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by

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Abstract

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This dissertation consists of four largely independent chapters. The first two chapters concern counterparts of classical theorems in modal logic in more general semantics: the Sahlqvist Correspondence Theorem *inter alia* for possibility semantics in Chapter 1 and the Goldblatt-Thomason Theorem and Fine's Canonicity Theorem for neighborhood semantics in Chapter 2. Chapter 3 contains various results on Heyting algebras, among which is the topological-dynamical study of the automorphism group of the smallest existentially closed Heyting algebra. The last chapter establishes choice-free duality between the category of ortholattices and a category of certain spectral spaces.

To My Family

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Introduction

The central objects of study in this dissertation are *Boolean algebra expansions* (BAEs). They are objects of the form $(B, \Box)$ where B is a Boolean algebra and $\Box : B \to B$ is an additional operation. The BAEs studied here are equipped with operations that satisfy additional axioms such as:

$$\Box x \leq \Box y \qquad \text{whenever } x \leq y, \qquad \text{(monotonicity)}$$
$$\bigwedge_{i<n} \Box x_i = \Box \bigwedge_{i<n} x_i \qquad \text{for } n < \omega. \qquad \text{(multiplicativity)}$$

If a BAE satisfies (monotonicity), it is called a *monotonic* Boolean algebra expansion (BAM); if it satisfies both of the axioms, it is a *Boolean algebra with an operator* (BAO).

The author's interest in BAEs comes from their relevance in modal and other nonclassical logics. Not only do BAEs arise as the Lindenbaum-Tarski algebras of modal logic [14, 17], but also they induce other algebraic structures relevant for nonclassical logics. We call BAOs that satisfy the following additional axioms *interior algebras*:

$$\Box x \leq x,$$
$$\Box\Box x = \Box x.$$

Interior algebras are related to *Heyting algebras* [30], which are expansions of bounded lattices important in intuitionistic logic. When regarded as posets, Heyting algebras are exactly the posets of fixed points of $\Box$ in interior algebras. Likewise, BAOs satisfying $x \leq \Box\neg\Box\neg x$ are related to *ortholattices*, which are of interest in an alternative foundation of quantum physics [13]. Ortholattices arise from such a BAO by taking the poset of the fixed points of $\Box\neg\Box\neg$ and expanding it with the operation $\Box\neg\Box$.

Representation theorems are important in algebra at large. Most often, we axiomatically define a class $\mathcal{V}$ of algebras, but paradigmatic examples X^+ in $\mathcal{V}$ are constructed in a specific manner from some mathematical object X in some other class $\mathcal{F}$. For instance, with the class of Boolean algebras, X is a pure set, and X^+ is the powerset of X; with the class of groups, X is again a pure set, and $X^+ = \mathrm{Sym}(X)$. One is then interested in obtaining an arbitrary member $A \in \mathcal{V}$ as a subalgebra of $(\mathrm{Cst}\, A)^+$ where $\mathrm{Cst}\, A$, the *canonical structure* of A, is in $\mathcal{F}$, or encoding which subalgebra of $(\mathrm{Cst}\, A)^+$ it is using an appropriate topology on $\mathrm{Cst}\, A$. In the most basic setting of this kind that is relevant to my research, $\mathcal{V}$ is the class of

Boolean algebras with an operator, and $\mathcal{F}$ is the class of sets with a binary relation on it. The following is a brief summary of classical results in this setting.

Definability One can use an equation E for $\mathcal{V}$ to define a subclass $\mathcal{K}$ of $\mathcal{F}$: $X \in \mathcal{F}$ belongs to $\mathcal{K}$ if and only if X^+ satisfies E. In *correspondence theory* [5], one compares this notion of definability with other means of defining a subclass of $\mathcal{F}$ such as definability in first-order logic. One classical result in this field is the *Sahlqvist Correspondence Theorem*, which gives a syntactic sufficient condition on E under which the corresponding subclass of $\mathcal{F}$ is also definable in first-order logic. Additionally, one may be interested in characterizing this notion of definability in a way that does not mention the syntax explicitly, just as first-order definability of classes can be characterized in terms of closure. The Goldblatt-Thomason theorem [40] does this in terms of closure properties under constructions characteristic of the class $\mathcal{F}$.

Completeness One important aspect of studying algebras motivated by logic is to examine subclasses of $\mathcal{V}$ that can be defined by a set of equations, i.e., *varieties*. An important property of a variety is *completeness*. A variety $\mathcal{V}'$ is complete if the least variety containing $\{X^+ \in \mathcal{V}' \mid X \in \mathcal{F}\}$ is $\mathcal{V}'$ itself. Complete varieties $\mathcal{V}'$ are easier to study in the sense that the study of $\mathcal{V}'$ reduces to its members of the form X^+, which are easier to analyze. One important sufficient condition for a variety to be complete is *canonicity*. A variety $\mathcal{V}'$ is canonical if for every $A \in \mathcal{V}'$ the algebra $(\mathrm{Cst}\, A)^+$ is also in $\mathcal{V}'$. Every canonical variety is complete; in fact, most completeness results have involved canonicity. Yet again, there is a classical result in this field, proved by Fine [31], concerning the relationship between first-order definability and canonicity.

In the first half of the thesis, we prove results similar to the classical results mentioned above in the setting of more general forms of duality theory.

In Chapter 1, we study correspondence theory in the setting of *possibility semantics* [48]. In possibility semantics, $\mathcal{F}$ is the class of certain structures with a partial order and an additional binary relation, which are used to represent complete Boolean algebras with completely multiplicative operators as structures of the form X^+. This is more general than the aforementioned framework, where structures of the form X^+ are necessarily atomic. We prove that the same syntactic condition on equations E as in Sahlqvist's result guarantees that the class $\{X \in \mathcal{F} \mid X^+ \text{ satisfies } E\}$ is also definable in first-order logic. After the publication of this chapter as [81], Zhao [82] generalized this result to a wider class of equations, i.e., that of inductive formulas.

In Chapter 2, we look at the problem of non-syntactic characterization of definability and canonicity in the setting where $\mathcal{V}$ is the class of *monotonic Boolean algebra expansions*, Boolean algebras with additional operations that are merely monotonic. In this setting, $\mathcal{F}$ is a class of certain higher-order structures. In proving a result like Fine's, one has to make sense of "first-order definability" within such a class. Incidentally, Chang [18] proposed a logic for $\mathcal{F}$, who motivated it as a formalism of social situations. This logic shares important properties with first-order logic, and we use it to establish a similar relationship between canonicity in $\mathcal{V}$ and definability of subclasses of $\mathcal{F}$ in that logic. In the proof, we use the duality between types and definable subsets in model theory as a guide. This solved a problem asked in [60]. By using the same method, we prove a non-syntactic characterization of definability of subclasses of $\mathcal{F}$ in the same style as the Goldblatt-Thomason Theorem.

The second half of this dissertation involves the application of the theory of BAOs to Heyting algebras and ortholattices, where ideas used in the development of possibility semantics are still relevant.

Chapter 3 contains various results on Heyting algebras. The chapter begins with the study of existentially closed Heyting algebras. The original motivation for studying this class of algebras comes from the nontrivial fact that the theory of Heyting algebras has a model-completion, or, equivalently, that intuitionistic propositional logic admits uniform interpolation [37]. The latter half of the chapter is a study of Beth semantics, a more general semantic framework than Kripke semantics in which more Heyting algebras can be represented. Beth semantics, as well as possibility semantics, is an example of *nuclear* semantics of intuitionistic logic [10]: one starts with the Heyting algebra of open sets in an Alexandroff space and constructs another by taking the set of the fixed points of a *nucleus*, a closure operator that distributes over meets.

Chapter 4 is on a novel duality theory for ortholattices that does not rely on choice principles. This chapter builds on Goldblatt's work [39] in which he represented an arbitrary ortholattice by an irreflexive symmetric relational structure consisting of all proper filters of the ortholattice, with a topology on it. One will see in this chapter how the use of choice principles can be dispensed by changing the topology on such a structure as suggested by Bezhanishvili and Holliday [12]. Again, one can see ideas prevalent in possibility semantics in this Chapter: considering all filters of a lattice as opposed to just maximal ones is reminiscent of choice-free construction of canonical extensions in possibility semantics [48], and the symmetric relations occurring in Goldblatt's representation theorem can be thought of as arising as the incompatibility relations of the partial orders in possibility semantics [50].

Chapter 1

Modal Correspondence Theory for Possibility Semantics

1.1 Introduction

Possibility semantics [49] (based on [52]) is a generalization of standard Kripke semantics that makes use of a concept of *possibility frames*.[1] Like Kripke frames, possibility frames have a set of states and binary accessibility relations for modalities. In addition, possibility frames have a refinement relation, which is a partial order between states. Some states in a possibility model may only partially determine the atomic propositions, in contrast to worlds in Kripke models, which completely determine each atomic proposition. Consequently, possibility frames have a close connection with intuitionistic modal frames, but the former yield *classical* modal logic. As is the case for intuitionistic modal semantics, a key issue for possibility semantics is the interaction between the refinement and accessibility relations. In this setting, modal axioms express properties not only of the accessibility relation but also of the interaction between accessibility and refinement.

While standard Kripke frames are semantically equivalent to complete, atomic and completely additive Boolean algebras with operators (BAOs), possibility frames are semantically equivalent to complete and completely additive, but not necessarily atomic, BAOs. As shown in [49, Theorem 5.27], for any complete and completely additive BAO, there exists a possibility frame that validates the same modal formulae as the BAO does, and vice versa, just as there exists such a modally equivalent Kripke frame for any complete, atomic and completely additive BAO. It follows from this and other results [59] that more normal modal logics are sound and complete with respect to some class of possibility frames than with respect to some class of Kripke frames. For other recent results on possibility semantics and mention related work, see [7, 11, 46, 45].

In the present chapter, we show how aspects of correspondence theory, as studied for

[1]What we call "possibility frames" in the present chapter are essentially the "*full* possibility frames" of [49].

standard Kripke semantics [4], can be extended to the more general setting of possibility semantics. In Section 1.2, we introduce possibility semantics briefly, referring to [49] for a more detailed account of the semantics. We define key concepts such as possibility frames, possibility models and the standard translation. In Section 1.3, we study syntactic sufficient conditions for local correspondence. In particular, we prove the analogue of Sahlqvist's Theorem for possibility semantics, namely, that every Sahlqvist formula locally corresponds to a first-order formula with respect to possibility frames. This extends a result [49, Proposition 6.23] which states that Lemmon-Scott formulae $\Diamond_{\bar{a}}\Box_{\bar{b}}p \to \Box_{\bar{c}}\Diamond_{\bar{d}}p$ have first-order correspondents over possibility frames. In Section 1.4, we study more model-theoretic aspects of correspondence theory. We prove a counterpart of van Benthem's characterization [5] of first-order definable modal formulae in terms of preservation by ultrapowers. Finally, in Section 1.5 we state an open problem for future research and related work.

1.2 Possibility semantics

Introduction to the semantics

Fix an enumeration $\Phi = \{p_i \mid i \in \kappa\}$ $(\kappa = |\Phi|)$ of propositional variables (whose indices we sometimes identify with the variables themselves) and a nonempty set I of modal operator indices. Then the modal language $\mathcal{L}(\Phi, I)$ is generated by the following grammar:

$$\phi ::= p \mid \phi \wedge \phi \mid \neg\phi \mid \phi \to \phi \mid \Box_a\phi,$$

where $\phi \in \mathcal{L}(\Phi, I)$, $p \in \Phi$ and $a \in I$. We assume that $\phi_1 \vee \phi_2$ and $\Diamond_a\phi$ are shorthand for $\neg(\neg\phi_1 \wedge \neg\phi_2)$ and $\neg\Box_a\neg\phi$, respectively.

We view a partially ordered set $\mathbb{P}$ as a topological space whose open sets are the downward closed sets. This is an Alexandrov topology. We denote by $\overline{X}$ and X° the closure and the interior of a set $X \subseteq \mathbb{P}$, so $\overline{X} = \{x \in \mathbb{P} \mid \exists x' \sqsubseteq x\, x' \in X\}$ and $X^\circ = \{x \in \mathbb{P} \mid \forall x' \sqsubseteq x\, x' \in X\}$, where $\sqsubseteq$ is the partial order of $\mathbb{P}$. We write $\mathrm{RO}(\mathbb{P})$ for the set of regular open subsets of $\mathbb{P}$, i.e., those subsets $X \subseteq \mathbb{P}$ such that $\overline{X}^\circ = X$. For $X \subseteq \mathbb{P}$, the least regular open set containing X is $\overline{(\Downarrow X)}^\circ$, where $\Downarrow X$ denotes the least downward closed set containing X. We write X^{ro} for $\overline{(\Downarrow X)}^\circ$. Note that $(\cdot)^{\mathrm{ro}}$ satisfies the axioms of closure operators: for any $X, Y \subseteq \mathbb{P}$, we have $X \subseteq X^{\mathrm{ro}}$, $X^{\mathrm{ro}} = (X^{\mathrm{ro}})^{\mathrm{ro}}$ and $X \subseteq Y \Rightarrow X^{\mathrm{ro}} \subseteq Y^{\mathrm{ro}}$. For $x, y \in \mathbb{P}$, we also write $x \between y$ to indicate that x and y are *compatible*, i.e., $\exists z\,(z \sqsubseteq x \wedge z \sqsubseteq y)$. We write $x \perp y$ to indicate that it is not the case that $x \between y$.

We give a definition of possibility frames in the following. Note that, in [49, Definition 2.21], the term "possibility frame" is used for a kind of general frame version of the structures defined in Definition 1.2.1.1 below, which are essentially the "*full* possibility frames" of [49, Definition 2.21]. The structures in Definition 1.2.1.1 are the possibility-semantic analogues of Kripke frames.

Definition 1.2.1.

1. A *possibility frame* is a triple $\mathfrak{F} = (F, \sqsubseteq, (R_a)_{a\in I})$ where $(F, \sqsubseteq)$ is a partially ordered set, each R_a is a binary relation on F, and the set $\mathrm{RO}(\mathfrak{F}) := \mathrm{RO}(F, \sqsubseteq)$ is closed under the map
$$l_a\colon X(\subseteq F) \mapsto \{y \in F \mid R_a[y] \subseteq X\}$$
for each $a \in I$. We refer to elements of F as *states* of the frame. We call $\sqsubseteq$ and each R_a the *refinement relation* and an *accessibility relation* of $\mathfrak{F}$, respectively.

2. A *possibility model* is a pair $\mathfrak{M} = (\mathfrak{F}, \pi)$ where π is a map $\Phi \to \mathrm{RO}(\mathfrak{F})$, called a *valuation* on the frame $\mathfrak{F}$.

When considering a possibility frame $\mathfrak{F}$, we regard l_a as a map $\mathrm{RO}(\mathfrak{F}) \to \mathrm{RO}(\mathfrak{F})$.

Definition 1.2.2. Let $\mathfrak{M} = (\mathfrak{F}, \pi)$ be a possibility model and $\phi \in \mathcal{L}(\Phi, I)$.

1. For $w \in \mathfrak{M}$, define the relation $\mathfrak{M}, w \Vdash \phi$ recursively as follows:[2]
$$\begin{gathered}\mathfrak{M}, w \Vdash p \Leftrightarrow w \in \pi(p) \quad (p \in \Phi);\\ \mathfrak{M}, w \Vdash \phi_1 \wedge \phi_2 \Leftrightarrow \mathfrak{M}, w \Vdash \phi_1 \text{ and } \mathfrak{M}, w \Vdash \phi_2;\\ \mathfrak{M}, w \Vdash \neg\phi \Leftrightarrow \forall v \sqsubseteq w\, (\mathfrak{M}, v \nVdash \phi);\\ \mathfrak{M}, w \Vdash \phi_1 \to \phi_2 \Leftrightarrow \forall v \sqsubseteq w\, (\mathfrak{M}, v \Vdash \phi_1 \Rightarrow \mathfrak{M}, v \Vdash \phi_2);\\ \mathfrak{M}, w \Vdash \Box\phi \Leftrightarrow \forall v\, (Rwv \Rightarrow \mathfrak{M}, v \Vdash \phi).\end{gathered}$$

2. Let $[\![\phi]\!]^{\mathfrak{M}} = \{w \in \mathfrak{M} \mid \mathfrak{M}, w \Vdash \phi\}$. Call this the *truth set* of ϕ in $\mathfrak{M}$.

3. For $w \in \mathfrak{F}$, we write $\mathfrak{F}, w \Vdash \phi$ and say that v *forces* ϕ in $\mathfrak{F}$ if and only if for every possibility model $(\mathfrak{F}, \pi)$, we have $(\mathfrak{F}, \pi), w \Vdash \phi$. $\mathfrak{F}$ *validates* ϕ if and only if for every $v \in \mathfrak{F}$, the formula ϕ is forced by v in $\mathfrak{F}$.

Note that since we define $\vee$ in terms of $\wedge$ and $\neg$, we have the following:
$$\mathfrak{M}, w \Vdash \phi_1 \vee \phi_2 \Leftrightarrow \forall w' \sqsubseteq w\, \exists w'' \sqsubseteq w' (\mathfrak{M}, w'' \Vdash \phi_1 \text{ or } \mathfrak{M}, w'' \Vdash \phi_2).$$

Remark. Let (W, R) be an arbitrary Kripke frame. Let $\mathbb{P} = (W, \sqsubseteq)$ be the discrete partial order on W, i.e., $x \sqsubseteq y \Leftrightarrow x = y$ for $x, y \in W$. Then $\mathfrak{F} = (\mathbb{P}, R)$ is a possibility frame. Let $\mathfrak{M}_0$ be a Kripke model that is an expansion of (W, R), and let $\mathfrak{M}$ be the possibility model that is an expansion of $\mathfrak{F}$ with the same valuation as $\mathcal{M}$. Let $\mathfrak{M}_0$ be a Kripke model that is an expansion of (W, R), and let $\mathfrak{M}$ be a possibility model that is an expansion of $\mathfrak{F}$ with the same valuation as $\mathfrak{M}_0$. Then we have
$$\mathfrak{M}_0 \Vdash_{\text{Kripke}} \phi \Leftrightarrow \mathfrak{M} \Vdash \phi$$
for any modal formula ϕ, where $\Vdash_{\text{Kripke}}$ is the forcing relation of Kripke semantics.

[2] Note that the clauses for $\neg$ and $\to$ scan the partial order *downward*. This is in line with a convention used in weak forcing (see, e.g., [65]), to which the present semantics is related. In contrast, in the literature on semantics for intuitionistic logic, the convention of going upward is more common.

In the present chapter, we are interested in the relationship between the validity of a modal formula over a possibility frame and the first-order properties of the accessibility and refinement relations in the frame. To see how familiar correspondences from Kripke semantics must be reconsidered in the setting of possibility semantics, it helps to consider a concrete example, such as the following.

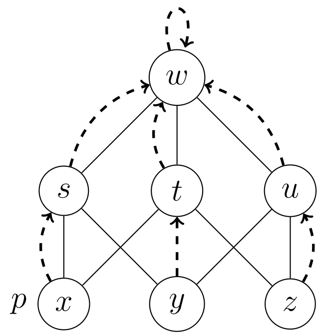


Figure 1.1: A possibility frame $\mathfrak{F}$ and a valuation π on it. The refinement relation of $\mathfrak{F}$ is shown by solid lines as in Hasse diagrams and the accessibility relation is shown by dashed arrows. The valuation π is such that $\pi(p) = \{x\}$.

Example 1.2.3. Consider the possibility frame $\mathfrak{F} = (F, \sqsubseteq, R)$ of Figure 1.1. It can be checked that $\mathfrak{F}$ satisfies the axioms for a possibility frame.[3] Note that for each state w in $\mathfrak{F}$ there exists exactly one v such that Rwv. This property of partial functionality is defined by the F axiom $\Diamond p \to \Box p$ over standard Kripke frames. However, it can be seen that for the state y we have $\mathfrak{F}, y \nVdash \Diamond p \to \Box p$. To see this, observe that the forcing clause for the defined operator $\Diamond$ works out to (see Figure 1.2):

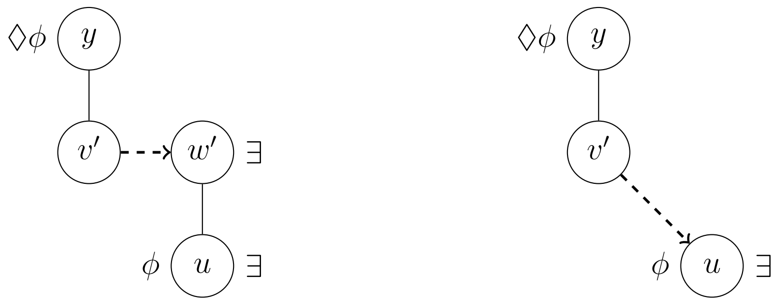
$$(\mathfrak{F}, \pi), y \Vdash \Diamond \phi \Leftrightarrow \forall v' \sqsubseteq y \, \exists w' \, (Rv'w' \wedge \exists u \sqsubseteq w' \, (\mathfrak{F}, \pi), u \Vdash \phi). \tag{1.1}$$

Consider the valuation π also shown in Figure 1.1. (It is easy to check that this is indeed a valuation on $\mathfrak{F}$, i.e., $\pi(p)^{\mathrm{ro}} = \pi(p)$.) Then we know $(\mathfrak{F}, \pi), y \Vdash \Diamond p$: in (1.1), the only possible value of v' is y itself, and one can pick w' to be t so that the right hand side holds. However, we also have $(\mathfrak{F}, \pi), y \nVdash \Box p$, since $t \notin \pi(p)$.

Example 1.2.4. Another example is the B axiom $p \to \Box \Diamond p$. This defines the symmetry of the accessibility relation over standard Kripke frames. The accessibility relation R of $\mathfrak{F}$ from Figure 1.1 is not symmetric. However, the B axiom is validated by $\mathfrak{F}$; indeed, as we will see later, $p \to \Box \Diamond p$ is validated by $\mathfrak{F}$ if and only if for every $u, v, v' \in \mathfrak{F}$:

$$(Rwv \wedge v' \sqsubseteq v) \Rightarrow \exists w' \, (Rv'w' \wedge w' \between w). \tag{1.2}$$

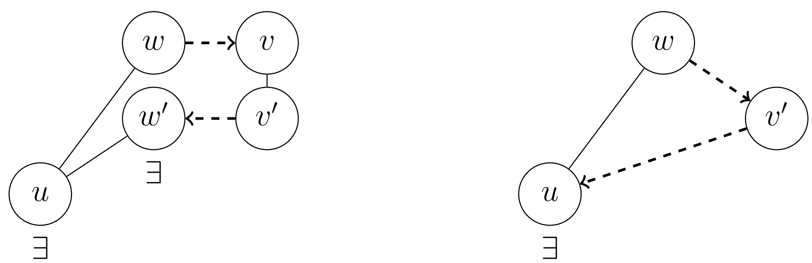
[3] The possibility frame $\mathfrak{F}$ is constructed from a Kripke frame $(\{0, 1, 2\}, R)$, where R is the symmetric closure of $\{(1, 0), (1, 2), (0, 0), (2, 2)\}$, by *functional powerset possibilization* as in [49, p. 53]. The observations made here follow from the construction.



(1.1) (1.4)

Figure 1.2: Forcing conditions for $\Diamond$. The same conventions as in Figure 1.1 apply.

(See Figure 1.3.) All the states in $\mathfrak{F}$ are compatible with one another except that x, y, z are pairwise incompatible. These states are not in the range of R, so (1.2) holds.



(1.2) (1.3)

Figure 1.3: Conditions equivalent to the validity of the B axiom. The same conventions as in Figure 1.1 apply.

To see why (1.2) is equivalent to the validity of B, suppose that (1.2) holds in $\mathfrak{F}$ and that $(\mathfrak{F}, \pi), w \Vdash p$, i.e., $w \in \pi(p)$. We show $(\mathfrak{F}, \pi), w \Vdash \Box\Diamond p$. It suffices to show $(\mathfrak{F}, \pi), v \Vdash \Diamond p$, for an arbitrary v such that Rwv. With (1.1) in mind, take an arbitrary $v' \sqsubseteq v$. By (1.2), there exist w' and u such that $Rv'w'$, $u \sqsubseteq w'$ and $u \sqsubseteq w$. Since $\pi(p)$ is open, i.e., downward closed, $u \in \pi(p)$. Then by (1.1), we have $(\mathfrak{F}, \pi), v \Vdash \Diamond p$. Conversely, suppose that (1.2) does not hold. For $w \in \mathfrak{F}$, let π be a valuation such that $\pi(p) = \{w\}^{\mathrm{ro}}$. Then $(\mathfrak{F}, \pi), w \Vdash p$. However, we see $(\mathfrak{F}, \pi), w \nVdash \Box\Diamond p$. Indeed, by the failure of (1.2), there exists v such that $(\mathfrak{F}, \pi), v \nVdash \Diamond p$. This is because if $w' \perp w$ then for all $u \sqsubseteq w'$ we have $u \perp w'$ and thus $u \notin \pi(p) = \{w\}^{\mathrm{ro}}$; for $u \in \{w\}^{\mathrm{ro}}$ if and only if $\forall u' \sqsubseteq u\, u' \between w$.

It is often the case that conditions on a possibility frame that are equivalent to validity of modal formulae can be simplified by imposing additional conditions on the interaction of the accessibility and the refinement relation in possibility frames. For instance, if we assume

$$(Rwv \wedge v' \sqsubseteq v) \Rightarrow Rwv',^{4} \tag{R-down}$$

[4] This condition is often assumed for frames for intuitionistic modal logic (see, e.g., [79]) with the refinement relation flipped. See also Footnote 2.

it is easily seen that (1.2) is equivalent to

$$Rwv' \Rightarrow \exists u\,(Rv'u \wedge u \sqsubseteq w), \tag{1.3}$$

which is much closer to the symmetry of R, the property that the B axiom defines over standard Kripe frames (see again Figure 1.3). In fact, many familiar modal axioms without $\Diamond$ define the same property over possibility frames satisfying (R-down) as over Kripke frames; for instance, the 4 axiom $\Box p \to \Box\Box p$ is validated by a possibility frame $(F, \sqsubseteq, R)$ satisfying (R-down) if and only if R is transitive. Moreover, (1.1) can be simplified if $\mathfrak{F}$ satisfies (R-down):

$$(\mathfrak{F}, \pi), y \Vdash \Diamond\phi \Leftrightarrow \forall v' \sqsubseteq y\, \exists u\,(Rv'u \wedge (\mathfrak{F}, \pi), u \Vdash \phi). \tag{1.4}$$

(See Figure 1.2.) We refer to [49, Section 2.3] for further discussion of (R-down) and other similar conditions.

A few points should be made about these conditions. First, in Definition 1.2.1.1 we stated a condition for a structure $(F, \sqsubseteq, (R_a)_{a\in I})$ to be a possibility frame in terms of $\mathrm{RO}(F, \sqsubseteq)$ and l_a; we will see in Section 1.2 that this condition, like (R-down), can be stated in a first-order manner. Second, as shown in [49, Section 2.3], we can assume (R-down) and other conditions on the interaction of R and $\sqsubseteq$ without loss of generality. That is, given a possibility frame $\mathfrak{F}$, we can construct a modally-equivalent possibility frame $\mathfrak{F}'$ that satisfies (R-down) and other interaction conditions (see also Example 1.3.13). Third, the main results of the present chapter hold without imposing these conditions; unless otherwise stated, we do not assume (R-down) and other interaction conditions on possibility frames, beyond those that follow from the definition of possibility frames (again see Section 1.2).

To develop correspondence theory for possibility semantics, we will take an algebraic perspective on possibility frames. An important consequence of the definitions above is that truth sets in an arbitrary possibility model $\mathfrak{M} := (\mathfrak{F}, \pi)$ are always in $\mathrm{RO}(\mathfrak{F})$. As is the case for $\mathrm{RO}(\mathbb{P})$ where $\mathbb{P}$ is an arbitrary partial order, $\mathrm{RO}(\mathfrak{F})$ is a complete Boolean algebra with respect to set inclusion, where the meet is the intersection, the complement is the interior of the set-theoretic complement, and the join is the interior of the closure of the union. One can show that $[\![\phi_1 \wedge \phi_2]\!]^{\mathfrak{M}} = [\![\phi_1]\!]^{\mathfrak{M}} \wedge [\![\phi_2]\!]^{\mathfrak{M}}$, $[\![\neg\phi]\!]^{\mathfrak{M}} = -[\![\phi]\!]^{\mathfrak{M}}$ and $[\![\phi_1 \to \phi_2]\!]^{\mathfrak{M}} = (-[\![\phi_1]\!]^{\mathfrak{M}}) \vee [\![\phi_2]\!]^{\mathfrak{M}}$, where $\wedge$, $-$ and $\vee$ on the right hand sides denote the meet, the complement and the join in $\mathrm{RO}(\mathfrak{F})$, respectively. We trust that no confusion will arise in using the same symbols for the logical connectives and the algebraic operations.

Definition 1.2.5.

1. A map $f\colon \mathrm{RO}(\mathfrak{F}) \to \mathrm{RO}(\mathfrak{F})$ is *completely additive* if it preserves arbitrary joins, i.e., for every family $S \subseteq \mathrm{RO}(\mathfrak{F})$ we have $f(\bigvee S) = \bigvee\{f(X) \mid X \in S\}$. We also say that a map $f\colon \mathrm{RO}(\mathfrak{F})^n \to \mathrm{RO}(\mathfrak{F})$ is *completely additive in i-th coordinate* for $i \in \{1, \ldots, n\}$ if for every $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n \in \mathrm{RO}(\mathfrak{F})$ the map

$$\begin{aligned} \mathrm{RO}(\mathfrak{F}) &\to \mathrm{RO}(\mathfrak{F}) \\ (X_1, \ldots, X_{i-1}, X, X_{i+1}, \ldots, X_n) &\mapsto f(X_1, \ldots, X_{i-1}, X, X_{i+1}, \ldots, X_n) \end{aligned}$$

is completely additive. $f\colon \mathrm{RO}(\mathfrak{F})^n \to \mathrm{RO}(\mathfrak{F})$ is *completely additive* if it is completely additive in i-th coordinate for every $i \in \{1, \ldots, n\}$. *Completely multiplicative* maps are defined similarly, but with joins replaced by meets.

2. We say that f is a *left adjoint* of g and that g is a *right adjoint* of f, if $f, g\colon \mathrm{RO}(\mathfrak{F}) \to \mathrm{RO}(\mathfrak{F})$ satisfy, for $X, Y \in \mathrm{RO}(\mathfrak{F})$,

$$f(X) \subseteq Y \Leftrightarrow X \subseteq g(Y).$$

Note that completely additive maps are order-preserving, and that if f and g both have left adjoints, so does the composite $f \circ g$.

The complete Boolean algebra $\mathrm{RO}(\mathfrak{F})$ becomes a BAO when equipped with the operators l_a for $a \in I$, which are completely multiplicative operators (see [49, Section 2] for more on the duality theory relating possibility frames and BAOs). It is easy to see that, in general, a completely multiplicative map g over a complete lattice $(L, \leq)$ has a left adjoint f of the form $X \mapsto \min\{Z \in L \mid Y \leq g(Z)\}$. In our setting, this implies that each l_a has a left adjoint of the form $Y \mapsto \min\{Z \in \mathrm{RO}(\mathfrak{F}) \mid Y \subseteq l_a(Z)\} = (R_a[Y])^{\mathrm{ro}}$.

Translation to classical logic

Let the signature $\tau = \{\sqsubseteq\} \cup \{R_a \mid a \in I\}$, where $\sqsubseteq$ is a first-order binary relation symbol and each R_a is a first-order binary relation symbol. We write $\mathcal{L}^1(\tau)$ for the first-order τ-language and $\mathcal{L}^2(\tau)$ for the monadic second-order counterpart. $\mathcal{L}^1(\tau)$ will be our first-order correspondence language. We use $x, y, z, \xi, \eta, \zeta$, etc. for first-order variables and P, Q, etc. for second-order monadic ones. In particular, let $\{P_i\}$ be a set of distinct monadic second-order variables, each P_i corresponding to the propositional variable p_i. Let $\bar{\tau}$ be the signature $\tau \cup \{P_i \mid i \in \kappa\}$.

We regard a possibility frame $\mathfrak{F} = (F, \sqsubseteq, (R_a)_{a \in I})$ as a structure for $\mathcal{L}^1(\tau)$, by letting $\mathrm{dom}\,\mathfrak{F} = F$, $\sqsubseteq^{\mathfrak{F}} = \sqsubseteq$ and $R_a^{\mathfrak{F}} = R_a$ for each $a \in I$. Likewise, we regard a possibility model $\mathfrak{M} = (\mathfrak{F}, \pi)$ as a structure $(\mathfrak{F}, (\pi(p))_{p \in \Phi})$ for $\mathcal{L}^1(\bar{\tau})$, as an expansion of $\mathfrak{F}$ with $P_i^{\mathfrak{M}} = \pi(p_i)$. In general, for a structure $\mathfrak{N}$, we use $\models$ for the satisfaction relation for first-order languages, and for parameters $a_1, \ldots, a_m \in \mathfrak{N}$ and a first-order formula $\beta(x; y_1, \ldots, y_m)$, we write $\beta(\mathfrak{N}; a_1, \ldots, a_m)$ for the set $\{b \in \mathfrak{N} \mid \mathfrak{N} \models \beta(b; a_1, \ldots, a_m)\}$.

We can view a possibility frame $\mathfrak{F}$ as a structure for $\mathcal{L}^2(\tau)$ in two different ways. In one view, which is employed in the rest of this section and Section 1.3, we consider a possibility frame $\mathfrak{F}$ as a *general prestructure* for $\mathcal{L}^2(\tau)$, with its one-place relational universe being $\mathrm{RO}(\mathfrak{F})$.[5] In the other view, which appears in Section 1.4, we consider a possibility frame $\mathfrak{F}$ as an (ordinary) structure for $\mathcal{L}^2(\tau)$, with no limitation on values that bound second-order monadic variables can assume. In each case, we again write $\models$ for the corresponding appropriate satisfaction relation for $\mathcal{L}^2(\tau)$.

[5] Our treatment of second-order logic follows [29].

Having defined classical languages and satisfaction relations, we can see, as in [49, Section 2.2], that the various conditions imposed on possibility frames are actually first-order. First, we can show that there exists a formula $\beta^Q_{\mathrm{ro}}(x) \in \mathcal{L}^1(\tau \cup \{Q\})$, where Q is a unary relation symbol, such that for every $X \subseteq \mathfrak{F}$, we have $\beta^Q_{\mathrm{ro}}((\mathfrak{F}, X)) = X^{\mathrm{ro}}$, where $(\mathfrak{F}, X)$ is an expansion of $\mathfrak{F}$ that interprets Q as X. Concretely, $\beta^Q_{\mathrm{ro}}(x)$ is the formula

$$\forall y \sqsubseteq x \, \exists z \sqsubseteq y \, \exists z' \sqsupseteq z \, Qz'$$

where $\sqsupseteq$ is the inverse of $\sqsubseteq$. With this in mind, it can further be shown [49, Proposition 2.30] that a structure $\mathfrak{F}$ for $\mathcal{L}^1(\tau)$ is a possibility frame if and only if it satisfies (in addition to the axiom of partial orders) the following pair of sentences in $\mathcal{L}^1(\tau)$ for each $a \in I$:

$$\begin{aligned} \beta^a_{R\text{-rule}} &:\equiv U((x' \sqsubseteq x \wedge R_a x'y' \wedge y' \between z) \to \exists y \, (R_a xy \wedge y \between z)); \\ \beta^a_{R \Rightarrow \mathrm{win}} &:\equiv U(R_a xy \to \forall y' \sqsubseteq y \, \exists x' \sqsubseteq x \, \forall x'' \sqsubseteq x' \, \exists y'' \between y \, R_a x''y''), \end{aligned}$$

where $U(\cdot)$ denotes the universal closure. Understanding the details of these conditions will not be necessary for the purposes of this chapter; what will be important for us in this chapter is just that the class of possibility frames is first-order definable. We refer to [49, Section 2.3] and [7] for further discussion of these conditions, as well as simpler versions that can be assumed without loss of generality.

We now give the analogue for possibility semantics of the standard translation of modal formulae into first-order formulae.

Definition 1.2.6. For $\phi \in \mathcal{L}(\Phi, I)$ and a variable x, we define $\mathrm{ST}_x(\phi) \in \mathcal{L}^2(\tau)$ inductively as follows:

$$\begin{aligned} \mathrm{ST}_x(p_i) &= P_i x, \\ \mathrm{ST}_x(\neg\phi) &= \forall y \sqsubseteq x \, \neg \, \mathrm{ST}_y(\phi), \\ \mathrm{ST}_x(\phi_1 \wedge \phi_2) &= \mathrm{ST}_x(\phi_1) \wedge \mathrm{ST}_x(\phi_2), \\ \mathrm{ST}_x(\phi_1 \to \phi_2) &= \forall y \sqsubseteq x \, (\mathrm{ST}_y(\phi_1) \to \mathrm{ST}_y(\phi_2)), \\ \mathrm{ST}_x(\Box_a \phi) &= \forall y \, (R_a xy \to \mathrm{ST}_y(\phi)). \end{aligned}$$

Recall that we are viewing a possibility frame as a general prestructure as explained above. The following definition is standard [4], and the lemmas following it can be proved in the usual way.

Definition 1.2.7. For $\phi \in \mathcal{L}(\Phi, I)$ and $\alpha(x) \in \mathcal{L}^1(\tau)$, we say that ϕ *locally corresponds* to $\alpha(x)$, or that $\alpha(x)$ is a *local correspondent* of ϕ, if for every possibility frame $\mathfrak{F}$ and $w \in \mathfrak{F}$, we have

$$\mathfrak{F}, w \Vdash \phi \Leftrightarrow \mathfrak{F} \models \alpha(w).$$

For a first-order sentence $\tilde{\alpha} \in \mathcal{L}^1(\tau)$, we say that ϕ *globally corresponds* to $\tilde{\alpha}$, or that $\tilde{\alpha}$ is a *global correspondent* of ϕ, if for every possibility frame $\mathfrak{F}$ we have

$$\mathfrak{F} \Vdash \phi \Leftrightarrow \mathfrak{F} \models \tilde{\alpha}.$$

Lemma 1.2.8. Given a possibility frame $\mathfrak{F}$, $w \in \mathfrak{F}$ and $\phi \in \mathcal{L}(\Phi, I)$, we have

$$\mathfrak{F}, w \Vdash \phi \Leftrightarrow \mathfrak{F} \models U^2(\mathrm{ST}_w(\phi)),^{6}$$

where $U^2(\phi)$ denotes the universal quantification by the monadic second-order variables P_i occurring in ϕ. (Recall that in this section the domain of monadic second-order quanitification is $\mathrm{RO}(\mathfrak{F})$ for a possibility frame $\mathfrak{F}$.)

Lemma 1.2.9. For $\phi \in \mathcal{L}(\Phi, I)$ and $\alpha(x) \in \mathcal{L}^1(\tau)$, the following are equivalent:

1. ϕ locally corresponds to $\alpha(x)$.
2. For arbitrary possibility frame $\mathfrak{F}$ and $w \in \mathfrak{F}$, we have

$$\mathfrak{F} \models U^2(\mathrm{ST}_w(\phi)) \Leftrightarrow \mathfrak{F} \models \alpha(w).$$

1.3 Sahlqvist theory

In this section, we prove the possibility-semantic version of Sahlqvist's Theorem.

For $\mathcal{L}(\Phi, I)$, *positive* and *negative* occurrences of propositional variables, and *positive* and *negative* formulae are defined recursively as follows. For $p \in \Phi$, the occurrence of p in $p \in \mathcal{L}(\Phi, I)$ is positive. Suppose an occurrence of p in $\phi \in \mathcal{L}(\Phi, I)$ is positive (respectively, negative) and $\psi \in \mathcal{L}(\Phi, I)$. Then the corresponding occurrences of p in $\phi \wedge \psi$, $\psi \wedge \phi$, $\psi \to \phi$ and $\Box_a\phi$ are positive (respectively, negative); and the corresponding occurrences of p in $\neg\phi$ and $\phi \to \psi$ are negative (respectively, positive). A modal formula is positive (respectively, negative) if all occurrences of all propositional variables in it are positive (respectively, negative).

We define *Sahlqvist antecedents*, *Sahlqvist implications* and *Sahlqvist formulae* in the standard way (see, e.g., [14]). More concretely, they are specified by the following grammar:

$$\begin{array}{lr}
B ::= p_i \mid \Box_a B & \text{(boxed atoms)} \\
A ::= B \mid \langle\text{negative formula}\rangle \mid \Diamond A \mid A \wedge A \mid A \vee A & \text{(Sahlqvist antecedents)} \\
I ::= A \to \langle\text{positive formula}\rangle & \text{(Sahlqvist implications)} \\
F ::= I \mid F \wedge F \mid F \vee F \mid \Box_a F & \text{(Sahlqvist formulae)}
\end{array}$$

where $i \in \Phi$, $a \in I$, and in the last clause the disjuncts do not have shared variables.

The following is the main theorem of the present section:

Theorem 1.3.1. Every Sahlqvist formula locally corresponds to a first-order formula in the setting of possibility semantics. Moreover, one can effectively calculate the first-order correspondent from a Sahlqvist formula.

[6] By $\mathfrak{F} \models U^2(\mathrm{ST}_w(\phi))$, we mean $U^2(\mathrm{ST}_x(\phi))$ is satisfied by $\mathfrak{F}$ and a variable assignment sending x to w.

The rest of the present section is devoted to developing a theory necessary to prove the theorem. The argument will be based on *algebraic correspondence theory* [24], although there will be slight changes in terminology and convention.

The key observation is as follows. Call a class function $\mathcal{V}$ a *definably enumerable class* if the domain of $\mathcal{V}$ is the class of possibility frames and there exists a formula $\beta(x; z_1, \ldots, z_k) \in \mathcal{L}^1(\tau)$ such that for every $\mathfrak{F}$ we have

$$\mathcal{V}(\mathfrak{F}) = \{\beta(\mathfrak{F}; w_1, \ldots, w_k) \mid w_1, \ldots, w_k \in \mathfrak{F}\} \cup \{\emptyset\}.$$

Lemma 1.3.2. Let $\phi(p_0, \ldots, p_{n-1}) \in \mathcal{L}(\Phi, I)$ and $\mathcal{V}_0, \ldots, \mathcal{V}_{n-1}$ be definably enumerable classes.[7] Assume for every possibility frame $\mathfrak{F}$ and $w \in \mathfrak{F}$, the following are equivalent:

$$\mathfrak{F} \models U^2(\mathrm{ST}_w(\phi)); \tag{1.5}$$

$$\forall P_0 \in \mathcal{V}_0(\mathfrak{F}) \cdots \forall P_{n-1} \in \mathcal{V}_{n-1}(\mathfrak{F})\,(\mathfrak{F}, P_0, \ldots, P_{n-1}) \models \mathrm{ST}_w(\phi). \tag{1.6}$$

Then, ϕ locally corresponds to a first-order formula.

Proof. Let $\beta_i(x; z_1^i, \ldots, z_{k_i}^i)$ witness $\mathcal{V}_i$ being definably enumerable. Let $\alpha(x)$ be the first-order formula obtained by replacing, in $U^2(\mathrm{ST}_x(\phi))$, each quantifier $\forall P_i$ by $\forall z_1^i \cdots \forall z_{k_i}^i$ and each occurrence of $P_i x$ by $\beta_i(x; z_1^i, \ldots, z_{k_i}^i)$, for each $i \in n$, where z_j^i are fresh variables. Moreover, let $\alpha_\emptyset(x)$ be the formula obtained by replacing, in $\mathrm{ST}_x(\phi)$, each occurrence of $P_i x$ with $x \neq x$. It can easily be seen that ϕ indeed locally corresponds to $\alpha(x) \wedge \alpha_\emptyset(x)$. $\square$

Remark. A statement similar to Lemma 1.3.2 is true of Kripke semantics, as proved by van Benthem [6, p. 9.15]. Modal formulae that satisfy the hypothesis of the Kripke-semantic version of the lemma belong to a class of formulae that van Benthem called M_1^{sub}, which is now commonly called the class of *van Benthem formulae* [22, Definition 30]. An anonymous reviewer remarked that this class, which includes the class of Sahlqvist formulae, the class of inductive formulae [42] and many more, is beyond the reach of current algorithmic correspondence techniques.

In what follows, by $\mathfrak{F}$ we mean a possibility frame.

Definition 1.3.3. A modal formula is *normative* if, for each $p \in \Phi$, the number of positive occurrences of p in it is at most one.[8]

In the following, we assume, without loss of generality, that negative propositional variables in a normative Sahlqvist antecedent are all towards the end of the enumeration $p_0, p_1, \ldots$ of the propositional variables occurring in the formula.

We will later associate with a normative Sahlqvist antecedent a certain kind of map, a *Sahlqvist map*, between partial orders. Below, n will be the number of propositional variables in a normative antecedent, and m will be the number of those that occur positively.

[7] By the notation like $\phi(p_0, \ldots, p_{n-1})$, we understand hereafter that all propositional variables occurring in the formula are present in the parentheses.

[8] In [24], a related but slightly different concept of 1-implications is used.

Definition 1.3.4. Let $n, m, l \in \omega$ ($m \leq n$) and $\bar{a}_1, \dots, \bar{a}_m \in I^{<\omega}$. A *Sahlqvist map of type* $(n, m, l; \bar{a}_1, \dots, \bar{a}_m)$ is a map of the form $f \circ \langle (g_1 \times \cdots \times g_m) \circ \pi_m, h_1, \dots, h_l \rangle \colon \mathrm{RO}(\mathfrak{F})^n \to \mathrm{RO}(\mathfrak{F})$ where

1. $f \colon \mathrm{RO}(\mathfrak{F})^{m+l} \to \mathrm{RO}(\mathfrak{F})$ is completely additive;
2. $\pi_m \colon \mathrm{RO}(\mathfrak{F})^n \to \mathrm{RO}(\mathfrak{F})^m$ is the projection onto the first m coordinates, i.e., $\pi_m(X_0, \dots, X_{n-1}) = (X_1, \dots, X_{m-1})$;
3. each $g_i \colon \mathrm{RO}(\mathfrak{F}) \to \mathrm{RO}(\mathfrak{F})$ has a left adjoint of the form
$$Y \mapsto R^{\mathrm{ro}}_{\bar{a}_i}[Y] := (R_{\bar{a}_i(0)}[(R_{\bar{a}_i(1)}[\cdots (R_{\bar{a}_i(|a_i|-1)}[Y])^{\mathrm{ro}} \cdots])^{\mathrm{ro}}])^{\mathrm{ro}};$$
4. each $h_i : \mathrm{RO}(\mathfrak{F})^n \to \mathrm{RO}(\mathfrak{F})$ is order-reversing.

Note that for a formula $\phi(p_0, \dots, p_{n-1}) \in \mathcal{L}(\Phi, I)$ and possibility models $(\mathfrak{F}, \pi)$ and $(\mathfrak{F}, \pi')$, we have $[\![\phi]\!]^{(\mathfrak{F},\pi)} = [\![\phi]\!]^{(\mathfrak{F},\pi')}$ if $\pi \restriction n = \pi' \restriction n$, where we identify propositional variables with their indices. Write $[\![\phi]\!]^{\mathfrak{F}}$ for the map $\mathrm{RO}(\mathfrak{F})^n \to \mathrm{RO}(\mathfrak{F})$ that maps $\tilde{\pi} \in \mathrm{RO}(\mathfrak{F})^n$ to the unique value of $[\![\phi]\!]^{(\mathfrak{F},\pi)}$ where $\pi \colon \Phi \to \mathrm{RO}(\mathfrak{F})$ extends $\tilde{\pi}$.

Lemma 1.3.5. Let $\phi \in \mathcal{L}(\Phi, I)$ be a positive (respectively, negative) formula. Then $[\![\phi]\!]^{\mathfrak{F}}$ is order-preserving (respectively, order-reversing).

Proof. By simultaneous induction. $\square$

For a sequence of modal indices $\bar{a} \in I^{<\omega}$ and a modal formula ϕ, we define the expression $\Box_{\bar{a}}\phi$ recursively as $\Box_{\langle\rangle}\phi = \phi$ and $\Box_{\bar{a}b}\phi = \Box_{\bar{a}}\Box_b\phi$. Let $r : I^{<\omega} \to I^{<\omega}$ be the string reversal; i.e., $r(\langle\rangle) = \langle\rangle$ and $r(b\bar{a}) = r(\bar{a})b$.

Lemma 1.3.6. If $\phi(p_0, \dots, p_{n-1})$ is a normative Sahlqvist antecedent, then $[\![\phi]\!]^{\mathfrak{F}}$ is a Sahlqvist map of type $(n, m, l; \bar{a}_0, \dots, \bar{a}_{m-1})$ for some $l \in \omega$, where m is the number of variables that occur positively in ϕ and, for each $i \in m$, the unique positive occurrence of p_i in ϕ follows $\Box_{r(\bar{a}_i)}$.

Proof. By induction. The properties used in the proof are that $\mathrm{RO}(\mathfrak{F})$, the underlying BAO of $\mathfrak{F}$, is a complete and completely additive BAO, making $\wedge$, $\vee$ and the operators for $\Diamond_a$ completely additive; that the operators l_a have left adjoints; and that if $l \colon \mathrm{RO}(\mathfrak{F}) \to \mathrm{RO}(\mathfrak{F})$ has a left adjoint of the form $Y \mapsto R^{\mathrm{ro}}_{\bar{a}}[Y]$, then $l_b \circ l$ has a left adjoint of the form $Y \mapsto R^{\mathrm{ro}}_{\bar{a}b}[Y]$. $\square$

For $X \in \mathrm{RO}(\mathfrak{F})$, we write $Y \leq_1 X$ if $Y = \{y\}^{\mathrm{ro}}$ for some $y \in X$. Note that if $Y \leq_1 X$ then $Y \subseteq X$.

Lemma 1.3.7. For $X \in \mathrm{RO}(\mathfrak{F}) \setminus \{\emptyset\}$, we have $X = \bigvee_{Y \leq_1 X} Y$.

Proof. Since $Y \leq_1 X \Rightarrow Y \subseteq X$, we have $\bigvee_{Y \leq_1 X} Y \subseteq X$.

Let $x \in X$ be arbitrary. Then $\{x\}^{\mathrm{ro}} \leq_1 X$, whence $x \in \{x\}^{\mathrm{ro}} \subseteq \bigvee_{Y \leq_1 X} Y$. Therefore, $X \subseteq \bigvee_{Y \leq_1 X} X$. □

Note that the lemma above is a consequence of $(\cdot)^{\mathrm{ro}}$ being a closure operator and $\mathrm{RO}(\mathfrak{F})$ being the set of fixed points of $(\cdot)^{\mathrm{ro}}$.

For $\bar{a} \in I^{<\omega}$, write $\mathbf{V}_1^{\bar{a}}(\mathfrak{F})$ for the family of regular open sets that are either empty or of the form $R_{\bar{a}}^{\mathrm{ro}}[\{z\}^{\mathrm{ro}}]$ where $z \in \mathfrak{F}$. Also, let $\mathbf{V}_0(\mathfrak{F}) := \mathbf{V}_0 := \{\emptyset\}$. For $\bar{a} \in I^{<\omega}$, write $R_{\bar{a}}xy$ if and only if there exist $z_1, \ldots, z_{|a|-1} \in \mathfrak{F}$ such that

$$R_{\bar{a}(0)}xz_1 \wedge R_{\bar{a}(1)}z_1z_2 \wedge \cdots \wedge R_{\bar{a}(|a|-1)}z_{|a|-1}y.$$

Lemma 1.3.8. For each $X \in \mathrm{RO}(\mathfrak{F})$ and for each $\bar{a}$,

$$R_{\bar{a}}^{\mathrm{ro}}[X] = (R_{\bar{a}}[X])^{\mathrm{ro}}.$$

Therefore, $\mathbf{V}_1^{\bar{a}}$ is a definably enumerable class as witnessed by the first-order formula $\beta_1^{\bar{a}}(x; z)$[9]

$$[\exists z' \, (\lambda y \, R_{\bar{a}}z'y \wedge [\lambda y' \, y' = z/Q]\beta_{\mathrm{ro}}^{Q}(z'))/Q]\beta_{\mathrm{ro}}^{Q}(x).$$

(Recall that $\beta_{\mathrm{ro}}^{Q}(x)$ is a first-order formula that defines X^{ro} in the expansion of $\mathfrak{F}$ that interprets Q as X.)

Proof. For $S \subseteq F \times F$, let us define the map l_S by, for $X \subseteq \mathfrak{F}$,

$$l_S(X) = \{x \in \mathfrak{F} \mid \forall y \, (Sxy \Rightarrow y \in X)\}.$$

Then $l_{R_a} = l_a$, and it can be shown that $l_{R_{b\bar{a}}} = l_b \circ l_{R_{\bar{a}}}$. Hence, by induction, we may regard $l_{R_{\bar{a}}}$ as a map $\mathrm{RO}(\mathfrak{F}) \to \mathrm{RO}(\mathfrak{F})$, for every $\bar{a} \in I^{<\omega}$.

It can easily be seen that $Y \mapsto R_{\bar{a}}^{\mathrm{ro}}[Y]$ is a left adjoint of $l_{a(|a|-1)} \circ \cdots \circ l_{a(0)}$. By reasoning similar to that in the case of l_a, we see that $Y \mapsto R_{\bar{a}}$ is a left adjoint of $l_{R_{r(\bar{a})}}$. Since $l_{a(|a|-1)} \circ \cdots \circ l_{a(0)} = l_{R_{r(\bar{a})}}$, we conclude $R_{\bar{a}}^{\mathrm{ro}}[X] = (R_{\bar{a}}[X])^{\mathrm{ro}}$ for any $X \in \mathrm{RO}(\mathfrak{F})$, by the uniqueness of the left adjoint. □

$\mathbf{V}_0$ is also a definably enumerable class trivially.

Lemma 1.3.9. Let $f \colon \mathrm{RO}(\mathfrak{F})^n \to \mathrm{RO}(\mathfrak{F})$ be a Sahlqvist map of type $(n, m, l; \bar{a}_0, \ldots, \bar{a}_{m-1})$ and $G \colon \mathrm{RO}(\mathfrak{F})^n \to \mathrm{RO}(\mathfrak{F})$ be order-preserving. Then for $w \in \mathfrak{F}$, the following are equivalent:

$$\begin{aligned}&\forall P_0 \in \mathrm{RO}(\mathfrak{F}) \cdots \forall P_{n-1} \in \mathrm{RO}(\mathfrak{F}) \\ &\qquad (w \in f(P_0, \ldots, P_{n-1}) \Rightarrow w \in G(P_0, \ldots, P_{n-1}));\end{aligned} \tag{1.7}$$

$$\begin{aligned}&\forall P_0 \in \mathbf{V}_1^{\bar{a}_0}(\mathfrak{F}) \cdots \forall P_{m-1} \in \mathbf{V}_1^{\bar{a}_{m-1}}(\mathfrak{F}) \, \forall P_m \in \mathbf{V}_0 \cdots \forall P_{n-1} \in \mathbf{V}_0 \\ &\qquad (w \in f(P_0, \ldots, P_{n-1}) \Rightarrow w \in G(P_0, \ldots, P_{n-1})).\end{aligned} \tag{1.8}$$

[9]We use the λ-notation and the notation for syntactic substitution as in [14].

Proof. For simplicity, assume $n = 2$, $m = 1$, and $l = 2$; it is straightforward to adapt the proof below for the general case.

($\Rightarrow$) is clear. Assume (1.8). Suppose $f = f_0 \circ \langle g \circ \pi_1, h\rangle$ where $f_0\colon \mathrm{RO}(\mathfrak{F})^2 \to \mathrm{RO}(\mathfrak{F})$ is completely additive, $g\colon \mathrm{RO}(\mathfrak{F}) \to \mathrm{RO}(\mathfrak{F})$ is the right adjoint of the map $Y \mapsto R^{\mathrm{ro}}_{\bar{a}_0}[Y]$ and $h\colon \mathrm{RO}(\mathfrak{F})^2 \to \mathrm{RO}(\mathfrak{F})$ is order-reversing. Take arbitrary $P_0, P_1 \in \mathrm{RO}(\mathfrak{F})$ and assume $w \in f(P_0, P_1)$. We will show $w \in G(P_0, P_1)$.

By the adjunction, we can show that if $P_0 = \emptyset$ then $g(P_0) = \emptyset$. Assume $g(P_0) = \emptyset$. Then $w \in f(P_0, P_1) = f_0(g(P_0), h(P_0, P_1)) = f_0(\emptyset, h(P_0, P_1)) = f_0(g(\emptyset), h(P_0, P_1))$. Since h is order-reversing and f_0 is order-preserving, $w \in f_0(g(\emptyset), h(\emptyset, P_1)) = f(\emptyset, P_1)$. By $\emptyset \in \mathbf{V}^{\bar{a}_0}_1$ and (1.10), we have $w \in G(P_0, P_1)$.

Assume $g(P_0) \neq \emptyset$. Since h is order-reversing, f_0 is completely additive, and $g(P_0) = \bigvee_{X \leq_1 g(P_0)} X$ (by Lemma 1.3.7), we have

$$\begin{aligned}
w &\in f(P_0, P_1) \\
&= f_0(g(P_0), h(P_0, P_1)) \\
&\subseteq f_0(\bigvee_{X \leq_1 g(P_0)} X, h(P_0, \emptyset)) \\
&= \bigvee_{\{x\}^{\mathrm{ro}} \subseteq g(P_0)} f_0(\{x\}^{\mathrm{ro}}, h(P_0, \emptyset)) \\
&= \bigvee_{R^{\mathrm{ro}}_{\bar{a}_0}[\{x\}^{\mathrm{ro}}] \subseteq P_0} f_0(\{x\}^{\mathrm{ro}}, h(P_0, \emptyset)),
\end{aligned}$$

where the last equality follows because g's left adjoint is $Y \mapsto R^{\mathrm{ro}}_{\bar{a}_0}[Y]$. For each $x \in \mathfrak{F}$, let $Q_x = R^{\mathrm{ro}}_{\bar{a}_0}[\{x\}^{\mathrm{ro}}]$. Note that $Q_x \in \mathbf{V}^{\bar{a}_0}_1(\mathfrak{F})$ and that $g(Q_x) \supseteq \{x\}^{\mathrm{ro}}$ (the latter is by the general fact that the composite of a right adjoint after its left adjoint is inflating). Then

$$\begin{aligned}
w &\in \bigvee_{Q_x \subseteq P_0} f_0(\{x\}^{\mathrm{ro}}, h(P_0, \emptyset)) \\
&\subseteq \bigvee_{Q_x \subseteq P_0} f_0(g(Q_x), h(Q_x, \emptyset)) && (1.9) \\
&\subseteq \bigvee_{Q_x \subseteq P_0} G(Q_x, \emptyset) && (1.10) \\
&\subseteq \bigvee_{Q_x \subseteq P_0} G(P_0, P_1) && (1.11) \\
&= G(P_0, P_1). && (1.12)
\end{aligned}$$

The inclusion (1.9) is by the order-reversing property of h and the order-preserving property of f_0; (1.10) is by (1.8); and (1.11) is because G is order-preserving. $\square$

Corollary 1.3.10. Let $\phi(p_0, \ldots, p_{n-1})$ be a normative Sahlqvist antecedent and $\psi(p_0, \ldots, p_{n-1})$ be positive. Assume that m is the number of propositional variables that occur positively in ϕ, and that for each $i \in m$ the unique positive occurrence of p_i in ϕ follows $\Box_{\bar{a}_i}$. Then for $w \in \mathfrak{F}$, the following are equivalent:

$$\mathfrak{F} \models U^2(\mathrm{ST}_w(\phi \to \psi)); \tag{1.13}$$

$$\begin{gathered}\forall P_0 \in \mathbf{V}_1^{\bar{a}_0}(\mathfrak{F}) \cdots \forall P_{m-1} \in \mathbf{V}_1^{\bar{a}_{m-1}}(\mathfrak{F})\, \forall P_m \in \mathbf{V}_0 \cdots \forall P_{n-1} \in \mathbf{V}_0 \\ (\mathfrak{F}, P_0, \ldots, P_{n-1}) \models \mathrm{ST}_w(\phi \to \psi).\end{gathered} \tag{1.14}$$

Proof. Note that, for $w \in \mathfrak{F}$, we have $(\mathfrak{F}, P_0, \ldots, P_{n-1}) \models \mathrm{ST}_w(\phi \to \psi)$ if and only if

$$\forall w' \sqsubseteq w\, (w' \in [\![\phi]\!]^{\mathfrak{F}}(P_0, \ldots, P_{n-1}) \Rightarrow w' \in [\![\psi]\!]^{\mathfrak{F}}(P_0, \ldots, P_{n-1})).$$

By Lemma 1.3.6, $[\![\phi]\!]^{\mathfrak{F}}$ is a Sahlqvist map of type $(n, m, l; \bar{a}_0, \ldots, \bar{a}_{m-1})$ for some $l \in \omega$. By Lemma 1.3.5, $[\![\psi]\!]^{\mathfrak{F}}$ is order-preserving. By applying Lemma 1.3.9 to each $w' \sqsubseteq w$, we obtain the equivalence between (1.13) and (1.14). □

Corollary 1.3.11. For any Sahlqvist implication χ with a normative antecedent, there exists a first-order formula $\alpha(x)$ such that χ corresponds to $\alpha(x)$.

Proof. By Corollary 1.3.10 and Lemma 1.3.2. □

We will now see that the general case reduces to that of normative formulae. For $V, V' \subseteq \mathrm{RO}(\mathfrak{F})$, write $V + V'$ for the family of regular open sets of the form $P \vee P'$, where $P \in V$ and $P' \in V$. Note that if both $\mathcal{V}$ and $\mathcal{V}'$ are definably enumerable classes, so is the class $\mathcal{V} + \mathcal{V}'$ which is defined by $(\mathcal{V} + \mathcal{V}')(\mathfrak{F}) = \mathcal{V}(\mathfrak{F}) + \mathcal{V}'(\mathfrak{F})$.

Lemma 1.3.12. Let $m \leq n$. Suppose $\phi(p_0, \ldots, p_{n-1})$ is a modal formula such that each p_i is positive for $i = 0, \ldots, m-1$. Let $\psi(p_m, \ldots, p_{n-1})$ be positive. Assume that for definably enumerable classes $\mathcal{V}_0, \ldots, \mathcal{V}_{m-1}$ the following are equivalent for each $w \in \mathfrak{F}$:

$$\begin{aligned}&\forall P_0 \in \mathrm{RO}(\mathfrak{F}) \cdots \forall P_{m-1} \in \mathrm{RO}(\mathfrak{F}) \\ &\qquad (w \in [\![\sigma(\phi)]\!]^{\mathfrak{F}}(P_0, \ldots, P_{m-1}) \Rightarrow w \in [\![\sigma(\psi)]\!]^{\mathfrak{F}}(P_0, \ldots, P_{m-1}));\end{aligned} \tag{1.15}$$

$$\begin{aligned}&\forall P_0 \in \mathcal{V}_0(\mathfrak{F}) \cdots \forall P_{m-1} \in \mathcal{V}_{m-1}(\mathfrak{F}) \\ &\qquad (w \in [\![\sigma(\phi)]\!]^{\mathfrak{F}}(P_0, \ldots, P_{m-1}) \Rightarrow w \in [\![\sigma(\psi)]\!]^{\mathfrak{F}}(P_0, \ldots, P_{m-1})),\end{aligned} \tag{1.16}$$

where

$$\sigma = \left[\bigvee_{0 \leq i < m} p_i \middle/ p_m, \ldots, \bigvee_{0 \leq i < m} p_i \middle/ p_{n-1} \right].$$

Then the following are also equivalent for each $w \in \mathfrak{F}$:

$$\forall P \in \mathrm{RO}(\mathfrak{F})\, (w \in [\![\sigma_0(\phi)]\!]^{\mathfrak{F}}(P) \Rightarrow w \in [\![\sigma_0(\psi)]\!]^{\mathfrak{F}}(P)); \tag{1.17}$$

$$\forall P \in (\mathcal{V}_0 + \cdots + \mathcal{V}_{m-1})(\mathfrak{F})\, (w \in [\![\sigma_0(\phi)]\!]^{\mathfrak{F}}(P) \Rightarrow w \in [\![\sigma_0(\psi)]\!]^{\mathfrak{F}}(P)), \tag{1.18}$$

where $\sigma_0 = [p_0/p_0, \ldots, p_0/p_{n-1}]$.

Proof. (1.17) $\Rightarrow$ (1.18) is clear. We will see (1.18) $\Rightarrow$ (1.16) $\Rightarrow$ (1.15) $\Rightarrow$ (1.17). (1.16) $\Rightarrow$ (1.15) is by assumption. (1.15) $\Rightarrow$ (1.17) is by instantiating (1.15) with $P_1 \dots, P_{n-1} := P_0$, by $[\![\bigvee_{0\leq i<m} p_i]\!]^{\mathfrak{F}}(P_0, \dots, P_{n-1}) = \bigvee_{0\leq i<m} P_i$, and by the definition of σ.

We show (1.18) $\Rightarrow$ (1.16). For simplicity, assume $n = 3$ and $m = 2$ (the proof can be adapted for other cases straightforwardly). Take arbitrary $P_0 \in V_0$ and $P_1 \in V_1$. Then

$$
\begin{aligned}
w \in\ & [\![\sigma(\phi)]\!]^{\mathfrak{F}}(P_0, P_1) \\
&\subseteq [\![\sigma(\phi)]\!]^{\mathfrak{F}}(P_0 \vee P_1, P_0 \vee P_1) && (1.19)\\
&= [\![\sigma_0(\phi)]\!]^{\mathfrak{F}}(P_0 \vee P_1) && (1.20)\\
&\Rightarrow && (1.21)\\
w \in\ & [\![\sigma_0(\psi)]\!]^{\mathfrak{F}}(P_0 \vee P_1) \\
&= [\![\sigma(\psi)]\!]^{\mathfrak{F}}(P_0, P_1). && (1.22)
\end{aligned}
$$

(1.19) holds because $\sigma(\phi)$ is positive in p_0, and p_1 and $\sigma(p_2) = p_0 \vee p_1$. (1.20) holds by the definition of σ and σ_0. (1.21) follows from (1.18). (1.22) holds because neither p_0 nor p_1 occurs in ψ. □

By the lemma above, correspondence theory for a Sahlqvist implication in which the only propositional variable is p_0 reduces to that for a Sahlqvist implication with normative antecedents. More concretely, the case for such an implication χ reduces to that for the formula one obtains by replacing in χ the positive occurrences of p_0 in the antecedent by distinct propositional variables and, simultaneously, the other occurrences of p_0 by the disjunction of those distinct variables. We can further show a similar lemma for multiple variables to reduce the case for general Sahlqvist implications to that for Sahlqvist implications with normative antecedents.

We are now ready to prove the main theorem of this section.

Proof of Theorem 1.3.1. As in the correspondence theory for the standard Kripke semantics, one can show that the set of modal formulae that locally correspond to first-order formulae are closed under these operations:

$$
\begin{aligned}
\chi &\mapsto \Box_{\bar{a}}\chi && (\bar{a} \in I^{<\omega})\\
(\chi, \chi') &\mapsto \chi \wedge \chi' \\
(\chi, \chi') &\mapsto \chi \vee \chi' && (\text{if no propositional variables occur in both } \chi \text{ and } \chi')
\end{aligned}
$$

Also by the observation above one only needs to prove the theorem for a Sahlqvist implication whose antecedent is normative. This follows from Corollary 1.3.11. □

For a better understanding of the methods of this section, let us apply them to a concrete example.

Example 1.3.13. Assume that I is a singleton, denote its only element by $*$, and let $R = R_*$ and $x \rhd y \leftrightarrow Ryx$. The B axiom from Example 1.2.4 has an equivalent form

$$\mathrm{B}^{\mathrm{op}} :\equiv \Diamond\Box p_0 \to p_0$$

which is a Sahlqvist implication. We will calculate a local correspondent of B^{op} as an example, by using the theorems in this section.

As we saw in Example 1.2.4, we can assume extra conditions on the interaction of R and $\sqsubseteq$ to make correspondents simpler, without loss of generality. In fact, something additional is true here: often, for an interaction condition C, if a first-order formula $\alpha(x)$ is a local correspondent of a modal formula ϕ over the possibility frames that satisfy C, i.e., for any possibility frame $\mathfrak{F} \models C$ and $w \in \mathfrak{F}$,

$$\mathfrak{F}, w \Vdash \phi \Leftrightarrow \mathfrak{F} \models \alpha(w),$$

then one can effectively obtain a first-order $\tilde{\alpha}(x)$ which is a local correspondent of ϕ. See [49, Section 6.3] for the details. To compute a local correspondent of B^{op} it is convenient to assume the following conditions, alongside (R-down):

$$x \sqsubseteq y \leftrightarrow \forall x' \sqsubseteq x\, x' \between y; \qquad \text{(separativity)}$$

$$(\forall y' \sqsubseteq y\, \exists y'' \sqsubseteq y'\, Rxy'') \to Rxy; \qquad (R\text{-dense})$$

$$(Rx'y \wedge x' \sqsubseteq x) \to Rxy. \qquad (\text{up-}R)$$

Again, we can assume these conditions without loss of generality, in the strong sense stated above. One of the major consequences of the extra conditions is

$$R^{\mathrm{ro}}[\{x\}^{\mathrm{ro}}] = R[\{x\}]. \tag{1.23}$$

We are now ready to calculate a local correspondent of B^{op}. Using the simplified forcing relation (1.4) for $\Diamond$, we see that $\mathrm{ST}_x(\mathrm{B}^{\mathrm{op}})$ is equivalent to

$$\forall x_1 \sqsubseteq x\, ((\forall x_2 \sqsubseteq x_1\, \exists x_3 \rhd x_2\, \forall x_4 \rhd x_3\, P_0 x_4) \to P_0 x_1).$$

Since p_0 follows exactly one $\Box$ in the antecedent of B^{op}, one can apply Lemma 1.3.9 where the range of $\forall P_0$ is restricted to $\mathbf{V}_1^*$. This class is defined by the first-order formula $\beta_1^*(x; z)$, where

$$\beta_1^*(x; z) \leftrightarrow Rzx$$

by (1.23). A local correspondent of B^{op} is then obtained by applying Lemma 1.3.2: $\alpha_{\mathrm{B}^{\mathrm{op}}}(x) \wedge \alpha_\emptyset(x)$ is a local correspondent of B^{op}, where $\alpha_{\mathrm{B}^{\mathrm{op}}}(x)$ is the first-order formula obtained by replacing

$$\forall P_0 \cdots P_0 x \cdots$$

by

$$\forall z_0 \cdots \qquad \underbrace{Rz_0 x}_{\text{equivalent to } \beta_1^*(x; z_0)} \qquad \cdots$$

in $U^2(\mathrm{ST}_x(\mathrm{B}^{\mathrm{op}}))$, and $\alpha_\emptyset(x) = [\lambda x\, x \neq x/P_0]\,\mathrm{ST}_x(\mathrm{B}^{\mathrm{op}})$. $\alpha_{\mathrm{B}^{\mathrm{op}}}(x)$ can be calculated to be

$$\forall z_0\, \forall x_1 \sqsubseteq x\, ((\forall x_2 \sqsubseteq x_1\, \exists x_3 \rhd x_2\, \forall x_4 \rhd x_3\, Rz_0x_4) \to Rz_0x_1),$$

and $\alpha_\emptyset(x)$ is

$$\forall x_1 \sqsubseteq x\, ((\forall x_2 \sqsubseteq x_1\, \exists x_3 \rhd x_2\, \forall x_4 \rhd x_3\, x_4 \neq x_4) \to x_1 \neq x_1).$$

One can check that, under the assumption of the extra conditions above, $\forall x\, (\alpha_{\mathrm{B}^{\mathrm{op}}}(x) \wedge \alpha_\emptyset(x))$ is equivalent to (1.3), the global correspondent of the B axiom given in Example 1.2.4.

Given that the analogue of the Sahlqvist Correspondence Theorem holds for possibility semantics, it is natural to ask whether an analogue of the Sahlqvist Completeness Theorem holds for possibility semantics as well. We will briefly discuss this question in Section 1.5.

1.4 Model-theoretic characterization

In this section, we examine model-theoretic aspects of correspondence theory for possibility frames, extending and adapting the classical work of van Benthem [5]. We will see that the standard results for Kripke semantics smoothly extend to the setting of possibility semantics.

First, we investigate a model-theoretic characterization of modal formulae that *globally* correspond to first-order formulae. Unlike in the previous sections, we now regard possibility frames as (ordinary) structures for $\mathcal{L}^2(\tau)$, i.e., with no restriction on the range of second-order variables. In this section, we use the term "structures" without qualifications to refer to this kind of structure for $\mathcal{L}^2(\tau)$. We also assume in this section that I, the set of modal indices, is finite.

Let $\mathrm{FR}(\phi)$ denote the set of possibility frames $\mathfrak{F}$ such that for every possibility model $\mathfrak{M} = (\mathfrak{F}, \pi)$ and every $w \in \mathfrak{F}$, we have $\mathfrak{M}, w \Vdash \phi$. Equivalently, $\mathrm{FR}(\phi)$ is the set $\mathrm{Mod}(\mathrm{SOT}(\phi))$ of structures that models the monadic second-order formula $\mathrm{SOT}(\phi)$, where:

- $\mathrm{SOT}(\phi) := \tilde{U}^2(\mathrm{ST}_x(\phi)) \wedge \beta_{\mathrm{po}} \wedge \bigwedge_{a \in I} \beta^a_{R \Rightarrow \mathrm{win}} \wedge \beta^a_{R\text{-rule}}$;
- β_{po} states $\sqsubseteq$ is a partial order;
- $\tilde{U}^2(\chi)$ denotes the universal quantification by the second-order monadic variables occurring in χ, but with the domain of the quantification restricted to $\mathrm{RO}(\mathfrak{F})$; concretely, $\tilde{U}^2(\chi) := \chi$ for $\chi \in \mathcal{L}^2(\tau)$ with no monadic second-order free variables and $\tilde{U}^2(\chi) := \tilde{U}^2(\forall P\, (\beta^P_{\mathrm{val}} \to \chi))$ for χ with a monadic second-order free variable P; and
- β^P_{val} is a sentence in $\mathcal{L}^1(\tau \cup \{P\})$ that says that P is a regular open set within a possibility frame; i.e.,
$$\beta^P_{\mathrm{val}} :\equiv \forall x\, (Px \leftrightarrow \beta^P_{\mathrm{ro}}(x)).$$

Definition 1.4.1. Let $\mathfrak{F}$ be a structure. A *generated substructure* $\mathfrak{G}$ of $\mathfrak{F}$ is a substructure of $\mathfrak{F}$ such that if $x \in \mathfrak{G}$ and $\mathfrak{F} \models \heartsuit xy$ for some $\heartsuit \in \{\sqsupseteq\} \cup \{R_a \mid a \in I\}$, then $y \in \mathfrak{G}$.

It can be shown that a generated substructure of a possibility frame as a structure is again a possibility frame (see [49, Proposition 5.50.2]).

Lemma 1.4.2. Let $\mathfrak{F}$ be a structure and $\mathfrak{G}$ be a generated substructure of $\mathfrak{F}$. Let π be an interpretation of P_i $(i \in \kappa)$. Then for each modal formula ϕ and each $w \in \mathfrak{G}$, we have $(\mathfrak{F}, \pi) \models \mathrm{ST}_w(\phi) \Leftrightarrow (\mathfrak{G}, \pi) \models \mathrm{ST}_w(\phi)$.

Proof. Obvious. □

The following result is originally due to Goldblatt [41]. For a family $(\mathfrak{N}_i)_{i\in J}$ of structures and an ultrafilter U over J, we write $\prod_{i\in J} N_i/U$ for the ultraproduct of the family using J (see, e.g., [64]).

Lemma 1.4.3. Let $(\mathfrak{F}_i)_{i\in J}$ and $(\mathfrak{G}_i)_{i\in J}$ be families of structures. Assume that each $\mathfrak{F}_i$ is a generated substructure of $\mathfrak{G}_i$. Let U be an ultrafilter over J. Then $\mathfrak{F} := \prod_i \mathfrak{F}_i/U$ is a generated substructure of $\mathfrak{G} := \prod_i \mathfrak{G}_i/U$.

Proof. This can be proved in the same way as over Kripke frames whose accessibility relations are $\heartsuit$'s as in Definition 1.4.1. □

Recall that an ultrapower $\mathfrak{F}^J/U$ is the ultraproduct $\prod_{i\in J} \mathfrak{F}_i/U$ of the family $(\mathfrak{F}_i)_{i\in J}$ where $\mathfrak{F}_i = \mathfrak{F}$ for all $i \in J$. Given a family $(\mathfrak{F}_i)_{i\in J}$ of structures, one can think of a new structure $\bigoplus_{i\in J} \mathfrak{F}_i$, their *disjoint union*, since the signature τ is relational. Note that, if $(\mathfrak{F}_i)_{i\in J}$ is a family of *possibility frames*, then $\bigoplus_{i\in J} \mathfrak{F}_i$ is again a possibility frame (see [49, Proposition 5.54.2]).

Corollary 1.4.4. Let $(\mathfrak{F}_i)_{i\in J}$ be a family of structures and $\mathfrak{F} := \bigoplus_{i\in J} \mathfrak{F}_i$. Let U be an ultrafilter over J and $\mathfrak{G} = \prod_i \mathfrak{F}_i/U$. Then $\mathfrak{G}$ is isomorphic to some generated substructure of the ultrapower $\mathfrak{F}^J/U$.

Lemma 1.4.5. For $\phi \in \mathcal{L}(\Phi, I)$, we have that $\mathrm{FR}(\phi) = \mathrm{Mod}(\forall x\, \mathrm{SOT}(\phi))$ is closed under generated substructures.

Proof. By induction on the complexity of ϕ. □

Lemma 1.4.6. For $\phi \in \mathcal{L}(\Phi, I)$, if $\mathrm{FR}(\phi)$ is closed under ultrapowers, then it is closed under ultraproducts.

Proof. Obvious from the preceding lemmas, since $\mathrm{FR}(\phi)$ is closed under disjoint unions. □

We can now see that van Benthem's [5] characterization of basic elementary classes of Kripke frames can be extended to possibility frames as well. Recall that a class $\mathcal{K}$ of structures is *basic elementary* if $\mathcal{K} = \mathrm{Mod}(\alpha)$ for some first-order α. By definition, for a modal formula ϕ, we have that $\mathrm{FR}(\phi)$ is basic elementary if and only if ϕ has a global correspondent

Theorem 1.4.7. For $\phi \in \mathcal{L}(\Phi, I)$, we have $\mathrm{FR}(\phi)$ is basic elementary if and only if it is closed under ultrapowers.

Proof. By a general model-theoretic fact (see, e.g., [19, Corollary 6.1.16 (ii)]), $\mathrm{FR}(\phi) = \mathrm{Mod}(\forall x\, \mathrm{SOT}(\phi))$ is basic elementary if and only if $\mathrm{Mod}(\forall x\, \mathrm{SOT}(\phi))$ and its complement are closed under ultraproducts. Since $\forall x\, \mathrm{SOT}(\phi)$ is Π^1_1 for any $\phi \in \mathcal{L}(\Phi, I)$, we know that $\mathrm{Mod}(\neg \forall x\, \mathrm{SOT}(\phi))$, the complement of $\mathrm{Mod}(\forall x\, \mathrm{SOT}(\phi))$, is always closed under ultraproducts, since Σ^1_1 sentences are preserved under ultraproducts. Then by the previous lemma, $\mathrm{Mod}(\forall x\, \mathrm{SOT}(\phi))$ is basic elementary if and only if it is closed under ultrapowers. $\square$

Let us now turn to *local* correspondence. An analogous result for Kripke semantics was also proved by van Benthem [5].

Theorem 1.4.8. For $\phi \in \mathcal{L}(\Phi, I)$, we have that ϕ locally corresponds to a first-order formula if and only if for every possibility frame $\mathfrak{F}$, every index set J and an ultrafilter U over J, we have

$$\forall i \in J\, \mathfrak{F} \models \mathrm{SOT}(\phi)(w_i) \Rightarrow \mathfrak{F}^J/U \models \mathrm{SOT}(\phi)((w_i)_i/U). \tag{†}$$

Proof. First observe that a modal formula ϕ locally corresponds to a first-order $\alpha(x)$ if and only if $\mathrm{Mod}([c/x]\, \mathrm{SOT}(\phi)) = \mathrm{Mod}(\alpha(c))$ where Mod is defined analogously for the language $\mathcal{L}^2(\tau \cup \{c\})$, and c is a new constant symbol. In addition, the quantifier-wise syntactic complexity of the sentence $[c/x]\, \mathrm{SOT}(\phi)$ remains Π^1_1 in the new language. Thus, a proof similar to the one above applies to this theorem.

To be more precise, one can show the following analogue of Corollary 1.4.4:

Claim. Let $((\mathfrak{F}_i, w_i))_{i \in J}$ be a family of structures for $\mathcal{L}^1(\tau \cup \{c\})$ and U be an ultrafilter over J. Then $\prod_i \mathfrak{F}_i/U$ can be embedded in the ultraproduct

$$\prod_{i \in J} (\bigoplus_{j \in J} \mathfrak{F}_j, w_i)/U, \tag{$*$}$$

and its image is a generated substructure of $(\bigoplus_{j \in J} F_j)^J/U$.

Moreover, if $(\mathfrak{F}, w) \models [c/x]\, \mathrm{SOT}(\phi)$, then for a generated substructure $\mathfrak{G}$ of $\mathfrak{F}$ containing w we have $(\mathfrak{G}, w) \models [c/x]\, \mathrm{SOT}(\phi)$, and if $(\mathfrak{F}_i, w_i) \models [c/x]\, \mathrm{SOT}(\phi)$ for all $i \in J$, an index set, then $(\bigoplus_{j \in J} F_j, w_i) \models [c/x]\, \mathrm{SOT}(\phi)$ for all i. Thus, $\mathrm{Mod}(\mathrm{SOT}(\phi)[c/x])$ is closed under ultraproducts of the form ($*$) if and only if ϕ locally corresponds to a first-order formula. This can easily seen to be equivalent to the condition (†). $\square$

A standard application of a result like Theorem 1.4.8 is to obtain a syntactic closure property of the set of formulae having first-order correspondents, as follows.

Theorem 1.4.9. If $\Box_a \phi$ locally corresponds to a first-order formula, so does ϕ.

Proof. For simplicity, we omit modal operator indices. Suppose that ϕ does not locally correspond to a first-order formula. Then by Theorem 1.4.8, there exist a structure $\mathfrak{F} = (W, R, \sqsubseteq)$, an index set J, an ultrafilter U over J, and $(w_i)_i \in \mathfrak{F}^J$ such that for every $i \in I$ we have $\mathfrak{F} \models \mathrm{SOT}(\phi)(w_i)$ but $\mathfrak{F}^J/U \not\models \mathrm{SOT}(\phi)((w_i)_i/U)$. Let π be the valuation that witnesses the latter fact. Let v be an object not in W. For each $i \in I$, let $\mathfrak{F}_i = (W \sqcup \{v\}, R \sqcup \{(v, w_i)\}, \sqsubseteq \sqcup \{(v, v)\})$. Since for every $i \in J$ we have $\mathfrak{F}_i \models \exists! x\, Rvx$ and $\mathfrak{F}_i \models Rvw_i$, by Łoś's Theorem, we know that $\prod_i \mathfrak{F}_i/U \models \exists! x\, R((v)_i/U)x$ and that the unique witness to the preceding formula is $(w_i)_i/U$. Note that $\mathfrak{F}^J/U$ is a generated substructure of $\prod_i \mathfrak{F}_i/U$. The valuation π is also a valuation for $\prod_i \mathfrak{F}_i/U$. Hence, $(\prod_i \mathfrak{F}_i/U, \pi) \not\models \mathrm{ST}_x(\phi)((w_i)_i/U)$ and $(\prod_i \mathfrak{F}_i/U, \pi) \not\models \mathrm{ST}_x(\Box\phi)((v)_i/U)$. However, by construction, for every $i \in J$ we have $\mathfrak{F}_i \models \mathrm{SOT}(\Box\phi)(v)$. Therefore, by Theorem 1.4.8, we know that $\Box\phi$ does not locally correspond to a first-order formula. $\square$

1.5 Conclusion

We have seen that despite the richer structure of possibility frames, involving not only the accessibility relation but also the refinement relation, central results of standard correspondence theory continue to hold in this more general setting. A natural question raised by our results is this: does *every* formula that has a first-order correspondent in the setting of Kripke semantics also have a first-order correspondent in the setting of possibility semantics?

A second open problem suggested by our results concerns the Sahlqvist Completeness Theorem, which states that every Sahlqvist formula is canonical. A natural question to ask here is how this theorem can be extended to our general setting of possibility semantics. In [48, Section 5.6], a possibility-theoretic view of canonical extensions of BAOs is developed, according to which, for a normal modal logic Λ, there is a *canonical possibility frame*[10] whose modal theory is included in Λ. Unlike a canonical Kripke frame, built from the *ultra*filters in the Lindenbaum algebra of a logic, a canonical possibility frame is built from proper filters in the Lindenbaum algebra. The possibility frame constructed from a BAO in this way is called a *filter frame*. Even for an uncountable modal language, the construction of the latter does not require the ultrafilter axiom, or equivalently, the Boolean prime ideal axiom. The possibility-semantic version of canonicity of a modal formula ϕ is then defined so that ϕ is *filter-canonical* if and only if, for every normal modal logic Λ containing ϕ, the logic's canonical possibility frame validates ϕ. Assuming the ultrafilter axiom, ϕ is filter-canonical if and only if ϕ is canonical in the standard Kripke-semantic sense [48, p. 122]. Holliday also proves [48], without use of the Boolean prime ideal axiom, that the underlying BAO of the filter possibility frame of a BAO A coincides with what Gehrke and Harding [33] construct as the "canonical extension" of A and what Conradie and Palmigiano [23] call the *constructive canonical extension* of A (see also Suzuki [74]). Consequences [48, Theorem 7.20] of Conradie and Palmigiano's results [23] are that every *inductive formula* is filter-canonical, and that

[10] Again, we are dropping the word "full" from the technical term defined in [49].

every normal modal logic axiomatized by inductive formulae is sound and complete with respect to its canonical possibility frame.

Another question raised by our results concerns the potential applicability to possibility semantics of ALBA [21], a group of general syntactic algorithms that can calculate correspondents of formulae in many non-classical logics that are interpreted in lattices with operators. A possible research direction is to adapt the techniques of ALBA in order to reformulate the argument in Section 1.3 as a variant of ALBA and to prove more general correspondence results for possibility semantics.

Chapter 2

Correspondence, Canonicity, and Model Theory for Monotonic Modal Logics

2.1 Introduction

Monotonic modal logics generalize normal modal logics by dropping the K axiom $\Box(p \to q) \to (\Box p \to \Box q)$ and instead requiring only that $\vdash \phi \to \psi$ imply $\vdash \Box\phi \to \Box\psi$. There are a number of reasons for relaxing the axioms of normal modal logics and considering monotonic modal logics. For instance, monotonic modal logics are considered more appropriate to describe the ability of agents or systems to make certain propositions true in the context of games and open systems [68, 69, 2]. The standard semantics for monotonic modal logics is provided by monotonic neighborhood frames (see, e.g., [43]).

Just as the first-order language with a relation symbol is a useful correspondence language for Kripke frames, it is natural to consider what would be a useful correspondence language for monotonic neighborhood frames. Litak et al. [60] studied *coalgebraic predicate logic* (CPL) as a logic that plays that role and proved a characterization theorem in the style of van Benthem and Rosen [72]. In this article, we continue that path for monotonic neighborhood frames and prove variants of the Goldblatt-Thomason theorem [40] and the Fine canonicity theorem [31] in the setting of coalgebraic predicate logic.

We will deal with a relativized notion of CPL-elementarity, relativized to subclasses of the class of monotonic neighborhood frames. There are several important subclasses to consider: the class of filter neighborhood frames, providing a more general semantics [35, 36] for normal modal logics than relational semantics; the class of quasi-filter neighborhood frames, providing a semantics for regular modal logics; the class of augmented quasi-filter neighborhood frames, providing a less general semantics for regular modal logics; and the class of augmented filter neighborhood frames, which are Kripke frames in disguise [20, 66].

The analogue of the Goldblatt-Thomason theorem in this article is that a class of

Subclass	Closed under ...
monotonic	supersets
quasi-filter	supersets, intersections of nonempty finite families of neighborhoods
augmented quasi-filter	supersets, intersections of nonempty families of neighborhoods
filter	supersets, intersections of finite families of neighborhoods
augmented filter	supersets, intersections of families of neighborhoods

Table 2.1: Classes of monotonic neighborhood frames and their definitions

monotonic neighborhood frames closed under CPL-elementarity relative to any of the classes of neighborhood frames in Table 2.1 is modally definable if and only if it is closed under disjoint unions, bounded morphic images, and generated subframes, and it reflects ultrafilter extensions; and the analogue of Fine's theorem we will prove states that a sufficient condition for the canonicity of a monotonic modal logic is that it is complete with respect to the class of monotonic neighborhood frames it defines and that that class is closed under CPL-elementarity relative to any of the classes of neighborhood frames in Table 2.1.

The relevance of coalgebraic predicate logic in this article is that many monotonic modal logics define classes of monotonic neighborhood frames that are CPL-elementary. For instance, the monotonic modal logics axiomatized by formulas of the form

$$\langle\text{purely propositional positive formula}\rangle \to \langle\text{positive formula}\rangle \tag{2.1}$$

are determined by CPL-elementary classes of monotonic neighborhood frames (see Remark 2.2). In addition, relative to the class of augmented quasi-filter frames, all monotonic modal logics axiomatized by Sahlqvist formulas are CPL-elementarily determined (see Example 2.2.5). Further discussion regarding the relevance of this language in the context of Fine's theorem is in Remark 2.4.

Since augmented filter frames are Kripke frames in disguise (see also Example 2.2.5), our result regarding classes elementary relative to the class of augmented filter frames generalizes the original, Kripke-semantic Goldblatt-Thomason theorem. Also, our Goldblatt-Thomason theorem concerns elementary classes like the original theorem, whereas some existing Goldblatt-Thomason theorems such as [44] or [58] deal with classes closed under ultrafilter extensions.

The article is organized as follows. In § 2.2, we recall standard concepts in the semantics of monotonic modal logic and introduce the language for neighborhood frames. In § 2.3, we give an overview of the model theory of neighborhood frames for this language. We also define a two-sorted first-order language (Definition 2.3.8) and a translation of coalgebraic predicate logic into it (Proposition 2.3.5), which are used later to explain the existence of $\aleph_0$-saturated models of languages of coalgebraic predicate logic (Proposition 2.3.6). In § 2.4, we prove the main lemmas of this articles. In § 2.5, we give the applications of the main lemmas, which are analogues of the Goldblatt-Thomason Theorem and Fine's Canonicity Theorem.

The presentation of the results in this article does not presuppose the reader's prior knowledge of coalgebras or coalgebraic predicate logic.

2.2 Preliminaries

Languages and structures

In this subsection, we recall standard definitions in neighborhood semantics of modal logic and the language coalgebraic predicate logic introduced in [18] and [60] to describe neighborhood frames.

We define languages of coalgebraic predicate logic relative to sets of nonlogical symbols here; this is so that we can use expansions of the smallest language in proofs in § 2.4.

Definition 2.2.1.

(i) Let σ be the set of atomic formulas of some language of first-order logic. The *language of coalgebraic predicate logic L based on σ* is the least set of formulas containing σ and closed under Boolean combinations, existential quantification, and formation of formulas of the form $x \,\Box \lceil y\colon \phi \rceil$, where $\phi \in L$, and x and y are variables. To save space, we sometimes write $x \,\Box_y\, \phi$ or even $x \,\Box\, \phi$ for $x \,\Box \lceil y : \phi \rceil$. For a language L_0 of first-order logic, the *language of coalgebraic predicate logic based on L_0* is defined to be the language of coalgebraic predicate logic based on the set of atomic formulas of L_0. We write $L_=$ for the language of coalgebraic predicate logic based on the empty language, i.e., the language with just the equality symbol.

(ii) Let L_0 be a language of first-order logic and L the language of coalgebraic predicate logic based on L_0. An *L-structure* $F = (F, N^F)$ is an L_0-structure F with an additional datum $N^F : F \to \mathscr{P}(\mathscr{P}(F))$, where $\mathscr{P}$ is the powerset operation. The map N^F is called the *neighborhood function* of F. A set $U \in N^F(w)$ is called a *neighborhood* of w. If L_0 is the empty first-order language, the L-structures are exactly the *neighborhood frames*.

(iii) A neighborhood frame F is *monotonic* if for every $w \in F$ the family $N^F(w)$ is closed under supersets. F is a *quasi-filter* neighborhood frame if for every $w \in F$ the family $N^F(w)$ is closed under intersections of nonempty finite families of neighborhoods. F is a *filter* neighborhood frame if it is a quasi-filter frame and for every $w \in F$ the family $N^F(w)$ is nonempty. F is an *augmented quasi-filter* neighborhood frame if for every $w \in F$ the family $N^F(w)$ is either empty or a *principal upset* in the Boolean algebra $\mathscr{P}(F)$, i.e., there exists $U_0 \subseteq F$ such that $U \in N^F(w) \iff U_0 \subseteq U$. Finally, F is an *augmented filter* neighborhood frame if for every $w \in F$ the family $N^F(w)$ is a principal upset.

One might object to calling the language in Definition 2.2.1(i) the language of "coalgebraic predicate logic" since this is essentially what Chang introduced in [18] whereas the language in Litak et al. [60] is applicable to general coalgebras. We use the name "coalgebraic predicate logic" in this article because Chang's language does not have a name, and technically speaking we do not use Chang's syntax, which imposes a more strict rule regarding variables bound by modal operators.

Example 2.2.2. For a topological space $X = (X, \tau)$, we associate a neighborhood frame $X^* = (X, N)$ defined by

$$U \in N(w) \iff w \in U^\circ,$$

where $^\circ$ denotes topological interior. We call a monotonic neighborhood frame of the form X^* a *topological neighborhood frame.* Recall the satisfaction predicate $\Vdash_{\text{top}}$ for topological semantics and the satisfaction predicate $\Vdash_{\text{nbhd}}$ for neighborhood semantics (see, e.g., [20, 9] for more details):

$$M, w \Vdash_{\text{top}} \Box\phi \iff w \in \{w' \mid M, w' \Vdash_{\text{top}} \phi\}^\circ$$

and

$$M', w \Vdash_{\text{nbhd}} \Box\phi \iff \{w' \mid M, w' \Vdash_{\text{nbhd}} \phi\} \in N(w),$$

where M is a topological model, M' is a neighborhood model, and N is the neighborhood function of the neighborhood frame of M'. It is then easy to see that for every $w \in X$, every modal formula ϕ, every topological model M based on X, and every neighborhood model N based on X^*, if the valuations of M and N are the same, then

$$M, w \Vdash_{\text{top}} \phi \iff N, w \Vdash_{\text{nbhd}} \phi.$$

Definition 2.2.3. Let L be a language of coalgebraic predicate logic and F an L-structure. We define the satisfaction predicate $F \models \phi$ for a sentence $\phi \in L$. It is convenient to define the predicate for the expanded language $L(F)$ of coalgebraic predicate logic. In general, for $A \subseteq F$, we define $L(A)$ to be the language of coalgebraic predicate logic that has all symbols of L and for each $w \in A$ a constant symbol w that is intended to be interpreted as w itself. Now, F is an $L(F)$-structure in the obvious way. We define the satisfaction predicate $F \models \phi$ for $\phi \in L(F)$. The predicate is defined by recursion on ϕ. For symbols of first-order logic in L, the predicate is defined in the usual way. For $\phi = w \,\Box_y\, \phi_0$, we define

$$F \models w \,\Box_y\, \phi_0(y) \iff \phi_0(F) \in N^F(w)$$

where

$$\phi_0(F) = \{v \in F \mid F \models \phi_0(v)\}$$

and $\phi_0(v)$ stands for the substitution instance of $\phi_0(y)$ with v substituted for y.

The use of constant symbols interpreted as themselves is standard practice in model theory (see, e.g., [64]); it makes the notation and definitions much simpler, particularly in later parts of this article where we deal with types with parameters.

Example 2.2.4. Consider the B axiom $p \to \Box\neg\Box\neg p$. We see that this modal formula has a local frame correspondent relative to the class of monotonic neighborhood frames in the language $L_=$. Consider the validity of the B axiom for a monotonic neighborhood frame F and $w \in F$. By the monotonicity of F, the usual minimum valuation argument (see, e.g., [14]) applies: the B axiom is valid at w if and only if its consequent is true under the minimum valuation that makes its antecedent true, which is the valuation that sends p to the set $\{w\}$. The latter condition is expressible by a formula in $L_=$:

$$w \,\Box_y (\neg y \,\Box_z\, z \neq w).$$

Remark. It can be shown likewise that modal formulas of the form (2.1) have frame correspondents relative to the class of monotonic neighborhood frames. A formula of the form (2.1) is what is called a KW formula in [43] and axiomatizes a monotonic modal logic complete with respect to the class of monotonic neighborhood frames that it defines. Hence, the monotonic modal logics axiomatized by such formulas are determined by CPL-elementary classes (see Definition 2.4.1) of monotonic neighborhood frames.

Example 2.2.5. Consider the 4 axiom $\Box p \to \Box\Box p$. We show that this modal formula has a local frame correspondent relative to the class of augmented quasi-filter neighborhood frames in the same language $L_=$ as above. Consider the validity of the 4 axiom for an augmented quasi-filter neighborhood frame F and $w \in F$. If $w \in F$ is *impossible*, i.e., $N^F(w) = \varnothing$, then the 4 axiom is valid at w. Note that by monotonicity w is impossible if and only if $F \notin N^F(w)$, i.e., $F \models \neg w \,\Box_y\, y = y$. Otherwise, we can again use the minimum valuation argument. Here, the minimum interpretation of p that makes the antecedent true is $R[w]$ because F is an augmented quasi-filter neighborhood frame, where $R \subseteq F \times F$ is the binary relation defined by

$$xRy \iff \{z \in F \mid z \neq y\} \notin N^F(x) \quad (\iff F \models \neg x \,\Box_z\, z \neq y). \tag{2.2}$$

To summarize, the 4 axiom has the local frame correspondent

$$\neg w \,\Box\lceil y\colon y = y\rceil \vee (w \,\Box\lceil y\colon y = y\rceil \wedge w \,\Box\lceil y_1\colon y_1 \,\Box\lceil y_2\colon \neg y_2 \,\Box\lceil z\colon z \neq w\rceil\rceil\rceil).$$

In fact, since the accessibility relation R and the set of impossible worlds are definable in $L_=$ as we have seen above, the first-order frame correspondence language in [67] translates into $L_=$, and thus all Sahlqvist formulas have frame correspondents in $L_=$ relative to the class of augmented quasi-filter neighborhood frames.

The displayed formula (2.2) can be used to define the class of augmented quasi-filter neighborhood frames by coalgebraic predicate logic as well. Write $R[x]$ for the set of $y \in F$ satisfying (2.2) for an arbitrary monotonic neighborhood frame F and $x \in F$. We see that a monotonic neighborhood frame F is augmented quasi-filter if and only if either $N^F(w)$ is impossible, or $R[w] \in N^F(w)$ for every $w \in F$, i.e., F satisfies the $L_=$-sentence

$$\forall x[(x \,\Box_y\, y = y) \to x \,\Box_y \neg(x \,\Box_z\, z \neq y)].$$

Indeed, we have seen the "only if" direction in the last paragraph; to see the "if" direction, observe that $R[w] = \bigcap N^F(w)$.

Example 2.2.6. Recall that for a topological space X the *specialization preorder* of X is the preorder $\lesssim$ on X defined by

$$x \lesssim y \iff x \in \overline{\{y\}},$$

where $\overline{(\cdot)}$ denotes topological closure. A space X is T_0 if and only if $\lesssim$ is a partial order, and X is T_1 if and only if $\lesssim$ is a discrete partial order. Note that the specialization preorder of a topological space X is "definable" in coalgebraic predicate logic in the sense that

$$x \lesssim y \iff X^* \models \neg x \,\Box_z\, z \neq y. \tag{2.3}$$

Hence, the images under * of the classes of T_0 and T_1 spaces are CPL-elementary relative to the class of topological neighborhood frames: X is T_0 if and only if $X^* \models \forall z\, \forall w(z \lesssim w \wedge w \lesssim z \to w = z)$, and X is T_1 if and only if $X^* \models \forall z\, \forall w(z \lesssim w \to w = z)$, where $x \lesssim y$ abbreviates the formula of coalgebraic predicate logic on the right-hand side of the displayed formula (2.3).

Definition 2.2.7. Let F and F' be neighborhood frames. A function $f : F \to F'$ is a *bounded morphism* if for each $w \in F$:

$$f^{-1}(U') \in N^F(w) \implies U' \in N^{F'}(f(w)) \tag{“forth”}$$

and

$$U' \in N^{F'}(f(w)) \implies f^{-1}(U') \in N^F(w). \tag{“back”}$$

Lemma 2.2.8 ([27])**.** Let F and F' be monotonic neighborhood frames and $f : F \to F'$ be a function that satisfies the "forth" condition. Suppose in addition that for all $U' \in N^{F'}(f(w))$ there exists $U \in N^F(w)$ such that $f(U) \subseteq U'$. Then f is a bounded morphism.

Proof. By assumption, if $U' \in N^{F'}(w)$, then there exists U such that $f^{-1}(U') \supseteq U \in N^G(w)$; by monotonicity, we have $f^{-1}(U') \in N^G(w)$. ☐

Note that bounded morphisms between monotonic neighborhood frames clearly satisfy the assumption of this lemma.

Algebraic concepts

In this subsection, we recall some standard definitions from the algebraic treatment of modal logic; for the standard notions that we do not define here, see [78].

First, we recall basic definitions regarding the algebraic treatment of monotonic modal logic.

Definition 2.2.9. A *monotonic Boolean algebra expansion* (*BAM* for short) $A = (A, \Box^A)$ is a Boolean algebra A with an additional datum $\Box^A : A \to A$, a function that is *monotonic*, i.e., for all $a, b \in A$ we have $a \leq b \implies \Box^A(a) \leq \Box^A(b)$.

Lemma 2.2.10. Let F be a monotonic neighborhood frame. The function $\Box^F : \mathscr{P}(F) \to \mathscr{P}(F)$ defined by

$$X \mapsto \{w \in F \mid X \in N^F(w)\}$$

is monotonic. $\square$

Definition 2.2.11 ([27])**.** The *complex algebra* F^+ of a monotonic neighborhood frame F is the BAM $(\mathscr{P}(F), \Box^F)$, where $\mathscr{P}(F)$ is the Boolean algebra of the powerset of F.

Proposition 2.2.1. Let F and F' be monotonic neighborhood frames. A function $f : F \to F'$ is a bounded morphism if and only if $f^+ : F'^+ \to F^+$ defined by $f^+(X) = f^{-1}(X)$ is a homomorphism. $\square$

Since this article concerns canonicity, we need to recall definitions regarding canonical extensions.

Definition 2.2.12. Let B be a Boolean algebra. The *canonical extension* B^σ of B is the Boolean algebra of the powerset of the set $\mathrm{Uf}(B)$ of ultrafilters in B. An element of B^σ of the form $[a] := \{u \in \mathrm{Uf}(B) \mid a \in u\}$ for a fixed $a \in B$ is called *clopen*. Meets and joins of clopen elements of B^σ are *closed* and *open*, respectively. The sets of closed and open elements of B^σ are denoted $K(B^\sigma)$ and $O(B^\sigma)$, respectively.

Proposition 2.2.2. For a Boolean algebra B, the map $[-] : B \to B^\sigma$ is an embedding.

Proof. See, e.g., [78]. $\square$

Definition 2.2.13 (see, e.g., [78])**.**

(i) Let $A = (A, \Box)$ be a BAM. The *canonical extension* $A^\sigma = (A^\sigma, \Box^\sigma)$ of A is the canonical extension of the Boolean algebra A expanded by the function $\Box^\sigma$, where

$$\Box^\sigma(u) = \bigvee_{u \supseteq x \in K(A^\sigma)} \bigwedge_{x \subseteq a \in A} \Box(a).$$

(ii) A set Δ of modal formulas is *canonical* if for every BAM $A \models \Delta$ we have $A^\sigma \models \Delta$.

Proposition 2.2.3. For a BAM $A = (A, \Box)$, the function $\Box^\sigma$ is monotonic, and thus the canonical extension $A^\sigma = (A^\sigma, \Box^\sigma)$ is again a BAM.

Proof. See, e.g., [78]. $\square$

Remark. Canonical extensions can be defined for larger classes of algebras. We stick to BAMs in this article since they admit the most natural definition for $\Box^\sigma$, among other reasons.

Definition 2.2.14 ([43])**.**

(i) Let A be a BAM. The *ultrafilter frame* of A is a neighborhood frame $(\mathrm{Uf}(A), N^\sigma)$ with N^σ defined by

$$U \in N^\sigma(u) \iff \exists K \subseteq U \, \forall a \in A([a] \supseteq K \Rightarrow \Box(a) \in u), \tag{2.4}$$

where $u \in \mathrm{Uf}(A)$, and K ranges over closed elements of $A^\sigma = \mathscr{P}(\mathrm{Uf} A)$. We denote the ultrafilter frame of A by $\mathrm{Uf}(A)$.

(ii) Let F be a monotonic neighborhood frame. The *ultrafilter extension* $\mathrm{ue}\, F$ of F is $\mathrm{Uf}(F^+)$.

Proposition 2.2.4. Let A be a BAM.

(i) $\mathrm{Uf}(A)$ is monotonic.

(ii) $(\mathrm{Uf}(A))^+ = A^\sigma$. $\square$

Finally, we define a few notions necessary to state our Goldblatt-Thomason theorem.

Definition 2.2.15 ([43])**.** For a disjoint family $((F_i, N^i) \mid i \in I)$ of monotonic neighborhood frames, the *disjoint union* of the family is (F, N), where $F = \bigsqcup_i F_i$ and N is a neighborhood function defined by $U \in N(w) \iff U \cap F_i \in N^i(w)$. A monotonic neighborhood frame F is a *bounded morphic image* of another F' if there is a surjective bounded morphism $F' \twoheadrightarrow F$. A monotonic neighborhood frame F is a *generated subframe* of another F' if $F \subseteq F'$, and the inclusion map $F \hookrightarrow F'$ is a bounded morphism.

2.3 Model theory of neighborhood frames

In this section, we recall as well as develop results in the model theory of neighborhood frames and coalgebraic predicate logic.

Standard concepts in first-order model theory

Here, we define concepts that have counterparts in classical first-order model theory.

Definition 2.3.1. Let L be a language of coalgebraic predicate logic, F an L-structure, and $A \subseteq F$. A subset $X \subseteq F$ is A-*definable in* F if there is an L-formula $\phi(x; \bar{y})$ and a tuple $\bar{a}$ of elements of A (notation: $\bar{a} \in A$) such that $X = \phi(F; \bar{a})$. A subset X is *definable in* F if it is F-definable in F.

Definition 2.3.2.

(i) A set of L-sentences is called an L-*theory*.

(ii) Let L be a language of coalgebraic predicate logic and F be an L-structure. The *full* L-*theory* $\mathrm{Th}_L(F)$ of F is the set of L-sentences ϕ such that $F \models \phi$.

(iii) Two L-structures F, F' are *L-elementarily equivalent*, or $F \equiv_L F'$, if $\mathrm{Th}_L(F) = \mathrm{Th}_L(F')$.

For the rest of this section, we fix a language L of coalgebraic predicate logic and a monotonic L-structure F. We also let $T = \mathrm{Th}_L(F)$.

Definition 2.3.3. Let $A \subseteq F$. We write $\mathrm{Def}(F/A)$ for the Boolean algebra of A-definable subset in F, its operations being the set-theoretic ones. We also think of $\mathrm{Def}(F/A)$ as a BAM whose monotone operation $\Box$ is defined by

$$\Box(\phi(F)) = (\Box\phi)(F)$$

for an $L(A)$-formula $\phi(x)$, where $(\Box\phi)(x)$ is the L-formula $x \Box_y \phi(y)$. (It is easy to see that $\Box : \mathrm{Def}(F/A) \to \mathrm{Def}(F/A)$ is well defined here. This is true of similar definitions that appear in later parts of the article.)

It is easy to see that $\mathrm{Def}(F/A)$ is a subalgebra of F^+ as a BAM.

Proposition 2.3.1. Assume $F' \models T$. Then $\mathrm{Def}(F/\varnothing)$ and $\mathrm{Def}(F'/\varnothing)$ are isomorphic as BAMs.

Types play an important rôle in the proof of the original theorem of Fine as well as in this article.

Definition 2.3.4.

(i) The *Stone space* $S_1(T)$ *of* 1*-types over* $\varnothing$ *for* T is the ultrafilter frame $\mathrm{Uf}(\mathrm{Def}(F/\varnothing))$ of $\mathrm{Def}(F/\varnothing)$. (Note that if F' is such that $\mathrm{Th}_L(F') = T$, then $\mathrm{Uf}(\mathrm{Def}(F'/\varnothing)) = \mathrm{Uf}(\mathrm{Def}(F/\varnothing))$ and that $S_1(T)$ is, therefore, defined uniquely regardless of the choice of $F \models T$.) We consider $S_1(T)$ as a topological space whose open subsets are exactly the open elements of $(\mathrm{Uf}(\mathrm{Def}(F/\varnothing)))^+ = (\mathrm{Def}(F/\varnothing))^\sigma$. An element $p \in S_1(T)$ is called a 1*-type over* $\varnothing$.

(ii) Likewise, we let $S_1^F(A) = \mathrm{Uf}(\mathrm{Def}(F/A))$. An element $p \in S_1^F(A)$ is called a 1*-type over* A.

(iii) A set $\Sigma(x)$ of $L(A)$-formulas with one variable, say, x, is called a *partial* 1*-type over* A. We write $\Sigma(F)$ for the set $\{w \in F \mid \forall \phi \in \Sigma\, F \models \phi(w)\}$.

Convention 2.3.2. We identify a 1-type p over A with the partial 1-type

$$\{\phi(x; \bar{a}) \mid \phi(F; \bar{a}) \in p, \bar{a} \in A\}$$

over A. In fact, this is closer to how types are usually defined in classical model theory and is what types are in [60]. Likewise, we write $[\phi]$ for the clopen set $[X]$ in a Stone space of 1-types if ϕ defines X.

Given a partial type $\Sigma(x)$, the intersection $\bigcap_{\phi\in\Sigma}[\phi]$ is a closed set in the Stone space of 1-types.

Definition 2.3.5.

(i) A partial 1-type $\Sigma(x)$ over A is *deductively closed* if $[\phi] \supseteq \bigcap_{\psi\in\Sigma}[\psi] \implies \phi \in \Sigma$.

(ii) For a deductively closed partial 1-type $\Sigma(x)$, we write E_Σ for the closed set

$$\{p \mid p \supseteq \Sigma\} = \bigcap_{\phi\in\Sigma}[\phi].$$

Proposition 2.3.3. Let $w \in F$ and $A \subseteq F$. The family $\mathrm{tp}^F(w/A)$ of A-definable subsets of F containing w is an ultrafilter in $\mathrm{Def}(F/A)$ and thus a 1-type over A. □

Definition 2.3.6.

(i) Let $A \subseteq F$. An element $w \in F$ *realizes* $p \in S_1^F(A)$, or $w \models p$, if $\mathrm{tp}^F(w/A) = p$. The 1-type p is *realized in* F if there is $w \in F$ with $w \models p$.

(ii) The L-structure F is $\aleph_0$*-saturated* if for every finite $A \subseteq F$, every $p \in S_1^F(A)$ is realized in F.

Model theory specific to neighborhood frames

In this section, we study the model theory of neighborhood frames while we relate it to the classical model theory.

Definition 2.3.7. Let L be a language of coalgebraic predicate logic based on L_0 and F an L-structure. The *essential part* F^{e} of F is the L-structure whose reduct to L_0 is the same as that of F and whose neighborhood function N^{e} is defined by

$$U \in N^{\mathrm{e}}(w) \iff U \text{ is definable in } F \text{ and } U \in N^F(w)$$

for $w \in F^{\mathrm{e}}$.

Proposition 2.3.4 ([18])**.** Let L be a language of coalgebraic predicate logic and F, G be L-structures. Suppose $F^{\mathrm{e}} \cong G^{\mathrm{e}}$.

(i) $F \equiv_L G$.

(ii) If F is $\aleph_0$-saturated, so is G. □

We define a class of languages of first-order logic, one for each language of coalgebraic predicate logic.

Definition 2.3.8 ([44],[16, Definition 9])**.** Let L be an arbitrary language of coalgebraic predicate logic and L_0 the language of first-order logic on which L is based. We define the language L^2 to be the two-sorted first-order language whose sorts are the *state sort* and *neighborhood sort* and whose atomic formulas are those in L_0, recast as formulas in which constants and variables belong to the state sort, together with xNU and $x \in U$, where x and U are variables for the state sort and the neighborhood sort, respectively. (In general, we will use lowercase variables for the state sort and uppercase variables for the neighborhood sort.)

Definition 2.3.9. Let L be a language of coalgebraic predicate logic and F an L-structure. Given a family $\mathcal{S} \subseteq \mathscr{P}(F)$ that contains all definable subsets of F, we write $(F, \mathcal{S})$ for the L^2-structure G. The domain of the state sort of G is that of F, and the domain of the neighborhood sort of G is $\mathcal{S}$. The L^2-structure G interprets all nonlogical symbols of L^2 but N and $\in$ in the same way as F. Finally, we have $(w, U) \in N^G \iff U \in N^F(w)$ and $(w, U) \in \in^G \iff w \in U$. A family $\mathcal{S}$ is *large for* F if $U \in \mathcal{S}$ whenever there is $w \in F$ with $U \in N^F(w)$.

Proposition 2.3.5 ([18, 60])**.** Let L be a language of coalgebraic predicate logic. Let $(-)^2 : L \to L^2$ be the translation that commutes with Boolean combinations and satisfies

$$\begin{aligned}(\exists x\, \phi)^2 &= \exists x(\phi^2)\\ (x \,\Box_y\, \phi)^2 &= \exists U[\forall y(y \in U \leftrightarrow \phi^2(y)) \wedge xNU].\end{aligned}$$

Let $\mathcal{S} \subseteq \mathscr{P}(F)$ be a family that contains all definable subsets of F. Then for every L-formula ϕ and $\bar{a} \in F$ we have

$$F \models \phi(\bar{a}) \iff (F, \mathcal{S}) \models \phi^2(\bar{a}).$$

Remark. Note that the same two-sorted language L^2 is considered in [44] even though their transformation of neighborhood frames into L^2-structures there is different from ours. While in [44] a neighborhood frame F is always associated with the structure M for L^2 whose neighborhood sort consists of those subsets of F that are neighborhoods of some state of F, we do not impose such a restriction here. In addition, there is a third language for neighborhood frames used before as a model correspondence language [16, Definition 12] for neighborhood and topological semantics of modal logic and for the study of model theory of topological spaces [32] in general. This is also a fragment of the two-sorted language introduced above and, in fact, contains the image of the embedding of coalgebraic predicate logic into the two-sorted language [83].

Lemma 2.3.10. Let L be a language of coalgebraic predicate logic and F an L-structure. Let G be an L^2-structure that is an elementary extension of $(F, \mathscr{P}(F))$. There exists an L-structure G' whose domain is that of the state sort of G and a family $\mathcal{S} \subseteq \mathscr{P}(G')$ that satisfies the following:

(i) $\mathcal{S}$ contains all definable subsets in G'.

(ii) $\mathcal{S}$ is large for G'.

(iii) $G \cong (G', \mathcal{S})$.

Proof. Note that F satisfies extensionality:

$$(F, \mathscr{P}(F)) \models \forall U\, \forall V [\forall x(x \in U \leftrightarrow x \in V) \to U = V].$$

By L^2-elementarity, so does G. Let G', S^G be the domains of the state sort and the neighborhood sort of G, respectively. Let $i : S^G \to \mathscr{P}(G')$ be defined by

$$i(U) = \{w \in G' \mid G \models w \in U\}.$$

By the extensionality of G, i is injective. Let $\mathcal{S}$ be the range of i. Define the neighborhood function $N^{G'}$ by

$$i(U) \in N^{G'}(w) \iff G \models wNU.$$

Let $\phi(x; \bar{y})$ be an L-formula and $X := \phi(G', \bar{a})$ be a definable set in G', where $\bar{a} \in G'$. Note that the L^2-structure $(F, \mathscr{P}(F))$ satisfies comprehension:

$$(F, \mathscr{P}(F)) \models \forall \bar{y}\, \exists U\, \forall x(\phi^2(x; \bar{y}) \leftrightarrow x \in U).$$

So does G. Let U witness the satisfaction by G of the existential formula $\exists U\, \forall x(\phi^2(x; \bar{a}) \leftrightarrow x \in U)$. It can easily be seen that $i(U) = \phi(G', \bar{a})$.

It is easy to see that $\mathcal{S}$ is large for G' and that $G \cong (G', \mathcal{S})$. $\square$

Proposition 2.3.6. Let L be a language of coalgebraic predicate logic and F an L-structure. There exists an L-structure G such that $G \equiv_L F$ and that G is $\aleph_0$-saturated.[1]

Proof. Consider the L^2-structure $(F, \mathscr{P}(F))$, and take an elementary extension G_0 of $(F, \mathscr{P}(F))$ that is $\aleph_0$-saturated. By Lemma 2.3.10(iii), take an L-structure G and $\mathcal{S} \subseteq \mathscr{P}(G)$ with $G_0 \cong (G, \mathcal{S})$. Suppose that $A \subseteq G$ is finite. Let $p \in S_1^G(A)$ be arbitrary. Let Σ^2 be the partial type $\{\phi^2 \mid \phi \in p\}$ over A in L^2. Since p is a proper filter in $\mathrm{Def}(F/A)$, the type Σ^2 is consistent by Proposition 2.3.5. Thus, by the $\aleph_0$-saturation of G_0, we can take $w \in G_0$ realizing Σ^2. By Proposition 2.3.5, we have $w \models p$. $\square$

We now introduce the notion of quasi-ultraproducts as we will use it to give a proof of Fine's theorem at the end of this article.

[1] Our use of both coalgebraic predicate logic and first-order logic makes phrases such as "elementarily equivalent" and "$\aleph_0$-saturation" potentially ambiguous because we have two different classes of definitions, one from the previous subsection and the other standard in classical model theory. Note, however, that (expansions of) neighborhood frames are never structures of any language of first-order logic and that first-order structures are never L'-structures for any language L' of coalgebraic predicate logic. Hence, for example, if L' is a language of coalgebraic predicate logic, and F is an L'-structure, then whenever we say that F is $\aleph_0$-saturated, we mean what we stated in Definition 2.3.6(ii), with L in the definition being L'.

Definition 2.3.11 ([18, 60]). Let L be a language of coalgebraic predicate logic based on L_0 and $(F_i)_{i\in I}$ be a family of monotonic L-structures. Suppose that D is an ultrafilter over I. Let $\prod_D F_i$ be the ultraproduct of $(F_i)_i$ as L_0-structures modulo D. A subset $A \subseteq \prod_D F_i$ is *induced by* a family $(A_i)_{i\in J}$ if $J \in D$, $A_i \subseteq F_i$ for $i \in J$, and

$$a \in A \iff a(i) \in A_i \text{ for all } i \in J.$$

A *quasi-ultraproduct of* $(F_i)_i$ *modulo* D is a monotonic L-structure that is the L_0-structure $\prod_D F_i$ equipped with a neighborhood function N that satisfies

$$A \in N(w) \iff A_i \in N^i(w(i)) \text{ for all } i \in J,$$

whenever $w \in \prod_D F_i$, and A is induced by $(A_i)_{i\in J}$. A class $\mathcal{K}$ of monotonic neighborhood frames *admits quasi-ultraproducts* if whenever $(F_i)_i$ is a family of neighborhood frames from $\mathcal{K}$, a quasi-ultraproduct of $(F_i)_i$ exists in $\mathcal{K}$.

Proposition 2.3.7 ([60, 18]).

1. Each class of the classes in Table 2.1 admits quasi-ultraproducts.

2. Let $(F_i)_{i\in I}$ be a family of monotonic L-structures for a language L of coalgebraic predicate logic. If F_i satisfies a theory T for all $i \in I$, so does a quasi-ultraproduct of $(F_i)_i$.

Proof.

1. By Remark 2.4, it suffices to prove this for the class of monotonic neighborhood frames, the class of quasi-filter frames. This could be done by using the machinery introduced in Litak et al. [60], but it is easy to prove it directly in the following way.

   Let $\mathcal{K}_0$ be either the class of monotonic neighborhood frames or the class of quasi-filter neighborhood frames. Let $(F_i)_i$ be a family of neighborhood frames in $\mathcal{K}_0$. Let N^i be the neighborhood function of F_i. Define the neighborhood function N on $\prod_D F_i$ as follows: a subset $U \subseteq \prod_D F_i$ is in $N(w)$ if and only if there is a set $A \subseteq U$ induced by $(A_i)_{i\in J}$ with $A_i \in N^i(w(i))$ for all $i \in J$. It is easy to see that this indeed defines a quasi-ultraproduct and that if each F_i is in $\mathcal{K}_0$ then so is the quasi-ultraproduct.

2. The usual argument by induction works; see Litak et al. [60]. □

2.4 Proof of the main lemmas

In this section we prove the main lemmas of this article. Recall that $L_=$ is the language of coalgebraic predicate logic based on the empty language of first-order logic.

Definition 2.4.1. Let $\mathcal{K}_0$ be a class of monotonic neighborhood frames. A class $\mathcal{K}$ of monotonic neighborhood frames is *CPL-elementary relative to* $\mathcal{K}_0$ if there is an $L_=$-theory T with

$$\mathcal{K} = \{F \in \mathcal{K}_0 \mid F \models T\}.$$

Two monotonic neighborhood frames F and F' are *CPL-elementarily equivalent relative to* $\mathcal{K}_0$ if $F, F' \in \mathcal{K}_0$ and $\mathrm{Th}_{L_=}(F) = \mathrm{Th}_{L_=}(F')$.

Remark. The class of filter frames is CPL-elementary relative to the class of quasi-filter frames (see Definition 2.4.1), and the class of augmented filter frames is CPL-elementary relative to the class of augmented quasi-filter frames; indeed, they are both defined by the same $L_=$-sentence $\forall x\, x \,\Box_y\, y = y$. Furthermore, by the second paragraph of Example 2.2.5, the class of augmented quasi-filter frames is CPL-elementary relative to the class of monotonic frames. Therefore, the main lemma in this section concerns the classes of monotonic and quasi-filter neighborhood frames, respectively, which suffice for the purpose of the main results (Theorems 2.5.1 and 2.5.2), which deal with any of the classes in Table 2.1.

Lemma 2.4.2. Let F be a monotonic neighborhood frame, and let G and G' be $(L_=)^2$- and $L_=$- structures, respectively, obtained by elementarily extending F as in Lemma 2.3.10.

(i) If F is monotonic, $X, Y \subseteq G'$ are definable, $X \subseteq Y$, and $X \in N^{G'}(w)$ for $w \in G'$, then $Y \in N^{G'}(w)$.

(ii) If F is an augmented filter frame, then for every $w \in G'$ either $N^{G'}(w)$ is empty or has a minimum element.

Proof. For (i), let $L(G')$-formulas $\phi(x; \bar{a})$ and $\psi(x; \bar{b})$ define X and Y, respectively. Since F is monotonic, we have

$$\begin{aligned}(F, \mathscr{P}(F)) \models \forall \bar{y}\, \forall \bar{z}\, \forall v [\, &\forall x(\phi(x; \bar{y}) \to \psi(x; \bar{z})) \\ &\wedge v \,\Box_x\, \phi(x; \bar{y}) \to v \,\Box_x\, \psi(x; \bar{z})].\end{aligned} \tag{2.5}$$

Since $(F, \mathscr{P}(F))$ e satisfies the $(-)^2$-translation of the right-hand side of the displayed formula (2.5) by Proposition 2.3.5, so does G. Again by Proposition 2.3.5,

$$G' \models \forall x(\phi(x; \bar{a}) \to \psi(x; \bar{b})) \wedge w \,\Box_x\, \phi(x; \bar{a}) \to w \,\Box_x\, \psi(x; \bar{b}).$$

Since $X \in N^{G'}(w)$, we have $\psi(G', \bar{b}) \in N^{G'}(w)$.

For (ii), first observe that the L^2-structure $(F, \mathscr{P}(F))$ satisfies the sentence

$$\forall x[\neg \exists U\, xNU \vee \exists U_0\, \forall U(xNU \to U_0 \subseteq U)],$$

where $\subseteq$ is an abbreviation of the obvious L^2-formula. Since G' satisfies the same L^2-formula, the claim follows. $\square$

We are now ready to prove the key lemmas used in the proof of our main result. Our lemmas are analogous to [3, 8.9 Theorem].

Lemma 2.4.3. Let F be a monotonic neighborhood frame. There exists $G \equiv_{L_=} F$ such that there is a surjective bounded morphism $f : G \twoheadrightarrow \mathrm{ue}\, F$. Moreover, if $\mathcal{K}_0$ is either the class of monotonic neighborhood frames or the class of quasi-filter neighborhood frames, and $F \in \mathcal{K}_0$, then we can take $G \in \mathcal{K}_0$.

The following is the outline of the proof, which comes after this paragraph. We follow the classical proof of [3, 8.9 Theorem] by taking an expansion L of the correspondence language so that every subset of the given frame F will be definable and taking an $\aleph_0$-saturated extension G in that language. However, we need to add more neighborhoods to the neighborhood frame G that is being constructed to make sure that the map from G to the ultrafilter frame of F is a bounded morphism. Much of the proof is dedicated to showing that this construction preserves elementary equivalence in L.

Proof. Let L be the language of coalgebraic predicate logic based on $\{P_S \mid S \subseteq F\}$, the unary predicates for the subsets of F. The neighborhood frame F can be made into an L-structure naturally. Let $G_0 \equiv_L F$ be an $\aleph_0$-saturated L-structure as obtained by Proposition 2.3.6. Let G_1 be the essential part of G_0. Let G_2 be the L-structure obtained from G_1 as follows: for each state $w \in G_1$, add as a neighborhood of w the set $\Sigma(G_1)$, where $\Sigma(x)$ is a partial type over a finite set $A \subseteq G_1$ such that $\Sigma(x)$ is deductively closed and that for every $\phi \in \Sigma$ we have $\phi(G_1) \in N^{G_1}(w)$. We call such a partial type *good at w*. Let G be the L-structure identical to G_2 except that its neighborhood function N^G is defined by $U \in N^G(w) \iff \exists U_0 \subseteq U\, U_0 \in N^{G_2}(w)$.

Note that a singleton partial type $\Sigma = \{\phi\}$ with $\phi(x) \in L(A)$ is always good at $w \in G_1$ if $\phi(G_1) \in N^{G_1}(w)$.

We show that $G \equiv_L F$. By Proposition 2.3.4, we have $G_1 \equiv_L G_0 \equiv_L F$, so it suffices to see that for every definable $X \subseteq G$ we have $X \in N^G(w) \iff X \in N^{G_1}(w)$. We show $\implies$ (the other direction is easy). By construction, there is either a definable set $Y \subseteq X$ with $Y \in N^{G_1}(w)$ or a partial type $\Sigma(x)$ over a finite set A good at w with $\Sigma(G_1) \subseteq X$. The former is a special case of the latter, so we assume the latter. Let A' be a finite set containing A and the parameters used in the definition of X. Let $f' : G_1 \twoheadrightarrow S_1^{G_1}(A')$ be defined by $f'(w) = \mathrm{tp}^{G_1}(w/A')$. By $\aleph_0$-saturation, f' is a surjection. We show that $f'(\Sigma(G_1)) = E_\Sigma \subseteq S_1^{G_1}(A')$. It is easy to show that $f'(\Sigma(G_1)) \subseteq E_\Sigma$; we show $f'(\Sigma(G_1)) \supseteq E_\Sigma$. Let $p \in E_\Sigma$ be arbitrary. By $\aleph_0$-saturation, take $w \in G_1$ with $f'(w) = p$. Since $p \supseteq \Sigma$, $w \in \Sigma(G_1)$. We have shown that $f'(\Sigma(G_1)) = E_\Sigma$. That $f'(X) = [X]$ easily follows from the $\aleph_0$-saturation of G as well. We have $E_\Sigma \subseteq [X]$. By the compactness of $S_1^{G_1}(A')$, we have a finite $\Sigma_0 \subseteq \Sigma$ for which $E_{\Sigma_0} \subseteq [X]$. Being the intersection of finitely many clopen sets,

$$E_{\Sigma_0} = \bigcap_{\phi \in \Sigma_0} [\phi] = \left[\bigwedge \Sigma_0\right]$$

is clopen. Since Σ is good at w, we have $(\bigwedge \Sigma_0)(G_1) \in N^{G_1}(w)$. We conclude that $X \in N^{G_1}(w)$ by Lemma 2.3.10 (i). (See Remark 2.4 for an alternate proof of this fact.)

Since $F^+ \cong \mathrm{Def}(F/\varnothing)$, we have ue $F \cong S_1(T)$, where T is the full L-theory of F, which is identical to $\mathrm{Th}_L(G)$. We show $f : G \twoheadrightarrow S_1(T)$ defined by $f(w) = \mathrm{tp}^G(w/\varnothing)$, which is surjective by $\aleph_0$-saturation, is a bounded morphism. In the rest of the proof, we write N^σ for the neighborhood function of $S_1(T)$.

The "forth" condition. Suppose that $U \in N^G(w)$. We show that $f(U) \in N^\sigma(\mathrm{tp}^G(w))$. By construction, we have either (I) $U \supseteq \phi(G, \bar{a}) \in N^G(w)$ or (II) $U \supseteq \Sigma(G) \in N^G(w)$, where $\phi(x, \bar{y})$ is an L-formula, $\bar{a} \in G$, and $\Sigma(x)$ is a partial type over a finite set A good at w. Since (I) is a special case of (II), we will just show (II).

For (II), assume that $U \supseteq \Sigma(G) \in N^G(w)$, where Σ is a partial 1-type over finite A good at w. Let $K = r(E_\Sigma)$, where $r : S_1^G(A) \to S_1(T)$ is the closed continuous map dual to the embedding $\mathrm{Def}(G/\varnothing) \hookrightarrow \mathrm{Def}(G/A)$. Note that $r(q) = q \cap \mathrm{Def}(G/\varnothing)$ for $q \in S_1^G(A)$. Being the image of a closed map of a closed set, K is closed. Recall the equation (2.4) that defines N^σ to see that it suffices to show (i) that for every $\chi(x) \in L$ we have $[\chi] \supseteq K \implies \chi(G) \in N^G(w)$ and (ii) that $K \subseteq f(U)$. For (i), assume that $[\chi] \supseteq r(E_\Sigma)$, where $\chi(x) \in L$, and $[\chi]$ denotes a subset in $S_1(T)$. Take an arbitrary $q \in E_\Sigma$. Then $r(q) \in r(E_\Sigma) \subseteq [\chi]$, so $\chi \in r(q) \subseteq q$. We have just shown that $[\chi] \supseteq E_\Sigma$, where $[\chi]$ denotes a subset in $S_1^G(A)$. By deductive closure $\chi \in \Sigma$. By construction, $\chi(G) \in N^G(w)$. For (ii), it suffices to show that arbitrary $q \in E_\Sigma$ can be realized by an element of U. Since q is a type over a finite set, by $\aleph_0$-saturation, we may take $v \models q$; this means $v \models \Sigma$, i.e., $v \in \Sigma(G) \subseteq U$.

The "back" condition. Suppose that $U' \subseteq S_1(T)$ is in $N^\sigma(\mathrm{tp}^G(w/\varnothing))$. We show that there is $U \subseteq G$ in $N^G(w)$ such that $f(U) \subseteq U'$. By the definition of N^σ, there is a partial type $\Sigma(x)$ over $\varnothing$ good at w such that $E_\Sigma \subseteq U'$. By construction, $\Sigma(G) \in N^G(w)$. Let $U := \Sigma(G)$. Then for every $v \in U$, the type $\mathrm{tp}^G(v/\varnothing)$ extends Σ and thus is in $E_\Sigma \subseteq U'$.

Closure in relatively CPL-elementary classes. By construction, G is monotonic.

Suppose that F is a quasi-filter neighborhood frame. Let $w \in G$ and $U, U' \in N^G(w)$ be arbitrary. By construction, there are deductively closed partial types $\Sigma(x), \Sigma'(x)$ over a finite set of parameters both of which are good at w with $\Sigma(G) \subseteq U$ and $\Sigma'(G) \subseteq U'$. The partial type $\Sigma \cup \Sigma'$ is also over a finite set, good at w. Moreover, $\Sigma \cup \Sigma'$ is deductively closed since F is a quasi-filter frame. Therefore, we have $(\Sigma \cup \Sigma')(G) = \Sigma(G) \cap \Sigma(G) \subseteq U \cap U'$, so $U \cap U' \in N^G(w)$. We have seen that G is a quasi-filter neighborhood frame. □

Remark. In the proof above, we obtain G not only by compactness but also by altering the neighborhoods in an ad-hoc way while maintaining elementary equivalence in $L_=$. There is no reason for us to believe that G has the same theory as F in ${L_=}^2$ or in the languages described in Remark 2.3. This is why we find it difficult to extend our main result to the more expressive languages.

The following is the alternate proof that I announced at the end of the third paragraph of the proof (the concepts that we have not defined have obvious definitions): Suppose X is definable by $\psi(x; A')$ where $\psi \in L$ and $A' \subseteq G$ is a finite set. By $\aleph_0$-saturation of G_1, we have $\mathrm{Th}_{L(A')}(G_1) \cup \Sigma(x) \models \psi(x, A')$ (otherwise, realize the type $\Sigma(x) \cup \{\neg\psi(x, A')\}$ by some element in G_1, which would be in $\Sigma(G_1) \setminus X$.) By compactness, there is finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0(G_1) \subseteq \psi(G_1, A')$. Since $\bigwedge \Sigma_0(x)$ is a single formula of L, by deductive closure $\bigwedge \Sigma_0(x) \in \Sigma(x)$. Hence $\bigwedge \Sigma_0(G_1) \in N^{G_1}(w)$. By Lemma 2.3.10(i), we have $X = \psi(G_1, A') \in N^{G_1}(w)$ as desired.

2.5 Applications of the main lemmas

The Goldblatt-Thomason Theorem

An algebraic argument essentially the same as the classical counterpart can be used to show that a class of monotonic neighborhood frames closed under ultrafilter extensions is modally definable if and only if it is closed under bounded morphic images, generated subframes, and disjoint unions, and it reflects ultrafilter extensions [58][43, Theorem 7.23]. By applying Lemma 2.4.3, we obtain the following theorem.

Theorem 2.5.1. Let $\mathcal{K}$ be a class of monotonic neighborhood frames that is closed under CPL-elementary equivalence relative to any of the classes in Table 2.1. $\mathcal{K}$ is modally definable if and only if it is closed under bounded morphic images, generated subframes, and disjoint unions, and it reflects ultrafilter extensions.

Proof. Let $\mathcal{K}$ be a class of monotonic neighborhood frames that is closed under CPL-elementary equivalence relative to a class $\mathcal{K}_0$ in Table 2.1. We show the "if" case. Suppose that $\mathcal{K}$ is closed under bounded morphic images, generated subframes, and disjoint unions, and it reflects ultrafilter extensions. Apply Lemma 2.4.3 and Remark 2.4 to conclude that $\mathcal{K}$ is closed under ultrafilter extensions. Note that the hypothesis of [43, Theorem 7.23] is satisfied, and we conclude that $\mathcal{K}$ is modally definable. □

Example 2.5.1. As an example, we show that the image $\mathcal{K}$ under * of the class of discrete topological spaces is modally definable. For a quasi-filter frame F, F is a *-image of a discrete topological space if and only if $F \models \forall x\, \neg x \,\Box_z\, y \neq x$ and $F \models \forall x\, x \,\Box_y\, y = x$. Hence, $\mathcal{K}$ is CPL-elementary relative to the class of quasi-filter frames, and the Goldblatt-Thomason Theorem is applicable to $\mathcal{K}$. It is easy to check that $\mathcal{K}$ is closed under bounded morphic images, generated subframes, and disjoint unions, so it suffices to show that $\mathcal{K}$ reflects ultrafilter extensions. Assume that for a neighborhood frame $F = (F, N)$ its ultrafilter extension $\mathrm{ue}\, F = (\mathrm{ue}\, F, N^\sigma)$ is in $\mathcal{K}$. We show that $F \in \mathcal{K}$. The class of topological frames is defined by modal formulas $\Box p \wedge \Box q \to \Box(p \wedge q)$, $\Box p \to p$, and $\Box\Box p \to \Box p$ [9], so we may assume that F is topological as ultrafilter extensions reflect modally definable classes. Let $w \in F$ be arbitrary, and let u be the principal ultrafilter generated by w, so $u \in \mathrm{ue}\, F$.

Note that $U \in N^\sigma(u) \iff u \in U$ since $\mathrm{ue}\, F$ is the *-image of a discrete space. Recall the definition of N^σ in (2.4). The singleton $\{u\}$ is in $N^\sigma(u)$, and this has to be witnessed by $K = \varnothing$ or $K = \{u\}$ according to (2.4) of Definition 2.2.14.(i). Suppose $K = \varnothing$. Then (2.4) implies that $\Box^{F^+}\varnothing \in u$ among other things (recall that A in (2.4) is F^+ here). However, since F is topological, $\Box^{F^+}\varnothing = \varnothing$, and it cannot belong to an ultrafilter u. Hence, $K = \{u\}$. Again by (2.4), for all $a \subseteq F$ such that $[a] \supseteq K = \{u\}$, i.e., $a \in u$, we have that $\Box^{F^+}a \in u$. Let $a = \{w\}$, so $a \in u$. Since the set u is an ultrafilter, we have $a \wedge \Box^{F^+}a \neq \varnothing$, that is, $a \cap \{w \in F \mid a \in N(w)\} \neq \varnothing$; this implies $\{w\} \in N(w)$. Since w was arbitrary, we conclude that $F \in \mathcal{K}$. We have shown that $\mathcal{K}$ is modally definable; in fact, it is defined by $p \to \Box p$ in addition to the definition of the class of topological neighborhood frames.

Fine's Canonicity Theorem

By the dual equivalence between monotonic modal logics and varieties of BAMs [43, Chapter 7], we will state our version of Fine's Canonicity Theorem in an algebraic manner. Our presentation of the proof of the theorem is modeled after that of the classical version of the theorem in [78].

For a class $\mathcal{K}$ of neighborhood frames, we write $\mathcal{K}^+$ for the class $\{F^+ \mid F \in \mathcal{K}\}$.

Lemma 2.5.2. Let $\mathcal{K}$ be a class CPL-elementary relative to any of the classes in Table 2.1. Let $\mathcal{S} \supseteq \mathcal{K}^+$ be the least class of BAMs closed under subalgebras.

1. $\mathcal{S}$ is closed under canonical extensions.

2. $\mathcal{S}$ is closed under ultraproducts.

Proof.

1. Let $A \in \mathcal{S}$. For some $F \in \mathcal{K}$ we have $A \hookrightarrow F^+$. By duality theory [33, Theorem 5.4], we have $A^\sigma \hookrightarrow (F^+)^\sigma$. By Lemma 2.4.3 and Remark 2.4, there is $G \in \mathcal{K}$ with $(F^+)^\sigma \hookrightarrow G^+$. Thus, we have $A^\sigma \in \mathcal{S}$ by definition.

2. It suffices to do the following: given an ultraproduct $\prod_D F_i^+$ where I is an index set, D is an ultrafilter over I, and $(F_i)_i$ is a family of neighborhood frames in $\mathcal{K}$, we show that the ultraproduct embeds into $(\prod_D F_i)^+$, where $\prod_D F_i$ is a quasi-ultraproduct of $(F_i)_i$ modulo D. In fact, we show that $\iota : \prod_D F_i^+ \to (\prod_D F_i)^+$ defined by

$$s \in \iota(a) \iff \{i \mid s(i) \in a(i)\} \in D,$$

where $s \in \prod_D F_i$ and $a \in \prod_D F_i^+$ is a BAM embedding (we do not write equivalence classes modulo D explicitly; it is easy to see that ι is well defined). It can easily be seen that ι is a Boolean algebra embedding. We show that $\iota \circ \Box^{\mathrm{pu}} = \Box^{\mathrm{cm}} \circ \iota$, where $\Box^{\mathrm{pu}}$ and $\Box^{\mathrm{cm}}$ are the operations of the domain and the target of ι, respectively. Let N be the neighborhood function of the quasi-ultraproduct. We write $\Box^i$ and N^i for the operation of F_i^+. Note that for all a the set $\iota(a)$ is an induced subset of the quasi-ultraproduct; if we let $\pi_i(A)$ be the projection of an induced subset A of the quasi-ultraproduct onto the coordinate i, then $\{i \mid \pi_i(\iota(a)) = a(i)\} \in D$. We now have

$$\begin{aligned} s \in (\iota \circ \Box^{\mathrm{pu}})(a) &\iff \{i \mid s(i) \in (\Box^{\mathrm{pu}}(a))(i)\} \in D \\ &\iff \{i \mid s(i) \in \Box^i(a(i))\} \in D \qquad (*) \\ &\iff \{i \mid s(i) \in \Box^i(\pi_i(\iota(a)))\} \in D \\ &\iff \{i \mid \iota(a) \in N^i(s(i))\} \in D \\ &\iff \iota(a) \in N(s) \\ &\iff s \in (\Box^{\mathrm{cm}} \circ i)(a), \end{aligned}$$

where we have the equivalence (*) since

$$\{i \mid (\Box^{\mathrm{pu}}(a))(i) = \Box^i(a(i))\} \in D. \qquad \Box$$

Theorem 2.5.2. Let $\mathcal{K}$ be a class CPL-elementary relative to any of the classes in Table 2.1. The variety of BAMs generated by $\mathcal{K}^+$ is canonical, i.e., closed under canonical extensions.

Proof. Recall Remark 2.4. Gehrke and Harding [33] showed that if $\mathcal{S}$ is a class of BAMs closed under ultraproducts and canonical extensions, then $\mathcal{S}$ generates a canonical variety. Apply this result for the class $\mathcal{S}$ in Lemma 2.5.2 to conclude that the variety generated by $\mathcal{K}^+$, which is identical to the variety generated by $\mathcal{S}$, is canonical. $\Box$

Note that Fine's original theorem follows as a special case concerning the classes of augmented neighborhood frames.

Example 2.5.3. Consider the B axiom $p \to \Box\neg\Box\neg p$, which we considered in Example 2.2.4. Recall that it defined a CPL-elementary class $\mathcal{K}$ relative to the class of monotonic neighborhood frames. By [43, Proposition 6.5], the variety $\mathcal{V}$ defined by the B axiom is canonical and hence generated by $\mathcal{K}^+$. By Theorem 2.5.2, the canonicity of $\mathcal{V}$ is explained by the CPL-elementarity of $\mathcal{K}$.

Remark. By Remark 2.2, Theorem 2.5.2 can be used to show the canonicity of the monotonic modal logic axiomatized by any formula of the form (2.1).

2.6 Open questions

As we mentioned in Remarks 2.3 and 2.4, one could attempt to use a different notion of elementarity in stating and proving the results of this article, but we stuck to coalgebraic predicate logic due to the limitation of the proof technique we used. A natural question to ask here would be whether there is a more expressive first-order-like logic that admits similar results possibly by a different kind of proof. Another question would be to characterize classes of monotonic neighborhood frames that admit analogues of the Goldblatt-Thomason theorem and Fine's theorem in the same sense as in the main result of this article. This question leads to another problem of showing results similar to ours for other coalgebras than those discussed in this article.

It was suggested to the author that our version of Fine's theorem could be proved by using an algebraic result [34], which implies the original, Kripke-semantic version of the theorem. The argument proposed contained a gap, and therefore it remains open whether the results in this article follow from the aforementioned algebraic theorem. Even if they can indeed be proved in that manner, we hope that the proof presented here serves our original purpose of investigating the role of coalgebraic predicate logic in the study of monotonic modal logics, especially in the spirit of van Benthem's program [8] of re-analyzing algebraic arguments occurring in modal logic from a model-theoretic perspective.

Chapter 3

Adventures on Heyting Algebras

This chapter consists of two independent parts.

In the first part, we examine countable ultrahomogeneous existentially closed (e.c.) Heyting algebras. The existence of model-completion T^* of the theory T of Heyting algebra [37] is of interest in its relation to second-order intuitionistic propositional logic. Countable ultrahomogeneous Heyting algebras are paradigmatic models of T^*, one of them being its prime model.

In the first section, we see that there are uncountably many countable ultrahomogeneous e.c. Heyting algebras. The remainder of the first part concerns the prime model L of T^*. In the second section, we study the countable atomless Boolean algebra definable in L. In the third section, we look at the automorphism group of L with the Kechris-Pestov-Todorcević correspondence in mind, where it will be proved *inter alia* that $\mathrm{Aut}(L)$ is not amenable. In the last section of this part, we study issues related to the axiomatization of T^*.

It is an important future task to investigate the combinatorics of the age $\mathrm{Age}(L)$ of L, in particular about the existence of order expansion of $\mathrm{Age}(L)$ with the Ramsey property and the ordering property, and the metrizability of $\mathrm{Aut}(L)$.

The second part of this chapter on Heyting algebras concerns Beth semantics for intuitionistic logic and nuclei on locales.

Preliminaries

Let T be the theory of Heyting algebras. The model completion T^* of T exists. It is axiomatized by

$$T \cup \{U(\theta' \to \theta) \mid \theta \text{ existential}\} \tag{3.1}$$

where U denotes universal closure, θ' is a quantifier-free formula such that $T \models U(\theta \to \theta')$, and $T + U(\theta' \to \theta)$ is a conservative extension with respect to the universal formulas (θ' is the result of applying the QE algorithm in [37] to θ).

We will study an ultrahomogeneous model L of T^* later in this part. Neither T nor T^* is locally finite, but L is. However, L is not uniformly locally finite, so T^* is not $\aleph_0$-categorical. T^* is not uncountably categorical either because of its instability. The Fraïssé limit L is the prime model of T^*. In many cases, the Fraïssé limit of a class of finite structures is pseudofinite. However, this is not the case for the complete theory $T^* + (0 \neq 1)$; there is a sentence ϕ implied by the theory that is not satisfied by any finite structure of the same signature. Indeed, take ϕ to be the conjunction of the density of the partial order (see Ghilardi and Zawadowski [37, Proposition 4.28]), $0 \neq 1$, and $\bigwedge T$.

We review an important construction of Heyting algebras (this material appears in, e.g., Chagrov and Zakharyaschev [17]). For an arbitrary poset $\mathbb{P}$, the poset of upward closed sets, or *up-sets*, of $\mathbb{P}$ ordered by inclusion has a Heyting algebra structure. We call this Heyting algebra is *dual* of $\mathbb{P}$. Conversely, if L is a *finite* Heyting algebra, then one can associate with L the poset $\mathbb{P}$ of join-prime elements of L with the reversed order. One can show that the dual of $\mathbb{P}$ is isomorphic to L.

Suppose that L and L' are the duals of $\mathbb{P}$ and $\mathbb{P}'$, respectively, and that $f : \mathbb{P} \to \mathbb{P}'$ is *bounded morphic*, i.e., f is monotonic with

$$\forall u \in \mathbb{P}\, \forall v \geq f(u)\, \exists w \geq u\, f(w) = u,$$

then the function f^* defined on L' that maps each up-set with its inverse image under f is a Heyting algebra homomorphism $L' \to L$. We call f^* the *dual* of f as well. If f is injective, then f^* is surjective; if f is surjective, then f^* is a Heyting algebra embedding.

3.1 Countable Ultrahomogeneous Heyting algebras

The model completion T^* is the theory of the Fraïssé limit L of finite nontrivial Heyting algebras, which exists [37]. The amalgamation property of T was proved by Maksimova [63]; in fact, her construction establishes the strong amalgamation property for the class of finite Heyting algebras. We introduce notation naming structures obtained by the strong amalgamation property: Let D be the diagram $B \hookleftarrow A \hookrightarrow C$ in $\mathrm{Age}(L)$, where $\mathrm{Age}(L)$ the age of L is regarded as a category whose morphisms are the embeddings. The strong amalgamation property for $\mathrm{Age}(L)$ gives rise to a subalgebra $\bigsqcup D$ of L such that there are embeddings $\iota^D_{\hookleftarrow} : B \hookrightarrow \bigsqcup D$ and $\iota^D_{\hookrightarrow} : C \hookrightarrow \bigsqcup D$ with $\iota^D_{\hookleftarrow}(B) \setminus \iota^D_{\hookleftarrow}(A)$ and $\iota^D_{\hookrightarrow}(C) \setminus \iota^D_{\hookrightarrow}(A)$ disjoint.

The following is a model-theoretic argument that L is e.c.:

Proof. Consider a quantifier-free formula $\phi_0(x, y)$ and a tuple $\overline{a} \in L$. Note that $\langle \overline{a} \rangle^L$ is finite by the construction of L, so there is a quantifier-free formula $\psi(\overline{y})$ such that for any Heyting algebra L'' and $b \in L$, we have

$$L'' \models \psi(\overline{b}) \iff \langle \overline{b} \rangle^{L''} \cong \langle \overline{a} \rangle^L.$$

Now suppose that there is $L' \supset L$ such that $L' \models \exists x\, \phi_0(x, \overline{a})$. This implies the formula

$$\exists x\, \exists \overline{y} [\phi_0(x, y) \wedge \psi(\overline{y})] \tag{3.2}$$

is satisfiable over T. By the extended form of the finite model property for T that works for equations as well as inequations [25], there is a finite Heyting algebra L_0 satisfying (3.2). By construction, without loss of generality $L_0 \subseteq L$. Let $\xi, \overline{b} \in L_0$ be the witness to $\exists x$, $\exists \overline{y}$, resp. The isomorphism $\langle \overline{b} \rangle^L \to \langle \overline{a} \rangle^L$ induces another $i : L \to L$ by ultrahomogeneity. It follows that $i(\xi)$ solves the formula $\phi_0(x, \overline{a})$ in L. □

Fact 3.1.1. There are continuum many finitely generated Heyting algebras up to isomorphism.

This fact is probably well known, but a proof is included for the sake of completeness.

Proof. Regard the chain $\omega + 1$ as a Heyting algebra. Note that every order-preserving injection $\omega + 1 \to \omega + 1$ is a Heyting algebra embedding. Since such functions are in bijective correspondence with strings in ω^ω, there are continuum many of them. Now consider the free Heyting algebra F with one generator. Construct an embedding $\iota : (\omega + 1) \to F$ recursively as follows: let $\iota(0) = 0$ and $\iota(\omega) = 1$; having defined $\iota(n)$ for $n < \omega$, define $\iota(n + 1)$ to be the join of the two successors of $\iota(n)$. It can be checked directly that ι is a Heyting algebra embedding. By Abogatma and Truss [1, Theorem 2.4], we conclude that there are continuum many finitely generated Heyting algebras up to isomorphism. □

Proposition 3.1.1. Let $\mathcal{K}$ be an inductive class of finitely generated structures with the amalgamation property, and let $A \in \mathcal{K}$. There exists an ultrahomogeneous structure $A^\sharp \in \mathcal{K}$ that is existentially closed in $\mathcal{K}$ and extends A.

Proof. We construct $A^\sharp$ as the union of an ω-chain $A_0 \subseteq A_1 \subseteq \cdots$ of structures in $\mathcal{K}$. Let $A_0 = A$. Fix a bijection $\pi : \omega \times \omega \to \omega$ such that $\pi(i, k) < i$ for $i, k < \omega$.

Having A_i constructed, we extend A_i to A_{i+1} as follows:

Case $i = 2i'$ Apply the well-known construction to A_i to obtain A_{i+1} so that $A_{i+1} \models \phi(\bar{a})$ whenever $\phi(\bar{x})$ is an existential formula, $\bar{a}$ is in A_i, and there exists $C \in \mathcal{K}$ such that $A_i \subseteq C$ and that $C \models \phi(\bar{a})$.

Case $i = 2i' + 1$ We do the construction in the proof of Abogatma and Truss [1, Lemma 2.3], which is included for the sake of completeness. There are at most countably many partial isomorphisms of A_i, i.e., isomorphisms between substructures of A_i; enumerate them as $(\varphi_{ik})_{k<\omega}$. Take (j, k) such that $\pi(j, k) = i'$. Let A'_{i+1} be the structure in $\mathcal{K}$ witnessing the amalgamation property for the diagram

$$A_i \overset{\iota_1}{\hookleftarrow} \operatorname{dom} \varphi_{jk} \overset{\iota_2 \circ \varphi_{jk}}{\hookrightarrow} A_i,$$

where ι_1, ι_2 are the inclusion maps of the correct types. Replace A_{i+1} with an isomorphic copy if need be so that $A_i \subseteq A_{i+1}$. Note that φ_{jk} is extended to a partial isomorphism $\tilde{\varphi}_{jk}$ of A'_{i+1}, where $\operatorname{dom} \tilde{\varphi}'_{jk} = A_i$. One can use a similar construction to obtain A_{i+1} with a partial isomorphism $\tilde{\varphi}_{jk}$ extending $\tilde{\varphi}'_{jk}$ such that $A_i \subseteq \operatorname{ran} \tilde{\varphi}_{jk}$.

That $A^\sharp$ is e.c. in $\mathcal{K}$ can be proved as usual. Let $\varphi : B \to C$ be an isomorphism where B, C are finitely generated substructures of $A^\sharp$. Let $j < \omega$ be such that the finitely many generators of B and C are contained in A_j; in fact, we have $B, C \subseteq A_j$. Take $k < \omega$ so that $\varphi = \varphi_{jk}$. By the construction of A_{i+1} from A_i, where $i = 2\pi(j,k) + 1$, φ is extended by a partial automorphism $\tilde{\varphi}_{jk}$ of $A^\sharp$, where $\operatorname{dom} \tilde{\varphi}_{jk} \cap \operatorname{ran} \tilde{\varphi}_{jk} \supseteq A_i$. Note that $i > j$. By repeating this, one obtains a chain $\varphi = \varphi_0 \subsetneq \varphi_1 \subsetneq \cdots$, where $\operatorname{dom} \varphi_m < \operatorname{dom} \varphi_n$ whenever $m < n$, so the union $\bigcup_{n<\omega} \varphi$ has the domain $A^\sharp$, which is evidently an isomorphism $A^\sharp \to A^\sharp$. We have seen that $A^\sharp$ is ultrahomogeneous. □

Corollary 3.1.2. There are continuum many countable ultrahomogeneous e.c. Heyting algebras.

Proof. This follows immediately from the preceding propositions as a single countable ultrahomogeneous e.c. Heyting algebra has at most countable substructures up to isomorphism. □

3.2 Definable Countable Atomless Boolean Algebras

In the next section where we study the topological group of automorphisms of L, first-order interpretations of B in L would be useful. Of course, the countable atomless Boolean algebra embeds in L by the weak homogeneity of L. However:

Proposition 3.2.1. No substructure of L that is a countable atomless Boolean algebra is a relativized reduct.

Proof. We show that for any countable atomless Boolean algebra $B \subseteq L$ there are an automorphism σ of L over $\bar{a}$ and a distinct countable atomless Boolean algebra $B' \subseteq L$ such that $\sigma(B) = B'$ setwise. (Then the domain of B will be seen to be undefinable.)

Recall that B is the union of an ω-chain $B_0 \subseteq B_1 \subseteq \ldots$ of finite Boolean algebras. We construct an ω-sequence $A_0, A_1, \ldots$ of finite Boolean algebras that are subalgebras of L and an ω-chain $B'_0 \subseteq B'_1 \subseteq \ldots$ such that $B_k, B'_k \subseteq A_k$, that $B_k \cong B'_k$, and that $B_k \neq B'_k$.

Let D_0 be the diagram $B_0 \hookleftarrow \mathbf{2} \hookrightarrow B_0$. Let $A'_0 = \bigsqcup D$. By the weak homogeneity of L, there is an embedding $i_0 : A'_0 \to L$ such that $(i_0 \circ \iota^D_{\hookleftarrow}) \restriction B_0$ is the identity. Let A_0 be $\operatorname{ran} i_0$ and B'_0 be $\operatorname{ran}(i_0 \circ \iota^D_{\hookrightarrow})$.

Having A_k and B'_k defined, we define A_{k+1} and B'_{k+1} as follows. the diagram $A_k \hookleftarrow B'_k \hookrightarrow B_{i+1}$. Let $\tilde{D}_{k+1}$ be the diagram $A_k \hookleftarrow B_i \hookrightarrow B_{i+1}$, and D_{k+1} be $B_{i+1} \hookleftarrow B_i \hookrightarrow \bigsqcup \tilde{D}_{k+1}$. By appealing to the weak homogeneity of L as before, take an embedding $i_{i+1} : \bigsqcup D_{k+1} \hookrightarrow L$ so that $B'_{i+1} := \operatorname{ran}(i_{i+1} \circ \iota^{D_{k+1}}_{\hookrightarrow} \circ \iota^{\tilde{D}_{k+1}}_{\hookrightarrow})$ extends B'_i and that $B_{i+1} = \operatorname{ran}(i_{i+1} \circ \iota^{D_{k+1}}_{\hookleftarrow})$. Finally, let $A_{k+1} = \operatorname{ran} i_{i+1}$.

By construction and by the ultrahomogeneity of L, the two substructures B_k and B'_k are conjugate under an automorphism of L. Let B and B' be the unions of B_k's and B'_k's, respectively. Then B and B' are conjugate under an automorphism of L, and $B \neq B'$. □

If we drop the requirement that a copy of B in L be a subalgebra of L, we do obtain a natural interpretation as follows:

Proposition 3.2.2. There is an atomless Boolean algebra which is a relativized reduct of L.

Proof. The set B of fixed points of $1-(1-\cdot)$ in L is a Boolean algebra by setting $a \wedge^B b = \neg\neg(a \wedge^L b)$ and the remaining operations of B the restrictions of the corresponding operations of L. (Note that B is not a substructure of L.)

Suppose that $a \in B$ is an atom of B. We show that a is also an atom of L. To see this, assume the contrary, and let b be such that $0 < b < a$, where $b \notin B$. Since $b \notin B$, we have $1-(1-b) \neq b$; since $1-(1-c) \leq c$ for all $c \in B$, we have $1-(1-b) < b$. Now $1-(1-b) \in B$ and $0 < 1-(1-b)$ (since $1-b < 1$), so we have $0 < 1-(1-b) < a$, contradicting the assumption that a is an atom of B.

We have seen that any atom in B is an atom of L. Since there is no join-irreducible elements (let alone atoms) in L [37, Proposition 4.28.(iii)], B is atomless. $\square$

3.3 Automorphism Group

In this last section, we look at the automorphism group of L with the Kechris-Pestov-Todorcević correspondence in mind.

Recall that the extreme amenability of topological groups are of interest only if they are not locally compact [54]. It is well known that $\mathrm{Aut}(M)$ for a countable ω-categorical M is not locally compact [62]. Even though L is not ω-categorical, we can show the following.

Proposition 3.3.1. The topological group $\mathrm{Aut}(L)$ is not locally compact.

Proof. It suffices to show that for every finite subset $S \subseteq L$ there is an infinite orbit in $\mathrm{Aut}(L)_{(S)} \curvearrowright L$. Note that for every subalgebra $A \subseteq L$, there exists $a \in L \setminus A$ such that a is join-prime in $\langle Aa \rangle^L$. By repeatedly using this, take an ω-sequence $(a_i)_{i<\omega}$ of elements of L such that $a_i \in L \setminus \langle Sa_0 a_1 \dots a_{i-1} \rangle^L$ is join-prime in $\langle Sa_0 a_1 \dots a_i \rangle^L$ (and *a fortiori* in $\langle Sa_i \rangle^L$) for $i < \omega$. By construction, there exists an automorphism $\phi_i : L \to L$ fixing S pointwise such that $\phi_i(a_i) = a_{i+1}$ for $i < \omega$. Hence, the orbit of a_0 under $\mathrm{Aut}(L)_{(S)}$ is infinite. $\square$

An obvious strategy to study $\mathrm{Aut}(L)$ is to relate it to $\mathrm{Aut}(B)$, where B is the countable atomless Boolean algebra. The following lemma gives rise to a topological embedding of the former into the latter.

Lemma 3.3.2.

1. Let $f : H \to H_1$ be a Heyting algebra homomorphism between finite algebras. There are finite Boolean algebras $B(H)$ and $B(H_1)$ and a Boolean algebra homomorphism $B(f) : B(H) \to B(H_1)$. There are interior operators $^\circ, ^{\circ_1}$ on $B(H), B(H_1)$ such that $B(H)^\circ \cong H$ and $B(H_1)^{\circ_1} \cong H_1$. If f is injective, so is $B(f)$; if f is surjective, so is $B(f)$.

2. There is an interior operator $^\circ$ on the countable atomless Boolean algebra B such that B° is isomorphic to the universal ultrahomogeneous countable Heyting algebra L.

3. An automorphism $L \to L$ can be extended (as a function between pure sets) to another $B \to B$. Moreover, there is an embedding $\mathrm{Aut}(L) \hookrightarrow \mathrm{Aut}(B)$ that is a homeomorphism onto its image.

Proof.

1. Let P and P' be the dual posets of H and H_1, respectively. There is a bounded morphism $D(f) : P_1 \to P$ that is the dual of f. $D(f)$ is surjective if f is injective. Let $B(H) = \mathscr{P}(P)$ and $B(H_1) = \mathscr{P}(P_1)$. $D(f)$ induces a Boolean algebra homomorphism $B(f) : B(H) \to B(H_1)$. $B(f)$ is injective if $D(f)$ is surjective. Likewise, $B(f)$ is surjective if f is. Let $^\circ, ^{\circ_1}$ be the operations that take a subset to the maximal up-set contained by that set.

2. Let $(L_i)_{i<\omega}$ be a chain of finite Heyting algebras used in the construction of L; so $\bigcup_i L_i = L$. Let $B_i = B(L_i)$ and $^{\circ_i}$ be an interior operator such that $B_i{}^{\circ_i} \cong L_i$. We may take $B_i \subseteq B_{i+1}$ for $i < \omega$. Then $^{\circ_{i+1}}$ extends $^{\circ_i}$. Let $B = \bigcup_i B_i$ and $^\circ = \bigcup {}^{\circ_i}$. Then $B^\circ = (\bigcup_i B_i)^\circ = \bigcup_i B_i{}^{\circ_i} = \bigcup_i H_i = H$. It remains to show that B is atomless. Take an arbitrary $a \in B$ that is nonzero. Take $i < \omega$ such that $a \in B_i$. Let P_i be the poset dual to L_i; then a is a nonempty subset of P_i. Take some $w \in a$. Let P' be the poset obtained from P_i by replacing w with the 2-chain $\{w_1 < w_2\}$. Let $\pi : P' \twoheadrightarrow P_i$ be the surjection that maps the chain to $\{w\}$ and is the identity elsewhere. This is a bounded morphism, and it induces $\iota : L_i \hookrightarrow L'$, where L' is the dual of P'. Take $k < \omega$ such that there is an embedding $\iota' : L' \hookrightarrow L_k$ such that $\iota' \circ \iota$ is the identity on L_i. Write L' for that image of L'. Let $b = (a \setminus \{w\}) \cup \{w_1\}$. Then $b \in B_k = B(L_k) \subseteq B$ and $0 < b < a$.

3. Let $f : L \to L$ be an automorphism. Let $f_k : L_k \to L'_k$ be the restriction of f to L_k where $L'_k = f(L_k)$. Each f_k is an automorphism. By the fact above, f_k induces a Boolean algebra automorphism $B(f_k) : B(L_k) \to B(L'_k)$ for each $k < \omega$; and by construction $B(f_j)$ extends $B(f_k)$ for each $k < j < \omega$. Let $\hat{f} = \bigcup_k B(f_k)$. Then $\hat{f}$ is an isomorphism $B \to B$.

   Let $g : L \to L$ be another automorphism. We need to show $\hat{f} \circ \hat{g} = (f \circ g)\hat{}$. Let $a \in B$ be arbitrary. It suffices to show that $\hat{f}(\hat{g}(a)) = (f \circ g)\hat{}(a)$. Take $i < \omega$ such that $g(a), a \in B_i = B(L_i)$. Then $(f \circ g)\hat{}(a) = B((f \circ g)|_{L_k})(a) = B(f_k)(B(g_k)(a)) = \hat{f}(\hat{g}(a))$.

   Let $\iota : \mathrm{Aut}(L) \to \mathrm{Aut}(B)$ be the map $f \mapsto \hat{f}$. The map ι is a group homomorphism as seen above, and it is clearly injective.

   Next, we show that ι is continuous. Let $\bar{b}$ be a tuple in B. It suffices to show that for an automorphism $f : L \to L$ the value of $\hat{f}(\bar{b})$ is determined by the value of $f(\bar{a})$ for a tuple $\bar{a}$ in L. There exists $k < \omega$ such that $\bar{b}$ is in $B_k = B(L_k)$. Let $f_k : L_k \to L'_k$

be an automorphism that is a restriction of f. Then $\hat{f}(\bar{b}) = B(f_k)(\bar{b})$. Let $\bar{a}$ be an enumeration of the finite algebra L_k; then $\bar{a}$ is what we needed.

Finally, we show that the image $\iota(U)$ is open in $\operatorname{ran}\iota \subseteq \operatorname{Aut}(B)$ for an arbitrary basic open set U of $\operatorname{Aut}(L)$. Indeed, let U be the set of $f : L \to L$ fixing the values of f at $\bar{a} \in L$; then $\hat{g} \in \iota(U)$ in and only if $\hat{g} \restriction B_0 = \hat{f} \restriction B_0$ for $g : L \to L$, where B_0 is the Boolean subalgebra of B generated by $\bar{a}$. □

Note that despite $L \subseteq B$, the structure L is not interpretable in B because the latter is $\aleph_0$-categorical whereas the former is not.

There is another way $\operatorname{Aut}(B)$ and $\operatorname{Aut}(L)$ can be related. Recall the interpretation of B in L from Proposition 3.2.2, and let $h_{\neg\neg} : \operatorname{Aut}(L) \to \operatorname{Aut}(B)$ be the continuous homomorphism that it induces.

Lemma 3.3.3. Consider the copy of B as a relativized reduct of L as before. Every element L is a finite join of elements of B.

Proof. Let $a \in L$ be arbitrary. Take a finite subalgebra $H \subseteq L$ so $a \in H$, and let $\mathbb{P}$ be the dual poset of H so we may identify an element of H with an up-set of $\mathbb{P}$. Possibly by replacing L by another finite Heyting algebra into which L embeds, we may assume that $\mathbb{P}$ is a forest. Furthermore, without loss of generality, we may assume that a is principal. Suppose that a is generated by $x \in \mathbb{P}$. If x is a root, then a itself is regular, so there remains nothing to show. Suppose not, and let x^- be the predecessor of x. Let $\mathbb{P}_1, \mathbb{P}_2$ be disjoint posets isomorphic to that induced by $a \subseteq \mathbb{P}$. Let $\mathbb{P}' := (\mathbb{P} \setminus a) \sqcup \mathbb{P}_1 \sqcup \mathbb{P}_2$ whose partial order is the least containing those of the summands and $x^- \leq \mathbb{P}_1$, $x^- \leq \mathbb{P}_2$. Consider the surjective bounded morphism $\mathbb{P}' \twoheadrightarrow \mathbb{P}$ that collapses $\{\min \mathbb{P}_1, \min \mathbb{P}_2\}$ into x, and let $i : H \hookrightarrow H'$ be the Heyting algebra embedding it induces. Note that $\mathbb{P}_i \in H'$ is regular for $i = 1, 2$ and that $i(a) = \mathbb{P}_1 \vee \mathbb{P}_2$. Let $\phi : H' \to H_r(a)$ be an isomorphism such that $H_r(a)$ is a subalgebra of L and $\phi \restriction H$ is the identity. Let $r_1(a) := \phi(\mathbb{P}_1)$ and $r_2(a) := \phi(\mathbb{P}_2)$. We have $a = r_1(a) \vee r_2(a)$ and $r_i(a) \in B$ $(i = 1, 2)$ as promised. □

Proposition 3.3.4. The continuous homomorphism $h_{\neg\neg}$ is injective and is a homeomorphism onto its image. However, $h_{\neg\neg}$ is not surjective, and its image is a non-dense non-open set.

Proof. The first claim is immediate. We show that $h_{\neg\neg}$ is not surjective.

Consider the 3-element chain C_3, which can be regarded as a Heyting algebra, and let $a \in C_3$ be such that $0 < a < 1$. Note that a is irregular and a principal up-set in the dual finite poset of C_3. Let D be the diagram $C_3 \hookleftarrow \mathbf{2} \hookrightarrow C_3$, where $\mathbf{2}$ is the 2-element Heyting algebra. Let $a_0 = \iota^D_{\hookleftarrow}(a)$, $a_{1.5} = \iota^D_{\hookrightarrow}(a)$, and $H = H_r(a_{1.5})$. Next, let D' be the diagram $H \hookleftarrow \iota^D_{\hookleftarrow}(C_3) \hookrightarrow H$. Let $a_{1i} = \iota^{D'}_{\hookleftarrow}(r_i(a_{1.5}))$, $a_{2i} = \iota^{D'}_{\hookrightarrow}(r_i(a_{1.5}))$ and $a_{0i} = r_i(a_0)$ for $i = 1, 2$.

The Boolean subalgebra B_6 generated by a_{ji} $(0 \leq j \leq 2, 1 \leq i \leq 2)$ in B has six atoms, each permutation of which extends to an automorphism of B. Consider the permutation $a_{ji} \mapsto a_{(j+1 \bmod 3)i}$, which extends to an automorphism of B_6, which in turn extends to

$\phi \in \mathrm{Aut}(B)$ by ultrahomogeneity of B. By construction,

$$\bigvee_L \phi(\{a_{11}, a_{12}\}) \neq \bigvee_L \phi(\{a_{21}, a_{22}\})$$

showing that ϕ is not in the range of $h_{\neg\neg}$.

The last paragraph also shows that the image of $h_{\neg\neg}$ is not dense. To see that $\operatorname{ran} h_{\neg\neg}$ is not open, let $\bar{b}$ be an arbitrary tuple in B, and we show that $\mathrm{Aut}(B)_{(\bar{b})} \setminus \operatorname{ran} h_{\neg\neg} \neq \varnothing$. Take a finite subalgebra H of L such that H generates $\langle \bar{a} \rangle^B$ as a Boolean algebra. Let D'' be the diagram[1] $H \hookleftarrow \mathbf{2} \hookrightarrow \bigsqcup D$. The image $\operatorname{ran} \iota^{D''}_{\hookrightarrow}$ generates a copy B_6' of B_6. Take an automorphism ψ_0 on $\bigsqcup D''$ $\psi_0 \restriction B_6'$ is as constructed in the preceding paragraph and that $\psi_0 \restriction \operatorname{ran} \iota^{D''}_{\hookleftarrow}$ is the identity.[2] The automorphism ψ_0 extends to another $\phi \in \mathrm{Aut}(B)$, which is in $\mathrm{Aut}(B)_{(\bar{b})} \setminus \operatorname{ran} h_{\neg\neg}$. □

We will show the non-amenability of $\mathrm{Aut}(L)$ later in this section. Before doing so, we find it interesting to see that $\mathrm{Aut}(L)$ is distinct from the automorphism groups of better-known ultrahomogeneous structures.

Lemma 3.3.5. Let N be a countable strongly 2-homogeneous structure, $p \in S_1(N)$, and M be an ω-categorical structure in a possibly different countable language. Let $f_M : \omega \to \omega$ be defined by $f_M(n) = |S_n^M(\varnothing)|$. Suppose that for every $n_0 < \omega$ there exist $m < \omega$ and a set X of m-types realized in N such that for every $q(x_1, \ldots, x_m)$ and $i < m$ we have $p(x_i) \subseteq q(x_1, \ldots, x_m)$ and that $f_M(n_0 m) < |X|$. Then:

1. The topological group $\mathrm{Aut}(N)$ is not topologically isomorphic to $\mathrm{Aut}(M)$.
2. The abstract group $\mathrm{Aut}(N)$ is not isomorphic to $\mathrm{Aut}(M)$ if $\mathrm{Aut}(M)$ has the small index property.

More generally, an analogous statement about a strongly $(\kappa+1)$-homogeneous N, a κ-type p, and sets X of $\kappa \cdot m$-types of N holds true.

Proof. The second claim is a corollary of the first (see, e.g., Hodges [47, Lemma 4.2.6]).

By way of contradiction, assume that $\mathrm{Aut}(M)$ and $\mathrm{Aut}(N)$ are topologically isomorphic. First, we see:

Claim. There exists $n_0 = n_0(p) < \omega$ and a function $c : p(N) \to M^{n_0}$ such that for any formula $\phi(x_1, \ldots, x_m)$ in the language of N there is a formula $\phi^*(\overline{x_1}, \ldots, \overline{x_m})$ in the language of M with

$$N \models \phi(b_1, \ldots, b_m) \iff M \models \phi^*(c(b_1), \ldots, c(b_m))$$

for every $b_1, \ldots, b_m \in N$ with $b_i \in p(N)$ $(1 \leq i \leq m)$.

[1] To be more precise, one can replace $\bigsqcup D$ by an appropriate copy by the weak homogeneity of L.

[2] The existence of such an automorphism can be proved in terms of the concrete representation of the $\bigsqcup D''$.

In other words, N is Ind-interpretable in M. Before proving this claim, we note that if $(a_1, \dots, a_m) \in \phi(N^m) \triangle \psi(N^m)$, then $c(a_1) \dots c(a_m) \in \phi^*(M^{n_0 m}) \triangle \psi^*(M^{n_0 m})$.

We adapt the proof of a well-known fact [47, Lemma 7.3.7] combined with the strong 2-homogeneity of N to prove this claim. Let $h : \mathrm{Aut}(M) \to \mathrm{Aut}(N)$ be a topological isomorphism. By the strong 2-homogeneity, the realizers of p in N form an orbit of $\mathrm{Aut}(N) \curvearrowright N$. Fix $b \models p$ in N. Since h is continuous, $h^{-1}(\mathrm{Aut}(N)_{(b)})$ is open and hence contains $\mathrm{Aut}(M)_{(\overline{a})}$ for some $n_0 < \omega$ and $\overline{a} \in M^{n_0}$. Define $c : p(N) \to M^{n_0}$ so that for $b' \in p(N)$ we have $c(b') = h^{-1}(\alpha) \cdot \overline{a}$, where α is the unique element of $\mathrm{Aut}(N)$ such that $\alpha(b) = b'$. Now let $\phi(x_1, \dots, x_m)$ be as in the assumption of the claim, and consider $X := \{c(\overline{d}) \in M^{n_0 m} \mid \overline{d} \in N^m, N \models \phi(\overline{d})\}$. Since h is a group homomorphism, we can easily show that X is an orbit of $\mathrm{Aut}(B) \curvearrowright M^{n_0 m}$. By the $\aleph_0$-homogeneity of B, X is $\varnothing$-definable, say, by ϕ^*.

Take $m < \omega$ and X as in the assumption. For $q \in X$ let p^* be the possibly partial $n_0 m$-type $\{\phi^* \mid \phi \in q^*\}$ over $\varnothing$ of M. By construction, if a tuple $\overline{a} \in N^m$ realizes q, we have $c(\overline{a}) \models q^*$. We conclude that $|\{q^* \mid q \in X\}| > f_M(n_0 m)$, a contradiction. □

Corollary 3.3.6. The topological group $\mathrm{Aut}(L)$ is not realized as the automorphism group of any of the following structures:

- the countable atomless Boolean algebra B,
- the Fraïssé limit D of finite distributive lattices, or
- countable ultrahomogeneous structures in finite relational languages.

Moreover, $\mathrm{Aut}(L)$ is not isomorphic to $\mathrm{Aut}(B)$ or $\mathrm{Aut}(D)$ as abstract groups.

Proof. We will handle the cases of B and D first. Recall that $\mathrm{Aut}(B)$ and $\mathrm{Aut}(D)$ have the small index property [76, 28]. Since $\mathrm{Th}(L)$, $\mathrm{Th}(B)$, and $\mathrm{Th}(D)$ eliminate quantifiers, we may replace "types" with "quantifier-free types" in applying the preceding lemma to these structures. Since f_D grows asymptotically faster than f_B, it suffices to prove the conclusion for D. Let $\mathbf{V}$ be the variety of Gödel algebras, i.e., Heyting algebras satisfying the equation $(x \to y) \vee (y \to x) = 1$. This is a locally finite variety. For a tuple of variables $\overline{x}$, write $F^{\mathbf{V}}_{\overline{x}}$ for the free $\mathbf{V}$-algebra generated by $\overline{x}$, and let p be $\mathrm{qftp}^{F^{\mathbf{V}}_x}(x/\varnothing)$. Let $m < \omega$ be arbitrary and $\overline{x} = x_1 \dots x_m$. Consider $H_a := F^{\mathbf{V}}_{\overline{x}} \times (F^{\mathbf{V}}_{\overline{x}}/\theta_a)$, where θ_a is the principal filter generated by $a \in F^{\mathbf{V}}_{\overline{x}}$. This is a $\mathbf{V}$-algebra. Now, let $X_m = \{\mathrm{qftp}^{H_a}(\overline{x'}/\varnothing) \mid a \in F^{\mathbf{V}}_m\}$, where $\overline{x'} := (x_1, x_1) \dots (x_m, x_m)$. By construction, we have $p((x_i, x_i)) \subseteq q(\overline{x'})$ whenever $q(\overline{x'}) \in X_m$ and $1 \le i \le m$. Moreover, as H_a is finite for every $a \in F^{\mathbf{V}}_m$, every type in X_m is realized in L.

Let $n_0 < \omega$ be given. We have

$$f_D(n) = \sum_{i=1}^{n} S(n,i) i! M(i) \le n n! M(n) \max_i S(n,i),$$

where $n = n_0 m$, $S(\cdot, \cdot)$ are Stirling numbers of the second kind, and $M(i)$ is the i-th Dedekind number. Furthermore, by $\log \max_i S(n,i) = O(n \log n)$ [71] and $\log_2 M(n) = O\left(\binom{n}{n/2}\right)$ [57], we have

$$\log f_D(n) = O(n^2) + O\left(\binom{n}{n/2}\right) + O(n \log n) = O\left(\binom{n}{n/2}\right),$$

where we assumed n_0 is even without loss of generality. On the other hand, Valota [77] showed that $|X_m| = |F_m^{\mathbf{V}}| = (d(m))^2 + d(m)$, where $d(0) = 1$, and

$$d(k) = \prod_{i=0}^{k-1} (d(i) + 1)^{\binom{n}{i}}.$$

Therefore, $\log |X_m| = O(d(m))$, and

$$\log d(m) \geq \sum_{i=0}^{m-1} \binom{n}{i} \log d(i).$$

One can show by induction that $\log d(m)$ is at least the m-th Fubini number, which is strictly greater than $m!$ asymptotically [73]. Therefore, there exists m such that $|X_m| > f_D(n_0 m)$ as $\binom{n_0 m}{n_0 m/2} \sim 4^{n_0 m}/\sqrt{\pi n_0 m}$.

Finally, it is known that for every countable ultrahomogeneous structure M in a finite relational language, f_M is bounded from above by the exponential of a polynomial [15], so the claim follows from the argument above. □

We now proceed to showing the non-amenability of $\mathrm{Aut}(L)$.

Definition 3.3.7. Let H be a finite nondegenerate Heyting algebra. We write $I(b)$ for the set of join-prime elements below or equal to b for $b \in H$. Let $\prec$ be an arbitrary linear extension of the partial order on $I(1)$ induced from H. We define a total order $\prec^{\mathrm{alex}}$ on H extending $\prec$ by the following:

$$a \prec^{\mathrm{alex}} a' \iff \max_{\prec}(I(a) \triangle I(a')) \in I(a').$$

This is clearly a total order, which is known as the anti-lexicographic order. We call this a *natural ordering* on H.

An expansion of a finite nondegenerate Heyting algebra H by a natural total order is called a *finite Heyting algebra with a natural ordering.*

It is easy to check that if $(H, \prec)$ is a finite Heyting algebra with a natural ordering, and H happens to be a Boolean algebra, then $(H, \prec)$ is a finite Boolean algebra with a natural ordering in the sense of Kechris, Pestov, and Todorcević [54].

Proposition 3.3.8. The class $\mathcal{K}^*$ of finite Heyting algebras with a natural ordering is a reasonable Fraïssé expansion of $\mathrm{Age}(L)$.

Proof. We show that $\mathcal{K}^*$ is reasonable and that $\mathcal{K}^*$ has the amalgamation property. (Other claims are clear.) In what follows, for a totally ordered set $(X, <)$ and $Y, Z \subseteq X$, we write $Y < Z$ to mean that $y < z$ whenever $y \in Y$ and $z \in Z$.

Let $H_1 \subseteq H_2$ be finite Heyting algebra, and let $\prec_1^{\mathrm{alex}}$ be an arbitrary admissible total order on H_1. We show that there exists an admissible order on H_2 extending $\prec_1^{\mathrm{alex}}$. Let $\pi : \mathbb{P}_2 \twoheadrightarrow \mathbb{P}_1$ be the surjective bounded morphism dual to the inclusion map $H_1 \hookrightarrow H_2$. Note that identifying $I(1_{H_i})$ with $\mathbb{P}_i$ as pure sets, an admissible total order of H_i extends the dual of the order of $\mathbb{P}_i$ for $i = 1, 2$.

Suppose that for $p, q \in \mathbb{P}_1$ we have $p \prec_1 q$. Since $\prec_1^{\mathrm{alex}}$ is admissible, $p \not\lesssim q$. Take arbitrary $p', q' \in \mathbb{P}_2$ such that $\pi(p') = p$ and that $\pi(q') = q$. Since π is order-preserving *a fortiori*, we have $p' \not\lesssim q'$.

Let $R = (\leq \setminus \Delta) \cup \{(p', q') \mid \pi(p') \prec_2 \pi(q')\}$ be a binary relation on $\mathbb{P}_2 = I(1_{H_2})$, where Δ is the diagonal relation. It can be shown by induction from the fact in the preceding paragraph that R contains no cycle. Therefore, R can be extended to a total order $\prec_2$. Furthermore, for $p, q \in \mathbb{P}_1$, we have $\pi^{-1}(p) \prec_2 \pi^{-1}(q)$; *a fortiori*, $\pi^{-1}(p) \prec_2^{\mathrm{alex}} \pi^{-1}(q)$. This shows that $\prec_2^{\mathrm{alex}}$ extends $\prec_1^{\mathrm{alex}}$.

Next, we show the amalgamation property for $\mathcal{K}^*$. Let D be the diagram $H_1 \hookleftarrow H_0 \hookrightarrow H_2$ in Age(L) and let $\prec_i^{\mathrm{alex}}$ be an arbitrary admissible ordering on H_i for $i = 1, 2$. Recall the dual poset $\mathbb{P}$ of $\bigsqcup D$ is a sub-poset of the product order $\mathbb{P}_1 \times \mathbb{P}_2$, where $\mathbb{P}_i$ is the dual of H_i $(i = 1, 2)$ [63]. Define a total order $\prec$ on $\mathbb{P}$ so it extends the product order of $\prec_1$ and $\prec_2$.

We first show that $\prec$ extends the dual of the order of $\mathbb{P}$. Assume that $(p_1, p_2) \leq (q_1, q_2)$ for $(p_i, q_j) \in \mathbb{P}$ and $1 \leq i, j \leq 2$. (Recall that $p_i, q_i \in \mathbb{P}_i$.) Since the order of $\mathbb{P}$ is induced by the product of those of $\mathbb{P}_1$ and $\mathbb{P}_2$, we have $p_i \leq q_i$ for $i = 1, 2$. Because $\prec_i$ extends the dual of the order of $\mathbb{P}_i$, we have $p_i \succ_i q_i$ $(i = 1, 2)$. By the construction of $\prec$, we have $(p_1, p_2) \succ (q_1, q_2)$ as desired.

We then show that $(\bigsqcup D, \prec^{\mathrm{alex}})$ witnesses the amalgamation property. Because of the strong amalgamation property of Age(L), it suffices to show that $\prec^{\mathrm{alex}}$ extends $\iota^D_{\hookleftarrow}(\prec_1^{\mathrm{alex}})$ and $\iota^D_{\hookrightarrow}(\prec_2^{\mathrm{alex}})$. Take $p, p' \in \mathbb{P}_1$, and assume that $p \prec p'$ (the other case can be handled in a similar manner). Since $\iota^D_{\hookleftarrow}$ is induced by the projection $\pi_1 : \mathbb{P} \twoheadrightarrow \mathbb{P}_1$, it suffices to show that $\pi^{-1}(p) \prec^{\mathrm{alex}} \pi^{-1}(p')$. Now, it is easy to see that, in fact, $\pi^{-1}(p) \prec \pi^{-1}(p')$ by the construction of $\prec$. □

Corollary 3.3.9. Aut(L) is not amenable.

Proof. Consider the Boolean algebras that witness the conditions (i) and (ii) of [56, Proposition 2.2] for the class of finite Boolean algebras with natural orderings [54, Remark 3.1]; call them A_1 and A_2. Since $A_1, A_2 \in \mathcal{K}$, and the Heyting algebra embeddings $A_1 \to A_2$ are exactly the Boolean algebra embeddings $A_1 \to A_2$, the pair A_1, A_2 witness the conditions (i) and (ii) of the same propositions for $\mathcal{K}^*$. □

Finally, we study the aspects of the combinatorics of Age(L) pertaining to the extreme amenability of Aut(L). The Kechris-Pestov-Todorcević correspondence concerns order expansions of the ages of ultrahomogeneous structures with the ordering property [54]. One can make an empirical observation that the ordering property of an order expansion of a Fraïssé class have been proved by two classes of arguments, one of which is based on a lower-dimensional Ramsey property, with the other argument rather trivially following from the order-forgetfulness of the expansion. The former is applied to many classes of relational structures such as graphs, whereas the latter is used with the countable atomless Boolean algebras and the infinite-dimensional vector space over a finite field. Our structure L is similar to the latter classes of structures. However, we see the following.

Proposition 3.3.10. There is no Fraïssé order class of isomorphism types that expands the class of finite Heyting algebras and is order-forgetful.

Proof. Suppose that such a class $\mathcal{K}^*$ exists. Let H be an arbitrary finite Heyting algebra, and consider the action of $\mathrm{Aut}(H)$ on the set of binary relations on H. Since $\mathcal{K}^*$ is closed under isomorphs, the set of admissible orderings A_L on H is a union of orbits. Since $\mathcal{K}^*$ is order-forgetful, A_L consists of a single orbit.

Now, consider the poset $\mathbb{P}'$ that is the disjoint union of two 2-chains, with its quotient $\mathbb{P}$ obtained by collapsing one of the 2-chains into a point. The canonical surjection $\mathbb{P}' \twoheadrightarrow \mathbb{P}$ is bounded morphic, which induces a Heyting algebra embedding $H \hookrightarrow H'$. Let $a, b \in H'$ correspond to the two 2-chains. Clearly, H is rigid whereas there is an automorphism $\phi : H' \to H'$ under which a and b are conjugates. Consider an admissible ordering $\prec$ on H'; without loss of generality, we may assume $a \prec b$. Writing the action of $\mathrm{Aut}(H')$ by superscripts, we have $b \prec^{\phi} a$. Since $\mathcal{K}^*$ is a Fraïssé class, the restrictions of $\prec$ and $\prec^{\phi}$ to H, respectively, are admissible orderings on H. Now, we have $\prec \cap H^2 \neq \prec^{\phi} \cap H^2$, as witnessed by $(a, b) \in H^2$. These cannot belong to the same orbit of A_H as H is rigid. $\square$

3.4 Axiomatization

Following Darnière and Junker [26], we follow the formalism of co-Heyting algebras, or cHAs for short. They are exactly the order-theoretic dual of Heyting algebras. Let T be the theory of co-Heyting algebras. This is a theory in the language of lattices expanded by a binary function symbol $-$, where $x - y$ is the supremum of elements z for which $y \vee z \geq x$, which always exists in a co-Heyting algebra. As before, we write T^* for the model-completion of T.

We write $y \ll x$ iff $y \leq x$ and $x - y = 0$. Darnière and Junker [26, Section 4] lists two axioms D1 and S1 that are satisfied by e.c. co-Heyting algebras:

D1 For every a, c such that $c \ll a \neq 0$ there exists a nonzero element b such that:

$$c \ll b \ll a.$$

S1 For every a, b_1, b_2 such that $b_1 \vee b_2 \ll a \neq 0$ there exists nonzero elements a_1 and a_2 such that:

$$\begin{aligned} a - a_2 &= a_1 \geq b_1 \\ a - a_1 &= a_2 \geq b_2 \\ a_1 \wedge a_2 &= b_1 \wedge b_2. \end{aligned}$$

D1 is of the form (3.1), but S1 is not; in particular, the consequent of D1 does not imply the antecedent over T. However, consider the following condition:

$$(b_1 = a \text{ and } b_2 = 0) \text{ or } (b_2 = a \text{ and } b_1 = 0) \text{ or } (b_1 < a \text{ and } b_2 < a \text{ and } b_1 \wedge b_2 \ll a). \quad (\text{AS1}')$$

The same construction as in [26, Lemma 4.2] shows that AS1′ implies the consequent of S1 in T^*. It can also be seen that the consquent of S1 implies AS1′ over T. I refer to the conditional obtained from S1 by replacing the antecedent with AS1′ as S1′.

Proposition 3.4.1. D1 does not imply S1′; a fortiori, it does not axiomatize T^*.

Proof. It suffices to show that, given a finite cHA L with $x, y \in L$ such that $x \ll y$ and $a, b_1, b_2 \in L$ witnessing the failure of S1′, there is a finite $L' \supset L$ such that $L' \models \exists z(x \ll z \ll y)$, and that a, b_1, b_2 still witness the failure of S1′. For let L_0 be a cHA as in the hypothesis of the claim; the usual argument gives rise to a chain $L_0 \subset L_1 \subset \dots$, where L_{n+1} is constructed by applying the claim to L_n, the union $\bigcup_n L_n$ of which will satisfy D1 and the negation of S1′.

In fact, the following construction in [26, Lemma 4.1] works. Let $y_1, \dots, y_r$ be the join-irreducible components of y in L. Let $\mathcal{I}_0$ be the poset of the join-irreducible elements of I; let $\mathcal{I}$ be the poset obtained from $\mathcal{I}_0$ by replacing each y_i by the chain $\{\eta_i < y_i\}$. The bounded morphism $\mathcal{I} \twoheadrightarrow \mathcal{I}_0$ that collapses each chain $\{\eta_i < y_i\}$ to y_i induces a cHA embedding $L \hookrightarrow L'$, where L' is the cHA of downsets of $\mathcal{I}$. An element $z \in L'$ is in (the image of) L if and only if there is $1 \leq i \leq r$ such that $\eta_i \in z$ and that $y_i \notin z$. Suppose that there are $a_1, a_2 \in L'$ witnessing the consequent of S1′. By hypothesis, one of them is in $L' \setminus L$; without loss of generality, assume a_1 is. There is $1 \leq i \leq r$ such that $\eta_i \in a_1$ and that $y_i \notin a_1$. By the consequent of S1′, $a = a_1 \vee a_2 \in L$. Since $\eta_i \in a_1 \cup a_2$, we have that $y_i \in a_1 \cup a_2$. Hence, $y_i \in a_2$, and thus $\eta_i \in a_2$. Therefore, $\eta_i \in a_1 \cap a_2$, and $y_i \notin a_1 \cap a_2$. However, $a_1 \wedge a_2 = b_1 \wedge b_2 \in L$, which leads to a contradiction. □

Lemma 3.4.2. For a finite cHA L and $a, b \in L$, we have $a \ll b$ if and only if for every join-irreducible component b' of b we have $a \wedge b' < b'$.

Proof. Note that to prove quantifier-free formulas one may just treat elements of a cHA as closed sets in a space. If concepts of higher quantifier complexity (e.g., irreducibility) are involved, care must be taken.

Let $(b_i)_{i<k}$ be the join-irreducible components of b. Then

$$\begin{aligned}
b - a = b &\iff \bigvee_i b_i - a = \bigvee_i b_i && \\
&\iff \bigvee_i (b_i - a) = \bigvee_i b_i && \text{identity in cHAs} \\
&\iff \bigvee_i (b_i - a) \geq \bigvee_i b_i && \\
&\iff \forall i \ \bigvee_j (b_j - a) \geq b_i && \text{definition of } \bigvee \\
&\iff \forall i \exists j \ b_j - a \geq b_i && \text{join-primality of } b_j \\
&\iff \forall i \ b_i - a \geq b_i && \text{no other } j \text{ than } i \text{ can satisfy that} \\
&\iff \forall i \ b_i - (a \wedge b_i) \geq b_i && \\
&\iff \forall i \ (a \wedge b_i) < b_i && \text{by join-primality of } b_i\text{; see [26].}
\end{aligned}$$

□

Proposition 3.4.3. S1′ does not imply D1.

Proof. We use a similar argument as before. We let L_0 be the minimal nontrivial cHA, and we apply to L_n the construction in [26, Lemma 4.2] to obtain L_{n+1}. Note that for $n < \omega$ there is no chain consisting of more than one element in the poset of join-irreducible elements of L_n with the induced order.

We claim that for $n < \omega$ there is no nonzero $z \in L_n$ such that $0 \ll z \ll 1$—that is, 0 and 1 witness the failure of D1. Indeed, suppose that there is such a $z \neq 0$. There exists a join-irreducible component u' of 1 such that $u' \wedge z \neq 0$ since $z \neq 0$ and by distributivity. Take a join-irreducible component z' of $z \wedge u'$. We now have a nontrivial chain $\{z' < u'\}$ of join-irreducible elements. □

3.5 Beth Semantics and Nuclei

The second part of this chapter on Heyting algebras concerns Beth semantics for intuitionistic logic and nuclei on locales. First, we examine several topologies associated to an arbitrary Beth frame and prove that those topologies all give rise to a locale isomorphic to that which Bezhanishvili and Holliday [10] associated to the Beth frame. We next study these topologies arising from Beth frames that are trees and see that the associated topologies are homeomorphic to the bounded topology [61] of the tree. Finally, we show that Beth semantics restricted to *separative* trees is as general as Beth semantics restricted to trees.

We briefly recall basic notions in Beth semantics and pointless topology (see Bezhanishvili and Holliday [10] for references). *Nuclei* on a *locale* L, or a complete Heyting algebra, are simply closure operators on L that preserve meets. For a nucleus $j : L \to L$, we write L_j for the *sublocale* induced by j, i.e., the poset of fixed points of j. A sublocale L_j can be equipped with a structure of a Heyting algebra induced by the order, albeit the Heyting algebra operations will be different from those of L.

For a poset X, we write $\mathrm{Up}(X)$ for the locale of upward closed subsets of X. A *path* in a poset X is simply a nonempty chain that is closed under upper bounds. A set $U \in \mathrm{Up}(X)$ is *fixed* if for every $x \in X$ and every path C containing x, x is in U whenever C intersects U nontrivially. The *Beth nucleus* j_{b} on the locale $\mathrm{Up}(X)$ for a poset X is defined such that for every $U \in \mathrm{Up}(X)$, $j_{\mathrm{b}}(U)$ is the least fixed set in $\mathrm{Up}(X)$ including U.

Bezhanishvili and Holliday constructed for a poset X a topological space, which we call $P(X)$, such that each point in $P(X)$ is a path in X, and the algebra $\Omega(P(X))$ of open sets of $P(X)$ is isomorphic to $\mathrm{Up}(X)_{j_{\mathrm{b}}}$ [10, Theorem 4.18]. One can consider a similar construction for *maximal chains* instead of paths. More specifically, for a poset X, we define $\tilde{P}(X)$ to be the topological space of maximal chains in X whose open sets are exactly the subsets of the form $[U] := \{\alpha \mid \alpha \cap U \neq \emptyset\}$ for $U \in \mathrm{Up}(X)_{j_{\mathrm{b}}}$.

Proposition 3.5.1. $\Omega(\tilde{P}(X)) \cong \mathrm{Up}(X)_{j_{\mathrm{b}}}$ for every poset X.

Proof. The proof of [10, Theorem 4.18] can be adapted for maximal chains. □

Proposition 3.5.2. Let $f : P(X) \to \tilde{P}(X)$ be a function that assigns to each path α the maximal path containing α in X. Then f is a quotient map between topological spaces.

Proof. Left as an exercise to the reader. □

Trees

In what follows, we assume that X is a tree, i.e., for every $x \in X$ the set of lower bounds of x is wellordered.

Proposition 3.5.3. $\tilde{P}(X)$ is homeomorphic to the Kolmogorov quotient of $P(X)$.

Proof. It suffices to show that paths $\alpha, \beta \in P(X)$ are topologically indistinguishable if and only if they are contained in the same maximal chain—i.e., ${\downarrow}\alpha = {\downarrow}\beta$. Suppose that ${\downarrow}\alpha = {\downarrow}\beta$. Let $U \in \mathrm{Up}(X)_{j_{\mathrm{b}}}$ be arbitrary, and assume that $\alpha \in [U]$ or, equivalently, that α intersects U nontrivially. Let $x \in \alpha \cap U$. Since $\alpha \subseteq {\downarrow}\alpha = {\downarrow}\beta$, there is $y \geq x$ such that $y \in \beta$. Since U is an upset, we also have $y \in U$. Hence β and U intersect nontrivially, and $\beta \in [U]$. Combined with a symmetric argument, this shows that α and β are topologically indistinguishable. To show the converse, suppose that α and β are topologically indistinguishable. We show that ${\downarrow}\alpha \subseteq {\downarrow}\beta$ (the other inclusion can be proved in an entirely symmetric manner). Let $y \in {\downarrow}\alpha$ be arbitrary. Then there exists $x \in \alpha$ with $x \geq y$. Note that $j_{\mathrm{b}}({\uparrow}y) \supseteq {\uparrow}y \ni x \in \alpha$. Since $j_{\mathrm{b}}({\uparrow}y) \in \mathrm{Up}(X)_{j_{\mathrm{b}}}$, and $\alpha \in [j_{\mathrm{b}}({\uparrow}y)]$, β also intersects $j_{\mathrm{b}}({\uparrow}y)$ nontrivially by assumption. Let $z \in j_{\mathrm{b}}({\uparrow}y) \cap \beta$. Then β is a path containing z, which is in turn in $j_{\mathrm{b}}({\uparrow}y)$. This implies that ${\uparrow}y$ and β intersect nontrivially—i.e., $y \in {\downarrow}\beta$. $\square$

Given the proposition above, we study $\tilde{P}(X)$ instead of $P(X)$ for convenience's sake.

There is another natural topology on the set of maximal chains in a tree X. This topological space, which we call $\hat{P}(X)$, has a basis $\{[{\uparrow}x] \mid x \in X\}$ (note that $[{\uparrow}x]$ is the set of maximal chains containing x). This is called the *branch space* of X by some (e.g., [75]). If X is a subtree of $\omega^{<\omega}$, then $\hat{P}(X) \cap \omega^\omega$ is a subspace of *the Baire space* ω^ω (see, e.g., [55]), which carries the product topology. This is a totally disconnected Polish space of size continuum. More generally, if $X \subseteq \kappa^{<\kappa}$ for a regular cardinal κ with $|\kappa^{<\kappa}| = \kappa$, then $\hat{P}(X) \cap \kappa^\kappa$ is a subspace of a *generalized Baire space*, whose topology is called the *bounded topology* (see [61]).

Proposition 3.5.4. $\tilde{P}(X)$ and $\hat{P}(X)$ are homeomorphic.

Proof. For $x \in X$ let $b(x)$ stand for the least $y \leq x$ such that $[{\uparrow}y] = [{\uparrow}x]$. (For instance, if $X \subseteq \omega^{<\omega}$, then $b(x)$ is the longest initial segment of x that is an endpoint, $\langle\rangle$, or an immediate successor of a node with two or more immediate successors.) By definition, $[{\uparrow}x] = [{\uparrow}b(x)]$. Moreover, ${\uparrow}b(x)$ is fixed for every $x \in X$. To see this, let $z \notin {\uparrow}b(x)$. Assume that z and $b(x)$ are comparable. Since $z \notin {\uparrow}b(x)$, we have $z < b(x)$. By the minimality of $b(x)$, we have $[{\uparrow}b(x)] \subsetneq [{\uparrow}z]$. Then a maximal chain in $[{\uparrow}z] \setminus [{\uparrow}b(x)]$ contains z and does not intersect ${\uparrow}b(x)$. Next, assume that z and x are incomparable. Then no maximal chain containing z intersects ${\uparrow}b(x)$ nontrivially; indeed, if w is both in a maximal chain containing z and in ${\uparrow}b(x)$, ${\downarrow}w$ contains two incomparable elements z and $b(x)$.

The topology of $\tilde{P}(X)$ is at least as fine as that of $\hat{P}(X)$; indeed, it suffices to see that $[{\uparrow}x]$ is open in $\hat{P}(X)$ for all $x \in X$, which follows from the observations in the paragraph above.

To show that the topology of $\hat{P}(X)$ is at least as fine as that of $\tilde{P}(X)$, let $[U]$ be an arbitrary open set in $\tilde{P}(X)$ with $U \in \mathrm{Up}(X)_{j_{\mathrm{b}}}$. We have

$$U = j_{\mathrm{b}}(U) = j_{\mathrm{b}} \bigcup_{x \in U} {\uparrow}x = j_{\mathrm{b}} \bigcup_{x \in U} {\uparrow}b(x).$$

By the same argument as in the proof of Proposition 3.5.1, we have

$$[U] = [j_{\mathrm{b}} \bigcup_{x \in U} b(\uparrow x)] = \bigcup_{x \in U} [\uparrow b(x)].$$

We conclude that U is the union of some basic open sets in $\hat{P}(X)$. $\square$

Proposition 3.5.5. Suppose that $X \subseteq \omega^{<\omega}$. Define a tree $X' \subseteq \omega^{<\omega}$ with no maximal nodes by

$$X' = \{x \mid x \in X, x \text{ is not maximal}\} \cup \{x * 0^n \mid x \in X, x \text{ is maximal}, n \in \omega\},$$

where $*$ stands for concatenation of strings. Then $\tilde{P}(X)$ and $\tilde{P}(X')$ are homeomorphic, and $\mathrm{Up}(X)_{j_{\mathrm{b}}} \cong \mathrm{Up}(X')_{j_{\mathrm{b}}}$.

Since $\tilde{P}(X')$ has no maximal nodes, $\tilde{P}(X') \subseteq \omega^{\omega}$. This shows that, with a locale arising from countable trees of height $\leq \omega$ in Beth semantics, we can associate a subspace of the Baire space that represents that locale as the algebra of open sets in it.

Proof. It suffices to work with $\hat{P}(X)$ and $\hat{P}(X')$. There is a bijection $f : \hat{P}(X) \to \hat{P}(X')$ such that $f(\alpha) = \alpha * 0^{\omega}$ for α finite, and $f(\alpha) = \alpha$ otherwise. Suppose that $\alpha \in \hat{P}(X)$ is finite. Then α is an isolated point in $\hat{P}(X)$; indeed, $[\uparrow_X \max \alpha]_{\hat{P}(X)}$ is open in $\hat{P}(X)$, and $[\uparrow_X \max \alpha]_{\hat{P}(X)} = \{\alpha\}$. $f(\alpha) = \alpha * 0^{\omega}$ is also isolated in $\hat{P}(X) \subseteq \omega^{\omega}$, as witnessed by the open set $[\uparrow_{X'} \max \alpha]_{\hat{P}(X')} = \{\alpha * 0^{\omega}\}$ in $\hat{P}(X')$. The restriction $f|_{\hat{P}(X) \cap \omega^{\omega}}$ is clearly a homeomorphism. We conclude that f is a homeomorphism. $\square$

Even if X is not a subset of $\kappa^{<\kappa}$, $\tilde{P}(X)$ still has a nice property of Polish spaces:

Proposition 3.5.6. $\tilde{P}(X)$ is Hausdorff.

Proof. Let $\alpha, \beta \in \tilde{P}(X)$, and assume that $\alpha \neq \beta$. Since α and β are maximal chains, neither of the two chains is contained in the other; one can take $x \in \alpha \setminus \beta$ and $y \in \beta \setminus \alpha$. Note that x and y are incomparable by maximality of α and β. Then $j_{\mathrm{b}}(\uparrow x)$ and $j_{\mathrm{b}}(\uparrow y)$ are open sets, and they are clearly neighborhoods of α and β, respectively. Suppose by way of contradiction that $j_{\mathrm{b}}(\uparrow x) \cap j_{\mathrm{b}}(\uparrow y) \neq \emptyset$; take $z \in j_{\mathrm{b}}(\uparrow x) \cap j_{\mathrm{b}}(\uparrow y)$. Let γ be a maximal chain containing z. Then γ nontrivially intersects both $\uparrow x$ and $\uparrow y$. This is a contradiction; for, if $z \in \gamma \cap \uparrow x$, and $w \in \gamma \cap \uparrow y$, then $\downarrow \max\{z, w\}$ contains two incomparable elements—namely, x and y. $\square$

The Beth nucleus and double negation

The results in the next subsection suggest that the study of $\mathrm{Up}(X)_{\neg\neg}$ is useful in that of $\mathrm{Up}(X)_{j_{\mathrm{b}}}$ and $\tilde{P}(X)$. In general, $\mathrm{Up}(X)_{\neg\neg} \subseteq \mathrm{Up}(X)_{j_{\mathrm{b}}}$. Moreover, we have the following.

Proposition 3.5.7. For any poset X, $\mathrm{Up}(X)_{j_{\mathrm{b}}} = \mathrm{Up}(X)_{\neg\neg}$ if and only if $\mathrm{Up}(X)_{j_{\mathrm{b}}}$ is boolean.

Proof. This is in fact true of any locale A and a dense nucleus $j : A \to A$: $A_j = A_{\neg\neg}$ if and only if A_j is boolean. This fact is well known, but we include the proof for completeness.

It suffices to show the "if" direction. Note that $0_A \in A_j$. Since A_j is boolean, $A_j = \{x \to a \mid x \in A\}$ for some $a \in A$ (see, e.g., [70, p. III.10.4]). In particular, for some $x \in A$, we have $x \to a = 0_A$. Since $0_A \geq (x \to a) \geq a$, we have $a = 0_A$. As is well known, $A_{\neg\neg} = \{x \to 0_A \mid x \in A\}$. □

Posets where every principal upset is fixed

As suggested by [10, Example 4.15], posets every principal upset of which is fixed often give rise to locales that are easy to study.

Proposition 3.5.8. Suppose that every principal upset in X is fixed. Let P be either $P(X)$ or $\tilde{P}(X)$. $\alpha, \beta \in P$ are topologically indistinguishable if and only if one of the two points is dense in the other—i.e.,

$$\forall x \in \alpha \, \exists y \geq x \, (y \in \beta), \qquad \text{and} \qquad \forall y \in \alpha \, \exists x \geq y \, (z \in \alpha). \tag{3.3}$$

Proof. Suppose that α and β are topologically indistinguishable. Let $x \in \alpha$ be arbitrary. By hypothesis, $\uparrow x$ is fixed. Since $\uparrow x$ and α intersect nontrivially, so do $\uparrow x$ and β by the topological indistinguishablity of α and β. This means $\exists y \geq x \, (y \in \beta)$. Likewise, we have $\forall y \in \alpha \, \exists x \geq y \, (z \in \alpha)$. Conversely, assume (3.3). Suppose that $U \in \mathrm{Up}(X)_{j_b}$ intersects α nontrivially; let $x \in \alpha \cap U$. By hypothesis, $\exists y \geq x \, (y \in \beta)$. Since U is an upset, $y \in U$. We have shown that U intersects β nontrivially. The other direction can be shown in the same way. □

We exhibit a class of posets the satisfies the assumption of the preceding proposition. (For set-theoretic ideas that appear later in this subsection, see [53].) A poset X is *separative* if for every $x, y \in X$ we have

$$x \geq y \iff \forall x' \geq x \, \exists z \geq x' \, (z \geq y).$$

For a boolean algebra of B, let B^+ be the subposet of B consisting of every element of B but $\max B$. This is called a *topless boolean algebra*. Topless boolean algebras are separative. In fact, they are special separative posets: every separative posets densely embeds into some topless boolean algebra.

Proposition 3.5.9. Let X be a separative poset. Then every principal upset in X is fixed.

Proof. Let B be a boolean algebra such that X is a dense subset of B^+. Let $x \in X$ be arbitrary. Let $y \notin \uparrow_X x$. It suffices to show that there is a path in X containing y that does not intersect $\uparrow_X x$ nontrivially. Since $y \not\geq x$, we have $z := \neg x \vee y \neq 1$ and $z \in B^+$, where the operations are those of B. Since X is dense in B^+, there exists $z' \in X$ such that $z' \geq z$. Take a path α in X starting at y and containing z', such that an end segment of α is in $\uparrow_X z'$.

Assume for contradiction that α and $\uparrow_X x$ intersect nontrivially; let $w \in \alpha \cap \uparrow_X x$. Suppose that $w \leq z'$. This implies $x \leq z' \geq \neg x$, and it contradicts $z' \in X \subseteq B^+$. Suppose, on the other hand, that $w \geq z'$. This implies $x \leq w \geq \neg x$, which contradicts $w \in \alpha \subseteq X \subseteq B^+$. We conclude that α does not intersect $\uparrow_X x$ nontrivially and that $\uparrow_X x$ is fixed. □

Our next goal is to prove that Beth semantics restricted to separative trees is as general as that restricted to trees.

Lemma 3.5.10. Let X be a tree, and let b be as in the proof of Proposition 3.5.4. Let X' be the image of X under b with the induced order. Then X' is (isomorphic to) the separative quotient of X.

Proof. It suffices to show that

$$x \leq y \to b(x) \leq b(y) \tag{3.4}$$

and that

$$x \between y \iff b(x) \between b(y), \tag{3.5}$$

where $w \between z$ for $w, z \in X$ denotes w and z being compatible. This is because the separative quotient of X is determined up to isomorphism as the image of a map that satisfies the properties above. Note that, since X is a tree, $w \between z$ if and only if w and z are comparable. It is clear that b satisfies the first condition by definition. Suppose that $x \between y$. Since $x \geq b(x)$, $y \geq b(x)$, and X is a tree, $x \between y$. Suppose that $b(x) \between b(y)$. The two points $b(x)$ and $b(y)$ are comparable; without loss of generality we may assume that $b(x) \leq b(y)$. Then

$$\uparrow y = \uparrow b(y) \subseteq \uparrow b(x) = \uparrow x.$$

This implies $y \leq x$ and $y \between x$. □

Proposition 3.5.11. Suppose that X is a tree and that X' is its separative quotient. Then $\tilde{P}(X) \cong \tilde{P}(X')$, and thus $\mathrm{Up}(X)_{j_b} \cong \mathrm{Up}(X')_{j_b}$.

Proof. By the previous lemma, we may assume that $X' = b(X)$. Note that X' is also the set of fixed-points in X with respect to b.

The map $b : X \to X'$ induces a bijection $b_* : \tilde{P}(X) \to \tilde{P}(X')$, where $b_*(\alpha)$ is the image $b``\alpha$[3] for $\alpha \in \tilde{P}(X)$. Indeed, suppose that $\alpha \in \tilde{P}(X)$. Since $X$ is a tree, and $b(x) = x$ for all $x \in X'$, the image $b``\alpha$ is equal to $\alpha \cap X'$. Thus, the image $b``\alpha$ is a chain in $X' \subseteq X$, and it is maximal in X' by the maximality of α in X. Hence, b_* is well-defined as a map to $\tilde{P}(X')$. Next, suppose that $\beta \in \tilde{P}(X')$. We show that there is a unique $\alpha \supseteq \beta$ that is a maximal chain in X, and that thus b_* is bijective. Assume otherwise; let $\alpha_0, \alpha_1 \supseteq \beta$ be distinct maximal chains in X. By the maximality of α_0 and α_1, there exist $x_0 \in \alpha_0$ and $x_1 \in \alpha_1$ such that x_0 and x_1 are incomparable. Since x_0 and x_1 are incomparable, so are $b(x_0)$ and $b(x_1)$ by (3.5). Since $x_0 \geq b(x_0)$, and $x_1 \geq b(x_0)$, for $i < 2$ the two points x_i and

[3]Since square brackets are used to denote basic open sets, we use this notation for the f-image of a subset of $\operatorname{dom} f$ for a function f.

$b(x_{1-i})$ are incomparable. By the maximality of β in X', exactly one of $b(x_0), b(x_1) \in X'$ must be in β; we may assume $b(x_0) \in \beta$ without loss of generality. However, a chain α_1 in X cannot contain incomparable points $b(x_0)$ and x_1.

We show that b_* is in fact a homeomorphism. By Proposition 3.5.4, it suffices to consider the topology of $\tilde{P}(X)$ and $\tilde{P}(X')$ and their basic open sets. Specifically, since $\uparrow_X x = \uparrow_X b(x)$ for every $x \in X$, it suffices to show for every $x \in X'$

$$\begin{aligned} b_*``\{\alpha \in \tilde{P}(X) \mid x \in \alpha\} &= \{\beta \in \tilde{P}(X') \mid x \in \beta\}, \\ \{\alpha \in \tilde{P}(X) \mid x \in \alpha\} &= b^*``\{\beta \in \tilde{P}(X') \mid x \in \beta\}, \end{aligned}$$

where $b^* : \tilde{P}(X') \to \tilde{P}(X)$ is the inverse of b_*. These follow because $b_*(\alpha) = \alpha \cap X'$, and $b^*(\beta)$ is the unique maximal chain $\alpha \supseteq \beta$ in X. □

It is easy to see that a tree is separative if and only if every principal upset in it is fixed. Therefore, the last proposition also shows the generality of Beth semantics restricted to trees whose principal upsets are all fixed.

Chapter 4

Choice-Free Duality for Ortholattices

In this chapter, we continue the study of the topological duality for ortholattices started by Goldblatt [39], comparing it with the choice-free duality sketched in Bezhanishvili and Holliday [12], which is examined here in detail. In both cases, we characterize the duals of ortholattices and extend the duality categorically with the suitable morphisms. Afterwards, we identify a separate, first-order definable way in which the original ortholattice may be obtained from its choice-free dual. An application of this is a nontrivial characterization of the duals of *orthomodular* lattices.

In this chapter, an *ortholattice* is an expansion of a bounded lattice L by an *orthocomplementation*, i.e., a function $(\cdot)^\perp : L \to L$ that is involutive and antitone such that $a^\perp$ is a complement of a for every $a \in L$. One can show that the class of ortholattice is a variety in the appropriate signature. An important class of ortholattices is that of the lattices of closed subspaces of Hilbert spaces in which the orthocomplementation is given by taking orthogonal complements. These ortholattices satisfy additional axiom called *orthomodularity*:

$$x \leq y \to x \vee (x^\perp \wedge y) = y.$$

The class of orthomodular ortholattices is also a variety.

4.1 Duals of Ortholattices

Definition 4.1.1.

(i) A relational structure $(X, \perp)$ with a irreflexive symmetric relation $\perp$, is called an *orthoframe*. The relation $\perp$ is the *orthogonality relation* of the orthoframe.

(ii) Let L be an ortholattice.

a) The space $X_L^\pm = (X_L^\pm, \perp)$ of proper filters of L has the topology generated by the sets of the form $\widehat{a} = \{x \in X_L^\pm \mid a \in x\}$ and their complements for $a \in L$ with the binary relation defined by

$$x \perp y \iff (\exists a \in L)[a \in x \,\&\, a^{\perp} \in y].$$

This appeared in Goldblatt [39].

b) The space $X_L^+ = (X_L^+, \perp)$ consists of the same points and the same binary relation, but its topology is generated by sets of the form $\widehat{a}$ only. This was briefly discussed in Holliday and N. Bezhanishvili [12].

Proposition 4.1.1. Every ortholattice is isomorphic L to $\mathrm{COR}(X_L^+)$, where $\mathrm{COR}(X)$ is the ortholattice of compact open $\perp$-regular subsets of X.

Proof. We show that the image of the ortholattice embedding $\widehat{\cdot}$ from L to the ortholattice of $\perp$-regular subsets of X_L^+ is $\mathrm{COR}(X_L^+)$. Suppose that $A \in \mathrm{COR}(X_L^+)$. Since it is open, it is a union of basic open sets, which are of the form $\widehat{a}$. Since it is compact, it is a finite union of basic open sets: $A = \widehat{a_1} \cup \cdots \cup \widehat{a_n}$ for $a_1, \ldots, a_n \in L$. Since it is $\perp$-regular, we have

$$(a_1 \vee \cdots \vee a_n)^{\widehat{}} = (\widehat{a_1} \cup \cdots \cup \widehat{a_n})^{\perp\perp} = A^{\perp\perp} = A,$$

so A is in the image of $\widehat{\cdot}$. ∎

The following is an analogue of the characterization of the duals of Boolean algebras studied in Bezhanishvili and Holliday [12, § 5].

Proposition 4.1.2. An orthoframe $(X, \perp)$ with topology is homeomorphic[1] to X_L^+ for some ortholattice L iff all of the following conditions are met:

1. X is T_0.
2. $\mathrm{COR}(X)$ is closed under $\cap$ and ${}^{\perp}$.
3. $\mathrm{COR}(X)$ is a basis of X.
4. Every proper filter of $\mathrm{COR}(X)$ is of the form $\mathrm{COR}^X(x)$ for some $x \in X$.
5. If $x \perp y$, then there is $U \in \mathrm{COR}(X)$ such that $x \in U$ and $y \in U^{\perp}$.

Here, $\mathrm{COR}^X(x) = \{U \in \mathrm{COR}(X) \mid x \in U\}$.

Proof.

[1] A homeomorphism between two such structures is a homeomorphism between the two topological spaces that is an isomorphism between their orthoframe reducts.

"Only if" direction Note that $x \leqslant y$ if and only if $x \subseteq y$. Condition (3) is the definition of the topology of X_L^+. We show Condition (1). Suppose that $x \nleqslant y$, i.e., $x \nsubseteq y$. Take then $a \in x \setminus y$. Note that $\widehat{a}$ is $\perp$-regular and open. It can also be shown that $\widehat{a}$ is compact. Indeed, Let $(\widehat{b_i})_{i \in I}$ be a cover of $\widehat{a}$ by basic open sets: $\widehat{a} \subseteq \bigcup_i \widehat{b_i}$. Since the principal filter $\uparrow a$ generated by a is in $\widehat{a}$, so is it in $\bigcup_i \widehat{b_i}$, i.e., for some $i \in I$ we have $\uparrow a \in \widehat{b_i}$. This means that $a \leq b_i$ and that $\widehat{a} \subseteq \widehat{b_i}$, the unary union $\widehat{b_i}$ being a finite subcover of $\widehat{a}$. Therefore, $\widehat{a} \in \mathrm{COR}(X_L^+)$, $x \in \widehat{a}$, and $y \notin \widehat{a}$. Condition (2) follows from $\mathrm{COR}(X_L^+)$ being isomorphic to L. For (4), let G be a proper filter of $\mathrm{COR}(X_L^+)$. Let x be the image of G under the isomorphism $\mathrm{COR}(X_L^+) \to L$. Then $x \in X_L^+$. It is easy to see that $G = \mathrm{COR}^{X_L^+}(x)$. Finally, Condition (5) is the definition of $\perp$.

"If" direction We will show that if the five conditions are met, then there is a homeomorphism $\epsilon : (X, \perp) \to X^+_{\mathrm{COR}(X)}$ given by $x \mapsto \mathrm{COR}^X(x)$. That ϵ is surjective follows from Condition (4). To see ϵ is injective, let $x \neq y$ be in X. By X being T_0, either $x \nleqslant y$ or $y \nleqslant x$. Assume the former (the other case can be addressed in the same manner). By (3), take $U \in \mathrm{COR}(X)$ with $x \in U$ and $y \notin U$. We have $U \in \mathrm{COR}^X(x)$ and $U \notin \mathrm{COR}^X(y)$, and we have established the injectivity of ϵ. For the continuity of ϵ, we show that the inverse image of every basic open set in $X^+_{\mathrm{COR}(X)}$ is open in X. An arbitrary basic open set in $X^+_{\mathrm{COR}(X)}$ is of the form $\widehat{U}$ for $U \in \mathrm{COR}(X)$. We have:

$$\begin{aligned}
\epsilon^{-1}(\widehat{U}) &= \{x \in X \mid \mathrm{COR}^X(x) \in \widehat{U}\} \\
&= \{x \in X \mid U \in \mathrm{COR}^X(x)\} \\
&= \{x \in X \mid x \in U\} \\
&= U,
\end{aligned}$$

which is open. Now, since $\mathrm{COR}(X)$ is a basis of U, we conclude that ϵ is a homeomorphism. Finally, we show that ϵ is an isomorphism between the orthoframe reduct. We have:

$$\begin{aligned}
\mathrm{COR}^X(x) \perp \mathrm{COR}^X(y) &\iff (\exists U \in \mathrm{COR}(X))\, U \in \mathrm{COR}^X(x) \ \&\ U \in \mathrm{COR}^X(y) \\
&\iff (\exists U \in \mathrm{COR}(X))\, x \in U \ \&\ y \in U^\perp \\
&\iff x \perp y,
\end{aligned}$$

where the last $\Leftarrow$ is Condition (5), and the $\Rightarrow$ follows from the definition of $(\cdot)^\perp$. □

Here is a result similar to above but for Goldblatt's original representaion.

Proposition 4.1.3 (AC)**.** A orthoframe $(X, \perp)$ with topology is homeomorphic to $X_L^\pm$ for some ortholattice L iff all of the following conditions are met:

1. X is T_0 and compact.
2. $\mathrm{ClopR}(X)$ is closed under $\cap$ and $^\perp$.

3. If $x \neq y$, then there is $U \in \mathrm{ClopR}(X)$ such that either $x \in U$ and $y \notin U$, or $x \notin U$ and $y \in U$.

4. Every proper filter of $\mathrm{ClopR}(X)$ is of the form $\mathrm{ClopR}(x)$ for some $x \in X$.

5. If $x \perp y$, then there is $U \in \mathrm{ClopR}(X)$ such that $x \in U$ and $y \in U^{\perp}$.

Here, $\mathrm{ClopR}(X)$ is the lattice of clopen $\perp$-regular subsets of X, and $\mathrm{ClopR}(x) = \{U \in \mathrm{ClopR}(X) \mid x \in U\}$.

Proof.

“Only if” direction Goldblatt [39] showed that $X_L^{\pm}$ is compact and that $\mathrm{ClopR}(X_L^{\pm}) \cong L$. Since the topology of $X_L^{\pm}$ is finer than that of X_L^{+}, the space $X_L^{\pm}$ is T_0 as well.

“If” direction As before, we will show that $\epsilon : (X, \perp) \to X^{\pm}_{\mathrm{ClopR}(X)}$ given by $x \mapsto \mathrm{ClopR}^X(x)$ is a homeomorphism. The injectivity of ϵ follows from (3), and its surjectivity follows from (4) as before. The continuity of ϵ can be proved in the same manner. Note that $X_L^{\pm}$ is Haudorff. Since ϵ is a continuous map from a compact space to a Hausdorff space, it is a homeomorphism. Finally, it can be shown from (5) that ϵ is an isomorphism between the orthoframe reducts. ☐

Corollary 4.1.4 (AC)**.** 1. The preceding proposition obtains if the conditions (1), (2), and (3) are replaced by the following respectively:

   (1′) X is compact and Hausdorff.

   (2′) $\mathrm{ClopR}(X)$ is closed under ${}^{\perp}$.

   (3′) If $x \not\leq y$, then there is $U \in \mathrm{ClopR}(X)$ such that either $x \in U$ and $y \notin U$, or $x \notin U$ and $y \in U$.

   In particular, the space $X_L^{\pm}$ for an ortholattice L is a Stone space, i.e., a compact Hausdorff zero-dimensional space.

2. The space $(X_L^{\pm}, \not\perp)$ with the complement of $\perp$ is a modal space, i.e., the dual of some BAO (B-algebra).

3. The clopens $B(L)$ of $X_L^{\pm}$, i.e., the Boolean algebra dual to the space, is generated by $\mathrm{COR}(X_L^{\pm})$.

Proof. 1. Easy.

2. $(\not\perp)[y] = \{x \mid \forall a \in y\, a' \notin x\} = \bigcap_{a \in L} \complement\phi(a')$, where $\complement$ denotes set-theoretic complement. Note that $\complement\phi(a')$ is clopen.

3. A clopen U of $X_L^{\pm}$ is compact, and hence a finite union of basic opens, each of which is of intersections of sets of the form either $\phi(a)$ or $\complement\phi(a)$. Note that $\complement$ and $\cup$ are the Boolean complement and join of the said Boolean algebra.

$\square$

We conclude this section by showing that Goldblatt's original representation indeed requires some choice principle.

Proposition 4.1.5. The following are equivalent:

1. PIT, the Prime Ideal Theorem for Boolean algebras.

2. The space $X_L^{\pm}$ is compact for all Boolean algebras L.

Proof. PIT proves the compactness of $X_L^{\pm}$ for all Boolean algebras L as the only choice principle used in Goldblatt [39] is the Alexander Subbasis Theorem, which is equivalent to PIT.

Now assume that $X_L^{\pm}$ is compact for all Boolean algebras L. To show PIT, it suffices [51] to prove (I) the existence of a choice function for an arbitrary family of nonempty *finite* sets and (II) the Weak Rado Selection Lemma (whose statement can be found below).

(I) Let $\mathcal{S} := (S_i)_{i\in I}$ be a family of nonempty finite sets. Let L be the Boolean algebra presented by $\langle \bigsqcup_{i\in I} S_i \mid \{a \wedge b = 0 \mid a \neq b \in S_i, i \in I\}\rangle$. Consider $X_L^{\pm}$. For $I' \subseteq_{\text{fin}} I$, let $F_{I'} = \{u \in X_L^{\pm} \mid \forall i \in I'\, \exists a \in S_i\, a \in u\}$. It can be shown that $\mathcal{F} := (F_{I'})_{I'\in\mathscr{P}_{\text{fin}}(I)}$ is a filter basis of $X_L^{\pm}$. Since $X_L^{\pm}$ is compact, $\mathcal{F}$ has a cluster point u^+. We show that $f := \{(i,a) \mid i \in I, a \in S_i, a \in u^+\}$ is a choice function for $\mathcal{S}$. Since u^+ is a proper filter of L, at most one $a \in S_i$ can belong to u^+ by the construction of L. This shows that f is a function. We now show that $\operatorname{dom} f = I$. Let $i \in I$ be arbitrary. Suppose by way of contradiction that $S_i \cap u^+ = \varnothing$. Then $\complement\widehat{a}$ is a neighborhood of u^+ for $a \in S_i$, and so is $U := \bigcap_{a\in S_i} \complement\widehat{a}$, which is open as S_i is finite. Since u^+ is a cluster point, $U \cap F_{\{i\}}$ is nonempty, i.e., $\forall a \in S_i\, \exists u \in F_{\{i\}}\, a \notin u$, contradicting the definition of $F_{\{i\}}$.

(II) We will show:

> Suppose that for a set Λ there is a family of functions $(\gamma_S)_{S\in\mathscr{P}_{\text{fin}}(\Lambda)}$ such that $\gamma_S : S \to \{\pm 1\}$. Then there is $f : \Lambda \to \{\pm 1\}$ such that for all $S \subseteq_{\text{fin}} \lambda$ there exists $T \subseteq \Lambda$ with $S \subseteq T$ and $f \upharpoonright S = \gamma_T \upharpoonright S$.

Let $(\gamma_S)_S$ be given. Let $L = \langle \lambda^+, \lambda^- \mid \lambda^+ = \neg\lambda^-\rangle_{\lambda\in\Lambda}$. For $S \subseteq_{\text{fin}} \lambda$, let u_S be the filter of L generated $\{\lambda^{\pm} \mid \lambda \in \Lambda, \gamma_S(\lambda) = \pm 1\}$. It can be shown that u_S is proper so $u_S \in X_L^{\pm}$. Consider the net $(u_S)_{S\in\mathscr{P}_{\text{fin}}(\Lambda)}$, where the indices are ordered by inclusion. Since $X_L^{\pm}$ is compact, the net has a cluster point u_∞. Now we have

$$\forall\lambda \in \Lambda\, \forall S \subseteq_{\text{fin}} \Lambda\, \exists T \supseteq S[u_\infty \in \widehat{\lambda^{\pm}} \to u_T \in \widehat{\lambda^{\pm}} \text{ and } u_\infty \in \complement\widehat{\lambda^{\pm}} \to u_T \in \complement\widehat{\lambda^{\pm}}],$$

i.e.,

$$\forall\lambda \in \Lambda\, \forall S \subseteq_{\text{fin}} \Lambda\, \exists T \supseteq S [\lambda^{\pm} \in u_\infty \iff \lambda^{\pm} \in u_T] \tag{†}$$

Let $f = \{(\lambda, \pm 1) \mid \lambda^{\pm} \in u_\infty\}$. By a similar argument as before, f is a function $\Lambda \to \{\pm 1\}$. Also, by (†), $\forall S \subseteq_{\text{fin}} \Lambda\, \exists T \supseteq S\; f \restriction T = \gamma_T$ (*a fortiori*, $f \restriction S = \gamma_T \restriction S$). $\square$

4.2 Morphisms

Definition 4.2.1. For orthoframes $X = (X, \perp)$ and $X' = (X', \perp')$ where X and X' are also topological spaces, a *bounded morphic spectral map* $f : X \to X'$ is a spectral map $X \to X'$ that is also a bounded morphism $(X, \not\perp) \to (X', \not\perp')$ between these Kripke frames.

Note that a bounded morphic spectral map need not be bounded morphic with respect to the specialization order of the topological spaces. This makes an interesting contrast to the situation in [12] where that condition is required to obtain the dual equivalence result, whereas we do not need it here:

Proposition 4.2.1. The category **UVO** whose objects are of the form X_L^+ for an ortholattice L and whose morphisms are bounded morphic spectral maps is dually equivalent to the category of ortholattices and homomorophisms.

Proof. Suppose that $f : (X, \perp) \to (X', \perp')$ is a bounded morphic spectral map. Given $U \in \mathrm{COR}(X')$, let $f^+(U)$ be the inverse f-image of U. Since f is a spectral map, $f^+(U)$ is compact open. We see that $f^+(U)$ is $\perp$-regular as well. Indeed, since U is $\perp'$-regular, we have $\Box'\Diamond' U = U^{\perp'\perp'} = U$, where $\Box'$ and $\Diamond'$ are the modal operators corresponding to $\not\perp'$ [38]. Since f is bounded morphic with respect to $\not\perp'$, the map f^+ preserves modal operators. Thus we have $f^+(U)^{\perp\perp} = \Box\Diamond f^+(U) = f^+(\Box'\Diamond' U) = f^+(U)$, where $\Box$ and $\Diamond$ are defined from $\not\perp$ likewise. We now have a map $f^+ : \mathrm{COR}(X') \to \mathrm{COR}(X)$. It is easy to see that $\mathrm{COR}(\cdot)$ and $(\cdot)^+$ combined give rise to a functor from **UVO** to the category of ortholattices.

Secondly, suppose that $h : L \to L'$ is an ortholattice homomorphism. Given $u' \in X_{L'}^+$, let $h_+(u) \subseteq L$ be the inverse h-image of $u' \subseteq L'$. It is easy to see that $h_+(u)$ is a proper filter, so $h_+(u) \in X_L^+$. We now have a function $h_+ : X_{L'}^+ \to X_L^+$. We show that h_+ is a bounded morphic spectral map. For each $U' \in \mathrm{COR}(X_L^+)$, the inverse h_+-image of U' is compact open; indeed, a routine calculation shows that the inverse h_+-image of $\widehat{a}$ is $\widehat{h(a)}$, which is compact open in $X_{L'}^+$. By Lemma 6.6[2] of [12], the map h_+ is a spectral map. It remains to show that h_+ is bounded morphic with respect to the complements of the orthogonality relations. Suppose that $u' \not\perp' v'$ in $X_{L'}^+$, where $\perp'$ is the orthogonality relation of $X_{L'}^+$, and that $h_+(u') \perp h_+(v')$ by way of contradiction. Then there is $a \in L$ such that $a \in h_+(u')$ and $a^{\perp} \in h_+(v')$. By definition, we have $h(a) \in u'$ and $h(a)^{\perp'} = h(a^{\perp}) \in v'$, where $\perp'$ is the orthocomplement operation of L'. This is a contradiction. Suppose next that $h_+(u') \not\perp v$ for some $u' \in X_{L'}^+$ and $v \in X_L^+$. Let v' be the filter generated by the h-image of v. It is easy to

[2] The proof of the lemma does not use the fact that the spaces are spectral.

see that v' is proper and thus in $X^+_{L'}$ and that $h_+(v') = v$. Suppose by way of contradiction that $u' \perp v'$, i.e., there is $a' \in L'$ such that $a' \in v'$ and $a'^{\perp'} \in u'$. By definition, there is $a \in L$ with $h(a) \leq a'$ and $a \in v$. We now have $a'^{\perp'} \leq h(a)^{\perp'} = h(a^{\perp}) \in u'$. This means that $a^{\perp} \in h_+(u')$, which contradicts $h_+(u') \not\perp v$. It easy to see that $X^+_{\cdot}$ and $(\cdot)_+$ combined give rise to a functor from the category of ortholattices to **UVO**.

Finally, it is not hard to show that the two functors consist a dual equivalence of the categories. ☐

Proposition 4.2.2 (AC)**.** The category of modal spaces of the form $(X^{\pm}_L, \not\perp)$ for an ortholattice L is dually equivalent to the category of ortholattices.

Proof. This can easily proved via the duality for the category of modal B-algebras that are generated by the fixed points of the composite $\Box\Diamond$ of their modal operators. ☐

4.3 Logical Considerations

Objects in **UVO** can be regarded as a two-sorted first-order structure $\mathfrak{X} = (X, \mathcal{B}, \perp, \in)$ given a basis $\mathcal{B}$ of X^+_L in the following manner:

- The first sort of $\mathfrak{X}$ consists of the points of X^+_L.
- The second sort of $\mathfrak{X}$ is $\mathcal{B}$.
- The binary relation symbol $\perp$ between elements of the first sort is interpreted as the orthogonality relation of X^+_L.
- The binary relation symbol $\in$ between elements of the first sort and the second sort, respectively, is interpreted as the membership relation.

Let $\mathcal{L}$ be the first-order language for such structures. We use lowercase and uppercase variables for the first and the second sort of $\mathcal{L}$, respectively.

We say that an $\mathcal{L}$-formula is *invariant* if for every quantification $\exists U$, atomic formulas of the form $x \in U$ occurs only negatively. Note that $(X, \mathcal{B}, \perp, \in) \models \phi$ if and only if $(X, \mathcal{B}', \perp, \in) \models \phi$ for bases $\mathcal{B}, \mathcal{B}'$ of the same topology on X and an invariant sentence ϕ. Invariant $\mathcal{L}$-formulas are essentially formulas of Ziegler's logic $(\mathcal{L}')^t_{\omega\omega}$ for $\mathcal{L}' = \{\perp\}$ [83]. The idea is that invariant sentences depend only on the intrinsic topological information of objects of **UVO**, not how it is presented.

A *bottomless ortholattice* is a first-order structure of the form L^- in the language of two binary relation symbols, where $L^- = (L \setminus \{0\}, \leq, G^-_{\perp})$, the relation $\leq$ is the partial order induced by that of L, and $G^-_{\perp}$ is the intersection of $(L^-)^2$ and the graph of the orthocomplement of L.

Proposition 4.3.1. There is an interpretation Γ (in the sense of Hodges [47, 5.4 (b)]) of the class of bottomless ortholattices in **UVO** with the following properties:

1. The interpretation Γ consists of invariant formulas. Consequently, for two bases $\mathcal{B}, \mathcal{B}'$ of $X \in \mathbf{UVO}$, we have $\Gamma((X, \mathcal{B}, \perp, \in)) = \Gamma((X, \mathcal{B}', \perp, \in))$. (We write $\Gamma(X)$ for that ortholattice.)

2. For $X \in \mathbf{UVO}$, each element of the carrier set of $\Gamma(X)$ is a point of X, i.e., an element of the first sort of X as an $\mathcal{L}$-structure.

3. $\Gamma(X_L^+) \cong L^-$ for every ortholattice L.

Furthermore, for every first-order sentence ϕ in the language of ortholattices, there exists an invariant $\mathcal{L}$-sentence ϕ^* such that for every ortholattice L we have $L \models \phi$ if and only if $X_L^+ \models \phi^*$.

Proof. Now the specialization order $\sqsubseteq$ of an object X in $\mathbf{UVO}$ is uniformly definable by an invariant $\mathcal{L}$-sentence:

$$x \sqsubseteq y \iff X \models \neg \exists U \ni x[y \notin U].$$

Furthermore, the set of principal filters of X_L^+ is defined by the invariant formula $\phi(x) := \exists U \ni x \, \forall y \sqsubset x[y \notin U]$, where $\sqsubset \, = \, \sqsubseteq \setminus =$. Indeed, it is easy to see that $X_L^+ \models \phi(\uparrow a)$ for every $a \in L$. On the other hand, assume that $u \in X_L^+$ is not principal and that $X_L^+ \models \phi(u)$. Let $U \subseteq X_L^+$ be an open set witnessing $\phi(u)$. Since $\{\widehat{a} \mid a \in L\}$ is a basis for X_L^+, there exists $S \subseteq L$ such that $U = \bigcup_{a \in S} \widehat{a}$. Since U is a neighborhood around u, there exists $b \in S$ such that $u \in \widehat{b}$, i.e., $b \in u$. Let $y = \uparrow b$. Since $y \sqsubset u$, and u is not principal, $y \sqsubset u$. However, we have $y \in \widehat{b} \subseteq U$, which contradicts $\phi(u)$.

Let $\Gamma(X_L^+) = (M, \leq^M, G_\perp^-)$ be defined as follows: $M := \phi(X_L^+)$, i.e., the set of principal filters in X_L^+; the order $\leq^M$ is induced by the order-theoretic dual of the specialization order $\sqsubseteq$ in X_L^+, which can be defined by an invariant formula; and $(x, y) \in G_\perp^-$ if and only if $x \perp y$ and $\forall y' \sqsubset y \, x \not\perp y'$.

It is clear that the interpretation given above satisfies the promised properties. The last claim follows from the fact that L and L^- are bi-interpretable for every ortholattice L. $\square$

This proposition may be used to obtain the characterizations of the duals of many important classes of ortholattices. For instance, let ϕ state orthomodularity in the language of ortholattices. Then, the invariant $\mathcal{L}$-sentence ϕ^* obtained as in the proposition defines the class of the duals of orthomodular lattices in $\mathbf{UVO}$.

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