Title
The Langlands-Kottwitz method and deformation spaces of p-divisible groups of abelian type

Author
Youcis, Alexander Frank

Publication Date
2019

Peer reviewed|Thesis/dissertation

The Langlands-Kottwitz method and deformation spaces of p-divisible groups of abelian type

by

Alexander Frank Youcis

A dissertation submitted in partial satisfaction of
the requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committe in charge

Professor Sug Woo Shin, Chair
Professor Kenneth Ribet
Professor Kenneth Wachter

Spring 2019

Abstract

The Langlands-Kottwitz Method and Deformation Spaces of Abelian Type

by

Alex Youcis

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Sug Woo Shin, Chair

In [Sch13c] Scholze describes a formula, similar to those classically known as Langlands-Kottwitz type formulas, describing the trace of the cohomology of certain PEL type Shimura varieties at sub-hyperspecial level as sums of volume factors, orbital integrals, and twisted orbital integrals. The contribution to this formula arising from the prescence of non-hyperspecial level is encapsulated in the twisted orbital integral, the integrand of which can be described entirely in terms of cohomology of certain locally defined deformation spaces of p-divisible groups. In this thesis we extend these results, with some mild restrictions, to the case of abelian type Shimura varieties.

This thesis is dedicated to:

Dad: for teaching me that curiosity is man's most valuable treasure.
Mom: for teaching me the importance of self-love.
Mom and Dad: for reminding me to pay my rent on time.

Contents

Acknowledgements

First of all, I would like to thank my advisor Sug Woo Shin. I do this not only for his technical help with my work, but for his seemingly uncanny ability to suggest just the correct research problem to his students.

I would like to heartily thank many mathematicians for which the material of this thesis benefited greatly (either directly or indirectly) from their willingness to listen and their sage teachings. In particular though, I would like to thank Thomas Haines, Kai-Wen Lan, and Martin Olsson. I would like to give special thanks to Brian Conrad with whom I not only had many fruitful discussions concerning algebraic groups, but whose writing (in various forms) has been an invaluable part of my mathematical education.

I would also like to thank my friends at Berkeley. Completing my PhD was hard in more ways than mathematical, and without them I would have been lost at sea. I want to particularly thank my friend Alexander Bertoloni Meli who has not only continually inspired me to be a better mathematician, but whose persistent and warm friendship has kept me sane over the last few years.

Chapter 1

Introduction

The goal of this thesis, and future updates thereof, is to derive a formula which computes the trace of certain operators on the cohomology of Shimura varieties in terms of data that is amenable to study using the Arthur-Selberg trace formula. Namely, we develop a formula expressing the trace of the Galois-Hecke action on the cohomology of abelian type Shimura varieties at sub-hyperspecial level as a sum of volume factors, orbital integrals, and twisted orbital integrals. This is another addition to the circle of ideas falling under the heading of the *Langlands-Kottwitz method*. To put this in the correct context we briefly recall the history of these ideas and their relevance to the Langlands program.

The work of Deligne, Shimura, and many other mathematicians has allowed us to associate to a pair (G, X) (where G is a reductive group over $\mathbb{Q}$, and X a certain quotient of $G(\mathbb{R})$) an object called a *Shimura variety*. Roughly, this is the data of a family of smooth quasi-projective varieties $\{\mathsf{Sh}_K(G,X)\}$ over a number field E indexed by sufficiently small compact open subgroups K of $G(\mathbb{A}_f)$ (where $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the topological ring of *finite rational adeles*) with finite étale surjective transition maps. This family has a natural action of the group $G(\mathbb{A}_f)$ which means, essentially, that the E-scheme $\mathsf{Sh}(G,X) := \varprojlim \mathsf{Sh}_K(G,X)$ carries a continuous action of the group $G(\mathbb{A}_f)$ (in a precise sense reviewed in Appendix C).

Included in the class of Shimura varieties are the classical examples of *modular curves*, associated to a *Shimura datum* (G, X) with $G = \mathrm{GL}_2$. More specifically, modular curves are decorated moduli spaces of elliptic curves E—the decoration being trivializations of certain torsion subgroups of E. More generally, modular curves are part of a larger class of Shimura varieties known as Shimura varieties of so-called *PEL type*. These are decorated moduli spaces of polarized (the 'P') abelian varieties A with the decorations roughly corresponding to certain fixed rings acting by endomorphisms (the 'E') on A as well as trivializations of certain torsion subgroups of A (the 'L', standing for 'level').

Much work has been done to study the cohomology of Shimura varieties of PEL type and their relationship and the Langlands program. To elucidate this, let us first recall some setup. Given a Shimura variety $\mathsf{Sh}(G,X)$ the commuting actions of $G(\mathbb{A}_f)$ and $\mathrm{Gal}(\overline{E}/E)$ allow one to define a, evidently admissible, action of $\mathrm{Gal}(\overline{E}/E) \times G(\mathbb{A}_f)$ on the étale cohomology groups

$$H^i(\mathsf{Sh}(G,X)_{\overline{E}}, \overline{\mathbb{Q}_\ell}) = \varinjlim H^i(\mathsf{Sh}_K(G,X)_{\overline{E}}, \overline{\mathbb{Q}_\ell}) \tag{1.1}$$

and

$$H^i_c(\mathsf{Sh}(G,X)_{\overline{E}},\overline{\mathbb{Q}_\ell}) := \varinjlim H^i_c(\mathsf{Sh}_K(G,X)_{\overline{E}},\overline{\mathbb{Q}_\ell}) \tag{1.2}$$

as well as analogues of these cohomology groups where we replace $\overline{\mathbb{Q}_\ell}$ by certain certain E-rationally defined sheaves $\mathcal{F}_\xi$ with $G(\mathbb{A}_f)$-action associated to algebraic $\overline{\mathbb{Q}_\ell}$-representations of G. One might then ask how to describe the irreducible subquotients of these admissible $G(\mathbb{A}_f)\times \mathrm{Gal}(\overline{E}/E)$ representations. Equivalently, since the underlying $G(\mathbb{A}_f)$-representation is admissible, one would like to understand the stable subspaces for the action of $\mathrm{Gal}(\overline{E}/E)\times \mathscr{H}(G(\mathbb{A}_f))$ where $\mathscr{H}(G(\mathbb{A}_f))$ is the $\overline{\mathbb{Q}_\ell}$-Hecke algebra for $G(\mathbb{A}_f)$.

For instance, assuming that $\mathsf{Sh}_K(G,X)$ is proper for all K (which is equivalent to the adjoint group G^{ad} being $\mathbb{Q}$-anisotropic—see Lemma 2.1.29) one can deduce from Matsushima's formula (e.g. see [BR94, §2.3] and [BC83]) that the these $G(\mathbb{A}_f)\times\mathrm{Gal}(\overline{E}/E)$-representations are semisimple, and thus we can write

$$H^i(\mathsf{Sh}(G,X)_{\overline{E}},\mathcal{F}_\xi) = H^i_c(\mathsf{Sh}(G,X)_{\overline{E}},\mathcal{F}_\xi) = \bigoplus_{\pi_f} \pi_f \boxtimes \sigma^i(\pi_f) \tag{1.3}$$

where π_f ranges over irreducible admissible $\overline{\mathbb{Q}_\ell}$-representations for $G(\mathbb{A}_f)$ and $\sigma^i(\pi_f)$ is a finite-dimensional $\overline{\mathbb{Q}_\ell}$-representation of $\mathrm{Gal}(\overline{E}/E)$. One goal might be to explicitly describe the representation $\sigma^i(\pi_f)$ in terms of π_f.

For example, by construction these Galois representations are 'geometric' in the sense of Fontaine-Mazur, and one might hope to describe the trace of Frobeni on $\sigma(\pi_f)$ for almost all places v of E in terms of representation theoretic data of π_f (or ambitiously there locally associated Weil-Deligne representations). More specifically, let us decompose π_f as a restricted tensor product $\displaystyle\bigotimes_{p\text{ prime}}{}' \pi_p$ (as in [Fla79]) with respect to some set of hyperspecial maximal compact subgroups $K_{0,p}=\mathcal{G}_p(\mathbb{Z}_p)\subseteq G(\mathbb{Q}_p)$, where p travels over a cofinite set of rational primes, and where $\mathcal{G}_p$ is a reductive model of $G_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$.

Then, for almost all p the irreducible admissible $G(\mathbb{Q}_p)$-representation π_p is unramified (with respect to $K_{0,p}$) and $\sigma^i(\pi_f)$ is unramified at all v dividing p. Since π_p is unramified it thus defines a semisimple $\widehat{G}$-conjugacy class in the L-group ${}^LG=\widehat{G}\rtimes W_{\mathbb{Q}_p}$ of the form $t\rtimes\Phi$ where Φ is a lift of Frobenius (e.g. see [BR94, §1.11, 1.12]). Moreover, associated to (G,X), or more specifically X, is a representation $r_{-\mu}:\widehat{G}\rtimes W_{E_v}\to \mathrm{GL}(V)$ where V is some $\overline{\mathbb{Q}_\ell}$-space (where μ is as in §8.3 and $r_{-\mu}$ is as in [Kot84, §2.1.1]) where $v\mid p$ is some place of E. One might then hope, to relate the characteristic polynomials $P(\sigma^i(\pi_f))$ of Φ_{E_v} on $\sigma^i(\pi_f)$ to the characteristic polynomial $P(\pi_p)$ of $r_{-\mu}(t\rtimes\Phi_{E_v})$ on V.

Historically the study of the decomposition of these cohomology groups for PEL type Shimura varieties has been couched in the language of L-functions. For example, let us suppose that we are still in the situation where $\mathsf{Sh}_K(G,X)$ is proper for all K. Then, for a fixed K one can associate to $\mathsf{Sh}_K(G,X)$ its Hasse-Weil zeta function

$$\zeta(\mathsf{Sh}_K(G,X),s) := \prod_v \zeta_v(\mathsf{Sh}_K(G,X),s) \tag{1.4}$$

where $\zeta_v(\mathsf{Sh}_K(G,X),s)$ is (by definition) the local Hasse-Weil factor of the proper E_v-variety $\mathsf{Sh}_K(G,X)_{E_v}$ (e.g. see [Sch11, Definition 7.1]). One can then try to phrase relationships between the characteristic polynomials of Φ_{E_v} on $\sigma^i(\pi_f)$, where we assume that $\pi_f^K \neq 0$ appears with non-zero multiplicity in some $H^i(\mathsf{Sh}_K(G,X),\overline{\mathbb{Q}_\ell})$, to the characteristic polynomial of $r_{-\mu}(t \rtimes \Phi_{E_v})$ on V (for a place v as in the last paragraph) in terms of the relationship between $\zeta_v(\mathsf{Sh}_K(G,X),s)$ and the local L-function $L(\pi_p, r_{-\mu}, s)$ (as in [AG91, §2]). For example, let us denote by

$$P(\sigma(\pi_f)) := \prod_{i=0}^{2d} P(\sigma^i(\pi_f))^{(-1)^i} \tag{1.5}$$

where $d := \dim \mathsf{Sh}_K(G,X)$ (note that this number is independent of K). Then, if there were some integer $a(\pi_f) \in \mathbb{Z}$ such that

$$P(\sigma(\pi_f)) = P(\pi_p)^{a(\pi_f)} \tag{1.6}$$

then we would deduce, in particular, that

$$\zeta_v(\mathsf{Sh}_K(G,X),s) = \prod_{\pi_f} L(\pi_p, r_{-\mu}, s)^{a(\pi_f)} \tag{1.7}$$

where π_f is as above.

Much work on showing this so-called 'automorphicity' of the Hasse-Weil zeta functions for $G = \mathrm{GL}_2$ (i.e. the modular curve case) was done by, amongst others, Eichler, Shimura, Kuga, Sato, and Ihara. But, it was Langlands who formulated the first somewhat general version of these automorphicity conjectures in the seminal series of papers, [Lan77], [Lan79a], and [Lan82]. Langlands also carried out work towards a proof of his conjectures in certain PEL cases in [Lan79b].

The next major contribution to these conjectures was due to Kottwitz. Namely, in [Kot90] Kottwitz gave an essentially fully general version of the conjecture for how the cohomology of proper Shimura varieties (or, in the non-proper case, the contribution to the interior of the intersection cohomology of the Shimura variety with middle perversity) should decompose into irreducible $G(\mathbb{A}_f) \times \mathrm{Gal}(\overline{E}/E)$-representations. He then went on to actually verify in [Kot92a], without any prerequisite conjectures, the equality (1.6) for a very specific family PEL type Shimura varieties and for the set of all places v lying over a cofinite set of rational primes p.

The workhorse result that Kottwitz needed in [Kot92a] (which also underlies the conjectures in [Lan82] and [Kot90]) are formulas describing in useful terms the trace of the $\mathscr{H}(G(\mathbb{A}_f)) \times W_{E_v}$-action (where v is some finite place of E) on

$$H_c^*(\mathsf{Sh}(G,X)_{\overline{E}}, \mathcal{F}_\xi) := \sum_{i=0}^{2d} (-1)^i H_c^i(\mathsf{Sh}(G,X)_{\overline{E}}, \mathcal{F}_\xi) \tag{1.8}$$

(where this sum is taken in the Grothendieck group of $\overline{\mathbb{Q}_\ell}$-representations of $G(\mathbb{A}_f) \times W_{E_v}$) where $\mathsf{Sh}(G,X)$ were certain types of proper Shimura varieties of PEL type.

Namely, in [Kot92b] Kottwitz shows the following:

Theorem 1.0.1 (Imprecise version of §19 of [Kot92b])**.** *For certain Shimura datum (G, X) of PEL type with proper Shimura varieties there is in an equality of the trace*

$$\operatorname{tr}\left(f^p \mathbb{1}_{K_{0,p}} \times \tau \mid H_c^*(\mathsf{Sh}(G,X)_{\overline{E}}, \mathcal{F}_\xi)\right) \tag{1.9}$$

with a sum of the form

$$\sum_{\substack{(\gamma_0,\gamma,\delta) \\ \alpha(\gamma_0,\gamma,\delta)=1}} c(\gamma_0,\gamma,\delta) O_\gamma(f^p) TO_\delta(\phi_j) \operatorname{tr}(\xi(\gamma_0)) \tag{1.10}$$

For full references for this notation see §4.4, but let us roughly indicate their meaning:

- The function f^p is an element of the $\overline{\mathbb{Q}_\ell}$-Hecke algebra for $G(\mathbb{A}_f^p)$ (where $\mathbb{A}_f^p$ are the ring of finite adles with trivial component at p).
- The integer j is the power of Frobenius that τ induces modulo p.
- The function $\mathbb{1}_{K_{0,p}}$ is the indicator function on the hyperspecial subgroup $K_{0,p}$.
- The data $(\gamma_0, \gamma, \delta)$ (so-called *Kottwitz triples*) travels over certain semisimple conjugacy classes $\gamma = (\gamma_\ell)$ in $G(\mathbb{Q}_p)$ for all $\ell \neq p$, δ a semisimple conjugacy class in $G(E_j)$ (where E_j is the unramified extension of E_v of degree j), and γ_0 is a semisimple conjugacy class in $G(\mathbb{Q})$ which is elliptic in $G(\mathbb{R})$ and for which (roughly) γ_0 is (stably) conjugate to γ_ℓ for all ℓ and $N(\delta)$ (the norm of δ) is (stably) conjugate to γ_0.
- The term $c(\gamma_0, \gamma, \delta)$ is essentially a volume term.
- The symbol $O_\gamma(f^p)$ denotes the oribtal integral of f^p (see [Kot84, §1.5]).
- The function ϕ_j is the indicator function on $\mathcal{G}_p(\mathcal{O}_{E_j})$.
- The symbol $TO_\delta(\phi_j)$ is the twisted orbital integral of ϕ_j (see loc. cit.).

The reason that such a formula is desirable is that it is in a form which is amenable to comparison to the Arthur-Selberg trace formula (e.g. see [Gel96]). More specifically, one can stabilize a formula such as (1.10) to make it appear as in the geometric side of the Arthur-Selberg trace formula. One can then, very roughly, use the Arthur-Selberg trace formula to relate such a quantity to a spectral quantity, allowing one to relate the trace (1.9) to traces of automorphic representations on endoscopic groups of G. Finally, one can then destabilize this sum of traces to equate (1.9) to a sum of traces of automorphic representations on G itself. This is then, in some sense, a trace analogue of a formula such as (1.6). This is all explained, in great detail, in [Kot90].

One generally calls a formula relating terms of the form (1.9) and (1.10) a *Langlands-Kottwitz type equation* and its uses to understand the Galois representations showing up in the cohomology of Shimura varieties the *Langlands-Kottwitz method*.

One of the major limitations of the original Langlands-Kottwitz method is the apparently inherent need to only take traces of functions in the Hecke algebra whose p^{th}-component is the indicator function on a hyperspecial. Namely, the aforementioned

Langlands-Kottwitz type equations essentially relied on interpreting the traces as in (1.9) as weighted point counts on the special fiber of these Shimura varieties (at places of good reduction), using the Fujiwara-Varshavsky trace formula, and then explicitly computing these point counts by exploiting the concrete nature of the moduli problem of these varieties (as moduli spaces of decorated abelian varieties) using a generalized notion of Honda-Tate theory.

Of course, for any of this to make sense one needs to essentially work with a finite level Shimura variety $\mathsf{Sh}_K(G,X)$ which has good reduction at p. Essentially, the restriction to functions of the form $f^p\mathbb{1}_{K_{0,p}}$ is equivalent to working with finite level Shimura varieties of the form $\mathsf{Sh}_{K^pK_{0,p}}(G,X)$ which do have good reduction at p. So, in some sense, one should roughly interpret the restriction of the p^{th}-component to be the indicator function of a hyperspecial as a restriction to only being able to study the cohomology of these Shimura varieties at levels of good reduction.

The desire to have a version of the Langlands-Kottwitz method without these types of good reduction hypotheses, was brought into sharp relief by the work of Harris and Taylor in [HT01]. Namely, therein the authors actually describe the Galois representation

$$\sigma(\pi_f) := \sum_{i=0}^{2d} (-1)^i \sigma^i(\pi_f) \tag{1.11}$$

(again the sum taking place in the Grothendieck group of $\overline{\mathbb{Q}_\ell}$-representatons of W_{E_v}) for (G,X) a very specific case of the Shimura datum as in [Kot92a], but not just at hyperspecial level (levels of good reduction) but for general level. They then use the results they obtain to prove the local Langlands conjecture (as in the introduction of loc. cit.) for the groups $\mathsf{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_{n,F}$ for an arbitrary finite extension $F/\mathbb{Q}_p$.

The methods Harris and Taylor used in [HT01] to extend the results of [Kot92a] to bad reduction is not a direct generalization of the Langlands-Kottwitz method. Instead, Harris and Taylor study the cohomology of Shimura varieties by relating them to the a priori simpler cohomology of Igusa varieties and explaining the decomposition of the cohomology of Shimura varieties in terms of the cohomology of Igusa varieties and Rapoport-Zink spaces (see the discussion in the introduction to [HT01]). This latter work would later be put in a general framework for PEL type Shimura varieties by Mantovan in [Man05].

Any attempt to have proven the local Langlands conjecture for the group $\mathsf{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_{n,F}$ using solely the Langlands-Kottwitz method is bound to fail. Indeed, as mentioned above this method can only access the cohomology of the relevant Shimura varieties at levels of good reduction. Consequently, the method will only obtain infromation about unramified Galois representations, a very small subset of the totality of Galois representations present in the full local Langlands conjecture.

It was not until the work of Scholze in the series of papers [Sch11], [Sch13b], and [Sch13c] that a true generalization of the Langlands-Kottwitz method to cases of bad (i.e. non-hyperspecial at p) level was obtained in some proper PEL cases. Namely, in this work Scholze shows the following:

Theorem 1.0.2 (Imprecise version of Theorem 5.7 of [Sch13c])**.** *For certain Shimura datum (G,X) of PEL type with proper Shimura varieties there is in an equality of the* *trace*

$$\operatorname{tr}\left(f^p h \times \tau \mid H_c^*(\mathsf{Sh}(G,X)_{\overline{E}}, \mathcal{F}_\xi)\right) \tag{1.12}$$

with a sum of the form

$$\sum_{\substack{(\gamma_0,\gamma,\delta)\\ \alpha(\gamma_0,\gamma,\delta)=1}} c(\gamma_0,\gamma,\delta) O_\gamma(f^p) TO_\delta(\phi_{\tau,h}) \operatorname{tr}(\xi(\gamma_0)) \tag{1.13}$$

Let us explain the discrepancies between this result, which we call a *Langlands-Kottwitz-Scholze type formula*, and Theorem 1.0.1.

- The function h is now allowed to be any compactly supported locally constant function $K_{0,p} \to \mathbb{Q}$ (opposed to the indicator function $\mathbb{1}_{K_{0,p}}$).
- The function $\phi_{\tau,h}$ is some locally constant compactly supported function $G(E_j) \to \mathbb{Q}$ that depends only on the local data of the pair τ, h (and, implicitly, the local data of $\mathcal{G}_p$ and a conjugacy class of cocharacters μ of $G_{\overline{\mathbb{Q}_p}}$ agreeing, up to conjugacy, with the conjugacy class of cocharacters coming from X).

Let us expound on this function $\phi_{\tau,h}$ and, in doing so, indicate the difference in method of proof for Theorem 1.0.2 and Theorem 1.0.1. As mentioned before, the restriction $h = \mathbb{1}_{K_{0,p}}$ in Theorem 1.0.1 allows one to work with the cohomology of a finite level Shimura variety $\mathsf{Sh}_{K^pK_0}(G,X)$ which has good reduction at p since $K_{0,p}$ is hyperspecial. Allowing an arbitrary locally constant and compactly supported function $h : K_{0,p} \to \mathbb{Q}$ essentially necessitates working with the cohomology of a finite level Shimura variety of the form $\mathsf{Sh}_{K^pK_p}(G,X)$ where $K_p \subseteq K_{0,p}$. This finite level Shimura variety does not necessarily have have good reduction. But, since we have a finite étale surjection $f : \mathsf{Sh}_{K^pK_p}(G,X) \to\ Sh_{K^pK_0}(G,X)$ one can essentially compute the étale cohomology of a sheaf on $\mathcal{F}$ on $\mathsf{Sh}_{K^pK_p}(G,X)$ in terms of the cohomology of its pushforward $f_*\mathcal{F}$. Scholze then shows that this cohomology can still be computed using a Fujiwara-Varshavsky trace formula argument essentially identical to that in [Kot92b] but with an extra weighting factor, visible as the term $TO_\delta(\phi_{\tau,h})$ in Theorem 1.0.2, essentially accounting for the extra data coming from working at the lower level $K_p \subseteq K_{0,p}$ which explicitly appears as a certain nearby cycle term.

Using the work of Berkovich and Huber on relating nearby cycles to the cohomology of their tubular neighborhood Scholze is then able to identify this extra weighting factor in terms of the cohomology of certain open adic subspaces of the analytification $\mathsf{Sh}_{K^pK_p}(G,X)^{\mathrm{an}}_{E_v}$ of $\mathsf{Sh}_{K^pK_p}(G,X)$ over $\operatorname{Spa}(E_v)$. Then, using methods similar to the uniformization results of Shimura varieties by Rapoport-Zink spaces, one is able to describe these open adic subspaces in purely local terms as decorated deformation spaces of certain p-divisible groups with level data. This shows that these functions $\phi_{\tau,h}$ have purely local descriptions as desired. Finally, one verifies that these decorated deformation spaces have reasonably well-behaved cohomology, showing that the function $\phi_{\tau,h}$ is itself well-behaved.

Scholze then went on to give an alternative proof of the local Langlands conjecture for $\mathsf{Res}_{F/\mathbb{Q}_p}\mathrm{GL}_{n,F}$ using his generalization of the Langlands-Kottwitz method in [Sch13d]. Moreover, he did so by first using Theorem 1.0.2 to give an explicit characterization of the local Langlands conjecture. In essence showing that the unique association $\pi \mapsto \mathsf{LL}(\pi)$ satisfying the usual axioms characterizing the local Langlands conjecture (e.g. as in the introduction [HT01]) is characterized by the property that

$$\mathrm{tr}(f_{\tau,h} \mid \pi) = \mathrm{tr}(\tau \mid \mathsf{LL}(\pi))\mathrm{tr}(h \mid \pi) \tag{1.14}$$

holds for all $\tau \in W_F$ and h a locally constant compactly supported function $K_{0,p}$ and where $f_{\tau,h}$ is essentially the transfer of the function $\phi_{\tau,h} : G(E_j) \to \mathbb{Q}$ to a function $f_{\tau,h} : G(\mathbb{Q}_p) \to \mathbb{Q}$.

This result would go on to be generalized by the work of Scholze and Shin in [SS13] where they extend (and prove in certain cases) the equation (1.14) to a conjectural relationship between the functions $f_{\tau,h}$ and the local Langlands correspondence for a larger class of groups other than GL_n (essentially those coming from certain global PEL type Shimura datum) where, of course, a more sophisticated version of (1.14) is required due to the presence of endoscopy (see [SS13, Conjecture 7.1] for the precise statement).

Essentially all of the above cases of Langlands-Kottwitz or Langlands-Kottwitz-Scholze type formulas have been restricted to proper Shimura varieties of PEL type (or very closely related types). It then seems desirable to want to extend such results to larger classes of Shimura varieties. The reason for this is twofold. The first is the obvious desire to not have to restrict our attention to the cohomology of the limited class of Shimura varieties which are both proper and for which the Shimura datum is PEL type. The less obvious is the fact that with an eye towards using the Langlands-Kottwitz-Scholze method to study the local Langlands conjecture (e.g. formulating and proving versions of the Scholze-Shin conjecture) having results like Theorem 1.0.2 for more general classes of Shimura varieties gives us access to these techniques for a wider range of groups. As a simple example, unitary groups have naturally associated Shimura varieties which are not of PEL type (e.g. see Example 2.1.8).

The most natural large class of Shimura varieties are the class of so-called *Shimura varieties of abelian type*. These were one of the main focuses of Deligne in his seminal article [Del79] where he pleasantly classifies abelian type Shimura datum (G, X) in terms of the group G (something not present for smaller classes of Shimura datum). Abelian type Shimura varieties should, admitting the Tate conjecture, have moduli interpretations in fairly general stituations (e.g. see [Mil94, §3]) but there are very few Shimura varieties of abelian type (not of PEL type) which admit unconditionally describable moduli problems. This presents a serious barrier to extending the Langlands-Kottwitz method, and thus the Langlands-Kottwitz-Scholze method, to this large class of varieties.

Indeed, when natural moduli interpretations exist for Shimura varieties it is usually not a difficult task to hypothesize what natural integral models $\mathscr{S}_{K^pK_{0,p}}$ of these Shimura varieties $\mathsf{Sh}_{K^pK_{0,p}}$ at hyperpspecial level should be (i.e. $\mathscr{S}_{K^pK_{0,p}}$ should have an integral analogue of the moduli interpretation of $\mathsf{Sh}_{K^pK_{0,p}}$) and, usually then it is conceivable to also formulate a method to parameterize the points $\mathscr{S}_{K^pK_{0,p}}(\overline{\mathbb{F}_p})$ (a 'generalized Honda-Tate theory'). But, since Shimura varieties of abelian type lack natural moduli interpretation it is not at all clear how to create 'good models' nor how do parameterize the $\overline{\mathbb{F}_p}$-points of such models.

Despite all of this, deep insights of many mathematicians has given a conjectural framework in which one can handle these issues. Namely, the work of Milne (e.g. see

[Mil92]) and others has defined a good candidate for what a 'good model' of an arbitrary Shimura variety is, a so-called *integral canonical model* (although beware that the original notion of integral canonical models was slightly flawed—see the discussion in [Moo98, §3]). In addition, work of Langlands and Rapoport in [LR87], now called the *Langlands-Rapoport conjecture*, suggests how to usefully parameterize the $\overline{\mathbb{F}_p}$-points of the special fiber of these integral canonical models in a way analogous to Honda-Tate theory (see [Mil92] for a nice introduction to these topics).

While significant incremental progress had been made on both the existence of integral canonical models and the Langlands-Rapoport conjecture it was not until a series of papers by Kisin that general results were known. Namely, in [Kis10] Kisin shows that integral canonical models exist for arbitrary Shimura varieties of abelian type. Then, in [Kis17] Kisin showed that the Langlands-Rapoport conjecture does in fact hold for Shimura varieties of abelian type, up to a certain twist (see [Kis17, Theorem 4.6.7] for a precise statement).

This opens up the possibility to prove a Langlands-Kottwitz type formula, an analogue of Theorem 1.0.1, in the situation of abelian type Shimura varieties. This is, up to twist introduced by Kisin's solution to the Langlands-Rapoport conjecture, precisely the content of the upcoming work [KSZ] of Kisin, Shin, and Zhu. Namely, we record this as the following imprecise statement:

Theorem 1.0.3 (Imprecise version of expected result, [KSZ])**.** *A result like Theorem 1.0.1 should hold for p odd when (G, X) is of abelian type and G^{der} is simply connected.*

We have chosen to actually state a less general version of what is expected out of [KSZ]. Namely, there the authors handle the case of arbitrary Shimura varieties. The restriction to the case when G^{der} is simply connected allows one to not have to change the notation in Theorem 1.0.1 wildly, whereas the version of Theorem 1.0.1 in the general setting requires a more sophisticated indexing set than triples $(\gamma_0, \gamma, \delta)$ (requiring, instead, so-called *cohomological Kottwitz triples*).

Admitting this expected result of Kisin, Shin, Zhu one could imagine that a version of Theorem 1.0.2 might be obtained by a similar technique utilized by Scholze to adapt the work of Kottwitz in [Kot92b]. Namely, as discussed after the statement of Theorem 1.0.2 one should be able to carry over the work of Kisin, Shin, and Zhu, exactly analagously to how Scholze carried over the work of Kottwitz in [Sch13c], to obtain a Langlands-Kottwitz-Scholze type formula in the abelian type case assuming that one can make work in this general situation the technical details in [Sch13c]. In particular, there is a need to overcome the following technical obstacles:

- The yoga of computing the trace in terms of a Fujiwara-Varshavsky type trace formula with weighting factors corresponding to local rigid geometry of the analytification of $\mathsf{Sh}_{K^pK_p})_{E_v}$ (including the extension to the non-proper case).
- The existence of towers of rigid analytic spaces definable in purely local terms which agree with the local rigid geometry of these Shimura varieties by some sort of Serre-Tate type theory
- The well-behavedness of the cohomology of these spaces.

Admitting all of this, one would then have the following imprecise expected theorem:

Theorem 1.0.4 (Imprecise expected theorem)**.** *A result like Theorem 1.0.2 should hold when p is odd, (G, X) is of abelian type, and G^{der} is simply connected.*

The results of this thesis are essentially verifying that the above bulleted technical obstacles can be overcome, making way to prove the above expected theorem once the results of [KSZ] are released.

Let us finally outline the organization of this thesis. In §2 we verify that the first bulleted obstacle can be overcome in the case of arbitary Shimura varieties of abelian type, developing a general Fujiwara-Varshavsky like trace formula (proven in even greater generality in Appendix C) of pushforward sheaves. In §3 we develop the theory of these locally defined rigid analytic towers (which we call 'deformation spaces') and verify that they uniformize the local rigid geometry of abelian type Shimura varieties in the desired way, thus overcoming the second bulleted obstacle. Finally, in §4 we verify that these towers of deformation spaces have reasonably well-behaved cohomology, overcoming the last bulleted obstacle, and formulate more precisely Theorem 1.0.4.

Chapter 2

Shimura varieties of abelian type and a trace formula

In this section we apply the machinery of Appendix C to Shimura varieties of abelian type, using the integral canonical models of such Shimura varieties which were shown to exist in [Kis10]. The result is a formula for the trace of a Hecke operator on a Shimura variety at bad level in terms of a weighted point count on the special fiber of the integral canonical model at lower good level.

The weighting factor is in terms of the local rigid geometry of the analytification of the Shimura variety. Versions of this formula are present in [Sch13c] and [Far04], but neither are general enough to account for what we need.

For the remainder of the section we assume that all rational primes p are odd, a choice of convenience not necessity.

2.1 Integral canonical models of Shimura varieties of abelian type

We begin here by recalling the theory of integral canonical models for Shimura varieties of abelian type at hyperspecial level whose existence was proven, in this level of generality, by Kisin in [Kis10] (with the assumption that $p > 2$). We do this mostly as a means of fixing notation, but also to emphasize some aspects of [Kis10] that will be of particular importance to us, and make more convenient the proof of some basic properties of these models necessary to apply the formalism of Appendix C.

We will attempt to make this section relatively self-contained, but this is nigh impossible. For a nice introduction, both to some of the technical background of integral canonical models, as well as their history, the reader should consult [Moo98] and [Mil92] in addition to the article [Kis10], as well as the article [Kis17, 1.3] which makes minor corrections to the contents of [Kis10].

Remark 2.1.1. In the following we will assume that the reader is decently well-versed in the classical theory of Shimura varieties. For a reminder on the aspects of Shimura varieties that we will be particularly focused, in particular their relationship to **G**-schemes (the technical formalism that our first trace formula is couched in) see Appendices C and D.

Shimura varieties of abelian type

Let us denote by (G, X) a pair consisting of:

- G a connected reductive group over $\mathbb{Q}$.
- X a left $G(\mathbb{R})$-conjugacy class of morphisms of $\mathbb{R}$-algebraic groups $\mathbb{S} \to G_{\mathbb{R}}$

Here $\mathbb{S}$ denotes Deligne torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$ which is thought of as the fundamental group of the neutralized Tannakian category of $\mathbb{R}$-Hodge structures. We assume that our pair satisfies the axioms SV1, SV2, and SV3 (in the parlance of [Mil04, pg. 55]) in which case we call it a *Shimura datum*.

We shall denote by $E := E(G, X)$ the reflex field of (G, X). Moreover, for all $K \in \mathcal{N}(G)$, where $\mathcal{N}(G)$ denotes the set of neat compact open subgroups of $G(\mathbb{A}_f)$, we shall denote the associated Shimura variety over E by $\mathsf{Sh}_K(G, X)$. In particular, when we say 'Shimura variety' we will really mean 'the canonical model of the Shimura variety over its reflex field', making it notationally clear when we we are thinking of the Shimura variety as a variety over $\mathbb{C}$ or a complex analytic space.

For K and L in $\mathcal{N}(G)$, and $g \in G(\mathbb{A}_f)$ such that $L \supseteq g^{-1}Kg$ we denote by $t_{K,L}(g)$ the usual map

$$\mathsf{Sh}_K(G, X) \to \mathsf{Sh}_L(G, X) \tag{2.1}$$

defined as projection to $\mathsf{Sh}_{gLg^{-1}}(G, X)$ and then action by the element g. We will shorten $t_{K,L}(\mathrm{id})$ to $\pi_{K,L}$ and $t_{K,g^{-1}Kg}(g)$ to $[g]_K$. In particular, every morphism $t_{K,L}(g)$ can be decomposed as $[g]_{gLg^{-1}} \circ \pi_{K,gLg^{-1}}$.

Let us next recall the conditions we will require our Shimura datum to satisfy in this article. Namely, recall that a Shimura datum (G, X) is called *of Hodge type* if there exists a closed embedding of Shimura datum

$$i : (G, X) \hookrightarrow (G(\psi), \mathfrak{h}_{\psi}^{\pm}) \tag{2.2}$$

for some symplectic space (V, ψ) over $\mathbb{Q}$ and where $(G(\psi), \mathfrak{h}_{\psi}^{\pm})$ denotes the standard Siegel datum attached to (V, ψ) (see [Mil04, Chapter 6]), where we have written $\mathrm{GSp}(V, \psi)$ as $G(\psi)$ for short.

Recall that the meaning of the embedding (2.2) is that i is a closed embedding of $\mathbb{Q}$-groups $G \hookrightarrow G(\psi)$ such that $i(\mathbb{R})$ takes (every element of) the $G(\mathbb{R})$-conjugacy class X of morphisms $\mathbb{S} \to G_{\mathbb{R}}$ into the $G(\psi)(\mathbb{R})$-conjugacy class $\mathfrak{h}_{\psi}^{\pm}$ of morphisms $\mathbb{S} \to G(\psi)_{\mathbb{R}}$.

An embedding as in (2.2) is certainly not unique, but we will often times implicitly fix such an embedding (perhaps with some conditions, see Lemma 2.1.15) without comment. If (G, X) is a Shimura datum of Hodge type, we will often times imprecisely refer to the associated Shimura variety as being *of Hodge type*.

For the convenience of the reader, we recall some classical examples of Shimura varieties of Hodge type:

Example 2.1.2. All Shimura varieties of PEL type (e.g. see [Mil04, Chapter 8]) are of Hodge type.

Example 2.1.3. Let (V, q) be a quadratic space over $\mathbb{Q}$ of signature $(n, 2)$. One can then attach to this a reductive group $\mathrm{GSpin}(V, q)$ over $\mathbb{Q}$ whose derived subgroup $\mathrm{Spin}(V, q)$ is the universal cover of $\mathrm{SO}(V, q)$ (see [Per16, §1]). One can show that $\mathrm{GSpin}(V, q)$ acts transitively on the space X of oriented negative definite 2-planes in $V_{\mathbb{R}}$, and that X can be identified with a $\mathrm{GSpin}(V, q)(\mathbb{R})$-conjugacy class of morphisms of $\mathbb{R}$-algebraic groups $\mathbb{S} \to \mathrm{GSpin}(V, q)_{\mathbb{R}}$. The pair $(\mathrm{GSpin}(V, q), X)$ is then a Shimura variety of Hodge type which is not of PEL type.

Shimura varieties of this type are of particular visibility amongst Hodge type Shimura varieties which are not PEL type considering their role in the proof of the Tate conjecture for K3 surfaces in odd characteristics by Madapusi Pera (see Example 2.1.7 below and [MP15]).

We shall be working with a larger class of Shimura varieties than those of Hodge type however. Namely, recall that a Shimura datum (G, X) is called *of abelian type* if there is a Shimura datum (G_1, X_1) of Hodge type and a (necessarily central) isogeny $G_1^{\mathrm{der}} \to G^{\mathrm{der}}$ inducing an isomorphism

$$(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \xrightarrow{\approx} (G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \tag{2.3}$$

of the associated adjoint Shimura datum. As with the case for Shimura data of Hodge type, if (G, X) is a Shimura datum of abelian type then we will often times call the associated Shimura variety *of abelian type*. If (G, X) is a Shimura datum of Hodge type, we shall call a Hodge type datum (G_1, X_1) for which there exists an isogeny $G_1^{\mathrm{der}} \to G^{\mathrm{der}}$ inducing an isomorphism as in (2.3) an *associated Hodge type datum*, leaving the isogeny implicit.

Again, for the convenience of the reader, let us recall some canonical examples of abelian type Shimura datum:

Example 2.1.4. Every Shimura variety of Hodge type is evidently of abelian type.

Example 2.1.5. One of the simplest non-obvious class of examples of Shimura varieties of abelian type but not of Hodge type are the Shimura curves. Namely, let F be a totally real field with $d := [F : \mathbb{Q}] > 1$. Let B be an F-quaternion algebra which is split at only one real place of F. Let G be the reductive $\mathbb{Q}$-group defined by the following moduli problem:

$$G(R) := (R \otimes_{\mathbb{Q}} B)^\times \tag{2.4}$$

for R a $\mathbb{Q}$-algebra. Note then that $G_{\mathbb{R}} \cong \mathrm{GL}_{2,\mathbb{R}} \times H^{d-1}$ where H is the $\mathbb{R}$-algebraic group with moduli problem

$$H(S) := (S \otimes_{\mathbb{R}} \mathbb{H})^\times \tag{2.5}$$

where S is an $\mathbb{R}$-algebra and $\mathbb{H}$ is the Hamiltonian quaternions.

Let $h_0 : \mathbb{S} \to G_{\mathbb{R}}$ be the morphism projecting trivially to H^{d-1} and having $\mathbb{S} \to \mathrm{GL}_{2,\mathbb{R}}$ define the complex structure on $\mathbb{R}^2$ with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let X be the $G(\mathbb{R})$-conjugacy class of h_0. Then, (G, X) is a Shimura datum (note that while h_0 does project trivially to a simple factor, this factor is not defined over $\mathbb{Q}$). The associated Shimura varieties are called *Shimura curves*. They are coarse moduli spaces of isogeny classes of abelian varieties with B-action.

More generally, if B is any F-quaternion algebra one can associated an F-Shimura datum (G_B, X_B) in much the same way as above. If B is not split at infinity (e.g. as in the case of Shimura curves) then this Shimura datum will be of abelian type, but not of Hodge type (e.g. see [Mil04, Page 94]).

Example 2.1.6. Let us suppose that (G, X) is a Shimura datum of Hodge type. We then claim that the adjoint Shimura datum $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is abelian type, but not Hodge type. Indeed, the natural isogeny $G^{\mathrm{der}} \to G^{\mathrm{ad}}$ evidently induces an isomorphism of adjoint Shimura datum (the identity) $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \xrightarrow{\approx} (G^{\mathrm{ad}}, X^{\mathrm{ad}})$ so that $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is evidently abelian type.

To see that it's not Hodge type let us make the following simple observation. If (H, Y) is a Shimura datum of Hodge type then, by definition, there exists an embedding of Shimura datum $\iota : (H, Y) \hookrightarrow (G(\psi), \mathfrak{h}_\psi^\pm)$ for some symplectic space (V, ψ) over $\mathbb{Q}$. Note then that $w_{(G(\psi),\mathfrak{h}_\psi^\pm)} = \iota \circ w_{(H,Y)}$ (where the w denotes the weight homomorphism for the Shimura datum) and, in particular, since $\mathfrak{h}_\psi^\pm$ consists of Hodge structures on $V_\mathbb{R}$ of type $\{(-1, 0), (0, -1)\}$ we see that that $w_{(G(\psi),\mathfrak{h}_\psi^\pm)}$, and thus $w_{(H,Y)}$, is injective. Thus, we deduce that $Z(H)_\mathbb{R}$ contains a copy of $\mathbb{G}_{m,\mathbb{R}}$. So, $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ can't be of Hodge type since $Z(G^{\mathrm{ad}})_\mathbb{R}$ is trivial, and thus cannot contain $\mathbb{G}_{m,\mathbb{R}}$.

So, for example, $(\mathrm{PSp}(\psi), (\mathfrak{h}_\psi^\pm)^{\mathrm{ad}})$ is an example of a Shimura datum that is of abelian type, but not of Hodge type.

Example 2.1.7. Let (V, q) be a quadratic space over $\mathbb{Q}$ of signature $(19, 2)$ and let X be as in Example 2.1.3. Then, the pair $(\mathrm{SO}(V, q), X)$ is a Shimura datum of abelian type not of Hodge type. Its associated Shimura datum of Hodge type can be taken to be the datum from Example 2.1.3. This Shimura variety, and in particular its integral canonical model, are of great importance in the aforementioned work of Madapusi Pera (see [MP15]) where it can be shown that the integral canonical model of such a Shimura variety contains as an open subscheme the moduli space of polarized K3 surfaces with level structure (see loc. cit. as well as [Riz10]).

Example 2.1.8. As a last example, let us explain the existence of a Shimura variety of abelian type (G, X) with G a unitary group over $\mathbb{Q}$. Namely, let F be a totally real extension of $\mathbb{Q}$ and let E be an imaginary quadratic extension of F. Let $(W, \langle -, - \rangle)$ be a hermitian space of dimension n relative to E/F and let $U := U(W, \langle -, - \rangle)$ be the associated unitary group over F. Note then that for all real places v_i of F there is evidently an isomorphism

$$U_{v_i} \cong U(p_i, q_i) \tag{2.6}$$

where $U(p_i, q_i)$ is the unitary group over $\mathbb{R}$ of signature (p_i, q_i). Let us say that U is of *non-compact type* if $U(\mathbb{R})$ is non-compact. Equivalently, for at least one place v_i we have that both $p_i \neq 0$ and $q_i \neq 0$ (i.e. that $U(p_i, q_i)(\mathbb{R})$ is non-compact).

So, let us assume that U is of non-compact type and set $G := \mathsf{Res}_{F/\mathbb{Q}} U$. Let

$$h : \mathbb{S} \to G_\mathbb{R} \cong \prod_i U(p_i, q_i) \tag{2.7}$$

(where we have a priori fixed this latter isomorphism) be defined in terms of its projections h_i defined as follows. If $p_i = 0$ or $q_i = 0$ we define h_i to be trivial. Otherwise, define h_i as follows:

$$h_i(z) := \begin{pmatrix} \frac{z}{\overline{z}} & & & & & \\ & \ddots & & & & \\ & & \frac{z}{\overline{z}} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \tag{2.8}$$

where there are p_i entries of $\frac{z}{\overline{z}}$ and q_i entries of 1. Set X to be the $G(\mathbb{R})$-conjugacy class of h. We claim that (G, X) is a Shimura datum of abelian type which is not of Hodge type.

The fact that (G, X) is a Shimura datum is elementary and left to the reader (the assumption that U is of non-compact type being used in Axiom SV3 of [Mil04]). To see that it's of abelian type, we must find an associated Hodge type datum. Let $GU(W, \langle -, - \rangle)$ denote the unitary similitude group of the hermitian space $(W, \langle -, - \rangle)$ and set $H := \mathsf{Res}_{F/\mathbb{Q}} GU(W, \langle -, - \rangle)$. We then define $H^{\mathbb{Q}}$ to be the fiber product $H \times_{\mathsf{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}} \mathbb{G}_{m,\mathbb{Q}}$ where the map $H \to \mathsf{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$ is the similitude character and the map $\mathbb{G}_{m,\mathbb{Q}} \to \mathsf{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$ is the usual inclusion. We define a morphism

$$h' : \mathbb{S} \to (H^{\mathbb{Q}})_{\mathbb{R}} \tag{2.9}$$

as follows. Begin by noting that

$$(H^{\mathbb{Q}})_{\mathbb{R}} = \left\{ (g_i) \in \prod_i GU(W_{v_i}, \langle -, - \rangle) : c(g_i) = c(g_j) \text{ for all } i, j \text{ and } c(g_i) \in \mathbb{R}^\times \right\} \tag{2.10}$$

Let us fix one such isomorphism. We then define h', via this fixed isomorphism, by its projections h'_i to each $GU(W_{v_i}, \langle -, - \rangle)$ by saying that if $U_{v_i} = U(p_i, q_i)$ then we define

$$h'_i(z) := \begin{pmatrix} z & & & & & \\ & \ddots & & & & \\ & & z & & & \\ & & & \overline{z} & & \\ & & & & \ddots & \\ & & & & & \overline{z} \end{pmatrix} \tag{2.11}$$

where there are p_i copies of z, and q_i copies of $\overline{z}$. One can then check that $(H^{\mathbb{Q}}, h')$ defines a PEL type Shimura datum (e.g. see [Mil04, Chapter 8]).

Note now that $(H^{\mathbb{Q}})^{\text{der}}$ is naturally isomorphic to $\mathsf{Res}_{F/\mathbb{Q}} SU(W, \langle -, - \rangle)$ which is, likewise, equal to G^{der}. Let $(H^{\mathbb{Q}})^{\text{der}} \to G^{\text{der}}$ be the identity map. It's not hard to see then that this induces an isomorphism of Shimura datum between $((H^{\mathbb{Q}})^{\text{ad}}, (h')^{\text{ad}})$ and $(G^{\text{ad}}, h^{\text{ad}})$. Thus, (G, X) is of abelian type.

Remark 2.1.9. One definitive advantage of working with the class of abelian type Shimura varieties is their simple classification due to Deligne (e.g. see [Mil04, Appendix B]) whereas the class of Hodge type Shimura varieties is not so easily parameterized.

Note that, almost by definition, if $\mathsf{Sh}_K(G,X)$ is of abelian type, with data as in (2.3), then every connected component of $\mathsf{Sh}_K(G,X)_{\overline{E}}$ admits a finite surjection from a connected component of the $\overline{\mathbb{Q}}$-scheme $\mathsf{Sh}_L(G_1,X_1)_{\overline{E_1}}$. We will note later that such a property holds also in the context of integral canonical models. Thanks to this observation, we are usually able to deduce basic structural results about Shimura varieties of abelian type from general structural results about Shimura varieties of Hodge type (e.g. see Lemma 2.1.29 below).

Some preliminaries on hyperspecial subgroups

The work of Kisin, completing a program based on the fundamental ideas of Langlands, Milne, and Vasiu (see [Moo98] for a history of the results before Kisin), produces integral canonical models, in the sense of §7.1, for the $G(\mathbb{A}_f^p)$-scheme associated to a Shimura variety of abelian type (base changed to an appropriate completion of its reflex field) by fixing the level at p to be some fixed hyperspecial subgroup of $G(\mathbb{Q}_p)$. In other words, this confirms the suspicion that hyperspecial levels are levels of 'good reduction' for Shimura varieties (at least in the case when the Shimura variety is of abelian type).

So, before we recall the definition of these integral canonical models it will be helpful to first remind ourselves of the definition, and basic properties, of hyperspecial subgroups of a p-adic reductive group. This will, in turn, necessitate a brief discussion of the theory of unramified group schemes which we do in this subsection.

For the definition of the titular objects as well as the characterization given by Lemma 2.1.11 we work in a more general context than p-adic groups (to clarify the ideas), but we will soon thereafter return to the more familiar setting of groups over $\mathbb{Q}_p$.

So, let us fix a Henselian discrete valuation ring $\mathcal{O}$ with fraction field F and finite residue field k and let H be a connected reductive group over F. We then recall that a compact open subgroup K_0 of $H(F)$ is called *hyperspecial* if there exists a connected reductive group scheme $\mathcal{H}$ over $\mathcal{O}$ such that $K_0 = \mathcal{H}(\mathcal{O})$. Here we recall that a group scheme $\mathcal{H}$ over $\mathcal{O}$ is called *reductive* if it's a smooth (equivalently flat) affine (equivalently linear by [DG70, VI$_B$.13]) group scheme over $\mathcal{O}$ with connected and reductive generic and special fibers $\mathcal{H}_k$ and $\mathcal{H}_F$.

Remark 2.1.10. In the case of p-adic local fields one can develop the theory of hyperspecial subgroups in the more general framework of Bruhat-Tits theory and, in particular, they can be characterized as stabilizers of certain points for the action of $G(F)$ on the Bruhat-Tits building. See [Tit79] as well as the fundamental papers [BT72] and [BT84] for a discussion of hyperspecial subgroups from this perspective. We have chosen to avoid Bruhat-Tits theory as much as possible to make the discussion easier to a non-expert in p-adic reductive groups.

By definition, if H is a connected reductive group over F, then $H(F)$ contains a hyperspecial subgroup if and only if H admits a connected reductive model over $\mathcal{O}$. It will be helpful for us (particularly in Lemma 2.1.17) to have a more concrete criterion for when such a reductive model exists. To this end, let us say that connected reductive group H over F is called *unramified* if it is quasi-split (i.e. has an F-rational Borel subgroup) and becomes split over a finite unramified extension of F.

We then have the following useful criterion for the existence of reductive models of a connected reductive group H:

Lemma 2.1.11. *Let H be a connected reductive group over F. Then, H is unramified if and only if there exists a reductive group scheme $\mathcal{H}$ over $\mathcal{O}$ such that $\mathcal{H}_F \cong H$.*

Proof. A proof can be inferred from [Tit79] in the case when F is a p-adic local field. We provide a proof here that obviates the usage of Bruhat-Tits theory in alignment with our avoidance of such theory in the definition of hyperspecial subgroups.

Suppose first that H admits a reductive model $\mathcal{H}$ over $\mathcal{O}$. Note then that by Lang's Theorem (see [Poo17, 5.12.6]) we have that $H^1(k, \mathrm{Inn}(\mathcal{H}_k)) = 0$ where $\mathrm{Inn}(\mathcal{H}_k) := \mathcal{H}_k/Z(\mathcal{H}_k)$ a connected reductive group over k. Thus, by the usual classification of $\mathrm{Inn}(\mathcal{H}_k)$-torsors all inner twists of $\mathcal{H}_k$ are trivial. But, as is well-known, over any field every reductive group has a unique quasi-split inner form. Thus, $\mathcal{H}_k$ must be quasi-split, and so must admit a Borel subgroup k-rationally.

Let us denote by $\underline{\mathrm{Bor}}_{\mathcal{H}/\mathcal{O}}$ the functor associating to an $\mathcal{O}$-scheme S the set of Borel subgroups of $\mathcal{H}_S$ (see [Con11, Definition 5.2.10]). One can show that $\underline{\mathrm{Bor}}_{\mathcal{H}/\mathcal{O}}$ is represented by a smooth proper $\mathcal{O}$-scheme which we call X (see [Con11, Theorem 5.2.11]). By Hensel's lemma (e.g. see [BLR12, Proposition 6, §2.3]) we know that $X(\mathcal{O}) \to X(k)$ is surjective. Since $\mathcal{H}_k$ is quasi-split, we know that $X(k)$ is not empty, thus neither is $X(\mathcal{O})$. So, let $\mathcal{B} \subseteq \mathcal{H}$ be a Borel subgroup. Then, by definition, $\mathcal{B}_F \subseteq H$ is a Borel subgroup. Thus, H is quasi-split.

Thus, what remains is to show that H becomes split after a finite unramified extension of F. Let us denote by $\underline{\mathrm{Tor}}_{\mathcal{H}/\mathcal{O}}$ the functor which associates to an $\mathcal{O}$-scheme S the set of maximal tori in $\mathcal{H}_S$ (e.g. see [Con11, Definition 3.2.1]). This is represented by a smooth quasi-affine $\mathcal{O}$-scheme Y (see [Con11, Theorem 3.2.6]). Now, let k' be a finite extension of k upon which $\mathcal{H}_k$ becomes split. Then, by definition, $Y(k')$ is non-empty. But, if F' denotes the unique unramified extension of F of degree $[k' : k]$ and $\mathcal{O}'$ its integer ring, then $\mathcal{O}'$ is a Henselian local $\mathcal{O}$-algebra with residue field k'. In particular, since $Y_{\mathcal{O}'}$ is smooth, we know that

$$Y(\mathcal{O}') = Y_{\mathcal{O}'}(\mathcal{O}') \to Y_{\mathcal{O}'}(k') = Y(k') \tag{2.12}$$

is surjective by Hensel's lemma. Thus, we see that $\mathcal{H}_{\mathcal{O}'}$ has some maximal torus $\mathcal{T}$ of fibral dimension d (which is constant since $\mathcal{O}'$ is connected) with split special fiber. That said, we note that this residual splitness already implies that $\mathcal{T}$ is split (i.e. isomorphic to $\mathbb{G}^d_{m,\mathcal{O}'}$). Indeed, the sheaf of isomorphisms $\underline{\mathrm{Isom}}(\mathcal{T}, \mathbb{G}^d_{m,\mathcal{O}'})$ sending an $\mathcal{O}'$-scheme S to the isomorphisms $\mathcal{T}_S \to \mathbb{G}^d_{m,S}$ is an $\underline{\mathrm{Aut}}(\mathbb{G}^d_{m,\mathcal{O}'}) = \underline{\mathrm{GL}_d(\mathbb{Z})}$-torsor, and thus representable (by affine descent) and smooth. Thus, again by Hensel's lemma the map

$$\underline{\mathrm{Isom}}(\mathcal{T}, \mathbb{G}^d_{m,\mathcal{O}'})(\mathcal{O}') \to \underline{\mathrm{Isom}}(\mathcal{T}, \mathbb{G}^d_{m,\mathcal{O}'})(k') \tag{2.13}$$

is surjective, from where the claim follows from the splitness of $\mathcal{T}_k$. Thus, $\mathcal{T}_{F'}$ is a split maximal torus of $H_{F'}$ and the claim follows.

Conversely, suppose that H is unramified. If H is split, then we are done by the work of Chevalley-Demazure (see [DG70, XXV]) which implies that H admits a split reductive model. So, let us now suppose that H is an arbitrary unramified group. Let H' be the unique split form of H and let $\mathcal{H}'$ be a split reductive model over $\mathcal{O}$. We know then (e.g. see [Ser13, Chapter III,§1]) that H is determined by a unique class $[H] \in H^1(F, \mathrm{Aut}(H'))$. Let β denote the image of $[H]$ in $H^1(F, \mathrm{Out}(H'))$. Since H is unramified it's not hard to see that β lies in the sub-pointed set $H^1_{\mathrm{ur}}(F, \mathrm{Out}(H'))$ which is equal to $H^1(\mathcal{O}, \mathrm{Out}(\mathcal{H}'))$.

Since $\mathcal{H}'$ is split, we can choose a pinning (see [Con11, Defintion 1.5.4]) for it. This pinning determines a section of the natural surjection of $\mathrm{Spec}(\mathcal{O})_{\text{ét}}$ sheaves $\mathrm{Aut}(\mathcal{H}') \to \mathrm{Out}(\mathcal{H}')$. This then determines a splitting of the associated map of pointed sets $H^1(\mathcal{O}, \mathrm{Aut}(\mathcal{H}')) \to H^1(\mathcal{O}, \mathrm{Out}(\mathcal{H}'))$, allowing us to view $H^1(\mathcal{O}, \mathrm{Out}(\mathcal{H}'))$ as a sub-pointed set of $H^1(\mathcal{O}, \mathrm{Aut}(\mathcal{H}'))$. Thus, having chosen a pinning, β can naturally be thought of as an element of $H^1(\mathcal{O}, \mathrm{Aut}(\mathcal{H}')$. Since affine group schemes over $\mathcal{O}$ are still a stack, we then see that this element of $H^1(\mathcal{O}, \mathrm{Aut}(\mathcal{H}'))$ determines a form $\mathcal{G}$ of $\mathcal{H}$ over $\mathcal{O}$.

Note then that $\mathcal{G}_F$ is a quasi-split (by the arguments from the first part of this proof) form of H' and has β as its image in $H^1(F, \mathrm{Out}(H'))$. Thus, H and $\mathcal{G}_F$ are two quasi-split forms of H' with the same image in $H^1(F, \mathrm{Out}(F'))$ they must be isomorphic. The result follows. □

Remark 2.1.12. Let us note that hyperspecial subgroups of $H(F)$ are not, in general, unique up to $H(F)$-conjugacy. Even if F is a p-adic local field (say with ring of integers $\mathcal{O}$) and H is a connected reductive unramified group over F, then the hyperspecial subgroups of $H(F)$ are not necessarily conjugate. For example, $\mathrm{SL}_n(F)$ has n distinct $\mathrm{SL}_n(F)$-conjugacy classes of hyperspecial subgroups. They correspond to the conjugates of $\mathrm{SL}_n(\mathcal{O})$ in $\mathrm{GL}_n(\mathcal{O})$ by the elements $\mathrm{diag}(\pi^i, 1 \ldots, 1)$ in $\mathrm{GL}_n(\mathcal{O})$.

That said, hyperspecial subgroups of $H(F)$ are unique up to $H(F)$-conjugacy when H is adjoint. Thus, if H is not adjoint, two hyperspecial subgroups of $H(F)$ become conjugate in $H^{\mathrm{ad}}(F)$. Thus, the question of whether they are conjugate essentially becomes a question of whether one can lift this conjugacy modulo the center. This can be reduced to a question about Galois cohomology.

For example, if H is a connected split reductive group over F then the $H(F)$-conjugacy classes of hyperspecial subgroups of $H(F)$ correspond bijectively to the finite abelian group $H^1(F, Z(H))/\mathrm{im}(\mathcal{H}^{\mathrm{ad}}(\mathcal{O}))$. Here $\mathcal{H}$ is some fixed model of H over $\mathcal{O}$, and $\mathrm{im}(\mathcal{H}^{\mathrm{ad}}(\mathcal{O}))$ denotes the image of $\mathcal{H}^{\mathrm{ad}}(\mathcal{O}) \subseteq H^{\mathrm{ad}}(F)$ under the natural map $H^{\mathrm{ad}}(F) \to H^1(F, Z(H))$ (e.g. see [Con11, Theorem 7.2.16]). So, for example, if H is connected reductive group over F with connected and split center (e.g. GL_n or GSp_{2n}), then there is a unique $H(F)$-conjugacy class of hyperspecial subgroups of $H(F)$ since $Z(H)$ is then a split torus, and thus $H^1(F, Z(H)) = 0$.

One is able to say much more, in particular cover the non-split case, if one is willing to work in the context of Bruhat-Tits theory (e.g. see [BT72, 3.3.5]).

The following observation, while trivial, will be useful at several places later on:

Lemma 2.1.13. *Let H_1 and H be reductive groups over F and suppose there is a surjection $f : H_1 \to H$ with central kernel. Then, if H_1 is unramified, so is H.*

Proof. It's clear that H is quasi-split over F since if $B \subseteq H$ is a Borel, then $f(B) \subseteq H$ is a Borel by our assumption that the kernel is central. Similarly, let F_1/F be a finite unramified extension such that $(H_1)_{F_1}$ is split. Then, evidently H_{F_1}, being a quotient of $(H_1)_{F_1}$, is also split. $\square$

Similarly, we shall use the following result later as well:

Lemma 2.1.14. *Let H be a reductive group over F which is unramified. Then, $Z(H)$ is also unramified.*

Proof. Let us note that $Z(H)$ is evidently quasi-split (since it, itself, is a Borel) and thus it suffices to show that $Z(H)$ is split over an unramified extension of F. But, note that since H is split over an unramified extension L of F we have that H_L contains a maximal split torus T. Since $Z(H)_L = Z(H_L)$ is contained in every maximal torus of H_L, we deduce that $Z(H)_L \subseteq T$, and thus that $Z(H)_L$ is split. The conclusion follows. $\square$

Let us now return to the setting of p-adic groups. Namely, we now assume that our F is a p-adic local field. The first result that we would like to recall is the condition on when a rational embedding of unramified groups $H_1 \hookrightarrow H_2$ over F has the property that a hyperspecial of H_1 is always contained in a hyperspecial of H_2. It's clear that this is the case when H_2 is the general linear group since, in this case, every maximal compact open subgroup of $H_2(F)$ is hyperspecial. In fact, Kisin proves something even stronger:

Lemma 2.1.15 (Kisin)**.** *Let H be a connected reductive unramified group over F and let $\mathcal{H}$ be a model of H over $\mathcal{O}$. Let $i : H \hookrightarrow \mathrm{GL}(V)$ be an embedding of F-groups where V is some finite-dimensional F-space. Then, there exists an $\mathcal{O}$-lattice $V_{\mathcal{O}} \subseteq V$ and an embedding of $\mathcal{O}$-groups $i_{\mathcal{O}} : \mathcal{H} \hookrightarrow \mathrm{GL}(V_{\mathcal{O}})$ whose generic fiber recovers i.*

Proof. This is [Kis10, Lemma 2.3.1], noting that nowhere in the proof is it specifically used that F is $\mathbb{Q}_p$. $\square$

When we are dealing with Shimura varieties, whose groups are defined over $\mathbb{Q}$, it's helpful to know that, in fact, unramified groups have models not only formally locally on $\mathbb{Z}$ but, in fact, Zariski locally. Namely, we have the following result:

Lemma 2.1.16. *Let E be a number field, and let $\mathfrak{p}$ be a finite place of E. Set $F := E_{\mathfrak{p}}$. Let G be a connected reductive group over E and let $i : G \hookrightarrow \mathrm{GL}(W)$ be an embedding where W is a finite-dimensional E-space. Suppose that G_F is unramified and $\mathcal{G}_{\mathcal{O}_F}$ is a model of G_F over $\mathcal{O}_F$. Let $i_{\mathcal{O}_F} : G_{\mathcal{O}_F} \hookrightarrow \mathrm{GL}(W_{\mathcal{O}_F})$ be an embedding as in Lemma 2.1.15. The choice of an $\mathcal{O}_E$-lattice $W_{\mathcal{O}_E} \subseteq W_{\mathcal{O}_F}$ determines a model $\mathcal{G}$ of G over $(\mathcal{O}_E)_{\mathfrak{p}}$ and an embedding $i_{(\mathcal{O}_E)_{\mathfrak{p}}} : \mathcal{G} \hookrightarrow \mathrm{GL}(W_{(\mathcal{O}_E)_{\mathfrak{p}}})$.*

Proof. Set $\mathcal{G}$ to be the Zariski closure of G in $\mathrm{GL}(W_{(\mathcal{O}_E)_{\mathfrak{p}}})$. The natural embedding $i_{(\mathcal{O}_E)_{\mathfrak{p}}} : \mathcal{G} \hookrightarrow \mathrm{GL}(W_{(\mathcal{O}_E)_{\mathfrak{p}}})$ gives upon base change to $\mathcal{O}_F$ the embedding $i_{\mathcal{O}_F} : \mathcal{G}_{\mathcal{O}_F} \hookrightarrow \mathrm{GL}(W_{\mathcal{O}_F})$. The only point which is not clear is that $\mathcal{G}$ is reductive. But, it's patently linear. It's connected and smooth because this can be checked on the flat cover $\mathrm{Spec}(\mathcal{O}_F) \to \mathrm{Spec}((\mathcal{O}_E)_{\mathfrak{p}})$ in which case it reduces to the fact that $\mathcal{G}_{\mathcal{O}_F}$ is connected and smooth. Thus, it remains to see why the fibers of $\mathcal{G}_{(\mathcal{O}_E)_{\mathfrak{p}}} \to \mathrm{Spec}((\mathcal{O}_E)_{\mathfrak{p}})$ have trivial unipotent radical. But, this is is clear since the special fiber coincides with that of $\mathcal{G}_{\mathcal{O}_F}$ and the general fiber is G. $\square$

Integral canonical models for Hodge type Shimura varieties

Let us now return to the setting of Shimura varieties and, in particular, to dealing with a Shimura datum (G,X). Let us assume that we are given a rational embedding $i : G \hookrightarrow \mathrm{GL}(W)$, for W some finite-dimensional $\mathbb{Q}$-space. Let p be a prime such that $G_{\mathbb{Q}_p}$ is unramified, and let us pick a model $\mathcal{G}_{\mathbb{Z}_p}$ of G over $\mathbb{Z}_p$. Let us then take lattices $W_{\mathbb{Z}_p} \subseteq W_{\mathbb{Q}_p}$ and $W_{\mathbb{Z}} \subseteq W$ as in Lemma 2.1.16. Then, as in that lemma, we also obtain a reductive model $\mathcal{G}$ of G over $\mathbb{Z}_{(p)}$. Let us set K_0 to be the hyperspecial $\mathcal{G}(\mathbb{Z}_p) = \mathcal{G}_{\mathbb{Z}_p}(\mathbb{Z}_p)$ in $G(\mathbb{Q}_p)$.

We can now more precisely state the result obtained in [Kis10]. Namely, therein Kisin shows the existence of an integral canonical model of the $G(\mathbb{A}_f^p)$-subscheme of $\{\mathsf{Sh}_{K^pK_0}(G,X)\}_{K^p\in\mathcal{N}^p(G)}$, base changed to an appropriate local field, when (G,X) is of abelian type. Here we denote by $\mathcal{N}^p(G)$ the set of neat open compact subgroups of $G(\mathbb{A}_f^p)$. In this section we will recall this construction in the case when (G,X) is of Hodge type (delaying the construction of the abelian type case until the next section).

One basic property relating the unramified properties of G to that of the reflex field of (G,X), which will greatly simplify notation in the latter, is the following well-known result:

Lemma 2.1.17. *Let (G,X) be a Shimura datum (not assumed of abelian type) with reflex field E. If $G_{\mathbb{Q}_p}$ is unramified, then p is unramified in E.*

Proof. Fix any prime $\mathfrak{p}$ of E lying over p. Following [Mil04] let us denote for any extension $M/\mathbb{Q}$ the set $C(M)$ of $G(M)$-conjugacy classes of cocharacters of G_M, which carries a natural $\mathrm{Aut}(M/\mathbb{Q})$-action. We know that, by definition, E is the fixed field of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\{\mu_h\}$ in $C(\mathbb{C}) \xrightarrow{\approx} C(\overline{\mathbb{Q}})$ where $\{\mu_h\}$ are the cocharacters $\mu_h : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$ induced by each $h \in X$. Choose a place $\overline{\mathfrak{p}}$ of $\overline{\mathbb{Q}}$ lying over $\mathfrak{p}$. Then, we get a distinguished embeddings $E_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{Q}_p}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. It's then easy to see that $E_{\mathfrak{p}}$ is the fixed field of the $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$-action on the image of $\{\mu_h\}$ under the bijection $C(\overline{\mathbb{Q}}) \to C(\overline{\mathbb{Q}_p})$. Since G is unramified, so split over an unramified extension, it's clear then that $E_{\mathfrak{p}}/\mathbb{Q}_p$ is unramified. The claim follows. □

Let us now begin, in earnest, the recollection of the construction of an integral canonical model over $\mathcal{O}$ of the $G(\mathbb{A}_f^p)$-scheme $\{\mathsf{Sh}_{K^pK_0}(G,X)_{E_{\mathfrak{p}}}\}_{K^p\in\mathcal{N}^p(G)}$ over $E_{\mathfrak{p}}$ in the case when (G,X) is of Hodge type. Before we begin, let us make a few notational conventions. Namely, let us fix a finite place $\mathfrak{p}$ of E lying over p. Let us shorten $E_{\mathfrak{p}}$ to F and $\mathcal{O}_F$ to $\mathcal{O}$, and let us write the residue field of $\mathcal{O}$ as k and set $q := \#k$. By Lemma 2.1.17 we know that $F \subseteq \mathbb{Q}_p^{\mathrm{ur}}$ and thus for any finite extension k'/k we know that the unique unramified extension of F of degree $[k':k]$ is $B(k') := \mathrm{Frac}(W(k'))$. We also know that $F^{\mathrm{ur}} = \mathbb{Q}_p^{\mathrm{ur}}$ so that the p-adic completion of F^{ur} is $\breve{\mathbb{Q}}_p := \mathrm{Frac}\,\breve{\mathbb{Z}}_p$ where $\breve{\mathbb{Z}}_p := W(\overline{k})$.

To form the construction of the integral canonical model in the Hodge type case we start by recalling the following lemma. To begin, let us assume that our embedding $i : G \hookrightarrow \mathrm{GL}(W)$ is actually induced by an embedding of Shimura datum $i : (G,X) \hookrightarrow (G(\psi), \mathfrak{h}_\psi^{\pm})$ where ψ is some symplectic form on W. Note then that, possibly replacing W by a larger vector space using Zarhin's trick, we may assume without loss of generality that ψ actually extends to a perfect symplectic pairing (also denoted) ψ on $W_{\mathbb{Z}_{(p)}}$. Let us denote by $\mathcal{G}(\psi)$ this reductive group scheme over $\mathbb{Z}_{(p)}$.

Let us set $K_0' := \mathcal{G}(\psi)(\mathbb{Z}_p)$. Note that $i(K_0)$ is contained in

$$G(\psi)(\mathbb{Q}_p) \cap \mathrm{GL}(W_{\mathbb{Z}_p}) = \mathcal{G}(\psi)(\mathbb{Z}_p) \tag{2.14}$$

We then have the following lemma of Kisin:

Lemma 2.1.18. *Let notation be as above. Then, for every $K^p \in \mathcal{N}^p(G)$ there exists some $K^{p\prime} \in \mathcal{N}^p(G(\psi))$ such that $K^pK_0 \subseteq K^{p\prime}K_0'$ and the induced map*

$$\mathsf{Sh}_{K^pK_0}(G,X) \to \mathsf{Sh}_{K^{p\prime}K_0'}(G(\psi),\mathfrak{h}_\psi^\pm)_E \tag{2.15}$$

is a closed embedding.

Proof. This is [Kis10, Lemma 2.1.2] except the claim that we can take $K^{p\prime}$ to be neat. But, this follows from the fact that the above lemma is insensitive to shrinking (as the argument in [Del79] cited in the proof of [Kis10, Lemma 2.1.2] shows). In reality, we need to be slightly more careful. Namely, let $K^{p\prime}$ be any compact open subgroup satisfying the claim. We can find some neat compact open subgroup N of $K^{p\prime}$ of finite index. Note then that $K^p/(K^p \cap N)$ is finite so that the group $K^{p\prime\prime}$generated by N and K^p in $K^{p\prime}$ is, topologically, a finite union of translates of N, thus open. Moreover, this group is neat since both N and K^p are. Thus, we can take $K^{p\prime\prime}$ as our desired subgroup. □

Now, by the discussion in §8.4 we know that the Siegel $G(\psi)(\mathbb{A}_f^p)$-scheme given as the tower $\{\mathsf{Sh}_{K^{p\prime}K_0'}(G(\psi),\mathfrak{h}_\psi^\pm)_{\mathbb{Q}_p}\}_{K^{p\prime}\in\mathcal{N}^p(G(\psi))}$ admits an integral canonical model, which we denote by $\{\mathscr{S}_{K^{p\prime}}(G(\psi),\mathfrak{h}_\psi^\pm)\}_{K^{p\prime}\in\mathcal{N}^p(G(\psi))}$, over $\mathbb{Z}_p$. For each K^p in $\mathcal{N}^p(G)$ and each $K^{p\prime}$ in $\mathcal{N}^p(G(\psi)(\mathbb{A}_f^p))$ containing K^p we obtain a morphism from $\mathsf{Sh}_{K^pK_0}(G,X)_F$ into the F-scheme $\mathsf{Sh}_{K^{p\prime}K_0'}(G(\psi),\mathfrak{h}_\psi^\pm)_F$. Lemma 2.1.18 then says that for $K^{p\prime}$ sufficiently small this morphism is a closed embedding. We also obtain a map $\mathsf{Sh}_{K^pK_0}(G,X)_F$ into $\mathscr{S}_{K^{p\prime}}(G(\psi),\mathfrak{h}_\psi^\pm)_\mathcal{O}$ defined by the obvious conditions. If $K^{p\prime}$ is sufficiently small then, again by Lemma 2.1.18, we have that this map is a locally closed embedding.

Let us define $\mathscr{S}^-_{K^p,K^{p\prime}}(G,X)$ to be the the scheme theoretic image of $\mathsf{Sh}_{K^pK_0}(G,X)_F$ in $\mathscr{S}_{K^{p\prime}}(G(\psi),\mathfrak{h}_\psi^\pm)_\mathcal{O}$. If $K^{p\prime}$ satisfies (2.1.18), so that we can identify $\mathsf{Sh}_{K^pK_0}(G,X)_F$ as a locally closed subset of $\mathscr{S}_{K^{p\prime}}(G(\psi),\mathfrak{h}_\psi^\pm)_\mathcal{O}$, this is the same thing as just taking the closure of $\mathsf{Sh}_{K^pK_0}(G,X)_F$ in the scheme $\mathscr{S}_{K^{p\prime}}(G(\psi),\mathfrak{h}_\psi^\pm)_\mathcal{O}$ and endowing it with the reduced subscheme structure. Also, let us denote by $\mathscr{S}_{K^p,K^{p\prime}}(G,X)$ the normalization of $\mathscr{S}^-_{K^p,K^{p\prime}}(G,X)$. If $K_1^{p\prime}$ and $K_2^{p\prime}$ are two compact open subgroups of $G(\psi)(\mathbb{A}_f^p)$ containing K^p such that $K_1^{p\prime} \subseteq K_2^{p\prime}$ it's evident that we obtain morphisms

$$\pi^-_{K_1^p,K_2^{p\prime}} : \mathscr{S}^-_{K^p,K_1^{p\prime}}(G,X) \to \mathscr{S}^-_{K^p,K_2^{p\prime}}(G,X) \tag{2.16}$$

and

$$\pi_{K_1^p,K_2^{p\prime}} : \mathscr{S}_{K^p,K_1^{p\prime}}(G,X) \to \mathscr{S}_{K^p,K_2^{p\prime}}(G,X) \tag{2.17}$$

induced by the maps

$$\mathscr{S}_{K_1^{p\prime}}(G(\psi),\mathfrak{h}_\psi^\pm)_\mathcal{O} \to \mathscr{S}_{K_2^{p\prime}}(G(\psi),\mathfrak{h}_\psi^\pm)_\mathcal{O} \tag{2.18}$$

obtained from the universal property of schematic image and the universal property of normalization respectively. We set

$$\mathscr{S}_{K^p}(G,X) := \varprojlim_{K^{p\prime}\supseteq K^p} \mathscr{S}_{K^p,K^{p\prime}}(G,X) \tag{2.19}$$

where the limit is taken in the category of $\mathcal{O}$-schemes. This limit exist since the transition maps are evidently finite, and thus affine (e.g. see [Sta18, Tag 01YX]).

The fundamental result of Kisin is the following which is, thanks to insights of Milne, the real impediment to the existence of integral canonical models for $\{\mathsf{Sh}_{K^pK_0}(G,X)_F\}_{K^p\in\mathcal{N}^p(G)}$:

Theorem 2.1.19 (Kisin)**.** *Let K^p be in $\mathcal{N}(G)$ and $K^{p\prime}$ be in $\mathcal{N}^p(G(\psi)(\mathbb{A}_f^p))$ be such that the conclusion of Lemma 2.1.18 holds. Then, $\mathscr{S}_{K^p,K^{p\prime}}(G,X)$ is smooth.*

Proof. This is [Kis10, Proposition 2.3.5]. □

To actually define the desired integral canonical models, as a system, we make the following observation:

Lemma 2.1.20. *The inverse system $\{\mathscr{S}_{K^p,K^{p\prime}}(G,X)\}_{K^{p\prime}\supseteq K^p}$ stabilizes (i.e. the transition maps are isomorphisms for sufficiently small $K^{p\prime}$).*

Proof. Let us fix a compact open subgroup $K_0^{p\prime}$ such that the result of Lemma 2.1.18 holds. Then, for any compact open subgroup $K_1^{p\prime}$ contained in $K_0^{p\prime}$ the result of Lemma 2.1.18 holds by the method of proof (see the original argument cited in [Del79]). We claim that the map

$$\pi_{K_1^{p\prime},K_0^{p\prime}} : \mathscr{S}_{K^p,K_1^{p\prime}}(G,X) \to \mathscr{S}_{K^p,K_0^{p\prime}}(G,X) \tag{2.20}$$

is an isomorphism.

To see this, note that since $\mathscr{S}_{K^p,K_i^{p\prime}} \to \operatorname{Spec}(\mathcal{O})$ is flat with reduced special fiber one has that the natural map

$$\pi_0(\mathsf{Sh}_{K^pK_0}(G,X)_F)) \to \pi_0(\mathscr{S}_{K^p,K_i^{p\prime}}(G,X)) \tag{2.21}$$

is a bijection for $i=0,1$ (see [Sta18, Tag 055J]). So, let C_0 be a connected component of $\mathscr{S}_{K^p,K_0^{p\prime}}(G,X)$. The above discussion then implies, since $\pi_{K_1^{p\prime},K_0^{p\prime}}$ is an isomorphism on the generic fiber, that $C_1 := \pi^{-1}_{K_1^{p\prime},K_0^{p\prime}}(C_0)$ is a connected component of $\mathscr{S}_{K^p,K_1^{p\prime}}(G,X)$. It thus suffices to show that

$$\pi_{K_1^{p\prime},K_0^{p\prime}} : C_1 \to C_0 \tag{2.22}$$

is an isomorphism.

We claim that the map (2.22) is finite. To see this we first verify that the map $\pi^{-}_{K_1^{p\prime},K_0^{p\prime}}$ from (2.16) is finite. To do this, note that the map is evidently quasi-finite since the map (2.18) is quasi-finite, and it's proper since (2.18) is proper and so we can apply the Cancellation Principle (e.g. see [Vak, Theorem 10.1.19]). Thus, $\pi_{K_1^{p\prime},K_0^{p\prime}}$ is finite since it's quasi-finite and proper by Zariski's Main Theorem (e.g. see [Sta18, Tag 02LS]).

To deduce that the map $\pi_{K_1^{p\prime},K_0^{p\prime}}$ from (2.17) is finite, it suffices, again by the Cancellation Theorem, to see that the normalization maps

$$\mathscr{S}_{K^p,K_i^{p\prime}}(G,X)\to\mathscr{S}_{K^p,K_1^{p\prime}}(G,X)^- \tag{2.23}$$

are finite. For this we need only observe that $\mathscr{S}_{K^p,K_1^{p\prime}}(G,X)^-$ is finite type over $\mathcal{O}$, and thus excellent (see [Sta18, Tag 07QW]) from where the claim follows from standard results about excellent rings (e.g. see [Sta18, Tag 035S] in conjunction with [Sta18, Tag 07QV]). The fact that the map (2.22) is finite then follows.

The claim that (2.22) is an isomorphism then follows because it's a finite birational map between integral schemes with normal target (e.g. see [Sta18, Tag 0AB1]). □

In particular, we can deduce from this lemma that $\mathscr{S}_{K^p}(G,X)$, as a limit, terminates at a finite stage. The reason to prefer the notation $\mathscr{S}_{K^p}(G,X)$ over any individual $\mathscr{S}_{K^p,K^{p\prime}}(G,X)$ for sufficiently small $K^{p\prime}$ is twofold. First it eliminates the dependence on an arbitrary choice of an ancillary compact open subgroup $K^{p\prime}$ in $G(\psi)(\mathbb{A}_f^p)$. Secondly, it makes the definition of $\mathscr{S}_{K^p}(G,X)$ as a $G(\mathbb{A}_f^p)$-scheme system over $\mathcal{O}$ much simpler. Namely, let's consider the maps

$$t_{K^pK_0,L^pK_0}(g^p):\mathsf{Sh}_{K^pK_0}(G,X)_F\to\mathsf{Sh}_{L^pK_0}(G,X)_F \tag{2.24}$$

for $K^p,L^p\in\mathcal{N}^p(G)$ and $g^p\in G(\mathbb{A}_f^p)$ such that $L^p\supseteq(g^p)^{-1}K^pg^p$. Fix one compact open subgroups $K_0^{p\prime}$ of $G(\psi)(\mathbb{A}_f^p)$ containing K^p and satisfying the conclusion of Lemma 2.1.18. Then, for all $L^{p\prime}$ a compact open subgroup of $G(\psi)(\mathbb{A}_f^p)$ satisfying the conclusion of Lemma 2.1.18 we note that $g^pL^p(g^p)^{-1}\cap {K'^p}_0$ contains K^p and is contained in $K_0^{p\prime}$ (and thus satisfies the conclusion of Lemma 2.1.18). It's clear then that the map (2.24) lifts to a map

$$\mathscr{S}_{K^p,g^pL^p(g^p)^{-1}\cap {K'^p}_0}(G,X)\to\mathscr{S}_{L^p,L^{p\prime}}(G,X) \tag{2.25}$$

and varying L^p gives a map $\mathscr{S}_{K^p}(G,X)\to\mathscr{S}_{L^p}(G,X)$ which, as one can easily check, is naturally independent of $K_0^{p\prime}$. Let us denote this map $t_{K^p,L^p}(g^p)$.

We then have the following result which, essentially, follows from Theorem 2.1.19:

Theorem 2.1.21. *The system $\{\mathscr{S}_{K^p}(G,X)\}_{K^p\in\mathcal{N}^p(G)}$, with the morphisms $\{t_{K^p,L^p}(g^p)\}$, is an integral canonical model over $\mathcal{O}$ of the $G(\mathbb{A}_f^p)$-scheme $\{\mathsf{Sh}_{K^pK_0}(G,X)_F\}_{K^p\in\mathcal{N}^p(G)}$ over F.*

Proof. This is [Kis10, Theorem 2.3.8]. See also [Moo98, Proposition 3.19] for an explanation of why Theorem 2.1.19 is the only real impediment. □

When we speak of the integral canonical model of $\{\mathsf{Sh}_{K^pK_0}(G,X)_F\}_{K^p\in\mathcal{N}^p(G)}$, which we know is unique up to isomorphism by Lemma 7.1.26, we will mean this explicit model constructed above.

Integral canonical models for abelian type Shimura varieties

Let us now explain how to extend the construction from the last subsubsection to also work if (G,X) is just assumed to be of abelian type. The rough idea for the construction is as follows. Let (G_1,X_1) be a Shimura datum of Hodge type such that there is an isogeny $G_1\to G$ inducing an isomorphism on the adjoint Shimura datum. As intimated before, the geometric connected components of $\mathsf{Sh}_K(G,X)$ are finite quotients of geometric connected components of $\mathsf{Sh}_L(G_1,X_1)$. One might then imagine creating $\mathscr{S}_K(G,X)$ as the 'same quotient' of $\mathscr{S}_L(G_1,X_1)$. Of course, this requires a serious undertaking to do rigorously.

So, let us fix notation as in the beginning of the last section, except now (G,X) is assumed only to be of abelian type. The first key observation of Kisin is the following:

Lemma 2.1.22. *Let (G,X) be a Shimura datum of abelian type. There then exists a Shimura datum (G_1,X_1) of Hodge type such that:*

1. *There exists an isogeny $f:G_1^{\mathrm{der}}\to G^{\mathrm{der}}$ inducing an isomorphism on adjoint Shimura datum.*
2. *The group $(G_1)_{\mathbb{Q}_p}$ is unramified.*
3. *There exists a reductive model $\mathcal{G}_1$ of G_1 over $\mathbb{Z}_{(p)}$ and a central isogeny $\mathcal{G}_1\to\mathcal{G}$ inducing an isomorphism $\mathcal{G}_1^{\mathrm{ad}}\to\mathcal{G}^{\mathrm{ad}}$ which models f.*

Proof. Take (G_1,X_1) as in [Kis10, Lemma 3.4.13(3)]. Then, properties (1) and (2) follow by construction (specifically [Kis10, Lemma 3.4.13(1)] and [Kis10, Lemma 3.4.13(2)]). To show that property (3) holds for this choice (G_1,X_1) we merely make use of [Vas16, Lemma 2.3.1]. □

Let us call a Hodge type Shimura datum (G_1,X_1) satisfying the conditions of Lemma 2.1.22 *good* with respect to (G,X) and implicitly fix, for such good data, the group $\mathcal{G}_1$ and isogeny $\mathcal{G}_1\to\mathcal{G}$ from above. We will also write L_0 for $\mathcal{G}_1(\mathbb{Z}_p)$. We will now try to construct the connected components of the integral canonical model of (G,X) as quotients of connected components of the integral canonical models of (G_1,X_1).

To make this precise, we begin with some group theoretic preliminaries, the notation of which we take from [Kis10]. Following [Del79] we have the following operation defined on groups. Namely, suppose that H is a group equipped with an action by group maps of the group Δ. Suppose that Γ is a Δ-invariant subgroup of H. Suppose we are given a map $\varphi:\Gamma\to\Delta$ which is Δ-invariant when one lets Δ act on itself by inner automorphisms, and which has the property that $\varphi(\gamma)$ acts on H by the inner automorphism γ. We then define the group $H*_\Gamma\Delta$ as follows: $H*_\Gamma\Delta:=(H\rtimes\Delta)/N$ where

$$N:=\{(\gamma,\varphi(\gamma)^{-1}):\gamma\in\Gamma\}\tag{2.26}$$

which, one can check, is well-defined (in the sense that N is a normal subgroup of $H\rtimes\Delta$).

Again, following [Del79], we use this construction to construct groups $\mathscr{A}(\mathcal{H})$ and $\mathscr{A}(\mathcal{H})^\circ$ associated to a reductive group $\mathcal{H}$ over $\mathbb{Z}_{(p)}$ with generic fiber H. Namely, we set

$$\mathscr{A}(\mathcal{H}):=H(\mathbb{A}_f^p)/\overline{Z(\mathcal{H})(\mathbb{Z}_{(p)})}*_{\mathcal{H}(\mathbb{Z}_{(p)})_+/Z(\mathcal{H})(\mathbb{Z}_{(p)})}\mathcal{H}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+\tag{2.27}$$

and

$$\mathscr{A}(\mathcal{H})^\circ := \overline{\mathcal{H}(\mathbb{Z}_{(p)})_+}/\overline{Z(\mathcal{H})(\mathbb{Z}_{(p)})} *_{\mathcal{H}(\mathbb{Z}_{(p)})_+/Z(\mathcal{H})(\mathbb{Z}_{(p)})} \mathcal{H}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+ \tag{2.28}$$

where we define the terms as follows. Here we write

$$\mathcal{H}(\mathbb{Z}_{(p)})_+ := \mathcal{H}(\mathbb{Z}_{(p)}) \cap H(\mathbb{Q})_+ \tag{2.29}$$

where $H(\mathbb{Q})_+$ is as in Appendix D. Similarly, we write

$$\mathcal{H}(\mathbb{Z}_{(p)})^+ := H(\mathbb{R})^+ \cap \mathcal{H}(\mathbb{Z}_{(p)}) \tag{2.30}$$

The closure of $Z(\mathcal{H})(\mathbb{Z}_{(p)})$ is taken in $Z(H)(\mathbb{A}_f^p)$. The closure of $\mathcal{H}(\mathbb{Z}_{(p)})_+$ is taken in $H(\mathbb{A}_f^p)$.

Here the construction in (2.27) is with $\mathcal{H}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ acting on $H(\mathbb{A}_f^p)/\overline{Z(\mathcal{H})(Z_{(p)})}$ via conjugation and in the construction in (2.28) the action is with $\mathcal{H}(\mathbb{Z}_{(p)})^+$ acting by conjugation on $\overline{\mathcal{H}(\mathbb{Z}_{(p)})_+}/\overline{Z(\mathcal{H})(\mathbb{Z}_{(p)})}$. In both cases the map

$$\varphi : \mathcal{H}(\mathbb{Z}_{(p)})_+/Z(\mathcal{H})(\mathbb{Z}_{(p)}) \to \mathcal{H}(\mathbb{Z}_{(p)})^+ \tag{2.31}$$

is the quotient map.

For a discussion of the rational analogues of these groups one can see [Del79, §2.1]. Contained in their is a proof that the rational analogue of $\mathscr{A}(\mathcal{H})^\circ$ only depends on the derived subgroup, the same result holds true here with essentially the same proof. Namely, one easily obtains the following:

Lemma 2.1.23. *Let $\mathcal{H}$ and $\mathcal{H}_1$ be reductive group schemes over $\mathbb{Z}_{(p)}$. Then, an isomorphism $\mathcal{H}_1^{\mathrm{der}} \to \mathcal{H}^{\mathrm{der}}$ naturally induces an isomorphism $\mathscr{A}(\mathcal{H}_1)^\circ \to \mathscr{A}(\mathcal{H})^\circ$.*

So, let us return to the setting of our abelian type Shimura datum (G, X) and a choice of a Hodge type Shimura datum (G_1, X_1) which is good with respect to (G, X). Let us choose a connected component X_1^+ of the topological space X_1. Such data is equivalent to giving a $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class of homomorphisms $\mathbb{S} \to (G_1^{\mathrm{ad}})_\mathbb{R}$.

Recall then that we can form, for all $L \in \mathcal{N}(G_1)$, the connected component of the Shimura variety $\mathsf{Sh}_L^\circ(G_1, X_1)_\mathbb{C}$ over $\mathbb{C}$ (supressing the dependence on the choice of X_1^+) by declaring it to be the Zariski connected component whose analytification contains the image of X_1^+ under the map $X_1 \to |\mathsf{Sh}_L(G_1, X_1)_\mathbb{C}^{\mathrm{an}}|$ given by Lemma 8.3.16. Note that evidently for all $L_1 \subseteq L_2$ the natural map

$$(\pi_{L_1,L_2})_\mathbb{C} : \mathsf{Sh}_{L_1}(G_1, X_1)_\mathbb{C} \to \mathsf{Sh}_{L_2}(G_1, X_1)_\mathbb{C} \tag{2.32}$$

maps $\mathsf{Sh}_{L_1}^\circ(G_1, X_1)_\mathbb{C}$ into $\mathsf{Sh}_{L_2}^\circ(G_1, X_1)_\mathbb{C}$. Let us set

$$\mathsf{Sh}^\circ(G_1, X_1)_\mathbb{C} = \varprojlim_{L \in \mathcal{N}(G_1)} \mathsf{Sh}_L^\circ(G_1, X_1)_\mathbb{C} \tag{2.33}$$

and

$$\mathsf{Sh}_{L_0}^\circ(G_1, X_1)_\mathbb{C} = \varprojlim_{L^p \in \mathcal{N}^p(G_1)} \mathsf{Sh}_{L^pL_0}(G_1, X_1)_\mathbb{C} \tag{2.34}$$

Note that $\mathsf{Sh}^\circ(G_1,X_1)_\mathbb{C}$ naturally includes into $\mathsf{Sh}(G_1,X_1)$ and $\mathsf{Sh}^\circ_{L_0}(G_1,X_1)_\mathbb{C}$ is naturally a quotient of $\mathsf{Sh}^\circ(G_1,X_1)_\mathbb{C}$.

Note that the system $\{\mathsf{Sh}^\circ_{L^pL_0}(G_1,X_1)_\mathbb{C}\}_{L^p\in\mathcal{N}^p(G_1)}$ does not come equipped with an action of $G_1(\mathbb{A}_f^p)$ since, of course, the morphisms $[g]_\mathbb{C}$, the action by g morphsims, needn't send these distinguished components into each other. That said, this system does have a natural right action by the group $\mathscr{A}(\mathcal{G}_1)^\circ$ defined in terms of the action at the space at infinite level $\mathsf{Sh}^\circ(G_1,X_1)_C$ given by

$$[x,g](h,q)=[q^{-1}x,\mathrm{ad}(q^{-1})(hg)] \tag{2.35}$$

which naturally descends to an action on $\mathsf{Sh}^\circ_{L_0}(G_1,X_1)_\mathbb{C}$.

We then have the following result of Kisin. To state it, let us denote by $E_1:=E(G_1,X_1)$ the reflex field of (G_1,X_1) and E_1^p the maximal unramified-at-p extension of E_1:

Lemma 2.1.24. *The system $\{\mathsf{Sh}_{L^pL_0}(G_1,X_1)_\mathbb{C}\}_{L^p\in\mathcal{N}^p(G_1)}$ with its $\mathscr{A}(\mathcal{G}_1)^\circ$ action naturally descend to a system $\{\mathsf{Sh}^\circ_{L^pL_0}(G_1,X_1)\}_{L^p\in\mathcal{N}^p(G_1)}$ with an action of $\mathscr{A}(\mathcal{G}_1)^\circ$ over E_1^p.*

Proof. This is [Kis10, Lemma 2.2.4]. □

Thanks to this lemma it makes sense to consider the system of varieties over $\breve{\mathbb{Q}}_p$ given by $\{\mathsf{Sh}^\circ_L(G_1,X_1)_{\breve{\mathbb{Q}}_p}\}_{L^p\in\mathcal{N}^p(G_1)}$ which, again, comes equipped with a natural action of $\mathscr{A}(\mathcal{G}_1)^\circ$. Now, from the previous subsubsection we know the existence of an integral canonical model $\{\mathscr{S}_{L^p}(G_1,X_1)\}$ of the $G_1(\mathbb{A}_f^p)$-scheme $\{\mathsf{Sh}_{L^pL_0}(G_1,X_1)_{(E_1)_{\mathfrak{p}_1}}\}_{L^p\in\mathcal{N}^p(G_1)}$ where $\mathfrak{p}_1$ is a prime of E_1 lying over p. Note that there is a natural bijection

$$\pi_0(\mathsf{Sh}_{L^pL_0}(G_1,X_1)_{\breve{\mathbb{Q}}_p})\to\pi_0(\mathscr{S}_{L^p}(G_1,X_1)_{\breve{\mathbb{Z}}_p}) \tag{2.36}$$

(see [Sta18, Tag 055J] and [Sta18, Tag035Q, (2)]), with the map being taking the closure of the component in $\mathscr{S}^-_{L^pL_0}(G_1,X_1)_{\breve{\mathbb{Z}}_p}$ and taking the normalization. We thus see that the system $\{\mathsf{Sh}^\circ_{L^pL_0}(G_1,X_1)_{\breve{\mathbb{Q}}_p}\}$ naturally lifts to a system $\{\mathscr{S}^\circ_{L^p}(G_1,X_1)_{\breve{\mathbb{Z}}_p}\}$ and that the $\mathscr{A}(\mathcal{G}_1)^\circ$ action also lifts.

As in the rational case, let us set

$$\mathscr{S}^\circ(G_1,X_1)_{\breve{\mathbb{Z}}_p}=\varprojlim_{L^p\in\mathcal{N}^p(G)}\mathscr{S}^\circ_{L^p}(G_1,X_1)_{\breve{\mathbb{Z}}_p} \tag{2.37}$$

which, by virtue of the action of $\mathscr{A}(\mathcal{G}_1)^\circ$ on the system $\{\mathscr{S}_{L^p}(G_1,X_1)_{\breve{\mathbb{Z}}_p}\}$, comes equipped with an action of the group $\mathscr{A}(\mathcal{G}_1)^\circ$.

Then, motivated by the results of [Del79, §2], let us define $\mathscr{S}(G,X)_{\breve{\mathbb{Z}}_p}$ as follows:

$$\mathscr{S}(G,X)_{\breve{\mathbb{Z}}_p}:=(\mathscr{A}(\mathcal{G})\times\mathscr{S}^\circ(G_1,X_1)_{\breve{\mathbb{Z}}_p})/\mathscr{A}^\circ(\mathcal{G}_1) \tag{2.38}$$

Let us explain what the action of $\mathscr{A}^\circ(\mathcal{G}_1)$ on this product is. For $h\in\mathscr{A}^\circ(\mathcal{G}_1)$ the action will be of the form $(g,p)h:=(h^{-1}g,pg)$ and so it suffices to explain what the action is on each of the individual terms. Note that $\mathscr{A}^\circ(\mathcal{G}_1)$ evidently acts on the right of $\mathscr{A}(\mathcal{G}_1)$ and Lemma 2.1.23 allows us to think of $\mathscr{A}^\circ(\mathcal{G}_1)$ as acting on $\mathscr{A}(\mathcal{G})$. The action of $\mathscr{A}^\circ(\mathcal{G}_1)$ on $\mathscr{S}^\circ(G_1,X_1)_{\breve{\mathbb{Z}}_p}$ is the standard one we have been using.

Note that $\mathscr{S}(G,X)_{\breve{\mathbb{Z}}_p}$ naturally comes with a right $G(\mathbb{A}_f^p)$-action by its action on the $\mathscr{A}(\mathcal{G}_2)$-factor. Let us set $\mathscr{S}_{K^p}(G,X)_{\breve{\mathbb{Z}}_p}$ to be $\mathscr{S}(G,X)_{\breve{\mathbb{Z}}_p}/K^p$. The result of Kisin is then the following:

Theorem 2.1.25 (Kisin)**.** *The system $\{\mathscr{S}_{K^p}(G,X)_{\breve{\mathbb{Z}}_p}\}_{K^p\in\mathcal{N}^p(G)}$ has a natural $\mathrm{Gal}(\overline{F}/F)$-action, and this system descends to an integral canonical model $\{\mathscr{S}_{K^p}(G,X)\}_{\mathcal{N}^p(G)}$ of $G(\mathbb{A}_f^p)$-scheme $\{\mathsf{Sh}_{K^pK_0}(G,X)\}_{K^p\in\mathcal{N}^p(G)}$.*

One important deduction we can make from the above discussion is that we can extend the fact that the connected components of Shimura varieties of abelian type are finite quotients of Shimura varieties of Hodge type. To do this, we begin by observing that our fixed isomorphism $\mathcal{G}_1^{\mathrm{ad}} \xrightarrow{\approx} \mathcal{G}^{\mathrm{ad}}$, in particular, gives us an identification of $(G_1^{\mathrm{ad}})_\mathbb{R} \xrightarrow{\approx} G_\mathbb{R}^{\mathrm{ad}}$. In particular, we can naturally identify the $G_1^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class X_1^+ of morphisms $\mathbb{S}\to(G_1^{\mathrm{ad}})_\mathbb{R}$ with a $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class X^+ of morphisms $\mathbb{S}\to G_\mathbb{R}^{\mathrm{ad}}$.

This choice of X^+ allows us to define the connected components $\mathsf{Sh}_K^\circ(G,X)_\mathbb{C}$ which form a natural projective system. Again by [Kis10, Lemma 2.2.4] this system has a model over E^p, the maximal unramified-at-p extension of $E=E(G,X)$. For each $K^p\in\mathcal{N}^p(G)$ let us denote by $\mathscr{S}_{K^p}^\circ(G,X)_{\breve{\mathbb{Z}}_p}$ the closure of $\mathsf{Sh}_{K^pK_0}^\circ(G,X)_{\breve{\mathbb{Q}}_p}$ in $\mathscr{S}_{K^p}(G,X)_{\breve{\mathbb{Z}}_p}$.

We then have the following result, almost by construction:

Lemma 2.1.26. *Let (G,X) and (G_1,X_1) be as above. Let $K^p\in\mathcal{N}^p(G)$. Then, there exists $L^p\in\mathcal{N}^p(G_1)$ such that $L^p\cap G_1^{\mathrm{der}}(\mathbb{A}_f^p)$ maps onto $K^p\cap G_1^{\mathrm{der}}(\mathbb{A}_f^p)$. Moreover, the choice of such an L^p determines an isomorphism*

$$\mathscr{S}_{L^p}^\circ(G_1,X_1)_{\breve{\mathbb{Z}}_p}/\Delta \xrightarrow{\approx} \mathscr{S}_{K^p}^\circ(G,X)_{\breve{\mathbb{Z}}_p} \tag{2.39}$$

where Δ is a finite group.

Properness properties of integral canonical models

Having reviewed the concrete construction of the integral canonical model for general abelian type Shimura varieties we can, using the deep work of Madapusi Pera in [Per12], deduce the following fundamental fact. To motivate it, recall that the integral canonical model was defined to provide meaning to the phrase of 'good model' in the abscence of other natural conditions to impose. That said, if the Shimura variety at hand is proper, the natural property to require that the model have is not the integral canonical property, but properness (even though this doesn't uniquely specify the model). Thus, it is reasonable to hope that in the situation where the Shimura variety is proper, then so is the integral canonical model.

To prove this, it will be helpful to first recall the following basic fact from the theory of algebraic groups which collects various equivalent group theoretic conditions that factor into the properness of Shimura varieties:

Lemma 2.1.27. *Let k be a field of characteristic 0 and H a connected reductive group over k. Then, the following conditions are equivalent:*

1. *The group H has no proper k-rational parabolic subgroups.*

2. *The group H^{der} has no proper k-rational parabolic subgroups.*

3. *The group H^{ad} has no proper k-rational parabolic subgroups.*

4. *The group H^{der} is k-anisotropic (i.e. H^{ad} contains no non-trivial split tori).*

5. *The group H^{ad} is k-anisotropic.*

6. *The group $H(k)$ contains no unipotent elements.*

7. *The group $H^{\mathrm{der}}(k)$ contains no unipotent elements.*

8. *The group $H^{\mathrm{ad}}(k)$ contains no unipotent elements.*

Remark 2.1.28. It has been pointed out to me by A. Bertoloni-Meli that this result is largely contained in the contents of [BT65, §8]. We leave the proof here in the hope that it may be interesting to the reader.

Proof. The equivalence of (1) and (2) is elementary, and well-known and the equivalence of (1) and (4) follows from the standard dynamic approach to algebraic groups (e.g. see [Spr10, §16.2]).

To see the equivalence of (2) and (3) we proceed as follows. Evidently if H^{der} has a k-rational parabolic P then the image of P under the isogeny $H^{\mathrm{der}} \to H^{\mathrm{ad}}$, which we denote P^{ad}, is evidently a k-rational parabolic of H^{ad}. Indeed, we have the evident surjection $H^{\mathrm{der}}/P \to H^{\mathrm{ad}}/P^{\mathrm{ad}}$. Since $H^{\mathrm{ad}}/P^{\mathrm{ad}}$ is reduced (since it's smooth) we know that the scheme-theoretic image of $H^{\mathrm{der}}/P^{\mathrm{der}}$ is $H^{\mathrm{ad}}/P^{\mathrm{ad}}$. Since H^{der}/P is projective, this implies that $H^{\mathrm{ad}}/P^{\mathrm{ad}}$ is projective. Thus, P^{ad} is parabolic. Conversely, suppose that Q is a k-rational parabolic subgroup of H^{ad}. Let P denote the connected component of the preimage of Q under the isogeny $H^{\mathrm{der}} \to H^{\mathrm{ad}}$. Note then that evidently the map $H^{\mathrm{der}}/P \to H^{\mathrm{ad}}/Q$ is finite. Since H^{ad}/Q is projective, this in turn implies that H^{der}/P is projective. Thus, P is a k-rational parabolic.

To see the equivalence of (4) and (5), one proceeds as follows. Evidently if H^{ad} is k-anisotropic, then so is H^{der} since the image of a non-trivial split torus $T \subseteq H^{\mathrm{der}}$ under the isogeny $H^{\mathrm{der}} \to H^{\mathrm{ad}}$ would also be a non-trivial split torus. Suppose now that H^{der} is anisotropic and that one has an embedding $\mathbb{G}_{m,k} \hookrightarrow H^{\mathrm{ad}}$. Let T be the connected component of the preimage of $\mathbb{G}_{m,k}$ under the isogeny $H^{\mathrm{der}} \to H^{\mathrm{ad}}$. Note then that T is a 1-dimensional connected algebraic group and thus, by well-known classification, either $\mathbb{G}_{a,k}$, $\mathbb{G}_{m,k}$, or a 1-dimensional unitary group U associated to a degree 2 quadratic extension of k. Since there are no non-trivial maps $\mathbb{G}_{a,k} \to \mathbb{G}_{m,k}$ or $U \to \mathbb{G}_{m,k}$ we deduce that $T = \mathbb{G}_{m,k}$ from where the conclusion follows.

To see the equivalence of (3) and (8) we proceed as follows. Suppose first that H^{ad} has a proper parabolic subgroup P. Note then that $U := R_u(P)$ is non-trivial. Indeed, if U is trivial then this implies that P is parabolic and reductive. But, note then that H^{ad}/P is projective and affine (see [New78, Theorem 3.5]) and thus $\mathrm{Spec}(k)$, since it's geometrically connected, reduced, projective and affine. This implies that P is non-proper, contradictory to assumption. Since k is characteristic 0, we know that U is isomorphic, as a k-scheme, to $\mathbb{A}_k^{\dim U}$ (e.g. see [Mil17, §14.d]) and so, in particular, we know that $U(k)$ is infinite. In particular, any non-trivial $u \in U(k)$ gives a non-trivial unipotent element of $H^{\mathrm{ad}}(k)$.

Conversely, let $u \in H^{\mathrm{ad}}(k)$ be a non-trivial. Let us denote by $\mathscr{U} := \mathscr{U}(H^{\mathrm{ad}})$ and $\mathscr{N} := \mathscr{N}(\mathrm{Lie}(H^{\mathrm{ad}}))$ the variety of unipotent elements of H^{ad}, and the variety of nilpotent elements of $\mathrm{Lie}(H^{\mathrm{ad}})$ respectively. Let $\varepsilon : \mathscr{N} \to \mathscr{U}$ denote the 'exponential isomorphism' (e.g. see [BR85, §9]) and let $X := \varepsilon^{-1}(u) \in \mathscr{N}(k)$. Recall that a cocharacter $\phi : \mathbb{G}_{m,\overline{k}} \to H^{\mathrm{ad}}_{\overline{k}}$ with the property that for all $t \in \overline{k}^{\times}$ one has that $(\mathrm{Ad}\phi_(t))(X) = t^2 X$ is called associated to X and there is a parabolic subgroup $P_\phi \subseteq H^{\mathrm{ad}}_{\overline{k}}$ with the property that $\mathrm{Lie}(P_\phi)$ is the direct sum of the positive weight spaces for $\mathrm{Ad}\phi$. We know (e.g. see [Jan04, §5] and [MS03, §3]) that there exists cocharacters ϕ associated to X and that $P_\phi =: P$ is independent of ϕ. Evidently $\mathrm{Gal}(\overline{k}/k)$ acts on the cocharacters associated to X and thus, by the independence of P_ϕ, allows us to descend P to a k-rational parabolic subgroup of H^{ad} from where the conclusion follows.

For the equivalence of (6) and (8) we proceed as follows. Suppose first that $H^{\mathrm{ad}}(k)$ contains no non-trivial unipotent elements and $u \in H(k)$ is a non-trivial unipotent element. Since u must map to the identity in $H^{\mathrm{ad}}(k)$ this implies that $u \in Z(H)(k)$ which is clearly a contradiction since $Z(H)$ is multiplicative since H is reductive. Suppose next that $H(k)$ contains no non-trivial unipotent elements and suppose that $u \in H^{\mathrm{ad}}(k)$ is a non-trivial unipotent group. Note that since $H^1(k, Z(H))$ is torsion, we know that u^n maps to 0 in $H^1(k, Z(H))$ for some $n \geqslant 0$. Note then that u^n is also a non-trivial unipotent element of $H^{\mathrm{ad}}(k)$. Since u^n maps to 0 in $H^1(k, Z(H))$ we know there exists some $h \in H(k)$ which maps to u^n in $H^{\mathrm{ad}}(k)$. Evidently h_u, the unipotent part of h, is non-trivial which is a contradiction.

For the equivalence of (7) and (8) one proceeds as follows. Evidently if $H^{\mathrm{ad}}(k)$ has no non-trivial unipotent elements then neither does $H^{\mathrm{der}}(k)$. Indeed, if $u \in H^{\mathrm{der}}(k)$ is non-trivial and unipotent, then the image in $H^{\mathrm{ad}}(k)$ is unipotent, thus trivial by assumption. This implies that $u \in Z(H^{\mathrm{der}})(k)$ which, since H is reductive so that $Z(H^{\mathrm{der}})$ is multiplicative, implies that u is trivial, which is a contradiction. Conversely, suppose that $H^{\mathrm{der}}(k)$ has no non-trivial unipotents. Let $u \in H^{\mathrm{ad}}(k)$ be a non-trivial unipotent element. Note that since the image of u in $H^1(k, Z(H^{\mathrm{der}}))$ is torsion, that u^n maps to 0 in $H^1(k, Z(H^{\mathrm{der}}))$ for some $n \geqslant 0$. But, since u is a non-trivial unipotent element, thus is u^n. Since u^n maps to zero in $H^1(k, Z(H))$ we know that there exists $h \in H(k)$ which maps to u^n. Clearly h_u, the unipotent part of h, is non-trivial which is a contradiction.

□

From this, we can now show the following where we have, again, retained notation from the previous subsubsections:

Theorem 2.1.29. *The following are equivalent:*

1. *The group G^{ad} is $\mathbb{Q}$-anisotropic .*
2. $\mathsf{Sh}_{K^pK_0}(G, X)$ *is proper (equivalently projective) for all $K^p \in \mathcal{N}^p(G)$.*
3. $\mathscr{S}_{K^p}(G, X)$ *is proper (equivalently projective) for all $K^p \in \mathcal{N}^p(G)$.*

Proof. The equivalence of (1) and (2) is precisely [Pau04, Lemma 3.1.5]. Indeed, $\mathsf{Sh}_{K^pK_0}(G,X)$ is proper if and only it's projective since it's quasi-projective. But, by standard descent we know that $\mathsf{Sh}_{K^pK_0}(G,X)$ is proper if and only if $\mathsf{Sh}_{K^pK_0}(G,X)_{\mathbb{C}}$ is proper. The claim then follows from loc. cit. using the standard fact that $\mathsf{Sh}_{K^pK_0}(G,X)_{\mathbb{C}}$ is proper if and only if its analytification is compact (e.g. see [Ser56, Proposition 6]) and the equivalence of (5) and (6) in Lemma 2.1.27.

The equivalence of (2) and (3) is precisely [Per12, Corollary 4.1.7] in the case when (G,X) is of Hodge type. Suppose now that (G,X) is of abelian type. To show the claim, it suffices to prove the claim with the objects replaced by $\mathsf{Sh}^{\circ}_{K^pK_0}(G,X)_{\breve{\mathbb{Q}}_p}$ and $\mathscr{S}^{\circ}_{K^pK_0}(G,X)_{\breve{\mathbb{Z}}_p}$ by descent and the fact that $\mathsf{Sh}_{K^pK_0}(G,X)_{\breve{\mathbb{Q}}_p}$ and $\mathscr{S}_{K^p}(G,X)_{\breve{\mathbb{Z}}_p}$ are covered by finitely many objects of this form. We are then reduced to the case of Hodge type by Lemma 2.1.26. ∎

2.2 A trace formula for Hecke correspondences at bad level

We now marry the machinery from Appendix C with the discussion of integral canonical models to give a formula for the trace of a Hecke operator at bad level (with some assumptions) on the cohomology at infinite level of a Shimura variety of abelian type in terms of a weighted point count on the special fiber of the integral canonical model of the Shimura variety at a lower good level, where the weighting factor is in terms of the rigid geometry of the analytification of the Shimura variety.

Remark 2.2.1. We will remind the reader of the barebones material from Appendix C necessary to make this section self-contained. That said, it is most likely easier to read this section having at least skimmed the material in Appendix C.

We keep the notation from the last section and so, in particular, assume that (G,X) is a Shimura datum of abelian type. Let us write W_F for the Weil subgroup of $\mathrm{Gal}(\overline{F}/F)$, and I_F for the inertia subgroup of W_F. Let us choose a Frobenius lift Frob_F under the surjection $\mathrm{Gal}(\overline{F}/F)\to\mathrm{Gal}(\overline{k}/k)$. Again, by Lemma 2.1.17, we know that $F^{\mathrm{un}}=\mathbb{Q}_p^{\mathrm{un}}$ and so we will sometimes make this identification implicitly.

Let us note that by Lemma 8.3.22 the system $\{\mathsf{Sh}_K(G,X)\}_{K\in\mathcal{N}(G)}$ is a $G(\mathbb{A}_f)$-scheme over E as in the parlance of Appendix C. Thus, all the materials therein apply.

Let us now fix an algebraic $\overline{\mathbb{Q}_\ell}$-representation ξ of G which we assume to be adapted to $\mathcal{N}(G)$ (see §7.2 for a discussion of various notions of adapted). If (G,X) is of Hodge type then ξ is adapted to every $K\in\mathcal{N}(G)$. Indeed, this follows from the following basic fact which is essentially says that $\{\mathsf{Sh}_K(G,X)\}_{K\in\mathcal{N}(G)}$ can be taken to be Shimura like with respect to the trivial group as in Lemma 8.3.14:

Lemma 2.2.2. *Let (G,X) be a Shimura variety of Hodge type. Then, $Z(\mathbb{Q})\cap K$ is trivial for $K\in\mathcal{N}^p(G)$ sufficiently small.*

Proof. It's clear that this is equivalent to saying that $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_f)$. Note that this is equivalent to the split rank of $Z(G)$ being equal to the split rank of $Z(G)_{\mathbb{R}}$

(e.g. see [Mil04, Theorem 5.26]). The fact that this holds true for G as in a Hodge type Shimura datum is classical (see [Del79, 2.3.2] and [Del79, 2.3.4]). $\square$

Note though that if (G, X) is of abelian type, then it's possible that ξ may not be adapted to $\mathcal{N}(G)$ (e.g. see Example 8.3.15 for an example which doesn't satisfy Lemma SV5 as in [Mil04]).

For all $K \in \mathcal{N}(G)$ we obtain a lisse $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}_{\xi,K}$ on $\mathsf{Sh}_K(G, X)$ which form a compatible family of sheaves for K in the set $\mathcal{N}(\xi)$ of ξ-adapted neat open compact subgroups of $G(\mathbb{A}_f)$. In other words, for any K and L in $\mathcal{N}(\xi)$ with $L \supseteq g^{-1}Kg$ we have a natural isomorphism

$$\varphi_{K,L}(g) : t_{K,L}(g)^*\mathcal{F}_{\xi,L} \xrightarrow{\approx} \mathcal{F}_{\xi,K} \tag{2.40}$$

of lisse $\overline{\mathbb{Q}_\ell}$-sheaves on the étale site of $\mathsf{Sh}_K(G, X)$. This follows from the material of §7.2.

Let us moreover assume that K^p in $\mathcal{N}^p(G)$ is such that K^pK_0 is in $\mathcal{N}(\xi)$. Then, for all $K \in \mathcal{N}(G)$ such that $K \subseteq K^pK_0$ we have that K is in $\mathcal{N}(\xi)$ and if $K = L^pK_0$ for some L^p in $\mathcal{N}^p(G)$ then $(\mathcal{F}_{\xi,L^pK_0})_F$ on $\mathsf{Sh}_{L^pK_0}(G, X)_F$ admits a model $\mathcal{F}_{\xi,L^p}$ on $\mathscr{S}_{L^p}(G, X)$ over $\mathcal{O}$.

Since ξ is adapted to $\mathcal{N}(G)$ we obtain a natural action of the group $\mathrm{Gal}(\overline{E}/E)\times G(\mathbb{A}_f)$, whose elements we denote $\tau \times f$, on

$$H^i_c(\overline{\mathsf{Sh}(G, X)}_{\text{ét}}, \overline{\mathcal{F}_\xi}) := \varinjlim_{K\in\mathcal{N}(\xi)} H^i_c(\overline{\mathsf{Sh}_K(G, X)}_{\text{ét}}, \overline{\mathcal{F}_{\xi,K}}) \tag{2.41}$$

(where the transition maps come from proper pullback since the transition maps are finite) where the overline denotes base change to the algebraic closure. We call the object on the left the *compactly supported cohomology at infinite level*, the individual terms on the right hand side the *compactly supported cohomology at level* K, and any (non-specific) term on the right hand side the *compactly supported cohomology at finite level*.

We will denote by $H^*_c(\overline{\mathsf{Sh}(G, X)}_{\text{ét}}, \overline{\mathcal{F}_\xi})$ the element of

$$\mathrm{Groth} := \mathrm{Groth}(\mathrm{Gal}(\overline{E}/E) \times G(\mathbb{A}_f)) \tag{2.42}$$

the usual Grothendieck group of $\overline{\mathbb{Q}_\ell}[\mathrm{Gal}(\overline{E}/E) \times G(\mathbb{A}_f)]$-modules, given by

$$H^*_c(\overline{\mathsf{Sh}(G, X)}_{\text{ét}}, \overline{\mathcal{F}_\xi}) := \sum_{i=0}^{2d}(-1)^i H^i_c(\overline{\mathsf{Sh}(G, X)}_{\text{ét}}, \overline{\mathcal{F}_{\xi,K}}) \tag{2.43}$$

where d is the constant dimension of the Shimura varieties $\mathsf{Sh}_K(G, X)$ (i.e. d is equal to the dimension of the complex manifold $G(\mathbb{R})/\mathrm{Stab}(h_0)$ for any $h_0 \in X$).

The action of $\mathrm{Gal}(\overline{E}/E) \times G(\mathbb{A}_f)$ on the compactly supported cohomology at infinite level is smooth in the sense of representations of locally profinite groups (see Lemma 7.3.3). We thus get a naturally induced action of $\mathrm{Gal}(\overline{E}/E) \times \mathscr{H}(G)$ on the compactly supported cohomology at infinite level, where by $\mathscr{H}(G)$ we mean the $\overline{\mathbb{Q}_\ell}$-Hecke algebra defined, for us, to be $C^\infty_c(G(\mathbb{A}_f), \overline{\mathbb{Q}_\ell})$, the $\overline{\mathbb{Q}_\ell}$-algebra of locally constant compactly supported functions $G(\mathbb{A}_f) \to \overline{\mathbb{Q}_\ell}$. We will denote the elements of $\mathrm{Gal}(\overline{E}/E) \times \mathscr{H}(G)$ by $\tau \times f$.

In particular, for every $K \in \mathcal{N}(\xi)$ we have that

$$H_c^*(\overline{\mathsf{Sh}(G,X)}_{\text{ét}}, \overline{\mathcal{F}_\xi})^K \xrightarrow{\approx} H_c^*(\overline{\mathsf{Sh}_K(G,X)}_{\text{ét}}, \overline{\mathcal{F}_{\xi,K}}) \tag{2.44}$$

(by Lemma 7.3.3) where the isomomorphism is as $\mathrm{Gal}(\overline{E}/E) \times \mathscr{H}(G,K)$-modules where, here $\mathscr{H}(G,K)$ denotes the subalgebra of the $\overline{\mathbb{Q}_\ell}$-Hecke algebra consisting of bi-K-invariant functions. Since the right hand side of (2.44) is finite-dimensional, we also see that, as a $G(\mathbb{A}_f)$-representation $H_c^*(\overline{\mathsf{Sh}(G,X)}_{\text{ét}}, \overline{\mathcal{F}_\xi})$ is admissible.

We are primarily interested in describing the traces of certain elements $\tau \times f$ in $\mathrm{Gal}(\overline{E}/E) \times \mathscr{H}(G)$ on this object $H_c^*(\overline{\mathsf{Sh}(G,X)}_{\text{ét}}, \overline{\mathcal{F}_\xi})$ in Groth as this gives us a concrete way of studying what representations show up in the compactly supported cohomology at infinite level. We would like to do this using a sort Grothendieck-Lefschetz type formalism to compute these traces in terms of weighted point counts over some scheme over a finite field. In particular, we would like to utilize the integral canonical models whose construction was recalled in the last section.

To do this we are evidently necessitated to work with, instead of $\mathsf{Sh}_K(G,X)$ the base change $\mathsf{Sh}_K(G,X)_F$ or, what amounts to the same thing, only considering $\tau \in \mathrm{Gal}(\overline{F}/F)$ where, here, we are implicitly choosing a prime $\overline{\mathfrak{p}}$ of $\overline{\mathbb{Q}}$ lying over $\mathfrak{p}$ in E to identify $\mathrm{Gal}(\overline{F}/F)$ with a subgroup of $\mathrm{Gal}(\overline{E}/E)$. Moreover, since Grothendieck-Lefschetz trace formulas only work to understand the trace of integral powers of Frobenii acting on the cohomology of a variety over a finite field, we need to further restriction to assuming that $\tau \in W_F$.

Now, for a compact open subgroup K of $G(\mathbb{A}_f)$ and an element g of $G(\mathbb{A}_f)$ let us denote by $e(K,g)$ the following element of $\mathscr{H}(G,K)$:

$$e(K,g) := \frac{1}{\mu(K)} \mathbb{1}_{KgK} \tag{2.45}$$

where μ is the (implicitly fixed) Haar measure on $G(\mathbb{A}_f)$. In other words, $e(K,g)$ is the normalized indicator function on the double coset KgK. The element $e(K,g)$ of $\mathscr{H}(G,K)$ is known as *Hecke operator* of level K and center g. Note that any element of $\mathscr{H}(G)$ can be written as a linear combination of Hecke operators (of different levels and centers). In particular, to understand the trace of an element $\tau \times f$ on $H_c^*(\overline{\mathsf{Sh}(G,X)}_{\text{ét}}, \overline{\mathcal{F}_\xi})$ it will suffices to understand the trace of elements of the form $\tau \times e(K,g)$. In fact, it suffices to understand the actions of these Hecke operators $e(K,g)$ when K is sufficiently small and so, in particular, when $K \in \mathcal{N}(\xi)$.

The first step in doing this is the following well-known result which explicitly relates the traces of such Hecke operators to more manageable cohomology groups at finite levels. This follows from the general formalism of **G**-schemes as in Appendix C (namely Lemma 7.3.6).

Lemma 2.2.3. *Let $\tau \in W_F$ and $e(K,g)$ be a Hecke opearator of level K and element g, and assume that $K \in \mathcal{N}(\xi)$. Then,*

$$\mathrm{tr}\left(\tau \times e(K,g) \mid H_c^*(\overline{\mathsf{Sh}_K(G,X)}_{\text{ét}}, \overline{\mathcal{F}_\xi})\right) \tag{2.46}$$

is equal to

$$\mathrm{tr}\left(\tau \times u_{g,K} \mid H_c^*(\overline{\mathsf{Sh}_K(G,X)_F}_{\text{ét}}, \overline{(\mathcal{F}_{\xi,K})_F})\right) \tag{2.47}$$

Let us explain the notation used in the latter half of the above statement. Namely, let us denote by $c_{g,K}$ the *Hecke correspondence* (over F)

$$\begin{tikzcd} & \mathsf{Sh}_{K^g}(G,X)_F \arrow[dl, "\pi_{K^g,K}"'] \arrow[dr, "t_{K^g,K}(g)"] & \\ \mathsf{Sh}_K(G,X)_F & & \mathsf{Sh}_K(G,X)_F \end{tikzcd} \tag{2.48}$$

where we have denoted by K^g the group $K \cap gKg^{-1}$. Using the isomorphisms given in (2.40) we then naturally obtain a cohomological correspondence $u_{g,K} \in \mathrm{Coh}(\mathcal{F}_{\xi,K})$ (see Appendix B for a reminder on correspondences) and thus an induced map $R\Gamma_c(\tau \times u_{g,K})$ on $H^*_c(\overline{\mathsf{Sh}_K(G,X)_F}_{\text{ét}}, \overline{(\mathcal{F}_{\xi,K})_F})$. The above claim is then, essentially, that this operator agrees with the operator defined by the action of $\tau \times e(K,g)$.

Proof. (Lemma 2.2.3) Applying Lemma 7.3.6 shows that the first term is equal to the second term, except without the base change to F. The proof then follows from the fact that the natural map

$$H^*_c(\overline{\mathsf{Sh}_K(G,X)}_{\text{ét}}, \overline{(\mathcal{F}_{\xi,K})}) \to H^*_c(\overline{\mathsf{Sh}_K(G,X)_F}_{\text{ét}}, \overline{(\mathcal{F}_{\xi,K})_F}) \tag{2.49}$$

is an isomorphism which is $\mathrm{Gal}(\overline{F}/F)$-equivariant, and which intertwines the cohomological Hecke correspondence over E and over F. □

We have thus reduced ourselves to needing to compute the traces of a Galois twisted cohomological operator on the compactly supported cohomology at some, possibly bad, level K. To exploit the integral canonical models, we need to make some assumptions on the Hecke operator, in particular its level, to be able to explicitly relate it to such a model. To do this, we assume that K can be decomposed as K^pK_p with $K^p \in \mathcal{N}^p(G)$ and K_p a normal compact open subgroup of K_0 as well as assuming that we can write $g = g^pg_p$ with $g^p \in G(\mathbb{A}_f^p)$ and $g_p \in K_0$.

The desired formula for this trace will then be a weighted point count on the special fiber of $\mathscr{S}_{K^p}(G,X)$, where the weighting factor will essentially be encoded in terms of the local rigid geometry of $\mathsf{Sh}_K(G,X)_F^{\mathrm{an}}$ where this denotes the analytification of $\mathsf{Sh}_K(G,X)_F$ thought of as an adic space locally of finite type over $\mathrm{Spa}(F)$ (see Appendix A for a reminder on the theory of rigid geometry in the language of Huber's adic spaces).

The formula is as follows:

Theorem 2.2.4. *Let $K^p \in \mathcal{N}^p(G)$ be such that $K^pK_0 \in \mathcal{N}(\xi)$, and let K_p be a compact open normal subgroup of K_0. Let us set $K := K^pK_p$ and let $g = g^pg_p$ for $g^p \in G(\mathbb{A}_f^p)$ and $g_p \in K_0$. Then, there exists some constant $j_0 \geqslant 0$, depending only on g^p, such that for all $\tau \in \mathrm{Frob}_F^j W_F$ with $j > j_0$ the following equality holds:*

$$\mathrm{tr}\left(\tau \times e(K,g) \mid H^*_c(\overline{\mathsf{Sh}(G,X)}_{\text{ét}}, \overline{\mathcal{F}_\xi})\right) \tag{2.50}$$

is equal to

$$\sum_{\substack{\overline{y} \in \mathscr{S}_{(K^p)g^p}(\overline{k}) \\ \Phi_q^j(c_1(\overline{y})) = c_2(\overline{y})}} \mathrm{tr}(u_{c_2(\overline{y})} \mid (\mathcal{F}_{\xi,K^p})_{c_2(\overline{y})})\mathrm{tr}\left(\tau \times g_p \mid H^*(\overline{\mathscr{Y}_{\overline{y}}}_{\text{ét}}, \overline{\mathbb{Q}_\ell})\right) \tag{2.51}$$

Moreover, if G^{ad} is $\mathbb{Q}$-anisotropic then one can take the constant j_0 to be 0.

Let us now explain the notation in the latter half of this theorem. To begin, let us set $q := \#k$ and denote by Φ_q^j the j^{th}-power of the q-Frobenius map

$$\Phi_q : \mathscr{S}_{K^p}(G,X)_{\overline{k}} \to \mathscr{S}_{K^p}(G,X)_{\overline{k}} \tag{2.52}$$

Next let us also make the following notational convention for convenience:

$$\pi := (\pi_{K,K^pK_0})^{\mathrm{an}}_{\breve{\mathbb{Q}}_p} : \mathsf{Sh}_K(G,X)^{\mathrm{an}}_{\breve{\mathbb{Q}}_p} \to \mathsf{Sh}_{K^pK_0}(G,X)^{\mathrm{an}}_{\breve{\mathbb{Q}}_p} \tag{2.53}$$

where the superscript 'an' denotes the analytification of a $\breve{\mathbb{Q}}_p$-scheme which is an adic space of finite type over $\mathrm{Spa}(\breve{\mathbb{Q}}_p)$. Next, given a point $\overline{y} \in \mathscr{S}_{(K^p)^{g_p}}(\overline{k})$ let us denote by $\mathfrak{X}_{\overline{y}}$ the formal scheme

$$\mathfrak{X}_{\overline{y}} := (\widehat{\mathscr{S}_{K^p}(G,X)}_{\breve{\mathbb{Z}}_p})_{/c_2(\overline{y})} \tag{2.54}$$

which is a formal scheme formally of finite type over $\mathrm{Spf}(\breve{\mathbb{Z}}_p)$. In words, $\mathfrak{X}_{\overline{y}}$ is completion of $\mathscr{S}_{K^p}(G,X)_{\breve{\mathbb{Z}}_p}$ along $c_2(\overline{y})$.

Let us then denote by $\mathscr{X}_{\overline{y}}$ the adic space locally of finite type over $\mathrm{Spa}(\breve{\mathbb{Q}}_p)$ defined as $(\mathfrak{X}_{\overline{y}})_\eta$. In other words, $\mathscr{X}_{\overline{y}}$ is the generic fiber of $\mathfrak{X}_{\overline{y}}$ which, since the latter is a formal scheme, is a rigid analytic space. Finally, note that since $\mathscr{S}_{K^p}(G,X)_{\breve{\mathbb{Z}}_p}$ is separated over $\mathrm{Spec}(\breve{\mathbb{Z}}_p)$ with generic fiber $\mathsf{Sh}_{K^pK_0}(G,X)_{\breve{\mathbb{Q}}_p}$ that $\mathscr{X}_{\overline{y}}$ is naturally identified with an open adic subspace of $\mathsf{Sh}_{K^pK_0}(G,X)^{\mathrm{an}}_{\breve{\mathbb{Q}}_p}$ (see Lemma 5.4.8), and we shall make this identification freely. We then set $\mathscr{Y}_{\overline{y}}$ to be $\pi^{-1}(\mathscr{X}_{\overline{y}})$ which is an open adic subspace of $\mathsf{Sh}_K(G,X)^{\mathrm{an}}_{\breve{\mathbb{Q}}_p}$.

By $\overline{\mathscr{Y}_{\overline{y}}}$ we mean the base change of $\mathscr{Y}_{\overline{y}}$ to $\mathbb{C}_p$. We define $H^*(\overline{\mathscr{Y}_{\overline{y}}}_{\text{ét}}, \overline{\mathbb{Q}_\ell})$ as an element of the Grothendieck group of $\overline{\mathbb{Q}_\ell}$-spaces given by

$$H^*(\overline{\mathscr{Y}_{\overline{y}}}_{\text{ét}}, \overline{\mathbb{Q}_\ell}) := \sum_{i=0}^{2d} (-1)^i H^i(\overline{\mathscr{Y}_{\overline{y}}}_{\text{ét}}, \overline{\mathbb{Q}_\ell}) \tag{2.55}$$

where $d = \dim(Y_{\overline{y}})$ which agrees with the previously defined d. Now, *a priori*, $H^*(\overline{\mathscr{Y}_{\overline{y}}}, \overline{\mathbb{Q}_\ell})$ only carries the structure of an (element of the Grothendieck group of) $\overline{\mathbb{Q}_\ell}[I_F]$-module. That said, the assumption that $\Phi_q^j(c_1(\overline{y})) = c_2(\overline{y})$ allows one to promote this to an action of $\mathrm{Frob}_F^j I_F$. Moreover, the fact that $\mathscr{Y}_{\overline{y}}$ is of the form $\pi^{-1}(\mathscr{X}_{\overline{y}})$ also allows one to endow this cohomology group with an action of g_p which naturally commutes with the $\mathrm{Frob}_F^j I_F$-action. Thus, in particular, $H^*(\overline{\mathscr{Y}_{\overline{y}}}_{\text{ét}}, \overline{\mathbb{Q}_\ell})$ inherits an action of $\tau \times g_p$ and thus the term $\mathrm{tr}\left(\tau \times g_p \mid H^*(\overline{\mathscr{Y}_{\overline{y}}}_{\text{ét}}, \overline{\mathbb{Q}_\ell})\right)$ is sensical.

Finally, let us clarify the meaning of the term $\mathrm{tr}(u_{c_2(\overline{y})} \mid (\mathcal{F}_{\xi,K^p})_{c_2(\overline{y})})$. Note that both the Hecke correspondence $c_{g,K}$ and $u_{g,K}$ have integral analogues c_{K^p,g^p} and u_{K^p,g^p} where the former is an algebraic correspondence on $\mathscr{S}_{K^p}(G,X)$ and the latter is an element of $\mathrm{Coh}(\mathcal{F}_{\xi,K^p})$. One can then base change this to the special fiber to obtain an algebraic correspondence on $\mathscr{S}_{K^p}(G,X)_k$, which we also denote $c_{g,K}$ and whose components $(c_{g,K})_i$ we have shortened to c_1 and c_2 in the above. We also obtain a cohomological Hecke correspondence $u_{g,K} \in \mathrm{Coh}((\mathcal{F}_{\xi,K})_k)$. Since $u_{g,K}$ is defined over k, it commutes with Φ_q^j, and since $\overline{y}$ is a fixed point of this Frobenius twisted correspondence one gets a naive local term $u_{c_2(\overline{y})}$ on $(\mathcal{F}_{\xi,K^p})_{c_2(\overline{y})}$ thus completing the definition of $\mathrm{tr}(u_{c_2(\overline{y})} \mid (\mathcal{F}_{\xi,K^p})_{c_2(\overline{y})})$.

Remark 2.2.5. We will not give a full proof of this theorem here. Namely, we will lean quite heavily on the more general formula developed for **G**-schemes in the appendix for the brunt of the argument. That said, for the reader's convenience, let us summarize the ingredients of a proof without recourse to the appendix. First one pushes forward the sheaf $\mathcal{F}_{\xi,K}$ to a sheaf on $\mathsf{Sh}_{K^pK_0}(G,X)_F$. One can then use the projection formula to decompose this into a tensor product of $\mathcal{F}_{\xi,K^pK_0}$ and $\pi'_*\overline{\mathbb{Q}_\ell}$ where $\pi' := (\pi_{K,K^pK_0})_F$.

Assuming that the compactly supported cohomology of this sheaf can be computed as the compactly supported cohomology of the nearby cycles special fiber (a non-trivial assumption) one can then apply a Fujiwara-Varshavsky trace formula to compute things in terms of a weighted sum of fixed points of the mod p Hecke correspondence. The weighting factors are in terms of naive local factors which, since our sheaf decomposes into a tensor product, essentially come down to computing separately the naive local factors on $(\mathcal{F}_{\xi,K^p})_k$ and $R\psi\pi'_*\overline{\mathbb{Q}_\ell}$. The first is the term $\mathrm{tr}(u_{c_2(\overline{y})} \mid (\mathcal{F}_{\xi,K^p})_{c_2(\overline{y})})$. The second is the term $\mathrm{tr}\left(\tau \times g_p \mid H^*(\overline{\mathscr{Y}_{\overline{y}}}_{\text{ét}}, \overline{\mathbb{Q}_\ell})\right)$ using a theorem of Berkovich and Huber to turn the stalk of the nearby cycle sheaf at $c_2(\overline{y})$ into the cohomology of $(\pi'_*\overline{\mathbb{Q}_\ell})^{\mathrm{an}}$ on $\mathscr{X}_{\overline{y}}$. Finally, this clearly coincides with the cohomology of $\overline{\mathbb{Q}_\ell}$ on $\mathscr{Y}_{\overline{y}}$ by the ayclicity of π.

Before we begin the rigorous proof of the above, let us further explain the key obstruction to applying the machinery of Appendix C without reservation. By Theorem 7.4.4 it suffices to check that the compactly supported cohomology of some pushforward sheaf, from a bad level to a good level, agrees with the compactly supported cohomology of its complex of nearby cycles. Namely, we need to prove that the natural map

$$H^i_c(\overline{\mathsf{Sh}_{K^pK_0}(G,X)_F}_{\text{ét}}, \pi'_*\overline{\mathcal{F}_{\xi,K,F}}) \to H^i_c(\overline{\mathscr{S}_{K^p}(G,X)_k}_{\text{ét}}, R\psi\pi'_*\overline{\mathcal{F}_{\xi,K,F}}) \tag{2.56}$$

which is Galois and Hecke equivariant (i.e. which is $\mathrm{Gal}(\overline{E}/E) \times \mathscr{H}(G,K)$-equivariant) is an isomorphism where, as in the above remark, we have shortened $(\pi_{K,K^pK_0})_F$ to π'. While this holds true in the proper case, the case when G^{ad} is $\mathbb{Q}$-anisotropic, by the proper base change theorem, when the Shimura variety is not proper there is no reason for an isomorphism like (2.56) to hold. Thus, the real content of Theorem 2.2.4, in addition to the material of Appendix C, is the following:

Lemma 2.2.6. *Let (G,X) be a Shimura datum of abelian type. Let K be as in Theorem 2.2.4. Then, there is a Hecke and Galois equivariant isomorphism*

$$H^i_c(\overline{\mathsf{Sh}_{K^pK_0}(G,X)_F}_{\text{ét}}, \pi'_*\overline{\mathcal{F}_{\xi,K,F}}) \cong H^i_c(\overline{\mathscr{S}_{K^p}(G,X)_k}_{\text{ét}}, R\psi\pi'_*\overline{\mathcal{F}_{\xi,K,F}}) \tag{2.57}$$

To show this, we first make the following basic observation:

Lemma 2.2.7. *Suppose that there exists an $\mathcal{O}$-model $\mathcal{X}$ of $\mathsf{Sh}_K(G,X)_F$ together with a finite map $f : \mathcal{X} \to \mathscr{S}_{K^p}(G,X)$ whose generic fiber is identified with π' and such that there exists a Galois and Hecke equivariant isomorphism*

$$H^i_c(\overline{\mathsf{Sh}_K(G,X)_F}_{\text{ét}}, \overline{\mathcal{F}_{\xi,K}}) \cong H^i_c(\overline{\mathcal{X}_k}_{\text{ét}}, R\psi\mathcal{F}_{\xi,K}) \tag{2.58}$$

Then, (2.56) holds.

Proof. This follows immediately from proper base change. Namely, we have a natural commutative diagram of $\overline{\mathbb{Q}}_\ell[\mathrm{Gal}(\overline{F}/F)]$-representations which is also evidently Hecke equivariant:

$$\begin{tikzcd} H^i_c(\overline{\mathsf{Sh}_K(G,X)_F}_{\text{ét}}, \overline{\mathcal{F}_{\xi,K}}) \arrow[r, "(2)"] \arrow[dd, "(1)"'] & H^i_c(\overline{\mathcal{X}}_{k\text{ét}}, R\psi\mathcal{F}_{\xi,K}) \arrow[d, "(3)"] \\ & H^i_c(\overline{\mathscr{S}_{K^p}(G,X)}_{k\text{ét}}, (f_{\overline{k}})_* R\psi\mathcal{F}_{\xi,K}) \arrow[d, "(4)"] \\ H^i_c(\overline{\mathsf{Sh}_{K^pK_0}(G,X)_F}_{\text{ét}}, \overline{\pi'_*\mathcal{F}_{\xi,K}}) \arrow[r] & H^i_c(\overline{\mathscr{S}_{K^p}(G,X)}_{k\text{ét}}, R\psi\pi'_*\mathcal{F}_{\xi,K}) \end{tikzcd} \tag{2.59}$$

Here (1) and (3) are the natural isomorphisms coming from the fact that π' and f are finite, (2) is our assumed isomorphism, and (4) is the isomorphisms given to us by proper base change. We thus deduce that the bottom arrow, which is always Galois and Hecke equivariant, is an isomorphism as desired. □

So, this lemma tells us that, in good situations, we can reduce (2.56) to the existence of models of $\mathsf{Sh}_K(G,X)_F$ with good properties as in Lemma 2.2.7. Now, one doesn't expect to have anything as nice as a theory of integral canonical models whose unicity (in terms of the Nèron lifting property) hinged upon the assumption that the models were smooth. Indeed, as simple examples like the modular curve show, these cohomology groups aren't unramified in general, so we cannot expect smooth models satisfying (2.56). That said, there is no, *a priori*, obstruction to having many non-smooth models which satisfy the conditions of Lemma 2.2.7, which is all we need.

In the case of Hodge type, such a result has been shown by Lan and Stroh using the compactifications of integral canonical models constructed by Madapusi-Pera. More precisely:

Theorem 2.2.8 (Lan-Stroh). *Let (G,X) be a Shimura datum of Hodge type. Let $\mathscr{S}_K(G,X)$ be the integral model of $\mathsf{Sh}(G,X)_F$ constructed in the introduction to [Per12]. Then there is a Hecke and Galois equivariant isomorphism*

$$H^i_c(\overline{\mathsf{Sh}_K(G,X)_F}_{\text{ét}}, \overline{\mathcal{F}_{\xi,K}}) \cong H^i_c(\overline{\mathscr{S}_K(G,X)}_{k\text{ét}}, R\psi\mathcal{F}_{\xi,K}) \tag{2.60}$$

Proof. This is stated explicitly in [LS17] when K_p is the pullback of a hyperspecial subgroup of $G(\psi_{\mathbb{Z}_p})(\mathbb{Z}_p)$, using the work in [Per12]. The main obstruction to extending this result to more general K_p was the contents of [Per12] not dealing with more general cases. That said, in the 2018 update to [Per12], such restrictions on K_p have been removed, allowing one to extend the results of [LS17] to general Hodge type Shimura varieties. This is commented on in [LS18, Remark 2.1.12]. □

Let us now prove Lemma 2.2.6:

Proof. (Lemma 2.2.6) Since the map

$$H^i_c(\overline{\mathsf{Sh}_{K^pK_0}(G,X)_F}_{\text{ét}}, \pi'_*\overline{\mathcal{F}_{\xi,K,F}}) \to H^i_c(\overline{\mathscr{S}_{K^p}(G,X)}_{k\text{ét}}, R\psi\pi'_*\overline{\mathcal{F}_{\xi,K,F}}) \tag{2.61}$$

is evidently Galois and Hecke equivariant, it suffices to prove that the map of underlying $\overline{\mathbb{Q}_\ell}$-spaces is an isomorphism. Moreover, it's clear that it suffices to show that for all geometric connected components $\mathsf{Sh}_{K^pK_0}(G,X)^\circ_{\overline{\mathbb{Q}_p}}$ of $\mathsf{Sh}_{K^pK_0}(G,X)_{\overline{\mathbb{Q}_p}}$ the natural map

$$H^i_c((\mathsf{Sh}^\circ_{K^pK_0}(G,X)_{\overline{\mathbb{Q}_p}})_{\text{ét}}, \pi'_*\overline{\mathcal{F}_{\xi,K,F}}) \cong H^i_c((\mathscr{S}^\circ_{K^p}(G,X)_{\overline{k}})_{\text{ét}}, R\psi\pi'_*\overline{\mathcal{F}_{\xi,K,F}}) \tag{2.62}$$

is an isomorphism. Let us now take an associated good Hodge type datum (G_1,X_1). We know then from Lemma 2.1.26 that we can find an isomorphism

$$\mathscr{S}_{K^p}(G,X)^\circ_{\overline{\mathbb{Z}_p}} \xrightarrow{\approx} \mathscr{S}_{L^p}(G_1,X_1)^\circ_{\overline{\mathbb{Z}_p}}(G_1,X_1)/\Delta \tag{2.63}$$

where Δ is some finite group. Let q denote the quotient map from $\mathscr{S}_{L^p}(G_1,X_1)^\circ_{\overline{\mathbb{Z}_p}}$ to $\mathscr{S}_{K^p}(G,X)^\circ_{\overline{\mathbb{Z}_p}}$. Let us denote by $\mathsf{Sh}_K(G,X)^{\circ_i}(G,X)_{\overline{\mathbb{Q}_p}}$, for $i=1,\ldots,n$, the connected components of the space $\pi^{-1}(\mathsf{Sh}_{K^pK_0}(G,X)^\circ_{\overline{\mathbb{Q}_p}}(G,X))$. We also know, from the classical rational versions of Lemma 2.1.26, that we can find a finite group Δ', a subgroup $L_p\subseteq G_1(\mathbb{A}^p_f)$, and an isomorphism

$$\coprod_{i=1}^n \mathsf{Sh}_{K^pK_p}(G,X)^{\circ_i}(G,X)_{\overline{\mathbb{Q}_p}} \xrightarrow{\approx} \left(\coprod_{i=1}^n \mathsf{Sh}_{L^pL_p}(G_1,X_1)^{\circ_i}_{\overline{\mathbb{Q}_p}}\right)/\Delta' \tag{2.64}$$

Again, we denote by q' the quotient map

$$\coprod_{i=1}^n \mathsf{Sh}_{L^pL_p}(G_1,X_1)^{\circ_i}_{\overline{\mathbb{Q}_p}} \to \coprod_{i=1}^n \mathsf{Sh}_{K^pK_p}(G,X)^{\circ_i}(G,X)_{\overline{\mathbb{Q}_p}} \tag{2.65}$$

Let us denote by π' the natural projection map

$$\coprod_{i=1}^n \mathsf{Sh}_{L^pL_p}(G_1,X_1)^{\circ_i}_{\overline{\mathbb{Q}_p}} \to \mathsf{Sh}_{L^pL_0}(G,X)^\circ_{\overline{\mathbb{Q}_p}} \tag{2.66}$$

We then have a diagram

$$\begin{tikzcd} \coprod_{i=1}^n \mathsf{Sh}^{\circ_i}_{L^pL_p}(G_1,X_1)_{\overline{\mathbb{Q}_p}} \arrow[r] \arrow[d] & \coprod_{i=1}^n \mathsf{Sh}_{K^pK_p}(G,X)^\circ_{\overline{\mathbb{Q}_p}} \arrow[d] \\ \mathsf{Sh}_{L^pL_0}(G_1,X_1)^\circ_{\overline{\mathbb{Q}_p}} \arrow[r] & \mathsf{Sh}_{K^pK_0}(G,X)^\circ_{\overline{\mathbb{Q}_p}} \end{tikzcd} \tag{2.67}$$

which, while not Cartesian, becomes Cartesian when one quotients the top left space by Δ/Δ'. Now, from the proper base change theorem, since the maps in (2.67) are finite, and the Hochschild-Serre spectral sequence we have the following series of identifications:

$$\begin{aligned} H^*_c(\mathsf{Sh}^\circ_{K^pK_0}(G,X)_{\overline{\mathbb{Q}_p}}, \pi_*\mathcal{F}_{\xi,K^pK_p}) &= H^*_c(\mathsf{Sh}^\circ_{L^pL_0}(G_1,X_1)_{\overline{\mathbb{Q}_p}}, q^*\pi_*\mathcal{F}_{\xi,K^pK_p})^\Delta \\ &= H^*_c\left(\coprod_{i=1}^n \mathsf{Sh}_{L^pL_p{}^\circ i}(G_1,X_1)_{\overline{\mathbb{Q}_p}}, \pi'_*(q')^*\mathcal{F}_{\xi,K^pK_p}\right)^\Delta \end{aligned} \tag{2.68}$$

Now, using the isogeny $G_1^{\mathrm{ad}} \to G^{\mathrm{ad}}$ we can find some automorphic sheaf $\mathcal{G}_{\xi',L^pL_p}$ on $\mathsf{Sh}_{L^pL_p}(G_1,X_1)$ such that we can identify $(q')^*\mathcal{F}_{K^pK_p}$ with $\mathcal{G}_{\xi',L^pL_p}$ on each $\mathsf{Sh}^{\circ_i}_{L^pL_p}(G_1,X_1)_{\overline{\mathbb{Q}_p}}$. Thus, this last term can be identified with

$$\left(\bigoplus_{i=1}^{n} H_c^*(\mathsf{Sh}^{\circ}_{L^pL_0}(G_1,X_1)_{\overline{\mathbb{Q}_p}}, \pi'_*\mathcal{G}_{L^pL_p})\right)^{\Delta} \tag{2.69}$$

By Theorem 2.2.8 we know that we can identify this with

$$\left(\bigoplus_{i=1}^{n} H_c^*(\mathscr{S}_{L^p}(G_1,X_1)^{\circ_i}(G_1,X_1)_{\overline{k}}, R\psi\pi'_*\mathcal{G}_{L^pL_p})\right)^{\Delta} \tag{2.70}$$

Then, again by Hocschild-Serre, we can identify this with $H^*(\mathscr{S}_{K^p}(G,X)^{\circ}_{\overline{k}}, R\psi\mathcal{F}_{K^pK_p})$ as desired. $\square$

Chapter 3

Deformation spaces and uniformzation

In this section we define rigid analytic spaces that will allow us to give purely local description of the spaces $\mathscr{Y}_{\overline{y}}$ with $g_p \times \tau$ action in Theorem 2.2.4, liberating this local weighting factor from its, *a priori*, global description. In the Hodge type setting these spaces can essentially be obtained by giving level structure to the deformation spaces with Tate tensors given by Faltings in [Fal99] but for which we adopt a more group theoretic description as given in [Kim13]. In the abelian type setting we obtain our spaces (which we imprecisely still call 'deformation spaces') from those in the Hodge type setting by group theoretic manipulations similar to the construction of integral canonical models of Shimura varieties of abelian type as discussed in the previous section.

3.1 Group theoretic preliminaries

Before we describe the construction of these deformation spaces it will be necessary to first establish some group theoretic notations and results.

Let us fix F to be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformizer π, and residue field k. For much of this subsection we will assume that $F/\mathbb{Q}_p$ is unramified, but we do not immediately need this assumption in this subsubsection. Let us also fix a reductive group $\mathcal{H}$ over $\mathcal{O}$ and denote its generic fiber H, with its special fiber being just denoted $\mathcal{H}_k$.

Maximal split tori and cocharacters

We would first like to pick some data concerning $\mathcal{H}$ that will be important for the rest of the section which amounts to showing that the notion of 'maximally split tori' exist integrally and is well-behaved. Namely, let us say that a torus $\mathcal{S} \subseteq \mathcal{H}$ is *maximally split* if $\mathcal{S}_F \subseteq H$ and $\mathcal{S}_k \subseteq \mathcal{H}_k$ are maximally split.

We then have the following lemma:

Lemma 3.1.1. *There exist closed subgroup schemes $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{B}$ in $\mathcal{H}$ with $\mathcal{S}$ a maximal split torus, $\mathcal{T}$ a maximal torus, and $\mathcal{B}$ a Borel subgroup. Moreover, $Z_{\mathcal{H}}(\mathcal{S}) = \mathcal{T}$ and all maximal split tori in $\mathcal{H}$ are $\mathcal{H}(\mathcal{O})$-conjugate.*

Before we prove this, it's worth recalling this following basic fact:

Lemma 3.1.2. *Let K be a perfect field and G a quasi-split group over K. Then, a maximal torus $T \subseteq G$ is contained in a Borel subgroup of G if and only if T contains a maximal split torus of G.*

Proof. Let T be a maximal torus and let S be a maximal split torus contained in T. There then exists a Borel subgroup B of G containing S (e.g. this follows quite easily from [Spr10, Theorem 14.1.7]). But, note that B contains a maximal split torus S' with $C_G(S')$ a Levi subgroup of B (e.g. see [Bor12, 20.6]). Now, by Grothendieck's theorem we know that S and S' are conjugate, say $S = gS'g^{-1}$. Note then that $T = C_G(S) = gC_G(S')g^{-1} \subseteq gBg^{-1}$. Thus, T is contained in a Borel.

Conversely, suppose that B is a Borel subgroup of G and $T \subseteq B$ is a maximal torus. Note then that there exists a maximal split torus S in B such that $T' := C_G(S)$ is a Levi subgroup of B (again see [Bor12, 20.6]). But, note that $C_G(S)$ is a maximal torus since it's solvable and reductive. But then, since $T \cap R_u(B)$ is trivial, we see that T maps isomorphically to $B/R_u(B) \cong T'$ and thus we see that T contains a maximal torus of the same rank S, and thus that T contains a maximal split torus of G. □

Proof. (Lemma 3.1.1) Let $\overline{\mathcal{S}}$ be a maximal split torus in $\mathcal{H}_k$. Choose a maximal torus $\overline{\mathcal{T}}$ containing $\overline{\mathcal{S}}$. Note that since $\overline{\mathcal{T}}$ contains a maximal split torus of $\mathcal{H}_k$ that there is some Borel subgroup $\overline{\mathcal{B}}$ contained in $\mathcal{H}_k$ containing $\overline{\mathcal{T}}$. Now, since the scheme of Borel subgroups of $\mathcal{H}$ is smooth (see [Con11, Theorem 5.2.11]) we can use Hensel's lemma to lift $\overline{\mathcal{B}}$ to a Borel subgroup $\mathcal{B}$ of $\mathcal{H}$. Now, since the moduli space of maximal tori in $\mathcal{B}$ is smooth (see [Con11, Theorem 3.2.6]) we can apply Hensel's lemma to lift $\overline{\mathcal{T}}$ to a maximal torus $\mathcal{T}$ in $\mathcal{B}$ such that $\mathcal{T}_k = \overline{\mathcal{T}}$. Next, recall though that the tori over a an irreducible normal scheme X are classified by continuous representations $\rho : \pi_1^{\text{ét}}(X, \overline{x}) \to \mathrm{GL}(\Lambda)$ where Λ is a finite free $\mathbb{Z}$-module given the discrete topology (e.g. see [Con11, Corollary B.3.6]) where $\overline{x}$ is any geometric point of X. In particular, since the map $\pi_1^{\text{ét}}(\mathrm{Spec}(\mathcal{O}), \overline{x}) \to \pi_1^{\text{ét}}(\mathrm{Spec}(k), \overline{x})$ (where $\overline{x}$ corresponds to any embedding $k \hookrightarrow \overline{k}$) is an isomorphism of topological groups, we see that the embedding $\overline{S} \hookrightarrow \mathcal{T}_k$ lifts uniquely to an embedding $\mathcal{S} \hookrightarrow \mathcal{T}$ where $\mathcal{S}_k = \overline{\mathcal{S}}$. Moreover, by construction the $\pi_1(\mathrm{Spec}(\mathcal{O}), \overline{x})$-action on $X_*(\mathcal{S})_{\overline{x}}$ is trivial, so that $\mathcal{S}$ is split.

To see that it's maximal split, we need only prove that $\mathcal{S}_F$ is a maximal split torus in H. Note though that since $\mathcal{S}_F$ is a subtorus of $\mathcal{T}_F$ which is contained in the Borel $\mathcal{B}_F$ we see from Lemma 3.1.2 that it suffices to show that $\mathcal{S}_F$ is a maximal split torus in $\mathcal{T}_F$. But, this is clear since a split subtorus of $\mathcal{T}_F$ can be lifted to a split subtorus of $\mathcal{T}$ (again by [Con11, Corollary B.3.6]).

The fact that $Z_{\mathcal{H}}(\mathcal{S}) = \mathcal{T}$ is clear by construction. Indeed, we know that $Z_{\mathcal{H}}(\mathcal{S})$ is a smooth closed subgroup scheme of $\mathcal{T}$ (see [Con11, Lemma 2.2.4]) with connected fibers (by classical theory) and so it suffices to see that this is an isomorphism on fibers. But, this is then clear by construction of $\mathcal{S}$ and $\mathcal{T}$.

To see the last claim, let us first assume that $\mathcal{S}'$ is another maximal torus in $\mathcal{H}$. Let $X := \underline{\mathrm{Transp}}_{\mathcal{H}}(\mathcal{S}, \mathcal{S}')$ be the transporter scheme (see [Con11, §2.1]). Then, since $\mathcal{S}$ and $\mathcal{S}'$ are of multiplicative type we know (see [Con11, Proposition 2.1.2]) that X is a smooth $\mathbb{Z}_p$-scheme. Now, by Grothendieck's theorem we know that $X(k)$ is non-empty, and thus $X(\mathcal{O})$ is non-empty by Hensel's lemma. □

Let us fix the objects $\mathcal{S}, \mathcal{T}$ and $\mathcal{B}$ from the above proposition. We'll denote their generic fibers as S, T, and B respectively. Recall that the *Weyl group* of $\mathcal{S}$ inside of $\mathcal{H}$ is the quotient group scheme

$$\mathcal{W} := N_{\mathcal{H}}(S)/Z_{\mathcal{H}}(S) = N_{\mathcal{H}}(S)/\mathcal{T} \tag{3.1}$$

Note then that

$$W := \mathcal{W}_F = N_H(S)/Z_H(S) = N_H(S)/T \tag{3.2}$$

Note that $\mathcal{W}$ is a constant group scheme since it embeds into the automorphism sheaf $\mathrm{Aut}(\mathcal{S}) \cong \underline{\mathrm{GL}_m(\mathbb{Z})}$ where m is the relative dimension of $\mathcal{S}$ over $\mathcal{O}$.

We would like to claim that the equalities

$$W(F) = N_H(S)(F)/T(F) \tag{3.3}$$

and

$$\mathcal{W}(\mathcal{O}) = N_{\mathcal{H}}(S)(\mathcal{O})/\mathcal{T}(\mathcal{O}) \tag{3.4}$$

hold. The claim over F follows from the classical theory of the relative Bruhat decomposition (e.g. see [CGP15, Corollary 3.4.3]) even though $H^1(F, T)$ may not be zero. The claim for $\mathcal{O}$ is actually of a simpler nature. Indeed, it suffices to show that $H^1_{\text{ét}}(\mathrm{Spec}(\mathcal{O}), \mathcal{T}) = 0$. We have a natural map

$$H^1_{\text{ét}}(\mathrm{Spec}(\mathcal{O}), \mathcal{T}) \to H^1(k, T) \tag{3.5}$$

Then, the conclusion follows from the following basic lemma:

Lemma 3.1.3. *Let $(A, \mathfrak{m})$ be a Henselian local ring with residue field l. Let $\mathcal{G}$ be a smooth affine group scheme over A. Then, the natural map $H^1_{\text{ét}}(\mathrm{Spec}(A), \mathcal{G}) \to H^1(l, \mathcal{G}_l)$ is injective. In particular, if l is finite and $\mathcal{G}$ has connected fibers then $H^1_{\text{ét}}(\mathrm{Spec}(A), \mathcal{G}) = 0$.*

Proof. Note that since $\mathcal{G}$ is a smooth affine group scheme that all $\mathcal{G}$-torsors are reprentable by some smooth affine A-scheme $\mathcal{X}$. A similar statement holds over l, and the map on cohomology sends $\mathcal{X}$ to $\mathcal{X}_l$. Suppose now that $\mathcal{X}_k$ is trivial. Then, $\mathcal{X}_k(k) \neq \varnothing$. But, by Hensel's lemma we know that $\mathcal{X}(\mathcal{O}) \to \mathcal{X}_k(k)$ is surjective, and thus $\mathcal{X}(\mathcal{O}) \neq \varnothing$. Thus, $\mathcal{X}$ is trivial as desired. This has shown that the kernel of the map $H^1_{\text{ét}}(\mathrm{Spec}(A), \mathcal{G}) \to H^1(l, \mathcal{G}_l)$ is trivial, and the general injectivity result follows from the standard twisting argument of Serre. The final claim is then clear from Lang's theorem. □

Combining all of this, we can now prove the following lemma which will justify our cavalierness concerning the distinction between conjugacy classes of rational and integral cocharacters for unramified groups:

Lemma 3.1.4. *The natural map*

$$\mathrm{Hom}(\mathbb{G}_{m,\mathcal{O}}, \mathcal{H})/\mathcal{H}(\mathcal{O}) \to \mathrm{Hom}(\mathbb{G}_{m,F}, H)/H(F) \tag{3.6}$$

(where the quotients indicate modding out by conjugacy) is a bijection.

Proof. Let us note that since all maximal split tori in $\mathcal{H}$ and H are conjugate, and any cocharacter lands in a split torus, one easily deduces that the natural maps

$$\mathrm{Hom}(\mathbb{G}_{m,\mathcal{O}}, \mathcal{S})/(N_{\mathcal{H}}(\mathcal{S})(\mathcal{O})/\mathcal{T}(\mathcal{O})) \to \mathrm{Hom}(\mathbb{G}_{m,\mathcal{O}}, \mathcal{H})/\mathcal{H}(\mathcal{O}) \tag{3.7}$$

and

$$\mathrm{Hom}(\mathbb{G}_{m,F}, S)/(N_H(S)(F)/T(F)) \to \mathrm{Hom}(\mathbb{G}_{m,F}, H)/H(F) \tag{3.8}$$

are bijections. So, it suffices to prove that

$$\mathrm{Hom}(\mathbb{G}_{m,\mathcal{O}}, \mathcal{S})/(N_{\mathcal{H}}(\mathcal{S})(\mathcal{O})/\mathcal{T}(\mathcal{O})) \to \mathrm{Hom}(\mathbb{G}_{m,F}, H)/(N_H(S)(F)/T(F)) \tag{3.9}$$

is a bijection. Note though that this follows from the equality

$$N_{\mathcal{H}}(\mathcal{S})(\mathcal{O})/\mathcal{T}(\mathcal{O}) = \mathcal{W}(\mathcal{O}) = W(F) = N_H(S)(F)/T(F) \tag{3.10}$$

□

Let us now consider any discretely valued extensions F' (with ring of integers $\mathcal{O}'$) of F the notation $C_{\mathcal{H}}(F')$ to denote the sets

$$C_{\mathcal{H}}(F') := \mathrm{Hom}(\mathbb{G}_{m,F'}, H_{F'})/H(F') = \mathrm{Hom}(\mathbb{G}_{m,\mathcal{O}'}, \mathcal{H}_{\mathcal{O}'})/\mathcal{H}(\mathcal{O}') \tag{3.11}$$

where the last equality follows from Lemma 3.1.4. We shall denote elements in $C_{\mathcal{H}}(F')$ by $\boldsymbol{\mu}$. Note that if $F_1' \supseteq F'$ then we get a natural map of sets $C_{\mathcal{H}}(F') \to C_{\mathcal{H}}(F_1')$ and we say that an element of $C_{\mathcal{H}}(F_1')$ is *defined over* F' if its in the image of this map. Note that the choice of a maximal split torus $\mathcal{S}'$ in $\mathcal{H}_{\mathcal{O}'}$ and a Borel $\mathcal{B}'$ containing it (with generic fiber S' and B') also gives us sections of the maps

$$\mathrm{Hom}(\mathbb{G}_{m,F'}, H_{F'}) \to C_{\mathcal{H}}(F') \tag{3.12}$$

and

$$\mathrm{Hom}(\mathbb{G}_{m,\mathcal{O}'}, \mathcal{H}_{\mathcal{O}'}) \to C_{\mathcal{H}}(\mathcal{O}') \tag{3.13}$$

in the form of the B (resp. $\mathcal{B}$) dominant cocharacters $X_*(S')_+$ (resp. $X_*(\mathcal{S}')_+$) respectively. For $\boldsymbol{\mu}$ we denote by $\boldsymbol{\mu}_+$ the unique element of $X_*(S)_+$ or $X_*(\mathcal{S})_+$ corresponding to $\boldsymbol{\mu}_+$ (which is meant will always be clear from context). Finally, note that writing the group $\mathrm{Hom}(\mathbb{G}_{m,\mathcal{O}'}, \mathcal{H}_{\mathcal{O}'})$ multiplicatively, for an element $\boldsymbol{\mu} \in C_{\mathcal{H}}(F')$ the set $\boldsymbol{\mu}^{-1}$ denotes the result of inverting every element of $\boldsymbol{\mu}$.

Let us now suppose that F'/F is algebraic. Note then that we get a natural discrete action of the Galois group $\mathrm{Gal}(F'/F)$ on $C_{\mathcal{H}}(F')$. We define the *reflex field* $E(\boldsymbol{\mu})$ of $\boldsymbol{\mu} \in C_{\mathcal{H}}(F')$ to be the unique subextension E of F'/F such that the stablizer of $\boldsymbol{\mu}$ in $\mathrm{Gal}(F'/F)$ is $\mathrm{Gal}(F'/M)$. Note that if $\boldsymbol{\mu}$ is defined over a subextension of M of F'/F then $E(\boldsymbol{\mu}) \subseteq M$. But since H is quasi-split we have conversely by a lemma of Kottwitz (see [Kot84, Lemma 1.1.3]) that $\boldsymbol{\mu}$ is defined over $E(\boldsymbol{\mu})$ and thus $E(\boldsymbol{\mu})$ is the minimal subextension of definition in F'/F.

Let now F'/F be any discretely valued extension and let $\boldsymbol{\mu} \in C_{\mathcal{H}}(F')$. Let us choose an element $\mu \in \boldsymbol{\mu}$ thought of as a cocharacter of $\mathcal{H}_{\mathcal{O}'}$. We would like to recall some basic groups constructed from μ using the so-called *dynamic method*. Namely, recall that from the cocharacter $\mu : \mathbb{G}_{m,\mathcal{O}'} \to \mathcal{H}_{\mathcal{O}'}$ we can associate a smooth split unipotent group $U(\mu)$ over $\mathcal{O}'$. Specifically, for a $\mathcal{O}'$-algebra R and an element $h \in \mathcal{H}(R)$ let us write $\mu(t,h)$ to mean the element $\mu(t)h\mu(t)^{-1}$ considered as an element of $\mathcal{H}(R[t,t^{-1}])$. Let us then say that $\lim_{t\to 0}\mu(t,h)$ *exists* if the element $\mu(t,h)$ of $\mathcal{H}(R[t,t^{-1}])$ actually lies in the subset $\mathcal{H}(R[t])$. One can then define a functor $P(\mu)$ on $\mathcal{O}'$-algebras R by declaring that

$$P(\mu)(R) := \{h \in \mathcal{H}(\mu)(R) : \lim_{t\to 0}\mu(t,h) \text{ exists}\} \tag{3.14}$$

Note that if $h \in P(\mu)(R)$ then $\lim_{t\to 0}\mu(t,h)$ (by which we mean the specialization of $\mu(t,h)$ along $R \to R[t]$) makes sense. We then set $U(\mu)(R)$ to be the following subset of $P(\mu)(R)$:

$$U(\mu)(R) := \{h \in P(\mu)(R) : \lim_{t\to 0}\mu(t,h) = 1\} \tag{3.15}$$

where, by 1, we mean the identity element of $\mathcal{H}(R)$.

Recall then that the functor $U(\mu)$ is represented by a smooth split unipotent group scheme over $\mathrm{Spec}(\mathcal{O}')$ with connected fibers and of relative dimension $d(\mu) := \dim \mathrm{Fil}^1_\mu(\mathfrak{h}_{\mathcal{O}'})$, the dimension of the first filtered piece of the filtration on $\mathfrak{h} := \mathrm{Lie}(\mathcal{H})$ induced by μ (e.g. see [Con11, Theorem 4.1.7]).

We record the following basic observations for later:

Lemma 3.1.5. *Let $\mu : \mathbb{G}_{m,\mathcal{O}'} \to \mathcal{H}_{\mathcal{O}'}$ be a cocharacter, and denote by $\mu^{\mathrm{ad}} : \mathbb{G}_{m,\mathcal{O}'} \to \mathcal{H}^{\mathrm{ad}}_{\mathcal{O}'}$ the induced adjoint cocharacter. We then get an induced map $U(\mu) \to U(\mu^{\mathrm{ad}})$. This map is an isomorphism of* $\mathrm{Spec}(\mathcal{O}')$*-groups.*

Proof. Note that evidently we get a map $f : U(\mu) \to U(\mu^{\mathrm{ad}})$ by definition. Note then that $\ker f = U(\mu) \cap Z_{\mathcal{H}}$. Since $U(\mu)$ is a split unipotent group and $Z_{\mathcal{H}}$ is of multiplicative type we deduce that $\ker f$ is trivial. Thus, we see by [Con11, Theorem 5.3.5] that f is a closed immersion. Since $U(\mu)$ and $U(\mu')$ are smooth with connected fibers it suffices to show that they have the same dimension. But, since $\mathrm{Ad} : \mathcal{H}_{\mathcal{O}'} \to \mathrm{GL}(\mathfrak{h}_{\mathcal{O}'})$ factors through $\mathcal{H}^{\mathrm{ad}}$ (see [Con11, Proposition 3.3.8]) we see that $\mathrm{Fil}^1_\mu(\mathfrak{h}_{\mathcal{O}'}) \cap Z(\mathfrak{h}_{\mathcal{O}'})$ is trivial. From this, it's easy to see that

$$\dim \mathrm{Fil}^1_\mu(\mathfrak{h}_{\mathcal{O}'}) = \dim \mathrm{Fil}^1_{\mu^{\mathrm{ad}}}(\mathfrak{h}^{\mathrm{ad}}_{\mathcal{O}'}) \tag{3.16}$$

and the conclusion follows. □

Lemma 3.1.6. *Let $\iota : \mathcal{H}_{\mathcal{O}'} \hookrightarrow \mathcal{G}$ be a closed embedding of reductive groups over* $\mathrm{Spec}(\mathcal{O}')$*. Then, for any cocharacter $\mu : \mathbb{G}_{m,\mathcal{O}'} \to \mathcal{H}_{\mathcal{O}'}$ there is an equality of smooth* $\mathrm{Spec}(\mathcal{O}')$*-group schemes $U(\iota \circ \mu) \cap \mathcal{H} = U(\mu)$.*

Proof. This is clear. See [Con11, Proposition 4.1.10]. □

The last thing we check is that the group scheme $U(\mu)$ doesn't depend on the choice of $\mu \in \boldsymbol{\mu}$:

Lemma 3.1.7. *Let $\mu_1, \mu_2 : \mathbb{G}_{m,\mathcal{O}'} \to \mathcal{H}_{\mathcal{O}'}$ be $\mathcal{H}(\mathcal{O}')$-conjugate cocharacters of $\mathcal{H}_{\mathcal{O}'}$. Then, there is an equality $U(\mu_1) = U(\mu_2)$ of* $\mathrm{Spec}(\mathcal{O}')$*-groups.*

Proof. Let $h_0 \in \mathcal{H}(\mathcal{O}')$ be such that $\mu_1 = h\mu_2 h^{-1}$. It's clear then that for a $\mathcal{O}'$-algebra R and some $g \in \mathcal{H}(R[t,t^{-1}])$ that $\mu_1(t)g\mu_1(t)^{-1} \in \mathcal{H}(R[t])$ if and only if $\mu_2(t)g\mu_2(t)^{-1} \in \mathcal{H}(R[t])$. Moreover, it's clear that $\lim_{t\to 0}\mu_1(t,g) = 1$ if and only if $\lim_{\to 0}\mu_2(t,g) = 1$. $\square$

For this reason we shall often times denote by $U(\boldsymbol{\mu})$ the smooth split unipotent $\mathrm{Spec}(\mathcal{O}')$-group $U(\mu)$ for any $\mu \in \boldsymbol{\mu}$. We shall often times also denote by $d(\boldsymbol{\mu})$ the integer $d(\mu)$ for any $\mu \in \boldsymbol{\mu}$.

The map κ_H

We would next like to define a variant of the Kottwitz map (as in [Kot85]) in our specific situation.

Let us first say that for all $j \geqslant 1$ we will denote by F_j (or $\mathbb{Q}_{p^j}$ if $F = \mathbb{Q}_p$) the unramified extension of F of degree j and $\mathcal{O}_j$ (or $\mathbb{Z}_{p^j}$ if $\mathcal{O} = \mathbb{Z}_p$) its ring of integers. Let us set F^{ur} (resp. $\mathcal{O}^{\mathrm{ur}}$) to be the maximal unramified extension of F (resp. its ring of integers). Let us set $\breve{F}$ and $\breve{\mathcal{O}}$ to be the completion of F^{ur} and $\mathcal{O}^{\mathrm{ur}}$ respectively. The notation $j \leqslant \infty$ means that either $j \in \mathbb{N}$ or $j = \infty$ (which is interpreted to mean $\breve{F}$ or $\breve{\mathcal{O}}$ depending on context). We denote by σ the Frobenius element in the group $\mathrm{Gal}(F^{\mathrm{ur}}/F) = \mathrm{Aut}_{\mathrm{cont.}}(\breve{F}/F)$ as well as its projection to $\mathrm{Gal}(F_j/F)$ for all j. We shall also use the symbol σ to denote the action of Frobenius on the F-points, $\mathcal{O}$-points, or k-points of any group over $\mathcal{O}$.

We first recall the following version of the Cartan decomposition for quasi-split groups:

Lemma 3.1.8 (Cartan decomposition)**.** *For any finite extension F' of F (or $F' = \breve{F}$) and π any uniformizer of F' then we have a decomposition*

$$H(F') = \bigsqcup_{\lambda \in \mathcal{C}} K\lambda(\pi)K \tag{3.17}$$

where $\mathcal{C}$ is any section of the map $\mathrm{Hom}(\mathbb{G}_{m,F'}, H_{F'}) \to C_{\mathcal{H}}(F')$, and this decomposition is independent of π. In particular, the choice of a maximal split torus S' in H'_F and and a Borel B containing S gives us a natural choice of section, namely $X_*(S')_+$.

Proof. If M/F is finite this follows from the classical Cartan decomposition (e.g. see [Tit79, §3.3.3]). If $M = \breve{F}$ it follows by passing to the limit and applying a simple density argument. Let F_j/F be a finite extension such that G is split, so that all cocharacters over $\breve{F}$ are defined over F_j. Note then that for any element $h \in \mathcal{H}(\breve{F})$ we have a sequence of elements h_j of $H(F_j)$ convering to h—this follows since H is unirational (e.g. see [Mil17, Theorem 17.93]). Now each h_j can be written as $h_j^1\lambda_j(\pi)h_j^2$ for some $h_j^1, h_j^2 \in \mathcal{H}(\mathcal{O}_j)$ and some cocharacter λ_j. It's clear that eventually the λ_j must stabilize so that we're obtaining $h_j^1\lambda(\pi)h_j^2$ and it's clear that since $\mathcal{H}(\breve{\mathcal{O}})$ is closed in $H(\breve{F})$ that h_j^i converge to elements of $\mathcal{H}(\breve{\mathcal{O}})$ from where the claim follows. $\square$

We begin defining κ_H by first defining a compatible family of maximal tori in H_M as M runs over the finite subextensions of F^{ur}/F (or $M=\breve{F}$). Namely, let us fix subgroups $\mathcal{S}\subseteq\mathcal{T}\subseteq\mathcal{B}$ of $\mathcal{H}$ where $\mathcal{S}$ is a split maximal torus of $\mathcal{H}$, $\mathcal{T}$ a maximal torus of $\mathcal{H}$, and $\mathcal{B}$ a Borel. Let us denote by $S\subseteq T\subseteq B$ the generic fibers of these objects. Let us also fix π to be a uniformizer of F. Then, for any such M let us denote S^M the split subtorus of T_M corresponding to the containment of Galois lattices $X_*(T_M)^{\mathrm{Gal}(F^{\mathrm{ur}}/M)}\subseteq X_*(T_M)$. In other words, S^M is the split maximal torus in T_M. It's clear from set-up that $X_*(S^M)$ naturally embeds into $X_*(S^{M'})$ for any M'/M contained in F^{ur}/F and, moreover, that with respect to the consistent family of Borels B_M we have that $X_*(S^M)_+$ embeds into $X_*(S^{M'})_+$.

Now, let us define for any such M the map

$$\kappa_M : H(M)\to X_*(S^M)_+ \tag{3.18}$$

by declaring that $\kappa_H(b)$, for an element $b\in H(M)$, is the unique $\mu\in X_*(S^M)_+$ such that b lies in $\mathcal{H}(\mathcal{O}_M)\mu(\pi)\mathcal{H}(\mathcal{O}_M)$. By construction it's clear that for any extension M'/M of such fields and any $b\in H(M)$ we have consistency of our κ-maps, in the sense that $\kappa_M(b)=\kappa_{M'}(b)$ under the inclusion $X_*(S^M)_+\subseteq X_*(S^{M'})_+$.

Let us now write $M=F_j$ for some $j\leqslant\infty$ and S^M as S_j. Recall that if $b,b'\in H(F_j)$ then b and b' are *σ-conjugate* if there exists some $h\in H(F_j)$ such that $b=hb'\sigma(h)^{-1}$. We say that b and b' are *integrally σ-conjugate* if there exists some $h\in\mathcal{H}(\mathcal{O}_j)$ such that $b=hb'\sigma(h)^{-1}$. Let us denote the set of σ-conjugacy classes in $H(F_j)$ by $B_j(\mathcal{H})$ (or $B_j(H)$) and the set of integral σ-conjugacy classes in $H(F_j)$ by $C_j(\mathcal{H})$. Note then that since $\mathcal{H}(\mathcal{O}_j)\mu(\pi)\mathcal{H}(\mathcal{O}_j)$ is stable under integral σ-conjugacy for each $\mu\in X_*(S_j)_+$ we have that κ_{F_j} factors through $C_j(\mathcal{H})$.

For an element $\boldsymbol{\mu}\in C_{\mathcal{H}}(E)$ for some unramified extension $E=F_r/F$ we denote by $C_j(\mathcal{H},\boldsymbol{\mu})$ the set of $[b]\in C_{rj}(\mathcal{H})$ such that $\kappa_{F_{rj}}^{-1}(\boldsymbol{\mu}_+)$ which makes sense since μ is defined over F_{rj}. Of course, note that F_{rj} is merely E_j.

For comparison to other sources, we recall the relationship of our $\kappa_{\breve{F}}$ to the Kottwitz map (as in [Kot85]). If we denote by $R^\vee$ the coroot lattice of the pair $(H_{\breve{F}},T_{\breve{F}})$ then the composition

$$H(F_\infty)\xrightarrow{\kappa_{\breve{F}}} X_*(T_{\breve{F}})_+\to (X_*(T_{\breve{F}})/R^\vee)^{\mathrm{ur}}_{\Gamma_F} \tag{3.19}$$

(where $(-)_{\Gamma_F^{\mathrm{ur}}}$ denotes the $\Gamma_F^{\mathrm{ur}}=\mathrm{Gal}(F^{\mathrm{ur}}/F)$-coinvariants) factors through $B(H)$ we shall denote κ_H.

Let $\pi_1(H)$ denote the abelian fundamental group of Borovoi (cf. [Bor98]). Then, under the identification of $\pi_1(H)_\Gamma$ with $(X_*(T)/R^\vee)_\Gamma$ (see loc. cit.) we have the following basic observation:

Lemma 3.1.9. *The map κ_H coincides with the Kottwitz map as defined in [Kot85].*

Proof. See [RV14, Remark 2.2, ii)]. □

Some miscellaneous lemmas

We record in here some miscellaneous results concerning reductive groups over $\mathcal{O}$.

We begin with the following two observations:

Lemma 3.1.10. *Let $\mathcal{H}_i$ be reductive* $\mathrm{Spec}(\mathcal{O})$*-groups with generic fibers H_i for $i = 1, 2$. Let $f : \mathcal{H}_1 \to \mathcal{H}_2$ be a map of reductive* $\mathrm{Spec}(\mathcal{O})$*-groups with* $\ker f$ *having connected fibers. Then, $\mathcal{H}_1(\mathcal{O}) \to \mathcal{H}_2(\mathcal{O})$ is surjective. If* $\ker f$ *is central then $H_1(F) \to H_2(F)$ is surjective.*

Proof. By definition we have a short exact sequence of $\mathrm{Spec}(\mathcal{O})$-groups

$$0 \to \ker f \to \mathcal{H}_1 \to \mathcal{H}_2 \to 0 \tag{3.20}$$

which gives an exact sequence of pointed sets

$$0 \to (\ker f)(\mathcal{O}) \to \mathcal{H}_1(\mathcal{O}) \to \mathcal{H}_2(\mathcal{O}) \to H^1_{\text{ét}}(\mathrm{Spec}(\mathcal{O}), \ker f) \tag{3.21}$$

But, this last pointed set is trivial by Lemma 3.1.3.

Suppose now that f has central kernel. To see that the map $H_1(F) \to H_2(F)$ is surjective we proceed as follows. Let $S_1 \subseteq T_1 \subseteq B_1$ be a maximal split torus of H_1, contained in a maximal torus of H_1, contained in a Borel of H_1. Then, $f(T_1)$ is a maximal torus of H_2 and $f(B_1)$ is a Borel of H_2. By Lemma 3.1.2 we see that $f(T_1)$ contains a maximal split S_2. It's then clear that $S_2 = f(S_1)$. Now, by Lemma 3.1.8 we know that every element of $H_2(F)$ is of the form $h_1\lambda(\pi)h_2$ with $h_i \in \mathcal{H}_2(\mathcal{O})$ and $\lambda \in X_*(S_2)$. Note that $X_*(S_1) \to X_*(S_2)$ is surjective, so choose some $\lambda' \in X_*(S_2)$ such that λ maps to λ'. Note that since $\mathcal{H}_1(\mathcal{O}) \to \mathcal{H}_2(\mathcal{O})$ is surjective there exists some $h_i' \in \mathcal{H}_1(\mathcal{O})$ such that $f(h_i') = h_i$. Clearly then $f(h_1'\lambda'(\pi)h_2') = h_1\lambda(\pi)h_2$. The claim follows. □

Corollary 3.1.11. *Let $\mathcal{H}$ be a reductive group over $\mathcal{O}$ and define $\mathcal{H}^{\mathrm{ab}} := \mathcal{H}/\mathcal{H}^{\mathrm{der}}$. Then, the map $\mathcal{H}(\mathcal{O}) \to \mathcal{H}^{\mathrm{ab}}(\mathcal{O})$ is surjective.*

Proof. This follows since the kernel of $\mathcal{H} \to \mathcal{H}^{\mathrm{ab}}$ is $\mathcal{H}^{\mathrm{der}}$ which has connected fibers (e.g. see [Con11, Theorem 5.3.1]. □

We next need the following lemma concerning the induced map $C_{\mathcal{H}_1}(F) \to C_{\mathcal{H}_2}(F)$ induced by some map $f : \mathcal{H}_1 \to \mathcal{H}_2$:

Lemma 3.1.12. *Let $f : \mathcal{H}_1 \to \mathcal{H}_2$ be a surjective morphism of reductive* $\mathrm{Spec}(\mathcal{O})$*-groups and assume that* $\ker f$ *is central and has connected fibers. Then, the following statements hold:*

1. *For any cocharacter μ of $\mathcal{H}_1$ the map $C_j(\mathcal{H}_1, \boldsymbol{\mu}) \to C_j(\mathcal{H}_2, f(\boldsymbol{\mu}))$ (also denoted f) is surjective.*

2. *Two elements $[b_1]$ and $[b_2]$ have the same image under f if and only if there exists some $z \in (\ker f)(\mathcal{O}_j)$ and some $h \in \mathcal{H}_1(\mathcal{O}_j)$ such that $b_1 = zhb_2\sigma(h)^{-1}$.*

3. *The map $C_j(\mathcal{H}_1, \boldsymbol{\mu}) \to C_j(\mathcal{H}_2, f(\boldsymbol{\mu}))$ is a bijection if $F/\mathbb{Q}_p$ is unramified and if $j = \infty$ because every element of $z \in (\ker f)(\breve{\mathbb{Z}}_p)$ can be written in the form $z = t\sigma(t)^{-1}$ for some $t \in (\ker f)(\breve{\mathbb{Z}}_p)$.*

Proof. Since $H_1(F_j) \to H_2(F_j)$ is surjective by Lemma 3.1.10 it's easy to see that the map $C_j(\mathcal{H}_1, \boldsymbol{\mu}) \to C_j(\mathcal{H}_2, \boldsymbol{\mu})$ is surjective. The second statement is clear by definition. To see the last claim it suffices to show that every $z \in (\ker f)(\breve{\mathbb{Z}}_p)$ is of the form $t\sigma(t)^{-1}$ for $t \in (\ker f)(\breve{\mathbb{Z}}_p)$. Since $\ker f$ is connected we know that, over $\breve{\mathbb{Z}}_p$, it's a split torus. Thus, it suffices to check the claim for $\ker f = \mathbb{G}_{m,\breve{\mathbb{Z}}_p}$ which can easily be done by hand (e.g. see [BC09, Lemma 9.3.3]). $\square$

We finish this section by mentioning some elementary results concerning fiber products of groups over a common adjoint group. Namely, we have the following:

Lemma 3.1.13. *Let $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}$ be reductive groups over $\mathcal{O}$ such that $\mathcal{Z}_i := Z(\mathcal{H}_i)$ are tori. Suppose further that we also have central isogenies $f_i : \mathcal{H}_i^{\mathrm{der}} \to \mathcal{H}^{\mathrm{der}}$ which induce isomorphisms*

$$\mathcal{H}_1^{\mathrm{ad}} \cong \mathcal{H}^{\mathrm{ad}} \cong \mathcal{H}_2^{\mathrm{ad}} \tag{3.22}$$

Then, the fibered product $\mathcal{H}_3 := \mathcal{H}_1 \times_{\mathcal{H}^{\mathrm{ad}}} \mathcal{H}_2$ is a reductive group over $\mathcal{O}$ and we have a short exact sequence of groups

$$0 \to \mathcal{Z}_3 \to \mathcal{H}_3 \to \mathcal{H} \to 0 \tag{3.23}$$

with $\mathcal{Z}_3 := \mathcal{Z}_1 \times_{\mathcal{O}} \mathcal{Z}_3$. Moreover, $\mathcal{Z}_3 = Z(\mathcal{H}_3)$ and thus $\mathcal{H}_3^{\mathrm{ad}}$ is naturally identified with $\mathcal{H}^{\mathrm{ad}}$. The natural inclusions $\mathcal{H}_3 \hookrightarrow \mathcal{H}_1 \times_{\mathcal{O}_F} \mathcal{H}_2$ as well as $\mathcal{H}_3 \hookrightarrow \mathcal{H}_1^{\mathrm{der}} \times \mathcal{H}_2^{\mathrm{der}}$ are closed embeddings. Finally, let us assume that f_i are isomorphisms. Then, $\mathcal{H}_3^{\mathrm{der}}$ naturally contains $\mathcal{H}^{\mathrm{der}}$ as a closed subgroup of $\mathcal{H}_1^{\mathrm{der}} \times_{\mathcal{O}} \mathcal{H}_2^{\mathrm{der}} \subseteq \mathcal{H}_1 \times_{\mathcal{O}} \mathcal{H}_2$.

Proof. Let us begin by noting that $\mathcal{H}_3$ is certainly an affine group scheme over $\mathcal{O}$. We claim then that we have an extension as in (3.23). Let us denote $\mathcal{Q} := \mathcal{H}_3/\mathcal{Z}_3$ and by $\mathcal{Q}^{\mathrm{pre}}$ the quotient presheaf $(\mathcal{H}_3/\mathcal{Z}_3)^{\mathrm{pre}}$. Note then that we have an evident injective map $\mathcal{Q}^{\mathrm{pre}} \to (\mathcal{H}_1/\mathcal{Z}_1)^{\mathrm{pre}}$ and thus we have an injective map $\mathcal{Q} \to \mathcal{H}$ since the latter is the sheafification of $(\mathcal{H}_1/\mathcal{Z}_1)^{\mathrm{pre}}$. This map is also surjective since the composition $\mathcal{H}_3 \to \mathcal{Q} \to \mathcal{H}$ can also be factorized as $\mathcal{H}_3 \to \mathcal{H}_1 \to \mathcal{H}$ which is surjective. Since $\mathcal{Z}_3$ are smooth with connected fibers, we deduce the same is true for $\mathcal{H}_3$. The fact that it's reductive then can be checked on fibers from where it follows from the classic result that extensions of reductive groups over a field are reductive. The fact that $\mathcal{Z}_3 = Z(\mathcal{H}_3)$ is also clear.

To prove the penultimate claim we note that the natural map $\iota : \mathcal{H}_3 \to \mathcal{H}_1 \times_{\mathcal{O}_F} \mathcal{H}_2$ is evidently a monomorphism, so that $\ker \iota$ is trivial. The claim then follows from [Con11, Theorem 5.3.5]. The last claim is clear. $\square$

Let the notation be as in the previous lemma. Let us now suppose that we have cocharacters $\mu_i : \mathbb{G}_{m,M} \to (H_i)_M$, where $F_j = M/F$ is unramified, such that the induced cocharacters $\mathbb{G}_{m,M} \to H$ are conjugate. Note that since $H_2(M) \to H(M)$ is surjective by Lemma 3.1.10 this means we can modify μ_2 by conjugation by an element of $H(M)$ such that the induced cocharacter of H_M actually agrees with the induced cocharacter of μ_1. Note then that we get a well-defined cocharacter $\mu_3 : \mathbb{G}_{m,M} \to (H_3)_M$. The $H_3(M)$ conjugacy class of this cocharacter depends only on the conjugacy classes $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$. For this reason, to any conjugacy classes $\boldsymbol{\mu}_i \in C_{\mathcal{H}_i}(M)$ for $i = 1, 2$ that are equalized to some conjugacy class $\boldsymbol{\mu} \in C_{\mathcal{H}}(M)$ we get a well-defined conjugacy class $\boldsymbol{\mu}_3 \in C_{\mathcal{H}_3}(M)$.

Lemma 3.1.14. *Let notation be as above. Let $\boldsymbol{\mu}_i \in C_{\mathcal{H}_i}(M)$ for $i = 1, 2$ whose induced conjugacy classes on H_M agree, and which we denote $\boldsymbol{\mu}$. Then, we have a natural bijection*

$$C_j(\mathcal{H}_1, \boldsymbol{\mu}_1) \times_{C_j(\mathcal{H}, \boldsymbol{\mu})} C_j(\mathcal{H}_2, \boldsymbol{\mu}_2) \to C_j(\mathcal{H}_3, \boldsymbol{\mu}_3) \tag{3.24}$$

Proof. This is clear. □

3.2 Deformation spaces of Hodge type

The construction of 'deformation spaces' in the abelian type setting will rely on the construction in the Hodge type setting, so we discuss this first. This construction is largely a recasting Faltings deformation theory of p-divisible groups with Tate tensors in more group-theoretic language which will be crucial to construct spaces in the abelian type setting. Specifically, instead of associating deformation rings to objects of the form $(\overline{X}, \{s_\alpha\})$, where $\overline{X}$ is a p-divisible group over $\overline{\mathbb{F}_p}$ and $\{s_\alpha\}$ is a set of Tate tensors on its Dieudonne module $D(\overline{X})(\breve{\mathbb{Z}}_o)$, we shall think of associating 'deformation rings' to linear algebraic data of the form $(\mathcal{H}, [b], \boldsymbol{\mu})$, the definition of which we soon make.

Before we begin we also make the convention that all crystalline theory will be relative to $\mathbb{Z}_p$ with its usual PD structure. If $\mathfrak{Z}$ is a formal scheme over $\mathrm{Spf}(\mathbb{Z}_p)$ and $\mathbb{X}$ is a p-divisible group over $\mathfrak{Z}$ we denote by $D(\mathbb{X})$ the (contravariant) Dieudonne crystal on $\mathfrak{Z}$ (see [dJ95, §2.4] for a recollection of p-divisible groups and Dieudonne crystals over formal schemes). For a crystal (of quasi-coherent $\mathcal{O}_{\mathfrak{Z}}^{\mathrm{crys}}$-modules) $\mathbb{E}$ on a formal scheme $\mathfrak{Z} \to \mathrm{Spf}(\mathbb{Z}_p)$ we denote by $\mathbb{E}_{\mathbb{Q}_p}$ its image in the category of isocrystals over $\mathfrak{Z}$.

Deformation datum and deformation spaces at base level

Let us keep the group theoretic notation from the previous subsection. We now require though that $F/\mathbb{Q}_p$ be unramified. Let us now define a *deformation datum (of degree j) over F* (with $j < \infty$) to be a triple $(\mathcal{H}, [b], \boldsymbol{\mu})$ where $\mathcal{H}$ is a connected reductive group over $\mathcal{O}$, $\boldsymbol{\mu}$ is a $\mathcal{H}(\mathcal{O}_E)$-conjugacy class of maps $\mathbb{G}_{m,\mathcal{O}_E} \to \mathcal{H}_{\mathcal{O}_E}$ for some finite subextension E of F^{ur}/F (i.e. an element of $C_{\mathcal{H}}(E)$), and $[b] \in C_j(\mathcal{H}, \boldsymbol{\mu}^{-1})$. Note that the field E mentioned above is part of the data of $\boldsymbol{\mu}$, one which we will often supress. Also, while it will often be the case that E is merely $E(\boldsymbol{\mu})$ having the extra flexibility of having E be a proper extension of $E(\boldsymbol{\mu})$ will be useful.

Remark 3.2.1. We have chosen to follow the sign conventions of [Kim13] which explains the inverse present in the above definition.

A *morphism* $f : (\mathcal{H}_1, [b_1], \boldsymbol{\mu}_1) \to (\mathcal{H}, [b], \boldsymbol{\mu})$ of deformation data (of degree j) is a morphism $f : \mathcal{H}_1 \to \mathcal{H}$ of reductive groups over $\mathcal{O}$ such that $f([b_1]) = f([b])$ and $f \circ (\boldsymbol{\mu}_1)_{EE_1} = \boldsymbol{\mu}_{EE_1}$. Note that if $f : \mathcal{H}_1 \to \mathcal{H}$ is any morphism of reductive groups over $\mathcal{O}$ and $(\mathcal{H}_1, [b_1], \boldsymbol{\mu}_1)$ is a deformation data, then $(\mathcal{H}, f([b_1]), f_{\mathcal{O}_E}(\boldsymbol{\mu}_1))$ is a deformation datum. This follows essentially from the previously mentioned compatibility of the κ-maps. With this, we define the *adjoint datum* of a deformation datum $\mathfrak{d}_1 := (\mathcal{H}_1, [b_1], \boldsymbol{\mu}_1)$, denoted by $(\mathcal{H}_1^{\mathrm{ad}}, [b_1]^{\mathrm{ad}}, \boldsymbol{\mu}_1^{\mathrm{ad}})$, to be $(\mathcal{H}_1^{\mathrm{ad}}, q([b_1]), q(\boldsymbol{\mu}_1))$ where $q : \mathcal{H}_1 \to \mathcal{H}_1^{\mathrm{ad}}$ is the usual quotient map. We shall denote this deformation datum by $\mathfrak{d}_1^{\mathrm{ad}}$.

Remark 3.2.2. Note that, really,we should likely call the above an *unramified deformation datum (of degree j)*. Of course, since we have no definition of deformation datum in the ramified setting this doesn't quite make sense.

Our ultimate goal is to attach to deformation data $\mathfrak{d} = (\mathcal{H}, [b], \boldsymbol{\mu})$ of degree j a tower of rigid analytic spaces $(\mathscr{D}_K(\mathfrak{d}))_{K \subseteq \mathcal{H}(\mathbb{Z}_p)}$ over $\mathrm{Spa}(E_j)$. To do this though we need to assume some conditions on the deformation datum. Namely, we will need to assume that the deformation datum is of so-called *abelian type*. The definition of of abelian type, to be given in the next section, is an extension of the notion of a deformation datum of *Hodge type* which we now define.

Suppose that $\mathfrak{d} = (\mathcal{H}, [b], \boldsymbol{\mu})$ is a deformation datum of degree j and we are given $\rho : \mathcal{H} \to \mathrm{GL}(\Lambda)$ a faithful representation of $\mathcal{H}$ where Λ is a finitely generated free $\mathcal{O}$-module. We denote by V the H-representation $\Lambda \otimes_{\mathcal{O}} F$. Let us then choose an element $b \in [b]$ and note then that V_{F_j} can be seen as an isocrystal over k_j with Frobenius $F_b := b \circ (1 \otimes \sigma)$. Let us then define an *enhanced deformation datum (of degree j)* to be a quadruple $(\mathcal{H}, b, \boldsymbol{\mu}, \rho)$ where $(\mathcal{H}, [b], \boldsymbol{\mu})$ is a deformation datum (of degree j) and ρ is a faithful representaiton $G \hookrightarrow \mathrm{GL}(\Lambda)$ such that $M(b, \rho) := \Lambda^\vee \otimes_{\mathcal{O}} \mathcal{O}_{E_j}$ is stable under $F_b^\vee$ in the isocrystal $V_{F_j}^\vee$. In this case we see that $M(b, \rho)$ is an F-crystal over k_{E_j}. We call $(\mathcal{H}, [b], \boldsymbol{\mu})$ the *underlying deformation datum* of the enhanced deformation datum.

Remark 3.2.3. The reason for dualizing here is largely notational, and is to make the compatibility with Shimura datum cleaner. This will be seen explicitly later on.

Let us then call a enhanced deformation datum $(\mathcal{H}, b, \boldsymbol{\mu}, \rho)$ *Dieudonne* if the F-crystal $M(b, \rho)$ is a Dieudonne crystal. We then say that a deformation datum $(\mathcal{H}, [b], \boldsymbol{\mu})$ is of *Hodge type* if there exists a Dieudonne enhanced deformation datum with associated deformation datum $(\mathcal{H}, [b], \boldsymbol{\mu})$. We now wish to explain how to associate to a Hodge type deformation datum $(\mathcal{H}, [b], \boldsymbol{\mu})$ a deformation ring of p-divisible groups with extra structure over $\mathcal{O}_{E_j}$.

Remark 3.2.4. Note that the Hodge type condition puts strong restrictions on the objects involved. For instance, it implies that $\boldsymbol{\mu}$ is minuscule.

To start, let us observe that since $\kappa_{E_j}(b) = (\boldsymbol{\mu}^{-1})_+$ that there is some element $\mu \in \boldsymbol{\mu}^{-1}$ (e.g. the dominant element) such that $b \in \mathcal{H}(\mathcal{O}_{E_j})\mu^{-1}(\pi)\mathcal{H}(\mathcal{O}_{E_j})$. In particular, we can write $b = h_1 \mu^{-1}(\pi) h_2$ for some $h_1, h_2 \in \mathcal{H}(\mathcal{O}_E)$. Note then that $b = h\nu^{-1}(\pi)$ where $h := h_1 h_2$ and $\nu = h_2^{-1} \mu h_2$. In particular, we can fix an element $\mu \in \boldsymbol{\mu}$ such that $b \in \mathcal{H}(\mathcal{O}_{E_j})\mu^{-1}(\pi)$. We call such a choice a *rigidification* of $\mathfrak{d}$, and we will write it as $(\mathcal{H}, b, \mu)$. Similarly, a *rigidification* of an enhanced deformation datum $(\mathcal{H}, b, \boldsymbol{\mu}, \rho)$ will merely be a quadruple $(\mathcal{H}, b, \mu, \rho)$ with $(\mathcal{H}, b, \mu)$ a rigidification of $(\mathcal{H}, b, \boldsymbol{\mu})$.

So, suppose now that $\widetilde{\mathfrak{d}} = (\mathcal{H}, b, \mu, \rho)$ is a ridified Dieudonne enhanced deformation datum. We then associate to this a formal scheme $\mathscr{D}^{\mathrm{fml}}(\widetilde{\mathfrak{d}})$ over $\mathrm{Spf}(\mathcal{O}_{E_j})$ by declaring it to be $\hat{U}(\mu^{-1}_{\mathcal{O}_j})$ which, by definition, is the completion (at the identity) of the group scheme $U(\mu^{-1}_{\mathcal{O}_{E_j}})$ over $\mathrm{Spec}(\mathcal{O}_{E_j})$. Let us write $R(\widetilde{\mathfrak{d}})$ for the global sections of $\mathscr{D}^{\mathrm{fml}}(\widetilde{\mathfrak{d}})$.

Note, in particular, that $\mathscr{D}^{\mathrm{fml}}(\widetilde{\mathfrak{d}})$ does not depend on anything other than the pair $(\mathcal{H}, \mu)$, and this is not a mistake. The purpose of the b and the ρ in the rigidified enhanced deformation datum $(\mathcal{H}, b, \mu, \rho)$ will be to specify a p-divisible group over $\mathscr{D}^{\mathrm{fml}}(\widetilde{\mathfrak{d}})$.

Let us begin by noting that since $U(\mu^{-1}_{\mathcal{O}_{E_j}})$ is a smooth split unipotent group scheme over $\mathrm{Spf}(\mathcal{O}_{E_j})$ that we have a (non-canonical isomorphism) $R(\widetilde{\mathfrak{d}}) \cong \mathcal{O}_{E_j}[\![t_1,\ldots,t_d]\!]$. Thus, we know that there is a lift of Frobenius along $\mathscr{D}^{\mathrm{fml}}(\widetilde{\mathfrak{d}}) \to \mathrm{Spf}(\mathcal{O}_{E_j})$. Specifically having chosen an identification of complete local rings $R(\widetilde{\mathfrak{d}}) \cong \mathcal{O}_{E_j}[\![t_1,\ldots,t_d]\!]$ we can choose a Frobenius unique lift ϕ so that $\phi(t_i) = t_i^p$. Let us fix this choice of lifting.

We will now define a p-divisible group $\mathbb{X}(\widetilde{\mathfrak{d}})$ on $\mathscr{D}^{\mathrm{fml}}(\widetilde{\mathfrak{d}})$. Indeed, let us start by observing that since $M(b,\rho)$ is an F-crystal over k_{E_j} that we obtain from classical Dieduonne theory an associated p-divisible group over k_j which we denote $\overline{X}(\widetilde{\mathfrak{d}})$. We would like to specify a lift of $\overline{X}(\widetilde{\mathfrak{d}})$ to a p-divisible group over $\mathrm{Spec}(\mathcal{O}_{E_j})$. But, by Grothendieck-Messing theory this is equivalent to specifying an admissible lift of the Hodge filtration on $M(b,\rho)_{k_{E_j}}$ to $M(b,\rho)$. The below then shows that $\mu_{\mathcal{O}_{E_j}}$ naturally provides such a lifting.

Namely, we have the following observation

Lemma 3.2.5. *Let ν be the cocharacter of $\mathrm{GL}(M(b,\rho))$ given by $\rho^\vee_{\mathcal{O}_{E_j}} \circ \mu_{\mathcal{O}_{E_j}}$. Then the filtration $\mathrm{Fil}^1_\nu(M(b,\mu)) \subseteq M(b,\mu)$ is an admissible lifting of the Hodge filtration on $M(b,\mu)_{k_{E_j}}$.*

Proof. See [Kim13, Lemma 2.5.8] for a proof that $\mathrm{Fil}^1_\nu(M(b,\mu))$ induces the Hodge filtration over k_{E_j}. Note that while this proof is working over $\mathcal{O}_{\breve{F}}$ the argument works verbatim over $\mathcal{O}_{E_j}$. The admissibility is clear since both $\mathrm{Fil}^1_\nu(M(b,\mu))$ and its quotient are free. $\square$

From this we see that we can use Grothendieck-Messing theory to associate to the filtration $\mathrm{Fil}^1_\nu(M(b,\mu)_{\mathcal{O}_{E_j}})$ a lifting $X(\widetilde{\mathfrak{d}})$ of $\overline{X}(\widetilde{\mathfrak{d}})$ over $\mathcal{O}_{E_j}$. Let us then define a quadruple of data

$$(\mathbb{M}(\widetilde{\mathfrak{d}}), \mathrm{Fil}^1(\mathbb{M}(\widetilde{\mathfrak{d}})), F(\widetilde{\mathfrak{d}}), \nabla(\widetilde{\mathfrak{d}})) \tag{3.25}$$

on $\mathscr{D}^{\mathrm{fml}}(\widetilde{\mathfrak{d}})$ as follows. The object $\mathbb{M}(\widetilde{\mathfrak{d}})$ is a $R(\widetilde{\mathfrak{d}})$-module and $\mathrm{Fil}^1(\mathbb{M}(\widetilde{\mathfrak{d}}))$ a direct factor of $\mathbb{M}(\widetilde{\mathfrak{d}})$, the definitions of which are merely the pullback along the map $\mathscr{D}^{\mathrm{fml}}(\widetilde{\mathfrak{d}}) \to \mathrm{Spf}(\mathcal{O}_{E_j})$ of $M(b,\mu)$ and $\mathrm{Fil}^1_\nu(M(b,\mu))$ respectively. We define F, to be be the Frobenius semilinear operator on $\mathbb{M}(\widetilde{\mathfrak{d}})$ defined by $u_t^{-1}(F_b^\vee \otimes 1)$ where, recall, F_b is the Frobenius semilinear operator on $M(b,\mu)$, and $u_t \in \mathscr{D}^{\mathrm{fml}}(\widetilde{\mathfrak{d}})(R(\widetilde{\mathfrak{d}}))$ is the tautological element.

The definition of $\nabla(\widetilde{\mathfrak{d}})$ then comes from the following result of Faltings:

Lemma 3.2.6 (Faltings)**.** *There exists a, necessarily unique, topologically quasi-nilpotent connection*

$$\nabla(\widetilde{\mathfrak{d}}) : \mathbb{M}(\widetilde{\mathfrak{d}}) \to \mathbb{M}(\widetilde{\mathfrak{d}}) \otimes \widehat{\Omega}^1_{R(\widetilde{\mathfrak{d}})/\mathcal{O}_{E_j}} \tag{3.26}$$

which is horizontal relative to F.

Proof. See [Fal99, §7]. $\square$

Note then that the triple $(\mathbb{M}(\widetilde{\mathfrak{d}}), F(\widetilde{\mathfrak{d}}), \nabla(\widetilde{\mathfrak{d}}))$ is a Dieudonne module in the sense of [dJ95, Definition 2.3.4] and thus (see [dJ95, Main Theorem 1]) corresponds to a p-divisible group $\overline{\mathbb{X}}(\widetilde{\mathfrak{d}})$ over $\mathrm{Spec}(R(\widetilde{\mathfrak{d}})/(p))$. But, the category of p-divisible groups over $\mathrm{Spec}(R(\widetilde{\mathfrak{d}}))$

is actually equivalent to the category of quadruples $(M, F, \nabla, \mathrm{Fil})$ where M is a finite free $R(\widetilde{\mathfrak{d}})$-module, $\mathrm{Fil} \subseteq M$, is a direction summand, ∇ is an integrable topologically quasi-nilpotent connection on M, and F is a ϕ-semilinear morphism on M which is horizontal with respect to ∇ (e.g see [Moo98, §4]). This is essentially just combining [dJ95, Main Theorem 1] with Grothendieck-Messing theory since $R(\widetilde{\mathfrak{d}})$ is a pro-PD thickening of of $R(\widetilde{\mathfrak{d}})/(p)$.

Thus, associated to $\mathrm{Fil}^1(\mathbb{M}(\widetilde{\mathfrak{d}}))$ there is a unique lift $\mathbb{X}(\widetilde{\mathfrak{d}})$ of $\overline{\mathbb{X}}(\widetilde{\mathfrak{d}})$ over $\mathrm{Spec}(R(\widetilde{\mathfrak{d}}))$. Note by [dJ95, Lemma 2.4.4] that p-divisible groups on $\mathrm{Spec}(R(\widetilde{\mathfrak{d}}))$ and $\mathrm{Spf}(R(\widetilde{\mathfrak{d}}))$ are equivalent, and we shall often confuse the two. We would now like to explain what universal properties the pair $(\mathrm{Spec}(R(\widetilde{\mathfrak{d}})), \mathbb{X}(\widetilde{\mathfrak{d}}))$ satisfies.

Let us first work in the case when $\mathcal{H} = \mathrm{GL}(\Lambda)$ and ρ is the identity map. Thus, we have that $\widetilde{\mathfrak{d}} = (\mathrm{GL}(\Lambda), b, \mu, \mathrm{id})$. Let $\mathscr{C}_{\mathcal{O}_{E_j}}$ denote the category of Artinian local $\mathcal{O}_{E_j}$-algebras with residue field k_{E_j}. We then define a functor

$$\mathsf{Def}_{\overline{X}(\widetilde{\mathfrak{d}})} : \mathscr{C}_{\mathcal{O}_{E_j}} \to \mathsf{Set} \tag{3.27}$$

by

$$\mathsf{Def}_{\overline{X}(\widetilde{\mathfrak{d}})}(R) := \left\{ (\mathbb{X}, \iota) : \mathbb{X} \in \mathsf{BT}_p(R) \text{ and } \iota : \mathbb{X}_{k_{E_j}} \xrightarrow{\approx} \overline{X}(\widetilde{\mathfrak{d}}) \right\} / \sim \tag{3.28}$$

where $\mathsf{BT}_p(R)$ is the category of p-divisible groups over R, and the equivalence relation $\sim$ is the obvious one. We then have the following

Theorem 3.2.7 (Faltings)**.** *The pair* $(\mathscr{D}^{\mathrm{fml}}(\widetilde{\mathfrak{d}}), \mathbb{X}(\widetilde{\mathfrak{d}}))$ *pro-represents* $\mathsf{Def}_{\overline{X}(\widetilde{\mathfrak{d}})}$.

Proof. See [Moo98, §4], noting that since $F/\mathbb{Q}_p$ was assumed unramified and $\mathcal{H}$ is unramified we have by the discussion in the last section that $\mathcal{O}_{E_j} = W(k_{E_j})$. □

In particular, we see that the pair $(\mathscr{D}^{\mathrm{fml}}(\widetilde{\mathfrak{d}}), \mathbb{X}(\widetilde{\mathfrak{d}}))$ only depends on the data $(\mathrm{GL}(\Lambda), b)$ and not on μ. The independence of μ can be easily explained by noting that by [Kim13, Lemma 2.5.8] the cocharacter μ is, in some sense, determined by the pair $(\mathrm{GL}(\Lambda), b)$.

We would now like to explain what sort of moduli-theoretic properties $(\mathscr{D}^{\mathrm{fml}}(\widetilde{\mathfrak{d}}), \mathbb{X}(\widetilde{\mathfrak{d}}))$ satisfies in the general situation. To start, we recall the following result of Kisin which says that the image of our faithful representation $\rho : \mathcal{H} \hookrightarrow \mathrm{GL}(\Lambda)$ can be defined as the scheme theoretic stabilizer of some tensors on Λ.

More precisely, let us set $\Lambda^\otimes$ to be the direct sum of all finite combinations of finite tensor powers, duals, symmetric, and alternating powers of Λ in the category of $\mathcal{O}$-modules. Note that $\mathrm{GL}(\Lambda)$ naturally acts on $\Lambda^\otimes$ and, moreover, under the natural identification $\Lambda = \Lambda^{\vee\vee}$ we have an identification $\Lambda^\otimes = (\Lambda^\vee)^\otimes$, an identification which we freely make. Finally, if R is any $\mathcal{O}$-algebra then the object $\Lambda_R^\otimes$ has the obvious meaning, and there is a natural identification $\Lambda_R^\otimes$ and $(\Lambda^\otimes)_R$.

We then have the following:

Lemma 3.2.8 (Kisin)**.** *There exists a finite set* $\{s_\alpha\}$ *of elements of* $\Lambda^\otimes$ *such that the (scheme theoretic) image* $\rho(\mathcal{H})$ *of* ρ *is the scheme theoretic stabilizer of* $\{s_\alpha\}$*. In other words, the inclusion of subfunctors* $\mathrm{Stab}(\{s_\alpha\}) \hookrightarrow \mathrm{GL}(\Lambda)$*, where*

$$\mathrm{Stab}(\{s_\alpha\})(R) := \{h \in \mathcal{H}(R) : h(s_\alpha \otimes 1) = s_\alpha \otimes 1 \textit{ for all } \alpha\} \tag{3.29}$$

is relatively representable, and its representing object is ρ.

Proof. See [Kis10, Proposition 1.3.2]. $\square$

Now, for any set $\{s_\alpha\}$ of tensors we see that $s_\alpha \in \Lambda^\otimes$ induce tensors $s_\alpha \otimes 1$ (shortened to s_α) on

$$D(\overline{\mathbb{X}}(\widetilde{\mathfrak{d}}))(R(\widetilde{\mathfrak{d}}))^\otimes = M(b,\rho)^\otimes \otimes_{\mathcal{O}_{E_j}} R(\widetilde{\mathfrak{d}}) = \Lambda^\otimes \otimes_{\mathcal{O}_F} R(\widetilde{\mathfrak{d}}) \tag{3.30}$$

These correspond to morphisms of crystals $s_\alpha^{\mathrm{crys}} : \mathbb{1} \to D(\overline{\mathbb{X}}(\widetilde{\mathfrak{d}}))$ on $\mathrm{Spf}(R(\widetilde{\mathfrak{d}}),(p))$ (where $\mathrm{Spf}(A,(p))$ means the formal spectrum where A is given the p-adic topology). Now, note that while $F(\widetilde{\mathfrak{d}})$ acts on $D(\overline{\mathbb{X}}(\widetilde{\mathfrak{d}}))$, it does not act on $D(\overline{\mathbb{X}}(\widetilde{\mathfrak{d}}))^\otimes$ since it doesn't even naturally act on $D(\overline{\mathbb{X}}(\widetilde{\mathfrak{d}}))^\vee$ (the un-twisted linear dual). That said, it is true that $F(\widetilde{\mathfrak{d}})$ does act on $D(\mathbb{X}(\widetilde{\mathfrak{d}}))^\otimes_{\mathbb{Q}_p}$. If we now assume that $\mathcal{H}$ fixes these tensors s_α it's easy to see that $s_\alpha^{\mathrm{crys}} : \mathbb{1} \to\in D(\overline{\mathbb{X}}(\widetilde{\mathfrak{d}}))^\otimes_{\mathbb{Q}_p}$ are evidently $F(\widetilde{\mathfrak{d}})$-fixed. Also, again assuming that s_α are fixed by $\mathcal{H}$, one can see that the tensors z_α lie in $\mathrm{Fil}^0_\mu D(\overline{\mathbb{X}}(\widetilde{\mathfrak{d}}))^\otimes$. Indeed, this can be checked over $\mathcal{O}_{E_j}$, and the fact that the Hodge filtration is defined by a cocharacter of $\mathcal{H}$, and $\mathcal{H}$ fixes the s_α, provides the claim.

Let us now fix a set $\{s_\alpha\}$ of tensors as in Lemma 3.2.8. Work of Faltings in [Fal99] shows that, at least for algebras of the form $\mathcal{O}_{E_j}[\![u_1,\ldots,u_v]\!]/(p^m)$ or $\mathcal{O}_{E_j}[\![u_1,\ldots,u_v]\!]$ for some $m,v \geqslant 1$, the choice of these tensors allow us to identify $\mathbb{X}(\widetilde{\mathfrak{d}})$ over $\mathscr{D}^{\mathrm{fml}}(\widetilde{\mathfrak{d}})$ as a 'universal object'. To explain this rigorously, let us define a set of *Tate tensors* on an $\mathbb{X} \in \mathsf{BT}_p(R)$ (where R is an object of $\mathscr{C}_{\mathcal{O}_{E_j}}$) to be a set of tensors $\{t_\alpha\}$ in $D(\mathbb{X})^\otimes$ which are in the 0^{th}-part of the Hodge filtration on $D(\mathbb{X})^\otimes$ and whose images in $D(\mathbb{X})^\otimes_{\mathbb{Q}_p}$ are Frobenius invariant.

We then have the following result of Faltings (appropriately reinterpreted by Kim in [Kim13]). To state it, note that our faithful representation ρ allows us (see Lemma 3.1.6) to embed $\mathscr{D}^{\mathrm{fml}}(\widetilde{\mathfrak{d}})$ into $\mathscr{D}^{\mathrm{fml}}(\rho(\widetilde{\mathfrak{d}}))$ where

$$\rho(\widetilde{\mathfrak{d}}) := (\mathrm{GL}(\Lambda), \rho(b), \rho_{\mathcal{O}_E} \circ \mu, \mathrm{id}) \tag{3.31}$$

Moreover, it's clear from construction that $\mathbb{X}(\widetilde{\mathfrak{d}})$ is just the pullback of $\mathbb{X}(\rho(\widetilde{\mathfrak{d}}))$. By Lemma 3.2.7 this gives us an embedding of functors $\mathscr{D}^{\mathrm{fml}}(\widetilde{\mathfrak{d}}) \hookrightarrow \mathsf{Def}_{\overline{\mathbb{X}}(\widetilde{\mathfrak{d}})}$. We can now explicitly describe the image of this functor, at least when we restrict our attention to algebras of the form $\mathcal{O}_{E_j}[\![u_1,\ldots,u_v]\!]$ or $\mathcal{O}_{E_j}[\![u_1,\ldots,u_v]\!]/(p^m)$ for some $v,m \geqslant 1$. Namely:

Theorem 3.2.9 (Faltings)**.** *Let A a ring of the form $\mathcal{O}_{E_j}[\![u_1,\ldots,u_v]\!]$ or $\mathcal{O}_{E_j}[\![u_1,\ldots,u_v]\!]/(p^m)$ for some $v,m \geqslant 1$. Then, $X \in \mathsf{Def}_{\overline{X}(\widetilde{\mathfrak{d}})}(\mathrm{Spf}(A))$ lies in $\mathscr{D}^{\mathrm{fml}}(\widetilde{\mathfrak{d}})(\mathrm{Spf}(A))$ if and only if the morphism of crystals $s_\alpha : \mathbb{1} \to D(\overline{X}(\widetilde{\mathfrak{d}}))^\otimes$ over k_{E_j} lift to Tate tensors $\mathbb{1} \to D(X)^\otimes$ on* $\mathrm{Spf}(A,(p))$.

Proof. This is contained in [Moo98, §4] in the case of $A = \mathcal{O}_{E_j}[\![u_1,\ldots,u_v]\!]$. In the case that $A = \mathcal{O}_{E_j}[\![u_1,\ldots,u_v]\!]/(p^m)$ for some $m \geqslant 1$ the idea of extension is simple, and is observed in [Kim13, Theorem 3.6]. $\square$

Of course, we can reinterpret this in terms of a moduli problem as follows. Let us denote by $\widehat{\mathscr{C}}^{\mathrm{sm}}_{\mathcal{O}_{E_j}}$ the category of local Artinian $\mathcal{O}_{E_j}$-algebras of the form $\mathcal{O}_{E_j}[\![u_1,\ldots,u_v]\!]$ or $\mathcal{O}_{E_j}[\![u_1,\ldots,u_v]\!]/(p^m)$ for some $v,m \geqslant 1$. Let us define

$$\mathsf{Def}^{(s_\alpha)}_{\widetilde{\mathfrak{d}}} : \widehat{\mathscr{C}}^{\mathrm{sm}}_{\mathcal{O}_{E_j}} \to \mathsf{Set} \tag{3.32}$$

by

$$\mathsf{Def}_{\widetilde{\mathfrak{d}}}^{(s_\alpha)}(A) = \{(\mathbb{X},\iota) \in \mathsf{Def}_{\widetilde{\mathfrak{d}}}(A) : s_\alpha^{\text{crys}} \text{ lift to Tate tensors on } D(\mathbb{X})\} \tag{3.33}$$

Then, of course, Theorem 3.2.9 can be stated as the claim that $\mathsf{Def}_{\widetilde{\mathfrak{d}}}^{(s_\alpha)}$ is represented by $(\mathscr{D}^{\text{fml}}(\widetilde{\mathfrak{d}}), \mathbb{X}(\widetilde{\mathfrak{d}}))$.

Another important implication of this result is that, by analyzing the moduli problem $\mathsf{Def}_{\widetilde{\mathfrak{d}}}^{(s_\alpha)}$, the pair $(\mathscr{D}^{\text{fml}}(\widetilde{\mathfrak{d}}), \mathbb{X}(\widetilde{\mathfrak{d}}))$ doesn't depend fully on $\widetilde{\mathfrak{d}}$ but really only depended on $\mathfrak{e} := (\mathcal{H}, b, \boldsymbol{\mu}, \rho)$ as well as the chosen tensors $(s_\alpha) \subseteq \Lambda^\otimes$. The reason for this is that having chosen any $\mu \in \boldsymbol{\mu}$ with $b \in \mathcal{H}(\mathcal{O}_j)\mu^{-1}(p)$ would have given the same moduli space, since the tensors lying in the 0^{th}-part of the Hodge filtration relies only on $\boldsymbol{\mu}$ (e.g. see [Kim13, Lemma 2.5.8, 3)]). Even more is true, which is that while pair $(\mathscr{D}^{\text{fml}}(\mathfrak{e}), \mathbb{X}(\mathfrak{e}))$ a priori actually depend on the on the chosen tensors (s_α), it does not (cf. [Kim13, Theorem 3.6] and [Kim13, Remark 3.7.4]). Note as well that since changing the chosen representative $[b]$ doesn't change the isomorphism class of the crystal $M(b,\mu)$ that $\mathbb{X}(\mathfrak{e})$ actually depends only on $\overline{\mathfrak{d}} := (\mathcal{H}, [b], \boldsymbol{\mu}, \rho)$. Finally, note that $\mathscr{D}^{\text{fml}}(\overline{\mathfrak{d}})$ really only depends up to isomorphism, in fact, on $\mathfrak{d}$ (it only depends on $(\mathcal{H}, \boldsymbol{\mu})$. Because of this, we can, up to natural isomorphism, unambiguously write the pair $(\mathscr{D}^{\text{fml}}(\mathfrak{d}), \mathbb{X}(\overline{\mathfrak{d}}))$.

We, in particular, record the following rephrasing of the above discussion:

Proposition 3.2.10. *Let $\mathfrak{d}$ be a deformation datum of Hodge type and let $\rho_i : \mathcal{H} \hookrightarrow \mathrm{GL}(\Lambda_i)$ be faithful representations so that $\widetilde{\mathfrak{d}}_i := (\mathcal{H}, b, \mu, \rho_i)$ for $i = 1, 2$ are rigidified enhanced Dieudonne deformation data. Then $\widetilde{\mathfrak{d}}_3 := (\mathcal{H}, b, \mu, \rho)$ where $\rho : \mathcal{H} \hookrightarrow \mathrm{GL}(\Lambda_1 \times \Lambda_2)$ is the obvious embedding is also a rigidified enhanced Dieudonne deformation data. Note then that for any choice of tensors $(s_{\alpha,i})$ on Λ_i defining the embedding ρ_i we get natural projection maps*

$$\mathsf{Def}_{\widetilde{\mathfrak{d}}_1}^{(s_{\alpha,1})} \longleftarrow \mathsf{Def}_{\widetilde{\mathfrak{d}}_3}^{(s_{\alpha,1} \times s_{\alpha,2})} \longrightarrow \mathsf{Def}_{\widetilde{\mathfrak{d}}_2}^{(s_{\alpha,2})} \tag{3.34}$$

The induced

$$\mathscr{D}^{\text{fml}}(\widetilde{\mathfrak{d}}_1) \longleftarrow \mathscr{D}^{\text{fml}}(\widetilde{\mathfrak{d}}_3) \longrightarrow \mathscr{D}^{\text{fml}}(\widetilde{\mathfrak{d}}_2) \tag{3.35}$$

are isomorphisms and, in fact, the identity maps.

Generic fibers and level structure

We would now like to pass from $\mathscr{D}^{\text{fml}}(\mathfrak{d})$ to a tower of rigid analytic varieties over $\mathrm{Spa}(E_j)$ indexed by the compact open subgroups of $\mathcal{H}(\mathbb{Z}_p)$. To do this, let us begin by defining $\mathscr{D}(\mathfrak{d})$ to be $\mathscr{D}^{\text{fml}}(\mathfrak{d})_\eta$, the rigid analytic fiber of $\mathscr{D}^{\text{fml}}(\mathfrak{d})$ over $\mathrm{Spa}(\mathcal{O}_{E_j})$ (as reviewed in §5.3). Since $\mathscr{D}^{\text{fml}}(\mathfrak{d})$ is abstractly isomorphic to $\mathcal{O}_{E_j}[\![t_1, \ldots, t_d]\!]$ we see that $\mathscr{D}(\mathfrak{d})$ is abstractly isomorphic to $\mathbb{D}_{E_j}^d$—the d-fold product of the open unit ball over $\mathrm{Spa}(E_j)$.

Let us note that for each $n \geqslant 1$ we have a finite flat group scheme given by $\mathbb{X}[p^n] \to \mathrm{Spec}(R(\mathfrak{d}))$. In particular, we can endow $\mathbb{X}[p^n]$ with the natural topology. In other words, if $B := \mathcal{O}(\mathbb{X}[p^n])$ is given by $(p, t_1, \ldots, t_n)B$-adic topology, for which it is complete (e.g. see [KH17, Corollary 1.1.15]). In particular, we see that the map $\mathrm{Spf}(B) \to \mathrm{Spf}(R(\mathfrak{d}))$ is a finite (adic) morphism of formal schemes. Let us denote by $\mathscr{X}[p^n]$ the generic fiber $\mathbb{X}[p^n]_\eta$ (thought of as a formal scheme in the above way).

The following result is basic, but since we don't know a reference we include its proof for the convenience of the reader:

Lemma 3.2.11. *Let $\mathcal{O}$ be a complete DVR with $F := \mathrm{Frac}(\mathcal{O})$ of characteristic $\ell \geqslant 0$. Let $\mathfrak{X} = \mathrm{Spf}(R)$ be a formal scheme over $\mathrm{Spf}(\mathcal{O})$ which is locally formally of finite type. Let $\mathbb{G} = \mathrm{Spf}(R')$ be a finite flat group scheme over $\mathfrak{X}$ (with R' endowed with the natural topology from R) with $\ell \nmid |\mathbb{G}|$. Then, $\mathbb{G}_\eta$ is a finite étale group adic space over $\mathfrak{X}_\eta$.*

Proof. Consider the morphism $\mathbb{G}_\eta \to \mathfrak{X}_\eta$. The fact that the morphism is finite is classic (e.g. see [dJ95, Proposition 7.2.1, d)]), and the fact that it's flat follows similarly since $\mathbb{G} \to \mathfrak{X}$ is flat. Note that since the generic fiber construction is functorial, we see that $\mathbb{G}_\eta$ is evidently a finite flat group adic space over $\mathfrak{X}_\eta$. In particular, for any affinoid open $\mathrm{Spa}(A, A^+)$ in $\mathfrak{X}_\eta$ its preimage $\mathrm{Spa}(B, B^+)$ we have that $\mathrm{Spa}(B, B^+)$ is a finite flat group adic space over $\mathrm{Spa}(A, A^+)$. We need only show that B is an étale A-algebra (e.g. see [Hub96, Corollary 1.7.3]). But, if

$$m : \mathrm{Spa}(B, B^+) \times_{\mathrm{Spa}(A,A^+)} \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(B, B^+) \tag{3.36}$$

is the multiplication map, it's clear that this gives rise to a comultiplication map $\Delta : B \to B\widehat{\otimes}_A B$. But, since B is a finite flat A-module, we have an equality $B\widehat{\otimes}_A B = B \otimes_A B$. A similar statement holds for the inverse map and identity section of $\mathrm{Spa}(B, B^+)$. Using this we show that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a finite flat group scheme and the order of $\mathrm{Spec}(B)$ is invertible in A. The fact that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is étale is now classical (e.g. see [Sha86, §4, Corollary 3]). □

It's clear then that since $\{\mathbb{X}[p^n]\}$ is a p-divisible group over $\mathrm{Spf}(R(\mathfrak{d}))$ that the projective system $\{\mathscr{X}[p^n]\}$ of étale sheaves on $\mathscr{D}(\mathfrak{d})_{\text{ét}}$ forms a lisse $\mathbb{Z}_p$-sheaf (as in [Sch13a, Definition 8.1]) on $\mathscr{D}(R(\mathfrak{d}))$. Let us denote this lisse $\mathbb{Z}_p$-sheaf by $T(\mathscr{X})$ and its image in the category of lisse $\mathbb{Q}_p$-sheaves as $V(\mathscr{X})$. We would first like to record the fact that any Tate tensors $z_\alpha : \mathbb{1} \to D(\overline{\mathbb{X}}(\mathfrak{d}))$ on the $\mathrm{Spec}(R(\mathfrak{d})/(p))$-crystal $D(\overline{\mathbb{X}}(\mathfrak{d}))$ have 'étale realizations' in $T(\mathscr{X})$.

To state this precisely, let C be an algebraically closed extension of E_j complete with respect to a rank 1 valuation and set $\mathcal{O}_C := C^\circ$. Note then that the underlying topological space of $\mathrm{Spa}(C)$ is a point. We note that morphisms $\mathrm{Spa}(C) \to \mathscr{D}(\mathfrak{D})$ are naturally in bijection with maps of formal schemes $\mathrm{Spf}(\mathcal{O}_C) \to \mathrm{Spf}(R(\mathfrak{d}))$. Indeed, note that since p is invertible in C that every map $\mathrm{Spa}(C) \to \mathscr{D}^{\mathrm{fml}}(\mathfrak{d})^{\mathrm{ad}}$ factors uniquely through $\mathscr{D}(\mathfrak{d})$. But, $\mathscr{D}(\mathfrak{d})^{\mathrm{ad}} = \mathrm{Spa}(R(\mathfrak{d}))$ and it's clear that maps $\mathrm{Spa}(C) \to \mathrm{Spa}(R(\mathfrak{d}))$ are in bijection with maps $\mathrm{Spa}(\mathcal{O}_C) \to \mathrm{Spa}(R(\mathfrak{d}))$ and such maps are, by construction, in unique bijection with maps $\mathrm{Spf}(\mathcal{O}_C) \to \mathrm{Spf}(R(\mathfrak{d}))$. We call such a point $\overline{x} : \mathrm{Spa}(C) \to \mathscr{D}(\mathfrak{d})$ a *geometric point* and denote by $\overline{x}^{\mathrm{fml}} : \mathrm{Spf}(\mathcal{O}_C) \to \mathrm{Spf}(R(\mathfrak{d}))$ the corresponding map of formal spectra.

Now, let $\overline{x} : \mathrm{Spa}(C, \mathcal{O}_C) \to \mathscr{D}(\mathfrak{d})$ be rooted at a classical point (i.e. if x is the image of $\overline{x}$ then $k(x)/E_j$ is finite). Then, $\overline{x}$ factors uniquely through $\mathrm{Spa}(\mathbb{C}_p)$ where $\mathbb{C}_p$ is the completion of an algebraic closure of $\overline{\mathbb{Q}_p}$. In particular, when discussing x is a classical point of $\mathscr{D}(\mathfrak{d})$, we assume that any geometric point $\overline{x} : \mathrm{Spa}(C) \to \mathscr{D}(\mathfrak{d})$ rooted at x has $C = \mathbb{C}_p$.

Let $\overline{x}$ be a geometric point of $\mathscr{D}$ rooted at the classical point x. Note then that we get a p-divisible group $\mathbb{X}_x$ over $\mathrm{Spf}(k(x)^+)$ as well as a continuous $\mathbb{Q}_p[\mathrm{Gal}(\overline{k(x)}/k(x))]$-module

$V(\mathscr{X})_{\overline{x}}$. Let D_{crys} denote Fontaine's crystalline realization functor. We then have the following result:

Theorem 3.2.12. *There is a natural isomorphism of filtered (φ, N)-isocrystals $D_{\text{crys}}(V(\mathscr{X})^{\vee}_{\overline{x}}) \cong D(\mathbb{X}_x)_{\mathbb{Q}_p}$.*

Proof. See [Kim14, Example 4.4] and note that $R(\mathfrak{d})$ evidently satisfies the 'refined almost étaleness' condition of loc. cit. (e.g. see [Kim14, §2.2.3]). □

With this, we can state precisely the claim that the crystalline tensors $z_\alpha : \mathbb{1} \to D(\mathbb{X})$ induce tensors $z^{\text{ét}}_\alpha : \mathbb{1} \to T(\mathscr{X})$. Namely, let us begin by fixing a rigidified Dieudonne enhanced deformation datum $\widetilde{\mathfrak{d}}$ with underlying deformation datum $\mathfrak{d}$. Let us fix an embedding $\rho : \mathcal{H} \hookrightarrow \text{GL}(\Lambda)$ giving rise to a datum $\overline{\mathfrak{d}}$. Let us also fix Tate tensors $z_\alpha : \mathbb{1} \to D(\mathbb{X}(\widetilde{\mathfrak{d}}))$. The following holds:

Theorem 3.2.13 (Kim)**.** *For each α there exists a unique morphism $z^{\text{ét}}_\alpha : \mathbb{1} \to T(\mathscr{X})$ of lisse $\mathbb{Z}_p$-sheaves on $\mathscr{D}(\mathfrak{d})$ such that for every geometric point $\overline{x}$ of $\mathscr{D}(\mathfrak{d})$ rooted at a classical point x we have that $(z^{\text{ét}}_\alpha)_{\overline{x}} \in T(\mathscr{X})_{\overline{x}} \subseteq V(\mathscr{X})_{\overline{x}}$ is matched with $(z_\alpha)_x \in D(\mathbb{X}_x) \subseteq D(\mathbb{X}_x)_{\mathbb{Q}_p}$ under the isomorphism in Theorem 3.2.12.*

Proof. The unicity is clear since maps of lisse $\mathbb{Z}_p$-sheaves on a connected base are determined by their value on a single geometric point. The existence follows from the contents of [Kim14]. This is explicitly explained in [Kim13, §8] in the case when $k_{E_j} = \overline{\mathbb{F}_p}$, but as remarked there this is unnecessary since the results of [Kim14] don't make this assumption. □

In particular, suppose we are given any set of tensors $\{s_\alpha\}$ in $\Lambda^\otimes$ which are invariant under $\mathcal{H}$ (e.g. those defining the embedding ρ as in Lemma 3.2.8). Then, as in the last section we obtain a corresponding set of Tate tensors $s^{\text{crys}}_\alpha : \mathbb{1} \to D(\mathbb{X}(\widetilde{\mathfrak{d}}))$. The above theorem allows us then to find étale realizations $s^{\text{ét}}_\alpha : \mathbb{1} \to T(\mathscr{X})$ of these tensors.

The last thing we would like to note is the following:

Lemma 3.2.14. *Let $z_\alpha : \mathbb{1} \to D(\mathbb{X})^\otimes$ be a set of Tate tensors. Then, for any geometric point $\overline{x}$ of $\mathscr{D}(\mathfrak{d})$ rooted at a classical point x there is an isomorphism of lisse $\mathbb{Z}_p$-sheaves $\underline{\Lambda} \to T(\mathscr{X})_{\overline{x}}$ taking s_α to $s^{\text{ét}}_\alpha$.*

Proof. This follows from [Kis10, Proposition 1.3.4] and [Kis10, Corollary 1.3.5]. □

We now use these observations to construct finite étale Galois covers $\mathscr{D}_K(\widetilde{\mathfrak{d}}) \to \mathscr{D}(\widetilde{\mathfrak{d}})$ for each compact open subgroup $K \subseteq \mathcal{H}(\mathcal{O}_F)$. Let us begin by fixing tensors $\{s_\alpha\}$ on $\Lambda^\otimes$ defining our fixed faithful representation ρ. For an integer $i \geqslant 0$ and consider the pre-sheaf $\underline{\text{Isom}}^{(s_\alpha)}_{i,\widetilde{\mathfrak{d}}}$ on $\mathscr{D}(\widetilde{\mathfrak{d}})_{\text{ét}}$ defined by associating to an object S of $\mathscr{D}(\widetilde{\mathfrak{d}})_{\text{ét}}$ the set

$$\underline{\text{Isom}}^{(s_\alpha)}_{i,\widetilde{\mathfrak{d}}} := \{\text{Isomorphisms } f : \underline{\Lambda/p^i\Lambda} \to \mathscr{X}[p^i]_S : f \circ s_\alpha = s^{\text{ét}}_\alpha\} \tag{3.37}$$

where we sloppily denote by $s^{\text{ét}}_\alpha$ the image of $s^{\text{ét}}_\alpha$ under the projection $T(\mathscr{X}) \to \mathscr{X}[p^i]$. We will denote shorten $\underline{\text{Isom}}^{(s_\alpha)}_{i,\widetilde{\mathfrak{d}}}$ to $\underline{\text{Isom}}_i$ when the other data is clear from construction.

We observe the following:

Lemma 3.2.15. *The pre-sheaf* $\underline{\mathrm{Isom}}_i$ *is a sheaf on* $\mathscr{D}(\widetilde{\mathfrak{d}})_{\text{ét}}$ *and, moreover, is represented by a finite étale cover of* $\mathscr{D}(\widetilde{\mathfrak{d}})$.

Proof. To see it's a sheaf we first note that $\mathrm{Hom}(\underline{\Lambda/p^i\Lambda}, \mathscr{X}[p^i])$ is an étale sheaf since $\mathscr{D}(\widetilde{\mathfrak{d}})$ is analytic (e.g. see [Hub96, §2.2.2]). The fact that $\underline{\mathrm{Isom}}_i$ is then a sheaf easily follows from the fact that being an isomorphism can be checked étale locally. To show that $\underline{\mathrm{Isom}}_i$ is representable, we first note that it's a $\mathcal{H}(\mathcal{O}/p^i\mathcal{O})$-torsor. The only non-obvious part is that it's non-empty, but this follows from Lemma 3.2.14. Representability then follows from a result of Huber (e.g. see [Hub96, §2.2.3]). The fact that it's finite étale and surjective over $\mathscr{D}$ is clear since it's a $\mathcal{H}(\mathcal{O}/p^i\mathcal{O})$-torsor. □

Let us set

$$K_i := \ker(\mathcal{H}(\mathcal{O}) \to \mathcal{H}(\mathcal{O}/p^i\mathcal{O})) \tag{3.38}$$

Of course, by Hensel's lemma we have that $\mathcal{H}(\mathcal{O})/K_i \cong \mathcal{H}(\mathcal{O}/p^i\mathcal{O})$. Let us set $\mathscr{D}_{K_i}(\widetilde{\mathfrak{d}}) \to \mathscr{D}(\widetilde{\mathfrak{d}})$ to be the finite étale cover of $\mathscr{D}(\widetilde{\mathfrak{d}})$ representing $\underline{\mathrm{Isom}}_i$. For any compact open subgroup K of $\mathcal{H}(\mathcal{O})$ we know that there exists some $i \geqslant 0$ such that $K_i \subseteq K$. Let us then define $\underline{\mathrm{Isom}}^{(s_\alpha)}_{K,\widetilde{\mathfrak{d}}}$ to be the quotient sheaf $\underline{\mathrm{Isom}}_i/(K/K_i)$. Let us note that $\underline{\mathrm{Isom}}_K$ is representable by a finite étale cover of $\mathscr{D}(\widetilde{\mathfrak{d}})$ (again by [Hub96, §2.2.3]) since $\underline{\mathrm{Isom}}_K$ is a K/K_i torsor. We denote this finite étale cover of $\mathscr{D}(\widetilde{\mathfrak{d}})$ by $\mathscr{D}_K(\widetilde{\mathfrak{d}})$. Note that since $\underline{\mathrm{Isom}}_i = \underline{\mathrm{Isom}}_{i'}/(K_i/K_{i'})$ evidently holds for any $i' \geqslant i$ that the definition of $\underline{\mathrm{Isom}}_K$ (and thus $\mathscr{D}_K(\widetilde{\mathfrak{d}}) \to \mathscr{D}(\widetilde{\mathfrak{d}})$) is independent of the choice of $K_i \subseteq K$. Also, note that $\mathscr{D}(\widetilde{\mathfrak{d}}) = \mathscr{D}_{\mathcal{H}(\mathcal{O})}(\widetilde{\mathfrak{d}})$.

Let us observe that if $K \subseteq K'$ are compact open subgroups of $\mathcal{H}(\mathcal{O})$ then we evidently have a natural projection map

$$\pi_{K,K'} : \mathscr{D}_K(\widetilde{\mathfrak{d}}) \to \mathscr{D}_{K'}(\widetilde{\mathfrak{d}}) \tag{3.39}$$

which is finite étale. Let us denote by $\mathscr{D}_\infty(\widetilde{\mathfrak{d}})$ the projective system $\{\mathscr{D}_K(\widetilde{\mathfrak{d}})\}$. Moreover, if $K \trianglelefteq K'$ it's evident that $\pi_{K,K'}$ is a finite Galois cover with Galois group K'/K. We would now like to explain how $\mathscr{D}_\infty(\widetilde{\mathfrak{d}})$ carries a natural action of $\mathcal{H}(\mathcal{O})$. Of course, by this, we mean an action in the sense of §7.

So, let K and K' be open subgroups of $\mathcal{H}(\mathcal{O})$ and suppose that $h \in \mathcal{H}(\mathcal{O})$ is such that $h^{-1}Kh = K'$. We then define an isomorphism

$$[h] : \mathscr{D}_K(\widetilde{\mathfrak{d}}) \to \mathscr{D}_{K'}(\widetilde{\mathfrak{d}}) \tag{3.40}$$

as follows. Choose some K_i contained in both K and K'. Then, by definition, for the natural action of $\mathcal{H}(\mathcal{O}/p^i\mathcal{O})$ on $\mathscr{D}_{K_i}(\widetilde{\mathfrak{d}})$ we have that $\mathscr{D}_K(\widetilde{\mathfrak{d}}) \cong \mathscr{D}_{K_i}(\widetilde{\mathfrak{d}})/(K/K_i)$ and $\mathscr{D}_{K'}(\widetilde{\mathfrak{d}})$. Thus, we see that for the natural isomorphism $[h] : \mathscr{D}_{K_i}(\widetilde{\mathfrak{d}}) \xrightarrow{\approx} \mathscr{D}_{K_i}(\widetilde{\mathfrak{d}})$ the assumption that $h^{-1}Kh = K'$ implies that $[h]$ induces an isomorphism

$$\mathscr{D}_K(\widetilde{\mathfrak{d}}) = \mathscr{D}_{K_i}(\widetilde{\mathfrak{d}})/(K/K_i) \xrightarrow{\approx} \mathscr{D}_{K_i}(\widetilde{\mathfrak{d}})/(K'/K_i) = \mathscr{D}_{K'}(\widetilde{\mathfrak{d}}) \tag{3.41}$$

as desired. This is a right action as one can quickly verify.

Remark 3.2.16. Let us note that we do not get a natural action of $H(F)$ on the system $\mathscr{D}_\infty(\widetilde{\mathfrak{d}})$. We will remark more on this later when we relate the tower $\mathscr{D}_\infty(\widetilde{\mathfrak{d}})$ to Rapoport-Zink spaces of Hodge type.

We would like to verify that the tower $\mathscr{D}_\infty(\widetilde{\mathfrak{d}})$ with $\mathcal{H}(\mathcal{O})$-action only depends on $\mathfrak{d}$ and not on $\widetilde{\mathfrak{d}}$ or the tensors (s_α). We do this by stating an analogue of Proposition 3.2.10:

Proposition 3.2.17. *Let $\mathfrak{d}$ be a deformation datum of Hodge type. Let $\rho_i : \mathcal{H} \hookrightarrow \mathrm{GL}_(\Lambda_i)$ be embeddings such that $\widetilde{\mathfrak{d}}_i := (\mathcal{H}, b, \boldsymbol{\mu}, \rho_i)$ are rigidified enhanced Dieudonne deformation data. Let $\widetilde{\mathfrak{d}}_3 := (\mathcal{H}, b, \boldsymbol{\mu}, \rho)$ with $\rho : \mathcal{H} \hookrightarrow \mathrm{GL}(\Lambda_1 \times \Lambda_2)$ the obvious embedding. Then, for any choice of tensors $(s_{\alpha,i}) \subseteq \Lambda_i^\otimes$ defining ρ_i and any compact open subgroup $K \subseteq \mathcal{H}(\mathcal{O}_F)$ the obvious morphisms of sheaves on $\mathscr{D}(\widetilde{\mathfrak{d}})$*

$$\underline{\mathrm{Isom}}_{K,\widetilde{\mathfrak{d}}_2}^{(s_{\alpha,i})} \longleftarrow \underline{\mathrm{Isom}}_{K,\widetilde{\mathfrak{d}}_3}^{(s_{\alpha,i} \times s_{\alpha,2})} \longrightarrow \underline{\mathrm{Isom}}_{K,\widetilde{\mathfrak{d}}_3}^{(s_{\alpha,3})} \tag{3.42}$$

is an isomorphism. Moreover, these isomorphisms are compatible with the $\mathcal{H}(\mathcal{O}_F)$-action and thus define isomorphisms of projective systems

$$\mathscr{D}_\infty(\widetilde{\mathfrak{d}}_1) \longleftarrow \mathscr{D}_\infty(\widetilde{\mathfrak{d}}_2) \longrightarrow \mathscr{D}_\infty(\widetilde{\mathfrak{d}}_3) \tag{3.43}$$

compatible with $\mathcal{H}(\mathcal{O}_F)$-action.

Proof. This is clear from combining Proposition 3.2.10 together with the fact that any K/K_i-equivariant morphism of K/K_i torsors is automatically an isomorphism. $\square$

So, we shall unabashedly can discuss the projective system $\mathscr{D}_\infty(\mathfrak{d})$ with $\mathcal{H}(\mathcal{O})$ for any deformation datum of Hodge type. We shall call this projective system the *deformation space at infinite level* associated to $\mathfrak{d}$.

Relationship to Rapoport-Zink spaces of Hodge type

Let us now assume that $F = \mathbb{Q}_p$. Let $\mathfrak{d}$ be a deformation datum of degree j and of Hodge type. We would like to make explicit the relationship between the deformation space $\mathscr{D}_\infty(\mathfrak{d})$ at infinite level and (a certain) Rapoport-Zink space at infinite level as in [Kim13] with $\mathcal{H}(\mathcal{O})$-actions.

To begin with, let us recall that if $\mathfrak{d} = (\mathcal{H}, [b], \boldsymbol{\mu})$ is a deformation datum of Hodge type then Kim in [Kim13] attaches to the pair $(\mathcal{H}, b)$ (where now $b \in [b]$ is thought of just as an element of $H(\breve{\mathbb{Q}}_p)$) a tower $\mathsf{RZ}_{\mathcal{H},b}^\infty = \{\mathsf{RZ}_{\mathcal{H},b}^K\}$ of rigid analytic spaces over $\mathrm{Spa}(\breve{\mathbb{Q}}_p)$ where K ranges over the compact open subgroups of $\mathcal{H}(\mathbb{Z}_p)$ as well as a formal model $\mathsf{RZ}_{\mathcal{H},b}$ of $\mathsf{RZ}_{\mathcal{H},b}^{\mathcal{H}(\mathbb{Z}_p)}$. This tower carries an action of $H(\mathbb{Q}_p)$ as well as a Weil descent datum Φ from $\mathrm{Spa}(\breve{\mathbb{Q}}_p)$ down to $\mathrm{Spa}(E)$ where E is the reflex field for $\boldsymbol{\mu}$. We claim that there is an embedding of towers $\mathscr{D}_\infty(\mathfrak{d})_{\mathrm{Spa}(\breve{\mathbb{Q}}_p)} \hookrightarrow \mathsf{RZ}_{\mathcal{H},b}^\infty$ which commutes with the $\mathcal{H}(\mathbb{Z}_p)$-action and the action of Φ^j where, of course, the action of Φ^j is the effective one coming from the model $\mathscr{D}_\infty(\mathfrak{d})$ of $\mathscr{D}_\infty(\mathfrak{d})_{\mathrm{Spa}(\breve{\mathbb{Q}}_p)}$ over $\mathrm{Spa}(E_j)$.

We begin observing that if we forget that $b \in H(\mathbb{Q}_{p^j})$ and instead identify it with its image in $H(\breve{\mathbb{Q}}_p)$ then we get a formal scheme $\mathsf{Def}_{\mathcal{H},b}$ over $\mathrm{Spf}(\breve{\mathbb{Z}}_p)$ as in [Kim13, Theorem 3.6] which is evidently canonically isomorphic (once the choice of the data involved in a ridified enchanced Dieudonne deformation datum $\widetilde{\mathfrak{d}}$ have been chosen) to $\mathscr{D}^{\mathrm{fml}}(\mathfrak{d})_{\breve{\mathbb{Z}}_p}$. But, as in [Kim13, Equation (4.8.5)] we have a natural identification $(\mathsf{RZ}_{\mathcal{H},b})^\wedge_{x_0} \cong \mathsf{Def}_{\mathcal{H},b}$ where x_0 denotes the identity of the ($\overline{\mathbb{F}_p}$-points of the) affine Deligne-Lusztig variety $X_{\boldsymbol{\mu}}^{\mathcal{H}}(b)(\overline{\mathbb{F}_p})$

(as in [Kim13, Definition 2.5.2]) which is identified with $\mathsf{RZ}_{\mathcal{H},b}(\overline{\mathbb{F}_p})$ via [Kim13, Proposition 2.5.10]. So, in particular, we get an isomorphism of formal schemes $\mathscr{D}^{\mathrm{fml}}(\mathfrak{d})_{\mathrm{Spf}(\breve{\mathbb{Z}}_p)} \xrightarrow{\approx} (\mathsf{RZ}_{\mathcal{H},b})^{\widehat{}}_{x_0}$.

Let us note that if Φ is the Weil descent datum down to $\mathrm{Spf}(\mathcal{O}_E)$ given on $\mathsf{RZ}_{\mathcal{H},b}$ as in [Kim13, Definition 7.3.5] then evidently x_0, being defined over $\mathbb{F}_{p^j}$, is fixed by Φ^j. Thus, we get a Weil descent datum Φ^j down to $\mathrm{Spf}(\mathcal{O}_{E_j})$ defined on $(\mathsf{RZ}_{\mathcal{H},b})^{\widehat{}}_{x_0}$. We claim that the aforementioned isomorphism $\mathscr{D}^{\mathrm{fml}}(\mathfrak{d})_{\mathrm{Spf}(\breve{\mathbb{Z}}_p)} \xrightarrow{\approx} (\mathsf{RZ}_{\mathcal{H},b})^{\widehat{}}_{x_0}$ is equivariant for the Weil descent datum on both sides (where the Weil descent datum on $\mathscr{D}^{\mathrm{fml}}(\mathfrak{d})_{\mathrm{Spf}(\breve{\mathbb{Z}}_p)}$ is the effective one coming from the model $\mathscr{D}^{\mathrm{fml}}(\mathfrak{d})$). But, this is clear from the explicit description of the Weil descent datum on $\mathsf{RZ}_{\mathcal{H},b}$ given in [Kim13, Definition 7.3.5] and the explicit moduli problem for $\mathscr{D}^{\mathrm{fml}}(\mathfrak{d})$ given in Theorem 3.2.9.

From this we see that we get a Φ^j-equivariant open embedding of adic spaces spaces $\mathscr{D}(\mathfrak{d})_{\mathrm{Spa}(\breve{\mathbb{Q}}_p)} \hookrightarrow \mathsf{RZ}^{\mathcal{H}(\mathbb{Z}_p)}_{\mathcal{H},b}$ with image $((\mathsf{RZ}_{\mathcal{H},b})^{\widehat{}}_{x_0})_\eta$. Note then that for each compact open subgroup $K \subseteq \mathcal{H}(\mathbb{Z}_p)$ we can define open adic subspaces U_K of $\mathsf{RZ}^K_{\mathcal{H},b}$ by taking the fibered product $\mathscr{D}(\mathfrak{d})_{\mathrm{Spa}(\breve{\mathbb{Q}}_p)} \times_{\mathsf{RZ}^{\mathcal{H}(\mathbb{Z}_p)}_{\mathcal{H},b}} \mathsf{RZ}^K_{\mathcal{H},b}$. Note that evidently U_K carries a Weil descent datum down to $\mathrm{Spa}(E_j)$ inherited from $\mathsf{RZ}^{\mathcal{H}(\mathbb{Z}_p)}_{\mathcal{H},b}$. We claim that there are isomorphisms of $\mathrm{Spa}(\breve{\mathbb{Q}}_p)$-spaces $(\mathscr{D}_K)_{\mathrm{Spa}(\breve{\mathbb{Q}}_p)} \xrightarrow{\approx} U_K$ compatible with the Weil descent datum down to $\mathrm{Spa}(E_j)$ and the $\mathcal{H}(\mathbb{Z}_p)$-action on the tower. But, again this is clear by choosing a rigidified enhanced Dieudonne deformation datum $\widetilde{\mathfrak{d}}$ with underlying datum $\mathfrak{d}$ as well as tensors (s_α) defining the embedding $\rho : \mathcal{H} \hookrightarrow \mathrm{GL}(\Lambda)$ and inspecting of the moduli problems defining $\mathsf{RZ}^K_{\mathcal{H},b}$ (as in [Kim13, §7.4]) and those defining $\mathscr{D}_K(\widetilde{\mathfrak{d}})$.

We record these observations as follows:

Proposition 3.2.18. *For every compact open subgroup $K \subseteq \mathcal{H}(\mathbb{Z}_p)$ there is an open embedding of $\mathrm{Spa}(\breve{\mathbb{Q}}_p)$-spaces $\mathscr{D}_K(\mathfrak{d})_{\mathrm{Spa}(\breve{\mathbb{Q}}_p)} \hookrightarrow \mathsf{RZ}^K_{\mathcal{H},b}$ induced (via taking the generic fiber and pulling back) from an identification of formal $\mathrm{Spf}(\breve{\mathbb{Z}}_p)$-schemes $\mathscr{D}^{\mathrm{fml}}(\mathfrak{d}) \cong (\mathsf{RZ}_{\mathcal{H},b})^{\widehat{}}_{x_0}$ which is equivariant for the natural Weil descent data down to $\mathrm{Spa}(E_j)$ on both sides. This gives an open embedding of $\mathrm{Spa}(\breve{\mathbb{Q}}_p)$-towers $\mathscr{D}_\infty(\mathfrak{d})_{\mathrm{Spa}(\breve{\mathbb{Q}}_p)} \hookrightarrow \mathsf{RZ}^\infty_{\mathcal{H},b}$ which is compatible with the $\mathcal{H}(\mathbb{Z}_p)$-action.*

Remark 3.2.19. We now can elucidate the content of Remark 3.2.16. Namely, it's truly the tower $\mathsf{RZ}^\infty_{\mathcal{H},b}$ which obtains a $H(\mathbb{Q}_p)$-action. Let us denote by $H(\mathbb{Q}_p)^+$ the subgroup of $H(\mathbb{Q}_p)$ stabilizing the subtower $\mathscr{D}_\infty(\mathfrak{d})_{\mathrm{Spa}(\breve{\mathbb{Q}}_p)}$. Then, the $\mathcal{H}(\mathbb{Z}_p)$-action on $\mathscr{D}_\infty(\mathfrak{d})_{\mathrm{Spa}(\breve{\mathbb{Q}}_p)}$ comes really from the observation that $\mathcal{H}(\mathbb{Z}_p) \subseteq H(\mathbb{Q}_p)^+$. The equality of these groups is something that will be explored in a future draft of this article.

A trivial corollary of this is the following:

Corollary 3.2.20. *There exists a unique pre-perfectoid space X over $\mathrm{Spa}(\breve{\mathbb{Q}}_p)$ with $\mathcal{H}(\mathbb{Z}_p)$-action and Weil descent datum to $\mathrm{Spa}(E_j)$ such that the equality $X \sim \varprojlim_K \mathscr{D}_K(\mathfrak{d})$ holds.*

Proof. Let $\mathsf{RZ}^\infty_{\mathcal{H},b}$ denote the pre-perfectoid space over $\mathrm{Spa}(\breve{\mathbb{Q}}_p)$ constructed in [Kim13, §7.6]. Denote by X the preimage of $\mathscr{D}(\mathfrak{d}) \subseteq \mathsf{RZ}^{\mathcal{H}(\mathbb{Z}_p)}_{\mathcal{H},b}$ under the projection map $\mathsf{RZ}^\infty_{\mathcal{H},b} \to$

$\mathsf{RZ}_{\mathcal{H},b}^{\mathcal{H}(\mathbb{Z}_p)}$. Since X is an open adic subspace of $\mathsf{RZ}_{\mathcal{H},b}^{\infty}$ it's evidently pre-perfectoid. It's then clear from the above discussion that $X \sim \varprojlim_K \mathscr{D}_K(\mathfrak{d})$ and that X posseses the desired Weil descent datum. $\square$

Remark 3.2.21. It's likely true that the Weil descent datum on X is effective. This will be explored in a later draft of this article.

Uniformization of Shimura varieties

We would now like to explain how the tower $\mathscr{D}_\infty(\mathfrak{d})$ can be used to study the space with $g_p \times \tau$-action $\mathscr{Y}_{\overline{y}}$ from Theorem 2.2.4.

Namely, let us fix a Hodge type Shimura datum (G, X) with G unramified at p and having a reductive model $\mathcal{G}$. Let's suppose further that we've fixed an embedding $\iota : \mathcal{G} \hookrightarrow \mathcal{G}(\psi) \subseteq \mathrm{GL}(\Lambda_\psi)$ for some symplectic space as in §2.1 and let us fix tensors (s_α) on Λ_ψ defining this faithful representation. We now fix an element $K^pK_p \in \mathcal{N}(G)$ such that $K_p \subseteq K_0 := \mathcal{G}(\mathbb{Z}_p)$ which is compact open and normal. We then choose an open compact subgroup $K^{p\prime} \subseteq \mathcal{G}(\psi)(\mathbb{A}_f^p)$ as in Lemma 2.1.18. From this we obtain maps

$$\mathscr{S}_{K^p}(G, X) \to \mathscr{S}_{K^p}^{-}(G, X) \hookrightarrow \mathscr{S}_{K^{p\prime}}(\mathcal{G}(\psi), \mathfrak{h}_\psi^{\pm}) \tag{3.44}$$

as discussed in §2.1.

Let $\mathcal{A}$ be the universal abelian variety over $\mathcal{G}(\psi)$ (having implicitly chosen a $\mathbb{Z}_{(p)}$-lattice inside of Λ_ψ). Then, for any point $\overline{x} \in \mathscr{S}_{K^p}(G, X)(\overline{\mathbb{F}_p})$ we denote by $\mathcal{A}_{\overline{x}}$ the pullback of $\mathcal{A}$ along the map $\overline{x} \to \mathscr{S}_{K^p}(G, X)$ composed with the maps in (3.44). Note in particular that we get a p-divisible group $\mathcal{A}_{\overline{x}}[p^\infty]$ over $\overline{\mathbb{F}}_p$. That said, one can identify $D(\mathcal{A}_{\overline{x}}[p^\infty])$ with $\Lambda_\psi \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$ in such a way that the Frobenius on $D(\mathcal{A}_{\overline{x}}[p^\infty])$ fixes the tensors $s_\alpha \otimes 1$ in $\Lambda_\psi \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$ (e.g. see [Kim18, Proposition 3.3.8]). Thus, $\mathcal{A}_{\overline{x}}[p^\infty]$ defines a p-divisible group with $\mathcal{G}$-structure over $\overline{\mathbb{F}_p}$ and thus gives rise to an element $b_x \in \mathcal{G}(\breve{\mathbb{Q}}_p)$.

Let us now assume take some $g^pg_p \in K^pK_p$ and assume that $\overline{x} = c_2(\overline{y})$ for an element $\overline{y} \in \mathscr{S}_{(K^p)^{g^p}}(\overline{\mathbb{F}_p})$ such that $\Phi^j(c_1(\overline{y})) = c_2(\overline{y})$ where c_i are the defining projections of the g^p-Hecke correspondence on $\mathscr{S}_{K^p,\overline{\mathbb{F}_p}}$. Note then that we have a Φ^j-twisted prime-to-p isogeny induced by g^p between $\mathcal{A}_{c_1(\overline{y})}$ and $\mathcal{A}_{c_2(\overline{y})}$ which are equal abelian varieties since the underlying abelian variety doesn't depend on level structure. This then induces a Φ^j-twisted isomorphism between $\mathcal{A}_{c_2(\overline{y})}[p^\infty]$ and itself which is necessarily effective by the effectivity of Galois descent on the affine schemes $\mathcal{A}_{c_2(\overline{y})}[p^n]$ for all n. This shows that, in fact, $b_{c_2(\overline{y})} \in G(E_j)$.

Finally, let us denote by E the reflex field of (G, X). Note then that when thinking of $b \in G(\breve{\mathbb{Q}}_p)$ we have that $\kappa(b) = \boldsymbol{\mu}_+^{-1}$ where $\boldsymbol{\mu}$ is the conjugacy class of cocharacters $\mathbb{G}_{m,\breve{\mathbb{Q}}_p} \to G_{\breve{\mathbb{Q}}_p}$ induced by X (e.g. see [Kim18, Lemma 3.3.14]). Since the reflex field of $\boldsymbol{\mu}$ contains $E_\mathfrak{p}$ we deduce that $\boldsymbol{\mu}$ is defined over $(E_\mathfrak{p})_j$. Thus, in conclusion, we see that from $\overline{y} \in \mathscr{S}_{(K^p)^{g^p}}(\overline{\mathbb{F}_p})$ such that $\Phi^j(c_1(\overline{y})) = c_2(\overline{y})$ we've produced a deformation datum $\mathfrak{d}_{\overline{y}} := (\mathcal{G}, b_{c_2(\overline{y})}, \boldsymbol{\mu})$. In fact, the datum $(\mathcal{G}, b_{c_2(\overline{y})}, \boldsymbol{\mu}, \iota)$ is an enhanced deformation datum of Hodge type from the discussion in [Kim18].

We can now state the precise uniformization theorem we would like to prove:

Theorem 3.2.22. *Let $\overline{y} \in \mathscr{S}_{(K^p)^{g^p}}(\overline{\mathbb{F}}_p)$ be such that $\Phi^j(c_1(\overline{y})) = c_2(\overline{y})$. Then, there is an isomorphism of* $\mathrm{Spa}(\breve{\mathbb{Q}}_p)$*-spaces $\mathscr{D}_{K_p}(\mathfrak{d}_{\overline{y}})_{\mathrm{Spa}(\breve{\mathbb{Q}}_p)} \cong \mathscr{Y}_{\overline{y}}$ (with $\mathscr{Y}_{\overline{y}}$ as in Theorem 2.2.4) compatible with $\Phi^j \times g_p$ actions on both sides.*

Proof. Let us deal with the case first when $K_p = K_0$. Then, this question is implied by showing that there is a Φ^j-equivariant isomorphism of formal $\mathrm{Spf}(\breve{\mathbb{Z}}_p)$-schemes $\mathscr{D}^{\mathrm{fml}}(\mathfrak{d}_{\overline{y}}) \xrightarrow{\approx} \widehat{(\mathscr{S}_{K^p})_{\breve{\mathbb{Z}}_p}}_{/c_2(\overline{y})}$. The fact that this is an isomorphism of formal schemes is a result of Kisin (see [Kim18, Proposition 4.1.6] and combine it with Proposition 3.2.18). To see that it's Φ^j-equivariant we note that $\mathscr{D}^{\mathrm{fml}}(\mathfrak{d}_{\overline{y}})_{\mathrm{Spf}(\breve{\mathbb{Z}}_p)} \hookrightarrow \mathsf{RZ}_{\mathcal{G},b_{\overline{y}}}$ is Φ^j equivariant by Proposition 3.2.18. But, if Θ^ϕ is the map of [Kim18, Proposition 4.3] then Θ^ϕ is Φ, and thus Φ^j-equivariant, and induces the isomorphism $\mathscr{D}^{\mathrm{fml}}(\mathfrak{d}_{\overline{y}})_{\mathrm{Spf}(\breve{\mathbb{Z}}_p)} \xrightarrow{\approx} \widehat{(\mathscr{S}_{K^p})_{\breve{\mathbb{Z}}_p}})_{/c_2(\overline{y})}$ by taking completions at $c_2(\overline{y})$ which implies the desired Weil invariance.

To deal with the case for arbitrary K_p we proceed similarly. By [Kim18, Theorem 5.4] we have an isomorphism of rigid spaces

$$\Theta^\phi_{K^pK_p} : I^\phi(\mathbb{Q})\backslash \mathsf{RZ}^{K_p}_{\mathcal{G},b_{\overline{y}}} \times G(\mathbb{A}^p_f)/K^p \xrightarrow{\approx} \mathsf{Sh}^{\mathrm{rig}}_{K^pK_p}(\phi) \tag{3.45}$$

compatible with $K_p \times \Phi$-action. It's clear that $\mathscr{D}_{K_p}(\mathfrak{d}_{\overline{y}})_{\mathrm{Spa}(\breve{\mathbb{Q}}_p)}$ open embeds into $I^\phi(\mathbb{Q})\backslash \mathsf{RZ}^{K_p}_{\mathcal{G},b_{\overline{y}}} \times G(\mathbb{A}^p_f)/K^p$ having picked the representative $\overline{x}$ for the isogeny class ϕ and that this embeding is $\Phi^j \times g_p$-equivariant. Similarly we see $\mathscr{Y}_{\overline{y}}$ embeds into $\mathsf{Sh}^{\mathrm{rig}}_{K^pK_p}(\phi)$ by noting that $\widehat{\mathscr{S}_{K^p}(\mathcal{G},X)}_{\backslash c_2(\overline{y})}$ is just a further completion at a closed point of $(\widehat{\mathscr{S}_{K^p}(\mathcal{G},X)}_{\breve{\mathbb{Z}}_p})_{\mathscr{I}^\phi}$ and thus $\mathscr{X}_{\overline{y}}$ embeds into $\mathsf{Sh}^{\mathrm{rig}}_{K^pK_0}(\phi)$ and so $\mathscr{Y}_{\overline{y}}$ open embeds into $\mathsf{Sh}^{\mathrm{rig}}_{K^pK_p}(\phi)$ by pullback. It's evident that this embedding is also $\Phi^j \times g_p$-equivariant. The claim then follows from [Kim18, Theorem 5.4] by noting that $\Theta_{K^pK_p}$ maps $\mathscr{D}_{K_p}(\mathfrak{d}_{\overline{y}})_{\mathrm{Spa}(\breve{\mathbb{Q}}_p)}$ to $\mathscr{Y}_{\overline{y}}$ (by inspecting their moduli problem) $\Phi^j \times g_p$-equivariantly (by the above argument) and must be an isomorphism because it's a map of K_0/K_p-torsors over the isomorphism at level K_0. $\square$

3.3 Deformation spaces of abelian type

We now venture to extend the ideas of the last section to account not only for deformation spaces of Hodge type, but those of abelian type as well. The theory will be much less satisfying than that of the case of Hodge type, in particular not being literal 'deformation spaces'. We also need to work with to a more restrictive case to show that these deformation spaces exist. We hope to, in a future draft of this article, give a more comprehensive study of deformation spaces of abelian type and, in particular, remove this mentioned condition.

Deformation datum of abelian type

Let us begin by saying that a deformation datum $\mathfrak{d} = (\mathcal{H}, [b], \boldsymbol{\mu})$ is of *abelian type* if there exists a deformation datum $\mathfrak{d}_1 = (\mathcal{H}_1, [b_1], \boldsymbol{\mu}_1)$ of Hodge type and a central isogeny $\mathcal{H}_1 \to \mathcal{H}$ which induces an isomorphism $\mathfrak{d}^{\mathrm{ad}} \cong \mathfrak{d}_1^{\mathrm{ad}}$. We shall call any Hodge type datum

$\mathfrak{d}_1$ satisfying these properties *associated to* $\mathfrak{d}$ (of course, there can be more than one such datum). Suppose that $\mathfrak{d}$ is abelian type. We then say that an associated Hodge type datum $\mathfrak{d}_1$ is *good* if $\mathcal{Z}_1 := Z(\mathcal{H}_1)$ has connected fibers, and the reflex fields of $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}$ agree. We say that $\mathfrak{d}$ is *good* if it posseses a good associated Hodge type deformation datum. Such data, while a priori quite strict, always exists in the setting of Shimura varieties (e.g. see [Kis17, Lemma 4.6.6]).

We shall associate to a good deformation datum of abelian type a deformation space in one of the two cases:

- (Type 1) The central isogeny $f : \mathcal{H}_1^{\mathrm{der}} \to \mathcal{H}$ can be taken to be an isomorphism (e.g. $\mathcal{H}^{\mathrm{der}}$ is simply connected or the datum is already Hodge).
- (Type 2) The group $\mathcal{H}$ is adjoint.

As mentioned above we hope to, in a future draft of this article, extend the construction to work for arbitrary $\mathcal{H}$.

The construction for Type 1 datum

Let $\mathfrak{d} = (\mathcal{H}, [b], \boldsymbol{\mu})$ be a good Type 1 Shimura datum of abelian type and let $\mathfrak{d}_1 = (\mathcal{H}_1, [b_1], \boldsymbol{\mu}_1)$ be a good Hodge type datum associated to $\mathfrak{d}$ such that $f : \mathcal{H}_1^{\mathrm{der}} \to \mathcal{H}^{\mathrm{der}}$ is an isomorphism. Let $\rho : \mathcal{H}_1 \hookrightarrow \mathrm{GL}(\Lambda)$ be a faithful embedding of $\mathcal{H}_1$ defined by the set of tensors $\{s_\alpha\}$ as in Lemma 3.2.8. As before, we denote by $\overline{\mathfrak{d}_1}$ the quadruple $(\mathcal{H}_1, [b_1], \boldsymbol{\mu}_1, \rho)$.

Since ρ is also a faithful embedding of $\mathcal{H}_1^{\mathrm{der}}$ into $\mathrm{GL}(\Lambda)$ we can use Lemma 3.2.8 to enlarge the set $\{s_\alpha\}$ of tensors to a set $\{s_\alpha, t_\beta\}$ of tensors that defines the scheme theoretic image of $\mathcal{H}_1^{\mathrm{der}}$. Let us then observe that

$$X_\infty := \underline{\mathrm{Isom}}((\Lambda, \{s_\alpha, t_\beta\}), (T(\mathscr{X}), \{s_\alpha^{\text{ét}}, t_\beta^{\text{ét}}\}) \tag{3.46}$$

is a $\mathcal{H}_1^{\mathrm{der}}(\mathbb{Z}_p) = \mathcal{H}(\mathbb{Z}_p)$-torsor over $\mathscr{D}(\mathfrak{d}_1)$. But, by Lemma 3.1.5 (and the assumptions on the reflex field coming from the fact that $\mathfrak{d}_1$ is good) we have a natural identification

$$\mathscr{D}(\mathfrak{d}_1) \xrightarrow{\approx} \mathscr{D}(\mathfrak{d}_1^{\mathrm{ad}}) \xleftarrow{\approx} \mathscr{D}(\mathfrak{d}^{\mathrm{ad}}) \tag{3.47}$$

We then set

$$\mathscr{D}_\infty(\mathfrak{d}, \mathfrak{d}_1) := (\mathcal{H}(\mathbb{Z}_p) \times X_\infty)/\sim \tag{3.48}$$

where $(h, f) \sim (h', f')$ if there exists some $g \in \mathcal{H}^{\mathrm{der}}(\mathbb{Z}_p)$ such that the equality $(h', f') = (hg, g^{-1}f)$ holds. In other words, $\mathscr{D}_\infty(\mathfrak{d}, \mathfrak{d}_1)$ is the image of X_∞ under the usual map of pointed

$$H^1_{\text{ét}}(\mathscr{D}(\mathfrak{d}^{\mathrm{ad}}), G) \to H^1_{\text{ét}}(\mathscr{D}(\mathfrak{d}^{\mathrm{ad}}), \mathcal{H}(\mathbb{Z}_p)) \tag{3.49}$$

induced by the inclusion $\mathcal{H}^{\mathrm{der}}(\mathbb{Z}_p) \hookrightarrow \mathcal{H}(\mathbb{Z}_p)$.

We would like to claim that the $\mathcal{H}(\mathbb{Z}_p)$-torsor $\mathscr{D}_\infty(\mathfrak{d}, \mathfrak{d}_1)$ doesn't depend on the choice of $\mathfrak{d}_1$:

Lemma 3.3.1. *Let $\mathfrak{d}_1$ and $\mathfrak{d}_2$ denote good Hodge type datum associated to $\mathfrak{d}$. Then, there is a non-canonical isomorphism of $\mathcal{H}(\mathbb{Z}_p)$-torsors on $\mathscr{D}(\mathfrak{d}^{\mathrm{ad}})$*

$$\mathscr{D}_\infty(\mathfrak{d},\mathfrak{d}_1) \to \mathscr{D}_\infty(\mathfrak{d},\mathfrak{d}_2) \tag{3.50}$$

Proof. Note that since $\mathfrak{d}_1^{\mathrm{ad}} \cong \mathfrak{d}^{\mathrm{ad}} \cong \mathfrak{d}_2^{\mathrm{ad}}$ that using the discussion suceeding Lemma 3.1.13 we obtain a third good Hodge type datum $(\mathcal{H}_3, [b_3], \boldsymbol{\mu}_3)$ together with maps $q_i : \mathcal{H}_3 \to \mathcal{H}_i$ such that $\mathfrak{d}_i = q_i(\mathfrak{d}_3)$. Let us also note that the embedding $\rho_1 \times \rho_2 : \mathcal{H}_1 \times_{\mathbb{Z}_p} \mathcal{H}_2 \hookrightarrow \mathrm{GL}(\Lambda_1 \oplus \Lambda_2)$ gives rise to an embedding of groups $\mathcal{H}_3 \hookrightarrow \mathrm{GL}(\Lambda_1 \oplus \Lambda_2)$. Let us denote by $\mathcal{T}^i$ the set of tensors defining $\mathcal{H}_i$ inside of $\mathrm{GL}(\Lambda_i)$. Note then that we can choose $\mathcal{T}^3$ a set of tensors in $(\Lambda_1 \oplus \Lambda_2)^{\otimes}$ containing the natural images $\mathcal{T}^i$ in $(\Lambda_1 \oplus \Lambda_2)^{\otimes}$. We then would like to construct $\mathcal{H}^{\mathrm{der}}(\mathbb{Z}_p)$-equivariant maps

$$\mathscr{D}_\infty(\mathfrak{d},\mathfrak{d}_3) \to \mathscr{D}_\infty(\mathfrak{d},\mathfrak{d}_i) \tag{3.51}$$

for $i = 1, 2$. This will, since both are $\mathcal{H}(\mathbb{Z}_p)$-torsors, imply that these maps are isomorphisms. But note that $T(\mathscr{X}(\overline{\mathfrak{d}_3}))$ is naturally identified with $T(\mathscr{X}(\overline{\mathfrak{d}_1})) \times T(\mathscr{X}(\overline{\mathfrak{d}_2}))$. Note then that if we have an isomorphism $\Lambda_1 \oplus \Lambda_2 \to T(\mathscr{X}(\overline{\mathfrak{d}_1})) \times T(\mathscr{X}(\overline{\mathfrak{d}_2}))$ preserving tensors, then the natural composition of the inclusion and projection give us an isomorphism $\Lambda_i \to T(\mathscr{X}(\overline{\mathfrak{d}_i}))$ preserving tensors. This gives us the desired map. $\square$

The construction for Case 2

Let us now assume that $\mathfrak{d}$ is an adjoint good abelian type deformation datum with associated good Hodge type datum $\mathfrak{d}_1$. Let us begin by noting that we have a short exact sequence of topological groups

$$1 \to Z_1(\mathbb{Z}_p) \to \mathcal{H}_1(\mathbb{Z}_p) \to \mathcal{H}(\mathbb{Z}_p) \to 1 \tag{3.52}$$

Indeed, this follows from Lemma 3.1.3 since Z_1 was assumed to have connected fibers. Note also that by the assumption that $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}$ have the same reflex field E we have a natural identification of $\mathscr{D}^{\mathrm{fml}}(\mathfrak{d}_1) \xrightarrow{\approx} \mathscr{D}^{\mathrm{fml}}(\mathfrak{d})$ by Lemma 3.1.5.

Note then that by the material discussed in the previous subsection we have that $\mathscr{D}_\infty(\mathfrak{d}_1) \to \mathscr{D}^{\mathrm{fml}}(\mathfrak{d}_1) = \mathscr{D}^{\mathrm{fml}}(\mathfrak{d})$ is a $\mathcal{H}_1(\mathbb{Z}_p)$-torsor. We then consider the $\mathcal{H}(\mathbb{Z}_p)$-torsor $\mathscr{D}_\infty(\mathfrak{d})/Z_1(\mathbb{Z}_p) \to \mathscr{D}^{\mathrm{fml}}(\mathfrak{d})$ which we denote $\mathscr{D}_\infty(\mathfrak{d},\mathfrak{d}_1)$. More specifically, for an open compact subgroup $K \subseteq \mathcal{H}_1(\mathbb{Z}_p)$ we have that its image, which we denote K^{ad}, under $\mathcal{H}_1(\mathbb{Z}_p) \to \mathcal{H}(\mathbb{Z}_p)$ is an open compact subgroup and such groups form a basis of open compact subgroups of $\mathcal{H}(\mathbb{Z}_p)$. If we assume that K is normal in $\mathcal{H}_1(\mathbb{Z}_p)$ then we can easily describe the level K^{ad}-cover $\mathscr{D}_{K^{\mathrm{ad}}}(\mathfrak{d},\mathfrak{d}_1)$ corresponding to $\mathscr{D}_\infty(\mathfrak{d})/Z_1(\mathbb{Z}_p) \to \mathscr{D}^{\mathrm{fml}}(\mathfrak{d})$ as the object

$$\mathscr{D}_K(\mathfrak{d})/(K \cap Z(\mathbb{Z}_p)) \to \mathscr{D}^{\mathrm{fml}}(\mathfrak{d}) \tag{3.53}$$

which exists since $K \cap Z(\mathbb{Z}_p)$ acts through the finite quotient group given by $\mathcal{H}_1(\mathbb{Z}_p)/(K \cap Z(\mathbb{Z}_p))$. Of course we have an isomorphism

$$\mathcal{H}_1(\mathbb{Z}_p)/(K \cap Z(\mathbb{Z}_p)) \cong \mathcal{H}(\mathbb{Z}_p)/K^{\mathrm{ad}} \tag{3.54}$$

and it's not hard to see that this presents $\mathscr{D}_{K^{\mathrm{ad}}}(\mathfrak{d},\mathfrak{d}_1)$ as a $\mathcal{H}(\mathbb{Z}_p)/K^{\mathrm{ad}}$ torsor over $\mathscr{D}(\mathfrak{d})$.

We would like to show that the $\mathscr{D}(\mathfrak{d})$-spaces $\mathscr{D}_{K^{\mathrm{ad}}}(\mathfrak{d},\mathfrak{d}_1)$ are independent of the choice of $\mathfrak{d}_1$. We give one proof, and then show that much more can be said in the situation where $\mathcal{H}_1$ is fixed.

Proposition 3.3.2. *Let $\mathfrak{d}_i=(\mathcal{H}_i,[b_i],\boldsymbol{\mu}_i)$ for $i=1,2$ be good Hodge type deformation datum (of degree j) for the good adjoint deformation datum $\mathfrak{d}=(\mathcal{H},[b],\boldsymbol{\mu})$. Then, for every compact open normal subgroup $K_p\subseteq\mathcal{H}(\mathbb{Z}_p)$ there exists a (non-canonical) isomorphism of $\mathscr{D}(\mathfrak{d})$-spaces $\mathscr{D}_{K_p}(\mathfrak{d},\mathfrak{d}_1)\xrightarrow{\approx}\mathscr{D}_{K_p}(\mathfrak{d},\mathfrak{d}_2)$ compatible with the natural $\mathcal{H}^{\mathrm{ad}}(\mathbb{Z}_p)/K_p$-action and which is compatible as K_p-varies.*

Proof. Let us set $Z_i:=Z(\mathcal{H}_i)$. Let us also fix faithful embeddings $\rho_i:\mathcal{H}_i\hookrightarrow\mathrm{GL}(\Lambda_i)$ such that $\Lambda_i^\vee\otimes_{\mathbb{Z}_p}\mathcal{O}_{E_j}$ is F_{b_i}-stable. Let us also fix tensors s_α defining ρ_1 and t_α defining ρ_2. It suffices to produce an isomorphism of $\mathcal{H}(\mathbb{Z}_p)$-torsors

$$\underline{\mathrm{Isom}}((\Lambda_1,s_\alpha),(T(\mathscr{X}_1),z_\alpha^{\text{ét}}))/Z_1(\mathbb{Z}_p)\xrightarrow{\approx}\underline{\mathrm{Isom}}((\Lambda_2,t_\alpha),(T(\mathscr{X}_2),w_\alpha^{\text{ét}}))/Z_2(\mathbb{Z}_p)\tag{3.55}$$

over $\mathscr{D}(\mathfrak{d}_1)\xrightarrow{\approx}\mathscr{D}(\mathfrak{d})\xleftarrow{\approx}\mathscr{D}(\mathfrak{d}_2)$ where this last set of isomorphisms follows from Lemma 3.1.5.

To do this, let us define the third group $\mathcal{H}_3$ as the group $\mathcal{H}_1\times_{\mathcal{H}}\mathcal{H}_2$. As in Lemma 3.1.13 we see that $\mathcal{H}_3$ is a reductive group over $\mathbb{Z}_p$. Moreover, we see that since $[b_1]^{\mathrm{ad}}=[b_2]^{\mathrm{ad}}$ we get, as in Lemma 3.1.14, a well-defined element $[(b_1,b_2)]\in C_j(\mathcal{H}_3,\boldsymbol{\mu}_3)$ where $\boldsymbol{\mu}_3$ is as in the referenced lemma. Let us now note that $\rho_1\times\rho_2$ is a faithful embedding of $\mathcal{H}_1\times_{\mathcal{O}_F}\mathcal{H}_2$ into $\mathrm{GL}(\Lambda_1\times\Lambda_2)$ whose image is defined by the tensors $\{(s_\alpha,1),(1,t_\beta),r_\gamma\}$ where r_γ are the obvious tensors defining the embedding $\mathrm{GL}(\Lambda_1)\times\mathrm{GL}(\Lambda_2)\hookrightarrow\mathrm{GL}(\Lambda_1\times\Lambda_2)$. Note then that since $\mathcal{H}_3\hookrightarrow\mathcal{H}_1\times)\mathcal{O}_F\mathcal{H}_2$ (as in Lemma 3.1.13) we see then that $\rho_1\times\rho_2$ is also a faithful representation $\mathcal{H}_3\hookrightarrow\mathrm{GL}(\Lambda_1\times\Lambda_2)$ whose image is cut out by some set of tensors $T:=\{(s_\alpha,1),(1,t_\alpha),r_\gamma,d_\delta\}$.

Let us then note that by construction $\Lambda_1\times\Lambda_2$ is stable under the action of (b_1,b_2). Thus, we get by the construction above a $\mathcal{H}_3^{\mathrm{ad}}(\mathbb{Z}_p)=\mathcal{H}(\mathbb{Z}_p)$ torsor

$$\underline{\mathrm{Isom}}((\Lambda_1\times\Lambda_2,T),(T(X_1)\times T(X_2),T^{\text{ét}}))\tag{3.56}$$

on the space

$$\hat{U}(\boldsymbol{\mu}_3)_\eta\xrightarrow{\approx}\hat{U}(\boldsymbol{\mu}_3^{\mathrm{ad}})_\eta=\hat{U}(\boldsymbol{\mu}_1^{\mathrm{ad}})_\eta=\mathscr{D}(\mathfrak{d})\tag{3.57}$$

where, as per usual, the first equality is by Lemma 3.1.5 and $T(x_i)$ is the p-divisible group constructed from the data $\overline{\mathfrak{d}_i}$. Let us note that the reason that the lisse $\mathbb{Z}_p$-sheaf in the above is $T(X_1)\times T(X_2)$ is because (b_1,b_2) acts on $\Lambda_1\times\Lambda_2$ via the action of b_i on Λ_i.

Let us denote by ι the embedding $\Lambda_1\hookrightarrow\Lambda_1\times\Lambda_2$ and by p_1 the projection $T(X_1)\times T(X_2)$. Then, we have the map

$$\underline{\mathrm{Isom}}((\Lambda_1\times\Lambda_2,T),(T(X_1)\times T(X_2),T^{\text{ét}}))\to\underline{\mathrm{Isom}}((\Lambda_1,s_\alpha),(T(X_1),z_\alpha^{\text{ét}}))\tag{3.58}$$

given by $\varphi\mapsto p_2\circ\varphi\circ\iota$. This map is well-defined (in the sense that $p_2\circ\varphi\circ\iota$ is an isomorphism preserving tensors) precisely because T and $T^{\text{ét}}$ contain the tensors whose preservation dictates that Λ_1 maps isomorphically to $T(x_1)$ and that it preserves the tensors s_α. This map is evidently equivariant for the projection map $\mathcal{H}_3(\mathbb{Z}_p) \to \mathcal{H}_1(\mathbb{Z}_p)$. We then get a well-defined map

$$\underline{\mathrm{Isom}}((\Lambda_1 \times \Lambda_2, T), (T(X_1) \times T(X_2), T^{\text{ét}}))/Z_3(\mathbb{Z}_p) \to \underline{\mathrm{Isom}}((\Lambda_1, s_\alpha), (T(X_1), z_\alpha^{\text{ét}}))/Z_1(\mathbb{Z}_p) \tag{3.59}$$

equivariant for the map $\mathcal{H}_3/Z_3(\mathbb{Z}_p) \to \mathcal{H}_1(\mathbb{Z}_p)/Z(\mathbb{Z}_p)$. But, this map of groups is an isomorphism (in fact, up to reidentification, it's the identity map on $\mathcal{H}(\mathbb{Z}_p)$). Thus, we see that (3.59) is an equivariant map of $\mathcal{H}(\mathbb{Z}_p)$-torsors on $\mathscr{D}(\mathfrak{d})$ and thus an isomorphism. We can repeat the procedure by replacing all instances of 1 with 2 to deduce that we have an isomorphism of $\mathcal{H}(\mathbb{Z}_p)$-torsors

$$\underline{\mathrm{Isom}}((\Lambda_1 \times \Lambda_2, T), (T(X_1) \times T(X_2), T^{\text{ét}}))/Z_3(\mathbb{Z}_p) \to \underline{\mathrm{Isom}}((\Lambda_2, t_\alpha), (T(X_2), z_\alpha^{\text{ét}}))Z_2(\mathbb{Z}_p) \tag{3.60}$$

and thus we get the desired isomorphism (3.55). $\square$

As promised, we remark that in the case when $\mathcal{H}_1 = \mathcal{H}_2$ (i.e. the case where we've merely) changed $[b_1]$ and $[b_2]$ we can say something much more concrete:

Proposition 3.3.3. *Let $\mathfrak{d}_1 = (\mathcal{H}_1, [b_1], \boldsymbol{\mu}_1)$ and $\mathfrak{d}_2 = (\mathcal{H}_1, [b_2], \boldsymbol{\mu}_1)$ be two good Hodge type data relative to $\mathfrak{d} = (\mathcal{H}, [b], \boldsymbol{\mu})$ such that $\mathfrak{d}_1^{\text{ad}} = \mathfrak{d} = \mathfrak{d}_2^{\text{ad}}$. Then, there exists an isomorphism of lisse $\mathbb{Z}_p$-sheaves*

$$T(\mathscr{X}_{b_1})_{\mathscr{D}(\mathfrak{d})_{\mathrm{Spa}(\mathbb{Q}_{p^\infty})}} \cong T(\mathscr{X}_{b_2})_{\mathscr{D}(\mathfrak{d})_{\mathrm{Spa}(\mathbb{Q}_{p^\infty})}} \tag{3.61}$$

preserving $z_\alpha^{\text{ét}}$ whose $Z(\mathbb{Z}_p)$-orbit is $\mathrm{Gal}(\mathbb{Q}_p^{\text{ur}}/E_j)$-stable.

Note that this proposition evidently allows us to create an isomorphism of quotient sheaves

$$\underline{\mathrm{Isom}}((\Lambda_1, s_\alpha), (T(X_{b_1}), z_\alpha))/Z_1(\mathbb{Z}_p) \xrightarrow{\approx} \underline{\mathrm{Isom}}((\Lambda_1, s_\alpha), (T(X_{b_2}), w_\alpha^{\text{ét}}))/Z_2(\mathbb{Z}_p) \tag{3.62}$$

but actually says much more by explicitly comparing the pairs $(T(X_{b_1}), z_\alpha)$ and $(T(X_{b_2}), w_\alpha^{\text{ét}})$.

Proof. Let us choose representatives $b_1 \in [b_1]$ and $b_2 \in [b_2]$. Note though that by Lemma 3.1.12 that there exists $z \in Z(\mathcal{O}_{E_j})$ and some $h \in \mathcal{H}_1(\mathbb{Z}_{p^j})$ such that $b_2 = zhb_1\sigma(h)^{-1}$. So, if we fix $\rho : \mathcal{H}_1 \hookrightarrow \mathrm{GL}(\Lambda)$ such that $\Lambda^\vee \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^j}$ is stable under F_{b_1} it will automatically be stable under F_{b_2}. Let us also fix tensors (s_α) defining ρ. Set $\overline{\mathfrak{d}_1} := \mathfrak{d}_1 = (\mathcal{H}_1, [b_1], \boldsymbol{\mu}_1, \rho)$ and $\overline{\mathfrak{d}_2} := (\mathcal{H}_1, [b_2], \boldsymbol{\mu}_1, \rho)$.

Let us set $\overline{X}_{b_i}$, X_{b_i}, $\mathbb{X}_{b_i}$, and $T(\mathscr{X}_{b_i})$ the objects from the previous subsection created from the datum $\overline{\mathfrak{d}_i}$. We now give an isomorphism of lisse $\mathbb{Z}_p$-sheaves

$$T(\mathscr{X}_{b_1})_{\mathscr{D}(\mathfrak{d})_{\mathrm{Spa}(\mathbb{Q}_{p^\infty})}} \cong T(\mathscr{X}_{b_2})_{\mathscr{D}(\mathfrak{d})_{\mathrm{Spa}(\mathbb{Q}_{p^\infty})}} \tag{3.63}$$

preserving $z_\alpha^{\text{ét}}$ whose $Z(\mathbb{Z}_p)$-orbit is $\mathrm{Gal}(\mathbb{Q}_p^{\text{ur}}/E_j)$-stable.

Moreover, again by Lemma 3.1.12, we know that there exists some $t \in Z(\mathbb{Z}_{p^\infty})$ such that $z = t\sigma(t)^{-1}$. Note then that we have that ht is an isomorphism of crystals $\mathbb{D}((\overline{X}_{b_1})_{\overline{\mathbb{F}_p}}) \to \mathbb{D}((\overline{X}_{b_2})_{\overline{\mathbb{F}_p}})$ which, because $\kappa_{E_j}(b_1) = \kappa_{E_j}(b_2)$, respects filtration. This then creates an isomorphism of p-divsible groups $(\overline{X}_{b_1})_{\overline{\mathbb{F}_p}} \xrightarrow{\approx} (\overline{X}_{b_2})_{\overline{\mathbb{F}_p}}$ and an isomorphism $(X_{b_1})_{\mathbb{Z}_{p^\infty}} \xrightarrow{\approx} (X_{b_2})_{\mathbb{Z}_{p^\infty}}$. We thus get, by construction, an isomorphism $(\mathbb{X}_{b_1})_{\mathscr{D}^{\mathrm{fml}}(\mathfrak{d})_{\mathbb{Z}_{p^\infty}}} \xrightarrow{\approx} (\mathbb{X}_{b_2})_{\mathscr{D}^{\mathrm{fml}}(\mathfrak{d})_{\mathbb{Z}_{p^\infty}}}$ and thus finally an isomorphism of lisse $\mathbb{Z}_p$-sheaves

$$T(\mathscr{X}_{b_1})_{\mathscr{D}(\mathfrak{d})_{\mathrm{Spa}(\mathbb{Q}_{p^\infty})}} \cong T(\mathscr{X}_{b_2})_{\mathscr{D}(\mathfrak{d})_{\mathrm{Spa}(\mathbb{Q}_{p^\infty})}} \tag{3.64}$$

Since $th \in \mathcal{H}_1(\mathbb{Z}_{p^\infty})$ we see also that this isomorphism preserves the tensors $z_\alpha^{\text{ét}}$. To see why the $Z(\mathbb{Z}_p)$-conjugacy class of this isomorphism is $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/E_j)$-stable, let us merely note that by definition $\sigma(t) = z^{-1}t$. Note then that for any $r \geqslant 1$ we have that $\sigma^r(t) = z\sigma(z)\cdots\sigma^{r-1}(z)t$. Note, in particular, that if $j \mid r$ then $z\sigma(z)\cdots\sigma^{r-1}(z) \in Z(\mathbb{Z}_p)$. Since $h \in Z(\mathbb{Z}_{p^j})$ and $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/E_j)$ is generated by σ^r with $r := j[E:\mathbb{Q}_p]$ the conclusion follows. □

This allows us to give a more concrete proof of Proposition 3.3.2 in the case where $\mathcal{H}_1 = \mathcal{H}_2$ in general. Namely, it's easy to adapt the above proof to also show that not only does $\mathscr{D}_{K^{\mathrm{ad}}}(\mathfrak{d}, \mathfrak{d}_1)$ not depend on the choice of $[b_1]$, but that it doesn't depend on the choice of $\boldsymbol{\mu}_1$ lying over $\boldsymbol{\mu}$.

Proposition 3.3.4. *Let $\mathfrak{d}_1 = (\mathcal{H}_1, [b_1], \boldsymbol{\mu}_1)$ and $\mathfrak{d}_2 = (\mathcal{H}_1, [b_2], \boldsymbol{\mu}_2)$ be two good Hodge type data relative to $\mathfrak{d} = (\mathcal{H}, [b], \boldsymbol{\mu})$ such that $\mathfrak{d}_1^{\mathrm{ad}} = \mathfrak{d} = \mathfrak{d}_2^{\mathrm{ad}}$. Then, for all compact open normal subgroups $K_p \subseteq \mathcal{H}^{\mathrm{ad}}(\mathbb{Z}_p)$ we have an isomorphism of $\mathscr{D}(\mathfrak{d})$-spaces $\mathscr{D}_{K_p}(\mathfrak{d}, \mathfrak{d}_1) \xrightarrow{\approx} \mathscr{D}_{K_p}(\mathfrak{d}, \mathfrak{d}_2)$ compatible with the natural $\mathcal{H}^{\mathrm{ad}}(\mathbb{Z}_p)/K_p$-action and which is compatible as K_p-varies.*

Proof. This follows essentially by the proof method of the previous proposition. Again, let us choose a faithful representation $\rho : \mathcal{H}_1 \hookrightarrow \mathrm{GL}(\Lambda)$ such that $\Lambda^* \otimes_{\mathbb{Z}_p} \mathcal{O}_{E_j}$ is stable under F_{b_1}. We then set $\overline{\mathfrak{d}_1} := (\mathcal{H}_1, [b_1], \boldsymbol{\mu}_1, \rho)$ and $\overline{\mathfrak{d}_2} := (\mathcal{H}_1, \underline{[b_1]}, \boldsymbol{\mu}_2, \rho)$. Note that by Theorem 3.2.9 we have that $\mathbb{X}(\overline{\mathfrak{d}_i})$ is the universal deformation of $\overline{X}(\overline{\mathfrak{d}_i})$ on $\mathscr{D}^{\mathrm{fml}}(\mathfrak{d}_i)$ preserving Tate tensors. Note though that by Lemma 3.1.5 we have a natural identification of $\mathscr{D}^{\mathrm{fml}}(\mathfrak{d}_i)$ which, since $\overline{X}(\overline{\mathfrak{d}_i})$ doesn't depend on μ_1 (it only depends on (b_1, ρ)), corresponds $\mathbb{X}(\overline{\mathfrak{d}_i})$ from where the claim follows. □

Uniformization results

We now show that uniformization results can be obtained for Shimura varieties from deformation spaces of abelian typese in certain situations. Namely, we will assume that our abelian type Shimura datum (G, X) has G^{der} simply connected (which we call the Type 1 case).

We begin by recalling the following construction result of Kisin:

Proposition 3.3.5 (Kisin). *There exists a Shimura datum (G_1, X_1) of Hodge type with the following properties:*

1. *There exists an isogeny $f : G_1^{\mathrm{der}} \to G^{\mathrm{der}}$ inducing an isomorphism $(G_1^{\mathrm{ad}}, X_1^{\mathrm{ad}}) \cong (G, X)$.*

2. *$Z_1 := Z(G_1)$ is a torus.*

3. *G_1 is unramified at p and we have an equality of residue fields $E(G_1, X_1)_p = E(G, X)_p$ (which means the completion of these fields at some compatible pair of prime ideals $\mathfrak{p}$ lying over p)*

Proof. See [Kis17, Lemma 4.6.6]. □

We then have the following observation concerning the group G_1 from this proposition:

Lemma 3.3.6. *There exists a reductive model $\mathcal{G}_1$ of G_1 such that $\mathcal{Z}_1 := Z(\mathcal{G}_1)$ is a torus, and $\mathcal{G}_1^{\mathrm{ad}} \cong \mathcal{G}$.*

Proof. Since $\mathrm{Spec}(\mathbb{Z}_p)$ is normal we can use [Vas16, Lemma 2.3.1] to lift the central isogeny $f : G_1^{\mathrm{der}} \to G^{\mathrm{der}}$ to a central isogeny $g : \mathcal{L}_1 \to \mathcal{G}$ in fact, as in loc. cit., we have that $\mathcal{L}$ is the normalization of $\mathcal{G}$ in G_1. Note that G_1 is unramified, the same is true for Z_1 from Lemma 2.1.14. Thus, there exists a unique model $\mathcal{Z}_1$ of Z_1 over $\mathrm{Spec}(\mathbb{Z}_p)$. In particular, it corresponds to the $\pi_1^{\text{ét}}(\mathrm{Spec}(\mathbb{Z}_p), \overline{x})$ (where $\overline{x} : \mathrm{Spec}(\overline{\mathbb{Q}_p}) \to \mathrm{Spec}(\mathbb{Z}_p)$ $X_*(Z_1)$, where the action is via the natural surjection $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p) \cong \pi_1^{\text{ét}}(\mathrm{Spec}(\mathbb{Z}_p), \overline{x})$ (see [Con11, Corollary B.3.6]). Let us note that $\ker g$ is a multiplicative group modeling $\ker f$. Note then that (again using the logic of the last sentence and [Con11, Corollary B.3.6]) we get a unique embedding $\ker g \hookrightarrow \mathcal{Z}_1$ extending the map $\ker f = Z_1 \cap G_1^{\mathrm{der}} \to Z_1$ sending x to x^{-1}.

Let us define $\mathcal{G}_1 := (\mathcal{L}_1 \times \mathcal{Z}_1)/\ker g$ where $\ker(g) \to \mathcal{Z}_1 \times \mathcal{L}_1$ is defined to have second projection the tautological embedding and the first projection is the map from the previous sentence. We claim then that $\mathcal{G}_1^{\mathrm{ad}} \cong \mathcal{G}$ and $\mathcal{G}_1$ is a model of G_1. This latter claim is clear since the generic fiber is $(Z_1 \times G_1^{\mathrm{der}})/\ker f$ which is classically known to be G_1. Note then that we have a natural surjection $\mathcal{G}_1 \to \mathcal{G}$ given by g on $\mathcal{L}_1$ and the trivial map on $\mathcal{Z}_1$. It's clear then that the kernel is $(\mathcal{Z}_1 \times \ker g)/\ker g \cong \mathcal{Z}_1$. Thus, it suffices to show that $Z(\mathcal{G}_1) = \mathcal{Z}_1$, but this is clear. Noting that $\mathcal{Z}_1$ is a torus, the proposition is proven. □

In particular, note that if $f : G_1^{\mathrm{der}} \to G^{\mathrm{der}}$ is an isomorphism, then the same holds true integrally—that the map $\mathcal{G}_1^{\mathrm{der}} \to \mathcal{G}^{\mathrm{der}}$ is an isomorphism.

Let us now consider (G, X) to be an abelian type Shimura datum of Type 1 which is unramified at p with reflex field E. Let us fix the following data. First, we fix a reductive model $\mathcal{G}$ of $G_{\mathbb{Q}_p}$ with associated hyperspecial subgroup $K_0 \subseteq G(\mathbb{Q}_p)$. Let us also fix an element $g^p \in G(\mathbb{A}_f^p)$, an element $g_p \in K_0$, a normal open compact subgroup $K_p \subseteq \mathcal{G}(\mathbb{Z}_p)$ and a $\tau \in W_{E_p}$. Let us set $j := v(\tau)$.

Let us also fix an element $\overline{y} \in \mathscr{S}_{(K^p)g^p}(G, X)$ such that $\Phi_q^j(c_1(\overline{y})) = c_2(\overline{y})$ where $q := \#\mathcal{O}_{E_p}/(p)$ where, as before, we are using c_i to denote the projections in the Hecke correspondence. We would like to claim that associated to $\overline{y}$ is a deformation datum of degree j.

We state this as the following lemma:

Lemma 3.3.7. *There exists a deformation datum$(\mathcal{G}, [b], \boldsymbol{\mu})$ of degree j over $E_{\mathfrak{p}}$ attached to the point $\overline{y}$.*

Proof. To do this, let us start by noting that by the main theorem in [Kis17], the discussion in [Kis17, §4.5], and [Kis17, Lemma 3.3.4] we can associate to $\overline{y}$ a σ-$\mathcal{G}(\mathbb{Z}_{p^j})$-conjugacy class whose associated σ-$\mathcal{G}((E_p)_j)$-conjugacy class $[b]$ lies in $C_j(\mathcal{H}, \boldsymbol{\mu})$, where $\boldsymbol{\mu}$ is the conjugacy class of cocharacters $\mathbb{G}_{m,E_p} \to G_{E_p}$ obtained from X. More precisely, from the cited results one can obtain from $\overline{y}$ a isogeny class ϕ (in the language of loc. cit.) and an element of the Φ^j-fixed points of $X_p(\phi)$. One can then use [Kis17, Lemma 3.3.4] to get a σ-$\mathcal{G}((E_j))$-conjugacy class and, by construction, it lies in $C_j(\mathcal{H}, \boldsymbol{\mu})$ by [Kim13, Lemma 2.5.8]. $\square$

We would As defined in the discussion following Theorem 2.2.4 let us consider the $\mathrm{Spa}(\mathbb{Q}_{p^\infty})$-space $\mathscr{Y}_{\overline{y}}$ contained in $\mathsf{Sh}_{K^pK_p}(G,X)^{\mathrm{an}}_{\mathbb{Q}_{p^\infty}}$ with its natural Φ_q^j and g_p actions. We then have the following:

Theorem 3.3.8. *There exists an isomorphism of* $\mathrm{Spa}(\mathbb{Q}_{p^\infty})$*-spaces*

$$\mathscr{D}_{K_p}(\mathfrak{d}) \xrightarrow{\approx} \mathscr{Y}_{\overline{y}} \tag{3.65}$$

which is $\tau \times g_p$-equivariant, where $\mathfrak{d} := (\mathcal{G}, [b], \boldsymbol{\mu})$. Moreover, these isomorphisms are compatible as K_p varies.

Proof. Let us fix an associated Hodge type datum (G_1, X_1) as in Proposition 3.3.5. Let us also consider a model $\mathcal{G}_1$ of G_1 over $\mathbb{Z}_p$ associated to the model $\mathcal{G}$ of G via Lemma 3.3.6. Let us note that since we have an isomorphism of connected Shimura vareties $\mathsf{Sh}(G,X)^+ \cong \mathsf{Sh}(G_1,X_1)^+$ that we can prove that the pullback of $\mathscr{X}_{\overline{y}}$, in the parlance of Theorem 2.2.4, (which doesn't depend on (G,X) but only on the connected Shimura datum) to $\mathsf{Sh}_{(K^p)'K_0'}(G_1,X_1)^{\mathrm{an}}$ agrees with that of $\mathscr{X}_{\overline{y}}$ pulled back to $\mathsf{Sh}_{K^pK_0}(G,X)$. The desired result then follows by applying Theorem 3.2.22 together with the results of [Del79, §2.7.11]. $\square$

Chapter 4

The test function and an expected formula

Using the material of the last section we define for every so-called *deformation pre-datum* of abelian type $\mathfrak{p} = (\mathcal{H}, \boldsymbol{\mu})$, integer j, and pair of elements $\tau \times h \in W_{E_j} \times C^\infty(\mathcal{H}(\mathbb{Z}_p))$ a 'test function' $\phi_{\tau,h} = \phi_{\tau,h}^{\mathfrak{p},j} \in C_c^\infty(\mathcal{H}(E_j), \mathbb{C})$ whose twisted orbital integrals will account, roughly, for the 'contribution at p' in the formula Theorem 2.2.4.

4.1 Deformation pre-data and the test function

Let us fix $\mathcal{H}$ to be a reductive group scheme over $\mathbb{Z}_p$ with generic fiber H and $\boldsymbol{\mu}$ a conjugacy class of cocharacters of H_E where $F_r = E/\mathbb{Q}_p$ is some finite unramified extension. We call the pair $\mathfrak{p} := (\mathcal{H}, \boldsymbol{\mu})$ a *deformation pre-datum*. Let us fix an integer $j \geqslant 0$ and for any $[b] \in C_{rj}(\mathcal{H})$ let us write $\mathfrak{p}([b])$ for the triple $(\mathcal{H}, [b], \boldsymbol{\mu})$. We say that $\mathfrak{p}$ is a (degree j) pre-deformation of *abelian type* if there exists $[b] \in C_{rj}(\mathcal{H})$ such that $\mathfrak{p}([b])$ is a deformation datum (of degree j and) of abelian type. Let us say that $[b] \in C_{rj}(\mathcal{H})$ is *adapted* to $\mathfrak{p}$ if $\mathfrak{p}([b])$ is a deformation datum of abelian type. We shall restrict ourselves to the case of abelian type datum of types 1 and 2, so that the material of the last subsection applies.

Let us first make a simple observation concerning this notion that will be helpful in the sequel:

Lemma 4.1.1. *Let $\mathfrak{p}$ be a deformation pre-datum of degree j. Suppose that for some $b \in H(E_j)$ the triple $\mathfrak{p}([b])$ is a deformation datum of Hodge type. Choose a faithful representation $\rho : \mathcal{H} \hookrightarrow \mathrm{GL}(\Lambda)$ such that F_b stabilizies $\Lambda^\vee \otimes_{\mathbb{Z}_p} \mathcal{O}_{E_j}$. Then, for all $b' \in H(E_j)$ such that $[b] \in C_j(\mathcal{H}, \boldsymbol{\mu})$ we have that $F_{b'}$ stabilizes $\Lambda^\vee \otimes_{\mathbb{Z}_p} \mathcal{O}_{E_j}$.*

In particular, if $\mathfrak{p}([b])$ is of Hodge type for some $[b] \in C_j(\mathcal{H})$ then a $[b'] \in C_j(\mathcal{H})$ has $\mathfrak{p}([b'])$ a deformation datum of abelian type if and only if $[b'] \in C_j(\mathcal{H}, \boldsymbol{\mu})$ and, moreover, this datum is of Hodge type.

Proof. Note that since $b \in C_j(\mathcal{H}, \boldsymbol{\mu})$ that for the dominant element $\mu^+ \in \boldsymbol{\mu}$ we can write $b = h_1\mu^+(p)h_2$ for some $h_1, h_2 \in \mathcal{G}(\mathcal{O}_{E_j})$. For this reason we see that F_b will stabilize $\Lambda^\vee \otimes_{\mathbb{Z}_p} \mathcal{O}_{E_j}$ if and only if $\mu^+(p)$ does. But, this then is clearly a property shared by b'. The second claim then easily follows form the definition of Hodge type datum. $\square$

From this we can define a (degree j) deformation pre-datum $\mathfrak{p} = (\mathcal{H}, \boldsymbol{\mu})$ to be of *Hodge type* if there exists (or equivalently by the above) a $[b] \in C_{rj}(\mathcal{H})$ such that $\mathfrak{p}([b])$ is of Hodge type.

From the contents of the last section, we know that for any $[b] \in C_j(\mathcal{H}, \boldsymbol{\mu})$ adapted to $\mathfrak{p}$ we get a tower of rigid analytic varieties $\{\mathscr{D}_{K_p}(\mathfrak{p}([b]))\}$ over $\mathrm{Spa}(E_j)$ where K_p range over the compact open subgroups of $\mathcal{H}(\mathbb{Z}_p)$. Before we define our test functions, it will be helpful to record the following result:

Lemma 4.1.2. *Let $\mathfrak{p}$ be a pre-deformation datum (of degree j) of abelian type and $[b]$ and element of $C_j(\mathcal{H}, \boldsymbol{\mu})$ adapted to $\mathfrak{p}$. Then, for all compact open subgroups $K_p \subseteq \mathcal{H}(\mathbb{Z}_p)$ and all primes $\ell \neq p$ the cohomology groups $H^i(\mathscr{D}_{K_p}(\mathfrak{p}([b]))_{\mathrm{Spa}(\mathbb{C}_p)}, \overline{\mathbb{Q}_\ell})$ are finite-dimensional $\overline{\mathbb{Q}_\ell}$-spaces for all $i \geqslant 0$.*

Proof. Let us shorten $\mathscr{D}_{K_p}(\mathfrak{p}([b]))_{\mathrm{Spa}(\mathbb{C}_p)}$ to X_{K_p}. Using standard arguments it suffices to prove that $H^*(X_{K_p}, \mathbb{Z}/\ell^n\mathbb{Z})$ is finite for every n. To do this, let $\pi : X_{K_p} \to X_{K_0}$ be the standard projection. Since π is a finite étale map we have that $H^i(X_{K_p}, \mathbb{Z}/\ell^n\mathbb{Z})$ is isomorphic to $H^i(X_{K_0}, \pi_*\mathbb{Z}/\ell^n\mathbb{Z})$. Thus, it suffices to prove that this latter group is finite. To see this, note that X_{K_0} is the base change to $\mathrm{Spa}(\mathbb{C}_p)$ of the generic fiber of a quasi-compact (in fact affine) formal scheme formally of finite type over $\mathrm{Spf}(\mathbb{Z}_{p^\infty})$, namely $\mathscr{D}^{\mathrm{fml}}(\mathfrak{d})$. Thus, by [Ber15, Corollary 3.1.2] it suffices to explain why the sheaves $\pi_*\mathbb{Z}/\ell^n\mathbb{Z}$ are $\mathscr{D}^{\mathrm{fml}}(\mathfrak{d})$-constructible (in the sense of loc. cit.). But, this is clear since $\pi_*\mathbb{Z}/\ell^n\mathbb{Z}$ is finite locally constant for the étale topology on X_{K_0} and, moreover, evidently has a discrete action of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. The claim then follows from loc. cit. □

The other basic lemma to establish is the following:

Lemma 4.1.3. *Let $\mathfrak{p}$ be a pre-deformation datum (of degree j) of abelian type and $[b]$ and element of $C_j(\mathcal{H}, \boldsymbol{\mu})$ adapted to $\mathfrak{p}$. Then, for all compact open subgroups $K_p \subseteq \mathcal{H}(\mathbb{Z}_p)$ and all primes $\ell \neq p$ the cohomology groups $H^i(\mathscr{D}_{K_p}(\mathfrak{p}([b]))_{\mathrm{Spa}(\mathbb{C}_p)}, \overline{\mathbb{Q}_\ell})$ vanish for $i > \dim \mathrm{Fil}^1_\mu(\mathfrak{g}_{\mathcal{O}_{E_j}})$.*

To do this, we first recall that an adic space X is called *weakly Stein* if X is a countable increasing union of affinoid open subspaces. We then make the following trivial observation:

Proposition 4.1.4. *Let X be a weakly Stein adic space, and let $f : Y \to X$ be a finite surjection. Then, Y is also weakly Stein.*

Proof. Suppose that $X = \bigcup_n U_n$ with $U_n \subseteq U_{n+1}$ and U_n affinoid. Note then that since f is finite we have that $f^{-1}(U_i)$ is open and affinoid (e.g. see [Hub96, Paragraph (1.4.4)]). Since $Y = \bigcup_n f^{-1}(U_n)$ and $f^{-1}(U_n) \subseteq f^{-1}(U_{n+1})$ the claim follows. □

From this, we deduce the following simple corollary:

Corollary 4.1.5. *Let $\mathfrak{p}$ be a pre-deformation datum (of degree j) of abelian type and $[b]$ and element of $C_j(\mathcal{H}, \boldsymbol{\mu})$ adapted to $\mathfrak{p}$. Then, for all compact open subgroups $K_p \subseteq \mathcal{H}(\mathbb{Z}_p)$ the spaces $\mathscr{D}_{K_p}(\mathfrak{p}([b]))$ are weakly Stein.*

Proof. Note that since $\mathscr{D}_{K_0}(\mathfrak{p}([b]))$ is a product of open disks it's evidently weakly stein (being a countable increasing union of affinoid open subspaces isomorphic to closed disks) and since we have finite surjections $\mathscr{D}_{K_p}(\mathfrak{p}([b])) \to \mathscr{D}_{K_0}(\mathfrak{p}([b]))$ the conclusion follows. □

With this, we can now easily prove Lemma 4.1.3:

Proof. (Lemma 4.1.3) Since $\mathscr{D}_{K_p}(\mathfrak{p}([b]))$ is weakly Stein this follows immediately from [Han17, Corollary 3.7] together with the observation that since we have finite étale surjections maps $\mathscr{D}_{K_p}(\mathfrak{p}([b])) \to \mathscr{D}_{K_0}(\mathfrak{p}([b])$ that

$$\dim \mathscr{D}_{K_p}(\mathfrak{p}([b])) = \dim \mathscr{D}_{K_0}(\mathfrak{p}([b]) = \dim \mathrm{Fil}^1_\mu(\mathfrak{g}_{\mathcal{O}_{E_j}}) \tag{4.1}$$

where this latter equality follows since it's the generic fiber of $\widehat{U}(\boldsymbol{\mu})$ which has relative dimension $\dim \mathrm{Fil}^1_\mu(\mathfrak{g}_{\mathcal{O}_{E_j}})$ by [Con11, Theorem 4.1.7]. □

With these basic observations out of the way, we can now define the functions of interest to us. To do this let us start by, in addition to the deformation pre-datum $(\mathcal{H}, \boldsymbol{\mu})$ and integer $j \geqslant 0$, fixing the following set of data:

- An element $\tau \in W_{E_j}$ (where, again, E is the field of definition of $\boldsymbol{\mu}$).
- A compact open normal subgroup $K_p \trianglelefteq K_0$.
- An element $g_p \in K_0$.

Let us then define the *test function* associated to the data $(\mathcal{H}, \boldsymbol{\mu}, \tau, g_p, K_p)$, abbreviated to just ϕ_{τ,g_p}, as the function $\phi_{\tau,g_p} : H(E_j) \to \overline{\mathbb{Q}_\ell}$ given as follows:

$$\phi_{\tau,g_p}(b) := \begin{cases} 0 & \text{if} \quad [b] \text{ is not adapted to } \mathfrak{p} \\ \mathrm{tr}(\tau \times g_p \mid H^*(\mathscr{D}_{K_p}(\mathfrak{p}([b]))_{\mathrm{Spa}(\mathbb{C}_p)}, \overline{\mathbb{Q}_\ell}) & \text{if} \quad \text{otherwise} \end{cases} \tag{4.2}$$

where the fact that this function is sensical (in the sense that we are taking a finite sum of finite numbers) follows from the discussion above.

In the subsequent subsections we verify that ϕ_{τ,g_p} satisfies various natural properties.

4.2 Fundamental properties of the function $\phi_{\tau,h}$

We first verify that $\phi_{\tau,g_p} \in C_c^\infty(G(\mathbb{Q}_{p^j}), \overline{\mathbb{Q}_\ell})$. To verify local constancy we begin by making the following definition. Let us fix a deformation datum $\mathfrak{d} = (\mathcal{H}, [b], \boldsymbol{\mu})$ of Hodge type. Let us also fix a faithful embedding $\rho : \mathcal{H} \hookrightarrow \mathrm{GL}(\Lambda)$ of $\mathcal{H}$ for which F_b stabilizes Λ and let s_α denote the set of associated tensors on $\mathbb{D}(\overline{X}(\mathfrak{d}))$. Let us say that an automorphism φ of $\overline{X}(\mathfrak{d})$ is emphadapted to $\mathfrak{d}$ if it fixes the tensors s_α on $\mathbb{D}(\overline{X}(\mathfrak{d}))$. Note then that φ acts on $\mathscr{D}_{K_p}(\mathfrak{d})$ for every compact open subgroup $K_p \subseteq \mathcal{G}(\mathbb{Z}_p)$. since it acts on $T(\mathscr{X})$ in a way that preserves $z_\alpha^{\text{ét}}$.

We then have the following adaptation of a lemma of Scholze:

Lemma 4.2.1. *Let $K_p \subseteq \mathcal{H}(\mathbb{Z}_p)$ be compact open. Then, there exists some $m \geqslant 1$ such that if φ is an adapted automorphism of $\overline{X}(\mathfrak{d})$ that acts trivially on $\overline{X}(\mathfrak{d})[p^m]$ then φ acts trivially on $H^i(\mathscr{D}_{K_p}(\mathfrak{d})_{\mathrm{Spa}(\mathbb{C}_p)}, \overline{\mathbb{Q}_\ell})$ for all $i \geqslant 0$.*

Proof. The proof is exactly the same as in [Sch13c, Proposition 3.15]. $\square$

With this we can prove the local constancy of ϕ_{τ,g_p} in the Hodge type case much as in [Sch13c]:

Lemma 4.2.2. *The function ϕ_{τ,g_p} is locally constant and compactly supported.*

Proof. Let us note that $\mathrm{Supp}(\phi_{\tau,g_p}) \subseteq \mathcal{H}(\mathcal{O}_{E_j})\mu^+(p)\mathcal{H}(\mathcal{O}_{E_j}) =: S$. Since this latter set is compact it's clear that it suffices to show that ϕ_{τ,g_p} is locally constant. Since S is closed (being compact) it's clear that if $b \notin S$ then $\phi_{\tau,g_p}(b) = 0$ and there is a neighborhood of b in $G(\mathbb{Q}_{p^j})$ for which ϕ_{τ,g_p} vanishes simultaneously. Thus, we restrict our attention to the case when $b \in S$.

Let us first assume that $(\mathcal{H}, \boldsymbol{\mu})$ is a deformation pre-datum of Hodge type. Let's fix $b \in S$ and let's fix a a faithful representation $\rho : \mathcal{H} \hookrightarrow \mathrm{GL}(\Lambda)$ for which $\Lambda^\vee \otimes_{\mathbb{Z}_p} \mathcal{O}_{E_j}$ is F_b-stable. Let us begin noting that the map $c_b : H(\breve{\mathbb{Q}}_p) \to H(\breve{\mathbb{Q}}_p)$ given by $g \mapsto gb\sigma(g)^{-1}$ is open (e.g. see [Sch13c, Lemma 4.4]). Let us set U_m to be $c_b(\ker(\mathcal{G}(\breve{\mathbb{Z}}_p) \to \mathcal{G}(W_m(\overline{\mathbb{F}_p})))$ which is an open neighborhood of b. Let us note that any $b' \in U$ has the property that it stabilizes $\Lambda^\vee \otimes_{\mathbb{Z}_{p^j}} \mathcal{O}_{E_j}$ since this can be checked over $\breve{\mathbb{Z}}_p$ where it follows from the fact that b' and b are σ-conjugate. Moreover, it's clear that the p-divisible groups with Tate tensors $(X(b), (s_\alpha))$ and $(X(b'), (s_\alpha))$ are isomorphic over $\overline{\mathbb{F}_p}$, and thus over some finite extension $\mathbb{F}_{p^s}$ of the residue field of $\mathcal{O}_{E_j}$. From this we immediately deduce that $\mathscr{D}_{K_p}(\mathfrak{p}([b])$ and $\mathscr{D}_{K_p}(\mathfrak{p}([b'])$ become isomorphic over $\mathrm{Spa}(\breve{\mathbb{Q}}_p)$ equivariant for the $W_{\mathbb{Q}_p} \times (K_0/K_p)$-actions on both sides. Let us choose m large enough so that Lemma 4.2.1 applies to $(X(b)_{\mathbb{F}_{p^s}}, (s_\alpha))$. Note then that for any $\tau \in W_{E_j}$ the differing actions of τ on $(X(b)_{\mathbb{F}_{p^s}}, (s_\alpha))$ and $(X(b')_{\mathbb{F}_{p^s}}, (s_\alpha)$ differ by an automorphism of $(X(b)_{\mathbb{F}_{p^s}}, (s_\alpha))$ that acts trivially on $X(b)[p^m]$. Thus, by this lemma we see that differing actions of τ coincide and thus we deduce that $\phi_{\tau,g_p}(b) = \phi_{\tau,g_p}(b')$ for all $b' \in U_m$.

Let us suppose now that $\mathfrak{p}$ is an adjoint deformation pre-datum of good abelian type. Let us take $[b]$ adapted to $\mathfrak{p}$ and take a good Hodge type datum $(\mathcal{H}_1, [b_1], \boldsymbol{\mu}_1)$ associated to $(\mathcal{H}, [b], \boldsymbol{\mu})$. Let $g_p' \in H_1(E_j)$ lie over g_p. Let U be a neighborhood of b_1 in $H(E_j)$ such that $\phi_{\tau,g_p'}$ is constant and let V be the product of $f(U \cap H_1^{\mathrm{der}}(\mathbb{Q}_{p^j})) \cdot O_m$ where O_m is the translation of $\ker(\mathcal{Z}(\mathcal{O}_{E_j}) \to \mathcal{Z}(\mathcal{O}_{E_j}/p^m))$ whose product with $f(U \cap H_1^{\mathrm{der}}(\mathbb{Q}_{p^j}))$ contains b and where m is sufficiently large. Then, ϕ_{τ,g_p} will be constant on V. $\square$

The last thing that we verify is that the function ϕ_{τ,g_p} takes values in $\mathbb{Q}$-independent of ℓ:

Proposition 4.2.3. *The function ϕ_{τ,g_p} takes values in $\mathbb{Q}$ independent of ℓ (with $\ell \neq p$).*

Proof. We need to show that the trace of $\tau \times g_p$ on $H^*_{\text{ét}}(\mathscr{D}_{K_p}(\mathfrak{d})_{\mathbb{C}_p}, \overline{\mathbb{Q}_\ell})$ takes values in $\mathbb{Q}$ independent of ℓ. Note that, as in the proof of [Sch13c, Proposition 4.2] we can twist $\mathscr{D}_{K_p}(\mathfrak{d})_{\mathbb{C}_p}$ by the Galois action $\mathrm{Gal}(\overline{E_j}/E_j)$ to obtain a different rigid analytic variety U over $\mathrm{Spa}(E_j)$ with the property that there is an isomorphism $U_{\mathbb{C}_p} \cong \mathscr{D}_{K_p}(\mathfrak{d})_{\mathbb{C}_p}$ which transports the τ action on $U_{\mathbb{C}_p}$ to the $\tau \times g_p$ action on $\mathscr{D}_{K_p}(\mathfrak{d})_{\mathbb{C}_p}$. Thus, it suffices to show that the trace of τon $H^*(U_{\mathbb{C}_p}, \overline{\mathbb{Q}_\ell})$ is an element of $\mathbb{Q}$ independent of the prime $\ell \neq p$.

In the proof of [Sch13c, Proposition 4.2] Scholze uses [Mie07, Theorem 7.1.6] by knowing that his rigid space has 'controlled cohomology' in his parlance. Now, while a priori U does not have controlled cohomology, so that the results of [Mie07] cannot be directly cited, because the cohomology groups $H^i_{\text{ét}}(U_{\mathbb{C}_p}, \overline{\mathbb{Q}_\ell})$ are finite-dimensional one can reduce to the result of [Mie07]. Namely, since U is weakly Stein we can take an increasing union of affinoid open subsets U_n whose union is n. We then know that $H^*_{\text{ét}}(U_{\mathbb{C}_p}, \overline{\mathbb{Q}_\ell})$ is $\varprojlim H^*((U_n)_{\mathbb{C}_p}, \overline{\mathbb{Q}_\ell})$ (e.g. see the discussion in the proof of [Han17, Corollary 3.7]). This isomorphism is τ-equivariant, as well as are the transition maps $H^*((U_{n+1})_{\mathbb{C}_p}, \overline{\mathbb{Q}_\ell}) \to H^*((U_n)_{\mathbb{C}_p}, \overline{\mathbb{Q}_\ell})$. Note then that since each of the spaces $H^*((U_n)_{\mathbb{C}_p}, \overline{\mathbb{Q}_\ell})$ is finite dimensional, and has rational τ-trace independent of ℓ (by [Mie07, Theorem 7.1.6]) the result also holds for $H^*(U_{\mathbb{C}_p}, \overline{\mathbb{Q}_\ell})$ from which the conclusion follows. $\square$

4.3 The function $\phi_{\tau,h}$

We now extend the function ϕ_{τ,g_p} to a function $\phi_{\tau,h}$ where $h \in C_c^\infty(\mathcal{H}(\mathbb{Z}_p), \mathbb{Q})$. Let us fix a $\mathbb{Q}$-valued Haar measure on $\mathcal{H}(\mathbb{Z}_p)$ with total mass 1. Namely, let us take a compact open normal subgroup K_p of $\mathcal{H}(\mathbb{Z}_p)$ under which h is bi-invariant. We know then that we can express h as

$$h = \mu(K_p) \sum_{K_p g K_p \in K_p \backslash \mathcal{H}(\mathbb{Z}_p)/K_p} e(g, K_p) \tag{4.3}$$

(where recall our definition of $e(K_p, g_i)$ from (2.45)) where, of course, only finitely many of these terms is non-zero. We then set

$$\phi_{\tau,h} := \mu(K_p) \sum_{K_p g K_p \in K_p \backslash \mathcal{H}(\mathbb{Z}_p)/K_p} \operatorname{tr}(\tau \times g \mid H^*(\mathscr{D}_{K_p}(\mathfrak{d})_{\mathbb{C}_p}, \overline{\mathbb{Q}_\ell})) \tag{4.4}$$

where ϕ_{τ,g_p}, again, is short for the test function associated to $(\mathcal{H}, \boldsymbol{\mu}, \tau, g_p, K_p)$.

We then have the following basic observation:

Proposition 4.3.1. *the function $\phi_{\tau,h} : H(E_j) \to \overline{\mathbb{Q}_\ell}$ is a well-defined (independent of all choices) function taking values in $\mathbb{Q}$ independent of $\ell \neq p$ and which is locally constant and compactly supported.*

Proof. The only thing that needs to be verified is that $\phi_{\tau,h}$ is independent of the choices made in its definition. It clearly suffices to show that if we take some smaller compact open normal subgroup $K_p' \subseteq K_p$ then the two sums

$$\mu(K_p) \sum_{K_p g K_p \in K_p \backslash \mathcal{H}(\mathbb{Z}_p)/K_p} h(g) \operatorname{tr}(\tau \times g \mid H^*(\mathscr{D}_{K_p}(\mathfrak{d})_{\mathbb{C}_p}, \overline{\mathbb{Q}_\ell})) \tag{4.5}$$

and

$$\mu(K_p') \sum_{K_p' g' K_p' \in K_p' \backslash \mathcal{H}(\mathbb{Z}_p)/K_p'} h(g') \operatorname{tr}(\tau \times g' \mid H^*(\mathscr{D}_{K_p'}(\mathfrak{d})_{\mathbb{C}_p}, \overline{\mathbb{Q}_\ell})) \tag{4.6}$$

are equal. Namely, since $\mathscr{D}_{K_p'}(\mathfrak{d}) \to \mathscr{D}_{K_p}(\mathfrak{d})$ is a Galois cover with covering group K_p/K_p' this is clear. $\square$

4.4 The expected trace formula

We now would like to state the expected formula that one is able to obtain by putting together the material from this section, together with §1. In a future update to this article, once a preprint of [KSZ] has been made public, we will include a fully correct formula with proof.

Let us assume that (G, X) is a Shimura datum of abelian type with G^{der} simply connected with reflex field E. Let us fix the following data:

- Let us fix an algebraic $\overline{\mathbb{Q}_\ell}$-representation ξ of G which is adapted to $\mathcal{N}(G)$ and let $\mathcal{F}_\xi$ be the lisse $\overline{\mathbb{Q}_\ell}$-sheaf on the $G(\mathbb{A}_f)$-scheme $\mathsf{Sh}(G, X)$ over E.
- Let us fix a prime p such that $G_{\mathbb{Q}_p}$ is unramified, choose a reductive model $\mathcal{G}$ of $G_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$, and let $K_0 \subseteq G(\mathbb{Q}_p)$ be the hyperspecial subgroup associated to $\mathcal{G}$.
- Let us fix functions $f^p \in C_c^\infty(G(\mathbb{A}_f^p), \overline{\mathbb{Q}_\ell})$ and $h \in C_c^\infty(K_0, \mathbb{Q})$.
- Let us fix a place v of E dividing p and let $\tau \in W_{E_v}$ and set $j := v(\tau)$.
- Let $\phi_{\tau,h}$ be the test function associated to the data $(\mathcal{G}, \boldsymbol{\mu}, \tau, h)$ as in §4.3.

We then have the following expected formula:

Theorem 4.4.1 (Expected result)**.** *Let notation be as above. There exists a $j_0 \geqslant 0$, depending only on $f^p h$, such that if $j > j_0$ then*

$$\mathrm{tr}(\tau \times f^p h \mid H_c^*(\overline{\mathsf{Sh}(G, X)}, \overline{\mathcal{F}_\xi})) = \sum_{\substack{(\gamma_0,\gamma,\delta) \\ \alpha(\gamma_0,\gamma,\delta)=1}} c(\gamma_0, \gamma, \delta)\mathrm{tr}(\xi(\gamma_0))O_\gamma(f^p)TO_\delta(\phi_{\tau,h}) \tag{4.7}$$

Moreover, if G^{ad} is $\mathbb{Q}$-anisotropic then we may take $j_0 = 1$.

Let us briefly indicate what the terms in the above formula are, leaving the details to a future update of this article. Here the set $(\gamma_0, \gamma, \delta)$ ranges over the set of so-called *punctual Kottwitz triples* for (G, X) as orginally defined in [Kot90, §2]. The object $\alpha(\gamma_0, \gamma, \delta)$ associated to a punctual Kottwitz triple $(\gamma_0, \gamma, \delta)$ is called the *Kottwitz invariant* and whose definition can be found in loc. cit. The term $c(\gamma_0, \gamma, \delta)$ is a constant term roughly corresponding to a volume factor, as well as the measurement of the difference in local-to-global principles for G and (an inner form) of the centeralizer Z_{γ_0} (e.g. see [Kot90, §3]).

Let us also briefly indicate the method of proof for this expected formula or, rather, what needs to change from the work contained in [KSZ]. In [KSZ] the authors prove a formula similar to (4.7) in the case when $h = \mathbb{1}_{K_0}$ by, in some sense, using a version of Theorem 2.2.4 in the good reduction case and the work towards proving the Langlands-Rapoport conjecture by Kisin in [Kis17] and by explicitly identifying $\phi_{\tau,h}$ to be the indicator function on $\mathcal{G}(\mathcal{O}_j)$. The proof of Theorem 4.4.1 will then essentially proceed by using Theorem 2.2.4 for a general level $K_p \subseteq K_0$ together with Kisin's work on the Langlands-Rapoport conjecture—the main work for this latter part already being performed in [KSZ]. This is all outlined, in more detail, in the introduction.

Bibliography

[AG91] James Arthur and Stephen Gelbart, *Lectures on automorphic l-functions*, L-functions and arithmetic (Durham, 1989) **153** (1991), 1–59.

[BB66] Walter L Baily and Armand Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Annals of mathematics (1966), 442–528.

[BC83] Armand Borel and William Casselman, *l^2-cohomology of locally symmetric manifolds of finite volume*, Duke Mathematical Journal **50** (1983), no. 3, 625–647.

[BC09] Olivier Brinon and Brian Conrad, *Cmi summer school notes on p-adic hodge theory.*

[Ber93] Vladimir G Berkovich, *Étale cohomology for non-archimedean analytic spaces*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **78** (1993), no. 1, 5–161.

[Ber96] ______, *Vanishing cycles for formal schemes. ii*, Inventiones mathematicae **125** (1996), no. 2, 367–390.

[Ber15] ______, *Finiteness theorems for vanishing cycles of formal schemes*, Israel Journal of Mathematics **210** (2015), no. 1, 147–191.

[BGR84] Siegfried Bosch, Ulrich Güntzer, and Reinhold Remmert, *Non-archimedean analysis, volume 261 of grundlehren der mathematischen wissenschaften [fundamental principles of mathematical sciences]*, 1984.

[BH06] Colin J Bushnell and Guy Henniart, *The local langlands conjecture for gl (2)*, vol. 335, Springer Science & Business Media, 2006.

[Bha17] BHARGAV Bhatt, *Math 679: Perfectoid spaces*, Notes by M. Stevenson. url: http://www-personal. umich. edu/˜ stevmatt/perfectoid2. pdf (2017).

[BL93] Siegfried Bosch and Werner Lütkebohmert, *Formal and rigid geometry*, Mathematische Annalen **295** (1993), no. 1, 291–317.

[BLR12] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, vol. 21, Springer Science & Business Media, 2012.

[Bor72] Armand Borel, *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem*, Journal of Differential Geometry **6** (1972), no. 4, 543–560.

[Bor98] Mikhail Borovoi, *Abelian galois cohomology of reductive groups*, no. 626, American Mathematical Soc., 1998.

[Bor12] Armand Borel, *Linear algebraic groups*, vol. 126, Springer Science & Business Media, 2012.

[BR85] Peter Bardsley and Roger W Richardson, *Étale slices for algebraic transformation groups in characteristic p*, Proceedings of the London Mathematical Society **3** (1985), no. 2, 295–317.

[BR94] Don Blasius and Jonathan D Rogawski, *Zeta functions of shimura varieties*, Motives (Seattle, WA, 1991) **55** (1994), no. Part 2, 525–571.

[BT65] Armand Borel and Jacques Tits, *Groupes réductifs*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **27** (1965), no. 1, 55–150.

[BT72] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. (1972), no. 41, 5–251. MR 0327923

[BT84] ———, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Inst. Hautes Études Sci. Publ. Math. (1984), no. 60, 197–376. MR 756316

[Car79] Pierre Cartier, *Representations of p-adic groups: a survey*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part, vol. 1, 1979, pp. 111–155.

[CGP15] Brian Conrad, Ofer Gabber, and Gopal Prasad, *Pseudo-reductive groups*, vol. 26, Cambridge University Press, 2015.

[Con99] Brian Conrad, *Irreducible components of rigid spaces*, ANNALES-INSTITUT FOURIER, vol. 49, Association des annales de l'institut Fourier, 1999, pp. 473–541.

[Con07] ———, *Deligne's notes on Nagata compactifications*, J. Ramanujan Math. Soc. **22** (2007), no. 3, 205–257.

[Con08] ———, *Several approaches to non-archimedean geometry*, p-adic geometry, 2008, pp. 9–63.

[Con11] ———, *Reductive group schemes*, Autour des schémas en groupes **1** (2011), 93–444.

[Con12] ———, *Weil and grothendieck approaches to adelic points*, Enseign. Math.(2) **58** (2012), no. 1-2, 61–97.

[Del71a] Pierre Deligne, *Théorie de hodge, ii*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **40** (1971), no. 1, 5–57.

[Del71b] ———, *Travaux de shimura*, Séminaire Bourbaki vol. 1970/71 Exposés 382–399, Springer, 1971, pp. 123–165.

[Del79] ———, *Variétés de shimura: interprétation modulaire, et techniques de construction de modeles canoniques*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part, vol. 2, 1979, pp. 247–289.

[Del82] ———, *Hodge cycles on abelian varieties*, Hodge cycles, motives, and Shimura varieties, Springer, 1982, pp. 9–100.

[DG70] Michel Demazure and Alexandre Grothendieck, *Schémas en groupes. tome iii, structure des schémas en groupes réductifs: Séminaire du bois marie, 1962/64, sga 3*.

[dJ95] Arthur J de Jong, *Crystalline dieudonné module theory via formal and rigid geometry*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **82** (1995), no. 1, 5–96.

[DR73] Pierre Deligne and Michael Rapoport, *Les schémas de modules de courbes elliptiques*, Modular functions of one variable II, Springer, 1973, pp. 143–316.

[EvdGM] Bas Edixhoven, Gerard van der Geer, and Ben Moonen, *Abelian varieties*, Book in preparation.

[Fal99] Gerd Faltings, *Integral crystalline cohomology over very ramified valuation rings*, Journal of the American Mathematical Society **12** (1999), no. 1, 117–144.

[Far04] Laurent Fargues, *Cohomologie des espaces de modules de groupes p-divisibles et correspondances de langlands locales*, Astérisque (2004), no. 291, 1–200.

[FC13] Gerd Faltings and Ching-Li Chai, *Degeneration of abelian varieties*, vol. 22, Springer Science & Business Media, 2013.

[FK13] Eberhard Freitag and Reinhardt Kiehl, *Etale cohomology and the weil conjecture*, vol. 13, Springer Science & Business Media, 2013.

[Fla79] D Flath, *Decomposition of representations into tensor products*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part, vol. 1, 1979, pp. 179–183.

[Fuj97] Kazuhiro Fujiwara, *Rigid geometry, lefschetz-verdier trace formula and deligne's conjecture*, Inventiones mathematicae **127** (1997), no. 3, 489–533.

[Gel96] Stephen S Gelbart, *Lectures on the arthur-selberg trace formula*, vol. 9, American Mathematical Soc., 1996.

[Gor02] Eyal Zvi Goren, *Lectures on hilbert modular varieties and modular forms*, no. 14, American Mathematical Soc., 2002.

[Han17] David Hansen, *Artin vanishing in rigid analytic geometry*, arXiv preprint arXiv:1708.07276 (2017).

[Hel01] Sigurdur Helgason, *Differential geometry and symmetric spaces*, vol. 341, American Mathematical Soc., 2001.

[Hel13] Eugen Hellmann, *On arithmetic families of filtered varphi-modules and crystalline representations*, Journal of the Institute of Mathematics of Jussieu **12** (2013), no. 4, 677–726.

[Hen16] Timo Henkel, *A comparison of adic spaces and berkovich spaces*, arXiv preprint arXiv:1610.04117 (2016).

[Hid12] Haruzo Hida, *p-adic automorphic forms on shimura varieties*, Springer Science & Business Media, 2012.

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple shimura varieties.(am-151)*, vol. 151, Princeton university press, 2001.

[Hub93a] Roland Huber, *Bewertungsspektrum und rigid geometrie*, Regensburger Mathematische Schriften 23 (1993).

[Hub93b] Roland Huber, *Continuous valuations*, Mathematische Zeitschrift **212** (1993), no. 1, 455–477.

[Hub94] ______, *A generalization of formal schemes and rigid analytic varieties*, Mathematische Zeitschrift **217** (1994), no. 1, 513–551.

[Hub96] ______, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, vol. 30, Vieweg+Teubner Verlag, 1996.

[Hub98] ______, *A finiteness result for direct image sheaves on the étale site of rigid analytic varieties*, Journal of Algebraic Geometry **7** (1998), no. 2, 359.

[Jan04] Jens Carsten Jantzen, *Nilpotent orbits in representation theory*, Lie theory, Springer, 2004, pp. 1–211.

[KH17] Kiran S Kedlaya and assisted by David Hansen, *Sheaves, stacks, and shtukas*.

[Kim13] Wansu Kim, *Rapoport-zink spaces of hodge type*, arXiv preprint arXiv:1308.5537 (2013).

[Kim14] ______, *The relative breuil–kisin classification of p-divisible groups and finite flat group schemes*, International Mathematics Research Notices **2015** (2014), no. 17, 8152–8232.

[Kim18] ______, *Rapoport–zink uniformization of hodge-type shimura varieties*, Forum of Mathematics, Sigma, vol. 6, Cambridge University Press, 2018.

[Kis10] Mark Kisin, *Integral models for shimura varieties of abelian type*, Journal of The American Mathematical Society **23** (2010), no. 4, 967–1012.

[Kis17] ______, *Mod p points on shimura varieties of abelian type*, Journal of the American Mathematical Society **30** (2017), no. 3, 819–914.

[KL13] Kiran S Kedlaya and Ruochuan Liu, *Relative p-adic hodge theory: Foundations*, arXiv preprint arXiv:1301.0792 (2013).

[KL16] ______, *Relative p-adic hodge theory, ii: Imperfect period rings*, arXiv preprint arXiv:1602.06899 (2016).

[Kot84] Robert E Kottwitz, *Shimura varieties and twisted orbital integrals*, Mathematische Annalen **269** (1984), no. 3, 287–300.

[Kot85] ______, *Isocrystals with additional structure*, Compositio Math **56** (1985), no. 2, 201–220.

[Kot90] ______, *Shimura varieties and-adic representations in automorphic forms, shimura varieties, and l-functions, vol. i (ann arbor, mi, 1988) perspect*, 1990.

[Kot92a] ______, *On the λ-adic representations associated to some simple shimura varieties*, Inventiones mathematicae **108** (1992), no. 1, 653–665.

[Kot92b] ______, *Points on some shimura varieties over finite fields*, Journal of the American Mathematical Society **5** (1992), no. 2, 373–444.

[KSZ] Mark Kisin, SugWoo Shin, and Yihang Zhu, *The stable trace formula for certain shimura varieties of abelian type*.

[KW13] Reinhardt Kiehl and Rainer Weissauer, *Weil conjectures, perverse sheaves and l'adic fourier transform*, vol. 42, Springer Science & Business Media, 2013.

[Lan76] Robert Langlands, *Some contemporary problems with origins in the jugendtraum*, Proceedings of symposia in pure mathematics, vol. 28, 1976, p. 401.

[Lan77] Robert P Langlands, *Shimura varieties and the selberg trace formula*, Canadian Journal of Mathematics **29** (1977), no. 6, 1292–1299.

[Lan79a] ______, *Automorphic representations, shimura varieties, and motives. ein märchen*, Proc. Symp. Pure Math, vol. 33, 1979, pp. 205–246.

[Lan79b] ______, *On the zeta-functions of some simple shimura varieties*, Canadian Journal of Mathematics **31** (1979), no. 6, 1121–1216.

[Lan82] ______, *Les débuts d'une formule des traces stable*, UER de mathématiques & LA 212 du CNRS, 1982.

[Lan12] Kai-Wen Lan, *Comparison between analytic and algebraic constructions of toroidal compactifications of pel-type shimura varieties*, Journal für die reine und angewandte Mathematik (Crelles Journal) **2012** (2012), no. 664, 163–228.

[Lan13] ______, *Arithmetic compactifications of pel-type shimura varieties*, no. 36, Princeton University Press, 2013.

[Lou17] João NP Lourenço, *The riemannian hebbarkeitss\" atze for pseudorigid spaces*, arXiv preprint arXiv:1711.06903 (2017).

[LR87] Robert P Langlands and Michael Rapoport, *Shimuravarietäten und gerben.*, Journal für die reine und angewandte Mathematik **378** (1987), 113–220.

[LS17] Kai-wen Lan and Benoît Stroh, *Nearby cycles of automorphic Étale sheaves, ii.*

[LS18] Kai-Wen Lan and Benoît Stroh, *Compactifications of subschemes of integral models of shimura varieties*, Forum of Mathematics, Sigma, vol. 6, Cambridge University Press, 2018.

[Man05] Elena Mantovan, *On the cohomology of certain pel-type shimura varieties*, Duke Mathematical Journal **129** (2005), no. 3, 573–610.

[MFK94] David Mumford, John Fogarty, and Frances Kirwan, *Geometric invariant theory*, vol. 34, Springer Science & Business Media, 1994.

[Mie07] Yoichi Mieda, *On ℓ-independence for the étale cohomology of rigid spaces over local fields*, Compositio Mathematica **143** (2007), no. 2, 393–422.

[Mie14] ______, *Zelevinsky involution and l-adic cohomology of the rapoport-zink tower*, arXiv preprint arXiv:1401.5135 (2014).

[Mil90] James S Milne, *Canonical models of (mixed) shimura varieties and automorphic vector bundles*, Automorphic forms, Shimura varieties, and L-functions **1** (1990), 283–414.

[Mil92] ______, *The points on a shimura variety modulo a prime of good reduction*, The zeta functions of Picard modular surfaces **151253** (1992).

[Mil94] JS Milne, *Shimura varieties and motives*, Proc. Symp. Pure Math, vol. 55, 1994, pp. 447–523.

[Mil04] James S Milne, *Introduction to shimura varieties*, Harmonic analysis, the trace formula, and Shimura varieties **4** (2004), 265–378.

[Mil11] James Stuart Milne, *Shimura varieties and moduli*, arXiv preprint arXiv:1105.0887 (2011).

[Mil17] James S Milne, *Algebraic groups: The theory of group schemes of finite type over a field*, vol. 170, Cambridge University Press, 2017.

[Moo98] Ben Moonen, *Models of shimura varieties in mixed characteristics*, London Mathematical Society Lecture Note Series (1998), 267–350.

[MP15] Keerthi Madapusi Pera, *The Tate conjecture for K3 surfaces in odd characteristic*, Invent. Math. **201** (2015), no. 2, 625–668.

[MRM74] David Mumford, Chidambaran Padmanabhan Ramanujam, and Jurij Ivanovič Manin, *Abelian varieties*, vol. 108, Oxford university press Oxford, 1974.

[MS03] George J McNinch and Eric Sommers, *Component groups of unipotent centralizers in good characteristic*, Journal of Algebra **260** (2003), no. 1, 323–337.

[New78] Peter E Newstead, *Introduction to moduli problems and orbit spaces, volume 51 of tata institute of fundamental research lectures on mathematics and physics*, Tata Institute of Fundamental Research, Bombay (1978).

[Ols16] Martin Olsson, *Algebraic spaces and stacks*, vol. 62, American Mathematical Soc., 2016.

[Pau04] Frédéric Paugam, *Galois representations, mumford-tate groups and good reduction of abelian varieties*, Mathematische Annalen **329** (2004), no. 1, 119–160.

[Per12] Keerthi Madapusi Pera, *Toroidal compactifications of integral models of shimura varieties of hodge type*, arXiv preprint arXiv:1211.1731 (2012).

[Per16] ———, *Integral canonical models for spin shimura varieties*, Compositio Mathematica **152** (2016), no. 4, 769–824.

[Pin90] Richard Pink, *Arithmetical compactification of mixed shimura varieties*, Rheinischen Friedrich-Wilhelms-Universitat Bonn, 1990.

[Poo17] Bjorn Poonen, *Rational points on varieties*, vol. 186, American Mathematical Soc., 2017.

[PR92] Vladimir Petrovich Platonov and Andrei Stepanovich Rapinchuk, *Algebraic groups and number theory*, Russian Mathematical Surveys **47** (1992), no. 2, 133–161.

[Ray74] Michel Raynaud, *Géométrie analytique rigide d'apres tate, kiehl*, Table ronde d'analyse non archimédienne (Paris, 1972) (1974), 319–327.

[Riz10] Jordan Rizov, *Kuga-Satake abelian varieties of K3 surfaces in mixed characteristic*, J. Reine Angew. Math. **648** (2010), 13–67.

[RV14] Michael Rapoport and Eva Viehmann, *Towards a theory of local shimura varieties*, arXiv preprint arXiv:1401.2849 (2014).

[Sch11] Peter Scholze, *The langlands–kottwitz approach for the modular curve*, International Mathematics Research Notices **2011** (2011), no. 15, 3368–3425.

[Sch12] ———, *Perfectoid spaces*, Publications mathématiques de l'IHÉS **116** (2012), no. 1, 245–313.

[Sch13a] ______, *p-adic hodge theory for rigid-analytic varieties*, Forum of Mathematics, Pi, vol. 1, Cambridge University Press, 2013.

[Sch13b] ______, *The langlands-kottwitz approach for some simple shimura varieties*, Inventiones mathematicae **192** (2013), no. 3, 627–661.

[Sch13c] ______, *The langlands-kottwitz method and deformation spaces of p-divisible groups*, Journal of the American Mathematical Society **26** (2013), no. 1, 227–259.

[Sch13d] ______, *The local langlands correspondence for gl n over p-adic fields*, Inventiones mathematicae **192** (2013), no. 3, 663–715.

[Sch13e] ______, *Perfectoid spaces: a survey*, arXiv preprint arXiv:1303.5948 (2013).

[Ser56] Jean-Pierre Serre, *Géométrie algébrique et géométrie analytique*, Annales de l'institut Fourier, vol. 6, Association des Annales de l'Institut Fourier, 1956, pp. 1–42.

[Ser13] ______, *Galois cohomology*, Springer Science & Business Media, 2013.

[Sha86] Steven Shatz, *Group schemes, formal groups, and p-divisible groups*, Arithmetic Geometry, Springer Verlag, 1986.

[Spr10] Tonny Albert Springer, *Linear algebraic groups*, Springer Science & Business Media, 2010.

[SS13] Peter Scholze and Sug Shin, *On the cohomology of compact unitary group shimura varieties at ramified split places*, Journal of the American Mathematical Society **26** (2013), no. 1, 261–294.

[ST68] Jean-Pierre Serre and John Tate, *Good reduction of abelian varieties*, Annals of Mathematics (1968), 492–517.

[Sta18] The Stacks Project Authors, *Stacks Project*, 2018.

[Stu17] Bernard Le Stum, *Rigid cohomology of locally noetherian schemes part 1: Geometry*, arXiv preprint arXiv:1707.02797 (2017).

[SW] P Scholze and J Weinstein, *p-adic geometry, notes from a course at berkeley.*

[SW12] Peter Scholze and Jared Weinstein, *Moduli of p-divisible groups*, arXiv preprint arXiv:1211.6357 (2012).

[Tem15] Michael Temkin, *Introduction to berkovich analytic spaces*, pp. 3–66, Springer International Publishing, 2015.

[Tit79] Jacques Tits, *Reductive groups over local fields*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part, vol. 1, 1979, pp. 29–69.

[Vak] Ravi Vakil, *The rising sea: Foundations of algebraic geometry (november 18, 2017 version)*.

[Var07] Yakov Varshavsky, *Lefschetz–verdier trace formula and a generalization of a theorem of fujiwara*, GAFA Geometric And Functional Analysis **17** (2007), no. 1, 271–319.

[Vas99] Adrian Vasiu, *Integral canonical models of shimura varieties of preabelian type*, Asian Journal of Mathematics **3** (1999), no. 2, 401–518.

[Vas16] ______, *Extension theorems for reductive group schemes*, Algebra & Number Theory **10** (2016), no. 1, 89–115.

[Wed] Torsten Wedhorn, *Adic spaces*.

[Wei17] Jared Weinstein, *Arizona winter school 2017: Adic spaces*.

Chapter 5

Appendix A: A review of adic geometry

In this first appendix we collect some foundational results concerning rigid analytic varieties for the convenience of the reader, with the appendix culminating in the recollection of formal schemes, their generic fibers, and the relationship to analytifications. We will be working with the language of Huber's theory of adic spaces and it is, of course, hopeless to try to include enough basic material about adic spaces to make this appendix fully self-contained in that regard. So, see [Hub94], [Hub96, Chapter 1], [SW], [Wed], or [Wei17] for an introduction to the subject, especially concerning the basic definitions. The goal here is, instead, to extract from [Hub96], and some other references, some more technical definitions and results that are useful to understand some of the results of this paper. We have certainly included significantly more than is strictly necessary (including many examples) with the hope that it may be useful didactic tool for the reader.

Before we begin, let us remark that we will be largely following the standard notation in [Hub96] except for the following two large exceptions. What Huber calls an 'f-adic ring' we call a *Huber ring*, and what Huber calls an 'affinoid ring' we call a *Huber pair* (this is in accordance with the vernacular used by Scholze and his collaborators). We will also assume, unless otherwise stated, that (A, A^+) is sheafy (in the sense of [SW]) which will be an essential non-issue for the type of spaces we will be dealing with. Finally, we assume, as a matter of convention, that unless stated otherwise all of our Huber pairs (A, A^+) are complete (in the sense that A is complete for its topology), so that if $X = \mathrm{Spa}(A, A^+)$ then

$$\mathcal{O}_X(X) = A, \qquad \mathcal{O}_X^+(X) = A^+ \tag{5.1}$$

Let us also shorten $\mathrm{Spa}(A, A^\circ)$ to $\mathrm{Spa}(A)$ for a Huber ring A.

Remark 5.0.1. As a final remark, we note that while we are *a priori* working in the category of general (honest) adic spaces (meaning objects of the category (V) locally isomorphic to $\mathrm{Spa}(A, A^+)$ where (A, A^+) is a sheafy Huber pair), in practice all of our results will be focused on the case of rigid analytic varieties or locally Noetherian formal schemes. Thus, one should assume that all of our stated results are stated in the context of the caveat given in [Hub96, (1.1.1), Page 37].

5.1 Initial definitions and examples

For an adic space X, let us denote by $\mathcal{O}_X$ its structure sheaf, $\mathcal{O}_X^+$ its sheaf of integral elements, and v_x the valuation on $\mathcal{O}_{X,x}$. We recall that a morphism of adic spaces $X \to Y$ can be thought of as a map of locally topologically ringed spaces

$$f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \tag{5.2}$$

such that the morphism $\mathcal{O}_Y \to f_* \mathcal{O}_X$ sends $\mathcal{O}_Y^+$ into $f_* \mathcal{O}_X^+$ and the resulting map $\mathcal{O}_Y^+ \to f_* \mathcal{O}_X^+$ is local (see [Wed, Lemma 8.14]).

We will denote by Adic the category of adic spaces, and by Adic_S the category of adic spaces over the fixed adic space S, which we shorten to Adic_R if $S = \mathrm{Spa}(R)$. Thus, if $S = \mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$, the final adic space, then $\mathsf{Adic}_S = \mathsf{Adic}$. Finally, we will denote by $\mathsf{AffAdic}_S$ the category of S-adic spaces which are affinoid.

Now, while we won't be discussing much of the basic theory of adic spaces, especially the foundational but difficult properties of Huber rings and $\mathrm{Spa}(A, A^+)$ (as in [Hub93b] and [Hub94]), we do mention the following pivotal theorem of Huber which underlies a lot of the theory, and which will be useful to us later on:

Lemma 5.1.1. *Let A be a (possibly non-complete) Huber ring. Then, there is a bijection*

$$\{\text{Open subgroups of } A\} \to \{\text{Open subgroups of } \widehat{A}\} \tag{5.3}$$

given by $G \mapsto \widehat{G}$ which restricts to a bijection on open subrings. Moreover, under this bijection the following holds:

1. $\widehat{A^\circ} = (\widehat{A})^\circ$.
2. $\widehat{A^{\circ\circ}} = (\widehat{A})^{\circ\circ}$.
3. *This bijection induces a bijection on rings of definition of A and rings of definition of $\widehat{A}$.*
4. *This bijection induces a bijection on rings of integral elements of A and rings of integral elements of $\widehat{A}$.*

Proof. See [Hub93a, Theorem 2.4.3]. □

While this lemma is implicit in a lot of Huber's work (e.g. the fact that $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is a Huber pair for an adic space X and an open subset U of X) it will be useful to us mainly later on in verifying some basic properties of adic spaces.

So, now as is standard in algebraic geometry, we will often times think of elements of Adic_S via the fully faithful Yoneda embedding

$$\mathsf{Adic}_S \hookrightarrow \mathbf{PSh}(\mathsf{Adic}_S) \tag{5.4}$$

given by sending an adic space X over S to the presheaf it represents. We will denote this presheaf by X as well. Of course, since the presheaf X is actually a sheaf for the big Zariski site on S (i.e. the site for which covers are usual open covers) and every adic space can be written as a (possibly non-filtered) colimit of affinoid adic spaces, we actually obtain an embedding

$$\mathsf{Adic}_S \hookrightarrow \mathbf{PSh}(\mathsf{AffAdic}_S) \tag{5.5}$$

In short, an adic space X over S is determined by its functorial values $X(\mathrm{Spa}(R, R^+))$ as $\mathrm{Spa}(R, R^+)$ travels over affinoid adic spaces over S. Let us shorten $X(\mathrm{Spa}(R, R^+))$ to $X(R, R^+)$ and let us shorten $X(\mathrm{Spa}(R))$ to $X(R)$.

Let us next recall that an *analytic field* E is a complete topological field whose topology is induced by a rank 1 valuation. We then note that (E, E°) is a Huber pair, and since E is a strongly Noetherian Tate ring (in the sense of [Hub94]) we know that it's sheafy. Thus $\mathrm{Spa}(E)$ is an adic space, consisting of a single point given by the rank 1-valuation inducing the topology of E.

Let us also, for notational convenience, fix π to be a pseudouniformizer (as in the language of [SW]) of E, so that $\pi E^\circ = E^{\circ\circ}$. Moreover, let's fix k to be the residue field $E^\circ/\pi E^\circ$ of E°.

Example 5.1.2. If $E/\mathbb{Q}_p$ is finite, then E is an analytic field. So is $\breve{E} := \widehat{E^{\mathrm{ur}}}$ (the completion of the maximal unramified extension of E), as is $\mathbb{C}_p$. In fact, given any analytic field E, any finite extension of E is analytic, as is the field $\mathbf{C}_E := \widehat{\overline{E}}$.

Note though that $\mathbb{Q}_p^{\mathrm{ur}}$ is not analytic for it is not complete. Also, any discrete field will not be analytic since its topology comes from a valuation that is not of rank 1.

The reason why non-archimedean fields are so important, comes down to the following result:

Lemma 5.1.3. *Let K be a topological field whose topology is induced by a valuation. Then, either K is analytic or K is discrete.*

Let us now recall some incredibly basic examples of adic spaces that will be useful to keep in mind. In the following we will fix an analytic field E:

Example 5.1.4. There is the final object of Adic given by $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$ where $\mathbb{Z}$ is endowed with the discrete topology.

Example 5.1.5. The somewhat ur-example of an adic space over $\mathrm{Spa}(E)$ is given by the *closed unit disk* $\mathbb{B}_E := \mathrm{Spa}(E\langle t\rangle, E^\circ\langle t\rangle)$. It represents the functor on $\mathsf{AffAdic}_E$ given by $(R, R^+) \mapsto R^+$. Thus, $\mathbb{B}_E$ is a ring object in Adic_E.

The actual underlying topological space of $\mathbb{B}_E$ is quite complicated (see [Sch12, Example 2.20] for a description when E is algebraically closed). We remark though that $\mathbb{B}_E$ is connected, as it should be, given that this is the main technical aspect of naive p-adic geometry that one needs to fix. To see this connectedness assume that $\mathbb{B}_E = U \sqcup V$, for opens U and V, then the fact that $\mathcal{O} := \mathcal{O}_{\mathbb{B}_E}$ is a sheaf would imply that $E\langle t\rangle = \mathcal{O}(U) \times \mathcal{O}(V)$ and since $E\langle t\rangle$ is integral, and so has no non-trivial idempotents, this implies that one of $\mathcal{O}(U)$ or $\mathcal{O}(V)$ is zero, assume without loss of generality that $\mathcal{O}(U) = 0$. This then implies that U is empty, and so the conclusion follows. To see that U is empty note that for any affinoid adic subspace $\mathrm{Spa}(A, A^+) \subseteq U$ we get a map of Huber pairs $(\mathcal{O}(U), \mathcal{O}^+(U)) \to (A, A^+)$. Since this former pair is the zero pair, and this is a unital ring map, the latter pair is also the zero pair so that $\mathrm{Spa}(A, A^+)$ is empty. Since U is covered by such affinoids we deduce that U is empty.

Note though that $\mathbb{B}_E$ is not irreducible. One can see this explicitly in the description of the points of $\mathbb{B}_E$ where, at least when E is algebracially closed, the irreducible closed subsets correspond exactly the the closures of the points of Type 2 (in the parlance of [Sch12, Example 2.20]. We will see a concrete realization of the non-irreducibility of $\mathbb{B}_E$ later.

Of course, we can also formulate n-dimensional analogues of the above in the form of $\mathbb{B}_E^n := \mathrm{Spa}(E\langle T_1, \ldots, T_n\rangle)$ which represents the obvious functor $(R, R^+) \mapsto (R^+)^n$. This is called the *closed unit n-disk* over E.

Example 5.1.6. Perhaps the next most important adic space over E one can consider is that of the open unit disk over E. Namely, let us note that for all $n \geqslant 1$ we get natural rational subsets U_n of $\mathbb{B}_E$ given by $U_n := U\left(\dfrac{t^n}{\pi}\right)$ (using the notation for rational domains as in [Hub96]). Note that U_n is a rational domain even though $t^n E\langle t\rangle$ is not open, since $\{\pi, t^n\}E\langle t\rangle$ is open (it's all of $E\langle t\rangle$) and U_n evidently coincides with $U\left(\dfrac{\{t^n, \pi\}}{\pi}\right)$. For all $n \geqslant m \geqslant 1$ we have open embeddings $U_m \hookrightarrow U_n$ merely because if $|t^m(x)| \leqslant |\pi| < 1$ then evidently $|t^n(x)| \leqslant |t^m(x)| \leqslant |\pi|$. given by just sending T to itself.

We then set the *open unit disk* over E, denoted $\mathbb{D}_E$, to be the colimit $\varinjlim U_n$, which exists since the U_n's form an increasing chain of opens. It is not hard to check that $\mathbb{D}_E$ represents the functor on $\mathsf{AffAdic}_E$ sending (R, R^+) to $R^{\circ\circ}$. Indeed, a map $\mathrm{Spa}(R, R^+) \to \mathbb{D}_E$ corresponds to an element $r \in R^+$ such that $r^n/\pi \in R^+$ for some $n \geqslant 1$ (see the argument at the end of Example 5.1.9 below), which is clearly the same as r being in $R^{\circ\circ}$.

For the benefit of the reader, let us make the following observation. Namely, we see that, by definition of rational open subsets, one has for all $n \geqslant 1$

$$U_n = \mathrm{Spa}\left(E\langle t\rangle, (E^\circ\langle t, s\rangle/(t^n - \pi s))^\sim\right) \tag{5.6}$$

where $(-)^\sim$ denotes the integral closure in $E\langle t\rangle$. Now, if E is algebraically closed and we denote an n^{th}-root of π by $\pi^{\frac{1}{n}}$, then this integral closure is just $E^\circ\left\langle \dfrac{t}{\pi^{\frac{1}{n}}}\right\rangle$. It's evident then that there is an isomorphism $\mathbb{B}_E \xrightarrow{\approx} U_n$ given on coordinate rings by sending t to $\pi^{\frac{1}{n}}t$. But if, for example, $E = \mathbb{Q}_p$ then $E^\circ\langle t, s\rangle/(t^n - \pi s)$ is integrally closed in $E\langle t\rangle$, and thus its integral closure is itself. From this we see that U_n is not isomorphic to $\mathbb{B}_E$. Indeed, if they were isomorphic then since then we would obtain an isomorphism of E°-algebras between $E^\circ\langle t\rangle$ and $E^\circ\langle t, s\rangle/(\pi s - t^n)$ which, of course, is impossible by considering the reduction modulo π. We will see later that they are twists over $\mathbf{C}_E$.

Again, we can formulate n-dimensional analogues as $\mathbb{D}_E^n$, the *open unit n-disk* over E. One can think of it as the n-fold self fiber product of $\mathbb{D}_E$, using the notion of fiber products from the next section, or by analogizing the above procedure and writing it as $\varinjlim W_m$ where $W_m := U\left(\dfrac{\{T_1^m, \ldots, T_n^m\}}{\pi}\right)$ which is a rational open in $\mathbb{B}_E^n$. It's clear that this represents the functor sending (R, R^+) to $(R^{\circ\circ})^n$.

Example 5.1.7. The *unit circle* or *torus* over E, denoted $\mathbb{T}_E$, is the rational open subset $U\left(\frac{1}{T}\right)$ of $\mathbb{B}_E$. As an adic space over E it is the affinoid $\mathrm{Spa}(E\langle t, t^{-1}\rangle)$ (where $(E\langle t, t^{-1}\rangle)^\circ = E^\circ\langle t, t^{-1}\rangle$). One then has a functorial identification of $\mathbb{T}_E(R, R^+)$ with $(R^+)^\times$ for (R, R^+) over $\mathrm{Spa}(E)$. Indeed, a morphism $\mathrm{Spa}(R, R^+) \to \mathbb{T}_E$ over $\mathrm{Spa}(E)$ is determined by its map on Huber pairs, and this map must send t to an element of R^+ which is invertible. Conversely, it's clear that any element of R^+ which is invertible gives rise to a morphism $\mathrm{Spa}(R, R^+) \to \mathbb{T}_E$. We will make explicit why this is called a torus, in relation to the classical torus, in a later section.

We now observe that $\mathbb{T}_E \cap \mathbb{D}_E = \varnothing$ in $\mathbb{B}_E$. Indeed, there is no valuation $|\cdot|_x \in \mathrm{Spa}(E\langle t\rangle)$ such that $1 \leqslant |t(x)|$ and $|t^n(x)| \leqslant |\pi| < 1$ for some $m \geqslant 1$. Thus, we see that $\mathbb{B}_E$ is not irreducible as claimed at the end of Example 5.1.5. That said, note that this does not actually contradict the fact that $\mathbb{B}_E$ is connected since $\mathbb{T}_E \cup \mathbb{D}_E \subsetneq \mathbb{B}_E$. Namely, there are valuations $|\cdot|_x$ such that $|t^n(x)| > |\pi|$ for all $n \geqslant 1$ but $|t(x)| < 1$ (see the points of type (5) in [Sch12, Example 2.20]).

We also have n-dimensional analogues $\mathbb{T}_E^n := \mathrm{Spa}(E\langle t_1, t_1^{-1}, \dots, t_n, t_n^{-1}\rangle)$ called the *n-dimensional unit circle* or *n-dimensional torus* over E. It represents the functor sending (R, R^+) to $((R^+)^\times)^n$.

Example 5.1.8. One also has the *punctured closed disk* $\mathbb{B}_E^*$ which is given by the open adic subspace of $\mathbb{B}_E$ defined as the set of $|\cdot|_x \in \mathbb{B}_E$ such that $|t(x)| \neq 0$. Note that this is actually the complement of a single closed point of $\mathbb{B}_E$ the one given by

$$\sum_{n=0}^{\infty} a_n T^n \mapsto |a_0| \tag{5.7}$$

Note that $\mathbb{B}_E$ is not a rational open in $\mathbb{B}_E$, in fact it's not affinoid, in fact it's not even quasi-compact! Namely, it's clear that for all $n \geqslant 1$ one has that $V_n := U\left(\frac{\pi^n}{T}\right)$ is rational domain in $\mathbb{B}_E$, and that $\mathbb{B}_E^* = \varinjlim V_n$ (with the obvious inclusions for increasing n), and that there is no $n \in \mathbb{N}$ such that $\mathbb{B}_E^* = V_n$, which shows that V_n is not quasi-compact. We can similarly define the *punctured open disk* over E defined by $\mathbb{D}_E^* := \mathbb{D}_E \cap \mathbb{B}_E^*$, where the intersection is taken within $\mathbb{B}_E$.

One can see that there is a functorial identification of $\mathbb{B}_E^*(R, R^+)$ with $R^+ \cap R^\times$ and $\mathbb{D}_E^*(R, R^+)$ with $R^{\circ\circ} \cap R^\times$ for (R, R^+) over $\mathrm{Spa}(E)$. Indeed, a map $\mathrm{Spa}(R, R^+) \to \mathbb{B}_E^*$ is a morphism $\mathrm{Spa}(R, R^+) \to \mathbb{B}_E$ whose image lands in $\mathbb{B}_E^*$. A morphism $\mathrm{Spa}(R, R^+) \to \mathbb{B}_E$ corresponds to an element $r \in R^+$. The map on topological spaces then sends $|\cdot|_x$ to the valuation in $\mathbb{B}_E$ given by

$$\sum_{n=0}^{\infty} a_n T^n \mapsto \left|\sum_{n=0} a_n r^n(x)\right| \tag{5.8}$$

and so this map will send every $|\cdot|_x$ to an element of $\mathbb{B}_E^*$ if and only if $|r|_x \neq 0$ for all $|\cdot|_x \in \mathrm{Spa}(R, R^+)$ which by [Hub93b, Proposition 3.6, i)] implies that $r \in R^\times$ as desired. The same argument works for $\mathbb{D}_E^*$.

Example 5.1.9. Let us now build an affine adic line over E. We do this by gluing together closed disks of larger and larger radius. Namely, for all $n \geqslant 1$ let us denote by U_n the adic space $\mathbb{B}_E$ over $\mathrm{Spa}(E)$. Note then that we have an open embedding $U_n \hookrightarrow U_{n+1}$ given by the map on Huber pairs dictated by $t \mapsto \pi t$, which identifies U_n with $U\left(\frac{T}{\pi}\right)$ in U_{n+1}. Note then that we can form the colimit $\varinjlim U_n$ (in the category (V) as in [Hub94]) which is evidently an adic space, which we denote $\mathbb{A}_E^{1,\mathrm{an}}$ and call the *affine adic line* over E.

We note then that we can functorially identify $\mathbb{A}_E^{1,\mathrm{an}}(R,R^+)$ with R for (R,R^+) over E. Indeed, given any $r \in R$ one has, from the fact that π is a pseudouniformizer, that $\pi^n r \in R^+$ for some $n \geqslant 1$, let n_0 be the minimal such n. This then gives us a morphism $\mathrm{Spa}(R,R^+) \to U_{n_0}$ by the discussion in Example 5.1.5, and thus a morphism $\mathrm{Spa}(R,R^+) \to \mathbb{A}_E^{1,\mathrm{an}}$ by composition with the map $U_{n_0} \to \mathbb{A}_E^{1,\mathrm{an}}$. Conversely, given a morphism $\mathrm{Spa}(R,R^+) \to \mathbb{A}_E^{1,\mathrm{an}}$, the fact that $\mathrm{Spa}(R,R^+)$ is quasi-compact implies that $\mathrm{Spa}(R,R^+)$ will factor through some U_n, let n_0 be the minimal such n. Note then that the morphism $\mathrm{Spa}(R,R^+) \to U_{n_0}$ gives us an element $r_0 \in R^+$. One can then associate to the morphism $\mathrm{Spa}(R,R^+) \to \mathbb{A}_E^{1,\mathrm{an}}$ the element $\frac{r_0}{\pi^{n_0}} \in R$. One can check that this is a functorial bijection between $\mathbb{A}_E^{1,\mathrm{an}}(R,R^+)$ and R.

Finally, let us compute the global sections of $\mathbb{A}_E^{1,\mathrm{an}}$. Since $\mathbb{A}_E^{1,\mathrm{an}} = \varinjlim U_n$ the sheaf condition says that $\mathcal{O}_{\mathbb{A}_E^{1,\mathrm{an}}}(\mathbb{A}_E^{1,\mathrm{an}})$ is $\varprojlim \mathcal{O}(U_n)$ which is easily seen to agree with the following sheaf ring

$$\left\{\sum_{j=0}^{\infty} a_n t^j : \lim |a_j \pi^{-nj}| = 0 \text{ for all } n > 0\right\} \tag{5.9}$$

which is the ring of 'everywhere convergent power series' on E. Note then that, in particular, we have a natural inclusion $E[t] \hookrightarrow \mathcal{O}_{\mathbb{A}_E^{1,\mathrm{an}}}(\mathbb{A}_E^{1,\mathrm{an}})$.

Example 5.1.10. We would now like to build an adic projective line over E. There are several ways we can do this. Perhaps the most naive would be to build it by gluing two copies of $\mathbb{A}_E^{1,\mathrm{an}}$ along the complement of the origin (by which we mean the classical point from equation (5.7)). We can take advantage of the analytic nature of our objects, and so define it more in line with the definitions one usually sees in topology. Namely, we define the *adic projective line* over E, denoted $\mathbb{P}_E^{1,\mathrm{an}}$, to be the adic space obtained by gluing two copies of the disk $\mathbb{B}_E$ along $\mathbb{T}_E$ where the attaching map is given by the automorphism of $\mathbb{T}_E$ given on Huber rings by $t \mapsto t^{-1}$.

We will explain later what functor $\mathbb{P}_E^{1,\mathrm{an}}$ represents, but it is the obvious one.

Example 5.1.11. We now want to define something like the 'closure' of the closed unit disk $\mathbb{B}_E$. This may seem counterintuitive since you expect the closed unit disk to be, well, closed. But, the obvious open embedding of $\mathbb{B}_E$ into $\mathbb{A}_E^{1,\mathrm{ad}}$ as U_1 (in the notation of Example 5.1.9) doesn't have closed image. So, we mean something like the closure of $\mathbb{B}_E$ in $\mathbb{A}_E^{1,\mathrm{an}}$. This will turn out to be an object of incredible importance later on.

To this end, let us define the subring $R^+ \subseteq E\langle t\rangle$ as follows:

$$R^+ = \left\{\sum_{n=0}^{\infty} a_n t^n : a_0 \in E^\circ,\, a_n \in E^{\circ\circ} \text{ for } n \geqslant 1\right\} \tag{5.10}$$

Note that, as the notation suggests, R^+ is a ring of integral elements of $E\langle t\rangle$. Indeed, it's obviously contained in $(E\langle t\rangle)^\circ = E^\circ\langle t\rangle$, it's integrally closed in $E\langle t\rangle$ (essentially because $E^{\circ\circ}$ is prime), and it's open (essentially beacuse $E^{\circ\circ}$ is). Let us then define the *closure of* $\mathbb{B}_E$ to be $\mathrm{Spa}(E\langle t\rangle, R^+)$, which we denote $\overline{\mathbb{B}_E}$.

Let us note that there is a natural open embedding $\mathbb{B}_E \hookrightarrow \overline{\mathbb{B}_E}$ given by the natural inclusion of Huber pairs $(E\langle t\rangle, R^+) \hookrightarrow (E\langle t\rangle, E^\circ\langle t\rangle)$. In fact, the complement of $\mathbb{B}_E$ in $\overline{\mathbb{B}_E}$ consists of a single point which is given as follows:

$$\sum_{n=0}^{\infty} a_n t^n \mapsto \sup a_n \gamma^n \tag{5.11}$$

which has values in $\Gamma = \mathbb{R}_{>-} \times \gamma^{\mathbb{Z}}$ which is viewed as an ordered abelian group by delcaring that $r > \gamma$ for all real numbers $r > 1$ but $\gamma > 1$. This is then a rank 2 valuation which is in $\mathrm{Cont}(E\langle t\rangle)$ (in the notation of [Hub93b]) but not in $\mathbb{B}_E$ since $t \in E^\circ\langle t\rangle$ and this valuation sends t to γ which is not less than or equal to $(1,0)$ (the neutral element of Γ).

The fact that, at least topologically, this is the closure of $\mathbb{B}_E$ in $\mathbb{A}_E^{1,\mathrm{an}}$ will be justified later. For now, let us remark what the functor of points of $\overline{\mathbb{B}_E}$ is. Namely, it's clear that a map $\mathrm{Spa}(R,R^+) \to \overline{\mathbb{B}_E}$ corresponds to an element $r \in R$ such that $\sum\limits_{n=0} a_n r^n \in R^+$ for all $\sum\limits_{n=0}^{\infty} a_n t^n \in R^+$. But, it's clear that this is equivalent to the statement that $pr^n \in R^+$ for all $n \geqslant 1$. That said, we can edulcorate this (in a way that will be useful later) by observing that this is equivalent to $pr^n \in R^\circ$ for all $n \geqslant 1$. Indeed, if $pr^n \in R^\circ$ for all n, then for a fixed n we have that $(pr^n)^2 = p(pr^{2n}) \in pR^\circ$. But, $pR^\circ \subseteq R^{\circ\circ} \subseteq R^+$. Thus, we see that $(pr^n)^2$ is actually in R^+. Since R^+ is integrally closed we then deduce that $pr^n \in R^+$, and thus the two conditions are equivalent.

Note that this object $\overline{\mathbb{B}_E}$ is not of the form $\mathrm{Spa}(A)$, even though it was naturally constructed from $\mathbb{B}_E$ which is of this form (it's $\mathrm{Spa}(E\langle t\rangle)$). This gives one insight, especially when the imporance of $\overline{\mathbb{B}_E}$ is discussed later in Example 5.2.9, as to why one needs to deal with adic spaces of the general form $\mathrm{Spa}(A,A^+)$ even when classical rigid varieties are all of the form $\mathrm{Spa}(A)$.

Example 5.1.12. Let us now give an example of an adic space which is not over $\mathrm{Spa}(E)$. Namely, let's assume momentarily that E's topology is induced from a discrete valuation. Then, E° is actually a complete DVR. This, in particular, implies that the ring $E^\circ[\![t]\!]$ is actually Noetherian, and thus the Huber pair $(E^\circ[\![t]\!], E^\circ[\![t]\!])$, where we give $E^\circ[\![t]\!]$ the (π,t)-adic topology, is actually sheafy (this is in contrast to something like $E = \mathbb{C}_p$ where E° is very non-Noetherian). From E° we can then obtain the adic space over $\mathrm{Spa}(E^\circ)$ given by the natural map $\mathrm{Spa}(E^\circ[\![t]\!]) \to \mathrm{Spa}(E^\circ)$ which is called the *open formal unit disk* over E°. One can easily see that on the category $\mathsf{AffAdic}_{E^\circ}$ the open formal unit disk represents the functor $(R,R^+) \mapsto R^{\circ\circ}$.

We also have n-dimension analagoues in the form of the *formal open n-disk* over E° given by $\mathrm{Spa}(E^\circ[\![t_1,\ldots,t_n]\!]$. It clearly represents the functor sending (R,R^+) to $(R^{\circ\circ})^n$.

We also have a *formal closed n-disk* given by $\mathrm{Spa}(E^\circ\langle t_1,\ldots,t_n\rangle)$ (again, this means the π-adic completion of $E^\circ[t_1,\ldots,t_n]$). This represents the functor that sends a Huber pair (R,R^+) over E° to $(R^+)^n$.

Example 5.1.13. As a last example, we give discuss an adic space of a type that won't really show up (explicitly at least!) in this paper, but which has been incredibly important in recent years. Namely, let's assume that E is actually algebraically closed. Then, we can consider $E^\circ\langle t^{\frac{1}{p^\infty}}\rangle$ which rigorously means the π-adic completion of $\varinjlim E^\circ[t]$ where the transition maps are given by $t\mapsto t^p$. We then set $E\langle t^{\frac{1}{p^\infty}}\rangle$ to be the localization of $E^\circ\langle t^{\frac{1}{p^\infty}}\rangle$ with respect to π. Somewhat surprisingly the pair $\left(E\langle t^{\frac{1}{p^\infty}}\rangle, E^\circ\langle t^{\frac{1}{p^\infty}}\rangle\right)$ (with the π-adic topology) is a sheafy Huber pair. This was proven by Scholze, in greater generality, in [Sch12, Theorem 6.3]. This the associated adic space of this Huber pair is called the *perfectoid unit disk* over E.

Let us finally remark that the class of adic spaces we will mostly be concerned with are those that can be built out of gluing together adic closed subspaces of n-disks or formal n-disks. We will eventually make this more rigorous, but for now, let us recall the definition of an adic closed subspace. Namely, given a Huber pair (A,A^+) and an ideal I of A one can show that $(A/I,(A^+/(I\cap A^+))^\sim)$ (where $(-)^\sim$ denotes integral closure in A/I) is a Huber pair where A/I is given the quotient topology.

Example 5.1.14. Consider the ideal $I\subseteq E\langle t_1,t_2\rangle$ generated by t_1t_2-1. Then, one can clearly see that the quotient adic space is given by the affinoid $\mathrm{Spa}\left(E\langle t_1,t_2\rangle/(t_1t_2-1)\right)$ which is evidently identified with $\mathbb{T}_E$.

Remark 5.1.15. One can formulate the notion of closed embedding more generally. The only reason that one must be careful in doing so is that the theory of coherent sheaves on adic spaces is not very well-behaved for non-Noetherian objects. For a discussion of this see [KH17]. For Noetherian objects, for example for the type of adic spaces we will encounter, the notion of coherent sheaves, and in particular ideal sheaves, is completely fine. We leave it to the reader to formulate the needed theory.

The last three things we intend to discuss in this subsection are the notions of points, dimensions, and fibers.

To begin, it's helpful to recall the notion of an analytic point of an adic space. Intuitively, the analytic points of an adic space X are those that look something like the classical theory of rigid geometry. The non-analytic points look something more like the theory of schemes or formal schemes. So, let us begin by recalling that a Huber ring A is *Tate* if it has a pseudouniformizer. We then call a point x of an adic space X *analytic* if there exists some open neighborhood U of x such that $\mathcal{O}_X(U)$ is Tate. Let us denote by X_a the open subset of X consisting of analytic points. We call X *analytic* if $X=X_a$.

One has the following elementary lemma concerning analytic points:

Lemma 5.1.16. *A point x in an adic space X is analytic if and only if its support* $\mathrm{supp}(v_x)\subseteq\mathcal{O}_{X,x}$ *is open.*

Proof. See [Wed, Proposition 8.35]. ∎

Recall that the *support* of a valuation v on a ring R, denoted $\mathrm{supp}(v)$, is the ideal of elements of R that map to 0 under v.

Example 5.1.17. Any adic space over an analytic field is evidently analytic. Thus, of the examples we have discussed the only that could possibly be non-analytic are the open and closed formal disks over E° when E has a topology induced by a discrete valuation, as in Example 5.1.12. One can then see that, in both cases, the adic spaces are not analytic. Indeed, $\mathrm{Spa}(E^\circ\langle t\rangle)$ contains the valuation

$$E^\circ\langle t\rangle \to E^\circ\langle t\rangle/(\pi) = (E^\circ/pi)[t] \to E^\circ/\pi \to \{0,1\} \tag{5.12}$$

where all the mappings are the obvious quotient maps except the last one which is just the indicator function on $E^\circ/\pi - \{0\}$. One can easily verify that this is continuous since it's kernel contains (π), and it's clear that this point is not analytic because its support is (π, t).

So, let us call a Huber pair (E, E^+), where E is either a discrete field or an analytic field, and E^+ is a valuation ring of E, an *affinoid field.* We call the former case a *discrete affinoid field* and the latter case an *analytic affinoid field.* Note that if (E, E^+) is an analytic affinoid field, then E's topology coincides with the valuation topology from E

Note that if X is an adic space and $x \in X$, then the valuation v_x on $\mathcal{O}_{X,x}$ endows the residue field $k(x)$ of $\mathcal{O}_{X,x}$ with a valuation and we denote by $k(x)^+$ the valuation ring of $k(x)$ (equivalently $k(x)^+$ is the image of $\mathcal{O}^+_{X,x}$ under the quotient map $\mathcal{O}_{X,x} \to k(x)$). We

Morphisms locally of finite type and fiber products

As we said in the previous section, one of our key goals is to define the class of adic spaces that have a good étale cohomology theory by the work of Huber. Probably the most key hypothesis is a finiteness condition which takes the places of 'locally of finite type' from the theory of schemes. Of course, one must be slightly careful in the definition of such a notion though considering the existence of the extra ring of integral elements R^+.

So, let us begin by recalling the definition of the Huber pair of *convergent power series (with radius of convergence* 1*)* over a Huber pair (R, R^+). Let us fix a ring of definition R_0 of R, and an ideal of definition I of R_0. Then, for any $n \geqslant 1$ let $R\langle t_1, \ldots, t_n\rangle$ be the completion of $R[t_1, \ldots, t_n]$ where we give $R[t_1, \ldots, t_n]$ the ring topology where $R_0[t_1, \ldots, t_n]$ is an open adic subring with ideal of definition $I[t_1, \ldots, t_n]$. We then let $R\langle t_1, \ldots, t_n\rangle^+$ be the integral closure of $R^+\langle t_1, \ldots, t_n\rangle$ in $R\langle t_1, \ldots, t_n\rangle$. Let us then define $(R, R^+)\langle t_1, \ldots, t_n\rangle$ to be the Huber pair $(R\langle t_1, \ldots, t_n\rangle, R^+\langle t_1, \ldots, t_n\rangle)$.

Remark 5.1.18. Of course, defining this is a little silly since the knowledge of its definition was necessary for the knowledge of the sheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ on an adic space X.

Recall then that we call a morphism of Huber rings $A \to B$ *topologically of finite type* if there exists a continuous, open, surjection of A-algebras (equivalently a morphism of A-algebras which is a quotient map as topological spaces)

$$q : A\langle t_1, \ldots, t_s\rangle \to B \tag{5.13}$$

We call a morphism $(A, A^+) \to (B, B^+)$ of Huber pairs *weakly topologically of finite type* if the morphism of Huber rings $A \to B$ is topologically finite type, and *topologically of finite type* if $A \to B$ is topologically finite type and, if in addition, there is a surjection as in (5.13) such that B^+ is the integral closure of $q(A\langle t_1, \ldots, t_n\rangle^+)$.

Now, while the notion of weakly topologically of finite type is the most naive definition of 'finite type' hypotheses of Huber pairs, it certainly seems non-optimal since it doesn't impose any condition whatsoever on the rings of integral elements and, in fact, most strong theorems need hypotheses stronger than weakly topologically of finite type. That said, being actually topologically of finite type is often times too strong. For example, we have the following:

Lemma 5.1.19. *Let A and B be Huber rings. Then, for any morphism $A \to B$ topologically of finite type, then $B^+ = B^\circ$ is the unique ring of integral elements of B such that $(A, A^\circ) \to (B, B^+)$ is topologically of finite type.*

Proof. See [Hub93a, 2.4.17]. □

So, for example, being topologically of finite type is restrictive enough to exclude examples like $(E, E^\circ) \to (E\langle t\rangle, R^+)$ as in Example 5.1.11 which, as we will see later, is pivotal for the theory of compactly supported étale cohomology.

So, to remedy this, we would like to intermediary notion of 'finite type' hypotheses that puts a condition on the morphism of $+$-rings, but which is not so restrictive as topologically of finite type. For this, we have the following definition. Say that a morphism of Huber pairs $f : (A, A^+) \to (B, B^+)$ to be *$+$-weakly topologically of finite type* if $A \to B$ is topologically of finite type, and there exists some finite set of elements $M \subseteq B$ such that $B+$ is the smallest ring of definition of B that contains $f(A^+)$ and M (i.e. B^+ is the integral closure of the $f(A^+)$-algebra generated by M and $B^{\circ\circ}$).

Remark 5.1.20. Intuitively this says that, up to integral closure, B^+ is 'finitely generated' over the 'closure of A^+ in B' where, by the 'closure', we mean the smallest ring of integral elements of B that contains $f(A^+)$, which is $f(A^+)[B^{\circ\circ}]$ (compare with [Wed, Remark 7.15]).

We then call a morphism of adic spaces $f : X \to Y$ *locally of weakly finite type* if for all affinoid opens $\mathrm{Spa}(A, A)$ of Y there exists an open affinoid cover $\{\mathrm{Spa}(B, B^+)\}$ of $f^{-1}(\mathrm{Spa}(A, A^+))$ such that the morphisms $(A, A^+) \to (B_i, B_i^+)$ are all weakly topologically of finite type, and the notions of *locally of finite type* and *locally of $+$-weakly finite type* similarly. As per usual, the removal of the word 'locally' indicates the addition of quasi-compactness assumptions.

We have the following inclusion of 'finite type' hypotheses:

Lemma 5.1.21. *The condition 'locally of finite type' implies the condition 'locally of $^+$weakly finite type' which, in turn, implies the condition 'locally of weakly finite type'.*

Remark 5.1.22. Note that our definition differs slightly from the one in [Hub94] and [Hub96]. That said, they coincide when X is an analytic adic space, which will be the case of primary concern to us, and the only case in which we will use results from [Hub96] concerning locally of finite type morphisms (see [Hub94, Lemma 3.5, iii)] for a proof that the two notions coincide for analytic adic spaces).

Let us now give some examples of morphisms with the above finiteness conditions.

Example 5.1.23. From the previous sections, Examples 5.1.5, 5.1.6, 5.1.7, 5.1.8, 5.1.9, and 5.1.10 are all locally of finite type over E, but only Examples 5.1.5, 5.1.7, and 5.1.10 are of finite type over E. The closed unit disk in Example 5.1.12 is finite type over $\mathrm{Spa}(E^\circ)$.

The space $\overline{\mathbb{B}_E}$ in Example 5.1.11 is $+$-weakly finite type over $\mathrm{Spa}(E)$, but is not finite type.

The formal open unit disk from Example 5.1.12 is not locally finite type over $\mathrm{Spa}(E^\circ)$, the reason for which we will explain below. Moreover, Example 5.1.13 is not locally of finite type over E since any topologically finite-type E-algebra has finite-dimension, and evidently the perfectoid disk does not.

We also record the following obvious facts concerning morphisms locally of finite type:

Lemma 5.1.24. *The following results are true:*

1. *Open embeddings of adic spaces are locally of finite type.*

2. *Morphisms topologically of finite type between Huber pairs have the cancellation property: let $f : (A, A^+) \to (B, B^+)$ and $g : (B, B^+) \to (C, C^+)$ be morphisms of Huber rings. If f and g are topologically finite type then so is $g \circ f$. If $g \circ f$ is topologically finite type then f is topologically finite type.*

3. *Morphisms locally of finite type satisfy the cancellation property: if $f : X \to Y$ and $g : Y \to Z$ are morphisms of adic spaces, and if $g \circ f$ is locally of finite type, then g is locally of finite type. In particular, if X and Y are locally of finite type adic spaces over S, and $f : X \to Y$ is a morphism over S, then f is locally of finite type.*

4. *Morphisms $+$-weakly topologically of finite type between Huber pairs have the cancellation property: let $f : (A, A^+) \to (B, B^+)$ and $g : (B, B^+) \to (C, C^+)$ be morphisms of Huber rings. If f and g are $+$-weakly topologically finite type then so is $g \circ f$. If $g \circ f$ is $+$-weakly topologically finite type then f is $+$-weakly topologically finite type.*

5. *Morphisms locally of $+$-weakly finite type satisfy the cancellation property: if $f : X \to Y$ and $g : Y \to Z$ are morphisms of adic spaces, and if $g \circ f$ is locally of $+$-weakly finite type, then g is locally of $+$-weakly finite type. In particular, if X and Y are locally of $+$-weakly finite type adic spaces over S, and $f : X \to Y$ is a morphism over S, then f is locally of $+$-weakly finite type.*

We also record the following non-trivial fact of Huber:

Theorem 5.1.25 (Huber)**.** *Let (A, A^+) and (B, B^+) be Huber pairs which are either strongly Noetherian (in the sense of [Hub94]) or which have a Noetherian ring of definition. Then, if we have a morphism $(A, A^+) \to (B, B^+)$ such that $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ is locally of finite type, then $(A, A^+) \to (B, B^+)$ is topologically of finite type.*

Proof. See [Hub93a, Theorem 3.8.15]. □

To describe the 'good' spaces for Huber's étale cohomology we will also need the notion of separatedness, which is most easily defined in terms of fiber products. So, to this end, we would now like to recall the definition of the fiber product of adic spaces. Of course, this is not so simple of an affair in general. And, in fact, the fiber product of arbitrary adic spaces (even affinoid adic spaces!) doesn't exist. Thankfully, the situation we are going to work in doesn't have this complication.

To make the statement concerning the existence of fiber products simpler, we record the following definition concerning morphisms, intuitively, asserts a strong topological similarity between the source and target adic spaces (that they have the "same topology"). Namely, let $f : X \to Y$ be a morphism of adic space. Then, f is called *adic* if for every pair of points $x \in X$ and $y \in Y$ there are affinoind open neighborhoods $\mathrm{Spa}(B, B+)$ of x and $\mathrm{Spa}(A, A^+)$ of y with $f(\mathrm{Spa}(B, B^+)) \subseteq \mathrm{Spa}(A, A^+)$ and satisfing the following condition on the morphism $(A, A^+) \to (B, B^+)$: there exists rings of definition A_0 of A and B_0 of B, and an ideal of definition I of A_0 such that $f(A_0) \subseteq B_0$ and $f(I)$ generates an ideal of definition in B_0.

Remark 5.1.26. If f is adic then for any pairs of rings of definition A_0 and B_0 such that $f(A_0) \subseteq B_0$, and any ideal of definition I of A_0, one has that the ideal in B_0 generated by $f(I)$ is an ideal of definition (see [Hub93b, Proposition 3.8]).

Example 5.1.27. Evidently any morphism locally of finite type is adic since any open continuous surjection is adic, and the convergent power series over A is an adic A-algebra. Thus, the locally of finite type spaces described in Example 5.1.23 give a multitude of examples.

Moreover, we can now explain why the morphism mentioned at the end of Example 5.1.23 $\mathrm{Spa}(E^\circ[\![u_1, \ldots, u_r]\!]) \to \mathrm{Spa}(E^\circ)$ is not locally of finite type. Indeed, it's not adic, since Remark 5.1.26 would imply that $E^\circ[\![u_1, \ldots, u_r]\!]$ is π-adic which it's not if $r > 0$ (it's $(\pi, u_1, \ldots, u_r)$-adic).

One can make the fact that an adic morphism is one that "respects topology" more precise as follows. Recall that for every point $x \in X$ the Huber pair $(k(x), k(x)^+)$ is a Huber pair which is either analytic or discrete, the distinction between an analytic and a non-analytic point. One might imagine that a morphism f should "respect topology" if it preserves this distinction. This is the content of the following:

Lemma 5.1.28. *Let $f : X \to Y$ be a morphism of adic spaces. Then, f is adic if and only if $f(X_a) \subseteq Y_a$. In particular, if Y is analytic, then f is automatically adic.*

Proof. See [Hub93b, Proposition 3.8]. □

So, with this setup we have the following observation of Huber:

Lemma 5.1.29. *Let X, Y, and Z be adic spaces with morphisms $f : X \to Y$ and $g : Z \to Y$. Assume that f is locally of weakly finite type and g is adic, then the fiber product $X \times_Y Z$ exists in the category of adic spaces exists. Moreover, if X, Y, and Z are affinoid so is $X \times_Y Z$.*

Proof. As per usual, it suffices to consider the case of affinoids $X = \mathrm{Spa}(B, B^+)$, $Y = \mathrm{Spa}(A, A^+)$, and $Z = \mathrm{Spa}(C, C^+)$. One then constructs $X \times_Y Z$ as the adic spectrum of the completion of the Huber pair (D, D^+) where $D = B \otimes_A C$, D^+ is the integral closure of the image of $B^+ \otimes_{A^+} C^+$ in D, and the topology is given by giving D the topology having $B_0 \otimes_{A_0} C_0$ (where A_0, B_0, C_0 are rings of definition) as a ring of definition with ideal of definition $I(B_0 \otimes_{A_0} C_0)$ where I is an ideal of deifnition of A_0. See [Hub96, Proposition 1.2.2]. □

Remark 5.1.30. One sees where the adicness of f and g came into play. Namely, to define a topology on the tensor product one wants to somehow relate the topologies of A, B, and C and this is precisely what the adic condition allows us to do.

Let us remark that fiber products exist in generalities beyond the above (e.g. see [Hub96, Proposition 1.2.2]), but can sometimes behave in unexpected way. In particular, we have the following example of a fiber product of affinoids which exists but is not affinoid and, in fact, is not even quasi-compact:

Example 5.1.31. Let E be an analytic field whose topology is induced by a discrete non-trivial valuation. Note then that $\mathrm{Spa}(E^\circ[\![t]\!]) \times_{\mathrm{Spa}(E^\circ)} \mathrm{Spa}(E)$ is the open unit disk $\mathbb{D}_E$. Indeed, this follows from Examples 5.1.6 and 5.1.12 which show that this fiber product and the open disk give the same functor on $\mathsf{AffAdic}_E$. Note that $\mathbb{D}_E$ is non-affinoid, and not even quasi-compact.

Before we give other examples, we record the following non-trivial result:

Lemma 5.1.32. *Let* $\mathrm{Spa}(A)$*,* $\mathrm{Spa}(B)$*, and* $\mathrm{Spa}(C)$ *be finite type over* $\mathrm{Spa}(E)$*, and suppose that we have morphisms* $\mathrm{Spa}(B) \to \mathrm{Spa}(A)$ *and* $\mathrm{Spa}(C) \to \mathrm{Spa}(A)$*. Then, the image of* $B^\circ \otimes_{A^\circ} C^\circ$ *in* $B \otimes_A C$ *has integral closure* $(B \otimes_A C)^\circ$*. Moreover, if* E *is algebraically closed then the image of* $B^\circ \otimes_{A^\circ} C^\circ$ *in* $B \otimes_A C$ *is equal to* $(B \otimes_A C)^\circ$*. In particular, in all cases one has that* $X \times_Y Z = \mathrm{Spa}(B \widehat{\otimes}_A C)$*, where* $\widehat{\otimes}$ *denotes the completed tensor product (i.e. the* π*-adic completion of the algebraic tensor product).*

Proof. This is essentially given in [BGR84, Section 6.3]. □

With this result in hand, we can now give some more concrete examples

Example 5.1.33. Let us suppose now that we have $A = E\langle t_1, \ldots, t_n\rangle/(f_i)$ and $B = E\langle u_1, \ldots, u_m\rangle/(g_j)$. Then,

$$\mathrm{Spa}(A) \times_{\mathrm{Spa}(E)} \mathrm{Spa}(B) = \mathrm{Spa}\left(E\langle t_1, \ldots, t_n, u_1, \ldots, u_m\rangle\right)/(f_i, g_j) \tag{5.14}$$

equals

$$\mathrm{Spa}(E\langle t_1, \ldots, t_n, w_1, \ldots, w_m\rangle/(f_i, g_j)) \tag{5.15}$$

So, for example, we now see that $\mathbb{B}^n_E = (\mathbb{B}_E)^n$ and $\mathbb{T}^n_E = (\mathbb{T}_E)^n$ as one would hope.

Moreover, since we are going to be focusing our attention on adic spaces locally of finite type over $\mathrm{Spa}(E)$ we see that (or mostly all) of the fiber products we're going to consider will be locally describable using equation (5.14).

Example 5.1.34. Evidently we see that $\mathbb{D}^n_E = (\mathbb{D}_E)^n$.

Example 5.1.35. Let's now assume again that E's valuation is discrete, and consider the formal closed disk $\mathrm{Spa}(E^\circ\langle t\rangle)$ over E° (which, by the assumptions on E make sense since $E^\circ\langle t\rangle$ is Noetherian). Then, this is of finite type over $\mathrm{Spa}(E^\circ)$. Moreover, the morphism $\mathrm{Spa}(E) \to \mathrm{Spa}(E^\circ)$ is adic. So then, one sees that

$$\mathrm{Spa}(E^\circ\langle t\rangle) \times_{\mathrm{Spa}(E^\circ)} \mathrm{Spa}(E) = \mathrm{Spa}(E\langle t\rangle) \tag{5.16}$$

and so the fiber product is closed disk over $\mathrm{Spa}(E)$. Of course, the reason this differed from Example 5.1.31 is that in this case the closed formal disk is actuall adic $\mathrm{Spa}(E^\circ)$, whereas the open formal unit disk is not.

We would also like to know that the adic spectrum behaves well under base change to analytic extensions:

Corollary 5.1.36. *Let E be an analytic field, and $E_0' \supseteq E$ an algebraic extension, and set $E' := \widehat{E_0'}$. Then, for any $\mathrm{Spa}(A)$ finite type over E, then $\mathrm{Spa}(A)_{E'}$ exists and is equal to $\mathrm{Spa}(A\widehat{\otimes}_E E')$.*

Proof. The existence follows from the fact that $\mathrm{Spa}(E') \to \mathrm{Spa}(E)$ is adic, and so we can apply Lemma 5.1.29. Moreover, for any finite extension $E' \supseteq E$ the claim follows immediately from Lemma 5.1.32. One then obtains the general claim from passing to the limit and Lemma 5.1.1 part (1). □

Let us give an example to see why the integral closure in the definition of the fiber product is necessary in the case of base extension:

Example 5.1.37. Consider the rational open domains $U_n \subseteq \mathbb{B}_{\mathbb{Q}_p}$ from Example 5.1.5. Note then that just computing the completed tensor products $\mathcal{O}(U_n)\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$ and $\mathcal{O}^+(U_n)\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$ we obtain $\mathbb{C}_p\langle t\rangle$ and $\mathbb{C}_p\langle t,s\rangle/(\pi s - t^n)$. Now, the latter is *not* integrally closed in the former, and so taking the integral closure we obtain $\mathbb{C}_p\left\langle \dfrac{t}{\pi^{\frac{1}{n}}}\right\rangle$ and thus, $U_n \times_{\mathrm{Spa}(\mathbb{Q}_p)} \mathrm{Spa}(\mathbb{C}_p)$ is actually $\mathrm{Spa}\left(\mathbb{C}_p\langle t\rangle, \mathbb{C}_p\left\langle \frac{t}{\pi^{\frac{1}{n}}}\right\rangle\right)$ which is, in fact, isomorphic to $\mathbb{D}_{\mathbb{C}_p}$ showing the claim in Example 5.1.6 that U_n is a twist of $\mathbb{D}_{\mathbb{Q}_p}$ (of course the same discussion works for any E).

5.2 Partially proper and separated morphisms

To have a good theory of compactly supported cohomology for adic spaces we will need, not surprisingly, a good notion of compactifications. This, as it turns out, is a more complicated subject than for schemes where, assuming we are working with quasi-compact and quasi-separated schemes, compactifyability of a morphism $X \to S$ really reduces down to the statement that $X \to S$ is separated and finite type (this result is known as Nagata compactification, see [Con07] for more details). One might then wonder what properties of adic spaces guarantee a good notion of compactification.

That said, what a compactification even *is* for rigid spaces is a little less clear. Namely, while one is usually happy to assume that you are working with S-schemes such that $X \to$

S is quasi-compact, as in the condition of Nagata cited above, many of our morphisms we wish to compactify, as in $\mathbb{D}_E \to \mathrm{Spa}(E)$ from Example 5.1.6, are not quasi-compact. It seems then too strong to hope for proper compactifications of $X \to S$ when this morphism is not quasi-compact. Indeed, if Z is a compactification of X which is assumed to be quasi-separated (see below for the definition of quasi-separated), then the open embedding $j : X \hookrightarrow Z$ is quasi-compact and thus, if Z were proper over S, so then must be the morphism $X \to S$.

Thus, if we are going to wish to have a theory of compactly supported cohomology, and thus compactifications, for adic spaces over $\mathrm{Spa}(E)$ that are not quasi-compact (e.g. the open disk) then we will need a notion of compactification that is weaker than proper. Namely, we will somehow need to define a class of morphisms that acts like proper morphisms sans the quasi-compactness condition.

The rough idea for defining this class of morphisms is the realization the classically a locally of finite type quasi-separated morphism of schemes $X \to S$ is proper if and only if it satisfies:

1. it is quasi-compact.
2. it satisfies the property that for any diagram of the form

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Y \end{tikzcd} \tag{5.17}$$

   where K is a field, and A is a valuation ring of K, there exists a unique morphism $\mathrm{Spec}(A) \to X$ making the following diagram commute

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur] & Y \end{tikzcd} \tag{5.18}$$

This is just the valuative criterion for properness (See [Sta18, Tag 0A40]). Writing this in terms of functors we see that this says that for all valuation rings A with field of fractions K the natural map $X(A) \to X(K)$ is bijective where we are thinking of all objects as Y-schemes, and so $X(A)$, for example, means morphisms $\mathrm{Spec}(A) \to X$ over Y.

One then might imagine that to obtain a notion of properness without quasi-compactness hypotheses, one would take locally of finite type quasi-separated morphisms which satisfy this valuative criterion. This is precisely what partially proper morphisms (defined below) are.

Remark 5.2.1. Despite the fact that the origin of partial properness is in rigid analytic geometry, Y. Mieda (in [Mie14, Section 3]) has since adapted the notion to schemes to great effect.

Before we define the notion of partial properness for a morphism, we need to recall the following basic topological preliminary. Namely, recall that a topological space X is called *quasi-separated* if the intersection of any two compact open subsets of X is still quasi-compact. We call a morphism $f : X \to Y$ of topological spaces *quasi-separated* if for any quasi-separated open subset $U \subseteq Y$ the subset $f^{-1}(U) \subseteq X$ is quasi-separated.

The next lemma shows that essentially all of the examples we've so-far encountered are quasi-separated:

Lemma 5.2.2. *The following results hold:*

1. *Let X be topological space such that each point has a neighborhood basis of quasi-compact opens. Then, if X is quasi-separated, so is any subspace of X.*

2. *If $f : X \to Y$ is a morphism of quasi-separated topological spaces, then f is quasi-separated.*

3. *If $\mathrm{Spa}(A)$ is an analytic affinoid adic space, then $\mathrm{Spa}(A)$ is quasi-separated*

Proof.

1. Obvious.

2. It suffices to check the claim that the intersection of quasi-compact opens is quasi-compact for a basis of quasi-compact opens. In particular, it suffices to check that $\mathrm{Spa}(B) \cap \mathrm{Spa}(C) \subseteq \mathrm{Spa}(A)$ is quasi-compact if $\mathrm{Spa}(B), \mathrm{Spa}(C)$ are affinoid open subsetes of $\mathrm{Spa}(A)$. But, note that this intersection is the fiber product $\mathrm{Spa}(B) \times_{\mathrm{Spa}(A)} \mathrm{Spa}(C)$ which, since open embeddings are locally of finite type (by Lemma 5.1.24 Part (1)) and are automaticaly adic since $\mathrm{Spa}(A)$ is (by Lemma 5.1.28) we deduce from Lemma 5.1.29 that this intersection is, in fact, affinoid.

□

So, let us now define a morphism of analytic adic spaces $f : X \to S$ which is locally of $+$-weakly finite type and quasi-separated to be *partially proper* if for all objects $\mathrm{Spa}(L, L^+)$ of $\mathsf{AffAdic}_S$, where (L, L^+) is an affinoid field (necessarily analytic), the natural map $X(L, L^+) \to X(L)$ is bijective where, as before, all objects are to be considered as objects over S and so, for example, $X(L)$ means morphisms $\mathrm{Spa}(L) \to X$ over S. We say that f is *proper* if it is partially proper and quasi-compact. It is clear that this then looks quite close to the valuative criterion for properness for schemes.

Let us now discuss some natural examples and non-examples of partially proper morphisms:

Example 5.2.3. The adic spaces over $\mathrm{Spa}(E)$ given in Examples 5.1.6, 5.1.9, the punctured open disk in 5.1.8 and 5.1.11 are partially proper over $\mathrm{Spa}(E)$. Only this last example is actually proper. One might wonder then whether this is a general phenomenon.

Note also that the Examples5.1.5, 5.1.7, and the closed punctured disk in 5.1.8 are not partially proper over $\mathrm{Spa}(E)$.

Now, as the open unit disk over $\mathrm{Spa}(E)$ shows, it is *not* true that partially proper morphisms are universally closed (with the obvious meaing), since if $X \to \mathrm{Spa}(E)$ is universally closed, then any morphism $X \to Y$ over $\mathrm{Spa}(E)$ is closed. But, evidently the natural open embedding $\mathbb{D}_E \hookrightarrow \mathbb{B}_E$ is not closed.

That said, for proper morphisms we do have closedness, and an equivalence with the more classical notion of closedness. To state this correctly, we need to first have a definition of separatednes. Namely, if $X \to S$ is a quasi-separated morphism locally of $+$-weakly finite type morphism of analytic adic spaces, let's call it *separated* if for all affinoid fields $\mathrm{Spa}(L, L^+)$ in $\mathsf{AffAdic}_S$ the natural morphism $X(L, L^+) \to X(L)$ is injective. In particular, all partially proper morphisms are separated.

We then have the following result showing that separatedness and properness agree with their more naturally definitions coming from algebraic geometry:

Lemma 5.2.4. *Let S and X be analytic adic spaces and let $f : X \to S$ be a quasi-separated morphism locally of $+$-weakly finite type. We then have the following:*

1. *The morphism f is separated if and only if the diagonal map (defined as usual) $\Delta_f : X \to X \times_S X$ has closed image.*

2. *The morphism f is proper if and only if it is separated and universally closed.*

Proof. See [Hub96, Lemma 1.3.10], noting that he first defines the notions of separatedness and properness as in this proposition, and then proves it's equivalent to the conditions we used to define the notions. □

Good spaces over $\mathrm{Spa}(E)$ and compactifications

Let us now define the type of adic spaces over $\mathrm{Spa}(E)$ that will have reasonable étale cohomology theories. These spaces will consist of a separatedness condition, a finiteness condition, and a topological condition. The first two will be completely standard coming from the theory of schemes, and the last is a technical condition known as 'tautness' which, as the conversation at the end of [Hub96, Page 272] seems to suggest, is one of technical convenience opposed to ideological necessity. That said, as we will see later, tautness is also a condition that shows up naturally in the comparison between Huber's theory of adic spaces and Berkovich's theory of k-analytic spaces.

So, let us call a topological space X *taut* if for any quasi-compact open subset $U \subseteq X$ the closure $\overline{U}$ in X is also quasi-compact. We call a morphism $f : X \to Y$ of topological spaces *taut* if for every taut open subspace $U \subseteq Y$, the preimage $f^{-1}(U)$ is a taut open subspace of X.

We would now like to give some examples of taut spaces. But, before we do so we state the following lemma of Huber:

Lemma 5.2.5. *Let S and X be analytic adic spaces, and $f : X \to S$ a morphism locally of $+$-weakly finite type and quasi-separated. Then:*

1. *If S is quasi-compact and quasi-separated it is taut.*

2. *If S is quasi-separated and f is partially proper then it is taut.*

Proof. The first claim is obvious, the second claim follows from applying [Hub96, Lemma 5.1.3 vii)] in conjuction with [Hub96, Lemma 1.3.13] by covering Y with affinoid opens. □

We can now easily state the following:

Example 5.2.6. The adic spaces from Example 5.1.5-5.1.11 are taut.

Remark 5.2.7. It is quite difficult to construct non-pathological examples of locally of finite type adic spaces over $\mathrm{Spa}(E)$ that are not taut. That said, a natural example shows up in the work of Hellman [Hel13] pertaining to the theory of p-adic period domains (see [Hel13]).

With all of this setup, we can now state the properties of an adic space over $\mathrm{Spa}(E)$ that make it amenable to Huber's theory of compactly supported étale cohomology. Namely, let S and X be analytic adic spaces. Then a morphism $f : X \to S$ is *good* if it is locally of $+$-weakly finite type, separated, and taut. We say that an adic space X over $\mathrm{Spa}(E)$ is *good* if the structure morphism $X \to \mathrm{Spa}(E)$ is good.

The reason that good spaces are useful is that they have natural theories of compactifications. To this end, if S and X and analytic adic spaces and if $f : X \to S$ is good, let us say that a *compactification* (j, Y, f') of f is a locally closed embedding $j : X \hookrightarrow Y$ and a partially proper morphism $f' : Y \to S$ such that $f' \circ j = f$.

We call a compactification (j, Y, f') a *universal compactification* if it is initial amongst factorizations of f through partially proper morphisms over S. More explicitly, the compactification (j, Y, f') is universal if given any partially proper morphism $p : P \to S$ and a morphism $h : X \to P$ over S, there exists a unique morphism $g : Y \to P$ such that $g \circ j = h$ and $f' = p \circ g$. Note that a universal compactification, if it exists, is obviously unique up to unique isomorphism, and we denote it $(j, \overline{X}, \overline{f})$.

We then have the following theorem of Huber that makes the notion of good reasonable from the perspective of compactifications:

Lemma 5.2.8. *Let S and X be analytic adic spaces and $f : X \to S$ a good morphism. Then, there exists a universal compactification $(j, \overline{X}, \overline{f})$ of f. Moreover, the morphism j is a quasi-compact open embedding, every point of $\overline{X}$ is a specialization of a point of $j(X)$, and the morphism $\mathcal{O}_{\overline{X}} \to j_*\mathcal{O}_X$ is an isomorphism of locally topologically ringed sheaves. Moreover, these properties characterize the universal compactification of X.*

Proof. This is [Hub96, Theorem 5.1.5] together with [Hub96, Lemma 5.1.7]. □

Using this lemma we can give a concrete example of a universal compactification and prove an earlier claim:

Example 5.2.9. The open embedding $j : \mathbb{B}_E \hookrightarrow \overline{\mathbb{B}_E}$ from Example 5.1.11 together with the natural structure map map $\overline{f} : \overline{\mathbb{B}_E} \to \mathrm{Spa}(E)$ is the universal compactification of $\mathbb{B}_E \to \mathrm{Spa}(E)$. Indeed, by Lemma 5.2.8 it suffices to show that every point of $\overline{\mathbb{B}_E}$ is a specialization of a point in $\mathbb{B}_E$, that j is a quasi-compact open embedding, and that the morphism $\mathcal{O}_{\overline{\mathbb{B}_E}} \to j_*\mathcal{O}_{\mathbb{B}_E}$ is an isomorphism of locally topologically ringed sheaves. The first claim follows from the explicit description of $\overline{\mathbb{B}_E}$ given in Example 5.1.11. Namely, $\overline{\mathbb{B}_E} - \mathbb{B}_E$ consists of a single valuation defined by

$$\sum_{n=0}^{\infty} a_n t^n \mapsto \sup a_n \gamma^n \tag{5.19}$$

where $\Gamma = \mathbb{R}^{>0} \times \gamma^{\mathbb{Z}}$ where the ordering on Γ is determined by the property that $r > \gamma > 1$ for all real numbers r such that $r > 1$. In particular, it's clear that this valuation is a specialization of the Gauss point (see the description of the points in [Sch12, Example 2.20]). The claim that j is a quasi-compact open embedding is obvious. Finally, we need to show that $\mathcal{O}_{\overline{\mathbb{B}_E}} \to j_* \mathcal{O}_{\mathbb{B}_E}$ is an isomorphism. Since $\overline{\mathbb{B}_E}$ is affinoid, it suffices to check this claim on global sections where it becomes the literal identity morphism $E\langle t\rangle \to E\langle t\rangle$, from where the claim follows.

Using this, we can justify the claim in Example 5.1.11 that topologically $\overline{\mathbb{B}_E}$ really is homeomorphic to the closure of $\mathbb{B}_E$ in $\mathbb{A}_E^{1,\mathrm{an}}$. To do this, note that by the universal property of the embedding $j : \mathbb{B}_E \hookrightarrow \overline{\mathbb{B}_E}$ the fact that $\mathbb{A}_E^{1,\mathrm{an}} \to \mathrm{Spa}(E)$ is partially proper implies that the embedding $\mathbb{B}_E \hookrightarrow \mathbb{A}_E^{1,\mathrm{an}}$ gives rise to a $\overline{\mathbb{B}_E} \to \mathbb{A}_E^{1,\mathrm{an}}$. Since $\overline{\mathbb{B}_E}$ is proper the image of this morphism is closed, and so must contain more than just $\mathbb{B}_E$. But, since $\overline{\mathbb{B}_E}$ contains only one more point of $\mathbb{B}_E$ this implies that $\overline{\mathbb{B}_E} \to \mathbb{A}_E^{1,\mathrm{an}}$ is injective. Its image is a closed set containing $\mathbb{B}_E$ and since it contains only one more point than $\mathbb{B}_E$ it's evidently the closure of $\mathbb{B}_E$ in $\mathbb{A}_E^{1,\mathrm{an}}$. Then, since $\overline{\mathbb{B}_E} \to \mathbb{A}_E^{1,\mathrm{an}}$ is continuous and closed and maps bijectively onto the closure of $\mathbb{B}_E$ in $\mathbb{A}_E^{1,\mathrm{an}}$ we conclude that it is homeomorphic to this closure as desired.

This example is not a mistake either. One can actually generalize the above to give a procedure to obtain the universal compactification of any good morphism of analytic adic spaces:

Theorem 5.2.10. *Let $f : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ be a good morphism of analytic adic spaces, and let us also denote by f the map on the associated Huber pairs. Let R^+ be the minimal ring of integral elements of B that contains the subring $f(A^\circ)$ (i.e. R^+ is the integral closure of $f(A^\circ) + B^{\circ\circ}$). Then, (B, R^+) is a Huber pair, and there is a partially proper morphism $f' : \mathrm{Spa}(B, R^+) \to \mathrm{Spa}(A, A^+)$ such that if j denotes the embedding $\mathrm{Spa}(B, B^+) \hookrightarrow \mathrm{Spa}(B, R^+)$, then the pair $(j, \mathrm{Spa}(B, R^+), f')$ is the universal compactification of f.*

Proof. See the proof of [Hub96, Proposition 5.1.5]. □

One then obtains the universal compactification of a general good morphism by gluing together constructions as in Theorem 5.2.10.

Example 5.2.9, and more generally Theorem 5.2.10, show the virtue of working with Huber pairs of the form (A, A^+) with $A^+ \subsetneq A$. Namely, even if one is only interested in studying Huber pairs of the form (R, R°) natural constructions, like the universal compactification, take you outside this realm.

We record here one more basic fact concerning universal compactifications, which should be intuitively obvious from the definition:

Lemma 5.2.11. *Suppose that S and X are analytic spaces and $f : X \to S$ is quasi-compact and good. Then, the universal compactification $(j, \overline{X}, \overline{f})$ has the property that $\overline{f} : \overline{X} \to S$ is quasi-compact and thus proper.*

Proof. See [Hub96, Lemma 5.1.6]. □

Finite, étale and smooth morphisms

We now come to the most important type of morphism for cohomology theory: étale morphisms. There are some pleasant surprising simplifications to the theory when one restricts themselves to the category of analytic adic spaces. But, most of these results rely heavily on Noetherian hypotheses, and so let us emphasize that for this section we really will be heavily relying on the assumptions discussed in Remark 5.0.1. Many of these results go through to the non-Noetherian setting, but require (in some instances) significant tweaks to the proofs.

Before we define étale morphisms in general, it will be helpful to define finite étale morphisms. So, to this end, we first define finite morphisms. As per usual, we will define a finite morphism of affinoids, and then use this to define the notion of finiteness in general:

Lemma 5.2.12. *Let $f : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ be a morphism of analytic affinoid adic spaces, and let f also denote the associated morphism of Huber pairs. Then, the following are equivalent:*

1. *The morphism f is of finite type, and the maps $f : A \to B$ and $f : A^+ \to B^+$ is integral.*

2. *There is an an isomorphism of Huber pairs $(B, B+) \to (C, C^+)$ over (A, A^+) where C is a finite A-algebra with the natural topology, and C^+ is the integral closure of A^+ in C.*

Moreover, given any finite A-algebra C the pair (C, C^+), where C is given the natural topology and C^+ is the integral closure of (the image of) A^+ in C, is a complete sheafy Huber pair.

The definition of the natural topology is as follows. Given an analytic affinoid Huber pair (A, A^+), and a finite A-module M a surjection of A-modules $A^{\oplus n} \to M$ endows M with a quotient topology. This topology is independent of surjection (see the discussion in [KH17, Definition 1.1.11]) and is called the *natural topology* on M.

Proof(Lemma 5.2.12). It's clear that (2) implies (1), and the only non-obvious part of (1) implies (2) is that B's topology must be equivalent to the natural one. For this see [KH17, Corollary 1.1.12].

Let us now explain why if $A \to C$ is a finite algebra morphism, and we endow C with the natural topology, then C is a Huber ring , why the integral closure C^+ of A^+ in A is a ring of integral elements, and why (C, C^+) is sheafy. We do this only in the case when A

is a strongly Noetherian Tate ring (which suffices for essentially all applications). Huber cites [Hub93a, 3.12.10] for the general case.

Let us begin by observing that the natural topology on C is a topological ring. We need to show that the two maps $C \times C$ given by addition and multiplication are continuous. The addition map is continuous by applying the open mapping theorem (see [KL16, Theorem 1.2.7]) to the A-linear map $C \times C \to C$. The multiplication natural map $C \times C \to C \otimes_A C$ (where the latter is given the natural topology) is evidently continuous, and the map $C \otimes_A C \to C$ given by multiplication is also continuous by the open mapping theorem. Thus, the composition $C \times C \to C \otimes_A C \to C$, which is the multiplication mapping, is also continuous. Thus, C is a topological ring which is easily seen to be Huber. □

We then define a morphism of affinoid adic spaces $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ to be *finite* if it satisfies any of the equivalent conditions Lemma 5.2.12. Note that one important, somewhat quirky, property of finite morphisms of analytic adic spaces is that a finite morphism $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ really only depends on the map $A \to B$. Namely, the category of finite adic spaces over $\mathrm{Spa}(A, A^+)$ is naturally equivalent to the category of finite schemes over $\mathrm{Spec}(A)$. This gives us a huge algebraic tie-in when dealing with the theory of analytic adic spaces.

Let us give a natural example of a finite morphism:

Example 5.2.13. Consider the morphism $\mathbb{B}_{\mathbb{C}_p} \to \mathbb{B}_{\mathbb{C}_p}$ given on Huber pairs by $t \mapsto t^p - t - 1$. This map is topologically of finite type by Theorem 5.1.24 Part (2), and it's evident that the maps $E\langle t\rangle \to E\langle t\rangle$ and $E^\circ\langle t\rangle \to E^\circ\langle t\rangle$ are integral.

We can then define a morphism $f : X \to Y$ of analytic adic spaces to be *finite* if there exists an affinoid cover $\{\mathrm{Spa}(A, A^+)\}$ of Y such that $f^{-1}(\mathrm{Spa}(A, A^+))$ is affinoid and the morphism $f^{-1}(\mathrm{Spa}(A, A^+)) \to \mathrm{Spa}(A, A^+)$ is a finite morphism of analytic affinoid adic spaces.

We then have the following non-trivial result of Huber:

Lemma 5.2.14. *Suppose that $f : X \to Y$ is a finite morphism of analytic adic spaces. Then, for any affinoid open $U \subseteq Y$ the open $f^{-1}(U)$ is affinoid, and the morphism $f^{-1}(U) \to U$ is finite.*

Proof. See [Hub93a, 3.6.20 and 3.12.12]. □

Remark 5.2.15. Note that this result must be more subtle than the usual proof in the theory of schemes. Indeed, recall that there is no good notion of "affinoid morphism" in rigid, or adic, geometry. Specifically, it's not true that if $f : X \to Y$ is a morphism of adic spaces (even good varieties over $\mathrm{Spa}(E)$!) such that $f^{-1}(V)$ is affinoid for some open affinoid cover $\{V\}$ of Y that $f^{-1}(U)$ is affinoid for all affinoid $U \subseteq Y$. See [SW, Example 9.1.2].

Let us now give a canonical example of a non-finite map

Example 5.2.16. The open embedding $\mathbb{D}_E \hookrightarrow \mathbb{B}_E$ is not finite.

We recall that one nice consequence of Zariski's main theorem is that a morphism of schemes $f : X \to Y$ is finite if and only if it's quasi-finite and proper. This still holds true for analytic adic spaces:

Lemma 5.2.17. *Let $f : X \to Y$ be morphism local of finite type between analytic adic spaces. Then, f is finite if and only if it's proper and for all $y \in Y$ the set $f^{-1}(y)$ is finite.*

Proof. See [Hub96, Proposition 1.5.5]. □

Remark 5.2.18. One of the most immediate application of this result is the finiteness of non-constant morphisms between dimension 1 proper curves over $\mathrm{Spa}(E)$.

Finally, we recall another criterion for finiteness between affinoid analytic adic spaces that mirrors the classical theory of schemes:

Lemma 5.2.19. *Let $f : X \to Y$ be a morphism of finite type between analytic affinoid adic spaces. Then, f is finite if and only if f is proper.*

Proof. This is [Hub96, Proposition 1.4.6]. □

Remark 5.2.20. Note that in Lemma 5.2.19 it was pivotal that f was of finite type, and not just $+$-weakly finite type. Indeed, the morphism $\overline{\mathbb{B}_E} \to \mathrm{Spa}(E)$ is a $+$-weakly finite type and proper morphism between analytic affinoid adic spaces, but is not finite.

We are now finally ready to talk about the notion of finite étale morphisms of adic spaces. Namely, let us call a morphism of analytic affinoid adic spaces $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ *finite étale* if it's finite and if the map of schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is étale (and thus finite étale). Note that, by the discussion after Lemma 5.2.12 the map $\mathrm{Spa}(B, B^+) \mapsto \mathrm{Spec}(B)$ is an equivalence beteween the category of finite étale covers of $\mathrm{Spa}(B, B^+)$ and $\mathrm{Spec}(B)$. Again, this has deep consequences in relating the étale cohomology of adic spaces to that of schemes.

Let us give a somewhat surprising example of a finite étale cover:

Example 5.2.21. The morphism from Example 5.2.13 is étale. Indeed, we need to show that the map $\mathbb{C}_p\langle t\rangle \to \mathbb{C}_p\langle t\rangle$ mapping t to $t^p - t - 1$ is étale. Since this map is evidently flat, it suffices to show that its cotangent sheaf vanishes. To do this must merely show that for all closed points $x \in \mathrm{Spec}(\mathbb{C}_p\langle t\rangle)$ the pullback of the cotangent sheaf to x is zero. That said, the classical theory of Tate algebras tells us that the closed points of $\mathbb{C}_p\langle t\rangle$ are precisely the ideals of the form $x = (t - \alpha)$ where $\alpha \in \mathcal{O}_{\mathbb{C}_p}$. In particular, given this closed point x we see that the pullback of the cotangent sheaf to x is the module $M_x := \mathbb{C}_p dt/(p\alpha^{p-1} - 1)dt$. But, note that since $\alpha \in \mathcal{O}_{\mathbb{C}_p}$ one has that $p\alpha^{p-1} - 1 \neq 0$ (for valuation reasons) and thus $p\alpha^{p-1} - 1$ is a unit of $\mathbb{C}_p$, and the module M_x vanishes. Thus, the morphism $t \mapsto t^p - t - 1$ is a finite étale map $\mathbb{B}_{\mathbb{C}_p} \to \mathbb{B}_{\mathbb{C}_p}$.

We also claim that this map is surjective. To do this it suffices to note that the map is proper, thus closed, and so it suffices to show it has dense image. Since the classical points of $\mathbb{B}_{\mathbb{C}_p}$ are dense it suffices to check surjectivity on classical points, which is obvious. Namely, given any $\beta \in \mathcal{O}_{\mathbb{C}_p}$ we want to show that $t^p - t - 1 = \beta$ has a root in $\mathcal{O}_{\mathbb{C}_p}$. But, it evidently has a root in $\mathbb{C}_p$ by algebraic closedness, and by the integrality of the polynomial, this root lies in $\mathcal{O}_{\mathbb{C}_p}$.

So, this is all very surprising. Indeed, we have just shown that $\mathbb{B}_{\mathbb{C}_p}$ has a connected finite étale cover! This implies that, once we define what this means, the adic space $\mathbb{B}_{\mathbb{C}_p}$ will have non-trivial (profinite) étale fundamental group and so, in particular, non-trivial first étale cohomology. So, a disk has non-trivial topology. This is the subtle nature of p-adic geometry and is related to, in a similar way to schemes, the fact that objects over p-adic fields have richer p-adic cohomology theory. We will see later that the ℓ-adic cohomology, for $\ell \neq p$, of $\mathbb{B}_{\mathbb{C}_p}$ is trivial. In fact, in a way that we will later make precise, the fact that $\mathbb{B}_{\mathbb{C}_p}$ has non-trivial (étale) topology comes from the fact that the scheme $\mathbb{A}^1_{\overline{\mathbb{F}_p}}$ has non-trivial (étale) topology.

Let us now define a morphism $f : X \to Y$ of analytic adic spaces to be *étale* if for every point $x \in X$ there are open neighborhoods V of $y := f(x)$ and $U \subseteq f^{-1}(V)$ of x such that the morphism $f : U \to V$ can be factored as follows:

$$\begin{tikzcd} U \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & W \arrow[dl, "g"] \\ & V & \end{tikzcd} \tag{5.20}$$

where i is an open embedding and g is finite étale.

As an example of an étale morphism which is not an open embedding or finite étale we have the following:

Example 5.2.22. The map $\mathbf{G}_m \to \mathbf{G}_m/q^{\mathbb{Z}}$ (see [Hub94, §5]).

Note that this is the 'naive' definition of an étale morphism: it's the smallest category of morphisms containing open embeddings and finite étale morphisms, which should obviously be étale. It is *not* true that this naive category of étale morphisms agrees with the category of morphisms in the category of schemes. That said, we will see that it agrees with the more algebraic definitions of étale morphisms coming from scheme theory.

Specifically, we have the following theorem of Huber:

Theorem 5.2.23. *Let $f : X \to Y$ be a morphism locally of finite type between analytic adic spaces. Then, the following are equivalent:*

1. *f is étale.*

2. *f satisfies the infinitesimal lifting criterion: given any object* $\mathrm{Spa}(A, A^+)$ *of* $\mathsf{AffAdic}_Y$ *and a square-zero ideal I of A the map* $X(\mathrm{Spec}(A, A^+)) \to X(\mathrm{Spec}(A, A+)/I)$ *(where, note, we are treating all adic spaces involved as living over Y) is a bijection.*

3. *f is flat and unramified.*

4. *f satisfies the Jacobian criterion: For any affinoid open* $\mathrm{Spa}(B, B^+) \subseteq X$ *and affinoid open* $\mathrm{Spa}(A, A^+) \subseteq Y$*, where both Huber pairs are Tate, such that* $f(\mathrm{Spa}(B, B^+)) \subseteq \mathrm{Spa}(A, A^+)$ *one has that* $\mathrm{Spa}(B, B^+)$ *is isomorphic over* $\mathrm{Spa}(A, A^+)$ *to an adic spaces of the form*
$$\mathrm{Spa}(A\langle t_1, \ldots, t_n\rangle, A\langle t_1, \ldots, t_n\rangle)/I \tag{5.21}$$

where $I = (f_1, \ldots, f_n)$ *and the Jacobian* $\left(\dfrac{\partial f_i}{\partial t_j}\right)$ *has invertible determinant in* $A\langle t_1, \ldots, t_n\rangle/I$. *Moreover, if this propert holds for a pair of affines, it holds for all smaller affines contained in either the source or target.*

Where, here, a morphism of analytic adic spaces $f : X \to Y$ of adic spaces is *flat* if it's flat as a morphism of locally ringed spaces (i.e. if for all $x \in X$ and $y := f(x)$ one has that the local ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is flat) and unramified means that, in addition, $\mathfrak{m}_y \mathcal{O}_{X,x} = \mathfrak{m}_x$ (where $\mathfrak{m}_x$ and $\mathfrak{m}_y$ are the maximal ideals of their respective local rings) and $k(y) \to k(x)$ is finite separable (it is equivalent to say that Δ_f is an open embedding).

Proof. This is putting together Example 1.6.6 Part ii), Corollary 1.7.2, Proposition 1.7.5, and Lemma 2.2.8 of [Hub96]. □

While we don't venture to prove 5.2.23, we do mention one major, and importance difference between the theory of analytic adic spaces and that of schemes that makes the above possible:

Lemma 5.2.24. *Let* X *be an analytic adic space, and let* $x \in X$*, and let* π *be a pseudouniformizer of* $\mathcal{O}(U)$ *for some neighborhood* U *of* x *such that* $\mathcal{O}(U)$ *is Tate. Then, the local pairs* $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$*,* $(\mathcal{O}^+_{X,x}, \mathfrak{m}_x)$*, and* $(\mathcal{O}_{X,x}, \pi)$ *are Henselian.*

For background on Henselian pairs see [Sta18, Tag 09XD].

Proof. See [KL13, Lemma 2.4.17] or [Bha17, Propostition 18.5]. □

We list here some obvious properties of étale morphisms of analytic adic spaces:

Lemma 5.2.25. *Let* X*,* Y*, and* Z *be analytic adic spaces. Then, the following statements are true:*

1. *If* $f : X \to Y$ *and* $g : Y \to Z$ *are étale, then* $g \circ f : X \to Z$ *is étale.*
2. *If* $f : X \to Y$ *and* $g : Y \to Z$ *are morphisms such that* g *and* $g \circ f$ *is étale, then* g *is étale.*

We end this section by defining the notion of a smooth morphism between analytic adic spaces. As one might imagine, one can proceed precisely as in the algebraic theory:

Theorem 5.2.26. *Let* $f : X \to Y$ *be morphism locally of finite type between analytic adic spaces. Then, the following properties of* f *are equivalent:*

1. *It has constant fiber rank* d *and satisfies the infinitesimal lifting criterion: given any object* $\mathrm{Spa}(A, A^+)$ *of* $\mathsf{AffAdic}_Y$*, and any square-zero ideal* I *of* A*, the natural morphism* $X(\mathrm{Spa}(A, A^+)) \to X(\mathrm{Spa}(A, A^+)/I)$ *is surjective.*

2. *It satisfies the Jacobian criterion: For any affinoid open* $\mathrm{Spa}(B, B^+) \subseteq X$ *and affinoid open* $\mathrm{Spa}(A, A^+) \subseteq Y$*, where both Huber pairs are Tate, such that* $f(\mathrm{Spa}(B, B^+)) \subseteq \mathrm{Spa}(A, A^+)$ *one has that* $\mathrm{Spa}(B, B^+)$ *is isomorphic over* $\mathrm{Spa}(A, A^+)$ *to an adic spaces of the form*
$$\mathrm{Spa}(A\langle t_1, \ldots, t_n\rangle, A\langle t_1, \ldots, t_n\rangle)/I \tag{5.22}$$
where $I = (f_1, \ldots, f_m)$ *and the Jacobian* $\left(\dfrac{\partial f_i}{\partial t_j}\right)$ *has rank* $n - d$*. Moreover, if this property holds for a pair of affines, it automatically holds for all smaller affines contained in either the source or target.*

3. *For any* $x \in X$ *there exists affinoid neibhborhoods* $U = \mathrm{Spa}(B, B^+)$ *of* x *and* $V = \mathrm{Spa}(A, A^+)$ *of* $y := f(x)$ *such that the morphism* $U \to V$ *factors as*
$$\begin{tikzcd} U \arrow[rr, "g"] \arrow[dr, "f"'] & & \mathrm{Spa}((A, A^+)\langle t_1, \ldots, t_d\rangle) \arrow[dl, "p"] \\ & V & \end{tikzcd} \tag{5.23}$$
where g *is étale and* p *is the natural projection map.*

We call a morphism $f : X \to Y$ locally of finite type between analytic adic spacex *smooth of relative dimension* d it it satisfies any of the equivalent conditions of Theorem 5.2.26. Note that a smooth morphism of relative dimension 0 is just an étale morphism. We call a morphism *smooth* if it is smooth of some relative dimension.

We record here the following nice fact concerning the topology of smooth maps that mirrors the same result in the theory of schemes

Lemma 5.2.27. *A smooth morphism of analytic adic spaces is open.*

Proof. This is [Hub96, Proposition 1.7.8]. □

5.3 Formal schemes and their generic fibers

We now review the notion of the generic fiber of formal schemes, due originally to Berthelot and Raynaud, in the language of Huber's adic spaces.

Suppose now that $\mathcal{O}$ is a complete DVR of mixed characteristic $(0, p)$ with uniformizer π, residue field k with cardinality q, and fraction field E (an analytic field). Let us fix an algebraic closure $\overline{E}$ with ring of integers $\overline{\mathcal{O}}$, the maximal unramified extension E^{ur} with ring of integers $\mathcal{O}^{\mathrm{ur}}$, the completion $\breve{E}$ of E^{ur} with ring of integers $\breve{\mathcal{O}}$, and the completion $\mathbf{C}_E$ of $\overline{E}$ (with respect to the unique extension of valuation from E) with ring of integers $\mathcal{O}_{\mathbf{C}_E}$.

Recall that a formal scheme (automatically locally Noetherian) $\mathfrak{X}$ over $\mathrm{Spf}(\mathcal{O})$ is called *locally topologically of finite type* over $\mathrm{Spf}(\mathcal{O})$ if there is an open cover by affines of the form $\mathrm{Spf}(A)$ where A is *topologically of finite type* which means that it admits a continuous open surjection of $\mathcal{O}$-algebras
$$\mathcal{O}\langle t_1, \ldots, t_s\rangle \to A \tag{5.24}$$

where, as per usual, $\mathcal{O}\langle t_1, \ldots, t_n\rangle$ is the ring of convergent power series over $\mathcal{O}$ (i.e. the π-adic completion of $\mathcal{O}[t_1, \ldots, t_s]$). We say that $\mathfrak{X}$ is *topologically of finite type* if it is, in addition, quasi-compact.

Similarly, we say that $\mathfrak{X}$ *locally formally of finite type* if there is an open cover of $\mathfrak{X}$ by affine opens $\mathrm{Spf}(A)$ where A is *formally of finite type*, which means that there exists a continuous open surjection of $\mathcal{O}$-algebras

$$\mathcal{O}[\![u_1, \ldots, u_r]\!]\langle t_1, \ldots, t_s\rangle \to A \tag{5.25}$$

We say that $\mathfrak{X}$ is *formally of finite type* if it is locally formally of finite type and compact.

Let us denote the underlying reduced subscheme of $\mathfrak{X}$ by $\mathfrak{X}^{\mathrm{red}}$ and its special fiber $\mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O})} \mathrm{Spec}(k)$ by $\mathfrak{X}_s$. Then, we have the following basic algebraic characteriation of topologically/formally of finite type:

Lemma 5.3.1. *A formal scheme $\mathfrak{X}$ over $\mathrm{Spf}(\mathcal{O})$ is (locally) topologically of finite type over $\mathrm{Spf}(\mathcal{O})$ if and only if its special fiber $\mathfrak{X}_s$ is (locally) of finite type over $\mathrm{Spec}(k)$. It is (locally) formally of finite type over $\mathrm{Spf}(\mathcal{O})$ if its reduced subscheme $\mathfrak{X}^{\mathrm{red}}$ is (locally) finite type over $\mathrm{Spec}(k)$.*

Proof. See [Ber96, Lemma 1.2]. □

Let us remark the following basic observation:

Lemma 5.3.2. *Let $\mathfrak{X}, \mathfrak{Y}$ and $\mathfrak{Z}$ be formal $\mathcal{O}$-schemes locally formally of finite type. Then, $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z}$ exists in the category of locally formally of finite type formal $\mathcal{O}$-schemes.*

Let us give the following basic examples of formal schemes:

Example 5.3.3. The *open formal disk* and *closed formal disk* over $\mathrm{Spf}(\mathcal{O})$, defined to be $\mathrm{Spf}(\mathcal{O}[\![u_1, \ldots, u_r]\!])$ and $\mathrm{Spf}(\mathcal{O}\langle t_1, \ldots, t_s\rangle)$ respectively, are evidently formally of finite type over $\mathrm{Spf}(\mathcal{O})$, with only the closed formal disk being topologically of finite type.

We then have the following comparison between the theory of formal schemes and adic spaces:

Lemma 5.3.4. *There is a fully faithful embedding*

$$\left\{\begin{matrix}\textit{Formal schemes formally}\\ \textit{of finite type over } \mathrm{Spf}(\mathcal{O})\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\textit{Adic spaces}\\ \textit{over } \mathrm{Spa}(\mathcal{O})\end{matrix}\right\} \tag{5.26}$$

(which is a special case of an embedding into adic spaces of all locally Noetherian formal schemes), denoted $\mathfrak{X} \mapsto \mathfrak{X}^{\mathrm{ad}}$, whose underlying adic space is characterized by the existence of a map of locally topologically ringed spaces

$$\lambda_{\mathfrak{X}} : (\mathfrak{X}^{\mathrm{ad}}, \mathcal{O}^+_{\mathfrak{X}^{\mathrm{ad}}}) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \tag{5.27}$$

which is universal amongst such maps from adic spaces. This functor enjoys the following nice properties:

1. *If $\mathfrak{X}$ is the affine formal scheme $\mathrm{Spf}(A)$ then $\mathfrak{X}^{\mathrm{ad}} = \mathrm{Spa}(A)$, and $\lambda_{\mathfrak{X}}(v)$, for a point $v \in \mathrm{Spa}(A)$, is defined to be $\{a \in A : v(a) < 1\}$.*

2. *On the level of topological spaces* $\lambda_{\mathfrak{X}}^{-1}(\mathrm{Spf}(A)) = \mathrm{Spa}(A)$ *for any affine open* $\mathrm{Spf}(A) \subseteq \mathfrak{X}$.

3. $\mathfrak{X}^{\mathrm{ad}}$ *is qcqs.*

4. *The natural map* $(\mathfrak{X}_{\breve{\mathcal{O}}})_\eta \to (\mathfrak{X}_\eta)_{\breve{\mathcal{O}}}$ *is an isomorphism.*

Proof. See [Hub94, Proposition 4.1] for the first part of the claim, and see Proposition 4.2 of loc. cit. for an essential characterization of the image of this functor.

The fact that $\mathrm{Spf}(A)^{\mathrm{ad}} = \mathrm{Spa}(A)$ would follow from some formal scheme analogue of the statement that $\mathrm{Spec}(A)$ represents the functor sending S to $\mathrm{Hom}(A, \mathcal{O}(S))$ on the category of locally topologically ringed spaces. Namely, we would want a statement such as maps $T \to \mathrm{Spf}(A)$ of locally topologically ringed spaces correspond bijectively to continuous ring maps $A \to \mathcal{O}(T)$. In fact, it would suffice to prove then when T is of the form $(Y, \mathcal{O}_Y^+)$ for an adic space Y. This is what Huber does in[Hub94, Proposition 4.1, (1)].

Let us now show why $\lambda_{\mathfrak{X}}^{-1}(\mathrm{Spf}(A)) = \mathrm{Spa}(A)$. This follows quite easily from the universal property. Indeed, if we consider $U := \lambda_{\mathfrak{X}}^{-1}(\mathrm{Spf}(A))$ as an open adic subspace of $\mathfrak{X}^{\mathrm{ad}}$ then it's evident that we obtain a map of locally topologically ringed spaces $(U, \mathcal{O}_U^+) \to (\mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)})$. Moreover, let Y be any adic space with a map of locally topologically ringed spaces $(Y, \mathcal{O}_Y^+) \to (\mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)})$. Note then that by the universal property of $t(\mathfrak{X})$ have a unique morphism $Y \to X$ of adic spaces factoring the map to $\mathrm{Spf}(A)$. Moreover since Y has image in $\mathrm{Spf}(A) \subseteq \mathfrak{X}$ we see that the map $Y \to t(\mathfrak{X})$ has image in $\lambda_{\mathfrak{X}}^{-1}(\mathrm{Spf}(A))$, from where the claim easily follows.

To see that $\mathfrak{X}^{\mathrm{ad}}$ is qc is just the simple observation that because $\mathfrak{X}$ is covered by the finitely many compact subsets $\mathrm{Spa}(A_i)$, for $\mathrm{Spf}(A_i)$ a finite affine open covering of $\mathfrak{X}$. To see that it's quasi-separated it suffices to see that the cover $\{\mathrm{Spf}(A_i)\}$ has quasi-separated preimage. This follows from the fact that $\lambda_{\mathfrak{X}}^{-1}(\mathrm{Spf}(A_i))$ is $\mathrm{Spa}(A_i)$ which is spectral.

This follows more generally from the fact that the functor $\mathfrak{X}^{\mathrm{ad}}$ can be extended to all locally Noetherian formal schemes with the same description. Namely, we know that if Y is an adic space, then maps $Y \to (\mathfrak{X}_\eta)_{\breve{\mathcal{O}}}$ correspond to maps $Y \to \mathfrak{X}_\eta$ and to maps $Y \to \mathrm{Spa}(\breve{\mathcal{O}})$. But, since $\mathrm{Spa}(\breve{\mathcal{O}}) = \mathrm{Spf}(\breve{\mathcal{O}})^{\mathrm{ad}}$ such data is the same as a pair of maps of locally topologically ringed spaces $(Y, \mathcal{O}_Y^+) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ and $(Y, \mathcal{O}_Y^+) \to (\mathrm{Spf}(\breve{\mathcal{O}}), \mathcal{O}_{\mathrm{Spf}(\breve{\mathcal{O}})})$ which is the same thing as a map of locally topologically ringed spaces $(Y, \mathcal{O}_Y^+) \to (\mathfrak{X}_{\breve{\mathcal{O}}}, \mathcal{O}_{\mathfrak{X}_{\breve{\mathcal{O}}}})$ which can be checked by reduction to the affine case and the aforementioned [Hub94, Proposition 4.1, (1)]. So, we see that $(\mathfrak{X}^{\mathrm{ad}})_{\breve{\mathcal{O}}}$ represents the fsame functor as $(\mathfrak{X}_{\breve{\mathcal{O}}})^{\mathrm{ad}}$, and the conclusion follows. $\square$

We call the adic spaces $\mathfrak{X}^{\mathrm{ad}}$ the *adification* of the formal scheme $\mathfrak{X}$.

For example, we have the following obvious correspondence:

Example 5.3.5. The open formal disk $\mathrm{Spf}([\![u_1, \ldots, u_r]\!])$ over $\mathrm{Spf}(\mathcal{O})$ has adification the open formal disk $\mathrm{Spa}([\![u_1, \ldots, u_s]\!])$ over $\mathrm{Spa}(\mathcal{O})$ from Example 5.1.12, and similarly for the closed formal disk.

We define the *generic fiber* (relative to $\mathcal{O}$) of $\mathfrak{X}$, a formal $\mathrm{Spf}(\mathcal{O})$-scheme locally formally of finite type, to be the fibered product

$$\mathfrak{X}_\eta := \mathfrak{X}^{\mathrm{ad}} \times_{\mathrm{Spa}(\mathcal{O})} \mathrm{Spa}(E) \tag{5.28}$$

Note that this exists since $\mathrm{Spa}(E)$ is an open adic subspace of $\mathrm{Spa}(\mathcal{O})$, given by the non-vanishing locus of π.

We then have the following examples of the formation of generic fiber:

Example 5.3.6. The generic fiber $\mathrm{Spf}(\mathcal{O}[\![u_1, \ldots, u_r]\!])_\eta$ is the open unit r-disk $\mathbb{D}_E^r$ over $\mathrm{Spa}(E)$ as observed in Example 5.1.31. Similarly, the generic fiber $\mathrm{Spf}(\mathcal{O}\langle t_s, \ldots, t_s\rangle)_\eta$ is the closed unit s-disk $\mathbb{B}_E^s$ over $\mathrm{Spa}(E)$.

Example 5.3.7. The generic fiber of $\mathrm{Spf}(\mathcal{O}[\![u_1, \ldots, u_r]\!]\langle t_1, \ldots, t_s\rangle)$ is $\mathbb{D}_E^r \times_{\mathrm{Spa}(E)} \mathbb{B}_E^s$. Indeed, it's clear that $\mathrm{Spf}(\mathcal{O}[\![u_1, \ldots, u_r]\!]\langle t_1, \ldots, t_s\rangle)$ is the fiber product $\mathrm{Spf}(\mathcal{O}[\![u_1, \ldots, u_r]\!]) \times_{\mathrm{Spf}(\mathcal{O})} \mathrm{Spf}(\mathcal{O}\langle t_1, \ldots, t_s\rangle)$ and the result follows from the obvious fact (recorded below in Lemma 5.3.9) that the generic fiber commutes with base change.

Example 5.3.8. Let us denote by $\widehat{\mathbf{G}_m}$ the formal scheme over $\mathrm{Spf}(\mathcal{O})$ obtained by taking the π-adic completion of the $\mathcal{O}$-scheme $\mathbf{G}_m = \mathrm{Spec}(\mathcal{O}[t, t^{-1}])$. In other words, $\widehat{\mathbf{G}_m}$ is $\mathrm{Spf}(\mathcal{O}\langle t, t^{-1}\rangle)$. It is called the *formal torus* over $\mathrm{Spf}(\mathcal{O})$. Then, the generic fiber $\widehat{\mathbf{G}_m}_\eta$ is the unit circle $\mathbb{T}_E$ over $\mathrm{Spa}(E)$. Indeed, it's clear that $\widehat{\mathbf{G}_m}^{\mathrm{ad}}$ represents the functor $(R, R^+) \mapsto (R^+)^\times$ on $\mathsf{AffAdic}_{\mathcal{O}}$, and thus the generic fiber $\widehat{\mathbf{G}_m}^{\mathrm{ad}}$ represents the same functor as $\mathbb{T}_E$ on $\mathsf{AffAdic}_E$, and thus they're isomorphic. Alternatively, one can just use the fact that $\mathrm{Spa}(\mathcal{O}\langle t, t^{-1}\rangle) \to \mathrm{Spa}(\mathcal{O})$ is actually topologically of finite type, and so Lemma 5.1.29 actually implies that $\mathrm{Spa}(\mathcal{O}\langle t, t^{-1}\rangle)_\eta$ is the affinoid obtained by the procedure in the proof of the lemma. It's clear that this affinoid is just $\mathrm{Spa}(E\langle t, t^{-1}\rangle) = \mathbb{T}_E$.

This justifies why $\mathbb{T}_E$ is called a torus, since it's the generic fiber of a formal torus (obtained by the completion of an algebraic torus over $\mathrm{Spec}(\mathcal{O})$).

Huber proves the following about the construction $\mathfrak{X}_\eta$:

Lemma 5.3.9. *Let $\mathfrak{X}$ be a formal scheme locally formally of finite type over $\mathrm{Spf}(\mathcal{O})$. Then, the following properties hold:*

1. *If $\mathfrak{X} \to \mathfrak{Y}$ is a morphism of formal schemes locally of finite type over $\mathrm{Spf}(\mathcal{O})$, and if $\mathfrak{Z} \to \mathfrak{Y}$ is a morphism locally of finite type then*

$$(\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z})_\eta \cong \mathfrak{X}_\eta \times_{\mathfrak{Y}_\eta} \mathfrak{Z}_\eta \tag{5.29}$$

2. *The natural morphism*

$$(\mathfrak{X}_{\breve{\mathcal{O}}})_\eta \to \mathfrak{X}_\eta \times_{\mathrm{Spa}(E)} \mathrm{Spa}(\breve{E}) \tag{5.30}$$

is an isomorphism.

Proof. See [Hub98, Lemma 3.13 i)] for the first claim.

For the last claim, let us merely note that

$$\begin{aligned}(\mathfrak{X}_{\breve{\mathcal{O}}})_\eta &= (\mathfrak{X}_{\breve{\mathcal{O}}})^{\mathrm{ad}} \times_{\mathrm{Spa}(\breve{\mathcal{O}})} \mathrm{Spa}(\breve{E}) \\ &= \left(\mathfrak{X}^{\mathrm{ad}} \times_{\mathrm{Spa}(\mathcal{O})} \mathrm{Spa}(\breve{\mathcal{O}})\right) \times_{\mathrm{Spa}(\breve{\mathcal{O}})} \mathrm{Spa}(\breve{E}) \\ &= \left(\mathfrak{X}^{\mathrm{ad}} \times_{\mathrm{Spa}(\mathcal{O})} \mathrm{Spa}(E)\right) \times_{\mathrm{Spa}(E)} \mathrm{Spa}(\breve{E}) \\ &= \mathfrak{X}_\eta \times_{\mathrm{Spa}(E)} \mathrm{Spa}(\breve{E})\end{aligned} \tag{5.31}$$

as desired where we have used Lemma 5.3.4 part (4) in the obvious place. □

Remark 5.3.10. One might wonder why we have not discussed the analogues of Lemma 5.3.4 part (4) (resp. Lemma 5.3.9 part (3)) with $\breve{\mathcal{O}}$ (resp. $\breve{E}$) replaced by $\widehat{\overline{\mathcal{O}}}$ (resp. $\widehat{\overline{E}}$). The reason is quite simple. Namely, $\widehat{\overline{\mathcal{O}}}$ is not Noetherian, and so without knowing some strong condition (like the strong sheafiness, and its formal power series analogue) we can't discuss $\mathfrak{X}_{\widehat{\overline{\mathcal{O}}}}$ in the language of Huber's adic spaces.

We then have the following structural theorem about $\mathfrak{X}_\eta$, which tells us that it falls into the reasonable category of adic spaces over $\mathrm{Spa}(E)$ discussed in the previous section:

Lemma 5.3.11. *Let $\mathfrak{X}$ be a separated formal scheme which is locally formally of finite type over* $\mathrm{Spf}(\mathcal{O})$*. Suppose first that $\mathfrak{X}$ is affine. Then there exists a filtration*

$$\mathfrak{X}_\eta = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \tag{5.32}$$

where X_i are finite type $\mathrm{Spa}(E)$*-adic spaces and $\overline{X_i}$ is quasi-compact in $\mathfrak{X}_\eta$. For general $\mathfrak{X}$, $\mathfrak{X}_\eta \to$* $\mathrm{Spa}(E)$ *is taut, locally of finite type, and separated (and so, in particular, good).*

Proof. Let us begin by showing that we have the lcaimed filtration when $\mathfrak{X} = \mathrm{Spf}(A)$, where A is formally of finite type, with some fixed surjection as in equation (5.25). So, then we have that $\mathfrak{X}^{\mathrm{ad}} = \mathrm{Spa}(A)$. Let us now denote the image of u_i under equation (5.25) by $f_i \in A$ then we can describe $\mathfrak{X}_\eta$ as the increasing union

$$\mathfrak{X}_\eta = \bigcup_{n\geqslant 1} U_n \tag{5.33}$$

where

$$U_n = U\left(\frac{f_1^n, \ldots, f_r^n}{\pi}\right) \tag{5.34}$$

where, now, here U_n is a rational open in $\mathrm{Spa}(A)$, and so isomorphic to $\mathrm{Spa}(\mathcal{O}(U_n), \mathcal{O}^+(U_n))$. It now suffices to show that U_n has quasi-compact closure in $\mathfrak{X}_\eta$. To see this, let us note that we have a closed embedding

$$\mathfrak{X} \hookrightarrow \mathrm{Spf}(\mathcal{O}[\![u_1, \ldots, u_r]\!]\langle t_1, \ldots, t_s\rangle) \tag{5.35}$$

which then obviously gives us a closed embedding

$$\mathfrak{X}_\eta \hookrightarrow \mathrm{Spf}(\mathcal{O}[\![u_1, \ldots, u_r]\!]\langle t_1, \ldots, t_s\rangle)_\eta \tag{5.36}$$

where, if we identify this latter rigid space, with $\mathbb{D}_E^r \times_{\mathrm{Spa}(E)} \mathbb{B}_E^s$ (where $\mathbb{D}_E$ is the open unit disk over $\mathrm{Spa}(E)$, and $\mathbb{B}_E$ the closed unit disk), then U_n closed embeds into $\mathbb{B}_E^r(0, |\pi|^{\frac{1}{n}}) \times_{\mathrm{Spa}(E)} \mathbb{B}_E^s$, where $\mathbb{B}_E^r(0, |\pi|^{\frac{1}{n}})$ denotes the concentric closed subdisk of $\mathbb{D}$ of radius $|\pi|^{\frac{1}{n}}$. Thus, it's clear that checking our desired properties about the U_n reduces to checking the analagous properties for the filtration $\mathbb{B}_E^r(0, |\pi||^{\frac{1}{n}}) \times_{\mathrm{Spa}(E)} \mathbb{B}_E^s$ of $\mathbb{D}_E^r \times_{\mathrm{Spa}(E)} \mathbb{B}_E^s$ which is easily done by hand.

Note that it easily follows from the above that $\mathfrak{X}_\eta$ is locally of finite type over $\mathrm{Spa}(A)$. Let us now explain why $\mathfrak{X}_\eta$ is separated. Let us denote by $f : \mathfrak{X} \to \mathrm{Spf}(\mathcal{O})$ the structure map and $f_\eta : \mathfrak{X}_\eta \to \mathrm{Spa}(E)$ the induced map on geierci fibers. We need to show that the map

$$\Delta_{f_\eta} : \mathfrak{X}_\eta \to \mathfrak{X}_\eta \times_{\mathrm{Spa}(E)} \mathfrak{X}_\eta \tag{5.37}$$

has closed image. That said, it's evident that under the identification of $(\mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O})} \times_{\mathrm{Spa}(\mathcal{O})} \mathfrak{X})_\eta$ with $\mathfrak{X}_\eta \times_{\mathrm{Spa}(E)} \mathfrak{X}_\eta$ that $(\Delta_f)_\eta = \Delta_{f_\eta}$. In particular, the separatedness of $\mathfrak{X}_\eta$ quickly follows from that of $\mathfrak{X}$.

Lastly, let's explain why this implies that $f_\eta : \mathfrak{X}_\eta \to \mathrm{Spa}(E)$ is reasonable. The latter two properties have already been explained, so it suffices to prove the first. But, one can easily reduce to the case of affine formal schemes for which it's obvious. $\square$

We can explicitly describe the functor of points $\mathfrak{X}_\eta$ on $\mathsf{AffAdic}_E$, namely we have the following:

Lemma 5.3.12. *Let (R, R^+) be an object of $AffAdic_E$. Then, R^+ is a filtered colimit $\varinjlim R_0$ where R_0 ranges over the $\mathcal{O}$-algebra rings of definition $R_0 \subseteq R^+$. Moreover $\mathfrak{X}_\eta$, as a sheaf on $\mathsf{AffAdic}_E$ (with the big Zariski topology) can be functorially identified with the sheafification of the presheaf that sends (R, R^+) to $\varinjlim \mathfrak{X}(\mathrm{Spf}(R_0))$, where R_0 is endowed with the π-adic topology.*

Proof. This is a special case of [SW12, Proposition 2.2.2]. $\square$

Let us note the following examples:

Example 5.3.13. If $\mathfrak{X} = \mathrm{Spf}(\mathcal{O}\langle t \rangle)$ then Lemma 5.3.12 just says that there is a functorial identification between $\mathfrak{X}_\eta(R, R^+) = R^+$ and $\varinjlim \mathfrak{X}(R_0) = \varinjlim R_0$, which is just the content of the first part of the lemma. If $\mathfrak{X} = \mathrm{Spf}(\mathcal{O}[\![t]\!])$ then this says that there is a functorial identification between $\mathfrak{X}_\eta(R, R^+) = R^{\circ\circ}$ and $\varinjlim R_0^{\circ\circ}$. But, it's evident that $\varinjlim R_0^{\circ\circ}$ can naturally be identified with $(\varinjlim R_0)^{\circ\circ}$ which is $(R^+)^{\circ\circ} = R^{\circ\circ}$, thus proving the claim in this case. In fact, for formal schemes formally of finite type over $\mathrm{Spf}(\mathcal{O})$ one can reduce Lemma 5.3.12 to these two cases, thus giving an alternate proof to Lemma 5.3.12.

We would like to know in what situations is $\mathfrak{X}_\eta$ partially proper. To this end, let $\mathfrak{X}$ and $\mathfrak{Y}$ be two formal $\mathcal{O}$-schemes locally formally of finite type over $\mathcal{O}$, and let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism. We call f *pseudo-proper* if for any irreducible components $\mathfrak{C} \subseteq \mathfrak{X}$ and $\mathfrak{D} \subseteq \mathfrak{Y}$ such that $f(\mathfrak{C}) \subseteq \mathfrak{D}$ the induced morphism $f^{\mathrm{red}} : \mathfrak{C}^{\mathrm{red}} \to \mathfrak{D}^{\mathrm{red}}$ is proper.

We then have the following result:

Theorem 5.3.14. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be pseudo-proper morphism of locally formally of finite type separated $\mathcal{O}$-schemes. Then, $f_\eta : \mathfrak{X}_\eta \to \mathfrak{Y}_\eta$ is partially proper and locally of finite type.*

Proof. Let us note that since $\mathfrak{X}_\eta$ and $\mathfrak{Y}_\eta$ are locally of finite type over $\mathrm{Spa}(E)$ and taut, that f_η is automatically taut and locally finite type. It's also clear that it's separated since this condition can be checked by having closed diagonal map, which descends evidently to the generic fiber. Thus, it suffices to check that it satisfies the existence part of the valuative criterion. Let us denote the embeddings $i_1 : \mathfrak{X}_\eta \hookrightarrow \mathfrak{X}$ and $i_2 : \mathfrak{Y}_\eta \hookrightarrow \mathfrak{Y}$. Suppose that (L, L^+) is an analytic affinoid field, and let $j : \mathrm{Spa}(L, L^\circ) \to \mathrm{Spa}(L, L^+)$ denote the natural morphism. Suppose that we have a morphism a morphism $g : \mathrm{Spa}(L, L^+) \to \mathfrak{Y}_\eta$ and a morphism $h : \mathrm{Spa}(L, L^\circ) \to \mathfrak{X}_\eta$ such that $f_\eta \circ h = g \circ j$. Note then that by [Stu17, Lemma 3.15] we obtain there exists a morphism $k : \mathrm{Spa}(L, L^+) \to \mathfrak{X}$ such that $f \circ k = i_2 \circ g$. Note though that since $\pi \neq 0$ in L, we have that the morphism k has image landing in $\mathfrak{X}_\eta$. Moreover, it's then clear that $f_\eta \circ k = g$. The conclusion follows. $\square$

As a corollary, we obtain the following:

Corollary 5.3.15. *Let $\mathfrak{X}$ be a formal scheme formally of finite type over* $\mathrm{Spf}(\mathcal{O})$. *Assume that $\mathfrak{X}_{\mathrm{red}}$ is proper over* $\mathrm{Spec}(k)$. *Then, $\mathfrak{X}_\eta$ is partially proper over* $\mathrm{Spa}(E)$.

As a matter of notational convention, we will denote by $\mathfrak{X}_{\eta^{\mathrm{ur}}}$ the adic space over $\mathrm{Spa}(\breve{E})$ obtained from base change of $\mathfrak{X}_\eta$ to $\mathrm{Spa}(\breve{E})$ and by $\mathfrak{X}_{\overline{\eta}}$ the adic space over $\mathrm{Spa}(\widehat{\overline{E}})$ obtained from base change of $\mathfrak{X}_\eta$ to $\mathrm{Spa}(\widehat{\overline{E}})$. Note that these are both still good over their base.

Let us note that composing $\lambda_{\mathfrak{X}}$ with the natural inclusion $\mathfrak{X}_\eta \hookrightarrow \mathfrak{X}^{\mathrm{ad}}$ we get a map of topological spaces

$$\mathrm{sp}_{\mathfrak{X}} : |\mathfrak{X}_\eta| \to |\mathfrak{X}| = |\mathfrak{X}^{\mathrm{red}}| \tag{5.38}$$

called the *specialization map* of $\mathfrak{X}$. The specialization map is spectral (see [Hub96, (1.1.13)] for the definition of spectral, and [Hub98, Lemma 3.13 i)] for the proof) which tells us, in particular, that for any closed subset Z of $|\mathfrak{X}^{\mathrm{red}}|$ the subset $\mathrm{sp}_{\mathfrak{X}}^{-1}(Z)$ is constructible and locally closed.

If $\mathfrak{X} = \mathrm{Spf}(A)$ then we can describe the specialization map explicitly using Lemma 5.3.4. Namely, if I is the maximal ideal of definition of $\mathrm{Spf}(A)$ (i.e. $I = A^{\circ\circ}$) then $\mathrm{sp}_{\mathfrak{X}}(v)$ is

$$\{a \in A : v(a) < 1\}/I \in \mathrm{Spec}(A/I) = \mathfrak{X}^{\mathrm{red}} \tag{5.39}$$

This map is deceptively deep, and will be useful not only in the study of the étale cohomology of generic fibers, but actually is part of a characterization of a class of formal schemes (as we will see in Theorem 5.3.22).

Before we explain this, let us give some examples:

Example 5.3.16. Let's fix $\mathfrak{X}$ to be $\mathrm{Spf}(\mathcal{O}\langle t\rangle)$, the formal closed disk, so that $\mathfrak{X}_\eta = \mathbb{B}_E$ and $\mathfrak{X}^{\mathrm{red}} = \mathbb{A}^1_k$. We then obtain a map of topological spaces $\mathrm{sp}_{\mathfrak{X}} : |\mathbb{B}_E| \to |\mathbb{A}^1_k|$. If x is a classical point of E, corresponding to $\alpha \in \mathcal{O}$, then one has that $\mathrm{sp}_{\mathfrak{X}}(x)$ is the point $\overline{\alpha} := (\alpha \mod \pi) \in \mathbb{A}^1(k)$. Indeed, one sees that $\widetilde{\lambda_{\mathfrak{X}}}$ is the open ideal of $\mathcal{O}\langle t\rangle$ consisting of those f such that $|f(x)| < 1$. Note that if we express $f(t) = \displaystyle\sum_{n=0}^{\infty} a_n(t-\alpha)^n$, then this is equivalent to saying that $\pi \mid a_0$. Thus, the reduction of this ideal is polynomials in

$(t-\overline{\alpha})$ with no-constant term. In other words, it corresponds to the ideal $(t-\overline{\alpha}) \in \mathbb{A}^1_k$ as claimed.

Whereas, if x is the Gauss point (in the parlance of [Sch12, Example 2.20]) then $\mathrm{sp}_{\mathfrak{X}}(x)$ is the generic point of $\mathbb{A}^1_k$. Indeed, the set $f(t) \in \mathcal{O}\langle t\rangle$ such that $|f(x)| < 1$ are precisely $\pi\mathcal{O}\langle t\rangle$. The reduction of this modulo π is just the zero ideal.

Example 5.3.17. If $\mathfrak{X}$ is the open formal disk $\mathrm{Spf}(\mathcal{O}[\![t]\!])$, then there is not much content to the specialization map. Namely, in this case $\mathfrak{X}^{\mathrm{red}} = \mathrm{Spec}(k)$, and so $|\mathfrak{X}^{\mathrm{red}}|$ is a singleton. Thus, the specialization map is the unique map $|\mathfrak{X}_\eta| \to |\mathfrak{X}^{\mathrm{red}}|$.

Let us now explain the importance of the specialization map in two steps, first the more concrete application to the study of a formal scheme, and its closed formal subschemes, and then the more claim that the specialization map has deep impacts on the structure of formal schemes.

To explain this first part, let us assume that $\mathfrak{X}$ is a formal scheme locally formally of finite type over $\mathrm{Spf}(\mathcal{O})$. Let us now assume that Z is a closed subscheme of $\mathfrak{X}$. Let us denote by $\widehat{\mathfrak{X}}_{/Z}$ the completion of $\mathfrak{X}$ along Z. We note that $\widehat{\mathfrak{X}}_{/Z}$ is a formal scheme locally formally of finite type over $\mathcal{O}$. Indeed, by Lemma 5.3.1 it suffices to check that $(\widehat{\mathfrak{X}}_{/Z})^{\mathrm{red}}$ is locally of finite type over $\mathrm{Spec}(k)$. That said, $(\widehat{\mathfrak{X}}_{/Z})^{\mathrm{red}} = Z^{\mathrm{red}}$, and Z^{red} is a closed subscheme of $\mathfrak{X}^{\mathrm{red}}$, and thus evidently locally of finite type over Z.

Let us note that we have the canonical morphism $f : \widehat{\mathfrak{X}}_{/Z} \to \mathfrak{X}$, and one might ask what properties the morphism f_η posseses. To this end, one has the following pretty result:

Lemma 5.3.18. *The morphism f_η is an open embedding with image precisely $\mathrm{sp}_{\mathfrak{X}}^{-1}(Z^{\mathrm{red}})^\circ$ (where, here, $(-)^\circ$ denotes the interior as a subspace of the topological space $\mathfrak{X}_\eta$).*

Proof. See [Hub98, Lemma 3.13]. □

So, we see that the specialization map plays a pivotal topological role in the study of generic fibers of certain closed formal subschemes of $\mathfrak{X}$. Namely, as remarked above, $\mathrm{sp}_{\mathfrak{X}}^{-1}(Z^{\mathrm{red}})$ is a closed constructible subset of $\mathfrak{X}_\eta$, and thus $(\widehat{\mathfrak{X}}_{/Z})_\eta$ is the interior of a closed constructible set of $\mathfrak{X}_\eta$. This turns out to have deep consequences for the cohomology of these spaces, as we will later discuss.

Let us give an example of the content of Lemma 5.3.18:

Example 5.3.19. Take $\mathfrak{X}$ to be the closed formal disk $\mathrm{Spf}(\mathcal{O}\langle t\rangle)$. Consider the origin (t) in the reduced subscheme $\mathbb{A}^1_k$. Then, the corresponding point of $|\mathfrak{X}|$, is the open ideal $J = (t,\pi)$. The completion of $\mathfrak{X}$ along $Z = V(J)$ is $\mathcal{O}[\![t]\!]$ and the morphism $f : \widehat{\mathfrak{X}}_{/Z} \to \mathfrak{X}$ is the morphism $\mathrm{Spf}(\mathcal{O}[\![t]\!]) \to \mathrm{Spf}(\mathcal{O}\langle t\rangle)$ coming from the inclusion $\mathcal{O}\langle t\rangle \to \mathcal{O}[\![t]\!]$. Thus, the morphism f_η is the natural inclusion of $\mathbb{D}_E$ into $\mathbb{B}_E$. Note then that a valuation $v \in \mathbb{B}_E$ is in $\mathrm{sp}_{\mathfrak{X}}^{-1}((t))$ if and only if $\{a \in \mathcal{O}\langle t\rangle : v(a) < 1\}/(\pi) = (t)$ (or, equivalently, if $\{a \in \mathcal{O}\langle t\rangle : v(a) < 1\} = J$).

Now, if $v \in \mathbb{D}_E$ then $v \in U_n$ for some n (in the parlance of Example 5.1.6) and thus for some $n \geqslant 1$ we have that $v(t^n) \leqslant v(\pi) < 1$. In particular, we see that $\{a \in \mathcal{O}\langle t\rangle : v(a) < 1\} \supseteq (\pi, t^n)$ and since this ideal is prime we have that $\{a \in \mathcal{O}\langle t\rangle : v(a) < 1\} \supseteq J$. Since this latter ideal is maximal we conclude that $\{a \in \mathcal{O}\langle t\rangle : v(a) < 1\} = J$. Thus, we see that if $v \in \mathbb{D}_E$ then $\mathrm{sp}_{\mathfrak{X}}(v) = (t)$ and so we see that $\mathbb{D}_E \subseteq \mathrm{sp}_{\mathfrak{X}}^{-1}((t))$ and, since $\mathbb{D}_E$ is open in $\mathbb{B}_E$, we see that $\mathbb{D}_E \subseteq \mathrm{sp}_{\mathfrak{X}}^{-1}((t))^\circ$. We leave it to the reader to see why this is an equality.

Now, to explain what we mean by the aforementioned cryptic claim that the specialization map has deep consequences on the geometry of the formal scheme, we first make an observation. The formal scheme $\mathfrak{X}$ over $\mathrm{Spf}(\mathcal{O})$ is an object of a fundamentally mixed characteristic, integral nature. Thus, not shockingly, $\mathfrak{X}_\eta$ captures nowhere near the entire geometry of $\mathfrak{X}$.

For example:

Example 5.3.20. Let $\mathfrak{X}$ be the closed formal disk $\mathrm{Spf}(\mathcal{O}\langle t\rangle)$ which is the π-adic completion of $\mathrm{Spec}(\mathcal{O}[t])$. Consider then $\widetilde{\mathfrak{X}}$ defined to be the π-adic completion of the blowup B of $\mathrm{Spec}(\mathcal{O}[t])$ along (π, t). One then gets a natural map $\widetilde{\mathfrak{X}} \to \mathfrak{X}$ by blowing up the natural map $B \to \mathrm{Spec}(\mathcal{O}[t])$. One can show that since (π, t) is principle over the generic fiber, that $\widetilde{\mathfrak{X}}_\eta \to \mathfrak{X}_\eta$ is an isomorphism of adic spaces over $\mathrm{Spa}(E)$. But, $\widetilde{\mathfrak{X}} \to \mathfrak{X}$ is certainly not an isomorphism of formal schemes since $\widetilde{\mathfrak{X}}^{\mathrm{red}}$ is non-smooth over k, but $\mathfrak{X}^{\mathrm{red}} = \mathbb{A}^1_k$.

Remark 5.3.21. While we won't discuss it here, Example 5.3.20 gives insight into yet another, beautiful, theory of 'rigid spaces' over E developed by Raynaud (see [Ray74]). Namely, one can show that if they restrict the generic fiber functor to a suitable class (quasi-compact and admissible) of formal schemes over $\mathrm{Spf}(\mathcal{O})$, then it essentially surjects onto the class of quasi-separated finite type adic spaces over $\mathrm{Spa}(E)$. Moreover, as Example 5.3.20 suggets, if one inverts the so-called 'admissible blowups' on this category of formal schemes the generic fiber functor becomes an equivalence. See [BL93], specifically Section 4, for a nice discussion of this in classical language. In particular, this gives one the very algebraic perspective that (certain) adic spaces over $\mathrm{Spa}(E)$ are the generic fibers of formal schemes.

So, given the above observation that the generic fiber functor is far from fully-faithful, one might wonder what the minimal extra data about $\mathfrak{X}$ is needed to remember the full structure of $\mathfrak{X}$. For example, as Example 5.3.20 shows, the one thing that is totally forgotten by the generic fiber is, not shockingly, the reduced subscheme. So, one should also keep track of the reduced subscheme $\mathfrak{X}^{\mathrm{red}}$. But, one also should have some sort of glue that binds the generic $\mathfrak{X}_\eta$ and $\mathfrak{X}^{\mathrm{red}}$ together to build the mixed characteristic object $\mathfrak{X}$. Astonishingly, the topological data of $\mathrm{sp}_{\mathfrak{X}}$ is enough, at least in the case when the formal scheme satisfies some mild conditions.

Namely, let us call a formal scheme $\mathfrak{X}$ locally of finite type over $\mathrm{Spf}(\mathcal{O})$ *normal* if for all $x \in \mathfrak{X}$ the local ring $\mathcal{O}_{\mathfrak{X},x}$ is normal. Similarly, we say that $\mathfrak{X}$ is *flat* over $\mathcal{O}$ if for all $x \in \mathfrak{X}$ the local $\mathcal{O}$-algebra $\mathcal{O}_{\mathfrak{X},x}$ is flat. Then, if $\mathfrak{X}$ is normal and flat over $\mathcal{O}$ then an observation of Scholze, using a result of Lourenço, shows that the knowedgthis is enough over $\breve{\mathcal{O}}$:

Theorem 5.3.22 (Scholze-Lourenço). *Let $\mathcal{O}$ have algebraically closed residue field, and let $\mathscr{C}$ denote the category of locally formally of finite type, normal, flat formal $\mathcal{O}$-schemes and $\mathscr{D}$ the category of triples (X, Y, p) where X is an adic space over* $\mathrm{Spa}(E)$*, Y is a perfect scheme over k, and p is a morphism of topological spaces $p : |X| \to |Y|$. Then, the association $\mathfrak{X} \mapsto (\mathfrak{X}_\eta, (\mathfrak{X}^{\mathrm{red}})^{\mathrm{perf}}, \mathrm{sp}_{\mathfrak{X}})$ is a fully faithful functor $\mathscr{C} \to \mathscr{D}$.*

Proof. See [Lou17, Corollary 6.6]. □

Recall here that if Y is a scheme over k, then its *perfection* Y^{perf} is obtained as the inverse limit over the absolute Frobenius (this the *inverse perfection* in the parlance of [KL13]). Note then that because the absolute Frobenius morphism is a homeomorphism the natural projection $p : Y^{\mathrm{perf}} \to Y$ is a homeomorphism. Thus, the morphism $|\mathfrak{X}_\eta| \to (\mathfrak{X}^{\mathrm{red}})^{\mathrm{perf}}$ is more correctly the composition $p^{-1} \circ \mathrm{sp}_{\mathfrak{X}}$.

5.4 Analytification and generic fibers of models

We would now like to discuss the classical theory of analytification of schemes over $\mathrm{Spa}(E)$ where, here, again E is any analytic field and how they relate to generic fibers of the completions models of these schemes over $\mathcal{O}$ in the case when E's topology is induced by a discrete valuation.

So, let us begin with taking X to be a locally of finite type E-scheme. We then have the following version of 'analytification' in the language of adic spaces:

Lemma 5.4.1. *There exists an adic space $\mathscr{X}$ over* $\mathrm{Spa}(E)$ *together with a map of locally ringed spaces $(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) \to (X, \mathcal{O}_X)$ over E which satisfies the following universal property: given any adic space $\mathscr{Y}$ over* $\mathrm{Spa}(E)$ *and morphism of locally ringed spaces $(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}}) \to (X, \mathcal{O}_X)$ over E there exists a unique morphism of adic spaces $\mathscr{Y} \to \mathscr{X}$ over E factorizing $(\mathscr{Y}, \mathcal{O}_{\mathscr{Y}}) \to (X, \mathcal{O}_X)$ through $(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) \to (X, \mathcal{O}_X)$.*

Moreover, given three locally of finite type E-schemes X, Y and Z and morphisms $X \to Z$ and $Y \to Z$ the natural morphism $(X \times_Y Z)^{\mathrm{an}} \to X^{\mathrm{an}} \times_{Y^{\mathrm{an}}} Z^{\mathrm{an}}$ is an isomorphism.

Proof. See [Hub94, Proposition 3.8]. □

Given a locally of finite type E-scheme X we denote the adic space over $\mathrm{Spa}(E)$ we denote the adic space over $\mathrm{Spa}(E)$ described in Lemma 5.4.1 X^{an} and call it the *analytification* of X. Note that evidently the association of X^{an} to X is functorial.

Remark 5.4.2. What we are calling X^{an} is often times called X^{ad} (e.g. see [SW]). This can be confusing though since there is a construction similar to Lemma 5.3.4 that associates to a scheme X its 'adification', which sends affine schemes $\mathrm{Spec}(A)$ to the adic spaces $\mathrm{Spa}(A)$ where A is given the discrete topology. For this reason, we have chosen to denote this adic space X^{an}.

Let us give the fundamental example of this construction:

Example 5.4.3. Consider $\mathbb{A}^n_E$. We then claim that $(\mathbb{A}^n_E)^{\mathrm{an}}$ is canonically isomorphic to $\mathbb{A}^{n,\mathrm{an}}_E$ where $\mathbb{A}^{n,\mathrm{an}}_E := (\mathbb{A}^{1,\mathrm{an}}_E)^n$ where $\mathbb{A}^{1,\mathrm{an}}_E$ is the adic space from Example 5.1.9. To do this we first need to define a morphism of locally ringed spaces $\mathbb{A}^{n,\mathrm{an}}_E \to \mathbb{A}^n_E$. But, by the universal property of affine schemes in the category of locally ringed spaces (see [Sta18, Tag 01I1]) it suffices to define a map of E-algebras $E[t_1, \ldots, t_n] \to \mathcal{O}_{\mathbb{A}^{n,\mathrm{an}}_E}(\mathbb{A}^{n,\mathrm{an}})_E)$. This can be done precisely as at the end of Example 5.1.9. We then claim that the morphism $\mathbb{A}^{n,\mathrm{an}}_E \to \mathbb{A}^n_E$ of locally ringed spaces satisfies the universal property as in Lemma 5.4.1.

Indeed, to see this we need to show that for any adic space Y over E and morphism of locally ringed spaces $f : (Y, \mathcal{O}_Y) \to \mathbb{A}^n_E$ over E there exists a unique morphism of adic spaces $Y \to \mathbb{A}_E^{n,\mathrm{an}}$ over E factoring f. Now, a morphism of locally ringed spaces over E is determined by its map on global sections $E[t_1, \ldots, t_n] \to \mathcal{O}_Y(Y)$ which, in turn, is determined by n sections $f_1, \ldots, f_n \in \mathcal{O}_Y(Y)$. Now, in Example 5.1.9 we verified that $\mathbb{A}_E^{1,\mathrm{an}}$ represents the functor on $\mathsf{AffAdic}_E$ given by $(R, R^+) \mapsto R$. Then, by the sheaf condition, we have that it represents the functor $Y \mapsto \mathcal{O}_Y(Y)^n$. Thus, the elements $f_1, \ldots, f_n$ determine a unique morphism of adic spaces $Y \to \mathbb{A}_E^{n,\mathrm{an}}$. Tracing through the identifications easy to see that this is the desired map

We can soup this up to handle the case of affine schemes as well:

Example 5.4.4. Let $g_1, \ldots, g_m \in E[t_1, \ldots, t_n]$. Let us define an ideal sheaf $\mathcal{I}$ on $\mathbb{A}_E^{n,\mathrm{an}}$ as follows. We can evidently write $\mathbb{A}_E^{n,an} = \varinjlim U_j$ where $U_j = \mathbb{B}_E^n$ in a way completely analagous to our construction in Example 5.1.9. Define the sheaf $\mathcal{I}$ so that $\mathcal{I}\mid_{U_j}$ is the sheaf associated to the $E\langle t_1, \ldots, t_n\rangle$-module $(g_1, \ldots, g_m) \subseteq E\langle t_1, \ldots, t_n\rangle$. One can check that this is compatible with the transition maps, and so defines a sheaf $\mathcal{I}$ on $\mathbb{A}_E^{n,\mathrm{an}}$. Consider Z, the closed adic subspace of $\mathbb{A}_E^{n,\mathrm{an}}$ defined by the ideal sheaf $\mathcal{I}$. It is then not hard to see that Z admits a natural map of locally ringed spaces to $\mathrm{Spec}(E[t_1, \ldots, t_n]/(g_i))$ and that this morphism satisfies the conditions of Lemma 5.4.1 for

Example 5.4.5. Let X be a locally of finite type E-scheme and let $j : U \subseteq X$ be an open subscheme. Let $f : X^{\mathrm{an}} \to X$ denote the definitional morphism of locally ringed spaces. Then, the open adic subspace $f^{-1}(U)$ of X^{an} together with the obvious map of locally ringed spaces $f^{-1}(U) \to U$ is the analytification of U.

Indeed, let Y be any adic space over E and let $g : (Y, \mathcal{O}_Y) \to (U, \mathcal{O}_U)$ be a morphism of locally ringed spaces. Then, we obtain the induced map $g \circ j : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ and thus a unique morphism of adic spaces $a : Y \to X^{\mathrm{an}}$ such that $a \circ f = g \circ j$. Clearly then a has image landing in $f^{-1}(U)$ and thus factors to give a map $a : Y \to f^{-1}(U)$ such that $a \circ f \mid_{f^{-1}(U)} = g$. The conclusion follows.

With this, we can now make the following observation concerning the structure of the adic space X^{an} over $\mathrm{Spa}(E)$:

Lemma 5.4.6. *Let X be a separated E-scheme locally of finite type over E. Then, X^{an} is locally of finite type and partially proper over* $\mathrm{Spa}(E)$*. In particular, X^{an} is good.*

Proof. The fact that X is locally of finite type over $\mathrm{Spa}(E)$ follows immediately from Example 5.4.4 by covering X with affine opens and applying Example 5.4.5. To show that X^{an} is separated over $\mathrm{Spa}(E)$ we need only to show that X^{an} is separated. To do this, it suffices to give an open cover of X^{an} by separated opens whose pairwise intersections are separated. To do this we take an affine open cover $\{U_i\}$ of X and note then that $\{f^{-1}(U_i)\}$ is a separated open cover of X^{an} (apply again Example 5.4.4). To see that the intersections are separated we merely note that the separatedness assumption on X implies that $U_i \cap U_j$ is affine, and thus so is $f^{-1}(U_i) \cap f^{-1}(U_j) = f^{-1}(U_i \cap U_j)$.

Finally, we check that X^{an} is partially proper. To do this, let $\mathrm{Spa}(L, L^+)$ be an (necessarily analytic) affinoid field over $\mathrm{Spa}(E)$ and let $i : \mathrm{Spa}(L) \to \mathrm{Spa}(L, L^+)$ be standard map. Suppose now that we are given a map $g : \mathrm{Spa}(L) \to X^{\mathrm{an}}$ over $\mathrm{Spa}(E)$. We want to show that there exists a map $f : \mathrm{Spa}(L, L^+) \to X^{\mathrm{an}}$ (necessarily unique by separatedness) over $\mathrm{Spa}(E)$ such that the $f \circ i = g$. Now, this is equivalent to showing that given a morphism $g : (\mathrm{Spa}(L)\, \mathcal{O}_{\mathrm{Spa}(L)}) \to X$ of locally ringed spaces over E that there exists a unique morphism $f : (\mathrm{Spa}(L, L^+), \mathcal{O}_{\mathrm{Spa}(L,L^+)}) \to X$ of locally ringed space over E. Note though that $\mathrm{Spa}(L)$ is a single point, and so if $\mathrm{Spec}(A)$ is any affine open of X containing the image of this point, we get a factorization of $(\mathrm{Spa}(L, \mathcal{O}_{\mathrm{Spa}(L)}) \to X$ through $\mathrm{Spec}(A)$. This then corresponds to a morphism $A \to L$ which, in turn, gives rise to a morphism $(\mathrm{Spa}(L, L^+), \mathcal{O}_{\mathrm{Spa}(L,L^+)}) \to \mathrm{Spec}(A)$ and thus a morphism of locally ringed spaces $(\mathrm{Spa}(L, L^+), \mathcal{O}_{\mathrm{Spa}(L,L^+)}) \to X$ over E. This is obviously independent of the choice of open. So, we have obtained a morphism $f : \mathrm{Spa}(L, L^+) \to X$ of locally ringed spaces over E such that $f \circ i = g$. □

We would now like to compare this theory to the discussion in the last section. So, let us now adopt the notation of $\mathcal{O}$, E, π, and k from the last section. Take $\mathcal{X}$ to be a separated locally of finite type $\mathcal{O}$-scheme with generic fiber X and special fiber $\mathcal{X}_k$. One can then associate to $\mathcal{X}$ its *π-adic completion* $\widehat{\mathcal{X}}$ which is the completion of of $\mathcal{X}$ along its special fiber $\mathcal{X}_k$. It's evident that $\widehat{\mathcal{X}}$ a formal $\mathrm{Spf}(\mathcal{O})$-scheme topologically of finite type by considering Lemma 5.3.1 since the reduced subscheme is $\mathcal{X}_k$. Therefore, we can form its generic fiber $\widehat{\mathcal{X}}_\eta$ which we shall denote $\mathscr{X}$. One can also take the analytification X^{an} of X. One may then wonder what the relationship between these two good adic spaces over $\mathrm{Spa}(E)$ are.

Note that one at least has a natural map $i : \mathscr{X} \to X^{\mathrm{an}}$. Indeed, it suffices to give a map of locally ringed spaces $(\mathscr{X}, \mathcal{O}_{\mathscr{X}}) \to (X, \mathcal{O}_X)$ over E. Now, note that there is a natural morphism of locally ringed spaces $(\mathfrak{X}^{\mathrm{ad}}, \mathcal{O}_{\mathfrak{X}^{\mathrm{ad}}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. In particular, we see that $\mathscr{X} \hookrightarrow \mathfrak{X}^{\mathrm{ad}}$ lands in the non-vanishing locus of π in $(\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ which is X. This gives us the desired map i.

Now, in general, we cannot expect i to be an isomorphism, as the following example shows:

Example 5.4.7. Let $\mathfrak{X}$ be the closed formal disk $\mathrm{Spf}(\mathcal{O}\langle t\rangle)$ over $\mathrm{Spf}(\mathcal{O})$. Then, $\mathfrak{X}_\eta = \mathbb{B}_E$ and $X = \mathbb{A}^1_E$, so that $X^{\mathrm{an}} = \mathbb{A}^{1,\mathrm{an}}_E$. The resulting morphism $i : \mathbb{B}_E \to \mathbb{A}^{1,\mathrm{an}}_E$ is just the open inclusion of $\mathbb{B}_E$ as U_1 in the parlance of Example 5.1.9.

In fact, we saw in Lemma 5.4.6 that X^{an} is always partially proper whereas we saw in Theorem 5.3.14 that $\mathfrak{X}_\eta$ is only guaranteed to be partially proper when one has some properness assumptions on $\mathfrak{X}$. This observation, and Example 5.4.7 lead us to speculate that maybe i is always an open embedding, and an isomorphism when we assume that $\mathfrak{X}$ has some properness conditions.

The following lemma shows that the former is true if we assume that $\mathcal{X}$ is proper:

Lemma 5.4.8. *The map i is an open embedding. Moreover, if $\mathcal{X}$ is assumed to be a proper $\mathcal{O}$-scheme, then i is an isomorphism.*

Proof. See [Hub94, Remark 4.6, iv)]) for an adic treatment, and see [Con99, Section A.3] for a more introductory discussion in classical language. □

The last thing we do is connect these ideas with the ideas surrounding Lemma 5.3.18. Namely, let Z be a closed subscheme of $\mathcal{X}_k$, and thus a closed subscheme of $\widehat{\mathcal{X}}$. Let us denote by $\mathscr{X}_Z$ denote the generic fiber of the completion of of $\widehat{\mathcal{X}}$ along Z, which is a locally finite type good adic space over $\mathrm{Spa}(E)$ by Lemma 5.3.11. We call $\mathscr{X}_Z$ the *tube around* Z. It is an adic space over $\mathrm{Spa}(E)$ that should, in some sense, model the properties of $Z \subseteq \mathcal{X}_k$ (we will give explicit examples of this later). By Lemma 5.3.18 we obtain an open embedding $\mathscr{X}_Z \hookrightarrow \mathscr{X} = \mathscr{X}_{\mathcal{X}_k}$ and thus, by Lemma 5.4.8 an embedding $\mathscr{X}_Z \hookrightarrow X^{\mathrm{an}}$.

5.5 Relationship to other geometric theories

Let E, again, denote a general analytic field. We would like to compare the theory of certain adic spaces over $\mathrm{Spa}(E)$ to more classical theory of rigid analytic geometry a la Tate and Berkovich's theory of k-analytic spaces. For an introduction to the former one can see [BGR84], for an introduction to the latter one can see [Ber93] or [Tem15]. A nice expository discussion of both theories, as well as others, can be found in [Con08].

Tate's rigid analytic spaces

Recall roughly that Tate's theory deals with spaces glued together from local model objects of the form $\mathrm{Sp}(A) := \mathrm{MaxSpec}(A)$, where A is a topologically of finite type E-algebra endowed with the quotient topology from any surjection $E\langle t_1, \ldots, t_n\rangle$ (the topology is independent of such choice). The 'topology' on $\mathrm{Sp}(A)$ is not a literal topology on the underlying set of $\mathrm{Sp}(A)$. The issue, essentially, is that one wants that spaces like the closed unit disk $\mathrm{Sp}(E\langle t\rangle)$ are connected, and one can see from Example 5.1.7 that the fact that $\mathbb{D}_E \cup \mathbb{T}_E$ was not all of $\mathbb{B}_E$, and thus a disconnection, could only be see on the level of higher rank valuations, which are not included in Tate's theory. Thus, the 'topology' on $\mathrm{Sp}(A)$ is a sort of Grothendieck topology called the *G-topology* of $\mathrm{Sp}(A)$.

One of the main advantages of Huber's theory is the miraculous fact that the natural inclusion $\mathrm{Sp}(A) \hookrightarrow \mathrm{Spa}(A)$ induces an equivalence of topoi $\mathrm{Sh}(\mathrm{Spa}(A)) \xrightarrow{\approx} \mathrm{Sh}(\mathrm{Sp}(A))$ (where $\mathrm{Spa}(A)$ is given the topology as a topological space, and $\mathrm{Sp}(A)$ this aforementioned G-topology). In fact, for a sober topological space X, the category of sheaves on X determines X up to homeomorphism. So, Huber actually constructed, using his theory of adic spaces, the *unique* sober topological space whose topos matches that of the G-topos on $\mathrm{Spa}(A)$, a truly remarkable thing.

To make the connection complete, one would like to extend the association of $\mathrm{Sp}(A)$ to $\mathrm{Spa}(A)$ functorial, and for it to fully and faithfully embed the category of rigid analytic varieties into the category of adic spaces over $\mathrm{Spa}(E)$. The functoriality can be shown by the easy observation that a morphism $A \to B$ of rings, of the form above, is automatically continuous and evidently maps A° into B°. Thus, the association $\mathrm{Sp}(A)$ to $\mathrm{Spa}(A)$ is at least functorial.

The fact that it extends to a fully faithful embedding of all rigid analytic spaces over E into adic spaces over E which, with mild conditions, has explicitly describable essential image is the following:

Theorem 5.5.1. *There is a fully faithful embedding*

$$r_E : \left\{\begin{matrix}\text{Rigid analytic}\\ \text{varieties over } E\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{Adic spaces}\\ \text{over } \mathrm{Spa}(E)\end{matrix}\right\} \tag{5.40}$$

given on affinoids by $r_E(\mathrm{Sp}(A)) = \mathrm{Spa}(A)$. Moreover, r_E restricts to an equivalence of categories

$$\left\{\begin{matrix}\text{Quasi-separated rigid analytic}\\ \text{varieties over } E\end{matrix}\right\} \xrightarrow{\approx} \left\{\begin{matrix}\text{Quasi-separated adic spaces locally}\\ \text{of finite type over } \mathrm{Spa}(E)\end{matrix}\right\} \tag{5.41}$$

Proof. See [Hub96, Paragraph (1.1.11)]. □

The need to restrict to quasi-separated spaces to get a nice description of the essential image is one of an essentially topological nature. Note also the slightly subtle application of Lemma 5.1.19 to know that every adic space X locally of finite type over $\mathrm{Spa}(E)$ has the property that any affinoid open $\mathrm{Spa}(A, A^+)$ of X must have $A^+ = A^\circ$, so that it is of the form $\mathrm{Spa}(A)$ and so in the essential image of r_E.

Now, this functor r_E, not shockingly, preserves all reasonable properties about morphisms. So, for example, when one defines the notion of an étale of rigid analytic varieties, one has that $f : X \to Y$ is étale if and only $r_E(f) : r_E(X) \to r_E(Y)$ is étale. We will not list all of these compatibilities here, but Huber's theory was carefully constructed so that in (essentially) all cases, properties of morphisms are faithfully reflected by r_E.

Let us finally remark that the points of X can be understood quite simply in terms of the points of $r_E(X)$. Namely, there is a natural inclusion of X into $r_E(X)$ (in fact, it's an inclusion of 'G-ringed spaces') whose image is precisely the set of $x \in r_E(X)$ such that $k(x)/E$ is finite. One thus often calls the points of $x \in X$ with $k(x)/E$ finite the *classical points* of X.

Example 5.5.2. Let E be algebraically closed. The classical version of the closed unit disk is $\mathrm{Sp}(E\langle t\rangle)$ whose points naturally coincide with the set E°, where the association is $\alpha \in E^\circ$ maps to the obvious point $(t - \alpha)$ in $\mathrm{MaxSpec}(E\langle t\rangle) = \mathrm{Sp}(E\langle t\rangle)$. The points of $\mathbb{B}_E = r_E(\mathrm{Sp}(E\langle t\rangle))$ that have residue field finite over E are precisely the Type 1 points in the parlance of [Sch12, Example 2.20], which are those valuations v of $E\langle t\rangle$ of the form v_α where $\alpha \in E^\circ$ and

$$v_\alpha(f(t)) = |f(\alpha)| \tag{5.42}$$

(where $|\cdot|$ here denotes the absolute value on E). The above inclusion $\mathrm{Sp}(E\langle t\rangle) \hookrightarrow r_E(\mathrm{Sp}(E\langle t\rangle)) = \mathbb{B}_E$ is then the one taking $(t - \alpha)$ to v_α.

Now, the disadvantage of working with Huber's formalism is its lack of concreteness. For example, it seems out of reach to explicitly describe the points of the underlying topological space of $\mathbb{B}^2_E$ explicitly like one did for $\mathbb{B}_E$ as in [Sch12, Example 2.20]. That said, this is really not so farfetched. For example, while one can explicitly describe the points of the scheme $\mathbb{A}^1_{\mathbb{C}}$ and $\mathbb{A}^2_{\mathbb{C}}$, actually describing all of the points of $\mathbb{A}^3_{\mathbb{C}}$. But, in both cases we are happy enough to know the classical points (e.g. for $\mathbb{B}^2_E$ they are corresponding to $(E^\circ)^2$ and for $\mathbb{A}^3_{\mathbb{C}}$ they are corresponding to $\mathbb{C}^3$).

The advantages with working in the language of adic spaces are many. One has the obvious advantage of being able work with a genuine topological space opposed to a G-topology. But, more seriously, as the example of the universal compactification of $\mathbb{B}_E$, as in Example 5.2.9, the category of classical rigid spaces is not closed (in the category of adic spaces over $\mathrm{Spa}(E)$) under certain natural, important constructions. This disparity has been made even more apparent in recent years thanks to Scholze's work on perfectoid spaces (see [Sch13e] for a somewhat dated summary) which provide adic spaces that act somewhat like 'universal covers' for classical rigid spaces. These cannot be discussed in the classical language (they aren't even locally Noetherian!) but can be dealt with mostly in the language of Huber's adic spaces.

Berkovich spaces

We would now like to have a result like Theorem 5.5.1 for Berkovich's theory of E-analytic spaces. Let us first very roughly explain the relationship between such spaces, Tate's rigid analytic spaces, and Huber's adic spaces over $\mathrm{Spa}(E)$. Intuitively, if one thinks of obtaining $\mathrm{Spa}(A)$ from $\mathrm{Sp}(A)$ from adding in 'missing points', then Berkovich's space $\mathcal{M}(A)$ is an intermediary object where doesn't add in *all* the 'missing valuations' but only those missing ones that are still of rank 1. So, for example, when $A = E\langle t\rangle$ (where E is algebraically closed) then $\mathcal{M}(A)$ should contain all points of $\mathbb{B}_E$ except those of Type 5 (in the parlance of [Sch12, Example 2.20]). The situation is slightly more subtle. Indeed, whereas one has an inclusion $\mathrm{Sp}(A) \hookrightarrow \mathrm{Spa}(A)$ one has a *quotient map* $\mathrm{Spa}(A) \to \mathcal{M}(A)$. In fact, one can identify $\mathcal{M}(A)$ with the maximal Hausdorff quotient of $\mathrm{Spa}(A)$.

While Huber's framework is often times more advantageous than Berkovich's for our discussion (e.g. for universal compactifications), Berkovich's theory has had incredible application. While we won't enumerate these here, we mention one significant advantage to Berkovich's theory over Huber's. Namely, because Berkovich's spaces are (usually) Hausdorff, they have a good theory of topology. In particular, one can show that (with mild conditions) Berkovich spaces are locally path connected. This allows one to apply calssical topological theory to Berkovich spaces, which has been used to great effect.

Let us now state the analogue of Theorem 5.5.1 for Berkovich spaces:

Theorem 5.5.3. *There is an equivalence of categories b_E*

$$\left\{\begin{matrix}\text{Taut adic spaces locally}\\ \text{of finite type over } \mathrm{Spa}(E)\end{matrix}\right\} \xrightarrow{\approx} \left\{\begin{matrix}\text{Hausdorf strictly } E\text{-analytic}\\ \text{Berkovich spaces over } \mathcal{M}(E)\end{matrix}\right\} \tag{5.43}$$

such that the underlying topological space of $b_E(X)$ is the maximal Hausdorff quotient of X. Moreover, this functor respects base change, in the sense that for any analytic extension $E' \supseteq E$ the natural map $b_{E'}(X_{E'}) \to b_E(X)_{E'}$ is an isomorphism.

Proof. See [Hen16]. $\square$

Chapter 6

Appendix B: correspondences, Fujiwara's trace formula, and nearby cycles

Let us now recall some basic definitions concerning correspondences, both algebraic and analytic. We will be loosely following the treatment in [Var07] and [Far04].

Algebraic correspondences

Fix a base scheme S, and finite type S-schemes C and X. Then, a *correspondence of C over X* is a morphism of S-schemes

$$c : C \to X \times_S X \tag{6.1}$$

where we denote by $c_1 : C \to X$ and $c_2 : C \to X$ the compositions of c with the two projection maps $X \times X$. We say that a correspondence c is *finite étale* if the two morphisms c_1 and x_2 are. In the sequel, we shall restrict ourselves to dealing only with finite étale correspondences without mention.

Suppose now that ℓ is invertible in $\mathcal{O}_S$ and let $\mathcal{F} \in D^b_c(X, \overline{\mathbb{Q}_\ell})$. We then call a morphism $u : c_1^*\mathcal{F} \to c_2^!\mathcal{F}$ a *cohomological correspondence on $\mathcal{F}$ relative to c*. We denote the $\overline{\mathbb{Q}_\ell}$-space of cohomological correspondences on $\mathcal{F}$ relative to c by $\mathrm{Coh}(\mathcal{F}, c)$. We will denote by $u^\sharp$ the unique map $c_{2!}c_1^*\mathcal{F} \to \mathcal{F}$ associated to our map u by the adjointness between $c_{2!}$ and $c_2^!$.

Observation 6.0.1. *Since c is finite étale we have natural identifications $c_2^! = c_2^*$ and $c_{2!} = c_{2*}$.*

We shall use this observation freely throughout the rest of this section without comment.

Let us now restrict ourselves to the case when S is the spectrum of a field, say $S = \mathrm{Spec}(L)$. We will denote by $\overline{C}$, $\overline{X}$, $\overline{c}$ and $\overline{\mathcal{F}}$ the result of base changing these objects to a fixed algebraic closure $\overline{L}$ of L. We note then that base change also produces a natural map of $\overline{\mathbb{Q}_\ell}$-spaces

$$\mathrm{Coh}(c, \mathcal{F}) \to \mathrm{Coh}(\overline{c}, \overline{\mathcal{F}}) : u \mapsto u_{\overline{L}} \tag{6.2}$$

which we also denote by $\overline{u}$. Let us note that we have a natural identification between $\overline{u^\sharp}$ and $\overline{u}^\sharp$ which we freely use.

From $u \in \mathrm{Coh}(c, \mathcal{F})$ we naturally obtain an endomorphism of cohomology

$$R\Gamma(u) : R\Gamma(\overline{X}, \overline{\mathcal{F}}) \to R\Gamma(\overline{X}, \overline{\mathcal{F}}) \tag{6.3}$$

obtained by the following composition:

$$R\Gamma(\overline{X}, \overline{\mathcal{F}}) \xrightarrow{\overline{c}_1^*} R\Gamma(\overline{C}, \overline{c_1}^*\overline{\mathcal{F}}) \xrightarrow{\approx} R\Gamma(\overline{X}, \overline{c}_{2*}\overline{c}_1^*\overline{\mathcal{F}}) \xrightarrow{R\Gamma(\overline{u}^\sharp)} R\Gamma(\overline{X}, \overline{\mathcal{F}}) \tag{6.4}$$

which we call the endomorphism *induced* by u. We can similarly define an endomorphism $R\Gamma_c(u)$ of $R\Gamma_c(\overline{X}, \overline{\mathcal{F}})$.

Let us note that since $\mathcal{F}$ is defined over L that for any $\tau \in \mathrm{Gal}(\overline{L}/L)$ we obtain a natural endomorphism

$$\tau : R\Gamma(\overline{X}, \overline{\mathcal{F}}) \to R\Gamma(\overline{X}, \overline{\mathcal{F}}) \tag{6.5}$$

which commutes naturally with $R\Gamma(u)$. We then make the following definition:

$$\mathrm{tr}\left(\tau \times u \mid H^*(\overline{X}, \overline{\mathcal{F}})\right) := \mathrm{tr}\left(\tau \circ R\Gamma(u)\right) \tag{6.6}$$

as is standard. Here the order of the composition between τ and $R\Gamma(u)$ doesn't matter since τ and $R\Gamma(u)$ commute. Considering compactly supported cohomology we can make a similar definition for $\mathrm{tr}\left(\tau \times u \mid H_c^*(\overline{X}, \overline{\mathcal{F}})\right)$

Remark 6.0.2. Here we are using $H^*(\overline{X}, \overline{\mathcal{F}})$ to loosely mean the natural element $\sum_{j\geqslant 0}(-1)^j H^j(\overline{X}, \overline{\mathcal{F}})$ of the Grothendieck group of finite-dimensional $\overline{\mathbb{Q}_\ell}$-vector spaces with continuous $\mathrm{Gal}(\overline{L}/L)$ action. A similar remark applies for $H_c^*(\overline{X}, \overline{\mathcal{F}})$. It is certainly more useful to think of the complex $R\Gamma(\overline{X}, \overline{\mathcal{F}})$ in the stead of $H^*(\overline{X}, \overline{\mathcal{F}})$.

Going back to the setting of a general base scheme let $c : C \to X \times_S X$ and $d : D \to Y \times_S Y$ be correspondences. Then by a *morphism* from c to d we just mean a commutative diagram of S-schemes

$$\begin{tikzcd} X \arrow[d, "f"'] & C \arrow[l, "c_1"'] \arrow[r, "c_2"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y & D \arrow[l, "d_1"] \arrow[r, "d_2"'] & Y \end{tikzcd} \tag{6.7}$$

which we denote by $m = (f, g)$. We call m *finite étale* if f and g are. Again, for our purposes it suffices to restrict our attention only to morphisms of correspondences which are finite étale which we implicitly do in the sequel. We wish now to explain how morphisms (f, g) of correspondences naturally us to manufacture new cohomological correspondences.

So, first, let us fix $\mathcal{G} \in D_c^b(Y, \overline{\mathbb{Q}_\ell})$ and let $v \in \mathrm{Coh}(d, \mathcal{G})$. We then define the *pullback of v along m*, denoted $m^*(v)$, to be the cohomological correspondence in $\mathrm{Coh}(c, f^*\mathcal{G})$ given as follows:

$$c_1^* f^* \mathcal{G} \xrightarrow{\mathrm{BC}} g^* d_1^* \mathcal{G} \xrightarrow{g^*(v)} g^* d_2^* \mathcal{G} \xrightarrow{\mathrm{BC}} c_2^* f^* \mathcal{G} \tag{6.8}$$

where BC denote the obvious base change isomorphisms.

Now assume that $\mathcal{F} \in D_c^b(X, \overline{\mathbb{Q}_\ell})$ and let u be a cohomological correspondence on $\mathcal{F}$ relative to c. We can then define the *pushforwards of u along m*, denoted $m_*(u)$, to be the cohomological correspondence in $\mathrm{Coh}(c, f_*\mathcal{F})$ given as follows:

$$d_1^* f_* \mathcal{F} \xrightarrow{\mathrm{BC}} g_* c_1^* \mathcal{F} \xrightarrow{g_*(u)} g_* c_2^* \mathcal{F} \xrightarrow{\mathrm{BC}} d_2^* g_* \tag{6.9}$$

where BC are the obvious base change isomorphisms coming from the fact that all of our morphisms are finite étale.

Moreover, let us make the following basic observation:

Lemma 6.0.3. *There is a canonical isomorphism $R\Gamma(\overline{X}, \overline{\mathcal{F}}) \xrightarrow{\approx} R\Gamma(\overline{Y}, \overline{f_*\mathcal{F}})$ carrying the endomorphism $\tau \circ R\Gamma(u)$ to the endomorphism $\tau \circ R\Gamma(m_*(u))$ for any $\tau \in \mathrm{Gal}(\overline{L}/L)$.*

Proof. This follows immediately from the assumption that m is finite étale. □

Let us suppose that $S = \mathrm{Spec}(L)$ and that $\overline{y} : \mathrm{Spec}(\Omega) \to C$ is a geometric point. We say that $\overline{y}$ is a *fixed point* of u if the geometric points $c_1 \circ \overline{y}$ and $c_2 \circ \overline{y}$ are equal. Suppose that $\overline{y}$ is a fixed point of u, and let $\mathcal{F} \in D_c^b(X, \overline{\mathbb{Q}_\ell})$ and $u \in \mathrm{Coh}(c, \mathcal{F})$. We then obtain a morphism $u_{\overline{y}} : \mathcal{F}_{c_2(\overline{y})} \to \mathcal{F}_{c_2(\overline{y})}$ as follows. Note that by our assumptions on c_2 (namely finiteness) we have a natural identification

$$(c_{2*} c_1^* \mathcal{F})_{c_2(\overline{y})} = \bigoplus_{c_2^{-1}(c_2(\overline{y}))} \mathcal{F}_{\overline{z}} \tag{6.10}$$

We then define $u_{\overline{y}}$ to be the composition

$$\mathcal{F}_{c_2(\overline{y})} \hookrightarrow \bigoplus_{\overline{z} \in c_2^{-1}(c_2(\overline{y}))} \mathcal{F}_{\overline{z}} = (c_{2*} c_1^* \mathcal{F})_{\overline{y}} \to \mathcal{F}_{c_2(\overline{y})} \tag{6.11}$$

where the last map is the stalk on $\overline{y}$ of $u : c_{2*} c_1^* \mathcal{F} \to \mathcal{F}$. We call $u_{\overline{y}}$ the *local factor* of u at the fixed point $\overline{y}$.

Remark 6.0.4. What we are calling here the 'local factor' at $\overline{y}$ is often times called the *naive local factor* (viz. [Var07]). The justification for our focus on the naive local term is justified by the Theorem of Fujiwara discussed below.

Correspondences over $\mathbb{F}_q$ and the Fujiwara-Varshavsky trace formula

We now recall a formula that allows one to compute the traces of correspondences over finite fields in terms of local factors in a way completely analagous to the generalized Grothendieck-Lefschetz trace formula.

The formula, at least in the case of proper schemes (and with a much less explicit description of the local factors), can be obtained by the general machinery of the Lefschetz-Verdier trace formula. We state here only the version of most use to us. This was (more or less) conjectured first by Deligne, proven first by Fujiwara in [Fuj97], and then made more explicit by Varshavsky in [Var07].

To explain this formula let's fix a prime p, and q a power of p. For any finite type separated $\mathbb{F}_q$-scheme X let us denote by $\Phi_X : X \to X$ its absolute q-Frobenius, a morphism over $\mathrm{Spec}(\mathbb{F}_q)$. Let us denote by $\Phi_{\overline{X}} : \overline{X} \to \overline{X}$ the morphism $\Phi_X \times \mathrm{id}_{\mathrm{Spec}(\overline{\mathbb{F}_q})}$—a morphism over $\mathrm{Spec}(\overline{\mathbb{F}_q})$ which 'remembers' the model X over $\mathbb{F}_q$. In particular, the set of $\Phi_{\overline{X}}$-fixed points of $\overline{X}(\overline{\mathbb{F}_q})$ are precisely $X(\mathbb{F}_q)$.

Suppose now that C is a second finite type $\mathbb{F}_q$-scheme, and we are given $c : C \to X \times_{\mathrm{Spec}(\mathbb{F}_q)} X$ a correspondence of C over X. For all $j \geqslant 0$ we define the j^{th} *Frobenius twist of* c, denoted by $c^{(j)}$, to be the correspondence written diagramatically as

$$\begin{tikzcd}
 & & C \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\
 & X \arrow[dl, "\Phi_X^j"'] & & X \\
X & & &
\end{tikzcd} \tag{6.12}$$

another correspondence of C over X. Note that $\overline{c^{(j)}}$ agrees with $\overline{c}^{(j)}$ where the latter means twist by $\Phi_{\overline{X}}^j$.

We call a pair $(\mathcal{F}, \varphi)$, where $\mathcal{F} \in D_c^b(\overline{X}, \overline{\mathbb{Q}_\ell})$ and $\varphi : \Phi_{\overline{X}}^* \mathcal{F} \xrightarrow{\approx} \mathcal{F}$, a *Weil complex*. Let us note that if $w \in \mathrm{Coh}(\overline{c}, \mathcal{F})$ then we get an induced $w^{(j)} \in \mathrm{Coh}(\overline{c}^{(j)}, \mathcal{F})$ given by

$$(\Phi_{\overline{X}}^j \circ \overline{c}_1)^* \mathcal{F} = \overline{c}_1^* (\Phi_{\overline{X}}^j)^* \mathcal{F} \xrightarrow{\varphi^n} \overline{c}_1^* \mathcal{F} \xrightarrow{w} \overline{c}_2^* \mathcal{F} \tag{6.13}$$

which we call the j^{th} *Frobenius twist* of the cohomological correspondence w.

For all $j \geqslant 0$ let us define $\mathrm{Fix}_j(c)$ to be the set of of fixed geometric points $\mathrm{Spec}(\overline{\mathbb{F}}_q) \to C$ relative to $c^{(j)}$. Suppose that we are given a Weil complex $(\mathcal{F}, \varphi)$ and a $w \in \mathrm{Coh}(\overline{c}, \mathcal{F})$. Then, as a generalization of the Grothendieck-Lefschetz trace formula, we would like to relate the trace $R\Gamma_c(w^{(j)})$ to traces of local factors at points of $\mathrm{Fix}_j(c)$.

To this end, we have the following theorem of Fujiwara-Varshavsky:

Theorem 6.0.5 (Fujiwara-Varshavsky)**.** *Let X be a finite type separated $\mathbb{F}_q$-scheme, $(\mathcal{F}, \varphi)$ a Weil complex, and $w \in \mathrm{Coh}(\overline{c}, \mathcal{F})$. Then, there exists some $N \geqslant 1$ such that for all $j \geqslant N$ the set $\mathrm{Fix}_j(c)$ is finite and the following equality holds:*

$$\mathrm{tr}\left(R\Gamma_c\left(w^{(j)}\right)\right) = \sum_{\overline{y} \in \mathrm{Fix}_j(c)} \mathrm{tr}(w_{\overline{y}}) \tag{6.14}$$

Moreover, if X is proper then we may take $N = 1$.

Proof. This is essentially Theorem 2.3.2 of [Var07] where we have used the fact that since c_1 and c_2 are assumed to be finite étale to automatically deduce the conditions of a) and the fact that, in c), we have that

$$\mathrm{ram}(c_2 \mid_{c_2^{-1}(U)}) = \mathrm{ram}(c_2, X - U) = 1 \tag{6.15}$$

from where the above statement easily follows. $\square$

Specialization

We now wish to discuss the relationship between cohomological correspondences on the special and generic fiber of a correspondence over $\mathcal{O}$ where $\mathcal{O}$ is as in the last section.

So, to this end, let us fix finite type separated $\mathcal{O}$-schemes $\mathcal{C}$ and $\mathcal{X}$, and let $c:\mathcal{C}\to\mathcal{X}\times_{\mathrm{Spec}(\mathcal{O})}\mathcal{X}$ be a correspondence of $\mathcal{C}$ over $\mathcal{X}$. Let us denote by C and X the generic fiber of $\mathcal{C}$ and $\mathcal{X}$ respectively, and by $\mathcal{C}_k$ and $\mathcal{X}_k$ their special fiber. Note that by base change we obtain correspondences of C over X and $\mathcal{C}_k$ over $\mathcal{X}_k$. We denote these by c_η and c_s respectively, and by $c_{\overline{\eta}}$ and $c_{\overline{s}}$ their base changes to $\overline{E}$ and $\overline{k}$—note that these are just the same as $\overline{c_\eta}$ and $\overline{c_s}$. Our goal is then to explain how to transport cohomological correspondences in the generic fiber to cohomological correspondences in the special fiber.

To do this we need to recall the basic setup of the theory of nearby cycles. So, suppose that $\mathcal{Z}$ is any finite type separated $\mathcal{O}$-scheme with generic fiber Z and special fiber $\mathcal{Z}_k$. Let $\widetilde{j}$ denote the composition

$$\overline{Z}\to Z_{\breve{E}}\hookrightarrow\mathcal{Z}_{\breve{\mathcal{O}}}\tag{6.16}$$

and let $\widetilde{i}$ denote the composition

$$\mathcal{Z}_{\overline{k}}\hookrightarrow\mathcal{Z}_{\breve{\mathcal{O}}}\tag{6.17}$$

Then, for any any $\ell\neq p$ we have the functor of *nearby cycles*

$$R\psi: D^b_c(Z,\overline{\mathbb{Q}_\ell})\to D^b_c(\mathcal{Z}_{\overline{k}}\times\overline{\eta},\overline{\mathbb{Q}_\ell})\tag{6.18}$$

given by

$$R\psi\mathcal{F}:=\widetilde{i}^*R\widetilde{j}_*\overline{\mathcal{F}}\tag{6.19}$$

where, here, the category $D^b_c(\mathcal{Z}_{\overline{k}}\times\overline{\eta},\overline{\mathbb{Q}_\ell})$ is the category of constructible $\overline{\mathbb{Q}_\ell}$-sheaves on $\mathcal{Z}_{\overline{k}}$ with $\mathrm{Gal}(\overline{E}/E)$-action. More rigorously it's the category consisting of sheaves $\mathcal{G}$ on $\mathcal{Z}_{\overline{k}}$ equipped with compatible isomorphisms $\phi_\tau^*\mathcal{G}\xrightarrow{\approx}\mathcal{G}$ where $\phi_\tau:(\mathcal{Z}_{\overline{k}})^\tau\to\mathcal{Z}_{\overline{k}}$ is the obvious morphism, where

$$(\mathcal{Z}_{\overline{k}})^\tau:=\mathcal{Z}_{\overline{k}}\times_{\mathrm{Spec}(\overline{k}),\tau}\mathrm{Spec}(\overline{k})\tag{6.20}$$

and $\tau:\mathrm{Spec}(\overline{k})\to\mathrm{Spec}(\overline{k})$ is the morphism induced by the natural surjection $\mathrm{Gal}(\overline{E}/E)\to\mathrm{Gal}(\overline{k}/k)$.

Going back to our original situation, we would like to explain a natural map of $\overline{\mathbb{Q}_\ell}$-spaces

$$R\psi:\mathrm{Coh}(c,\mathcal{F})\to\mathrm{Coh}(c_{\overline{s}},R\psi(\mathcal{F}))\tag{6.21}$$

Namely, given $u\in\mathrm{Coh}(c,\mathcal{F})$ we define a morphism $R\psi(u)\in\mathrm{Coh}(c_{\overline{s}},R\psi(\mathcal{F}))$ as coming from the following map

$$\begin{aligned}c_{2\overline{s}*}c_1\overline{s}^*R\psi(\mathcal{F})&\to c_{2\overline{s}*}R\psi(c_{1\overline{s}}^*\mathcal{F})\\&\xrightarrow{\approx}R\psi(c_{2\overline{s}*}c_{1\overline{s}}^*\mathcal{F})\\&\to R\psi(\mathcal{F}_2)\end{aligned}\tag{6.22}$$

where the last map was obtained by the functoriality of $R\psi$ applied to the map $u^\sharp$.

We would then like to compare the action of $\tau\times R\Gamma(u)$ for certain $\tau\in\mathrm{Gal}(\overline{E}/E)$ to something involving $R\psi(u)$, at least in the case when X is proper. To this end, let's assume that $\tau \in \mathrm{Gal}(\overline{E}/E)$ is such that its projection under the map $\mathrm{Gal}(\overline{E}/E) \to \mathrm{Gal}(\overline{k}/k)$ is equal to Frob_q^j (where Frob_q is the geometric Frobenius). Let us note that the action of τ on $R\psi(\mathcal{F})$ defines a Weil complex which we will denote $(\tau, R\psi(\mathcal{F}))$.

We then have the following comparison result:

Observation 6.0.6. *Suppose that $\mathcal{X}$ is a finite type separated $\mathcal{O}$-scheme. Then, there is a canonical morphism*

$$R\Gamma_c(\overline{X}, \overline{\mathcal{F}}) \to R\Gamma_c(\mathcal{X}_{\overline{k}}, R\psi(\mathcal{F})) \tag{6.23}$$

which carries $\tau \circ R\Gamma(u)$ to $R\psi(u)^{(j)}$. Moreover, if $\mathcal{X}$ is a proper $\mathcal{O}$-scheme this is an isomorphism.

Proof. This follows from just tracing through the definitions and applying the proper base change theorem in the proper case. For more details see [Far04, Section 6.3.2]. □

Thus, to compute the trace of the generic fiber of a correspondence over $\mathcal{O}$ it suffices to compute the trace of the corresponding correspondence over the special fibers involving the nearby cycles, at least in the proper case.

Of course, one might hope that the above observation holds true for arbitrary separated finite type $\mathcal{X}$ if one puts compactly supported cohomology on both sides. This is, in general, not the case. That said, for the objects of interest to us we will, in fact, have such an equality (e.g. see Lemma 2.2.6).

Nearby cycles and some comparison theorems

The last result that we will need to recall concerning nearby cycles is a comparison result of Berkovich and Huber which allows one to compute the stalk of a nearby cycle sheaf in terms of the cohomology of its tubular neighborhood. So, let $\mathcal{O}$ be as before and let's assume that $\mathcal{X}$ is a finite type separated $\mathcal{O}$-scheme. Moreover, let $\mathcal{F} \in D_c^b(X, \mathbb{Q}_\ell)$.

For a closed subscheme $\mathcal{Z}$ of $\mathcal{X}_k$ let us denote, as in §5.3, by $\mathscr{Z}$ the rigid analytic space $(\widehat{\mathcal{X}}_{/\mathcal{Z}})_\eta$. We then have the following result of Berkovich in terms of Berkovich spaces, and Huber in terms of adic spaces:

Theorem 6.0.7 (Berkovich-Huber)**.** *There is a canonical isomorphism*

$$R\Gamma(\overline{\mathscr{Z}}, \overline{\mathcal{F}^{\mathrm{an}}}) \cong R\Gamma(\overline{\mathcal{Z}}, R\psi\mathcal{F}) \tag{6.24}$$

compatible with Galois actions. In particular, if $\mathcal{Z}$ is a point z then we get an isomorphism

$$R\Gamma(\overline{\mathscr{Z}}, \overline{\mathcal{F}^{\mathrm{an}}}) \cong (R\psi\mathcal{F})_{\overline{z}} \tag{6.25}$$

Proof. This is [Ber96, Corollary 3.5]. □

Chapter 7

Appendix C: G-schemes and a Lefchetz like trace formula

The goal of this appendix is to recall the formalism of **G**-schemes, for **G** a locally profinite group, and write down a formula for the trace of a cohomological Hecke operator at a bad (i.e. a level without a good integral model) level K in terms of the rigid geometry of the K-level and the reduction of a model at a lower good level K_0 (i.e. a level with a good integral model).

The rough idea of this method is that one can implement the Fujiwara trace formula for the special fiber of models at a lower good p-level, and then relate this to rigid geometry via theorems of Berkovich and Huber. This formula will be one of the main driving technical results needed for the contents of this paper. This material is largely contained in [Far04, §6, §7] in the case of good reduction, and in [Sch13c] in very specific cases of bad reduction. Our result can be seen as a common generalization of both these sources in the context of Hecke correspondences.

For much of the below we have chosen to work here in an odd level of generality, and this choice should be explained. Namely, we have chosen to work in the setting of $G(\mathbb{A}_f)$-schemes and/or $G(\mathbb{A}_f^p)$-schemes which are 'Shimura like', which, as the name suggests, mimics quite closely the setup existent in the theory of Shimura varieties. In fact, the only version of this setup that we shall employ *is* the setting of Shimura varieties. Much of what we say will work with $G(\mathbb{A}_f)$ and/or $G(\mathbb{A}_f^p)$ replaced by **G**, an arbitrary locally profinite group. We have chosen to work in this specific setting of $G(\mathbb{A}_f)$-schemes which are Shimura like to emphasize the purely geometric nature of the trace formulas we discuss, but whose transition to Shimura varieties is straightforward.

Much of the foundational material concerning **G**-schemes and/or $G(\mathbb{A}_f)$-schemes is directly inspired by work of Deligne (e.g. in [Del79]), but is phrased slightly different than how we have chose to approach it here. We have provided proofs for some well-known results that seem to lack a canonical source, as well as extended the results to general affine bases. Such results are certainly well-known to the experts (e.g. such ideas are implicitly used in papers such as [Kis10]) but, to the knowledge of the author, not been explicitly written down.

7.1 G-schemes

We begin by defining $\mathbf{G}$-schemes, in the context in which we use them, together with some terminology that is unique to this paper.

Basic definitions

Let us begin by recalling that a topological group $\mathbf{G}$ is *locally profinite* if it has a neighborhood basis of the identity consisting of profinite subgroups. It is easy to see then that an open subgroup of $\mathbf{G}$ is profinite if and only if it is compact.

Example 7.1.1. Any profinite group is locally profinite. So, for example, one can take the absolute Galois group $\mathbf{G} = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ for some field F.

Example 7.1.2. If G is an algebraic group over $\mathbb{Q}$, then $G(\mathbb{A}_f)$, $G(\mathbb{A}_f^p)$, and $G(\mathbb{Q}_p)$ are locally profinite groups. See §7.1 for more details.

Let us now fix $\mathbf{G}$ to be a locally profinite group. We wish to study schemes equipped with a 'continuous' right $\mathbf{G}$-action. Namely, let us fix an affine base scheme S_0. Suppose that X is an S_0 scheme with a right action of $\mathbf{G}$ by S_0-morphisms. Suppose moreover that for any compact open subgroup K of $\mathbf{G}$ a geometric quotient (see below for a recollection of geometric quotients) $X \to X/K$ exists in the category of S_0-schemes. We then say that the action of $\mathbf{G}$ on X is *continuous* if X/K is quasi-projective over S_0 for all K, and one has that the natural morphisms $X \to X/K$ identify X as the projective limit $\varprojlim_K X/K$ in the category of S_0-schemes.

Remark 7.1.3. For the reader's edification, it's helpful to compare the definition of a scheme with a continuous $\mathbf{G}$-action to the following basic fact. Let M be a left $\mathbf{G}$-module. Then, the action map $\mathbf{G} \times M \to M$ is continuous, when M is endowed with the discrete topology, if and only if $M = \varinjlim_K M^K$, where K travels over the open compact subgroups of $\mathbf{G}$, and M^K is the fixed points of M by K. Supposing then, purely informally, that one that M is a ring, and carrying this definition through the spectrum functor, one obtains something that looks quite like the above definition (recall that, at least in the category of affine schemes, $\mathrm{Spec}(M^U) = \mathrm{Spec}(M)/U$).

The inclusion of the quasi-projective hypotheses is of a technical nature relating to quotients of schemes by finite group actions (see Lemma 7.1.6).

Let X be an S_0 scheme and Γ an abstract group acting on the right of X by automorphisms over S_0. Recall that a *geometric quotient* of X by Γ is a morphism of S_0-schemes $p : X \to Y$ such that:

- The map $|p| : |X| \to |Y|$ (where $|\cdot|$ denotes the underlying topological space) identifies $|Y|$ with $|X|/\Gamma$.
- By nature of the action of Γ on X the sheaf $p_*\mathcal{O}_X$ has a natural left action by the group Γ, and we have a natural map of sheaves of rings $\mathcal{O}_Y \to (p_*\mathcal{O}_X)^\Gamma$. We then demand that this map is an isomorphism.

It's clear that if a geometric quotient of X by Γ exists, it's unique up to isomorphism over S_0.

Remark 7.1.4. Note that the construction $(|X|/\Gamma, (p_*\mathcal{O}_X)^\Gamma)$ always produces a categorical quotient (see Lemma 7.1.5 below) in the category RS_{S_0} of ringed spaces over S_0. Thus, the existence of a geometric quotient comes down, ultimately, to the question of whether the ringed space $(|X|/\Gamma, (p_*\mathcal{O}_X)^\Gamma)$ is a scheme (in particular, whether it is even locally ringed) and whether the map $p : (X, \mathcal{O}_X) \to (|X|/\Gamma, (p_*\mathcal{O}_X)^\Gamma)$ is locally ringed.

We have the following basic lemma which alternatively gives a Yoneda characterization of Y:

Lemma 7.1.5. *Let X be an S_0-scheme, and let Γ be an abstract group acting on the right of X by automorphisms over S_0. Then, if $p : X \to Y$ is a geometric quotient, then it is a* categorical quotient. *In other words, for all S_0-schemes Z, the natural map*

$$\mathrm{Hom}_{S_0}(X, Z)^K \to \mathrm{Hom}_{S_0}(Y, Z) \tag{7.1}$$

is an isomorphism.

Proof. This is obvious. Since a K-invariant map $X \to Z$ over S_0 is the same thing as a K-invariant map of topological spaces, and the induced map of sheaves being K-invariant. It's then clear that it must factor through uniquely through the geometric quotient. □

The reason for the assumptions that S_0 is affine and X_K is quasi-projective are made clear by the following lemma:

Lemma 7.1.6. *Let S_0 be an affine scheme, and X a quasi-projective S_0-scheme. Then, for any finite abstract group Γ acting on the right of X by S_0-automorphisms, a geometric quotient Y exists. Moreover, the obvious morphism $[X/\Gamma] \to Y$ is a coarse moduli space.*

Here by $[X/\Gamma]$ we mean the quotient stack on the fppf site of S_0 (see [Sta18, Tag 04UV]), and we mean a coarse moduli space in the sense that the morphism $[X/\Gamma] \to Y$ is initial amongst maps from $[X/\Gamma]$ to schemes over S_0, and for every geometric point $\overline{s}$ of S_0 the map $\pi_0([X/\Gamma])(\overline{s}) \to Y(\overline{s})$ is a bijection (see [Ols16, Definition 11.1.1]).

Remark 7.1.7. The fact that $[X/\Gamma] \to Y$ is a coarse moduli space will not factor into our discussion to heavily. One should just take it as a reassurance that the functor of points of a geometric quotient (at least in the setting of Lemma 7.1.6) is not too farfetched. In good situations (see Lemma 7.1.13) this coarse moduli space will be fine, in which case this tells us precisely what the functor of points of Y is.

Also, the assumptions that S_0 are affine and that X is quasi-projective really are to avoid having to talk about algebraic spaces. If one is willing to work with such objects, then these assumptions can (largely) be removed.

Proof. (Lemma 7.1.6) This follows from [EvdGM, Theorem 4.16] and the subsequent [EvdGM, Remarks 4.17, i)]. □

Now, if one has an S_0-scheme X with a continuous right $\mathbf{G}$-action, then one obtains a family of schemes X/K indexed by the compact open subgroups of $\mathbf{G}$. Let us formalize what it should mean to have such a family with good properties, and to compare the relationship between S_0-schemes with a continuous right $\mathbf{G}$-action and such good families.

So, let us say that a collection $\mathcal{N}$ of compact open subgroups of $\mathbf{G}$ is *reasonable* if it is a downward closed (with respect to inclusion) and conjugation closed set and is such that for all $H \subseteq \mathbf{G}$ compact open, there exists $K \leqslant H$ such that $K \in \mathcal{N}$.

Let us observe the following basic fact about reasonable collections of compact open subgroups of $\mathcal{N}$:

Lemma 7.1.8. *If $\mathcal{N}$ is a reasonable collection of compact open subgroups of $\mathbf{G}$, then given any compact open subgroup K of $\mathbf{G}$ there exists a subset $\{K_i\} \subseteq \mathcal{N}$ such that $K_i \trianglelefteq K$ and $\bigcap_i K_i$ is trivial.*

Proof. This follows easily from the following observation. For any compact open subgroup H of K there exists some $K' \in \mathcal{N}$ such that $K' \trianglelefteq K$ and $K' \subseteq H$. Indeed, by the assumption that $\mathcal{N}$ is reasonable we know that there exists some $K'' \in \mathcal{N}$ such that $K'' \leqslant H$. Note that since K'' is a compact open subgroup of K, which is profinite, we must have that $[K : K'']$ is finite. Thus, we see that $[K : N_K(K'')]$ is finite, where $N_K(K'')$ denotes the normalizer or K'' in K. One can then take

$$K' = \bigcap_{g_j} g_j K'' g_j^{-1} \tag{7.2}$$

where g_j runs over coset representatives of $K/N_K(K'')$. Evidently $K' \leqslant K'' \leqslant H$ is compact open, and is normal in K. Since $\mathcal{N}$ is downward closed we also have that $K' \in \mathcal{N}$. $\square$

Let us now define a *(right)* $\mathbf{G}$*-scheme over* S_0 to be triples of data $(\mathcal{N}, \{X_K\}_{K\in\mathcal{N}}, \{t_{K,L}(g)\})$ where

- $\mathcal{N}$ is a reasonable set of compact open subgroups of $\mathbf{G}$.
- X_K is a quasi-projective S_0 scheme for all $K \in \mathcal{N}$
- For each $K, L \in \mathcal{N}$ and $g \in \mathbf{G}$ such that $L \supseteq g^{-1}Kg$ we have a finite surjective morphism

  $$t_{K,L}(g) : X_K \to X_L \tag{7.3}$$

  satisfying the following criteria:

  1. If $L, K, M \in \mathcal{N}$ and $g, h \in \mathbf{G}$ are such that $M \supseteq g^{-1}Lh$ and $L \supseteq h^{-1}Kh$ then we have the following equality:

     $$t_{L,M}(g) \circ t_{K,L}(h) = t_{K,M}(hg) \tag{7.4}$$

  2. If $K \in \mathcal{N}$ and $k \in K \subseteq \mathbf{G}$ then $t_{K,K}(k)$ is the identity.

3. If $L, K \in \mathcal{N}$ and $K \trianglelefteq L$ then conditions a) and b) imply that the finite group L/K acts on X_K on the right with $l + K$ in L/K acting by $t_{K,L}(l)$. We then demand that $t_{K,L}(\mathrm{id}) : X_K \to X_L$ identifies X_L as a geometric quotient of X_K by L/K.

We will often denote a $\mathbf{G}$-scheme by $\{X_K\}_{K\in\mathcal{N}}$ letting the maps $t_{K,L}(g)$ be implicit. We shall also shorten $t_{K,L}(\mathrm{id})$ to $\pi_{K,L}$ and $t_{K,g^{-1}Kg}(g)$ to $[g]_K$. Thus, we have an equality

$$t_{K,L}(g) = \pi_{g^{-1}Kg,L} \circ [g]_K \tag{7.5}$$

as easily follows from the definition of a $\mathbf{G}$-scheme. One can also check that the morphisms $\pi_{K,L}$ turn $\{X_K\}_{K\in\mathcal{N}}$ into a projective system of S_0-schemes. We call $\mathcal{N}$ the *indexing set* of the $\mathbf{G}$-scheme $\{X_K\}_{K\in\mathcal{N}}$.

We now show that there is a way to pass between schemes over S_0 with a right $\mathbf{G}$-action and $\mathbf{G}$-schemes. Namely, given a scheme X with a continuous right $\mathbf{G}$-action we obtain maps $t_{K,L}(g) : X/K \to X/L$, for compact open subgroups $K \subseteq L$ of $\mathbf{G}$ and $g \in \mathbf{G}$ such that $L \supseteq g^{-1}Kg$, defined as follows. Since $X \to X/K$ is a geometric quotient by K, to give $X/K \to X$ is the same as to give a K-invariant map $X \to X/L$ by Lemma 7.1.5. Define this map to be the map defined by the right action of g followed by the quotient map to X/L. This is right K-invariant since $L \supseteq gKg^{-1}$. For any reasonable $\mathcal{N}$, let us denote by $Q_{\mathcal{N}}(X)$ the triple $(\mathcal{N}, \{X/K\}_{K\in\mathcal{N}}, \{t_{K,L}(g)\})$.

Conversely, consider a $\mathbf{G}$-scheme $\{X_K\}_{K\in\mathcal{N}}$ and consider $\varprojlim_{K\in\mathcal{N}} X_K$, which exists in the category of schemes since the transition maps are finite (see [Sta18, Tag 01YX]). This inherits a right action by $\mathbf{G}$ by declaring that $g \in \mathbf{G}$ acts on the right via the projective limit of the morphisms

$$X \to X_K \xrightarrow{[g]_K} X_{g^{-1}Kg} \tag{7.6}$$

Let us denote this S_0-scheme with a right $\mathbf{G}$-action by $I(\{X_K\})$.

We then have the following claim:

Lemma 7.1.9. *For any S_0-scheme X with a continuous right $\mathbf{G}$-action and any reasonable $\mathcal{N}$ the triple $Q_{\mathcal{N}}(X)$ is a $\mathbf{G}$-scheme. Conversely, given any $\mathbf{G}$-scheme $\{X_K\}_{K\in\mathcal{N}}$ the right $\mathbf{G}$-action on the S_0-scheme $I(\{X_K\})$ is continuous.*

Proof. Suppose first that $\{X_K\}_{K\in\mathcal{N}}$ is a set of data as in (2). For all $K, L \in \mathcal{N}$ with $K \subseteq L$, set $\pi_{K,L} := t_{K,L}(\mathrm{id})$. It's easy then to see that $\{X_K\}_{K\in\mathcal{N}}$ becomes a projective system relative to the transition maps $\pi_{K,L}$. Set $X := \varprojlim_K X_K$, and let $\pi_K : X \to X_K$ be the projection morphism. Note that this inverse limit is exists in the category of S_0-schemes, since the transition morphisms $\pi_{K,L}$ are finite and so, in particular, affine (see [Sta18, Tag 01YW]). We claim that X carries a natural right $\mathbf{G}$-action. Indeed, for each $g \in \mathbf{G}$ define the map

$$t(g) : X \to X \tag{7.7}$$

as follows. For each $K \in \mathcal{N}$ define $\pi_K \circ t(g)$ to be the composition

$$X \xrightarrow{\pi_{gKg^{-1}}} X_{gKg^{-1}} \xrightarrow{t_{K,gKg^{-1}}} X_K \tag{7.8}$$

It is then easy to see that these morphisms satisfy the compatibility conditions with the projective system $\{X_K\}$ (using (a)), and thus does glue to a morphism $t(g)$. It's then easy to see (again using (a)) to see that this defines a right action of $\mathbf{G}$ on X.

Let us now show that this action is continuous. Let us start by showing that if $K \in \mathcal{N}$ then the morphism $\pi_K : X \to X_K$ is a geometri quotient in the category of S_0-schemes. To do this, let us begin by observing that there exists a cofinal set $\mathcal{N}'$ of groups K' in $\mathcal{N}$ which are normal in K. Note then that, by assumption in (c), the map $\pi_{K',K} : X_{K'} \to X_K$ is a geometric quotient. The fact then that $\pi : X \to X_K$ is a geometric quotient follows from the fact that $X = \varprojlim_{K' \in \mathcal{N}'} X_{K'}$ so that $|X| = \varprojlim_{K' \in \mathcal{N}'} |X_{K'}|$ (see [Sta18, Tag 0CUF]).

It remains to show that if K is any compact open subgroup of $\mathbf{G}$, not necessarily in $\mathcal{N}$, then the geometric quotient X/K exists and is quasi-projective. That said, there exists some $K' \in \mathcal{N}$ such that $K' \trianglelefteq K$. We know that X/K' is the quasi-projective scheme $X_{K'}$, and the above action of $G(\mathbb{A}_f)$ on X descends to an action of K/K' on $X_{K'}$. Now, by the remarks preceding this proof, we know that $X_{K'}$ has a geometric quotient by K/K' which is quasi-projective for the obvious reasons (e.g. [Sta18, Tag 0C4N]).

Conversely, suppose that we are given an S_0-scheme X equipped with a continuous right action of $\mathbf{G}$. It's then almost clear that $\{X_K\}$ forms a system as in (1). The only non-obvious fact is that the morphisms $\pi_{K,L}$ are all finite. But, by basic theory (e.g. see [Ols16, Corollary 6.2.9]) the map $\pi_{K,L}$ is integral, but since the map is also an S_0-morphism of finite presentation S_0-schemes, it's also finite. $\square$

We would like to make this association functorial. But, to do so, we first need to define morphisms both of S_0-schemes with continuous right $\mathbf{G}$-actions and morphisms of $\mathbf{G}$-schemes. So, to begin with, let us suppose that X_1 and X_2 are schemes with a continuous right $\mathbf{G}$-action over S_0. Then, a *morphism* $X_1 \to X_2$ is a morphism of S_0-schemes $\phi : X_1 \to X_2$ which is $\mathbf{G}$-equivariant. We call ϕ an isomorphism (resp. closed embedding) if it is an isomorphism (resp. closed embedding) of S_0-schemes.

Suppose now that $\{X^1_K\}_{K\in\mathcal{N}}$ and $\{X^2_K\}_{K\in\mathcal{N}}$ be two $\mathbf{G}$-schemes with morphisms $t^i_{K,L}(g)$ for $i = 1, 2$. Then, a *morphism* $\{X^1_K\}_{K\in\mathcal{N}} \to \{X^2_K\}_{K\in\mathcal{N}}$ is a set of morphisms of S_0-schemes $\phi_K : X^1_K \to X^2_K$ for all $K \in \mathcal{N}$, such that for all $K, L \in \mathcal{N}$ with $L \supseteq g^{-1}Kg$ the diagram

$$\begin{tikzcd} X^1_K \arrow[r, "\phi_K"] \arrow[d, "t^1_{K,L}(g)"'] & X^2_K \arrow[d, "t^2_{K,L}(g)"] \\ X^2_L \arrow[r, "\phi_L"'] & X^2_L \end{tikzcd} \tag{7.9}$$

commutes. We call $\{\phi_K\}$ an *isomorphism* (resp. *closed embedding*) if each ϕ_K is an isomorphism (resp. *closed embedding*).

We then have the following obvious extension of Lemma 7.1.9:

Lemma 7.1.10. *The associations $X \mapsto Q_{\mathcal{N}}(X)$ and $\{X_K\}_{K\in\mathcal{N}} \mapsto I(\{X_K\})$ are functorially (in the obvious way) and define essentially inverse equivalences of categories between S_0-schemes with a continuous right $\mathbf{G}$-action and $\mathbf{G}$-schemes with indexing set $\mathcal{N}$. Moreover, the functors I and $Q_{\mathcal{N}}$ reflect closed embeddings.*

One might assume, given Lemma 7.1.10, that $\mathbf{G}$-schemes over S_0 and S_0 schemes with a continuous right $\mathbf{G}$-action are essentially identical notions. This is not true in general since many properties of a $\mathbf{G}$-scheme $\{X_K\}_{K\in\mathcal{N}}$ with indexing set $\mathcal{N}$ might not hold for the system $Q_{\mathcal{M}}(I(\{X_K\}))$ for a different indexing set $\mathcal{M}$ (e.g. see the notion of smoothness below). That said, the ambiguity is all coming from the association of a $\mathbf{G}$-scheme to a scheme with a right $\mathbf{G}$-action. The S_0-scheme with continuous right $\mathbf{G}$-action $I(\{X_K\})$, which we will often just denote X when $\{X_K\}_{K\in\mathcal{N}}$ is clear from context, is unambiguous and called the *space at infinite level* associated to $\{X_K\}_{K\in\mathcal{N}}$. We shall then, in contrast, call X_K the *space at level* K and the space at some level K for some $K\in\mathcal{N}$ a *space at finite level*.

We shall need a more generalized notion of morphism between $\mathbf{G}$-schemes and schemes with a continuous right $\mathbf{G}$-action over S_0. Namely, suppose that $\mathbf{G}_1$ and $\mathbf{G}_2$ are two (possibly equal) locally profinite groups and let $f:\mathbf{G}_1\to\mathbf{G}_2$ be a continuous group map. Then, if X_i is an S_0-scheme with continuous right $\mathbf{G}_i$-action for $i=1,2$ then, a *morphism* $\phi: X_1\to X_2$ *relative to* f is a morphism $\phi: X_1\to X_2$ of S_0-schemes which intertwines the actions relative to f (i.e. for all $g\in\mathbf{G}_1$ one has that $\phi(x\cdot g)=\phi(x)\cdot f(g)$). We say that ϕ is an isomorphism (resp. closed embedding) if f is an isomorphism (resp. closed embedding) and ϕ is an isomorphism (resp. closed embedding) of S_0-schemes. If $\mathbf{G}_1=\mathbf{G}_2$ and no f is specified, then one can assume that $f=\mathrm{id}$ in which case this reduces to the previously defined notion of a morphism of $\mathbf{G}$-schemes.

Suppose now that $\{X^i_{K_i}\}_{K_i\in\mathcal{N}_i}$ is a $\mathbf{G}_i$-scheme over S_0 for $i=1,2$. Then, the data of a *morphism relative to* f from $\{X^1_{K_1}\}_{K_1\in\mathcal{N}_1}$ to $\{X^2_{K_2}\}_{K_2\in\mathcal{N}_2}$ consist of the following:

- An inclusion preserving map $j:\mathcal{N}_2\to\mathcal{N}_1$ such that for all $K_2\in\mathcal{N}$ one has that $j(K_2)\subseteq f^{-1}(K_1)$ and $j(f(g)K_2f(g)^{-1})=gj(K_2)g^{-1}$.
- Morphisms $\phi_{K_2}: X_{j(K_2)}\to Y_{K_2}$ of schemes over S_0, such that for all $g\in\mathbf{G}$ and $L^2,K^2\in\mathcal{N}_2$ such that $L^2\supseteq f(g)^{-1}K^2f(g)$ we have that the diagram

$$\begin{tikzcd} X^1_{j(K_2)} \arrow[r, "\phi_{K_2}"] \arrow[d, "t^1_{j(K_2),j(L_2)}(g)"'] & X^2_{K_2} \arrow[d, "t^2_{K_2,L_2}(f(g))"] \\ X^1_{j(L_2)} \arrow[r, "\phi_{L_2}"'] & X^2_{L_2} \end{tikzcd} \tag{7.10}$$

  commutes.

We say that $\{\phi_{K_2}\}$ is an *isomorphism* (resp. *closed embedding*) if f is an isomorphism (resp. closed embedding), $j(K_2)=f^{-1}(K_2)$ and each ϕ_{K_2} is an isomorphism (resp. closed embedding) of S_0-schemes.

Example 7.1.11. As an example of this theory that might be familiar to the reader, we have the following rephrasing of the classical theory of Galois descent for quasi-projective varieties. Namely, let F be a field. Then, there is a natural equivalence of categories

$$\left\{\begin{matrix}\text{Quasi-projective}\\ F\text{-schemes}\end{matrix}\right\} \xrightarrow{\approx} \left\{\begin{matrix}F\text{-schemes } X \text{ with continuous right}\\ \mathrm{Gal}(F^{\mathrm{sep}}/F)\text{-action together with an}\\ \text{equivariant morphism } X\to\mathrm{Spec}(F^{\mathrm{sep}})\end{matrix}\right\} \tag{7.11}$$

where we consider $\mathrm{Spec}(F^{\mathrm{sep}})$ as a $\mathrm{Spec}(F)$-scheme with continuous right $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-action in the obvious way. The equivalence sends X_0 in the left hand side to $X := (X_0)_{F^{\mathrm{sep}}}$ with the obvious $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-action and equivariant map $X \to \mathrm{Spec}(F^{\mathrm{sep}})$, and conversely sends X in the right hand side to $X/\mathrm{Gal}(F^{\mathrm{sep}}/F)$.

Almost Shimura like G-schemes

The goal of this subsection is to whittle down the collection **G**-schemes to a more managable class, which is representative of the type of **G**-schemes which we will exclusively deal with.

To start, we would like to define what it means for a **G**-scheme to be *smooth* over its base S_0. Following [Mil92, Definition 2.2], we call a **G**-scheme $\{X_K\}_{K\in\mathcal{N}}$ *smooth* if each X_K is a smooth quasi-projective scheme over S_0 and the transition morphisms $\pi_{K,L}$ are étale.

Note that if $\{X_K\}_{K\in\mathcal{N}}$ is a smooth **G**-scheme, then $X \to S_0$ will not be smooth in general, simply because it will not even be of finite type. That said, we do have the following which will be of use in our discussion of integral canonical models:

Lemma 7.1.12. *Let $\{X_K\}_{K\in\mathcal{N}}$ be a smooth* **G***-scheme. Suppose that S_0 is Noetherian and regular, then X has Noetherian stalks, X is regular, and the morphism $X \to S_0$ is formally smooth.*

Proof. Let $x \in X$ be arbitrary and for all $K \in \mathcal{N}$ let us denote by x_K the projection $\pi_K(x)$. Note then that since the morphisms $\pi_{K,L}$ are affine allows us to deduce that

$$\mathcal{O}_{X,x} = \varinjlim_{K\in\mathcal{N}} \mathcal{O}_{X_K,x_K} = \varinjlim_{K\subseteq K_0} \mathcal{O}_{X_K,x_K} \tag{7.12}$$

for any choice of some of $K_0 \in \mathcal{N}$. Let us denote by $\mathfrak{m}_K$ the maximal ideal of $\mathcal{O}_{X_K,x_K}$. Note then that since each $\pi_{K,L}$ is assumed étale that we have $\mathfrak{m}_K = \mathfrak{m}_{K_0}\mathcal{O}_{X_K,x_K}$. Thus, by passing to the limit, we see that $\mathfrak{m}_{K_0}\mathcal{O}_{X,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$. Since X_{K_0} is quasi-projective over S_0, which is Noetherian, we know that $\mathfrak{m}_{K_0}$ is finitely generated, and the Noetherianess of $\mathcal{O}_{X,x}$ follows. The fact that $\mathcal{O}_{X,x}$ is regular then follows from the standard fact that a a Noetherian ring which is a colimit of regular local subrings is automatically regular (see [Sta18, Tag 07DX]).

To see that $X \to S$, let $\mathrm{Spec}(A) \to S$ be a morphism and I a square-zero ideal of A. Then, we need to verify that the natural map

$$\mathrm{Hom}_S(\mathrm{Spec}(A), X) \to \mathrm{Hom}_S(\mathrm{Spec}(A/I), X) \tag{7.13}$$

is surjective. That said, we know that $X = \varprojlim_{K\in\mathcal{N}} X_K$. In fact, fixing some sufficiently small compact open subgroup $K_0 \in \mathcal{N}$ and some countable cofinal system of compact open subgroups $K_i \trianglelefteq K$ we have that $\varprojlim_i X_{K_i}$. The maps $\mathrm{Hom}_S(\mathrm{Spec}(A), X_{K_i}) \to \mathrm{Hom}_S(\mathrm{Spec}(A/I)/X_{K_i})$ are surjections since X_{K_i} are smooth over S_0. Thus, we see that (7.13) is surjective since this system clearly satisfies the Mittag-Leffler criterion (e.g. see [Sta18, Tag0594]). $\square$

Whereas the regularity condition of smoothness was a purely geometric assumption, and have little do with the structure of $\mathbf{G}$-action, we would like to mention a complication of the $\mathbf{G}$-action for general $\mathbf{G}$-schemes that are even present in the situation of Shimura varieties.

To begin this discussion, recall the following well-known result:

Lemma 7.1.13. *Let X be an S_0 scheme and Γ a finite abstract group acting on X. Let $\pi : X \to Y$ be the quotient of X by Γ. Then, the following are equivalent:*

1. *The morphism π is finite étale.*
2. *The natural map of stacks $[X/\Gamma] \to Y$ is an isomorphism.*
3. *The map π is an $\mathrm{Aut}(Y/X)$-torsor.*

In this case Y is also the fppf sheafification of the quotient presheaf sending T in S_{fppf} to $X(T)/\Gamma$.

So, if X is an S_0 scheme with a right action by the finite constant group Γ, then we call a quotient mapping $\pi : X \to Y$ *Galois* if it satisfies any of the equivalent properties (1), (2), or (3) of Lemma 7.1.13. We define the *Galois group* $\mathrm{Gal}(X/Y)$ to be $\mathrm{Aut}(X/Y)^{\mathrm{op}}$. Note that $\mathrm{Aut}(X/Y)$ acts on X on the left, and $\mathrm{Gal}(X/Y)$ acts on X on the right.

So, let us assume that $\{X_K\}_{K\in\mathcal{N}}$ is a smooth $\mathbf{G}$-scheme over S_0. Note then that each morphism $\pi_{K,L}$ with $K \trianglelefteq L$ is a finite quotient mapping and étale, and thus Galois. We thus obtain a surjective homomorphism

$$L/K \to \mathrm{Gal}(X_K/X_L) \tag{7.14}$$

but there is no need, in general, for this homomorphism to be injective.

So, for the sake of simplicity, we will restrict our attention to $\mathbf{G}$-schemes where we can uniformly describe the kernel of the map (7.14). To this end, let us call a $\mathbf{G}$-scheme *almost Shimura* if it is smooth and there exists some central subgroup $Z \subseteq Z(\mathbf{G})$ (where $Z(\mathbf{G})$ denotes the center of $\mathbf{G}$) such that for all $L, K \in \mathcal{N}$ with $K \trianglelefteq L$ the morphism (7.14) has kernel KZ_L, where $Z_L := Z \cap L$, thus giving an isomorphism $L/KZ_L \xrightarrow{\approx} \mathrm{Gal}(X_K/X_L)$. When we want to emphasize a choice of Z, we may say that $\{X_K\}_{K\in\mathcal{N}}$ is almost Shimura like *relative to Z*.

Remark 7.1.14. The assumption that Z is central is mainly for convenience. It is easily conceivable that this assumption is unnecessary. But, since all of our examples satisfy this, we make our lives easier by adopting this convention. Note, also, that Z is, by no means, unique. Moreover, in good situations one can take Z to be the trivial subgroup in the above definition. That said, we give an example below (see Example 8.3.15) where one cannot take Z to be trivial.

Until we get to this example, it is worth comparing the above definition to the following classical situation, which is (essentially) the model for almost Shimura like $\mathbf{G}$-schemes. Recall that in the classical theory of modular curves (see §8.1 for a reminder on the adelic version of this theory, and §8.3 for the relationship to the classical theory) one creates smooth connected affine curves $Y(\Gamma) = \Gamma\backslash\mathfrak{h}$ where $\mathfrak{h}$ is the upper half-plane, Γ a congruence subgroup $\mathrm{SL}_2(\mathbb{Z})$, and the action is by fractional linear transformation. Then, if $\Gamma \trianglelefteq \Gamma'$ then one obtains a left action of Γ'/Γ on $Y(\Gamma)$ such that $Y(\Gamma')$ can be identified with $(\Gamma'/\Gamma)\backslash Y(\Gamma)$. But, note that $\mathrm{SL}_2(\mathbb{Z})$ actually acts on $\mathfrak{h}$ through $\mathrm{PSL}_2(\mathbb{Z})$. Thus, in general, the Galois group of $Y(\Gamma) \to Y(\Gamma')$ will be the image of Γ'/Γ in $\mathrm{PSL}_2(\mathbb{Z})$ which is just $\Gamma'/\Gamma Z_{\Gamma'}$ where $Z_{\Gamma'} = \Gamma' \cap Z$ and where $Z = \{\pm 1\}$.

$G(\mathbb{A}_f)$-schemes and $G(\mathbb{A}_f^p)$-schemes

We would now like to discuss some more specific notions relating to **G**-schemes where **G** is the most common choice of locally profinite, but not profinite, group that shows up in this paper, in the form of the adelic points of a connected reductive group over $\mathbb{Q}$.

Let us begin by fixing our notation concerning the adeles over $\mathbb{Q}$. Namely, let $\mathbb{A} := \mathbb{A}_\mathbb{Q}$ denote the ring of adeles over $\mathbb{Q}$, and let us denote by $\mathbb{A}_f := \mathbb{A}_\mathbb{Q}^\infty$ the ring of finite adeles of $\mathbb{Q}$. The ring $\mathbb{A}_f$ is a locally compact (in fact, locally profinite) topological ring, which can be identified algebraically with $\widehat{\mathbb{Z}} \otimes_\mathbb{Z} \mathbb{Q}$ and which can be identified, as a topological ring, as the restricted direct product

$$\mathbb{A}_f = \prod_\ell{}' \mathbb{Q}_\ell \tag{7.15}$$

where the restricted direct product is taken relative to the open subrings $\mathbb{Z}_\ell \subseteq \mathbb{Q}_\ell$. For a fixed rational prime p let us then denote by $\mathbb{A}_f^p$ the subring of $\mathbb{A}_f$ with trivial p-component, or, in other words, the subring of (7.15) defined by

$$\mathbb{A}_f^p = \prod_{\ell \neq p}{}' \mathbb{Q}_\ell \tag{7.16}$$

The ring $\mathbb{A}_f^p$ is also a locally compact (locally profinite) topological ring, which can be algebraically identified with $\widehat{\mathbb{Z}}^p \otimes_\mathbb{Z} \mathbb{Q}$ where $\widehat{\mathbb{Z}}^p := \prod_{\ell \neq p} \mathbb{Z}_\ell$.

Remark 7.1.15. The algebraic identification $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes_\mathbb{Z} \mathbb{Q}$ can also be made to be a topological identification. Indeed, writing $\mathbb{Q} = \varinjlim_S \mathbb{Z}_S$ at S, where S ranges over the set of all finite subsets of rational primes, and $\mathbb{Z}_S$ is the localization of $\mathbb{Z}$, one obtains a presentation of $\mathbb{A}_f$ as $\varinjlim \mathbb{A}_{f,S}$ where

$$\mathbb{A}_{f,S} := \widehat{\mathbb{Z}} \otimes_\mathbb{Z} \mathbb{Z}_S = \prod_{\ell \in S} \mathbb{Q}_\ell \times \prod_{\ell \notin S} \mathbb{Z}_\ell \tag{7.17}$$

Giving $\mathbb{A}_{f,S}$ the product topology (obtained from the right most term of (7.17)), we obtain a direct limit topology on $\mathbb{A}_f$ which coincides with the topology from (7.15) under the natural ring isomorphism $\mathbb{A}_f \cong \widehat{\mathbb{Z}} \otimes_\mathbb{Z} \mathbb{Q}$.

Let us now fix $G/\mathbb{Q}$ a (connected) reductive group. Recall that the group $G(\mathbb{A}_f)$ has the natural structure of a locally profinite topological group obtained by giving $G(\mathbb{A}_f)$ the subspace topology of $\mathbb{A}_f^m$ obtained from a closed embedding of affine $\mathbb{Q}$-schemes

$G \hookrightarrow \operatorname{Spec}(\mathbb{Q}[x_1,\ldots,x_m])$ (see [Con12, Proposition 2.1] for details, in particular the independence of affine embedding). The same method works to endow $G(\mathbb{Q}_\ell)$ and $G(\mathbb{A}_f^p)$ with the structure of locally profinite groups for all rational primes ℓ and p.

One can also (non-canonically) describe the locally profinite group $G(\mathbb{A}_f)$ as a restricted product. Namely, suppose that S is a finite set of rational primes, and suppose that $\mathcal{G}$ is a group scheme model of G over $D(S) \subseteq \operatorname{Spec}(\mathbb{Z})$ (such models always exist by basic 'spreading out' principles, see [Poo17, Section 3.2]). Given such a model $\mathcal{G}$, one can describe the topological group $G(\mathbb{A}_f)$ as the following restricted direct product:

$$G(\mathbb{A}_f) = {\prod_{p\notin S}}' G(\mathbb{Q}_p) \times \prod_{p\in S} G(\mathbb{Q}_p) \tag{7.18}$$

Here the restricted direct product is taken relative to the open subgroups $\mathcal{G}(\mathbb{Z}_\ell) \subseteq G(\mathbb{Q}_\ell)$ for $\ell \notin S$. Because any two models $\mathcal{G}/D(S)$ and $\mathcal{G}'/D(S')$ are isomorphic over $D(T)$, for some third finite set of rational primes T, the above definition is independent of the choice of model (the fact that it agrees with our initial definition of the topological group $G(\mathbb{A}_f)$ follows from [Con12, Section 3]). One can mirror the arguments in loc. cit. to show that $G(\mathbb{A}_f^p)$ has the obvious presentation analagous equation (7.18).

Remark 7.1.16. The choice to work with G connected and reductive is not necessary at many points below, but we do so since all of our examples will have G connected reductive, and are unaware of interesting examples where G is not connected and reductive.

We recall that, almost by definition, the topologization of $G(\mathbb{A}_f)$ is functorial in G (i.e. if $G \to H$ is a morphism of groups over $\mathbb{Q}$, then $G(\mathbb{A}_f) \to H(\mathbb{A}_f)$ is continuous). Moreover, if $G \hookrightarrow H$ is a closed embedding of group schemes, then the induced map of topological groups $G(\mathbb{A}_f) \hookrightarrow H(\mathbb{A}_f)$ is a closed embedding of topological spaces. The same follows, *mutatis mutandis*, for the $\mathbb{A}_f^p$-points or the $\mathbb{Q}_\ell$-points.

Note also that we have natural continuous projection operators $G(\mathbb{A}_f) \to G(\mathbb{Q}_\ell)$ (resp. $G(\mathbb{A}_f^p) \to G(\mathbb{Q}_\ell)$) for all primes ℓ (resp. all primes $\ell \neq p$). This can either be thought of as the map induced by functoriality from the ring map $\mathbb{A}_f \to \mathbb{Q}_\ell$ or, equivalently, the projection operator when $G(\mathbb{A}_f)$ is given the presentation as a restricted direct product as above (and similarly for $G(\mathbb{A}_f^p)$).

So, applying the discussion of the last section to $G(\mathbb{A}_f)$ or $G(\mathbb{A}_f^p)$ allows us to define the notion of $G(\mathbb{A}_f)$-schemes and $G(\mathbb{A}_f^p)$-schemes, as well as schemes with a continuous right action of $G(\mathbb{A}_f)$ or $G(\mathbb{A}_f^p)$. So, for the remainder of this section, let $\mathbf{G}$ denote either $G(\mathbb{A}_f)$ or $G(\mathbb{A}_f^p)$.

It will be helpful in practice to fix notation concerning the most commonly used reasonable indexing sets of compact open subgroups of $G(\mathbb{A}_f)$ and $G(\mathbb{A}_f^p)$ which consists (essentially) of the class of open compact subgroups K of $\mathbf{G}$ which, in the case of Shimura varieties, act without fixed points (on the Shimura variety at infinite level). Let us now define rigorously this class of open compact subgroups of $\mathbf{G}$.

Begin by fixing an algebraic representation $\rho : G \to \mathrm{GL}_n$ over $\mathbb{Q}$. For an element $g = (g_\ell) \in \mathbf{G}$ (where the indices ℓ vary over all rational primes ℓ, or all rational primes $\ell \neq p$ depending on whether $\mathbf{G}$ is $G(\mathbb{A}_f)$ or $G(\mathbb{A}_f^p)$) let $t_\ell(g)$ denote the order of the torsion subgroup that the eigenvalues of $\rho(g_\ell) \in \mathrm{GL}_n(\mathbb{Q}_\ell)$ generate in $\overline{\mathbb{Q}_\ell}^\times$. We say that

g is *neat relative to* ρ if the $t_\ell(g)$ form a relatively prime set (i.e. the ideal in $\mathbb{Z}$ generated by the $t_\ell(g)$ is the unit ideal). We then call a subgroup K of $\mathbf{G}$ *neat relative to* ρ if all of its elements are.

We then have the following elementary lemma:

Lemma 7.1.17. *Let $g \in \mathbf{G}$ be arbitrary. Then, the following are equivalent:*

1. *The element g is neat relative to every representation ρ.*

2. *The element g is neat relative to a fixed faithful representation ρ.*

Proof. This follows immediately from the well-known fact (see [Del82, Proposition 3.1, a)]) that if $\rho : G \to \mathrm{GL}(V)$ is a faithful algebraic representation, then $V \oplus V^\vee$ generates the Tannakian category $\mathsf{Rep}(G)$. Indeed, it's clear that g is neat relative to $\rho \oplus \rho^\vee$ and that it's neat relative to any faithful subrepresentation of a representation generated by $\rho \oplus \rho^\vee$ in $\mathsf{Rep}(G)$. The aforementioned result of Deligne completes the assertion. $\square$

Thus, we define an element $g \in \mathbf{G}$ to be *neat* if it's neat relative to all representations ρ of G, or equivalently relative to a fixed faithful representation, and we define a subgroup K of $\mathbf{G}$ to be *neat* if all of its elements are. Note that a subgroup of $G(\mathbb{A}_f^p)$ is neat if and only if it's neat as a subgroup of $G(\mathbb{A}_f)$ so, most questions concerning neatness for subgroups of either $G(\mathbb{A}_f^p)$ or $G(\mathbb{A}_f)$ reduce readily to questions about subgroups of $G(\mathbb{A}_f)$.

Remark 7.1.18. Of course, we will be most interested in neat *open compact* subgroups of $\mathbf{G}$, in which case the distinction between $G(\mathbb{A}_f^p)$ and $G(\mathbb{A}_f)$ is important for topological reasons.

It will also be useful to mention a relationship between the two most common versions of neatness for an open compact subgroup of $G(\mathbb{A}_f)$ defined in the literature coincide. Namely, the definition we have adopted here is the one used in references like [Pin90] and [Lan13]. That said, many authors (e.g. see [Mil04]) use a notion of neatness adopted from the notion of a neat congruence subgroup. Namely, if $G/\mathbb{Q}$ is a reductive group, let us call an element $g \in G(\mathbb{Q})$ *neat* if all representations (equivalently for some faithful representation, the proof being the same as in Lemma 7.1.17) $\rho : G \to \mathrm{GL}_n(V)$ one has that the eigenvalues of $\rho(g)$ generate a torsion free subgroup of $\overline{\mathbb{Q}}^\times$. A subgroup $\Gamma \subseteq G(\mathbb{Q})$ is called *neat* if all of its elements are.

One knows that a subgroup $K \subseteq G(\mathbb{A}_f)$ is open compact if and only if $K \cap G(\mathbb{Q})$ is congruence (e.g. see [Mil04, Proposition 4.1]), and so one might ask what the relationship between neat congruence subgroups of $G(\mathbb{Q})$, and neat open compact subgroups of $G(\mathbb{A}_f)$ is. We have the following result in this direction:

Lemma 7.1.19. *Let $K \subseteq G(\mathbb{A}_f)$ be a neat compact open subgroup. Then, $G(\mathbb{Q}) \cap gKg^{-1}$ is a neat congruence subgroup for all $g \in G$.*

Proof. Suppose that K is neat. Then, note that $K \cap G(\mathbb{Q})$ is neat. Indeed, suppose not. Then, we can find a faithful representation $\rho : G \hookrightarrow \mathrm{GL}_n$ and some $g \in G(\mathbb{Q}) \cap K$ such that the eigenvalues $\rho(g)$ generates a torsion subgroup of $\overline{\mathbb{Q}}^\times$. Let $m > 1$ be the order of this torsion subgroup. Then, note that $(g)_\ell$ is a (diagonal) element of K. Moreover, for all ℓ one has that $t_\ell(g) = m$ relative to ρ, and so we see that the $t_\ell(g)$ generate (m), a non-unit ideal of $\mathbb{Z}$. Thus, K is not neat. To see that $K \cap gKg^{-1}$ is neat for all g we note that evidently if K is neat then so is gKg^{-1} since conjugate elements have the same eigenvalues. So, we can apply the argument just given to show that $G(\mathbb{Q}) \cap gKg^{-1}$ is neat. □

Remark 7.1.20. It's clear that one only needs to check for $g \in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$ which is a finite set (see [PR92, Chapter 8]). Also, the author suspects that the converse to Lemma 7.1.19 doesn't hold, but this is not important to our current work.

The vast majority of instantaneous instances of the phrase 'neat' can be replaced by the phrase 'torsion free' (evidently neat implies torsion free). The reason that neat is then preferred to the simpler notion of torsion free is that it is more functorial as the following simple lemma shows:

Lemma 7.1.21. *Let $G/\mathbb{Q}$ be a connected reductive algebraic group, and let $K \subseteq G(\mathbb{A}_f)$ be a neat subgroup and $\Gamma \subseteq G(\mathbb{Q})$ a neat subgroup. Then, for any closed connected reductive group H the subgroups $K \cap H(\mathbb{A}_f)$ and $\Gamma \cap H(\mathbb{Q})$ are neat (relative to H). Moreover, if G' is another connected reductive group over $\mathbb{Q}$ and $f : G \to G'$ is a homomorphism, then $f(K) \subseteq G'(\mathbb{A}_f)$ and $f(\Gamma) \subseteq G'(\mathbb{Q})$ are neat.*

Finally, for the notion of neat open compact subgroups of $\mathbf{G}$ be useful, we need the following classical lemma which shows that they are quite abundant. Before we state the result, let us denote by $\mathcal{N}(G)$ the neat open compact subgroups of $G(\mathbb{A}_f)$, and by $\mathcal{N}^p(G)$ the neat open compact subgroups of $G(\mathbb{A}_f^p)$. When we don't need to distinguish between $G(\mathbb{A}_f^p)$ and $G(\mathbb{A}_f)$, which we have then been writing as $\mathbf{G}$ in this section, we will shorten these two $\mathcal{N}(\mathbf{G})$:

Lemma 7.1.22. *The set $\mathcal{N}(\mathbf{G})$ satisfies the conditions of (2) from Lemma 7.1.9.*

Proof. The fact that $\mathcal{N}(\mathbf{G})$ is downward closed is evident. The fact that $\mathcal{N}(\mathbf{G})$ is conjguation closed has already been explained, and follows from the easy observation that conjugate elements of GL_n have the same eigenvalues. Thus, it remains to show that for every compact open subgroup K' of $\mathbf{G}$ there exists some $K \in \mathcal{N}(\mathbf{G})$ with $K \leqslant K'$. We follow the argument in [Pin90, Section 0.6].

Let us first assume that $G = \mathrm{GL}_n$. Let $N \geqslant 3$ be any integer. Let us denote by $\widehat{\Gamma}(N)$ the following subgroup of $\mathrm{GL}_n(\widehat{\mathbb{Z}})$:

$$\widehat{\Gamma}(N) = \ker(\mathrm{GL}_n(\widehat{\mathbb{Z}}) \to \mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z})) \tag{7.19}$$

We claim that the assumption $N \geqslant 3$ implies that $\widehat{\Gamma}(N)$ is neat. It suffices to assume that $N = p^i$ since if $N \geqslant 3$ then there exists some $p^i \mid N$ with $p^i \geqslant 3$, and since neat subgroups are downward closed it suffices to prove the claim for $\widehat{\Gamma}(p^i)$ since $\widehat{\Gamma}(N) \subseteq \widehat{\Gamma}(p^i)$. The claim for $\widehat{\Gamma}(p^i)$ is trivial and left to the reader. So, let K' be a compact open subgroup of $\mathrm{GL}_n(\mathbb{A}_f)$. Note that since the $\widehat{\Gamma}(N)$, for $N \geqslant$, evidently form a cofinal system in $\mathrm{GL}_n(\mathbb{A}_f)$, and so we have that $\widehat{\Gamma}(N) \subseteq K'$ for N sufficiently large.

For general G one can take a linear embedding $G \hookrightarrow \mathrm{GL}_n$ for some n and then consider the pullback of the groups $\widehat{\Gamma}(N)$ as defined above, which will be open compact subgroups of $G(\mathbb{A}_f)$. $\square$

So, now that we have all of this setup, we can state the one of the main classes of $\mathbf{G}$-schemes of interest to us. Namely, a $\mathbf{G}$-scheme $\{X_K\}_{K\in\mathcal{N}}$ is called *Shimura like* if it is almost Shimura like and $\mathcal{N} = \mathcal{N}(\mathbf{G})$. In other words, a $\mathbf{G}$-scheme is Shimura like if it is almost Shimura like and the indexing subgroups are precisely the neat open compact subgroups of $\mathbf{G}$.

As a last observation, we note that if K_0 is an arbitrary compact open subgroup of $G(\mathbb{A}_f)$, then for any $K^p \in \mathcal{N}^p(G)$, one has that $K^pK_0 \in \mathcal{N}(G)$. Indeed, it's clear that K^pK_0 is compact open in $G(\mathbb{A}_f)$. To see that it's neat we fix a faithful representation $\rho : G \hookrightarrow \mathrm{GL}_n$ and note that by assumption that K^p is neat we have that for any $g = (g_\ell) \in K^pK_0$ that $(t_\ell(\rho(g))_{\ell\neq p}$ is the unit ideal in $\mathbb{Z}$, and so evidently $(t_\ell(\rho(g))_\ell$ is the unit ideal in $\mathbb{Z}$. Thus, in a sense, for any compact open subgroup $K_0 \subseteq G(\mathbb{A}_f)$ we can 'embed' $\mathcal{N}^p(G)$ into $\mathcal{G}$ as the set of subgroups of the form K^pK_0 for $K^p \in \mathcal{N}^p(G)$.

Base change and models

We would now like to discuss the notion of base change for $\mathbf{G}$-schemes where we are back, again, to assuming $\mathbf{G}$ is an aribtrary locally profinite group. This will allow us to talk about models of $\mathbf{G}$-schemes over DVRs, and we then seek to define the notion of integral canonical models in the sense of [Mil04] and [Kis10].

So, let us fix a locally profinite group $\mathbf{G}$, an affine base scheme S_0, and a $\mathbf{G}$-scheme $\{X_K\}_{K\in\mathcal{N}}$ over S_0. Suppose that T_0 is an affine scheme, and $f : T_0 \to S_0$ is a morphism. Then, we can form the *pull back system* of $\{X_K\}_{K\in\mathcal{N}}$ along f by which we mean the set of schemes $\{X_K \times_{S_0} T_0\}_{K\in\mathcal{N}}$ together with the morphisms

$$t_{K,L}(g)_{T_0} : X_K \times_{S_0} T_0 \to X_K \times_{S_0} T_0 \tag{7.20}$$

The are no claims being made that this system is a $\mathbf{G}$-scheme over T_0. The only issue being that the pull back of a geometric quotient needn't be, in general, a geometric quotient.

That said, we have the following result which gives us situations in which the pull back of a geometric quotient is a geometric quotient:

Lemma 7.1.23. *Let S_0 be an affine base scheme and X an S_0-scheme. Let Γ be a finite abstract group and suppose that X is equipped with a right Γ-action by S_0-automorphisms. Suppose further that $\pi : X \to Y$ is a quotient of X by Γ. Then, if $T_0 \to S_0$ is a morphism of affine schemes, then $\pi_{T_0} : X_{T_0} \to Y_{T_0}$ is a geometric quotient with X_{T_0} given the induced right Γ-action if either:*

1. *The morphism $T_0 \to S_0$ is flat.*
2. *The morphism π is étale.*

Proof. The claim then follows from [EvdGM, Theorem 4.6, Chapter IV]. $\square$

Of course, we deduce from this the following set of examples where one has that the pull back system is a $\mathbf{G}$-scheme:

Lemma 7.1.24. *The pull back system of $\{X_K\}_{K\in\mathcal{N}}$ is a $\mathbf{G}$-scheme over T_0 if either:*

1. *The morphism $f : T_0 \to S_0$ is flat.*

2. *The $\mathbf{G}$-scheme $\{X_K\}_{K\in\mathcal{N}}$ is smooth.*

Proof. As mentioned above, the only question is about whether or not geometric quotients are preserved by pull back from which this follows from Lemma 7.1.23. □

As a corollary, we deduce the following:

Corollary 7.1.25. *If $\{X_K\}_{K\in\mathcal{N}}$ is an almost Shimura like $\mathbf{G}$-scheme over S_0 relative to Z, then the pull back of $\{X_K\}_{K\in\mathcal{N}}$ along a flat map $f : T_0 \to S_0$ is a Shimura like $\mathbf{G}$-scheme over T_0 relative to Z.*

Note that if $\{X_K\}_{K\in\mathcal{N}}$ is a $\mathbf{G}$-scheme over S_0, and $T_0 \to S_0$ is a map such that the pull back of the system is a $\mathbf{G}$-scheme then there is a natural description of the space at infinite level of the pull back. Namely, it's clear that if X is the space at infinite level for $\{X_K\}_{K\in\mathcal{N}}$ then the space at infinite level for the pull back is X_{T_0} with the induced right $\mathbf{G}$-action.

We would now like to talk about the notion of models. Namely, let's assume that $\mathcal{O}$ is a DVR of mixed characteristic $(0, p)$ and with fraction field F. If $\{X_K\}_{K\in\mathcal{N}}$ is a $\mathbf{G}$-scheme over F, then a *model over $\mathcal{O}$* is a $\mathbf{G}$-scheme $\{\mathcal{X}_K\}_{K\in\mathcal{N}}$ together with an isomorphism

$$\{X_K\}_{K\in\mathcal{N}} \xrightarrow{\approx} \{\mathcal{X}_K \times_{\mathrm{Spec}(\mathcal{O})} \mathrm{Spec}(F)\}_{K\in\mathcal{N}} \tag{7.21}$$

of $\mathbf{G}$-schemes over F. In practice we will suppress this isomorphism over the generic fiber.

We would like to define a notion of 'good model' of $\mathbf{G}$-schemes in a relatively narrow situation which encompasses the theory of good reduction of Shimura varieties called an *integral canonical model* which, roughly, satisfy a 'Nèron lifting property' for a class of test schemes. Namely, let us say that a model $\{\mathcal{X}_K\}_{K\in\mathcal{N}}$ over $\mathrm{Spec}(\mathcal{O})$ of an almost Shimura like $\mathbf{G}$-scheme $\{X_K\}_{K\in\mathcal{N}}$ is an *integral canonical model* if it satisfies the following two criterion:

1. It is almost Shimura like.

2. Its space at infinite level satisfies the *Nèron lifting property*: for all regular and formally smooth $\mathcal{O}$-schemes $\mathcal{Y}$ the natural map

$$\mathrm{Hom}(\mathcal{Y}_F, X) \to \mathrm{Hom}(\mathcal{Y}, \mathcal{X}) \tag{7.22}$$

is a bijection.

there is no need, in general, for an integral canonical model to exist.

Let us note the following observation that an integral canonical model, if it exists, is unique. Note that there is some content to this statement since the Nèron lifting property only characterizes the $\mathcal{O}$-scheme $\mathcal{X}$ a priori, and not the scheme with a right $\mathbf{G}$-action:

Lemma 7.1.26. *Suppose that $\{\mathscr{X}^i_K\}_{K\in\mathcal{N}}$ are integral canonical models of $\{X_K\}_{K\in\mathcal{N}}$ over* $\mathrm{Spec}(\mathcal{O})$ *for $i=1,2$. Then, there is a canonical isomorphism*

$$\{\mathscr{X}^1_K\}_{K\in\mathcal{N}} \xrightarrow{\approx} \{\mathscr{X}^2_K\}_{K\in\mathcal{N}} \tag{7.23}$$

of $\mathbf{G}$*-schemes over* $\mathrm{Spec}(\mathcal{O})$*.*

Proof. Let $\mathscr{X}^1$ and $\mathscr{X}^2$ denote the spaces at infinite level associated to these two $\mathbf{G}$-schemes. It suffices to show, by Lemma 7.1.10 to show that there is an isomorphism of $\mathcal{O}$-schemes $\mathscr{X}^1 \to \mathscr{X}^2$ equivariant for the right $\mathbf{G}$-actions. Since these two $\mathbf{G}$-schemes over $\mathcal{O}$ are models of $\{X_K\}_{K\in\mathcal{N}}$ we can choose an isomorphism $\phi_F : \mathscr{X}^1_F \to \mathscr{X}^2_F$ of $\mathbf{G}$-schemes over F. By the Nèron lifting property this lifts to an isomorphism of $\mathcal{O}$-schemes $\phi : \mathscr{X}^1 \to \mathscr{X}^2$ with generic fiber ϕ_F. To show that it is $\mathbf{G}$-equivariant let $g \in \mathbf{G}$ be arbitrary. We want to show that the two maps maps $f_i : \mathscr{X}^1 \to \mathscr{X}^2$ given by $f_1(x) := \phi\circ[g]$ and $f_2 = [f(g)] \circ \phi$ (where the brackets indicate the action map) are equal. But, note that $\mathscr{X}^1$ is reduced (by Lemma 7.1.12), $\mathscr{X}^2$ is separated over $\mathcal{O}$ (by Lemma 7.1.26), and since $\mathscr{X}^1_F$ is dense in $\mathscr{X}^1$ the conclusion follows. □

Let us prove the elementary observation used above:

Lemma 7.1.27. *Let $\mathbf{G}$ be any locally profinite group, S_0 any affine scheme, and let X be an S_0-scheme with a continuous right $\mathbf{G}$-action. Then, X is separated over S.*

Proof. Fix any compact open subgroup $K \in \mathcal{N}$. Then, we evidently have that X is also the projective limit $\varprojlim_{K'\subseteq K} X/K'$. The map $X/K \to S$ is quasi-projective by hypothesis, and so separated. The map $X \to X/K$ is affine, since the transition maps $X/K'' \to X/K'$ are finite, and thus is also separated. Thus, $X \to S$, being the composition $X \to X/K \to S$ is separated. □

Remark 7.1.28. The history of the notion of integral canonical models is slightly treacherous, and so it might be helpful to the reader to briefly recall it. The original notion of integral canonical models was informally suggested to exist by Langlands in [Lan76] and was first rigorously defined in [Mil92] in the context of integral canonical models of Shimura varieties. That said, Milne originally used a more lax class of test schemes than those listed above (i.e. regular and formally smooth over $\mathcal{O}$).

Unfortunately, the key example of integral canonical models (discussed in §8.4) was shown to satisfy this lifting property for Milne's suggested class of test schemes assuming the result [FC13, Cor. V.6.8] of Faltings-Chai, which was later shown to be incorrect by Raynaud-Ogus-Gabber, which seems to have first been noted in print by Vasiu in [Vas99] who suggested an alternate class of test schemes. Later Moonen (see [Moo98, Comment 3.4]) suggested an alternate, more practical, class of test schemes, altering slightly a class suggested by Vasiu. Ultimately the class of regular and formally smooth $\mathcal{O}$-schemes was settled upon by Kisin in [Kis10] who showed the existence of integral canonical models for Shimura varieties of abelian type.

7.2 Associated ℓ-adic sheaves

We now discuss how to associate ℓ-adic sheaves to $\mathbf{G}$-schemes in a systematic way.

General theory

Let us fix a locally profinite group $\mathbf{G}$. Suppose that we have an affine base scheme S_0, assumed Noetherian for simplicity, and an almost Shimura like $\mathbf{G}$-scheme $\{X_K\}_{K\in\mathcal{N}}$ over S_0 with respect to $Z \subseteq Z(\mathbf{G})$. We wish to explain how given a continuous representation r of $\mathbf{G}$ and a $K \in \mathcal{N}$ one can build a lisse sheaf $\mathcal{F}_{r,K}$ on X_K, and discuss various properties of this sheaf. This is loosely based on the beginning of [HT01, III.2] as well as the discussion in [Kot92b, §6]. In essence, this is an adaptation of the classical correspondence between lisse sheaves and representations of the fundamental group to the situation of disconnected Galois covers.

Remark 7.2.1. Let us explicitly mention the relationship between the contents of this section and the sources [HT01, III.2] and [Kot92b, §6]. One can view this section as largely an explication and edulceration of the content of [HT01, III.2] with a more explicit discussion of these sheaves in the context of $\mathbf{G}$-schemes that will be useful for our purposes. The content of [Kot92b, §6] has two issues that make it difficult to make rigorous in general situations. First, since we are generally dealing with disconnected schemes, arguments using the étale fundamental group are very, very mildly imprecise. Second, in the situation that Kottwitz is dealing with Shimura varieties for which one can take Z, in the definition of Shimura like $G(\mathbb{A}_f)$-schemes, to be trivial. This is not the case for all the Shimura varieties we wish to consider, namely not for all abelian type Shimura varieties, but is true for Hodge type Shimura varieties as considered in [Kot92b].

So, let us begin by fixing a prime ℓ invertible on S_0 and a finite extension M of $\mathbb{Q}_\ell$ with uniformizer π. Let us further fix $n \geqslant 1$ and $m \geqslant 1$. Assume now that we have a continuous representation

$$r_m : \mathbf{G} \to \mathrm{GL}_{\mathcal{O}_M/\pi^m\mathcal{O}_M}(\Lambda_m) \tag{7.24}$$

where Λ_m is a free finite rank $\mathcal{O}_M/\pi^m\mathcal{O}_M$-module. Let us fix $K \in \mathcal{N}$ and let us fix a cofinal subset $\mathcal{S} \subseteq \mathcal{N}$ consisting of normal subgroups of K. We will define a lisse $\mathcal{O}_M/\pi^m\mathcal{O}_M$-sheaf $\mathcal{F}_{r_m,K,\mathcal{S}}$ on X_K which will be canonically independent of our choice of $\mathcal{S}$. In practice we will think of $\mathcal{S}$ as being $\mathcal{S}_K$ which is defined to be the set of all normal subgroups of K that are in $\mathcal{N}$. Note that $\mathcal{S}_K$ really does form a cofinal family by Lemma 7.1.8.

Remark 7.2.2. In the discussion in [HT01, III.2] Harris and Taylor assume that $\mathcal{S}$ is a linear order or, in other words, that we have chosen a sequence

$$K = K_1 \trianglerighteq K_2 \trianglerighteq K_3 \cdots \tag{7.25}$$

with $K_i \in \mathcal{N}$ and such that $\bigcap_i K_i$ is the trivial subgroup. Let us note that this is, indeed, always possible. It evidently suffices to show that there is a cofinal, linearly ordered, set of open compact subgroups of $\mathbf{G}$ since one can then intersect these with K and proceed as in Lemma 7.1.8.

Finally, it suffices to prove this last claim in the case of GL_n since choosing an embedding $G \hookrightarrow \mathrm{GL}_n$ allows one to pull back such a cofinal, linearly ordered family in $\mathrm{GL}_n(\mathbb{A}_f)$ to $G(\mathbb{A}_f)$. For $N \geqslant 1$ let us set

$$\widehat{\Gamma}(N) := \ker(\mathrm{GL}_n(\widehat{\mathbb{Z}}) \to \mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z})) \tag{7.26}$$

We can then take $\widehat{\Gamma}(N!)$ as our cofinal, linearly ordered set of compact open subgroups of $\mathrm{GL}_n(\mathbb{A}_f)$. As this example shows, it is much more natural to allow arbitary cofinal systems. For example, in the case of GL_n, one can take $\widehat{\Gamma}(N)$ for all N.

So, let us define the presheaf $\mathcal{F}_{r_m,K,\mathcal{S}}$ on the étale site of X_K as follows:

$$\mathcal{F}_{r_m,K,\mathcal{S}}(U) := \varinjlim_{K' \in \mathcal{S}} \mathrm{Fun}_K(\pi_0(U_{K'}), \Lambda_m) \tag{7.27}$$

where $U_{K'} := U \times_{X_K} X_{K'}$ and $\mathrm{Fun}_K(\pi_0(U_{K'}), \Lambda_m)$ denotes the $\mathcal{O}_M/\pi^m\mathcal{O}_M$-module of (right) K-invariant functions

$$f : \pi_0(U_{K'}) \to \Lambda_m \tag{7.28}$$

where $\pi_0(U_{K'})$ is given the action via the right action of K on $U_{K'}$ and Λ_m is given the right action induced by r_m. Less cryptically, a function $f : \pi_0(U_{K'}) \to \Lambda_m$ is K-invariant if for all $\gamma \in K$ and all $C \in \pi_0(U_{K'})$ one has that

$$f(C \cdot \gamma) = r_m(\gamma)^{-1} f(C) \tag{7.29}$$

which is easily seen to be a $\mathcal{O}_M/\pi^m\mathcal{O}_M$-module since $\mathbf{G}$-acts $\mathcal{O}_M/\pi^m\mathcal{O}_M$-linearly only Λ_m. Here, to be explicit, the action of K on $U_{K'}$ is the pull back of the action of K on $X_{K'}$ which is defined by having $k \in K$ act on $X_{K'}$ by $[k]_{K'}$ which is an automorphism of $X_{K'}$ since $kK'k^{-1} = K'$ since $K' \trianglelefteq K$.

Of course, the direct limit in (7.27), which is with respect to the directed set $\mathcal{N}^{\mathrm{op}}$, is the one with transition for $K_1 \trianglelefteq K_2$ in $\mathcal{S}$ being taken to the map

$$\mathrm{Fun}_K(\pi_0(U_{K_2}), \Lambda_m) \to \mathrm{Fun}_K(\pi_0(U_{K_1}), \Lambda_m) \tag{7.30}$$

taking f to $f \circ (\pi_{K_1,K_2})_U$ where we conflate $(\pi_{K_1,K_2})_U$ with the map it induces on sets of connected components. Also, note that if we have a morphism $U' \to U$ in the étale site of X_K, one naturally obtains a morphism $\mathcal{O}_M/\pi^m\mathcal{O}_M$-modules $\mathcal{F}_{r_m,K,\mathcal{S}}(U) \to \mathcal{F}_{r_m,K\mathcal{S}}(U')$ coming from the natural maps $\pi_0(U'_{K'}) \to \pi_0(U_{K'})$ for all $K' \in \mathcal{S}$. Thus, $\mathcal{F}_{r_m,K,\mathcal{S}}$ really is a presheaf of $\mathcal{O}_M/\pi^m\mathcal{O}_M$-modules.

We then have the following elementary lemma:

Lemma 7.2.3. *The presheaf $\mathcal{F}_{r_m,K,\mathcal{S}}$ is a sheaf naturally independent of $\mathcal{S}$. Let $K'_0 \in \mathcal{S}$ be such that $K'_0 \subseteq \ker r_m$. Then, $(\mathcal{F}_{r_m,K,\mathcal{S}}) |_{X_{K'_0}}$ is constant with global sections the module $\Lambda_m^{Z_K}$ of Z_K-invariants of Λ_m. In particular, $\mathcal{F}_{r_m,K,\mathcal{S}}$ is a lisse $\mathcal{O}_M/\pi^m\mathcal{O}_M$-sheaf.*

Proof. Let us first verify that $\mathcal{F}_{r_m,K,\mathcal{S}}$ is a sheaf. To do this it evidently suffices to show that for each $K' \in \mathcal{S}$ that the association

$$U \mapsto \mathrm{Fun}_K(\pi_0(U_{K'}), \Lambda_m) \tag{7.31}$$

is a sheaf, since the presheaf inverse limit of sheaves is a sheaf.

Let us now show that if $K_0' \subseteq \ker r_m$, then $(\mathcal{F}_{r_m,K,\mathcal{S}}) \mid_{X_{K_0'}}$ is constant. We start by noting that for any $U \to X_{K_0'}$ an étale morphism, one has that

$$\mathcal{F}_{r_m,K,\mathcal{S}}(U) = \varinjlim_{K' \subseteq K_0'} \mathrm{Fun}_K(\pi_0(U_{K'}), \Lambda_m) \tag{7.32}$$

But, note that by the discussion in Lemma 7.1.24 the map $U_{K'} \to U_{K_0'}$ is a quotient mapping for K_0'/K'. But, since K_0'/K' acts trivially on Λ_m by the assumption that $K_0' \subseteq \ker r_m$ this implies that any K-invariant function f will necessarily factor through $\pi_0(U_{K'})/(K_0'/K') = \pi_0(U_{K_0'})$. Thus, this inverse limit is just $\mathrm{Fun}_K(\pi_0(U_{K_0'}), \Lambda_m)$.

But, note that

$$U_{K_0'} = U \times_{X_K} X_{K_0'} = U \times_{X_{K_0'}} X_{K_0'} \times_{X_K} X_{K_0'} = U \times \underline{K/K_0' Z_K} \tag{7.33}$$

where the last claim followed from the fact that $X_{K_0'} \to X_K$ is a Galois cover with Galois group $K/K_0'Z_K$. So then, we deduce that $\pi_0(U_{K_0'})$ is just $\pi_0(U) \times K/K_0'Z_K$ and that the K action is entirely on this second factor. Thus, we see that $\mathcal{F}_{r,K,\mathcal{S}}(U)$ is just $\mathrm{Fun}_K(K/K_0'Z_K, \Lambda_m)^{\pi_0(U)}$. Thus, we conclude that $\mathcal{F}_{r,K,\mathcal{S}} \mid_{X_{K_0'}}$ is constant.

The last claim then follows from the observation that the $\mathcal{O}_M/\pi^m\mathcal{O}_M$-module $\mathrm{Fun}_K(K/K_0'Z_K, \Lambda_m)$ can be identified with the module of Z_K-invariants of Λ_m since $K_0' \subseteq \ker r_m$. □

Due to the independence of $\mathcal{F}_{r_m,K,\mathcal{S}}$ we shall always denote it by $\mathcal{F}_{r_m,K}$ where, as before mentioned, we are implicitly taking $\mathcal{S}_K$ as our canonical choice for $\mathcal{S}$.

Suppose now that we are given a continuous representation $r : \mathbf{G} \to \mathrm{GL}_{\mathcal{O}_M}(\Lambda)$ where Λ is a free $\mathcal{O}_M$-module of finite rank. For all $m \geqslant 1$ let us denote by r_m the induced continuous representation $\mathbf{G} \to \mathrm{GL}_{\mathcal{O}_M/\pi^m\mathcal{O}_M}(\Lambda_m)$ where $\Lambda_m := \Lambda/\pi^m\Lambda$.

We then have the following obvious observation:

Lemma 7.2.4. *There are canonical isomorphisms*

$$\mathcal{F}_{r_{m+1},K,\mathcal{S}} \otimes_{\Lambda_{m+1}} \Lambda_m \xrightarrow{\approx} \mathcal{F}_{r_m,K,\mathcal{S}} \tag{7.34}$$

Proof. Choose $K_0' \subseteq \ker r_{m+1}$, then evidently $K_0' \subseteq \ker r_m$. A byproduct of the proof of Lemma 7.2.3 is that $\mathcal{F}_{r_i,K,\mathcal{S}}$ is the same as the sheaf associating to U the $\mathcal{O}_M/\pi^i\mathcal{O}_M$-module $\mathrm{Fun}_K(\pi_0(U_{K_0'}), \Lambda_i)$ for $i = m, m+1$. The claim is then obvious. □

Thus we obtain a lisse $\mathcal{O}_M$-sheaf (called locally constant in [FK13, I.12] or smooth in [KW13, Appendix A]) from the A-R π-adic sheaf $(\mathcal{F}_{r_m,K,\mathcal{S}})$ which we denote $\mathcal{F}_{r,K,\mathcal{S}}$. It's clear that this lisse $\mathcal{O}_M$-sheaf is naturally independent of the choice of $\mathcal{S}$, and so we shall denote it $\mathcal{F}_{r,K}$ where, again, we are implicitly thinking of $\mathcal{S}$ as $\mathcal{S}_K$ when the choice needs to be made.

Would now like to understand this lisse $\mathcal{O}_M$-sheaf $\mathcal{F}_{r,K}$ in terms of a fundamental group representation. To this end, let us fix a geometric point $\overline{x}$ of the space at infinite level X. Denote for all $K' \in \mathcal{N}$ the geometric point $\overline{x}_{K'}$ obtained by composing $\overline{x}$ with $\pi_{K'} : X \to X_{K'}$. For all $K' \in \mathcal{N}$ us denote by $C_{\overline{x},K'}$ the connected component of $X_{K'}$ containing $\overline{x}_{K'}$.

Fixing K as above, let us denote by $H_{\overline{x}}(K/K')$, for every $K' \in \mathcal{S}_K$, the stabilizer of $C_{\overline{x},K'}$ in $\mathrm{Gal}(X_{K'}/X_K) = K/K'Z_K$. Note that then that we obtain a pointed pro-finite-étale cover $\{C_{\overline{x},K'} \to C_{\overline{x},K}\}_{K' \in \mathcal{S}_K}$ with Galois group

$$H_{\overline{x}}(K) := \varprojlim_{K' \in \mathcal{S}_K} H_{\overline{x}}(K/K') \tag{7.35}$$

and thus a natural continuous surjection

$$\rho_{\overline{x},K} : \pi_1^{\text{ét}}(C_{\overline{x},K}, \overline{x}_K) \to H_{\overline{x}}(K) \tag{7.36}$$

in the usual way.

We then have the following claim:

Lemma 7.2.5. *Let $r : \mathbf{G} \to \mathrm{GL}_{\mathcal{O}_M}(\Lambda)$ be a continuous representation where Λ is a finite free $\mathcal{O}_M$-module. Then, the action of $\pi_1^{\text{ét}}(C_{\overline{x},K}, \overline{x}_K)$ on $(\mathcal{F}_{r,K})_{\overline{x}_K}$ is isomorphic to the representation*

$$\pi_1^{\text{ét}}(C_{\overline{x},K}, \overline{x}_K) \xrightarrow{\rho_{\overline{x},K}} H_{\overline{x}}(K) \xrightarrow{r} \mathrm{GL}_{\mathcal{O}_M}(\Lambda^{Z_K}) \tag{7.37}$$

Proof. This follows quite easily from Lemma 7.2.3 by considering. □

Finally, suppose now that $r : \mathbf{G} \to \mathrm{GL}_{\overline{\mathbb{Q}_\ell}}(V)$ is any continuous representation where V is a finite dimensional $\overline{\mathbb{Q}_\ell}$-space. Since K is compact we know that $r(K)$ is contained in $\mathrm{GL}_n(\mathcal{O}_M)$ for some finite extension $M/\mathbb{Q}_\ell$. The exact procedure above takes this representation $r : K \to \mathrm{GL}_n(\mathcal{O}_M)$ and produces a lisse $\mathcal{O}_M$-sheaf $\mathcal{F}_{r,K}$ which, technically, depends on the choice of $\mathcal{O}_M$. That said, the corresponding lisse $\overline{\mathbb{Q}_\ell}$-sheaf (see [KW13, Appendix A] where it's called a smooth $\overline{\mathbb{Q}_\ell}$-sheaf) does not depend on this choice. We thus obtain a well-defined lisse $\overline{\mathbb{Q}_\ell}$-sheaf which we denote $\mathcal{F}_{r,K}$.

We then have the following obvious corollary of Lemma 7.2.5:

Lemma 7.2.6. *Let $r : \mathbf{G} \to \mathrm{GL}_{\overline{\mathbb{Q}_\ell}}(V)$ be a continuous finite-dimensional representation. Then, with the notation as above, the $\pi_1^{\text{ét}}(C_{\overline{x},K}), \overline{x}_K)$-action on $(\mathcal{F}_{r,K})_{\overline{x}_K}$ is isomorphic to the representation*

$$\pi_1^{\text{ét}}(C_{\overline{x},K}, \overline{x}_K) \xrightarrow{\rho_{\overline{x},K}} H_{\overline{x}}(K) \xrightarrow{r} \mathrm{GL}_{\overline{\mathbb{Q}_\ell}}(V^{Z_K}) \tag{7.38}$$

So that we don't have to worry about distingushing between these three cases above, when we say 'continuous representation' in what follows, let us understand a continuous representation on $\mathbf{G}$ to valued in a group of the form $\mathrm{GL}_{\mathcal{O}_M/\pi^m\mathcal{O}_M}(\Lambda_m)$, $\mathrm{GL}_{\mathcal{O}_M}(\Lambda)$, or $\mathrm{GL}_{\overline{\mathbb{Q}_\ell}}(V)$. When we don't wish to specify which of these three types, but we want to use the notation of a representation, we will write $r : \mathbf{G} \to \mathrm{GL}_R(W)$.

The existence of the Z_K-invariants in the statements of the above lemmas is quite disquieting, and makes it difficult to relate these sheaves between different K. So, let us say that a representation r called *adapted to* K if $Z_K \subseteq \ker r$. Let us say that r is *adapted to* $\mathcal{N}$ if it is adapted to some $K \in \mathcal{N}$. Let us denote by $\mathcal{N}(r)$ the subset of those $K \in \mathcal{N}$ for which r is K-adapated. For example, if $\{X_K\}_{K \in \mathcal{N}}$ is alomst Shimura like with respect to the trivial subgroup (which is often the case) one has that $\mathcal{N}(r) = \mathcal{N}$.

In general, we have the following trivial observation:

Lemma 7.2.7. *Let $r : \mathbf{G} \to \mathrm{GL}_R(W)$ be a continuous representation which is adapted to $\mathcal{N}$. Then, the set $\mathcal{N}(r)$ forms a reasonable subset of $\mathcal{N}$.*

Thus, if r is adapted to $\mathcal{N}$ we may, up to replacing our indexing set $\mathcal{N}$ by a smaller indexing set $\mathcal{N}(r)$, one can assume that any representation is adapted to every element of the indexing set.

The assumption that r is K-adapted also has the nice property that it allows us to build a compatible family of sheaves on X_K. Namely, we have the following simple lemma:

Lemma 7.2.8. *Let r be a continuous representation adapted to $\mathcal{N}$. Then, for any $g \in \mathbf{G}$ and $K \in \mathcal{N}(r)$ we have natural isomorphisms*

$$\varphi_{g,K} : [g]_K^* \mathcal{F}_{r,g^{-1}Kg} \xrightarrow{\approx} \mathcal{F}_{r,K} \tag{7.39}$$

such that for any $\overline{x}$ a geometric point of X, the map $(\varphi_g)_{\overline{X}_K}$ is the automorphism $r(g)$ of W under the identifications

$$([g]_K^* \mathcal{F}_{r,g^{-1}Kg})_{\overline{x}_K}) = (\mathcal{F}_{r,g^{-1}Kg})_{\overline{x}_{g^{-1}Kg}} = W, \qquad (\mathcal{F}_{r,K})_{\overline{x}_K} = W \tag{7.40}$$

from the above lemmas.

Proof. This is obvious from Lemma 7.2.3 by noting that the $K/K_0'Z_K$-action is via r. □

Similarly, we have the following:

Lemma 7.2.9. *Let r be a continous representation of $\mathbf{G}$ adapted to $\mathcal{N}$, and let $K' \subseteq K$ be elements of $\mathcal{N}(r)$. Then, there are canonical isomorphisms*

$$\varphi_{K',K} : \pi_{K',K}^* \mathcal{F}_{r,K} \xrightarrow{\approx} \mathcal{F}_{r,K'} \tag{7.41}$$

such that under the identifications

$$(\pi_{K',K} \mathcal{F}_{r,K})_{\overline{x}_{K'}} = (\mathcal{F}_{r,K})_{\overline{x}_K} = W, \qquad (\mathcal{F}_{r,K'})_{\overline{x}_{K'}} = W \tag{7.42}$$

the induced map $(\varphi_{K',K})_{\overline{x}_{K'}}$ is the identity.

Proof. This is also obvious from Lemma 7.2.3. □

We can also use the above machinery to understand the pull back of a $\mathcal{N}$-adpated sheaf to the space X at infinite level along the projection map $\pi_K : X \to X_K$. Namely, we have the following:

Lemma 7.2.10. *Let r be a continuous representation of $\mathbf{G}$ adapted to $\mathcal{N}$. Then, for $K \in \mathcal{N}(r)$ the pullback $\mathcal{F}_r := \pi_K^* \mathcal{F}_{r,K}$ doesn't depend on the choice of $K \in \mathcal{N}(r)$. The isomorphisms $\varphi_{g,K}$ for $K \in \mathcal{N}(r)$ define a left action of $\mathbf{G}$ on $\mathcal{F}_r$. The sheaf $\mathcal{F}_r$, together with the action of $\mathbf{G}$, can be identified with the constant sheaf $\underline{W}$ together with the action of W induced by r.*

7.3 Cohomology at infinite level and Hecke operators

We would now like to discuss the (compactly supported) étale cohomology of $\mathbf{G}$-schemes with coefficients in the ℓ-adic sheaves we constructed in the last section. So, let us fix a locally profinite group $\mathbf{G}$, a field F, and an almost Shimura like $\mathbf{G}$-scheme $\{X_K\}_{K\in\mathcal{N}}$ over F with respect to $Z \leqslant Z(\mathbf{G})$. Suppose further that $r:\mathbf{G}\to \mathrm{GL}_{\overline{\mathbb{Q}_\ell}}(V)$ is a continuous $\overline{\mathbb{Q}_\ell}$-representation for some ℓ invertible in F which is $\mathcal{N}$-adpated.

From the discussion in the previous section we obtain the data of a reasonable subset $\mathcal{N}(r)\subseteq\mathcal{N}$ and for each $K'\subseteq K$ with $K\in\mathcal{N}(r)$ a lisse $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}_{r,K}$ on X_K together with compatible isomorphisms $\varphi(K',K)$ and $\varphi_{g,K}$. We also obtained the lisse $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}_r$ on the space at infinite level X.

Thus, for all $K'\subseteq K$ in $\mathcal{N}(r)$ these compatibilities (and the fact that $\pi_{K',K}$ is finite, thus proper, for the case of compactly supported cohomology) implies that we obtain morphisms of finite-dimensional continuous $\overline{\mathbb{Q}_\ell}$-representations of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$

$$\pi^*_{K',K}: H^i(\overline{X_K}_{\text{ét}},\overline{\mathcal{F}_{r,K}})\to H^i_{\text{ét}}(\overline{X_{K'}},\overline{\mathcal{F}_{r,K'}}) \tag{7.43}$$

and

$$\pi^*_{K',K}: H^i_c(\overline{X_K}_{\text{ét}},\overline{\mathcal{F}_{r,K}})\to H^i_c(\overline{X_{K'}}_{\text{ét}},\overline{\mathcal{F}_{r,K'}}) \tag{7.44}$$

where, as per usual, the overline indicates base change to F^{sep}.

We note that then that these morphisms make $\{H^i(\overline{X_K}_{\text{ét}},\overline{\mathcal{F}_{r,K}})\}_{K\in\mathcal{N}(r)}$ as well as $\{H^i_c(\overline{X_K}_{\text{ét}},\overline{\mathcal{F}_{r,K}})\}_{K\in\mathcal{N}(r)}$ into directed systems of finte-dimensional continuous $\overline{\mathbb{Q}_\ell}$-representations of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$. Let us then make the following definitions:

$$H^i(\overline{X}_{\text{ét}},\overline{\mathcal{F}_r}) := \varinjlim_{K\in\mathcal{N}(r)} H^i(\overline{X_K}_{\text{ét}},\overline{\mathcal{F}_{r,K}}) \tag{7.45}$$

and

$$H^i_c(\overline{X}_{\text{ét}},\overline{\mathcal{F}_r}) := \varinjlim_{K\in\mathcal{N}(r)} H^i_c(\overline{X_K}_{\text{ét}},\overline{\mathcal{F}_{r,K}}) \tag{7.46}$$

Note, moreover, that the $\mathbf{G}$-action on the system $\{X_K\}_{K\in\mathcal{N}(r)}$ together with the compatible isomorphisms $\varphi_{g,K}$ endow $H^i(\overline{X}_{\text{ét}},\overline{\mathcal{F}_r})$ and $H^i_c(\overline{X}_{\text{ét}},\overline{\mathcal{F}_r})$ with a left action of $\mathbf{G}$ which, since it was defined F-rationally, commutes with the action of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$and thus we obtain a left linear action of $\mathrm{Gal}(F^{\mathrm{sep}}/F)\times\mathbf{G}$. We would like to explain some basic properties of this action.

Remark 7.3.1. Let us note that while the notation (7.46) is largely symbolic, since there is not a good notion of compactly supported cohomology for $\overline{X}$ which is not finite type over $\mathrm{Spec}(F^{\mathrm{sep}})$, the notation (7.45) is not so symbolic. Namely, one can show that this direct limit actually coincides with the étale cohomology of the space $\overline{X}$ with respect to the sheaf $\overline{\mathcal{F}}$ (see [Sta18, Tag 09YQ]).

To study the structure of the $\mathrm{Gal}(F^{\mathrm{sep}}/F)\times\mathbf{G}$-modules $H^i(\overline{X}_{\text{ét}},\overline{\mathcal{F}_r})$ and $H^i_c(\overline{X}_{\text{ét}},\overline{\mathcal{F}_r})$ it will be helpful to recall some of the basic definitions and properties of representations of the most commonly studied class of representations of $\mathbf{G}$. The so-called category of ‘smooth representations’ of $\mathbf{G}$.

Remark 7.3.2. For more information concerning the following, including the proofs of many of the stated results, see [Car79] or [BH06, Chapter 1].

Let us begin by fixing k to be an arbitrary algebraically closed characteristic 0 field (which will be $\overline{\mathbb{Q}_\ell}$ for some ℓ in practice) and U a k-vector space (possibly of infinite dimension). Recall that a linear left action of $\mathbf{G}$ on U is called a *smooth representation* if the following equality holds:

$$U = \bigcup_K U^K \tag{7.47}$$

as K ranges over all compact open subgroups of K and, as per usual, U^K denotes the K-invariants of U. Of course, it suffices to check this for K sufficiently small in particular, it suffices to check the claim holds where K ranges over $\mathcal{N}$ for any reasonable set of compact open subgroups of $\mathbf{G}$. If U is a smooth representation it is called *admissible* if, in addition to (7.47) one has that $\dim_k U^K$ is finite for all compact open subgroups K. Again, it evidently suffices to check this for K in a reasonable set $\mathcal{N}$ of compact opens.

We then have the following nice result:

Lemma 7.3.3. *The* $\mathbf{G}$*-representations* $H^i(\overline{X}_{\text{ét}}, \overline{\mathcal{F}_r})$ *and* $H^i_c(\overline{X}_{\text{ét}}, \overline{\mathcal{F}_r})$ *are smooth* $\overline{\mathbb{Q}_\ell}$*-representations of* $\mathbf{G}$*. Moreover, for every* $K \in \mathcal{N}(r)$ *one has that*

$$H^i(\overline{X}_{\text{ét}}, \overline{\mathcal{F}_r})^K = H^i(\overline{X_K}_{\text{ét}}, \overline{\mathcal{F}_{r,K}}) \tag{7.48}$$

and

$$H^i_c(\overline{X}_{\text{ét}}, \overline{\mathcal{F}_r})^K = H^i_c(\overline{X_K}_{\text{ét}}, \overline{\mathcal{F}_{r,K}}) \tag{7.49}$$

so that $H^i(\overline{X}_{\text{ét}}, \overline{\mathcal{F}_r})$ *and* $H^i_c(\overline{X}_{\text{ét}}, \overline{\mathcal{F}_r})$ *are actually admissible* $\overline{\mathbb{Q}_\ell}$*-representations of* $\mathbf{G}$*.*

Proof. This is clear since

$$\begin{aligned}
H^i(\overline{X}_{\text{ét}}, \overline{\mathcal{F}_r})^K &= \left(\varinjlim_{K' \trianglelefteq K} H^i(\overline{X}_{K'}, \overline{\mathcal{F}_{r,K'}})\right)^K \\
&= \varinjlim_{K' \trianglelefteq K} H^i(\overline{X}_{K'}, \overline{\mathcal{F}_{r,K'}})^K \\
&\overset{(*)}{=} \varinjlim_{K' \trianglelefteq K} H^i(\overline{X}_K, \overline{\mathcal{F}_{r,K}}) \\
&= H^i(\overline{X}_K, \overline{\mathcal{F}_{r,K}})
\end{aligned} \tag{7.50}$$

where $(*)$ follows from the fact that $X_{K'} \to X_K$ is a Galois cover with Galois group K/K'. A similar argument holds for compactly supported cohomology. □

We would now like to recall the analogue of the group algebra for smooth representations of locally profinite groups. Since $\mathbf{G}$ is locally profinite, and thus locally compact, we know that $\mathbf{G}$ has a left Haar measure $d_l g$ which is unique up to scaling. Similarly, we know that $\mathbf{G}$ has a right Haar measure $d_r g$ unique up to scaling. To alleviate some notational difficulties let us assume that $\mathbf{G}$ is *unimodular* which means that $d_l g$ is also right $\mathbf{G}$-invariant or, in other words, that every left Haar measure is also a right Haar measure. Let us then fix a Haar measure on $\mathbf{G}$ which we unambiguously denote by dg or μ.

Remark 7.3.4. We note that if $G/\mathbb{Q}$ is a connected reductive group then the locally profinite groups $G(\mathbb{Q}_\ell)$, $G(\mathbb{A}_f^p)$, and $G(\mathbb{A}_f)$ are all unimodular (see [Car79] for a discussion in the p-adic case, but the same method using the algebraic modulus character works for all three cases). This is one case in which the reductiveness of G is pivotal. For example, if B_n is the standard Borel in GL_n then $B_n(\mathbb{Q}_\ell)$ is not unimodular.

Let us denote by $\mathscr{H}(\mathbf{G})$ (or $\mathscr{H}_k(\mathbf{G})$ when we want to emphasize the coefficients) the set of locally constant compactly supported functions $f : \mathbf{G} \to k$ (where k is our fixed algebraically closed field from before). This becomes a k-algebra under the convolution operation:

$$(f_1 * f_2)(g) := \int_{\mathbf{G}} f_1(gh^{-1}) f_2(h) dh \tag{7.51}$$

which converges since f_1 and f_2 are compactly supported on $\mathbf{G}$. This algebra is not unital in general since, unless $\mathbf{G}$ is compact, the constant function 1 is not compactly supported.

For all compact open subgroups K of $\mathbf{G}$ let us denote by $\mathscr{H}(\mathbf{G}, K)$ (or $\mathscr{H}_k(\mathbf{G}, K)$) the subalgebra of functions $f : \mathbf{G} \to k$ which are bi-invariant under K (i.e. $f(k_1 g k_2) = f(g)$ for all $g \in \mathbf{G}$ and $k_1, k_2 \in K$). This subalgebra is unital with unit

$$e_K := \frac{1}{\mu(K)} \mathbb{1}_K \tag{7.52}$$

where $\mathbb{1}_K$ denotes the indicator function on K. Note that if $K_1 \subseteq K_2$ then there is a natural map $\mathscr{H}(\mathbf{G}, K_2) \to \mathscr{H}(\mathbf{G}, K_1)$. Since every f in $\mathscr{H}(\mathbf{G})$ is evidently in $\mathscr{H}(G, K)$ for some K we deduce that

$$\mathscr{H}(\mathbf{G}) = \varinjlim_{K} \mathscr{H}(\mathbf{G}, K) \tag{7.53}$$

for any cofinal set K of compact open subgroups of $\mathbf{G}$ (in particular a reasonable set of compact open subgroups of $\mathbf{G}$). Moreover, one can easily see that $\mathscr{H}(\mathbf{G}, K) = e_K \mathscr{H}(\mathbf{G}) e_K$.

Given a smooth $\mathbf{G}$-representation U, we can endow U with the structure of a left $\mathscr{H}(\mathbf{G}, K)$-module as follows. Given any $f \in \mathscr{H}(\mathbf{G})$ and $u \in U$ we define fu as follows:

$$fu := \int_{\mathbf{G}} f(g) g(u) dg \tag{7.54}$$

which the reader can easily check does define the structure of a left $\mathscr{H}(G)$-module on U. Moreover, let us note that U thought of as a left $\mathscr{H}(G)$-module is *non-degenerate* in the sense that $\mathscr{H}(\mathbf{G})U = U$ (i.e. for every $u \in U$ there is some $f \in \mathscr{H}(G)$ such that $fu = u$) this is not guaranteed for general modules since $\mathscr{H}(\mathbf{G})$ is not unital. Let us denote by $M(U)$ the k-space U thought of as a left $\mathscr{H}(\mathbf{G})$-module.

We then have the following trivial lemma:

Lemma 7.3.5. *The association of $M(U)$ to U defines an equivalence of categories between smooth $\mathbf{G}$-representations and left non-degenerate $\mathscr{H}(\mathbf{G})$-modules. A quasi-inverse is given by taking a non-degenerate left $\mathscr{H}(G)$-module M to the representation whose underlying k-space is M and where one defines gm, for $g \in \mathbf{G}$ and $m \in M$, as $\mathbb{1}_{KgK} m$ for any compact open $K \subseteq \mathbf{G}$ such that $e_K m = m$.*

Proof. This is [BH06, 4.3, Prop(2)]. $\square$

Now, let us come back to the setting where $\{X_K\}_{K\in\mathcal{N}}$ is an almost Shimura like $\mathbf{G}$-scheme over the field F and $\mathcal{F}$ is a $\mathcal{N}$-adpated continuous representation of $\mathbf{G}$. Since $H^i(\overline{X}_{\text{ét}},\overline{\mathcal{F}_r})$ and $H^i_c(\overline{X}_{\text{ét}},\overline{\mathcal{F}_r})$ are smooth $\overline{\mathbb{Q}_\ell}$-representations of $\mathbf{G}$ they, in particular, have the structure of a non-degenerate left $\mathscr{H}(G)$-module (where the field is now $k=\overline{\mathbb{Q}_\ell}$). Note that evidently the action of $\mathscr{H}(G)$ commutes with the action of $\mathrm{Gal}(F^{\text{sep}}/F)$ thus giving us a left action of $\mathrm{Gal}(F^{\text{sep}}/F)\times\mathscr{H}(G)$. We will denote the elements of $\mathrm{Gal}(F^{\text{sep}}/F)\times\mathscr{H}(G)$ by $\tau\times f$. We would like to understand the traces of this action in a more understandable way.

Namely, given $\tau\in\mathrm{Gal}(F^{\text{sep}}/F)$ and $f\in\mathscr{H}(G,K)$ we have that the image of $\tau\times f$ on $H^i(\overline{X}_{\text{ét}},\overline{\mathcal{F}_r})$ and $H^i_c(\overline{X}_{\text{ét}},\overline{\mathcal{F}_r})$ lands inside the space of K-invariants of these vector spaces which, by admissibility, are finite-dimensional. Thus, the trace of the action of $\tau\times f$ makes sense. It is these traces we would like to understand. Since every $f\in\mathscr{H}(G)$ is in some $\mathscr{H}(G,K)$, and every $f\in\mathscr{H}(G,K)$ is an $\overline{\mathbb{Q}_\ell}$-linear combination of functions of the form $\mathbb{1}_{KgK}$ for $g\in\mathbf{G}$ it really suffices to understand the traces of the operators of the form $\tau\times\mathbb{1}_{KgK}$.

To do this, let us begin by noting that associated to a $g\in\mathbf{G}$ and a $K\in\mathcal{N}$ is a correspondence, in the sense of §6. Namely, we have the following diagram:

$$\begin{tikzcd} & X_{K^g} \arrow[dl, "\pi_{K^g,K}"'] \arrow[dr, "t_{K^g,K}(g)"] & \\ X_K & & X_K \end{tikzcd} \tag{7.55}$$

where $K^g:=K\cap gKg^{-1}$. We call this the *Hecke correspondence.* Note that if $K\in\mathcal{N}(r)$, so that $K^g\in\mathcal{N}(r)$ as well, and thus we have natural isomorphisms

$$(c^1_{g,K})^*\mathcal{F}_{r,K}\xrightarrow{\approx}(c^2_{g,K})^*\mathcal{F}_{r,K} \tag{7.56}$$

as in Lemma 7.2.9. This then gives us a morphism

$$u_{g,K}:(c^2_{g,K})_*(c^1_{g,K})^*\mathcal{F}_{r,K}\to\mathcal{F}_{r,K} \tag{7.57}$$

called the *cohomological Hecke correspondence* at level K and element g.

We then have the following geometric way to understand the action of the operator $e(g,K):=\dfrac{1}{\mu(K)}\mathbb{1}_{KgK}$:

Lemma 7.3.6. *Let $K\in\mathcal{N}(r)$ and $g\in\mathbf{G}$. Then the operators*

$$e(g,K):H^i(\overline{X_K}_{\text{ét}},\overline{\mathcal{F}_{r,K}})\to H^i(\overline{X_K}_{\text{ét}},\overline{\mathcal{F}_{r,K}}) \tag{7.58}$$

and

$$R\Gamma(u_{g,K}):H^i(\overline{X_K}_{\text{ét}},\overline{\mathcal{F}_{r,K}})\to H^i(\overline{X_K}_{\text{ét}},\overline{\mathcal{F}_{r,K}}) \tag{7.59}$$

agree. Similarly, the operators

$$e(g,K):H^i_c(\overline{X}_{\text{ét}},\overline{\mathcal{F}_r})\to H^i_c(\overline{X}_{\text{ét}},\overline{\mathcal{F}_r}) \tag{7.60}$$

and

$$R\Gamma_c(u_{g,K}) : H^i_c(\overline{X_{K\text{ét}}}, \overline{\mathcal{F}_{r,K}}) \to H^i_c(\overline{X_{K\text{ét}}}, \overline{\mathcal{F}_{r,K}}) \tag{7.61}$$

agree.

Proof. This is clear since, by the definition of the action of $e(g,K)$, we see that for $v \in H^i(\overline{X_{K\text{ét}}}, \overline{\mathcal{F}_{r,K}})$

$$e(g,K)(v) = \int_{\mathbf{G}} e(g,K)(h)hv\,dh = \int_{KgK} hv\,dv \tag{7.62}$$

Note that for $k_1gk_2 \in KgK$ we have that $k_2v = v$ (by Lemma 7.3.3) and gv is just the action which lands in the $K \cap g^{-1}Kg$ invariants of $H^i(\overline{X}_\infty, \overline{\mathcal{F}_r})$ which is equal to (again by Lemma 7.3.3) $H^i(\overline{X_{K^g}}, \overline{\mathcal{F}_{r,K^g}})$ and k_2 projects back down to $H^i(\overline{X_K}, \overline{\mathcal{F}_r})$. It's not hard to see that this agrees with $R\Gamma(u_{g,K})$. A similar argument works for compactly supported cohomology. □

Thus, to understand the $\mathbf{G}$-action on $H^i(\overline{X}_{\text{ét}}, \overline{\mathcal{F}_r})$ and $H^i_c(\overline{X}_{\text{ét}}, \overline{\mathcal{F}_r})$ it suffices to understand the operators $R\Gamma(u_{g,K})$ for all $K \in \mathcal{N}(r)$ and $g \in \mathbf{G}$.

7.4 The Lefschetz like trace formula

We are now finally able to start discussing the actual Lefschetz type trace formula that we are ultimately after. Let us fix the following data/notation:

- Let $\mathcal{O}$ denote the ring of integers of a p-adic local field $F = \mathrm{Frac}(\mathcal{O})$ to be its fraction field and k to be its residue field.
- Let G be a reductive group over $\mathbb{Q}$.
- Let $\{X_K\}_{K\in\mathcal{N}}$ be a Shimura like $G(\mathbb{A}_f)$-scheme $\{X_K\}_{K\in\mathcal{N}}$ over F.
- Let ξ be an algebraic $\overline{\mathbb{Q}_\ell}$-representation of G on some $\overline{\mathbb{Q}_\ell}$-space V and let us denote also by ξ the induced representation $G(\mathbb{A}_f) \to \mathrm{GL}_{\overline{\mathbb{Q}_\ell}}(V)$ which we suppose is adapted to $\mathcal{N}$.
- Fix a compact open subgroup $K_0 \subseteq G(\mathbb{Q}_p)$ and suppose that the $G(\mathbb{A}^p_f)$-scheme $\{X_{K_0K^p}\}$ has an integral canonical model $\{\mathcal{X}_{K^p}\}$ over $\mathrm{Spec}(\mathcal{O})$ and the sheaves $\mathcal{F}_{\xi,K_0K^p}$ have models $\mathcal{F}_{\xi,K^p}$ over $\mathcal{X}_{K^p}$.

All of the above assumptions are fairly benign in practice, and can be easily arranged. That said, we come to the main assumption on which will require serious work to verify holds in the setting of Shimura varieties:

- (GC) Assume that for all compact open normal subgroups $K_p \trianglelefteq K_0$ and all $K^p \in \mathcal{N}^p$ that the natural map

  $$R\Gamma_c(\overline{X_{K^pK_0}}, \overline{\pi_*\mathcal{F}_{\xi,K^pK_p}}) \to R\Gamma_c((\mathcal{X}_{K^p})_{\overline{k}}, R\psi(\pi_*\mathcal{F}_{\xi,K^pK_p})) \tag{7.63}$$

  is an isomorphism where $\pi := \pi_{K^pK_p,K^pK_0}$.

Now, given the discussion in Section 7.3 we know that computing the trace $\mathrm{Gal}(\overline{E}/E) \times G(\mathbb{A}_f)$-action on $H^*(\overline{X}_\infty, \overline{\mathcal{F}_\infty})$ in terms of other known quantities reduces to computing the traces of Hecke correspondences. If we were trying to compute the trace of the Hecke correspondence of the form $c_{K^pK_0}(g^p)$ we have a clear way forward using the model of this correspondence over $\mathcal{O}$ and the Fujiwara trace formula. Our goal is to explain how we can do this for correspondences of the form $c_{K^pK_p}(g^pg_p)$ when $K_p \trianglelefteq K_0$ may be a proper subgroup.

The rough idea is that while we may not have models of the correspondence at the K_p-level, we always have a map of correspondences to a Hecke correspondence at a good level. Namely, let us set $g := g_pg^p$, $K := K^pK_p$, $H := K^pK_0$, $\pi_1 := \pi_{K,H}$, $\pi_2 := \pi_{K^g,H^{g_p}}$, and $m = (\pi_1, \pi_2)$. We then have a natural commutative diagram:

$$
\begin{tikzcd}
 & X_{K^g} \arrow[dl, "c_K(g)_1"'] \arrow[d, "\pi_2"] \arrow[dr, "c_K(g)_2"] & \\
X_{K^pK_p} \arrow[d, "\pi_1"'] & X_{(K^pK_0)^{g^p}} \arrow[dl, "c_H(g^p)_1"] \arrow[dr, "c_H(g^p)_2"'] & X_{K^pK_p} \arrow[d, "\pi_1"] \\
X_{K^pK_0} & & X_{K^pK_0}
\end{tikzcd}
\tag{7.64}
$$

which then gives us a clear plan of attack. Namely, while we may don't have models for the Hecke correspondence at level K, we do at level H. We could then use Lemma 6.0.3 to compute the trace of the cohomological correspondence $u_K(g)$ in terms of the trace of the pushforward correspondence $m_*(u_K(g))$.

To see why this is helpful, let us note that because m is finite étale we may apply the projection formula to obtain the following decomposition of sheaves that underlies the whole discussion:

Observation 7.4.1. *There is a canonical isomorphism of lisse $\mathbb{Q}_\ell$-sheaves on X_H:*

$$\pi_{1*}\mathcal{F}_{\xi,K} \xrightarrow{\approx} \mathcal{F}_{\xi,H} \otimes_{\overline{\mathbb{Q}_\ell}} \pi_{1*}\overline{\mathbb{Q}_\ell} \tag{7.65}$$

Now, note that $\pi_{1*}\overline{\mathbb{Q}_\ell}$ has a natural cohomological correspondence induced by the action of g_p. Namely, we want to give an isomorphism

$$c_H(g^p)_1^*\pi_{1*}\overline{\mathbb{Q}_\ell} \xrightarrow{\approx} c_H(g^p)_2^*\pi_{1*}\overline{\mathbb{Q}_\ell} \tag{7.66}$$

Now, base change tells us that this is naturally the same thing as giving an isomorphism

$$\pi_{2*}\overline{\mathbb{Q}_\ell} \xrightarrow{\approx} \pi_{2*}\overline{\mathbb{Q}_\ell} \tag{7.67}$$

which we define to be the action of $t_{K^g,K^g}(g_p)$ (which makes sense since $K_p \trianglelefteq K_0$). Let us denote this cohomological correspondence on $\pi 1 * \overline{\mathbb{Q}_\ell}$ relative to $c_H(g^p)$ by v.

We then have the following:

Lemma 7.4.2. *The isomorphism equation (7.65) carries the cohomological correspondence $m_*(u_{\xi,K}(g))$ to $u_{\xi,H}(g^p) \otimes v$.*

Proof. It suffices to check that the induced action on fibers is the same. So, let $\overline{x}$ be a geometric point of X_H. Then, we have the following identification

$$(\pi_{1*}\mathcal{F}_{\xi,K})_{\overline{x}} = \bigoplus_{\overline{y}\in\pi_1^{-1}(\overline{x})} (\mathcal{F}_{\xi,K})_{\overline{y}} \tag{7.68}$$

Here we can identify $(\mathcal{F}_{\xi,K})_{\overline{y}} = V_{\mathbb{Q}_\ell}$, and the action of g is via g_ℓ on the factors $V_{\mathbb{Q}_\ell}$ and by g_p permuting the set $\pi_1^{-1}(\overline{x})$. Similarly, we have the identification

$$\begin{aligned}(\mathcal{F}_{\xi,H}\otimes\pi_{1*}\overline{\mathbb{Q}_\ell})_{\overline{x}} &= (\mathcal{F}_{\xi,H})_{\overline{x}}\otimes(\pi_{1*}\overline{\mathbb{Q}_\ell})_{\overline{x}} \\ &= (\mathcal{F}_{\xi,H})_{\overline{x}}\otimes\left(\bigoplus_{\overline{y}\in\pi_1^{-1}(x)}\overline{\mathbb{Q}_\ell}\right)\end{aligned} \tag{7.69}$$

where, similarly, we may identify $(\mathcal{F}_{\xi,H})_{\overline{x}}$ with $V_{\mathbb{Q}_\ell}$. One then has that g acts on this tensor product by g_ℓ on the first factor, and by permuting the coordinates of the second factor by g_p. The claim then readily follows. $\square$

Let us now fix some $\tau\in W_F$ with $j := v(\tau)$. Combining this and Lemma 6.0.3 we see that if we set $\mathcal{G} := \mathcal{F}_{\xi,H}\otimes\pi_{1*}\overline{\mathbb{Q}_\ell}$ then

$$\operatorname{tr}\left(\tau\times u_{\xi,K}(g)\mid H_c^*(\overline{X_K},\overline{\mathcal{F}_{\xi,K}})\right) = \operatorname{tr}\left(\tau\times(u_{\xi,H}(g^p)\otimes v)\mid H_c^*(\overline{X_H},\overline{\mathcal{G}})\right) \tag{7.70}$$

and thus we're reduced to computing the trace of the cohomological correspondence $u_{\xi,H}(g^p)\otimes p$. Now, by assumption (GC) we have that the terms in equation (7.70) are equal to

$$\operatorname{tr}\left(\tau\times R\psi(u_{\xi,H}(g^p)\otimes v)\mid H_c^*((\mathcal{X}_{K^p})_{\overline{k}},\overline{R\psi(\mathcal{G})})\right) \tag{7.71}$$

which puts in a situation where we are able to, in theory, apply Fujiwara's trace formula.

Before we do this though, let us note the following trivial observation obtained by noticing that smooth base change implies that $R\psi\mathcal{F}_{\xi,K^pK_0}\cong\mathcal{F}_{\xi,K^p}$:

Observation 7.4.3. *There is a natural isomorphism*

$$R\psi(\mathcal{F}_{\xi,H}\otimes\pi_{1*}\overline{\mathbb{Q}_\ell})\cong(\mathcal{F}_{\xi,K^p})_k\otimes R\psi\pi_{1*}\overline{\mathbb{Q}_\ell} \tag{7.72}$$

which carries $R(u_{\xi,H}(g^p)\otimes v)$ to $u_{\xi,H}(g^p)_k\otimes R\psi(v)$.

Thus, we see that the terms showing up in equation (7.70) are equal to

$$\operatorname{tr}\left(\tau\times(u_{\xi,H}(g^p)_k\otimes R\psi(v))\mid H_c^*((\mathcal{X}_{K^p})_{\overline{k}},\overline{(\mathcal{F}_{\xi,K^p})_k\otimes R\psi\pi_{1*}\overline{\mathbb{Q}_\ell}})\right) \tag{7.73}$$

which looks like something to which the Fujiwara trace formula might be applicable. Of course, to be able to do this fully we need to transform this trace into purely the trace of a Frobenius twisted correspondence. But, this happens by assumption (GC) and Lemma 6.0.6. Thus, we see that our desired term is equal to

$$\operatorname{tr}\left((u_{\xi,H}(g^p)_k\otimes R\psi(v))^{(j)}\mid H_c^*((\mathcal{X}_{K^p})_{\overline{k}},\overline{(\mathcal{F}_{\xi,K^p})_k\otimes R\psi\pi_{1*}\overline{\mathbb{Q}_\ell}})\right) \tag{7.74}$$

which is something that we can apply Theorem 6.0.5 to.

Namely, we see that for j sufficiently large we can write this trace as

$$\sum_{\substack{\overline{y}\in\mathcal{X}_{(K^p)g^p}(\overline{k})\\ \Phi_q^j(c_1(\overline{y}))=c_2(\overline{y})}} \mathrm{tr}((u_{\xi,H}(g^p)_k\otimes R\psi(v))_{\overline{y}}) \tag{7.75}$$

But, noting that since we have decompose our cohomological correspondence as a tensor product, we know that we can factor these traces. Namely, the above is equal to

$$\sum_{\substack{\overline{y}\in\mathcal{X}_{(K^p)g^p}(\overline{k})\\ \Phi_q^j(c_1(\overline{y}))=c_2(\overline{y})}} \mathrm{tr}((u_{\xi,H}(g^p)_k)_{\overline{y}})\mathrm{tr}(R\psi(v))_{\overline{y}}) \tag{7.76}$$

Note though that if we set $\mathscr{X}_{\overline{y}}$ to be $(\widehat{\mathcal{X}_{/c_2(\overline{y})}})_\eta$ a rigid analytic space over $\mathrm{Spa}(F^{\mathrm{ur}})$ sitting inside $X^{\mathrm{an}}_{K^pK_0}$ then since $\Phi_q^j(c_1(\overline{y}))=c_2(\overline{y})$ that $\mathscr{X}_{\overline{y}}$ has a natural action of τ (e.g. see [Far04, §6.10]. Note then that

$$H^i(\mathscr{X}_{\overline{y}},(\pi_{1*}\overline{\mathbb{Q}_\ell})^{\mathrm{an}})\cong H^i(\mathscr{X}_{\overline{y}},(\pi_1^{\mathrm{an}})_*\overline{\mathbb{Q}_\ell}) \tag{7.77}$$

carries a natural action of g_p inherited from its action on $\pi_{1*}\mathcal{F}_{\xi,K}$.

We then use Theorem 6.0.7 we get an isomorphism

$$H^i(\overline{\mathscr{X}_{\overline{y}}},(\pi_{1*}\overline{\mathbb{Q}_\ell})^{\mathrm{an}})\cong(R^i\psi\pi_{1*}\overline{\mathbb{Q}_\ell})_{c_2(\overline{y})} \tag{7.78}$$

which is equivariant for the I_F-action on both sides. But, one can check by tracing through the definitions that it is actually $\tau\times g_p$ equivariant (e.g. see the discussion in [Far04, §6.10] and, in particular, [Far04, Proposition 6.7.1]). Finally, let us set $\mathscr{Y}_{\overline{y}}$ to be $(\pi_K^{\mathrm{an}})^{-1}(\mathscr{X}_{\overline{y}})$. Then, we obviously have a $g_p\times\tau$-equivariant isomorphism

$$H^i(\overline{\mathscr{X}_{\overline{y}}},(\pi_{1*}\overline{\mathbb{Q}_\ell}^{\mathrm{an}})\cong H^i(\overline{\mathscr{Y}_{\overline{y}}},\overline{\mathbb{Q}_\ell}) \tag{7.79}$$

which gives us a $g_p\times\tau$-equivariant isomorphism

$$H^i(\mathscr{Y}_{\overline{y}},\overline{\mathbb{Q}_\ell})\cong(R^i\psi\pi_{1*}\overline{\mathbb{Q}_\ell})_{c_2(\overline{y})} \tag{7.80}$$

by comparing with (7.78).

Summarizing all of this, we obtain the following main theorem:

Theorem 7.4.4. *Let the notation be as above, and assume condition (GC). Then,*

$$\begin{aligned}\mathrm{tr}(\tau\times e(K,g)\mid H_c^*(\overline{X_\infty},\overline{\mathcal{F}_\xi}))&=\mathrm{tr}(\tau\times u_{\xi,g}\mid H_c^*(\overline{X_K},\overline{\mathcal{F}_{\xi,K}})\\ &=\sum_{\substack{\overline{y}\in\mathcal{X}_{(K^p)g^p}(\overline{k})\\ \Phi_q^j(c_1(\overline{y}))=c_2(\overline{y})}} \mathrm{tr}((u_{\xi,H}(g^p)_k)_{\overline{y}})\mathrm{tr}\left(H^*(\overline{\mathscr{Y}_{\overline{y}}},\overline{\mathbb{Q}_\ell})\right)\end{aligned} \tag{7.81}$$

Chapter 8

Appendix D: A recollection of Shimura varieties

In this section we recall the basic setup of Shimura varieties. The reason for this is two-fold. First, we hope that it will be useful to the reader as a review of the classical theory of Shimura varieties in the language used in the literature surrounding the contents of this article. Second, it will be helpful to verify that a Shimura variety associated to a Shimura datum (G, X) forms a $G(\mathbb{A}_f)$-scheme in the parlance of Appendix C, so that we can apply the machinery therein.

In an attempt to make this material more easily readable to a non-expert in Shimura varieties, we ramp up the difficulty of the material gradually by first considering modular curves (the case of "GL_2 Shimura varieties"), then the more general setting of Siegel modular varieties (the case of "GSp_{2n} Shimura varieties), and finally Shimura varieties in their maximal generality.

Throughout this section we assume that all schemes are locally Noetherian. This makes our discussions of the moduli problems associated to various Shimura varieties easier to deal with since locally Noetherian schemes are locally connected and so, in particular, have open connected components. This creates no real restriction since Shimura varieties are locally Noetherian, so that they are determined by their functor of points on the category of locally Noetherian schemes (or locally Noetherian schemes relative to some base scheme).

8.1 Modular curves

We begin by discussing modular curves with "generalized level structure" which, roughly, classify (families of) elliptic curves together with trivializations of certain torsion subgroups (when interpreted correctly).

Let us begin by defining some notation. If T is a $\mathbb{Q}$-scheme and $f : E \to T$ is an elliptic scheme over T (i.e. a E is a T-group whose structure map is smooth, proper, and has connected fibers of dimension 1), let us denote by $V(E)$ the following lisse $\mathbb{A}_f$-sheaf

$$V(E) := \prod{}' (R^1 f_* \mathbb{Q}_\ell)^\vee \tag{8.1}$$

where the prime denotes restricted direct product relative to $(R^1 f_* \mathbb{Z}_\ell)^\vee$. If T is connected and $\bar{t}$ is a geometric point of T, then we can use smooth proper base change to identify this sheaf as the total Tate module:

$$V(E_{\bar{t}}) := \prod{}' V_\ell(E_{\bar{t}}) \tag{8.2}$$

(where the restricted direct product is taken with respect to the integral Tate module $T_\ell(E_{\bar{t}}) \subseteq V_\ell(E_{\bar{t}})$) along with the associated action of $\pi_1^{\text{ét}}(T, \bar{t})$.

In particular, if T is integral and normal with generic point η, then there is a surjection $\mathrm{Gal}(\overline{\eta}/\eta) \to \pi_1^{\text{ét}}(T, \overline{\eta})$ and we can think of $V(E)$ as the Galois representation

$$\mathrm{Gal}(\overline{\eta}/\eta) \to \mathrm{GL}(V(E_{\overline{\eta}})) = \mathrm{GL}_2(\mathbb{A}_f) \tag{8.3}$$

(where the last equality is merely as topological groups) which, of course, factors through $\pi_1^{\text{ét}}(T, \overline{\eta})$.

Fix $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ a compact open subgroup. Given a $\mathbb{Q}$-scheme T and an elliptic scheme E over T denote by $\underline{\mathrm{Isom}}(\mathbb{A}_f^2, V(E))$ the sheaf on the étale site of T which associates to a U the set $\underline{\mathrm{Isom}}(\mathbb{A}_f^2, V(E))(U)$ of isomorphisms of lisse $\mathbb{A}_f$-sheaves $\mathbb{A}_f^2 \xrightarrow{\approx} V(E_U)$ (where $\mathbb{A}_f^2$ is thought of as the constant $\mathbb{A}_f$-sheaf on U). Note that K naturally acts on the right of this sheaf by precomposition (thinking of K as a subgroup of the group of automorphism of the sheaf $\mathbb{A}_f^2$). Let $\underline{\mathrm{Isom}}(\mathbb{A}_f^2, V(E))/K$ denote the quotient sheaf on $T_{\text{ét}}$.

Remark 8.1.1. One should note that, as one might expect, this quotient sheaf does not coincide with the quotient presheaf. For example, suppose that E is an elliptic curve over $\mathbb{Q}$. Then, $\Gamma(F, \underline{\mathrm{Isom}}(\mathbb{A}_f^2, V(E)))$ is empty, and thus the quotient presheaf $\Gamma(F, (\underline{\mathrm{Isom}}(\mathbb{A}_f^2, V(E))/K)^{\mathrm{pre}})$ is empty, for any algebraic $F/\mathbb{Q}$ not containing $\mathbb{Q}^{\mathrm{ab}}$ and so, in particular, it's empty for any finite extension $F/\mathbb{Q}$. Indeed, since $\det(V(E_F))$ is, as a $\mathrm{Gal}(\overline{F}/F)$-module, precisely $\mathbb{A}_f(1) \cong \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$ (this is, in the obvious sense, the 'total' cyclotomic character of F).

So if $\underline{\mathrm{Isom}}(\mathbb{A}_f^2, V(E))(F)$ were non-empty, then $\mathbb{A}_f(1)$ as a $\mathrm{Gal}(\overline{F}/F)$-module would be trivial, which would imply that $F \supseteq \mathbb{Q}^{\mathrm{ab}}$. That said, $\underline{\mathrm{Isom}}(\mathbb{A}_f^2, V(E))/\widehat{\Gamma}(3)$ (using the notation below) is precisely the sheaf of $\Gamma(3)$-structures $\underline{\mathrm{Isom}}((\mathbb{Z}/3\mathbb{Z})^2, E[3])$ which, of course, may have F-points for a finite extension $F/\mathbb{Q}$.

Given a connected $\mathbb{Q}$-scheme T and a geometric point $\bar{t}$ of T, we can identify the values of this quotient sheaf with something more concrete:

Lemma 8.1.2. *Let T be a connected $\mathbb{Q}$-scheme and $\bar{t}$ a point of T. If $(U, \overline{u}) \to (T, \bar{t})$ is a connected pointed étale map, then there is a functorial (in such data) isomorphism $\Gamma(U, \underline{\mathrm{Isom}}(\mathbb{A}_f^2, V(E))/K)$ with the set of right K-orbits of isomorphisms $\lambda : \mathbb{A}_f^2 \xrightarrow{\approx} V(E_{\overline{u}})$ such that the right K-orbit λK is stable under the left action of $\pi_1^{\text{ét}}(U, \overline{u})$.*

So, if T is integral and normal with generic point η, then we can think of an element $\lambda K \in \Gamma(T, \underline{\mathrm{Isom}}({\mathbb{A}_f}^2, V(E))/K)$ as a right K-orbit of isomorphisms $\mathbb{A}_f^2 \xrightarrow{\approx} V(E_{\overline{\eta}})$ that is stable under the left action of $\mathrm{Gal}(\overline{\eta}/\eta)$.

Remark 8.1.3. This explains the phenomenon mentioned in Remark 8.1.1. Indeed, an element of $(\underline{\mathrm{Isom}}(\mathbb{A}_f^2, V(E))/K)^{\mathrm{pre}}$ is a right K-orbit of $\pi_1^{\text{ét}}(U,\overline{u})$-invariant isomorphisms $\mathbb{A}_f^2 \xrightarrow{\approx} V(E_{\overline{u}})$ which is very different than a $\pi_1^{\text{ét}}(U,\overline{u})$-stable right K-orbit of isomorphisms $\mathbb{A}_f^2 \xrightarrow{\approx} V(E_{\overline{u}})$!

Let us define a functor

$$M(K) : (\mathsf{Sch}/\mathbb{Q})^{\mathrm{op}} \to \mathsf{Set} : T \mapsto \{(E, \lambda K)\}/\sim \tag{8.4}$$

where E/T is an elliptic scheme and $\lambda K \in \Gamma(T, \underline{\mathrm{Isom}}(\mathbb{A}_f{}^2, V(E))/K)$. Here we write $(E,\lambda K) \sim (E', \lambda' K)$ if there exists a quasi-isogeny (see §8.2 for a recollection of quasi-isogenies in greater generality) $E \to E'$ carrying λK to $\lambda' K$.

We then have the following classical result:

Lemma 8.1.4. *If K is neat, then the functor $M(K)$ is representable by a smooth affine curve over $\mathbb{Q}$.*

Proof. If $K \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ then this is shown in [DR73, Section 3.11], and for general K one can find some $K' \trianglelefteq K$ such that $K' \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ and then K/K' acts without fixed points on $M(K')$ and $M(K) = M(K')/(K/K')$. □

Remark 8.1.5. Here is an alternative way that one can approach the proof of Lemma 8.1.4, as well as how to modify it for K compact open but not neat.

For general compact open subgroups K of $\mathrm{GSp}(V, \mathbb{A}_f)$, one can can define a category fibered in groupoids $\mathcal{M}_{1,1}(K)$ which associates to a $\mathbb{Q}$-scheme T the groupoid of pairs $(E, \lambda K)$ with isomorphisms being quasi-isogenies preserving the right K-cosets. So, clearly, our functor $Y(K)$ is just $\pi_0(\mathcal{M}_{1,1,K})$. One can show (see [DR73, Section 3.11] again) that $\mathcal{M}_{1,1}(K)$ is a one-dimensional smooth Deligne-Mumford stack over $\mathbb{Q}$.

When, K is neat one can show that the inertia stack of $\mathcal{M}_{1,1}(K)$ is trivial, so that $\mathcal{M}_{1,1}(K)$ is an algebraic space. But, since it's 1-dimensional it's automatically a scheme (see [Sta18, 0ADD]), which is just the scheme $M(K)$ from Lemma 8.1.4. This gives an alternative proof of the representability in Lemma 8.1.4. Also, for general K one can then deduce that if $K' \trianglelefteq K$ is a neat open compact subgroup, then the geometric quotient $M(K')/(K/K')$ (which exists since K/K' is finite and $M(K')$ is quasi-projective) together with the obvious map $\mathcal{M}_{1,1}(K) \to M(K')/(K/K')$ is a coarse moduli space.

We call this smooth affine curve $M(K)$ the *generalized modular curve with level structure K*.

Suppose that $K, L \in \mathcal{N}(\mathrm{GL}_2)$ and $g \in \mathrm{GL}_2(\mathbb{A}_f)$ such that $L \supseteq g^{-1}Kg$. We define a map

$$t_{K,L}(g) : M(K) \to M(L) \tag{8.5}$$

by declaring that on T-points it sends

$$(E, \lambda K) \mapsto (E, \lambda g L) \tag{8.6}$$

which makes sense since, as one can easily see, the assumption $L \supseteq g^{-1}Kg$ implies that this map is well-defined (i.e. independent of the equivalence class of $(E, \lambda K)$).

One then has the following basic result:

Lemma 8.1.6. *The system $\{M(K)\}_{K\in\mathcal{N}(\mathrm{GL}_2)}$, with the above defined $t_{K,L}(g)$, is a Shimura like $\mathrm{GL}_2(\mathbb{A}_f)$-scheme over $\mathbb{Q}$ relative to the trivial subgroup of $Z(\mathrm{GL}_2(\mathbb{A}_f))$.*

This is fairly easy to show (most of it is shown in loc. cit.) but, we will discuss this in greater generality below, so let us delay the discussion of the proof of this fact.

If $N \geqslant 1$ we can define the following subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \subseteq \mathrm{GL}_2(\mathbb{A}_f)$, called the *completed principal congruence subgroup (for* GL_2*) of level* N:

$$\widehat{\Gamma}(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\} \tag{8.7}$$

which can easily be seen to be a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$ and which is neat if $N \geqslant 3$ (this was already observed in Lemma 7.1.22). As one might expect, this is the profinite completion of the classical congruence subgroup $\Gamma(N) \subseteq \mathrm{GL}_2(\mathbb{Z})$.

Taking $K = \widehat{\Gamma}(N)$ one has an identification of $M(\widehat{\Gamma}(N))$ with the classic modular curve $M(N)$ associating to a $\mathbb{Q}$-scheme T the set of isomorphism classes of elliptic curves (E, α) where $\alpha : E[N] \xrightarrow{\approx} \underline{\mathbb{Z}/N\mathbb{Z}}^2$. One can find a proof in [DR73, Section 3.11], but it also follows quite readily from Lemma 8.1.2.

One can easily see that the set of completed principal congruence subgroups $\widehat{\Gamma(N)}$ form a cofinal subset of $\mathcal{N}(\mathrm{GL}_2)$. Thus, if we denote by M the space at infinite level associated to the $\mathrm{GL}_2(\mathbb{A}_f)$-scheme $\{M(K)\}_{K\in\mathcal{N}(\mathrm{GL}_2)}$ then one has an identification

$$M := \varprojlim_K M(K) = \varprojlim_N M(N) \tag{8.8}$$

But, while this latter presentation for M is psychologically comforting it really obfuscates the $\mathrm{GL}_2(\mathbb{A}_f)$-action. When presented this way, it looks as M only has a $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ action. The $\mathrm{GL}_2(\mathbb{A}_f)$-action on $\varprojlim_N Y(N)$ can roughly be interpreted on the finite levels $M(N)$ by as the classical Hecke correspondences.

Remark 8.1.7. The above gives a natural example of why one might not want to impose such strong conditions on the indexing set $\mathcal{N}$ in the definition of $\mathbf{G}$-schemes. Namely, in a looser definition of $\mathbf{G}$-scheme (with more general $\mathcal{N}$) one would allow $\mathcal{N} := \{\widehat{\Gamma}(N)\}_{N\geqslant 3}$ as an admissible indexing set, so that $\{M(N)\}_{N\geqslant 3}$ would actually be a $\mathrm{GL}_2(\mathbb{A}_f)$-scheme over $\mathbb{Q}$. But, as the above comments concerning the obfuscation of $\mathrm{GL}_2(\mathbb{A}_f)$-action indicate, in practice it's better to work with the larger indexing set $\mathcal{N}(\mathrm{GL}_2)$ anyways.

8.2 Siegel modular varieties

We now recall the higher-dimensional analogues of modular curves, the so-called Siegel modular varieties. These parameterize polarized abelian varieties with generalized level structure which, again, roughly record trivializations of torsion subgroups (when interpreted correctly).

Recall that a *symplectic space over* R, where R is any ring, consists of a pair (V, ψ) where V is a finite free R-module and $\psi : V \times V \to R$ is a non-degenerate alternating pairing. Recall that what it means for ψ to be non-degenerate is that the natural R-linear map $V \to V^\vee$ given by mapping x to $\psi(x,-)$ is injective. If this map is an isomorphism we call ψ *perfect*.

We have the following elementary observations about symplectic spaces over fields and the integers:

Lemma 8.2.1. *Let V be a symplectic space over R. Then, the following holds:*

1. *If R is a field (assumed of characteristic not 2 for simplicity), then V is automatically perfect. Moreover, if $\dim V = 2n$ then (V,ψ) is isomorphic to $(R^{2n}, \psi_{\mathrm{std}})$ where*
$$\psi_{\mathrm{std}}(x,y) := x^\top \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} y \tag{8.9}$$
where I_n is the $n \times n$ identity matrix.

2. *If $R = \mathbb{Z}$ then, up to isomorphism, the symplectic spaces over R of rank $2n$ are of the form $(\mathbb{Z}^{2n}, \psi_D)$ for some $D = (d_1, \ldots, d_n) \in (\mathbb{Z}_{>0})^n$ satisfying $d_i \mid d_{i+1}$ and where ψ_D is the pairing*
$$\psi_D(x,y) = x^\top M_D y \tag{8.10}$$
where M_D is the matrix $\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$ where we conflate the n-tuple D with the diagonal matrix with those entries.

We call the symplectic space $(F^{2n}, \psi_{\mathrm{std}})$ on F^d, where F is a field, the *standard symplectic space* of dimension $2n$. We call a symplectic space (V,ψ) over $\mathbb{Z}$ isomorphic to $(\mathbb{Z}^{2n}, \psi_D)$ of *type D*.

Now, associated to a symplectic space (V,ψ) over R are two natural groups:

$$\mathrm{Sp}(V,\psi) := \{g \in \mathrm{GL}_R(V) : \psi(gx,gy) = \psi(x,y), \text{ for all } x,y \in V\} \tag{8.11}$$

and

$$\mathrm{GSp}(V,\psi) := \left\{(g,c) \in \mathrm{GL}_R(V) \times R^\times : \psi(gx,gy) = c\psi(x,y), \text{ for all } x,y \in V\right\} \tag{8.12}$$

called, respectively, the *symplectic group of* (V,ψ) and the *symplectic similitude group of* (V,ψ). Note that there is a natural map

$$\nu : \mathrm{GSp}(V,\psi) \to R^\times \tag{8.13}$$

given by $\nu(g,c) = c$, called the *similitude character* of $\mathrm{GSp}(V,\psi)$.

Suppose now that (V,ψ) is a symplectic space over $\mathbb{Q}$. Then, for any $\mathbb{Q}$-algebra R the base change (V_R, ψ_R) is a symplectic space over R. One can then see that the associations

$$\mathsf{Alg}_{\mathbb{Q}} \to \mathsf{Set} : R \mapsto \mathrm{Sp}(V_R, \psi_R) \tag{8.14}$$

and

$$\mathsf{Alg}_{\mathbb{Q}} \to \mathsf{Set} : R \mapsto \mathrm{GSp}(V_R, \psi_R) \tag{8.15}$$

are functorial and, as one can easily show, are representable by connected reductive algebraic groups $S(\psi)$ and $G(\psi)$.

One then easily sees that the similitude character is also functorial in R, and thus gives rise to a similitude character $\nu : G(\psi) \to \mathbb{G}_{m,\mathbb{Q}}$. This identifies $\mathbb{G}_{m,\mathbb{Q}}$ with the abelianization $G(\psi)^{\mathrm{ab}}$. Also, the inclusion $S(\psi) \hookrightarrow G(\psi)$ identifies $S(\psi)$ with the derived subgroup $G(\psi)^{\mathrm{der}}$. It is worth recording, for future reference, that $S(\psi)$ is simply connected (in the sense of semisimple algebraic groups), as can be easily seen from the root datum of $S(\psi)$ (e.g. see [Mil17]).

Remark 8.2.2. What we are denoting here by $G(\psi)$ and $S(\psi)$ are usually denoted by $\mathrm{GSp}(V, \psi)$ and $\mathrm{Sp}(V, \psi)$. The reason we have chosen to use the notation $G(\psi)$ and $S(\psi)$ is two-fold. Firstly, it eliminates the notational ambiguity between the symplectic (similitude) group and its group of $\mathbb{Q}$-points. But, much more importantly, it eliminates a huge notational burden that comes with writing $\mathrm{GSp}(V, \psi)$ for every instance of the symplectic similitude group.

Note that (V, ψ) is the standard 2-dimensional symplectic space over $\mathbb{Q}$ then $G(\psi) \hookrightarrow \mathrm{GL}_2$ is actually an isomorphism—one can just check that every invertible matrix preserves this ψ up to a scalar factor, or one can just argue for dimension reasons. Thus, while one naturally thinks of higher-dimensional analogues of GL_2 as being GL_n it turns out that, in fact, the correct higher-dimensional analogues of GL_2 from the perspective of generalizing modular curves are the group of symplectic similitudes associated to higher-dimensional symplectic spaces. We now endeavor to define these higher-dimensional analogues of modular curves.

Remark 8.2.3. In the following we will assume the reader is familiar with parts of the general theory of abelian schemes over an arbitrary base. If not, one can glean most of the needed material from [Lan13, Section 1.3]. One can also consult the book project [EvdGM] or the classical references [MRM74] and [MFK94, Chapter 6].

We begin, as in the case of modular curves, with the discussion of various $\mathbb{A}_f$-sheaves associated to an abelian schemes over a $\mathbb{Q}$-scheme. So, let T be a $\mathbb{Q}$-scheme and $f : A \to T$ an n-dimensional abelian scheme. Let us denote by $V(A)$ the following lisse $\mathbb{A}_f$-sheaf on T:

$$V(A) := \prod_{\ell}{}' (R^1 f_* \mathbb{Q}_\ell)^\vee \tag{8.16}$$

where, as in the case of elliptic schemes, the restricted direct product is taken relative to $(R^1 f_* \mathbb{Z}_\ell)^\vee$. And, again as for elliptic schemes, if T is connected and $\bar{t}$ is a geometric point of T, then we can describe this lisse $\mathbb{A}_f$-sheaf as the continuous $\pi_1^{\text{ét}}(T, \bar{t})$-module $V(A_{\bar{t}})$ where $V(A_{\bar{t}})$ is the total Tate module

$$V(A_{\bar{t}}) := \prod_{\ell}{}' V_\ell(A_{\bar{t}}) \tag{8.17}$$

where $V_\ell(A_{\bar{t}})$ is the rational ℓ-adic Tate module of $A_{\bar{t}}$ and the restricted direct product is taken with respect to the integral ℓ-adic Tate modules $T_\ell(A_{\bar{t}}) \subseteq V_\ell(A_{\bar{t}})$. Finally, again as for elliptic schemes, if T is integral and normal with geometric generic point $\overline{\eta}$ then we can think of $V(A)$ as corresponding to the Galois representation

$$\mathrm{Gal}(\overline{\eta}/\eta) \to \mathrm{GL}_{\mathbb{A}_f}(V(A_{\overline{\eta}})) = \mathrm{GL}_{2n}(\mathbb{A}_f) \tag{8.18}$$

(where the last equality is merely as topological groups) which, of course, factors through the quotient $\mathrm{Gal}(\overline{\eta}/\eta) \twoheadrightarrow \pi_1^{\text{ét}}(T, \overline{\eta})$.

Let us denote by $\mathsf{AV}(T)$ the category of abelian schemes over T, and by $\mathsf{AV}_{\mathbb{Q}}(T)$ the *(quasi-)isogeny category of abelian schemes over* T, by which mean the category whose objects are abelian schemes over T and whose morphisms $A_1 \to A_2$ are global sections of the fppf sheaf on T given by $\underline{\mathrm{Hom}}(A_1, A_2) \otimes_{\mathbb{Z}} \mathbb{Q}$. In other words, if T is connected then

$$\mathrm{Hom}_{\mathsf{AV}_{\mathbb{Q}}(T)}(A_1, A_2) := \mathrm{Hom}_{\mathsf{AV}(T)}(A_1, A_2) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{8.19}$$

and if T is disconnected, one then extends to disconnected T in the obvious manner.

Remark 8.2.4. The above sounds overly complicated, but it just says that if T has connected components $\{T_\alpha\}$ then a homomorphism $A_1 \to A_2$ of abelian schemes over T is a collection $\{q_\alpha \varphi_\alpha\}$ where $q_\alpha \in \mathbb{Q}$ and φ_α is a morphism of abelian schemes $(A_1)_{T_\alpha} \to (A_2)_{T_\alpha}$. This differs from just $\mathrm{Hom}(A_1, A_2) \otimes_{\mathbb{Z}} \mathbb{Q}$ when the number of components is infinite (since, similar to why $\mathbb{Z}[\![t]\!] \otimes_{\mathbb{Z}} \mathbb{Q}$ differs from $\mathbb{Q}[\![t]\!]$, the denominators would be forced to be bounded).

Recall that an *isogeny* of abelian schemes $\varphi' : A_1 \to A_2$ is a surjective morphism of T-group schemes whose kernel is finite (equivalently quasi-finite) over T. We have the following elementary lemma concerning the definition of isogenies:

Lemma 8.2.5. *Let $\varphi' : A_1 \to A_2$ be an isogeny of abelian schemes over T. Then, the function*

$$\deg(\varphi') : |T| \to \mathbb{Z} \tag{8.20}$$

given by $\deg(\varphi')(t) := \mathrm{rank}(\ker(\varphi')_t)$ (where the rank of a finite group scheme over a field k is the k-dimension of its global sections) is locally constant on T.

Proof. Begin by noting that an isogeny φ' is automatically flat. Indeed, using the fibral criterion for flatness (see [Sta18, Tag 039C]) it suffices to prove the claim when T is a field. But, this then follows from Miracle Flatness (see [Sta18, Tag 00R4]) since φ' is evidently a quasi-finite map between equi-dimensional smooth schemes over T. Thus, $\ker(\varphi')$ is a finite flat group scheme over T. Let us denote by $g : \ker(\varphi') \to T$ the structure morphism of $\ker(\varphi')$. It's clear that $g_* \mathcal{O}_{\ker(\varphi')}$ is locally free, and $\deg(\varphi')$ is nothing but the rank function for the vector bundle $g_* \mathcal{O}_{\ker(\varphi')}$, and thus it's evidently locally constant. □

In particular, if T is connected then Lemma 8.2.5 implies that $\deg(\varphi')$ is a constant which we call the *degree* of φ'. The following lemma is also elementary, but useful in what follows:

Lemma 8.2.6. *Let T be a $\mathbb{Q}$-scheme, and $\varphi' : A_1 \to A_2$ a morphism of abelian schemes. Then, φ' is an isogeny if and only if there exists some $N \geqslant 1$ and some $\beta : A_2 \to A_1$ such that $\varphi' \circ \beta = [N]_{A_2}$ and $\beta \circ \varphi' = [N]_{A_1}$ where the brackets denote the multiplication by N map. This N is then the degree of φ'.*

Proof. See [EvdGM, Proposition 5.12]. They assume that T is irreducible, but this is unnecessary. □

Now, let us define a *quasi-isogeny* $\varphi : A_1 \to A_2$ to be morphism φ in $\mathsf{AV}_{\mathbb{Q}}(T)$ such that for all connected components T_α of T one has a factorization $\varphi_{T_\alpha} = q\varphi'$ for some isogeny $\varphi' : (A_1)_{T_\alpha} \to (A_2)_{T_\alpha}$ and some $q_\alpha \in \mathbb{Q}^\times$. It's not hard to see (applying Lemma 8.2.6 connected component by connected component) that the quasi-isogenies are the isomorphisms in the isogeny category. Thus, one easily sees that a quasi-isogeny $\varphi : A_1 \to A_2$ induces an isomorphism of lisse $\mathbb{A}_f$-sheaves $V(\varphi) : V(A_1) \to V(A_2)$.

Recall that one always has a canonical perfect alternating pairing

$$\psi_{\text{Weil}} : V(A) \times V(A^\vee) \to \mathbb{A}_f(1) \tag{8.21}$$

where, here, $\mathbb{A}_f(1)$ is the lisse $\mathbb{A}_f$-sheaf $V(\mathbb{G}_{m,T})$ defined precisely the same as $V(A)$. In other words, it's the sheaf

$$\left(\varprojlim_N \mu_{n,T}\right) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{8.22}$$

This pairing ψ_{Weil} known as the *Weil pairing.*

Remark 8.2.7. The Weil pairing is, as far as the author sees, not explicitly defined in [Lan13]. So, we just remark here that one can construct it as follows. One can show (see [EvdGM, Proposition 7.5]) that if A is an abelian scheme over T then $A[N]$ and $A^\vee[N]$ are Cartier dual, and so the Weil pairing at level N is just the natural evaluation homomorphism

$$A[N] \times A^\vee[N] \to \mu_{n,T} \tag{8.23}$$

and the Weil pairing at the $\mathbb{A}_f$-level is obtained via the limit, and then tensoring with $\mathbb{Q}$.

More generally, the theory of theta groups (see [EvdGM, Chapter VIII]) gives an isomorphism of fppf sheaves between $A^\vee$ and $\mathcal{E}xt^1(A, \mathbb{G}_{m,T})$ which is just an edulceration of [Lan13, Remark 1.3.2.3]. One can then derive the duality $A^\vee[N] \cong (A[N])^\vee$ mentioned above by examining the long exact sequence in fppf sheaves associated to the short exact sequence

$$0 \to A[N] \to A \xrightarrow{[N]_A} A \to 0 \tag{8.24}$$

which we leave to the reader to formalize.

Finally, in good settings (e.g. if the base is Noetherian), then one can also interpret it as follows. One can find a natural identification of $V(A^\vee)$ with $V(A)^\vee(1)$ and the Weil pairing is then the determinant of the relative cup product combined with relative Poincare duality combined with the fact that $\wedge^n R^2 f_* \mathbb{Q}_\ell \cong R^{2n} f_* \mathbb{Q}_\ell$.

Suppose now that we are given a *quasi-polarization* on A, which means a quasi-isogeny $\varphi : A \to A^\vee$ such that for every connected component T_α of T one has a factorization $\varphi_{T_\alpha} = q\varphi'_\alpha$ for some polarization (see [Lan13, §1.3.2]) $\varphi'_\alpha : A_{T_\alpha} \to A^\vee_{T_\alpha}$. As we have already observed, the quasi-polarization φ induces an isomorphism of $\mathbb{A}_f$-sheaves $V(\varphi) : V(A) \xrightarrow{\approx} V(A^\vee)$ and thus we get the *φ-adapted Weil pairing* $\psi_\varphi : V(A) \times V(A) \to \mathbb{A}_f(1)$ given by the following composition:

$$V(A) \times V(A) \xrightarrow{(\text{id}, V(\varphi))} V(A) \times V(A') \xrightarrow{\psi_{\text{Weil}}} \mathbb{A}_f(1) \tag{8.25}$$

which is still a perfect alternating pairing.

Let us call a pair (A,φ), where A is an abelian scheme over T and φ is a quasi-polarization on A, a *quasi-polarized* abelian scheme over T. We define a sheaf $\underline{\mathrm{Isom}}((V_{\mathbb{A}_f},\psi_{\mathbb{A}_f}),(V(A),\psi_\varphi$ on the étale site of T by associating to any étale $U\to T$ the set of isomorphisms $\lambda : V_{\mathbb{A}_f}\xrightarrow{\approx} V(A_U)$ such that for some trivialization $\alpha:\mathbb{A}_f(1)\xrightarrow{\approx}\mathbb{A}_f$ one has that

$$\alpha(\psi_\varphi(\lambda x,\lambda y)) = c\psi_{\mathbb{A}_f}(x,y) \tag{8.26}$$

for all $x,y\in V_{\mathbb{A}_f}$ and some $c\in\mathbb{A}_f^\times$. Note that any two isomorphisms α will only change the constant c in (8.26), and so whether or not λ is a symplectic similitude (i.e. whether (8.26) holds for some c) is independent of the trivialization. Note that there is a natural right action of the group $G(\psi)(\mathbb{A}_f) = \mathrm{GSp}(V_{\mathbb{A}_f},\psi_{\mathbb{A}_f})$ on this sheaf given by precomposition. So, for K a compact open subgroup of $G(\psi)(\mathbb{A}_f)$ let us denote by $\underline{\mathrm{Isom}}((V(\mathbb{A}_f),\psi_{\mathbb{A}_f}),(V(A),\psi_\varphi))/K$ the quotient sheaf.

Remark 8.2.8. As in Remark 8.1.1 one really does need to sheafify the quotient presheaf. In particular, $\underline{\mathrm{Isom}}((V_{\mathbb{A}_f},\psi_{\mathbb{A}_f}),(V(A),\psi_\varphi))$ will have no sections on U if $\mathbb{A}_f$ is not isomorphic to $\mathbb{A}_f(1)$, but the quotient sheaf will, in general, have sections on U.

Just as in the case of elliptic schemes, we have the following description of the values of this quotient sheaf on a connected scheme:

Lemma 8.2.9. *Suppose that T is a connected $\mathbb{Q}$-scheme and (A,φ) is a quasi-polarized abelian scheme over T. Pick $\overline{t}$ a geometric point of T, and let $(U,\overline{u})\to(T,\overline{t})$ be a connected pointed étale map. Then, there is a functorial (in the data $(U,\overline{u})$) isomorphism between $\Gamma(U,\underline{\mathrm{Isom}}((V_{\mathbb{A}_f},\psi_{\mathbb{A}_f}),(V(A),\psi_\varphi))/K)$ and the set of right K-orbits of isomorphisms $\lambda : V_{\mathbb{A}_f}\xrightarrow{\approx} V(A_{\overline{u}})$ which is a symplectic isomorphism for some trivialization $\alpha:\mathbb{A}_f(1)\xrightarrow{\approx}\mathbb{A}_f$ which are stable under the left action of $\pi_1^{\text{ét}}(U,\overline{u})$ by post composition (i.e. λK is left $\pi_1^{\text{ét}}(U,\overline{u})$-stable).*

So, again, if T is normal and connected and $\overline{\eta}$ is a geometric generic point of T, then we can identify $\Gamma(T,\underline{\mathrm{Isom}}((V_{\mathbb{A}_f},\psi_{\mathbb{A}_f}),(V(A),\psi_\varphi))/K)$ with the set of right K-orbits of isomorphisms $\lambda: V_{\mathbb{A}_f}\to V(A_{\overline{\eta}})$ which are left stable under the action of $\mathrm{Gal}(\overline{\eta}/\eta)$.

With all of this setup, we can now define the natural $G(\psi)(\mathbb{A}_f)$-scheme over $\mathbb{Q}$ alluded to before. Namely, given any K a compact open subgroup of $G(\psi)(\mathbb{A}_f)$ let us define a functor

$$Y(K):(\mathsf{Sch}/\mathbb{Q})^{\mathrm{op}}\to\mathsf{Set} \tag{8.27}$$

given by

$$Y(K)(T) := \{(A,\varphi,\lambda K)\}/\sim \tag{8.28}$$

where (A,φ) is a quasi-polarized abelian scheme over T and λK is an element of $\Gamma(T,\underline{\mathrm{Isom}}((V_{\mathbb{A}_f},\psi_{\mathbb{A}_f}),(V$ where two such pairs $(A,\varphi,\lambda K)$ and $(A',\varphi',\lambda' K)$ are equivalent if there exists a quasi-isogeny $A\to A'$ carrying φ to φ' and λK to $\lambda' K$.

We then have the following famous result of Mumford:

Theorem 8.2.10 (Mumford)**.** *For all $K\in\mathcal{N}(G(\psi))$ the functor $Y(K)$ is representable by a smooth quasi-projective variety over $\mathbb{Q}$.*

Proof. This is essentially shown in [MFK94, Theorem 7.9], but it's couched in different language, as well as not evidently applying for all neat open compact subgroups K. One can see [Lan13, Theorem 1.4.1.13] for a more comprehensive proof. One can also see [Hid12, Chapter 6] for an alternative discussion discussion. $\square$

Remark 8.2.11. As in Remark 8.1.5 one can show that for all compact open K in $\mathrm{GSp}(V, \mathbb{A}_f)$ there is a Deligne-Mumford stack $\mathscr{A}_{n,1}(K)$ over the fppf site of $\mathbb{Q}$, such that $Y(K) = \pi_0(\mathscr{A}_{n,1}(K))$ and show that for all K there exists a coarse moduli space of $\mathscr{A}_{n,1}(K)$ which will be fine (and thus be $Y(K)$) when K is neat. Even though for any neat K we have that the $\mathscr{A}_{n,1}(K)$ can easily be shown to have trivial inertia stack, and thus an algebraic space, this doesn't immediately imply the result of Mumford, as it did in the modular curve case, because we don't have the trick that one-dimensional algebraic spaces (over a field) are schemes.

We call these spaces $Y(K)$, for $K \in \mathcal{N}(G(\psi))$ *Siegel modular varieties with level structure* K associated to (V, ψ).

Note that if $K, L \in \mathcal{N}(G(\psi))$ and $g \in G(\psi)(\mathbb{A}_f)$ is such that $L \supseteq g^{-1}Kg$ then one has a natural morphism

$$t_{K,L}(g) : Y(K) \to Y(L) \tag{8.29}$$

given by sending $(A, \varphi, \lambda K)$ to $(A, \varphi, \lambda g K)$. This is well-defined (i.e. independent of the choice of $(A, \varphi, \lambda K)$ in its representative class) by the assumption that $L \supseteq g^{-1}Kg$.

We then have the following, predictable result:

Lemma 8.2.12. *Let (V, ψ) be a symplectic space over $\mathbb{Q}$. Then, the collection $\{Y_K\}_{K \in \mathcal{N}(G(\psi))}$, with the described maps $t_{K,L}(g)$, is a Shimura like $G(\psi)(\mathbb{A}_f)$-scheme over $\mathbb{Q}$ relative to the trivial subgroup of $Z(G(\psi)(\mathbb{A}_f))$.*

Again, we will explain this in greater generality in the §8.3, so we delay the justification.

Let us remark how one can generalize the notion of principle congruence subgroups as in the case of modular curves. The natural set of such generalizations will be greater in number since we now have an extra parameter to contend with (the polarization). So, let us begin by defining a *model* of (V, ψ) over $\mathbb{Z}$ to be a symplectic space $(V_\mathbb{Z}, \psi_\mathbb{Z})$ over $\mathbb{Z}$ together with an isomorphism $(V, \psi) \xrightarrow{\approx} ((V_\mathbb{Z})_\mathbb{Q}, (\psi_\mathbb{Z})_\mathbb{Q})$.

So, suppose that $(V_\mathbb{Z}, \psi_\mathbb{Z})$ is a model of (V, ψ) of type D. Let $N \geqslant 1$ be any integer coprime to d_n, and define the following subgroup of $G(\psi)(\mathbb{A}_f)$:

$$\widehat{\Gamma}_{(V_\mathbb{Z}, \psi_\mathbb{Z})}(N) := \ker(\mathrm{GSp}(V_{\widehat{\mathbb{Z}}}, \psi_{\widehat{\mathbb{Z}}}) \to \mathrm{GSp}(V_{\mathbb{Z}/N\mathbb{Z}}, \psi_{\mathbb{Z}/N\mathbb{Z}})) \tag{8.30}$$

which we call the *completed congruence subgroup of level N* of $G(\psi)(\mathbb{A}_f)$ associated to the model $(V_\mathbb{Z}, \psi_\mathbb{Z})$. Note that the condition that N was coprime to d_n ensures that $\psi_{\mathbb{Z}/N\mathbb{Z}}$ is still non-degenerate. Note that any two models of type D are isomorphic, so define conjugate subgroups of $G(\psi)(\mathbb{A}_f)$. When we wish to reference such a completed congruence subgroups we call them *completed congruence subgroups of level N and type D*. In the special case when $D = (1, \ldots, 1)$ we call these *principal congruence subgroups of level N*.

The completed congruence subgroups of level N and type D are neat if N is sufficiently large relative to D. For example, if $D = (1, \ldots, 1)$ then it suffices to take $N \geqslant 3$. The neat completed congruence subgroups of level N and type D (as N and D vary) form a cofinal system inside of $\mathcal{N}(G(\psi))$, and the associated moduli functors are relatively simple.

Namely, let us suppose that $(V_{\mathbb{Z}}, \psi_{\mathbb{Z}})$ is a model of (V, ψ) of type D, and N is large enough so that $\widehat{\Gamma}_{(V_{\mathbb{Z}},\psi_{\mathbb{Z}})}(N)$ is neat. Let us define a moduli problem

$$Y((V_{\mathbb{Z}}, \psi_{\mathbb{Z}}), N) : (\mathsf{Sch}/\mathbb{Q})^{\mathrm{op}} \to \mathsf{Set} \tag{8.31}$$

that associates to a $\mathbb{Q}$-scheme T the set

$$Y((V_{\mathbb{Z}}, \psi_{\mathbb{Z}}), N)(T) := \{(A, \varphi, \lambda)\}/\sim \tag{8.32}$$

where A is an abelian scheme over T, φ is a polarization (not a quasi-polarization!) of type D, and λ is an isomorphism of group schemes such that $V_{\mathbb{Z}/N\mathbb{Z}} \xrightarrow{\approx} A[N]$ (where $V_{\mathbb{Z}/N\mathbb{Z}}$ is considered as a finite constant group scheme) which is a symplectic similitude (i.e. preserves the respective pairings up to scalars) relative to some (equivalently any) trivialization of groups schemes $\mu_{N,T} \xrightarrow{\approx} \mathbb{Z}/N\mathbb{Z}$ (where $\mathbb{Z}/N\mathbb{Z}$ is the trivial group scheme over T). Here, a *polarization of type* D is a polarization $\varphi : A \to A^\vee$ such that $\ker(\varphi)$, thought of as a finite flat group scheme over T, is étale locally isomorphic to the constant group scheme $(\mathbb{Z}/d_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/d_n\mathbb{Z})$ where $D = (d_1, \ldots, d_n)$.

As one might expect, one has natural isomorphisms between $Y(\widehat{\Gamma}_{(V_{\mathbb{Z}},\psi_{\mathbb{Z}})}(N))$ and $Y((V_{\mathbb{Z}}, \psi_{\mathbb{Z}}), N)$. This follows, essentially, from Lemma 8.2.9 as the reader can easily verify.

Let us fix a model (V_0, ψ_0) of (V, ψ) of type $(1, \ldots, 1)$. Then, one can show that the principal congruence subgroups $\widehat{\Gamma}_{(V_0,\psi_0)}(N)$, for $N \geqslant 3$, form a cofinal subset of $\mathcal{N}(G(\psi))$. So, we then have the following description of the space Y at infinite level associated to $G(\psi)(\mathbb{A}_f)$-scheme $\{Y(K)\}_{K \in \mathcal{N}(G(\psi))}$:

$$Y := \varprojlim_{K \in \mathcal{N}(G(\psi))} Y(K) = \varprojlim_{((V_{\mathbb{Z}},\psi_{\mathbb{Z}}),N)} Y((V_{\mathbb{Z}}, \psi_{\mathbb{Z}}), N) = \varprojlim_{N} Y((V_0, \psi_0), N) \tag{8.33}$$

where the second limit is over all pairs $((V_{\mathbb{Z}}, \psi_{\mathbb{Z}}), N)$ consisting of a model $(V_{\mathbb{Z}}, \psi_{\mathbb{Z}})$ of (V, ψ) and N a sufficiently large positive integer (relative to the type of $(V_{\mathbb{Z}}, \psi_{\mathbb{Z}})$), and the last limit ranges over $N \geqslant 3$. The first two descriptions of Y are limits over reasonable subsets of compact open subgroups of $G(\psi)(\mathbb{A}_f)$ and so the $G(\psi)(\mathbb{A}_f)$-action is the obvious one. The latter presentation obscures the $G(\psi)(\mathbb{A}_f)$-action, and only obviously admits an action by $\mathrm{GSp}(V_0, \psi_0)$. As in the case of modular curves the entire $G(\psi)(\mathbb{A}_f)$-action manifests itself on this latter limit by Hecke correspondences on the system $\{Y((V_0, \psi_0), N)\}_{N \geqslant 3}$.

Example 8.2.13. As remarked before, if (V, ψ) is the sympletcic $\mathbb{Q}$-space $(\mathbb{Q}^2, \psi_{\mathrm{std}})$, then the natural inclusion $f : G(\psi) \hookrightarrow \mathrm{GL}_2$ is an isomorphism. One can then show that fixing this isomorphism f, that there is an isomorphism $\{Y(K)\}_{K \in \mathcal{N}(G(\psi))} \xrightarrow{\approx} \{M(K)\}_{K \in \mathcal{N}(\mathrm{GL}_2)}$ relative to f. So, as already intimated several times, Siegel modular varieties are higher dimensional analogues of modular curves.

Remark 8.2.14. One might wonder why we have chosen to work with a general symplectic space (V, ψ) over $\mathbb{Q}$ considering that they are all isomorphic to $(\mathbb{Q}^{2n}, \psi_{\text{std}})$ for $2n = \dim(V)$. First, if one wishes to start by thinking of models of (V, ψ) over $\mathbb{Z}$, then the fact that all models are *not* isomorphic makes this notation more flexible. The second advantage is that allowing for arbitrary symplectic spaces allows one to define congruence subgroups relative to any symplectic space, and its model, which makes this set of congruence subgroups closed under conjugation, which is not something that happens if one restricts to strict choices of isomorphy classes of the symplectic space over $\mathbb{Q}$ and its models.

That said, the moduli spaces associated to these more canonical choices of symplectic spaces and models are more commonly focused on in most texts on the subject (e.g. in [MFK94] or [Hid12]). So, let us just mention that if we take $(V, \psi) = (\mathbb{Q}^{2n}, \psi_{\text{std}})$ then one usually denotes $G(\psi)$ as GSp_{2n} and calls it *the symplectic similitude group* over $\mathbb{Q}$. Then, if one takes the model $(V_D, \psi_D) := (\mathbb{Z}^{2g}, \psi_D)$ the spaces $Y((V_D, \psi_D), N)$ are usually denoted $\mathscr{A}_{n,D,n}$. In particular, taking $(V_0, \psi_0) := (V_{(1,\ldots,1)}, \psi_{(1,\ldots,1)})$ one has that $Y((V_0, \psi_0), N)$ is what is usually called $\mathscr{A}_{n,1,N}$ and called the *moduli space of principally polarized abelian varieties with level N structure.* So, for example, if Y denotes the space at infinite level for the Shimura like $\mathrm{GSp}_{2n}(\mathbb{A}_f)$-scheme over $\mathbb{Q}$ given by $\{Y(K)\}_{K \in \mathcal{N}(\mathrm{GSp}_{2n})}$ then (8.33) just says that

$$Y = \varprojlim_{N \geqslant 3} \mathscr{A}_{n,1,N} \tag{8.34}$$

as used at several points in our discussion of integral canonical models.

8.3 General Shimura varieties

In this subsection we discuss the ultimate generalization of the examples from the last two sections in the form of Shimura varieties. Of course, we don't aim to give even a semblance of a comprehensive discussion of the theory, only a brief outline of the results important to us. We explain here in more detail only the results which are difficult to glean from the main resources on the topic, and which are of utmost relevance for us here. These aforementioned references are the original papers of Deligne [Del71b] and [Del79], the notes [Mil04], and the text [Hid12].

The complex theory

Let G be a connected reductive group over $\mathbb{Q}$ and X a class of homomorphims $h : \mathbb{S} \to G_{\mathbb{R}}$ of $\mathbb{R}$-groups where $\mathbb{S}$ is the *Deligne torus* $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ (i.e. it's the fundamental group of the neutralized Tannakian category of real Hodge structures, see [Del82]) which is an orbit under the left conjugation action of $G(\mathbb{R})$ on $G_{\mathbb{R}}$. We assume that (G, X) satisfies the first three axioms of Deligne from [Del79, Section 2.1], which we write SV1, SV2, and SV3 following [Mil04, Defintion 5.5], in which case we call the pair (G, X) a *Shimura datum.*

Let us note that if $h_0 \in X$ is arbitrary, and if K_{h_0} denotes the stabilizer of $h_0 \in X$, then K_{h_0} is a maximal compact subgroup of $G_{\mathbb{R}}$ and therefore $X = G(\mathbb{R})/K_{h_0}$. This naturally endows X with the structure of a real manifold, naturally independent of the choice of h_0, and the left action of $G(\mathbb{R})$ on X is by smooth automorphisms. More surprising is the fact that X has the natural structure of a complex manifold, which we now recall.

Suppose that $\rho : G_{\mathbb{R}} \to \mathrm{GL}(V)$ is a representation of $G_{\mathbb{R}}$. Then, for any $h \in X$ we obtain a real Hodge structure V_h given by the underlying vector space V, and the Hodge structure given by $\rho \circ h$. Axiom SV1 implies that the weight decomposition $V = \bigoplus_{n\in\mathbb{Z}} V_{n,h}$ is independent of h. Let us denote this common $\mathbb{R}$-subspace of V by V_n.

We then have the following beautiful theorem of Deligne:

Lemma 8.3.1. *The real manifold X has the unique structure of a complex manifold, compatible with its real manifold structure, such that for any representation $\rho : G \to \mathrm{GL}(V)$ and any integer $n \in \mathbb{Z}$ the trivial holomorphic vector bundle $(V_n)_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{O}_X$ has a descending filtration by holomorphic subbundles $\mathcal{F}^p$ such that for all $h \in X$ the fiber $\mathcal{F}^p_h$ is $\mathrm{Fil}^p(V_{n,h})_{\mathbb{C}}$, the p^{th} graded piece of the Hodge filtration of $(V_{n,h})_{\mathbb{C}}$.*

Proof. This is [Del79, Proposition 1.1.14, i)]. See [Mil04, Theorem 2.14, a)] as well. □

A careful reading of the proof of [Mil04, Theorem 2.14] gives an explicit description of the (integrable) almost complex structure on X. Indeed, for each $h \in X$ we get the aforementioned identification of X as $G(\mathbb{R})/K_h$. Thus, we see that $T_h(X)$ (the tangent space of X at h) can naturally be identified with $\mathfrak{g}/\mathfrak{k}_h$ where $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R}))$ and $\mathfrak{k}_h = \mathrm{Lie}(K_h)$. Now, SV1 explicitly states that the Hodge structure on $\mathfrak{g}$ inherited from $\mathrm{Ad}\circ h$ is of type $\{(-1,1),(0,0),(1,-1)\}$. Since K_h is the stabilizer of h, it's clear that the natural inclusion $\mathfrak{k}_h \subseteq \mathfrak{g}$ identifies $\mathfrak{k}_h$ as $\mathfrak{g}^{0,0}$. Thus, we have a natural identification of $T_h(X)$ with $\mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1}$. Note that $\mathrm{Ad}(h(i))$, by the definition of the Hodge decomposition, acts by $i^{-1} \cdot \bar{i} = -1$ on $\mathfrak{g}^{-1,1}$ and $i\bar{i}^{-1} = -1$ on $\mathfrak{g}^{1,-1}$. Thus, it acts by -1 on all of $T_h(X)$. Thus, if ζ_8 denotes a fixed choice of a primitive 8^{th} root of unity, then $J_h := \mathrm{Ad}(h(\zeta_8))$ acts on $T_h(X)$ in such a way that J_h^2 is negative of the identity. Thus, defines a complex structure on $T_h(X)$. Then, the aforementioned careful reading tells us that $(J_h)_{h\in X}$ is precisely the (integrable) almost complex structure on X defined by Lemma 8.3.1.

One important consequence of this is the following:

Corollary 8.3.2. *The transitive action of $G(\mathbb{R})$ on X is by holomorphic automorphisms.*

Proof. Since $G(\mathbb{R})$ acts on X by smooth automorphisms it suffices to show that for all $h \in X$ and $g \in G(\mathbb{R})$ the $\mathbb{R}$-linear map $\mathrm{Ad}(g) : T_h(X) \to T_{g\cdot h}(X)$ is actually $\mathbb{C}$-linear where they are given their respective complex structures. But, note that evidently

$$\begin{aligned} \mathrm{Ad}(g) \circ J_h \circ \mathrm{Ad}(g)^{-1} &= \mathrm{Ad}(g) \circ \mathrm{Ad}(h(\zeta_8)) \circ \mathrm{Ad}(g)^{-1} \\ &= \mathrm{Ad}(gh(\zeta_8)g^{-1}) \\ &= \mathrm{Ad}((g\cdot h)(\zeta_8)) \\ &= J_{g(h)} \end{aligned} \tag{8.35}$$

from where the claim follows. □

Next, let us suppose that (G,X) is a Shimura datum, and let us fix some $h_0 \in X$ and denote K_{h_0} by K_∞. Then, for any K a compact open subgroup of $G(\mathbb{A}_f)$ one can form the topological space

$$\mathsf{Sh}_K(G,X)^{\mathrm{an}}_{\mathbb{C}} := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K = G(\mathbb{Q})\backslash G(\mathbb{A})/(K \cdot K_\infty) \tag{8.36}$$

where $G(\mathbb{Q})$ acts diagonally (by left conjugation on X and by left multiplication on $G(\mathbb{A}_f)$ or $G(\mathbb{A})$) and K (resp. $K \cdot K_\infty$) acts only on $G(\mathbb{A}_f)$ (resp. $G(\mathbb{A})$) by right multiplication. Here we are endowing $G(\mathbb{A}_f)$ (resp. $G(\mathbb{A})$) with the natural adelic topologies. Using the first presentation of $\mathsf{Sh}_K(G,X)^{\mathrm{an}}_{\mathbb{C}}$ it is also equivalent to endow $G(\mathbb{A}_f)$ with the discrete topology (the proof of Lemma 8.3.5 doesn't depend on which topology we pick as the reader can quickly check). We shall denote a point $G(\mathbb{Q})(h,g)K$ of $\mathsf{Sh}_K(G,X)^{\mathrm{an}}_{\mathbb{C}}$ by $[h,g]_K$.

Our first goal is to recall why this topological space is not too unreasonable. The key point is the following observation:

Lemma 8.3.3. *The double coset space $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$ is finite.*

Proof. See the discussion in the proof of [Mil04, 5.12]. □

Thus, we might imagine that $\mathsf{Sh}_K(G,X)^{\mathrm{an}}_{\mathbb{C}}$ is just a finite disjoint union of quotients of X. This is essentially correct, but the fact that $G(\mathbb{Q})$ acts diagonally means that one must pay slightly more care to make this precise, this is what we now do.

So, let us denote the connected component of X that contains our chosen $h_0 \in X$ by X^+. It's fairly easy to see that X^+ is just the left $G(\mathbb{R})^+$ conjugacy class of h_0, where $G(\mathbb{R})^+$ denotes the connected component of the real manifold $G(\mathbb{R})$. Of course, since $G(\mathbb{R})^+$ is acting by conjugation, this action will factor through $G^{\mathrm{ad}}(\mathbb{R})^+$. Let us then denote by $G(\mathbb{R})_+$ the preimage of $G^{\mathrm{ad}}(\mathbb{R})^+$ under the map $G(\mathbb{R}) \to G^{\mathrm{ad}}(\mathbb{R})$, and denote by $G(\mathbb{Q})_+$ the intersection $G(\mathbb{R})_+ \cap G(\mathbb{Q})$.

Remark 8.3.4. One must be slightly careful in the above discussion. A priori the conjugation action of $G(\mathbb{R})^+$ on X^+ will factor through the group $G(\mathbb{R})^+/Z(G(\mathbb{R})^+)$ which has no obvious reason to be equal to

$$G^{\mathrm{ad}}(\mathbb{R})^+ = (G/Z(G))(\mathbb{R})^+ \tag{8.37}$$

That said, by [Mil04, Proposition 5.1] the natural map $G(\mathbb{R})^+ \to G^{\mathrm{ad}}(\mathbb{R})^+$ is surjective with kernel $Z(G)(\mathbb{R}) \cap G(\mathbb{R})^+$ which is contained in $Z(G(\mathbb{R})^+)$. Thus, the action of $G(\mathbb{R})^+$ really does factor through $G^{\mathrm{ad}}(\mathbb{R})^+$. This is one reason why it's often more convenient often to work with X^+ than X directly.

For each $g \in G(\mathbb{A}_f)$ let us denote by $\Gamma_g(K)$ the group $G(\mathbb{Q})_+ \cap gKg^{-1}$. Note then that we have a natural left action of $\Gamma_g(K)$ on X^+ coming from the left conjugation action of $G(\mathbb{R})^+$ on X^+. We also have a natural map

$$\Gamma_g(K)\backslash X^+ \to \mathsf{Sh}_K(G,X)^{\mathrm{an}}_{\mathbb{C}} \tag{8.38}$$

given by sending $\Gamma_g(K)h$ to $[h,g]_K$, which one can easily check is well-defined (i.e. independent of the representative h of $\Gamma_g(K)h$). Moreover, one can check that the image of this map only depends on the value of g in

$$\mathcal{C}_K := G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K \tag{8.39}$$

which is a finite set (see [Mil04, Lemma 5.12]).

One then has the following elementary description of the topological space $\mathsf{Sh}_K(G,X)^{\mathrm{an}}_{\mathbb{C}}$:

Lemma 8.3.5. *The map*

$$\bigsqcup_{g\in\mathcal{C}_K}(\Gamma_g(K)\backslash X^+)\to \mathsf{Sh}_K(G,X)^{\mathrm{an}}_{\mathbb{C}} \tag{8.40}$$

is a homeomorphism.

Proof. Consider first the topological space

$$G(\mathbb{Q})_+\backslash X^+\times G(\mathbb{A}_f)/K \tag{8.41}$$

Then, the map

$$\bigsqcup_{g\in\mathcal{C}_K}\Gamma_g(K)\backslash X^+\to G(\mathbb{Q})_+\backslash X^+\times G(\mathbb{A}_f)/K \tag{8.42}$$

is easily seen to be a homeomorphism. Note though that we have an obvious continuous inclusion

$$G(\mathbb{Q})_+\backslash X^+\times G(\mathbb{A}_f)/K\to \mathsf{Sh}_K(G,X)^{\mathrm{an}}_{\mathbb{C}} \tag{8.43}$$

Axiom SV3 allows us to apply real approximation to then show that this is a bijection. See[Mil04, Lemma 5.13] for more details. □

We would like to endow, at least in the case when K is neat, the topological space $\mathsf{Sh}_K(G,X)^{\mathrm{an}}_{\mathbb{C}}$ with the structure of a complex manifold, as the notation suggests. To explain this, for K a compact open subgroup of $G(\mathbb{A}_f)$ and $g\in G(\mathbb{A}_f)$, let us denote by $\overline{\Gamma_g(K)}$ the image of $\Gamma_g(K)$ in the group $G^{\mathrm{ad}}(\mathbb{Q})^+ := G^{\mathrm{ad}}(\mathbb{R})^+\cap G^{\mathrm{ad}}(\mathbb{Q})$, which is just the image of $\Gamma_g(K)$ under the map $G(\mathbb{Q})\to G^{\mathrm{ad}}(\mathbb{Q})$. We then have the following observation:

Lemma 8.3.6. *If $K\in\mathcal{N}(G)$ then $\overline{\Gamma_g(K)}$ is torsion free for all $g\in G(\mathbb{A}_f)$. Thus, $\overline{\Gamma_g(K)}$ acts without fixed points on X^+ for all $g\in G(\mathbb{A}_f)$.*

Proof. Since K is neat we know that $G(\mathbb{Q})\cap gKg^{-1}$ is neat by Lemma 7.1.19 and thus, consequently, $\Gamma_g(K)$ (being a subgroup of $G(\mathbb{Q})\cap gKg^{-1}$) is neat. Thus, by Lemma 7.1.21 the image of $\Gamma_g(K)$ under the map $G(\mathbb{Q})\to G^{\mathrm{ad}}(\mathbb{Q})$, which is $\overline{\Gamma_g(K)}$, is also neat and thus torsion free. To prove that $\overline{\Gamma_g(K)}$ acts without fixed points on X^+ we proceed as follows.

Note that, as we have already discussed, we can interpret the action of $G(\mathbb{R})^+$ on X^+ as factoring through the action of $G^{\mathrm{ad}}(\mathbb{R})^+$, in which case we can think of it as a $G^{\mathrm{ad}}(\mathbb{R})^+$ conjugacy class of homomorphisms $\mathbb{S}\to G^{\mathrm{ad}}_{\mathbb{R}}$. But, SV1 implies that the natural embedded copy $\mathbb{G}_{m,\mathbb{R}}\subseteq\mathbb{S}$ (which picks out the weight filtration of the $\mathbb{R}$-Hodge structure for representations of $\mathbb{S}$) has central image in $G_{\mathbb{R}}$ and so the morphisms $\mathbb{S} \to G_{\mathbb{R}}^{\mathrm{ad}}$ factor through $\mathbb{S}/\mathbb{G}_{m,\mathbb{R}}$ which is isomorphic to the algebraic group $U(1)$.

Thus, X^+ can be thought of as a $G^{\mathrm{ad}}(\mathbb{R})^+$ conjugacy class of homomorphisms $U(1) \to G_{\mathbb{R}}^{\mathrm{ad}}$. Moreover, the assumption that (G, X) is a Shimura datum immediately implies that conditions (a), (b), and (c) of [Mil04, Theorem 1.21] apply. Thus, X^+ is a Hermitian symmetric domain, and a reading of the proof loc. cit. shows that it has the same complex structure we have already endowed it with. Now, loc. cit. also implies that $G^{\mathrm{ad}}(\mathbb{R})^+$ is the group of holomorphic automorphisms of X^+ and, by assumption, $\overline{\Gamma_K(g)}$ is a torsion free subgroup. The claim that it acts without fixed points on X^+ then follows from [Mil04, Proposition 3.1]. □

From this we immediately deduce that for each $K \in \mathcal{N}(G)$ and $g \in G(\mathbb{A}_f)$ one has that $\Gamma_g(K)\backslash X^+$ has the natural structure of a complex manifold and thus Lemma 8.3.5 implies that $\mathsf{Sh}_K(G, X)_{\mathbb{C}}^{\mathrm{an}}$ has a natural structure of a complex manifold.

Remark 8.3.7. Since X^+ is a Hermitian symmetric domain, it is simply connected (see [Hel01, Theorem 4.6, Chapter VIII]). Thus, we see that $\pi_1(X^+, h_0) = \overline{\Gamma_g(K)}$.

Of course, as the notation suggests, $\mathsf{Sh}_K(G, X)_{\mathbb{C}}^{\mathrm{an}}$ is not only a complex manifold but, in fact, is algebraic. Suprisingly, it's algebraic in essentially only one way as the following beautiful theorem of Borel and Baily-Borel shows:

Theorem 8.3.8 (Borel, Baily-Borel)**.** *Let $K \in \mathcal{N}(G)$. Then, there exists a (necessarily unique) smooth equidimensional quasi-projective variety $\mathsf{Sh}_K(G, X)_{\mathbb{C}}$ over $\mathbb{C}$ with analytification $\mathsf{Sh}_K(G, X)_{\mathbb{C}}^{\mathrm{an}}$ such that for any smooth quasi-projective variety X over $\mathbb{C}$ the natural map*

$$\mathrm{Hom}(X, \mathsf{Sh}_K(G, X)_{\mathbb{C}}) \to \mathrm{Hom}(X^{\mathrm{an}}, \mathsf{Sh}_K(G, X)_{\mathbb{C}}^{\mathrm{an}}) \tag{8.44}$$

is bijective.

Proof. To show that $\mathsf{Sh}_K(G, X)_{\mathbb{C}}^{\mathrm{an}}$ is algebraizable, it suffices to show that each $\Gamma_g(K)\backslash X^+$ is algebraizable as in Lemma 8.3.5. This follows [BB66, Theorem 10.11], which also shows its the analytification of a quasi-projective variety. Thus, there exists a quasi-projective variety $\mathsf{Sh}_K(G, X)_{\mathbb{C}}$ with analytification $\mathsf{Sh}_K(G, X)_{\mathbb{C}}^{\mathrm{an}}$. This works for any compact open subgroup K of $G(\mathbb{A}_f)$. But, the assumption that K is neat implies that $\mathsf{Sh}_K(G, X)_{\mathbb{C}}^{\mathrm{an}}$ is a complex manifold, and thus $\mathsf{Sh}_K(G, X)_{\mathbb{C}}$ is necessarily smooth. The second claim follows from the extension theorem [Bor72, Theorem A]. It is explicitly stated in [Bor72, Theorem 3.10]. Again, one really needs to work on each of the $\Gamma_g(K)\backslash X^+$ piece by piece. □

Note that if K and K' are neat open compact subgroups of $G(\mathbb{A}_f)$ satisfying $K \subseteq K'$ then we get obvious maps of topological spaces

$$(\pi_{K,K'})_{\mathbb{C}}^{\mathrm{an}} : \mathsf{Sh}_K(G, X)_{\mathbb{C}}^{\mathrm{an}} \to \mathsf{Sh}_{K'}(G, X)_{\mathbb{C}}^{\mathrm{an}} \tag{8.45}$$

sending $[h, g]_K$ to $[h, g]_{K'}$.

The following result is fundamental:

Lemma 8.3.9. *For all $K, K' \in \mathcal{N}(G)$ with $K \subseteq K'$ the map $(\pi_{K,K'})_{\mathbb{C}}^{\mathrm{an}}$ is a finite holomorphic covering map. If $K \trianglelefteq K'$ then $(\pi_{K,K'})_{\mathbb{C}}^{\mathrm{an}}$ is a Galois covering space with Galois group $K'/K(Z(G)(\mathbb{Q}) \cap K')$*

Proof. Let us begin by determining the identity of the map $(\pi_{K,K'})_{\mathbb{C}}^{\mathrm{an}}$ under the bijection from Lemma 8.3.5. Namely, let us note that we have a natural surjection

$$\mathcal{C}_K \twoheadrightarrow \mathcal{C}_{K'} \tag{8.46}$$

given by taking $G(\mathbb{Q})_+ gK$ to $G(\mathbb{Q})_+ gK'$. For each representative $g \in \mathcal{C}_K$ let us denote by $\overline{g}$ its image in $\mathcal{C}_{K'}$. Then, it's clear that under the identification of Lemma 8.3.5 the map

$$\bigsqcup_{g \in \mathcal{C}_K} \left(\Gamma_g(K)\backslash X^+\right) = \mathsf{Sh}_K(G,X)_{\mathbb{C}}^{\mathrm{an}} \to \mathsf{Sh}_{K'}(G,X)_{\mathbb{C}}^{\mathrm{an}} = \bigsqcup_{h \in \mathcal{C}_{K'}} \left(\Gamma_h(K')\backslash X^+\right) \tag{8.47}$$

is defined on the component $\Gamma_g(K)\backslash X^+$ as the natural quotient mapping

$$\Gamma_g(K)\backslash X^+ \to \Gamma_{\overline{g}}(K')\backslash X^+ \tag{8.48}$$

To check that this is holomorphic it suffices to check that the composition

$$X^+ \to \Gamma_g(K)\backslash X^+ \to \Gamma_{\overline{g}}(K')\backslash X^+ \tag{8.49}$$

is holomorphic, but this is evident since this composition defines the complex structure on $\Gamma_{\overline{g}}\backslash X^+$.

To see that $(\pi_{K,K'})_{\mathbb{C}}^{\mathrm{an}}$ is a covering map it suffices to check it for each map of the form (8.49). But, note that both $X^+ \to \Gamma_g(K)\backslash X^+$ and $X^+ \to \Gamma_{\overline{g}}(K')\backslash X^+$ are covering maps. The fact that it's finite follows from the fact that the fiber over any point has a natural surjection from $\Gamma_{\overline{g}}(K')/\Gamma_g(K)$ which has a surjection from K'/K, which is finite.

The last claim concerning the Galois group is clear since we have a natural surjection from K'/K to the Galois group whose kernel is clearly equal to $Z(\mathbb{Q}) \cap K'$. □

Of course, this together with Theorem 8.3.8 immediately implies:

Corollary 8.3.10. *Let $K, K' \in \mathcal{N}(G)$ with $K \subseteq K'$. Then, there exists a unique finite étale cover*

$$(\pi_{K,K'})_{\mathbb{C}} : \mathsf{Sh}_K(G,X)_{\mathbb{C}} \to \mathsf{Sh}_{K'}(G,X)_{\mathbb{C}} \tag{8.50}$$

of smooth quasi-projective varieties over $\mathbb{C}$ whose analytification is $(\pi_{K,K'})_{\mathbb{C}}^{\mathrm{an}}$. If $K \trianglelefteq K'$ then $\pi_{K,K'}$ is a Galois cover of schemes over $\mathbb{C}$, with Galois group $K'/K(Z(G)(\mathbb{Q}) \cap K')$.

Similarly, we have the homeomorphism

$$([g]_K)_{\mathbb{C}}^{\mathrm{an}} : \mathsf{Sh}_K(G,X)_{\mathbb{C}} \to \mathsf{Sh}_{g^{-1}Kg}(G,X)_{\mathbb{C}} \tag{8.51}$$

defined by

$$([g]_K)_{\mathbb{C}}^{\mathrm{an}}[h,g']_K = [h,g'g]_{g^{-1}Kg} \tag{8.52}$$

This is evidently well-defined, since if $[h,g']_K = [h,g'']_K$ then $g'K = g''K$ then evidently

$$g'g(g^{-1}Kg) = g''g(g^{-1}Kg) \tag{8.53}$$

so $[h,g'g]_{g^{-1}Kg} = [h,g''g]_{g^{-1}Kg}$.

Again, we have the following elementary observation:

Lemma 8.3.11. *For all $K \in \mathcal{N}(G)$ and $g \in G(\mathbb{A}_f)$ the map $([g]_K)^{\mathrm{an}}_{\mathbb{C}}$ is biholomorphic.*

Proof. Since $([g]^{\mathrm{an}}_K)_{\mathbb{C}}$ is a homeomorhism, it suffices to show that it's holomorphic. But, in the notation of Lemma 8.3.5 its evident that $([g]_K)^{\mathrm{an}}_{\mathbb{C}}$ comes from bijections

$$\Gamma_{g'}(K)\backslash X^+ \to \Gamma_{g'g}(gKg^{-1})\backslash X^+ \tag{8.54}$$

which are obviously induced from the right action by g on X^+. By Corollary 8.3.2 we know that this map is holomorphic thus, by definition, so is (8.54) and thus so is $([g]_K)^{\mathrm{an}}_{\mathbb{C}}$. $\square$

Again appealing to Theorem 8.3.8 show the following:

Corollary 8.3.12. *For all $K \in \mathcal{N}(G)$ and $g \in G(\mathbb{A}_f)$ there is a unique isomorphism*

$$([g]_K)_{\mathbb{C}} : \mathsf{Sh}_K(G,X)_{\mathbb{C}} \to \mathsf{Sh}_{g^{-1}Kg}(G,X)_{\mathbb{C}} \tag{8.55}$$

of smooth quasi-projective schemes over $\mathbb{C}$ whose analytification is $([g]_K)^{\mathrm{an}}_{\mathbb{C}}$.

If L and K are compact neat open subgroups of $G(\mathbb{A}_f)$ such that the containment $L \supseteq g^{-1}Kg$ holds then let us define $t_{K,L}(g)_{\mathbb{C}}$ as follows:

$$t_{K,L}(g)_{\mathbb{C}} := (\pi_{L,g^{-1}Kg})_{\mathbb{C}} \circ ([g]_K)_{\mathbb{C}} \tag{8.56}$$

Then, our discussion up until this point shows the following:

Lemma 8.3.13. *The system $\{\mathsf{Sh}_K(G,X)_{\mathbb{C}}\}_{K\in\mathcal{N}(G)}$ with morphisms $t_{K,L}(g)_{\mathbb{C}}$ is a smooth $G(\mathbb{A}_f)$-scheme over $\mathbb{C}$ which is Shimura like with respect to $Z := Z(G)(\mathbb{Q})$.*

That said, we have made the claim many times before that, in good situations, we should be able to take Z to be the trivial subroup. We give one instance in which this is the case:

Lemma 8.3.14. *Let (G,X) be a Shimura datum satisfying SV5 as in [Mil04]. Then, $\{\mathsf{Sh}_K(G,X)_{\mathbb{C}}\}_{K\in\mathcal{N}(G)}$ is Shimura like where one can take Z to be the trivial subgroup.*

Proof. By Lemma 8.3.13 it suffices to show that SV5 implies that for K neat one has that $Z(G)(\mathbb{Q}) \cap K$ is trivial. That said, SV5 says precisely that $Z(G)(\mathbb{Q})$ is discrete in $Z(G)(\mathbb{A}_f)$. Thus, we know that $Z(G)(\mathbb{Q})\cap K$ is discrete in K and so, since K is compact, finite. But, since K is neat it's torsion free and so finite subgroups are trivial. $\square$

Let us give an example where one cannot take Z to be trivial:

Example 8.3.15. Consider (G,X) where $G = \mathrm{Res}_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}\mathrm{GL}_{2,\mathbb{Q}(\sqrt{2})}$ and X is the $G(\mathbb{R})$-conjugacy class of the homomorphism $h_0 : \mathbb{S} \to G_{\mathbb{R}}$ given as follows. Recall that one model of $\mathbb{S}$ as an $\mathbb{R}$-algebraic group is the following group functor sending an $\mathbb{R}$-algebra R to

$$\left\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathrm{GL}_2(R)\right\} \tag{8.57}$$

which, in particular, gives us a natural embedding $\mathbb{S} \hookrightarrow \mathrm{GL}_{2,\mathbb{R}}$. Note though that we also have a natural isomorphism

$$G_{\mathbb{R}} \cong \prod_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})} \mathrm{GL}_{2,\mathbb{R}} \tag{8.58}$$

We then let h_0 be the diagonal embedding of $\mathbb{S}$ into $G_{\mathbb{R}} = (\mathrm{GL}_{2,\mathbb{R}})^2$. This pair (G, X) is a Shimura datum, and is an example of a Shimura datum of *Hilbert type*. The Shimura varieties $\mathsf{Sh}_K(G, X)$ are examples of so-called *Hilbert modular varieties*. For more information on the theory of such varieties see [Gor02].

It will be helpful to recall the structure of the topological space at infinite level for the $G(\mathbb{A}_f)$-scheme $\{\mathsf{Sh}_K(G, X)_{\mathbb{C}}\}_{K \in \mathcal{N}(G)}$ over $\mathbb{C}$. Namely, we have the following:

Lemma 8.3.16. *There is a natural identification of topological spaces with a right $G(\mathbb{A}_f)$-action*

$$|\mathsf{Sh}(G, X)^{\mathrm{an}}_{\mathbb{C}}| := \varprojlim_{K \in \mathcal{N}(G)} |\mathsf{Sh}_K(G, X)^{\mathrm{an}}_{\mathbb{C}}| = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/Z(\mathbb{Q})^- \tag{8.59}$$

where $Z(\mathbb{Q})^-$ denotes the closure of $Z(\mathbb{Q})$ in $Z(\mathbb{A}_f)$. When (G, X) satisfies SV5 this right hand side reduces to $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)$.

Proof. See [Mil90, Proposition 10.1]). □

Canonical models

We would now like to remove the last decoration (the subscript $\mathbb{C}$) on the $G(\mathbb{A}_f)$-scheme $\{\mathsf{Sh}_K(G, X)_{\mathbb{C}}\}_{K \in \mathcal{N}(G)}$ by defining a model of each $\mathsf{Sh}_K(G, X)_{\mathbb{C}}$ over an appropriate number field E.

Let us begin by recalling that associated to the Shimura datum (G, X) is its *reflex field* $E(G, X)$. To define this, one observes that each $h \in X$ defines a cocharacter $\mu_h : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$. Indeed, one has the following decomposition of $\mathbb{S}_{\mathbb{C}}$ (which holds true more generally for any Weil restriction from a Galois extension):

$$\mathbb{S}_{\mathbb{C}} = (\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}})_{\mathbb{C}} \cong \prod_{\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})} \mathbb{G}_{m,\mathbb{C}} \tag{8.60}$$

We then let μ_h be the composition $\mathbb{G}_{m,\mathbb{C}} \hookrightarrow \mathbb{S}_{\mathbb{C}} \xrightarrow{h} G_{\mathbb{C}}$ where the $\mathbb{G}_{m,\mathbb{C}}$ is the factor (8.60) corresponding to $\mathrm{id} \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Remark 8.3.17. This cocharacter μ_h is such that for any real representation $\rho : G_{\mathbb{R}} \to \mathrm{GL}(V)$ endowing V with a pure $\mathbb{R}$-Hodge structure, the composition $\rho_{\mathbb{C}} \circ \mu_h$ is the cocharacter of $\mathrm{GL}(V_{\mathbb{C}})$ giving the grading of $V_{\mathbb{C}}$ corresponding to the Hodge filtration on $V_{\mathbb{C}}$.

We now seek to define the reflex field of a Shimura datum (G, X), denoted $E(G, X)$, as follows. For all fields $F \supseteq \mathbb{Q}$ let us denote by $C(F)$ the set

$$\mathrm{Hom}(\mathbb{G}_{m,F}, G_F)/G(F) \tag{8.61}$$

where the right action of $G(F)$ is by postcomposing any homomorphism $\mathbb{G}_{m,F} \to G_F$ with conjugation by an element of $G(F)$. Note that for any $\sigma \in \mathrm{Aut}(F/\mathbb{Q})$ we get an action of σ on $\mathrm{Hom}(\mathbb{G}_{m,F}, G_F)$ defined by

$$\sigma \cdot \mu := \sigma_{G_F} \circ \mu \circ \sigma_{\mathbb{G}_{m,F}}^{-1} \tag{8.62}$$

where σ_{G_F} and $\sigma_{\mathbb{G}_{m,F}}$ are the usual morphisms on G_F and $\mathbb{G}_{m,F}$. Note then that this actually descends to an action of $\mathrm{Aut}(F/\mathbb{Q})$ on $C(F)$ since if $\mu = g\mu g^{-1}$ then $\sigma \cdot \mu = \sigma(g)(\sigma \cdot \mu')\sigma(g)^{-1}$.

We have the following elementary observation:

Lemma 8.3.18. *Let $F \subseteq F'$ be an extension of algebraically closed fields containing $\mathbb{Q}$. Then, the natural map $C(F) \to C(F')$ is a bijection.*

Proof. Let us fix a maximal torus T of G_F, so that $T_{F'}$ is a maximal torus of $G_{F'}$. Note that since every cocharacter to G_F takes value in some maximal torus in G_F that up to $G(F)$-conjugation, we have that every cocharacter to G_F takes values in T. Moreover, two cocharacters with values in T are conjugate if and only if, by definition, they are conjugate by an element of $W_T(F)$ where $W_T := N_G(T)/T$ is the Weyl group of T. Thus, in conclusion, we obtain a natural bijection

$$C(F) \xrightarrow{\approx} \mathrm{Hom}(\mathbb{G}_{m,F}, T)/W_T(F) \tag{8.63}$$

The exact same argument shows that we have a bijection

$$C(F') \xrightarrow{\approx} \mathrm{Hom}(\mathbb{G}_{m,F'}, T_{F'})/W_{T_{F'}}(F') \tag{8.64}$$

The map $C(F) \to C(F')$ is, under these bijections, precisely the natural map

$$\mathrm{Hom}(\mathbb{G}_{m,F}, T)/W_T(F) \to \mathrm{Hom}(\mathbb{G}_{m,F'}, T_{F'})/W_{T_{F'}}(F') \tag{8.65}$$

That said, W_T is a finite group scheme over F and thus, since F is algebraically closed, is constant. Since $W_{T_{F'}}$ is evidently $(W_T)_F$ it follows that the map $W_T(F) \to W_{T_{F'}}(F')$ is a bijection. Moreover, since T is split, the natural map

$$\mathrm{Hom}(\mathbb{G}_{m,F}, T) \to \mathrm{Hom}(\mathbb{G}_{m,F'}, T_{F'}) \tag{8.66}$$

is a bijection. Thus, it's easy to conclude that (8.65) is a bijection, and thus so is $C(F) \to C(F')$. □

So, let us begin by noting that the set of cocharacters $\{\mu_h\}_{h\in X}$ are conjugate cocharacters to $G_{\mathbb{C}}$ and so give a well-defined element c of $C(\mathbb{C})$. But, by Lemma 8.3.18 it defines an element of $C(\overline{\mathbb{Q}})$, which we also denote by c. We then define the *reflex field* of (G, X) to be the fixed field of $c \in C(\overline{\mathbb{Q}})$ under the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action (which is necessarily a number field). We denote the reflex field by $E(G, X)$.

Remark 8.3.19. One needs to be careful in noting that the functorial association $F \mapsto C(F)$ is *not* a sheaf on the étale site of $\mathbb{Q}$! In other words, it is not true that if $F_1 \subseteq F_2$ are algebraic extensions of $\mathbb{Q}$, that one has an equality

$$C(F_1) = C(F_2)^{\mathrm{Aut}(F_2/F_1)} \tag{8.67}$$

In conrete terms, let us say that an element $c \in C(\overline{\mathbb{Q}})$ is *defined* over F if $\mathrm{Aut}(\overline{\mathbb{Q}}/F)$ acts trivially on c. Then, it is not true that a conjugacy class being defined over F means that some element of the conjugacy class is defined over F, but the converse is obviously true. So, the reflex field is not the smallest number field such that some element of $\{\mu_h\}$ is defined over that number field.

That said, it is an observation of Kottwitz (see [Kot84, Lemma 1.1.3]) that this is true if G happens to be quasi-split (i.e. has a Borel defined over $\mathbb{Q}$). Essentially one can mimic the proof of Lemma 8.3.18 with T replaced by any maximal torus over F. Thus, when G is quasi-split the reflex field of a Shimura datum (G, X) really is just the minimal field of definition for some element μ_h for $h \in X$.

Deligne in [Del71b, §3] defined the notion of what it meant for a model of the scheme $\mathsf{Sh}_K(G, X)_{\mathbb{C}}$ over a field $E \supseteq E(G, X)$ to be *canonical*, and showed that a canonical model, if it exists, is unique up to unique isomorphism. We won't define canonical models rigorously here referring the reader instead to loc. cit., [Mil04, Section 12], or [Moo98, §2] for a thorough discussion.

That said, let us informally summarize the idea as follows. One first notes that if (T, X) is a Shimura datum where T is a torus, then X is a singleton and $\mathsf{Sh}_K(T, X)_{\mathbb{C}}$ is a finite set for any $K \in \mathcal{N}(T)$. So, to give a model of $\mathsf{Sh}_K(T, X)_{\mathbb{C}}$ over $E(T, X)$ is to give an action of $\mathrm{Aut}(\mathbb{C}/E(T, X))$ intertwining the action on the structure map to $\mathrm{Spec}(\mathbb{C})$. One give this action explicitly using class field theory (e.g. think about the natural action one gets when T is split!). For general (G, X) one defines a point $h \in X$ to be *special* if h factors through $T_{\mathbb{R}}$ for some rational torus $T \subseteq G$. One then defines a model to be canonical if for all special points, the point is defined over $E(G, X)^{\mathrm{ab}}$ and that the action of the Galois group on this special point is a (modification) of the action one defined above using class field theory.

A surprising and somewhat difficult fact is that there exists enough special points on a Shimura variety to make the notion of canonical model unique. In other words, one needs to show that there is a density of special points on any Shimura variety. In even more specific terms, one needs to show that for a reductive group over $\mathbb{Q}$ the conjugacy $G(\mathbb{R})$-conjugacy classes of tori in $G_{\mathbb{R}}$ defined over $\mathbb{Q}$ is large in some sense. This is [Del71b, §5.1].

Remark 8.3.20. There is a well-known 'sign error' in Deligne's original article [Del71b] concerning this explicit action for tori. See the discussion in [Mil04] for a discussion of this.

We then have the following result due to many people including, notably, Borovoi, Deligne, and Milne:

Theorem 8.3.21. *For any Shimura datum (G, X) and any $K \in \mathcal{N}(G)$ a canonical model of $\mathsf{Sh}_K(G, X)_{\mathbb{C}}$ over $E(G, X)$ exists.*

Proof. See [Moo98, §2] for a discussion of the literature concerning this result, as well as a fix to a known error in the literature. □

We denote the canonical model of $\mathsf{Sh}_K(G,X)_\mathbb{C}$ over $E(G,X)$ by $\mathsf{Sh}_K(G,X)$. We then have the following result which shows us that the theory of $G(\mathbb{A}_f)$-schemes applies well to Shimura varieties:

Lemma 8.3.22. *The morphisms*

$$t_{K,L}(g)_\mathbb{C} : \mathsf{Sh}_K(G,X)_\mathbb{C} \to \mathsf{Sh}_L(G,X)_\mathbb{C} \tag{8.68}$$

(for $K, L \in \mathcal{N}(G)$ and $g \in G(\mathbb{A}_f)$ satisfying $L \supseteq g^{-1}Kg$) have models

$$t_{K,L}(g) : \mathsf{Sh}_K(G,X) \to \mathsf{Sh}_L(G,X) \tag{8.69}$$

over $E(G,X)$. Moreover, the system $(\mathcal{N}(G), \{\mathsf{Sh}_K(G,X)\}_{K\in\mathcal{N}(G)}, \{t_{K,L}(g)\})$ is a model of the $G(\mathbb{A}_f)$-scheme $\{\mathsf{Sh}_K(G,X)_\mathbb{C}\}$ over $E(G,X)$. This model is Shimura like relative to $Z = Z(G)(\mathbb{Q})$ for any (G,X) and, if (G,X) satisfies SV5 from [Mil04], also relative to Z the trivial group.

Proof. This follows essentially from [Del71b, Corollaire 5.7]. The only thing to remark is why we can take Z the same as over $\mathbb{C}$. But, this is obvious. Namely, if $K \trianglelefteq K'$ then we still have a surjection

$$K'/K \to \mathrm{Gal}(\mathsf{Sh}_K(G,X)/\mathsf{Sh}_{K'}(G,X)) \tag{8.70}$$

Moreover, it's clear (e.g. by flat descent) that $k' \in K'/K$ acts trivially on $\mathsf{Sh}_K(G,X)$ if and only if it acts trivially on $\mathsf{Sh}_K(G,X)_\mathbb{C}$ from where the conclusion follows. □

So, if (G,X) is a Shimura datum with reflex field $E(G,X)$ we call the $E(G,X)$ scheme $\mathsf{Sh}_K(G,X)$ for any $K \in \mathcal{N}(G)$ the *Shimura variety of level K* associated to (G,X). We shall call the space at infinite level associated to the $G(\mathbb{A}_f)$-scheme $\{\mathsf{Sh}_K(G,X)\}_{K\in\mathcal{N}(G)}$ the *Shimura variety* associated to the datum (G,X) and denote it $\mathsf{Sh}(G,X)$. We shall often conflate the Shimura variety $\mathsf{Sh}(G,X)$ and the system $\{\mathsf{Sh}_K(G,X)\}_{K\in\mathcal{N}(G)}$ and call both the Shimura variety. This is permissible by Lemma 7.1.10.

If S_0 is any affine $E(G,X)$-scheme, then can pull back the Shimura variety $\{\mathsf{Sh}_K(G,X)\}_{K\in\mathcal{N}}$ along $S_0 \to \mathrm{Spec}(E(G,X))$ to obtain a system $\{\mathsf{Sh}_K(G,X)_{S_0}\}_{K\in\mathcal{N}}$ which by Lemma 7.1.24 is actually a Shimura like $G(\mathbb{A}_f)$-scheme over S_0 relative to Z to $Z(G)(\mathbb{Q})$ (or the trivial group if (G,X) satisfies SV5). The space at infinite level is then $\mathsf{Sh}(G,X)_{S_0}$ with the induced right $G(\mathbb{A}_f)$-action.

The last information that we shall need concerning Shimura varieties is the notion of how morphisms of Shimura datum relate to morphisms of their associated Shimura varieties. Namely, let us define a *morphism* of Shimura datum $f : (G_1,X_1) \to (G_2,X_2)$ to be a morphism of algebraic groups $f : G_1 \to G_2$ such that for all $h \in X_1$ one has that $f \circ h \in X_2$.

We then have the fundamental observation of Deligne:

Lemma 8.3.23. *Let $f:(G_1,X_1)\to(G_2,X_2)$ be a morphism of Shimura data. If $K_i\in\mathcal{N}(G_i)$ and $f(K_1)\subseteq K_2$ then the natural map*

$$f(K_1,K_2)^{\mathrm{an}}_{\mathbb{C}}:\mathsf{Sh}_{K_1}(G_1,X_1)^{\mathrm{an}}_{\mathbb{C}}\to\mathsf{Sh}_{K_2}(G_2,X_2)^{\mathrm{an}}_{\mathbb{C}}\tag{8.71}$$

given by

$$f(K_1,K_2)^{\mathrm{an}}_{\mathbb{C}}([h,g]_{K_1})=[f\circ h,f(g)]_{K_2}\tag{8.72}$$

has a unique algebraic model over $E:=E(G_1,X_1)E(G_2,X_2)$ (the compositum). In other words, there exists a unique morphism of E-schemes

$$f(K_1,K_2):\mathsf{Sh}_{K_1}(G_1,X_1)_E\to\mathsf{Sh}_{K_2}(G_2,X_2)_E\tag{8.73}$$

such that $f(K_1,K_2)^{\mathrm{an}}_{\mathbb{C}}$ agrees with (8.71).

Proof. This follows by checking that the morphism $f(K_1,K_2)^{\mathrm{an}}_{\mathbb{C}}$ is equivariant with respect to the $\mathrm{Aut}(\mathbb{C}/E(G_1,X_1))$-action defining the canonical models. See [Mil04, Theorem 13.6] for details. □

As a corollary of this, we deduce the following:

Corollary 8.3.24. *Let $f:(G_1,X_1)\to(G_2,X_2)$ be a morphism of Shimura daitum. Let $j:\mathcal{N}(G_2)\to\mathcal{N}(G_1)$ be any map as in the definition of morphisms of* **G***-schemes in §7.1. Then, for each $K_2\in\mathcal{N}(G_2)$ consider the map $f(j(K_2),K_2)$ from Lemma 8.3.23. Then, the system $\{f(j(K_2),K_2)\}$ is a morphism*

$$\{\mathsf{Sh}_{K_1}(G_1,X_1)\}_{K_1\in\mathcal{N}(G_1)_E}\to\{\mathsf{Sh}_{K_2}(G_2,X_2)_E\}_{K_2\in\mathcal{N}(G_2)}\tag{8.74}$$

relative to $f:G_1(\mathbb{A}_f)\to G_2(\mathbb{A}_f)$.

Moreover, Deligne observed that these morphisms were compatible with the notion of closed embedding. Namely, let us call a morphism $f:(G_1,X_1)\to(G_2,X_2)$ of Shimura datum a *closed embedding* if the map $f:G_1\to G_2$ is a closed embedding of algebraic groups over $\mathbb{Q}$.

We have the following elementary, simple observation concerning closed embeddings of Shimura datum:

Lemma 8.3.25. *Let $f:(G_1,X_1)\to(G_2,X_2)$ be morphism of Shimura datum. Then, $E(G_1,X_1)\supseteq E(G_2,X_2)$.*

Then, Deligne proved the following:

Theorem 8.3.26. *Let $f:(G_1,X_1)\to(G_2,X_2)$ be a closed embedding of Shimura datum. Then, the induced map $f:\mathsf{Sh}(G_1,X_1)\to\mathsf{Sh}(G_2,X_2)$ is a closed embedding of $E(G_1,X_1)$ which is equivariant for the $G_1(\mathbb{A}_f)$-action on both sides.*

Proof. This is [Del71b, Proposition 1.15]. □

8.4 Mumford model for Siegel modular varieties

We would now like to explain how one can modify the construction in §8.2 to give an integral canonical model for a natural subsystem (to be defined later) of a Shimura variety associated to the symplectic similitude group $G(\psi)$ of rational symplectic space (V,ψ). Before we discuss integral canonical models though, let us explicitly define this Shimura variety and relate it to the Siegel modular varieties from §8.2.

So, let us fix, for the rest of this section, a symplectic space (V,ψ) over $\mathbb{Q}$ of dimension $2n$. We then define the *Siegel upper and lower half spaces* associated to (V,ψ), denoted $\mathfrak{h}_\psi^\pm$ to be the set of all maps of algebraic $\mathbb{R}$-groups $\mathbb{S}\to G(\psi)_\mathbb{R}$ such that the following two conditions hold:

1. Under the tautological embedding $G(\psi)\hookrightarrow \mathrm{GL}(V)$, h defines a Hodge structure on V of type $\{(-1,0),(0,-1)\}$.
2. The bilinear form $V_\mathbb{R}\times V_\mathbb{R}\to\mathbb{R}$ defined by $(x,y)\mapsto \psi(x,h(i)y)$ is definite (i.e. positive or negative definite).

Equivalently, in the parlance of [Mil04, page 28], the two conditions (1) and (2) above are equivalent to saying that h defines an $\mathbb{R}$-Hodge structure on V such that $\pm\psi$ is a polarization.

Let us fix a *symplectic basis* of (V,ψ) by which we mean a vector basis $e_1,\ldots,e_n,f_1,\ldots,f_n$ of V such that

$$\psi(e_i,e_j)=0=\psi(f_i,f_j),\quad \psi(e_i,f_j)=\delta_{ij},\qquad \text{for all } i,j=1,\ldots,n \tag{8.75}$$

Define $J_0\in\mathrm{GL}(V)$ by the equation

$$J_0(e_i)=f_i,\;\; J_0(f_i)=-e_i \tag{8.76}$$

and define $h_0:\mathbb{S}\to G(\psi)_\mathbb{R}$ to be such that on $\mathbb{R}$-points it corresponds to the map $\mathbb{C}^\times\to G(\psi)(\mathbb{R})$ taking $a+bi$ to $a+bJ_0$. One can check that h_0 does have image in $G(\psi)$ and that X really is the left $G(\mathbb{R})$-conjugacy class of h_0 (e.g. see [Mil04, Chapter 6]). Moreover, one can show(G,X) is a Shimura datum satifying axioms SV1-SV6 (e.g. loc. cit.). We say a Shimura datum of the form $(G(\psi),\mathfrak{h}_\psi^\pm)$ for some symplectic space (V,ψ) is of *Siegel type.*

To try to understand the Shimura variety associated to a Shimura datum of Siegel type, let us begin by noting that the reflex field of such a datum is $\mathbb{Q}$. Indeed, for our fixed symplectic space as in the last paragraph, let $V_1\subseteq V$ be the span of the e_i's and V_2 the span of the f_j's. Then, one can easily show that μ_{h_0} is the cocharacter of $G(\psi)_\mathbb{C}$ whose composition with the natural embedding $G(\psi)_C\hookrightarrow \mathrm{GL}(V_\mathbb{C})$ with weight decomposition $V_\mathbb{C}$ as $(V_1)_\mathbb{C}\oplus(V_2)_\mathbb{C}$ where $(V_1)_\mathbb{C}$ has weight 1, and $V_\mathbb{C}$ has weight 0. This is evidently defined over $\mathbb{Q}$ and thus (see Remark 8.3.19) this implies that $E(G(\psi),\mathfrak{h}_\psi^\pm)=\mathbb{Q}$. So, by Lemma 8.3.22 we get a $G(\psi)(\mathbb{A}_f)$-scheme $\{\mathsf{Sh}_K(G(\psi),\mathfrak{h}_\psi^\pm)\}_{K\in\mathcal{N}(G(\psi))}$ over $\mathbb{Q}$.

We then have the following deep, foundational result:

Theorem 8.4.1. *The $G(\psi)(\mathbb{A}_f)$-scheme $\{\mathsf{Sh}_K(G(\psi))\}_{K\in\mathcal{N}(G(\psi))}$ and the $G(\psi)(\mathbb{A}_f)$-scheme $\{Y(K)\}_{K\in\mathcal{N}(G(\psi))}$ (from §8.2) are isomorphic.*

Proof. The content of this statement follows, essentially, from the discussion in [Mil11] or [Lan12, Section 2.3].

We give a rough, intuitive idea of the steps though. One first needs to show that the $\mathbb{C}$-schemes $\mathsf{Sh}_K(G(\psi),\mathfrak{h}_\psi^\pm)_\mathbb{C}$ and $Y(K)_\mathbb{C}$ are isomorphic. First one relativizes the discussion in [Mil04, page 70] to get an understanding of the points of $\mathsf{Sh}_K(G(\psi),\mathfrak{h}_\psi^\pm)_\mathbb{C}^{\mathrm{an}}$ for any smooth analytic space in terms of polarized variations of Hodge structure of type $\{(-1,0),(0,-1)\}$.

One then uses this together with Theorem 8.3.8 to understand what the points $\mathsf{Sh}_K(G(\psi),\mathfrak{h}_\psi^\pm)_\mathbb{C}$ look like for smooth complex varieties (which characterizes $\mathsf{Sh}_K(G(\psi),\mathfrak{h}_\psi^\pm)_\mathbb{C}$ by Yoneda's lemma). One then uses [Del71a, Rappel 4.4.3] to show that $\mathsf{Sh}_K(G(\psi),\mathfrak{h}_\psi^\pm)_\mathbb{C}$ represent the same functor as $Y(K)_\mathbb{C}$ on smooth $\mathbb{C}$-schemes. A more careful analysis then shows that, in fact, the $G(\psi)(\mathbb{A}_f)$-scheme $\{\mathsf{Sh}_K(G(\psi))_\mathbb{C}\}_{K\in\mathcal{N}(G(\psi))}$ and the $G(\psi)(\mathbb{A}_f)$-scheme scheme $\{Y(K)_\mathbb{C}\}_{K\in\mathcal{N}(G(\psi))}$ are isomorphic.

Finally, one shows that the $Y(K)$ is a canonical model for the Shimura variety $\mathsf{Sh}_K(G(\psi),\mathfrak{h}_\psi^\pm)_\mathbb{C}$ by using the theory of complex multiplication of abelian varieties (see the discussion starting in [Del71b, §4] or the discussion starting on [Mil04, page 122]). □

So, let us fix a prime p. We now wish to modify the construction in §8.2 to give an examples of an integral canonical model of a $G(\psi)(\mathbb{A}_f^p)$-scheme associated to the $G(\psi)(\mathbb{A}_f)$-scheme $\{Y(K)\}_{K\in\mathcal{N}(G(\psi))}$. To do this, we fix to fix a model $(V_\mathbb{Z},\psi_\mathbb{Z})$ of (V,ψ) as discussed in §8.2. We assume, without loss of any real generality, that $(V_\mathbb{Z},\psi_\mathbb{Z})$ is of type $(1,\ldots,1)$. We can then consider the subgroup of $G(\psi)(\mathbb{Q}_p)$ given as follows:

$$K_0 := \mathrm{GSp}(V_{\mathbb{Z}_p},\psi_{\mathbb{Z}_p}) \subseteq G(\psi)(\mathbb{Q}_p) \tag{8.77}$$

which is compact open but not neat (as a subgroup of $G(\psi)(\mathbb{A}_f)$).

Remark 8.4.2. We really only need to fix a model of (V,ψ) over $\mathbb{Z}_{(p)}$, but there is no harm in fixing a model over $\mathbb{Z}$.

By the discussion at the end of §7.1 we know that

$$\{\mathsf{Sh}_{K^pK_0}(G(\psi),\mathfrak{h}_\psi^\pm)\}_{K^p\in\mathcal{N}^p(G(\psi))} \tag{8.78}$$

is a smooth Shimura like $G(\psi)(\mathbb{A}_f^p)$-scheme over $\mathbb{Q}$ relative to the trivial subgroup of $Z(G(\mathbb{A}_f^p))$. We would like to find an integral canonical model (in the sense of §7.1) for this $G(\psi)(\mathbb{A}_f^p)$-scheme over $\mathbb{Z}_{(p)}$. We do this by directly altering the moduli description of $Y(K)$ from §8.2 to work over any $\mathbb{Z}_{(p)}$-scheme.

Let us begin by defining some integral (i.e. over $\mathbb{Z}_{(p)}$) analogues of the objects defined in §8.2. So, suppose that T is a $\mathbb{Z}_{(p)}$-scheme, and let A be an abelian scheme over T with structure map f. Then, we define a lisse $\mathbb{A}_f^p$-sheaf on T as follows:

$$V^p(A) := \prod_{\ell\neq p}{}' (R^1f_*\mathbb{Q}_\ell)^\vee \tag{8.79}$$

where the restricted direct product is taken with respect to $(R^1f_*\mathbb{Z}_\ell)^\vee$ contained in $(R^1f_*\mathbb{Q}_\ell)^\vee$. So, if T is a $\mathbb{Q}$-scheme, then we have the relationship

$$V(A) = V^p(A) \times (R^1f_*\mathbb{Q}_p)^\vee \tag{8.80}$$

where $V(A)$ is as in §8.2.

Just as in §8.2 We can give more concrete descriptions of $V^p(A)$ in when T is connected in terms of the fundamental group. Namely, if T is connected and $\bar{t}$ is a geometric point of T, then we can describe this $\mathbb{A}_f^p$-sheaf as the continuous $\pi_1^{\text{ét}}(T,\bar{t})$-module $V^p(A_{\bar{t}})$ where $V^p(A_{\bar{t}})$ is the $\mathbb{A}_f^p$-Tate module

$$V^p(A_{\bar{t}}) := \prod_{\ell \neq p}{}' V_\ell(A_{\bar{t}}) \tag{8.81}$$

where $V_\ell(A_{\bar{t}})$ is the rational ℓ-adic Tate module of $A_{\bar{t}}$ and the restricted direct product is taken with respect to the integral ℓ-adic Tate modules $T_\ell(A_{\bar{t}}) \subseteq V_\ell(A_{\bar{t}})$. So, again as in §8.2, if T is integral and normal with geometric generic point $\overline{\eta}$ then we can think of $V^p(A)$ as corresponding to the Galois representation

$$\mathrm{Gal}(\overline{\eta}/\eta) \to \mathrm{GL}_{\mathbb{A}_f}(V^p(A_{\overline{\eta}})) = \mathrm{GL}_{2n}(\mathbb{A}_f^p) \tag{8.82}$$

(where the last equality is merely as topological groups) which, of course, factors through the quotient $\mathrm{Gal}(\overline{\eta}/\eta) \twoheadrightarrow \pi_1^{\text{ét}}(T,\overline{\eta})$.

Let us now define the $\mathbb{Z}_{(p)}$-*category of abelian schemes over* T, denoted $\mathsf{AV}_{\mathbb{Z}_{(p)}}(T)$ as follows. The objects of $\mathsf{AV}_{\mathbb{Z}_{(p)}}(T)$ are just the abelian schemes over T, and a morphism $A_1 \to A_2$ in $\mathsf{AV}_{\mathbb{Z}_{(p)}}(T)$ is a global section of the fppf sheaf $\underline{\mathrm{Hom}}(A_1,A_2)\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$. Again, if T has connected components $\{T_\alpha\}$ such an object just corresponds to an object of the form $a_\alpha\varphi_\alpha$ over each T_α where $\varphi_\alpha : (A_1)_{T_\alpha} \to (A_2)_{T_\alpha}$ is a morphism of abelian schemes, and $a_\alpha \in \mathbb{Z}_{(p)}$.

We define a $\mathbb{Z}_{(p)}$-*isogeny* $A_1 \to A_2$, for A_1 and A_2 abelian schemes over T, to be a morphism in $\mathsf{AV}_{\mathbb{Z}_{(p)}}(T)$ such that for every connected component T_α of T one has that $\varphi\,|_{T_\alpha} = a_\alpha\varphi_\alpha$ for some $a_\alpha \in \mathbb{Z}_{(p)}^\times$ and $\varphi_\alpha : (A_1)_{T_\alpha} \to (A_2)_{T_\alpha}$ an isogeny of abelian schemes of degree prime-to-p. We then define a $\mathbb{Z}_{(p)}$-*polarization* of an abelian scheme A over T to be a $\mathbb{Z}_{(p)}$-isogeny $\varphi : A \to A^\vee$ such that for every connected component T_α of T one has that $\varphi\,|_{T_\alpha} = a_\alpha\varphi_\alpha$ for $a_\alpha \in \mathbb{Z}_{(p)}^\times$ and $\varphi_\alpha : A \to A^\vee$ a polarization of degree prime-to-p. We call a pair (A,φ) of an abelian scheme over T and a $\mathbb{Z}_{(p)}$-polarization φ a $\mathbb{Z}_{(p)}$-*polarized scheme* over T.

Note that if T is a $\mathbb{Z}_{(p)}$-scheme and $\varphi : A_1 \to A_2$ is a a $\mathbb{Z}_{(p)}$-isogeny, then the induced map $V^p(\varphi) : V^p(A_1) \to V^p(A_2)$ is an isomorphism. Thus, in the same way we define the adapted Weil pairing in §8.2 we can define a φ-*adapted Weil pairing* on $V^p(A)$ for any $\mathbb{Z}_{(p)}$-polarized abelian scheme (A,φ). This is a pairing $\psi_\varphi : V^p(A)\times V^p(A) \to \mathbb{A}_f^p(1)$ obtained in the same way as the normal adapted Weil pairing from §8.2.

So, let us now assume that K^p is a compact open subgroup of $G(\psi)(\mathbb{A}_f^p)$, T is a $\mathbb{Z}_{(p)}$-scheme, and (A,φ) is a $\mathbb{Z}_{(p)}$-polarized abelian scheme over T. We define a sheaf $\underline{\mathrm{Isom}}((V_{\mathbb{A}_f^p},\psi_{\mathbb{A}_f^p}),(V^p(A),\psi_\varphi))$ on the étale site of T in the analogous way we did in §8.2, by saying that its sections over a T-scheme U are the isomorphisms of $V_{\mathbb{A}_f^p} \xrightarrow{\approx} V^p(A_U)$ that are symplectic similitudes relative to some (equivalently any) trivialization $\mathbb{A}_f^p(1) \xrightarrow{\approx} \mathbb{A}_f^p$. This sheaf inherits a right action of K^p by precomposition when K^p is thought of as symplectic automorphisms of the sheaf $V_{\mathbb{A}_f^p}$ with the pairing $\psi_{\mathbb{A}_f^p}$. Let us denote by $\underline{\mathrm{Isom}}((V_{\mathbb{A}_f^p},\psi_{\mathbb{A}_f^p}),(V^p(A),\psi_\varphi))/K^p$ the quotient sheaf.

We can describe sections of this quotient sheaf over connected T in much the same way we did in §8.2:

Lemma 8.4.3. *Let T be a connected $\mathbb{Z}_{(p)}$-scheme and $\bar{t}$ a geometric point of T. Let $(U,\overline{u}) \to (T,\bar{t})$ be a pointed connected étale morphism. Then, there is a functorial (in the data $(U,\overline{u})$) identification of the set of sections $\underline{\mathrm{Isom}}((V_{\mathbb{A}_f^p},\psi_{\mathbb{A}_f^p}),(V^p(A),\psi_\varphi))(U)$ with the set of right K^p-orbits of isomorphisms $\lambda : V_{\mathbb{A}_f} \xrightarrow{\approx} V(A_{\overline{u}})$ which are symplectic with respect to some (equivalently any) trivialization $\mathbb{A}_f^p(1) \xrightarrow{\approx} \mathbb{A}_f^p$ and such that the right K^p-orbit λK^p is left stable under the action of $\pi_1^{\text{ét}}(U,\overline{u})$.*

So, let us fix a compact open subgroup K^p of $G(\psi)(\mathbb{A}_f^p)$. Consider the functor

$$\mathscr{S}_{K^p}(G(\psi),\mathfrak{h}_\psi^\pm) : (\mathsf{Sch}/\mathbb{Z}_{(p)})^{\mathrm{op}} \to \mathsf{Set} \tag{8.83}$$

defined as

$$\mathscr{S}_{K^p}(G(\psi),\mathfrak{h}_\psi^\pm)(T) = \{(A,\varphi,\lambda K^p)\}\,/\sim \tag{8.84}$$

where (A,φ) is a $\mathbb{Z}_{(p)}$-polarized scheme over T and λK^p is an element of $\Gamma(T,\underline{\mathrm{Isom}}(V_{\mathbb{A}_f^p},\psi_{\mathbb{A}_f^p}),(V(A),\psi_\varphi))$ and where two pairs $(A,\varphi,\lambda K^p)$ and $(A',\varphi',\lambda' K^p)$ are *equivalent* if there exists a $\mathbb{Z}_{(p)}$-isogeny $A \to A'$ carrying φ to φ', and λK^p to $\lambda' K^p$.

We then have the following result:

Theorem 8.4.4. *For all $K^p \in \mathcal{N}^p(G(\psi))$ the set valued functor $\mathscr{S}_{K^p}(G(\psi),\mathfrak{h}_\psi^\pm)$ is representable by a smooth quasi-projective $\mathbb{Z}_{(p)}$-scheme.*

Proof. This is, in fact, what Mumford showed in the book [MFK94]. So, the same references discussed in the proof of Theorem 8.2.10 apply here. □

Suppose now that $K^p, L^p \in \mathcal{N}^p(G(\psi))$ and $g^p \in G(\psi)(A_f^p)$ are such that $L^p \supseteq (g^p)^{-1}K^p g^p$. Then we can define a morphism

$$t_{K^p,L^p}(g^p) : \mathscr{S}_{K^p}(G(\psi),\mathfrak{h}_\psi^\pm) \to \mathscr{S}_{L^p}(G(\psi),\mathfrak{h}_\psi^\pm) \tag{8.85}$$

given on T-points by sending $(A,\varphi,\lambda K^p)$ to $(A,\varphi,\lambda g^p L^p)$ which, because of the assumption that $L^p \supseteq (g^p)^{-1}K^p g^p$, is well-defined (i.e. independent of representatives).

We then have the following beautiful observation of Milne:

Theorem 8.4.5 (Milne)**.** *The system*

$$\{\mathscr{S}_{K^p}(G(\psi),\mathfrak{h}_\psi^\pm)\}_{K^p\in\mathcal{N}^p(G(\psi))} \tag{8.86}$$

is a Shimura like $G(\psi)(\mathbb{A}_f^p)$-scheme over $\mathrm{Spec}(\mathbb{Z}_{(p)})$ *where we can take Z to be the trivial subgroup. Moreover, its base change to $\mathbb{Z}_p$ an integral canonical model of the $G(\psi)(\mathbb{A}_f^p)$-scheme*

$$\{\mathsf{Sh}_{K^pK_0}(G(\psi),\mathfrak{h}_\psi^\pm)_{\mathbb{Q}_p}\}_{K^p\in\mathcal{N}^p(G(\psi))} \tag{8.87}$$

Before we briefly indicate the idea of the proof of the above theorem, let us mention what the space at infinite level for the system of $\mathscr{S}_{K^p}(G(\psi),\mathfrak{h}_\psi^\pm)$ looks like. This is evidently important since, after all, the Nèron lifting property of a model of $G(\psi)(\mathbb{A}_f^p)$-scheme is a property entirely determined by the scheme-theoretic properties of its space at infinite level.

To do this, let us note that for any $N\geqslant 3$ with $(N,p)=1$, the moduli problems $Y((V_\mathbb{Z},\psi_\mathbb{Z}),N)$ from §8.2 actually make sense for an $\mathbb{Z}_{(p)}$-scheme. In particular, the same method that shows that these functors are representable over $\mathbb{Q}$ and can be identified with $Y(\widehat{\Gamma}_{(V_\mathbb{Z},\psi_\mathbb{Z})}(N))$ also works over the ring $\mathbb{Z}_{(p)}$. Namely, let us denote by $\mathcal{Y}((V_\mathbb{Z},\psi_\mathbb{Z}),N)$ the representing object of the set valued functor $Y((V_\mathbb{Z},\psi_\mathbb{Z}),N)$ naturally extended to $\mathbb{Z}_{(p)}$-schemes.

Define the following subgroups of $G(\psi)(\mathbb{A}_f^p)$ for all $N\geqslant 1$ and $(N,p)=1$:

$$\widehat{\Gamma}^p_{(V_\mathbb{Z},\psi_\mathbb{Z})}(N):=\ker\left(\mathrm{GSp}(V_{\widehat{\mathbb{Z}}^p},\psi_{\widehat{\mathbb{Z}}^p})\to\mathrm{GSp}(V_{\mathbb{Z}/N\mathbb{Z}},\psi_{\mathbb{Z}/N\mathbb{Z}})\right) \tag{8.88}$$

where $\widehat{\mathbb{Z}}^p:=\prod_{\ell\neq p}\mathbb{Z}_\ell$. It's not hard to see that $\widehat{\Gamma}^p_{(V_\mathbb{Z},\psi_\mathbb{Z})}(N)$ is compact open for all $N\geqslant 1$ with $(N,p)=1$ and neat if $N\geqslant 3$.

We then have the following:

Lemma 8.4.6. *The $\mathbb{Z}_{(p)}$-schemes $\mathcal{Y}((V_\mathbb{Z},\psi_\mathbb{Z}),N)$ are smooth and quasi-projective. In fact, they are isomorphic to $\mathscr{S}_{\widehat{\Gamma}^p_{(V_\mathbb{Z},\psi_\mathbb{Z})}(N)}(G(\psi),\mathfrak{h}_\psi^\pm)$.*

Let $\mathscr{S}_p(G(\psi),\mathfrak{h}_\psi^\pm)$ denote the space at infinite level of the $G(\psi)(\mathbb{A}_f)$-scheme $\{\mathscr{S}_{K^p}(G(\psi),\mathfrak{h}_\psi^\pm)\}_{K^p\in\mathcal{N}^p(G$ and $\mathsf{Sh}_p(G(\psi),\mathfrak{h}_\psi^\pm)$ its generic fiber. Since the $\widehat{\Gamma}^p_{(V_\mathbb{Z},\psi_\mathbb{Z})}(N)$ are easily seen to be cofinal in $G(\psi)(\mathbb{A}_f^p)$ we deduce the following.

Corollary 8.4.7. *Let $\mathscr{S}_p(G(\psi),\mathfrak{h}_\psi^\pm)$ denote the space at infinite level of the $G(\psi)(\mathbb{A}_f)$-scheme $\{\mathscr{S}_{K^p}(G(\psi),\mathfrak{h}_\psi^\pm)\}_{K^p\in\mathcal{N}^p(G(\psi))}$. Then, we have the following identification of $\mathbb{Z}_{(p)}$-schemes:*

$$\mathscr{S}_p(G(\psi),\mathfrak{h}_\psi^\pm)=\varprojlim_{\substack{N\geqslant 3\\(N,p)=1}}\mathcal{Y}((V_\mathbb{Z},\psi_\mathbb{Z}),N) \tag{8.89}$$

One can then intuitively think of the statement Theorem 8.4.5, which amounts to the claim that $\mathscr{S}_p(G(\psi),\mathfrak{h}^\pm)_{\mathbb{Z}_p}$ has the extension property, as follows. Suppose that we want to show that

$$\mathrm{Hom}(\mathrm{Spec}(\mathcal{O}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p),\mathsf{Sh}_p(G(\psi),\mathfrak{h}_\psi^\pm)_{\mathbb{Q}_p})\to\mathrm{Hom}(\mathrm{Spec}(\mathcal{O}),\mathscr{S}_p(G(\psi),\mathfrak{h}_\psi^\pm)_{\mathbb{Z}_p}) \tag{8.90}$$

is a bijection for some p-adically complete DVR R over $\mathbb{Z}_p$. Note though that if $F:=\mathrm{Frac}(\mathcal{O})=\mathcal{O}\otimes_{\mathbb{Z}_p}\mathbb{Q}_o$a point of the left hand side would consist of an abelian scheme A_F over F together with compatible $\widehat{\Gamma}^p_{(V_\mathbb{Z},\psi_\mathbb{Z})}(N)$-level structure for all $N\geqslant 3$ with $(N,p)=1$. In particular that this would imply that $A_F[N]$ is the constant group scheme over F for all $N\geqslant 3$ with $(N,p)=1$. In particular, choosing $\ell\neq p$ a prime, we see that the ℓ-adic Tate module $T_\ell(A_F)$ has trivial Galois action and so, in particular, is unramified. Thus, by the Nèron-Ogg-Shafarevich theorem (see [ST68, Section 1]) A_F has an abelian scheme model A over $\mathcal{O}$. It's clear that the level structure also lifts for all $(N,p)=1$ and $N \geqslant 3$ and so we do, in fact, get a point of the right hand side.

One can then think of Theorem 8.4.5 as the observation that a generalized Nèron-Ogg-Shafarevich's theorem holds for formally smooth and regular schemes over $\mathbb{Z}_p$. This is where the erroneous result of Faltings-Chai (mentioned in Remark 7.1.28 was used) and where the observations of Vasiu, Moonen, and Kisin (from the same remark) came into play.