Title
Representations of (Degenerate) Affine and Double Affine Hecke Algebras of Type C

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Representations of (Degenerate) Affine and Double Affine Hecke Algebras of Type C

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To...

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Representations of (Degenerate) Affine and Double Affine Hecke Algebras of Type C

Abstract

We compute the images of polynomial GL_N-modules and the coordinate algebra under the Etingof-Freund-Ma functor [**5**]. These yield $\mathcal{Y}$-semisimple representations of degenerate affine and double affine Hecke algebra of type C. We give a combinatorial description of the image in terms of standard tableaux on a collection of skew shapes and analyze weights of the image in terms of contents. For the nondegenerate case, we consider Jordan-Ma functor [**8**]. We compute the images of finite dimensional irreducible $U_q(\mathfrak{gl}_N)$-modules and the quantum coordinate algebra under the Jordan-Ma functor, which are also $\mathcal{Y}$-semisimple representations of affine and double affine Hecke algebras respectively.

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CHAPTER 1

Introduction

Schur-Weyl duality connects polynomial representations of $GL_N = GL_N(\mathbb{C})$ and representations of the symmetric group S_n. Let $V = \mathbb{C}^N$ denote the vector representation of GL_N. Then $V^{\otimes n}$ has a GL_N-action. Let S_n be the symmetric group on n indices. The tensor $V^{\otimes n}$ also has a natural right S_n-action which commutes with the left GL_N action. By Schur-Weyl duality, we have the decomposition

$$V^{\otimes n} = \bigoplus_{|\lambda|=n,\ell(\lambda)\leq N} V^{\lambda} \boxtimes S_{\lambda},$$

where $n \leq N$, λ is a partition of n with at most N rows, S_λ runs through all irreducible representations S_n and V^λ is the irreducible GL_N-module with highest weight λ. Moreover, the actions of Jucys-Murphy elements are diagonalizable. In [**1**], Arakawa and Suzuki constructed a functor from the category of $U(\mathfrak{gl}_N)$-modules to the category of representations of the degenerate affine Hecke algebra of type A_{n-1}. In [**2**], Calaque, Enriquez and Etingof generalized this functor to the category of representations of degenerate double affine Hecke algebra of type A_{n-1}. Etingof, Freund and Ma [**5**] extended the construction to the category of representations of degenerate affine and double affine Hecke algebra of type BC_n by considering the classical symmetric pair $(\mathfrak{gl}_N, \mathfrak{gl}_p \times \mathfrak{gl}_{N-p})$. As a quantization of the functors by Etingof-Freund-Ma, Jordan and Ma in [**8**] constructed functors from the category of $U_q(\mathfrak{gl}_N)$-modules to the category of representations of affine Hecke algebra of type C_n and from the category of quantum $\mathcal{D}$-modules to the category of representations of the double affine Hecke algebra of type $C^\vee C_n$. The construction in [**8**] used the theory of quantum symmetric pair $(U_q(\mathfrak{gl}_N), B_\sigma)$ where B_σ is a coideal subalgebra. This is a quantum analogue of the classical symmetric pair.

On the other hand, in [**18**], Reeder did the classification of irreducible representations of affine Hecke algebra of type C_2 with equal parameters. In [**9**], Kato indexed and analyzed the weights of representations of affine Hecke algebra of type C_n. In [**12**], Ma analyzed the image of principal series modules under the Etingof-Freund-Ma functor. Moreover, the combinatorial description of Young diagrams is used to describe irreducible representations of the symmetric group and Hecke algebra of type A with standard tableaux on the Young diagram indexing the bases. Similarly, the skew shape and standard tableaux on it describes certain irreducible representations of the affine Hecke algebra of type A. Moreover, in [**19**], Suzuki and Vazirani introduced a description of some irreducible representations of the double affine Hecke algebra of type A by periodic skew Young diagrams and periodic standard tableaux on it. In [**16**], Ram introduced the chambers and local regions and described some representations of the affine Hecke algebra. In [**3**], Daugherty introduced the combinatorial description of representations of degenerate extended two-boundary Hecke algebra. In [**4**], Daugherty and Ram gave a Schur-Weyl type duality approach to the affine Hecke algebra of type C_n.

This paper focuses on the representations of (degenerate) affine and double affine Hecke algebras of type C_n under the Schur-Weyl type duality and explores the combinatorial descriptions. In the second chapter, we talk about representations of degenerate affine Hecke algebras of type C_n and give a combinatorial description which is similar to the combinatorial description in [**3**] and [**4**] but is via a different structure, the Etingof-Freund-Ma functor. In the third chapter, we consider the image of coordinate algebra and its combinatorial description under Etingof-Freund-Ma functor, which is a representation of degenerate double affine Hecke algebra of type C_n. In the fourth chapter, we consider the quantum case: images under Jordan-Ma functor, which are representations of affine and double affine Hecke algebras of type C_n.

CHAPTER 2

Degenerate affine Hecke algebras of type C and Etingof-Freund-Ma functor

2.1. Definitions and notations

2.1.1. Root system of type C_n. Let $\mathfrak{h}^*$ be a finite-dimensional real vector space with basis $\{\epsilon_i | i = 1, \cdots, n\}$ and a positive definite symmetric bilinear form $(\cdot, \cdot)$ such that $(\epsilon_i, \epsilon_j) = \delta_{ij}$. Let R_n be an irreducible root system of type C_n with

$$R_n = \{\epsilon_i + \epsilon_j | i, j = 1, \cdots, n\} \cup \{\epsilon_i - \epsilon_j | i, j = 1, \cdots, n \text{ and } i \neq j\},$$

and the positive roots are

$$R_{n+} = \{\epsilon_i + \epsilon_j | i, j = 1, \cdots, n\} \cup \{\epsilon_i - \epsilon_j | 1 \leq i < j \leq n\}.$$

For any root $\alpha \in R_n$, the coroot is $\alpha^\vee = \dfrac{2\alpha}{(\alpha, \alpha)}$. Let Q be the root lattice and $Q^\vee$ be the coroot lattice. Let $\alpha_i = \epsilon_i - \epsilon_{i+1}$, for $i = 1, \cdots, n-1$ and $\alpha_n = 2\epsilon_n$. Then the collection of simple roots are

$$\Pi_n = \{\alpha_i | i = 1, \cdots, n\}.$$

For each simple root α_i, define the reflection $s_i := s_{\alpha_i}$,

$$s_{\alpha_i}(\lambda) = \lambda - (\lambda, \alpha_i^\vee)\alpha_i,$$

where $\lambda \in \mathfrak{h}^*$. Then the finite Weyl group W of type C_n is generated by the generators

$$s_1, \cdots, s_{n-1}, s_n$$

with the relations

$$s_i^2 = 1, \text{ for } i = 1, \cdots, n, \tag{2.1}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \text{ for } i = 1, \cdots, n-1, \tag{2.2}$$

$$s_{n-1} s_n s_{n-1} s_n = s_n s_{n-1} s_n s_{n-1}, \tag{2.3}$$

$$s_i s_j = s_j s_i, \text{ for } |i-j| > 1, \tag{2.4}$$

where the generator s_n is also denoted by γ_n in some cases.

2.1.2. Affine Weyl group of type C_n. For any $\iota \in \mathfrak{h}^*$, where $\iota = \iota_1\epsilon_1 + \cdots + \iota_n\epsilon_n$ and $\iota_k \in \mathbb{Z}$, let $y^\iota = y_1^{\iota_1} \cdots y_n^{\iota_n}$ and the action of $w \in W_0$ by $w.y^\iota = y^{w(\iota)}$. Let $W_a = W \ltimes Q^\vee$ and the affine Weyl group of type C_n is generated by $s_1, \cdots, s_{n-1}, s_n$ and $Y_i^{\pm}$, for $i = 1, \cdots, n$ with the following additional relations to (2.1)-(2.4),

$$s_i Y_j = Y_j s_i, \text{ for } j \neq i, i+1, \tag{2.5}$$

$$Y_i Y_j = Y_j Y_i, \tag{2.6}$$

$$s_i Y_i s_i = Y_{i+1}, \text{ for } i = 1, \cdots, n-1, \tag{2.7}$$

$$s_n Y_n s_n = Y_n^{-1}. \tag{2.8}$$

2.1.3. Definition of degenerate affine Hecke algebra of type C_n. Let κ_1 and κ_2 be two parameters. The trigonometric degenerate affine Hecke algebra $H_n(\kappa_1, \kappa_2)$, which we denote also by $dAHA$, is an algebra generated over $\mathbb{C}$ by $s_1, \cdots, s_{n-1}, \gamma_n$, where we take $\gamma_n = s_n$, and $y_1, \cdots, y_n$ with relations (2.1)-(2.6) and the following relations

$$s_i y_i - y_{i+1} s_i = \kappa_1, \text{ for } i = 1, \cdots, n-1, \tag{2.9}$$

$$\gamma_n y_n + y_n \gamma_n = \kappa_2. \tag{2.10}$$

2.1.4. $\mathcal{Y}$-semisimple degenerate affine Hecke algebra representations. Now let define what we mean by $\mathcal{Y}$-semisimple. Let $\mathcal{Y} = \mathbb{C}[y_1, \cdots, y_n]$ be the commutative subalgebra of the degenerate affine Hecke algebra $H_n(\kappa_1, \kappa_2)$. Let L be a representation of $H_n(\kappa_1, \kappa_2)$. For a function $\zeta : \{1, \cdots, n\} \to \mathbb{C}$, let ζ_i denote $\zeta(i)$ and $\zeta = [\zeta_1, \cdots, \zeta_n]$. Define the simultaneous generalized eigenspace as

$$L_\zeta^{gen} = \{v \in L | (y_i - \zeta_i)^k v = 0 \text{ for some } k \gg 0 \text{ and for all } i = 1, \cdots, n\}.$$

Since the polynomial algebra $\mathcal{Y}$ is commutative, the restriction of L on $\mathcal{Y}$ decomposes to a sum of simultaneous generalized eigenspace, i.e. $L = \oplus_\zeta L_\zeta^{gen}$. Similarly, define the simultaneous eigenspace

$$L_\zeta = \{v \in L | y_i v = \zeta_i v \text{ for all } i = 1, \cdots, n\}.$$

DEFINITION 2.1.1. *If the restriction of L on $\mathcal{Y}$ decomposes to a sum of simultaneous eigenspaces, i.e. $L = \oplus_\zeta L_\zeta$, then call L is $\mathcal{Y}$-semisimple. The function ζ is called a weight and L_ζ is the weight space of weight ζ.*

2.2. Etingof-Freund-Ma Functor

We recall the definition of the Etingof-Freund-Ma functor $F_{n,p,\mu}$ in [**5**]. Let N be a positive number and V be the vector representation of $\mathfrak{gl}_N$. Let p, q be positive integers such that $N = p+q$. Let $\mathfrak{t} = \mathfrak{gl}_p \times \mathfrak{gl}_q$ and $\mathfrak{t}_0$ be the subalgebra in $\mathfrak{t}$ consisting of all the traceless elements in $\mathfrak{t}$. Let χ is a character defined on $\mathfrak{t}$ as

$$\chi\left(\begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}\right) = q \cdot tr(S) - p \cdot tr(T), \tag{2.11}$$

where $S \in \mathfrak{gl}_p$ and $T \in \mathfrak{gl}_q$. For a given $\mu \in \mathbb{C}$, define a functor $F_{n,p,\mu}$ from the category of $\mathfrak{gl}_N$-modules to the category of representations of degenerate affine Hecke algebra $H_n(1, p-q-\mu N)$

$$F_{n,p,\mu}(M) = (M \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu},$$

where the $(\mathfrak{t}_0, \mu)$-invariant corresponds $A.v = \mu\chi(A)v$, for all $A \in \mathfrak{t}_0$.

Let M be the 0-th tensor factor. Let V_i be the i-th tensor factor with $V_i = V$ being the vector representation for $i = 1, \cdots, n$. In [**8**], the action of the degenerate affine Hecke algebra $H_n(1, p-q-\mu N)$ is the quasi classical limit of the action of the affine Hecke algebra $\mathcal{H}_n(q, q^\sigma, q^{(p-q-\tau)})$ generated by $T_1, \cdots, T_{n-1}, T_n$ and $Y_1^\pm, \cdots, Y_n^\pm$. In the following figures, V_i is the vector representation for $i = 1, \cdots, n$. In [**8**], the action of T_i for $i = 1, \cdots, n-1$ was defined by $\tau_{V_i, V_{i+1}} \circ R_{i,i+1}$, where the flip operator $\tau_{V_i,V_{i+1}} : V_i \otimes V_{i+1} \to V_{i+1} \otimes V_i$ is defined by $v_i \otimes v_{i+1} \mapsto v_{i+1} \otimes v_i$ and $R_{i,i+1}$ is the R matrix acting on $V_i \otimes V_{i+1}$ as in Figure 2.1. Let $T_i = s_i e^{\hbar s_i/2}$. Proposition 39 in [**7**] and section

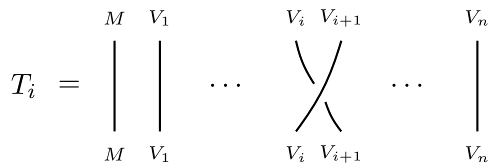


FIGURE 2.1. Action of T_i, $i = 1, \cdots, n-1$.

10.7 of [**8**] computed the action of s_i, i.e. s_i acts on $F_{n,p,\mu}(M)$ by exchanging the i-th and $(i+1)$-th tensor factors.

The action of T_n was defined as the diagram in Figure 2.2, where the matrix J_V is a right-handed

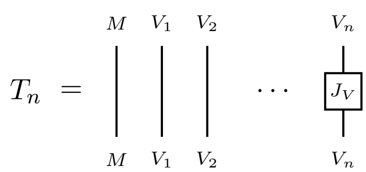


FIGURE 2.2. Action of T_n

numerical solution of the reflection equation $R_{21}(J_V)_1 R_{12}(J_V)_2 = (J_V)_2 R_{21}(J_V)_1 R_{12}$ in section 7 of [**8**]. Section 10.7 of [**8**] compute the quasi classical limit of T_n. Then γ_n acts on $F_{n,p,\mu}(M)$ by multiplying the n-th tensor factor by $J = diag(I_p, -I_q)$.

The action of Y_1 was define by Let $Y_1 = e^{y_1 \hbar}$. By Proposition 10.13 in [**8**],

$$y_1 = -\sum_{s,t} (E_s^t)_0 \otimes (E_t^s)_1 + \frac{n}{N} + \frac{\mu(q-p)}{2} - \frac{N}{2}, \tag{2.12}$$

where E_s^t is the $N \times N$ matrix with the (s,t) entry being 1 and other entries being 0 and $(E_s^t)_i$ means E_s^t acting on the i-th tensor factor. Let $s_{k,l}$ denote the transposition $(k,l) \in S_n$ and $\gamma_k \in W$ denote the action multiplying the k-th factor by J. In [**5**], the action of y_1 is given by

$$-\sum_{s|t} (E_s^t)_0 \otimes (E_t^s)_1 + \frac{p-q-\mu N}{2}\gamma_1 + \frac{1}{2}\sum_{l>1} s_{1,l} + \frac{1}{2}\sum_{l\neq 1} s_{1,l}\gamma_1\gamma_l, \tag{2.13}$$

where $\sum_{s|t} = \sum_{s=1}^{p} \sum_{t=p+1}^{n} + \sum_{t=1}^{p} \sum_{s=p+1}^{n}$. In Section 2.5.3, we show that the computation via (2.13) agrees with (2.12). By the relation $y_k = s_{k-1}y_{k-1}s_{k-1} - s_{k-1}$, we compute the action of y_k for $k = 1, \cdots, n$.

2.3. GL_N-module

We consider images of polynomial GL_N-modules under Etingof-Freund-Ma functor. Recall the facts about polynomial GL_N-modules. Let M be a polynomial GL_N-module and $H \subset GL_N$ be the collection of invertible diagonal matrices. Let $v \in M$ satisfy

$$x.v = x_1^{\lambda_1} \cdots x_N^{\lambda_N} v,$$

for any $x = diag(x_1, \cdots, x_N) \in H$. Then v is a weight vector of H-weight $\lambda = (\lambda_1, \cdots, \lambda_N)$. The subspace

$$M(\lambda) = \{v \in M | x.v = x_1^{\lambda_1} \cdots x_N^{\lambda_N} v, x \in H\}$$

is called the weight space of weight λ. Then the polynomial GL_N-module M is a direct sum of weight spaces

$$M = \bigoplus_{\lambda} M(\lambda).$$

Let $B \subset GL_N$ be the collection of all invertible upper triangular matrices. Let $v \in M$ be a generator of M. If v satisfies $x.v = c(x)v$ for some function $c(x)$ and any $x \in B$, then v is called a highest weight vector. If M has the unique highest weight vector up to a scalar of the highest weight ξ, then M is a highest weight module with the highest weight ξ and let us denote M by V^{ξ}. A GL_N-module M is irreducible if and only if M is a highest weight GL_N-module. Furthermore, two highest weight GL_N-modules are isomorphic if and only if they have the same highest weight. Let $\xi = \sum_{i=1}^{N} \xi_i \epsilon_i$ satisfying $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_N$ and $\xi_i \in \mathbb{Z}$ for $i = 1, \cdots, N$. Then ξ is an integral dominant weight of GL_N. Let P^+ denote the collection of all integral dominant weights and $P^+_{\geq 0}$ denote the collection of all integral dominant weights $\xi = \sum_{i=1}^{N} \xi_i \epsilon_i$ with $\xi_i \in \mathbb{N}$, for $i = 1, \cdots, N$. Then the highest weight modules with highest weights $\xi \in P^+_{\geq 0}$ are all the irreducible polynomial GL_N-modules. Let M be a rational GL_N-module. Then $M = det^m \otimes N$ for some $m \in \mathbb{Z}$ and a polynomial GL_N-module N. Then the highest weight modules with integral dominant highest weights are all the irreducible rational GL_N-modules.

The collection $P^+_{\geq 0}$ has a one-to-one correspondence with the collection of partitions with at most N parts and thus the one-to-one correspondence with Young diagrams with at most N rows. For the ease of writing, for each irreducible polynomial GL_N-module V^ξ with highest weight $\xi \in P^+_{\geq 0}$, let us denote the corresponding partition $(\xi_1, \cdots, \xi_N)$ and Young diagram also by ξ. Moreover, define $|\xi| = \sum_{i=1}^N \xi_i$ for $\xi \in P^+$.

For a highest weight GL_N-module V^ξ, $\xi \in P^+_{\geq 0}$, with weight space decomposition $V^\xi = \bigoplus V^\xi(\lambda)$, the character of V^ξ

$$\chi_{V^\xi} = \sum_\lambda dim(V^\xi(\lambda)) x_1^{\lambda_1} \cdots x_N^{\lambda_N}$$

is the Schur polynomial $s_\xi(x_1, \cdots, x_N)$ of shape ξ.

By Pieri's rule,

$$s_\xi e_1 = \sum_\nu s_\nu,$$

where $\nu \in P^+_{\geq 0}$ runs through all the shapes obtained by adding a cell to some row of ξ. Observe that $e_1 = s_\xi$, where $\xi = (1)$, is the character of the vector representation V of GL_N. This fact indicates how the tensor product of an irreducible polynomial GL_N-module and vector representation decomposes into a sum of irreducible polynomial GL_N-modules.

2.4. Invariant space

In this section, we compute the underlying vector space $F_{n,p,\mu}(V^\xi) = (M \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ by finding a special basis of it and then index the basis elements by a collection of standard tableaux.

2.4.1. Definition of the invariant space.

Let M be a GL_N-module, then M has a $\mathfrak{gl}_N$-module structure. For any $X \in \mathfrak{gl}_N$ and $v \in M$,

$$X.v = \frac{d}{dt}(e^{tX}.v)_{t=0}.$$

Recall the notations, $K = GL_p \times GL_q$, $Lie(K) = \mathfrak{t}$ and $\mathfrak{t}_0 \subset \mathfrak{t}$ which is the collection of traceless matrices in $\mathfrak{t}$.

Proposition 2.4.1. *The underlying vector space is invariant under tensoring powers of the determinant representation, i.e.* $(det^m \otimes M \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu} \cong (M \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$, *for any* $m \in \mathbb{C}$.

PROOF. Take any element from $(det^m \otimes M \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$, we can denote it by $\mathbb{1} \otimes w$, where $w \in M \otimes V^{\otimes n}$. According to the definition of invariant space

$$\begin{aligned}&(det^m \otimes M \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}\\ =&\{\mathbb{1} \otimes w | A.(\mathbb{1} \ \otimes w) = \mu\chi(A)(\mathbb{1} \otimes w), \text{ for any } A \in \mathfrak{t}_0\}.\end{aligned}$$

Compute the action of $A \in \mathfrak{t}_0$

$$\begin{aligned}A.\mathbb{1} &= \frac{d}{dt}(e^{tA}.\mathbb{1})_{t=0}\\ &= \frac{d}{dt}(det^m(e^{tA}))_{t=0}.\mathbb{1}\\ &= \frac{d}{dt}(e^{m\cdot tr(tA)})_{t=0}.\mathbb{1} = 0,\end{aligned}$$

since $tr(A) = 0$. Then it follows

$$\begin{aligned}A.(\mathbb{1} \otimes w) &= (A.\mathbb{1}) \otimes w + \mathbb{1} \otimes (A.w)\\ &= \mathbb{1} \otimes (A.w).\end{aligned}$$

Hence

$$\begin{aligned}&(det^m \otimes M \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}\\ =&\{\mathbb{1} \otimes w | \mathbb{1} \otimes (A.w) = \mu\chi(A)(\mathbb{1} \otimes w), \text{ for any } A \in \mathfrak{t}_0\}\\ \cong&\{w | A.w = \mu\chi(A)w, \text{ for any } A \in \mathfrak{t}_0\}\\ =&(M \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}.\end{aligned}$$

$\square$

REMARK 2.4.2. *For an irreducible rational GL_N-module M, we write $M = det^m \otimes V^{\xi}$ for some integer m and some highest weight module V^{ξ} with the highest weight $\xi \in P^+_{\geq 0}$ such that $\xi_N = 0$. Then $(M \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu} = (V^{\xi} \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$. So it is enough to consider highest weight module V^{ξ} with highest weight $\xi \in P^+_{\geq 0}$ such that $\xi_N = 0$, which is associated to partitions ξ of length at most $N-1$.*

2.4.2. Computation of the $(\mathfrak{t}_0, \mu)$ invariant space.

PROPOSITION 2.4.3. *The $(\mathfrak{t}_0, \mu)$ invariant space $F_{n,p,\mu}(V^\xi) = (V^\xi \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$, for $\mu \in \mathbb{C}$ and $\xi \in P^+_{\geq 0}$.*

$$\begin{aligned}(V^\xi \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu} \cong & Hom_{\mathfrak{t}_0}(\mathbb{1}_{\mu\chi}, Res^{\mathfrak{gl}_N}_{\mathfrak{t}_0} V^\xi \otimes V^{\otimes n}) \\ \cong & Hom_{\mathfrak{t}}(\mathbb{1}_\theta, Res^{\mathfrak{gl}_N}_{\mathfrak{t}} V^\xi \otimes V^{\otimes n}),\end{aligned}$$

where $\mathbb{1}_\theta$ is a one-dimensional $\mathfrak{t}$-module and

$$\mathbb{1}_\theta = (\mu q + \frac{|\xi| + n}{N}) tr_{\mathfrak{gl}_p} + (-\mu p + \frac{|\xi| + n}{N}) tr_{\mathfrak{gl}_q}.$$

PROOF. The $(\mathfrak{t}_0, \mu)$ invariant space $F_{n,p,\mu}(V^\xi) = (V^\xi \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ is defined to be the subspace

$$\{v \in V^\xi \otimes V^{\otimes n} | Av = \mu\chi(A)v \text{ for any A } \in \mathfrak{t}_0\}.$$

To compute this subspace, we lift it to a $\mathfrak{t}$-invariant space. Let $\mathbb{1}_\psi$ the one-dimensional $\mathfrak{t}$-module such that

$$\begin{aligned}&(V^\xi \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu} \\ =&(Res^{\mathfrak{gl}_N}_{\mathfrak{t}}(V^\xi \otimes V^{\otimes n}) \otimes \mathbb{1}_\psi)^{\mathfrak{t}}.\end{aligned}$$

Let $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathbb{C}\{I_N\}$. For any $P \in \mathfrak{t}$, there is a unique decomposition $P = A + B$ such that $A \in \mathfrak{t}_0$ and $B = bI_N$ for some $b \in \mathbb{C}$. So the $\mathfrak{t}$-invariant corresponds to

$$\{v \in V^\xi \otimes V^{\otimes n} | Pv + \mathbb{1}_\psi(P)v = 0\}$$

. Then $Pv + \mathbb{1}_\psi(P)v = Av + Bv + \mathbb{1}_\psi(P)v = 0$. And $B = bI_N$ acts by the scalar

$$b(|\xi| + n) = (|\xi| + n)\frac{tr(B)}{N}$$

. Also, we have $\chi(P) = \chi(A) + \chi(B) = \chi(A)$, since $\chi(B) = qbp - pbq = 0$. So

$$\begin{aligned} &\{v \in V^{\xi} \otimes V^{\otimes n} | Pv + \mathbb{1}_{\psi}(P)v = 0\} \\ =&\{v \in V^{\xi} \otimes V^{\otimes n} | Av = \mu\chi(A)v\}. \end{aligned}$$

For any $P \in \mathfrak{t}$ with

$$P = \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}$$

where $S \in \mathfrak{gl}_p$ and $T \in \mathfrak{gl}_q$, we have

$$\begin{aligned} \mathbb{1}_{\psi}(P) &= -\mu\chi(A) - \frac{|\xi| + n}{N} tr(B) \\ &= -\mu\chi(P) - \frac{|\xi| + n}{N} tr(P) \\ &= (-\mu q - \frac{|\xi| + n}{N}) tr_{\mathfrak{gl}_p}(S) + (\mu p - \frac{|\xi| + n}{N}) tr_{\mathfrak{gl}_q}(T). \end{aligned}$$

Hence it follows that the one dimensional $\mathfrak{t}$-module

$$\mathbb{1}_{\theta} = (\mu q + \frac{|\xi| + n}{N}) tr_{\mathfrak{gl}_p} + (-\mu p + \frac{|\xi| + n}{N}) tr_{\mathfrak{gl}_q}.$$

□

Remark 2.4.4. *The $(\mathfrak{t}, \mathbb{1}_{\theta})$ invariant space above is equivalent to the following K invariant space.*

$$\begin{aligned} (V^{\xi} \otimes V^{\otimes n})^{\mathfrak{t}_0, \mu} \cong& Hom_{\mathfrak{t}_0}(\mathbb{1}_{\mu\chi}, Res_{\mathfrak{t}_0}^{\mathfrak{gl}_N} V^{\xi} \otimes V^{\otimes n}) \\ \cong& Hom_{\mathfrak{t}}(\mathbb{1}_{\theta}, Res_{\mathfrak{t}}^{\mathfrak{gl}_N} V^{\xi} \otimes V^{\otimes n}) \\ \cong& Hom_K(det^a \boxtimes det^b, V^{\xi} \boxtimes V^{\otimes n}), \end{aligned}$$

where $a = \mu q + \frac{|\xi|+n}{N}$ and $b = -\mu p + \frac{|\xi|+n}{N}$.

2.4.3. A basis of invariant space and standard tableaux.

The characters of irreducible polynomial GL_N-modules are Schur functions. So we could consider the restriction of $V^{\xi} \otimes V^{\otimes n}$ by exploring Schur functions. Recall the following fact of Schur functions.

PROPOSITION 2.4.5. *Let $s_\nu(x_1,\cdots,x_p,z_{p+1},\cdots,z_N)$ be the character of V^ν, then*

$$s_\nu(x_1,\cdots,x_p,z_{p+1},\cdots,z_N)=\Sigma c^\nu_{\omega_1,\omega_2}s_{\omega_1}(x_1,\cdots,x_p)s_{\omega_2}(z_{p+1},\cdots,z_N),$$

where ω_1 is a highest weight of GL_p and ω_2 is a highest weight of GL_q, $c^\nu_{\omega_1,\omega_2}$ is the Littlewood-Richardson coefficient.

The Littlewood-Richardson coefficient $c^\nu_{\omega_1,\omega_2}$ is the multiplicity of the K-module $V^{\omega_1}\boxtimes V^{\omega_2}$ in the restriction of GL_N-module V^ν. Let $V^\xi\otimes V^{\otimes n}=\bigoplus_\nu m_\nu V^\nu$ as GL_N-modules, where $\nu\in P^+_{\geq 0}$ and $m_\nu\in\mathbb{N}$ is the multiplicity of V^ν in V^ξ. Then the $(\mathfrak{t}_0,\mu)$ invariant space

$$F_{n,p,\mu}(V^\xi)=Hom_K(det^a\boxtimes det^b,Res^{GL_N}_K V^\xi\otimes V^{\otimes n}) \tag{2.14}$$

$$=\bigoplus_\nu m_\nu Hom_K(det^a\boxtimes det^b,Res^{GL_N}_K V^\nu). \tag{2.15}$$

Since $\nu\in P^+_{\geq 0}$, to guarantee $Hom_K(det^a\boxtimes det^b,Res^{GL_N}_K V^\nu)\neq 0$ for each ν in (13), it suffices to consider $a,b\in\mathbb{N}$, otherwise $F_{n,p,\mu}(V^\xi)=(V^\xi\otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}=0$. Our goal is to compute the ν such that the multiplicity of $det^a\boxtimes det^b$ in the K restriction of the GL_N-module V^ν is nonzero. To do this, we need Okada's theorem [**15**].

THEOREM 2.4.6. *For any two rectangular shapes (a^p) and (b^q), where a and b are nonnegative integers and $p\leq q$, then*

$$s_{a^p}\cdot s_{b^q}=\sum c^\nu_{(a^p)(b^q)}s_\nu,$$

where $c^\nu_{(a^p)(b^q)}=1$ when ν satisfies the condition

$$\nu_i+\nu_{p+q-i+1}=a+b,\quad i=1,\cdots,p, \tag{2.16}$$

$$\nu_p\geq max(a,b) \tag{2.17}$$

$$\nu_i=b,\quad i=p+1,\cdots,q \tag{2.18}$$

and $c^\nu_{(a^p)(b^q)}=0$ otherwise.

COROLLARY 2.4.7. *Now we have the following fact, the* $(\mathfrak{t}_0, \mu)$ *invariant space*

$$F_{n,p,\mu}(V^\xi) = (V^\xi \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu} \tag{2.19}$$

$$= \bigoplus_{\nu} Hom_{GL_N}(V^\nu, V^\xi \otimes V^{\otimes n}), \tag{2.20}$$

where $\nu \in P^+_{\geq 0}$ *runs through all partitions satisfying* (2.16)-(2.18).

Moreover, by Pieri's rule, the vector space $Hom_{GL_N}(V^\nu, V^\xi \otimes V^{\otimes n})$ has a basis indexed by standard tableaux T such that the shape of T is ν/ξ and the dimension of this vector space

$$m_\nu = dimHom_{GL_N}(V^\nu, V^\xi \otimes V^{\otimes n})$$

equals the number of standard tableaux T with the shape of T being ν/ξ. If $m_\nu \neq 0$, then $\xi \subset \nu$ and $|\nu| = |\xi| + n$.

THEOREM 2.4.8. *The* $(\mathfrak{t}_0, \mu)$ *invariant space* $F_{n,p,\mu}(V^\xi) = (V^\xi \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ *has a one to one correspondence to the set of standard tableaux* T *such that the shape of* T *is* ν/ξ *for* $\nu \in P^+_{\geq 0}$ *with* $|\nu| = |\xi| + n$, ν *runs through all the partitions satisfying* (2.16)-(2.18) *and* $\xi \subset \nu$.

Let us consider the following example of $(\mathfrak{t}_0, \mu)$ invariant space.

EXAMPLE 2.4.9. *Let* $M = V^\xi$ *be a* GL_3*-module,* $\xi = 2\epsilon_1 + \epsilon_2$, $n = 3$, $p = 1$ *and* $\mu = 0$.

Then $(a^p) = (2^1)$ *and* $(b^q) = (2^2)$.

*By Okada's theorem in [**15**], we could compute the shapes* ν *such that the invariant space is nonzero.*

Then a basis of the invariant space is indexed by standard tableaux on skew shapes obtained by the

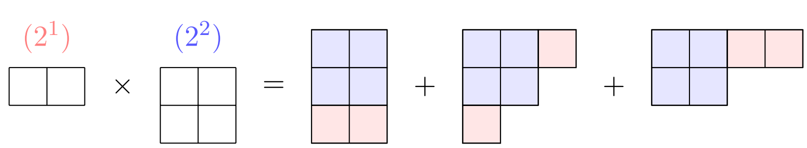


FIGURE 2.3. Shapes ν such that $Hom_K(det^a \boxtimes det^b, V^\nu \neq 0)$

shapes above skewed by ξ.

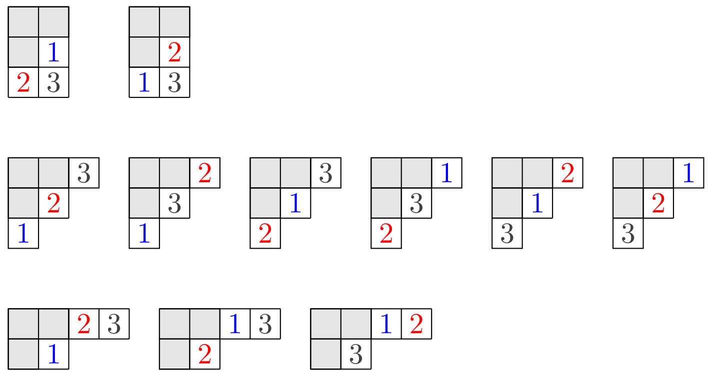


FIGURE 2.4. The collection of standard tableaux indexing a basis of $F_{3,1,0}(V^{\xi})$

In this example, we obtain an invariant space $F_{3,1,0}(V^{\boxplus})$ *of* 11 *dimensions.*

2.4.4. One skew shape. In this subsection, we associate a skew shape $\varphi^{\xi}_{n,p,\mu}$ to the image $F_{n,p,\mu}(V^{\xi})$ under Etingof-Freund-Ma functor and we call $\varphi^{\xi}_{n,p,\mu}$ the minimal shape of $F_{n,p,\mu}(V^{\xi})$. Let $\xi = \sum_{i=1}^{N} \xi_i \epsilon_i \in P^+_{\geq 0}$. The corresponding Young diagram $\xi = (\xi_1, \cdots, \xi_N)$. The first q rows of ξ forms a Young diagram denoted by $\xi^{(1)}$ and the last p rows of ξ forms a Young diagram denoted by $\xi^{(2)}$. Fix a parameter μ, we have a pair of rectangles (a^p) and (b^q) denoting the K-module $det^a \boxtimes det^b$, where $a = \mu q + \frac{|\xi|+n}{N}$ and $b = -\mu p + \frac{|\xi|+n}{N}$.
Suppose $p \leq q$. Placing the northwestern corner the rectangle (a^p) next to the northeastern corner of the rectangle (b^q) forms a Young diagram β. Delete the Young diagram $\xi^{(1)}$ from northwestern corner of β. Let $\rotatebox[origin=c]{180}{$\xi^{(2)}$}$ denote the skew shape obtained by rotating $\xi^{(2)}$ by π. Delete the rotated $\xi^{(2)}$ from the southeastern corner of β, i.e. the skew shape $\varphi^{\xi}_{n,p,\mu}$ is defined by $\varphi^{\xi}_{n,p,\mu} = \nu/\xi^{(1)}$, where $\nu_i = a + b - \xi_{N-i+1}$ for $i = 1, \cdots, p$ and $\nu_i = b$ for $i = p+1, \cdots, q$.

Let $\varphi = \varphi^{\xi}_{n,p,\mu}$. If a cell (i,j) of the skew shape φ satisfy $(i+1,j) \notin \varphi$ and $(i,j+1) \notin \varphi$, then call (i,j) a corner of φ. Define γ-move on a skew shape φ: delete a corner $(i,j) \in \varphi$ such that $j > max(a,b)$ and $1 \leq i \leq p$, and add the cell $(p+q-i+1, a+b-j+1)$. The condition $j > max(a,b)$ guarantees the new shape after γ-move is still a skew shape. Denote the γ-move by $\varphi \to \varphi'$ where $\varphi' = \varphi \setminus (i,j) \cup (p+q-i+1, a+b-j+1)$. Note that for a given φ, the γ-move

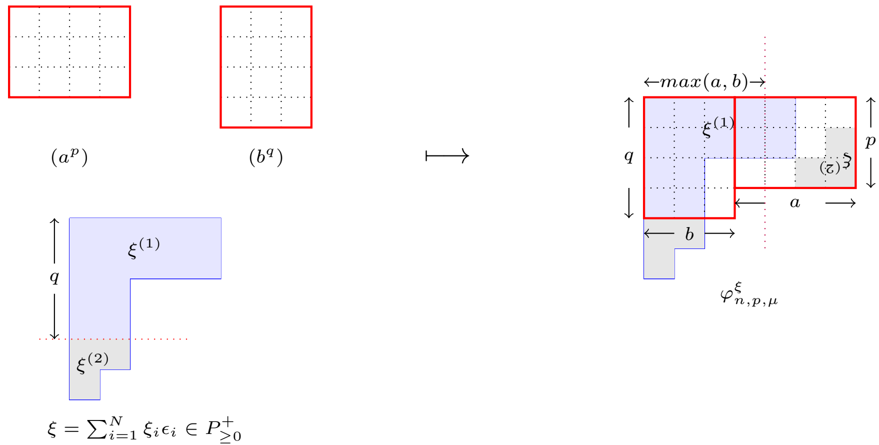


FIGURE 2.5. One skew shape

stops when there is no cell (i,j) such that $j > max(a,b)$. Given the skew shape $\varphi^{\xi}_{n,p,\mu}$, a collection $D(\varphi^{\xi}_{n,p,\mu})$ of skew shapes consists of $\varphi^{\xi}_{n,p,\mu}$ and all the skew shapes obtained by applying γ-moves on $\varphi^{\xi}_{n,p,\mu}$ for finitely many times. The shape $\varphi^{\xi}_{n,p,\mu}$ is called the minimal shape of the representation $F_{n,p,\mu}(V^{\xi})$.

Continue Example 2.4.9, the representation $F_{3,1,0}(V^{\boxplus})$ is index by the following skew shape φ. The collection $D(\varphi)$ of skew shapes is obtained as follows:

2.4.5. Skew shapes and standard tableaux. For the ease of description, let us use the following definition of skew shapes and standard tableaux. Given a partition $\xi = (\xi_1, \cdots, \xi_l)$, the corresponding Young diagram ξ is a subset of $\mathbb{Z}^2$, consisting of (i,j) such that $1 \leq i \leq l$ and $1 \leq j \leq \xi_i$. Let $\nu = (\nu_1, \cdots, \nu_l)$ and $\xi = (\xi_1, \cdots, \xi_l)$ such that $\nu_i \geq \xi_i$ for $1 \leq i \leq l$, then for the corresponding Young diagrams $\xi \subset \nu$ holds. A skew shape ν/ξ is the subset $\nu \setminus \xi$ of $\mathbb{Z}^2$. For example, let $\nu = (7,6,5,3,2,1)$ and $\xi = (5,5,2,2,2,1)$, then Young diagrams ν and ξ and the skew shape ν/ξ are the following subsets of $\mathbb{Z}^2$.

$$\nu = \{(i,j) | 1 \leq i \leq 6, 1 \leq j \leq \nu_i\},$$

$$\xi = \{(i,j) | 1 \leq i \leq 6, 1 \leq j \leq \xi_i\}$$

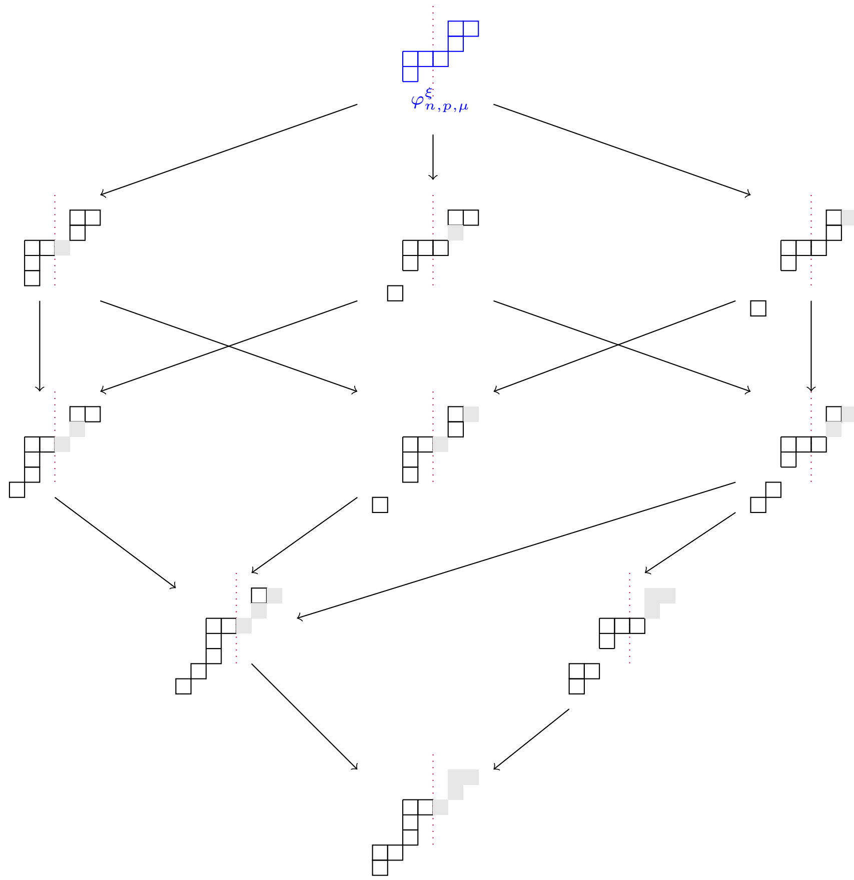


FIGURE 2.6. All skew shapes obtained by γ-move

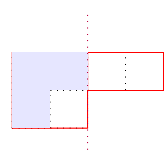


FIGURE 2.7. The minimal skew shape of $F_{3,1,0}(V^{\xi})$

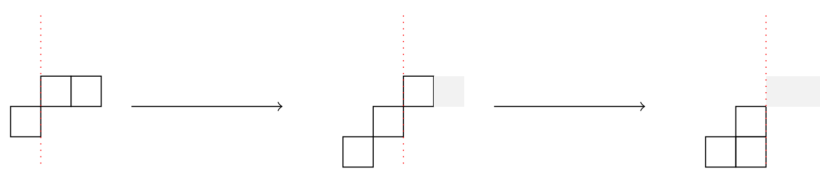


FIGURE 2.8. All skew shapes of $F_{3,1,0}(V^{\xi})$

and

$$\nu/\xi = \{(1,6),(1,7),(2,6),(3,3),(3,4),(3,5),(4,3)\}.$$

Define a tableau T on n-indices $\{1, \cdots, n\}$ to be an injective map T

$$\begin{aligned} T : \{1, \cdots, n\} &\to \mathbb{Z}^2 \\ k &\mapsto (\mathfrak{i}(k), \mathfrak{j}(k)) \end{aligned}$$

where $\mathfrak{i}$ and $\mathfrak{j}$ being two maps from $\{1, \cdots, n\}$ to $\mathbb{Z}$ and the image $Im(T)$ of T being a skew shape. The image $Im(T)$ is also called the shape of the tableaux T. Let $cont_T$ be a map

$$\begin{aligned} cont_T : \{1, \cdots, n\} &\to \mathbb{Z} \\ k &\mapsto \mathfrak{j}(k) - \mathfrak{i}(k), \end{aligned}$$

call $cont_T(k)$ is the content of k in the tableau T. If

$$T^{-1}(i+1, j) > T^{-1}(i, j)$$

and

$$T^{-1}(i, j+1) > T^{-1}(i, j)$$

hold for each cell $(i, j) \in Im(T)$, then call T is a standard tableau.
Let

$$Tab_c^{\lambda,\mu} = \{T | T \text{ is a standard tableau and } Im(T) \in D(\varphi^{\xi}_{n,p,\mu})\}.$$

The invariant space $F_{n,p,\mu}(V^{\xi}) = (V^{\xi} \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ has a basis indexed by a collection of standard tableaux on the skew shapes in $D(\varphi^{\xi}_{n,p,\mu})$, i.e. all the tableaux in $Tab_c^{\lambda,\mu}$. Let v_T denote the basis vector indexed by $T \in Tab_c^{\lambda,\mu}$. Then as a vector space

$$\begin{aligned} F_{n,p,\mu}(V^{\xi}) &= (V^{\xi} \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu} \\ &= span_{\mathbb{C}}\{v_T | T \in Tab_c^{\lambda,\mu}\}. \end{aligned}$$

2.5. $\mathcal{Y}$- semisimplicity

2.5.1. Action of $\mathcal{Y}$. In this subsection let us computer the $\mathcal{Y}$-actions on the invariant space $F_{n,p,\mu}(V^{\xi}) = (V^{\xi} \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$. In [**7**], Jordan computed the action of y_1 and used the fact that Etingof-Freund-Ma functor is a trigonometric degeneration of the quantum case. Now let us review the computation and conduct it in the degenerate case. Let us use the following notations in [**5**] for sums

$$\sum_{s,t} = \sum_{s=1}^{N}\sum_{p=1}^{N}, \tag{2.21}$$

$$\sum_{s|t} = \sum_{s=1}^{p}\sum_{t=p+1}^{N} + \sum_{t=1}^{p}\sum_{s=p+1}^{N}, \tag{2.22}$$

$$\sum_{st} = \sum_{s=1}^{p}\sum_{t=1}^{p} + \sum_{s=p+1}^{N}\sum_{t=p+1}^{N}. \tag{2.23}$$

It is easy to observe that the sum of (2.22) and (2.23) equals (2.21).

Review the definition of y_1 on the $(\mathfrak{t}_0, \mu)$-invariant space $F_{n,p,\mu}(V^\xi) = (V^\xi \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ in [**5**],

$$y_1 = -\sum_{s|t}(E_s^t)_0 \otimes (E_t^s)_1 + \frac{p-q-\mu N}{2}\gamma_1 + \frac{1}{2}\sum_{l>1} s_{1,l} + \frac{1}{2}\sum_{l\neq 1} s_{1,l}\gamma_1\gamma_l.$$

Compute the last two terms of y_1, we have

$$\begin{aligned}
&\frac{1}{2}\sum_{l>1} s_{1,l} + \frac{1}{2}\sum_{l\neq 1} s_{1,l}\gamma_1\gamma_l \\
=&\frac{1}{2}\sum_{l>1}\sum_{s,t}(E_s^t)_1 \otimes (E_t^s)_l + \frac{1}{2}\sum_{l>1}\sum_{s,t}(E_s^tJ)_1 \otimes (E_t^sJ)_l \\
=&\sum_{l>1}\sum_{st}(E_s^t)_1 \otimes (E_t^s)_l \\
=&\sum_{st}(E_s^t)_1(\sum_{l>1} 1 \otimes (E_t^s)_l) \\
=&\sum_{st}(E_s^t)_1(\Delta^{(n)}(E_t^s) - (E_t^s)_0 - (E_t^s)_1)
\end{aligned}$$

The last step follows the fact that $\sum_{l>1} 1 \otimes (E_t^s)_l = \Delta^{(n)}(E_t^s) - (E_t^s)_0 - (E_t^s)_1$, where Δ denotes the comultiplication of Lie algebra $\mathfrak{gl}_N$ and $\Delta^{(n)}(E_t^s) = \sum_{l=0}^{n}(E_t^s)_l$.

Applying the fact that y_1 preserves on the $(\mathfrak{t}_0, \mu)$-invariant space $F_{n,p,\mu}(V^\xi) = (V^\xi \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$, the computation of the last two terms of y_1 above continues as follows.

$$\begin{aligned}
&\sum_{st}(E^t_s)_1(\Delta^{(n)}(E^s_t)-(E^s_t)_0-(E^s_t)_1)\\
=&\sum_{s=1}^{p}(\mu q+\frac{|\xi|+n}{N})(E^s_s)_1+\sum_{s=p+1}^{N}(-\mu p+\frac{|\xi|+n}{N})(E^s_s)_1\\
&-\sum_{s=1}^{p}p(E^s_s)_1-\sum_{s=p+1}^{N}q(E^s_s)_1-\sum_{st}(E^t_s)_1\otimes(E^s_t)_0\\
=&(\mu q-p+\frac{|\xi|+n}{N})\sum_{s=1}^{p}(E^s_s)_1+(-\mu p-q+\frac{|\xi|+n}{N})\sum_{s=p+1}^{N}(E^s_s)_1\\
&-\sum_{st}(E^s_t)_0\otimes(E^t_s)_1
\end{aligned}$$

Combining other terms in the definition of y_1,

$$\begin{aligned}
y_1=&-\sum_{s,t}(E^t_s)_0\otimes(E^s_t)_1+\frac{p-q-\mu N}{2}\gamma_1\\
&+(\mu q-p+\frac{|\xi|+n}{N})\sum_{s=1}^{p}(E^s_s)_1+(-\mu p-q+\frac{|\xi|+n}{N})\sum_{s=p+1}^{N}(E^s_s)_1\\
=&-\sum_{s,t}(E^t_s)_0\otimes(E^s_t)_1+(\mu q-p+\frac{|\xi|+n}{N}+\frac{p-q-\mu N}{2})\sum_{s=1}^{p}(E^s_s)_1\\
&+(-\mu p-q+\frac{|\xi|+n}{N}-\frac{p-q-\mu N}{2})\sum_{s=p+1}^{N}(E^s_s)_1\\
=&-\sum_{s,t}(E^t_s)_0\otimes(E^s_t)_1+(\frac{|\xi|+n}{N}+\frac{\mu q-\mu p}{2}-\frac{N}{2})\sum_{s=1}^{N}(E^s_s)_1\\
=&-\sum_{s,t}(E^t_s)_0\otimes(E^s_t)_1+\frac{|\xi|+n}{N}+\frac{\mu q-\mu p}{2}-\frac{N}{2},
\end{aligned}$$

REMARK 2.5.1. *Since the action in [**8**] was define on $F_{n,p,\mu}(M)$ for M is a $\mathcal{D}$-module, there is a difference between equation* (2.12) *and the above result. If we input a $\mathcal{D}$-module instead of V^ξ, the above result will be the same with equation* (2.12).

Moreover, the action of y_k for $k>1$ is computed by induction.

Proposition 2.5.2. *The action of* y_k*, for* $k = 1, \cdots, n$*, on the invariant space* $(V^{\xi} \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ *is computed by*

$$y_k = -\sum_{s,t} (\Delta^{(k-1)} E_s^t)_{(0,k)} \otimes (E_t^s)_k + \frac{|\xi| + n}{N} + \frac{\mu q - \mu p}{2} - \frac{N}{2},$$

where $(E_s^t)_{(0,k)}$ *denotes the tensor product* $(V^{\xi} \otimes V^{\otimes(k-1)})$ *and hence* $\Delta^{(k-1)} E_s^t$ *acting on* $(E_s^t)_{(0,k)}$.

Proof. We verified the action of y_1 above. Suppose the statement is true for y_i, $i < k$. Let compute the action of y_k. By the relation $s_{k-1}y_{k-1} - y_k s_{k-1} = \kappa_1 = 1$ and the inductive hypothesis, it follows

$$\begin{aligned}
y_k &= s_{k-1} y_{k-1} s_{k-1} - s_{k-1} \\
&= -\sum_{s,t,j,l} (\Delta^{(k-2)} E_s^t)_{(0,k-1)} \otimes (E_l^t E_t^s E_s^j)_{k-1} \otimes (E_t^l E_j^s)_k \\
&\quad - \sum_{s,t} (E_s^t)_{k-1} \otimes (E_t^s)_k + \frac{|\xi| + n}{N} + \frac{\mu q - \mu p}{2} - \frac{N}{2} \\
&= -\sum_{s,t,j} (\Delta^{(k-2)} E_s^t)_{(0,k-1)} \otimes (E_j^j)_{k-1} \otimes (E_t^s)_k \\
&\quad - \sum_{s,t} (E_s^t)_{k-1} \otimes (E_t^s)_k + \frac{|\xi| + n}{N} + \frac{\mu q - \mu p}{2} - \frac{N}{2}
\end{aligned}$$

Take the fact $\sum_j (E_j^j)_{k-1} = (I_N)_{k-1}$. The above computation continues

$$\begin{aligned}
&= -\sum_{s,t} (\Delta^{(k-2)} E_s^t)_{(0,k-1)} \otimes (I_N)_{k-1} \otimes (E_t^s)_k \\
&\quad - \sum_{s,t} (E_s^t)_{k-1} \otimes (E_t^s)_k + \frac{|\xi| + n}{N} + \frac{\mu q - \mu p}{2} - \frac{N}{2} \\
&= -\sum_{s,t} (\Delta^{(k-1)} E_s^t)_{(0,k)} \otimes (E_t^s)_k + \frac{|\xi| + n}{N} + \frac{\mu q - \mu p}{2} - \frac{N}{2}.
\end{aligned}$$

□

The Lie algebra $\mathfrak{gl}_N$ has a basis $\{E_s^t | 1 \le s, t \le N\}$ with the dual basis $\{E_t^s\}$ with respect to the Killing form. Let C denote the Casimir element of $U(\mathfrak{gl}_N)$, then $C = \sum_{s,t} E_s^t E_t^s$. The following computation follows

$$\begin{aligned}\Delta(C) =& \sum_{s,t} \Delta(E_s^t)\Delta(E_t^s) \\ =& \sum_{s,t} (E_s^t \otimes 1 + 1 \otimes E_s^t)(E_t^s \otimes 1 + 1 \otimes E_t^s) \\ =& (\sum_{s,t} E_s^t E_t^s) \otimes 1 + 1 \otimes (\sum_{s,t} E_s^t E_t^s) + 2 \sum_{s,t} E_s^t \otimes E_t^s.\end{aligned}$$

Thus

$$\sum_{s,t} E_s^t \otimes E_t^s = \frac{\Delta(C) - C \otimes 1 - 1 \otimes C}{2}.$$

2.5.2. Weights and contents. In [**17**], Ram talked about the standard tableaux and representations of affine Hecke algebra of type C and analyzed the weights in terms of boxes. Now let us analyze the weights of $F_{n,p,\mu}(V^{\xi})$ in terms of contents. In section 5, we obtain a basis of the $(\mathfrak{t}_0, \mu)$-invariant space $F_{n,p,\mu}(V^{\xi}) = (V^{\xi} \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ indexed by $Tab(\varphi^{\xi}_{n,p,\mu})$, i.e. standard tableaux on a family of skew shapes ν/ξ where ν are obtained by Okada's theorem. The action of y_k on the basis element indexed by standard tableau T is by a scalar. Moreover, this scalar is computed in terms of the content of the box fixed by k.

Theorem 2.5.3. *Let v_T denote the basis element of the invariant space indexed by standard tableau T. Then v_T is an eigenvector of y_k and the eigenvalue is computed as*

$$-cont_T(k) + \mathfrak{S},$$

where $\mathfrak{S} = \dfrac{|\xi| + n}{N} + \dfrac{\mu q - \mu p}{2} - \dfrac{N}{2}$.

Proof. Let us $T \in Tab(\varphi^{\xi}_{n,p,\mu})$. Since T is a standard tableau, then T corresponds to a sequence $(\nu^{(k)})_{k=0}^{k=n}$ of Young diagrams, where

$$\begin{aligned}
&\nu^{(0)} = \xi,\\
&\nu^{(1)} = \xi \cup T(\{1\}),\\
&\nu^{(2)} = \xi \cup T(\{1,2\}),\\
&\cdots\\
&\nu^{(n)} = \xi \cup T(\{1,2,\cdots,n\}),
\end{aligned}$$

where $T(\{1,\cdots,k\})$ is the collection of cells filled by numbers $1,\cdots,k$, i.e. the Young diagram $\nu^{(k)}$ is formed by adding the cells filled by numbers $1,\cdots,k$ to the Young diagram ξ. So it follows, for $k = 1,\cdots,n$,

$$v_T \in (V^{\xi} \otimes V^{\otimes k})[\nu^{(k)}] \otimes V^{\otimes(n-k)},$$

where $(V^{\xi} \otimes V^{\otimes k})[V^{\nu^{(k)}}]$ denotes the V^{ν_k}-isotopic component of the tensor product $V^{\xi} \otimes V^{\otimes k}$.

By the previous subsection 2.5.1, it follows that the term $\sum_{s,t}(\Delta^{(k-1)}(E^t_s))_{(0,k)} \otimes (E^s_t)_k$ acts on v_T by

$$\frac{C_{(0,k+1)} - C_{(0,k)} \otimes 1_k - 1_{(0,k)} \otimes C_k}{2}.$$

Moreover, the Casimir element acts on the highest weight module V^{ν} by the scalar $\langle \nu, \nu + 2\rho \rangle$, where the weight $2\rho = \sum_{i=1}^{N}(N - 2i + 1)\epsilon_i$. So for each k such that $1 \le k \le N$, $C_{(0,k+1)}$ acts on $V^{\nu^{(k)}}$ by the scalar $\langle \nu^{(k)}, \nu^{(k)} + 2\rho \rangle$, $C_{(0,k)}$ acts on $V^{\nu^{(k-1)}}$ by the scalar $\langle \nu^{(k-1)}, \nu^{(k-1)} + 2\rho \rangle$ and C_k acts on V by the scalar $\langle \epsilon, \epsilon + 2\rho \rangle = N$, namely

$$\frac{C_{(0,k+1)} - C_{(0,k)} \otimes 1_k - 1_{(0,k)} \otimes C_k}{2}$$

acts by

$$\frac{1}{2}(\langle \nu^{(k)}, \nu^{(k)} + 2\rho \rangle - \langle \nu^{(k-1)}, \nu^{(k-1)} + 2\rho \rangle - \langle \epsilon, \epsilon + 2\rho \rangle).$$

Let $T(k)$ be the cell $(\mathfrak{i}(k),\mathfrak{j}(k))$, then $\nu^{(k)}_{\mathfrak{i}(k)}=\mathfrak{j}(k)=\nu^{(k-1)}_{\mathfrak{i}(k)}+1$ and $\nu^{(k)}_i=\nu^{(k-1)}_i$, for $i\neq \mathfrak{i}(k)$.

$$\begin{aligned}&\frac{1}{2}(\langle\nu^{(k)},\nu^{(k)}+2\rho\rangle-\langle\nu^{(k-1)},\nu^{(k-1)}+2\rho\rangle-\langle\epsilon,\epsilon+2\rho\rangle)\\=&\frac{1}{2}((\mathfrak{j}(k)+N-2\mathfrak{i}(k)+1)(\mathfrak{j}(k))-(\mathfrak{j}(k)+N-2\mathfrak{i}(k))(\mathfrak{j}(k)-1)-N)\\=&\mathfrak{j}(k)-\mathfrak{i}(k).\end{aligned}$$

Then the statement follows. □

THEOREM 2.5.4. *Let $F_{n,p,\mu}(V^\xi)$ denote the image of the irreducible GL_N-module V^ξ, for some $\xi\in P^+$, under Etingof-Freund-Ma functor. Then $F_{n,p,\mu}(V^\xi)$ has a basis indexed tableaux in $Tab_c^{\lambda,\mu}$, i.e. $\{v_T|T\in Tab_c^{\lambda,\mu}\}$. This basis is a weight basis with each basis vector v_T is a weight vector of weight $\zeta_T=-cont_T+\mathfrak{S}$. So $F_{n,p,\mu}(V^\xi)$ is a $\mathcal{Y}$-semisimple representation of $H_n(1,p-q-\mu N)$. Moreover, it is obvious different standard tableaux give different weights. Hence each weight space is one dimensional.*

2.6. Intertwining operators

2.6.1. Definition of intertwining operators.

DEFINITION 2.6.1. *For $i=1,\cdots,n-1$, define the intertwining operators*

$$\phi_i=[s_i,y_i],$$

and for γ_n, define

$$\phi_n=[\gamma_n,y_n].$$

PROPOSITION 2.6.2. *The intertwining operators ϕ_i satisfy the braid relations*

$$\phi_i\phi_{i+1}\phi_i=\phi_{i+1}\phi_i\phi_{i+1}, i=1,\cdots,n-1,$$

$$\phi_i\phi_j=\phi_j\phi_i, |i-j|>1,$$

$$\phi_{n-1}\phi_n\phi_{n-1}\phi_n=\phi_n\phi_{n-1}\phi_n\phi_{n-1}.$$

Since the operators ϕ_i's satisfy the same braid relations with s_i's and γ_n, it makes sense to define the following.

DEFINITION 2.6.3. *Let W denote the finite Weyl group of type C_n, for each $w \in W$, it has a reduced expression $w = s_{i_1} s_{i_2} \dots s_{i_m}$, $l(w) = m$, here we take the convention $s_n = \gamma_n$. Define*

$$\phi_w = \phi_{i_1} \phi_{i_2} \dots \phi_{i_m}.$$

2.6.2. Properties of intertwining operators.

Some computations on intertwining operators:

(1) $\phi_i = s_i(y_i - y_{i+1}) - 1$,

$\phi_n = 2\gamma_n y_n - \kappa_2$.

(2) $\phi_i^2 = (1 - y_i + y_{i+1})(1 + y_i - y_{i+1})$,

$\phi_n^2 = (\kappa_2 - 2y_n)(\kappa_2 + 2y_n)$.

DEFINITION 2.6.4. *Define the actions of W on weight $\zeta = [\zeta_1, \cdots, \zeta_n]$: for an arbitrary $w \in W$, the action of w is*

$$w.\zeta = \zeta \circ w^{-1},$$

where we take $\zeta_{-k} = -\zeta_k$.

THEOREM 2.6.5. *Let L be a $\mathcal{Y}$-semisimple module and L_ζ denote the weight space of weight ζ, then*

$$\phi_w L_\zeta \subset L_{w.\zeta}.$$

PROOF. It suffices to show the statement is true for each operator ϕ_i.

Case 1. When $1 \le i \le n-1$. We have the following facts that

$$y_i \phi_i = \phi_i y_{i+1},$$

$$y_{i+1} \phi_i = \phi_i y_i,$$

and

$$y_j\phi_i = \phi_i y_j, j \neq i \text{ or } i+1.$$

Case 2. Consider ϕ_n. We have facts that

$$y_n\phi_n = -\phi_n y_n,$$

$$y_j\phi_n = \phi_n y_j, j \neq n.$$

$\square$

REMARK 2.6.6. *Since each weight space of $F_{n,p,\mu}(V^{\xi})$ is one dimensional, so the action of ϕ_i is either 0 or an isomorphism.*

LEMMA 2.6.7. *If $\zeta_i - \zeta_{i+1} \neq \pm 1$ for some $i \in \{1, 2, \cdots, n-1\}$, then $\phi_i v_\zeta \neq 0$, where v_ζ is the weight vector of the weight ζ.*

PROOF. Suppose that $\phi_i v_\zeta = 0$. Then $\phi_i^2 v_\zeta = 0$. By the computation above

$$\phi_i^2 = (1 - y_i + y_{i+1})(1 + y_i - y_{i+1}).$$

Then $\phi_i^2 v_\zeta = (1 - \zeta_i + \zeta_{i+1})(1 + \zeta_i - \zeta_{i+1})v_\zeta = 0$. Then we have that $\zeta_i - \zeta_{i+1} = \pm 1$. $\square$

Similarly, we have the following fact.

LEMMA 2.6.8. *If $\zeta_n \neq \pm\frac{\kappa_2}{2}$, then $\phi_n v_\zeta \neq 0$, where v_ζ is the weight vector of the weight ζ.*

PROOF. Suppose that $\phi_n v_\zeta = 0$. Then $\phi_n^2 v_\zeta = 0$. By the computation above

$$\phi_n^2 = (\kappa_2 - 2y_n)(\kappa_2 + 2y_n).$$

Then $\phi_n^2 v_\zeta = \phi_n^2 = (\kappa_2 - 2\zeta_n)(\kappa_2 + 2\zeta_n)v_\zeta = 0$. Then we have that $\zeta(n) = \pm\frac{\kappa_2}{2}$. $\square$

2.6.3. Properties of irreducible $\mathcal{Y}$-semisimple representations. Let L be an irreducible $\mathcal{Y}$-semisimple representation of $H_n(\kappa_1, \kappa_2)$. Let $\zeta = [\zeta_1, \cdots, \zeta_n]$ is a weight L.

THEOREM 2.6.9. *If $\zeta_i = \zeta_{i+1}$ for some $1 \leq i \leq n-1$, then $L_\zeta = 0$.*

Proof. Let ζ be a weight such that $\zeta_i = \zeta_{i+1}$. Suppose there exists a nonzero element $v \in L_\zeta$. Consider the vector $s_i v$. Since $\phi_i = s_i(y_i - y_{i+1}) - 1 = (y_{i+1} - y_i)s_i + 1$, we have $\phi_i v = -v$.

$$
\begin{aligned}
(y_i - y_{i+1})s_i v =&(1-\phi_i)v \\
=&2v \neq 0.
\end{aligned}
$$

And act again by $y_i - y_{i+1}$,

$$
\begin{aligned}
&(y_i - y_{i+1})^2 s_i v \\
=&2(y_i - y_{i+1})v = 0.
\end{aligned}
$$

This means $s_i v$ belongs to the generalized eigenspace of $y_i - y_{i+1}$ and does not belong to the eigenspace of $y_i - y_{i+1}$, which contradicts $\mathcal{Y}$-semisimplicity. $\square$

Theorem 2.6.10. *Let $\kappa_2 \neq 0$. If $\zeta_n = 0$, then $L_\zeta = 0$.*

Proof. Let ζ be a weight such that $\zeta_n = 0$. Suppose there exists a nonzero element $v \in L_\zeta$. Consider the vector $\gamma_n v$. Since $\phi_n = 2\gamma_n y_n - \kappa_2 = -2y_n\gamma_n + \kappa_2$, we have $\phi_n v = -\kappa_2 v$.

$$
\begin{aligned}
2y_n\gamma_n v =&(\kappa_2 - \phi_n)v \\
=&2\kappa_2 v \neq 0.
\end{aligned}
$$

Act again by y_n, we have

$$
\begin{aligned}
&2{y_n}^2\gamma_n v \\
=&2\kappa_2 y_n v = 0.
\end{aligned}
$$

his means $s_i v$ belongs to the generalized eigenspace of y_n and does not belong to the eigenspace of y_n, which contradicts $\mathcal{Y}$-semisimplicity. $\square$

Remark 2.6.11. *When $\kappa_2 = 0$, it is possible for an irreducible $\mathcal{Y}$-semisimple module L to contain a nonzero weight space L_ζ with $\zeta_n = 0$. In this case, $\gamma_n v \in \mathbb{C}v$. Otherwise, the vector $v + \gamma_n v$ generalizes a nonzero proper submodule of L, which contradicts the irreducibility.*

LEMMA 2.6.12. *For any arbitrary $w \in W$, the intertwining operator*

$$\phi_w = w\Pi_{\alpha_{ij}\in R(w)}(y_i - y_j) + \sum_{x<w} xQ(y),$$

where $Q(y)$ is a polynomial of $y_1, \cdots, y_n$.

THEOREM 2.6.13. *Let ζ be a weight of L such that $L_\zeta \neq 0$. Let v be a nonzero weight vector in L_ζ. Then the set $\{\phi_w v | w \in W\}$ spans the irreducible representation L.*

PROOF. We need to show $w.v$ lies in the span of $\{\phi_w v | w \in W\}$ for any arbitrary $w \in W$. We prove by induction on the length of w. When the length of w is zero, the statement is trivial. Now assume for w with $l(w) < k$, the statement holds, i.e. $w.v$ can be expressed by a linear combination of elements in $\{\phi_w v | w \in W\}$. Set w is of length k and $w = s_{i_1} \cdots s_{i_k}$. Then by Lemma 2.6.12, we have $\phi_w \cdot v = \Pi_{\alpha_{ij}\in R(w)}(\zeta_i - \zeta_j) \cdot w \cdot v + \Sigma_{x<w} c_x x \cdot v$. Since $l(x) < k$, the terms $x \cdot v$ can be express by $\{\phi_w v | w \in W\}$. As long as the coefficient $\Pi_{\alpha_{ij}\in R(w)}(\zeta_i - \zeta_j) \neq 0$, $w \cdot v$ can be express by $\{\phi_w v | w \in W\}$. So it is reduced to consider only the case when $\Pi_{\alpha_{ij}\in R(w)}(\zeta_i - \zeta_j) = 0$.
In this case, there exists $p \in [1, k]$ such that

$$\Pi_{\alpha_{ij}\in R(s_{i_{p+1}} \cdots s_{i_k})}(\zeta_i - \zeta_j) \neq 0$$

and

$$\Pi_{\alpha_{ij}\in R(s_{i_p} \cdots s_{i_k})}(\zeta_i - \zeta_j) = 0.$$

Set $u = s_{i_{p+1}} \cdots s_{i_k}$. When $i_p \in [1, n-1]$, this implies $(y_{i_p} - y_{i_{p+1}})\phi_u v = 0$ and hence $\phi_u v = 0$ by Theorem 2.6.9. And when $i_p = n$, this implies $2y_n \phi_u v = 0$ and hence $\phi_u v = 0$ by Theorem 2.6.10. It follows $\Pi_{a_{ij}\in R(u)}(\zeta_i - \zeta_j)u.v = \sum_{x<u} xQ(y)v$ and hence

$$\Pi_{a_{ij}\in R(u)} w.v = \sum_{x<u} s_{i_1} \cdots s_{i_p} xQ(y).$$

Since $l(s_{i_1} \cdots s_{i_p} x) < k$, then $(s_{i_1} \cdots s_{i_p} x).v$ and hence $w.v$ can be expressed by a linear combination of elements in $\{\phi_w v | w \in W\}$. $\square$

THEOREM 2.6.14. *Let ζ be a weight such that $L_\zeta \neq 0$. Let $w \neq 1 \in W$ such that $w.\zeta = \zeta$. Then $\phi_w v = 0$.*

Proof. Let $w = s_{i_1} \cdots s_{i_k}$. since $w.\zeta = \zeta$, there is $1 \leq p \leq k$ such that $s_{i_1} \cdots s_{i_p} = (hm)$ where $\zeta_h = \zeta_m$. Consider $\phi_{i_{p-1}} \cdots \phi_{i_1} \phi_w v = \Pi_{1 \leq j \leq p}(1 - \zeta_{i_j} + \zeta_{i_j+1})(1 + \zeta_{i_j} - \zeta_{i_j+1})\phi_u v$. It follows $\phi_u v = 0$ and hence $\phi_w v = 0$. □

Corollary 2.6.15. *Let ζ be a weight such that $L_\zeta \neq 0$. Then it follows $dim(L_\zeta) = 1$.*

Proposition 2.6.16. *(1) Let v be a nonzero weight vector of weight ζ such that*

$$|\zeta_i - \zeta_{i+1}| = 1.$$

Then $\phi_i v = 0$.

(2) Let v be a nonzero weight vector of weight ζ such that $\zeta_n = \pm\frac{\kappa_2}{2}$. Then $\phi_n v = 0$.

Remark 2.6.17. *Some similar results also happen in degenerate affine Hecke algebra of type A_{n-1}. Let $H_n(1)$ be the degenerate affine Hecke algebra generated by $s_i(i = 1, \cdots n-1)$ and $y_i(i = 1 \cdots n)$ with the following relations:*

$$s_i^2 = 1, i = 1, \cdots, n-1,$$

$$s_i s_j = s_j s_i, |i-j| > 1,$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, i = 1, \cdots, n-1,$$

$$y_i y_j = y_j y_i,$$

$$s_i y_i - y_{i+1} s_i = 1,$$

$$s_i y_j = y_j s_i, j \neq i, i+1.$$

There is the same definition of $\mathcal{Y}$-semisimple representation. And for any $\mathcal{Y}$-semisimple representation M, if a weight ζ with $\zeta_i = \zeta_{i+1}$, then $M_\zeta = 0$.
*Furthermore, we still define the intertwining operator $\phi = s_i y_i - y_i s_i$, then we will also have $\phi_i^2 = (1 - y_i + y_{i+1})(1 + y_i - y_{i+1})$. This also implies the fact that if $\phi_i v_\zeta = 0$ then we have $\zeta_i - \zeta_{i+1} = \pm 1$. For the double affine Hecke algebra of type A, [**19**] explored similar properties in details.*

2.7. Combinatorial moves

2.7.1. Moves among standard tableaux.

Let $Tab_c^{\lambda,\mu}$ denote the collection of standard tableaux indexing the basis of $F_{n,p,\mu}(V^{\xi})$ in Section 2.4. We define a set of moves $\mathfrak{m}_1, \cdots, \mathfrak{m}_n$ on $Tab_c^{\lambda,\mu} \sqcup \{\mathfrak{o}\}$ as follows. The move $\mathfrak{m}_i$ for $i-1, \cdots, n-1$ is defined as

$$\mathfrak{m}_i(T) = \begin{cases} T', & T' \text{ is a standard tableau} \\ \mathfrak{o}, & \text{otherwise,} \end{cases}$$

where $T'(k) = T(s_i(k))$. The move $\mathfrak{m}_n$ is defined to be

$$\mathfrak{m}_n \cdot T = \begin{cases} \mathfrak{o}, & \mathfrak{i}(n) \leq max(p,q) \text{ and } \mathfrak{j}(n) \leq max(a,b) \\ T'', & \text{otherwise,} \end{cases}$$

where $T''(j) = T(j)$ for each $j \neq n$ and $T''(n) = (N - \mathfrak{i}(n) + 1, a + b - \mathfrak{j}(n) + 1)$.

Remark 2.7.1. *There is a straightforward observation. For any shape $\varphi' \in D(\varphi^{\xi}_{n,p,\mu})$ and any $i \leq min(p,q)$, the sum of the column number of the last cell of the i-th row and the column number of the last cell of the $(N-i+1)$-th row equal $a+b$. So $T''(n) = (N - \mathfrak{i}(n) + 1, a + b - \mathfrak{j}(n) + 1)$ means that the $\mathfrak{m}_n$-move takes the cell filled by n to the end of the $(N - \mathfrak{i}(n) + 1)$-th row.*

here m be the column number of the last cell of the $(N - \mathfrak{i}(n) + 1)$-th row of $Im(T)$.

2.7.2. Correspondence between algebraic actions and combinatorial moves.

Let v_T denote the basis vector indexed by $T \in Tab_c^{\lambda,\mu}$ and ζ_T denote the weight of v_T, namely $\zeta_T = -cont_T + \mathfrak{S}$.

Proposition 2.7.2. *(1) For $i = 1, \cdots, n-1$, if $\mathfrak{m}_i(T) \neq \mathfrak{o}$ holds, then $\mathfrak{m}_i(T) \in Tab_c^{\lambda,\mu}$ and the common eigenbasis vector $v_{\mathfrak{m}_i(T)}$ is of weight $\zeta_{\mathfrak{m}_i(T)} = s_i.\zeta_T$.*

(2) If $\mathfrak{m}_n(T) \neq \mathfrak{o}$, then $\mathfrak{m}_n(T) \in Tab_c^{\lambda,\mu}$ and the common eigenbasis vector $v_{\mathfrak{m}_n(T)}$ is of weight $\zeta_{\mathfrak{m}_i(T)} = \gamma_n.\zeta_T$

Proof. First, for $i = 1, \cdots, n-1$, if $\mathfrak{m}_i(T) \neq \mathfrak{o}$, then by the definition of the move $\mathfrak{m}_i$, $T \in Tab_c^{\lambda,\mu}$ and we want to show $\zeta_{\mathfrak{m}_i(T)} = s_i.\zeta_T$.

Then let us consider the case when $w = \gamma$. In this case w moves the box filled by n in the $\mathfrak{i}$-th row of tableau T to the end of the $(N - \mathfrak{i} + 1)$-th row. So the only box in the new tableau

$\gamma.T$ with a different position comparing with the tableau T is the box filled by n. Thus the only difference in the new weight associated to $\gamma.T$ comparing with ζ_T is the eigenvalue of y_n. Let $(\mathfrak{i},\mathfrak{j})$ denote the coordinates of the box filled by n in the tableau T. Then the coordinates of the box filled by n in the new tableau $\gamma.T$ is $(N-\mathfrak{i}+1, \mu(q-p)+2\frac{|\xi|+n}{N}-\mathfrak{j}+1)$. Then the eigenvalue of y_n in the new weight $\zeta_{\gamma.T}$ associated to $\gamma.T$ is $\mathfrak{j}-\mathfrak{i}-\frac{|\xi|+n}{N}+\frac{N}{2}+\frac{\mu(p-q)}{2}$. So the new weight equals $\gamma.\zeta_T$.

□

Proposition 2.7.3. *If $w.T \neq 0$ for some $w \in W$, then $\phi_w v_T \neq 0$.*

Proof. It is enough to verify the statement when w is the transposition s_i or γ_n.
First, consider the case when $w = s_i$, $i = 1, \cdots, n-1$. Suppose $\phi_i v_T = 0$ for some $1 \leq i \leq n-1$ implies that $\phi_i^2 v_T = 0$ and $\phi_i^2 = (1-y_i+y_{i+1})(1+y_i-y_{i+1})$. Then $\zeta_T(i) - \zeta_T(i+1) = \pm 1$. In this case the contents of boxes filled by i and $i+1$ differ by 1 and hence the two boxes are adjacent and in the same row or in the same column. We have $s_i.T = 0$ in this case. This contradicts the condition. So we have $\phi_i v_T \neq 0$.
Second, consider the case when $w = \gamma_n$. Suppose $\phi_n v_T = 0$ which implies the eigenvalue of y_n is $\pm\frac{\kappa_2}{2}$. Since $\phi_n^2 v_T = 0$ in this case and $\phi_n^2 = (\kappa_2 - 2y_n)(\kappa_2 + 2y_n)$. Then the box filled by n is either $(p, \mu q + \frac{|\xi|+n}{N})$ or $(q, -\mu p + \frac{|\xi|+n}{N})$. But by the definition of action of γ_n on the tableau T, we have in both cases that $\gamma_n.T = 0$. This contradicts the condition. Hence we have that $\phi_n v_T \neq 0$. □

Remark 2.7.4. *(1) If $\mathfrak{m}_i(T) \neq \mathfrak{o}$, then $\phi_i v_T = c v_{\mathfrak{m}_i(T)}$ up to a nonzero scalar $c \in \mathbb{C}$ for $i = 1, \cdots, n$.*
(2) If $\mathfrak{m}_i(T) = \mathfrak{o}$, then $\phi_i v_T = 0$ for $i = 1, \cdots, n$.

Example 2.7.5. *In Example 2.4.9, the action of intertwining operators are as follows. The diagonals give the eigenvalue of y_i's.*

Let k be the filling of the cell (q,b),we could compute that the eigenvalue of y_k is $-\frac{\kappa_2}{2}$. Similarly, let k be the filling of the cell (p,a), it follows the eigenvalue of y_k is $\frac{\kappa_2}{2}$. Furthermore, $\kappa_2 = p-q-a+b$.

2.8. Irreducible representations

2.8.1. The image $F_{n,p,\mu}(V^\xi)$ is irreducible.

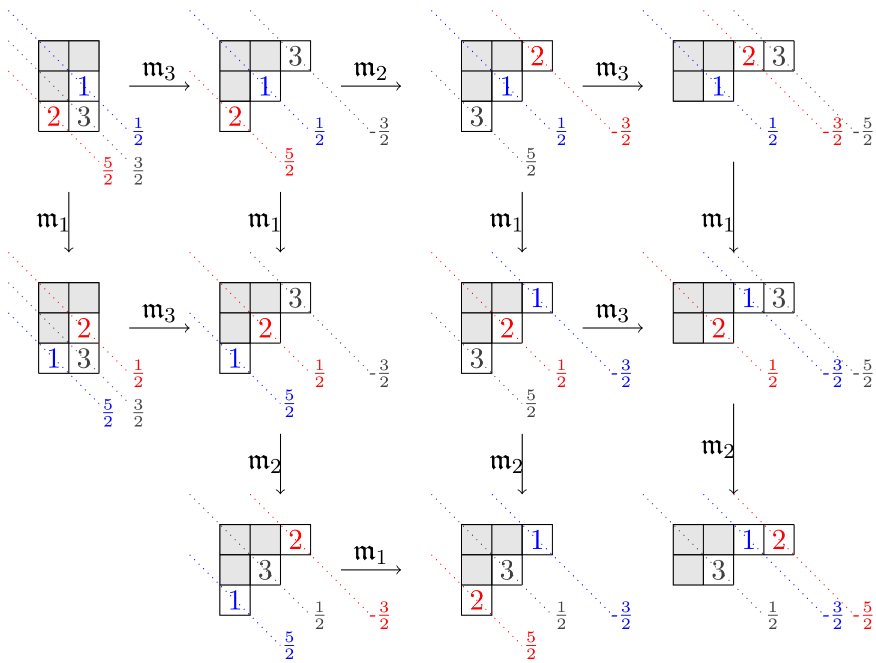


FIGURE 2.9. Moves among weight basis vectors of $F_{3,1,0}(V^{\xi})$

LEMMA 2.8.1. *Let φ_1 and φ_2 be two skew shapes in $D(\varphi)$ with $\varphi_1 \to \varphi_2$. Then there exist standard tableaux T_1 and T_2 with $Im(T_1) = \varphi_1$ and $Im(T_2) = \varphi_2$ such that $\gamma_n(T_1) = T_2$.*

PROOF. The $\varphi_1 \to \varphi_2$ implies that φ_2 is obtained by moving a corner (i, φ_i) of φ_1 to the end of the $(N - i + 1)$-th row of φ_1. Since (i, φ_1) is a corner of φ_1, there exists a standard tableau T_1 such that (i, φ_1) is filled by n. Applying the γ_n move to T_1, let $T_2 = \gamma_n(T_1)$. Then T_2 is a standard tableau with $Im(T_2) = \varphi_2$. $\square$

We show in the following the representation of degenerate affine Hecke algebra obtained through Etingof-Freund-Ma functor is irreducible.

THEOREM 2.8.2. *The image $F_{n,p,\mu}(V^{\xi})$ of a finite dimensional irreducible $\mathfrak{gl}_N$-module V^{ξ} under the Etingof-Freund-Ma functor is irreducible.*

PROOF. A basis of $F_{n,p,\mu}(V^{\xi})$ is indexed by

$$\mathcal{T}^{\xi}_{\mu,p} = \{T | T \text{ is a standard tableau and } Im(T) \in D(\varphi^{\xi}_{n,p,\mu})\}.$$

It's obvious to see that the underlying vector space of $F_{n,p,\mu}(V^{\xi})$ is isomorphic to the vector space $span_{\mathbb{C}}\{v_T | T \in \mathcal{T}^{\xi}_{\mu,p}\}$. Let N be a submodule of $F_{n,p,\mu}(V^{\xi})$. Then N contains at least one weight vector of $F_{n,p,\mu}(V^{\xi})$. Let v_T be a weight vector associated to the tableau $T \in \mathcal{T}^{\xi}_{\mu,p}$ and the submodule N contains v_T.

We show in the following we get every other weight vector from an arbitrary weight vector v_T. Consider the actions of signed permutations on standard tableaux since the actions of signed permutations on standard tableaux are compatible with the actions of intertwining operators on weight vectors.

Case 1. For any the standard tableau T' with the same shape of the tableau T, there exists $w \in S_n$ such that $T' = w.T$. Equivalently $v_{T'} = c\phi_{\omega} v_T$ where $c \in \mathbb{C}$ is nonzero.

Case 2. For standard T_1 and T_2 with $Im(T_1) \to Im(T_2)$, combining Proposition 2.7.2 and Case 1, it follows $T_2 = \omega(T_1)$ for some $\omega \in W(BC_n)$ and hence $v_{T_2} = c\phi_{\omega} v_{T_1}$ where $c \in \mathbb{C}$ is nonzero.

Furthermore, consider two arbitrary standard tableaux T_1 and T_2 in $\mathcal{T}^{\xi}_{\mu,p}$. Let T be a standard tableaux of shape φ. There is a path $\varphi \to \varphi_1 \to \cdots \to Im(T_1)$ and hence $v_{T_1} = c_1\phi_{\omega} v_{T_0}$.

□

2.8.2. Irreducible representation associated to a skew shape $\varphi^{\xi}_{n,p,\mu}$. Define a representation $L^{\varphi^{\xi}_{n,p,\mu}}$ of $H_n(1, p-q-\mu N)$ as follows. Let the underlying vector space be

$$span_{\mathbb{C}}\{w_T | T \in \mathcal{T}^{\xi}_{\mu,p}\}.$$

The action of $H_n(1, p-q-\mu N)$ is defined by

$$y_k w_T = (-cont_T(k) + \mathfrak{S}) w_T, \tag{2.24}$$

$$s_i w_T = \frac{(1 - cont_T(i) + cont_T(i+1)) w_{s_i(T)}}{cont_T(i) - cont_T(i+1)} + \frac{1}{cont_T(i) - cont_T(i+1)} w_T, \tag{2.25}$$

$$\gamma_n w_T = \frac{(p-q-\mu N - 2cont_T(n)) w_{\gamma_n(T)}}{2cont_T(n)} + (p-q-\mu N)\frac{1}{2cont_T(n)} w_T. \tag{2.26}$$

THEOREM 2.8.3. *The representation $F_{n,p,\mu}(V^{\xi})$ is isomorphic to $L^{\varphi^{\xi}_{n,p,\mu}}$.*

Proof. Fix a $T \in \mathcal{T}^{\xi}_{\mu,p}$. Define a map $f : F_{n,p,\mu}(V^{\xi}) \to L^{\varphi^{\xi}_{n,p,\mu}}$ by

$$f(v_T) = w_t$$

and $f(\phi_i v_T) = (1 - cont_T(i) + cont_T(i+1))w_{s_i(T)}$. $\square$

2.9. Combinatorial description

In this section, we first discuss some properties of a representation of the degenerate affine Hecke algebra $H_n(1, \kappa_2)$ obtained via the Etingof-Freund-Ma functor, where $\kappa_2 = p - q - \mu N$, and then we show that any representation satisfying these properties is the image of some irreducible polynomial representation of GL_N via the Etingof-Freund-Ma functor.

2.9.1. Some facts of $F_{n,p,\mu}(V^{\xi})$. Let $\xi \in P^+$ and $F = F_{n,p,\mu}(V^{\xi})$ be a representation $H_n(1, p - q - \mu N)$ obtained through Etingof-Freund-Ma functor and $\zeta = (\zeta_1, \cdots, \zeta_n)$ be weight of F such that $F_\zeta \neq 0$. For $i = 1, \cdots, n$, if there is an increasing sequence $i = i_0 < i_1 < \cdots < i_m \leq n$ such that $|\zeta_{i_k} - \zeta_{k+1}| = 1$ for $k = 0, \cdots, m-1$ and $\zeta_{i_m} = \pm\frac{\kappa_2}{2}$, then we call the coordinate ζ_i is fixed. It is easy to observe the following two properties.

Property 2.9.1. *For $i = 1, \cdots, n$, if $|\zeta_i| \leq |\frac{\kappa_2}{2}|$, then ζ_i is fixed, i.e. there is an increasing sequence $i = i_0 < i_1 < \cdots < i_m \leq n$ such that $|\zeta_{i_k} - \zeta_{k+1}| = 1$ for $k = 0, \cdots, m-1$ and $\zeta_{i_m} = \pm\frac{\kappa_2}{2}$.*

Property 2.9.2. *The parameter κ_2 is an integer. If κ_2 is even, then all ζ_i's, for $i = 1, \cdots, n$, are integers. If κ_2 is odd, then all ζ_i's, for $i = 1, \cdots, n$ are half integers.*

Recall that the the cell (p, a) in $\varphi^{\xi}_{n,p,\mu}$ gives the eigenvalue $\frac{\kappa_2}{2}$ and that the cell (q, b) gives the eigenvalue $-\frac{\kappa_2}{2}$. Then Property 2.9.2 follows.

In [**17**], Ram explored the facts of weights of a semisimple affine Hecke algebra representation. Now let us explore facts of weights in the degenerate case. Let L be an irreducible and $\mathcal{Y}$-semisimple representation of $H_n(1, \kappa_2)$ satisfying Property 2.9.1 and Property 2.9.2 above and ζ be a weight such that $L_\zeta \neq 0$. Then ζ satisfies the following proposition.

PROPOSITION 2.9.3. *If there exist $1 \leq i < j \leq n$ such that $\zeta_i = \zeta_j$, then there exist $i < k_1 < j$ such that $\zeta_{k_1} = \zeta_i + 1$ and $i < k_2 < j$ such that $\zeta_{k_2} = \zeta_i - 1$.*

PROOF. Let ζ be a weight such that $L_\zeta \neq 0$. Suppose there exist $1 \leq i < j \leq n$ such that $\zeta_i = \zeta_j$ and there is no $i < k < j$ such that $\zeta_k = \zeta_i$. We proof by induction on $j - i$.
First, if $j - i = 1$, then $\zeta_i = \zeta_{i+1}$ which contradicts Theorem 2.6.9.
Second, if $j - i = 2$, by Theorem 2.6.9 and Lemma 2.6.7, it follows $\zeta_{i+1} = \zeta_i \pm 1 = \zeta_{i+2} \pm 1$. Let v be a nonzero weight vector of weight ζ. Proposition 2.6.16 implies $\phi_i v = \phi_{i+1} v = 0$. Combining the definition of the intertwining operators, it follows $s_i v = \mp v$ and $s_{i+1} v = \pm v$ and hence

$$\pm v = s_i s_{i+1} s_i v = s_{i+1} s_i s_{i+1} v = \mp v,$$

which is a contradiction.
So the base case of the induction is $j - i = 3$. If $\zeta_i \neq \zeta_{i+1} \pm 1$ or $\zeta_{j-1} \neq \zeta_j \pm 1$. Lemma 2.6.7 implies the existence of a weight satisfying the condition in the case $j - i = 2$, which is a contradiction. So it hold $|\zeta_i - \zeta_{i+1}| = 1$ and $|\zeta_{j-1} - \zeta_j| = 1$. If $\zeta_i = \zeta_{i+1} + 1$ and $\zeta_{j-1} = \zeta_j + 1$, then $k_1 = j - 1$ and $k_2 = i + 1$. Similarly, if $\zeta_i = \zeta_{i+1} - 1$ and $\zeta_{j-1} = \zeta_j - 1$, then $k_1 = i + 1$ and $k_2 = j - 1$. If $\zeta_i = \zeta_{i+1} \pm 1$ and $\zeta_{j-1} = \zeta_j \mp 1$, then $\zeta_{i+1} = \zeta_{i+2}$ which contradicts Theorem 2.6.9.
Suppose the statement is true for all $j - i < m$, consider the case $j - i = m$.
Case1. If $|\zeta_i - \zeta_{i+1}| \neq 1$ or $|\zeta_{j-1} - \zeta_j| \neq 1$ and v is a nonzero weight vector of weight ζ, then $\phi_i v$ (or $\phi_{j-1} v$ respectively) is a nonzero weight vector of weight $s_i \zeta$ (or $s_{j-1}\zeta$ respectively) with $s_i \zeta$ (or $s_{j-1}\zeta$ respectively) has $\zeta_{i+1} = \zeta_j$ (or $\zeta_i = \zeta_{j-1}$ respectively). Then the k_1 and k_2 exist by the inductive hypothesis.
Case 2. If $\zeta_i = \zeta_{i+1} \pm 1$ and $\zeta_{j-1} = \zeta_j \mp 1$, this implies $\zeta_{i+1} = \zeta_{j-1}$, the statement still holds by inductive hypothesis.
Case 3. If $\zeta_i = \zeta_{i+1} + 1$ and $\zeta_{j-1} = \zeta_j + 1$, then $k_1 = j - 1$ and $k_2 = i + 1$.
Case 4. If $\zeta_i = \zeta_{i+1} - 1$ and $\zeta_{j-1} = \zeta_j - 1$, then $k_1 = i + 1$ and $k_2 = j - 1$.

□

Next let us explore another fact of L.

LEMMA 2.9.4. *Let $\zeta = [\zeta_1, \cdots, \zeta_n]$ be a weight of L such that $L_\zeta \neq 0$ and ζ satisfies $\zeta_i > \frac{|\kappa_2|}{2}$ for $i = k, \cdots, n$. Then there is weight*

$$\zeta' = [\zeta_1, \cdots, \zeta_{k-1}, -\zeta_n, -\zeta_{n-1}, \cdots, -\zeta_{k+1}, -\zeta_k]$$

such that $L_{\zeta'} \neq 0$.

PROOF. Let v be a nonzero weight vector of ζ. Consider the element

$$h = \phi_n(\phi_{n-1}\phi_n)\cdots(\phi_k\phi_{k+1}\cdots\phi_n),$$

then $hv \in L_{\zeta'}$ and $hv \neq 0$ by Lemma 2.6.7 and Lemma 2.6.8. □

DEFINITION 2.9.5. *Let $\zeta = [\zeta_1, \cdots, \zeta_n]$ be a weight of L such that $L_\zeta \neq 0$ and ζ satisfies the condition: if a coordinate $\zeta_i > 0$, then ζ_i is fixed, i.e. there exists an increasing sequence $i = i_0 < i_1 < \cdots < i_m \leq n$ such that $|\zeta_{i_k} - \zeta_{i_{k+1}}| = 1$ and $\zeta_{i_m} = \pm\frac{\kappa_2}{2}$. Then we call ζ is a minimal weight of L.*

PROPOSITION 2.9.6. *There exists at least one minimal weight $\zeta = [\zeta_1, \cdots, \zeta_n]$ of L such that $L_\zeta \neq 0$.*

PROOF. Let ζ be any weight such that $L_\zeta \neq 0$. If $0 < \zeta_i \leq \frac{|\kappa_2|}{2}$, then ζ_i is fixed since L satisfies Property 2.9.1. So it suffices to consider the coordinate $\zeta_i > \frac{|\kappa_2|}{2}$. We want to show that starting with any weight ζ such that $L_\zeta \neq 0$, there is an algorithm to obtain a weight ζ' such that $L_{\zeta'} \neq 0$ and ζ' satisfies the condition: if a coordinate $\zeta'_i > 0$, then ζ'_i is fixed.

Suppose $\{\zeta_{r_1}, \zeta_{r_2}, \cdots, \zeta_{r_l}\}$ is the collection of all the coordinates such that $\zeta_{r_i} > \frac{|\kappa_2|}{2}$ and ζ_{r_i} is not fixed, for $1 \leq r_1 < r_2 < \cdots < r_l \leq n$. Let v be a nonzero weight vector of weight ζ. We start with the rightmost coordinate ζ_{r_l} in this collection. If $r_l \neq n$, there are only the following two cases.

Case 1. There exists an increasing sequence $r_l+1 = j_0 < j_1 < \cdots < j_l \leq n$ such that $|\zeta_{j_{k+1}} - \zeta_{j_k}| = 1$ and $\zeta_{j_l} = \pm\frac{\kappa_2}{2}$. Then $|\zeta_{r_l} - \zeta_{r_l+1}| \neq 1$, otherwise there is an increasing sequence $r_l = j_{-1} < j_1 < j_1 < \cdots < j_l \leq n$ such that $|\zeta_{j_{k+1}} - \zeta_{j_k}| = 1$ and $\zeta_{j_l} = \pm\frac{\kappa_2}{2}$. So $\phi_{r_l}v$ is a nonzero vector of weight $\zeta^{(1)} = s_{r_l}\zeta$.

Case 2. If $\zeta_{r_l+1} < -\frac{|\kappa_2|}{2}$, then $|\zeta_{r_l} - \zeta_{r_l+1}| > 1$ and hence $\phi_{r_l}v$ is a nonzero weight vector of weight

$\zeta^{(1)} = s_{r_l}\zeta$.

Then we consider $\zeta^{(1)}_{r_l+1}$ and we are in the same situation. Hence we repeat this process for $(n - r_l)$ times and obtain a nonzero weight vector $(\phi_{n-1} \cdots \phi_{r_l+1}\phi_{r_l})v$ of weight

$$\zeta^{(n-r_l)} = (s_{n-1} \cdots s_{r_l+1}s_{r_l})\zeta.$$

Next, we deal with the second rightmost coordinate $\zeta_{r_{l-1}} = \zeta^{(n-r_l)}_{r_{l-1}}$ in the collection above and repeat the process above for $(n - 1 - r_{l-1})$ times. We obtain a nonzero weight vector

$$(\phi_{n-2} \cdots \phi_{r_{l-1}+1}\phi_{r_{l-1}})(\phi_{n-1} \cdots \phi_{r_l+1}\phi_{r_l})v$$

of weight

$$\zeta^{(2n-1-r_{l-1}-r_l)} = (s_{n-2} \cdots s_{r_{l-1}+1}s_{r_{l-1}})(s_{n-1} \cdots , s_{r_l+1}s_{r_l})\zeta.$$

Next, we continue to deal with other coordinates in the collection in the order of $\zeta_{r_{l-2}}, \zeta_{r_{l-3}}, \cdots , \zeta_{r_1}$ and repeat the process for $(n - k - r_k)$ times for the coordinate ζ_{r_k} for $k = 1, \cdots , l$. We obtain a nonzero weight vector

$$(\phi_{n-l} \cdots \phi_{r_1+1}\phi_{r_1})(\phi_{n-l+1} \cdots \phi_{r_2+1}\phi_{r_2}) \cdots (\phi_{n-1} \cdots \phi_{r_l+1}\phi_{r_l})v$$

of weight

$$\zeta^{(ln-l(l-1)/2-r_1-r_2\cdots-r_l)} = (s_{n-l} \cdots s_{r_1+1}s_{r_1})(s_{n-l+1} \cdots s_{r_2+1}s_{r_2}) \cdots (s_{n-1} \cdots s_{r_l+1}s_{r_l})\zeta.$$

The weight $\zeta^{(ln-l(l-1)/2-r_1-r_2\cdots-r_l)}$ satisfies the condition that

$$\zeta_i^{(ln-l(l-1)/2-r_1-r_2\cdots-r_l)} > \frac{|\kappa_2|}{2}$$

for $i = n - l + 1, \cdots , n$. Moreover, for $i = 1, \cdots , n - l$, it follows either

$$\zeta_i^{(ln-l(l-1)/2-r_1-r_2\cdots-r_l)} < 0$$

or the coordinate $\zeta_i^{(ln-l(l-1)/2-r_1-r_2\cdots-r_l)}$ is fixed. Applying Lemma 2.9.4, there is a weight

$$\zeta' = \gamma_n(s_{n-1}\gamma_n)\cdots(s_{n-l+1}\cdots s_{n-1}\gamma_n)\zeta^{(ln-l(l-1)/2-r_1-r_2\cdots-r_l)}$$

such that $L_{\zeta'} \neq 0$ and satisfying the condition: if

$$\zeta'_i > 0,$$

then ζ'_i is fixed for any $i = 1, \cdots, n$. □

REMARK 2.9.7. *Lemma 2.9.4 and Proposition 2.9.6 indicate that for any weight ζ such that $L_\zeta \neq 0$ and a nonzero $v \in L_\zeta$, there is a nonzero weight vector $\phi_\omega v \in L_{\zeta'}$ such that ζ' satisfies the condition in Proposition 2.9.6.*

EXAMPLE 2.9.8. *Let $\zeta = [-2, 2, 4, 5, 6, -3, 1]$ and $v \in L$ is a nonzero weight vector of weight ζ. Locate the collection of all the coordinates which are positive and not fixed: $\{\zeta_3 = 4, \zeta_4 = 5, \zeta_5 = 6\}$, i.e. there are three coordinates with $r_1 = 3, r_2 = 4$ and $r_3 = 5$. We deal with these coordinates from right to left. First, we deal with the rightmost coordinate $\zeta_5 = 6$ in this collection and apply the step for $(n - r_3) = 2$ times. We obtain a nonzero weight vector*

$$(\phi_{n-1}\cdots\phi_{r_3})v = (\phi_6\phi_5)v$$

of weight

$$\zeta^{(n-r_3)} = \zeta^{(2)} = (s_6 s_5)\zeta = [-2, 2, 4, 5, -3, 1, 6].$$

Then we work on with the coordinate $\zeta_4 = \zeta_4^{(2)} = 5$ and apply the step for $(n - 1 - r_2)$ times. We obtain a nonzero weight vector

$$(\phi_{n-2}\cdots\phi_{r_2})(\phi_{n-1}\cdots\phi_{r_3})v = (\phi_5\phi_4)(\phi_6\phi_5)v$$

of weight

$$\zeta^{(2n-1-r_1-r_2)} = \zeta^{(4)} = (s_5 s_4)\zeta^{(2)} = (s_5 s_4)(s_6 s_5)\zeta = [-2, 2, 4, -3, 1, 5, 6].$$

Finally, we deal with the coordinate $\zeta_3 = \zeta_3^{(4)} = 4$ *and apply the step for* $n-2-r_3$ *times. We obtain a nonzero weight vector*

$$(\phi_{n-3}\cdots\phi_{r_1})(\phi_{n-2}\cdots\phi_{r_2})(\phi_{n-1}\cdots\phi_{r_3})v = (\phi_4\phi_3)(\phi_5\phi_6)(\phi_6\phi_5)v$$

of weight

$$\zeta^{(3n-3-r_1-r_2-r_3)} = \zeta^{(6)} = (s_4s_3)\zeta^{(4)} = [-2,2,-3,1,4,5,6].$$

Now the weight $\zeta^{(6)}$ *satisfies the condition in Lemma 2.9.4 with* $\zeta_i^{(6)} > \frac{|\kappa_2|}{2}$ *for* $i = 5,6,7$*. Moreover, for each* $i = 1,\cdots,4$*, either* $\zeta_i^{(6)} < 0$ *or that* $\zeta_i^{(6)}$ *is fixed.*
Applying Lemma 2.9.4, we obtain a nonzero weight vector

$$\phi_7(\phi_6\phi_7)(\phi_5\phi_6\phi_7)(\phi_4\phi_3)(\phi_5\phi_6)(\phi_6\phi_5)v$$

of weight

$$\zeta' = \gamma_7(s_6\gamma_7)(s_5s_6\gamma_7)\zeta^{(6)} = [-2,2,-3,1,-6,-5,-4].$$

Example 2.9.9. *Let* $\zeta = [0,4,-1,6,-2,5,1]$ *and* $v \in L$ *is a nonzero weight vector of weight* ζ*. There are three coordinates* $\zeta_2 = 4$*,* $\zeta_4 = 6$ *and* $\zeta_6 = 5$ *satisfying the condition that* $i = 2,4,6$*, there is no increasing sequence* $i < i_1 < \cdots < i_l \le n$ *such that* $|\zeta_{i_{k+1}} - \zeta_{i_k}| = 1$ *and* $|\zeta_{i_l}| = \pm\frac{\kappa_2}{2}$*. Starting with the coordinate with maximal index* $i = 6$ *and applying the intertwining operators, it follows*

$$[0,4,-1,6,-2,5,1] \xrightarrow{s_6} [0,4,-1,6,-2,1,5] \xrightarrow{s_5s_4} [0,4,-1,-2,1,6,5] \xrightarrow{s_4s_3s_2} [0,-1,-2,1,4,6,5]$$

and by Lemma 2.9.4, it follows

$$[0,-1,-2,1,4,6,5] \xrightarrow{s_5s_6\gamma_7} [0,-1,-2,1,-5,4,6] \xrightarrow{s_6\gamma_7} [0,-1,-2,1,-5,-6,4] \xrightarrow{\gamma_7} [0,-1,-2,1,-5,-6,-4]$$

Let $\zeta' = [0,-1,-2,1,-5,-6,-4]$*. Then there is a nonzero weight vector*

$$\phi_7(\phi_6\phi_7)(\phi_5\phi_6\phi_7)(\phi_4\phi_3\phi_2)(\phi_5\phi_4)\phi_6 v \in L_{\zeta'}.$$

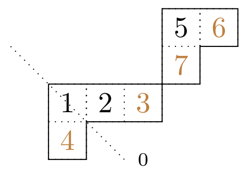
Remark 2.9.10. *For any minimal weight* ζ *of* $F = F_{n,p,\mu}(V^\xi)$ *such that* $F_\zeta \neq 0$*, let* T_ζ *be the corresponding standard tableau. Then* $Im(T_\zeta)$ *is the minimal shape* $\varphi^\xi_{n,p,\mu}$ *of* $F_{n,p,\mu}(V^\xi)$*.*

Before introducing the third property of $F_{n,p,\mu}(V^\xi)$, we need the following definition and lemma.

Definition 2.9.11. *Let $\zeta = [\zeta_1, \cdots, \zeta_n]$ be a weight. If a coordinate ζ_i, $i = 1, 2, \cdots, n$, satisfies the condition that there is no $i < k \leq n$ such that $\zeta_k = \zeta_i \pm 1$, then the coordinate ζ_i is a corner of ζ.*

Remark 2.9.12. *Let $\zeta = [\zeta_1, \cdots, \zeta_n]$ and T_ζ is the corresponding standard tableau. For $i = 1, \cdots, n$, ζ_i is a corner of ζ if and only if $T(i)$ is a southeastern corner of $Im(T_\zeta)$.*

Example 2.9.13. *Let $\zeta = [0, -1, -2, 1, -5, -6, -4]$. Then $\zeta_3 = -2$, $\zeta_4 = 1$, $\zeta_6 = -6$ and $\zeta_7 = -4$ are corners of ζ. The corresponding standard tableau T_ζ has southeastern corners $3, 4, 6$ and 7.*



Lemma 2.9.14. *Let L be an irreducible and $\mathcal{Y}$-semisimple representation of $H_n(1, \kappa_2)$ satisfying Property 2.9.1. Let ζ be a minimal weight of L such that $L_\zeta \neq 0$. For $i = 1, \cdots, n$, if the coordinate ζ_i is a corner of ζ, then $\zeta_i = \pm\frac{\kappa_2}{2}$ or $\zeta < -\frac{|\kappa_2|}{2}$.*

Proof. First, since L satisfies Property 2.9.1, if $|\zeta_i| < \frac{|\kappa_2|}{2}$, then ζ_i is fixed, i.e. there is an increasing sequence $i = i_0 < i_1 < \cdots < i_m \leq n$ such that $|\zeta_{i_k} - \zeta_{k+1}| = 1$ for $k = 0, \cdots, m-1$ and $\zeta_{i_m} = \pm\frac{\kappa_2}{2}$. This contradicts the fact that ζ_i is a corner of ζ.
Second, suppose $\zeta_i > \frac{|\kappa_2|}{2}$. Since ζ is a minimal weight, ζ_i if fixed, which again contradicts the fact that ζ_i is a corner. $\square$

Now we introduce the third property of $F_{n,p,\mu}(V^\xi)$.

Property 2.9.15. *Let ζ be a minimal weight such that $F_\zeta \neq 0$. If ζ_k is the rightmost coordinate equal to $\frac{|\kappa_2|}{2}$ and ζ_r is the rightmost coordinate equal to $-\frac{|\kappa_2|}{2}$, then at least one of these two coordinates is not a corner.*

Proof. Let T_ζ be the corresponding standard tableau of weight ζ. Since ζ is a minimal weight, the shape $Im(T_\zeta)$ is the minimal shape $\varphi = \varphi^\xi_{n,p,\mu}$. So it suffices to show that it is impossible for T_ζ to have $T_\zeta(k)$ and $T_\zeta(r)$ at southeastern corners simultaneously, equivalently, it is impossible for φ to have a southeast corner at eigenvalue $\frac{\kappa_2}{2}$ and a southeastern corner at eigenvalue $-\frac{\kappa_2}{2}$ simultaneously. Let $p \leq q$,

$$a = \mu q + \frac{|\xi| + n}{N}$$

and

$$b = -\mu p + \frac{|\xi| + n}{N}.$$

Suppose φ simultaneously has a southeastern corner at eigenvalue $\frac{\kappa_2}{2}$ and a southeastern corner at eigenvalue $-\frac{\kappa_2}{2}$, then $p < q$ and $a > b$ follow. In this case, φ has cell (p, a) at eigenvalue $-\frac{|\kappa_2|}{2}$ and cell (q, b) at eigenvalue $\frac{|\kappa_2|}{2}$. Furthermore, the fact that cell (p, a) is a southeastern corner indicates $\xi_1^{(2)} = \xi_{q+1} = b$. The fact that cell $(q, b) \in \varphi$ indicates $\xi_q^{(1)} = \xi_q < b$. This contradicts $\xi \in P_{\geq 0}^+$. $\square$

2.9.2. Combinatorial description of irreducible representations in $\mathcal{M}$. In the following sections, let $\mathcal{M}(H_n(1, \kappa_2))$ be collection of $\mathcal{Y}$-semisimple representations of $H_n(1, \kappa_2)$ satisfying Properties 2.9.1-2.9.15. In this subsection, we show that any irreducible representation in $\mathcal{M}(H_n(1, \kappa_2))$ is isomorphic to the image $F_{n,p,\mu}(V^\xi)$ for a tuple of n, p, μ and some $\xi \in P_{\geq 0}^+$.

Let $L \in \mathcal{M}(H_n(1, \kappa_2))$ be irreducible and ζ be a minimal weight such that $L_\zeta \neq 0$. Recall, if $\zeta_i \geq 0$, then there is an increasing sequence $k_1 < \cdots < k_m$ such that $\zeta_{k_{i+1}} = \zeta_{k_i} \pm 1$ and $\zeta_{k_m} = \pm\frac{\kappa_2}{2}$. The weight ζ gives a standard tableau T_ζ such that $\zeta_k = -cont_{T_\zeta}(k) + s$ for some fixed number s where $s - \kappa_2$ is an integer. Let $Im(T_\zeta) = \nu/\beta$ such that $\beta_1 < \nu_1$ and $\beta_{\ell(\nu)} < \nu_{\ell(\nu)}$. Let us explore in different cases depending on corners. According to Lemma 2.9.14, if ζ_i is a corner of ζ, for some $i = 1, \cdots, n$, then $\zeta_i = \pm\frac{\kappa_2}{2}$ or $\zeta_i < -\frac{|\kappa_2|}{2}$. For any minimal ζ, there is at least one corner of ζ. Let the coordinate ζ_{r_1} be the corner of ζ such that $\mathfrak{i}(r_1)$ is the maximal of $\{\mathfrak{i}(i) | \zeta_i \text{ is corner of } \zeta\}$ and the coordinate ζ_{r_2} is the corner of ζ such that $\mathfrak{i}(r_2)$ is the second largest number in $\{\mathfrak{i}(i) | \zeta_i \text{ is corner of } \zeta\}$ if ζ_{r_2} exists. It is obvious $\zeta_{r_2} < \zeta_{r_1}$. There are the following cases. If $\zeta_{r_1} = \frac{|\kappa_2|}{2}$, then $\zeta_{r_2} < -\frac{|\kappa_2|}{2}$ or ζ_{r_2} doesn't exist. By Lemma 2.9.14, if $\zeta_{r_1} = \frac{|\kappa_2|}{2}$ and $\zeta_{r_2} = -\frac{|\kappa_2|}{2}$, then ζ violates Property 2.9.15. When $\zeta_{r_1} = -\frac{|\kappa_2|}{2}$, $\zeta_{r_2} < -\frac{|\kappa_2|}{2}$ or there is no ζ_{r_2}. When $\zeta_{r_1} < -\frac{|\kappa_2|}{2}$, $\zeta_{r_2} < -\frac{|\kappa_2|}{2}$ or ζ_{r_2} doesn't exist. So let us discuss in five cases.

Case 1. The corner $\zeta_{r_1} = \frac{|\kappa_2|}{2}$ and the corner $\zeta_{r_2} < -\frac{|\kappa_2|}{2}$.

Denote $T_\zeta(r_1) = (i_1, j_1)$ and $T_\zeta(r_2) = (i_2, \nu_{i_2})$. Let $j_2 = i_2 + s + \frac{|\kappa_2|}{2}$. In this case, set two rectangles

$$(a^p) = ((\nu_1 - j_1)^{i_2})$$

and

$$(b^q) = ((\nu_1 - j_2)^{i_1}).$$

Claim 2.9.16. *Following the setting above, the number $\nu_{i_2} - j_1 - j_2 \geq 0$.*

Proof. Since ζ_{r_2} is a corner, there exists a weight $\tilde{\zeta}$ such that $L_{\tilde{\zeta}} \neq 0$, $Im(T_{\tilde{\zeta}}) = Im(T_\zeta)$ and $T_{\tilde{\zeta}}(n) = (i_2, \nu_{i_2})$, where $T_{\tilde{\zeta}}$ denotes the standard tableau given by the weight $\tilde{\zeta}$. Let v be a nonzero weight vector of weight $\tilde{\zeta}$. Since $\tilde{\zeta}_n \neq \pm\frac{\kappa_2}{2}$, it follows that $\phi_n v$ is a nonzero weight vector of weight $\gamma_n\tilde{\zeta}$. Moreover, the standard tableau $T_{\gamma_n\tilde{\zeta}}$ given by $\gamma_n\tilde{\zeta}_n$ satisfies that

$$Im(T_{\gamma_n\tilde{\zeta}})) = Im(T_\zeta) \setminus \{(i_2, \nu_{i_2})\} \cup \{(i_1+1, j_1+j_2-\nu_{i_2}+1)\}$$

since $(\gamma_n\tilde{\zeta})_n = -\tilde{\zeta}_n$. It follows that $T_{\gamma_n\tilde{\zeta}}$ is a standard tableau and hence $Im(T_{\gamma_n\tilde{\zeta}})$ is a skew shape. This fact forces $j_1 + j_2 - \nu_{i_2} + 1 \leq 1$ and thus

$$\nu_{i_2} - j_1 - j_2 \geq 0.$$

□

Set $\xi^{(1)} = (\xi_1^{(1)}, \cdots, \xi_{i_1}^{(1)})$ with

$$\xi_k^{(1)} = \beta_k + \nu_1 - j_1 - j_2,$$

for $k = 1, \cdots, i_1$ and $\xi^{(2)} = (\xi_1^{(2)}, \cdots, \xi_{i_2}^{(2)})$ with

$$\xi_k^{(2)} = \nu_1 - \nu_{i_2-k+1},$$

for $k = 1, \cdots, i_2$. Furthermore, set $\xi = (\xi_1, \cdots, \xi_{i_1+i_2})$ with

$$\xi_k = \xi_k^{(1)},$$

for $k = 1, \cdots, i_1$ and

$$\xi_k = \xi_{k-i_1}^{(2)},$$

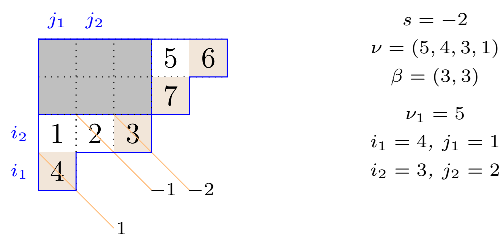
for $k = i_1 + 1, \cdots, i_1 + i_2$.

Remark 2.9.17. *Claim 2.9.16 implies the following two facts.*

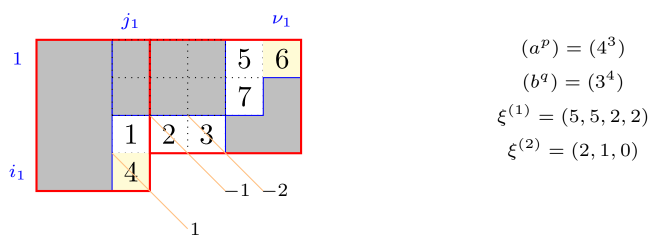
(1) *It follows* $\nu_1 - j_1 - j_2 \geq 0$.

(2) *The inequality* $\nu_1 - \nu_{i_2} = \xi_1^{(2)} \leq \xi_{i_1}^{(1)} = \beta_{i_1} + \nu_1 - j_1 - j_2$ *holds and hence* $\xi \in P^+$.

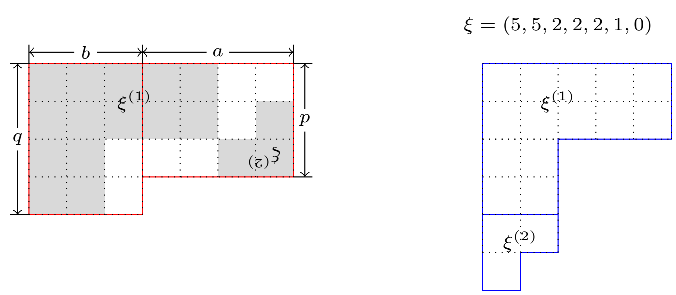
Example 2.9.18. *Continue Example 2.9.13. An irreducible representation* L *in* $\mathcal{M}(H_7(1,-2))$, *we start with a minimal weight* $\zeta = [0, -1, -2, 1, -5, -6, -4]$ *and the standard tableau of* ζ. *The corners of* ζ *are* $\zeta_3 = -2$, $\zeta_4 = 1$,$\zeta_6 = -6$ *and* $\zeta_7 = -4$. *Furthermore,* $\zeta_{r_1} = \zeta_4 = 1$ *and* $\zeta_{r_2} = \zeta_3 = -2$



Place the southeastern corner of $((\nu_1 - j_2)^{i_1})$ *at the cell* (i_1, j_1) *and northeastern corner of* $(\nu_1 - j_1)^{i_2}$ *at the cell* $(1, \nu_1)$. *The gray part on the left forms* $\xi^{(1)}$ *and the gray part on the right forms* $\xi^{(2)}$.



Furthermore, we obtain other parameters of Etingof-Freund-Ma functor as $N = p + q = 7$, $p = 3$ *and* $\mu = \frac{a-b}{N} = \frac{1}{7}$.



Case 2. The corner $\zeta_{r_1} = -\frac{|\kappa_2|}{2}$ and the corner $\zeta_{r_2} < -\frac{|\kappa_2|}{2}$.
Denote $T_\zeta(r_1) = (i_1, j_1)$ and $T_\zeta(r_2) = (i_2, \nu_{i_2})$. Let $j_2 = i_2 + s - \frac{|\kappa_2|}{2}$. In this case, set two rectangles

$$(a^p) = ((\nu_1 - j_1)^{i_2})$$

and

$$(b^q) = ((\nu_1 - j_2)^{i_1}).$$

We have a similar claim to Claim 2.9.16.

CLAIM 2.9.19. *Following the setting above, the number $\nu_{i_2} - j_1 - j_2 \geq 0$.*

The proof is the same with the proof of Claim 2.9.16.
Similarly, let $\xi^{(1)} = (\xi_1^{(1)}, \cdots, \xi_{i_1}^{(1)})$ with

$$\xi_k^{(1)} = \beta_k + \nu_1 - j_1 - j_2,$$

for $k = 1, \cdots, i_1$ and $\xi^{(2)} = (\xi_1^{(2)}, \cdots, \xi_{i_2}^{(2)})$ with

$$\xi_k^{(2)} = \nu_1 - \nu_{i_2-k+1},$$

for $k = 1, \cdots, i_2$. Furthermore, set $\xi = (\xi_1, \cdots, \xi_{i_1+i_2})$ with

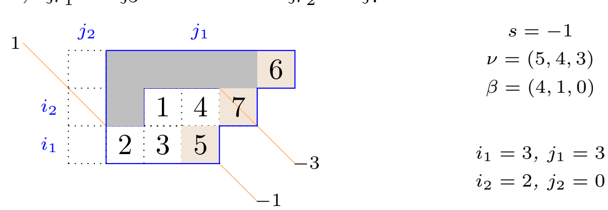
$$\xi_k = \xi_k^{(1)},$$

for $k = 1, \cdots, i_1$ and

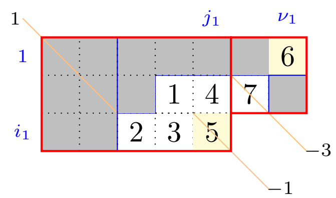
$$\xi_k = \xi_{k-i_1}^{(2)},$$

for $k = i_1 + 1, \cdots, i_1 + i_2$.

EXAMPLE 2.9.20. *Let L be an irreducible representation in $\mathcal{M}(H_7(1, -2))$ with a minimal weight $\zeta = [-1, 1, 0, -2, -1, -5, -3]$ and the standard tableau of ζ. The corners of ζ are $\zeta_4 = -6$,$\zeta_6 = -4$ and $\zeta_7 = -2$. Furthermore, $\zeta_{r_1} = \zeta_5 = -1$ and $\zeta_{r_2} = \zeta_7 = -3$*

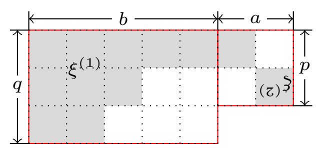


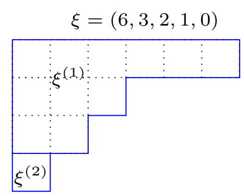
Place the southeastern corner of (b^q) at the cell (i_1, j_1) and northeastern corner of (a^p) at the cell $(1, \nu_1)$. The gray part on the left forms $\xi^{(1)}$ and the gray part on the right forms $\xi^{(2)}$.



$$(a^p) = (2^2)$$
$$(b^q) = (5^3)$$
$$\xi^{(1)} = (6,3,2)$$
$$\xi^{(2)} = (1,0)$$

Furthermore, we obtain other parameters of Etingof-Freund-Ma functor as $N = q + p = 5$, $q = 3$ *and* $\mu = \frac{b-a}{N} = \frac{3}{5}$.





Case 3. The corner $\zeta_{r_1} = \frac{|\kappa_2|}{2}$ and the corner ζ_{r_2} doesn't exist. Let $j = s + \frac{|\kappa_2|}{2}$. Then the cell $(0, j)$ on the diagonal of weight $-\frac{|\kappa_2|}{2}$. We explore the following in two subcases.

Case 3a. $j \geq 1$. Set two rectangles

$$(a^p) = (j^1)$$

and

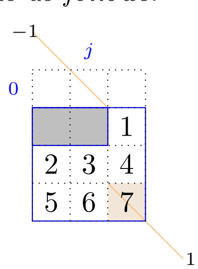
$$(b^q) = (\nu_1^{\ell(\nu)+1}).$$

Moreover, $\xi = (\xi_1, \cdots, \xi_{\ell(\nu)})$ with $\xi_1 = \nu_1 + j$ and $\xi_k = \beta_{k-1}$.

Example 2.9.21. *Let* L *be an irreducible representation in* $\mathcal{M}(H_7(1,-2))$ *with a minimal weight* $\zeta = [-1, 2, 1, 0, 3, 2, 1]$ *such that* $L_\zeta \neq 0$. *There is only one corner* $\zeta_7 = 1$. *So*

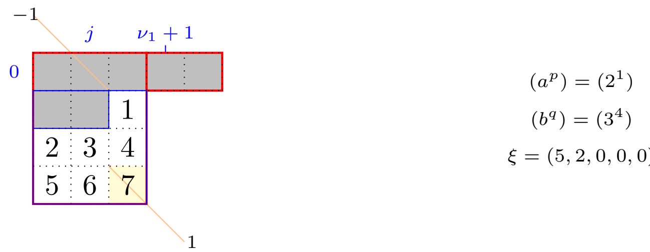
$$\zeta_{r_1} = \zeta_7 = 1 = \frac{|\kappa_2|}{2}.$$

The standard tableau of ζ *is as follows.*

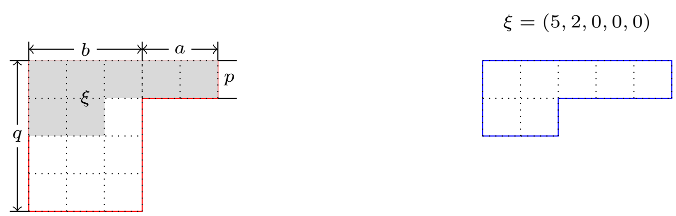


$$s = 1$$
$$\nu = (3,3,3)$$
$$\beta = (2,0,0)$$
$$\ell(\nu) = 3$$
$$j = 2$$

The two rectangles are $(a^p) = (2^1)$ *and* $(b^q) = (3^4)$. *Place the southeastern corner of* (b^q) *at* $T_\zeta(r_1) = T_\zeta(7)$ *and the northwestern corner of* (a^p) *at the cell* $(0, \nu_1 + 1)$. *The gray area forms* ξ.



Furthermore, we obtain other parameters of Etingof-Freund-Ma functor as $N = p + q = 5$, $p = 1$ *and* $\mu = \frac{a-b}{N} = -\frac{1}{5}$.



Case 3b. $j \leq 0$. Set two rectangles

$$(a^p) = (1^1)$$

and

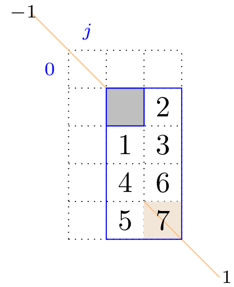
$$(b^q) = ((\nu_1 - j + 1)^{\ell(\nu)+1}).$$

Moreover, $\xi = (\xi_1, \cdots, \xi_{\ell(\nu)})$ with $\xi_1 = \nu_1 - j + 2$ and $\xi_k = \beta_{k-1} - j + 1$.

Example 2.9.22. *Let* L *be an irreducible representation in* $\mathcal{M}(H_7(1, -2))$ *with a minimal weight* $\zeta = [0, -2, -1, 1, 2, 0, 1]$ *such that* $L_\zeta \neq 0$. *There is only one corner* $\zeta_7 = 1$. *So*

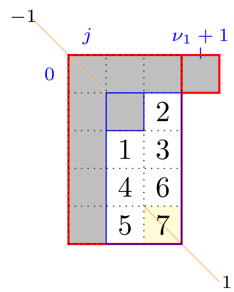
$$\zeta_{r_1} = \zeta_7 = 1 = \frac{|\kappa_2|}{2}.$$

The standard tableau of ζ *is as follows.*



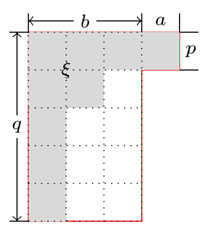
$$s=-1$$
$$\nu=(2,2,2,2)$$
$$\beta=(1,0,0,0)$$
$$\ell(\nu)=4$$
$$j=0$$

The two rectangles are $(a^p)=(1^1)$ *and* $(b^q)=(3^5)$. *Place the southeastern corner of* (b^q) *at* $T_\zeta(r_1)=T_\zeta(7)$ *and the northwestern corner of* (a^p) *at the cell* $(0,\nu_1+1)$. *The gray area forms* ξ.

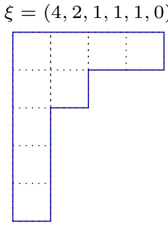


$$(a^p)=(1^1)$$
$$(b^q)=(3^5)$$
$$\xi=(4,2,1,1,1,0)$$

Furthermore, we obtain other parameters of Etingof-Freund-Ma functor as $N=p+q=6$, $p=1$ *and* $\mu=\frac{a-b}{N}=-\frac{1}{3}$.



$\xi=(4,2,1,1,1,0)$



Case 4. The corner $\zeta_{r_1}=-\frac{|\kappa_2|}{2}$ and there is no corner ζ_{r_2}. Set $j=s-\frac{|\kappa_2|}{2}$. Then the cell $(0,j)$ is on the diagonal of weight $\frac{|\kappa_2|}{2}$. Let us discuss in two subcases.

Case 4a. When $j\geq 1$. Set two rectangles

$$(a^p)=(j^1)$$

and

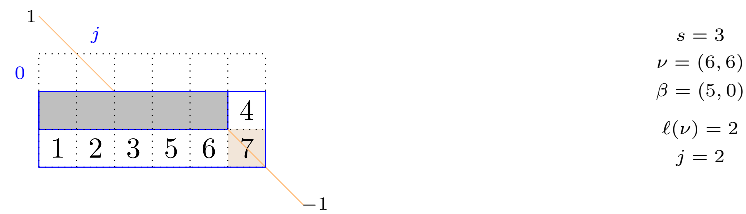
$$(b^q)=(\nu_1^{\ell(\nu)+1}).$$

Moreover, $\xi = (\xi_1, \cdots, \xi_{\ell(\nu)})$ with $\xi_1 = \nu_1 + j$ and $\xi_k = \beta_{k-1}$.

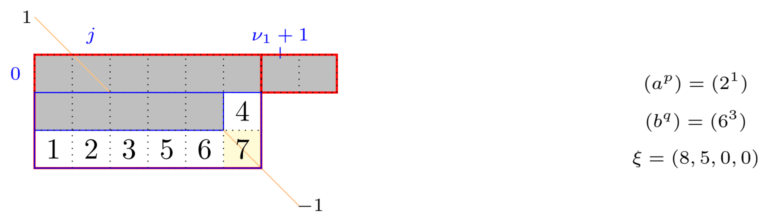
Example 2.9.23. *Let L be an irreducible representation in $\mathcal{M}(H_7(1, -2))$ with a minimal weight $\zeta = [4, 3, 2, -2, 1, 0, -1]$ such that $L_\zeta \neq 0$. There is only one corner $\zeta_7 = -1$. So*

$$\zeta_{r_1} = \zeta_7 = -1 = -\frac{|\kappa_2|}{2}.$$

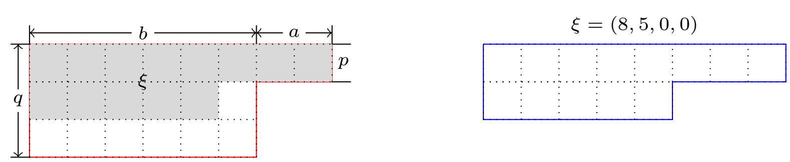
The standard tableau of ζ is as follows.



The two rectangles are $(a^p) = (2^1)$ and $(b^q) = (6^3)$. Place the southeastern corner of (b^q) at $T_\zeta(r_1) = T_\zeta(7)$ and the northwestern corner of (a^p) at cell $(0, \nu_1 + 1) = (0, 7)$. The gray area forms ξ.



Furthermore, we obtain other parameters of Etingof-Freund-Ma functor as $N = q + p = 4$, $q = 3$ and $\mu = \frac{b-a}{N} = 1$.



Case 4b. When $j \leq 0$. Set two rectangles

$$(a^p) = (1^1)$$

and

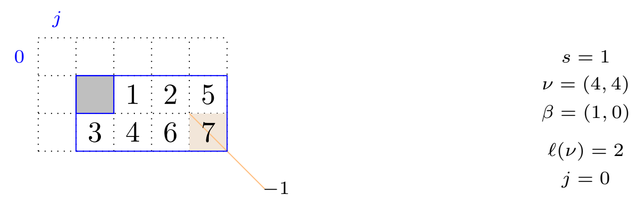
$$(b^q) = ((\nu_1 - j + 1)^{\ell(\nu)+1}).$$

Moreover, $\xi = (\xi_1, \cdots, \xi_{\ell(\nu)})$ with $\xi_1 = \nu_1 - j + 2$ and $\xi_k = \beta_{k-1} - j + 1$.

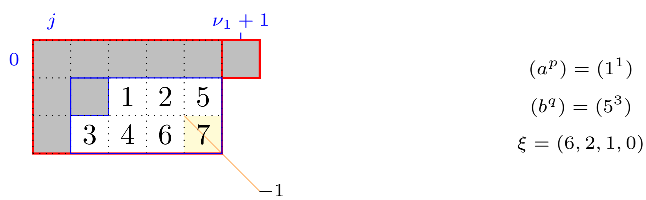
EXAMPLE 2.9.24. *Let L be an irreducible representation in $\mathcal{M}(H_7(1,-2))$ with a minimal weight $\zeta = [0, -1, 2, 1, -2, 0, -1]$ such that $L_\zeta \neq 0$. There is only one corner $\zeta_7 = -1$. So*

$$\zeta_{r_1} = \zeta_7 = -1 = -\frac{|\kappa_2|}{2}.$$

The standard tableau of ζ is as follows.



The two rectangles are $(a^p) = (1^1)$ and $(b^q) = (5^3)$. Place the southeastern corner of (b^q) at $T_\zeta(r_1) = T_\zeta(7)$ and the northwestern corner of (a^p) at the cell $(0, \nu_1 + 1)$. The gray area forms ξ.



Furthermore, we obtain other parameters of Etingof-Freund-Ma functor as $N = q + p = 4$, $q = 3$ and $\mu = \frac{b-a}{N} = 1$.

Case 5. The corner $\zeta_{r_1} < -\frac{|\kappa_2|}{2}$. Let $j_1 = \nu_{\ell(\nu)+\frac{|\kappa_2|}{2}} + \zeta_{r_1}$ and $j_2 = \nu_{\ell(\nu)-\frac{|\kappa_2|}{2}} + \zeta_{r_1}$. Set two rectangles

$$(a^p) = ((\nu_1 - j_1)^{\ell(\nu)})$$

and

$$(b^q) = ((\nu_1 - j_2)^{\ell(\nu)}).$$

Claim 2.9.25. *According to the setting above, the number $\nu_{\ell(\nu)} - j_1 - j_2 \geq 0$*

Proof. There exist a weight $\tilde{\zeta}$ such that $L_{\tilde{\zeta}} \neq 0$, $Im(T_{\tilde{\zeta}}) = Im(T_\zeta)$ and $T_{\tilde{\zeta}}(n) = (\ell(\nu), \nu_{\ell(\nu)})$. Let v be a nonzero weight vector of weight $\tilde{\zeta}$. Since $\zeta_{r_1} < -\frac{|\kappa_2|}{2}$, we obtain a nonzero weight vector $\phi_n v$ of weight $\gamma_n \tilde{\zeta}$. Moreover,

$$Im(T_{\gamma_n \tilde{\zeta}}) = Im(T_\zeta) \setminus \{(\ell(\nu), \nu_{\ell(\nu)})\} \cup \{(\ell(\nu)+1, 2\ell(\nu) - \nu_{\ell(\nu)} + 2s + 1)\}.$$

Since $Im(T_{\gamma_n \tilde{\zeta}})$ is a skew shape, it follows $2\ell(\nu) - \nu_{\ell(\nu)} + 2s + 1 \leq 1$. Applying $j_1 = \nu_{\ell(\nu)} + \frac{|\kappa_2|}{2} + \zeta_{r_1}$ and $j_2 = \nu_{\ell(\nu)} - \frac{|\kappa_2|}{2} + \zeta_{r_1}$, the statement $\nu_{\ell(\nu)} - j_1 - j_2 \geq 0$ follows. $\square$

Set $\xi^{(1)} = (\xi_1^{(1)}, \cdots, \xi_{\ell(\nu)}^{(1)})$ with

$$\xi_k^{(1)} = \beta_k + \nu_1 - j_1 - j_2$$

for $k = 1, \cdots, \ell(\nu)$, $\xi^{(2)} = (\xi_1^{(2)}, \cdots, \xi_{\ell(\nu)}^{(2)})$ with

$$\xi_k^{(2)} = \nu_1 - \nu_{\ell(\nu)-k+1}$$

for $k = 1, \cdots, \ell(\nu)$ and $\xi = (\xi_1, \cdots, \xi_{2\ell(\nu)})$ with

$$\xi_k = \xi_k^{(1)}$$

for $k = 1, \cdots, \ell(\nu)$ and

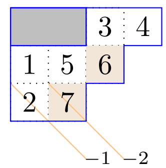
$$\xi_k = \xi_{k-\ell(\nu)}^{(2)}$$

for $k = \ell(\nu)+1, \cdots, 2\ell(\nu)$.

Remark 2.9.26. *Claim 2.9.25 implies the following two facts.*

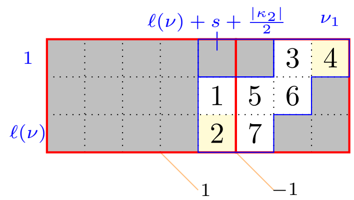
(1) It follows $\nu_1 - j_1 - j_2 \geq 0$.
(2) The inequality $\nu_1 - \nu_{\ell(\nu)} = \xi_1^{(2)} \leq \xi_{\ell(\nu)}^{(1)} = \nu_1 - j_1 - j_2$ holds and hence $\xi \in P^+$.

Example 2.9.27. *Let L be an irreducible representation in $\mathcal{M}(H_7(1,-2))$ with a minimal weight $\zeta = [-2,-1,-5,-6,-3,-4,-2]$ such that $L_\zeta \neq 0$. The corners of ζ are $\zeta_4 = -6$, $\zeta_6 = -4$ and $\zeta_7 = -2$. So $\zeta_{r_1} = \zeta_7 = -2$. The standard tableau of ζ is as follows.*



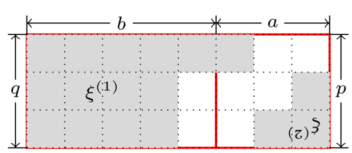
$s = -3$
$\nu = (4,3,2)$
$\beta = (2,0,0)$
$\ell(\nu) = 3$

The two rectangles $(a^p) = (3^3)$ and $(b^q) = (5^3)$ follow. Place the northeastern corner of $(a^p) = (3^3)$ at the cell $(1,\nu_1)$ and the southeastern corner of $(b^q) = (5^3)$ at the cell $(\ell(\nu), \ell(\nu) + \frac{|\kappa_2|}{2} + s$. The gray area on the left forms $\xi^{(1)}$ and the gray area on the right forms $\xi^{(2)}$.

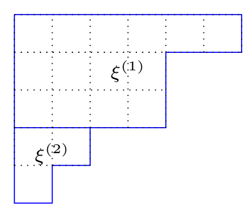


$(a^p) = (3^3)$
$(b^q) = (5^3)$
$\xi^{(1)} = (6,4,4)$
$\xi^{(2)} = (2,1,0)$

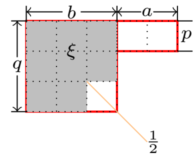
So the three shapes (a^p), (b^q) and ξ are set as follows. The other parameters of Etingof-Freund-Ma functor are set as $N = 6$, $p = 3$ and $\mu = 1/3$.



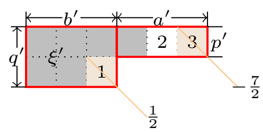
$\xi = (6,4,4,2,1,0)$



Remark 2.9.28. *When we fix the number n, for different input (ξ, N, p, μ), we get isomorphic H_n-modules. Consider the following example of representations of $H_3(1,-1)$.*
Let $\xi = (3,3,2)$, $N = 4$, $p = 1$ and $\mu = -\frac{1}{4}$.
In this case,$a = \mu q + \frac{|\xi|+n}{N} = 2$ and $b = -\mu p + \frac{|\xi|+n}{N} = 3$. Then the image $F = F_{3,1,-\frac{1}{4}}(V^\xi)$ is an $H_3(1,-1)$-module with the following minimal shape $\varphi^\xi_{3,1,-\frac{1}{4}} = (5,3,3)/(3,3,2)$.



Then the basis is indexed by the standard tableaux on the skew shapes: $(5,3,3)/\xi$*,* $(4,3,3,1)/\xi$ *and* $(3,3,3,2)/\xi$*. There is a minimal weight* $\zeta = [\frac{1}{2}, -\frac{5}{2}, -\frac{7}{2}]$ *such that* $F_\zeta \neq 0$*. Now let us recover a functor* $F_{n,p',\mu'}$ *such that* $F_{n,p',\mu'}(V^{\xi'})$ *is an* $H_3(1,-1)$*-module with a minimal weight* $\zeta = [\frac{1}{2}, -\frac{5}{2}, -\frac{7}{2}]$*. According to Case 1,* $(a'^{p'}) = (3^1)$*,* $(b'^{q'}) = (3^2)$*,* $\xi' = (4,2,0)$ *and* $\mu' = 0$*.*



2.9.3. Other $\mathcal{Y}$-semisimple representations. The image of the Etingof-Freund-Ma functor does not exhaust all the $\mathcal{Y}$-semisimple representations. The following are two examples of $\mathcal{Y}$-semisimple $H_n(1,\kappa_2)$ representation which are not in $\mathcal{M}(H_n(1,\kappa_2))$.

Example 2.9.29. *Obviously, the representation obtained under the Etingof-Freund-Ma does not contain a weight vector of weight* ζ *with* $-\frac{|\kappa_2|}{2} < \zeta_n < \frac{|\kappa_2|}{2}$.

Consider the representation of $H_3(1,-6)$ *generated by the weight vector of weight* $[1,2,-3]$*. This representation has the following characters:*

$$
\begin{tikzcd}
 & {[-3,-2,-1]} \arrow[d, "\mathfrak{m}_3"'] & \\
 & {[-3,-2,1]} \arrow[d, "\mathfrak{m}_2"'] & \\
 & {[-3,1,-2]} \arrow[dl, "\mathfrak{m}_1"'] \arrow[dr, "\mathfrak{m}_3"] & \\
{[1,-3,-2]} \arrow[dr, "\mathfrak{m}_3"'] & & {[-3,1,2]} \arrow[dl, "\mathfrak{m}_1"] \\
 & {[1,-3,2]} \arrow[d, "\mathfrak{m}_2"'] & \\
 & {[1,2,-3]} &
\end{tikzcd}
$$

CHAPTER 3

Degenerate double affine Hecke algebras of type C

3.1. Generators and relations of dDAHA

The degenerate double affine Hecke algebra $H_n(u, k_1, k_2, k_3)$, which we also denote by $dDAHA$, is an algebra over $\mathbb{C}$ with parameters $u, k_1, k_2, k_3 \in \mathbb{C}$, generated by

$$s_1, \cdots, s_{n-1}, \gamma_n, X_1^{\pm}, \cdots, X_n^{\pm}, y_1, \cdots, y_n$$

with the following relations in addition to relations (2.1)-(2.4).

$$[X_i, X_j] = [y_i, y_j] = 0, \tag{3.1}$$

$$[s_i, X_j] = [s_i, y_j] = 0, \text{ for } j \neq i, i+1, \tag{3.2}$$

$$[\gamma_n, X_j] = [\gamma_n, y_j] = 0, \text{ for } j \neq n, \tag{3.3}$$

$$s_i X_i = X_{i+1} s_i, \text{ for } i = 1, \cdots, n-1, \tag{3.4}$$

$$\gamma_n X_n = X_n^{-1} \gamma_n, \tag{3.5}$$

$$s_i y_i - y_{i+1} s_i = k_1, \text{ for } i = 1, \cdots, n-1, \tag{3.6}$$

$$\gamma_n y_n + y_n \gamma_n = k_2 + k_3, \tag{3.7}$$

$$[y_j, X_i] = k_1 X_j s_{ij} - k_1 X_j^{-1} s_{ij} \gamma_i \gamma_j, \text{ for } i < j, \tag{3.8}$$

$$[y_i, X_i] = uX_i - k_1 \sum_{k>i} X_k s_{ik} - k_1 \sum_{k>i} X_k^{-1} s_{ik} \gamma_i \gamma_k - (k_2 + k_3) X_i^{-1} \gamma_i - k_2 \gamma_i, \tag{3.9}$$

where s_{ij} denotes the element in Weyl group W which flips ϵ_i and ϵ_{i+1} and γ_i denotes the element in W sending ϵ_i to $-\epsilon_i$.

3.1.1. $\mathcal{Y}$-semisimple representation of degenerate DAHA. We define the definition of $\mathcal{Y}$-semisimple representations of a degenerate double affine Hecke algebra $H_n(u, k_1, k_2, k_3)$ as follows: Let $\mathcal{Y} = \mathbb{C}[y_1, \cdots, y_n]$ be the commutative subalgebra of the degenerate affine Hecke algebra

$H_n(u, k_1, k_2, k_3)$. Let L be a representation of $H_n(u, k_1, k_2, k_3)$. For an n-tuple $\zeta = (\zeta_1, \cdots, \zeta_n)$, define the simultaneous generalized eigenspace as

$$L_\zeta^{gen} = \{v \in L | (y_i - \zeta_i)^k v = 0 \text{ for some } k \gg 0 \text{ and for all } i = 1, \cdots, n\}.$$

Since the polynomial algebra $\mathcal{Y}$ is commutative, its representation L decomposes to a sum of simultaneous generalized eigenspace, i.e. $L = \oplus_\zeta L_\zeta^{gen}$. Similarly, define the simultaneous eigenspace

$$L_\zeta = \{v \in L | y_i v = \zeta_i v \text{ for all } i = 1, \cdots, n\}.$$

DEFINITION 3.1.1. *If a degenerate double affine Hecke algebra representation L decomposes to a sum of simultaneous eigenspaces as a $\mathcal{Y}$-module, i.e. $L = \oplus_\zeta L_\zeta$, then L is $\mathcal{Y}$-semisimple. If the subspace $L_\zeta \neq 0$, then call ζ weight of the representation L, L_ζ the corresponding weight space and any nonzero element $v \in L_\zeta$ weight vector of weight ζ.*

3.1.2. Another set of generators of dDAHA. Let $\gamma_i = s_i s_{i+1} \cdots s_{n-1} \gamma_n s_{n-1} \cdots s_{i+1} s_i$ for $i = 1, \cdots, n$.

LEMMA 3.1.2. *It follows that $X_1 \gamma_1 = \gamma_1 X_1^{-1}$.*

PROOF. Applying $X_i s_i = s_i X_{i+1}$, $X_{i+1}^{-1} s_i = s_i X_i^{-1}$ and $X_n \gamma_n = \gamma_n X_n^{-1}$, then it holds that

$$\begin{aligned} X_1 \gamma_1 &= X_1 s_1 \cdots \gamma_n \cdots s_1 \\ &= s_1 X_2 s_2 \cdots \gamma_n \cdots s_1 \\ &= s_1 \cdots s_{n-1} X_n \gamma_n \cdots s_1 \\ &= s_1 \cdots \gamma_n X_n^{-1} s_{n-1} \cdots s_1 \\ &= s_1 \cdots \gamma_n \cdots s_1 X_1^{-1} \\ &= \gamma_1 X_1^{-1}. \end{aligned}$$

□

LEMMA 3.1.3. *It holds $s_j \gamma_1 = \gamma_1 s_j$, $j \geq 2$.*

Proof. Apply the relation $s_j s_{j-1} s_j = s_{j-1} s_j s_{j-1}$, it follows

$$\begin{aligned}
s_j\gamma_1 &= s_j(s_1\cdots\gamma_n\cdots s_1)\\
&= s_1\cdots s_{j-2}(s_j s_{j-1} s_j)s_{j+1}\cdots\gamma_n\cdots s_1\\
&= s_1\cdots s_{j-2}(s_{j-1} s_j s_{j-1})s_{j+1}\cdots\gamma_n\cdots s_1\\
&= s_1\cdots s_{j-2}s_{j-1}s_j s_{j+1}\cdots\gamma_n\cdots s_{j+1}(s_{j-1}s_j s_{j-1})s_{j-2}\cdots s_1\\
&= s_1\cdots\gamma_n\cdots s_{j+1}(s_j s_{j-1}s_j)s_{j-2}\cdots s_1\\
&= s_1\cdots\gamma_n\cdots s_{j+1}s_j s_{j-1}s_{j-2}\cdots s_1 s_j\\
&= \gamma_1 s_j.
\end{aligned}$$

$\square$

Lemma 3.1.4. $\gamma_1 y_j = y_j\gamma_1 - k_1\gamma_1 s_{1,j} + k_1 s_{1,j}\gamma_1,\ j \geq 2.$

Proof. First, applying the relation $y_{j-1}s_{j-1} - s_{j-1}y_j = k_1$, it follows

$$\begin{aligned}
\gamma_1 y_j &= s_1\cdots\gamma_n\cdots(s_{j-1}y_j)s_{j-2}\cdots s_1\\
&= s_1\cdots\gamma_n\cdots(y_{j-1}s_{j-1})s_{j-2}\cdots s_1 - k_1 s_1\cdots\gamma_n\cdots s_j \hat{s}_{j-1}s_{j-2}\cdots s_1\\
&= s_1\cdots s_{j-2}(s_{j-1}y_{j-1})s_j\cdots\gamma_n\cdots s_1 - k_1\gamma_1 s_{1,j},
\end{aligned}$$

where $s_1\cdots\gamma_n\cdots s_j\hat{s}_{j-1}s_{j-2}\cdots s_1 = \gamma_1(s_1\cdots s_{j-1}\cdots s_1) = \gamma_1 s_{1,j}$.
Applying the relation $s_{j-1}y_{j-1} - y_j s_{j-1} = k_1$, it follows

$$\begin{aligned}
&s_1\cdots s_{j-2}(s_{j-1}y_{j-1})s_j\cdots\gamma_n\cdots s_1 - k_1\gamma_1 s_{1,j}\\
=&s_1\cdots s_{j-2}(y_j s_{j-1})s_j\cdots\gamma_n\cdots s_1 + k_1 s_1\cdots s_{j-2}\hat{s}_{j-1}s_j\cdots\gamma_n\cdots s_1 - k_1\gamma_1 s_{1,j}\\
=&y_j\gamma_1 + k_1 s_{1,j}\gamma_1 - k_1\gamma_1 s_{1,j}.
\end{aligned}$$

$\square$

Lemma 3.1.5. $\gamma_1 y_1 = -y_1\gamma_1\gamma_1 + k_1\sum_{j=2}^n s_{1,j}\gamma_1 + k_1\sum_{j=2}^n \gamma_1 s_{1,j} + (k_2+k_3).$

PROOF. Applying $s_j y_j - y_{j+1} s_j = k_1$ for $j = 1, \cdots, n-1$,

$$\begin{aligned}\gamma_1 y_1 &= s_1 \cdots \gamma_n \cdots s_2 y_2 s_1 + k_1 \gamma_1 s_{1,2} \\ &= s_1 \cdots \gamma_n y_n s_{n-1} \cdots s_1 + \sum_{j=2}^{n} k_1 \gamma_1 s_{1,j}.\end{aligned}$$

Applying the relation $\gamma_n y_n + y_n \gamma_n = k_2 + k_3$, the above computation continues as

$$= -s_1 \cdots s_{n-1} y_n \gamma_n \cdots s_1 + (k_2 + k_3) + \sum_{j=2}^{n} k_1 \gamma_1 s_{1,j}.$$

Applying the relation $s_{j-1} y_j - y_{j-1} s_{j-1} = -k_1$ for $j = 2, \cdots, n$, it follows that

$$\begin{aligned}&- s_1 \cdots s_{n-1} y_n \gamma_n \cdots s_1 + (k_2 + k_3) + \sum_{j=2}^{n} k_1 \gamma_1 s_{1,j} \\ =& - y_1 \gamma_1 + \sum_{j=2}^{n} k_1 s_{1,j} \gamma_1 + (k_2 + k_3) + \sum_{j=2}^{n} k_1 \gamma_1 s_{1,j}.\end{aligned}$$

□

Let $s_0 := X_1 \gamma_1 = X_1 s_1 \cdots \gamma_n \cdots s_1$. Then we have the following relations.

LEMMA 3.1.6. *The element s_0 satisfies*

$$s_0^2 = 1, \tag{3.10}$$

$$[s_0, s_j] = [s_0, y_j] = 0, \ \textit{for } j = 2, \cdots, n \tag{3.11}$$

$$s_0 y_1 - (u - y_1) s_0 = -k_2. \tag{3.12}$$

PROOF. By Lemma 3.1.2, $s_0^2 = X_1 \gamma_1 X_1 \gamma_1 = \gamma_1 X_1^{-1} X_1 \gamma_1 = 1$.
By Lemma 3.1.3, $s_0 s_j = X_1 \gamma_1 s_j = X_1 s_j \gamma_1$. Moreover, by (7), $X_1 s_j \gamma_1 = s_j X_1 \gamma_1 = s_j s_0$, for $j \geq 2$.
By Lemma 3.1.4,

$$\begin{aligned}s_0 y_j &= X_1 \gamma_1 y_j \\ &= X_1 y_j \gamma_1 + k_1 X_1 s_{1,j} \gamma_1 - k_1 X_1 \gamma_1 s_{1,j}.\end{aligned}$$

Applying the relation $y_jX_1 - X_1y_j = k_1X_1s_{1,j} - k_1X_1s_{1,j}\gamma_1\gamma_j$, the above computation continues

$$\begin{aligned}
&X_1y_j\gamma_1 + k_1X_1s_{1,j}\gamma_1 - k_1X_1\gamma_1s_{1,j} \\
=&y_jX_1\gamma_1 - k_1X_1s_{1,j}\gamma_1 + k_1X_1X_1s_{1,j}\gamma_1\gamma_j\gamma_1 + k_1X_1s_{1,j}\gamma_1 - k_1X_1\gamma_1s_{1,j} \\
=&y_js_0,
\end{aligned}$$

applying $\gamma_1\gamma_j = \gamma_j\gamma_1$ and $\gamma_1s_{1,j} = s_{1,j}\gamma_j$ in the last step.

By Lemma 3.1.5,

$$\begin{aligned}
s_0y_1 &= X_1\gamma_1y_1 \\
&= -X_1y_1\gamma_1 + k_1X_1\sum_{j=2} s_{1,j}\gamma_1 + k_1X_1\sum_{j=2}^{n}\gamma_1s_{1,j}.
\end{aligned}$$

Applying the relation

$$y_1X_1 - X_1y_1 = uX_1 - k_1X_1\sum_{j=2}^{n}s_{1,j} - k_1X_1\sum_{j=2}^{n}s_{1,j}\gamma_j\gamma_1 - (k_2+k_3)X_1\gamma_1 - k_2\gamma_1,$$

the computation above continues

$$\begin{aligned}
=&-y_1X_1\gamma_1 + uX_1\gamma_1 - k_1X_1(\sum_{j=2}^{n}s_{1,j})\gamma_1 - k_1X_1\sum_{j=2}^{n}s_{1,j}\gamma_j - (k_2+k_3)X_1 - k_2 \\
&+k_1\sum_{j=2}^{n}X_1s_{1,j}\gamma_1 + (k_2+k_3)X_1 + k_1\sum_{j=2}^{n}X_1\gamma_1s_{1,j} \\
=&-y_1s_0 + us_0 - k_2.
\end{aligned}$$

Then (3.12) follows. $\square$

PROPOSITION 3.1.7. *Degenerate double affine Hecke algebra $H_n(u, k_1, k_2, k_3)$ is generated by $s_0, s_1, \cdots, s_{n-1}, \gamma_n, y_1, \cdots, y_n$ with relations.*

Proof. Define a homomorphism of algebras f from degenerate double affine Hecke algebra $H_n(u, k_1, k_2, k_3)$ to itself.

$$f : H_n(u, k_1, k_2, k_3) \longrightarrow H_n(u, k_1, k_2, k_3)$$
$$s_i \mapsto s_i$$
$$y_i \mapsto y_i$$
$$X_i \mapsto s_{i-1} \cdots s_1 s_0 s_1 \cdots \gamma_n \cdots s_i$$

□

3.1.3. Etingof-Freund-Ma Functor. Let $\lambda \in \mathbb{C}$ and L_x denote the vector field on GL_N generated by the left action of $x \in \mathfrak{gl}_N$. Let $\mathcal{D}^\lambda(G/K)$ be the sheaf of differential operators on G/K twisted by the character $\lambda\chi$ of $\mathfrak{t} = \mathfrak{gl}_p \times \mathfrak{gl}_q$. The Etingof-Freund-Ma functor $F^\lambda_{n,p,\mu}$ sends a $\mathcal{D}^\lambda(G/K)$- module M to a representation of degenerate double affine Hecke algebra $H_n(u, k_1, k_2, k_3)$, where μ is a parameter in $\mathbb{C}$. The underlying space $F^\lambda_{n,p,\mu}(M)$ of the representation of $H_n(u, k_1, k_2, k_3)$ is constructed as

$$F^\lambda_{n,p,\mu}(M) = (M \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}.$$

Then for $k = 1, \cdots, n$, define the actions of X_k and y_k as follows:

$$X_k = \sum_{i,j} (AJA^{-1}J)_{ij} \otimes (E_{ij})_k,$$

where $(AJA^{-1}J)_{ij}$ is a function of A for $A \in G/K$, taking the ij-th entry of $AJA^{-1}J$, instead of y_k, define the action of $\tilde{y}_k = y_k - \dfrac{k_2 + k_3}{2}\gamma_k - \dfrac{k_1}{2}\sum_{i>k} S_{ki} + \dfrac{k_1}{2}\sum_{i<k} S_{ki} - \dfrac{k_1}{2}\sum_{i\neq k} S_{ki}\gamma_k\gamma_i$ as

$$\tilde{y}_k = \sum_{i|j} L_{E_{ij}} \otimes (E_{ji})_k.$$

Theorem 3.1.8. *[**5**]The actions of W, X_k and $\tilde{y}_k$ defined above makes the invariant space $F^\lambda_{n,p,\mu}(M)$ a representation of degenerate double affine Hecke algebra $H_n(u, k_1, k_2, k_3)$ with parameters*

$$\xi = \frac{2n}{N} + (\lambda + \mu)(q - p), \quad k_1 = 1, \quad k_2 = p - q - \lambda N, \quad k_3 = (\lambda - \mu)N.$$

We will compute the image of $\mathcal{A}^\lambda(G/K)$ under the functor in the following sections.

3.2. Invariant space

Before computing the invariant space $F^{\lambda}_{n,p,\mu}(\mathcal{A}^{\lambda}(G/K))$, we introduce the combinatorial tools we use, skew shapes and standard tableaux.

3.2.1. Integral dominant weights and skew shape. Now let us identity a pair of integral dominate weights ν/β with a skew shape. Let $\nu = (\nu_1, \cdots, \nu_N)$ and $\beta = (\beta_1, \cdots, \beta_N)$ with

$$\nu_1 \geq \cdots \geq \nu_N;$$

$$\beta_1 \geq \cdots \geq \beta_N;$$

$$\nu_i \geq \beta_i, i = 1, \cdots, N.$$

Let τ be the skew shape with $\tau \subset \mathbb{Z} \times \mathbb{Z}$ and

$$\tau = \{(l, m) | 1 \leq l \leq N, \beta_l + 1 \leq m \leq \nu_l\}.$$

Furthermore, define the content of a cell (l, m) to be $m - l$. For instance, a pair of integral dominant weights ν/β denotes a basis element with $\nu = (2, 2, -2)$ and $\beta = (1, 1, -3)$, then define the skew τ be the collection of cells $(1, 2)$, $(2, 2)$, $(3, 2)$ and $(4, -2)$. Let $|\nu| = \sum_i^N \nu_i$ and $|\beta| = \sum_{i=1}^N \beta_i$. Let

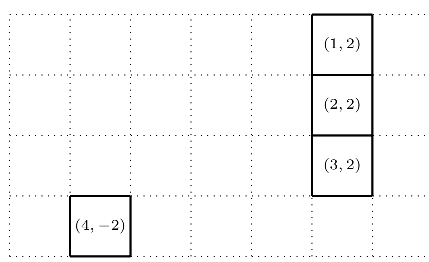


FIGURE 3.1. Skew shape $\tau = (2, 2 - 2)/(1, 1, -3)$.

$|\tau|$ denote the cardinality of τ, then $|\tau| = |\nu| - |\beta| = \sum_{i=1}^N (\nu_i - \beta_i)$.

Let τ be a skew shape defined above with $|\tau| = n$. Let $T : \{1, \cdots, n\} \to \tau$ be a bijective map

$$T : \{1, \cdots, n\} \to \tau$$

$$k \mapsto T(k) = (\mathfrak{i}(k), \mathfrak{j}(k)),$$

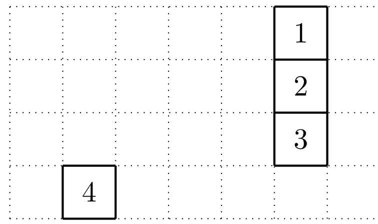


FIGURE 3.2. A standard tableaux on the skew shape $\tau = (2, 2-2)/(1, 1, -3)$

where $\mathfrak{i} : \{1, \cdots, n\} \to \mathbb{Z}$ is a function denoting the row number and $\mathfrak{j} : \{1, \cdots, n\} \to \mathbb{Z}$ is a function denoting the column number. Then T is called a tableau on τ, namely $Im(T) = \tau$. If both $\mathfrak{i}$ and $\mathfrak{j}$ are increasing, then T is called a standard tableau on τ.

3.2.2. Computation of $\mathcal{A}^\lambda(G/K)$. Let $\mathcal{A}(G)$ be the collection of all the analytic functions f on a small open set $U \subset G$. Then $\mathcal{A}(G)$ has a $G \times G$-module structure and it follows

$$\mathcal{A}(G) = \bigoplus_{\beta \in P^+} V^\beta \otimes V^{\beta^*},$$

where β^* is the dual of β, i.e. $\beta_i^* = -\beta_{N-i+1}$. Let $|\beta| = \sum_{i=1}^N \beta_i$. Then $|\beta^*| = -|\beta|$. Let $\mathcal{A}^\lambda(G/K)$ be the collection of all the analytic functions f on a small open set $U \subset G$ such that $\frac{d}{dt}|_{t=0} f(Ae^{tz}) = \lambda\chi(z)f(A)$ for any $z \in \mathfrak{t}_0$, where $\mathfrak{t}_0$ denotes the space of traceless matrices in $\mathfrak{t}$ and $A \in G/K$. Then as a left G-module, we have the following decomposition for $\mathcal{A}^\lambda(G/K)$,

$$\mathcal{A}^\lambda(G/K) = \bigoplus_{\beta \in P^+} V^\beta \otimes (V^{\beta^*})^{\mathfrak{t}_0, \lambda\chi},$$

where the G acts only on V^β and $(V^{\beta^*})^{\mathfrak{t}_0,\lambda\chi}$ only gives multiplicities.

Moreover, by Proposition 2.4.3, it follows that $(V^{\beta^*})^{\mathfrak{t}_0,\lambda\chi} \cong Hom_{\mathfrak{t}}(\mathbb{1}_\varphi, V^{\beta^*})$, where $\mathbb{1}_\varphi$ is a one-dimensional character of $Lie(K) = \mathfrak{t}$ and $\mathbb{1}_\varphi = (\lambda q + \frac{|\beta^*|}{N})tr_p + (-\lambda p + \frac{|\beta^*|}{N})tr_q$. According to Okada's theorem [**15**], the dimension of the space $(V^{\beta^*})^{\mathfrak{t}_0,\lambda\chi}$ is either 1 or 0 and the dimension is nonzero only when the dominant integral weight β^* satisfies the following conditions:

$$\beta_i^* \geq max(s,t), \quad i = 1, 2, \cdots, p;$$

$$\beta_i^* = t, \quad i = p+1, \cdots, q;$$

$$\beta_{N-i+1}^* = s + t - \beta_i^* \leq min(s,t), \quad i = 1, \cdots, p.$$

where $s = \lambda q - \frac{|\beta|}{N}$ and $t = -\lambda p - \frac{|\beta|}{N}$. Then β satisfies the conditions accordingly:

$$\beta_i \geq -min(s,t), \quad i = 1, 2, \cdots, p; \tag{3.13}$$

$$\beta_i = -t, \quad i = p+1, \cdots, q; \tag{3.14}$$

$$\beta_{N-i+1} = -s - t - \beta_i \leq -max(s,t), \quad i = 1, \cdots, p. \tag{3.15}$$

Remark 3.2.1. *(1) It suffices to consider the case when both s and t are integers, otherwise $(V^{\beta^*})^{\mathfrak{t}_0, \lambda\chi} = 0$.*

(2) The character $\mathbb{1}_\theta$ of $\mathfrak{t}$ depends on $|\beta^*| = -|\beta|$. For each given number $|\beta|$ such that both s and t are integers, we compute β satisfying (3.13)-(3.15).

Let B_c denote the collection of dominant integral weights β such that $|\beta| = c$ and β satisfies (3.13)-(3.15). Let $B = \sqcup_{c \in C} B_c$, where C denotes the collection of numbers c such that both $\lambda q - \frac{c}{N}$ and $-\lambda p - \frac{c}{N}$ are integers. Then we conclude that $\mathcal{A}^\lambda(G/K)$ decomposes as follows

$$\mathcal{A}^\lambda(G/K) = \bigoplus_{c \in C} \big(\bigoplus_{\beta \in B_c} V^\beta \big).$$

We will show in the following sections that for each $c \in C$, $((\bigoplus_{\beta \in B_c} V^\beta) \otimes V^{\otimes n})^{\mathfrak{t}_0, \mu}$ forms a representation of $H_n(u, k_1, k_2, k_3)$.

Let us see the following example of computation of $\beta \in B_c$ for some $c \in C$.

Example 3.2.2. *Let G be GL_4 and $p = 1$, i.e. $K = GL_1 \times GL_3$ and $\mathfrak{t} = \mathfrak{gl}_1 \times \mathfrak{gl}_3$. Consider $\mathcal{D}^1(G/K)$ be the sheaf of differential operators on G/K, twisted by the character χ, i.e. local sections of $\mathcal{D}^1(G/K)$ act on χ-twisted functions on G/K which are analytic functions f on a small open set $U \subset G$ such that $\frac{d}{dt}|_{t=0} f(Ae^{tz}) = \chi(z) f(A)$, for $z \in \mathfrak{t}$. Now compute β^* such that $(V^{\beta^*})^{\mathfrak{t}_0, \chi}$*

is nonzero.

Fix $c = |\beta^*| = \sum_{i=1}^4 \beta_i^* = 0$, *then*

$$s = \lambda q + \frac{|\beta^*|}{N} = 3, t = -\lambda p + \frac{|\beta^*|}{N} = -1.$$

Thus we obtain β^* *satisfying the following conditions:*

$$\beta_1^* \geq 3;$$

$$\beta_2^* = \beta_3^* = -1;$$

$$\beta_4^* = 2 - \beta_1^* \leq -1.$$

Then the corresponding β *satisfies:*

$$\beta_1 \geq 1; \tag{3.16}$$

$$\beta_2 = \beta_3 = 1; \tag{3.17}$$

$$\beta_4 = -2 - \beta_1 \leq -3. \tag{3.18}$$

So $B_0 = \{\beta \in P^+ | \beta \text{ satisfies } (3.16) - (3.18)\}$.

3.2.3. Computation of the invariant space $F^{\lambda}_{n,p,\mu}(\mathcal{A}^{\lambda}(G/K))$. From last subsection we obtain $\mathcal{A}^{\lambda}(G/K) = \bigoplus_{c\in C}(\bigoplus_{\beta\in B_c} V^{\beta})$. In this subsection, we compute for each $c \in C$ the $(\mathfrak{t}_0, \mu)$ invariant space

$$(V^{\beta} \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu},$$

then the image $F^{\lambda}_{n,p,\mu}(\mathcal{A}^{\lambda}(G/K)) = \bigoplus_{c\in C}(\bigoplus_{\beta\in B_c}(V^{\beta} \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu})$.

According to Proposition 2.4.3, for each $\beta \in B_c$, the $(\mathfrak{t}_0, \mu)$ invariant space $(V^{\beta} \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ is computed by

$$\begin{aligned}(V^{\beta} \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu} \cong& Hom_{\mathfrak{t}_0}(\mathbb{1}_{\mu\chi}, Res^{\mathfrak{gl}_N}_{\mathfrak{t}_0} V^{\beta} \otimes V^{\otimes n}) \\ \cong& Hom_{\mathfrak{t}}(\mathbb{1}_{\theta}, Res^{\mathfrak{gl}_N}_{\mathfrak{t}} V^{\beta} \otimes V^{\otimes n}),\end{aligned}$$

where $\mathbb{1}_\theta$ is a one-dimensional $\mathfrak{t}$-module related to the character $\mu\chi$ of $\mathfrak{t}_0$ and

$$\mathbb{1}_\theta = ((\mu q + \frac{c+n}{N})tr_p + (-\mu p + \frac{c+n}{N})tr_q.$$

Then the integral dominant weight ν such that the irreducible summand V^ν of $V^\beta \otimes V^{\otimes n}$ with $Hom_{\mathfrak{t}}(\mathbb{1}_\theta, V^\nu) \neq 0$ satisfies the following conditions:

$$\nu_i \geq max(a,b), \quad i = 1, 2, \cdots, p; \tag{3.19}$$

$$\nu_i = b, \quad i = p+1, \cdots, q; \tag{3.20}$$

$$\nu_{N-i+1} = a + b - \nu_i \leq min(a,b), \quad i = 1, \cdots, p, \tag{3.21}$$

where $a = \mu q + \frac{c+n}{N}$ and $b = -\mu p + \frac{c+n}{N}$. Then there exists a basis of the invariant space

$$\bigoplus_{\beta \in B_c} (V^\beta \otimes V^{\otimes n})^{\mathfrak{t}_0, \mu}$$

which is indexed by the collection of standard tableaux on skew shapes ν/β such that $\beta \subset \nu$, β satisfies (3.13)-(3.15) and ν satisfies (3.19)-(3.21).

Continue with Example 3.2.2, where $p = 1$, $\mu = -1$ for each β satisfying

$$\beta_1 \geq 1;$$

$$\beta_2 = \beta_3 = 1;$$

$$\beta_4 = -2 - \beta_1 \leq -3,$$

we compute the $(\mathfrak{t}_0, \mu)$ invariant space

$$\begin{aligned} &(V^\beta \otimes V^{\otimes n})^{\mathfrak{t}_0, \mu} \\ =&Hom_{\mathfrak{t}}(atr_p + btr_q, Res_{\mathfrak{t}}^{\mathfrak{gl}_N} V^\beta \otimes V^{\otimes n}), \end{aligned}$$

where $a = \mu q + \frac{n}{N} = -2$ and $b = -\mu p + \frac{n}{N} = 2$. Then by Okada's theorem [**15**], the dominant weight ν such that the irreducible summand V^ν of $V^\beta \otimes V^{\otimes n}$ with $Hom_K(det^a \boxtimes det^b, V^\nu) \neq 0$

satisfies the following conditions:

$$\nu_1 \geq 2, \tag{3.22}$$

$$\nu_2 = \nu_3 = 2, \tag{3.23}$$

$$\nu_4 \leq -2, \tag{3.24}$$

and thus there exists a basis of the invariant space $\bigoplus_{\beta \in B_c}(V^\beta \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ indexed by the collection of standard tableaux on skew shapes ν/β such that β satisfies (3.16)-(3.18) and ν satisfies (3.22)-(3.24).

Remark 3.2.3. *We have the following facts for the vector space* $\bigoplus_{\beta \in B_c}(V^\beta \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$.

(1) *The number of cells in row i such that $p+1 \leq i \leq q$ equals $b+t$ and the sum of the numbers of cells in row i and row $N-i+1$ equals $a+b+s+t$ for $1 \leq i \leq p$. Moreover, the numbers $b+t$ and $a+b+s+t$ do not depend on $|\beta| = c$.*

(2) *We only consider the image when $b+t \geq 0$, i.e. $-p(\mu+\lambda) + \frac{n}{N} \geq 0$. Otherwise $\beta \not\subset \nu$ for all β satisfying* (3.13)-(3.15) *and ν satisfying* (3.19)-(3.21) *and hence the invariant space $F^\lambda_{n,p,\mu}(\mathcal{A}^\lambda(G/K)) = 0$.*

(3) *Similarly, we consider the image when $a+b+s+t \geq 0$, i.e. $(q-p)(\mu+\lambda) + \frac{2n}{N} \geq 0$. Otherwise $\beta \not\subset \nu$ for all β satisfying* (3.13)-(3.15) *and ν satisfying* (3.19)-(3.21) *and hence the invariant space $F^\lambda_{n,p,\mu}(\mathcal{A}^\lambda(G/K)) = 0$.*

3.2.4. A skew shape. For the functor $F^\lambda_{n,p,\mu}$ and a number $c \in C$, we associate a skew shape τ_c to the vector space $\bigoplus_{\beta \in B_c}(V^\beta \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$.
Let us define τ_c in different cases.

Case 1. If $t \leq s$ and $b \leq a$, then $-t \leq a$ and $-s \leq b$. Set ν_c and β_c as follows.

$$(\nu_c)_i = a, \quad i = 1, 2, \cdots, p;$$

$$(\nu_c)_i = b, \quad i = p+1, \cdots, q;$$

$$(\nu_c)_{N-i+1} = b, \quad i = 1, \cdots, p$$

and

$$(\beta_c)_i = -t, \quad i = 1, 2, \cdots, p;$$

$$(\beta_c)_i = -t, \quad i = p+1, \cdots, q;$$

$$(\beta_c)_{N-i+1} = -s, \quad i = 1, \cdots, p.$$

Let $\tau_c = \nu_c/\beta_c$.

Case 2. If $s < t$ and $a < b$, then we have three subcases.

Case 2a. If $b + s \geq 0$ and $a + t \geq 0$, set ν_c and β_c as follows.

$$(\nu_c)_i = b, \quad i = 1, 2, \cdots, p;$$

$$(\nu_c)_i = b, \quad i = p+1, \cdots, q;$$

$$(\nu_c)_{N-i+1} = a, \quad i = 1, \cdots, p$$

and

$$(\beta_c)_i = -s, \quad i = 1, 2, \cdots, p;$$

$$(\beta_c)_i = -t, \quad i = p+1, \cdots, q;$$

$$(\beta_c)_{N-i+1} = -t, \quad i = 1, \cdots, p.$$

Let $\tau_c = \nu_c/\beta_c$.

Case 2b. If $b+s<0$ and $a+t>0$, set ν_c and β_c as follows.

$$(\nu_c)_i = -s, \quad i = 1, 2, \cdots, p;$$

$$(\nu_c)_i = b, \quad i = p+1, \cdots, q;$$

$$(\nu_c)_{N-i+1} = a+b+s, \quad i = 1, \cdots, p$$

and

$$(\beta_c)_i = -s, \quad i = 1, 2, \cdots, p;$$

$$(\beta_c)_i = -t, \quad i = p+1, \cdots, q;$$

$$(\beta_c)_{N-i+1} = -t, \quad i = 1, \cdots, p.$$

Let $\tau_c = \nu_c/\beta_c$.

Case 2c. If $b+s>0$ and $a+t<0$, set ν_c and β_c as follows.

$$(\nu_c)_i = b, \quad i = 1, 2, \cdots, p;$$

$$(\nu_c)_i = b, \quad i = p+1, \cdots, q;$$

$$(\nu_c)_{N-i+1} = a, \quad i = 1, \cdots, p$$

and

$$(\beta_c)_i = -a-s-t, \quad i = 1, 2, \cdots, p;$$

$$(\beta_c)_i = -t, \quad i = p+1, \cdots, q;$$

$$(\beta_c)_{N-i+1} = a, \quad i = 1, \cdots, p.$$

Let $\tau_c = \nu_c/\beta_c$.

Case 3. If $s<t$ and $b \le a$, then we have the following two subcases since $b+t \ge 0$ and $a+b+s+t \ge 0$.

Case 3a. If $a + s \geq 0$, set ν_c and β_c as follows.

$$(\nu_c)_i = a, \quad i = 1, 2, \cdots, p;$$

$$(\nu_c)_i = b, \quad i = p+1, \cdots, q;$$

$$(\nu_c)_{N-i+1} = b, \quad i = 1, \cdots, p$$

and

$$(\beta_c)_i = -s, \quad i = 1, 2, \cdots, p;$$

$$(\beta_c)_i = -t, \quad i = p+1, \cdots, q;$$

$$(\beta_c)_{N-i+1} = -t, \quad i = 1, \cdots, p.$$

Let $\tau_c = \nu_c/\beta_c$.

Case 3b. If $a + s < 0$, set ν_c and β_c as follows.

$$(\nu_c)_i = -s, \quad i = 1, 2, \cdots, p;$$

$$(\nu_c)_i = b, \quad i = p+1, \cdots, q;$$

$$(\nu_c)_{N-i+1} = a + b + s, \quad i = 1, \cdots, p$$

and

$$(\beta_c)_i = -s, \quad i = 1, 2, \cdots, p;$$

$$(\beta_c)_i = -t, \quad i = p+1, \cdots, q;$$

$$(\beta_c)_{N-i+1} = -t, \quad i = 1, \cdots, p.$$

Let $\tau_c = \nu_c/\beta_c$.

case 4. If $t \leq s$ and $b \leq a$, then we have the following two subcases since $b+t \geq 0$ and $a+b+s+t \geq 0$.

Case 4a. If $a + s \geq 0$, set ν_c and β_c as follows.

$$(\nu_c)_i = b, \quad i = 1, 2, \cdots, p;$$

$$(\nu_c)_i = b, \quad i = p+1, \cdots, q;$$

$$(\nu_c)_{N-i+1} = a, \quad i = 1, \cdots, p$$

and

$$(\beta_c)_i = -t, \quad i = 1, 2, \cdots, p;$$

$$(\beta_c)_i = -t, \quad i = p+1, \cdots, q;$$

$$(\beta_c)_{N-i+1} = -s, \quad i = 1, \cdots, p.$$

Let $\tau_c = \nu_c/\beta_c$.

Case 4b. If $a + s < 0$, set ν_c and β_c as follows.

$$(\nu_c)_i = b, \quad i = 1, 2, \cdots, p;$$

$$(\nu_c)_i = b, \quad i = p+1, \cdots, q;$$

$$(\nu_c)_{N-i+1} = a, \quad i = 1, \cdots, p$$

and

$$(\beta_c)_i = -a - s - t, \quad i = 1, 2, \cdots, p;$$

$$(\beta_c)_i = -t, \quad i = p+1, \cdots, q;$$

$$(\beta_c)_{N-i+1} = a, \quad i = 1, \cdots, p.$$

Let $\tau_c = \nu_c/\beta_c$.

3.2.5. Moves on τ_c. Our goal is to recover from τ_c all the skew shapes ν/β such that $\beta \subset \nu$, β satisfies (3.13)-(3.15) and ν satisfies (3.19)-(3.21). Now let us define two moves on a skew shape $\tau = \nu/\beta$ with N rows:

β-move Let $\beta' \in P^+$ and $\beta' = \beta + \epsilon_i - \epsilon_{N-i+1}$. The β-move on $\tau = \nu/\beta$ gives a new skew shape $\tau' = \nu/\beta'$. We denote β-move by

$$\tau \xrightarrow{\quad \beta \quad} \tau'$$

ν-move Let $\nu' \in P^+$ and $\nu' = \nu + \epsilon_i - \epsilon_{N-i+1}$. The ν-move on $\tau = \nu/\beta$ gives a new skew shape $\tau' = \nu'/\beta$. We denote the ν-move by

$$\tau \xrightarrow{\quad \nu \quad} \tau'$$

EXAMPLE 3.2.4. *Continue with Example 3.2.2 $N = 4$, $p = 1$, $\lambda = 1$ and $\mu = -1$.*

$$\tau_0 = (2, 2, 2, -2)/(1, 1, 1, -3)$$

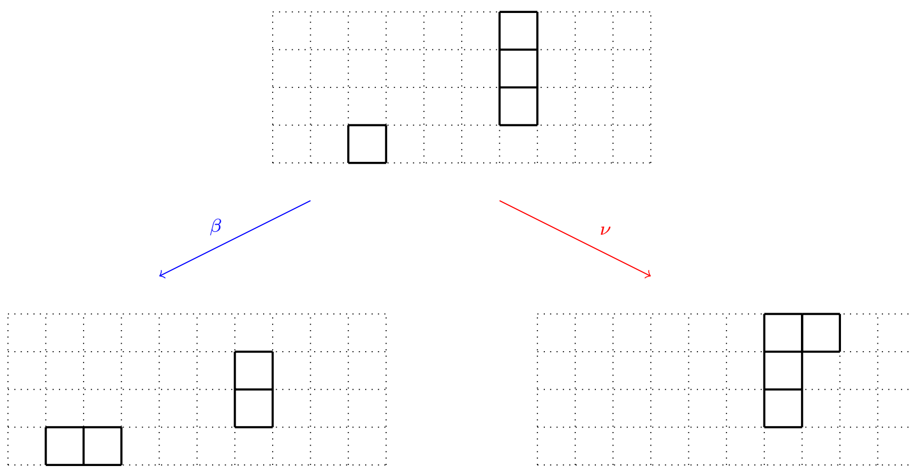


FIGURE 3.3. β-move and ν-move

Let $D_c^{\lambda,\mu}$ denote the set of skew shapes obtained by applying β-moves and ν-moves on τ_c for finitely many times. Then $D_c^{\lambda,\mu}$ consists of all the skew shapes ν/β such that $\beta \subset \nu$, β satisfies (3.13)-(3.15) and ν satisfies (3.19)-(3.21).

THEOREM 3.2.5. *Let $Tab_c^{\lambda,\mu}$ denote the collection of standard tableaux T such that the shape $Im(T) \in D_c^{\lambda,\mu}$. There is a basis of the invariant space*

$$\bigoplus_{\beta \in B_c} (V^\beta \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$$

which is indexed by $Tab_c^{\lambda,\mu}$.

3.3. $\mathcal{Y}$-actions

In [**5**], the linear operator $\tilde{y_k}$ on the invariant space is defined by $\tilde{y}_k = \sum_{i|j} L_{ij} \otimes (E_{ji})_k$. Consider $\mathcal{A}^\lambda(G/K)$ as a left G-module, then we have $g \cdot f(A) = f(g^{-1}A)$ for each $g \in G$ and $A \in G/K$. The action of L_{ij} is defined as

$$\begin{aligned} L_{ij} \cdot f(A) &= L_{E_{ij}} \cdot f(A) \\ &= \frac{d}{dt}|_{t=0} f(e^{tE_{ij}}A) \\ &= \frac{d}{dt}|_{t=0} e^{-tE_{ij}} \cdot f(A) \\ &= -E_{ij} \cdot f(A). \end{aligned}$$

Then the linear operator $\tilde{y}_k$ acts on $F^\lambda_{n,p,\mu}(\mathcal{A}^\lambda(G/K))$ by $-E_{ij} \otimes (E_{ji})_k$, which is the same of the action of $\tilde{y}_k$ in the degenerate affine Hecke algebra case. Thus we apply Theorem 2.5.3 to compute the $\mathcal{Y}$-action. Let $T \in Tab_c^{\lambda,\mu}$ and v_T is a basis element indexed by the standard tableau T. It follows that y_k acts on v_T by the scalar

$$\zeta_k^T = -cont_T(k) + \frac{c+n}{N} - \frac{N}{2} - \frac{\mu(p-q)}{2}.$$

So v_T is weight vector of weight $\zeta^T = [\zeta_1^T, \cdots, \zeta_n^T]$ and we conclude that $\bigoplus_{\beta \in B_c} (V^\beta \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ is a $\mathcal{Y}$-semisimple representation of $H_n(u, k_1, k_2, k_3)$.

Example 3.3.1. *Let us consider Example 3.2.2 and $T \in Tab^{1,0}_{4,1,-1}$. Then the action of y_k is computed by the content of k.*

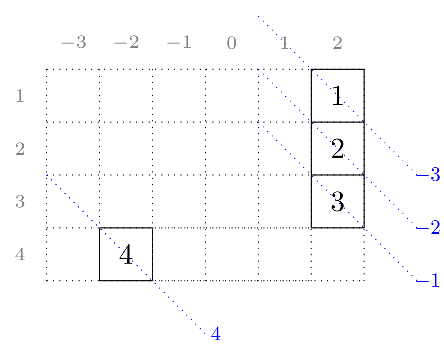


Figure 3.4. A standard tableau T of shape $\tau_0 = (2, 2, 2, -2)/(1, 1, 1, -3)$

3.3.1. Degenerate double affine Hecke algebra $\hat{H}_n(u, k_1, k_2, k_3)$. For the ease of combinatorial description, we take a different presentation of the degenerate double affine Hecke algebra. Let W_a be the affine Weyl group of type C_n generated by $\gamma_1, s_1, \cdots, s_{n-1}, s_n$ with the following relations:

$$s_i^2 = 1, \text{ for } i = 0, 1, \cdots, n, \tag{3.25}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \text{ for } i = 1, \cdots, n-1, \tag{3.26}$$

$$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0, \tag{3.27}$$

$$s_n s_{n-1} s_n s_{n-1} = s_{n-1} s_n s_{n-1} s_n, \tag{3.28}$$

$$s_i s_j = s_j s_i, \text{ for } |i - j| > 1, \tag{3.29}$$

where we take the notation $\gamma_1 = s_0$.

Let $\hat{H}_n(u, k_1, k_2, k_3)$ be the degenerate double affine Hecke algebra of type C_n generated by

$$s_0, s_1, \cdots, s_{n-1}, s_n, y_1, \cdots, y_n$$

with the relations (3.25)-(3.29) and additional relations:

$$s_0 y_1 + y_1 s_0 = k_2 + k_3, \tag{3.30}$$

$$s_i y_i - y_{i+1} s_i = -k_1, i = 1, \cdots, n-1 \tag{3.31}$$

$$s_n y_n - (u - y_n) s_n = -k_2, \tag{3.32}$$

$$y_i y_j = y_j y_i, \tag{3.33}$$

$$s_i y_j = y_j s_i, j \neq i, i+1, \tag{3.34}$$

$$s_0 y_j = y_j s_0, j \neq 1, \tag{3.35}$$

$$s_n y_j = y_j s_n, j \neq n. \tag{3.36}$$

There is an isomorphism between $H_n(u,k_1,k_2,k_3)$ and $\hat{H}_n(u,k_1,k_2,k_3)$

$$\begin{aligned}\sigma :&H_n(u,k_1,k_2,k_3) \to \hat{H}_n(u,k_1,k_2,k_3)\\ &s_i \mapsto s_{n-i}, i = 1,\cdots,n-1\\ &s_0 \mapsto s_n,\\ &s_n \mapsto s_0\\ &y_i \mapsto y_{n-i+1}, i = 1,\cdots,n.\end{aligned}$$

Moreover, we take the following notations:

$$y_{-i} = -y_i, \tag{3.37}$$

$$y_{k(2n+1)\pm i} = k\cdot u \pm y_i, \tag{3.38}$$

where $i = 1,\cdots,n$. Let $\mathcal{Z}_n = \mathbb{Z}\setminus\{k\cdot(2n+1)|k\in\mathbb{Z}\}$. In this way we define y_i for $i\in\mathcal{Z}_n$. In particular, $u - y_n = y_{(2n+1)-n} = y_{n+1}$. We also take the convention $y_{k\cdot(2n+1)} = y_{k\cdot(2n+1)-1}$. Then the relations (3.30)-(3.32) are written by

$$s_i y_i - y_{i+1} s_i = -u_i, \tag{3.39}$$

where

$$u_i = \begin{cases} k_2 + k_3, & i = 0\\ k_1, & i = 1,\cdots,n-1\\ k_2, & i = n.\end{cases} \tag{3.40}$$

3.3.2. Representations of $\hat{H}_n(u,k_1,k_2,k_3)$. A representation ρ of $H_n(u,k_1,k_2,k_3)$ on M induces a representation $\hat{\rho}$ of $\hat{H}_n(u,k_1,k_2,k_3)$ on M by

$$\begin{aligned}&\hat{\rho}(s_i) = \rho(s_{n-i}), i = 0,1,\cdots,n\\ &\hat{\rho}(y_i) = \rho(y_{n-i+1}), i = 1,\cdots,n.\end{aligned}$$

Moreover, if M is a $\mathcal{Y}$-semisimple representation of $H_n(u, k_1, k_2, k_3)$, then M is also $\mathcal{Y}$-semisimple as a representation of $\hat{H}_n(u, k_1, k_2, k_3)$. Let $v \in M$ be a weight vector of weight $\zeta = [\zeta_1, \zeta_2, \cdots, \zeta_n]$ as a representation of $H_n(u, k_1, k_2, k_3)$, then $v \in M$ is also a weight vector of weight $\hat{\zeta} = [\hat{\zeta}_1, \cdots, \hat{\zeta}_n]$ as a representation of H, where $\hat{\zeta}_i = \zeta_{n-i+1}$.

Hence the $\mathcal{Y}$-semisimple representation $\bigoplus_{\beta \in B_c}(V^\beta \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ of $H_n(u, k_1, k_2, k_3)$ is also a $\mathcal{Y}$-semisimple representation of $\hat{H}_n(u, k_1, k_2, k_3)$. Let $\tau = \nu/\beta$ be the shape $Im(T)$ of the standard tableau T with $\nu = (\nu_1, \cdots, \nu_N)$ and $\beta = (\beta_1, \cdots, \beta_N)$. Now we associate a standard tableau $\hat{T}$ to T as follows. Let $\hat{\nu} \in P^+$ such that $\hat{\nu}_i = -\nu_{N-i+1}$ and $\hat{\beta} \in P^+$ such that $\hat{\beta}_i = -\beta_{N-i+1}$. Then $\hat{\nu} \subset \hat{\beta}$ and set a new skew shape $\hat{\tau} = \hat{\beta}/\hat{\nu}$. Define a tableau $\hat{T}$ by

$$\begin{aligned} \hat{T}: \qquad & \{1, 2, \cdots, n\} \to \hat{\tau} \\ & \qquad k \mapsto (N - \mathfrak{i}(n-k+1) + 1, -\mathfrak{j}(n-k+1) + 1). \end{aligned}$$

It is not hard to see that $\hat{T}$ is also a standard tableau. Let $\hat{D}_c^{\lambda,\mu}$ be the collection of skew shapes $\{\hat{\tau} | \tau \in D_c^{\lambda,\mu}\}$ and $\widehat{Tab}_c^{\lambda,\mu}$ be the collection of standard tableaux $\{\hat{T} | T \in Tab_c^{\lambda,\mu}\}$ which consists of standard tableau $\hat{T}$ such that $Im(\hat{T}) \in \hat{D}_c^{\lambda,\mu}$. Then $\bigoplus_{\beta \in B_c}(V^\beta \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ as a representation of $\hat{H}_n(u, k_1, k_2 k_3)$ has a weight basis indexed by $\widehat{Tab}_c^{\lambda,\mu}$. Similarly, we define $\hat{\beta}$-move and $\hat{\nu}$-moves:

$\hat{\beta}$-move Let $\hat{\beta}' \in P^+$ and $\hat{\beta}' = \hat{\beta} + \epsilon_i - \epsilon_{N-i+1}$. The $\hat{\beta}$-move on $\hat{\tau} = \hat{\nu}/\hat{\beta}$ gives a new skew shape $\hat{\tau}' = \hat{\nu}/\hat{\beta}'$. We denote $\hat{\beta}$-move by

$$\hat{\tau} \xrightarrow{\quad \hat{\beta} \quad} \hat{\tau}'.$$

$\hat{\nu}$-move Let $\hat{\nu}' \in P^+$ and $\hat{\nu}' = \hat{\nu} + \epsilon_i - \epsilon_{N-i+1}$. The ν-move on $\hat{\tau} = \hat{\nu}/\hat{\beta}$ gives a new skew shape $\hat{\tau}' = \hat{\nu}'/\hat{\beta}$. We denote the $\hat{\nu}$-move by

$$\hat{\tau} \xrightarrow{\quad \hat{\nu} \quad} \hat{\tau}'.$$

Then $\hat{D}_c^{\lambda,\mu}$ is the collection of shapes obtained by applying $\hat{\beta}$-move and $\hat{\nu}$-move for finitely many times on $\hat{\tau}_c$. And $\widehat{Tab}_c^{\lambda,\mu}$ consists of all the standard tableaux $\hat{T}$ with $Im(\hat{T}) \in \hat{D}_c^{\lambda,\mu}$.

Let $v_{\hat{T}} \in \bigoplus_{\beta \in B_c}(V^\beta \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ be a weight vector corresponding to the standard tableau $\hat{T}$. Then $v_{\hat{T}}$ is a weight vector of weight $\zeta^{\hat{T}} = [\zeta_1^{\hat{T}}, \cdots, \zeta_n^{\hat{T}}]$, where

$$\zeta_k^{\hat{T}} = cont_{\hat{T}}(k) + \frac{c+n}{N} + \frac{\mu(q-p)+N}{2}.$$

FIGURE 3.5. A standard tableau T and the corresponding standard tableau $\hat{T}$

EXAMPLE 3.3.2. *Consider Example 3.2.2. The invariant space* $\bigoplus_{\beta\in B_0}(V^{\beta}\otimes V^{\otimes 4})^{\mathfrak{t}_0,-1}$ *is representation of* $H_4(2,1,-6,8)$ *which has a weight basis indexed by* $Tab^{1,0}_{4,1,-1}$. *The invariant space* $\bigoplus_{\beta\in B_0}(V^{\beta}\otimes V^{\otimes 4})^{\mathfrak{t}_0,-1}$ *has an* $\hat{H}_4(2,1,-6,8)$*-representation structure which has a weight basis indexed by* $\widehat{Tab}^{1,0}_{4,1,-1}$. *Figure 3.5 is a standard tableau* $T\in Tab^{1,0}_{4,1,-1}$ *and* $\hat{T}\in\widehat{Tab}^{1,0}_{4,1,-1}$ *is the corresponding standard tableau. Let* ζ^{T} *and* $\zeta^{\hat{T}}$ *be weights corresponding to* T *and* $\hat{T}$ *respectively. Then* $\zeta^{T}=[-3,-2,-1,4]$ *and* $\zeta^{\hat{T}}=[4,-1,-2,-3]$.

3.4. Intertwining operators

We define the intertwining operators in $\hat{H}_n(u,k_1,k_2,k_3)$ for $i=0,1,\cdots,n$:

$$\phi_0=-2s_0y_1+k_2+k_3, \tag{3.41}$$

$$\phi_i=s_i(y_i-y_{i+1})+k_1, i=1,\cdots,n-1, \tag{3.42}$$

$$\phi_n=s_n(2y_n-u)+k_2. \tag{3.43}$$

With notations (3.37), (3.38) and (3.40), we write (3.41)-(3.43) by

$$\phi_i=s_i(y_i-y_{i+1})+u_i \tag{3.44}$$

$$=-(y_i-y_{i+1})s_i-u_i. \tag{3.45}$$

By straightforward computation, we have

$$\phi_0^2 = (k_2 + k_3 - 2y_1)(k_2 + k_3 + 2y_1), \tag{3.46}$$

$$\phi_i^2 = (k_1 - y_i + y_{i+1})(k_1 + y_i - y_{i+1}), i = 1, \cdots, n-1, \tag{3.47}$$

$$\phi_n^2 = (k_2 - 2y_n + u)(k_2 + 2y_n - u). \tag{3.48}$$

Hence we write (3.46)-(3.48) by

$$\phi_i^2 = (u_i + y_i - y_{i+1})(u_i - y_i + y_{i+1}). \tag{3.49}$$

PROPOSITION 3.4.1. *The intertwining operators defined above satisfy the same braid relations as relations* (3.26)-(3.29), *namely*

$$\phi_i\phi_{i+1}\phi_i = \phi_{i+1}\phi_i\phi_{i+1}, \text{ for } i = 1, \cdots, n-1,$$

$$\phi_0\phi_1\phi_0\phi_1 = \phi_1\phi_0\phi_1\phi_0,$$

$$\phi_{n-1}\phi_n\phi_{n-1}\phi_n = \phi_n\phi_{n-1}\phi_n\phi_{n-1},$$

$$\phi_i\phi_j = \phi_j\phi_i, \text{ for } |i-j| > 1.$$

So for each $\omega \in W_a$, let $\omega = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression. We define the intertwining operator

$$\phi_\omega = \phi_{i_1} \cdots \phi_{i_\ell}.$$

The affine Weyl group W_a has an action on $\mathcal{Z}_n$. For $k \in \mathcal{Z}_n$ and $m \in \mathbb{Z}$,

$$s_0(k) = \begin{cases} -k, & k = \pm 1 + m(2n+1) \\ k, & \text{otherwise}, \end{cases}$$

for $i = 1, \cdots, n-1$,

$$s_i(k) = \begin{cases} k \pm 1, & k = \pm i + m(2n+1) \\ k \mp 1, & k = \pm(i+1) + m(2n+1) \\ k, & \text{otherwise} \end{cases}$$

and

$$s_n(k) = \begin{cases} k \pm 1, & k = \pm n + m(2n+1) \\ k, & \text{otherwise.} \end{cases}$$

We verify the following fact.

PROPOSITION 3.4.2. *For each $\omega \in W_a$, let $\omega = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression. Then*

$$\phi_\omega = \omega \prod_{p=1}^{\ell} (y_{\omega_p(i_p)} - y_{\omega_p(i_p+1)}) + \sum_{x<\omega} xP(y),$$

where $\omega_p = s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_{p+1}}$ and $P(y)$ is some polynomial on $y_1, \cdots, y_n$.

Now let us explore properties of these intertwining operators.

PROPOSITION 3.4.3. *The intertwining operators satisfy the following:*

(1) $y_1\phi_0 = -\phi_0 y_1$,

(2) $y_i\phi_i = \phi_i y_{i+1}$ *and* $y_{i+1}\phi_i = \phi_i y_i$, *for* $i = 1, \cdots, n-1$,

(3) $y_n\phi_n = \phi_n(u - y_n)$,

(4) $y_i\phi_j = \phi_j y_i$, *for* $i \neq j, j+1$.

PROOF. We write (1) – (4) by $y_i\phi_j = \phi_j y_{s_j(i)}$ for $i \in \mathcal{Z}_n$ and $j = 0, 1, \cdots, n$. Applying (3.44) and then (3.39), we have for each $i = 0, 1, \cdots, n$,

$$\begin{aligned} y_i\phi_i &= y_i s_i(y_i - y_{i+1}) + u_1 y_i \\ &= s_i y_{i+1}(y_i - y_{i+1}) - u_1(y_i - y_{i+1}) + u_1 y_i \\ &= s_i(y_i - y_{i+1})y_{i+1} + u_1 y_{i+1} \\ &= \phi_i y_{i+1}. \end{aligned}$$

Similarly we show

$$\begin{aligned}
y_i\phi_{i-1} &= y_i s_{i-1}(y_{i-1}-y_i)+u_1 y_i \\
&= s_{i-1}y_{i-1}(y_{i-1}-y_i)+u_1(y_{i-1}-y_i)+u_1 y_i \\
&= s_{i-1}(y_{i-1}-y_i)y_{i-1}+u_1 y_{i-1} \\
&= \phi_{i-1}y_{i-1}.
\end{aligned}$$

By (3.34)-(3.36), we verify (4). $\square$

COROLLARY 3.4.4. *For $\omega \in W_a$, it follows that $y_i\phi_\omega = \phi_\omega y_{\omega^{-1}(i)}$.*

For a weight $\zeta = [\zeta_1, \cdots, \zeta_n]$, we define for $i = 1, \cdots, n$ with

$$\zeta_{-i} = -\zeta_i$$

$$\zeta_{k(2n+1)+i} = k \cdot u + \zeta_i.$$

Then we extend ζ for $i \in \mathcal{Z}_n$ which is signed periodic, namely $\zeta_{i+k(2n+1)} = \zeta_i + k \cdot u$ $-\zeta_i = \zeta_{-i}$ for $i \in \mathcal{Z}_n$. Then the action of $\omega \in W_a$ on a weight $\zeta = [\zeta_1, \cdots, \zeta_n]$ is written by

$$(\omega\zeta)_i = \zeta_{\omega^{-1}(i)}. \tag{3.50}$$

COROLLARY 3.4.5. *Let L be a representation of the degenerate double affine Hecke algebra $\hat{H}_n(u, k_1, k_2, k_3)$ and $v \in L_\zeta$ is a weight vector of weight ζ, then $\phi_\omega.v \in L_{\omega\zeta}$ is 0 or a weight vector of weight $\omega\zeta$ for any $\omega \in W_a$.*

3.4.1. Properties of representations of $\hat{H}_n(u, k_1, k_2, k_3)$. In [**19**], several properties of $\mathcal{Y}$-semisimple representations of a double affine Hecke algebra of type A_n are explored. Now let us review these properties in the case of degenerate double affine Hecke algebra of type C_n.

LEMMA 3.4.6. *Let M be a $\mathcal{Y}$-semisimple representation of $\hat{H}_n(u, k_1, k_2, k_3)$ with $k_1, k_2, k_3 \neq 0$. Let M_ζ denote the weight space of weight ζ. If $\zeta_1 = 0$ or $\zeta_i = \zeta_{i+1}$ for $i \in \mathcal{Z}_n$, then $M_\zeta = 0$.*

PROOF. Suppose $v \in M_\zeta$ and $v \neq 0$. Consider the vector $s_i v$. Applying (44), we have

$$\begin{aligned}(y_i - y_{i+1})s_i v &= (-s_i(y_i - y_{i+1}) - 2u_i)v \\ &= -2u_i v \\ &\neq 0.\end{aligned}$$

Acting $(y_i - y_{i+1})$ again, we have

$$\begin{aligned}(y_i - y_{i+1})^2 s_i v &= -2u_i(y_i - y_{i+1})v \\ &= 0,\end{aligned}$$

So $v \in M_\zeta^{gen} \setminus M_\zeta$, which contradicts the fact that M is $\mathcal{Y}$-semisimple. Hence we conclude $M_\zeta = 0$ if $\zeta_i = \zeta_{i+1}$ for some $i \in \mathcal{Z}_n$. Similarly, we show $M_\zeta = 0$ if $\zeta_1 = 0$. $\square$

PROPOSITION 3.4.7. *Let M be a $\mathcal{Y}$-semisimple representation of $\hat{H}_n(u, k_1, k_2, k_3)$ and $v \in M$ is nonzero weight vector of weight ζ. Let $\omega \in W_a$ be an element such that $\omega \neq id$ and $\omega\zeta = \zeta$. It follows that $\phi_\omega v = 0$.*

PROOF. The fact $\omega\zeta = \zeta$ implies $\zeta_{\omega^{-1}(k)} = \zeta_k$ for all $k = 1, \cdots, n$. Since $\omega \neq id$, there is a number k such that $\omega^{-1}(k) \neq k$. Let $\omega = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression of ω. Then there is a number p such that $\omega_p s_{i_p} \omega_p^{-1}$ is a transposition $(k, \omega^{-1}(k))$. Consider $\phi_{\omega_p^{-1}} v$, which is weight vector of weight $\omega_p^{-1}\zeta$. Then $(\omega_p^{-1}\zeta)_{i_p} - (\omega_p^{-1}\zeta)_{i_p+1} = \zeta_{\omega_p(i_p)} - \zeta_{\omega_p(i_p+1)} = \pm(\zeta_k - \zeta_{\omega^{-1}(k)}) = 0$. By Lemma 3.4.6, the vector $\phi_{\omega_p^{-1}} v = 0$. Hence

$$\phi_\omega v = \phi_{s_{i_1} \cdots s_{i_p}} \phi_{\omega_p^{-1}} v = 0.$$

$\square$

PROPOSITION 3.4.8. *Let L be an irreducible $\mathcal{Y}$-semisimple representation of $\hat{H}_n(u, k_1, k_2, k_3)$ and $v \in L$ is a nonzero weight vector of weight ζ. Then $L = span_{\mathbb{C}}\{\phi_\omega v | \omega \in W_a\}$.*

PROOF. We use the same idea in [**19**] to verify this fact. It suffices to show that each $\omega v \in span_{\mathbb{C}}\{\phi_\omega v | \omega \in W_a\}$. Let us show by induction on the length $\ell(\omega)$ of ω.

In the case $\ell(\omega)=1$, $\omega=s_i$ for some $i=0,1,\cdots,n$. Then

$$\phi_\omega v=\phi_i v=s_i(y_i-y_{i+1})v+u_iv=(\zeta_i-\zeta_{i+1})s_iv+u_iv.$$

By Lemma 3.4.6 $\zeta_i\neq\zeta_{i+1}$, $s_iv=(\zeta_i-\zeta_{i+1})^{-1}\phi_iv-u_i(\zeta_i-\zeta_{i+1})^{-1}v$.

Suppose $\omega v\in span_{\mathbb{C}}\{\phi_\omega v|\omega\in W_a\}$ for all $\omega\in W_a$ such that $\ell(\omega)<\ell$. Let $\omega\in W_a$ with $\ell(\omega)=\ell$ and $\omega=s_{i_1}\cdots s_{i_\ell}$. By Proposition 3.4.2, $\phi_\omega=\omega\prod_{p=1}^{\ell}(y_{\omega_p(i_p)}-y_{\omega_p(i_p+1)})+\sum_{x<\omega}xP(y)$. If $\prod_{p=1}^{\ell}(\zeta_{\omega_p(i_p)}-\zeta_{\omega_p(i_p+1)})\neq 0$, then $\omega v\in span_{\mathbb{C}}\{\phi_\omega v|\omega\in W_a\}$. Now let us consider the case $\prod_{p=1}^{\ell}(\zeta_{\omega_p(i_p)}-\zeta_{\omega_p(i_p+1)})=0$. Let k be the maximal number such that $(\zeta_{\omega_k(i_k)}-\zeta_{\omega_k(i_k+1)})=0$ and thus $\prod_{p=k+1}^{\ell}(\zeta_{\omega_p(i_p)}-\zeta_{\omega_p(i_p+1)})\neq 0$. Consider the vector $\phi_{\omega_k^{-1}}v$, which is a weight vector of weight ${\omega_k}^{-1}\zeta$. Since $({\omega_k}^{-1}\zeta)_{i_k}-({\omega_k}^{-1}\zeta)_{i_k+1}=\zeta_{\omega_k(i_k)}-\zeta_{\omega_k(i_k+1)}$=0, it follows that $\phi_{\omega_k^{-1}}v=0$ by Lemma 3.4.6. Namely

$$\prod_{p=k+1}^{\ell}(\zeta_{\omega_p(i_p)}-\zeta_{\omega_p(i_p+1)})s_{i_{k+1}}\cdots s_{i_\ell}v+(\sum_{x<\omega_k^{-1}}xP(y)).v=0 \tag{3.51}$$

Multiplying $s_{i_1}\cdots s_{i_k}$ on both sides of (3.51), we have

$$\prod_{p=k+1}^{\ell}(\zeta_{\omega_p(i_p)}-\zeta_{\omega_p(i_p+1)})\omega v+s_{i_1}\cdots s_{i_k}(\sum_{x<\omega_k^{-1}}xQ(y)).v=0,$$

which implies $\omega v\in span_{\mathbb{C}}\{\phi_\omega v|\omega\in W_a\}$. □

PROPOSITION 3.4.9. *Let L be an irreducible $\mathcal{Y}$-semisimple representation of $\hat{H}_n(u,k_1,k_2,k_3)$. Let $v\in L$ is a nonzero weight vector of weight ζ. Then $\phi_i^2v=0$ implies $\phi_iv=0$ for $i=0,1,\cdots,n$.*

PROOF. By (3.49), the fact that $\phi_i^2v=0$ implies $(u_i+\zeta_i-\zeta_{i+1})(u_i-\zeta_i+\zeta_{i+1})=0$, namely $\zeta_i-\zeta_{i+1}=\pm u_i$. We want to show $\phi_iv=0$ in this case. Suppose the opposite, i.e. $\phi_iv\neq 0$. Then ϕ_iv is a nonzero weight vector of weight $s_i\zeta$. According to Proposition 3.4.8, $L=span_{\mathbb{C}}\{\phi_\omega\phi_iv|\omega\in W_a\}$. Then $v=\sum_{\omega\in W_a}c_\omega\phi_\omega\phi_iv$ for some numbers $c_\omega\in\mathbb{C}$. The vector is a weight vector of weight $\omega s_i\zeta$. Hence $c_\omega\neq 0$ implies $\omega s_i\zeta=\zeta$. Let us explore in two cases. First, in the case $\ell(\omega)<\ell(\omega s_i)$, $\phi_\omega\phi_i=\phi_{\omega s_i}$. The fact $\omega s_i\zeta=\zeta$ implies that $\phi_{\omega s_i}v=\phi_\omega\phi_iv=0$ by Proposition 3.4.7. Second, in the case $\ell(\omega)>\ell(\omega s_i)$, $\phi_\omega\phi_iv=\phi_{\omega s_i}\phi_i^2v=\phi_{\omega s_i}(u_i-\zeta_i+\zeta_{i+1})(u_i+\zeta_i-\zeta_{i+1})v=0$. So we have $v=0$, which contradicts the fact $v\neq 0$. □

REMARK 3.4.10. *The following three conditions are equivalent:* $\phi_i^2 v = 0$, $\zeta_i - \zeta_{i+1} = \pm u_i$ *and* $\phi_i v = 0$.

- $\phi_i v = 0$ *if and only if* $\zeta_i - \zeta_{i+1} = \pm 1$ *for* $i = 1, \cdots, n-1$.

- $\phi_0 v = 0$ *if and only if* $\zeta_1 = \pm \frac{k_2+k_3}{2}$.

- $\phi_n v = 0$ *if and only if* $\zeta_n = \frac{u \pm k_2}{2}$.

Proposition 3.4.7 and 3.4.8 imply the following fact about irreducible $\mathcal{Y}$-semisimple representations.

COROLLARY 3.4.11. *Let* L *be an irreducible* $\mathcal{Y}$*-semisimple representation of* $\hat{H}_n(u, k_1, k_2, k_3)$. *For each weight* ζ, *we have* $dim L_\zeta = 1$ *or* 0.

3.5. Combinatorial moves and irreducibility

3.5.1. Moves on standard tableaux. From last two sections, we obtain a basis of the invariant $\bigoplus_{\beta \in B_c} (V^\beta \otimes V^{\otimes n})^{\mathfrak{t}_0, \mu}$ and this basis is a common $\mathcal{Y}$-eigenbasis which is indexed by $\widehat{Tab}_c^{\lambda,\mu}$. Now we define a series of moves $\mathfrak{m}_0, \mathfrak{m}_1, \cdots, \mathfrak{m}_n$ on $\widehat{Tab}_c^{\lambda,\mu} \sqcup \{\mathfrak{o}\}$.
The move $\mathfrak{m}_i$ for $i - 1, \cdots, n-1$ is defined as

$$\mathfrak{m}_i(T) = \begin{cases} T', & T' \text{ is a standard tableau} \\ \mathfrak{o}, & \text{otherwise,} \end{cases}$$

where T' is defined via $T'(k) = T(s_i(k))$. The move $\mathfrak{m}_n$ is defined to be

$$\mathfrak{m}_n(T) = \begin{cases} T'', & p+1 \leq \mathfrak{i}(1) \leq q \text{ and } T'' \text{ is a standard tableau} \\ \mathfrak{o}, & \text{otherwise,} \end{cases}$$

where T'' is defined via $T''(j) = T(j)$ for each $j \neq n$ and $T''(n) = (N - \mathfrak{i}(n) + 1, -a - b - \mathfrak{j}(n) + 1)$.
The move $\mathfrak{m}_0$ is defined to be

$$\mathfrak{m}_0(T) = \begin{cases} T''', & p+1 \leq \mathfrak{i}(n) \leq q \text{ and } T''' \text{ is a standard tableau} \\ \mathfrak{o}, & \text{otherwise,} \end{cases}$$

where T''' is defined via $T'''(j) = T(j)$ for each $j \neq 1$ and $T'''(1) = (N - \mathfrak{i}(1) + 1, s + t - \mathfrak{j}(1) + 1)$.

REMARK 3.5.1. *The move $\mathfrak{m}_i$ preserves the shape of T, i.e. $Im(T) = Im(\mathfrak{m}_i(T))$ for $i = 1, \cdots, n-1$. The moves $\mathfrak{m}_0$ and $\mathfrak{m}_n$ do change the shape of T.*

3.5.2. Correspondence between the algebraic action and moves. Recall that the parameters in $\hat{H}_n(u, k_1, k_2, k_3)$ in Etingof-Freund-Ma functor [**5**] are computed by

$$u = \frac{2n}{N} + (\lambda + \mu)(q - p), \quad k_1 = 1, \quad k_2 = p - q - \lambda N, \quad k_3 = (\lambda - \mu)N.$$

Let $T \in \widehat{Tab}_c^{\lambda,\mu}$ and ζ^T denote the corresponding weight with $\zeta^T = [\zeta_1^T, \cdots, \zeta_n^T]$ where for $k = 1, \cdots, n$,

$$\zeta_k^T = cont_T(k) + \frac{c+n}{N} + \frac{\mu(q-p)+N}{2}.$$

Let v_T denote a weight vector of weight ζ^T. Next we verify the correspondence between the algebraic action and moves on $\widehat{Tab}_c^{\lambda,\mu} \sqcup \{\mathfrak{o}\}$.

PROPOSITION 3.5.2. *For $i = 0, 1, \cdots, n$ and $T \in \widehat{Tab}_c^{\lambda,\mu}$, $\mathfrak{m}_i(T) = \mathfrak{o}$ if and only if $\phi_i v_T = 0$.*

PROOF. We verify this proposition in three cases depending on i.

Case 1. $i = 1, \cdots, n-1$. The positions of i and $i+1$ in a standard tableau T might be: $i+1$ is adjacent to i and is on the right of i; $i+1$ is adjacent to i and is below i; $i+1$ is not adjacent to i. So it lies to the northeast or southwest of i.

$$\begin{array}{|c|c|}\hline i & i+1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline i \\ \hline i+1 \\ \hline\end{array} \qquad \begin{array}{cc} & \boxed{i+1} \\ \boxed{i} & \end{array} \qquad \begin{array}{cc} & \boxed{i} \\ \boxed{i+1} & \end{array}$$

According to the move $\mathfrak{m}_i$, $\mathfrak{m}_i(T) = 0$ if and only if i and $i+1$ are adjacent. The fact that i and $i+1$ are adjacent is equivalent to the fact that $cont_T(i) - cont_T(i+1) = \pm 1$ and thus $\zeta_i^T - \zeta_{i+1}^T = \pm 1$ which, by Remark 3.4.10, is equivalent to $\phi_i v_T = 0$.

We use the similar idea to verify the other two cases.

Case 2. $i = 0$. The tableau T'' is not a standard tableau if and only if (i) $p+1 \leq \mathfrak{i}(1) \leq q$ or (ii) T'' is not a skew Young diagram. The row number $p+1 \leq \mathfrak{i}(1) \leq q$ if and only if $(\mathfrak{i}(1), \mathfrak{j}(1)) = (p+1, -b+1)$

which corresponds to

$$\zeta_1^T = -b + 1 - (p+1) + \frac{c+n}{N} + \frac{\mu(q-p)+N}{2} = -\frac{k_2+k_3}{2}$$

and thus implies $\phi_0 v_T = 0$ by Remark 5.10. The subset T'' is not a Young diagram if and only if $(\mathfrak{i}(1), \mathfrak{j}(1)) = (q+1, -a+1)$ which corresponds to

$$\zeta_1^T = -a + 1 - (q+1) + \frac{c+n}{N} + \frac{\mu(q-p)+N}{2} = \frac{k_2+k_3}{2}$$

and thus implies $\phi_0 v_T = 0$ by Remark 5.10.

Case 3. $i = n$. The tableau T''' is not a standard tableau if and only if (i) $p+1 \leq \mathfrak{i}(1) \leq q$ or (ii) T''' is not a skew Young diagram. The row number $p+1 \leq \mathfrak{i}(1) \leq q$ if and only if $(\mathfrak{i}(n), \mathfrak{j}(n)) = (q, t)$ which corresponds to

$$\zeta_n^T = t - q + \frac{c+n}{N} + \frac{\mu(q-p)+N}{2} = \frac{u+k_2}{2}$$

and thus implies $\phi_n v_T = 0$ by Remark 5.10. The subset T''' is not a Young diagram if and only if $(\mathfrak{i}(n), \mathfrak{j}(n)) = (p, s)$ which corresponds to

$$\zeta_n^T = s - p + \frac{c+n}{N} + \frac{\mu(q-p)+N}{2} = \frac{u-k_2}{2}$$

and thus implies $\phi_n v_T = 0$ by Remark 5.10. $\square$

Moreover, we have the following proposition.

PROPOSITION 3.5.3. *Under the condition $\phi_i v_T \neq 0$, the nonzero weight vector $\phi_i v_T$ is of weight $s_i \zeta_T$. We have $s_i \zeta_T = \zeta^{\mathfrak{m}_i(T)}$.*

PROOF. We still verify this fact in three cases.

Case 1. $i = 1, \cdots, n-1$. By the definition of $\mathfrak{m}_i(T) = T'$, $\zeta_k^T = \zeta_k^{T'}$ for $k \neq i$ or $i+1$, $\zeta_i^T = \zeta_{i+1}^{T'}$ and $\zeta_{i+1}^T = \zeta_i^{T'}$. Namely $s_i \zeta^T = \zeta^{\mathfrak{m}_i(T)}$.

Case 2. $i = 0$. By the definition of $\mathfrak{m}_0(T) = T''$, $\zeta_k^T = \zeta_k^{T''}$ for $k \neq 1$ and

$$\begin{aligned}
&\zeta_1^T + \zeta_1^{T''} \\
=&\mathfrak{j}(1) - \mathfrak{i}(1) - a - b - \mathfrak{j}(1) + 1 - (N - \mathfrak{i}(1) + 1) + 2\frac{c+n}{N} + \mu(q-p) + N \\
=&- a - b - N + a + b - N + 2N \\
=&0.
\end{aligned}$$

Namely $s_0\zeta^T = \zeta^{\mathfrak{m}_0(T)}$.

Case 3. $i = n$. By the definition of $\mathfrak{m}_n(T) = T'''$, $\zeta_k^T = \zeta_k^{T'''}$ for $k \neq n$ and

$$\begin{aligned}
&\zeta_n^T + \zeta_n^{T''} \\
=&\mathfrak{j}(n) - \mathfrak{i}(n) + s + t - \mathfrak{j}(n) + 1 - (N - \mathfrak{i}(n) + 1) + 2\frac{c+n}{N} + \mu(q-p) + N \\
=&s + t - N + a + b - N + 2N \\
=&u.
\end{aligned}$$

Namely $s_n\zeta^T = \zeta^{\mathfrak{m}_n(T)}$. $\square$

EXAMPLE 3.5.4. *Continue with Example 3.2.2, when G be GL_4, $p = 1$, $\lambda = 1$ and $\mu = -1$, we denote the image by just the skew shape $\tau = \nu/\beta$ with $\beta = (1,1,1,-3)$ and $\nu = (2,2,2,-2)$. From a standard tableau on it we obtain other standard tableaux in $Tab^{1,0}_{4,1,-1}$.*

3.5.3. Irreducibility of $\bigoplus_{\beta\in B_c}(V^\beta \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ as a representation of $\hat{H}_n(u, k_1, k_2, k_3)$.

LEMMA 3.5.5. *Let τ^1 and τ^2 be two skew shapes in $\hat{D}_c^{\lambda,\mu}$ with*

$$\tau^1 \xrightarrow{\hat{\beta}} \tau^2$$

Then there exist standard tableaux T_1 and T_2 with $Im(T_1) = \tau^1$ and $Im(T_2) = \tau^2$ such that $\mathfrak{m}_0(T_1) = T_2$. Similarly, let τ^3 and τ^4 be two skew shapes in $\hat{D}_c^{\lambda,\mu}$ with

$$\tau^3 \xrightarrow{\hat{\nu}} \tau^4$$

Then there exist standard tableaux T_3 and T_4 with $Im(T_3) = \tau^3$ and $Im(T_4) = \tau^4$ such that $\mathfrak{m}_n(T_3) = T_4$.

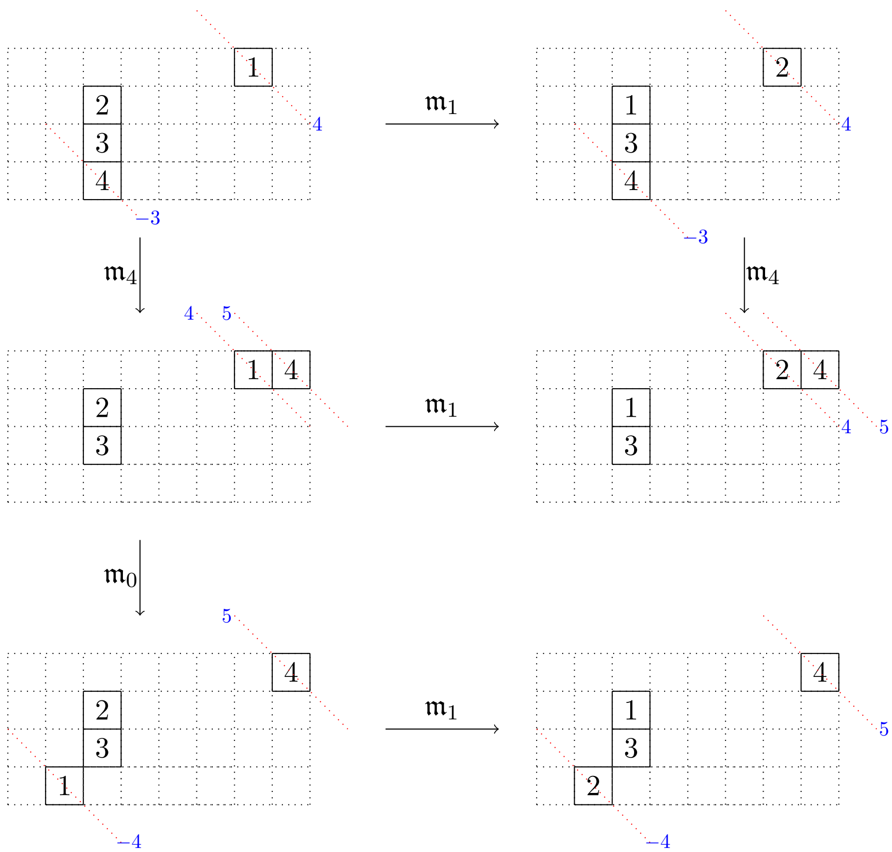


FIGURE 3.6. Moves on tableaux

PROOF. The fact that

$$\tau^1 \xrightarrow{\quad\hat{\beta}\quad} \tau^2$$

implies that τ^2 is obtained by moving a northwestern corner (i, j) of τ^1 to $(N-i+1, -a-b-j+1)$. Since (i, j) is a northwestern corner of τ^1, there exists a standard tableau $T_1 \in \widehat{Tab}_c^{\lambda,\mu}$ with $Im(T_1) = \tau^1$ such that (i, j) is filled by 1. Applying the move $\mathfrak{m}_0$ on T_1, let $T_2 = \mathfrak{m}_0(T_1)$. Then T_2 is a standard tableau with $Im(T_2) = \tau^2$. Similarly we verify the $\hat{\nu}$-move: if

$$\tau^3 \xrightarrow{\quad\hat{\nu}\quad} \tau^4$$

then there exist standard tableaux T_3 and T_4 with $Im(T_3) = \tau^3$ and $Im(T_4) = \tau^4$ such that $\mathfrak{m}_n(T_3) = T_4$. $\square$

We show in the following $\bigoplus_{\beta\in B_c}(V^\beta\otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ as a representation of $\hat{H}_n(u,k_1,k_2,k_3)$ is irreducible.

THEOREM 3.5.6. *The space*

$$\bigoplus_{\beta\in B_c}(V^\beta\otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$$

is irreducible as a representation of $\hat{H}_n(u,k_1,k_2,k_3)$.

PROOF. A basis of $L_c^{\lambda,\mu}=\bigoplus_{\beta\in B_c}(V^\beta\otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ is indexed by

$$\widehat{Tab}_c^{\lambda,\mu}=\{T|T \text{ is a standard tableau and } Im(T)\in\hat{D}_c^{\lambda,\mu}\}.$$

It's obvious to see that the underlying vector space of $\bigoplus_{\beta\in B_c}(V^\beta\otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ is isomorphic to $span_{\mathbb{C}}\{v_T|T\in\widehat{Tab}_c^{\lambda,\mu}\}$. Let N be a submodule of $\bigoplus_{\beta\in B_c}(V^\beta\otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$. Consider the intersection $N\cap(L_c^{\lambda,\mu})_{\zeta_T}$ for each $T\in\widehat{Tab}_c^{\lambda,\mu}$, where ζ_T is the weight associated to T. The intersection $N\cap(L_c^{\lambda,\mu})_{\zeta_T}$ is of dimension 0 or 1 since $(L_c^{\lambda,\mu})_{\zeta_T}$ is of dimension 1 and it is not possible that $N\cap(L_c^{\lambda,\mu})_{\zeta_T}=0$ for any $T\in\widehat{Tab}_c^{\lambda,\mu}$ since $N=0$ otherwise. Then N contains at least one weight vector of $\bigoplus_{\beta\in B_c}(V^\beta\otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$. Let v_T be a weight vector associated to the standard tableau $T\in\widehat{Tab}_c^{\lambda,\mu}$ and assume the submodule N contains v_T.

We show in the following we get every other weight vector from an arbitrary weight vector v_T with $T\in\widehat{Tab}_c^{\lambda,\mu}$. Consider the moves $\mathfrak{m}_i$ since the moves $\mathfrak{m}_i$ are compatible with the actions of intertwining operators.

Case 1. For any the standard tableau T' with the same shape of the tableau T, there exists $\omega\in W$ and $\omega=s_{i_1}\cdots s_{i_\ell}$ such that $T'=\mathfrak{m}_{i_1}(\cdots\mathfrak{m}_{i_\ell}(T))$. Equivalently $v_{T'}=c\phi_\omega v_T$ where $c\in\mathbb{C}$ is nonzero.

Case 2. For standard tableaux T_1 and T_2 with

$$\tau^1 \xrightarrow{\hat{\beta}} \tau^2$$

By Lemma 3.5.5 and Case 1, it follows $T_2=\omega(T_1)$ for some $\omega\in W_a$ and hence $v_{T_2}=c\phi_\omega v_{T_1}$ where $c\in\mathbb{C}$ is nonzero. Similarly, for standard tableaux T_3 and T_4 with

$$\tau^3 \xrightarrow{\hat{\nu}} \tau^4$$

By Lemma 3.5.5 and Case 1, it follows $T_4 = \omega(T_3)$ for some $\omega \in W_a$ and hence $v_{T_4} = c\phi_\omega v_{T_3}$ where $c \in \mathbb{C}$ is nonzero.

Furthermore, consider two arbitrary standard tableaux T and T' in $\widehat{Tab}_c^{\lambda,\mu}$. Let T_c be a standard tableau of shape $\hat{\tau}_c$. There is a path $\hat{\tau}_c \to \tau^1 \to \cdots \to Im(T)$ and hence $v_T = c\phi_\omega v_{T_c}$. There is also a path $\hat{\tau}_c \to \tau^{1'} \to \cdots \to Im(T')$ and hence $v_{T'} = c'\phi_{\omega'} v_{T_c}$. Then $v_{T'} = c''\phi_{\omega'}\phi_{\omega^{-1}} v_T$. $\square$

3.6. Another combinatorial description

The $\hat{H}_n(u, k_1, k_2, k_3)$ representation $L_c^{\lambda,\mu} = \bigoplus_{\beta \in B_c^\lambda} (V^\beta \otimes V^{\otimes n})^{\mathfrak{t}_0,\mu}$ has a weight basis indexed by $\widehat{Tab}_c^{\lambda,\mu}$ which consists of standard tableaux T with $Im(T) \in \hat{D}_c^{\lambda,\mu}$ and $\hat{D}_c^{\lambda,\mu}$ consists of skew shapes obtained by applying $\hat{\beta}$-moves and $\hat{\nu}$-moves on the skew shape $\hat{\tau}_c^{\lambda,\mu}$. Now we introduce valid pictures on $\hat{\tau}_c^{\lambda,\mu}$ such that the collection of all the valid pictures on $\hat{\tau}_c^{\lambda,\mu}$ indexes the weight basis.

3.6.1. The skew shape $\hat{\tau}_c^{\lambda,\mu}$ and a collection of pictures on $\hat{\tau}_c^{\lambda,\mu}$. Let $R_1 \subset \hat{\tau}_c^{\lambda,\mu}$ be the first p rows of the skew shape $\hat{\tau}_c^{\lambda,\mu}$, $R_2 \subset \hat{\tau}_c^{\lambda,\mu}$ be the $(p+1)$-th row through q-th row of the skew shape $\hat{\tau}_c^{\lambda,\mu}$ and $R_3 \subset \hat{\tau}_c^{\lambda,\mu}$ be the last p rows of the skew shape $htau$. So the skew shape $\hat{\tau}_c^{\lambda,\mu}$ is the union of R_1, R_2 and R_3. For any integer $x \in \mathcal{Z}_n$, there is a unique $q_x \in \mathbb{Z}$ and a unique r_x such that

$$r_x \in \{-n, \cdots, -1, 1, \cdots, n\}$$

and $x = (2n+1)q_x + r_x$. Now we define a valid picture P on the skew shape $\hat{\tau}_c^{\lambda,\mu}$.

Definition 3.6.1. *A valid picture P on $\hat{\tau}_c^{\lambda,\mu}$ is an injective map $P : \hat{\tau}_c^{\lambda,\mu} \to \mathcal{Z}_n$ satisfying the following condition:*

(1) The picture P is row increasing and column increasing;

(2) The collection $\{|r_x| \mid x \in Im(P)\}$ is exactly the set $\{1, 2, \cdots, n\}$;

(3) The image of R_2, $\{x \mid P^{-1}(x) \in R_2\} \subset \{1, 2, \cdots, n\}$;

(4) It holds that $0 < x_1 + x_2 < 2n+1$, for x_1 and x_2 such that $P^{-1}(x_1)$ lies in row k of $\hat{\tau}_c^{\lambda,\mu}$ and $P^{-1}(x_2)$ lies in row $N-k+1$ of $\hat{\tau}_c^{\lambda,\mu}$, where $k = 1, \cdots, p$;

(5) If $P^{-1}(x) \in R_1$, then $x \leq n$;

(6) If $P^{-1}(x) \in R_3$, then $x > 0$.

And we denote the collection of all the valid pictures on $\hat{\tau}_c^{\lambda,\mu}$ by $\mathscr{P}_c^{\lambda,\mu}$. Moreover, let $(\mathfrak{i}_P(x), \mathfrak{j}_P(x))$ denote the cell $P^{-1}(x)$ filled with x, namely $\mathfrak{i}_P(x)$ and $\mathfrak{j}_P(x)$ are the row number and the column number respectively of the cell $P^{-1}(x)$.

Example 3.6.2. *For instance, let $\lambda = -1$, $\mu = -1$, $n = 10$, $p = 2$ and $N = 5$. We consider the $H_{10}(2,1,4,0)$-representation $L_0^{-1,-1} = \bigoplus_{\beta \in B_0^{-1}} (V^\beta \otimes V^{\otimes 10})^{\mathfrak{t}_0,-1}$. Then $\hat{\tau}_0^{-1,-1} =$* $(2,2,2,-3,-3)/(1,1,-4,-4,-4)$.

Figure 3.7 is a valid picture P on $\hat{\tau}_0^{-1,-1}$, where $-11 = -21+10$, $-6 = 0-6$, $13 = 21-8$ and

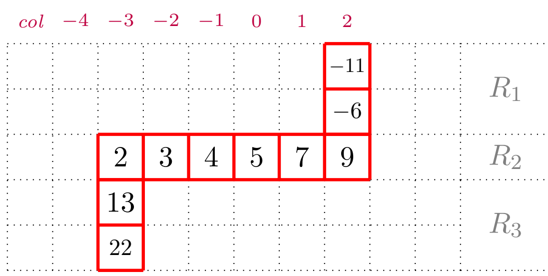


Figure 3.7. A valid picture on $\hat{\tau}_0^{-1,-1} = (2,2,2,-3,-3)/(1,1,-4,-4,-4)$

$22 = 21 + 1$.

3.6.2. The basis indexed by the collection of valid pictures on $\hat{\tau}$. We will verify that there is a one-to-one correspondence between $\widehat{Tab}_c^{\lambda,\mu}$ and $\mathscr{P}_c^{\lambda,\mu}$ and hence the weight basis indexed by $\widehat{Tab}_c^{\lambda,\mu}$ is indexed by $\mathscr{P}_c^{\lambda,\mu}$ correspondingly.

3.6.2.1. *From a valid picture P to a standard tableau T.* Now define a map f from the collection $\mathscr{P}_c^{\lambda,\mu}$ to the collection $\widehat{Tab}_c^{\lambda,\mu}$. Before the definition of the map, we introduce the periodic picture associated to a valid picture P.

Definition 3.6.3. *Given a valid picture P on $\hat{\tau}_c^{\lambda,\mu}$, we define a periodic picture $\mathcal{P}$ associated to P. Let $I_p = \{1, \cdots, p, q+1, \cdots, N\}$ and $R_c^{\lambda,\mu}$ be the subset of $\mathbb{Z} \times \mathbb{Z}$*

$$R_c^{\lambda,\mu} = \{(i,j) | i \in I_p \text{ and } j \in \mathbb{Z}\} \cup R_2.$$

The periodic picture $\mathcal{P}$ is a bijective map

$$\mathcal{P} : R_c^{\lambda,\mu} \to \mathcal{Z}_n \setminus \{(2n+1)k + x | k \in \mathbb{Z}, k \neq 0, x \in Im(P) \text{ and } P^{-1}(x) \in R_2\}$$

such that

(1) $\mathcal{P}((i,j)) = P((i,j))$ *for* $(i,j) \in \hat{\tau}_c^{\lambda,\mu}$;

(2) $\mathcal{P}((N-i+1, -a-b-j+1)) = -P((i,j))$ *for* $(i,j) \in \hat{\tau}_c^{\lambda,\mu}$;

(3) $\mathcal{P}((i, j+k\cdot u)) = \mathcal{P}((i,j)) + k\cdot(2n+1)$ *for* $i \in I_p$ *and* $j,k \in \mathbb{Z}$.

Moreover, let $(\mathfrak{i}_{\mathcal{P}}(x), \mathfrak{j}_{\mathcal{P}}(x))$ denote the cell $\mathcal{P}^{-1}(x)$ filled with x in the periodic picture $\mathcal{P}$. Namely $\mathfrak{i}_{\mathcal{P}}(x)$ and $\mathfrak{j}_{\mathcal{P}}(x)$ are the row number and the column number respectively of the cell $\mathcal{P}^{-1}(x)$.

Remark 3.6.4. *From the definition of an periodic picture, it is easy to see the following facts:*

(1) *Equivalently to Definition 3.6.3, given a valid picture P, we get the periodic picture $\mathcal{P}$ by adding numbers in the following way.*

 (i) *Fill the cell $(N - \mathfrak{i}_P(x) + 1, -a-b-\mathfrak{j}_P(x)+1)$ by $-x$, for each x such that*

$$P^{-1}(x) = (\mathfrak{i}_P(x), \mathfrak{j}_P(x)) \in R_1 \sqcup R_3;$$

 (ii) *Fill the cell $(\mathfrak{i}_P(x), \mathfrak{j}_P(x) \pm u)$ by $x \pm (2n+1)$, for each x such that*

$$P^{-1}(x) = (\mathfrak{i}_P(x), \mathfrak{j}_P(x)) \in R_1 \sqcup R_3.$$

(2) *Definition 3.6.3 is well-defined since for each $i = 1, \cdot, p$, the sum of the number of cells in the i-th row and the number of cells in the $N-i+1$-th row equals u and thus there are u*

cells filled in i-th row for each $i \in I_P$ after Step (i).

(3) The periodic picture $\mathcal{P}$ is row increasing and column increasing.

EXAMPLE 3.6.5. *For instance, take a valid picture P above in Example 3.6.2, we have the periodic picture $\mathcal{P}$ as follows in Figure 3.8. Applying the steps in Remark 3.6.4, in Step (i) we add $-13, -22, 6$ and 11. In Step (ii), let each color represent a period. We get $-15, -10, -8, 1$ by subtracting $2n+1 = 21$ from $6, 11, 13, 22$ and $-1, 8, 10, 15$ by adding $2n+1 = 21$ to $-22, -13, -11, -6$. Continue Step (ii), we get $-43, -34, -32, -27$ by subtracting $2n+1 = 21$ from $-22, -13, -11, -6$ and $27, 32, 34, 43$ by adding $2n+1 = 21$ to $6, 11, 13, 22$. We get $-64, -55, -53, -48$ by subtracting $2 \cdot (2n+1) = 42$ from $-22, -13, -11, -6$ and $48, 53, 55, 64$ by adding $2 \cdot (2n+1) = 42$ to $6, 11, 13, 22$. Continue Step (ii) for infinitely many times, then we get the periodic picture $\mathcal{P}$.*

		col	−4	−3	−2	−1	0	1	2			
		⋯		−64	−53	−43	−32	−22	−11	−1	10	⋯
		⋯		−55	−48	−34	−27	−13	−6	8	15	⋯
				2	3	4	5	7	9			
⋯	−15	−8	6	13	27	34	48	55	⋯			
⋯	−10	1	11	22	32	43	53	64	⋯			

FIGURE 3.8. The periodic picture $\mathcal{P}$ of P.

Given a valid picture P, we extended it to a periodic picture $\mathcal{P}$. Then we take the set consisting of cells of $\mathcal{P}$ filled by $\{1, \cdots, n\}$. Let us denote the shape by τ and let T be a tableau on τ such that $k \mapsto (\mathfrak{i}_{\mathcal{P}}(k), \mathfrak{j}_{\mathcal{P}}(k))$ for each $k = 1, \cdots, n$.

EXAMPLE 3.6.6. *Continue Example 3.6.5, we take the collection of cells filled by $\{1, 2, \cdots, 10\}$ and then obtain T.*

PROPOSITION 3.6.7. *The subset τ is a skew shape and T is a standard tableau on τ.*

PROOF. Let $\tau^1 \subset \tau$ be the first p rows of τ, $\tau^2 \subset \tau$ be the $(p+1)$-th row to q-th row of τ and $\tau^3 \subset \tau$ be the last p rows of τ. So τ^2 is a rectangle R_2 $\{(i,j) | p+1 \leq i \leq q$ and $-a-b+1 \leq j \leq s+t\}$.

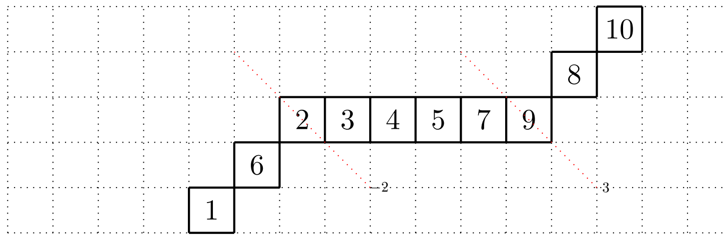


FIGURE 3.9. The tableau T obtained from a valid picture P

First, we verify τ^1 and τ^3 are skew shapes. We use the approach in [**16**] to show τ^1 is a skew shape. Let the cell $(i,j) \in \tau^1$ be filled with x_1 such that $1 \leq x_1 \leq n$ and the cell $(i+1, j+1) \in \tau^1$ be filled with x_2 such that $1 \leq x_2 \leq n$. Since the periodic picture $\mathcal{P}$ is row increasing and column increasing, the cell $(i, j+1)$ is filled with x_3 and $x_1 < x_3 < x_2$. Similarly, the cell $(i+1, j)$ is filled by x_4 and $x_1 < x_4 < x_2$. It follow that $x_1, x_2 \in \{1, \cdots, n\}$ and hence $(i, j+1), (i+1, j) \in \tau^1$. Namely, τ^1 is a skew shape. We verify that τ^3 is also a skew shape in a similar way.

Next we want to show $\tau^1 \cup \tau^2 \cup \tau^3$ is a skew shape. Let $\{(p,j) | j_1 \leq j \leq j_2\}$ be the last row of τ^1 and $\{(q+1, j) | j_3 \leq j \leq j_4\}$ be the first row of τ^3. It suffices to show that $j_1 \geq -b+1$, $j_2 \geq t$, $j_3 \leq -b+1$ and $j_4 \leq t$.

Suppose $j_2 < t$. Then $(p+1, j_2+1) \in R_2$. Let (p, j_2) be filled with x_1 such that $1 \leq x_1 \leq n$ and the cell $(p+1, j_2+1)$ be filled with x_2 such that $1 \leq x_2 \leq n$. Since the periodic picture $\mathcal{P}$ is row increasing and column increasing, the cell (p, j_2+1) is filled with x_3 and $x_1 < x_3 < x_2$. This contradict the fact that $\{(p,j) | j_1 \leq j \leq j_2\}$ be the last row of τ^1. So we have $j_2 \geq t$. We show similarly that $j_3 \leq -a-b+1$.

Let j' be the first column of the rectangle R_1, then the first column of the rectangle R_3 is $-a-b-j'+2$. Let the cell $(p, j'-1)$ be filled with x_5. By part (2) of Definition 7.3, the cell $(q+1, -a-b-j'+2)$ is filled with $-x_5$ in the periodic picture $\mathcal{P}$. Since $(q+1, -a-b+j'+2) \in R_3$, we have $-x_5 > 0$ and thus $x_5 < 0$. So we have $j_1 \geq j'$ and thus $j_1 \geq -a-b+1$.

Let j'' be the last column of the rectangle R_3, then the last column of the rectangle R_1 is $s+t-j''$. Let the cell $(q+1, j''+1)$ be filled by x_6. Then the cell $(p, s+t-j'') = (p, -a-b+u-j'')$ is filled with $2n+1-x_6$. The fact that $(p, s+t-j'') \in R_1$ implies that $2n+1-x-6 \leq n$ and thus $x_6 \geq n+1$. Hence we have $j_4 \leq j'' \leq s+t$.

So we have τ is a skew shape since $j_1 \geq -a-b+1$, $j_2 \geq s+t$, $j_3 \leq -a-b+1$ and $j_4 \leq s+t$.

Moreover, T is row increasing and column increasing by the fact that $\mathcal{P}$ is row increasing and column increasing. □

3.6.2.2. *From a standard tableau T to a valid picture P.* We define a map g from $\widehat{Tab}_c^{\lambda,\mu}$ to $\mathcal{P}_c^{\lambda,\mu}$. Let $T \in \widehat{Tab}_c^{\lambda,\mu}$. We associate each standard tableau T a periodic tableau $\mathcal{T}$ by adding numbers to T as follows.

DEFINITION 3.6.8. *Let the shape $Im(T)$ of T be $\hat{\tau} = \hat{\beta}/\hat{\nu}$*

(1) For a cell $(i,j) \in \hat{\tau}$ with $i = 1, \cdots, p$ or $q+1, \cdots, N$, let $x = T^{-1}((i,j))$. Fill the cell $(N-i+1, -a-b-j+1)$ by $-x$.

(2) For $i = 1, \cdots, p$ or $q+1, \cdots, N$ and a cell (i,j) filled by x, fill the cells $(i, j \pm u)$ by $x \pm (2n+1)$.

REMARK 3.6.9. *After the step* (1), *for each row i such that $i = 1, \cdots, p$ or $q+1, \cdots, N$, there are exactly u cells filled by numbers. So the periodic $\mathcal{T}$ is well-defined and all the cells in the row i are filled.*

EXAMPLE 3.6.10. *Let $\lambda = -1$, $\mu = -1$, $n = 10$, $p = 2$ and $N = 5$. With these parameters, we will have a representation of $H_{10}(2,1,4,0)$ which is indexed by the following region $\hat{\tau}_0^{-1,-1}$. We have a standard tableau T in Figure 10 which represents a weight vector. From standard tableau T*

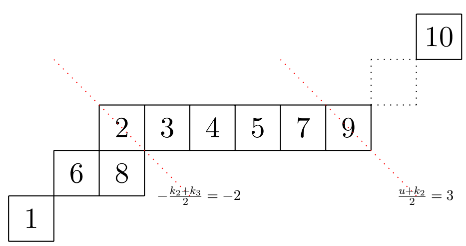


FIGURE 3.10. A standard tableau $T \in \widehat{Tab}_0^{-1,-1}$

in Figure 3.10, we get the periodic tableau $\mathcal{T}$. First fill in -6, -8, -1 and -10 and then the whole i-th row for $i \in I_2$. The periodic tableau $\mathcal{T}$ is as Figure 3.11.

LEMMA 3.6.11. *The periodic tableau $\mathcal{T}$ is row increasing and column increasing.*

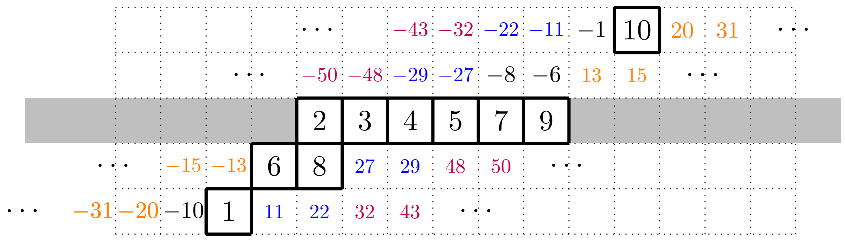


FIGURE 3.11. The periodic tableau $\mathcal{T}$ associated to T

Next we need to find out the skew shape $\hat{\tau}_c^{\lambda,\mu}$ by the parameters λ, μ, n and p. And the filling on the skew shape $\hat{\tau}_c^{\lambda,\mu}$ is a picture denoted by P.

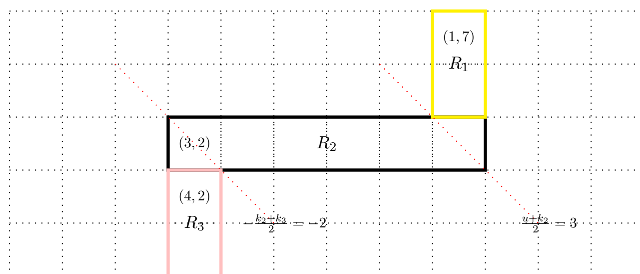


FIGURE 3.12. The skew shape $\hat{\tau}_0^{-1,-1}$

EXAMPLE 3.6.12. *Continue Example 3.6.5. We figure out the skew shape $\hat{\tau}_0^{-1,-1}$ in Figure 3.12. Take a standard tableau $T \in \widehat{Tab}_0^{-1,-1}$. Extend the standard tableau T to $\mathcal{T}$. The red region is the skew shape $\hat{\tau}_0^{-1,-1}$. Then the filling on the skew shape $\hat{\tau}_c^{\lambda,\mu}$ is the picture P as in Figure 3.13.*

PROPOSITION 3.6.13. *The picture P we obtained from the standard tableau T is a valid picture on $\hat{\tau}_c^{\lambda,\mu}$.*

PROOF. Let us show the picture we obtained from T satisfies the conditions in Definition 3.6.1. (i) Let x_1 be a filling in the i-th row of R_1 and x_2 be a filling in the $(p-i+1)$-th row of R_3 for $1 \leq i \leq p$, suppose $x_1 + x_2 \geq 2n+1$. We need the following notation. For a subset $\tau \in \mathbb{Z} \times \mathbb{Z}$, let $-\tau$ be the subset of $\mathbb{Z} \times \mathbb{Z}$

$$-\tau = \{(N-i+1, -a-b-j+1) | (i,j) \in \tau\}$$

and

$$\tau^{(k)} = \{(i, j+k \cdot u) | (i,j) \in \tau\}.$$

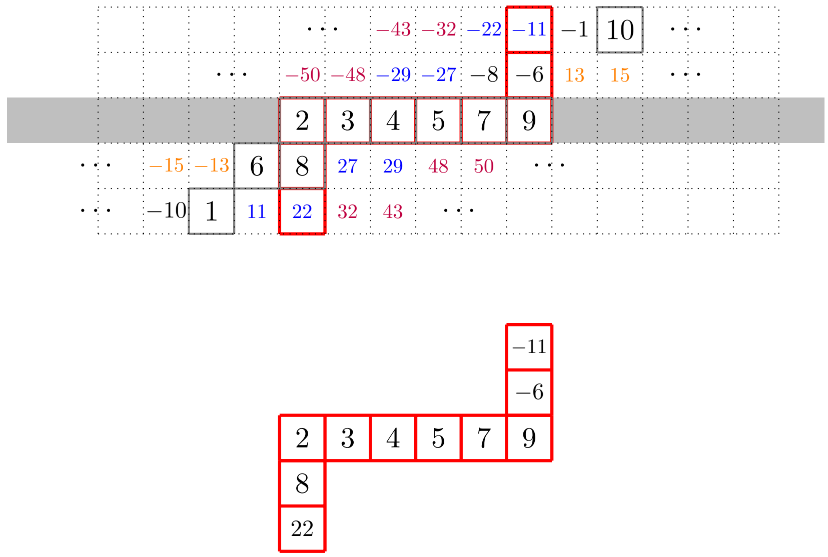


FIGURE 3.13. From an extended tableau to a valid picture

Since $\mathcal{T}$ is periodic, there is a cell in the $(p-i+1)$-th row of $-R_1$ filled by $-x_1$ and thus a cell in the $(p-i+1)$-th row of $(-R_1)^{(1)}$ filled by $-x_1+2n+1$. Let the first column of R_3 be j_1 and the last column of R_3 be j_2. Then the last column of $-R_1$ is j_1-1 and the first column of $(-R_1)^{(1)}$ is j_2+1. On the other hand, $x_1+x_2 \geq 2n+1$ implies $x_2 \geq -x_1+2n+1$, which contradicts the row increasing fact. So we have the fact that $x_1+x_2 < 2n+1$.

Suppose $x_1+x_2 \leq 0$. Then we have $-x_1 \geq x_2$, which contradicts the row increasing condition.

(ii) Let x be a filling in the i-th row of R_1 for $1 \leq i \leq p$.

First, consider the case $s \leq t$, then R_1 is above R_2, namely the last column of R_1 is less or equal to t. Since the fillings in R_2 are from $\{1, \cdots, n\}$ and the column increasing fact of $\mathcal{T}$, x is forced to be strictly less than n.

Second, consider the case $s > t$ and suppose $x > n$. In this case, the last column of R_3 is t. There is a cell in $(p-i+1)$-th row of $-R_1$ filled by $-x$ and a cell $(N-i+1, j_3)$ in $(p-i+1)$-th row of $(-R_1)^{(1)}$ filled by $-x+2n+1$. It follows that $j_3 > t$. On the other hand, $x > n$ implies $-x+2n+1 < n+1$. This fact contradicts the fact that the last column of R_3 is t. So we conclude $x \leq n$.

(iii) Let x be a filling in the i-th row of R_3. First, consider the case $a \leq b$. In this case, R_3 is below R_2, namely $j_1 \geq -b+1$, thus we have $x > 0$ by the column increasing property of $\mathcal{T}$. Second, consider the case $a > b$ and suppose $x < 0$. Then there is a cell in the $(p-i+1)$-th row of $-R_3$ filled by $-x$. We have the fact $-x < n$ since any filling y satisfies $y \leq n$ for y lying in the $(p-i+1)$-th row of R_1 and $\mathcal{T}$ is row increasing. This forces x to be $-n < x < 0$ and hence $0 < -x < n$, which contradicts the shape τ is a skew shape. So we still have $x > 0$ in the $a > b$ case. □

3.6.2.3. *One to one correspondence between* $\mathscr{P}_c^{\lambda,\mu}$ *and* $\widehat{Tab}_c^{\lambda,\mu}$.

THEOREM 3.6.14. *The weight basis of* $L_c^{\lambda,\mu}$ *is indexed by* $\mathscr{P}_c^{\lambda,\mu}$ *the collection of all the valid pictures* P *on a fixed shape* $\hat{\tau}_c^{\lambda,\mu}$.

PROOF. We prove the theorem by constructing a one-to-one correspondence between the collection $\widehat{Tab}_c^{\lambda,\mu}$ of standard tableaux T indexing the basis of invariant space and the collection $\mathscr{P}_c^{\lambda,\mu}$ of pictures P on a fixed region $\hat{\tau}_c^{\lambda,\mu}$. In Section 3.6.2.1, we define a map $f : \mathscr{P}_c^{\lambda,\mu} \to \widehat{Tab}_c^{\lambda,\mu}$ and in Section 3.6.2.2, we define a map $g : \widehat{Tab}_c^{\lambda,\mu} \to \mathscr{P}_c^{\lambda,\mu}$. Let us consider $g \circ f$. For any valid picture $P \in \mathscr{P}_c^{\lambda,\mu}$, extend P to the periodic picture $\mathcal{P}$ by Definition 3.6.1 and get $f(P) = T \in \widehat{Tab}_c^{\lambda,\mu}$. By Definition 3.6.8, the periodic tableau $\mathcal{T}$ associated to T is exactly $\mathcal{P}$, namely $\mathcal{T} = \mathcal{P}$. Hence $g \circ f(P) = P$. So $g \circ f = id_{\mathscr{P}_c^{\lambda,\mu}}$. Similarly, we show $f \circ g = id_{\widehat{Tab}_c^{\lambda,\mu}}$. Now we have a one-to-one correspondence between $\mathscr{P}_c^{\lambda,\mu}$ and $\widehat{Tab}_c^{\lambda,\mu}$.

Thus the weight basis is indexed by the following picture P on region $\hat{\tau}_c^{\lambda,\mu}$.

□

3.6.3. Moves on $\mathscr{P}_c^{\lambda,\mu}$. In Section 3.5 we defined moves on $\widehat{Tab}_c^{\lambda,\mu} \sqcup \{\mathfrak{o}\}$ which has a correspondence to the actions of intertwining operators on weight vectors. Now we extend the definition of moves $\mathfrak{m}_i$ for $i = 0, 1, \cdots, n$ to $\mathscr{P}_c^{\lambda,\mu} \sqcup \{\mathfrak{o}\}$ as follows. Let $T \in \widehat{Tab}_c^{\lambda,\mu}$ be a standard tableau and P be the corresponding valid picture in $\mathscr{P}_c^{\lambda,\mu}$. Let x denote the image $P((\mathfrak{i}, \mathfrak{j}))$ of the cell $(\mathfrak{i}, \mathfrak{j})$ and $x = (2n+1)q_x + r_x$ with $q_x \in \mathbb{Z}$ and $r_x \in \{-n, \cdots -1, 1, \cdots, n\}$. Then we have moves on $\mathscr{P}_c^{\lambda,\mu} \sqcup \{\mathfrak{o}\}$ defined as follows:

(1) For $i = 1, \cdots, n-1$, if $P' \in \mathscr{P}_c^{\lambda,\mu}$, set $\mathfrak{m}_i(P) = P'$ and

$$P'((\mathfrak{i},\mathfrak{j})) = \begin{cases} (2n+1)q_x + r_x & r_x \neq \pm i \text{ or } \pm(i+1) \\ (2n+1)q_x + r_x + 1 & r_x = i \text{ or } -(i+1) \\ (2n+1)q_x + r_x - 1 & r_x = i+1 \text{ or } -i; \end{cases}$$

Otherwise, $\mathfrak{m}_i(P) = \mathfrak{o}$.

(2) If $P'' \in \mathscr{P}_c^{\lambda,\mu}$, set $\mathfrak{m}_0(P) = P''$ and

$$P''((\mathfrak{i},\mathfrak{j})) = \begin{cases} (2n+1)q_x + r_x & r_x \neq \pm 1 \\ (2n+1)q_x - r_x + 1 & r_x = \pm 1; \end{cases}$$

Otherwise, $\mathfrak{m}_0(P) = \mathfrak{o}$.

(3) If $P''' \in \mathscr{P}_c^{\lambda,\mu}$, set $\mathfrak{m}_n(P) = P'''$ and

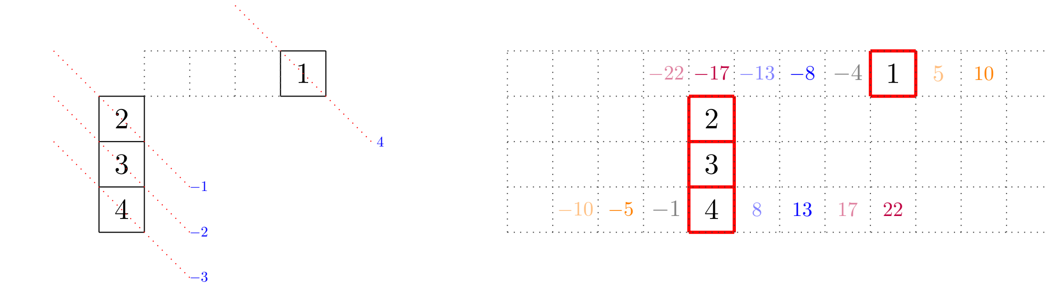
$$P'''((\mathfrak{i},\mathfrak{j})) = \begin{cases} (2n+1)q_x + r_x & r_x \neq \pm n \\ (2n+1)(q_x \pm 1) - r_x & r_x = \pm n. \end{cases}$$

Otherwise, $\mathfrak{m}_n(P) = \mathfrak{o}$.

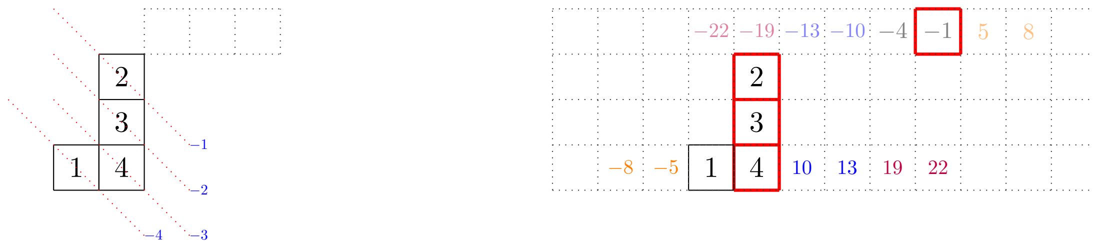
Let $\mathcal{P}_c^{\lambda,\mu}$ denote the collection of periodic picture $\mathcal{P}$ associated to $P \in \widehat{Tab}_c^{\lambda,\mu}$. Since the one-to-one correspondence between $\mathcal{P}_c^{\lambda,\mu}$ and $\mathscr{P}_c^{\lambda,\mu}$, the collection $\mathcal{P}_c^{\lambda,\mu}$ indexes the weight basis of $L_c^{\lambda,\mu}$. Check the moves on $\mathcal{P}_c^{\lambda,\mu} \sqcup \{\mathfrak{o}\}$.

(1) The move $\mathfrak{m}_0$ exchanges fillings $-1 + k(2n+1)$ and $1 + k(2n+1)$ in $\mathcal{P}$ for $k \in \mathbb{Z}$, if the new picture lies in $\mathcal{P}_c^{\lambda,\mu} \sqcup \{\mathfrak{o}\}$ and $\mathfrak{m}_0(\mathcal{P}) = \mathfrak{o}$ if otherwise;
(2) The move $\mathfrak{m}_i$ exchanges fillings $\pm i + k(2n+1)$ and $\pm(i+1) + k(2n+1)$ in $\mathcal{P}$ for $i = 1, \cdots, n-1$ and $k \in \mathbb{Z}$, if the new picture lies in $\mathcal{P}_c^{\lambda,\mu} \sqcup \{\mathfrak{o}\}$ and $\mathfrak{m}_0(\mathcal{P}) = \mathfrak{o}$ if otherwise;
(3) The move $\mathfrak{m}_n$ exchanges fillings $n + k(2n+1)$ and $n + 1 + k(2n+1)$ in $\mathcal{P}$ for $k \in \mathbb{Z}$ if the new picture lies in $\mathcal{P}_c^{\lambda,\mu} \sqcup \{\mathfrak{o}\}$ and $\mathfrak{m}_n(\mathcal{P}) = \mathfrak{o}$ if otherwise.

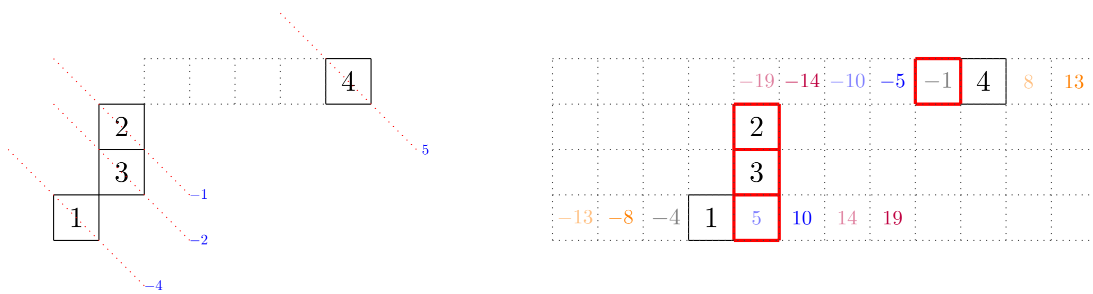
EXAMPLE 3.6.15. *Now let us look at several moves on standard tableaux and valid pictures. We start with the following standard tableau* T *on* $\hat{\tau}_0^{-1,-1}$.



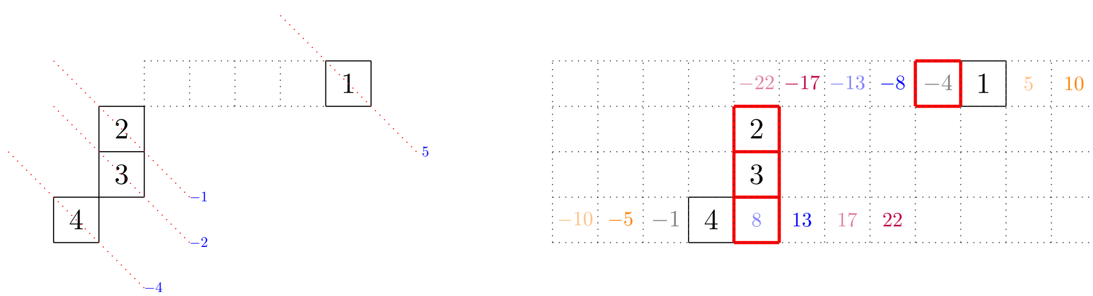
Applying $\mathfrak{m}_0$, *then we get*



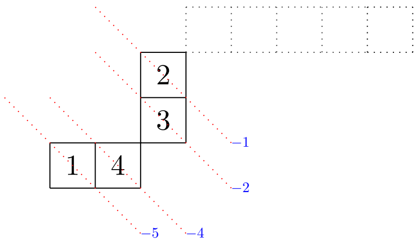
Applying $\mathfrak{m}_4$, *then we get*

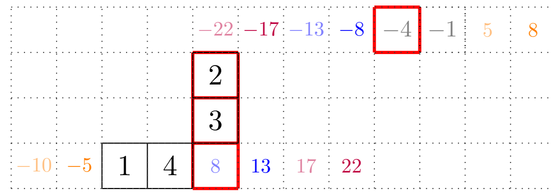


Applying $\mathfrak{m}_3\mathfrak{m}_2\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3$, *then we get*

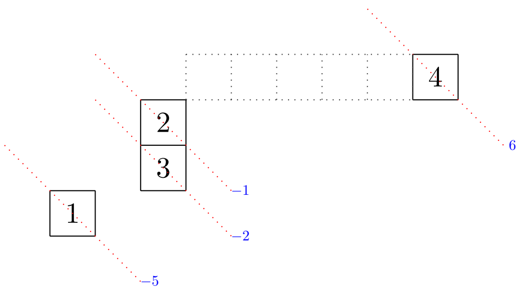


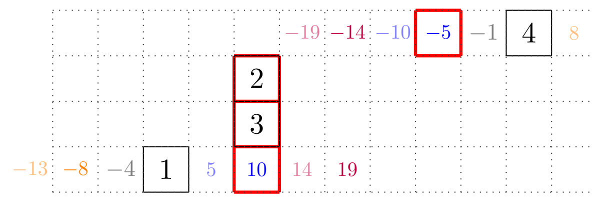
Applying $\mathfrak{m}_0$, *then we get*





Applying $\mathfrak{m}_4$*, then we get*





The combinatorial description by $\widehat{Tab}_c^{\lambda,\mu}$ consists of standard tableaux on a collection of skew shapes and the moves $\mathfrak{m}_0$ and $\mathfrak{m}_n$ on $\widehat{Tab}_c^{\lambda,\mu}$ move the cells filled with 1 and n respectively, whereas the combinatorial description by $\mathcal{P}_c^{\lambda,\mu}$ consists of valid pictures on a fixed region $\hat{\tau}_c^{\lambda,\mu}$ and moves on $\mathcal{P}_c^{\lambda,\mu}$ only changes the fillings. So we associate $\hat{\tau}_c^{\lambda,\mu}$ to the representation $L_c^{\lambda,\mu}$.

CHAPTER 4

Affine and double affine Hecke algebras of type C and Jordan-Ma functor

In this chapter, we consider the quantum cases, i.e. affine Hecke algebras and double affine Hecke algebra of type C. We consider the representations of affine Hecke algebras which are images of $U_q(\mathfrak{gl}_N)$ under the Jordan-Ma functor [**8**] and representations of double affine Hecke algebras which are images of the quantum coordinate algebra $A_q(GL_N)$ under the Jordan-Ma functor.

4.1. Affine and Double Affine Hecke Algebras

In [**8**], Jordan and Ma mentioned the following definitions of affine and double affine Hecke algebras of type C.

DEFINITION 4.1.1. *The affine Hecke algebra $\mathcal{H}_n(t, t_0, t_n)$ of type C is a unital associative algebra over $\mathbb{C}$ with three parameters t, t_0, t_n generated by $T_0, T_1, \cdots, T_{n-1}, T_n$ with the relations:*

$$T_iT_j = T_jT_i, |i-j| > 1, \tag{4.1}$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \tag{4.2}$$

$$T_{n-1}T_nT_{n-1}T_n = T_nT_{n-1}T_nT_{n-1}, \tag{4.3}$$

$$T_0T_1T_0T_1 = T_1T_0T_1T_0, \tag{4.4}$$

$$(T_i - t)(T_i + t^{-1}) = 0, \quad i = 1, \cdots, n-1, \tag{4.5}$$

$$(T_n - t_n)(T_n + t_n^{-1}) = 0, \tag{4.6}$$

$$(T_0 - t_0)(T_0 + t_0^{-1}) = 0. \tag{4.7}$$

The double affine Hecke algebra $\tilde{\mathcal{H}}_n(t, t_0, t_n, u_0, u_n, w)$ of type C is a unital associative algebra over $\mathbb{C}$ with six parameters generated by $T_0, T_1, \cdots, T_{n-1}, T_n$ and K_0 with relations (4.1)-(4.7) *and* *additional relations:*

(4.8) $$K_0T_i = T_iK_0, \quad i = 2, \cdots, n,$$

(4.9) $$T_1K_0T_1K_0 = K_0T_1K_0T_1,$$

(4.10) $$T_0T_1^{-1}K_0T_1 = T_1^{-1}K_0T_1T_0,$$

(4.11) $$(K_0 - u_n)(K_0 + u_n^{-1}) = 0,$$

(4.12) $$(wK_0P_1T_0 - u_0^{-1})(wK_0P_1T_0 + u_0) = 0,$$

where $P_i = T_iT_{i+1}\cdots T_n \cdots T_{i+1}T_i$, *for* $i = 1, \cdots, n$.

Set X_i, for $i = 1, \cdots, n$.

$$X_i = T_i^{-1}\cdots T_n^{-1}\cdots T_1^{-1}K_0^{-1}T_1\cdots T_{i-1}.$$

Set Y_i, for $i = 1, \cdots, n$.

$$Y_i = T_i\cdots T_n\cdots T_1T_0T_1^{-1}\cdots T_{i-1}^{-1}.$$

Now we explore the relations involving K_0 and Y_i's.

Lemma 4.1.2. *It follows*

$$K_0Y_j = Y_jK_0,$$

for $j = 2, \cdots, n$.

Proof. For $j = 2, \cdots, n$, we have $K_0T_j = T_jK_0$ and $K_0T_j^{-1} = T_j^{-1}K_0$ by (4.8).

$$\begin{aligned}
K_0Y_j &= K_0T_j\cdots T_n\cdots T_1T_0T_1^{-1}\cdots T_{j-1}^{-1}\\
&= T_j\cdots T_n\cdots T_2K_0T_1T_0T_1^{-1}\cdots T_{j-1}^{-1}\\
&= T_j\cdots T_n\cdots T_2T_1(T_1^{-1}K_0T_1T_0)T_1^{-1}\cdots T_{j-1}^{-1},
\end{aligned}$$

By (4.10), we deduce that

$$\begin{aligned}
&T_j \cdots T_n \cdots T_2T_1(T_1^{-1}K_0T_1T_0)T_1^{-1}\cdots T_{j-1}^{-1}\\
=&T_j \cdots T_n \cdots T_2T_1(T_0T_1^{-1}K_0T_1)T_1^{-1}\cdots T_{j-1}^{-1}\\
=&T_j \cdots T_n \cdots T_2T_1T_0T_1^{-1}K_0T_2^{-1}\cdots T_{j-1}^{-1}\\
=&T_j \cdots T_n \cdots T_2T_1T_0T_1^{-1}T_2^{-1}\cdots T_{j-1}^{-1}K_0\\
=&Y_jK_0.
\end{aligned}$$

□

Instead of $K_0, T_0, T_1, \cdots, T_n$, [**8**] mentioned the generators

$$T_1, \cdots, T_n, Y_1^{\pm}, \cdots, Y_n^{\pm}, X_1^{\pm}, \cdots, X_n^{\pm}.$$

In this paper, let us use the generators $K_0, T_1, \cdots, T_n$ and $Y_1^{\pm}, \cdots, Y_n^{\pm}$. The following definition is equivalent to Definition 4.1.1.

PROPOSITION 4.1.3. *The affine Hecke algebra* $\mathcal{H}_n(t, t_0, t_n)$ *is generated by*

$$T_1, \cdots, T_n, Y_1^{\pm}, \cdots, Y_n^{\pm}$$

with relations (4.1)-(4.3), (4.5)-(4.6) *and the following relations:*

$$Y_iY_j = Y_jY_i, \tag{4.13}$$

$$T_iY_{i+1}T_i = Y_i, \quad i = 1, \cdots, n-1, \tag{4.14}$$

$$T_iY_j = Y_jT_i, \quad i = 1, \cdot, n-1 \textit{ and } j \neq 0, i+1, \tag{4.15}$$

$$T_nY_j = Y_jT_n. \quad j \neq n, \tag{4.16}$$

$$(T_n^{-1}Y_n - t_0)(T_n^{-1}Y_n + t_0^{-1}) = 0. \tag{4.17}$$

The double affine Hecke algebra $\tilde{\mathcal{H}}_n(t, t_0, t_n, u_0, u_n, w)$ *is generated by*

$$K_0, T_1, \cdots, T_{n-1}, T_n, Y_1^{\pm}, \cdots, Y_n^{\pm}$$

with relations (4.1)-(4.3)*,* (4.5)-(4.6)*,* (4.8)-(4.9)*,* (4.11)*,* (4.13)-(4.17) *and the additional relations:*

$$K_0 Y_j = Y_j K_0, \quad j \geq 2, \tag{4.18}$$

$$(wK_0Y_1 - u_0^{-1})(wK_0Y_1 + u_0) = 0. \tag{4.19}$$

4.2. Intertwining Operators

In this section, we define a set of elements in double affine Hecke algebra $\tilde{\mathcal{H}}_n(t, t_0, t_n, u_0, u_n, w)$, which we called by intertwining operators.

Set $Y_0 = wY_1$, define the following operators:

$$\Phi_i = T_i(Y_i - Y_{i+1}) - (t - t^{-1})Y_i, \quad i = 1, \cdots, n-1, \tag{4.20}$$

$$\Phi_n = T_n(Y_n - Y_n^{-1}) - (t_n - t_n^{-1})Y_n - (t_0 - t_0^{-1}), \tag{4.21}$$

$$\Phi_0 = K_0(Y_0 - Y_0^{-1}) + (u_n - u_n^{-1})Y_0 + (u_0 - u_0^{-1}). \tag{4.22}$$

We verify these operators satisfying the same braid relations with T_i's. This verification is straightforward. So we omit the proof here. As a result, we define the intertwining operator Φ_z for each $z \in W_a$. Let $z = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression, then

$$\Phi_z = \Phi_{i_1} \cdots \Phi_{i_\ell}.$$

The squares of intertwining operators are computed as follows.

$$\Phi_i^2 = -(Y_i - t^2 Y_{i+1})(Y_i - t^{-2} Y_{i+1}), \quad i = 1, \cdots, n-1,$$

$$\Phi_n^2 = (t_n Y_n - t_n^{-1} Y_n^{-1} + t_0 - t_0^{-1})(t_n Y_n^{-1} - t_n^{-1} Y_n + t_0 - t_0^{-1}),$$

$$\Phi_0^2 = (u_n Y_0 - u_n^{-1} Y_0^{-1} + u_0 - u_0^{-1})(u_n Y_0^{-1} - u_n^{-1} Y_0 + u_0 - u_0^{-1}).$$

Moreover, the following relations of Y_i and Φ_i hold.

PROPOSITION 4.2.1.

$$Y_i\Phi_i = \Phi_i Y_{i+1}.$$

$$Y_n\Phi_n = \Phi_n Y_n^{-1},$$

$$Y_0\Phi_0 = \Phi_0 Y_0^{-1}.$$

REMARK 4.2.2. *Proposition 4.2.1 implies that the intertwining operators Φ_z moves one $\mathcal{Y}$-weight space to another $\mathcal{Y}$-weight space, i.e.*

$$\Phi_z L_\zeta \subset L_{z\zeta},$$

where $z \in W_a$, ζ and $z\zeta$ are defined as (3.50).

4.3. Quantum General Linear Groups

4.3.1. Quantum Group $U_q(\mathfrak{gl}_N)$. We use the definition of quantum group $U_q(\mathfrak{gl}_N)$ in [**13**]. Let $q \in \mathbb{C}$ be a nonzero complex and q is not a root of unity. Let P be the weight lattice and $P^\vee$ be the dual weight lattice with a symmetric bilinear pairing $(,) : P \times P \to \mathbb{Z}$ such that $(\epsilon_i, \epsilon_j) = \delta_{ij}$. The quantized enveloping algebra $U_q(\mathfrak{gl}_N)$ is generated by $e_1, \cdots, e_{N-1}$, $f_1, \cdots, f_{N-1}$ and q^h, $h \in P^\vee$ with relations:

$$q^{h_1} \cdot q^{h_2} = q^{h_1+h_2}, \tag{4.23}$$

$$q^h e_j q^{-h} = q^{\langle h, \epsilon_j - \epsilon_{j+1}\rangle} e_j, \tag{4.24}$$

$$q^h f_j q^{-h} = q^{\langle h, -\epsilon_j + \epsilon_{j+1}\rangle} f_j, \tag{4.25}$$

$$e_i f_j - f_j e_i = \delta_{i,j} \frac{q^{\epsilon_i - \epsilon_{i+1}} - q^{-\epsilon_i + \epsilon_{i+1}}}{q - q^{-1}}, \tag{4.26}$$

$$e_i e_j = e_j e_i, |i-j| > 1, \tag{4.27}$$

$$f_i f_j = f_j f_i, |i-j| > 1, \tag{4.28}$$

$$e_i^2 e_{i\pm1} - (q+q^{-1}) e_i e_{i\pm1} e_i + e_{i\pm1} e_i^2 = 0, \tag{4.29}$$

$$f_i^2 f_{i\pm1} - (q+q^{-1}) f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0. \tag{4.30}$$

The Hopf structure on $U_q(\mathfrak{gl}_N)$ is as follows: comultiplication Δ, counit ε and antipode S

$$\Delta(q^h) = q^h \otimes q^h, \tag{4.31}$$

$$\Delta(e_i) = e_i \otimes 1 + q^{\epsilon_i - \epsilon_{i+1}} \otimes e_i, \tag{4.32}$$

$$\Delta(f_i) = f_i \otimes q^{-\epsilon_i + \epsilon_{i+1}} + 1 \otimes f_i, \tag{4.33}$$

$$\varepsilon(q^h) = 1, \tag{4.34}$$

$$\varepsilon(e_i) = \varepsilon(f_i) = 0, \tag{4.35}$$

$$S(q^h) = q^{-h}, \tag{4.36}$$

$$S(e_i) = -q^{-\epsilon_i + \epsilon_{i+1}} e_i, \tag{4.37}$$

$$S(f_i) = -f_i q^{\epsilon_i - \epsilon_{i+1}}. \tag{4.38}$$

Let $V = \mathbb{C}^N$ be an N-dimensional vector space over $\mathbb{C}$ with v_i being the standard basis element. Let $E_{ij} \in End(V)$ be the $N \times N$ matrix with (i,j)-entry being 1 and other entries 0. Define the $U_q(\mathfrak{gl}_N)$-module structure by $\rho_V : U_q(\mathfrak{gl}_N) \to End(V)$

$$\rho_V(q^{\epsilon_i}) = qE_{ii} + \sum_{j \neq i} E_{jj}, i = 1, \cdots, N, \tag{4.39}$$

$$\rho_V(e_i) = E_{i,i+1}, i = 1, \cdots, N-1, \tag{4.40}$$

$$\rho_V(f_i) = E_{i+1,i}, i = 1, \cdots, N-1. \tag{4.41}$$

The vector space V together with the $U_q(\mathfrak{gl}_N)$-module structure is the vector representation of $U_q(\mathfrak{gl}_N)$.

4.3.2. L-operators. Let $\mathcal{R}$ be the universal R-matrix. The R-matrix under the vector representation is as follows:

$$R = (\rho_V \otimes \rho_V)(\mathcal{R}) = \sum_{i,j} q^{\delta_{i,j}} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i>j} E_{ij} \otimes E_{ji}. \tag{4.42}$$

$$R^{-} = (\rho_V \otimes \rho_V)(\mathcal{R}^{-1}) = \sum_{i,j} q^{\delta_{ij}} E_{ii} \otimes E_{jj} - (q - q^{-1}) \sum_{i>j} E_{ij} \otimes E_{ji}. \tag{4.43}$$

Definition 4.3.1. *The l-operators of $U_q(\mathfrak{gl}_N)$ are elements of $U_q(\mathfrak{gl}_N)$ satisfying*

$$(1 \otimes \rho_V)(\mathcal{R}) = \sum_{ij} l_{ij}^{+} \otimes E_{ij} \tag{4.44}$$

$$(\rho_V \otimes 1)(\mathcal{R}^{-1}) = \sum_{ij} E_{ij} \otimes l_{ij}^{-}. \tag{4.45}$$

In [**13**], there is a family of elements e_{ij}, for $1 \le i, j \le N$ and $i \neq j$ defined as follows. Take $i < j$,

$$e_{i,i+1} = e_i, e_{ij} = e_{ik}e_{kj} - qe_{kj}e_{ij}, \text{ for an arbitrary } i < k < j;$$

$$e_{i+1,i} = f_i, e_{ji} = e_{jk}e_{ki} - q^{-1}e_{ki}e_{jk}, \text{ for an arbitrary } i < k < j.$$

The l-operators are expressed in terms of e_{ij} as follows, for $i < j$,

$$l_{ij}^{+} = (q - q^{-1})q^{\epsilon_i}e_{ji};$$

$$l_{ji}^{-} = -(q - q^{-1})e_{ij}q^{-\epsilon_i}$$

and

$$l_{ii}^{\pm} = q^{\pm\epsilon_i}. \tag{4.46}$$

We use the following notations. Let $L^{\pm} = (l_{ij}^{\pm})$, $L_1^{\pm} = L^{\pm} \otimes id$ and $L_2^{\pm} = id \otimes L^{\pm}$.

Theorem 4.3.2. *Klimyk and Schmudgen [**10**] proved the algebra $U_q(\mathfrak{gl}_N)$ is generated by $l_{ij}^{\pm}$, $i, j = 1, \cdots, N$ with relations:*

$$L_1^{\pm}L_2^{\pm}R = RL_2^{\pm}L_1^{\pm} \tag{4.47}$$

$$L_1^{-}L_2^{+}R = RL_2^{+}L_1^{-} \tag{4.48}$$

$$l_{ii}^{+}l_{ii}^{-} = l_{ii}^{-}l_{ii}^{+} = 1, \quad i = 1, \cdots, N, \tag{4.49}$$

$$l_{ij}^{+} = l_{ji}^{-} = 0, \quad i > j. \tag{4.50}$$

The antipode S, comultiplication Δ and counit ε on l-operators are given by

$$S(L^{\pm}) = (L^{\pm})^{-1}, \tag{4.51}$$

$$\Delta(l_{ij}^{\pm}) = \sum_k l_{ik}^{\pm} \otimes l_{kj}^{\pm}, \tag{4.52}$$

$$\varepsilon(l_{ij}^{\pm}) = \delta_{ij}. \tag{4.53}$$

4.3.3. Hopf $*$ algebra structure and right modules. In [**13**], Noumi explained the Hopf $*$ structure. The quantum group $U_q(\mathfrak{gl}_N)$ has a Hopf $*$ algebra structure, where $* : U_q(\mathfrak{gl}_N) \to U_q(\mathfrak{gl}_N)$ is an involution and an algebra anti-automorphism, with

$$(q^h)^* = q^h, h \in P^{\vee}, \quad e_k^* = q^{-1} f_k q^{\epsilon_k - \epsilon_{k+1}}, \quad f_k^* = q^{-1} e_k q^{-\epsilon_k + \epsilon_{k+1}}, 1 \leq k \leq N-1.$$

The comultiplications of e_i^* and f_i^* are as follows,

$$\Delta(e_i^*) = e_i^* \otimes 1 + q^{\epsilon_i - \epsilon_{i+1}} \otimes e_i^* \tag{4.54}$$

$$\Delta(f_i^*) = f_i^* \otimes q^{-\epsilon_i + \epsilon_{i+1}} + 1 \otimes f_i^*. \tag{4.55}$$

And the $*$-operation on L-operators is

$$(l_{ij}^{\pm})^* = S(l_{ji}^{\mp}).$$

With the Hopf $*$ algebra structure, there is a one-to-one correspondence between left $U_q(\mathfrak{gl}_N)$-modules and right $U_q(\mathfrak{gl}_N)$-modules. Let M be left $U_q(\mathfrak{gl}_N)$-module and we define a right $U_q(\mathfrak{gl}_N)$-module structure on M and denote the right module by M°,

$$v.x = x^*.v, \quad x \in U_q(\mathfrak{gl}_N) \text{ and } v \in M.$$

Conversely, let N be a right $U_q(\mathfrak{gl}_N)$-module, we define the left $U_q(\mathfrak{gl}_N)$-module structure on N by

$$x.v = v.x^*, \quad x \in U_q(\mathfrak{gl}_N) \text{ and } v \in N.$$

LEMMA 4.3.3. *The comultiplication and $*$-operation commute.*

$$\Delta \circ * = (* \otimes *) \circ \Delta.$$

REMARK 4.3.4. *Let M and K be two left $U_q(\mathfrak{gl}_N)$-modules. Then $M\otimes K$ is a left $U_q(\mathfrak{gl}_N)$-module and $(M \otimes K)^\circ = M^\circ \otimes K^\circ$ is the corresponding right $U_q(\mathfrak{gl}_N)$-module.*

4.4. Jordan-Ma functor and representations of AHA and DAHA

4.4.1. Coideal subalgebras. The following is the definitions of coideal subalgebras in [**8**]. Let J^σ be the $N \times N$ complex matrix with $\sigma \in \mathbb{R}$

$$J^\sigma = \sum_{1\le k\le p} (q^\sigma - q^{-\sigma})E_{k,k} - \sum_{p+1\le k\le N-p} q^{-\sigma}E_{k,k} + \sum_{1\le k\le p} E_{k,N-k+1} + \sum_{1\le k\le p} E_{N-k+1,k}.$$

Let D_p be the $p \times p$ anti-diagonal matrix with each entry on the anti-diagonal is 1. Then we have

$$J^\sigma = \begin{pmatrix} (q^\sigma - q^{-\sigma})I_p & 0 & D_p \\ 0 & -q^{-\sigma}I_{N-2p} & 0 \\ D_p & 0 & 0 \end{pmatrix}$$

Define the elements c_{il} and c'_{il} of $U_q(\mathfrak{gl}_N)$, $i, l = 1, \cdots, N$ as follows:

$$c_{il} = \sum_{j,k=1}^{N} l^+_{ij} J^\sigma_{jk} S(l^-_{kl}),$$

$$c'_{il} = \sum_{j,k=1}^{N} S(l^-_{ij})(J^\psi)^{-1}_{jk} l^+_{kl}.$$

Let B_σ and B'_ψ be subalgebras of $U_q(\mathfrak{gl}_N)$ generated by $\{c_{il}|i, l = 1, \cdots, N\}$ and $\{c'_{il}|i, l = 1, \cdots, N\}$ respectively. It is easy to check that

REMARK 4.4.1. *(1) It follows that B_σ is a left coideal subalgebra from the comultiplication*

$$\begin{aligned}\Delta(c_{il}) &= \sum_{m,h=1}^{N} l^+_{im} S(l^-_{hl}) \otimes (\sum_{j,k=1}^{N} l^+_{mj} J^\sigma_{jk} S(l^-_{kh})) \\ &= \sum_{m,h=1}^{N} l^+_{im} S(l^-_{hl}) \otimes c_{mh}.\end{aligned}$$

(2) It follows that B'_ψ is a right coideal subalgebra from the comultiplication

$$\begin{aligned}\Delta(c'_{il}) &= \sum_{m,h=1}^{N} (\sum_{j,k=1}^{N} S(l^-_{mj})(J^\psi)^{-1}_{jk} l^+_{kh}) \otimes (S(l^-_{im}) l^+_{hl}) \\ &= \sum_{m,h=1}^{N} c'_{mh} \otimes (S(l^-_{im}) l^+_{hl}).\end{aligned}$$

4.4.2. Characters and invariant spaces. Let M be a $U_q(\mathfrak{gl}_N)$-module and $\mathbb{M}$ be a D_U-module, where D_U is the algebra of quantum differential operators defined in [**8**]. The characters of B_σ and B'_ψ are used to define the invariant spaces $F_n^{\sigma,\eta,\tau}(M)$ and $F_{n,\psi,\omega,\iota}^{\sigma,\eta,\tau}(M)$ which are the underlying vector spaces of the representations of affine Hecke algebra and double affine Hecke algebra in [**8**]. Let χ_τ^η be the character of the left coideal subalgebra B_σ such that

$$\chi_\tau^\eta(c_{il}) = q^\eta J_{il}^\tau.$$

Let λ_ι^ω be the character of the right coideal subalgebra B'_ψ such that

$$\lambda_\iota^\omega(c'_{il}) = q^\omega (J^\iota)^{-1}_{il}.$$

The invariant spaces are defined as

$$F_n^{\sigma,\eta,\tau}(M) = Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, M \otimes V^{\otimes n})$$

and

$$F_{n,\psi,\omega,\iota}^{\sigma,\eta,\tau}(\mathbb{M}) = Hom_{B'_\psi, B_\sigma}(\mathbb{1}_{\lambda_\iota^\omega} \boxtimes \mathbb{1}_{\chi_\tau^\eta}, \mathbb{M} \otimes_2 (\mathbb{1} \boxtimes V)^{\otimes_2 n}),$$

for $\sigma, \tau, \eta, \psi, \iota, \omega \in \mathbb{R}$. In [**8**], Jordan and Ma showed that $\mathbb{M}$ has a $U_q(\mathfrak{gl}_N) \otimes U_q(\mathfrak{gl}_N)$-module structure.

4.4.3. Computation of the invariant space. In this subsection, let us compute the invariant space $Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^\nu)$ for any $\nu \in P^+$ in the case $\sigma - \tau$ is an even number.
By Theorem 4.3.2, we define a character of $U_q(\mathfrak{gl}_N)$ in terms of l-operators. Let $\mathbb{1}_\eta$ be the one

dimensional character of $U_q(\mathfrak{gl}_N)$ with

$$\mathbb{1}_\eta(l_{ij}^{\pm}) = \begin{cases} 0, & i \neq j \\ q^{\mp\eta/2}, & i = j. \end{cases}$$

It is straightforward to check that the definition of $\mathbb{1}_\eta$ is compatible with relations (4.47)-(4.50). Moreover, the one dimensional character $\mathbb{1}_\eta$ is of highest weight $(-\eta/2)\sum_{i=1}^N \epsilon_i$ by (4.46).

LEMMA 4.4.2. *It holds that*

$$Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^\nu) \cong Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^0}, \mathbb{1}_\eta \otimes V^\nu)$$

PROOF. We want to show the following two vector spaces are isomorphic to each, i.e.

$$Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^\nu) \cong Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^0}, \mathbb{1}_\eta \otimes V^\nu).$$

Since

$$\begin{aligned} &Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^\nu) \\ \cong&\{v \in V^\nu | c_{il}.v = \chi_\tau^\eta(c_{il})v, i, j = 1, \cdots, N\} \end{aligned}$$

and

$$\begin{aligned} &Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^0}, \mathbb{1}_\eta \otimes V^\nu) \\ \cong&\{v \in V^\nu | \Delta(c_{il}).(1 \otimes v) = \chi_\tau^0(c_{il})(1 \otimes v), i, j = 1, \cdots, N\}, \end{aligned}$$

It suffices to show that

$$\begin{aligned} &\{v \in V^\nu | c_{il}.v = \chi_\tau^\eta(c_{il})v, i, j = 1, \cdots, N\} \\ =&\{v \in V^\nu | \Delta(c_{il}).(1 \otimes v) = \chi_\tau^0(c_{il})(1 \otimes v), i, j = 1, \cdots, N\}. \end{aligned}$$

By the definition of $\mathbb{1}_\eta$, we have

$$\mathbb{1}_\eta(S(l_{ij}^{\pm})) = \begin{cases} 0, & i \neq j \\ q^{\pm\eta/2}, & i = j. \end{cases}$$

We compute the action of c_{il} on $\mathbb{1}_\eta \otimes V^\nu$ and we have

$$\begin{aligned} \Delta(c_{il}).(1 \otimes v) &= \sum_{m,h=1}^{N} (l_{im}^{+} S(l_{hl}^{-}) \otimes c_{mh}).(1 \otimes v) \\ &= \sum_{m,h=1}^{N} \mathbb{1}_\eta(l_{im}^{+} S(l_{hl}^{-})) \otimes (c_{mh}.v) \\ &= q^{-\eta} \otimes (c_{il}.v) \end{aligned}$$

Let $v \in \{v \in V^\nu | c_{il}.v = \chi_\tau^\eta(c_{il})v, i, j = 1, \cdots, N\}$, then

$$\begin{aligned} \Delta(c_{il}).(1 \otimes v) &= q^{-\eta} \otimes (c_{il}.v) \\ &= q^{-\eta} \otimes (q^\eta J_{il}^\tau v) \\ &= J_{il}^\tau (1 \otimes v) \\ &= \chi_\tau^0(c_{il})(1 \otimes v) \end{aligned}$$

and hence $v \in \{v \in V^\nu | \Delta(c_{il}).(1 \otimes v) = \chi_\tau^0(c_{il})(1 \otimes v), i, j = 1, \cdots, N\}$.
On the contrary, suppose $v \in \{v \in V^\nu | \Delta(c_{il}).(1 \otimes v) = \chi_\tau^0(c_{il})(1 \otimes v), i, j = 1, \cdots, N\}$, then

$$\begin{aligned} \Delta(c_{il}).(1 \otimes v) =& \chi_\tau^0(c_{il})(1 \otimes v) \\ q^{-\eta} \otimes (c_{il}.v) =& J_{il}^\tau (1 \otimes v). \end{aligned}$$

This implies $1 \otimes (c_{il}.v) = q^\eta J_{il}^\tau (1 \otimes v)$ and hence $c_{il}.v = q^\eta J_{il}^\tau v = \chi_\tau^\eta(c_{il})v$. So we have

$$Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^\nu) = Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^0}, \mathbb{1}_\eta \otimes V^\nu).$$

$\square$

Now it suffices for us to compute in the case $\eta = 0$. To compute the space $Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^0}, V^\nu)$, we deduce the following fact.

$$
\begin{aligned}
& Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^0}, V^\nu) \\
\cong & \{v \in V^\nu | (L^+ J^\sigma S(L^-))_{ij}.v = J^\tau_{ij} v, 1 \leq i,j \leq N\} \\
= & \{v \in V^\nu | ((L^+ J^\sigma S(L^-))_{ij} - J^\tau_{ij}).v = 0, 1 \leq i,j \leq N\} \\
= & \{v \in V^\nu | ((J^\sigma S(L^-))_{ij} - (S(L^+) J^\tau)_{ij}).v = 0, 1 \leq i,j \leq N\} \\
= & \{v \in (V^\nu)^\circ | v.(J^\sigma S(L^-))_{ij} - (S(L^+) J^\tau)_{ij})^* = 0, 1 \leq i,j \leq N\} \\
= & \{v \in (V^\nu)^\circ | v.((L^+ J^\sigma)_{ij} - (J^\tau L^-)_{ij}) = 0, 1 \leq i,j \leq N\}.
\end{aligned}
$$

Let $\mathfrak{k}_\sigma^\tau$ denote the subalgebra of $U_q(\mathfrak{gl}_N)$ generated by

$$\sum_{k=1}^{N} (l^+_{ik} J^\sigma_{kj} - J^\tau_{ik} l^-_{kj}),$$

for $i, j = 1, \cdots, N$. The fact above allows us to compute $Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^0}, V^\nu)$ by computing the vectors in V^ν killed by the right action of $\mathfrak{k}_\sigma^\tau$. Next we consider the invariant space $Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^\nu)$.

Theorem 4.4.3. *In the case that $\sigma - \tau$ is an even number, the invariant space*

$$Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^\nu)$$

is either 0 or a one-dimensional vector space. The invariant space is nonzero if and only if $\nu \in P^+$ and

$$\nu_i = \frac{\eta + \sigma - \tau}{2}, \quad i = p+1, \cdots, N-p, \tag{4.56}$$

$$\nu_i + \nu_{N-i+1} = \eta, \quad i = 1, \cdots, p., \tag{4.57}$$

Proof. Consider the invariant space $Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^0}, \mathbb{1}_\eta \otimes V^\nu)$. Since this invariant space corresponds to $\mathfrak{k}_\sigma^\tau$-invariants, by Proposition A.2.2, the invariant space $Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^0}, \mathbb{1}_\eta \otimes V^\nu)$ is either 0 or one-dimensional.

Moreover, the $U_q(\mathfrak{gl}_N)$-module $\mathbb{1}_\eta \otimes V^\nu$ is the irreducible highest weight module of highest weight

$\nu-(\eta/2)(\sum_{i=1}^N \epsilon_i)$. By Theorem A.1.1, the invariant space $Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^0}, \mathbb{1}_\eta \otimes V^\nu)$ is nonzero if and only if $\nu \in P^+$ and

$$\nu_i - \frac{\eta}{2} = \frac{\sigma-\tau}{2}, \quad i = p+1, \cdots, N-p,$$
$$\nu_i - \frac{\eta}{2} + \nu_{N-i+1} - \frac{\eta}{2} = 0, \quad i = 1, \cdots, p,$$

which are equivalent to

$$\nu_i = \frac{\eta+\sigma-\tau}{2}, \quad i = p+1, \cdots, N-p,$$
$$\nu_i + \nu_{N-i+1} = \eta, \quad i = 1, \cdots, p.$$

By Lemma 4.4.2, we have the invariant space $Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^\nu)$ is nonzero if and only if ν satisfies conditions (4.56)-(4.57).

$\square$

Remark 4.4.4. *By A.2.12, the condition $\nu_i = \frac{\eta+\sigma-\tau}{2}$ for $i = p+1, \cdots, N-p$, is necessary for $Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^\nu) \neq 0$. In the case that $\eta+\sigma-\tau$ is not an even number, ν_i is not an integer, then $Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^\nu) = 0$ for $\nu \in P^+$.*

4.4.4. A basis of the invariant space.

4.4.4.1. *Tensor product of $U_q(\mathfrak{gl}_N)$-modules.* Let us consider the tensor product of an irreducible highest weight $U_q(\mathfrak{gl}_N)$-module V^ξ and the vector representation V.
Let $\chi(\xi)$ denote the character of the irreducible highest weight $U_q(\mathfrak{gl}_N)$-module with highest weight $\xi \in P^+$. Since ϵ_1 is a minuscule dominant weight, we apply Lemma 5A.9 in [**6**] in the case $\xi_0 = \epsilon_1$ and then

$$\chi(\xi)\chi(\epsilon_1) = \sum_{1\le i\le N, \xi+\epsilon_i \in P^+} \chi(\xi+\epsilon_i).$$

The vector representation is an irreducible highest weight representation $V = V^{\xi_0}$ of highest weight $\xi_0 = \epsilon_1$. We have

$$V^\xi \otimes V = \bigoplus_{\nu^{(1)}} V^{\nu^{(1)}},$$

the direct sum runs through $\nu^{(1)} \in P^+$ and $\nu^{(1)} = \xi + \epsilon_{k_1}$ for some $k_1 = 1, \cdots, N$.

Continue tensoring the vector representation V. Let $(\nu^{(0)}, \nu^{(1)}, \cdots, \nu^{(n)})$ be a sequence of integral dominant weights such that $\nu^{(0)} = \xi$ and $\nu^{(i)} = \nu^{(i-1)} + \epsilon_{k_i}$ for some $k_i = 1, \cdots, N$. We have

4.4.4.2. *A combinatorial basis.* Consider the invariant space

$$
\begin{aligned}
&Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^\xi \otimes V^{\otimes n}) \\
= &\bigoplus_{(\nu^{(0)},\nu^{(1)},\cdots,\nu^{(n)})} Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^{\nu^{(n)}}).
\end{aligned}
$$

By Theorem 4.4.3, $\dim Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^{\nu^{(n)}}) = 1$ if and only if $\nu^{(n)}$ satisfies conditions (4.56)-(4.57). Otherwise, $Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^{\nu^{(n)}}) = 0$. So the collection of sequences $(\nu^{(0)}, \nu^{(1)}, \cdots, \nu^{(n)})$ such that $T_{(\nu^{(0)},\nu^{(1)},\cdots,\nu^{(n)})}$ is a standard tableau and $\nu^{(n)}$ satisfies (4.56)-(4.57),i.e.

$$\{(\nu^{(0)}, \nu^{(1)}, \cdots, \nu^{(n)}) | T_{(\nu^{(0)},\nu^{(1)},\cdots,\nu^{(n)})} \text{ is a standard tableau and } \nu^{(n)} \text{ satisfies } (4.56)-(4.57)\}$$

indexes a basis of the invariant space $Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^\xi \otimes V^{\otimes n})$. Equivalently we have the following result.

THEOREM 4.4.5. *The invariant space $Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^\xi \otimes V^{\otimes n})$ has a basis indexed by the collection of standard tableaux T of shape ν/ξ such that ν satisfies* (4.56)-(4.57).

4.5. $\mathcal{Y}$-actions

4.5.1. Definition of $\mathcal{Y}$-action on the invariant space. Jordan and Ma defined the action of T_0 in [**8**]. With the relation $Y_1 = T_1 \cdots T_n \cdots T_1 T_0$, we deduce the action of Y_1 is given by

$$q^{N-\eta}(R^{-1}_{V^\xi,V} \circ \tau_{V,V^\xi} \circ R^{-1}_{V,V^\xi} \circ \tau_{V^\xi,V})_{(0,1)} \otimes id_{V_2} \otimes \cdots \otimes id_{V_n},$$

where $(R^{-1}_{V^\xi,V} \circ \tau_{V,V^\xi} \circ R^{-1}_{V,V^\xi} \circ \tau_{V^\xi,V})_{(0,1)}$ means $R^{-1}_{V^\xi,V} \circ \tau_{V,V^\xi} \circ R^{-1}_{V,V^\xi} \circ \tau_{V^\xi,V}$ acting on the tensor product $V^\xi \otimes V_1$. Here $V_1 = V_2 = \cdots = V_n = V$. We use the following diagram in Figure 4.1 to express Y_1. By (4.14) $T_i Y_{i+1} T_i = Y_i$, we deduce the action of Y_i, $i = 1, 2, \cdots, n$ as the following diagram in Figure 4.2. Consider the action of Y_i as diagram in Figure 4.3.

The category of finite dimensional complex representations of $U_q(\mathfrak{gl}_N)$ is a ribbon category. Here we denote the universal R-matrix by $\mathcal{R} = \sum x_i \otimes y_i$. For any finite dimensional representation

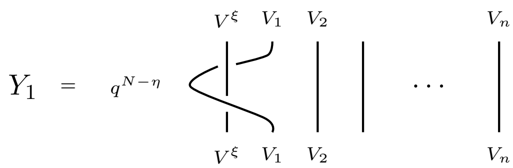


FIGURE 4.1. The action of Y_1

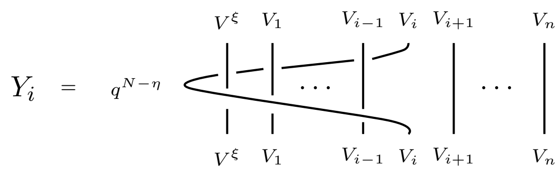


FIGURE 4.2. The action of Y_i

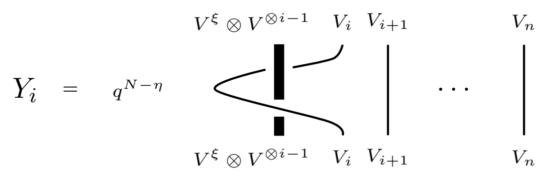


FIGURE 4.3. The action of Y_i

M of $U_q(\mathfrak{gl}_N)$, let θ_M denote the twist on M. The ribbon element is $q^{-2\rho}u$ with $u = \sum S(y_i)x_i$. Then the twist $\theta_M : M \to M$ is given via acting by $q^{-2\rho}u$. Then we have Y_i acts by

$$q^{N-\eta}(R^{-1}_{V_{(0,i-1)},V_i} \circ \tau_{V_i,V_{(0,i-1)}} \circ R^{-1}_{V_i,V_{(0,i-1)}} \circ \tau_{V_{(0,i-1)},V_i})_{(0,i)} \otimes id_{V_{i+1}} \otimes \cdots \otimes id_{V_n} \tag{4.58}$$

$$=q^{N-\eta}(\tau_{V_i,V_{(0,i-1)}} \circ R_{V_i,V_{(0,i-1)}} \circ \tau_{V_{(0,i-1)},V_i} \circ R_{V_{(0,i-1)},V_i})^{-1}_{(0,i)} \otimes id_{V_{i+1}} \otimes \cdots \otimes id_{V_n} \tag{4.59}$$

$$=q^{N-\eta}(\theta_{V_{(0,i-1)}\otimes V_i} \circ (\theta^{-1}_{V_{(0,i-1)}} \otimes \theta^{-1}_{V_i}))^{-1}_{(0,i)} \otimes id_{V_{i+1}} \otimes \cdots \otimes id_{V_n} \tag{4.60}$$

$$=q^{N-\eta}((\theta_{V_{(0,i-1)}} \otimes \theta_{V_i}) \circ \theta^{-1}_{V_{(0,i-1)}\otimes V_i})_{(0,i)} \otimes id_{V_{i+1}} \otimes \cdots \otimes id_{V_n}. \tag{4.61}$$

Here $V_{(0,i-1)} = V^\xi \otimes V^{\otimes(i-1)}$ and the subscript $(0,i)$ means action on $V^\xi \otimes V^{\otimes i}$.

4.5.2. Action of Y_i on $V_{(\nu^{(0)},\nu^{(1)},\nu^{(2)},\cdots,\nu^{(n)})}$. By Theorem 4.4.3, we have a basis of the invariant space

$$Hom_{B_\sigma}(\mathbb{1}_{\chi^\eta_\tau}, V^\xi \otimes V^{\otimes n})$$

indexed by the collection of sequences $(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}, \cdots, \nu^{(n)})$ such that $\nu^{(0)} = \xi$ and $\nu^{(n)}$ satisfies (4.56)-(4.57). Let $T_{(\nu^{(0)},\nu^{(1)},\nu^{(2)},\cdots,\nu^{(n)})}$ denote the standard tableau corresponding to the sequence

$(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}, \cdots, \nu^{(n)})$. Now let us denote by

$$V_{(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}, \cdots, \nu^{(i)})},$$

$i = 1, \cdots, n$, the irreducible summand of $V^{\xi} \otimes V^{\otimes i}$ indexed by the sequence

$$(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}, \cdots, \nu^{(i)}).$$

This corresponds to a basis element of the invariant space $Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^{\xi} \otimes V^{\otimes n})$. We compute the action of Y_i on $V_{(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}, \cdots, \nu^{(n)})}$ to compute how Y_i acts on the corresponding basis element.

Let ρ denote the half sum of positive roots. In [**11**], Leduc and Ram computed the action of $q^{-2\rho}$ on an irreducible representation V^{ξ} as multiplying by the scalar $q^{-(\xi, \xi+2\rho)}$. So we deduce the action of Y_i on $V_{(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}, \cdots, \nu^{(n)})}$.

Theorem 4.5.1. *The action of Y_i on $V_{(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}, \cdots, \nu^{(n)})}$ is multiplying by the scalar*

$$q^{(\nu^{(i)}, \nu^{(i)}+2\rho) - (\nu^{(i-1)}, \nu^{(i-1)}+2\rho) - (\epsilon_1, \epsilon_1+2\rho) - \eta + N}.$$

Proof. Since $V_{(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}, \cdots, \nu^{(n)})} \subset V_{(\nu^{(0)}, \nu^{(1)}, \cdots, \nu^{(i)})} \otimes V_{i+1} \otimes \cdots \otimes V_n$. By (4.61), we compute the $(\theta^{-1}_{V_{(0,i-1)} \otimes V_i})_{(0,i)}$ action on the summand $V_{(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}, \cdots, \nu^{(n)})}$ of the tensor product $V^{\xi} \otimes V^{\otimes n}$ via computing action $\theta^{-1}_{V_{(0,i-1)} \otimes V_i}$ on the irreducible summand

$$V_{(\nu^{(0)}, \nu^{(1)}, \cdots, \nu^{(i)})}$$

of the tensor product $V^{\xi} \otimes V_1 \otimes \cdots \otimes V_i$. By in [**11**], $(\theta^{-1}_{V_{(0,i-1)} \otimes V_i})_{(0,i)}$ acts by the scalar

$$q^{(\nu^{(i)}, \nu^{(i)}+2\rho)}.$$

Similarly, since $V_{(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}, \cdots, \nu^{(n)})} \subset V_{(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}, \cdots, \nu^{(i-1)})} \otimes V_i \otimes \cdots \otimes V_n$. We compute the $(\theta_{V_{(0,i-1)}})_{(0,i-1)}$ action on the summand $V_{(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}, \cdots, \nu^{(n)})}$ of the tensor product $V^{\xi} \otimes V^{\otimes n}$ via computing action $\theta_{V_{(0,i-1)}}$ on the irreducible summand

$$V_{(\nu^{(0)},\nu^{(1)},\cdots,\nu^{(i-1)})}$$

of the tensor product $V^\xi \otimes V_1 \otimes \cdots \otimes V_{i-1}$. By in [**11**], $(\theta_{V_{(0,i-1)}})_{(0,i-1)}$ acts by the scalar

$$q^{-(\nu^{(i-1)},\nu^{(i-1)}+2\rho)}.$$

Moreover, θ_V acts on V by the scalar $q^{-(\epsilon_1,\epsilon_1+2\rho)}$.

Hence Y_i acts on $V_{(\nu^{(0)},\nu^{(1)},\nu^{(2)},\cdots,\nu^{(n)})}$ by the scalar

$$q^{(\nu^{(i)},\nu^{(i)}+2\rho)-(\nu^{(i-1)},\nu^{(i-1)}+2\rho)-(\epsilon_1,\epsilon_1+2\rho)-\eta+N}.$$

$\square$

Corollary 4.5.2. *Let* $v_{(\nu^{(0)},\nu^{(1)},\nu^{(2)},\cdots,\nu^{(n)})}$ *be the basis vector corresponding to*

$$(\nu^{(0)},\nu^{(1)},\nu^{(2)},\cdots,\nu^{(n)}).$$

Then it follows

$$Y_i.v_{(\nu^{(0)},\nu^{(1)},\nu^{(2)},\cdots,\nu^{(n)})} = q^{(\nu^{(i)},\nu^{(i)}+2\rho)-(\nu^{(i-1)},\nu^{(i-1)}+2\rho)-(\epsilon_1,\epsilon_1+2\rho)-\eta+N} v_{(\nu^{(0)},\nu^{(1)},\nu^{(2)},\cdots,\nu^{(n)})}.$$

4.5.3. Y-actions in terms of contents. Let $T = T_{(\nu^{(0)},\nu^{(1)},\nu^{(2)},\cdots,\nu^{(n)})}$ be the standard tableau corresponding to the sequence $(\nu^{(0)},\nu^{(1)},\nu^{(2)},\cdots,\nu^{(n)})$ which corresponds to a basis element

$$v_T = v_{(\nu^{(0)},\nu^{(1)},\nu^{(2)},\cdots,\nu^{(n)})}$$

of the invariant space $F_n^{\sigma,\eta,\tau}(V^\xi)$. We deduce the fact that

$$\begin{aligned}&(\nu^{(i)},\nu^{(i)}+2\rho)-(\nu^{(i-1)},\nu^{(i-1)}+2\rho)-(\epsilon_1,\epsilon_1+2\rho)\\ =&2cont_T(i).\end{aligned}$$

So we have the action of Y_i on v_T is computed by

$$q^{2cont_T(i)-\eta+N}.$$

4.6. Image of the quantized coordinate ring $\mathcal{A}_q(GL_N)$

Let $\mathcal{A}_q(GL_N)$ be the quantized coordinate ring. We consider the image of $\mathcal{A}_q(GL_N)$ under the Jordan-Ma functor and we use the $U_q(\mathfrak{gl}_N) \otimes U_q(\mathfrak{gl}_N)$-structure of $\mathcal{A}_q(GL_N)$

$$\mathcal{A}_q(GL_N) = \bigoplus_{\beta \in P^+} V^{\beta^\vee} \boxtimes V^{\beta},$$

where $\beta^\vee$ is the dual of β. If $\beta = \sum_{i=1}^N \beta_i \epsilon_i$, then $\beta^\vee = \sum_{i=1}^N -\beta_{N-i+1}\epsilon_i$.

4.6.1. The invariant space $F^{\sigma,\eta,\tau}_{n,\psi,\omega,\iota}(\mathcal{A}_q(GL_N))$. First, we compute the invariant space

$$Hom_{B'_\psi}(\mathbb{1}_{\chi^\omega_\iota}, V^{\beta^\vee})$$

for any $\beta \in P^+$.

Let $\mathbb{1}_\omega$ be the one dimensional character of $U_q(\mathfrak{gl}_N)$ with

$$\mathbb{1}_\omega(l^{\pm}_{ij}) = \begin{cases} 0, & i \neq j \\ q^{\mp\omega/2}, & i = j. \end{cases}$$

It is straightforward to check that the definition of $\mathbb{1}_\omega$ is compatible with relations (4.47)-(4.50).

Lemma 4.6.1. *It holds that*

$$Hom_{B'_\psi}(\mathbb{1}_{\lambda^\omega_\iota}, V^\nu) \cong Hom_{B'_\psi}(\mathbb{1}_{\lambda^0_\iota}, V^\nu \otimes \mathbb{1}_\omega).$$

The proof of Lemma 4.6.1 is similar to Lemma 4.4.2. With Lemma 4.6.1, it suffices for us to compute in the case $\omega = 0$. To compute the invariant space $Hom_{B'_\psi}(\mathbb{1}_{\chi^0_\iota}, V^{\beta^\vee})$, we deduce the following fact.

$$\begin{aligned}
&Hom_{B'_\psi}(\mathbb{1}_{\lambda_\iota^0}, V^{\beta^\vee})\\
\cong&\{v \in V^{\beta^\vee} | (S(L^-)(J^\psi)^{-1}L^+)_{ij}.v = (J^\iota)^{-1}_{ij}v, 1 \le i,j \le N\}\\
=&\{v \in V^{\beta^\vee} | ((S(L^-)(J^\psi)^{-1}L^+)_{ij} - (J^\iota)^{-1}_{ij}).v = 0, 1 \le i,j \le N\}\\
=&\{v \in V^{\beta^\vee} | (((J^\psi)^{-1}L^+)_{ij} - (L^-(J^\iota)^{-1})_{ij}).v = 0, 1 \le i,j \le N\}\\
=&\{v \in V^{\beta^\vee} | ((L^+J^\iota)_{ij} - (J^\psi L^-)_{ij}).v = 0, 1 \le i,j \le N\}.
\end{aligned}$$

So we compute $Hom_{B'_\psi}(\mathbb{1}_{\lambda_\iota^0}, V^{\beta^\vee})$ by computing the vectors in $V^{\beta^\vee}$ killed by the left action of $\mathfrak{t}_\iota^\psi$. Then we have the theorem for the invariant space $Hom_{B'_\psi}(\mathbb{1}_{\lambda_\iota^\omega}, V^{\beta^\vee})$.

THEOREM 4.6.2. *In the case $\psi - \iota$ is an even integer. The vector space $Hom_{B'_\psi}(\mathbb{1}_{\lambda_\iota^\omega}, V^{\beta^\vee})$ is either one dimensional or zero. Moreover, it is nonzero if and only if $\beta = \sum_{i=1}^N \beta_i\epsilon_i$, where $\beta \in P^+$ and*

$$\beta_i = \frac{-\omega + \psi - \iota}{2}, \quad i = p+1, \cdots, N-p, \tag{4.62}$$

$$\beta_i + \beta_{N-i+1} = -\omega, \quad i = 1, \cdots, p. \tag{4.63}$$

PROOF. This theorem is verified Lemma 4.6.1, Proposition A.2.2 and Theorem A.1.1 in terms of $\beta^\vee$. Then (4.62) and (4.63) follow. □

Let $\mathcal{B}_\psi \subset P^+$ denote the collection of $\beta \in P^+$ satisfying (4.62)-(4.63). Then the vector space

$$\begin{aligned}
&F^{\sigma,\eta,\tau}_{n,\psi,\omega,\iota}(A_q(GL_N))\\
=&Hom_{B'_\psi \boxtimes B_\sigma}(\mathbb{1}_{\lambda_\iota^\omega} \boxtimes \mathbb{1}_{\chi_\tau^\eta}, (\bigoplus_{\beta\in P^+} V^{\beta^\vee} \boxtimes V^\beta) \otimes_2 (\mathbb{1} \boxtimes V_1) \otimes_2 \cdots \otimes_2 (\mathbb{1} \boxtimes V_n))\\
\cong&\bigoplus_{\beta\in\mathcal{B}_\psi} Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^\beta \otimes V_1 \otimes \cdots \otimes V_n).
\end{aligned}$$

On the other hand, according to Theorem 4.4.3, for each $\beta \in \mathcal{B}_\psi$, the vector space

$$Hom_{B_\sigma}(\mathbb{1}_{\chi_\tau^\eta}, V^\beta \otimes V_1 \otimes \cdots \otimes V_n)$$

has a basis indexed by the collection of sequences $(\nu^{(0)}, \nu^{(1)}, \cdots, \nu^{(n)})$ such that $\nu^{(0)} = \beta$ and $\nu^{(n)}$ satisfying (4.56)-(4.57). Equivalently, a basis indexed by the collection of standard tableaux T of shape ν/β such that ν satisfying (4.56)-(4.57).

THEOREM 4.6.3. *The invariant space* $F^{\sigma,\eta,\tau}_{n,\psi,\omega,\iota}(\mathcal{A}_q(GL_N))$ *has a basis indexed by the collection of sequences* $(\nu^{(0)}, \nu^{(1)}, \cdots, \nu^{(n)})$ *such that* $\nu^{(0)}$ *satisfying* (4.62)-(4.63) *and* $\nu^{(n)}$ *satisfying* (4.56)-(4.57). *Equivalently, a basis indexed by the collection of standard tableaux* T *of shape* $\nu^{(n)}/\beta$ *such that* $\nu^{(n)}$ *satisfying* (4.56)-(4.57) *and* β *satisfying* (4.62)-(4.63). *In the case* $\omega - \psi + \iota$ *or* $\eta + \sigma - \tau$ *is not an even integer, the invariant space* $F^{\sigma,\eta,\tau}_{n,\psi,\omega,\iota}(\mathcal{A}_q(GL_N)) = 0$.

APPENDIX A

Coideal subalgebras and invariants spaces

The theorem we proof here is an analogue of the theory in [**14**].

A.1. Main result

THEOREM A.1.1. *Take $\sigma - \tau$ to be an even integer. There exist nonzero vectors $v \in V^\nu$ such that $\mathfrak{t}^\tau_\sigma.v = 0$ or $(\mathfrak{t}^\tau_\sigma)^*.v = 0$ if and only if $\nu = \sum_{i=1}^N \nu_i \epsilon_i$, where $\nu \in P^+$ and*

$$\nu_i = \frac{\sigma - \tau}{2}, i = p+1, \cdots, N-p, \tag{A.1}$$

$$\nu_i + \nu_{N-i+1} = 0, i = 1, \cdots, p. \tag{A.2}$$

A.2. Proof of the main theorem

A.2.1. Properties of the invariants.

LEMMA A.2.1. *Let V^ν be an irreducible highest weight $U_q(\mathfrak{gl}_N)$-module with highest weight $\nu \in P^+$ and $v \in V^\nu$ be a nonzero $\mathfrak{t}^\tau_\sigma$-invariant vector, i.e. $\mathfrak{t}^\tau_\sigma.v = 0$. Let v_ν denote the highest weight component of v. Then $v_\nu \neq 0$.*

PROOF. Let $\tilde{m}_{ij} = ((J^\tau)^{-1}L^+J^\sigma - L^-)_{ij} = -l^-_{ij} + ((J^\tau)^{-1}L^+J^\sigma)_{ij}$, which are generators of $\mathfrak{t}^\tau_\sigma$. Consider the action of $\tilde{m}_{ij}$ for $i < j$,

$$\tilde{m}_{ij}.v = 0.$$

Let v_μ be the maximal weight component of v such that $v_\mu \neq 0$. Then take the $\mu+\epsilon_i-\epsilon_j$ component of both sides of the equation above, we have $l^-_{ij}.v_\mu = 0$ for any $i < j$. This implies μ is the highest weight of the module V^ν and hence $\mu = \nu$.

$\square$

PROPOSITION A.2.2. *Let $v \in V^\nu$ be a nonzero $\mathfrak{t}^\tau_\sigma$-invariant. Then for any $w \in V^\nu$ such that $\mathfrak{t}^\tau_\sigma.w = 0$, $w = kv$ for some $k \in \mathbb{C}$.*

PROOF. Let v and w be nonzero $\mathfrak{t}_\sigma^\tau$ invariants of the highest weight module V^ν. Then we have $v_\nu \neq 0$ and $w_\nu \neq 0$. Since the ν-component of V^ν is one dimensional. we have $w_\nu = kv_\nu$ for some $k \in \mathbb{C}$. Consider the vector $kv - w \in V^\nu$, this is also a $\mathfrak{t}_\sigma^\tau$ invariant. But the ν-component $(kv-w)_\nu = kv_\nu - w_\nu = 0$, which forces $kv - w = 0$ and hence $w = kv$.

□

REMARK A.2.3. *The dimension of the* $\mathfrak{t}_\sigma^\tau$*-invariant subspace of* V^ν *is either* 0 *or* 1*.*

PROPOSITION A.2.4. *Let* $v \in V^\nu$ *be a* $\mathfrak{t}_\sigma^\eta$*-invariant and* $w \in V^\mu$ *be a* $\mathfrak{t}_\eta^\tau$*-invariant. Then the tensor* $w \otimes v \in V^\mu \otimes V^\nu$ *is a* $\mathfrak{t}_\sigma^\tau$*-invariant.*

PROOF. This fact is verified by the following computation.

$$
\begin{aligned}
& \Delta(L^+J^\sigma - J^\tau L^-) \\
=& L^+ \otimes L^+J^\sigma - J^\tau L^- \otimes L^- \\
=& L^+ \otimes L^+J^\sigma - L^+ \otimes J^\eta L^- + L^+ \otimes J^\eta L^- - J^\tau L^- \otimes L^- \\
=& L^+ \otimes L^+J^\sigma - L^+ \otimes J^\eta L^- + L^+J^\eta \otimes L^- - J^\tau L^- \otimes L^- \\
=& L^+ \otimes (L^+J^\sigma - J^\eta L^-) + (L^+J^\eta - J^\tau L^-) \otimes L^-
\end{aligned}
$$

□

REMARK A.2.5. *By Proposition A.2.1, the image of* $w \otimes v$ *under the canonical map*

$$V^\mu \otimes V^\nu \to V(\nu + \mu)$$

is a $\mathfrak{t}_\sigma^\tau$*-invariant in* $V(\nu+\mu)$*. It suffices to show Theorem A.1.1 in the case* $\sigma - \tau = \pm 2$*.*

Let $\mathfrak{t}_\sigma^\sigma$ be the subalgebra generated by the entries of the matrix $L^+J^\sigma - J^\sigma L^-$. We have the following fact in [**14**].

THEOREM A.2.6. *[14] There exist a nonzero vector* $v \in V^\nu$ *such that* $\mathfrak{t}_\sigma^\sigma.v = 0$ *if and only if*

$$\nu_i = 0, i = p+1, \cdots, N-p,$$

$$\nu_i + \nu_{N-i+1} = 0, i = 1, \cdots, p.$$

COROLLARY A.2.7. *Let $v \in V^\nu$ be a $\mathfrak{t}_\sigma^\tau$-invariant and $w \in V^\mu$ is a $\mathfrak{t}_\sigma^\sigma$-invariant. Then*

$$w \otimes v \in V^\mu \otimes V^\nu$$

is a $\mathfrak{t}_\sigma^\tau$-invariant.

REMARK A.2.8. *It suffices to show there is a nonzero vector in V^ν which is killed by $\mathfrak{t}_\sigma^\tau$ where $\sigma - \tau = 2$ and $\nu = \sum_{i=1}^{N-p} \epsilon_i - \sum_{j=N-p+1}^{N} \epsilon_j$.*

A.2.2. Actions of e_i^* and f_i^*. Let $V = V^{\epsilon_1}$ be the vector representation of $U_q(\mathfrak{gl}_N)$ and V^* is the dual representation of the vector representation. We take the basis $\{v_1, \cdots, v_N\}$ of V and the dual basis $\{v_1^*, \cdots, v_N^*\}$ of V^*. We have

$$e_i.v_{i+1} = v_i, \quad e_i.v_j = 0, , j \neq i+1,$$

$$f_i.v_i = v_{i+1}, \quad f_i.v_j = 0, j \neq i,$$

$$e_i.v_i^* = -q^{-1} v_{i+1}^*, \quad e_i.v_j^* = 0, j \neq i,$$

$$f_i.v_{i+1}^* = -q v_i^*, \quad f_i.v_j^* = 0, j \neq i+1.$$

We compute the actions of e_i and f_i on the representation $(\bigwedge^{N-p} V) \otimes (\bigwedge^p V^*)$. Let

$$I = \{1 \leq i_1 < \cdots < i_{N-p} \leq N\}$$

and $v_I = v_{i_1} \wedge \cdots \wedge v_{i_{N-p}}$, then $\{v_I | I \subset \{1, \cdots, N, |I| = N-p\}\}$ forms a basis of $\bigwedge^{N-p} V$. Similarly, let

$$J = \{1 \leq j_1 < \cdots < j_p \leq N, |J| = p\}$$

and $v_J^* = v_{j_1}^* \wedge \cdots \wedge v_{j_p}^*$, then $\{v_J^* | J \subset \{1, \cdots, N\}\}$ forms a basis of $\bigwedge^p V^*$. The actions of e_i and f_i are computed as follows.

$$e_i.v_I = v_{I-\{i+1\}\cup\{i\}}, (i \notin I, i+1 \in I), \quad e_i.v_I = 0 \text{ otherwise},$$

$$f_i.v_I = v_{I-\{i\}\cup\{i+1\}}, (i+1 \notin I, i \in I), \quad f_i.v_I = 0 \text{ otherwise},$$

$$e_j.v_J^* = -q^{-1} v_{J-\{j\}\cup\{j+1\}}, (j+1 \notin J, j \in J), \quad e_j.v_J^* = 0 \text{ otherwise},$$

$$f_j.v_J^* = -q v_{J-\{j+1\}\cup\{j\}}, (j \notin J, j+1 \in J), \quad f_j.v_J^* = 0 \text{ otherwise}.$$

Recall the Hopf $*$ structure of $U_q(\mathfrak{gl}_N)$. The actions of e_i^* and f_i^* on $(\bigwedge^{N-p} V) \otimes (\bigwedge^p V^*)$ are as follows.

$$f_i^*.v_I = v_{I-\{i+1\}\cup\{i\}}, (i \notin I, i+1 \in I), \quad f_i^*.v_I = 0 \text{ otherwise}, \tag{A.3}$$

$$e_i^*.v_I = v_{I-\{i\}\cup\{i+1\}}, (i+1 \notin I, i \in I), \quad e_i^*.v_I = 0 \text{ otherwise}, \tag{A.4}$$

$$f_j^*.v_J^* = -q^{-1} v_{J-\{j\}\cup\{j+1\}}, (j+1 \notin J, j \in J), \quad f_j^*.v_J^* = 0 \text{ otherwise}, \tag{A.5}$$

$$e_j^*.v_J^* = -q v_{J-\{j+1\}\cup\{j\}}, (j \notin J, j+1 \in J), \quad e_j^*.v_J^* = 0 \text{ otherwise}. \tag{A.6}$$

Similarly, we compute the actions of e_i and f_i on the representation $(\bigwedge^p V) \otimes (\bigwedge^{N-p} V^*)$. Let $I = \{1 \leq i_1 < \cdots < i_p \leq N\}$ and $v_I = v_{i_1} \wedge \cdots \wedge v_{i_p}$, then $\{v_I | I \subset \{1, \cdots, N\}\}$ forms a basis of $\bigwedge^p V$. Similarly, let $J = \{1 \leq j_1 < \cdots < j_{N-p} \leq N\}$ and $v_J^* = v_{j_1}^* \wedge \cdots \wedge v_{j_{N-p}}^*$, then $\{v_J^* | J \subset \{1, \cdots, N\}\}$ forms a basis of $\bigwedge^{N-p} V^*$.

We compute the actions of e_i^* and f_i^* on $(\bigwedge^p V) \otimes (\bigwedge^{N-p} V^*)$ as follows.

$$f_i^*.v_I = v_{I-\{i+1\}\cup\{i\}}, (i \notin I, i+1 \in I), \quad f_i^*.v_I = 0 \text{ otherwise},$$

$$e_i^*.v_I = v_{I-\{i\}\cup\{i+1\}}, (i+1 \notin I, i \in I), \quad e_i^*.v_I = 0 \text{ otherwise},$$

$$f_j^*.v_J^* = -q^{-1} v_{J-\{j\}\cup\{j+1\}}, (j+1 \notin J, j \in J), \quad f_j^*.v_J^* = 0 \text{ otherwise},$$

$$e_j^*.v_J^* = -q v_{J-\{j+1\}\cup\{j\}}, (j \notin J, j+1 \in J), \quad e_j^*.v_J^* = 0 \text{ otherwise}.$$

A.2.3. $\mathfrak{t}_\sigma^\tau$-invariant vector. With the actions of e_i^* and f_i^* on $(\bigwedge^{N-p} V) \otimes (\bigwedge^p V^*)$, we compute the actions of the generators m_{ij} of $\mathfrak{t}_\sigma^\tau$ on $(\bigwedge^{N-p} V) \otimes (\bigwedge^p V^*)$. And we are going to show the following fact that.

Theorem A.2.9. *The vector $v = \sum_{I,J} t_{IJ} v_I \otimes v_J^* \in (\bigwedge^{N-p} V) \otimes (\bigwedge^p V^*)$, with I satisfies*

$$(1)\{p+1, \cdots, N-p\} \subset I$$

$$(2)|\{i, N-i+1\} \cap I| = 1, i = 1, \cdots, p$$

J satisfies

$$(1)\{p+1, \cdots, N-p\} \cap J = \emptyset,$$

$$(2)|\{i, N-i+1\} \cap J| = 1, i = 1, \cdots, p$$

and $t_{IJ} = \Pi_{i=1}^p k_i$, where

$$k_i = \begin{cases} 1, & i \in I \text{ and } i \notin J \\ (-1)^{p-i} q^{2N+\sigma+1-p-3i}, & i \in I \text{ and } i \in J \\ (-1)^{N-p-i-1} q^{N-p-i-1+\sigma}, & i \notin I \text{ and } i \notin J \\ (-1)^{N-1} q^{3N-2p-4i+2\sigma}, & i \notin I \text{ and } i \in J \end{cases}$$

in $(\bigwedge^{N-p} V) \otimes (\bigwedge^p V^*)$ is a $\mathfrak{t}_\sigma^\tau$-invariant, where $\sigma - \tau = 2$.

Remark A.2.10. *By Theorem A.2.9 above, we verify the existence of a nonzero $\mathfrak{t}_\sigma^\tau$-invariant in V^ν, where $\sigma - \tau = 2$ and $\nu = \sum_{i=1}^{N-p} \epsilon_i - \sum_{j=N-p+1}^{N} \epsilon_j$.*

Moreover, we have a similar fact in the case $\sigma - \tau = -2$.

Theorem A.2.11. *The vector $v = \sum_{I,J} t_{IJ} v_I \otimes v_J^* \in (\bigwedge^p V) \otimes (\bigwedge^{N-p} V^*)$, with I satisfies*

$$(1)\{p+1, \cdots, N-p\} \cap I = \emptyset$$

$$(2)|\{i, N-i+1\} \cap I| = 1, i = 1, \cdots, p$$

J satisfies

$$(1)\{p+1,\cdots,N-p\}\subset J,$$

$$(2)|\{i,N-i+1\}\cap J|=1, i=1,\cdots,p$$

and $t_{IJ}=\Pi_{i=1}^{p}k_i$, *where*

$$k_i=\begin{cases}1, & i\in I \text{ and } i\notin J\\ (-1)^{N-p-i+1}q^{N+p-\tau+3-3i}, & i\in I \text{ and } i\in J\\ (-1)^{p-i}q^{p-i+1+\tau}, & i\notin I \text{ and } i\notin J\\ (-1)^{N-1}q^{N+2p-4i-2\tau}, & i\notin I \text{ and } i\in J\end{cases}$$

in $(\bigwedge^p V)\otimes(\bigwedge^{N-p}V^*)$ *is a* $\mathfrak{t}_\sigma^\tau$*-invariant, where* $\sigma-\tau=-2$.

We will compute the coefficient t_{IJ} in the following subsections.

A.2.4. Proof of Theorem A.1.1. Let p be a positive integer such that $p<\frac{N}{2}$.

LEMMA A.2.12. *Let* $v\in V^\nu$ *be a* $\mathfrak{t}_\sigma^\tau$*-invariant and let* v_λ *be the weight* λ*-component of* v*. If* $v_\lambda\neq 0$*, then* λ *satisfies*

$$\lambda_i=\frac{\sigma-\tau}{2}, i=p+1,\cdots,N-p, \tag{A.7}$$

$$\lambda_i+\lambda_{N-i+1}=0, i=1,\cdots,p. \tag{A.8}$$

PROOF. Consider the action of the (i,i)-entry m_{ii} of $L^+J^\sigma-J^\tau L^-$ on v, where $p+1\leq i\leq N-p$. We have

$$\begin{aligned} m_{ii}&=-q^{-\sigma}l_{ii}^{+}+q^{-\tau}l_{ii}^{-}\\ &=-q^{-\sigma}q^{\epsilon_i}+q^{-\tau}q^{-\epsilon_i}. \end{aligned}$$

and $m_{ii}.v=0$. Let $\lambda=\sum_{i=1}^{N}\lambda_i\epsilon_i$. So each λ-component $(m_{ii}.v)_\lambda$ of $m_{ii}.v$ is also zero and we have $(m_{ii}.v)_\lambda=m_{ii}.v_\lambda$. Then $m_{ii}.v_\lambda=-q^{-\sigma}q^{\lambda_i}+q^{-\tau}q^{-\lambda_i}=0$, which implies

$$\lambda_i=\frac{\sigma-\tau}{2}$$

for $i = p+1, \cdots, N-p$. This proves (A.7).

Then consider the $(N-i+1, i)$-entry $m_{N-i+1,i}$, where $1 \leq i \leq p$ with

$$m_{N-i+1,i} = l^+_{N-i+1,N-i+1} - l^-_{ii} = q^{\epsilon_{N-i+1}} - q^{-\epsilon_i}.$$

This proves (A.8). $\square$

Next we show Theorem A.1.1 in the case $\sigma - \tau = 2$. Hence we compute nonzero $\mathfrak{t}^\tau_\sigma$-invariant vectors in $(\bigwedge^{N-p} V) \otimes (\bigwedge^p V^*)$.

Remark A.2.13. *Suppose $v \in (\bigwedge^{N-p} V) \otimes (\bigwedge^p V^*)$ is a nonzero $\mathfrak{t}^\tau_\sigma$-invariant vector. Since each weight λ-component v_λ is a linear combination of $v_I \otimes v^*_J$, we have $\lambda_i = 1$ for $i = p+1, \cdots, N-p$, $\lambda_i = \pm 1$ and $\lambda_i + \lambda_{N-i+1} = 0$ for $i = 1, \cdots, p$, according to Lemma A.2.12.*

Lemma A.2.14. *Let v be a $\mathfrak{t}^\tau_\sigma$-invariant vector in $(\bigwedge^{N-p} V) \otimes (\bigwedge^p V^*)$ and v is a linear combination of $v = \sum_{IJ} t_{IJ} v_I \otimes v^*_J$. If the coefficient t_{IJ} of $v_I \otimes v^*_J$ is nonzero in the linear combination, then we have $\{p+1, \cdots, N-p\} \subset I$ and $\{p+1, \cdots, N-p\} \cap J = \emptyset$.*

Proof. Let λ denote the weight of the tensor $v_I \otimes v^*_J$, which is the sum of the weight of v_I and the weight of v^*_J. The weight of v_I is $\sum_{i \in I} \epsilon_i$ and the weight of v^*_J is $-\sum_{j \in J} \epsilon_j$. For each i such that $p+1 \leq i \leq N-p$, the fact that $\lambda_i = 1$ implies $i \in I$ and $i \notin J$. Then it follow that $\{p+1, \cdots, N-p\} \subset I$ and $\{p+1, \cdots, N-p\} \cap J = \emptyset$. $\square$

Lemma A.2.15. *Let t_{IJ} denote the coefficient of $v_I \otimes v^*_J$ in the $\mathfrak{t}^\tau_\sigma$-invariant v. Let $\omega = \sum_{i=1}^N \omega_i \epsilon_i$ be the weight of $v_I \otimes v^*_J$ and k is the largest integer less or equal to p such that $\omega_k \neq 1$. If $t_{IJ} \neq 0$, then the coefficient $t_{I \cup \{k\} - \{N-k+1\}, J \cup \{N-k+1\} - \{k\}} \neq 0$.*

Proof. By Lemma A.2.13, $\omega_k = 0$ or $\omega_k = -1$. If $\omega_k = 0$, then there are two possibilities: (1)$k \notin I$ and $k \notin J$ or (2) $k \in I$ and $k \in J$. Consider the action of the $(k, N-p)$-entry $m_{k,N-p}$ of the matrix $L^+ J^\sigma - J^\tau L^-$, where

$$\begin{aligned} m_{k,N-p} &= -q^{-\sigma} l^+_{k,N-p} - l^-_{N-k+1,N-p} \\ &= -q^{-\sigma}(q - q^{-1}) q^{\epsilon_k} e_{N-p,k} + (q - q^{-1}) e_{N-p,N-k+1} q^{-\epsilon_{N-p}} \end{aligned}$$

and the condition $m^*_{k,N-p}.v = 0$.

Case 1. $\omega_k = 0$, $k \in I$ and $k \in J$.

The left action of $e^*_{N-p,k}$ on $v_I \otimes v^*_J$ gives a nonzero vector:

$$\begin{aligned} e^*_{N-p,k}.(t_{IJ} v_I \otimes v^*_J) &= (-q)^{-N+p+k+1}(1 \otimes f^*_{N-p-1} \cdots f^*_k).(t_{IJ} v_I \otimes v^*_J) \\ &= (-q)^{-2N+2p+2k+3} t_{IJ}(v_I \otimes v^*_{J\cup\{N-p\}-\{k\}}) \\ &\neq 0. \end{aligned}$$

Apart from $v_I \otimes v^*_J$, the tensor $v_I \otimes v^*_{J\cup\{N-k+1\}-\{k\}}$ is the only weight component of v gives nonzero $v_I \otimes v^*_{J\cup\{N-p\}-\{k\}}$ under the left action of $m^*_{k,N-p}$.

$$\begin{aligned} e^*_{N-p,N-k+1}.(v_I \otimes v^*_{J\cup\{N-k+1\}-\{k\}}) &= (q^{\epsilon_{N-p}-\epsilon_{N-k+1}} \otimes e^*_{N-k} \cdots e^*_{N-p}).(v_I \otimes v^*_{J\cup\{N-k+1\}-\{k\}}) \\ &= q(-q)^{p-k+1}(v_I \otimes v^*_{J\cup\{N-p\}-\{k\}}). \end{aligned}$$

The condition $m^*_{k,N-p}.v = 0$ implies the coefficient

$$t_{I,J\cup\{N-k+1\}-\{k\}} = t_{I\cup\{k\}-\{N-k+1\},J\cup\{N-k+1\}-\{k\}} = -t_{IJ}(-q)^{-2N+p+3k+1} \neq 0.$$

Case 2. $\omega_k = 0$, $k \notin I$ and $k \notin J$. The left action of $e^*_{N-p,k}$ on $v_I \otimes v^*_J$ gives a nonzero vector:

$$\begin{aligned} e^*_{N-p,k}.(t_{IJ} v_I \otimes v^*_J) &= (-q)^{-N+p+k+1}(f^*_{N-p-1} \cdots f^*_k \otimes q^{-\epsilon_k+\epsilon_{N-p}}).(t_{IJ} v_I \otimes v^*_J) \\ &= (-q)^{-N+p+k+1} t_{IJ}(v_{I\cup\{k\}-\{N-p\}} \otimes v^*_J) \\ &\neq 0. \end{aligned}$$

Apart from $v_I \otimes v^*_J$, the tensor $v_{I\cup\{k\}-\{N-k+1\}} \otimes v^*_J$ is the only weight component of v gives nonzero $v_{I\cup\{k\}-\{N-p\}} \otimes v^*_J$ under the left action of $m^*_{k,N-p}$.

$$\begin{aligned} e^*_{N-p,N-k+1}.(v_{I\cup\{k\}-\{N-k+1\}} \otimes v^*_J) &= (q^{\epsilon_{N-p}-\epsilon_{N-k+1}} \otimes e^*_{N-k} \cdots e^*_{N-p}).(v_{I\cup\{k\}-\{N-k+1\}} \otimes v^*_J) \\ &= q(-q)^{p-k+1}(v_{I\cup\{k\}-\{N-p\}} \otimes v^*_J). \end{aligned}$$

The condition $m^*_{k,N-p}.v = 0$ implies the coefficient

$$t_{v_{I\cup\{k\}-\{N-k+1\}},J} = t_{I\cup\{k\}-\{N-k+1\},J\cup\{N-k+1\}-\{k\}} = -t_{IJ}(-q)^{-N+2k-1} \neq 0.$$

Case 3. $\omega_k = -1$. Then in this case, $k \notin I$ and $k \in J$.

The left action of $e^*_{N-p,k}$ on $v_I \otimes v^*_J$ gives two nonzero vectors:

$$\begin{aligned} e^*_{N-p,k}.(t_{IJ}v_I \otimes v^*_J) &= (-q)^{-N+p+k+1}(f^*_{N-p-1}\cdots f^*_k \otimes q^{-\epsilon_k+\epsilon_{N-p}}).(t_{IJ}v_I \otimes v^*_J) \\ &= q(-q)^{-N+p+k+1}t_{IJ}(v_{I\cup\{k\}-\{N-p\}} \otimes v^*_J) \\ &\neq 0 \end{aligned}$$

and

$$\begin{aligned} e^*_{N-p,k}.(t_{IJ}v_I \otimes v^*_J) &= (-q)^{-N+p+k+1}(1 \otimes f^*_{N-p-1}\cdots f^*_k).(t_{IJ}v_I \otimes v^*_J) \\ &= (-q)^{-2N+2p+2k+3}t_{IJ}(v_I \otimes v^*_{J\cup\{N-p\}-\{k\}}) \\ &\neq 0. \end{aligned}$$

Apart from $v_I \otimes v^*_J$, the tensor $v_{I\cup\{k\}-\{N-k+1\}} \otimes v^*_J$ is the only weight component of v gives nonzero $v_{I\cup\{k\}-\{N-p\}} \otimes v^*_J$ under the left action of $m^*_{k,N-p}$.

$$\begin{aligned} e^*_{N-p,N-k+1}.(v_{I\cup\{k\}-\{N-k+1\}} \otimes v^*_J) &= (q^{\epsilon_{N-p}-\epsilon_{N-k+1}} \otimes e^*_{N-k}\cdots e^*_{N-p}).(v_{I\cup\{k\}-\{N-k+1\}} \otimes v^*_J) \\ &= q(-q)^{p-k+1}(v_{I\cup\{k\}-\{N-p\}} \otimes v^*_J). \end{aligned}$$

The condition $m^*_{k,N-p}.v = 0$ implies the coefficient

$$t_{I\cup\{k\}-\{N-k+1\},J} \neq 0.$$

Apart from $v_I \otimes v^*_J$, the tensor $v_I \otimes v^*_{J\cup\{N-k+1\}-\{k\}}$ is the only weight component of v gives nonzero $v_I \otimes v^*_{J\cup\{N-p\}-\{k\}}$ under the left action of $m^*_{k,N-p}$.

$$\begin{aligned} e^*_{N-p,N-k+1}.(v_I \otimes v^*_{J\cup\{N-k+1\}-\{k\}}) &= (q^{\epsilon_{N-p}-\epsilon_{N-k+1}} \otimes e^*_{N-k}\cdots e^*_{N-p}).(v_I \otimes v^*_{J\cup\{N-k+1\}-\{k\}}) \\ &= (-q)^{p-k+1}(v_I \otimes v^*_{J\cup\{N-p\}-\{k\}}). \end{aligned}$$

The condition $m^*_{k,N-p}.v = 0$ implies the coefficient

$$t_{I,J\cup\{N-k+1\}-\{k\}} \neq 0.$$

Applying case 1 and case 2, we have

$$t_{I\cup\{k\}-\{N-k+1\},J\cup\{N-k+1\}-\{k\}} \neq 0.$$

$\square$

LEMMA A.2.16. *Let the vector $v = \sum_{I,J} t_{IJ} v_I \otimes v_J^*$ be a $\mathfrak{k}^\tau_\sigma$-invariant. Then the coefficient t_{IJ} is nonzero if and only if I satisfies $|\{i, N-i+1\} \cap I| = 1, i = 1, \cdots, p$ and J satisfies $|\{i, N-i+1\} \cap J| = 1, i = 1, \cdots, p$.*

PROOF. Let us prove this lemma by contradiction. Let k be the maximal integer less or equal to p such that $|\{k, N-k+1\} \cap I| \neq 1$. Then consider two cases: (1) $|\{k, N-k+1\} \cap I| = 0$ or (2) $|\{k, N-k+1\} \cap I| = 2$. The $(k, N-p)$-entry $m_{k,N-p}$ of the matrix $L^+ J^\sigma - J^\tau L^-$ equals $-q^{-\sigma} l^+_{k,N-p} - l^-_{N-k+1,N-p}$. The fact $m^*_{k,N-p}.v = 0$ implies $(m^*_{k,N-p}.v)_\omega = 0$ for every ω-component. By Lemma A.2.15, we assume $\{k+1, \cdots, p\} \subset I$ and $\{N-p+1, \cdots, N-k\} \subset J$.

Case 1. $|\{k, N-k+1\} \cap I| = 0$. By (A.3)-(A.6), the action of $e^*_{N-p,N-k+1}$ is as follows.

$$\begin{aligned} e^*_{N-p,N-k+1}.(v_I \otimes v_J^*) &= (e^*_{N-k} \cdots e^*_{N-p+1} e^*_{N-p} \otimes 1).(v_I \otimes v_J^*) \\ &= v_{I-\{N-p\}\cup\{N-k+1\}} \otimes v_J^* \neq 0. \end{aligned}$$

Let λ denote the weight of $v_I \otimes v_J^*$. Then $\lambda_k = 0$ by $|\{k, N-k+1\} \cap I| = 0$ and Lemma A.2.12. The fact $m^*_{k,N-p}.v = 0$ forces the existence of a term which gives $v_{I-\{N-p\}\cup\{N-k+1\}} \otimes v_J^*$ after the action of $e^*_{N-p,k}$. The possible vector $v_{I'} \otimes v^*_{J'}$ is of weight $\lambda - \epsilon_k + \epsilon_{N-k+1}$ and hence $k \notin I'$, $N-k+1 \in I'$, $k \in J'$ and $N-k+1 \notin J'$. Then the action of $e^*_{N-p,k}$ on $v_{I'} \otimes v^*_{J'}$ gives a linear combination of $v_{I'-\{N-p\}\cup\{k\}} \otimes v^*_{J'}$ and $v_{I'} \otimes v^*_{J'-\{k\}\cup\{N-p\}}$, neither of which is $v_{I-\{N-p\}\cup\{N-k+1\}} \otimes v_J^*$. So there is no vector gives $v_{I-\{N-p\}\cup\{N-k+1\}} \otimes v_J^*$ after the action of $e^*_{N-p,k}$, which contradicts $m^*_{k,N-p}.v = 0$.

Case 2. $|\{k, N-k+1\} \cap I| = 2$. The action of $e^*_{N-p,N-k+1}$ gives the vector

$$\begin{aligned} e^*_{N-p,N-k+1}.(v_I \otimes v_J^*) &= (q^{\epsilon_{N-p}-\epsilon_{N-k+1}} \otimes e^*_{N-k} \cdots e^*_{N-p}).(v_I \otimes v_J^*) \\ &= (-q)^{p-k}(v_I \otimes v^*_{J-\{N-k+1\}\cup\{N-p\}}) \neq 0. \end{aligned}$$

Let λ denote the weight of $v_I \otimes v_J^*$. Then $\lambda_k = 0$ by $|\{k, N-k+1\} \cap I| = 2$ and Lemma A.2.12. The fact $m^*_{k,N-p}.v = 0$ forces the existence of a term which gives $v_I \otimes v^*_{J\cup\{N-p\}-\{N-k+1\}}$ after the action of $e^*_{N-p,k}$. The possible vector $v_{I'} \otimes v^*_{J'}$ is of weight $\lambda - \epsilon_k + \epsilon_{N-k+1}$ and hence $k \notin I'$, $N-k+1 \in I'$, $k \in J'$ and $N-k+1 \notin J'$. Then the action of $e^*_{N-p,k}$ on $v_{I'} \otimes v^*_{J'}$ gives a linear combination of $v_{I'-\{N-p\}\cup\{k\}} \otimes v^*_{J'}$ and $v_{I'} \otimes v^*_{J'-\{k\}\cup\{N-p\}}$, neither of which is $v_{I-\{N-p\}\cup\{N-k+1\}} \otimes v_J^*$. So there is no vector gives $v_{I-\{N-p\}\cup\{N-k+1\}} \otimes v_J^*$ after the action of $e^*_{N-p,k}$, which contradicts $m^*_{k,N-p}.v = 0$.

□

Remark A.2.17. *In summary, if the $v_I \otimes v_J^*$ with nonzero coefficient t_{IJ} in the linear combination of a $\mathfrak{t}^\tau_\sigma$-invariant v, then the index I satisfies*

(A.9) $$\{p+1, \cdots, N-p\} \subset I,$$

(A.10) $$|\{i, N-i+1\} \cap I| = 1, i = 1, \cdots, p$$

J satisfies

(A.11) $$\{p+1, \cdots, N-p\} \cap J = \emptyset,$$

(A.12) $$|\{i, N-i+1\} \cap J| = 1, i = 1, \cdots, p.$$

So it suffices to compute the coefficients of the vectors with indices I, J satisfying (A.9)-(A.12). Moreover, let

$$\nu = \sum_{i=1}^{N-p} \epsilon_i - \sum_{j=N-p+1}^{N} \epsilon_j.$$

By Proposition A.2.1, the highest weight ν-component v_ν of v is nonzero, namely $t_{I_0,J_0} \neq 0$. Without loss of generality, take $v_\nu = v_{\{1,2,\cdots,N-p\}} \otimes v^*_{\{N-p+1,\cdots,N\}}$. Let $I_0 = \{1, \cdots, N-p\}$ and $J_0 = \{N-p+1, \cdots, N\}$. And we denote $v_\nu = v_{I_0} \otimes v^*_{J_0}$.

Lemma A.2.18. *Let $v \in (\bigwedge^{N-p} V) \otimes (\bigwedge^{p} V^*)$ be a $\mathfrak{t}_\sigma^\tau$-invariant with nonzero highest weight component $v_\nu = v_{I_0} \otimes v^*_{J_0}$. Then for $1 \le i \le p$, the $\nu - \epsilon_i + \epsilon_{N-i+1}$-component of v is a linear combination of $v_{I_0} \otimes v^*_{J_0 - \{N-i+1\} \cup \{i\}}$ and $v_{I_0 - \{i\} \cup \{N-i+1\}} \otimes v^*_{J_0}$. In particular, the coefficient of $v_{I_0} \otimes v^*_{J_0-\{N-i+1\}\cup\{i\}}$ is $(-1)^{p-i} q^{2N+\sigma+1-p-3i}$ and the coefficient of $v_{I_0-\{i\}\cup\{N-i+1\}} \otimes v^*_{J_0}$ is $(-1)^{N-p-i-1} q^{N-p-i-1+\sigma}$.*

Proof. The $(i, N-p)$-entry $m_{i,N-p}$ of the matrix $L^+ J^\sigma - J^\tau L^-$ is

$$\begin{aligned} m_{i,N-p} &= -q^{-\sigma} l^+_{i,N-p} - l^-_{N-i+1,N-p} \\ &= q^{-\sigma}(q - q^{-1}) q^{\epsilon_i} e_{N-p,i} + (q-q^{-1}) e_{N-p,N-i+1} q^{-\epsilon_{N-p}}. \end{aligned}$$

The condition $m^*_{i,N-p}.v = 0$ implies $(m^*_{i,N-p}.v)_\omega = 0$, for any weight ω-component. Consider the case when $\omega = \nu - \epsilon_{N-p} + \epsilon_{N-i+1}$. Let x_i denote the coefficient of $v_{I_0} \otimes v^*_{J_0-\{N-i+1\}\cup\{i\}}$ and y_i denote the coefficient of $v_{I_0-\{i\}\cup\{N-i+1\}} \otimes v^*_{J_0}$. There are two terms of weight ω, i.e.

$$\begin{aligned} (m^*_{i,N-p}.v)_\omega =& -(l^-_{N-i+1,N-p})^*.(v_{I_0} \otimes v^*_{J_0}) \\ & - q^{-\sigma}(l^+_{i,N-p})^*.[x_i(v_{I_0} \otimes v^*_{J_0-\{N-i+1\}\cup\{i\}}) + y_i(v_{I_0-\{i\}\cup\{N-i+1\}} \otimes v^*_{J_0})] \\ =& 0. \end{aligned}$$

To compute the first term,

$$\begin{aligned} & (l^-_{N-i+1,N-p})^*.(v_{I_0} \otimes v^*_{J_0}) \\ =& -(q-q^{-1}) q^{-\epsilon_{N-p}} e^*_{N-p,N-i+1}.(v_{I_0} \otimes v^*_{J_0}) \\ =& -(q-q^{-1}) q^{-\epsilon_{N-p}} [((e_{N-p} \cdots e_{N-i})^* \otimes 1).(v_{I_0} \otimes v^*_{J_0}) \\ & + (q^{\epsilon_{N-p} - \epsilon_{N-i+1}} \otimes (e_{N-p} \cdots e_{N-i})^*).(v_{I_0} \otimes v^*_{J_0})] \\ =& -(q-q^{-1})[(v_{I_0-\{N-p\}\cup\{N-i+1\}} \otimes v^*_{J_0}) \\ & - (-q)^{p-i+2}(v_{I_0} \otimes v^*_{J_0-\{N-i+1\}\cup\{N-p\}})]. \end{aligned}$$

The second term is compute as follows. We have

$$
\begin{aligned}
& q^{-\sigma}(l^{+}_{i,N-p})^{*}.x_i(v_{I_0}\otimes v^{*}_{J_0-\{N-i+1\}\cup\{i\}})\\
=& q^{-\sigma}(q-q^{-1})e^{*}_{N-p,i}q^{\epsilon_i}.x_i(v_{I_0}\otimes v^{*}_{J_0-\{N-i+1\}\cup\{i\}})\\
=& x_i q^{-\sigma}(q-q^{-1})(-q^{-1})^{N-p-i-1}(1\otimes(f_i\cdots f_{N-p-1})^{*})(v_{I_0}\otimes v^{*}_{J_0-\{N-i+1\}\cup\{i\}})\\
=& x_i q^{-\sigma}(q-q^{-1})(-q^{-1})^{2N-2p-2i-1}(v_{I_0}\otimes v^{*}_{J_0-\{N-i+1\}\cup\{N-p\}})
\end{aligned}
$$

and

$$
\begin{aligned}
& q^{-\sigma}(l^{+}_{i,N-p})^{*}.y_i(v_{I_0-\{i\}\cup\{N-i+1\}}\otimes v^{*}_{J_0})\\
=& q^{-\sigma}(q-q^{-1})e^{*}_{N-p,i}q^{\epsilon_i}.y_i(v_{I_0-\{i\}\cup\{N-i+1\}}\otimes v^{*}_{J_0})\\
=& y_i q^{-\sigma}(q-q^{-1})(-q^{-1})^{N-p-i-1}((f_i\cdots f_{N-p-1})^{*}\otimes q^{-\epsilon_i+\epsilon_{N-p}})(v_{I_0-\{i\}\cup\{N-i+1\}}\otimes v^{*}_{J_0})\\
=& y_i q^{-\sigma}(q-q^{-1})(-q^{-1})^{N-p-i-1}(v_{I_0-\{N-p\}\cup\{N-i+1\}}\otimes v^{*}_{J_0}).
\end{aligned}
$$

Hence we obtain the following equations

$$
\begin{aligned}
&(-q)^{p-i+2}+q^{-\sigma}(-q^{-1})^{2N-2p-2i-1}x_i=0,\\
&-1+q^{-\sigma}(-q^{-1})^{N-p-i-1}y_i=0.
\end{aligned}
$$

We have

$$x_i=(-1)^{p-i}q^{2N+\sigma-p-3i+1} \tag{A.13}$$

and

$$y_i=(-1)^{N-p-i-1}q^{N-p-i+\sigma-1}. \tag{A.14}$$

□

LEMMA A.2.19. *Let $v\in(\bigwedge^{N-p}V)\otimes(\bigwedge^{p}V^{*})$ be a $\mathfrak{k}^{\tau}_{\sigma}$-invariant with nonzero highest weight component $v_{I_0}\otimes v^{*}_{J_0}$. Then for $1\le i\le p$, the only basis element of weight $\nu-2\epsilon_i+2\epsilon_{N-i+1}$ is*

$v_{I_0-\{i\}\cup\{N-i+1\}} \otimes v^*_{J_0-\{N-i+1\}\cup\{i\}}$, *the coefficient of which in v is*

$$t_{I_0-\{i\}\cup\{N-i+1\},J_0-\{N-i+1\}\cup\{i\}} = (-1)^{N-1} q^{3N-2p-4i+2\sigma}.$$

Proof. Consider the right action of $(i, N-p)$-entry $m_{i,N-p}$ of the matrix $L^+J^\sigma - J^\tau L^-$ on v. The condition $m^*_{i,N-p}.v = 0$ implies $(m^*_{i,N-p}.v)_\omega = 0$, for any weight ω-component. Consider the case when $\omega = \nu - \epsilon_i - \epsilon_{N-p} + 2\epsilon_{N-i+1}$. Let z_i denote the coefficient $t_{I_0-\{i\}\cup\{N-i+1\},J_0-\{N-i+1\}\cup\{i\}}$ of

$$v_{I_0-\{i\}\cup\{N-i+1\}} \otimes v^*_{J_0-\{N-i+1\}\cup\{i\}}.$$

There are two terms of weight ω, i.e.

$$\begin{aligned}(m^*_{i,N-p}.v)_\omega =& -(l^-_{N-i+1,N-p})^*.[x_i(v_{I_0} \otimes v^*_{J_0-\{N-i+1\}\cup\{i\}}) + y_i(v_{I_0-\{i\}\cup\{N-i+1\}} \otimes v^*_{J_0})]\\ &- q^{-\sigma}(l^+_{i,N-p})^*.z_i(v_{I_0-\{i\}+\{N-i+1\}} \otimes v^*_{J_0-\{N-i+1\}\cup\{i\}})\\ =&0.\end{aligned}$$

Computing the first term, we have

$$\begin{aligned}&(l^-_{N-i+1,N-p})^*.x_i(v_{I_0} \otimes v^*_{J_0-\{N-i+1\}\cup\{i\}})\\ =&-(q-q^{-1})q^{-\epsilon_{N-p}} e^*_{N-p,N-i+1}.x_i(v_{I_0} \otimes v^*_{J_0-\{N-i+1\}\cup\{i\}})\\ =&-x_i(q-q^{-1})((e^*_{N-p}\cdots e^*_{N-i})^* \otimes 1).(v_{I_0} \otimes v^*_{J_0-\{N-i+1\}\cup\{i\}})\\ =&-x_i(q-q^{-1})(v_{I_0-\{N-p\}+\{N-i+1\}} \otimes v^*_{J_0-\{N-i+1\}\cup\{i\}})\end{aligned}$$

and

$$\begin{aligned}&(l^-_{N-i+1,N-p})^*.y_i(v_{I_0-\{i\}\cup\{N-i+1\}} \otimes v^*_{J_0})\\ =&-(q-q^{-1})q^{-\epsilon_{N-p}} e^*_{N-p,N-i+1}.y_i(v_{I_0-\{i\}\cup\{N-i+1\}} \otimes v^*_{J_0})\\ =&-y_i(q-q^{-1})(q^{\epsilon_{N-p}-\epsilon_{N-i+1}} \otimes (e_{N-p}\cdots e_{N-i})^*).(v_{I_0-\{i\}\cup\{N-i+1\}} \otimes v^*_{J_0})\\ =&-y_i(q-q^{-1})(-q)^{p-i+1}(v_{I_0-\{i\}+\{N-i+1\}} \otimes v^*_{J_0-\{N-i+1\}\cup\{N-p\}}).\end{aligned}$$

The second term is computed as follows. We obtain

$$
\begin{aligned}
&q^{-\sigma}(l^{+}_{i,N-p})^{*}.z_i(v_{I_0-\{i\}\cup\{N-i+1\}} \otimes v^{*}_{J_0-\{N-i+1\}\cup\{i\}})\\
=&q^{-\sigma}(q-q^{-1})e^{*}_{N-p,i}q^{\epsilon_i}.z_i(v_{I_0-\{i\}\cup\{N-i+1\}} \otimes v^{*}_{J_0-\{N-i+1\}\cup\{i\}})\\
=&z_i q^{-\sigma}(q-q^{-1})q^{-1}(-q^{-1})^{N-p-i-1}\\
&[((f_i\cdots f_{N-p-1})^{*} \otimes q^{-\epsilon_i+\epsilon_{N-p}}).(v_{I_0-\{i\}\cup\{N-i+1\}} \otimes v^{*}_{J_0-\{N-i+1\}\cup\{i\}})\\
&+(1\otimes (f_i\cdots f_{N-p-1})^{*}).(v_{I_0-\{i\}\cup\{N-i+1\}} \otimes v^{*}_{J_0-\{N-i+1\}\cup\{i\}})]\\
=&z_i q^{-\sigma}(q-q^{-1})(-q^{-1})^{N-p-i-1}(v_{I_0-\{N-p\}\cup\{N-i+1\}} \otimes v^{*}_{J_0-\{N-i+1\}\cup\{i\}})\\
&-z_i q^{-\sigma}(q-q^{-1})(-q^{-1})^{2N-2p-2i}(v_{I_0-\{i\}\cup\{N-i+1\}} \otimes v^{*}_{J_0-\{N-i+1\}\cup\{N-p\}}).
\end{aligned}
$$

Then we obtain the equations

$$-x_i + q^{-\sigma}(-q^{-1})^{N-p-i-1}z_i = 0 \tag{A.15}$$

$$(-q)^{p-i+1}y_i + q^{-\sigma}(-q^{-1})^{2N-2p-2i}z_i = 0 \tag{A.16}$$

and thus we obtain from (A.13)-(A.14)

$$z_i = (-1)^{N-1}q^{3N-2p-4i+2\sigma}.$$

$\square$

Bibliography

[1] T. Arakawa and T. Suzuki, *Duality between $\mathfrak{sl}_n(\mathbf{C})$ and the degenerate affine Hecke algebra*, J. Algebra, 209 (1998), pp. 288–304.

[2] D. Calaque, B. Enriquez, and P. Etingof, *Universal KZB equations: the elliptic case*, in Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, vol. 269 of Progr. Math., Birkhäuser Boston, Boston, MA, 2009, pp. 165–266.

[3] Z. Daugherty, *Degenerate two-boundary centralizer algebras*, Pacific J. Math., 258 (2012), pp. 91–142.

[4] Z. Daugherty and A. Ram, *Two boundary Hecke Algebras and combinatorics of type C*, arXiv e-prints, (2018), p. arXiv:1804.10296.

[5] P. Etingof, R. Freund, and X. Ma, *A Lie-theoretic construction of some representations of the degenerate affine and double affine Hecke algebras of type BC_n*, Represent. Theory, 13 (2009), pp. 33–49.

[6] J. Jantzen, *Lectures on quantum groups*, (2021).

[7] D. Jordan, *Quantum D-modules, elliptic braid groups, and double affine Hecke algebras*, Int. Math. Res. Not. IMRN, (2009), pp. 2081–2105.

[8] D. Jordan and X. Ma, *Quantum symmetric pairs and representations of double affine Hecke algebras of type $C^\vee C_n$*, Selecta Math. (N.S.), 17 (2011), pp. 139–181.

[9] S. Kato, *An exotic Deligne-Langlands correspondence for symplectic groups*, Duke Math. J., 148 (2009), pp. 305–371.

[10] A. Klimyk and K. Schmudgen, *Quantum groups and their representations*, 1997.

[11] R. Leduc and A. Ram, *A ribbon hopf algebra approach to the irreducible representations of centralizer algebras: The brauer, birman–wenzl, and type a iwahori–hecke algebras*, Advances in Mathematics, 125 (1997), pp. 1–94.

[12] X. Ma, *On some representations of degenerate affine hecke algebras of type bc_n*, 2008.

[13] M. Noumi, *Macdonald's symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces*, Advances in Mathematics, 123 (1995), pp. 16–77.

[14] M. Noumi, M. Dijkhuizen, and T. Sugitani, *Multivariable askey-wilson polynomials and quantum complex grassmannians*, Fields Inst. Commun., 14 (1996).

[15] S. Okada, *Applications of minor summation formulas to rectangular-shaped representations of classical groups*, J. Algebra, 205 (1998), pp. 337–367.

[16] A. Ram, *Affine Hecke algebras and generalized standard Young tableaux*, vol. 260, 2003, pp. 367–415. Special issue celebrating the 80th birthday of Robert Steinberg.

[17] A. Ram, *Calibrated representations of affine hecke algebras*, arXiv: Representation Theory, (2004).

[18] M. Reeder, *Nonstandard intertwining operators and the structure of unramified principal series representations*, Forum Math., 9 (1997), pp. 457–516.

[19] T. Suzuki and M. Vazirani, *Tableaux on periodic skew diagrams and irreducible representations of the double affine Hecke algebra of type A*, Int. Math. Res. Not., (2005), pp. 1621–1656.