Title
On Linked Quaternionic Pairings

Author
Wright, Kayla

Publication Date
2018

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA
Santa Barbara

On Linked Quaternionic Pairings

A Thesis submitted in partial satisfaction
of the requirements for the degree of

Master's of Art

in

Mathematics

by

Kayla Wright

Committee in Charge:

Professor William Jacob, Chair

Professor Yitang Zhang

Professor Jon M^{c}Cammond

June 2018

The Thesis of
Kayla Wright is approved:

Professor Yitang Zhang

Professor Jon McCammond

Professor William Jacob, Committee Chairperson

June 2018

On Linked Quaternionic Pairings

Abstract

On Linked Quaternionic Pairings

Kayla Wright

In this paper, we study linked bilinear pairings $G \times G \longrightarrow Q$ and their associated Witt rings. An open problem is to classify all linked quaterionic pairings. The Elementary Type Conjecture [1] asserts that every finite linked quaternionic pairing can be built from symplectic pairings using direct sums and group extensions iteratively. We investigate the validity of this conjecture by studying an infinite quaternionic pairing and its sub-pairings motivated by certain structures arising in Henselian dyadic valued fields.

Contents

1 Introduction

The study of quadratic forms initially emerged from number theory in attempts to answer problems regarding quadratic diophantine equations, equivalence of lattices over $\mathbb{Z}$, and other problems that had been open for many years. In the 19^{th} century, mathematicians began to realize it was easier to solve equations with coefficients over a field rather than a general integral domain. They also realized that solutions to such equations can be interpreted in the associated field of fractions. This idea led Minkowski to develop a general theory of quadratic forms with coefficients in $\mathbb{Q}$.

Minkowski's ideas led German number theorist, Hasse, to explore the subject and investigate its connection to algebraic number theory. He was able to introduce Hensel's p-adic numbers, $\mathbb{Q}_p$, into the theory of quadratic forms.

After the development of this general theory of quadratic forms, an abstract version was developed by Witt. He took the concrete theory of quadratic forms over a field F and introduced a ring structure on isometry classes of anisotropic quadratic forms. Witt was able to recapture the theory of quadratic forms from this point of view, and proved a fundamental theorem in quadratic form theory known as Witt Cancellation. Namely, Witt constructed the commutative ring, known as the Witt ring, whose elements are anisotropic quadratic forms. In addition, Witt carried over Minkowski's work in the case where the field has characteristic not equal to 2.

In this thesis, we will explore the connection between quadratic forms over a field and the abstract Witt ring associated to the group $F^\bullet/F^{\bullet^2}$ for special fields F arising as the Henselization of $\mathbb{Q}_2(i)(t)$. The Elementary Type Conjecture is studied in this context.

2 Background

In this section, we follow the definitions given by Marshall [1].

2.1 Quadratic Forms

Assume that F denotes a field of characteristic not equal to 2. Also assume that $\vec{x} \in F^n$ denotes a row vector. In this subsection, we will define a quadratic form over F and discuss the basic objects used in quadratic form theory.

Definition 2.1.1. *A* ***quadratic form*** $f : F^n \longrightarrow F$ *of dimension* n *over* F *is a function defined by a second degree homogeneous polynomial* $f(\vec{x})$ *in* n *variables over* F*, which has the form*

$$f(\vec{x}) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j \qquad \text{where } a_{ij} \in F.$$

We say that f *is* ***isotropic*** *if there exists some nonzero* $\vec{v} \in F^n$ *such that* $f(\vec{v}) = 0$*. If* f *is not isotropic,* f *is called* ***anisotropic****.*

From another point of view, one can think of a quadratic form f over F through its matrix representation. Let $M_f = (b_{ij})$ be an $n \times n$ matrix whose ij^{th} entry be given by

$$b_{ij} = \begin{cases} a_{ij} & \text{if } i = j \\ \frac{1}{2} a_{ij} & \text{if } i < j \\ \frac{1}{2} a_{ji} & \text{if } i > j \end{cases} \qquad \text{where } \frac{1}{2} \text{ exists because } \mathrm{char}(F) \neq 2$$

Notably, this construction ensures that M_f is symmetric. From this, we have that

$$f(\vec{x}) = \sum_{i,j=1}^{n} b_{ij} x_i x_j = \vec{x} M_f \vec{x}^T$$

We say that f is *degenerate* if $\det(M_f) = 0$. For the remainder of the thesis, we will assume that f is *nondegenerate*.

Using these two notions of a quadratic form, it can be shown that there is a one-to-one correspondence between symmetric $n \times n$ matrices and quadratic forms of dimension n over F, whenever $\mathrm{char}(F) \neq 2$.

Definition 2.1.2. *We say that two quadratic forms f and g of the same dimension n over F are **isometric**, denoted $f \cong g$, if there exists $B \in GL_n(F)$, $n \times n$ invertible matrices over F, such that $g(\vec{x}) = f(\vec{x}B)$, where $\vec{x} = (x_1, \dots, x_n)$ a row vector.*

*Similarly, we say that the matrix representations M_f, M_g of f, g respectively are **similar** when there exists $B \in GL_n(F)$ such that $M_g = BM_fB^T$.*

Remark 2.1.1. *It can be checked using the definitions that M_f is similar to M_g if and only if $f \cong g$.*

Remark 2.1.2. *Isometry of quadratic forms is an equivalence relation.*

Note that if $f \cong g$, then f is isotropic if and only if g is isotropic. This observation motivates the consideration of isometry classes of quadratic forms as opposed to looking at one particular quadratic form.

2.2 Basic Theorems

In this section, we will state some basic results about quadratic forms, see for example (Marshall, 1980) or (Law, 2004), without proof. This first result says that quadratic forms can be diagonalized. Recall we are assuming the characteristic of F is different from two.

Theorem 2.2.1. *Every quadratic form over F is isometric to one of the type $f = a_1x_1^2 + \cdots + a_nx_n^2$, where $a_1, \ldots, a_n$ are nonzero elements of F.*

We will refer to such a representation of f as a **diagonalized** quadratic form. It is denoted $f = \langle a_1, \ldots, a_n \rangle$. Using this notation we have the following:

Proposition 2.2.1. *Let f, g be quadratic forms over F and let $a_i, b_i \in F^\bullet$ (where $F^\bullet$ is the set of units in F) for all $i \in \{1, \ldots, n\}$. Then,*

1. *If $f \cong g$, then $af \cong ag$ for any $a \in F^\bullet$*
2. $\langle a_1b_1^2, \ldots, a_n, b_n^2 \rangle \cong \langle a_1, \ldots, a_n \rangle$
3. *For any $\pi \in S_n$ (symmetric group), $\langle a_{\pi(1)}, \ldots, a_{\pi(n)} \rangle \cong \langle a_1, \ldots, a_n \rangle$*
4. *If $\langle a_1 \ldots, a_k \rangle \cong \langle b_1, \ldots, b_k \rangle$ and $\langle a_{k+1}, \ldots, a_n \rangle \cong \langle b_{k+1}, \ldots, b_n \rangle$, then $\langle a_1, \ldots, a_n \rangle \cong \langle b_1, \ldots, b_n \rangle$.*

The next result characterizes one and two-dimensional quadratic forms up to isometry.

Theorem 2.2.2. *Let $a, b, c, d \in F^\bullet$, then*

1. *$\langle a \rangle \cong \langle b \rangle$ if and only if $a \equiv b \mod F^{\bullet^2}$.*
2. *$\langle a, b \rangle \cong \langle c, d \rangle$ if and only if $ab \equiv cd \mod F^{\bullet^2}$ and there exists $x, y \in F$ such that $c = ax^2 + by^2$.*

Corollary 2.2.1. *For all $a \in F^\bullet$, $\langle a, -a \rangle \cong \langle 1, -1 \rangle$.*

In particular, these above results allow us to establish another characterization of isotropicity for diagonal quadratic forms.

Theorem 2.2.3. *Let f be a quadratic form over F of dimension $n \geq 2$. Then, f is isotropic if and only if there exists $b_3, \ldots, b_n \in F^\bullet$ such that $f \cong \langle 1, -1, b_3, \ldots, b_n \rangle$.*

The next result is key to the development of the algebraic theory of quadratic forms. It is essential to the subsequent definition of the Witt Ring.

Theorem 2.2.4 (Witt's Cancellation Theorem)**.** *Suppose that $\langle a_1, \ldots, a_n \rangle \cong \langle b_1, \ldots, b_n \rangle$ and $a_1 = b_1$, then $\langle a_2, \ldots, a_n \rangle \cong \langle b_2, \ldots, b_n \rangle$.*

This theorem allows us to formulate the following definitions:

Definition 2.2.1. *Let $f = \langle a_1, \ldots, a_n \rangle, g = \langle b_1, \ldots, b_m \rangle$ be quadratic forms with $a_i, b_j \in F$. The* ***direct sum*** *of f and g, denoted $\oplus$, is given by*

$$f \oplus g := \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle$$

The ***tensor product*** *of f and g, denoted $\otimes$, is given by*

$$f \otimes g := \langle a_1 b_1, \ldots, a_1 b_m, \ldots, a_n b_1, \ldots, a_n b_m \rangle$$

And we scale by some $a \in F$ as follows:

$$af := \langle aa_1, \ldots, aa_n \rangle$$

With these two operations, we are able to form the following ring:

Definition 2.2.2. *Let ϕ_i, ψ_i be quadratic forms over F. Then we define the* ***Witt-Grothendeick ring of* F**, *denoted $\hat{W}(F)$, to be formal differences of $\phi - \psi$ with the following notion of equivalence:*

$$\phi_1 - \psi_1 \sim \phi_2 - \psi_2 \iff \phi_1 \oplus \psi_2 \cong \phi_2 \oplus \psi_1$$

Remark 2.2.1. *Note that the notion of formal difference mimics the behavior of definition $\mathbb{Z}$ as an equivalence relation on $\mathbb{N}$. Moreover, $\hat{W}(F)$ is well defined because of Witt Cancellation Theorem.*

With the definition of $\hat{W}(F)$, we can define the Witt ring of a field.

Definition 2.2.3. *The* ***Witt ring of* F**, *denoted WF, is given by $WF = \hat{W}(F)/(\langle 1, -1\rangle)$.*

Definition 2.2.4. *One ideal of WF that is of particular importance later on, is the ideal IF of all even-dimensional forms.*

Remark 2.2.2. *The elements of WF correspond to the classes of anisotropic quadratic forms.*

2.3 Quaternionic Pairings

In this section, we define a quaternionic structure, state some basic results, and explain how the study of quadratic forms can be viewed as the study of quaternionic structures over a field F. Quaternionic pairings enable the development of an abstract theory of quadratic forms without referencing a field.

Recall that an **elementary abelian 2-group** is an abelian group such that every nontrivial element has order 2. Also recall that every elementary abelian 2-group carries a vector space structure over $\mathbb{Z}_2$.

Definition 2.3.1. *Let* $(G,\cdot),(Q,+)$ *be elementary abelian 2-groups, where* $(G,\cdot)$ *has a distinguished element* $-1 \in G$. *A* ***quaternionic pairing*** *is a map* $q : G \times G \longrightarrow Q$ *that satisfies the following four conditions:*

1. *For any* $g_1, g_2 \in G$, $q(g_1,g_2) = q(g_2,g_1)$.

2. *For any* $g_1, g_2, g_3, g_4 \in G$, $q(g_1 \cdot g_3, g_2) = q(g_1,g_2) + q(g_2,g_3)$ *and* $(g_1, g_2 \cdot g_4) = q(g_1,g_2) + q(g_1,g_4)$.

3. *For any* $g_1, g_2 \in G$, $q(1,g_2) = q(g_1,1) = 0$.

4. *For any* $g \in G, q(g,(-1)g) = 0$.

For notational convenience, we for any $a,b \in G$, *we will denote* $q(a,b)$ *as* (a,b). *We also abbreviate the structure above as a 3-tuple* (q,G,Q).

In order to connect quaternionic pairings to quadratic forms, Marshall requires one further axiom of quaternionic pairings called **linkage**.

Definition 2.3.2. *Given a quaternionic pairing* $q : G \times G \longrightarrow Q$, *we say that this pairing is* ***linked*** *if for every* $z \in Q$, *if* $z = (g_1,h_1) = (g_2,h_2)$ *for* $g_1, g_2, h_1, h_2 \in G$, *then there exists an* $\ell \in G$ *such that* $z = (g_1,\ell) = (g_2,\ell)$.

2.3.1 The Quaternionic Pairing Associated to a Field

The addition of the linkage axiom allows us to develop an abstract theory of quadratic forms in terms of quaternionic pairings.

Define G_F to be the quotient $F^\bullet/F^{\bullet^2}$, which is notably an elementary abelian

2-group. Using proposition 2.2.1 (2), we can view quadratic forms over F as n-tuples over F. Now, we define Q_F to be the set of isometry classes $\langle 1, -a, -b, ab\rangle$ with $a, b \in Q_F$. With this, we can define the following:

Definition 2.3.3. *The* ***quaternionic structure associated to F*** *is the map $q_F : G_F \times G_F \longrightarrow Q_F$ where (a, b) is mapped to the isometry class of $\langle 1, -a, -b, ab\rangle$.*

Remark 2.3.1. *The quaternionic structure associated to F, q_F, is a quaternionic pairing that satisfies the linkage property.*

Viewing these quadratic forms as n-tuples of G_F elements, we can use the prior notion of isometry for quadratic forms over fields to define isometry for quadratic forms of a quaternionic pairing.

Definition 2.3.4. *We say that two quadratic forms of dimension one and two are* ***isometric*** *if the following condition is satisfied:*

$$\langle a\rangle \cong \langle b\rangle \iff a = b \quad \text{and} \quad \langle a, b\rangle \cong \langle c, d\rangle \iff q(a, b) = q(c, d)$$

From the above, we can inductively define isometry for forms of dimension $n \geq 3$:

$$\langle a_1, \ldots, a_n\rangle \cong \langle b_1, \ldots, b_n\rangle \iff \exists a, b, c_3, \ldots, c_n$$

such that the following conditions are met:

1. $\langle a_2, \ldots, a_n\rangle \cong \langle a, c_3, \ldots, c_n\rangle$

2. $\langle a_1, a\rangle \cong \langle b_1, b\rangle$

3. $\langle b_2, \ldots, b_n\rangle \cong \langle b, c_3, \ldots, c_n\rangle$

Proposition 2.3.1. *Isometry is an equivalence relation on the set of quadratic forms associated with a linked quaternionic pairing.*

With this notion of isometry, we define the following operations:

Definition 2.3.5. *Let $f = \langle a_1, \dots, a_n \rangle, g = \langle b_1, \dots, b_m \rangle$ be quadratic forms with $a_i, b_j \in G_F$. The **direct sum** of f and g, denoted $\oplus$, is given by*

$$f \oplus g := \langle a_1, \dots, a_n, b_1, \dots, b_m \rangle$$

*The **tensor product** of f and g, denoted $\otimes$, is given by*

$$f \otimes g := \langle a_1 b_1, \dots, a_1 b_m, \dots, a_n b_1, \dots, a_n b_m \rangle$$

And we scale by some $a \in G_F$ as follows:

$$af := \langle aa_1, \dots, aa_n \rangle$$

Given this definition of isometry along with the definitions of the direct sum and tensor product operations, we may state the Witt Cancellation Theorem in this setting. This is essential to constructing a Witt ring associated with a linked quaternionic pairing.

Theorem 2.3.1 (Witt Cancellation Theorem, Marshall's Abstract Version)**.** *Let f, g, g' be arbitrary forms over $q : G \times G \longrightarrow Q$. If $f \oplus g \cong f \oplus g'$, then $g \cong g'$.*

2.4 The Abstract Witt Ring of a Linked Quaternionic Pairing

Let $\tilde{R}$ be the set of isometry classes of quadratic forms over G_F. $\tilde{R}$ equipped with the two binary operations $\oplus, \otimes$, nearly satisfy all the algebraic axioms for a ring. However, $(\tilde{R}, \oplus, \otimes)$ fails to have additive inverses. So, in this section, we remedy this flaw. Analogous to definition 2.2.3, we may define the Witt-Grothendeick ring of a linked quaternionic pairing.

Definition 2.4.1. *Let ϕ_i, ψ_i be quadratic forms over G. Let (q, G, Q) be a linked quaternionic pairing. Then we define the* ***Witt-Grothendeick ring of*** $(\mathbf{q}, \mathbf{G}, \mathbf{Q})$*, denoted $\hat{W}(q, G, Q)$, to be formal differences of $\phi - \psi$ with the following notion of equivalence:*

$$\phi_1 - \psi_1 \sim \phi_2 - \psi_2 \quad \Longleftrightarrow \quad \phi_1 \oplus \psi_2 \cong \phi_2 \oplus \psi_1$$

Remark 2.4.1. *As in remark 2.21, we have $\hat{W}(q, G, Q)$ is well defined because of Witt Cancellation Theorem.*

With the definition of $\hat{W}(q, G, Q)$, we can define the Witt ring of an LQP.

Definition 2.4.2. *The* ***Witt ring of a*** $(\mathbf{q}, \mathbf{G}, \mathbf{Q})$*, denoted $W(q, G, Q)$, is given by $W(q, G, Q) = \hat{W}(q, G, Q)/(\langle 1, -1 \rangle)$.*

We note that the equivalence class of $\langle 1, -1 \rangle$ is the additive identity 0 in our ring, and $\langle 1 \rangle$ is the multiplicative identity. Generally speaking, the additive inverse of the class $f = \langle a_1, \ldots, a_n \rangle$ is $-f = \langle -a_1, \ldots, -a_n \rangle$.

Remark 2.4.2. *Let F be a field and let $F^\bullet$ be the multiplicative group of F. Let $q : F^\bullet / F^{\bullet^2} \longrightarrow I^2F/I^3F$ be the associated linked quaternionic pairing. Then the WF and $W(q, F^\bullet/F^{\bullet^2}, I^2F/I^3F)$ are the same.*

2.5 Elementary Type Conjecture

Definition 2.5.1. *A direct sum of two Witt rings R_1, R_2 is defined by the fiber product i.e.*

$$R_1 \coprod_{\mathbb{Z}/2\mathbb{Z}} R_2 := \{(a, b) \mid a \in R_1, b \in R_2, \dim(a) \equiv \dim(b) \mod 2\}$$

where dim(a), dim(b) are the dimensions of $a \in R_1$, and $b \in R_2$ respectively.

Definition 2.5.2. *Let $\Delta_n = (\mathbb{Z}/2\mathbb{Z})^n$, then a group extension of a Witt ring R is $R[\Delta_n]$, the usual group ring.*

If the Witt ring of a quaternionic pairing can be built from direct sums and group extensions by a finite iteration starting with basic Witt rings $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}$, and the (finite) Witt ring WK of a local fields K where $[K, \mathbb{Q}_p] < \infty$, then we say that the Witt ring is of *elementary type.* Murray Marshall (1980) proposed the following conjecture:

Conjecture 2.5.1. *The Witt ring of every finite linked quaternionic pairing is of elementary type.*

2.6 Previous Work on the Elementary Type Conjecture

Many people have worked on proving this conjecture. So far, it is still an open problem to prove the Elementary Type Conjecture in its full generality, but some have made progress adding different hypotheses to weaken the conjecture. It is known that all known finitely generated Witt rings are of elementary type. It was also proven by A. Carson and M. Marshall in *Decomposition of Witt rings* [8] that the Elementary Type Conjecture holds for $|G| \leq 32$. Another proof of this fact is given in *Combinatorial Techniques and*

Abstract Witt Rings III Fitzgerald [5]. This paper gives a four step outline to proving the Elementary Type Conjecture. When the Witt ring , R, satisfies a property called being "reduced," it is verified by a theorem [4] given in Marshall's survey that states that if G is finite and "reduced", then G is of elementary type. Fitzgerald's paper [5] uses the formulation of Witt rings through linked quaternionic pairings and value sets to investigate the conjecture. When R is reduced, it gives a simpler proof of Marshall's result when $|G| \leq 32$.

In Marshall's survey paper [4], he says that the reduced case was proven using valuation theoretic techniques by L. Bröcker and T. Cavern [4], which shows that people have been working with valuation theory to attack this problem. We will see later that these techniques are relevant to the work done in this thesis.

People also care about Witt rings because they can give us information on field orderings and connections to Galois cohomology. In particular, if the Elementary Type Conjecture is true, then we have a complete classification of fields with $|F^{\bullet}/F^{\bullet^2}| = 2^n$ for any $n \geq 0$ up to quadratic equivalence.

In Marshall's *Classification of Finite Spaces of Orderings Marshall* [9], he solves in affirmative Elementary Type Conjecture for one important class of abstract Witt rings. Namely, this is the class of abstract Witt rings in which the field is called Pythagorean. This means that $|F^{\bullet}/F^{\bullet^2}| < \infty$ and -1 is not the sum of squares, but the sum of squares is a square. This is the opposite extreme to what we have been investigating, but it is particularly interesting because it can give us information about abstract Witt rings of spaces of orderings.

More recent work (2016) on investigating the Elementary Type Conjecture has been done in J.K. Arson's *The Witt group of a discretely valued field* [3]. In this paper, Arson uses of filtrations in the study of Witt rings of Henselian dyadic valued fields. In his paper, he argues that this is an important tool in his description. In particular, Arason has recently used them in a recent paper on the "wild part" of the Witt ring to work on the Elementary Type conjecture. This is very closely related to the approach this thesis takes on the exploration of the validity of the conjecture.

3 Valuations

In order to study the validity of the Elementary Type Conjecture, we will explore some quaterninoic pairings over valued fields.

Valuations can be described in multiple ways. The two descriptions that we work with can be formulated in the following two ways following the definitions given by Efrat [2].

3.1 Valuations as Subrings

We begin with the first notion of a valuation as a subring. Let F be a field.

Definition 3.1.1. *A **valuation ring** on F is a subring O of F such that for every $x \in F^{\bullet}$, at least one of $x, x^{-1} \in O$. The **group of units** is the group of all $x \in F^{\bullet}$ with both $x, x^{-1} \in O$ and is denoted $O^{\bullet}$.*

Proposition 3.1.1. *Every valuation ring is a local ring.*

Note that this is a basic fundamental result of the subject. As the proof is short, we will include it.

Proof. Let O be a valuation ring. In order to show that O is a local ring, we must show that it has a unique maximal ideal. We claim that $\mathfrak{m} = O \setminus O^{\bullet}$ is the unique maximal ideal of O. By definition of $\mathfrak{m}$, every element outside of $\mathfrak{m}$ must be a unit. So, it suffices to prove that $\mathfrak{m}$ is an ideal of O because then it must be maximal. It is clear that given any $r \in O$ and any $a \in \mathfrak{m}$, that $ra \in \mathfrak{m}$. So, we just need to verify that if $a, b \in \mathfrak{m}$, then $a + b \in \mathfrak{m}$. Assume that $a, b \in \mathfrak{m}$ are nonzero elements. Because O is a valuation ring, we have that a/b or b/a must be an element of R. Moreover, this gives that either

$a+b = b(1+a/b)$ or $a+b = a(1+b/a)$ is in $\mathfrak{m}$. Note that $\mathfrak{m}$ is unique because if it is was properly contained in another ideal $I \leq O$, it would contain a unit which means that $I = R$. ■

The above proposition gives us that $\mathfrak{m}$ is the unique maximal ideal of O. Using this fact, consider the following definition:

Definition 3.1.2. *The* ***residue field*** *of O is the field $\bar{F} := O/\mathfrak{m}$.*

Example 3.1. *Let $F = \mathbb{Q}$ and let $p \in \mathbb{Q}$ be prime. Define the valuation ring*

$$O = O_p := \{\frac{p^r n}{m} \in \mathbb{Q} \quad : \quad r \geq 0, n, m \in \mathbb{Z} \ \ \textit{such that} \ \ p \nmid n, p \nmid m\} \cup \{0\}$$

This is known as the p-adic valuation ring on $\mathbb{Q}$. And we see that its unique maximal ideal must be the elements of O with $r \geq 1$. In particular, $\bar{F} = \mathbb{F}_p$.

3.2 Valuations as Homomorphisms

The second formulation of a valuation is the notion of a valuation as a homomorphism from a group onto an ordered abelian group. Let F be a field and let $(\Gamma, \leq)$ be an ordered abelian group.

Definition 3.2.1. *A* ***valuation*** *of F is a group homomorphism $v : F^\bullet \longrightarrow (\Gamma, \leq)$ that satisfies the* ***ultrametric inequality*** *i.e. for any $x, y \in F^\bullet$ such that $x \neq y$, we have that*

$$v(x+y) \geq \min\{v(x), v(y)\}$$

Proposition 3.2.1. *For a valuation v on F, we have the following consequences:*

1. $v(-1) = 0$

2. If $x, y \in F^\bullet$ *such that* $v(x) < v(y)$*, then* $v(x+y) = v(x)$.

3. If $x_1, x_2, \dots x_n \in F^\bullet$ *with distinct valuations, then*

$$v\left(\sum_{i=1}^{n} x_i\right) = \min_{1 \leq i \leq n} v(x_i)$$

4. Let $m \in \mathbb{Z}$ *be greater than 1. If* $x, y \in F^\bullet$ *such that* $x^m - x = y$ *and* $v(y) < 0$*, then* $v(y) = mv(x)$.

Example 3.2. *Let* $p \in \mathbb{Z}$ *be prime. Let* $r \geq 0$ *and* $q \in \mathbb{Z}$ *such that gcd*$(p, q) = 1$. *Then the group homomorphism* $v_p : \mathbb{Q} \longrightarrow \mathbb{Z}$ *given by* $v_p(p^r q) = r$ *is called the* **p−*adic valuation***.

Note that this is clearly connected to the previous example given in the first notion of valuation. The two notions are connected in the following sense:

Proposition 3.2.2. *Let* F *be a field and* $(\Gamma, \leq)$ *be an ordered abelian group. Let* $v : F^\bullet \longrightarrow (\Gamma, \leq)$ *be a valuation on* F*, then*

$$O = \{x \in F^\bullet \ : \ v(x) \geq 0\}$$

is a valuation ring of F.

One particular type of valuation that is of interest is called a discrete valuation.

Definition 3.2.2. *A valuation* v *on a field* F *is called* ***discrete*** *if* $v(F^\bullet) \cong \mathbb{Z}$.

Note that the p-adic valuation on $\mathbb{Q}$ is a discrete valuation with residue field $\mathbb{F}_p$. It is known that the only non-trivial valuations on $\mathbb{Q}$ up to equivalence are the p-adic valuations.

3.3 The Completion of $\mathbb{Q}$ Under v_p

The ability to uniquely extend such valuations depends on properties of our field. In particular, one of the hypotheses needed to extend a valuation is the idea of completeness.

Let $p \in \mathbb{Q}$ be some fixed prime. Then, we will using the p-adic valuation on $\mathbb{Q}$, in order to define a notion of distance with respect to this valuation. For a general set X, recall the following definition:

Definition 3.3.1. *A* ***metric*** *on X is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that for any $x, y, z \in X$, d satisfies*

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ *if and only if* $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(y, z)$

In our case, we use the p-adic valuation on $\mathbb{Q}$ to define the following metric:

Definition 3.3.2. *For any $q_1, q_2 \in \mathbb{Q}$, we define the* **p-*adic metric*** *as $d_p(q_1, q_2) = p^{-v_p(q_1 - q_2)}$.*

Proposition 3.3.1. *The p-adic metric on $\mathbb{Q}$ satisfies all the metric axioms.*

Note that $\mathbb{Q}$ is a metric space with respect to the p-adic metric. Consider its usual Cauchy completion with respect to the p-adic metric, which we denote $\mathbb{Q}_p$. Since the operations on $\mathbb{Q}$ are continuous with respect to the metric, $\mathbb{Q}_p$ becomes a field too by continuity.

This field has nice algebraic structure. Let $\mathbb{Q}_p^{\bullet}$ be the multiplicative group of $\mathbb{Q}_p$ and let $\mathbb{Q}_p^{\bullet^2}$ be the subgroup of squares in $\mathbb{Q}_p^{\bullet}$. Observe the following about the quotient group $\mathbb{Q}_p^{\bullet}/\mathbb{Q}_p^{\bullet^2}$:

Proposition 3.3.2. *If $p > 2$, then $|\mathbb{Q}_p^{\bullet}/\mathbb{Q}_p^{\bullet^2}| = 4$. If $p = 2$, then $|\mathbb{Q}_p^{\bullet}/\mathbb{Q}_p^{\bullet^2}| = 8$.*

The idea of the proof will be sketched here. One can see this because when $p \neq 2$, we can figure out if an element $\alpha \in \mathbb{Q}_p^{\bullet}$ is a square using Hensel's Lemma and Newton's method on the polynomial $f(x) = x^2 - \alpha$.

Theorem 3.3.1. ***Hensel's Lemma.*** *Let K be a complete field with respect to a discrete valuation v. Let O_K be the valuation ring on K and let $\mathfrak{m}_K$ be its unique maximal ideal. Let k be the residue field of O_K given by $k \cong O_K/\mathfrak{m}_K$. Assume that $f \in O_K[x]$ be a polynomial. If $\bar{f}(x) \in k[x]$ has a simple root (i.e. there exists some $k_0 \in K$ such that $\bar{f}(k_0) = 0$ and $\bar{f}'(k_0) \neq 0$), then there exists a unique $a \in O_K$ such that $f(a) = 0$ and $\bar{a} = k_o \in k$.*

In the quadratic case, Hensel's Lemma can be proved using Newton's method. Recall Newton's method is a root-finding algorithm. The method starts with a differentiable function f and the function's derivative f', and an initial guess x_0 for a zero of the f. Newton's method gives a better approximation x_1 given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This process is repeated recursively for any $n \in \mathbb{N}$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

Note that in order to guarantee convergence of our series, we need x_0 to be "close enough" to our new root in question.

Next, we investigate the quotient $\mathbb{Q}_5^\bullet/\mathbb{Q}_5^{\bullet^2}$. First, we investigate $\alpha \in O_5$ such that $v_5(\alpha) = 0$. Note that since we can write any element in O_5 as a power series with coefficients $\{0, 1, 2, 3, 4\}$, so if such an α has 5-adic value 0, we know that $a_0 \in \{1, 2, 3, 4\}$ (else the first nonzero coefficient would be on a higher power). Then, evaluating $v_5(f(\beta)/f'(\beta))$ and observing that it is greater than 0, we can apply Newton's method.

Now, let $\beta = 1$ if $a_0 \in \{1, 4\}$ and let $\beta = 2$ if $a_0 \in \{2, 3\}$. Apply Newton's method to see that $\mathbb{Q}_5^\bullet/\mathbb{Q}_5^{\bullet^2} = \langle 2, 5\rangle = \mathbb{Z}_2/\mathbb{Z} \times \mathbb{Z}_2/\mathbb{Z} \cong \mathbb{F}_5/\mathbb{F}_5^2$.

Note that this process should generalize to any $p \geq 2$.

For the case of $p = 2$, recall that any element α in O_2 can be written as a power series:

$$\alpha = \sum_{i=0}^{\infty} a_i 2^i \qquad\qquad a_i \in \{0, 1\}$$

So, we look at the case where some square has $a_0 = 1$, so we look at

$$(1+a_1\cdot 2+a_2\cdot 2^2+a_3\cdot 2^3)^2 = 1+4(a_1+a_1^2)+8a_2+16(a_1a_2+a_3+a_2^2)+32a_1a_3+64a_2a_3$$

This implies that $O_2^2 \subset 1 + 8O_2$. Furthermore, this gives that $\mathbb{Q}_2^\bullet/(\mathbb{Q}_2^\bullet)^2 = \langle 2, 3, 5\rangle$ which implies that $|\mathbb{Q}_2^\bullet/(\mathbb{Q}_2^\bullet)^2| = 8$.

Corollary 3.3.1. *From here, we may extend $\mathbb{Q}_2$ by adjoining $i = \sqrt{-1}$. Now, we investigate the algebraic structure of quotient groups of $\mathbb{Q}_2(\sqrt{-1})$. We claim that $|\mathbb{Q}_2(\sqrt{-1})^\bullet/(\mathbb{Q}_2(\sqrt{-1})^\bullet)^2| = 16$.*

We can apply information about the structure of $\mathbb{Q}_2$ to see this. Applying Hensel's lemma, given some polynomial, we can lift any factorization of the polynomial to another factorization in a field extension.

To make this process more concrete, consider the following examples:

Example 3.3. *As an example, we demonstrate that $-1 \in \mathbb{Q}_5^{\bullet^2}$.*

Firstly, we know that any element in O_p can be expressed as a power series of the form:

$$\alpha = \sum_{i=0}^{\infty} a_i x^i \qquad\qquad a_i \in \{0, 1, \ldots, p-1\}$$

So, we found that -1 can be expressed as follows:

$$\begin{aligned}
-1 &= 4 - 5 \\
&= 4 + 4 \cdot 5 - 5^2 \\
&= 4 + 4 \cdot 5 + 4 \cdot 5^2 - 5^3 \\
&\vdots \\
&= \sum_{i=0}^{\infty} 4 \cdot 5^i
\end{aligned}$$

To see it's a square, observe that

$$\begin{aligned}
-1 &= 4 + 4 \cdot 5 + 4 \cdot 5^2 + \dots \\
&\approx (2 + 1 \cdot 5)^2 \\
&= 4 + 4 \cdot 5 + 5^2 \\
&\approx (2 + 1 \cdot 5 + 2 \cdot 5^2)^2 \\
&\vdots \\
&= (2 + 1 \cdot 5 + 2 \cdot 5^2 + 1 \cdot 5^3 + 2 \cdot 5^4 + \dots)^2
\end{aligned}$$

We can also use Newton's method on the polynomial $f(x) = x^2 + 1$ using $a_0 = 2$. Note that Newton's method requires that $v_p(\frac{f(a_n)}{f'(a_n)}) > 0$ when $p \geq 2$. Note that we compute $a_1, a_2, \dots$ using this formula:

$$a_1 = 2 - \frac{5}{4} = \frac{3}{4}$$
$$a_2 = \frac{3}{4} - \frac{\frac{25}{16}}{\frac{3}{2}} = \frac{-7}{24}$$

This approximation eventually gives that $\sqrt{-1} \in \mathbb{Q}_5$.

Example 3.4. *Consider* x^2+1 *over* $\mathbb{F}_5$*. Factoring in the base field, we obtain* $x^2+1 = (x+2)(x+3) \mod 5$*. So, there must exist* $u_1, u_2 \in O_5$ *such that* $x^2+1 = (x+u_1)(x+u_2)$ *such that* $u_1 \equiv 2 \mod 5$ *and* $u_2 \equiv 3 \mod 5$*.*

This example motivates the idea of extending valuations.

3.4 Extending Valuations

Theorem 3.4.1. *Let* F *be a field, let* Γ *be an ordered abelian group and let* $v : F \longrightarrow \Gamma$ *be a valuation on* F*. Let* L *be a field extension of* F*. Then there exists some* $\Gamma' \supseteq \Gamma$ *and* $v' : L \longrightarrow \Gamma'$ *an extension of our valuation* v*. We can only say that this is unique (up to isomorphism) when* F *is complete with respect to the given valuation.*

Corollary 3.4.1. *The p-adic valuation extends uniquely to* $\mathbb{Q}_p(\sqrt{-1})$*.*

Now, we look at $\mathbb{Q}_2(\sqrt{-1})$ and establish some notation. Let $i = \sqrt{-1}$. Let $\pi = 1 + i$. Note that it is advantageous to adjoin i as $-1 \notin \mathbb{Q}_2^2$. So, now look at $(1+i)^2$. We see that $(1+i)^2 = 1 + 2i - 1 = 2i$. Now, we want to know what the value of π is. Note that

$$\pi^2 = 2i = 2(\pi - 1)$$

So, we look at the values of this factorization coupled with the ultrametric inequality:

$$\begin{array}{rll}
v(2) = 1 & v(-1) = 0 & v(\pi - 1) \geq 0 \\
v(\pi) \geq 0 & v(\pi) \geq 1 & v(\pi^2) \geq 0 \\
v(\pi - 1) = 0 & v(\pi^2) = v(2(\pi - 1)) = 1 &
\end{array}$$

This gives that $v(\pi) = 1/2$. So, in particular, we have that:

$$\begin{tikzcd}
\mathbb{Q}_2(i) \arrow[r, "v"] & \frac{1}{2}\mathbb{Z} \subset \mathbb{Q} \\
\mathbb{Q}_2 \arrow[u] \arrow[r, "v_2"] & \mathbb{Z} \arrow[u]
\end{tikzcd}$$

This gives that

$$O_{\mathbb{Q}_2(i)} = \{\sum_{i=0}^{\infty} a_i \pi^i \ \mid \ a_i \in \{0, 1\}\}$$

After this extension of our valuation, we can further extend our valuation to functions in some variable t.

3.5 Extending the Valuation to Rational Functions

Let t be a variable and and define $v(t) = 0$. Consider a polynomial in $\mathbb{Q}_2(i)[t]$.

$$f(t) = \sum_{i=0}^{N} a_i t_i \quad \text{where } a_i \in \mathbb{Q}_2(i)$$

We define $v(f(t)) = \min\{v(a_i)\} \in \mathbb{Z}$. Note that if $v(a_i) \geq 0$, then $f(t) \in O_F$. We extend the valuation to rational functions with coefficients in $\mathbb{Q}_2(i)$ by defining

$$v(f(t)/g(t)) := v(f(t)) - v(g(t))$$

Now, we have a well-defined valuation on $\mathbb{Q}_2(i)(t)$. Given this, we can better understand the residue field of F under the valuation. Recall that the residue field of F is given by the quotient of the valuation ring $O_F = \{x \in F \ : \ v(x) \geq 0\}$ by the unique maximal ideal $\mathfrak{m}_F = \{x \in F \ : \ v(x) > 0\}$. In particular, we see that since $\overline{\mathbb{Q}_2(i)} = \mathbb{F}_2$. So when we adjoin the variable t, we obtain that the residue of F is given by $\bar{F} = \mathbb{F}_2(t)$.

4 Exploring the Quotient Group $F^{\bullet}/F^{\bullet^2}$

Let $F_0 = \mathbb{Q}_2(i)(t)$ and let v be the extension of the 2-adic valuation on $\mathbb{Q}$ where $v(t) = 0$. Let F be the completion of F_0 with respect to the metric induced by v in which Hensel's lemma will hold.

We are now going to study substructures of our quaternionic pairing $q : F^{\bullet}/F^{\bullet^2} \times F^{\bullet}/F^{\bullet^2} \longrightarrow Q$, where Q is some elementary abelian 2-group that satisfy a Poincare duality-like condition.

Definition 4.0.1. *Let G_1, G_2 be groups. A* ***perfect pairing*** *is a map $G_1 \times G_2 \longrightarrow \mathbb{Z}/2\mathbb{Z}$ where whenever nonzero $g_1 \in G_1$, we have that*

$$(g_1, *) : G_1 \longrightarrow Hom(G_2, \mathbb{F}_2)$$

is a group isomorphism. Or equivalently, for all nonzero $g_1 \in G_1$, there exists $g_2 \in G_2$ such that $(g_1, g_2) = 1$ and for all nonzero $g_2 \in G_2$, there exists a $g_1 \in G_1$ such that $(g_1, g_2) = 1$.

Example 4.1. *Let L be a local field of characteristic not equal to 2. For convenience, assume that $\sqrt{-1} \in L$. Then*

$$L^{\bullet}/L^{\bullet^2} \times L^{\bullet}/L^{\bullet^2} \longrightarrow Q \cong \mathbb{Z}/2\mathbb{Z}$$

is a perfect pairing. In particular, $L^{\bullet}/L^{\bullet^2}$ is even-dimensional and the pairing has a symplectic base which means that there exists a basis for $L^{\bullet}/L^{\bullet^2} = \{e_1 f_1, \ldots, e_n f_n\}$ such that for all $i, j \in \{1, \ldots, n\}$, we have that $(e_i, e_j) = (f_i, f_j) = 0$, $(e_i, f_i) = 1$ and for $i, j \in \{1, \ldots, n\}$ distinct, $(e_i, f_j) = 0$.

Example 4.2. *In the example studied below, we will have that $I^3F \cong \mathbb{Z}/2\mathbb{Z}$ and that $F^\bullet/F^{\bullet^2} \times I^2/I^3 \longrightarrow I^3/I^4 \cong \mathbb{Z}/2\mathbb{Z}$ is a perfect pairing.*

4.1 Quaternionic Pairing of $\mathbb{Q}_2(i)$

Now, we investigate a specific symplectic base in order to try to figure out a way to iteratively "fix" linkage problems by adding new group elements into our basis in order to correct the linkage errors in the previous submatrices.

Definition 4.1.1. *(Field Case:) Let F be an arbitrary field. We define the* ***representation set of a binary form*** *as follows:*

$$D_F\langle[a],[b]\rangle := \{[ax^2+by^2] \quad | \quad x,y \in F\}$$

Remark 4.1.1. *It follows from the representation theory that $(a,b) = 0 \in Q$ if and only if $[-b] \in D_F\langle[1],[a]\rangle$.*

This motives the following abstract definition:

Definition 4.1.2. *Define $[-b] \in D\langle[1],[a]\rangle$ if and only if $(a,b) = 0$ in the quaternionic pairing.*

(Case of $\mathbb{Q}_2(i)$) : We saw that order of the quotient group $\mathbb{Q}_2(i)/\mathbb{Q}_2(i)^2$ is 16. Using valuation theory, one can see that we can represent the square classes in $\mathbb{Q}_2(i)/\mathbb{Q}_2(i)^2$ with the following set

$$\frac{\mathbb{Q}_2(i)}{\mathbb{Q}_2(i)^2} = \{[1],[1+\pi],[1+\pi^3],[1+\pi^4],[1+\pi+\pi^3],[1+\pi-4],[1+\pi+\pi^3-4],[1+\pi^3-4],$$

$$[\pi],[\pi][1+\pi],[\pi][1+\pi^3],[\pi][1+\pi^4],[\pi][1+\pi+\pi^3],[\pi][1+\pi-4],[\pi][1+\pi+\pi^3-4],[\pi][1+\pi^3-4]\}.$$

We will consider these generators of $\mathbb{Q}_2(i)/\mathbb{Q}_2(i)^2$ in our next computation as a linear combination of the following ordered basis:

$$\beta_0 = \{u_1, u_2, u_3, u_4\} = \{[1+\pi], [1+\pi^3], [1+\pi^4], [\pi]\}.$$

We want to determine the subgroups of $\mathbb{Q}_2(i)/\mathbb{Q}_2(i)^2$ by computing $D\langle [1], [a]\rangle$ for $a \in \mathbb{Q}_2(i)/\mathbb{Q}_2(i)^2$. Since the pairing was perfect, these correspond to computing hyperplanes, we can accomplish this computation via a matrix $S \in SL_4(\mathbb{Z}_2)$ where

$$s_{i,j} = \begin{cases} 0 & \text{if } [i] \notin D\langle [1], [j]\rangle \\ 1 & \text{if } [i] \in D\langle [1], [j]\rangle \end{cases}.$$

Firstly, we note that S must be symmetric as we have that condition that

$$[x] \in D\langle [1], [y]\rangle \quad \Longleftrightarrow \quad [y] \in D\langle [1], [x]\rangle.$$

This gives the (symplectic) matrix:

$$S_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

as we noted must occur in example 4.1 at the beginning of this chapter. With this matrix, we have the following representation sets:

$$D\langle[1],[1+\pi]\rangle = \text{span}\langle[1+\pi],[\pi],[1+\pi^4]\rangle$$
$$D\langle[1],[1+\pi^3]\rangle = \text{span}\langle[1+\pi^3],[\pi],[1+\pi^4]\rangle$$
$$D\langle[1],[1+\pi^4]\rangle = \text{span}\langle[1+\pi^4],[1+\pi],[1+\pi^3]\rangle$$
$$D\langle[1],[\pi]\rangle = \text{span}\langle[\pi],[1+\pi],[1+\pi^3]\rangle.$$

To demonstrate the arithmetic in this field, notice that the class of $[1+\pi^2] \notin \beta_0$ as it can be generated by:

$$\begin{aligned}
1+\pi^2 &\equiv (1+\pi^2)(1+\pi)^2 && \text{congruent via multiplication by a square} \\
&\equiv (1+\pi^2)(1+\pi^2+2\pi) && \text{by expansion of } (1+\pi)^2 \\
&\equiv 1+\pi^2+2\pi+\pi^2+\pi^4+2\pi^3 && \\
&\equiv 1+\pi^4+2(\pi+\pi^2+\pi^3) && \\
&\equiv 1+2\pi \mod 1+\pi^5 && \text{using the } \pi\text{-adic expansion for 2} \\
&\equiv (1-\pi^3-\pi^4) && \text{using the } \pi\text{-adic expansion for -1} \\
&\equiv (1+\pi^3)(1+\pi^4). &&
\end{aligned}$$

Moreover, after exploring this example, we want to explore the larger quaternionic pairing $q : (F^\bullet/F^{\bullet^2} \times F^\bullet/F^{\bullet^2}) \longrightarrow Q$ where Q is some elementary abelian 2-group that is yet to be determined. In order to understand this complicated pairing, we will look at the structure of our domain and codomain as well as the pairings between such substructures to obtain information about the pairing as a whole.

4.2 Using Filtration to Investigate $F^\bullet/F^{\bullet^2}$

Now, for $i \geq 1$, define $U_i := \{1 + \pi^i g \ : \ g \in O_F\}$. We also identify U_0 with $O_F^\bullet$. These subgroups of $F^\bullet$ will give us a filtration of our group. Recall that

Definition 4.2.1. *Given a group G, we say that a sequence $\{G_n\}_{n\in\mathbb{N}}$ of nested normal subgroups of G is a **filtration** given that for any $n \in \mathbb{N}$, we have that $G_{n+1} \supseteq G_n$.*

Note that $F^\bullet$ is abelian which gives that any subgroup is certainly normal. Then, note that we have a filtration of $F^\bullet$, namely:

$$F^\bullet \supset U_0 = O_F^\bullet \supset U_1 \supset U_2 \supset U_3 \supset \dots \tag{1}$$

and a filtration of $F^\bullet/F^{\bullet^2}$:

$$\frac{F^\bullet}{F^{\bullet^2}} \supset \frac{U_0}{F^2 \cap U_0} \supset \frac{U_1}{F^2 \cap U_1} \supset \frac{U_2}{F^2 \cap U_2} \tag{2}$$

We want to understand U_i/U_{i+1} for any $i \geq 0$ and the quotients of the form:

$$\frac{U_0/(F^2 \cap U_0)}{U_1/(F^2 \cap U_1)} \cong \frac{U_0}{(F^2 \cap U_0) \cdot U_1}$$
$$\frac{U_1/(F^2 \cap U_1)}{U_2/(F^2 \cap U_2)} \cong \frac{U_1}{(F^2 \cap U_1) \cdot U_2}$$
$$\frac{U_2/(F^2 \cap U_2)}{U_3/(F^2 \cap U_3)} \cong \frac{U_2}{(F^2 \cap U_2) \cdot U_3}$$

To understand the structure of these quotients, we use the following isomorphisms. By basic valuation theory, we have

$$F^\bullet/U_0 \cong \mathbb{Z} \text{ (the value group of } F) \tag{4.2.1a}$$

This makes sense as we quotient out by elements of value 0.

Also, we have that

$$U_0/U_1 \cong \bar{F}^\bullet \tag{4.2.1b}$$

This can be seen as we know that $O_F^\bullet = U_0 \twoheadrightarrow \bar{F}^\bullet$. Moreover, the kernel of this surjection must necessarily be $U_1 = \{1 + \pi y \ : \ y \in O_F\}$ as these are elements of value 1 plus something from the ideal $\mathfrak{m}_F$.

Next, for any fixed $i \in \mathbb{N}$, then

$$\psi : U_i/U_{i+1} \longrightarrow (\bar{F}, +) \quad \text{given by} \quad 1 + \pi^i x \mapsto \bar{x} \tag{4.2.1c}$$

is a group isomorphism. Note that any element in the quotient U_i/U_{i+1} can be represented by $1 + \pi^i x$ for $x \in O_F$. We outline the proof that ψ is an isomorphism.

Proof. (*Sketch.*) Let $x, y \in O_F^\bullet$ such that $1 + \pi^i x \equiv 1 + \pi^i y \mod U_{i+1}$. This implies that for some $z \in O_F^\bullet$, we can express $1 + \pi^i x$ as

$$1 + \pi^i x = (1 + \pi^i y)(1 + \pi^{i+1} z) = 1 + \pi^i y + \pi^{i+1} z + \pi^{2i+1} yz$$

$$= 1 + \pi^i (y + \pi y + \pi^{i+1} yz)$$

This implies that $x = y + \pi y + \pi^{i+1} yz$. But since $v(\pi y + \pi^{i+1} yz) > 0$, we have that $\pi y + \pi^{i+1} yz \in \mathfrak{m}_F$. So, we obtain that $\bar{x} = \bar{y}$ which gives the well definition of ψ.

To see that ψ is a group homomorphism, let $1+\pi^i x, 1+\pi^i y \in U_i/u_{i+1}$. Then observe that

$$\psi((1+\pi^i x)(1+\pi^i y)) = \psi(1+\pi^i(x+y)) = \overline{x+y}$$

$$\psi(1+\pi^i x)\psi(1+\pi^i y) = \bar{x}+\bar{y}$$

Note that these two expressions are the same simply because the residue is an additive homomorphism from O_F to $O_F/\mathfrak{m}_F$ giving that ψ is a homomorphism.

To see that ψ is injective, let $\bar{x} = \bar{y} \in \bar{F}^+$. Then, we have that $\bar{x} - \bar{y} = 0 \in \bar{F}^+$. Then, we have that $1+\pi^i(x-y) \in U_{i+1}$ as $\overline{x-y} \in \mathfrak{m}_F$. So, since $1+\pi^i(x-y) = (1+\pi^i x)(1+\pi^i y)$ this implies that $1+\pi^i x \cong 1+\pi^i y \mod U_{i+1}$ giving the injectivity of ψ.

Lastly, ψ must be surjective because any element in $\bar{F}^+$ can be expressed as some $g+\mathfrak{m}_F$ for some $g \in O_F$. So, we see that $1+\pi^i x \mapsto \bar{x}$ is a surjection. Therefore, ψ is a well-defined group isomorphism ■

We discuss the squares in the field. By valuation theory, $U_5 \subseteq F^2$ as $1+\pi^5 * \in F^2$ and we can view the quotient group of the residue mod squares as follows:

$$F^\bullet/F^{\bullet^2} \cong O_F^\bullet/O_F^{\bullet^2} \oplus \langle\pi\rangle \tag{4.2.2a}$$

To see this, note that we automatically have that

$$O_F^\bullet/O_F^{\bullet^2} \oplus \langle\pi\rangle/F^{\bullet^2} \hookrightarrow F^\bullet/F^{\bullet^2}$$

via subgroup inclusion. So, it suffices to verify is that this is also a surjection. Consider some $x \in F^\bullet$. Then, we consider the case where x has even or odd value. Firstly, suppose that $v(x)$ is even. Then, there exists some $y \in O_F$ such that $v(y^2) = v(y) + v(y) = v(x)$. Working mod squares, we have that $v(x) = v(xy^{-2}) = 0$ which implies that the coset $x + \bar{F}^{\bullet^2}$ is the image of $O_F^\bullet / O_F^{\bullet^2}$ since $xy^{-2} \in O_F^\bullet$. Now, suppose that $v(x)$ is odd. Then there exists some $y \in O_F$ such that $v(y^2) - 1 = v(x)$. So, we have that $v(x) = v(\pi^{-1}xy^2) = 0$ which implies that the coset $x + \bar{F}^{\bullet^2}$ is the image of $\pi \cdot O_F^\bullet / O_F^{\bullet^2}$ since $x + F^{\bullet^2}$ is $\pi \cdot$ some unit. Thus, we see that this map is surjective and moreover a group isomorphism giving 4.2.2a.

Another basic result using the definition of U_1 and basic valuation theory is

$$O_F^\bullet / (O_F^{\bullet^2} \cdot U_1) \cong \bar{F}^\bullet / \bar{F}^{\bullet^2} \quad \text{via} \quad f \mod O_F^{\bullet^2} \cdot U_1 \mapsto \bar{f} \mod \bar{F}^{\bullet^2} \qquad (4.2.2\text{b.})$$

From here, we want to examine the behavior of these quotient groups modulo the squares of the field. We go through levels of the filtration to accomplish this. We begin at U_1. In particular, observe

$$U_1 / ((U_1 \cap \bar{F}^{\bullet^2}) \cdot U_2) \cong U_1/U_2 \cong (\bar{F}, +) \qquad (4.2.3\text{a})$$

This can be seen as $U_1 \cap \bar{F}^{\bullet^2} \subset U_2$. Note that we have this inclusion as for any $1 + \pi x = f^2$, we have that $\bar{f} = 1$ as $f^2 \in U_1$. Because $\text{char} F = 2$, this implies that we can express $f = 1 + \pi y$. Therefore, we see that $f^2 = 1 + \pi^2 y^2 + 2\pi y \in U_2$.

On the next level, consider U_2. This behaves differently than the previous level because $U_2 \cap \bar{F}^2 \nsubseteq U_3$ since

$$\begin{aligned}(1+\pi s)^2 &= 1+\pi^2 s^2 + 2\pi s \\ &\equiv 1+\pi^2 s^2 \mod U_3\end{aligned}$$

So, in this case, we have

$$U_2/((U_2 \cap \bar{F}^{\bullet^2}) \cdot U_3) \cong (\bar{F}^+/\bar{F}^{+^2}) \tag{4.2.3b}$$

On the next level, consider U_3. Notice this is the same case as in U_1 i.e.

$$U_3/((U_3 \cap \bar{F}^{\bullet^2}) \cdot U_4) \cong (\bar{F}, +) \tag{4.2.3c}$$

because $U_3 \cap F^2 \subseteq U_4$. Note that any element of U_3 is some $1+\pi^3 g$ for $g \in O_F$ and $\bar{F^{\bullet 2}}$ must have an even power of π.

The last level we consider is U_4. Since $U_5 \subseteq F^2$, it is redundant to multiply that subgroup in the denominator of the quotient, giving

$$U_4/((U_4 \cap \bar{F}^{\bullet^2}) \cdot U_5) \cong U_4/(U_4 \cap \bar{F}^{\bullet^2})$$

But in the case where we are looking at $U_4 \cap F^2$, we see that the squares $(1+2s)^2 = 1+4(s^2+s)$ will be of the form x^2+x. We define $\wp(F) = \langle x^2 + x \ : \ x \in F \rangle$ and see that

$$U_4/(U_4 \cap \bar{F}^{\bullet^2}) \cong F^+/\wp(F)^+ \tag{4.2.3d}$$

because if we consider the surjection $U_4/(U_4 \cap \bar{F}^{\bullet^2}) \twoheadrightarrow \bar{F}^+$, it has kernel generated by $x^2 + x$ which is $\wp(F)$.

The obtained isomorphisms can help us break down the structure of the elements in $F^\bullet/F^{\bullet^2}$. In particular, we can see that we can generate $F^\bullet/F^{\bullet^2}$ will the following:

$$F^\bullet/F^{\bullet^2} \quad \cong \quad \langle \mathbb{Z}/2\mathbb{Z} \times \bar{F}^\bullet/\bar{F}^{\bullet^2} \times \bar{F}^+ \times t\bar{F}^{+^2} \times \bar{F}^+ \times \frac{\bar{F}^+}{\wp(F)^+} \rangle \tag{4.2.4}$$

in other words, we can generate anything in $F^\bullet/F^{\bullet^2}$ by a product of:

$$\{(\pi^i)_{i\in\{0,1\}}, f \in O_F^\bullet/(O_F^{\bullet^2} \cdot U_1), (1+\pi f), (1+\pi^2 t f), (1+\pi^3 f), ((1+\pi^4 f) \mod \wp(F))\}$$

This gives us a way that we can approximate fixing linkage issues by the layers in the filtration. But in order to investigate any sort of linkage, we need to also understand how these filtrations are affecting the codomain of our pairing.

4.3 Using Filtration to Understand Quotient Subgroups of Q

Now, we want to look on the other side of our quaternionic pairing. Define

$$V_0 := (\pi, O_F^\bullet)$$

$$\forall i \in \mathbb{N} \quad V_i := (\pi, U_i) = (\pi, 1 + \pi^i f) \quad f \in O_F^\bullet$$

This gave us the following filtration:

$$(\pi, F^\bullet) \supseteq V_0 \supseteq V_1 \supseteq V_2 \supseteq V_3 \supseteq V_4$$

Using this filtration, we aim to understand V_i/V_{i+1}. In general, we will be able to see the following isomorphisms by looking at kernels of the following set of surjections:

$$\varphi_i : V_i \twoheadrightarrow \bar{F}^+ \qquad \text{via} \qquad (\pi, 1 + \pi^i f) \mapsto \bar{f}$$

First, note that

$$(\pi, F^\bullet)/(\pi, O_F^\bullet) \cong \{0\}$$

because $F^\bullet = \pi \cdot O_F^\bullet$ as $(\pi, \pi^i u) = (\pi, u)$ for any $u \in O_F^\bullet$.

We want to understand V_i/V_{i+1}. In particular, we want to use the information obtained by U_i/U_{i+1} to give us information on the other side of the quaternionic pairing. Note that

$$\bar{F}^\bullet = O_F^\bullet/U_1 \longrightarrow V_0/V_1 \qquad \text{via} \qquad u \mapsto (\pi, u)$$

will induce a map

$$\bar{F}^\bullet/\bar{F}^{\bullet^2} \longrightarrow V_0/V_1$$

as for any x, we have that $(\pi, x^2) = 0$. This gives us that this map must be well-defined. Moreover, it gives us a group isomorphism

$$V_0/V_1 \cong \bar{F}^\bullet/\bar{F}^{\bullet^2} \tag{4.3.0}$$

Next, we consider $i > 0$. We claim that

$$V_1/V_2 \cong \bar{F}^+/\bar{F}^{+^2} \tag{4.3.1a}$$

We look at $\ker \varphi_1$. We claim that $\ker \varphi_1 = \bar{F}^{+^2}$. Note that a quaternionic pairing evaluates to 0 if and only if the quadratic form is 0 i.e.

$$(\pi, V_i) = 0 \iff \langle\langle \pi, V_i \rangle\rangle = 0$$

Observe that

$$\begin{aligned} 1 + \pi f &= x^2 + \pi y^2 \mod \pi^2 \\ &= (1 + \pi g)^2 + \pi y^2 && (v(y) > 0) \\ &= 1 + \pi^2 g^2 && (\text{as } v(2\pi g) \geq 3) \\ &\implies f \in \bar{F}^{+^2} \end{aligned}$$

In the next layer, we claim that

$$V_2/V_3 \cong \bar{F}^+/\bar{F}^{+^2} \tag{4.3.1b}$$

We look at $\ker \varphi_2$. We claim that $\ker \varphi_2 = \bar{F}^{+^2}$. Observe that

$$\begin{aligned} 1 + \pi^2 f &= x^2 + \pi y^2 \mod \pi^3 \\ &= (1 + \pi g)^2 + \pi y^2 && (v(y) > 0) \\ &= 1 + \pi^2 g^2 && (\text{as } v(2\pi g) \geq 3) \\ &\implies f \in \bar{F}^{+^2} \end{aligned}$$

In the next layer, we have

$$V_3/V_4 \cong \bar{F}^+/\bar{F}^{+^2} \tag{4.3.1c}$$

We look at $\ker \varphi_3$. We claim that $\ker \varphi_3 = \bar{F}^{+^2}$. Observe that

$$\begin{aligned} 1 + \pi^3 f &= 1 + \pi(\pi g^2) \\ &= 1 + \pi^3 g^2 \mod \pi^4 \\ &\implies f \in \bar{F}^{+^2} \end{aligned}$$

In the last level, we have

$$V_4/V_5 \cong \bar{F}^+/\wp(\bar{F})^+ \tag{4.3.1f}$$

We look at $\ker \varphi_4$ to see that $\ker \varphi_4 = \wp(\bar{F})^+$. Observe that

$$\begin{aligned} 1 + \pi^4 f &= x^2 + \pi y^2 \\ &= (1 + \pi^2 g)^2 + \pi y^2 & \text{(where } v(y) > 0\text{)} \\ &= 1 + \pi^4(g^2 + g) \mod \pi^5 \end{aligned}$$

Thus, we see that $\ker \varphi_4$ is generated by the polynomials of the form $g^2 + g$ which is precisely saying that $\ker \varphi_4 = \wp(\bar{F})^+$.

Remark 4.3.1. *In a generalized form, these quotients are computed by Arason in theorem 2 of [3]. These quotients are called the wild part of WF. Note that in his computations it does not require the field to be complete.*

5 Linkage Within Our Specific Pairing

Using the notation from the previous section, let $G = \langle \pi \rangle \oplus U_1/U_1^2$ and let $Q = (G, G)$. The quaternionic pairing under consideration $q : G \times G \to Q = V_0 = (\pi, U)$. In order to show this is linked, we first classify all the possible $g_i \in G$ such that $q(g_1, g_2) = q(g_3, g_4) = q \in Q$. Via the filtration, we can look layer by layer to see the possible g_i combinations:

5.1 Possibilities for Needing a Linking Element

- $\mathbf{V_4 = (\pi, 1 + \pi^4 f)}$ **Layer:** The equation we want to satisfy is

  $$q(g_1, g_2) = q(g_3, g_4) = (\pi, 1 + \pi^4 f) \in V_4$$

  The question is for which $g_1 \in G$, does there exist a $g_2 \in G$ such that $q(g_1, g_2) = (\pi, 1 + \pi^4 f)$? Recall that G is generated by

  $$\{(\pi^i)_{i \in \{0,1\}}, f \in O_F^\bullet/(O_F^{\bullet^2} \cdot U_1), (1+\pi f), (1+\pi^2 t f), (1+\pi^3 f), ((1+\pi^4 f) \mod \wp(F))\}$$

  Our options for g_1 are generators of G. Below is a list of the possible pairs $(g_1, g_2) \in G \times G$ such that $q(g_1, g_2) = (\pi, 1 + \pi^4 f)$:

  1. $g_1 = \pi u_i$. For $i \geq 1$, then g_2 will exist. Namely, take $g_2 = 1 + \pi^4 f$.
  2. $g_1 = u_1 = 1 + \pi h$. Take $g_2 = 1 + \pi^3 h'$ with the constraint that $h \in F^{\bullet^2}$ and $h' = (f + \wp(g))h^{-1}$.
  3. $g_1 = 1 + \pi^2 h$. Take $g_2 = 1 + \pi^2 h'$. Need condition for this case.
  4. $g_1 = 1 + \pi^3 h$. Take $g_2 = 1 + \pi h'$ with the constraint that $h' \in F^{\bullet^2}$ and $h = (f + \wp(g))h'^{-1}$.

- $\mathbf{V_3/V_4} = (\pi, \mathbf{1} + \pi^{\mathbf{3}}\mathbf{tf^2})$ **Layer:** The equation we want to satisfy is

$$q(g_1, g_2) = q(g_3, g_4) = (\pi, 1 + \pi^3 t f^2) \in V_3/V_4$$

The question is for which $g_1 \in G$, does there exist a $g_2 \in G$ such that $q(g_1, g_2) = (\pi, 1 + \pi^3 t f^2)$? Below is a list of the possible pairs $(g_1, g_2) \in G \times G$ such that $q(g_1, g_2) = (\pi, 1 + \pi^3 t f^2)$:

1. $g_1 = \pi u_i$. For $i \geq 2$, then g_2 will exist. Namely, take $g_2 = 1 + \pi^3 t f^2$.
2. $g_1 = u_1 = 1 + \pi h$. Take $g_2 = 1 + \pi^2 h'$ with the constraint that $h \in F^{\bullet^2}$ and $h' \in tF^{\bullet^2}$.
3. $g_1 = 1 + \pi^2 h$. Take $g_2 = 1 + \pi h'$ with the constraint that $h \in tF^{\bullet^2}$ and $h' \in F^{\bullet^2}$.
4. $g_1 = 1 + \pi^3 h$. Take $g_2 = \pi u_i$ for $i \geq 2$ and with the constraint that $h \in tF^{\bullet^2}$.

- $\mathbf{V_2/V_3} = (\pi, \mathbf{1} + \pi^{\mathbf{2}}\mathbf{tf^2})$ **Layer:** The equation we want to satisfy is

$$q(g_1, g_2) = q(g_3, g_4) = (\pi, 1 + \pi^2 t f^2) \in V_2/V_3$$

The question is for which $g_1 \in G$, does there exist a $g_2 \in G$ such that $q(g_1, g_2) = (\pi, 1 + \pi^2 t f^2)$? Below is a list of the possible pairs $(g_1, g_2) \in G \times G$ such that $q(g_1, g_2) = (\pi, 1 + \pi^2 t f^2)$:

1. $g_1 = \pi u_i$. For $i \geq 3$, then g_2 will exist. Namely, take $g_2 = 1 + \pi^2 t f^2$.
2. $g_1 = u_1 = 1 + \pi h$. Take $g_2 = 1 + \pi^2 h'$ with the possible constraint that $h \in F^{\bullet^2}$ and $h' \in tF^{\bullet^2}$.

3. $g_1 = 1 + \pi^2 h$. Take $g_2 = \pi u_i$ with $i \geq 3$ and $h \in tF^{\bullet^2}$.

4. $g_1 = 1 + \pi^3 h$. Then no g_2 will exist as $\pi, 1 + \pi^3 h) \in V_3$.

- $\mathbf{V_1/V_2 = (\pi, 1 + \pi tf^2)}$ **Layer:** The equation we want to satisfy is

$$q(g_1, g_2) = q(g_3, g_4) = (\pi, 1 + \pi t f^2) \in V_1/V_2$$

The question is for which $g_1 \in G$, does there exist a $g_2 \in G$ such that $q(g_1, g_2) = (\pi, 1 + \pi t f^2)$? Below is a list of the possible pairs $(g_1, g_2) \in G \times G$ such that $q(g_1, g_2) = (\pi, 1 + \pi t f^2)$:

1. $g_1 = \pi u_i$. For $i \geq 4$, then g_2 will exist. Namely, take $g_2 = 1 + \pi t f^2$.
2. $g_1 = u_1 = 1 + \pi h$. Take $g_2 = \pi u_i$ with $i \geq 4$ and $h \in tF^{\bullet^2}$.

5.2 Linking These Elements

With this classification of the possible pairs that will get us into each layer, we now aim to look at all possible combinations of these pairs and identify the field element that can link each. In other words, whenever we have that $q(a, b) = q(c, d) = q \in V_i/V_{i+1}$, we must find and $\ell \in G$ such that $q(a, \ell) = q(c, \ell) = q$. So, we look at all the possible g_i combinations and will find an $\ell \in G$ that links them.

- $\mathbf{V_4 = (\pi, 1 + \pi^4 f)}$ **Layer:** The g_i possibilities are

$$(\pi u_i, 1 + \pi^4 f) = (1 + \pi h, 1 + \pi^3 h')$$

that we identify with

$$(\alpha_1, \alpha_2) = (\alpha_3, \alpha_4)$$

This gives 6 cases to link:

1. $(\alpha_1, \ell) = (\alpha_2, \ell)$
2. $(\alpha_1, \ell) = (\alpha_3, \ell)$
3. $(\alpha_1, \ell) = (\alpha_4, \ell)$
4. $(\alpha_2, \ell) = (\alpha_3, \ell)$
5. $(\alpha_2, \ell) = (\alpha_4, \ell)$
6. $(\alpha_3, \ell) = (\alpha_4, \ell)$

One example of linking such a possibility is case 2. With methods that we discussed in fuller detail in a later section, note that if we choose $\ell = (1 + \pi^3 f^2 h^{-1})(1 + \pi^4 h)$, we obtain that indeed

$$(\alpha_1, (1+\pi^3 f^2 h^{-1})(1+\pi^4 h)) = (\alpha_3, (1+\pi^3 f^2 h^{-1})(1+\pi^4 h)) = (\pi, 1+\pi^4 f)$$

because

$$\begin{aligned}
&(\alpha_1, (1 + \pi^3 f^2 h^{-1})(1 + \pi^4 h)) \\
&= (\pi u_i, 1 + \pi^3 f^2 h^{-1})(1 + \pi^4 h)) \\
&= (\pi, 1 + \pi^3 f^2 h^{-1}) + (\pi, 1 + \pi^4 h) + (u_i, 1 + \pi^3 f^2 h^{-1}) + (u_i, 1 + \pi^4 f) \\
&\equiv (\pi, 1 + \pi^4 h) \mod V_5
\end{aligned}$$

where we get various cancellations with formulae that will be given in a later section. Also observe that

$$
\begin{aligned}
&(\alpha_3, (1+\pi^3 f^2 h^{-1})(1+\pi^4 h)) \\
&= (1+\pi h, (1+\pi^3 f^2 h^{-1})(1+\pi^4 h)) \\
&= (1+\pi h, 1+\pi^3 f^2 h^{-1}) + (1+\pi h, 1+\pi^4 h) \\
&= (1+\pi h, 1+\pi^3 f^2 h^{-1}) \\
&\equiv (\pi, 1+\pi^4 h) \mod V_5
\end{aligned}
$$

- $\mathbf{V_3/V_4} = (\pi, \mathbf{1} + \pi^{\mathbf{3}}\mathbf{tf^2})$ **Layer:** The g_i possibilities are

$$(\pi u_i, 1+\pi^3 t f^2) = (1+\pi h, 1+\pi^2 h')$$

that we identify with

$$(\alpha_1, \alpha_2) = (\alpha_3, \alpha_4)$$

This gives 6 cases to link:

1. $(\alpha_1, \ell) = (\alpha_2, \ell)$
2. $(\alpha_1, \ell) = (\alpha_3, \ell)$
3. $(\alpha_1, \ell) = (\alpha_4, \ell)$
4. $(\alpha_2, \ell) = (\alpha_3, \ell)$
5. $(\alpha_2, \ell) = (\alpha_4, \ell)$
6. $(\alpha_3, \ell) = (\alpha_4, \ell)$

An example of finding linking elements in this case can be seen in case

6. Consider $\ell = (1+\pi h)(1+\pi^2 h')$. Then observe that

$$(\alpha_3, (1+\pi h)(1+\pi^2 h')) = (\alpha_4, (1+\pi h)(1+\pi^2 h'))$$

as

$$\begin{aligned}
&(\alpha_3, (1+\pi h)(1+\pi^2 h')) \\
&= (1+\pi h, (1+\pi h)(1+\pi^2 h')) \\
&= (1+\pi h, 1+\pi h) + (1+\pi h, 1+\pi^2 h') \\
&= (1+\pi h, 1+\pi^2 h') \\
&\equiv (\pi, 1+\pi^3 t f^2) \mod V_4
\end{aligned}$$

also

$$\begin{aligned}
&(\alpha_4, (1+\pi h)(1+\pi^2 h')) \\
&= (1+\pi^2 h' h, (1+\pi h)(1+\pi^2 h')) \\
&= (1+\pi^2 h', 1+\pi h) + (1+\pi^2 h', 1+\pi^2 h') \\
&= (1+\pi^2 h', 1+\pi h) \\
&\equiv (\pi, 1+\pi^3 t f^2) \mod V_4
\end{aligned}$$

Note that this layer is particularly interesting because it comes up in a later section which breaks linkage of a quaternionic pairing with many linked subpairings.

- $\mathbf{V_2/V_3} = (\pi, \mathbf{1+\pi^2 tf^2})$ **Layer:** The g_i possibilities are

$$(\pi u_i, 1+\pi^2 tf^2) = (1+\pi h, 1+\pi h')$$

that we identify with

$$(\alpha_1, \alpha_2) = (\alpha_3, \alpha_4)$$

This gives 6 cases to link:

1. $(\alpha_1, \ell) = (\alpha_2, \ell)$
2. $(\alpha_1, \ell) = (\alpha_3, \ell)$
3. $(\alpha_1, \ell) = (\alpha_4, \ell)$
4. $(\alpha_2, \ell) = (\alpha_3, \ell)$
5. $(\alpha_2, \ell) = (\alpha_4, \ell)$
6. $(\alpha_3, \ell) = (\alpha_4, \ell)$

On example of finding linking elements in this layer, consider linking case case 3. Consider $\ell = (1+\pi h)(1+\pi^2 tf^2)$. Then, observe that

$$(\alpha_1, (1+\pi h)(1+\pi^2 tf^2) = (\alpha_4, (1+\pi h)(1+\pi^2 tf^2) = (\pi, 1+\pi^2 tf^2)$$

because

$$\begin{aligned}
&(\alpha_1, (1+\pi h)(1+\pi^2 tf^2)) \\
&= (\pi u_i, (1+\pi h)(1+\pi^2 tf^2)) \\
&= (\pi, 1+\pi h) + (\pi, 1+\pi^2 tf^2) + (u_i, 1+\pi h) + (u_i, 1+\pi^2 tf^2) \\
&\equiv (\pi, 1+\pi^2 tf^2) \mod V_3
\end{aligned}$$

as well as

$$\begin{aligned}
&(\alpha_4, (1+\pi h)(1+\pi^2 t f^2)) \\
&= (1+\pi h', (1+\pi h)(1+\pi^2 t f^2)) \\
&= (1+\pi h', 1+\pi h) + (1+\pi h', 1+\pi^2 t f^2) \\
&\equiv (\pi, 1+\pi^2 t f^2) \mod V_3
\end{aligned}$$

- For the case of V_1/V_2, we have trivial linkage as $(\pi u_i, 1+\pi h')$ is the only possibility.

5.3 Strategies for Finding Linking Elements

In order to find such $\ell \in G$, we used the following relations that we can derive using the arithmetic of the field, the theory of quadratic forms and our knowledge of valuations.

In the theory of quadratic forms, a four dimensional quadratic form $\langle 1, a, b, ab\rangle := \langle\langle a, b\rangle\rangle$ is called a **two-fold Pfister form**. This forms determine the quaternionic structure of a field. Namely, note that $\langle 1, a, b, ab\rangle = \langle 1, c, d, cd\rangle$ if and only if $\langle a, b, ab\rangle = \langle c, d, cd\rangle$ and that for all x, y $\langle x, y\rangle = \langle x+y, (x+y)xy\rangle$. Note that this is where the linkage axiom comes from because if $\langle a, b, ab\rangle = \langle c, d, cd\rangle$, then $\langle a, b, ab\rangle - \langle c, d, cd\rangle$ is a hyperbolic 6-dimensional form. Moreover, its 3-dimensional totally isotropic subspace must intersect the supporting space of the four dimensional subform $\langle b, ab\rangle - \langle d, cd\rangle$. But note that if ℓ is represented by both $\langle b, ab\rangle$ and $\langle d, cd\rangle$ (such exists by the isotropic vector) we find that $\langle b, ab\rangle = \langle \ell, m\rangle$ and $\langle d, cd\rangle = \langle \ell, n\rangle$. This ultimately gives that

$\langle\langle a, b\rangle\rangle = \langle\langle a, \ell\rangle\rangle = \langle\langle c, \ell\rangle\rangle = \langle\langle c, d\rangle\rangle$ furthermore giving linkage!

With this as motivation, we return to the structure of quadratic forms over $\mathbb{Q}_2(i)$. Consider the representation set of $\langle 1+\pi a, 1+\pi b, 1+\pi(a+b)+\pi^2 ab\rangle$ which is the "pure part" of $\langle\langle 1+\pi a, 1+\pi b\rangle\rangle$. Note that we have

$$\langle 1+\pi a, 1+\pi b\rangle = \langle 2+\pi(a+b), (1+\pi a)(1+\pi b)(2+\pi(a+b))\rangle$$

from which we find (using $-1 \in \mathbb{Q}_2(i)^{\bullet^2}$) $\langle 1+\pi a, 1+\pi b, 1+\pi(a+b)+\pi^2 ab\rangle =$

$$\begin{aligned} &= \langle 2+\pi(a+b), (1+\pi a)(1+\pi b)(2+\pi(a+b)), -(1+\pi(a+b)+\pi^2 ab)\rangle \\ &= \langle 1+\pi^2 ab, (1+\pi a)(1+\pi b)(2+\pi(a+b)), *\rangle \end{aligned}$$

for some value $*$. In terms of quaternions, this means

$$\langle\langle 1+\pi a, 1+\pi b\rangle\rangle = \langle\langle 1+\pi^2 ab, (1+\pi a)(1+\pi b)(2+\pi(a+b))\rangle\rangle$$

which can be used to study the quaternionic structure. Note that if $a = 1, \pi$ and $b = \pi^3$ which arises in considering $\langle\langle 1+\pi, 1+\pi^4\rangle\rangle$ and $\langle\langle 1+\pi^3, 1+\pi^4\rangle\rangle$,we have that a $1+\pi^5, 1 = \pi^6$ in the new forms which are squares. This gives the vanishing relations. From this, we find the other relation where we set $a = 1$ and $b = \pi^2$ to obtain $\langle\langle 1+\pi, 1+\pi^3\rangle\rangle = \langle\langle 1+\pi^4, \pi\rangle\rangle$. This can be seen using the multiplicative property of Pfister forms in the Quternionic group and can say that $2+\pi(1+\pi^2) = \pi+2+\pi^4 = \pi(1+m)$ where $m \in \pi\mathbb{Z}_2[i]$ so $\langle\langle 1+\pi^4, (1+m)\rangle\rangle = 0$.

Now that we have some control over the arithmetic in $\mathbb{Q}_2(i)$, we can turn to the fields F_n with $\mathbb{Q}_2(i) \subset F_n$ with that complete discrete valuation extending the one on $\mathbb{Q}_2(i)$. Note that this has residue field $\overline{F_m} = \mathbb{F}_2(t_1, \ldots, t_n)$ instead of $\mathbb{F}_2$.

Even when $n = 1$, the field F_1 will have infinite square class group; but this group is generated by five types of generators:

multiplicative lifts $[a]$ of elements $a \in \overline{F_1}^{\bullet}/\overline{F_1^{\bullet 2}}$

elements $1 + \pi a$ where $a \in \overline{F_1}^{+}$

elements $1 + \pi^2 a$ where $a \in \overline{F_1}^{+} - (\overline{F_1}^{2})^{+}$

elements $1 + \pi^3 a$ where $a \in \overline{F_1}^{+}$

elements $1 + \pi^4 a$ where $a \in \overline{F_1}^{+}/\wp(\overline{F_1}^{+})$

Here $\wp(x) = x^2 + x$ is the Artin-Schreier operator (which is additive and therefore its image is a subgroup of $\overline{F_1}^{+}$.)

Finally, one has to check that in F_1, if x, y are units i.e. $v(x) = v(y) = 0$. Note that one has

$$\langle\langle x+y, 1-\pi\rangle\rangle \equiv \langle\langle x, 1-\pi(\frac{x}{x+y})\rangle\rangle + \langle\langle y, 1-\pi(\frac{y}{x+y})\rangle\rangle \mod \langle\langle G, 1+\pi^2 *\rangle\rangle.$$

with this generators for Q will have have t slots once $1+\pi*$ terms are reached.

We note as $1 + \pi(a + b) \equiv (1 + \pi a)(1 + \pi b) \mod 1 + \pi^2 *$, so that by the bilinearity of Pfister forms, we have

$$\langle\langle 1 + \pi(a + b), c\rangle\rangle \equiv \langle\langle 1 + \pi a, c\rangle\rangle + \langle\langle 1 + \pi b, c\rangle\rangle \mod \langle\langle G, 1 + \pi^2 *\rangle\rangle.$$

Next, using $\frac{x}{x+y} + \frac{y}{x+y} = 1$, we know by earlier relations that

$$\langle\langle x+y, 1-\pi\rangle\rangle \equiv \langle\langle x+y, 1-\pi(\frac{x}{x+y})\rangle\rangle + \langle\langle x+y, 1-\pi(\frac{y}{x+y})\rangle\rangle \mod \langle\langle G, 1+\pi^2 *\rangle\rangle.$$

This latter sum is equal to

$$\langle\langle \pi x, 1-\pi(\frac{x}{x+y})\rangle\rangle + \langle\langle \pi y, 1-\pi(\frac{y}{x+y})\rangle\rangle$$

using the fact that $\langle\langle a, 1-b\rangle\rangle \cong \langle\langle ab, 1-b\rangle\rangle$. Using the bilinearity of Pfister forms, we conclude that

$$\langle\langle \pi, 1-\pi(\frac{x}{x+y})\rangle\rangle + \langle\langle \pi, 1-\pi(\frac{y}{x+y})\rangle\rangle \equiv \langle\langle \pi, 1-\pi\rangle\rangle = 0 \mod \langle\langle G, 1+\pi^2 *\rangle\rangle.$$

5.4 Deriving Linking Formulae

Let $[x]$ denote the square class of x over any field F of characteristic not two with $-1 \in F^{\bullet^2}$. Whenever $x, y \in F^{\bullet}$, we have

$$(1+x, 1+y) = (x(1+y), 1+x) = (y(1+x), 1+y)$$

We develop a strategy to find ℓ such that $(x(1+y), \ell) = (y(1+x), \ell) = (1+x, 1+y)$. For this, we have to find a common value of the forms $\langle 1+x, x(1+x)(1+y)\rangle$ and $\langle 1+y, y(1+x)(1+y)\rangle$. The first form represents (multiplying $x(1+x)(1+y)$

by the square $\frac{1}{(1+y)^2}$).

$$\begin{aligned}
(1+x)+\frac{x(1+x)}{1+y} &= \frac{(1+x+y+xy)+(x+x^2)}{1+y} \\
&= \frac{(1+x)^2+y(1+x)}{1+y} \\
&= \frac{(1+x)^2\left(1+\frac{y}{(1+x)}\right)}{(1+y)} \\
&= (1+x)^2\frac{1+x+y}{(1+x)(1+y)}.
\end{aligned}$$

The same argument shows the second form represents (multiplying $y(1+x)(1+y)$ by the square $\frac{1}{(1+x)^2}$.

$$\begin{aligned}
(1+y)+\frac{y(1+y)}{1+x} &= \frac{(1+x+y+xy)+(y+y^2)}{1+x} \\
&= \frac{(1+y)^2+x(1+y)}{1+x} \\
&= \frac{(1+y)^2\left(1+\frac{x}{(1+y)}\right)}{(1+x)} \\
&= (1+y)^2\frac{1+x+y}{(1+x)(1+y)}.
\end{aligned}$$

This solves the generic linkage problem for these elements with

$$\ell = \frac{1+x+y}{(1+x)(1+y)}$$

which is symmetric in x and y.

Checking this calculation directly we see that

$$\begin{aligned}
(x(1+y),\ell) &= \left(x(1+y),(1+x)\frac{1+x+y}{1+y}\right)\\
&= \left(x(1+y),(1+x)\left(1+\frac{x}{1+y}\right)\right)\\
&= (x(1+y),1+x) = (1+y,1+x)
\end{aligned}$$

which by symmetry is a check on what we require, namely that

$$(1+x,1+y) = (x(1+y),\ell) = (y(1+x),\ell).$$

Example 5.1. *As an example, we see what happens in our special case where $x = t$ and $y = \pi$. We find, using the arithmetic of our field the appropriate ell is:*

$$\ell = \frac{1+t+\pi}{(1+t)(1+\pi)} = \frac{1}{(1+\pi)} + \frac{\pi}{(1+t)(1+\pi)} = \left(1+\pi\frac{1}{(1+t)}\right)\frac{1}{(1+\pi)}.$$

Using the power series $\frac{1}{(1+\pi)} = 1 - \pi + \pi^2 - \pi^3 + \pi^4 - \cdots$ *and using* $-1 =$

$1+\pi^2+\pi^3+\pi^4$ we find mod $F^{\bullet^2}$

$$
\begin{aligned}
\ell &= \left(1+\pi\frac{1}{(1+t)}\right)(1-\pi+\pi^2-\pi^3+\pi^4) \\
&= 1+\pi\left(-1+\frac{1}{(1+t)}\right)+\pi^2\left(1-\frac{1}{(1+t)}\right)+\pi^3\left(-1+\frac{1}{(1+t)}\right)+\pi^4 \\
&= 1-\pi\left(\frac{t}{(1+t)}\right)+\pi^2\left(\frac{t}{(1+t)}\right)-\pi^3\left(\frac{t}{(1+t)}\right)+\pi^4 \\
&= 1+\pi\left(\frac{t}{(1+t)}\right)+\pi^2\left(\frac{t}{(1+t)}\right)+\pi^4\left(1+\frac{t}{(1+t)}\right) \\
&= \left(1+\pi\frac{t}{(1+t)}\right)\left(1+\pi^2\frac{t}{(1+t)}\right)\left(1+\pi^3\frac{t^2}{(1+t)^2}\right)\cdot \\
&\qquad \left(1+\pi^4(1+\frac{t}{(1+t)}+\frac{t^3}{(1+t)^3})\right).
\end{aligned}
$$

Moreover, we see that

$$
\begin{aligned}
\left(1+\pi^2\frac{t}{(1+t)}\right) &= \left(1+\pi^2(\frac{t^2}{(1+t)^2}+t\frac{1}{(1+t)^2})\right) \\
&\equiv \left(1+\pi^2 t\frac{1}{(1+t)^2}\right) \\
&\equiv \left(1+\pi^2(\frac{1}{(1+t)^2}+t(\frac{1}{(1+t)^2})\right) \quad \mod \ (1-\pi^2 O_F) \\
&= \left(1+\pi^2\frac{1}{(1+t)}\right).
\end{aligned}
$$

Finally, we can rewrite

$$
\begin{aligned}
\left(1+\pi^4(1+\frac{t}{(1+t)}+\frac{t^3}{(1+t)^3})\right) &= \left(1+\pi^4(\frac{1+t^2+t^3}{(1+t)^3})\right) \\
&= \left(1+\pi^4(\frac{t}{(1+t)^3})\right).
\end{aligned}
$$

Recall the relations used in calculating values of the pairing on generators for

G. Firstly, we have (this much (mod $1+\pi^3 O_F$)):

$(x+y, 1+\pi) =$

$$
\begin{aligned}
&= \left(x+y, (1+\pi\frac{x}{x+y})(1+\pi\frac{y}{x+y})(1+\pi^2\frac{xy}{(x+y)^2})(1+\pi^3 e)\right) \\
&= (x\pi, 1+\pi\frac{x}{x+y}) + (y\pi, 1+\pi\frac{y}{x+y}) \\
&\quad +(x+y, 1+\pi^2\frac{xy}{(x+y)^2}) + (x+y, 1+\pi^3 e) \\
&= (x, 1+\pi\frac{x}{x+y}) + (y, 1+\pi\frac{y}{x+y}) + (x+y, 1+\pi^3 e) \\
&\quad + \left(x+y, (1+\pi^2\frac{xy}{(x+y)^2}\frac{x}{x+y})(1+\pi^2\frac{xy}{(x+y)^2}\frac{y}{x+y})(1+\pi^4 e')\right) \\
&= (x, 1+\pi\frac{x}{x+y}) + (y, 1+\pi\frac{y}{x+y}) + (x+y, 1+\pi^3 e) \\
&\quad +(x, 1+\pi^2\frac{x^2 y}{(x+y)^3}) + (y, 1+\pi^2\frac{xy^2}{(x+y)^3}) + (x+y, 1+\pi^4 e')
\end{aligned}
$$

where we note in the third line we use $(\pi, 1+\pi\frac{x}{x+y})+(\pi, 1+\pi\frac{y}{x+y}) = (\pi, 1+\pi) = 0$.

5.5 General Formulae

Using similar techniques as in the previous section, we have the following general linking formulae:

Formula A1. Whenever $x, y \in F^\bullet$ and

$$(1+x, 1+y) = (x(1+y), 1+x) = (y(1+x), 1+y)$$

Then the linking element ℓ_1 for which $(x(1+y), \ell_1) = (y(1+x), \ell_1) = (1+x, 1+y)$ is given by

$$\ell_1 = (1+x)(1+y)(1+x+y).$$

Formula A2. Whenever $x, y \in F^\bullet$ and

$$(1+x, 1+y) = (y(1+x), 1+y) = (x(1+x)(1+y), 1+x)$$

Then the linking ℓ_2 for which $(y(1+x), \ell_2) = (x(1+x)(1+y), \ell_2) = (1+x, 1+y)$ is given by

$$\ell_2 = (1+y)(1+y+xy).$$

Formula A3. Whenever $x, y \in F^\bullet$ and

$$(1+x, 1+y) = (x(1+x)(1+y), 1+x) = (y(1+x)(1+y), 1+y)$$

Then the linking ℓ_3 for which $(x(1+x)(1+y), \ell_3) = (y(1+x)(1+y), \ell_3) = (1+x, 1+y)$ is given by

$$\ell_3 = 1 - xy.$$

Formula B1. Whenever $x, y \in F^\bullet$ and

$$(x, y) = (x(1+y), y) = (y(1+x), x)$$

Then the linking element ℓ_{B1} for which $(x(1+y), \ell_{B1}) = (y(1+x), \ell_{B1}) = (x, y)$ comes as a common slot from

$$\langle y, xy(1+y)\rangle \quad \text{and} \quad \langle x, xy(1+x)\rangle$$

and is given by

$$\ell_{B1} = yx^2 + (xy + xy^2) = xy^2 + (xy + yx^2).$$

Formula B2. Whenever $x, y \in F^\bullet$ and

$$(x, y) = (x(1+y), y) = (xy(1+x), x)$$

Then the linking ℓ_{B2} for which $(x(1+y), \ell_{B2}) = (xy(1+x), \ell_{B2}) = (x, y)$ comes as a common slot from

$$\langle y, xy(1+y) \rangle \quad \text{and} \quad \langle x, y(1+x) \rangle$$

and is given by

$$\ell_{B2} = y + (xy + xy^2) = xy^2 + (y + xy).$$

Formula B3. Whenever $x, y \in F^\bullet$ and

$$(x, y) = (xy(1+y), y) = (xy(1+x), x)$$

Then the linking ℓ_{B3} for which $(xy(1+y), \ell_{B3}) = (xy(1+x), \ell_{B3}) = (x, y)$ comes as a common slot from

$$\langle y, x(1+y) \rangle \quad \text{and} \quad \langle x, y(1+x) \rangle$$

and is given by

$$\ell_{B3} = x + y + xy.$$

Formula C1. Whenever $x, y \in F^\bullet$ and

$$(1+x, y) = (xy, 1+x) = ((1+x)(1+y), y)$$

Then the linking element ℓ_{C1} for which $(xy, \ell_{C1}) = ((1+x)(1+y), \ell_{C1}) = (1+x, y)$ comes as a common slot from

$$\langle 1+x, xy(1+x)\rangle \quad \text{and} \quad \langle y, y(1+x)(1+x)\rangle$$

and is given by

$$\ell_{C1} = (1+x)y^2 - (xy + x^2y) = -y(1+x)^2 + (y + xy + y^2 + xy^2).$$

Formula C2. Whenever $x, y \in F^\bullet$ and

$$(1+x, y) = (xy, 1+x) = (y(1+x)(1+y), y)$$

Then the linking ℓ_{C2} for which $(xy, \ell_{C2}) = (y(1+x)(1+y), \ell_{C2}) = (1+x, y)$ comes as a common slot from

$$\langle 1+x, xy(1+x)\rangle \quad \text{and} \quad \langle y, (1+x)(1+y)\rangle$$

and is given by

$$\ell_{C2} = (1+x) - (xy + yx^2) = -y(1+x)^2 + (1 + x + y + xy).$$

Formula C3. Whenever $x, y \in F^\bullet$ and

$$(1+x, y) = (xy(1+x), 1+x) = (y(1+x)(1+y), y)$$

Then the linking ℓ_{C3} for which $(xy(1+x), \ell_{C3}) = (y(1+x)(1+y), \ell_{C3}) = (1+x, y)$ comes as a common slot from

$$\langle 1+x, xy\rangle \text{ and } \langle y, (1+x)(1+y)\rangle$$

and is given by

$$\ell_{C3} = 1 + x + xy = -y + (1 + x + y + xy).$$

5.5.1 Verifying These Formulae

To check these formulae in Case A, recall we have the following:

$$\begin{aligned}
(x(1+y), \ell_{A1}) &= \left(x(1+y), (1+x)\frac{1+x+y}{1+y}\right) \\
&= \left(x(1+y), (1+x)\left(1+\frac{x}{1+y}\right)\right) \\
&= (x(1+y), 1+x) = (1+y, 1+x)
\end{aligned}$$

which by symmetry is a check on what we require, namely that

$$(1+x, 1+y) = (x(1+y), \ell_{A1}) = (y(1+x), \ell_{A1}).$$

In the second case, we check that

$$\begin{aligned}
(y(1+x), \ell_{A2}) &= (y(1+x), (1+y)(1+y+xy)) \\
&= (y(1+x), 1+y) + (y+xy, 1+y+xy) \\
&= (1+x, 1+y)
\end{aligned}$$

while also

$$
\begin{aligned}
(y(1+x), \ell_{A2}) &= (y(1+x), (1+y)(1+y+xy)) \\
&= \left(y(1+x), 1+\frac{xy}{1+y}\right) \\
&= \left(\frac{x(1+x)}{1+y}, 1+\frac{xy}{1+y}\right) \\
&= (x(1+x)(1+y), \ell_{A2})
\end{aligned}
$$

where by the preceding we know $(y(1+x), \ell_{A2}) = (1+x, 1+y)$ which gives formula 2.

For the third case, we check that

$$
\begin{aligned}
(x(1+x)(1+y), \ell_{A3}) &= (x(1+x)(1+y), 1-xy) \\
&= (x(1+x)(1+y), 1+x-x(1+y)) \\
&= (x(1+x)(1+y), (1+x)(1-\frac{x(1+y)}{1+x}) \\
&= (x(1+x)(1+y), 1+x) \\
&= (1+y, 1+x)
\end{aligned}
$$

and by symmetry we have

$$
((1+x)(1+y), \ell_{A3}) = (x(1+x)(1+y), 1-xy) = (1+x, 1+y)
$$

as required, checking the three formulas for arbitrary fields containing a square root of -1.

6 Investigating Substructures of the Pairing

In this section, we will be investigating different layers of our filtration in the quaternionic pairing in question. Let $G = F^\bullet/F^{\bullet^2}$ and recall the quaternionic pairing we seek to understand is $q : G \times G \longrightarrow Q$. When restricting our maps to subgroups of G, we can obtain linked quaternionic subpairings. In particular, we see that they are elementary type when we break down the structure into direct sums and group extensions of the associated abstract Witt rings.

6.1 Examples of Linked Quaternionic Subpairings

In order to establish the idea of elementary type with these sub-quaternionic pairings, we give the following definitions for quaternionic pairings which give the same objects we defined before for Witt rings.

Recall that given an abstract Witt ring, R, and an elementary abelian 2-group, Δ, the **group extension** of R is the group ring $R[\Delta]$. In terms of quaternionic pairings, we have the following corresponding definition:

Definition 6.1.1. *Let $q : G \times G \longrightarrow Q$ be a quaternionic pairing and let Δ be an elementary abelian 2-group. The* ***group extension*** *of q is the quaternionic pairing*

$$q_\Delta : (G \times \Delta) \times (G \times \Delta) \longrightarrow Q \oplus (G \otimes \Delta) \oplus \wedge^2 \Delta$$

$$\textit{given by} \quad ((g_1, \delta_1), (g_2, \delta_2)) \mapsto (q(g_1, g_2), (g_1 \otimes \delta_2 + g_2 \otimes \delta_1), \delta_1 \wedge \delta_2)$$

Next, we define the other elementary type decomposition. Recall for R_1, R_2 abstract Witt rings, we defined the **direct sum of** R_1 and R_2 to be the fiber product over $\mathbb{Z}/2\mathbb{Z}$ i.e. $R_1 \coprod_{\mathbb{Z}/2\mathbb{Z}} R_2 = \{(r_1, r_2) \in R_1 \oplus R_2 \ : \ d_1(r_1) = d_2(r_2)\}$

which is also given by

$$\begin{tikzcd}
R_1 \coprod_{\mathbb{Z}/2\mathbb{Z}} R_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & R_1 \arrow[d, "d_2"] \\
R_1 \arrow[r, "d_1"] & \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}$$

where d_i are maps that compute the parity of the dimension of R_i.

On the quaternionic pairing side, we can formulate the following corresponding definition:

Definition 6.1.2. *Let $q_1 : G_1 \times G_1 \longrightarrow Q_1$ and $q_2 : G_2 \times G_2 \longrightarrow Q_2$ be quaternionic pairings. The direct sum of q_1, q_2 is given by*

$$q_1 \oplus q_2 : (G_1 \oplus G_2) \times (G_1 \oplus G_2) \longrightarrow Q_1 \oplus Q_2$$

$$\text{given by} \qquad ((g_1, g_2), (g_3, g_4)) \mapsto (q_1(g_1, g_3), q_2(g_2, g_4))$$

Now that we have formulated these definitions, we investigate restrictions of the quaternionic pairing in question.

Example 6.1. *Consider the associated abstract Witt ring R_0 given by the quaternionic pairing $q_0 : U_i \times U_i \longrightarrow Q$ for $i \geq 3$. Note that this is a "totally radical pairing" as for any $f, g \in O_F$, we have that $q_0(1 + \pi^i f, 1 + \pi^i g) = 0$. This means that the pairing is identically 0 which gives trivial linkage.*

Building off of the previous example, we look at the associated abstract Witt ring R_1 given by the quaternionic pairing $q_1 : (\langle \pi \rangle \times U_4) \times (\langle \pi \rangle \times U_4) \longrightarrow Q$.

Notice that the map given by

$$(\pi, *) : U_4 \hookrightarrow Q$$

is an injection as $(\pi, 1 + 4(x^2 + x)) = 0$ *and we know that* $1 + 4(x^2 + x) \in F^2$ *for any* $x \in O_F$*. Therefore, the quaternionic pairing is a group extension of* R_0 *which means that* $R_1 = R_0 \times \langle \pi \rangle$*.*

Example 6.2. *Now consider the associated abstract Witt ring* R_2 *given by the quaternionic pairing* $q_2 : (\langle \pi \rangle \times U_3) \times (\langle \pi \rangle \times U_3) \longrightarrow Q$*. This is a bit more complicated to deal with as*

$$(\pi, *) : U_4 \hookrightarrow Q$$

has nontrivial kernel; namely, we have that $(\pi, 1 + \pi^3 x^2) = 0$ *for all* $x \in O_F$*. In order to examine* q_2*, decompose* U_3 *as follows:*

$$U_3 = U_{31} \oplus U_{32} \oplus U_4$$

where $U_{31} = \langle 1 + \pi^3 x^2 \rangle$ *and* $U_{32} = \langle 1 + \pi^3 t y^2 \rangle$*. Now, we can say that*

$$(\pi, *) : U_{32} \oplus U_4 \hookrightarrow Q$$

is an injection and $q_2(\pi, U_{31}) = 0$*. Let* S_2 *be the associated abstract Witt ring to the totally radical pairing* $U_{32} \oplus U_4 \longrightarrow 0$*, and let* S_1 *be the associated abstract Witt ring to the totally radical pairing of* U_{31}*. Then, we see that the quaternionic pairing is* $R_2 = S_1 \coprod_{\mathbb{Z}/2\mathbb{Z}} (S_2 \times \langle \pi \rangle)$*.*

Example 6.3. *Consider the associated abstract Witt ring* R_3 *given by the* *quaternionic pairing* $q_3 : (\langle \pi, t\rangle \times U_4) \times (\langle \pi, t\rangle \times U_4) \longrightarrow Q$. *Decompose* U_4 *into* $U_4 = U_{41} \oplus U_{42}$ *where* $U_{41} = \langle 1 + \pi^4 z\rangle$ *where* z *is independent from the subgroup* $\wp(\bar{F}) + t\bar{F}^2$ *and* $U_{42} = \langle 1 + \pi^4 t x^2\rangle$. *Now, notice that* $(t, U_{42}) = 0$,

$$(\pi, *) : \langle t\rangle \times U_4 \hookrightarrow Q$$

is an injection, and

$$(t, *) : U_{41} \hookrightarrow Q$$

is also an injection. With these observations, if we let S_4 *be the associated abstract Witt ring to the radical generated by* U_{42} *and* S_3 *be the associated abstract Witt ring to the radical generated by* U_{41}, *then we have that*

$$R_3 = \left((S_3 \times \langle t\rangle) \coprod_{\mathbb{Z}/2\mathbb{Z}} S_4 \right) \times \langle \pi\rangle$$

6.2 Looking for a Counterexample to Elementary Type Conjecture

Thus far, we have considered examples of subpairings that can be built from an iterated process of group extensions and direct sums of smaller linked substructures. However, not all the subpairings of q are linked. Consider the following subpairing:

Example 6.4. *Consider the associated abstract Witt ring* R_4 *given by the quaternionic pairing* $q_4 : (\langle \pi, t\rangle \times U_3) \times (\langle \pi, t\rangle \times U_3) \longrightarrow Q$. *In order to decompose this into a sequence of direct sums and group extensions, observe the following conditions that must be satisfied:*

Decompose U_3, U_4 into the same way from Examples 6.2 and 6.3 i.e. $U_3 = U_{31} \oplus U_{32} \oplus U_{41} \oplus U_{42}$. Then, because we have the following interaction between $(\pi, 1+\pi^3 ty^2) = (t, 1+\pi^3 x^2)$, we would need

$$\begin{aligned} 0 &= q_4(t, U_{42}) \\ &= q_4(\pi, U_{31}) \\ &= q_4(t\pi, U_{32}) \end{aligned}$$

We also would need the following maps to be injective:

$$(\pi, *) : \langle t \rangle \times U_{32} \times U_4 \hookrightarrow Q$$
$$(t, *) : \langle \pi \rangle \times U_{31} \oplus U_{32} \oplus U_{41} \hookrightarrow Q$$

It turns out that this is our first example of a substructure of our pairing that fails to be linked. This can be seen via the following example. Consider

$$q_4(t, u_{32}) = q_4(\pi, u_{32}) = q_4(t, u_{32}u_{42}) = q_4(\pi, u_{32}u_{31})$$

where $u_{ij} \in U_{ij}$ represent any element in these subgroups. Then, there does not exist an $\ell \in \langle \pi, t \rangle \times U_3$ such that

$$q_4(\ell, u_{32}u_{42}) = q_4(\ell, u_{32}u_{31}) = q_4(t, u_{32}). \tag{*}$$

Proof. In order to show that no such ℓ can exist, we consider the possibilities of ℓ, namely $\ell = u, \ell = \pi u, \ell = tu$ or $\ell = t\pi u$ where $u \in U_3$. We aim to solve

$$(\ell, u_{32}u_{42}) = (\ell, u_{32}u_{31}) = (t, u_{32}) = (\pi, u_{32}).$$

We know that $(\pi t, u_{32}) = 0$, $(\pi, U_{31}) = 0$ and $(t, U_{42}) = 0$. Moreover, we know the maps

$$(\pi, *) : \langle t \rangle \times U_{32} \times U_4 \hookrightarrow Q$$

$$(t, *) : \langle \pi \rangle \times U_{31} \times U_{32} \times U_{41} \hookrightarrow Q$$

are injective as stated above. Since $(U_3, U_3) = 0$, it follows that the three remaining cases to consider, namely $\ell = \pi, t, \pi t$. Notice that if $\ell = \pi$ then

$$(\ell, u_{32}u_{42}) = (\pi, u_{32}u_{42}) = (\pi, u_{32}) + (\pi, u_{42}) \neq (\pi, u_{32})$$

where the inequality follows from the injectivity of the $(\pi, *)$ map. If $\ell = t$ then

$$(\ell, u_{32}u_{31}) = (t, u_{32}u_{31}) = (t, u_{32}) + (t, u_{31}) \neq (t, u_{32})$$

where the inequality follows from the injectivity of the $(t, *)$ map. Finally if $\ell = \pi t$ then

$$(\ell, u_{32}u_{42}) = (\pi t, u_{32}u_{42}) = (\pi t, u_{32}u_{42}) + (\pi t, u_{42}) = (\pi t, u_{42}) \neq (\pi, u_{32})$$

where the inequality follows as $(\pi, u_{32}) \notin V_4$ by formula (4.3.1c) since $u_{32} = 1 + \pi^3 t$ and $t \notin \overline{F}^2$. Thus, no linking element exists. ■

As we have found an a linking problem in the previous pairing, we should try to view this problem in a different light to see how the other substructures within Example 6.4 i.e. Examples 6.2 and 6.3 are twisting together to create a pairing that fails to be linked. More specifically, we need to investigate why linkage fails at this level. Recall that we can represent these pairings via matrices as discussed in section 4.1. This representation can potentially help us see what exactly is failing in the last example.

We can represent the pairing from Example 6.2 as the following matrix:

$$M_2 = \begin{array}{c|cccc|c} & u_{31} & u_{32} & u_{41} & u_{42} & \pi \\ \hline u_{31} & 0 & 0 & 0 & 0 & 0 \\ u_{32} & 0 & 0 & 0 & 0 & \beta_1 \\ u_{41} & 0 & 0 & 0 & 0 & \gamma_1 \\ u_{42} & 0 & 0 & 0 & 0 & \gamma_2 \\ \hline \pi & 0 & \beta_1 & \gamma_1 & \gamma_2 & 0 \end{array}$$

where we see that $\beta_1 = (u_{32}, \pi)$ and $\gamma_1 = (u_{41}, \pi), \gamma_2 = (u_{42}, \pi)$ are all independent classes in Q. Viewing this pairing as a matrix makes it very clear that this is a group extension by π as all the elements in the last row/column are distinct.

We can represent the pairing from Example 6.3 as the following matrix:

$$M_3 = \begin{array}{c|cc|c|c} & u_{41} & u_{42} & t & \pi \\ \hline u_{41} & 0 & 0 & \gamma_3 & \gamma_1 \\ u_{42} & 0 & 0 & 0 & \gamma_2 \\ \hline t & \gamma_3 & 0 & 0 & \alpha \\ \hline \pi & \gamma_1 & \gamma_2 & \alpha & 0 \end{array}$$

where $\alpha = (\pi, t)$ and $\gamma_1 = (u_{41}, \pi), \gamma_2 = (u_{41}, t), \gamma_3 = (u_{42}, \pi$ are all independent classes in Q.

Now, we look at the matrix representation of the pairing from Example 6.4:

$$M_4 = \left[\begin{array}{c|cccc|cc} & u_{31} & u_{32} & u_{41} & u_{42} & \pi & t \\ \hline u_{31} & 0 & 0 & 0 & 0 & 0 & \beta_2 \\ u_{32} & 0 & 0 & 0 & 0 & \beta_1 & \beta_1 \\ u_{41} & 0 & 0 & 0 & 0 & \gamma_1 & \gamma_3 \\ u_{42} & 0 & 0 & 0 & 0 & \gamma_2 & 0 \\ \hline \pi & 0 & \beta_1 & \gamma_1 & \gamma_2 & 0 & \alpha \\ t & \beta_2 & \beta_1 & \gamma_3 & 0 & \alpha & 0 \end{array}\right]$$

We see that with this basis, it is neither a group extension nor a direct sum of the previous two matrices associated to the pairings in Example 6.2, 6.3. In particular, the asymmetry between the β_i and γ_i seems to be a main contributor for this pairing not being of elementary type.

In order to remedy the linkage failure stated in $(*)$, we must find an element $\ell \in F$ (that notably must exist in the bigger field as we know the Witt ring of the entire field must be linked) to link equation $(*)$. If we then consider the extension $q_5 : (\langle \pi, t\rangle \times U_3 \oplus \langle \ell \rangle) \times (\langle \pi, t\rangle \times U_3 \oplus \langle \ell \rangle) \longrightarrow Q$, is this subpairing linked? Does adjoining this ℓ solve all the linking problems with this pairing? What linking problems can it create?

Moreover, if adjoining this ℓ creates more linking problems outside of the subpairing q_4, can we repeat this process of adjoining linking elements finitely many times to obtain a finite substructure that is linked? I believe this process can be be terminated with adjoining finitely many elements without getting the whole field. Since we know that the pairing will eventually be linked if we adjoin everything (as it would correspond to a Witt ring of a field), but I am convinced we do not need everything. This is potentially where I believe a counterexample to the Elementary Type Conjecture lies.

7 Bibliography

References

[1] M. Marshall. Abstract Witt Rings. Queen's Papers in Pure and Applied Mathematics # 57, 1980.

[2] I. Efrat. Valuations, Orderings, and Milnor's K-Theory, American Math Society, Vol. 124, 2006.

[3] J. Arason. The Witt group of a discretely valued field, University of Iceland, Dunhaga 3, IS 107 Reykjavk (2006).

[4] M. Marshall. The elementary type conjecture in quadratic form theory. Talca, 1991.

[5] R. Fitzgerald. Combinatorial Techniques and Abstract Witt Rings III. Pacific Journal of Mathematics, Vol. 148, 1991.

[6] M. Marshall. Open Questions in the Theory of Spaces of Orderings. Journal of Symbolic Logic, Vol. 67, No. 1. 2002.

[7] M. Marshall and J. Yucas. Linked Quaternionic Mappings and their Associated Witt Rings. Pacific Journal of Mathematics, Vol. 95, No. 2, 1981.

[8] A. Carson and M. Marshall. Decomposition of Witt rings. Canad. J. Math. 1928.

[9] M. Marshall. Classification of Finite Spaces of Orderings. Canad. J. Math. 31. 1979.