chi} \rangle$ is a $\mathbb{P}_{\omega_2} \restriction (\gamma)_1$-name. Note that $\langle \dot{S}, \dot{\chi} \rangle$ is an element of $H(\omega_2,\gamma)$ since, by Remark 5.4, γ is, and also notice that

$H(\omega_2, \gamma)$ satisfies that $\langle \dot{S}, \dot{\chi} \rangle$ is a $\mathbb{P}_{\omega_2} \restriction \gamma$-name. Let j denote the transitive collapse map of $H(\omega_2, \gamma)$ and let $\pi := j^{-1}$ denote the anticollapse map. Set $\bar{\gamma} := j(\gamma)$ and $\kappa := j(\omega_2)$, and observe that by Lemma 5.6, j collapses $H(\omega_2, \gamma)$ onto $L_{\delta_{\bar{\gamma}}(\kappa)}$.

We will be done if we can show that forcing with $\dot{\mathbb{U}}_\gamma$ adds a partition of ω_1 into countably-many $\dot{\chi}$-homogeneous sets, and towards this end, let G be V-generic over $\mathbb{P}_{\omega_2}$. We use G_γ to denote the generic G adds for $\dot{\mathbb{U}}_\gamma[G \restriction \gamma]$ over $V[G \restriction \gamma]$. Set $\bar{G}$ to be $j\left[(G \restriction \gamma) \cap H(\omega_2, \gamma)\right]$, and observe that $\bar{G}$ is generic for the poset $j(\mathbb{P}_{\omega_2} \restriction \gamma) = \mathbb{P}_\kappa \restriction \bar{\gamma} = \mathbb{P}_\kappa$ over $L_{\delta_{\bar{\gamma}}(\kappa)}$. Since $\mathbb{P}_\kappa$ is c.c.c. and $L_{\delta_{\bar{\gamma}}(\kappa)}$ is countably closed, $\bar{G}$ is also V-generic over $\mathbb{P}_\kappa$. In particular, π extends to an elementary embedding

$$\pi^* : L_{\delta_{\bar{\gamma}}(\kappa)}[\bar{G}] \longrightarrow L_{\delta_\gamma(\omega_2)}[G],$$

and since $\mathrm{crit}(\pi^*) > \omega_1$, we see that $\dot{S}[G] = j(\dot{S})[\bar{G}]$ and $\dot{\chi}[G] = j(\dot{\chi})[\bar{G}]$.

By the elementarity of j, $\langle j(\dot{S}), j(\dot{\chi}) \rangle$ is the $(\bar{\gamma})_0$-th pair of $\mathbb{P}_\kappa$ names where the first coordinate names a countable basis for a second countable, Hausdorff topology on ω_1 and the second names a coloring which is open with respect to the topology generated by that basis. By the construction of $\dot{\mathbb{Q}}_\kappa$, this means that $\dot{\mathbb{Q}}_\kappa$ names the poset to decompose ω_1 into countably-many $j(\dot{\chi})$-homogeneous sets with respect to the preassignment $\dot{f}_\kappa$. Thus forcing with $\dot{\mathbb{Q}}_\kappa[\bar{G}]$ adds a decomposition of ω_1 into countably-many $j(\dot{\chi})[\bar{G}] = \dot{\chi}[G]$-homogeneous sets. We will be done if we can show that G adds a generic for $\dot{\mathbb{Q}}_\kappa[\bar{G}]$.

To see this, we recall from Case 2 of the construction that $\dot{\mathbb{U}}_\gamma$ is the $\pi \restriction \rho_\kappa$-rearrangement of $\dot{\mathbb{Q}}_\kappa$. Moreover, as also described in Case 2, Lemma 2.16 applies. Thus $\dot{\mathbb{Q}}_\kappa[\bar{G}] = \dot{\mathbb{U}}_\gamma[G]$. G_γ is therefore $V[G \restriction \gamma]$-generic for $\dot{\mathbb{Q}}_\kappa[\bar{G}]$, which finishes the proof. □

We wrap up by sketching a proof of Theorem 1.4.

Proof Sketch of Theorem 1.4. We first describe how to build the names on the sequence $\underline{\dot{\mathbb{Q}}}$. The only modification to the construction for the previous theorem is that if, in Case 1 above, $(\gamma)_0$ names a Knaster poset of size $\aleph_1$, then we set $\dot{\mathbb{Q}}_\kappa$ to be this Knaster poset. With this modification to the sequence $\underline{\dot{\mathbb{Q}}}$, we still maintain the recursive assumption that for each $\kappa \in C$, any partition product based upon $\underline{\mathbb{P}} \restriction \kappa$ and $\underline{\dot{\mathbb{Q}}} \restriction \kappa$ is c.c.c.; this follows by Lemma 2.20, Lemma 2.13, and since the product of Knaster and c.c.c. posets is still c.c.c.

Now we want to see that forcing with this modified $\mathbb{P}_{\omega_2}$ gives the desired model. The proof that the extension satisfies OCA_{ARS} and $2^{\aleph_0} = \aleph_3$ is the same as before. To prove that it satisfies $\mathsf{FA}(\aleph_2, \text{Knaster}(\aleph_1))$, suppose that $\dot{\mathbb{K}}$ is forced in $\mathbb{P}_{\omega_2}$ to be a Knaster poset of size $\aleph_1$. We may assume without loss of generality that $\dot{\mathbb{K}}$ is forced to be a subset of ω_1. Fix γ so that $(\gamma)_0$ codes $\dot{\mathbb{K}}$, making γ large enough so that $\dot{\mathbb{K}}$ is a $(\mathbb{P}_{\omega_2} \restriction \gamma)$-name and so that all the dense sets we need to meet belong to $V[G \restriction \gamma]$. Next, arguing as in the proof of Theorem 1.3, we have $\kappa < \omega_2$, $j \colon H(\omega_2, \gamma) \longrightarrow L_{\delta_{\bar{\gamma}}(\kappa)}$, and an extension

$$\pi^* \colon L_{\delta_{\bar{\gamma}}(\kappa)}[\bar{G}] \longrightarrow L_{\delta_\gamma(\omega_2)}[G \restriction \gamma]$$

of the inverse π of j. By the modified Case 1 construction we have that $\dot{\mathbb{Q}}_\kappa = j(\dot{\mathbb{K}})$. By Case 2 in the construction of $\mathbb{P}_{\omega_2}$, $\dot{\mathbb{U}}_\gamma$ is the rearrangement of $\dot{\mathbb{Q}}_\kappa$ by $\pi \restriction \rho_\kappa$. However, by the final clause in Lemma 2.16, and since $\dot{\mathbb{Q}}_\kappa$ names a poset contained in $\omega_1 < \kappa = \mathrm{crit}(\pi)$, this rearrangement is exactly $\pi(\dot{\mathbb{Q}}_\kappa) = \dot{\mathbb{K}}$. So G_γ is generic for

$\dot{\mathbb{K}}[G \restriction \gamma]$ over $V[G \restriction \gamma]$, and hence G_γ is a filter in $V[G]$ for $\dot{\mathbb{K}}[G \restriction \gamma]$ which meets the desired dense sets. $\square$

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Thomas Gilton, Department of Mathematics, University of California, Los Angeles, Box 951555, Los Angeles, CA 90095-1555

E-mail address: `tdgilton@math.ucla.edu`

Itay Neeman, Department of Mathematics, University of California, Los Angeles, Box 951555, Los Angeles, CA 90095-1555

E-mail address: `ineeman@math.ucla.edu`