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# Obstructions to Deformation Quantization of Bundles

DISSERTATION

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for the degree of

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in Mathematics

by

Gregory Gordon Huey

Dissertation Committee:
Professor Vladimir Baranovsky, Chair
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# VITA

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# ABSTRACT OF THE DISSERTATION


Obstructions to Deformation Quantization of Bundles

By

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Doctor of Philosophy in Mathematics

University of California, Irvine, 2022

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The necessary conditions for a quantization of a module over an algebra on a symplectic manifold to exist are investigated. Considered is a symplectic algebraic variety $M$ with a fixed deformation quantization $\mathcal{O}_\hbar$ of its sheaf of regular functions, and a vector bundle $E$ on $M$ with a deformation quantization of order $k$ (as a module over $\mathcal{O}_\hbar$). It is found that range of cohomology classes must vanish if this order admits an extension to quantization of order $\ell > k$. For $\ell < 2k+2$ these conditions are also sufficient. For $\ell \geq 2k+2$ a previously unknown obstruction class is found. To construct an explicit form of the obstruction class, one employs a Gelfand-Fuks map from the Lie algebra cohomology to the de Rham cohomology of $M$. The properties of the Gelfand-Fuks map imply that if a lift of quantization from order $k$ to $\ell$ exists, then any element in the kernel of Lie algebra extension - an obstruction class - is mapped to an element in the image that is equivalent to zero. To illustrate the mechanism behind this statement the the Fedosov connection approach is generalized to realize this class via explicit expressions. The generalized Fedosov connection is treated in a manner analogous to the method employed in Tsygan and Nest [11], wherein the quantization of complex manifolds are studied. It is shown how Gelfand-Fuks classes may be obtained as brackets of the Fedosov connection forms.

# Introduction

Let $E \to M$ be a vector bundle over symplectic manifold $M$. Let the quantization of $E$ exist to order $\hbar^k$, with $\mathfrak{g}_k$ the Lie algebra of the automorphism group. Suppose that the quantization of $E$ can be extended to order $\hbar^\ell$ for some $\ell > k$, with $\mathfrak{g}_\ell$ the Lie algebra of the automorphism group. Let the class $\sigma_{k,\ell} \circ C$ be in $\ker(H^2(\mathfrak{g}_k, L) \xrightarrow{\nu} H^2(\mathfrak{g}_\ell, L))$, where $L$ is a module over the Lie algebras $\mathfrak{g}_\ell$ and $\mathfrak{g}_k$, $\sigma_{k,\ell}$ is the Abelianization operator of definition 1.4 and $\nu$ is a map defined in eq 1.19. $\sigma_{k,\ell} \circ C$ is then an obstruction class to the extension of the quantization from $k$ to $\ell$. For fixed $k, \ell$ an explicit form of $\sigma_{k,\ell} \circ C$ is computed. The primary conjecture of this thesis is:

> The obstruction classes in the Lie algebra cohomology complex for $\mathfrak{g}/\mathfrak{h}_k$ have trivial Gelfand-Fuks image if order $k$ quantization extends to order $\ell$ quantization. Conversely, if $\ell < 2k+2$ and $\sigma_{k,\ell} \circ C$ has trivial Gelfand-Fuks image then order $k$ quantization extends to order $\ell$ quantization.

In chapter 1 basic definitions are presented concerning Lie algebra cohomology as well as relations with de-Rham cohomology, the Frölicher-Nijenhuis bracket (Lie bracket generalized to vector-valued forms), the Lyndon–Hochschild–Serre spectral sequence and how it can be used to compute obstruction classes in Lie algebra cohomology. In chapter 2 basic definitions are presented concerning Transitive Harish-Chandra torsors and the Gelfand-Fuks construction. A necessary condition is for the lifting of the Gelfand-Fuks construction is presented, as well as sufficient conditions in a special (Abelian) case. The curvature class of a sub-algebra projection operator is discussed. In chapter 3 deformation quantization formalism of functions and modules is defined, and the physical motivation of this formalism is presented. Specializing to the vector bundle case, the Fedosov connection is defined, and how the required flatness of the Fedosov connection relates to the triviality of the Gelfand-Fuks image of the Lie algebra cohomology obstruction classes is explained. The primary conjecture of this thesis is proven as theorem 3.3. In chapter 4 details of how flatness of the Fedosov construction are presented, and how the existence of the necessary adjustment requires triviality of the Gelfand-Fuks image of the obstruction class is shown in explicit detail. In the appendices the necessary details of the calculations of the prior chapters are explained: the explicit computation of $\sigma_{k,\ell} \circ C$, the proof of exactness of a portion of the expression for the curvature of the Fedosov connection, and finally a proof of the exactness of certain expressions involving the Frölicher-Nijenhuis bracket.

# Chapter 1

# Lie algebra cohomology

## 1.1 Lie algebra cohomology and its Lie differential

If Lie group $G$ is complex semi-simple and simply connected, then it is completely determined by its Lie algebra $Lie\,(G)$ (and some version of the theory can be extended to reductive groups). In such a case one can calculate the cohomology of $G$ by working only with $Lie\,(G)$ - it is the de Rham cohomology of the complex of differential forms on $G$. One maps this complex of differential forms to a complex of left-invariant differential forms via a quasi-isomorphism $\eta$ that intuitively corresponds to an averaging process. Because these left-invariant differential forms are completely determined by their values at the identity element of $G$, they can be identified with the exterior algebra of the Lie algebra, with a suitable differential. The construction of this differential on an exterior algebra makes sense for any Lie algebra, so it is used to define Lie algebra cohomology for all Lie algebras. More generally one uses a similar construction to define Lie algebra cohomology with coefficients in a module. This leads to the Chevalley–Eilenberg complex. Historically, the Chevalley–Eilenberg complex was first investigated through the study of left-invariant differential forms on Lie groups. For complex semi-simple simply-connected groups this complex will yield the de Rham cohomology of the group. Thus, as one might expect, the differential of the Chevalley–Eilenberg complex in some sense imitates the de Rham differential.

**Definition 1.1.** Let $\kappa$ be a field of zero characteristic. In Chapter 4 it will be assumed $\kappa = \mathbb{C}$.

**Definition 1.2.** The $k$th cohomology group of Lie Algebra $\mathfrak{g}$, with coefficients in the left $\mathfrak{g}$-module $W$, is defined by

$$H^k(\mathfrak{g}, W) := Ext^k_{U(\mathfrak{g})}(\kappa, W) \tag{1.1}$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and $\kappa$ is the ground field (a trivial $U(\mathfrak{g})$-module) Fuks [6] section 9. Equivalently, the cohomology of Lie Algebra $\mathfrak{g}$ can also be defined via the Chevalley–Eilenberg complex. The cochains are defined by

$$C^k(\mathfrak{g}, W) := Hom_\kappa\left(\wedge^k \mathfrak{g}, W\right)$$

One can then define the Chevalley–Eilenberg differential $\delta_{Lie}$ via

$$\begin{aligned} \omega &\in C^k(\mathfrak{g}, W) \\ a_j &\in \mathfrak{g} \\ (\delta_{Lie}\omega)(a_1, \cdots, a_{k+1}) &= \sum_{j=1}^{k+1} (-1)^{j+1} a_j \omega(a_1, \cdots, \widehat{a_j}, \cdots, a_{k+1}) + \\ &\quad \sum_{j=1}^{k+1}\sum_{i=1}^{j-1} (-1)^{j+i} \omega([a_i, a_j], a_1, \cdots, \widehat{a_i}, \cdots, \widehat{a_j}, \cdots, a_{k+1}) \end{aligned} \tag{1.2}$$

The cohomology groups then quantify the deviation from exactness of the Chevalley–Eilenberg complex with the maps $\delta_{Lie} : C^k \to C^{k+1}$

$$H^k(\mathfrak{g}, W) = \frac{\ker\left(\delta_{Lie} : C^k \to C^{k+1}\right)}{Im\left(\delta_{Lie} : C^{k-1} \to C^k\right)}$$

## 1.2 A Lie algebra-like formula for the de Rham differential

The de Rham differential $d_{dR}$ evaluated at a set of vector fields is given by a similar formula to the Lie algebra differential in eq 1.2 Kolar et al. [7] section 7.8.

$$
\begin{aligned}
&\varpi \in \Omega^k(M) \\
&v_j \in \Gamma(TM) \\
(d_{dR}\varpi)(v_1, \cdots, v_{k+1}) \quad &= \quad \sum_{j=1}^{k+1} (-1)^{j+1} v_j \varpi(v_1, \cdots, \widehat{v_j}, \cdots, v_{k+1}) + \\
&\sum_{j=1}^{k+1}\sum_{i=1}^{j-1} (-1)^{j+i} \varpi([v_i, v_j], v_1, \cdots, \widehat{v_i}, \cdots, \widehat{v_j}, \cdots, v_{k+1})
\end{aligned} \tag{1.3}
$$

where $M$ is a manifold. The structural similarity between eq 1.2 and eq 1.3 is expected due to the set of vector fields $\Gamma(TM)$ being a Lie algebra when the Lie bracket is defined to be the commutator of composition of differential operators.

## 1.3 The Frölicher-Nijenhuis bracket

The Frölicher-Nijenhuis bracket is a generalization of the Lie bracket of vectors to a bracket of vector-valued $k$-forms Kolar et al. [7] section 8 and is denoted by $[,]_{FN}$. This bracket is defined by

$$[\cdot,\cdot]_{FN} \quad : \quad \Omega^{k}(M,TM)\times\Omega^{\ell}(M,TM)\to\Omega^{k+\ell}(M,TM)$$

$$X,Y\in\Gamma(TM)$$

$$\phi\in\Omega^{k}(M)\quad \psi\in\Omega^{\ell}(M)$$

$$\begin{aligned}
[\phi\otimes X,\psi\otimes Y]_{FN} &= \phi\wedge\psi\otimes[X,Y]+\Big(-i_{Y}d_{dR}\phi\wedge\psi\otimes X+(-1)^{k\ell}\,i_{X}d_{dR}\psi\wedge\phi\otimes Y\Big)\\
&\quad+\Big(-d_{dR}\left(i_{Y}\phi\wedge\psi\right)\otimes X+(-1)^{k\ell}\,d_{dR}\left(i_{X}\psi\wedge\phi\right)\otimes Y\Big)\\
&= \phi\wedge\psi\otimes[X,Y]+\phi\wedge\mathcal{L}_{X}\psi\otimes Y-\mathcal{L}_{Y}\phi\wedge\psi\otimes X\\
&\quad+(-1)^{k}\left(d_{dR}\phi\wedge i_{X}\psi\otimes Y+i_{Y}\phi\wedge d_{dR}\psi\otimes X\right)
\end{aligned}$$

where $[\cdot,\cdot]$ is the standard vector Lie bracket, $i_X$ is the insertion operator and $\mathcal{L}_X$ is the Lie derivative with respect to a vector. The motivation for this definition is such that the Lie derivative satisfies the familiar a commutation relation relation

$$[\mathcal{L}_{K},\mathcal{L}_{L}]_{FN}=\mathcal{L}_{[K,L]_{FN}}\quad K\in\Omega^{k}(M,TM)\,,\ L\in\Omega^{\ell}(M,TM)$$

The Frölicher-Nijenhuis bracket is denoted by $[,]_{FN}$ and satisfies the following relations

$$[K,L]_{FN} = -(-1)^{k\ell}[L,K]_{FN}$$

$$\begin{aligned} [K_1,[K_2,K_3]_{FN}]_{FN} &= [[K_1,K_2]_{FN},K_3]_{FN} + (-1)^{k_1k_2}[K_2,[K_1,K_3]_{FN}]_{FN} \\ \Rightarrow \quad [K,[K,K]_{FN}]_{FN} &= 0 \end{aligned} \tag{1.4}$$

*also*

if $|K|, |L|$ both odd

or both even

$$\begin{aligned} [L,[K,K]_{FN}]_{FN} &+ [K,[L,K]_{FN}]_{FN} + [K,[K,L]_{FN}]_{FN} = 0 \\ [L,[K,K]_{FN}]_{FN} &= \left((-1)^{k\ell}-1\right)[K,[K,L]_{FN}]_{FN} \end{aligned}$$

The Frölicher-Nijenhuis bracket can be specialized to vector-valued 1-forms as follows

$$K,\, L \in \Omega^1(M,TM), \quad a,\, b \in \Gamma(TM)$$

$$\begin{aligned} [K\,,\,L]_{FN}(a,b) = [Ka\,,\,Lb] + [La\,,\,Kb] + (KL+LK)[a\,,\,b] - K([La\,,\,b] + [a\,,\,Lb]) \\ -L([a\,,\,Kb] + [Ka\,,\,b]) \end{aligned} \tag{1.5}$$

## 1.4 The Lyndon–Hochschild–Serre spectral sequence

### 1.4.1 Calculation of the Lie Algebra Cohomology Classes

**Definition 1.3.** $\mathcal{D}_n$ is the Weyl algebra on $n$ conjugate variable pairs generated over $\kappa$ by $x_j,\, y_j,\, \hbar \;\; 1 \leq j \leq n$ and $Der\left(\mathcal{D}_n\right)$ are derivations on $\mathcal{D}_n$

$$\mathcal{D}_n := \kappa\left[\overrightarrow{x}, \overrightarrow{y}, \hbar\right] / \left\{y_j x_k - x_k y_j = \left[y_j, x_k\right] = \delta_{jk}\hbar\right\}$$

$$\begin{gathered} Der_k\left(\mathcal{D}_n\right) := \left\{ \tfrac{1}{\hbar}\left[f\left(x_j, y_i, \hbar\right), -\right] \middle| \deg\left(\tfrac{1}{\hbar} f\right) \geq k \right\} \\ Der\left(\mathcal{D}_n\right) := Der_{-1}\left(\mathcal{D}_n\right) = \left\{ D : \mathcal{D}_n \to \mathcal{D}_n \middle| D\left(fg\right) = D\left(f\right) g + f D\left(g\right),\; D\left(c\right) = 0 \,\forall c \in \kappa \right\} \end{gathered} \tag{1.6}$$

with $\overrightarrow{x} = (x_1, \ldots, x_n)$, $\overrightarrow{y} = (y_1, \ldots, y_n)$ and the multiplication between $x_j$ and $y_j$ being non-commuting. Assign the degrees

$$\begin{aligned} \deg\left(x_k\right) = \deg\left(y_j\right) &= 1 \\ \deg\left(\hbar\right) &= 2 \\ \deg\left(\tfrac{1}{\hbar}\right) &= -2 \end{aligned} \tag{1.7}$$

Note that the Lie bracket defined in eq 1.6 is compatible with this grading - in that the degree of the result is equal to the sum of the degrees of the arguments. Thus the Weyl algebra is a differential graded Lie algebra.

Let $\mathcal{A}_n := \mathcal{D}_n/\hbar\mathcal{D}_n$. One then defines

$$\begin{aligned}
\mathfrak{g} &= Der\left(\mathcal{D}_n\right) \ltimes gl\left(e, \mathcal{D}_n\right) \\
G &= Aut\left(\mathcal{D}_n\right) \ltimes GL\left(e, \mathcal{D}_n\right) = \exp\left(\deg_{\geq 0}\left(\mathfrak{g}\right)\right) \\
G_0 &= Sp\left(2n, \kappa\right) \times GL\left(e, \kappa\right) \\
\mathfrak{h}_k &\cong \{0\} \ltimes \hbar^{k+1} gl\left(e, \mathcal{D}_n\right) \\
\mathfrak{h}\left(k, \ell\right) &:= \mathfrak{h}_k/\mathfrak{h}_\ell \quad \mathfrak{g}_k := \mathfrak{g}/\mathfrak{h}_k \\
G_{k\geq 1} &:= \exp\left(\mathfrak{g}_k\right)
\end{aligned} \tag{1.8}$$

and the filtration

$$F_k\mathcal{D}_n := \left\{ \frac{1}{\hbar} f\left(x_j, y_i, \hbar\right) \middle| \deg\left(\frac{1}{\hbar} f\right) \geq k \right\} \tag{1.9}$$

where $\deg_m()$ is a projection operator that projects out degrees other than $m$, and $\deg_{\geq 0}()$ is a projection operator that projects out negative degrees, with the degree values defined in eq 1.7. Thus

$$\begin{aligned}
gl\left(e, \mathcal{D}_n\right) &\cong \bigoplus_{j=0}^{j=\infty} \hbar^j gl\left(e, \mathcal{A}_n\right) \\
\mathfrak{h}_k &\cong \{0\} \ltimes \bigoplus_{j=k+1}^{j=\infty} \hbar^j gl\left(e, \mathcal{A}_n\right) \\
\mathfrak{g}/\mathfrak{h}_k &\cong Der\left(\mathcal{D}_n\right) \ltimes gl\left(e, \mathcal{D}_n\right)/\hbar^{k+1} gl\left(e, \mathcal{D}_n\right) \cong Der\left(\mathcal{D}_n\right) \ltimes \bigoplus_{j=0}^{j=k} \hbar^j gl\left(e, \mathcal{A}_n\right) \\
\mathfrak{h}_k/\mathfrak{h}_\ell &\cong \{0\} \ltimes \bigoplus_{j=k+1}^{j=\ell} \hbar^j gl\left(e, \mathcal{A}_n\right) \qquad k < \ell
\end{aligned} \tag{1.10}$$

Lifting the Weyl algebra from modulus $\hbar^k$ to modulus $\hbar^\ell$ corresponds to the short exact sequence

$$\begin{gathered}
k < \ell \\
0 \to \quad \mathfrak{h}_k/\mathfrak{h}_\ell \quad \hookrightarrow \quad \mathfrak{g}/\mathfrak{h}_\ell \quad \longrightarrow \quad \mathfrak{g}/\mathfrak{h}_k \quad \to 0
\end{gathered} \tag{1.11}$$

One then has the short exact sequence Tsygan and Nest [11]

$$\begin{array}{ccccccccc} 0 \to & \frac{1}{\hbar}\mathbb{C}\left[\!\left[\hbar\right]\!\right] & \hookrightarrow & \frac{1}{\hbar}\mathcal{D}_n & \xrightarrow{\zeta} & Der\left(\mathcal{D}_n\right) & \to 0 \\ & & & \zeta\left(f\right) & := & \left[f,-\right] & \end{array} \tag{1.12}$$

Consider the Lie algebra $\mathfrak{g}$ and subalgebra $\mathfrak{h}$, where $\mathfrak{g}$ is a Lie Algebra over ring $R$ with Universal Enveloping Algebra $U\left(\mathfrak{g}\right)$. It is useful to compute the obstruction class(es) (elements of the Cohomology groups) for the extension of the Weyl algebra $\mathfrak{g}$ from modulus $\hbar^k$ to modulus $\hbar^\ell$. A useful computational tool for this purpose is the Lyndon–Hochschild–Serre spectral sequence McCleary [10], Fuks [6] chapter $8^{bis}.2$, section 5 respectively. $L$ is a module over the quotient algebra $\mathfrak{g}/\mathfrak{h}_k$ and hence also over the Lie algebra $\mathfrak{g}/\mathfrak{h}_\ell$ (such as $L = gl_r\left(\mathcal{A}_n\right)$). Note that $\mathfrak{h}_k/\mathfrak{h}_\ell$ is a Lie ideal of $\mathfrak{g}/\mathfrak{h}_\ell$. By Theorem 12.6 of McCleary [10]there is a Lyndon–Hochschild–Serre spectral sequence initialized at the second page

$$E_2^{p,q} = H^p\left(\mathfrak{g}/\mathfrak{h}_k, H^q\left(\mathfrak{h}_k/\mathfrak{h}_\ell, L\right)\right) \to H^{p+q}\left(\mathfrak{g}_\ell, L\right) \tag{1.13}$$

with differentials

$$\begin{array}{c} E_2^{p,q} = H^p\left(\mathfrak{g}/\mathfrak{h}_k, H^q\left(\mathfrak{h}_k/\mathfrak{h}_\ell, L\right)\right) \\ d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1} \quad r \geq 2 \end{array} \tag{1.14}$$

One is particularly interested in the $q = 0$ spectral sequence terms (the edge maps)

$$d_2^{p,0} : H^p_{Lie}\left(\mathfrak{g}/\mathfrak{h}_k, H^0_{Lie}\left(\mathfrak{h}_k/\mathfrak{h}_\ell, L\right)\right) \to E_2^{p,-1} \cong \{0\} \tag{1.15}$$

thus

$$\ker\left(d_2^{p,0}\right) = H^p_{Lie}\left(\mathfrak{g}/\mathfrak{h}_k, H^0_{Lie}\left(\mathfrak{h}_k/\mathfrak{h}_\ell, L\right)\right)$$

For $p = 0, 1$ the domain of the differential mapping into $E_2^{p,0}$ is trivial, and thus the image of that differential is trivial.

$$\begin{aligned} E_3^{0,0} &\cong H^0_{Lie}\left(\mathfrak{g}/\mathfrak{h}_k, H^0_{Lie}\left(\mathfrak{h}_k/\mathfrak{h}_\ell, L\right)\right) \\ E_3^{1,0} &\cong H^1_{Lie}\left(\mathfrak{g}/\mathfrak{h}_k, H^0_{Lie}\left(\mathfrak{h}_k/\mathfrak{h}_\ell, L\right)\right) \\ E_3^{p\geq 2,0} &\cong \frac{H^p_{Lie}\left(\mathfrak{g}/\mathfrak{h}_k, H^0_{Lie}(\mathfrak{h}_k/\mathfrak{h}_\ell, M)\right)}{Im\left(d_2^{p-2,1}: H^{p-2}_{Lie}\left(\mathfrak{g}/\mathfrak{h}_k, H^1_{Lie}(\mathfrak{h}_k/\mathfrak{h}_\ell, L)\right) \to H^p_{Lie}\left(\mathfrak{g}/\mathfrak{h}_k, H^0_{Lie}(\mathfrak{h}_k/\mathfrak{h}_\ell, L)\right)\right)} \end{aligned} \tag{1.16}$$

Using $H^0\left(\mathfrak{h}_k/\mathfrak{h}_\ell, L\right) \cong L^{\mathfrak{h}_k/\mathfrak{h}_\ell}$ one has

$$\begin{aligned} E_3^{0,0} &\cong H^0_{Lie}\left(\mathfrak{g}/\mathfrak{h}_k, L^{\mathfrak{h}_k/\mathfrak{h}_\ell}\right) \\ E_3^{1,0} &\cong H^1_{Lie}\left(\mathfrak{g}/\mathfrak{h}_k, L^{\mathfrak{h}_k/\mathfrak{h}_\ell}\right) \\ E_3^{p\geq 2,0} &\cong \frac{H^p_{Lie}\left(\mathfrak{g}/\mathfrak{h}_k, L^{\mathfrak{h}_k/\mathfrak{h}_\ell}\right)}{Im\left(d_2^{p-2,1}: H^{p-2}_{Lie}\left(\mathfrak{g}/\mathfrak{h}_k, H^1_{Lie}(\mathfrak{h}_k/\mathfrak{h}_\ell, L)\right) \to H^p_{Lie}\left(\mathfrak{g}/\mathfrak{h}_k, L^{\mathfrak{h}_k/\mathfrak{h}_\ell}\right)\right)} \end{aligned} \tag{1.17}$$

Of particular interest will be the differential

$$\begin{gathered} d_2^{0,1} : E_2^{0,1} \to E_2^{2,0} \\ d_2^{0,1} : H^0_{Lie}\left(\mathfrak{g}/\mathfrak{h}_k, H^1_{Lie}\left(\mathfrak{h}_k/\mathfrak{h}_\ell, L\right)\right) \to H^2_{Lie}\left(\mathfrak{g}/\mathfrak{h}_k, H^0_{Lie}\left(\mathfrak{h}_k/\mathfrak{h}_\ell, L\right)\right) \end{gathered} \tag{1.18}$$

Since by assumption the action of $\mathfrak{h}_\mathfrak{k}/\mathfrak{h}_\ell$ on $L$ is trivial, $L^{\mathfrak{h}_k/\mathfrak{h}_\ell} = L$ and in junior degrees the spectral sequence reduces to the long exact sequence

$$0 \to H^1\left(\mathfrak{g}/\mathfrak{h}_k, L\right) \to H^1\left(\mathfrak{g}/\mathfrak{h}_\ell, L\right) \to Hom_\kappa\left(\left(\mathfrak{h}_k/\mathfrak{h}_\ell\right)^{ab}, L\right)^{\mathfrak{g}_k} \xrightarrow{d_2^{0,1}} H^2\left(\mathfrak{g}/\mathfrak{h}_k, L\right) \xrightarrow{\nu} H^2\left(\mathfrak{g}/\mathfrak{h}_\ell, L\right) \tag{1.19}$$

where $ab$ denotes Abelianization: $\left(\mathfrak{h}_k/\mathfrak{h}_\ell\right)^{ab} \cong \left(\mathfrak{h}_k/\mathfrak{h}_\ell\right)/\left[\mathfrak{h}_k/\mathfrak{h}_\ell, \mathfrak{h}_k/\mathfrak{h}_\ell\right]$.

**Definition 1.4.** $\sigma_{k,\ell}$ is the quotient map: $\sigma_{k,\ell}\left(\mathfrak{k}\right) := \left(\mathfrak{k}\right)^{ab} = \mathfrak{k}/\left[\mathfrak{h}_k/\mathfrak{h}_\ell, \mathfrak{h}_k/\mathfrak{h}_\ell\right]$. $\sigma_{k,\ell}\left(\mathfrak{h}_k/\mathfrak{h}_\ell\right)$ is the Abelianization of $\mathfrak{h}_k/\mathfrak{h}_\ell$.

### 1.4.2 Obstruction cocycles

Let $\phi \in Hom_\kappa (\mathfrak{h}_k/\mathfrak{h}_\ell, L)^{\mathfrak{g}_k}$, then $\phi \circ \sigma_{k,\ell} \in Hom_\kappa \left( (\mathfrak{h}_k/\mathfrak{h}_\ell)^{ab}, L \right)^{\mathfrak{g}_k}$. The differential $d_2^{0,1}$ converts the $\mathfrak{g}_k$-invariant $\phi \circ \sigma_{k,\ell} : \mathfrak{h}_\mathfrak{k}/\mathfrak{h}_\ell \to L$ to a map of the form $(\mathfrak{g}/\mathfrak{h}_k) \wedge (\mathfrak{g}/\mathfrak{h}_k) \to L$ in a homomorphic manner. This implies that $d_2^{0,1}$ precomposes with the map

$$C : (\mathfrak{g}/\mathfrak{h}_k) \wedge (\mathfrak{g}/\mathfrak{h}_k) \to \mathfrak{h}_k/\mathfrak{h}_\ell \tag{1.20}$$

where

$$C(a, b) := [\Phi_H(a), \Phi_H(b)] - \Phi_H([a, b]) \tag{1.21}$$

and the projection operator $\Phi_H : \mathfrak{g}/\mathfrak{h}_k \to \mathfrak{g}/\mathfrak{h}_\ell$ is a vector space splitting with the kernel $\mathfrak{h}_k/\mathfrak{h}_\ell$. This yields

$$d_2^{0,1}(\phi \circ \sigma_{k,\ell}) = \phi \circ \sigma_{k,\ell} \circ C \in H^2(\mathfrak{g}/\mathfrak{h}_k, L) \tag{1.22}$$

One can take $\phi = Id$, which gives the following result.

**Theorem 1.1.** *The class* $\sigma_{k,\ell} \circ C$ is in $\ker(H^2(\mathfrak{g}/\mathfrak{h}_k, L) \xrightarrow{\nu} H^2(\mathfrak{g}/\mathfrak{h}_\ell, L))$.

*Proof.* Consider $\phi = Id \in Hom_\kappa \left( (\mathfrak{h}_k/\mathfrak{h}_\ell)^{ab}, L \right)^{\mathfrak{g}_k}$. By eq 1.22 one has $d_2^{0,1}(\phi \circ \sigma_{k,\ell}) = Id \circ \sigma_{k,\ell} \circ C = \sigma_{k,\ell} \circ C \in H^2(\mathfrak{g}/\mathfrak{h}_k, L)$ which implies $\sigma_{k,\ell} \circ C \in Im\left(d_2^{0,1}\right)$. By exactness of eq 1.19 one then has $\sigma_{k,\ell} \circ C \in \ker\left(H^2(\mathfrak{g}/\mathfrak{h}_k, L) \xrightarrow{\nu} H^2(\mathfrak{g}/\mathfrak{h}_\ell, L)\right)$ □

# Chapter 2

# Harish-Chandra torsors and Gelfand-Fuks maps

## 2.1 Harish-Chandra torsors

### 2.1.1 Torsors

**Definition 2.1.** A $G$-torsor $\pi : P \to M$ over an algebraic group $G$ is a non-empty scheme $P$, with an action $G \times P \to P$, given by $(g, p) \mapsto g(p)$ such that:

1. The stabilizer group is trivial for each $p \in P$. Thus, for each $p \in P$, the map $g \mapsto g(p)$ is a bijection.

2. $\pi : P \to M$ is a morphism of schemes such that $\pi(g(p)) = \pi(p)$. Thus, the $G$-action on $P$ preserves the fibers.

3. The $G$-action on $P$ is locally trivial, meaning that there exists an open cover $\{U_\lambda | \lambda \in \Lambda\}$ of $M$ such that for each $\lambda \in \Lambda$, there exist an isomorphism of schemes such that

$\pi^{-1}(U_\lambda) \cong G \times U_\lambda$ commuting with the $G$-action.

Note that a similar definition can be given in the category of smooth manifolds. A popular intuitive description of a torsor (when $M$ is a point) is that it is a group that has 'forgotten' its identity element.

### 2.1.2 Harish-Chandra Pair

**Definition 2.2.** The tuple $(G, \mathfrak{f})$ is a Harish-Chandra pair if $\mathfrak{f}$ is a Lie Algebra and $G$ is a Lie group such that $Lie(G) \hookrightarrow \mathfrak{f}$ where the adjoint action of $G$ on $Lie(G)$ is extended to an action of $G$ on $\mathfrak{f}$, and the differential of this action gives adjoint action of $Lie(G)$ on $\mathfrak{f}$.

**Definition 2.3.** A Harish-Chandra module $V$ over Harish-Chandra pair $(G, \mathfrak{f})$ is the $G$-module $V$, with a $G$-equivalent Lie-homomorphism $\mathfrak{f} \to End_\kappa(V)$ that is an extension of the tangent Lie-homomorphism on $Lie(G)$.

### 2.1.3 Transitive Harish-Chandra torsors

While any torsor over a unipotent group $H$ is trivial in the $\mathcal{C}^\infty$ topology, the situation is more complicated in the holomorphic case.

**Definition 2.4.** A Transitive Harish-Chandra Torsor over the Harish-Chandra Pair $(G, \mathfrak{f})$ is a $G$-torsor $\pi : P \to M$ such that $M$ is a smooth variety, together with a Lie Algebra homomorphism $\mathfrak{f} \to H^0(P, TP)$ that induces a vector bundle isomorphism $\mathfrak{f} \otimes_\kappa \mathcal{O}_P \cong TP$.

The Lie Algebra homomorphism of a Transitive Harish-Chandra torsor can be written as a

$G$-equivarient, $\mathfrak{f}$-valued 1-form

$$\gamma : TP \overset{\cong}{\to} \mathfrak{f} \otimes_{\kappa} \mathcal{O}_P \tag{2.1}$$

Note that $\gamma$ satisfies the Maurer-Cartan equation and restricts to the canonical Maurer-Cartan form on the vector fields tangent to to the fibers of $\pi : P \to M$ Baranovsky [1].

## 2.2 Curvature class of a sub-algebra projection

In what follows, the focus will be on a $(G, \mathfrak{g})$-Harish-Chandra $\pi : P \to M$ and its tangent bundle $TP$, with $\Omega^j(P, TP)$ vector-valued $j$-forms on $P$.

**Definition 2.5.** Let $\Phi_H \in \Omega^1(P, TP)$ be a "horizontal" projection operator on $(G, \mathfrak{g})$-Harish-Chandra torsor $P$, and let Lie algebra $\mathfrak{g}$ have the fixed vector space-splitting $\mathfrak{g} \cong (\mathfrak{g}/\mathfrak{h}) \oplus \mathfrak{h}$, $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie sub-algebra, such that $\Phi_H$ induces the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$, $a \mapsto a/\mathfrak{h}$. The complement is the "vertical" projection operator, defined as $\Phi_V := Id - \Phi_H$. Note that the "horizontal" and "vertical" are presented in quotation marks because these should be treated as labels, and are not the same as the usual horizontal and vertical forms on a principal bundle. Intuitively, these labels represent a subdivision of directions along the Lie algebra $\mathfrak{g}$ into degrees of freedom within some subalgebra $\mathfrak{h}$ (V) and without $(H)$.

The $\Phi_H, \Phi_V := Id - \Phi_H$ are elements in a graded Lie algebra in which the Lie bracket is given by the Frölicher–Nijenhuis bracket: $[\cdot,\cdot]_{FN} : \Omega^1(P, TP) \wedge \Omega^1(P, TP) \to \Omega^2(P, TP) : (K, L) \mapsto [\cdot,\cdot]_{FN}$, the generalization of the Lie bracket from vectors to vector-valued forms defined in eq 1.5.

**Definition 2.6.** $R_V$ is the "vertical" curvature, the obstruction to $\ker(\Phi_H)$ being integrable, and $R_H$ is the "horizontal" curvature, the obstruction to $\ker(\Phi_V)$ being integrable defined as follows

$$
\begin{aligned}
& [\Phi_H\,,\,\Phi_H]_{FN}(a,b) \\
= \quad & 2\left[\Phi_H a\,,\,\Phi_H b\right]-2\Phi_H\left(\left[\Phi_H a\,,\,b\right]-\left[\Phi_H b\,,\,a\right]\right)+2\Phi_H\left[a\,,\,b\right] \\
= \quad & 2\,R_H\left(a\,,\,b\right)+2\,R_V\left(a\,,\,b\right) \qquad (2.2) \\
& \textit{where}: \\
& R_H\left(a\,,\,b\right):=\Phi_V\left[\Phi_H a\,,\,\Phi_H b\right] \qquad R_V\left(a\,,\,b\right):=\Phi_H\left(\left[\Phi_V a\,,\,\Phi_V b\right]\right)
\end{aligned}
$$

In what follows, $\Phi_H$ will be specialize to be a projection into $\mathfrak{g}/\mathfrak{h}_k$, where $\mathfrak{g}$ and $\mathfrak{h}_k$ are defined in eq 1.8. In that case $R_H$ will give a class that is the obstruction to extending the deformation quantization of the vector bundle from $\hbar^k$ to $\hbar^\ell$. It will be shown that $R_V$ vanishes by construction.

## 2.3 Finite Dimensional Reductive Lie Algebra Version of a Gelfand-Fuks map

Let $(G,\mathfrak{f})$ be a Harish-Chandra pair, $P\to M$ a Transitive Harish-Chandra Torsor over $(G,\mathfrak{f})$ and $V$. Let us further assume that $G\cong G_U\rtimes G_R$, where $G_U$ is pro-unipotent, and $G_R$ is connected, finite-dimensional, reductive.

**Definition 2.7.** The Gelfand-Fuks map is defined as follows

$$
\begin{array}{cccc}
GF\ : & C^\bullet\left(\mathfrak{f};V\right) & \to & \Gamma\left(P,\Omega^\bullet\left(P\right)\otimes_\kappa V\right) \\
 & \cup & & \cup \\
 & C^\bullet\left(\mathfrak{f},Lie\left(G_R\right);V\right) & \to & \Gamma\left(P,\Omega^\bullet\left(P\right)\otimes_\kappa V\right)^{G_R-basic} \\
 & \alpha & \mapsto & \beta
\end{array}
$$

$\alpha\in C^\bullet\left(\mathfrak{f},Lie\left(G_R\right);V\right)$ is an $\ell$-cochain $\alpha:\wedge^\ell\mathfrak{f}\to V$, while $\beta\in\Gamma\left(P,\Omega_P^\bullet\otimes_\kappa V\right)^{G_R-basic}$ is a $G_R - basic$ $\ell$-cochain $\beta : \wedge^{\ell} TP \to V \otimes_{\kappa} \mathcal{O}_P$. Recall that $\beta \in \Gamma\left(P, \Omega_P^{\bullet} \otimes_{\kappa} V\right)^{G_R - basic}$ if $\iota_{\xi}\beta = \iota_{\xi} d_{dR}\beta = 0 \;\; \forall \xi \in Lie\left(G_R\right)$. Note that the same notation $GF$ is used for the restriction to the relative cochains, and in that case it takes values in the subspace of basic forms. The Gelfand-Fuks map pre-composes each $\alpha$ 'input' with the Transitive Harish-Chandra 1-form $\gamma$ defined in eq 2.1

$$\begin{array}{ccccc} \wedge^{\ell} TP & \xrightarrow{\gamma} & \wedge^{\ell} \mathfrak{f} \otimes_{\kappa} \mathcal{O}_P & \xrightarrow{\alpha} & V \otimes_{\kappa} \mathcal{O}_P \end{array}$$

$$\beta\left(a_1, \cdots, a_{\ell}\right) \;\; := \;\; \alpha\left(\gamma\left(a_1\right), \cdots, \gamma\left(a_{\ell}\right)\right)$$

Give some $\alpha \in C^{\bullet}\left(\mathfrak{f}; V\right)$, the resulting $\beta = GF\left(\alpha\right)$ can be written as a $V$-valued $\ell$-form on $P$, and the fact that $\gamma$ satisfies the Maurer-Cartan equation implies that $GF$ is consistent with the de Rham differential Fuks [6], Baranovsky [1].

## 2.4 The Gelfand-Fuks Lifting Diagram

Let $G_k := \exp\left(\deg_{\geq 0}\left(\mathfrak{g}/\mathfrak{h}_k\right)\right)$ and $H\left(k, \ell\right) := \exp\left(\deg_{\geq 0}\left(\mathfrak{h}_k/\mathfrak{h}_{\ell}\right)\right)$, where $\deg_{\geq 0}()$ is the previously-defined projection operator on degree, and $\mathfrak{g}$, $\mathfrak{h}_k$ are as defined in eq 1.8. It restricts to identity on the non-negative degrees of the Weyl algebra, while mapping negative degrees to zero. We then have the Lie algebra extension sequence

$$\begin{array}{c} 1 \to H\left(k, \ell\right) \hookrightarrow G_{\ell} \to G_k \to 1 \\ 0 \to \mathfrak{h}_k/\mathfrak{h}_{\ell} \hookrightarrow \mathfrak{g}/\mathfrak{h}_{\ell} \to \mathfrak{g}/\mathfrak{h}_k \to 0 \end{array} \tag{2.3}$$

$P_k \to M$ (respectively $P_{\ell} \to M$) is the $\left(G_k, \mathfrak{g}/\mathfrak{h}_k\right)$ (respectively $\left(G_{\ell}, \mathfrak{g}/\mathfrak{h}_{\ell}\right)$) Harish-Chandra torsor over base-space $M$ that will be associated with the quantization (as will be explained in chapter 3) to order $\hbar^k$ (respectively $\hbar^{\ell}$) with $\ell > k$. $P_k$ is induced from $P_{\ell}$ via the map

$G_\ell \to G_k$ in eq 2.3. $E$ is a module over $G_k$.

The two Gelfand-Fuks maps can be compared via the diagram

$$\begin{array}{ccc} C^\bullet\left(\mathfrak{g}/\mathfrak{h}_k,\, \mathfrak{sp}\left(2n,\kappa\right)\oplus\mathfrak{gl}\left(\mathfrak{e},\kappa\right),\, E\right) & \stackrel{GF}{\longrightarrow} & \Gamma\left(P_k,\Omega\left(P_k\right)\otimes E\right)^{G_k-basic} & \\ \downarrow & & \phi\downarrow & \phi\,\text{a pullback map} \\ C^\bullet\left(\mathfrak{g}/\mathfrak{h}_\ell,\, \mathfrak{sp}\left(2n,\kappa\right)\oplus\mathfrak{gl}\left(\mathfrak{e},\kappa\right),\, E\right) & \stackrel{GF}{\longrightarrow} & \Gamma\left(P_\ell,\Omega\left(P_\ell\right)\otimes E\right)^{G_\ell-basic} & \end{array} \tag{2.4}$$

where $\deg_{\leq 0}(\mathfrak{g})$ is the reductive part of $\mathfrak{g}$. Consider the quotient map: $q_{k\ell} : P_\ell \to P_k = P_\ell/\exp\left(\mathfrak{h}_k/\mathfrak{h}_\ell\right)$. Since $H\left(k,\ell\right)$ is unipotent, $q_{k\ell}$ is a homotopy equivalence which implies $\phi$ is a quasi-isomorphism of de Rham complexes.

## 2.5 Torsor lifting conditions

### 2.5.1 Necessary condition for the torsor to lift

Let $L$ be a module over $\mathfrak{g}_k$ with trivial $\mathfrak{h}_k/\mathfrak{h}_\ell$ action. Via a right-derived functor, or the Lyndon–Hochschild–Serre spectral sequence, the short exact sequence of the Lie algebra extension from $\hbar^k$ to $\hbar^\ell$, $\ell > k$, from eq 1.11 gives rise to the following long exact sequence Weibel [13]

$$\begin{array}{llllllll} 0\to & H^1\left(\mathfrak{g}_k,L\right) & \to & H^1\left(\mathfrak{g}_\ell,L\right) & \to & H^0\left(\mathfrak{g}_k,H^1\left(\mathfrak{h}_k/\mathfrak{h}_\ell,L\right)\right) & \stackrel{d_2^{0,1}}{\to} & H^2\left(\mathfrak{g}_k,L\right) \stackrel{\nu}{\to} \\ \stackrel{\nu}{\to} & H^2\left(\mathfrak{g}_\ell,L\right) & \to\cdots & & & & & \end{array}$$

The group $H^2\left(\mathfrak{g}_k,L\right)$ is the set of equivalence classes of extensions by module $L$. $\ker\left(\nu\right)$ are the obstructions classes for extending quantization from order $k$ to order $\ell$, and by exactness of the sequence, this is isomorphic to $Im\left(d_2^{0,1}\right)$. Note that this $d_2^{0,1}$ is the same as that which appears in eq 1.11. Thus, a non-trivial $\ker(\nu)$ is the reason for the failure of some quantizations to carry over from order $k$ to order $\ell$, and if $Im\left(d_2^{0,1}\right)$ is in the kernel of the Gelfand-Fuks map that implies all quantizations of order $k$ carry over to $\ell$. Thus, we have a key statement:

> If the torsor lifts as in eq 2.4, then it must be true that $Im\left(d_2^{0,1}\right)$ is in the kernel of the Gelfand-Fuks map.

From a mathematical point of view the existence of such a non-trivial class is an obstruction to the extension from order $\hbar^k$ to order $\hbar^\ell$. The class must be trivial for the extension to be self-consistent. From a physics point of view, there is a classical symmetry that holds to order $\hbar^k$, but a divergent amplitude - an anomaly - (for example, a divergent Feynman diagram) occurs at an order in $(k, \ell)$, breaking a classical symmetry at the quantum level. Either way, the result is a pathological theory. For the model to be self-consistent out to $\hbar^\ell$, all of the obstruction classes must be trivial.

### 2.5.2 A sufficient condition for the torsor to lift for Abelian extensions

In eq 2.3 assume that $\mathfrak{h}_k/\mathfrak{h}_\ell$ is Abelian. Set $L = \mathfrak{h}_k/\mathfrak{h}_\ell$. Then $H^2(\mathfrak{g}_k, L)$ has a class $C$ representing the extension of eq 2.3. We use the same notation for cocycle representing the class.

**Proposition 2.1.** *$GF(C)$ is exact if and only if $P_k$ lifts to some $(G_\ell, \mathfrak{g}/\mathfrak{h}_\ell)$ Harish-Chandra torsor $P_\ell$.*

*Proof.* This statement is equivalent to proposition 2.7 in Bezrukavnikov and Kaledin [4]. □

# Chapter 3

# Deformation Quantization

## 3.1 Introduction to Deformation Quantization of functions and modules

One can not discuss deformation quantization without first a discussion of the classical mechanics that the former is a deformation of. To that end, presented first below is the physics-based motivation for the process of deformation quantization.

**Definition 3.1.** The Configuration Space of a physical system is a manifold $W$ of $n$ canonical coordinates $W = \{(q_1, \cdots, q_n)\}$. The value of a tuple $(q_1, \cdots, q_n)$ represents one possible configuration of the system. It will additionally be assumed that $W$ is smooth.

**Definition 3.2.** The velocity phase space - also referred to as the Lagrangian picture - is the tangent bundle $TW$ of the configuration space $W$ is the velocity phase space and consists of the canonical coordinates and velocities $\{(q_1, \dot{q}_1, \cdots, q_n, \dot{q}_n)\}$. The value of a tuple $(q_1, \dot{q}_1, \cdots, q_n, \dot{q}_n)$ represents one possible state of the system. The Lagrangian $L(q_1, \dot{q}_1, \cdots, q_n, \dot{q}_n)$ is a functional of the coordinates and velocities that gives the instantaneous action of the specified state. The total action of a system is the Lagrangian integrated along the trajectory of the system through configuration space.

It is a fundamental principal of classical mechanics that this trajectory is such that action attains a minimum value across all possible trajectories. In quantum system, the system can be treated as following a weighted average of all possible trajectories in configuration space that satisfy specified constraints. The weight of each trajectory decreases as the action of the trajectory increases.

**Definition 3.3.** The momentum phase space - also referred to as the Hamiltonian picture - is the cotangent bundle $T^*W$.

Elements of the total space of $T^*W$ are the canonical coordinates and momenta $(q_1, p_1, \cdots, q_n, p_n)$ where the $q_j$ are the canonical coordinates on $W$ and the $p_j$ are the fiberwise coordinates with respect to the cotangent vectors $dq^j$. Intuitively, this means that the canonical velocities and momenta are colinear, differing by a scale factor.

**Definition 3.4.** Hamiltonian mechanics: Using the canonical variables $q_j$, $p_j$ with $p_j := \frac{\delta L}{\delta \dot{q}_j}$, $\dot{q}_j = \dot{q}_j(\overrightarrow{q}, \overrightarrow{p}, t)$, one defines the Hamiltonian and Hamiltonian mechanics

$$H = \overrightarrow{p} \cdot \dot{\overrightarrow{q}} - L\left(\overrightarrow{q}, \dot{\overrightarrow{q}}, t\right)$$
$$\dot{q}_j = \frac{\delta H}{\delta p_j} \qquad \dot{p}_j = -\frac{\delta H}{\delta q_j}$$

In the treatment that follows, the manifold $M$ will be identified as $M = T^*W$.

**Definition 3.5.** The Poisson bivector $P \in H^0(M, \Lambda^2 TM)$ defines a Poisson bracket $\{\cdot, \cdot\}$ via $\{f, g\} = P(df \wedge dg)$

Note that $[P, P]_{SN} = 0$, where $[\cdot, \cdot]_{SN}$ is the Schouten-Nijenhuis bracket. Coupled with the Poisson bracket $\{\cdot, \cdot\}$, the structure $(M, \{\_, \_\})$ is a Poisson manifold.

**Definition 3.6.** A symplectic manifold $(M, \omega)$ is a smooth manifold $M$ paired with a symplectic 2-form $\omega \in H^0(M, \Lambda^2 T^* M)$ that is closed and non-degenerate.

Note any symplectic 2-form induces an isomorphism between $TM$ and $T^*M$. The symplectic form $\omega : TM \overset{\cong}{\to} T^*M$ is essentially the inverse of the corresponding bivector: $P : T^*M \to TM$.

In the classical limit one begins with the associative multiplication

$$\cdot : \mathcal{O} \times \mathcal{O} \to \mathcal{O} \qquad (a, b) \mapsto a \cdot b \tag{3.1}$$

where $\mathcal{O}$ is the structure sheaf. Additionally, the $\mathcal{O}$-module $E$ on $M$ can be considered. $\mathcal{O}(M)$ acts on module $E$ over $M$ via the associative action

$$\begin{gathered} \cdot : \mathcal{O} \times E \to E \qquad (a, m) \mapsto a \cdot m \\ a \cdot (b \cdot m) = (a \cdot b) \cdot m \end{gathered} \tag{3.2}$$

One then postulates the existence of a quantization structure that is a deformation of these structures. This is accomplished defining a new algebra and a new module product, along with a quantized structure sheaf $\mathcal{O}_\hbar$ and module $E_\hbar$ such that, when expanded over powers of a smallness parameter $\hbar$, the terms of order $\hbar^0$ are precisely the "classical structure" defined in eqs 3.13.2. The smallness parameter $\hbar$ is identified as Planck's Constant and is in the center of $\mathcal{O}_\hbar$. One then has the deformed algebra, module and products

$$\begin{aligned}
* : \mathcal{O}_\hbar \times \mathcal{O}_\hbar \to \mathcal{O}_\hbar \qquad & \mathcal{O}_\hbar := \bigoplus_{j=0}^{\infty} \hbar^j \mathcal{O} \\
a * b = a \cdot b + \sum_{j=1}^{\infty} \hbar^j \beta_j (a, b) \qquad & \beta_j : \mathcal{O}_\hbar \times \mathcal{O}_\hbar \to \mathcal{O}_\hbar \\
* : \mathcal{O}_\hbar \times E_\hbar \to E_\hbar \qquad & E_\hbar := \bigoplus_{j=0}^{\infty} \hbar^j E \\
a * m = a \cdot m + \sum_{j=1}^{\infty} \hbar^j \alpha_j (a, m) \qquad & \alpha_j : A_\hbar \times E_\hbar \to E_\hbar
\end{aligned} \tag{3.3}$$

where the $\beta_j,\ \alpha_j$ are $\hbar$-linear.

Note that the algebra and module products of the structure defined in eq 3.3 are not a-priori associative - associativity must be imposed. This takes the form of constraints upon $\alpha_j,\ \beta_j$ - they are not completely arbitrary. These constraints take the form of the Maurer-Cartan equation. It was shown by Kontsevich [8] that such a deformation quantization is always possible for the algebra alone (for the $(A_\hbar, \beta)$), and that the set of equivalence classes of the $(A_\hbar, \beta)$ are in one-to-one correspondence with the set of equivalence classes of Poisson structures $(M, \{\cdot, \cdot\})$ (or equivalently $(M, \omega)$), modulo diffeomorphisms. One then finds that consistency with Poisson Bracket implies $\beta_1 (a, b) = \frac{1}{2} \{a, b\} = \frac{1}{2} P (da \wedge db)$. Thus, one should focus attention upon the module case.

It is instructive to consider the first few orders of power in $\hbar$ explicitly. Associativity of the algebra and module products requires the following, written as a term-by-term equality in powers of $\hbar$:

$$a*(b*c)=(a*b)*c \quad a*(b*m)=(a*b)*m$$
$$m\in E,\ a,b,c\in A$$

$$\begin{aligned}
\hbar^0: &\quad a\cdot b\cdot m - a\cdot b\cdot m &= 0\\
\hbar^1: &\quad \alpha_1(a,b\cdot m)+a\cdot\alpha_1(b,m)-\alpha_1(a\cdot b,m)-\beta_1(a,b)\cdot m &= 0\\
\hbar^2: &\quad \alpha_2(a,b\cdot m)+\alpha_1(a,\alpha_1(b,m))+a\cdot\alpha_2(b,m)-\alpha_2(a\cdot b,m)\\
&\quad -\alpha_1(\beta_1(a,b),m)-\beta_2(a,b)\cdot m &= 0\\
\hbar^3: &\quad \alpha_3(a,b\cdot m)+\alpha_2(a,\alpha_1(b,m))+\alpha_1(a,\alpha_2(b,m))+a\cdot\alpha_3(b,m)-\alpha_3(a\cdot b,m)\\
&\quad -\alpha_2(\beta_1(a,b),m)-\alpha_1(\beta_2(a,b),m)-\beta_3(a,b)\cdot m &= 0
\end{aligned}$$

Let $\gamma=\alpha\oplus\beta$, where $\gamma$ is an element of the direct sum of the Hochschild cochain complex of the algebra and the Hochschild cochain complex of the module. One defines a differential on this direct sum of cochain complexes

$$\begin{aligned}
(d_{N+1}\gamma_j)(a_0,\cdots,a_N,m)= &\quad a_0\cdot\gamma_j(a_1,\cdots,a_N,m)\\
&+\sum_{i=0}^{N-1}(-1)^{i+1}\gamma_j(a_0,\cdots,a_i\cdot a_{i+1},\cdots,a_N,m)\\
&+(-1)^{N+1}\gamma_j(a_0,\cdots,a_N\cdot m)+(-1)^N\gamma_j(a_0,\cdots,a_N)\cdot m
\end{aligned}$$

In the case of interest $N=1$

$$(d_2\gamma_j)(a,b,m)=a\cdot\gamma_j(b,m)-\gamma_j(a\cdot b,m)+\gamma_j(a,b\cdot m)-\gamma_j(a,b)\cdot m$$

Thus associativity requires

$$\begin{aligned}
\hbar^1: &\quad (d_2\gamma_1)(a,b,m) &= 0\\
\hbar^2: &\quad (d_2\gamma_2)(a,b,m) - \gamma_1(\gamma_1(a,b),m) + \gamma_1(a,\gamma_1(b,m)) &= 0\\
\hbar^3: &\quad (d_2\gamma_3)(a,b,m) - \gamma_2(\gamma_1(a,b),m) - \gamma_1(\gamma_2(a,b),m) + \gamma_2(a,\gamma_1(b,m)) &\\
&\quad +\gamma_1(a,\gamma_2(b,m)) &= 0
\end{aligned}$$

Employing the composition bracket $\frac{1}{2}[\gamma,\gamma] = -\gamma \circ_1 \gamma + \gamma \circ_2 \gamma$ with $\gamma = \sum_{j=1}^{\infty} \hbar^j \gamma_j$ such that $\frac{1}{2}[\gamma,\gamma](a,b,m) = -\gamma(\gamma(a,b),m) + \gamma(a,\gamma(b,m))$ one finds

$$\begin{aligned}
\hbar^2: &\quad \tfrac{1}{\hbar^2}\pi_2\left(\tfrac{1}{2}[\gamma,\gamma](a,b,m)\right) = & -\gamma_1(\gamma_1(a,b),m) + \gamma_1(a,\gamma_1(b,m))\\
\hbar^3: &\quad \tfrac{1}{\hbar^3}\pi_3\left(\tfrac{1}{2}[\gamma,\gamma](a,b,m)\right) = & -\gamma_1(\gamma_2(a,b),m) - \gamma_2(\gamma_1(a,b),m) + \gamma_1(a,\gamma_2(b,m))\\
& & +\gamma_2(a,\gamma_1(b,m))
\end{aligned}$$

where $\pi_m\left(\sum_{j=0}^{\infty} \hbar^j B_j\right) := \hbar^m B_m$ . Up to order $\hbar^3$, this yields

$$\begin{aligned}
\hbar^1 &: \quad \pi_1((d_2\gamma)(a,b,m)) + \pi_1\left(\tfrac{1}{2}[\gamma,\gamma](a,b,m)\right) &= 0\\
\hbar^2 &: \quad \pi_2((d_2\gamma)(a,b,m)) + \pi_2\left(\tfrac{1}{2}[\gamma,\gamma](a,b,m)\right) &= 0\\
\hbar^3 &: \quad \pi_3((d_2\gamma)(a,b,m)) + \pi_3\left(\tfrac{1}{2}[\gamma,\gamma](a,b,m)\right) &= 0
\end{aligned}$$

Thus one has found that an expansion of the associativity condition in powers of $\hbar$ up to cubic order is precisely the Maurer-Cartan equation

$$(d_2\gamma)(a,b,m) + \frac{1}{2}[\gamma,\gamma](a,b,m) = 0$$

**Definition 3.7.** Given a quantization of affine symplectic $M$, the Maurer-Cartan element $\hbar\beta_1 + \hbar^2\beta_2 + \hbar^3\beta_3 + \cdots$ may be converted to a cohomology class in $\hbar H^2_{dR}(M)[[\hbar]] = \hbar\omega_1 + \hbar^2\omega_2 + \hbar^3\omega_3 + \cdots$. This is the Deligne class Bezrukavnikov and Kaledin [4]. In the non-affine case, the Deligne class still exists - however it additionally incorporates data about gluing affine quantizations on double overlaps.

## 3.2 Deformation quantization of a vector bundle

Deformation quantization of a vector bundle over a symplectic algebraic variety is an important area of active research in mathematical physics. Some success has been had with special cases Baranovsky and Chen [2], however the necessary and sufficient conditions of the most general case has thus far proved elusive. In the interest of gaining some insight into what the latter might require, necessary conditions are investigated. In this thesis a smooth symplectic algebraic variety $M$ over a field $\kappa$ of characteristic zero is considered, with an algebraic symplectic form $\omega$ and a fixed deformation quantization $\mathcal{O}_\hbar$ of its sheaf of regular functions (see the next section for the definitions). For a vector bundle $E \to M$ (or rather its sheaf of regular sections which is denoted by the same letter) one would like to construct its deformation quantization $E_\hbar$, a locally free sheaf of $\mathcal{O}_\hbar$ modules reducing to $E$ modulo $\hbar$, or its partial order $k$ analogue $E_k$ which is a locally free sheaf over the quotient $\mathcal{O}_\hbar/\langle \hbar^{k+1} \rangle$ (in particular, one can identify $E$ with $E_0$). Given a partial quantization $E_k$ with $0 \le k < \infty$ one would like to find cohomological obstructions to existence of quantization $E_\ell$ of order $\ell > k$ (including the case $\ell = \infty$). The methods employed are based on computations in Lie algebra cohomology and the Gelfand-Fuks map in formal geometry. One finds that the vanishing of these obstructions classes is a necessary (but perhaps not sufficient) condition for the existence of quantization $E_\ell$.

Recall definition 1.3

$$\mathfrak{g} = Der\left(\mathcal{D}_n\right) \ltimes gl\left(e, \mathcal{D}_n\right)$$

$$G = Aut\left(\mathcal{D}_n\right) \ltimes GL\left(e, \mathcal{D}_n\right) = \exp\left(\deg_{\ge 0}\left(\mathfrak{g}\right)\right)$$

$$G_0 = Sp\left(2n, \kappa\right) \times GL\left(e, \kappa\right)$$

$$\mathfrak{h}_k \cong \{0\} \ltimes \hbar^{k+1} gl\left(e, \mathcal{D}_n\right)$$

$$\mathfrak{h}\left(k, \ell\right) := \mathfrak{h}_k / \mathfrak{h}_\ell \quad \mathfrak{g}_k := \mathfrak{g} / \mathfrak{h}_k$$

$$G_{k \ge 1} := \exp\left(\mathfrak{g}_k\right)$$

Recall the short exact sequence of eq 1.11

$$0 \to \mathfrak{h}\,(k,\ell) \hookrightarrow \mathfrak{g}_\ell \overset{P_k}{\to} \mathfrak{g}_k \to 0$$

with $L$ a module over the quotient Lie algebra $\mathfrak{g}_k \cong \mathfrak{g}_\ell/\mathfrak{h}\,(k,\ell)$ as well as over $\mathfrak{g}_\ell$. Let $P_k$ be a projection of the Weyl algebra modulus $\hbar^{k+1}$. One is interested in obstruction cocycles - cohomology classes which are non-trivial over $\mathfrak{g}_k$, but become trivial over $\mathfrak{g}_\ell$. In section 1.4 the Lyndon–Hochschild–Serre spectral sequence was employed to investigate the kernel of a map of cohomology induced by a lift from $\mathfrak{g}_k \to \mathfrak{g}_\ell$. Using this spectral sequence, the element $\sigma_{k,\ell} \circ C \in \ker(\nu)$ in the long exact sequence of eq 1.19 was found. This is the curvature class of $P_k$ of the Lie algebra of the twisted Weyl bundle, and is an element of the second vector-valued cohomology group of the Lie algebra $\mathfrak{g}_k$ of the twisted Weyl bundle.

For every bundle and quantization up to order $k$ one may construct a torsor over an infinite dimensional group $G_k$, for which the Lie algebra contains $\mathfrak{g}_k = Lie\,(G_k)$. If there exists an order $k$ quantization, it can be described via a Harish-Chandra torsor over $G_k$. The existence of this torsor permits the construction of a Gelfand-Fuks map $GF$, which sends Lie algebra cochains to de Rham cohomology on the base manifold. The properties of the Gelfand-Fuks map imply that if a lift of quantization from order $k$ to $\ell$ exists, then any element in the kernel of $\nu : H^2(\mathfrak{g}_k, L) \to H^2(\mathfrak{g}_\ell, L)$ - an obstruction class as described in subsection 2.5.1 - is mapped to an element in the image that is equivalent to zero ($\ker(GF) \supseteq \ker(\nu : H^2(\mathfrak{g}_k, L) \to H^2(\mathfrak{g}_\ell, L))$). Based on this, one may now articulate the following important theorem, though it will be proved at the end of this chapter as theorem 3.3.

The obstruction classes in the Lie algebra cohomology complex for $\mathfrak{g}_k$ have trivial Gelfand-Fuks image if order $k$ quantization extends to order $\ell$ quantization.

## 3.3 Basics of deformation quantization of functions and bundles.

Recall from definitions 1.3

$$\mathcal{D}_n := \kappa\left[\!\left[\overrightarrow{x}, \overrightarrow{y}, \hbar\right]\!\right] / \left\{y_j x_k - x_k y_j = [y_j, x_k] = \delta_{jk}\hbar\right\}$$

$$Der_k\left(\mathcal{D}_n\right) := \left\{ \tfrac{1}{\hbar}\left[f\left(x_j, y_i, \hbar\right), -\right] \middle| \deg\left(\tfrac{1}{\hbar} f\right) \geq k \right\}$$
$$Der\left(\mathcal{D}_n\right) := Der_{-1}\left(\mathcal{D}_n\right) = \left\{ D : \mathcal{D}_n \to \mathcal{D}_n \middle| D\left(fg\right) = D\left(f\right) g + f D\left(g\right),\ D\left(c\right) = 0\, \forall c \in \kappa \right\}$$

**Definition 3.8.** $\mathcal{M} := \mathcal{D}_n^{\oplus e}$ is a left $\mathcal{D}$-module

Recall the short exact sequence of eq 1.12

$$0 \to \ \tfrac{1}{\hbar}\kappa\left[\!\left[\hbar\right]\!\right] \ \hookrightarrow \ \tfrac{1}{\hbar}\mathcal{D}_n \ \stackrel{\zeta \otimes Id}{\longrightarrow} \ Der\left(\mathcal{D}_n\right) \ \to 0$$

$$\zeta\left(f\right) \quad := \quad \left[f, -\right]$$

Note that $\ker\left(\zeta\right)$ is the center of $\mathcal{D}_n$. Note also that $G = Aut\left(\mathcal{D}_n\right) \ltimes GL\left(e, \mathcal{D}_n\right)$ is a proalgebraic group, which has a reductive part $G_0 = Sp\left(2n, \kappa\right) \times GL\left(e, \kappa\right)$. The pro-unipotent part is $G/G_0$, which has Lie algebra $Der_1\left(\mathcal{D}_n\right) \ltimes \mathfrak{gl}\left(e, \mathcal{D}_n \setminus \kappa\right)$. This is a sub-Lie algebra of $Der_{-1}\left(\mathcal{D}_n\right) \ltimes \mathfrak{gl}\left(e, \mathcal{D}_n\right)$.

To define the completion $\mathcal{O}_h$, one promotes the Weyl algebra to power series

$$\widehat{\mathcal{D}}_n := \kappa\left[\!\left[\overrightarrow{x}, \overrightarrow{y}, \hbar\right]\!\right] / \left\{y_j x_k - x_k y_j = \delta_{jk}\hbar\right\}$$
$$\widehat{\mathcal{A}} := \widehat{\mathcal{D}_n} / \hbar\widehat{\mathcal{D}_n}$$

For every $z \in M$ one can define completions of $\mathcal{O}_h$ and $E_h$ at $z$, which will be denoted $\widehat{\mathcal{O}_h}$ and $\widehat{E_h}$ respectively as the extension of the former to such power series.

**Theorem 3.1.** *For every point $x \in M$, the completion $\widehat{\mathcal{O}_h}$ is locally isomorphic to $\mathcal{D}$ in some Zariski neighborhood containing $x$, and the completion $\widehat{E_h}$ is locally isomorphic to $\mathcal{M}$. Furthermore, any two choices of such isomorphisms are related by action of $G$.*

*Proof.* There exists a Zariski neighborhood containing $x$ such that $\widehat{\mathcal{O}_h} \cong \mathcal{D}$ locally follows from a proposition in section 5.1 of Baranovsky and Ginzburg [3]. The module $\widehat{E_h}/\hbar\mathcal{D}\widehat{E_h}$ is locally free, and thus is locally isomorphic to $\mathcal{M}/\hbar\mathcal{D}_n\mathcal{M} \cong \mathcal{A}^{\oplus e}$ in some Zariski neighborhood. As a locally free module, there exists a local basis $\left\{ s_{0,j} \in \widehat{E_h}/\hbar\widehat{E_h} \middle| 1 \leq j \leq e \right\}$ of $\widehat{E_h}/\hbar\widehat{E_h}$ in that neighborhood. One would then like to prove there exists a lift of $\left\{ s_{0,j} \in \widehat{E_h}/\hbar\widehat{E_h} \middle| 1 \leq j \leq e \right\}$ to $\left\{ s_j \in \widehat{E_h} \middle| 1 \leq j \leq e \right\}$ that is a local basis of $\widehat{E_h}$ in this Zariski neighborhood.

Before going further, one must prove that the $\left\{ s_j \in \widehat{E_h} \middle| 1 \leq j \leq e \right\}$ are linearly independent over $\mathcal{D}$. It will be shown that linear independence of $\left\{ s_j \in \widehat{E_h} \middle| 1 \leq j \leq e \right\}$ is implied by it being a lift of linearly independent $\left\{ s_{0,j} \in \widehat{E_h}/\hbar\widehat{E_h} \middle| 1 \leq j \leq e \right\}$. Suppose, in contradiction, that $\left\{ s_j \in \widehat{E_h} \middle| 1 \leq j \leq e \right\}$ are linearly dependent. Then there would exist $f_j \in \mathcal{D}$ such that $\sum_{j=1}^{e} f_j s_j = 0$. Let $p_j$ be the smallest power of $\hbar$ in $f_j$ that has a non-zero coefficient, and define $p := \min(p_j)$. Define $\overline{f_j} := f_j/\hbar^p$. By construction, one or more $\overline{f_j} \mod \hbar \neq 0$. One then has $\hbar^p \sum_{j=1}^{e} \overline{f_j} s_j = 0$. The module $\widehat{E_h}$ is assumed to be flat over $\mathbb{C}[[\hbar]]$, hence has no $\hbar$ torsion, meaning that one can divide out the $\hbar^p$ - ie: $\hbar^p \sum_{j=1}^{e} \overline{f_j} s_j = 0 \Rightarrow \sum_{j=1}^{e} \overline{f_j} s_j = 0$. Thus: $0 = \left( \sum_{j=1}^{e} \overline{f_j} s_j \right) \mod \hbar = \sum_{j=1}^{e} \left( \overline{f_j} \mod \hbar \right) (s_j \mod \hbar) = \sum_{j=1}^{e} \left( \overline{f_j} \mod \hbar \right) s_{0,j}$. Note that it is not possible for all of the $\left( \overline{f_j} \mod \hbar \right)$ to be zero. Thus, one has a linear dependence among the $\left\{ s_{0,j} \in \widehat{E_h}/\hbar\widehat{E_h} \middle| 1 \leq j \leq e \right\}$ with the $\left( \overline{f_j} \mod \hbar \right)$ as coefficients. This contradicts the $\left\{ s_{0,j} \in \widehat{E_h}/\hbar\widehat{E_h} \middle| 1 \leq j \leq e \right\}$ being a local basis. Thus, no such $f_j$ can exist, and it must be true that $\left\{ s_j \in \widehat{E_h} \middle| 1 \leq j \leq e \right\}$ are linearly independent over $\mathcal{D}$. With this one may conclude that $\left\{ s_j \in \widehat{E_h} \middle| 1 \leq j \leq e \right\}$ is a local basis of $\widehat{E_h}$ in this Zariski neighborhood.

Define the $\mathcal{D}_n$-submodule $Q \subseteq \widehat{E_h}$ generated by the basis $\left\{ s_j \in \widehat{E_h} \middle| 1 \leq j \leq e \right\}$. Since $Q$ is the span of $\mathcal{D}$ of $e$ number of linearly independent $s_j$, one has

$$Q \cong \mathcal{D}_n^{\oplus e} \cong \mathcal{M} \tag{3.4}$$

Consider the composite map

$$\alpha : \quad Q \quad \hookrightarrow \quad \widehat{E_h} \quad \twoheadrightarrow \quad \widehat{E_h}/\hbar\widehat{E_h}$$

Since $\alpha$ surjects $Q$ onto $\widehat{E_h}/\hbar\widehat{E_h}$, one may rewrite

$$\widehat{E_h} \cong Q + \hbar\widehat{E_h} \tag{3.5}$$

Consider now the $\mathcal{D}$-module $\widehat{E_h}/Q$ - using eq 3.5, one has

$$\widehat{E_h}/Q \cong \left(\hbar\widehat{E_h} + Q\right)/Q \cong \left(\hbar\widehat{E_h}\right)/\left(Q \cap \hbar\widehat{E_h}\right) \tag{3.6}$$

Eq 3.4 together with $Q \cong (Q/\hbar Q) \oplus \hbar Q$ and the fact that $\hbar\widehat{E_h}$ contains no elements proportional to a power of $\hbar$ less than 1 implies that

$$\left(\hbar\widehat{E_h}\right)/\left(Q \cap \hbar\widehat{E_h}\right) \cong \left(\hbar\widehat{E_h}\right)/\left(\hbar Q \cap \hbar\widehat{E_h}\right) \cong \hbar\left(\widehat{E_h}/\left(Q \cap \widehat{E_h}\right)\right) \cong \hbar\left(\widehat{E_h}/Q\right)$$

Combined with eq 3.6 yields

$$\widehat{E_h}/Q \cong \hbar\left(\widehat{E_h}/Q\right) \tag{3.7}$$

Let $\overrightarrow{m} = (m_1, \cdots, m_e) \in \widehat{E_h}/Q$ be an arbitrary element. Then eq 3.7 implies there exist matrix $A = [A_{ij}]$, $A_{ij} \in \mathcal{D}$ such that $m = \hbar A m$ which implies

$$(I - \hbar A)\overrightarrow{m} = \overrightarrow{0} \tag{3.8}$$

Operating from the left on both sides of eq 3.8 by $I+\sum_{j=1}^{\infty}\hbar^j A^j$ and using the fact that $\hbar$ is in the center of $\mathcal{D}$ yields

$$\overrightarrow{m}=\left(I+\sum_{j=1}^{\infty}\hbar^j A^j\right)\overrightarrow{0}=\overrightarrow{0}$$

Thus $\widehat{E_h}/Q=\left\{\overrightarrow{0}\right\}$. It follows that $\left\{s_j\in\widehat{E_h}\middle|1\leq j\leq e\right\}$ is a local basis of $\widehat{E_h}$ in this Zariski neighborhood, and thus $\widehat{E_h}$ is locally isomorphic to $\mathcal{M}$.

It will be shown that any two choices of such isomorphisms are related by action of $G$. The different choices of the isomorphisms correspond to different lifted basis sets. Let $\left\{s_j\in\widehat{E_h}\middle|1\leq j\leq e\right\}$ and $\left\{s'_j\in\widehat{E_h}\middle|1\leq j\leq e\right\}$ be two different lifted basis sets. The latter can be written in terms of the former: $s'_k=\sum_{j=1}^{e}t_{kj}s_j$ with $t_{kj}\in\mathcal{D}$. The matrices $[t_{kj}]$ are by construction elements of $GL\left(e,\mathcal{D}\right)$. Thus the action of an element of $G$ relates the different isomorphisms. It should be noted that $\widehat{E_h}$ could be either a left or right module - this difference would be reflected as either $t_{kj}\in\mathcal{D}$ or $t_{kj}\in\mathcal{D}_n^{op}$, which in turn would mean that $[t_{kj}]$ are in $GL\left(e,\mathcal{D}_n\right)$ or $GL\left(e,\mathcal{D}_n^{op}\right)$. However, the action of a matrix in $GL\left(e,\mathcal{D}_n^{op}\right)$ can be represented as an action from the other direction by the transpose of that matrix. Since $t^T\in GL\left(e,\mathcal{D}_n\right)\Leftrightarrow t\in GL\left(e,\mathcal{D}_n\right)$, one can restrict attention to $GL\left(e,\mathcal{D}_n\right)$ instead of $GL\left(e,\mathcal{D}_n^{op}\right)$ without loss of generality. □

For readability and cleanliness of notation, in all that follows $\mathcal{D}_n$ will be be used in place of $\widehat{\mathcal{D}}_n$, $\mathcal{O}_h$ in place of $\widehat{\mathcal{O}_h}$ and $E_h$ in place of $\widehat{E_h}$.

### 3.3.1 How different choices of completion-isomorphisms create the torsor $\mathcal{P}$

**Definition 3.9.** Let $\mathcal{P}_0$ denote a principal $G$-bundle of frames

$$\begin{aligned} \mathcal{P}_0 &= (\text{symplectic frames in } TM) \times_M (\text{usual frames in } E) \\ \mathcal{P}_0 &= \mathrm{Fr}_{Sp(2n,\kappa)}(TM) \times_M \mathrm{Fr}_{GL(e,\kappa)}(E) \end{aligned} \tag{3.9}$$

Denote by $\boldsymbol{W}_E$ the associated bundle

$$\begin{aligned} \boldsymbol{W}_E &= \mathcal{P}_0 \underset{G_0}{\times} \left(\hat{\mathbb{A}}^{\hbar} \oplus \left(\hat{\mathbb{A}}^{\hbar}\right)^e\right) = \left(\mathcal{P}_0 \times \left(\hat{\mathbb{A}}^{\hbar} \oplus \left(\hat{\mathbb{A}}^{\hbar}\right)^e\right)\right) \Big/ G_0 \\ G_0 &:= Sp(2n,\kappa) \times GL(e,\kappa) \end{aligned} \tag{3.10}$$

The Weyl bundle is twisted by the action of $G_0 := Sp(2n,\kappa) \times GL(e,\kappa)$, which gives rise to the semi-direct product structure in the Lie algebra $\mathfrak{g}$. Note that $G_0$ in eq 3.10 acts via $Sp(2n,\kappa)$ as a symplectic automorphism on every copy of $\hat{\mathbb{A}}^{\hbar}$ and $GL(e,\kappa)$ acts as a vector space transformation.

Different choices of the isomorphisms in the second statement of theorem 3.1 comprise a torsor that will be denoted as $\mathcal{P}$. The points of torsor $\mathcal{P}$ parameterize maps onto the base manifold $M$, and the fiber parameterizes all possible isomorphisms of the completions $\widehat{\mathcal{O}_h}$ to $\mathcal{D}$, and $\widehat{E_h}$ to $\mathcal{M}$. By the second statement of theorem 3.1 this fiber has a transitive action. One requires the existence of an equivariant $\mathcal{C}^{\infty}$ section $F$ to be used to pull-back the Deligne class $\omega_\hbar$ from $\mathcal{P}$ to $\mathcal{P}_0$.

$$\begin{array}{lll} \mathcal{P} & & \omega_\hbar \in \Omega^1(\mathcal{P},\mathfrak{g}) \\ \downarrow & \uparrow F & \\ \mathcal{P}_0 & & F^*\omega_\hbar \in \Omega^1(\mathcal{P}_0,\mathfrak{g}) \\ \downarrow & & \\ M & & \end{array} \tag{3.11}$$

**Theorem 3.2.** *There exists a $G_0$-equivariant $\mathcal{C}^\infty$ section $F : \mathcal{P}_0 \to \mathcal{P}$ (note that this does use the fact that $\kappa = \mathbb{C}$)*

*Proof.* Define for $Q := \{(x,y)|\, x \in \mathcal{P}_0,\, y \in G_0(x)\}$, $q : Q \to G_0$, $q(x,y) := a \in G_0\, st\, a \cdot x = y$, where $G_0(x)$ is the orbit of $x$. This is well-defined because the action of $G_0$ on $\mathcal{P}_0$ is free. Let $t \in \mathcal{P}_0$ and $v \in \mathcal{P}$ be arbitrary. One can then define a equivariant section $\widetilde{F} : G_0(t) \to G_0(v)$ as $\widetilde{F}(x) := q(t,x) \cdot v \Rightarrow \widetilde{F}(t) = v$ and $\widetilde{F}(x) := q(t,x) \cdot \widetilde{F}(t)$. Because the Lie group action of $G_0$ on $\mathcal{P}_0$ and $\mathcal{P}$ is smooth $\widetilde{F} : G_0(t) \to G_0(v)$ is a $\mathcal{C}^\infty$ section. It is also equivariant because $\widetilde{F}(b \cdot x) = q(t, b \cdot x) \cdot v = b \cdot q(t,x) \cdot v = b \cdot \widetilde{F}(x)\ \forall b \in G_0$. Note that be freeness of the $G_0$ action and the fact that $\mathcal{P}$ is metrizable via the Fubini–Study metric for quasi-projective spaces, one can apply theorem 3.1.11 of Feragen [5]. Thus there exists a neighborhood of $G_0(t)$ such that there exists a $G_0$-equivariant $\mathcal{C}^\infty$ extension of $\widetilde{F}$ over this neighborhood. One can apply this process to each $G_0$-orbit in $\mathcal{P}_0$, creating thereby an open cover of $\mathcal{P}_0$. Each element in the open cover is the $G_0$-equivariant $\mathcal{C}^\infty$ extension-neighborhood of a $G_0$-orbit. Using this open cover, one can then apply proposition 2.3 of Lashof [9]. The result is that any one of the above $G_0$-equivariant $\mathcal{C}^\infty$ sections defined on a $G_0$-orbit in $\mathcal{P}_0$ can be extended to the entire $\mathcal{P}_0$. One picks one of those extensions, and calls it $F$. Thus one has found a $G_0$-equivariant $\mathcal{C}^\infty$ section $F : \mathcal{P}_0 \to \mathcal{P}$. □

One can show, via an argument structurally similar to section 6.2 of Van den Bergh [12], that $\mathcal{P}$ is a transitive Harish-Chandra torsor with Harish-Chandra Pair $(G, \mathfrak{g})$. This can be understood as a projective limit of schemes. Though not finite dimensional, the group is a projective limit of unipotent finite dimensional groups.

### 3.3.2 Curvature class of the $\mathfrak{h}_k$ sub-algebra projection operator

Recall the definitions of $\Phi_H$, $\Phi_V := Id - \Phi_H$ from definition 2.5 and
$R_H(a\,,\,b) := \Phi_V\,[\Phi_H a\,,\,\Phi_H b]$, $R_V(a\,,\,b) := \Phi_H\,([\Phi_V a\,,\,\Phi_V b])$ from definition 2.6.

**Definition 3.10.** Specialize to $\Phi_H$ projects $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}_k$, where Lie algebra $\mathfrak{g}$ has the fixed canonical vector space-splitting $\mathfrak{g} \cong (\mathfrak{g}/\mathfrak{h}) \oplus \mathfrak{h}$, $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie sub-algebra. This agrees with the $\Phi_H$ used in eq 1.21.

**Lemma 3.1.** *$R_V$ vanishes.*

*Proof.* Note that the minimal power of $\hbar$ in $Im\,(\Phi_V)$ is $\hbar^{k+1}$, and thus the minimal power of $\hbar$ in $Im\,([\Phi_V\,,\,\Phi_V])$ is $\hbar^{2k+2}$. However, $\Phi_H$ projects to zero any power of $\hbar$ of $k+1$ or greater. Thus $\Phi_H\,([\Phi_V a\,,\,\Phi_V b]) = 0\ \forall a,\,b\, \in \mathfrak{g}$ which implies $R_V = 0$, or vanishing co-curvature. □

Thus one finds

$$\begin{aligned} [\Phi_H\,,\,\Phi_H]_{FN} &= 2\,R_H \\ R_H(a\,,\,b) &= \Phi_V\,[\Phi_H a\,,\,\Phi_H b] = C\,(a,b) \\ R_V(a\,,\,b) &= \Phi_H\,([\Phi_V a\,,\,\Phi_V b]) = 0 \end{aligned} \tag{3.12}$$

where $\sigma_{k,\ell} \circ C\,(a,b)$ is the obstruction class to extending the deformation quantization of the vector bundle from from $\hbar^k$ to $\hbar^\ell$ defined in eq 1.20

$$C : \frac{\mathfrak{g}}{\mathfrak{h}_k} \wedge \frac{\mathfrak{g}}{\mathfrak{h}_k} \longrightarrow \mathfrak{h}_k/\mathfrak{h}_\ell$$

Recall the semidirect product structure of $\mathfrak{g}$ from eq 1.8 and the implied structure of its Lie bracket

$$\begin{gathered} a, b \in \mathfrak{g} = Der\left(\mathcal{D}_n\right) \ltimes gl_e\left(\mathcal{D}_n\right) \\ a = \left(\tfrac{1}{\hbar}\left[f_a, -\right], \, B_a\right) \quad b = \left(\tfrac{1}{\hbar}\left[f_b, -\right], \, B_b\right) \end{gathered} \tag{3.13}$$

$$[a, b] = \left(\tfrac{1}{\hbar}\left[\left[f_a, f_b\right], -\right], \, \left[B_a, B_b\right] + \tfrac{1}{\hbar}\left[I_e f_a, B_b\right] - \tfrac{1}{\hbar}\left[I_e f_b, B_a\right]\right)$$

where $I_e$ is the $e \times e$ identity matrix. Utilizing the bracket of eq 3.13 one finds

$$\begin{gathered} C\left(a, b\right) := \Phi_V\left(\left[\Phi_H\left(a\right), \Phi_H\left(b\right)\right]\right) = \left[\Phi_H\left(a\right), \Phi_H\left(b\right)\right] - \Phi_H\left(\left[a, b\right]\right) \\ \\ C\left(a, b\right) \cong \{0\} \ltimes \pi_{k+1,\ell}\left(\left[\pi_{0,k}\left(B_a\right), \pi_{0,k}\left(B_b\right)\right]\right. \\ \left. -\tfrac{1}{\hbar}\left[\pi_{0,k}\left(Id_e f_b\right), \pi_{0,k}\left(B_a\right)\right] + \tfrac{1}{\hbar}\left[\pi_{0,k}\left(Id_e f_a\right), \pi_{0,k}\left(B_b\right)\right]\right) \end{gathered} \tag{3.14}$$

where $\pi_{k+1,\ell}$ projects out term proportional to powers of $\hbar$ from $\hbar^{k+1}$ to $\hbar^{\ell}$ inclusive

$$\pi_{k+1,\ell}\left(\sum_{j=0}^{\infty} \hbar^j \left\{Q_j\right\}_{RYO}\right) := \sum_{j=k+1}^{\ell} \hbar^j \left\{Q_j\right\}_{RYO} \tag{3.15}$$

where $\{\cdots\}_{RYO}$ indicates that the conjugate variables contained therein are assumed to be ordered such that all $y_i$ are always to the right of all $x_j$. Recall the quotient map $\sigma_{k,\ell}\left(\mathfrak{k}\right) := \left(\mathfrak{k}\right)^{ab} = \mathfrak{k}/\left[\mathfrak{h}_k/\mathfrak{h}_\ell, \mathfrak{h}_k/\mathfrak{h}_\ell\right]$ of definition 1.4. It is shown in appendix A that:

If $\ell < 2k+2$ then

$$\sigma_{k,\ell}\left(C\left(a, b\right)\right) \cong \pi_{k+1,\ell}\left(\mathbb{D}\left(B_a, B_b\right) + \mathbb{D}\left(B_a, \frac{1}{\hbar} Id_e f_b\right) + \mathbb{D}\left(\frac{1}{\hbar} Id_e f_a, B_b\right)\right) \tag{3.16}$$

If $\ell \geq 2k+2$ then

$$\begin{gathered} \sigma_{k,\ell}\left(C\left(a, b\right)\right) \cong \\ \pi_{k+1,2k+1}\left(\mathbb{D}\left(B_a, B_b\right) + \mathbb{D}\left(B_a, \tfrac{1}{\hbar} Id_e f_b\right) + \mathbb{D}\left(\tfrac{1}{\hbar} Id_e f_a, B_b\right)\right) \\ -\tfrac{1}{\hbar}\tfrac{1}{e} I_e \mathrm{Tr}\left(\pi_{2k+2,2k+2}\left(\mathbb{D}\left(B_a, B_b\right) + \mathbb{D}\left(B_a, \tfrac{1}{\hbar} Id_e f_b\right) + \mathbb{D}\left(\tfrac{1}{\hbar} Id_e f_a, B_b\right)\right)\right) \end{gathered} \tag{3.17}$$

where

$$\mathbb{D}(X,Y) := \left\{ \left( \pi_{0,k}(X) \exp\left( \hbar \sum_{i=1}^{n} \frac{\overleftarrow{\partial}}{\partial y_i} \frac{\overrightarrow{\partial}}{\partial x_i} \right) \pi_{0,k}(Y) \right) - \left( \pi_{0,k}(Y) \exp\left( \hbar \sum_{i=1}^{n} \frac{\overleftarrow{\partial}}{\partial y_i} \frac{\overrightarrow{\partial}}{\partial x_i} \right) \pi_{0,k}(X) \right) \right\}_{RYO}$$

The term in eq 3.17 of the form

$$\frac{1}{\hbar}\frac{1}{e} I_e \mathrm{Tr}\left(\cdots\right) \tag{3.18}$$

gives a new obstruction class that has not been found in prior literature.

### 3.3.3 Structure of the Fedosov Connection

In order to further elucidate the mechanism by which non-vanishing of $\sigma_{k,\ell} \circ C$ obstructs the existence of an extension of the quantization of the vector bundle from order $\hbar^k$ to order $\hbar^\ell$ it will be necessary to examine the Fedosov connection in detail. Define the connection on the vector bundle portion of $\boldsymbol{W}_E$ as

$$\nabla := Id_M \otimes \nabla_E + \nabla_M \otimes Id_E \tag{3.19}$$

whereas the total connection (which is on all of $\boldsymbol{W}_E$) is

$$\nabla_{tot} := \nabla_M \oplus \nabla = \nabla_M \oplus \left( Id_M \otimes \nabla_E + \nabla_M \otimes Id_E \right) \tag{3.20}$$

$\nabla_M$ is the Fedosov connection constructed in section 5.3 of Tsygan and Nest [11], and $\nabla_E$ is the new connection to be constructed here

$$
\begin{aligned}
\nabla_E &= \nabla_E^{1,0} + \nabla_E^{0,1} \\
\nabla_E^{1,0} &= \nabla_{WE} + A_{E0} + A_E \qquad && A_E = \sum_{j=1}^{\infty} A_{E\,j} \\
\nabla_E^{0,1} &= \bar{\partial} + B_E \qquad && B_E = \sum_{j=1}^{\infty} B_{E\,j} \\
A_{E\,j} &\in \Omega^{1,0}\left(M,\, \left(\mathcal{P}_0 \times \mathfrak{gl}\left(e, \mathcal{D}_n\right)\right)/G_0\right) \qquad && \forall j \geq 1 \\
B_{E\,j} &\in \Omega^{0,1}\left(M,\, \left(\mathcal{P}_0 \times \mathfrak{gl}\left(e, \mathcal{D}_n\right)\right)/G_0\right) \qquad && \forall j \geq 1
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
\nabla_M &= \nabla_M^{1,0} + \nabla_M^{0,1} \\
\nabla_M^{1,0} &= A_{-1} + \nabla_{WM} + A_{M0} + A_M \qquad && A_M = \sum_{j=1}^{\infty} A_{M\,j} \\
\nabla_M^{0,1} &= \bar{\partial} + B_M \qquad && B_M = \sum_{j=1}^{\infty} B_{M\,j}
\end{aligned}
$$

where $A_E$ and $B_E$ act by matrix multiplication (not by $\mathfrak{ad}$ action), $\partial$ is the $(1,0)$ part of the de Rham differential $d_{dR}$ and $\bar{\partial}$ is the $(0,1)$ part of $d_{dR}$, so that $d_{dR} = \partial + \bar{\partial}$. The $\nabla_{WM}$ and $\nabla_{WE}$ are the initial connections of $M$, $E$ respectively - no assumptions should be made regarding their flatness. $A_{E\,0}$ is a degree-zero correction to $\nabla_{WE}$ which one may choose - it is equivalent to the freedom to adjust the connection form by a global $1,0$ form (of degree zero or that contains no powers of $\hbar$). Note that the $Sp\left(2n, \kappa\right)$ acts on the $\mathcal{D}_n$ in $\mathfrak{gl}\left(e, \mathcal{D}_n\right)$ inside $\left(\mathcal{P}_0 \times \mathfrak{gl}\left(e, \mathcal{D}_n\right)\right)$ as well as the frame bundles in $\mathcal{P}_0$. Note that $A_{-1}$ is defined identically to as in section 5.1 of Tsygan and Nest [11] - as the canonical $\left(\mathcal{P}_0 \times \mathfrak{gl}\left(e, \mathcal{D}_n\right)\right)/G_0$-valued 1-form of degree $-1$ on $M$ such that

$$
A_{-1} : TM \xrightarrow{\omega} T^{*}M \tag{3.22}
$$

## 3.4 Vanishing of obstructions to deformation quantization of $E$

**Theorem 3.3.** *The obstruction classes in the Lie algebra cohomology complex for* $\mathfrak{g}/\mathfrak{h}_k$ *have trivial Gelfand-Fuks image if order* $k$ *quantization extends to order* $\ell$ *quantization. Conversely, if* $\ell < 2k+2$ *and* $\sigma_{k,\ell} \circ C$ has *trivial Gelfand-Fuks image then order* $k$ *quantization extends to order* $\ell$ *quantization.*

*Proof.* Assume that order $k$ quantization extends to order $\ell$ quantization. Recall the long exact sequence of eq 1.19

$$0 \to H^1(\mathfrak{g}/\mathfrak{h}_k, L) \to H^1(\mathfrak{g}/\mathfrak{h}_\ell, L) \to Hom_\kappa\left((\mathfrak{h}_k/\mathfrak{h}_\ell)^{ab}, L\right)^{\mathfrak{g}_k} \xrightarrow{d_2^{0,1}} H^2(\mathfrak{g}/\mathfrak{h}_k, L) \xrightarrow{\nu} H^2(\mathfrak{g}/\mathfrak{h}_\ell, L)$$

Order $\ell$ quantization exists implies that the Gelfand-Fuks Lifting Diagram of eq 2.4 exists and commutes

$$\begin{array}{ccc} C^\bullet(\mathfrak{g}/\mathfrak{h}_k, \mathfrak{sp}(2n,\kappa) \oplus \mathfrak{gl}(\mathfrak{e},\kappa), E) & \xrightarrow{GF} & \Gamma(\mathcal{P}_k, \Omega^\bullet(\mathcal{P}_k) \otimes E)^{G_k-basic} \\ \downarrow & & \phi \downarrow \qquad \phi \text{ a pullback map} \\ C^\bullet(\mathfrak{g}/\mathfrak{h}_\ell, \mathfrak{sp}(2n,\kappa) \oplus \mathfrak{gl}(\mathfrak{e},\kappa), E) & \xrightarrow{GF} & \Gamma(\mathcal{P}_\ell, \Omega^\bullet(\mathcal{P}_\ell) \otimes E)^{G_\ell-basic} \end{array}$$

Let the cochain $a \in H^2(\mathfrak{g}/\mathfrak{h}_k, L)$ be an arbitrary element in the kernel of $\nu$ - ie: $a \in \ker(\nu)$. Then $a$ is exact in the lower-left corner of eq 2.4. Let its image there be $\delta_{Lie} b$. That order $k$ quantization extends to order $\ell$ quantization implies that the torsor $\mathcal{P}_\ell$ exists, which then implies that the lower Gelfand-Fuks map is compatible with the differential: $GF(\delta_{Lie} b) = \mathrm{d}_{dR}(GF(b))$. Thus the Gelfand-Fuks image of $\delta_{Lie} b$ is in the zero cohomology class in the lower-right corner of diagram eq 2.4 (on $\mathcal{P}_\ell$). However, $\phi$ is a quasi-isomorphism, thus the Gelfand-Fuks image of $a$ is also trivial in the cohomology class in the upper-right corner (on $\mathcal{P}_k$). Since $a$ was arbitrary, it must be true that the Gelfand-Fuks image of

$\ker(\nu)$ is cohomologically trivial on $\mathcal{P}_k$. Theorem 1.1 proved that the obstruction class $\sigma_{k,\ell} \circ C \in \ker(\nu)$. This implies that $GF(\sigma_{k,\ell} \circ C)$ is trivial on $\mathcal{P}_k$.

Now the "converse" part of theorm 3.3 will be proven. Suppose that $\ell < 2k + 2$ and $\sigma_{k,\ell} \circ C$ has trivial Gelfand-Fuks image. Because $\ell < 2k + 2$, the extension to order $\ell$ is Abelian. Let $L = \mathfrak{h}_k/\mathfrak{h}_\ell$. The triviality of $GF(\sigma_{k,\ell} \circ C)$ by Proposition 2.7 of Bezrukavnikov and Kaledin [4] is equivalent to the existence of a lift from torsor $\mathcal{P}_k$ to torsor $\mathcal{P}_\ell$. The existence of the transitive Harish-Chandra torsor $\mathcal{P}_\ell$ would then imply the existence of a flat connection $\nabla$ (curvature of $\nabla$ vanishes to power $\hbar^\ell$) on the associated vector bundle of truncated Weyl algebras tensored with $E$ (the associated vector bundle of the torsor $\mathcal{P}_\ell$) via a general construction presented in Bezrukavnikov and Kaledin [4]. Intuitively this can be understood as follows: the existence of the transitive Harish-Chandra torsor $\mathcal{P}_\ell$ enlarges the Lie algebra acting on the associated vector bundle to include derivations that are isomorphic to the tangent bundle of the base manifold ($TM$ in this case). This guarantees the existence of a flat connection on that associated vector bundle. Then one could take the $E_\hbar$ up to power $\hbar^\ell$ to be the flat sections of $\nabla$ (that is, sections in $\ker(\nabla)$). This defines the quantization to order $\ell$. □

To illustrate the mechanism behind theorem 3.3 relating to the Gelfand-Fuks map, in Chapter 4 the Fedosov connection approach is generalized to realize this class via an explicit expression. The generalized Fedosov connection is treated in a manner analogous to the method employed in Tsygan and Nest [11], wherein the quantization of complex manifolds are studied. It is shown how Gelfand-Fuks classes may be obtained as Frölicher–Nijenhuis brackets of the Fedosov connection forms.

# Chapter 4

# Obstructions to Deformation Quantization of Bundles

The existence of a flat (Fedosov) connection on the associated vector bundle implies the existence of the quantization of that vector bundle, because the quantization can be taken as the kernel of the Fedosov connection. By assumption, quantization exists and flatness holds for $\nabla$ up to and including order $\hbar^k$ (and also flatness holds to all order in $\hbar$ for $\nabla_M$ via the usual Fedosov construction). One wishes to extend the quantization to order $\hbar^\ell$ for some $\ell > k$. Thus flatness must hold to order $\hbar^k$. The existence of quantization from order $\hbar^{k+1}$ to order $\hbar^\ell$ would require that it is possible to extend the triviality of $[\nabla, \nabla]_{FN}$ from order $\hbar^{k+1}$ to order $\hbar^\ell$, for all degrees.

## 4.1 Imposing Flatness of the Fedosov connection

The curvature of the connection has the following structure

$$\begin{aligned}
[\nabla,\nabla]_{FN} &= [Id_M\otimes\nabla_E, Id_M\otimes\nabla_E]_{FN} + [Id_M\otimes\nabla_E, \nabla_M\otimes Id_E]_{FN} \\
&\quad + [\nabla_M\otimes Id_E, Id_M\otimes\nabla_E]_{FN} + [\nabla_M\otimes Id_E, \nabla_M\otimes Id_E]_{FN} \\
&= Id_M\otimes[\nabla_E,\nabla_E]_{FN} + [\nabla_M,\nabla_M]_{FN}\otimes Id_E + 2\,[Id_M\otimes\nabla_E, \nabla_M\otimes Id_E]_{FN}
\end{aligned} \tag{4.1}$$

The term $[Id_M\otimes\nabla_L, \nabla_M\otimes Id_L]_{FN}$ does not *identically* vanish. That is, it is not trivial by construction, nor by any axioms of the Frölicher-Nijenhuis bracket. It will turn out, however, that via our construction and the choices one makes for adjustments to the connection the vanishing of this term may be imposed. One will impose a flatness condition - that $[\nabla,\nabla]_{FN}$ be trivial. To achieve this, one breaks $[\nabla,\nabla]_{FN}$ into pieces of differing holomorphiticity, and impose that each should vanish separately.

$$[\nabla,\nabla]_{FN} = \begin{pmatrix} [\nabla^{1,0},\nabla^{1,0}]_{FN} \\ 2\,[\nabla^{1,0},\nabla^{0,1}]_{FN} \\ [\nabla^{0,1},\nabla^{0,1}]_{FN} \end{pmatrix} \tag{4.2}$$

$$\begin{aligned}
[\nabla^{1,0},\nabla^{1,0}]_{FN} &= Id_M\otimes\left[\nabla_E^{1,0},\nabla_E^{1,0}\right]_{FN} + \left[\nabla_M^{1,0},\nabla_M^{1,0}\right]_{FN}\otimes Id_L \\
&\quad +2\left[Id_M\otimes\nabla_E^{1,0}, \nabla_M^{1,0}\otimes Id_L\right]_{FN} \\
[\nabla^{1,0},\nabla^{0,1}]_{FN} &= Id_M\otimes\left[\nabla_E^{1,0},\nabla_E^{0,1}\right]_{FN} + \left[\nabla_M^{1,0},\nabla_M^{0,1}\right]_{FN}\otimes Id_E \\
&\quad + \left[Id_M\otimes\nabla_E^{1,0}, \nabla_M^{0,1}\otimes Id_E\right]_{FN} + \left[Id_M\otimes\nabla_E^{0,1}, \nabla_M^{1,0}\otimes Id_E\right]_{FN} \\
[\nabla^{0,1},\nabla^{0,1}]_{FN} &= Id_M\otimes\left[\nabla_E^{0,1},\nabla_E^{0,1}\right]_{FN} + \left[\nabla_M^{0,1},\nabla_M^{0,1}\right]_{FN}\otimes Id_E \\
&\quad +2\left[Id_M\otimes\nabla_E^{0,1}, \nabla_M^{0,1}\otimes Id_E\right]_{FN}
\end{aligned} \tag{4.3}$$

One does not require that $[\nabla,\nabla]_{FN}$ be identically zero - instead one demands it lie in the center of the Weyl algebra.

$$\begin{aligned}
[\nabla,\nabla]_{FN} &= \theta+\phi+\tau \\
&= Id_M\otimes(0)+(\theta_M+\phi_M+\tau_M)\otimes Id_E+\nu^{2,0}+\nu^{1,1}+\nu^{0,2} \\
\theta &= \theta_M\otimes Id_E+\nu^{2,0} \\
\phi &= \phi_M\otimes Id_E+\nu^{1,1} \\
\tau &= \tau_M\otimes Id_E+\nu^{0,2}
\end{aligned} \tag{4.4}$$

with

$$\begin{aligned}
\theta_M &\in \Omega^{2,0}\left(M,\tfrac{1}{\hbar}\mathbb{C}\left[[\hbar]\right]\right) \\
\phi_M &\in \Omega^{1,1}\left(M,\mathbb{C}\left[[\hbar]\right]\right) \\
\tau_M &\in \Omega^{0,2}\left(M,\mathbb{C}\left[[\hbar]\right]\right)
\end{aligned} \tag{4.5}$$

Let it be assumed that the process of Tsygan and Nest [11], section 5.3 is applied to create $\nabla_M$ from $\nabla_{WM}$, and thus one already has $A_{Mj}$, $B_{Mj}$ chosen such that $[\nabla_M,\nabla_M]_{FN}=\theta_M+\phi_M+\tau_M$ where $\theta_M+\phi_M+\tau_M$ are the $(2,0)$, $(1,1)$, $(0,2)$ components of curvature of their constructed connection $\nabla_M$ that lie in the center of the Lie algebra: One imposes

$$[Id_M\otimes\nabla_E\,\boldsymbol{,}\,Id_M\otimes\nabla_E]_{FN} = 0 \tag{4.6}$$

similarly for the cross term

$$\begin{aligned}
\nu^{2,0} &= 0 \\
\nu^{1,1} &= 0 \\
\nu^{0,2} &= 0
\end{aligned} \tag{4.7}$$

which is equivalent to

$$2\left[Id_M\otimes\nabla_E,\nabla_M\otimes Id_E\right]_{FN} = 0 \tag{4.8}$$

To impose the desired flatness condition upon $[\nabla,\nabla]_{FN}$ is thus choosing $A_{Ej}$, $B_{Ej}$ such that the following is identically zero

$$[\nabla,\nabla]_{FN} - \begin{pmatrix} \theta \\ \phi \\ \tau \end{pmatrix} = \begin{pmatrix} [\nabla^{1,0},\nabla^{1,0}]_{FN} - \theta \\ 2\,[\nabla^{1,0},\nabla^{0,1}]_{FN} - \phi \\ [\nabla^{0,1},\nabla^{0,1}]_{FN} - \tau \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.9}$$

This proceeds exactly analogously to the ordinary Fedosov construction. Eq 4.9 is further broken into degrees. One then has an expression for each combination of holomorphicity and degree that must separately vanish. One uses these expression to set the quantities $A_{E\,j}$, $B_{E\,j}$, while the $A_{M\,j}$, $B_{M\,j}$ have already been determined prior. In each of these expressions, one sets the $A_{E\,j}$, $B_{E\,j}$ of greatest $j$ value, which appear as a differentiated term. Thus, for a given $j$, the $A_{E\,j}$, $B_{E\,j+1}$ are found as functions of $A_{M\,i}$, $B_{M\,i}$ for all $i$, and $A_{E\,i}$, $B_{E\,i+1}$ for $i < j$. The cross term $[Id_M \otimes \nabla_E, \nabla_M \otimes Id_E]_{FN}$ is crucial to this process - only through it does $A_{-1}$ occur in brackets with $A_{E\,j}$, $B_{E\,j}$. Since the bracket preserves degree, these terms contain the $A_{E\,j}$, $B_{E\,j}$ - which appear inside of a differential - of greatest $j$ in each expression. To proceed further, one requires the definition of the differential in question.

### 4.1.1 The differential

Consider the complex with the differential $\mathrm{d}_B$

$$\Upsilon^m := \bigoplus_{\substack{p,q \\ p+q=m}} \Omega^{p,q}\left(M,\, \left(\mathcal{P}_0 \times \mathfrak{gl}\left(e, \mathcal{D}_n\right)\right)/G_0\right)$$

$$\mathrm{d}_B := \mathfrak{ad}A_{-1} \otimes Id_E + Id_M \otimes \bar{\partial} + \bar{\partial} \otimes Id_E = \begin{pmatrix} \mathfrak{ad}A_{-1} \otimes Id_E \\ Id_M \otimes \bar{\partial} + \bar{\partial} \otimes Id_E \end{pmatrix}$$

$$\mathrm{d}_B : \Upsilon^m \to \Upsilon^{m+1} \tag{4.10}$$

Impose $[\mathrm{d}_E, \mathrm{d}_E]_{FN} = 0$ (where $[,]_{FN}$ is the Frölicher-Nijenhuis bracket)

$$[\mathrm{d}_B, \mathrm{d}_B]_{FN} = \begin{pmatrix} [A_{-1} \otimes Id_E, A_{-1} \otimes Id_E]_{FN} \\ 2\left[A_{-1} \otimes Id_E, Id_M \otimes \bar{\partial} + \bar{\partial} \otimes Id_E\right]_{FN} \\ \left[Id_M \otimes \bar{\partial} + \bar{\partial} \otimes Id_E, Id_M \otimes \bar{\partial} + \bar{\partial} \otimes Id_E\right]_{FN} \end{pmatrix} \tag{4.11}$$

The $(2,0)$, $(1,1)$ and $(0,2)$ components of $[\mathrm{d}_B, \mathrm{d}_B]_{FN}$ each vanish independently. The $(2,0)$ component of $[\mathrm{d}_B, \mathrm{d}_B]_{FN}$ is $[A_{-1} \otimes Id_E, A_{-1} \otimes Id_E]_{FN} = \frac{-i}{\hbar}\omega \otimes Id_E$. Although this is not zero, it is in the center of the Lie algebra of $\mathfrak{g}$-valued forms, and thus vanishes in a bracket. Furthermore, the Fedosov connection construction previously imposed $\left[A_{-1} \otimes Id_E, \bar{\partial} \otimes Id_E\right]_{FN} = 0$ and $\left[\bar{\partial} \otimes Id_E, \bar{\partial} \otimes Id_E\right]_{FN} = 0$. It would be natural to also assume $\left[Id_M \otimes \bar{\partial}+, Id_M \otimes \bar{\partial}\right]_{FN} = 0$. Thus one finds

$$\begin{array}{rcll}
[A_{-1} \otimes Id_E, A_{-1} \otimes Id_E]_{FN} & = & \frac{-i}{\hbar}\omega \otimes Id_E & \\
\left[A_{-1} \otimes Id_E, \bar{\partial} \otimes Id_E\right]_{FN} & = & \left[A_{-1} \otimes Id_E, Id_M \otimes \bar{\partial}\right]_{FN} & = 0 \\
\left[\bar{\partial} \otimes Id_E, \bar{\partial} \otimes Id_E\right]_{FN} & = & \left[Id_M \otimes \bar{\partial}, Id_M \otimes \bar{\partial}\right]_{FN} & = 0 \\
\left[Id_M \otimes \bar{\partial}, \bar{\partial} \otimes Id_E\right]_{FN} & = 0 & &
\end{array} \tag{4.12}$$

It will turn out that in order for the construction of the flat connection to proceed at each degree, the complex $\Omega^\bullet\left(M, \left(\mathcal{P}_0 \times \mathfrak{gl}\left(e, \mathcal{D}_n\right)\right)/G_0\right)$ should be acyclic at that degree. It will turn out that requiring this to be so will be equivalent (via the Gelfand-Fuks map) to the obstruction class (including the new class of interest) being coboundaries.

It turns out that for each degree one finds

$$\begin{pmatrix} \deg_{j-1}\left([\nabla^{1,0}, \nabla^{1,0}]_{FN} - \theta\right) \\ \deg_j\left(2\,[\nabla^{1,0}, \nabla^{0,1}]_{FN} - \phi\right) \\ \deg_{j+1}\left([\nabla^{0,1}, \nabla^{0,1}]_{FN} - \tau\right) \end{pmatrix} = 2\mathrm{d}_B \begin{pmatrix} Id_M \otimes A_{E\,j} \\ Id_M \otimes B_{E\,j+1} \end{pmatrix} + \begin{pmatrix} \Lambda^{2,0}_{j-1} \\ \Lambda^{1,1}_{j} \\ \Lambda^{0,2}_{j+1} \end{pmatrix} \tag{4.13}$$

where $\deg_j()$ projects out the $j$-th degree part of its argument and $\deg\left(\Lambda^{2,0}_{j-1}\right) = j-1$ , $\deg\left(\Lambda^{1,1}_{j}\right) = j$ , $\deg\left(\Lambda^{0,2}_{j+1}\right) = j+1$. Note that

$$\Lambda = \left[\nabla^{1,0}_{\geq 0} + \nabla^{0,1}_{\geq 1}, \nabla^{1,0}_{\geq 0} + \nabla^{0,1}_{\geq 1}\right]_{FN} + 2\mathrm{d}_B \begin{pmatrix} \nabla^{1,0}_{M\,\geq 0} \otimes Id_E \\ \nabla^{0,1}_{M\,\geq 1} \otimes Id_E \end{pmatrix} - \begin{pmatrix} \theta \\ \phi \\ \tau \end{pmatrix} \tag{4.14}$$

where the $\geq j$ subscript means that only terms with degree $\geq j$ are included. It is proven in Appendix B that $\Lambda_m := \begin{pmatrix} \mathit{\Lambda}^{2,0}_{m-1} \\ \mathit{\Lambda}^{1,1}_{m} \\ \mathit{\Lambda}^{0,2}_{m+1} \end{pmatrix}$ is closed, ie:

$$\mathrm{d}_B \begin{pmatrix} \mathit{\Lambda}^{2,0}_{m-1} \\ \mathit{\Lambda}^{1,1}_{m} \\ \mathit{\Lambda}^{0,2}_{m+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.15}$$

### 4.1.2 Exactness of $\Lambda$

The method that will be used to prove the existence of $A_{Ej}$, $B_{Ej}$ such that eq 4.9 holds will depend on acyclicity of $(\Upsilon^*, \mathrm{d}_B)$ in positive degrees. Here "degrees" refers to the $m$ in $\Upsilon^m$, not the degree of the Weyl algebra. This acyclicity allows one to infer that an expression is exact due to it being closed. Intuitively, this is essentially solving a differential equation by integrating out the derivative.

**Theorem 4.1.** *The cohomology complex* $(\Upsilon^*,\ d_B)$ *is acyclic in positive cohomological degrees.*

*Proof.* Recall from eq 4.10 that $\mathrm{d}_B$ consists of the $(1,0)$ component $\mathfrak{a}\mathfrak{d}A_{-1} \otimes Id_E$ and the $(0,1)$ component $Id_M \otimes \bar{\partial} + \bar{\partial} \otimes Id_E$. The full complex is essentially a tensor product of a Koszul complex and the Dolbeault complex. In what follows, the differentials are precisely the Koszul and Dolbeault differentials. With $\mathrm{d}_B$ broken into the holomorphic and anti-holomorphic terms, the full complex becomes a bi-complex, and the proof is straightforward via a spectral sequence.

$$\begin{aligned} \mathrm{d}_r^{p,q} &: \quad E_r^{p,q} \to E_r^{p+r,q-r+1} \\ E_{r+1}^{p,q} &\cong \quad Z_r^{p,q}/B_r^{p,q} \\ Z_r^{p,q} &:= \ker\left(\mathrm{d}_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}\right) \\ B_r^{p,q} &:= \mathrm{Im}\left(\mathrm{d}_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \to E_r^{p,q}\right) \end{aligned} \tag{4.16}$$

The differential $\mathrm{d}_r^{p,q}$ has bi-degree $(r, 1-r)$. Initialize the spectral sequence at the $r = 0$ page

$$\begin{aligned} E_0^{p,q} &= \Omega^{p,q}\left(M,\, \left(\mathcal{P}_0 \times \mathfrak{gl}\left(e, \mathcal{D}_n\right)\right)/G_0\right) \\ \mathrm{d}_0^{p,q} &= \mathfrak{ad} A_{-1} \otimes Id_E \end{aligned} \tag{4.17}$$

$$\mathrm{d}_0^{p,q} : \Omega^{p,q}\left(M,\, \left(\mathcal{P}_0 \times \mathfrak{gl}\left(e, \mathcal{D}_n\right)\right)/G_0\right) \to \Omega^{p+1,q}\left(M,\, \left(\mathcal{P}_0 \times \mathfrak{gl}\left(e, \mathcal{D}_n\right)\right)/G_0\right)$$

$\mathrm{d}_0^{p,q} = \mathfrak{ad} A_{-1} \otimes Id_E : \Omega^{p,q} \to \Omega^{p+1,q}$ yields a Koszul complex. The Koszul differential is dual to a homotopy map $\Omega^{p+1,q} \to \Omega^{p,q}$, and thus the cohomology in positive degree is quasi-isomorphic to degree zero cohomology group Weibel [13]. Thus one finds that for the $r = 1$ page $E_1^{p,q}$ is trivial except for $p = 0$. Next, on the $r = 1$ page one has a Dolbeault resolution

$$\begin{aligned} E_1^{p,q} &= \Omega^{p,q}\left(M,\, \left(\mathcal{P}_0 \times \mathfrak{gl}\left(e, \mathcal{D}_n\right)\right)/G_0\right) \\ \mathrm{d}_1^{p,q} &= Id_M \otimes \bar{\partial} + \bar{\partial} \otimes Id_E \end{aligned} \tag{4.18}$$

$$\mathrm{d}_1^{p,q} : \Omega^{p,q}\left(M,\, \left(\mathcal{P}_0 \times \mathfrak{gl}\left(e, \mathcal{D}_n\right)\right)/G_0\right) \to \Omega^{p+1,q}\left(M,\, \left(\mathcal{P}_0 \times \mathfrak{gl}\left(e, \mathcal{D}_n\right)\right)/G_0\right)$$

Thus for the $r = 2$ page $E_2^{p,q}$ is trivial except for $(p, q) = (0, 0)$. Intuitively, this is basically a anti-holomorphic version of the Poincare Lemma. Restrict further consideration to $r \geq 2$, and suppose that $E_2^{p,q}$, $(p,q) \neq (0,0)$ is trivial. One finds that for $(p,q) \neq (0,0)$ $\ker\left(\mathrm{d}_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}\right)$ is trivial, thus $E_{r+1}^{p,q}$ is trivial. However, for $(p,q) = (0,0)\,,\ r > 0$ $E_r^{p+r,q-r+1}$ is trivial, implying $\ker\left(\mathrm{d}_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}\right) \cong E_r^{p,q}$. Similarly, $E_r^{p-r,q+r-1}$ is trivial, implying $\mathrm{Im}\left(\mathrm{d}_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \to E_r^{p,q}\right)$ is trivial. Thus one has

$$\begin{gathered} E_{r+1}^{p,q} \cong E_r^{p,q} \quad \forall r \geq 2 \\ E_\infty^{p,q} \cong E_2^{p,q} \end{gathered} \tag{4.19}$$

Overall, it follows that the spectral sequence eq 4.16 degenerates beginning with the second page. Note that

$$\bigoplus_{\substack{p,q \\ p+q=m}} E_\infty^{p,q} \tag{4.20}$$

has an isomorphic filtration to

$$H^m\left(\Upsilon^*\right) \tag{4.21}$$

yielding

$$\begin{aligned} H^0\left(\Upsilon^*\right) &\cong E_\infty^{0,0} \\ H^{m>0}\left(\Upsilon^*\right) &\cong \{0\} \end{aligned} \tag{4.22}$$

thus the cohomology complex $(\Upsilon^*, \mathrm{d}_B)$ is acyclic in positive degrees. □

### 4.1.3 Construction of the flatness adjustment of the Fedosov connection

The cohomological acyclicity proven in Theorem 4.1 implies that $\Lambda_m$ must be exact since it is closed. Thus

$$\begin{pmatrix} \Lambda_{j-1}^{2,0} \\ \Lambda_{j}^{1,1} \\ \Lambda_{j+1}^{0,2} \end{pmatrix} = \mathrm{d}_B \begin{pmatrix} \lambda_{j}^{1,0} \\ \lambda_{j+1}^{0,1} \end{pmatrix} \tag{4.23}$$

Thus one has

$$\begin{pmatrix} \deg_{j-1}\left(\left[\nabla^{1,0}, \nabla^{1,0}\right]_{FN} - \theta\right) \\ \deg_{j}\left(2\left[\nabla^{1,0}, \nabla^{0,1}\right]_{FN} - \phi\right) \\ \deg_{j+1}\left(\left[\nabla^{0,1}, \nabla^{0,1}\right]_{FN} - \tau\right) \end{pmatrix} = 2\mathrm{d}_B \begin{pmatrix} Id_M \otimes A_{E\,j} \\ Id_M \otimes B_{E\,j+1} \end{pmatrix} + \mathrm{d}_B \begin{pmatrix} \lambda_{j}^{1,0} \\ \lambda_{j+1}^{0,1} \end{pmatrix} \tag{4.24}$$

By choosing

$$\begin{pmatrix} Id_M \otimes A_{E\,j} \\ Id_M \otimes B_{E\,j+1} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \lambda_{j}^{1,0} \\ \lambda_{j+1}^{0,1} \end{pmatrix} \tag{4.25}$$

the desired flatness condition is imposed at that degree. It is worth noting that for degree $j$ the closed condition of eq 4.15 depends on eq 4.13 vanishing for degrees less than $j$, as well as the second Bianchi identity holding. While one does make use of the second Bianchi identity, it is worth noting that the second Bianchi identity is a simple consequence of the axioms of the Frölicher-Nijenhuis bracket. Thus, all one is doing is making use of those axioms.

### 4.1.4 A second Gelfand-Fuks map

Let

$$\begin{aligned} \nabla &= Id_M \otimes d_{dR} + d_{dR} \otimes Id_E + S \\ S &= Id_M \otimes S_E + S_M \otimes Id_E \end{aligned} \tag{4.26}$$

be the connection of eq 3.19. The second Gelfand-Fuks map (not to be confused with the first Gelfand-Fuks map, introduced in section 2.4) that will be employed is constructed from the $S$

$$\begin{gathered} GF : \left(C^{\bullet}_{Lie}\left(\mathfrak{g}_{\ell},\, \mathfrak{sp}\left(2n,\mathbb{C}\right)\times\mathfrak{gl}\left(e,\mathbb{C}\right);\, \mathfrak{gl}\left(e,\mathcal{A}_n\right)\right),\, d_{Lie}\right)\to\left(\Omega^{\bullet}\left(M;\, End\left(E\right)\right),\mathrm{d}_{dR}\right) \\ GF\left(\eta\right)\left(X_1,\cdots,X_j\right)=\eta\left(S\left(X_1\right),\cdots,S\left(X_j\right)\right) \end{gathered} \tag{4.27}$$

where

$$\begin{gathered} \Omega^{j}\left(M;\, End\left(E\right)\right) \\ =\left\{\beta\in\left(\Omega^{j}\left(\mathcal{P}_0\right)\otimes\mathfrak{gl}\left(e,\mathcal{A}_n\right)\right)^{G_0}\middle|\, i_X\beta=i_Xd\beta=0\ \forall X\in\mathfrak{sp}\left(2n,\mathbb{C}\right)\times\mathfrak{gl}\left(e,\mathbb{C}\right)\right\} \end{gathered} \tag{4.28}$$

and the $\{X_1,\cdots,X_j\}$ are vector fields on $\mathcal{P}_0$. The $GF$ of eq 4.27 can be used to map the class $C$ of eq 3.12to de Rham cohomology on $M$, eq 4.28.

## 4.2 Putting it all together

To illustrate the mechanism behind theorem 1.1, it will now be shown there is an explicit equivalency between the cohomology class containing $\Lambda$ of eq 4.14, and the Gelfand-Fuks image of the obstruction class of theorem 3.1 and eq 3.12.

**Proposition 4.1.** *$\pi_{k+1,\ell}\left(\zeta\left(\Lambda\right)\right)$ is exact if and only if the obstruction class $\sigma_{k,\ell}\left(GF\circ C\right)$ is trivial.*

*Proof.* The goal is to show that

$$-2\sigma_{k,\ell}\left(GF\circ C\right)=\pi_{k+1,\ell}\left(\zeta\left(\Lambda\right)\right)+\left(\text{exact term}\right) \tag{4.29}$$

where $\zeta$ is defined in eq 1.12. The $GF$ of eq 4.27 can be used to map the class $C$ of eq 3.12

to de Rham cohomology on $M$, eq 4.28. Thus

$$\begin{aligned} GF(C)(a,b) &= C(S(a), S(b)) \\ &= \Phi_V [\Phi_H S(a) \ , \ \Phi_H S(b)] \end{aligned} \tag{4.30}$$

From eq 4.14 one can use eq 4.26 re-express $\Lambda$ as a bracket of $S$ and an exact term

$$\Lambda = [S,S]_{FN} - \begin{pmatrix} \theta_{\geq 0} \\ \phi \\ \tau \end{pmatrix} + (\text{exact term})$$

As the Lie algebra value of $\theta_{\geq 0}$, $\phi$, $\tau$ are in the center of $\mathfrak{g}_\ell$, they are in $\ker(\zeta)$.

To facilitate the proof, break the connection form $S$ into pieces, according to the projection operators $\Phi_H$ and $\Phi_V$

$$\begin{aligned} S = & \quad (\Phi_H + \Phi_V) S (\Phi_H + \Phi_V) \\ = & \quad \Phi_H S \Phi_H + \Phi_V S \Phi_H + \Phi_V S \Phi_V \\ = & \quad S_{HH} + S_{VH} + S_{VV} \end{aligned} \tag{4.31}$$

$$\Phi_H S \Phi_V = 0$$

Note that $\hbar$ is in the center of the Lie algebra, and $S$ contains only non-negative powers of $\hbar$, thus $(\Phi_H S \Phi_V)(a) = 0 \ \forall a \in \mathfrak{g}_\ell$. Additionally, the Lie bracket can not decrease the power of $\hbar$.

$$(GF \circ C)(a,b) \ = \ C(S(a), S(b)) \ = \ \Phi_V [S_{HH}(a) \ , \ S_{HH}(b)] \tag{4.32}$$

One has also

$$\Phi_V\left[S_{HH}+S_{VH}+S_{VV}\,,S_{HH}+S_{VH}+S_{VV}\right]_{FN}(a,b)=$$
$$\Phi_V\left[S_{HH}\,,S_{HH}\right]_{FN}(a,b)$$
$$+2\Phi_V\left[S_{HH}\,,S_{VH}+S_{VV}\right]_{FN}(a,b)$$
$$+\Phi_V\left[S_{VH}\,,S_{VH}+2S_{VV}\right]_{FN}(a,b)$$
$$+\Phi_V\left[S_{VV}\,,S_{VV}\right]_{FN}(a,b)+(\text{exact term})$$

$$a,b\in\mathfrak{g}_\ell$$

By assumption, quantization exists up to and including order $\hbar^k$, thus flatness holds for $\nabla$, and also to all order in $\hbar$ for $\nabla_M$ (via the Fedosov construction). One wishes to extend the quantization to order $\hbar^\ell$ for some $\ell > k$. Thus eq 4.9 holds to order $\hbar^k$. The existence of quantization from order $\hbar^{k+1}$ to order $\hbar^\ell$ would require that it is possible to extend the solution of

$$2\mathrm{d}_B\begin{pmatrix} Id_M\otimes S_E^{1,0}\\ Id_M\otimes S_{E\geq 1}^{0,1}\end{pmatrix}+\Lambda=0 \tag{4.33}$$

from order $\hbar^{k+1}$ to order $\hbar^\ell$, for all degrees. It is fruitful to imagine a basis for $\mathfrak{g}/\mathfrak{h}_\ell$ involving $\hbar^\rho x_i^\mu y_j^\nu$ with $\rho,\mu,\nu\in\mathbb{N}$ and having degree $2\rho+\mu+\nu$. Then, for powers or $\hbar$ from $k+1$ to $\ell$, and iteratively for degree from $-1$ to $\infty$ one would require $\Lambda$ to be exact up order $\hbar^\ell$. This in turn implies the existence of a solution to eq 4.33 at each degree which holds up to order $\hbar^\ell$.

By assumption, it is already true that

$$\Phi_H\left(2\mathrm{d}_B\begin{pmatrix} Id_M\otimes S_E^{1,0}\\ Id_M\otimes S_{E\geq 1}^{0,1}\end{pmatrix}+\Lambda\right)=0 \tag{4.34}$$

Thus the goal is to show that

$$\begin{gathered}\sigma_{k,\ell}\left(\Phi_V\left[S_{HH}\left(a\right)\,,S_{HH}\left(b\right)\right]\right) = \\ \gamma\pi_{k+1,\ell}\left(\zeta\left(\Phi_V\left[S_{HH}+S_{VH}+S_{VV}\,,S_{HH}+S_{VH}+S_{VV}\right]_{FN}\left(a,b\right)\right)\right)+\left(\text{exact term}\right)\end{gathered} \tag{4.35}$$

where $\gamma \in \mathbb{C}$ (recall that in this treatment $\hbar \notin \mathbb{C}$). Since the left-hand side of eq 4.35 is independent of $S_{VH}$ and $S_{VV}$, it must be shown that terms involving these components are exact.

Thanks to the structural similarity in differentials presented in section 1.2, $GF \circ \delta_{Lie} = d_L \circ GF$, where $d_L$ is the differential on the complex formed from eq 4.28. Thus a straightforward computation reveals that

$$\begin{gathered}\left[W\left(a\right),U\left(b\right)\right]+\left[U\left(a\right),W\left(b\right)\right]=U\left(\left(\delta_{Lie}W\right)\left(a,b\right)\right) \\ +\left(\delta_{Lie}U\right)\left(W\left(a\right),b\right)+\left(\delta_{Lie}U\right)\left(a,W\left(b\right)\right)-\left(\delta_{Lie}\left(UW\right)\right) \\ \\ \left[U,W\right]_{FN}\left(a,b\right)=-\left[U\left(a\right),W\left(b\right)\right]-\left[W\left(a\right),U\left(b\right)\right]+ \\ +\left(\left(\delta_{Lie}U\right)\left(W\left(a\right),b\right)+\left(\delta_{Lie}U\right)\left(a,W\left(b\right)\right)+\left(\delta_{Lie}W\right)\left(U\left(a\right),b\right)+\left(\delta_{Lie}W\right)\left(a,U\left(b\right)\right)\right. \\ \left.-\left(\delta_{Lie}\left(UW\right)\right)\left(a,b\right)-\left(\delta_{Lie}\left(WU\right)\right)\left(a,b\right)\right)\end{gathered} \tag{4.36}$$

for arbitrary $U, W \in \Omega^1\left(\mathcal{P},\, T\mathcal{P}\right)$. Thus

$$\begin{gathered}\left[W\left(a\right),U\left(b\right)\right]+\left[U\left(a\right),W\left(b\right)\right]=U\left(\left(\delta_{Lie}W\right)\left(a,b\right)\right)+\left(\text{exact term}\right) \\ \\ \left[U,W\right]_{FN}\left(a,b\right)=-\left[U\left(a\right),W\left(b\right)\right]-\left[W\left(a\right),U\left(b\right)\right]+\left(\text{exact term}\right)\end{gathered} \tag{4.37}$$

for $U, W \in \{S_{HH},\, S_{VH},\, S_{VV}\}$.

In particular, note that if $U\left((\delta_{Lie}W)\right)=0$, then $[U,W]_{FN}$ is exact. Thus, by eq 4.36

$$[S_{HH}\,,S_{VH}+S_{VV}]_{FN}\,(a,b)=(\text{exact term})$$

$$[S_{VH}\,,S_{VH}+2S_{VV}]_{FN}\,(a,b)=(\text{exact term}) \tag{4.38}$$

$$\Phi_V\,[S_{HH}\,,S_{HH}]_{FN}\,(a,b)=-2\Phi_V\,[S_{HH}\,(a)\;,S_{HH}\,(b)]+(\text{exact term})$$

Hence

$$\begin{gathered}\Phi_V\,[S_{HH}+S_{VH}+S_{VV}\,,S_{HH}+S_{VH}+S_{VV}]_{FN}\,(a,b)=\\ -2\Phi_V\,[S_{HH}\,(a)\;,S_{HH}\,(b)]\\ +\Phi_V\,[S_{VV}\,,S_{VV}]_{FN}\,(a,b)+(\text{exact term})\end{gathered} \tag{4.39}$$

To exhibit the $\sigma_{k,\ell}$ on the right-hand side of eq 4.39 (and hence eliminate the $\Phi_V\,[S_{VV}\,,S_{VV}]_{FN}$ term), consider the quantity

$$\begin{gathered}[\Phi_V W\Phi_H\,,\Phi_V W\Phi_H]_{FN}=\\ -2\,[\Phi_V W\Phi_H\,,\Phi_V W\Phi_H]+(\text{exact term})=\\ -\Phi_V W\Phi_H\left((\delta_{Lie}\Phi_V W\Phi_H)\,(a,b)\right)+(\text{exact term})=\\ -\Phi_V W\Phi_H\left([(\Phi_V W\Phi_H)\,(a)\,,b]+[a,(\Phi_V W\Phi_H)\,(b)]-\Phi_V W\Phi_H\,[a,b]\,(a,b)\right)+(\text{exact term})=\\ -\Phi_V W\Phi_H\left([(\Phi_V W\Phi_H)\,(a)\,,b]+[a,(\Phi_V W\Phi_H)\,(b)]\right)+(\text{exact term})\end{gathered} \tag{4.40}$$

where $W$ is an arbitrary Lie algebra-valued one form. The expression $(\Phi_V W\Phi_H)\,(a)$ will produce only terms of power of $\hbar$ greater than $k$. However, since the Lie bracket can not reduce the power of $\hbar$ of a term, the output of the bracket will be terms that all have a power of $\hbar$ greater than $k$. The $\Phi_H$ will kill all such terms. Thus
$\Phi_H\left([(\Phi_V W\Phi_H)\,(a)\,,b]+[a,(\Phi_V W\Phi_H)\,(b)]\right)=0$, leaving the result

$$[\Phi_V W\Phi_H\,,\Phi_V W\Phi_H]_{FN}=(\text{exact term}) \tag{4.41}$$

Note that due to the $\Phi_V$ in both arguments of the bracket of the left-hand side of eq 4.41, the minimum power of $\hbar$ of the output of the bracket is $2k+2$. Thus the $End\,(E)$ component of eq 4.41 is an arbitrary element of $\hbar^{2k+2}\mathfrak{sl}\,(e, \mathcal{A}_n) + \hbar^{2k+3}\mathfrak{gl}\,(e, \mathcal{D}_n)$. Since this is exact, it may freely be added to eq 4.39 without changing its class. This has the effect of quotienting by $\hbar^{2k+2}\mathfrak{sl}\,(e, \mathcal{A}_n) + \hbar^{2k+3}\mathfrak{gl}\,(e, \mathcal{D}_n)$, yielding

$$\begin{gathered}\Phi_V \left[S_{HH} + S_{VH} + S_{VV} \,\mathbf{,}\, S_{HH} + S_{VH} + S_{VV}\right]_{FN} (a, b) = \\ -2\sigma_{k,\ell} \left(\Phi_V \left[S_{HH}\,(a) \,\mathbf{,}\, S_{HH}\,(b)\right] + (\text{exact term})\right)\end{gathered} \tag{4.42}$$

Thus, one has at last

$$\pi_{k+1,\ell}\left(\zeta\left(\Phi_V \left[S \,\mathbf{,}\, S\right]_{FN}(a, b)\right)\right) = -2\,(GF \circ C)\,(a, b) + (\text{exact term}) \tag{4.43}$$

Thus

$$\pi_{k+1,\ell}\left(\zeta\,(\Lambda)\right) = -2\sigma_{k,\ell}\,(GF \circ C) + (\text{exact term}) \tag{4.44}$$

□

Eq 4.44 serves as an explicit illustration of the following: Suppose the vector bundle $E \to M$ has a quantization that extends from order $k$ to order $\ell > k$. This implies there exists a Fedosov connection $\nabla$ of the form eq 3.19, which, by definition satisfies the flatness condition of eq 4.9 up to order $\hbar^\ell$. This, in turn, implies exactness of the $\Lambda$ via eq 4.24 up to order $\hbar^\ell$ for all degrees. If $\ell \geq 2k+2$, and if there is a non-trivial trace contribution to $\sigma_{k,\ell}\,(C\,(a, b))$ at order $\hbar^{2k+2}$ via eq 3.17, this would be mapped into $\Omega^\bullet\,(M;\, End\,(E))$ via the $GF$ of eq 4.27, implying the obstruction class is non-trivial at order $\hbar^{2k+2}$, which in turn contraindicates exactness of $\Lambda$, which contradicts the starting assumptions, since $2k+2 \leq \ell$. Thus this class indeed serves as an obstruction to order $\ell$ quantization extension.

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# Appendix A

# Explicit formulas for cocyles

## A.1 Obstruction classes: Explicit Exhibition of the form of $\sigma_{k,\ell} \circ C$

Appendix A serves as a detailed exposition of the calculations necessary to arrive at eqs A.9, A.10, A.11.

When writing explicit forms of elements of the Weyl algebra and their products it simplifies expressions to choose an ordering convention of the conjugate variables. To that end, the standardized ordering is defined such that $y_j$ is always written to the right of $x_j$. This ordering will be denoted via $\{\}_{RYO}$, which re-orders its contents according to this convention. As an example, $\left\{x_1^\alpha y_1^\beta\right\}_{RYO} = \left\{y_1^\beta x_1^\alpha\right\}_{RYO} = x_1^\alpha y_1^\beta$. Let $\overrightarrow{\frac{\partial}{\partial x_i}}$ act to the right, and $\overleftarrow{\frac{\partial}{\partial x_i}}$ act to the left. With this ordering convention, it can be shown that

$$\{f\}_{RYO}\{g\}_{RYO} = \left\{ f \ \exp\left( \hbar \sum_{i=1}^{n} \frac{\overleftarrow{\partial}}{\partial y_i} \frac{\overrightarrow{\partial}}{\partial x_i} \right) g \right\}_{RYO} \tag{A.1}$$

$$f,\ g,\ \{f\}_{RYO},\ \{g\}_{RYO} \in \mathcal{D}_n$$

We then have

$$\begin{aligned} \left[\{f\}_{RYO}, \{g\}_{RYO}\right] &= \left\{ \left( f \ \exp\left( \hbar \sum_{i=1}^{n} \frac{\overleftarrow{\partial}}{\partial y_i} \frac{\overrightarrow{\partial}}{\partial x_i} \right) g \right) - \left( g \ \exp\left( \hbar \sum_{i=1}^{n} \frac{\overleftarrow{\partial}}{\partial y_i} \frac{\overrightarrow{\partial}}{\partial x_i} \right) f \right) \right\}_{RYO} \\ &= \left\{ f \left( \exp\left( \hbar \sum_{i=1}^{n} \frac{\overleftarrow{\partial}}{\partial y_i} \frac{\overrightarrow{\partial}}{\partial x_i} \right) - \exp\left( \hbar \sum_{i=1}^{n} \frac{\overleftarrow{\partial}}{\partial x_i} \frac{\overrightarrow{\partial}}{\partial y_i} \right) \right) g \right\}_{RYO} \end{aligned} \tag{A.2}$$

where we have used the fact that $\{fg - gf\}_{RYO} = 0$. Constructing the matrix commutator yields

$$\left[\{A\}_{RYO}, \{B\}_{RYO}\right] = \left\{ \left( A \ \exp\left( \hbar \frac{\overleftarrow{\partial}}{\partial y} \frac{\overrightarrow{\partial}}{\partial x} \right) B \right) - \left( B \ \exp\left( \hbar \frac{\overleftarrow{\partial}}{\partial y} \frac{\overrightarrow{\partial}}{\partial x} \right) A \right) \right\}_{RYO} \tag{A.3}$$

where in the final line we have suppressed the matrix indices. In general, $\sigma_{k,\ell} \circ C(a,b)$ will have no powers of $\hbar$ less than or equal to $k$ or greater than $\ell$

$$\begin{gathered} C : \mathfrak{g}_k \wedge \mathfrak{g}_k \to \mathfrak{h}(k,\ell) \\ C(a,b) := \Phi_V\left(\left[\Phi_H(a), \Phi_H(b)\right]\right) = \left[\Phi_H(a), \Phi_H(b)\right] - \Phi_H\left([a,b]\right) \\ C(a,b) \cong \{0\} \ltimes \pi_{k+1,\ell}\left(\left[\pi_{0,k}(B_a), \pi_{0,k}(B_b)\right]\right. \\ \left. -\tfrac{1}{\hbar}\left[\pi_{0,k}(Id_e f_b), \pi_{0,k}(B_a)\right] + \tfrac{1}{\hbar}\left[\pi_{0,k}(Id_e f_a), \pi_{0,k}(B_b)\right]\right) \end{gathered} \tag{A.4}$$

$$\begin{gathered} a, b \in \mathfrak{g} = Der(\mathcal{D}_n) \ltimes gl_r(\mathcal{D}_n) \\ a = \left(\tfrac{1}{\hbar}\left[f_a, -\right], B_a\right) \quad b = \left(\tfrac{1}{\hbar}\left[f_b, -\right], B_b\right) \end{gathered}$$

where$\pi_m()$ filters out all terms in the argument with a power of $\hbar$ different from $m$ and $\pi_{m,n}()$ filters out all terms in the argument with a power of $\hbar$ less than $m$ or greater than $n$.

Thus

$$\pi_{k+1,\ell}\left(\sum_{j=0}^{\infty}\hbar^{j}\left\{Q_j\right\}_{RYO}\right):=\sum_{j=k+1}^{\ell}\hbar^{j}\left\{Q_j\right\}_{RYO}$$

.

To understand the Abelianization map $\sigma_{k,\ell}$ first observe

$$\left[\mathfrak{h}\left(k,\ell\right),\mathfrak{h}\left(k,\ell\right)\right]\cong\{0\}\ltimes\left(\hbar^{2k+2}\mathfrak{sl}\left(e,\mathcal{A}_n\right)\oplus\prod_{j=2k+3}^{j=\ell}\hbar^{j}\mathfrak{gl}\left(e,\mathcal{A}_n\right)\right)\tag{A.5}$$

which implies

$$\frac{\mathfrak{h}\left(k,\ell\right)}{\left[\mathfrak{h}\left(k,\ell\right),\mathfrak{h}\left(k,\ell\right)\right]}\cong\left\{\begin{array}{ccc}\{0\}\ltimes\left(\prod_{j=k+1}^{j=\ell}\hbar^{j}\mathfrak{gl}\left(e,\mathcal{A}_n\right)\right) & if & \ell<2k+2\\ \{0\}\ltimes\left(\prod_{j=k+1}^{j=2k+1}\hbar^{j}\mathfrak{gl}\left(e,\mathcal{A}_n\right)\oplus\hbar^{2k+2}\mathcal{A}_n Id_e\right) & if & \ell\geq 2k+2\end{array}\right\}\tag{A.6}$$

$$C\left(a,b\right)\cong\pi_{k+1,\ell}\left(\mathbb{D}\left(B_a,B_b\right)+\mathbb{D}\left(B_a,\tfrac{1}{\hbar}Id_e f_b\right)+\mathbb{D}\left(\tfrac{1}{\hbar}Id_e f_a,B_b\right)\right)\tag{A.7}$$

where

$$\mathbb{D}\left(X,Y\right):=\left\{\left(\pi_{0,k}\left(X\right)\exp\left(\hbar\sum_{i=1}^{n}\frac{\overleftarrow{\partial}}{\partial y_i}\frac{\overrightarrow{\partial}}{\partial x_i}\right)\pi_{0,k}\left(Y\right)\right)-\left(\pi_{0,k}\left(Y\right)\exp\left(\hbar\sum_{i=1}^{n}\frac{\overleftarrow{\partial}}{\partial y_i}\frac{\overrightarrow{\partial}}{\partial x_i}\right)\pi_{0,k}\left(X\right)\right)\right\}_{RYO}$$

The effect of taking the quotient of $C\left(a,b\right)$ with $\left[\mathfrak{h}\left(k,\ell\right),\mathfrak{h}\left(k,\ell\right)\right]$ depends on if $\ell<2k+2$. If it is, the Lie Algebra $\mathfrak{h}\left(k,\ell\right)$ is already Abelian, and hence there is no effect. However, if $\ell\geq 2k+2$ there is no effect on grades $\hbar^k$ to $\hbar^{2k+1}$ (these are already Abelian), eliminates the traceless part of the $\hbar^{2k+2}$ grade, and truncates all powers of $\hbar$ of $2k+3$ and above. Thus the matrix $A$ proportional to $\hbar^{2k+2}$ may be replaced by a matrix proportional to the identity matrix with the same trace: $\frac{1}{e}I_e\mathrm{Tr}\left(A\right)$. Again one takes $\phi=\frac{1}{\hbar}Id$. For a general matrix $B = \sum_{j=0}^{\infty} \hbar^j B_j$ one has

$$\sigma_{k,\ell}\left(\pi_{k+1,\ell}\left(B\right)\right) \cong \left\{ \begin{array}{ccc} \pi_{k+1,\ell}\left(B\right) & if & \ell < 2k+2 \\ \\ \pi_{k+1,2k+1}\left(B\right) + \frac{1}{e} I_e \mathrm{Tr}\left(\pi_{k+1,2k+2}\left(B\right)\right) & if & \ell \geq 2k+2 \end{array} \right\} \tag{A.8}$$

Thus

$$\sigma_{k,\ell}\left(C\left(a,b\right)\right) \cong C\left(a,b\right) / \left(\hbar^{2k+2}\mathfrak{sl}\left(e,\mathcal{A}_n\right) \oplus \hbar^{2k+3}\mathfrak{gl}\left(e,\mathcal{A}_n\right) \oplus \cdots\right)$$

If $\ell < 2k+2$ then

$$\sigma_{k,\ell}\left(C\left(a,b\right)\right) \cong \pi_{k+1,\ell}\left(\mathbb{D}\left(B_a,B_b\right) + \mathbb{D}\left(B_a, \frac{1}{\hbar} I d_e f_b\right) + \mathbb{D}\left(\frac{1}{\hbar} I d_e f_a, B_b\right)\right) \tag{A.9}$$

If $\ell \geq 2k+2$ then

$$\begin{gathered} \sigma_{k,\ell}\left(C\left(a,b\right)\right) \cong \\ \pi_{k+1,2k+1}\left(\mathbb{D}\left(B_a,B_b\right) + \mathbb{D}\left(B_a, \tfrac{1}{\hbar} I d_e f_b\right) + \mathbb{D}\left(\tfrac{1}{\hbar} I d_e f_a, B_b\right)\right) \\ -\tfrac{1}{\hbar}\tfrac{1}{e} I_e \mathrm{Tr}\left(\pi_{2k+2,2k+2}\left(\mathbb{D}\left(B_a,B_b\right) + \mathbb{D}\left(B_a, \tfrac{1}{\hbar} I d_e f_b\right) + \mathbb{D}\left(\tfrac{1}{\hbar} I d_e f_a, B_b\right)\right)\right) \end{gathered} \tag{A.10}$$

The new contribution in the above is the

$$\frac{1}{\hbar}\frac{1}{e} I_e \mathrm{Tr}\left(\cdots\right) \tag{A.11}$$

# Appendix B

# Imposing Flatness on the Fedosov Connection

Recall the structure defined for a Fedosov connection in subsection 3.3.3. the One begins with:

$$\nabla = Id_M \otimes \nabla_E + \nabla_M \otimes Id_E \tag{B.1}$$

with

$$\begin{aligned}
\nabla_E &= \nabla_E^{1,0} + \nabla_E^{0,1} \\
\nabla_E^{1,0} &= \nabla_{WE} + A_{E0} + A_E \qquad A_E = \sum_{j=1}^{\infty} A_{E\,j} \\
\nabla_E^{0,1} &= \bar{\partial} + B_E \qquad B_E = \sum_{j=1}^{\infty} B_{E\,j} \\
A_{E\,j} &\in \Omega^{1,0}\left(M, \left(\mathcal{P}_0 \times \mathfrak{gl}\left(e, \mathcal{D}_n\right)\right)/G_0\right) \qquad \forall j \geq 1 \\
B_{E\,j} &\in \Omega^{0,1}\left(M, \left(\mathcal{P}_0 \times \mathfrak{gl}\left(e, \mathcal{D}_n\right)\right)/G_0\right) \qquad \forall j \geq 1
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
\nabla_M &= \nabla_M^{1,0} + \nabla_M^{0,1} \\
\nabla_M^{1,0} &= A_{-1} + \nabla_{WM} + A_{M0} + A_M \qquad A_M = \sum_{j=1}^{\infty} A_{M\,j} \\
\nabla_M^{0,1} &= \bar{\partial} + B_M \qquad B_M = \sum_{j=1}^{\infty} B_{M\,j}
\end{aligned}$$

where the $M$ subscript indicates items defined in Tsygan and Nest [11]. A pair of integer superscripts of the form $p, q$ on a operator or form indicates the component with holmorphicity $p$ and anti-holomorphicity $q$. An integer subscript is used to denote the component of that particular degree. Thus $A_m^{p,q}$ would be the component of $A$ with holmorphicity $p$, anti-holomorphicity $q$ and degree $m$.

The curvature of the connection $\nabla$ is given by the Frölicher-Nijenhuis bracket of $\nabla$ with itself Kolar et al. [7]. Expanding the expression for the curvature by substituting eq B.1 yields:

$$\begin{aligned} [\nabla, \nabla]_{FN} &= [Id_M \otimes \nabla_E, Id_M \otimes \nabla_E]_{FN} + [Id_M \otimes \nabla_E, \nabla_M \otimes Id_E]_{FN} \\ &\quad + [\nabla_M \otimes Id_E, Id_M \otimes \nabla_E]_{FN} + [\nabla_M \otimes Id_E, \nabla_M \otimes Id_E]_{FN} \\ &= Id_M \otimes [\nabla_E, \nabla_E]_{FN} + [\nabla_M, \nabla_M]_{FN} \otimes Id_E \\ &\quad +2\,[Id_M \otimes \nabla_E, \nabla_M \otimes Id_E]_{FN} \end{aligned} \tag{B.3}$$

Note that the term

$$[Id_M \otimes \nabla_E, \nabla_M \otimes Id_E]_{FN}$$

does not *trivialy* vanish. By “trivialy” in this case it is meant a-priori - via construction or by an axiom of eq 1.4. It can be chosen to vanish by suitable choices of the adjustments to the connection. This is a condition that will be imposed.

$$[\nabla, \nabla]_{FN} = \begin{pmatrix} [\nabla^{1,0}, \nabla^{1,0}]_{FN} \\ 2\,[\nabla^{1,0}, \nabla^{0,1}]_{FN} \\ [\nabla^{0,1}, \nabla^{0,1}]_{FN} \end{pmatrix} \tag{B.4}$$

Note that the holomorphicity components have been written seperately as rows in a column vector, in the order of $(2, 0)$, $(1, 1)$ and $(0, 2)$ from top to bottom.

One assumes that the process of section 5.3 of Tsygan and Nest [11] is applied to $\nabla_M$, and thus one has

$$[\nabla_M, \nabla_M]_{FN} = \begin{pmatrix} \theta_M \\ \phi_M \\ \tau_M \end{pmatrix}$$

$$\begin{aligned} \theta_M &\in \Omega^{2,0}\left(M, \tfrac{1}{\hbar}\mathbb{C}\left[[\hbar]\right]\right) \\ \phi_M &\in \Omega^{1,1}\left(M, \mathbb{C}\left[[\hbar]\right]\right) \\ \tau_M &\in \Omega^{0,2}\left(M, \mathbb{C}\left[[\hbar]\right]\right) \end{aligned} \tag{B.5}$$

where $\theta_M,\ \phi_M,\ \tau_M$ are the $(2,0)$, $(1,1)$, $(0,2)$ components of curvature of their constructed connection $\nabla_M$. This is imposed by making a suitable choices of connection adjustments $A_M,\ B_M$. This is performed via an inductive process on the degree. Note that only $\theta_M$ has negative degree terms, and only of degree $-2$.

Analogously, in this computation the goal is to impose

$$[\nabla_E, \nabla_E]_{FN} = 0 \tag{B.6}$$

The cross term $[Id_M \otimes \nabla_E, \nabla_M \otimes Id_E]_{FN}$ will be represented by $\nu$. The vanishing of this cross term will be imposed by making suitable choices in the connection adjustments $A_E,\ B_E$ such that $\nu = 0$.

$$2\left[Id_M \otimes \nabla_E, \nabla_M \otimes Id_E\right]_{FN} = \nu^{2,0} + \nu^{1,1} + \nu^{0,2}$$

$$\begin{aligned} \nu^{2,0} &\in \Omega^{2,0}\left(M, \mathbb{C}\left[[\hbar]\right]\right) \\ \nu^{1,1} &\in \Omega^{1,1}\left(M, \mathbb{C}\left[[\hbar]\right]\right) \\ \nu^{0,2} &\in \Omega^{0,2}\left(M, \mathbb{C}\left[[\hbar]\right]\right) \end{aligned} \tag{B.7}$$

Note that only $\theta_M$ has a negative degree term, and only $-2$. The desired flatness of $\nabla$ is equivalent to imposing

$$\begin{aligned} [\nabla,\nabla]_{FN} &= \theta+\phi+\tau \\ &= Id_M\otimes(0)+(\theta_M+\phi_M+\tau_M)\otimes Id_E+\nu^{2,0}+\nu^{1,1}+\nu^{0,2} \\ \theta &= \theta_M\otimes Id_E+\nu^{2,0} \\ \phi &= \phi_M\otimes Id_E+\nu^{1,1} \\ \tau &= \tau_M\otimes Id_E+\nu^{0,2} \end{aligned} \tag{B.8}$$

Note that all odd degrees of $\theta,\ \phi,\ \tau$ vanish, since they take values in $\frac{1}{\hbar}\mathbb{C}\left[[\hbar]\right]$ or $\mathbb{C}\left[[\hbar]\right]$, and $\deg(\hbar)=2$. It will be shown that this Fedosov construction is consistent with

$$\begin{aligned} \nu^{2,0} &= 0 \\ \nu^{1,1} &= 0 \\ \nu^{0,2} &= 0 \end{aligned} \tag{B.9}$$

The terms $[\nabla^{1,0},\nabla^{1,0}]_{FN}-\theta$, $2\,[\nabla^{1,0},\nabla^{0,1}]_{FN}-\phi$ and $[\nabla^{0,1},\nabla^{0,1}]_{FN}-\tau$ are the deviation of the curvature from its desired value. The connection will be adjusted so that these deviations vanish.

$$\rho=\begin{pmatrix}\rho^{2,0}\\ \rho^{1,1}\\ \rho^{0,2}\end{pmatrix}:=[\nabla,\nabla]_{FN}-\begin{pmatrix}\theta\\ \phi\\ \tau\end{pmatrix}=\begin{pmatrix}[\nabla^{1,0},\nabla^{1,0}]_{FN}-\theta\\ 2\,[\nabla^{1,0},\nabla^{0,1}]_{FN}-\phi\\ [\nabla^{0,1},\nabla^{0,1}]_{FN}-\tau\end{pmatrix} \tag{B.10}$$

The imposition of flatness is equivalent to $\rho=0$. That is, values for the adjustments $A_E$, $B_E$ will be chosen such that the connection $\nabla$ to yield $\rho=0$. Since each degree must separately vanish, one divides eq B.10 by degree, yielding

$$\begin{aligned}
\rho_m &= \begin{pmatrix} \rho^{2,0}_{m-1} \\ \rho^{1,1}_{m} \\ \rho^{0,2}_{m+1} \end{pmatrix} \\
&= \begin{pmatrix} \deg_{m-1}\left(\rho^{2,0}\right) \\ \deg_{m}\left(\rho^{1,1}\right) \\ \deg_{m+1}\left(\rho^{0,2}\right) \end{pmatrix} \\
&= \begin{pmatrix} \deg_{m-1}\left([\nabla^{1,0}, \nabla^{1,0}]_{FN} - \theta\right) \\ \deg_{m}\left(2\,[\nabla^{1,0}, \nabla^{0,1}]_{FN} - \phi\right) \\ \deg_{m+1}\left([\nabla^{0,1}, \nabla^{0,1}]_{FN} - \tau\right) \end{pmatrix} \\
&= \begin{pmatrix} 2\left[\nabla^{1,0}_{-1}, \nabla^{1,0}_{m}\right]_{FN} + \deg_{m-1}\left(\left[\nabla^{1,0}_{\geq 0}, \nabla^{1,0}_{\geq 0}\right]_{FN} - \theta\right) \\ 2\left[\nabla^{1,0}_{-1}, \nabla^{0,1}_{m+1}\right]_{FN} + 2\left[\nabla^{0,1}_{0}, \nabla^{1,0}_{m}\right]_{FN} + \deg_{m}\left(2\left[\nabla^{1,0}_{\geq 0}, \nabla^{0,1}_{\geq 1}\right]_{FN} - \phi\right) \\ 2\left[\nabla^{0,1}_{0}, \nabla^{0,1}_{m+1}\right]_{FN} + \deg_{m+1}\left(\left[\nabla^{0,1}_{\geq 1}, \nabla^{0,1}_{\geq 1}\right]_{FN} - \tau\right) \end{pmatrix}
\end{aligned} \tag{B.11}$$

where the $\deg_j()$ operator is defined to project out the degree $j$ component of its argument. The goal is to pick values for $A_{E\,m}$, $B_{E\,m+1}$ such that one achieves

$$\rho_m = \begin{pmatrix} \deg_{m-1}\left(\rho^{2,0}\right) \\ \deg_{m}\left(\rho^{1,1}\right) \\ \deg_{m+1}\left(\rho^{0,2}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

One can define the differential $\mathrm{d}_B$ analogously to the $\mathrm{d}_M$ utilized in Tsygan and Nest [11] (there appearing as $\mathrm{d} = \mathfrak{ad}A_{-1} + \bar{\partial}$)

$$\mathrm{d}_B \ := \ \mathfrak{ad}A_{-1} \otimes Id_E + Id_M \otimes \bar{\partial} + \bar{\partial} \otimes Id_E \ = \begin{pmatrix} \mathfrak{ad}A_{-1} \otimes Id_E \\ Id_M \otimes \bar{\partial} + \bar{\partial} \otimes Id_E \end{pmatrix}$$

$$\mathrm{d}_M := \mathfrak{ad}A_{-1} \otimes Id_E + \bar{\partial} \otimes Id_E \tag{B.12}$$
$$\mathrm{d}_e^{0,1} := Id_M \otimes \bar{\partial}$$
$$\mathrm{d}_B = \mathrm{d}_M + \begin{pmatrix} 0 \\ \mathrm{d}_e^{0,1} \end{pmatrix} = \begin{pmatrix} \mathrm{d}_M^{1,0} \\ \mathrm{d}_M^{0,1} + \mathrm{d}_e^{0,1} \end{pmatrix}$$

It must be true that $[\mathrm{d}_B, \mathrm{d}_B]_{FN} = 0$ and the resulting complex must be acyclic when our class of interest is a coboundary. One needs

$$[\mathrm{d}_B, \mathrm{d}_B]_{FN} \ = \begin{pmatrix} [A_{-1} \otimes Id_E, A_{-1} \otimes Id_E]_{FN} \\ 2\left[A_{-1} \otimes Id_E, Id_M \otimes \bar{\partial} + \bar{\partial} \otimes Id_E\right]_{FN} \\ \left[Id_M \otimes \bar{\partial} + \bar{\partial} \otimes Id_E, Id_M \otimes \bar{\partial} + \bar{\partial} \otimes Id_E\right]_{FN} \end{pmatrix} = \begin{pmatrix} \frac{-i}{\hbar}\omega \otimes Id_E \\ 0 \\ 0 \end{pmatrix} \tag{B.13}$$

The $(2,0)$, $(1,1)$, $(0,2)$ components of $[\mathrm{d}_E, \mathrm{d}_E]_{FN}$ must each achieve their required values independently. It is shown in Tsygan and Nest [11] that $\left[A_{-1} \otimes Id_E, \bar{\partial} \otimes Id_E\right]_{FN} = 0$ and $\left[\bar{\partial} \otimes Id_E, \bar{\partial} \otimes Id_E\right]_{FN} = 0$. An analogous calculation shows that $\left[A_{-1} \otimes Id_E, Id_M \otimes \bar{\partial}\right]_{FN} = 0$, $\left[Id_M \otimes \bar{\partial}+, Id_M \otimes \bar{\partial}\right]_{FN} = 0$ and $\left[Id_M \otimes \bar{\partial}+, \bar{\partial} \otimes Id_E\right]_{FN} = 0$. Thus we find

$$\begin{aligned} [A_{-1} \otimes Id_E, A_{-1} \otimes Id_E]_{FN} &= \frac{-i}{\hbar}\omega \otimes Id_E \\ \left[A_{-1} \otimes Id_E, \bar{\partial} \otimes Id_E\right]_{FN} &= \left[A_{-1} \otimes Id_E, Id_M \otimes \bar{\partial}\right]_{FN} = 0 \\ \left[\bar{\partial} \otimes Id_E, \bar{\partial} \otimes Id_E\right]_{FN} &= \left[Id_M \otimes \bar{\partial}, Id_M \otimes \bar{\partial}\right]_{FN} = 0 \\ \left[Id_M \otimes \bar{\partial}, \bar{\partial} \otimes Id_E\right]_{FN} &= 0 \end{aligned} \tag{B.14}$$

It is shown in Tsygan and Nest [11] that $[A_{-1} \otimes Id_E, A_{-1} \otimes Id_E]_{FN} = \frac{-i}{\hbar}\omega \otimes Id_E$. Although this is not zero, it is in the center of the algebra. As it will only appear inside another Lie algebra bracket, and thus where it appears it is effectively zero.

To proceed analogously to Tsygan and Nest [11], one must recast eq B.11 to form in which the vanishing of $\rho$ depends upon the exactness of a remainder term. One separates the deviations of the connection curvature from the imposed values into a component that is manifestly exact, and the remainder $\Lambda$

$$\begin{aligned}
\rho_m = & \begin{pmatrix} \rho^{2,0}_{m-1} \\ \rho^{1,1}_{m} \\ \rho^{0,2}_{m+1} \end{pmatrix} = \begin{pmatrix} \deg_{m-1}\left([\nabla^{1,0},\nabla^{1,0}]_{FN} - \theta\right) \\ \deg_{m}\left(2\,[\nabla^{1,0},\nabla^{0,1}]_{FN} - \phi\right) \\ \deg_{m+1}\left([\nabla^{0,1},\nabla^{0,1}]_{FN} - \tau\right) \end{pmatrix} \\
= & \; 2\mathrm{d}_B \begin{pmatrix} \nabla^{1,0}_{m} \\ \nabla^{0,1}_{m+1} \end{pmatrix} + \begin{pmatrix} \deg_{m-1}\left(\left[\nabla^{1,0}_{\geq 0},\nabla^{1,0}_{\geq 0}\right]_{FN} - \theta\right) \\ \deg_{m}\left(2\left[\nabla^{1,0}_{\geq 0},\nabla^{0,1}_{\geq 1}\right]_{FN} - \phi\right) \\ \deg_{m+1}\left(\left[\nabla^{0,1}_{\geq 1},\nabla^{0,1}_{\geq 1}\right]_{FN} - \tau\right) \end{pmatrix} \\
= & \; 2\mathrm{d}_B \begin{pmatrix} \nabla^{1,0}_{E\,m} \\ \nabla^{0,1}_{E\,m+1} \end{pmatrix} + 2\mathrm{d}_B \begin{pmatrix} \nabla^{1,0}_{M\,m} \\ \nabla^{0,1}_{M\,m+1} \end{pmatrix} + \begin{pmatrix} \deg_{m-1}\left(\left[\nabla^{1,0}_{\geq 0},\nabla^{1,0}_{\geq 0}\right]_{FN} - \theta\right) \\ \deg_{m}\left(2\left[\nabla^{1,0}_{\geq 0},\nabla^{0,1}_{\geq 1}\right]_{FN} - \phi\right) \\ \deg_{m+1}\left(\left[\nabla^{0,1}_{\geq 1},\nabla^{0,1}_{\geq 1}\right]_{FN} - \tau\right) \end{pmatrix}
\end{aligned} \tag{B.15}$$

Note that the holomorphicity components of $\rho_m$ are of shifted degrees.

## B.1 Existence of adjustment to connection forms that impose flatness

By construction $\rho_m = 0 \;\; \forall m \leq -2$. For $m = -1$ one has

$$\rho_{-1} = 2\mathrm{d}_B\mathrm{d}_B - \begin{pmatrix} \theta_{-2} \\ 0 \\ \tau_0 \end{pmatrix}$$

With prescribed value of $\theta_{-2} = \frac{-2i}{\hbar}\omega \otimes Id_E$ one can pick $\tau_0 = \nu_{-2}^{2,0} = \nu_{-1}^{1,1} = \nu_0^{0,2} = 0$ and achieve $\rho_{-1} = 0$. Thus in what follows it will be assumed that $m \geq 0$.

For $m = 0$

$$\rho_0 = \begin{pmatrix} \rho_{-1}^{2,0} \\ \rho_0^{1,1} \\ \rho_1^{0,2} \end{pmatrix} = 2\mathrm{d}_B \begin{pmatrix} Id_M \otimes A_{E\,0} \\ Id_M \otimes B_{E\,1} \end{pmatrix} + \Lambda_0$$

$$\Lambda_0 = 2\mathrm{d}_B \begin{pmatrix} Id_M \otimes \nabla_{WE} + \nabla_{WM} \otimes Id_E + A_{M0} \otimes Id_E \\ B_{M1} \otimes Id_E \end{pmatrix} - \begin{pmatrix} \theta_{-1} \\ \phi_0 \\ \tau_1 \end{pmatrix}$$

$$= 2\mathrm{d}_B \begin{pmatrix} Id_M \otimes \nabla_{WE} + \nabla_{WM} \otimes Id_E + A_{M0} \otimes Id_E \\ B_{M1} \otimes Id_E \end{pmatrix} - \begin{pmatrix} 0 \\ \phi_0 \\ 0 \end{pmatrix}$$

Clearly $\mathrm{d}_B\Lambda_0 = 0$ and thus $\Lambda_0$ is closed in the complex of eq 4.10. Since $\Lambda$ is in the second cohomology group, by theorem 4.1 it is exact. Thus

$$\Lambda_0 = \mathrm{d}_B \begin{pmatrix} Id_M \otimes \lambda_0^{1,0} \\ Id_M \otimes \lambda_1^{0,1} \end{pmatrix}$$

One may then choose

$$\begin{pmatrix} A_{E\,0} \\ B_{E\,1} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \lambda_0^{1,0} \\ \lambda_1^{0,1} \end{pmatrix}$$

which then achieves $\rho_0 = 0$.

For $m \geq 1$

$$\rho_m = 2\mathrm{d}_B \begin{pmatrix} A_{E\,m} \\ B_{E\,m+1} \end{pmatrix} + 2\mathrm{d}_B \begin{pmatrix} \nabla^{1,0}_{M\,m} \\ \nabla^{0,1}_{M\,m+1} \end{pmatrix} + \begin{pmatrix} \deg_{m-1}\left(\left[\nabla^{1,0}_{\geq 0}, \nabla^{1,0}_{\geq 0}\right]_{FN} - \theta\right) \\ \deg_m\left(2\left[\nabla^{1,0}_{\geq 0}, \nabla^{0,1}_{\geq 1}\right]_{FN} - \phi\right) \\ \deg_{m+1}\left(\left[\nabla^{0,1}_{\geq 1}, \nabla^{0,1}_{\geq 1}\right]_{FN} - \tau\right) \end{pmatrix}$$

$$= 2\mathrm{d}_B \begin{pmatrix} A_{E\,m} \\ B_{E\,m+1} \end{pmatrix} + \Lambda_m$$

$$\Lambda_m := \begin{pmatrix} \mathit{\Lambda}^{2,0}_{m-1} \\ \mathit{\Lambda}^{1,1}_{m} \\ \mathit{\Lambda}^{0,2}_{m+1} \end{pmatrix} = 2\mathrm{d}_B \begin{pmatrix} \nabla^{1,0}_{M\,m} \\ \nabla^{0,1}_{M\,m+1} \end{pmatrix} + \begin{pmatrix} \deg_{m-1}\left(\left[\nabla^{1,0}_{\geq 0}, \nabla^{1,0}_{\geq 0}\right]_{FN} - \theta\right) \\ \deg_m\left(2\left[\nabla^{1,0}_{\geq 0}, \nabla^{0,1}_{\geq 1}\right]_{FN} - \phi\right) \\ \deg_{m+1}\left(\left[\nabla^{0,1}_{\geq 1}, \nabla^{0,1}_{\geq 1}\right]_{FN} - \tau\right) \end{pmatrix}$$

Where $\nabla^{1,0}_{\geq j}$ is defined to be the projection of $\nabla^{1.0}$ to degrees $\geq j$. Thus $\nabla^{1,0}_{\geq 0}$ and $\nabla^{0,1}_{\geq 1}$ contain no components of $\mathrm{d}_B$.

Note that the holomorphicity components of $\Lambda_m$ are of shifted degrees: $\mathit{\Lambda}^{2,0}_{m-1} = \left(\deg_{m-1}(\Lambda_m)\right)^{2,0}$ has degree $m-1$, $\mathit{\Lambda}^{1,1}_{m} = \left(\deg_m(\Lambda_m)\right)^{1,1}$ has degree $m$ and $\mathit{\Lambda}^{0,2}_{m+1} = \left(\deg_{m+1}(\Lambda_m)\right)^{0,2}$ has degree $m+1$. Thus the subscript of $\rho$ and $\Lambda$ is a label of degree rather that a numerical index. To proceed as above one must prove $\Lambda_m$ is closed. This is achieved by showing that $\mathrm{d}_B\Lambda_m = 0$. It will turn out that $\mathrm{d}_B\Lambda_m = 0$ will require $\rho_j = 0 \;\; \forall j < m$.

Thus one must proceed inductively. Assume now that indeed $\rho_j = 0 \ \ \forall j < m$, then

$$\begin{aligned}
\mathrm{d}_B\Lambda_m = & \quad \mathrm{d}_B \begin{pmatrix} \deg_{m-1}\left(\left[\nabla^{1,0}_{\geq 0}, \nabla^{1,0}_{\geq 0}\right]_{FN}\right) \\ 2\deg_m\left(\left[\nabla^{1,0}_{\geq 0}, \nabla^{0,1}_{\geq 1}\right]_{FN}\right) \\ \deg_{m+1}\left(\left[\nabla^{0,1}_{\geq 1}, \nabla^{0,1}_{\geq 1}\right]_{FN}\right) \end{pmatrix} \\
= & \begin{pmatrix} \left[\nabla^{1,0}_{-1}, \deg_{m-1}\left(\left[\nabla^{1,0}_{\geq 0}, \nabla^{1,0}_{\geq 0}\right]_{FN}\right)\right]_{FN} \\ 2\left[\nabla^{1,0}_{-1}, \deg_m\left(\left[\nabla^{1,0}_{\geq 0}, \nabla^{0,1}_{\geq 1}\right]_{FN}\right)\right]_{FN} + \left[\nabla^{0,1}_{0}, \deg_{m-1}\left(\left[\nabla^{1,0}_{\geq 0}, \nabla^{1,0}_{\geq 0}\right]_{FN}\right)\right]_{FN} \\ \left[\nabla^{1,0}_{-1}, \deg_{m+1}\left(\left[\nabla^{0,1}_{\geq 1}, \nabla^{0,1}_{\geq 1}\right]_{FN}\right)\right]_{FN} + 2\left[\nabla^{0,1}_{0}, \deg_m\left(\left[\nabla^{1,0}_{\geq 0}, \nabla^{0,1}_{\geq 1}\right]_{FN}\right)\right]_{FN} \\ \left[\nabla^{0,1}_{0}, \deg_{m+1}\left(\left[\nabla^{0,1}_{\geq 1}, \nabla^{0,1}_{\geq 1}\right]_{FN}\right)\right]_{FN} \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} \sum_{j=0}^{m-1}\left[\nabla^{1,0}_{-1}, \left[\nabla^{1,0}_{j}, \nabla^{1,0}_{m-1-j}\right]_{FN}\right]_{FN} \\ -------------------------- \\ 2\sum_{j=0}^{m-1}\left[\nabla^{1,0}_{-1}, \left[\nabla^{1,0}_{j}, \nabla^{0,1}_{m-j}\right]_{FN}\right]_{FN} \\ +\sum_{j=0}^{m-1}\left[\nabla^{0,1}_{0}, \left[\nabla^{1,0}_{j}, \nabla^{1,0}_{m-1-j}\right]_{FN}\right]_{FN} \\ -------------------------- \\ \sum_{j=0}^{m-1}\left[\nabla^{1,0}_{-1}, \left[\nabla^{0,1}_{j+1}, \nabla^{0,1}_{m-j}\right]_{FN}\right]_{FN} \\ +2\sum_{j=0}^{m-1}\left[\nabla^{0,1}_{0}, \left[\nabla^{1,0}_{j}, \nabla^{0,1}_{m-j}\right]_{FN}\right]_{FN} \\ -------------------------- \\ \sum_{j=0}^{m-1}\left[\nabla^{0,1}_{0}, \left[\nabla^{0,1}_{j+1}, \nabla^{0,1}_{m-j}\right]_{FN}\right]_{FN} \end{pmatrix}$$

$$
= \left( \begin{array}{c}
\sum_{j=0}^{m-1} \left[ \left[ \nabla_{-1}^{1,0}, \nabla_{j}^{1,0} \right]_{FN}, \nabla_{m-1-j}^{1,0} \right]_{FN} - \sum_{j=0}^{m-1} \left[ \nabla_{j}^{1,0}, \left[ \nabla_{-1}^{1,0}, \nabla_{m-1-j}^{1,0} \right]_{FN} \right]_{FN} \\
\hdashline
2 \sum_{j=0}^{m-1} \left[ \left[ \nabla_{-1}^{1,0}, \nabla_{j}^{1,0} \right]_{FN}, \nabla_{m-j}^{0,1} \right]_{FN} - 2 \sum_{j=0}^{m-1} \left[ \nabla_{j}^{1,0}, \left[ \nabla_{-1}^{1,0}, \nabla_{m-j}^{0,1} \right] \right]_{FN} \\
+ \sum_{j=0}^{m-1} \left[ \left[ \nabla_{0}^{0,1}, \nabla_{j}^{1,0} \right], \nabla_{m-1-j}^{1,0} \right]_{FN} - \sum_{j=0}^{m-1} \left[ \nabla_{j}^{1,0}, \left[ \nabla_{0}^{0,1}, \nabla_{m-1-j}^{1,0} \right]_{FN} \right]_{FN} \\
\hdashline
\sum_{j=0}^{m-1} \left[ \left[ \nabla_{-1}^{1,0}, \nabla_{j+1}^{0,1} \right]_{FN}, \nabla_{m-j}^{0,1} \right]_{FN} - \sum_{j=0}^{m-1} \left[ \nabla_{j+1}^{0,1}, \left[ \nabla_{-1}^{1,0}, \nabla_{m-j}^{0,1} \right]_{FN} \right]_{FN} \\
+2 \sum_{j=0}^{m-1} \left[ \left[ \nabla_{0}^{0,1}, \nabla_{j}^{1,0} \right]_{FN}, \nabla_{m-j}^{0,1} \right]_{FN} - 2 \sum_{j=0}^{m-1} \left[ \nabla_{j}^{1,0}, \left[ \nabla_{0}^{0,1}, \nabla_{m-j}^{0,1} \right]_{FN} \right]_{FN} \\
\hdashline
\sum_{j=0}^{m-1} \left[ \left[ \nabla_{0}^{0,1}, \nabla_{j+1}^{0,1} \right]_{FN}, \nabla_{m-j}^{0,1} \right]_{FN} - \sum_{j=0}^{m-1} \left[ \nabla_{j+1}^{0,1}, \left[ \nabla_{0}^{0,1}, \nabla_{m-j}^{0,1} \right]_{FN} \right]_{FN}
\end{array} \right)
$$

$$
= \left( \begin{array}{c}
2 \sum_{j=0}^{m-1} \left[ \left[ \nabla_{-1}^{1,0}, \nabla_{j}^{1,0} \right]_{FN}, \nabla_{m-1-j}^{1,0} \right]_{FN} \\
\hdashline
2 \sum_{j=0}^{m-1} \left[ \left[ \nabla_{-1}^{1,0}, \nabla_{j}^{1,0} \right]_{FN}, \nabla_{m-j}^{0,1} \right]_{FN} + 2 \sum_{j=0}^{m-1} \left[ \left[ \nabla_{-1}^{1,0}, \nabla_{m-j}^{0,1} \right]_{FN}, \nabla_{j}^{1,0} \right]_{FN} \\
+2 \sum_{j=0}^{m-1} \left[ \left[ \nabla_{0}^{0,1}, \nabla_{j}^{1,0} \right], \nabla_{m-1-j}^{1,0} \right]_{FN} \\
\hdashline
2 \sum_{j=0}^{m-1} \left[ \left[ \nabla_{-1}^{1,0}, \nabla_{j+1}^{0,1} \right]_{FN}, \nabla_{m-j}^{0,1} \right]_{FN} \\
+2 \sum_{j=0}^{m-1} \left[ \left[ \nabla_{0}^{0,1}, \nabla_{j}^{1,0} \right]_{FN}, \nabla_{m-j}^{0,1} \right]_{FN} + 2 \sum_{j=0}^{m-1} \left[ \left[ \nabla_{0}^{0,1}, \nabla_{m-j}^{0,1} \right]_{FN}, \nabla_{j}^{1,0} \right]_{FN} \\
\hdashline
2 \sum_{j=0}^{m-1} \left[ \left[ \nabla_{0}^{0,1}, \nabla_{j+1}^{0,1} \right]_{FN}, \nabla_{m-j}^{0,1} \right]_{FN}
\end{array} \right)
$$

$$
= \left( \begin{array}{c}
\sum_{j=0}^{m-1} \left[ 2 \left[ \nabla_{-1}^{1,0}, \nabla_{j}^{1,0} \right]_{FN}, \nabla_{m-1-j}^{1,0} \right]_{FN} \\
-------------------------------- \\
\sum_{j=0}^{m-1} \left[ 2 \left[ \nabla_{-1}^{1,0}, \nabla_{j}^{1,0} \right]_{FN}, \nabla_{m-j}^{0,1} \right]_{FN} \\
+ \sum_{j=0}^{m-1} \left[ 2 \left[ \nabla_{-1}^{1,0}, \nabla_{j+1}^{0,1} \right]_{FN}, \nabla_{m-1-j}^{1,0} \right]_{FN} + \sum_{j=0}^{m-1} \left[ 2 \left[ \nabla_{0}^{0,1}, \nabla_{j}^{1,0} \right], \nabla_{m-1-j}^{1,0} \right]_{FN} \\
-------------------------------- \\
\sum_{j=0}^{m-1} \left[ 2 \left[ \nabla_{-1}^{1,0}, \nabla_{j+1}^{0,1} \right]_{FN}, \nabla_{m-j}^{0,1} \right]_{FN} + \sum_{j=0}^{m-1} \left[ 2 \left[ \nabla_{0}^{0,1}, \nabla_{j}^{1,0} \right]_{FN}, \nabla_{m-j}^{0,1} \right]_{FN} \\
+2 \sum_{j=0}^{m-1} \left[ \left[ \nabla_{0}^{0,1}, \nabla_{m-j}^{0,1} \right]_{FN}, \nabla_{j}^{1,0} \right]_{FN} \\
-------------------------------- \\
\sum_{j=0}^{m-1} \left[ 2 \left[ \nabla_{0}^{0,1}, \nabla_{j+1}^{0,1} \right]_{FN}, \nabla_{m-j}^{0,1} \right]_{FN}
\end{array} \right)
$$

However, by assumption $\rho_i = 0 \ \ \forall i < m$ which implies

$$
\begin{gathered}
2 \left[ \nabla_{-1}^{1,0}, \nabla_{i}^{1,0} \right]_{FN} + \deg_{i-1} \left( \left[ \nabla_{\geq 0}^{1,0}, \nabla_{\geq 0}^{1,0} \right]_{FN} - \theta \right) = 0 \\
2 \left[ \nabla_{-1}^{1,0}, \nabla_{i+1}^{0,1} \right]_{FN} + 2 \left[ \nabla_{0}^{0,1}, \nabla_{i}^{1,0} \right]_{FN} + \deg_{i} \left( 2 \left[ \nabla_{\geq 0}^{1,0}, \nabla_{\geq 1}^{0,1} \right]_{FN} - \phi \right) = 0 \\
2 \left[ \nabla_{0}^{0,1}, \nabla_{i+1}^{0,1} \right]_{FN} + \deg_{i+1} \left( \left[ \nabla_{\geq 1}^{0,1}, \nabla_{\geq 1}^{0,1} \right]_{FN} - \tau \right) = 0
\end{gathered}
$$

$$
\mathrm{d}_B\Lambda_m = \begin{pmatrix} -\sum_{j=0}^{m-1} \left[\deg_{j-1}\left(\left[\nabla^{1,0}_{\geq 0}, \nabla^{1,0}_{\geq 0}\right]_{FN} - \theta\right), \nabla^{1,0}_{m-1-j}\right]_{FN} \\ ------------------------------- \\ -\sum_{j=0}^{m-1} \left[\deg_{j-1}\left(\left[\nabla^{1,0}_{\geq 0}, \nabla^{1,0}_{\geq 0}\right]_{FN} - \theta\right), \nabla^{0,1}_{m-j}\right]_{FN} \\ -\sum_{j=0}^{m-1} \left[\deg_{j}\left(2\left[\nabla^{1,0}_{\geq 0}, \nabla^{0,1}_{\geq 1}\right]_{FN} - \phi\right), \nabla^{1,0}_{m-1-j}\right]_{FN} \\ ------------------------------- \\ -\sum_{j=0}^{m-1} \left[\deg_{j}\left(2\left[\nabla^{1,0}_{\geq 0}, \nabla^{0,1}_{\geq 1}\right]_{FN} - \phi\right), \nabla^{0,1}_{m-j}\right]_{FN} \\ -\sum_{j=0}^{m-1} \left[\deg_{j+1}\left(\left[\nabla^{0,1}_{\geq 1}, \nabla^{0,1}_{\geq 1}\right]_{FN} - \tau\right), \nabla^{1,0}_{m-1-j}\right]_{FN} \\ ------------------------------- \\ -\sum_{j=0}^{m-1} \left[\deg_{j+1}\left(\left[\nabla^{0,1}_{\geq 1}, \nabla^{0,1}_{\geq 1}\right]_{FN} - \tau\right), \nabla^{0,1}_{m-j}\right]_{FN} \end{pmatrix}
$$

$$
= \begin{pmatrix} -\sum_{j=0}^{m-1} \left[\deg_{j-1}\left(\left[\nabla^{1,0}_{\geq 0}, \nabla^{1,0}_{\geq 0}\right]_{FN}\right), \nabla^{1,0}_{m-1-j}\right]_{FN} \\ ------------------------------- \\ -\sum_{j=0}^{m-1} \left[\deg_{j-1}\left(\left[\nabla^{1,0}_{\geq 0}, \nabla^{1,0}_{\geq 0}\right]_{FN}\right), \nabla^{0,1}_{m-j}\right]_{FN} \\ -2\sum_{j=0}^{m-1} \left[\deg_{j}\left(\left[\nabla^{1,0}_{\geq 0}, \nabla^{0,1}_{\geq 1}\right]_{FN}\right), \nabla^{1,0}_{m-1-j}\right]_{FN} \\ ------------------------------- \\ -2\sum_{j=0}^{m-1} \left[\deg_{j}\left(\left[\nabla^{1,0}_{\geq 0}, \nabla^{0,1}_{\geq 1}\right]_{FN}\right), \nabla^{0,1}_{m-j}\right]_{FN} \\ -\sum_{j=0}^{m-1} \left[\deg_{j+1}\left(\left[\nabla^{0,1}_{\geq 1}, \nabla^{0,1}_{\geq 1}\right]_{FN}\right), \nabla^{1,0}_{m-1-j}\right]_{FN} \\ ------------------------------- \\ -\sum_{j=0}^{m-1} \left[\deg_{j+1}\left(\left[\nabla^{0,1}_{\geq 1}, \nabla^{0,1}_{\geq 1}\right]_{FN}\right), \nabla^{0,1}_{m-j}\right]_{FN} \end{pmatrix}
$$

$$
= \begin{pmatrix}
-\deg_{m-2}\left(\left[\left[\nabla^{1,0}_{\geq 0}, \nabla^{1,0}_{\geq 0}\right]_{FN}, \nabla^{1,0}_{\geq 0}\right]_{FN}\right) \\
------------------------------ \\
-\deg_{m-1}\left(\left[\left[\nabla^{1,0}_{\geq 0}, \nabla^{1,0}_{\geq 0}\right]_{FN}, \nabla^{0,1}_{\geq 1}\right]_{FN}\right) \\
-2\deg_{m-1}\left(\left[\left[\nabla^{1,0}_{\geq 0}, \nabla^{0,1}_{\geq 1}\right]_{FN}, \nabla^{1,0}_{\geq 0}\right]_{FN}\right) \\
------------------------------ \\
-2\deg_{m}\left(\left[\left[\nabla^{1,0}_{\geq 0}, \nabla^{0,1}_{\geq 1}\right]_{FN}, \nabla^{0,1}_{\geq 1}\right]_{FN}\right) \\
-\deg_{m}\left(\left[\left[\nabla^{0,1}_{\geq 1}, \nabla^{0,1}_{\geq 1}\right]_{FN}, \nabla^{1,0}_{\geq 0}\right]_{FN}\right) \\
------------------------------ \\
-\deg_{m+1}\left(\left[\left[\nabla^{0,1}_{\geq 1}, \nabla^{0,1}_{\geq 1}\right]_{FN}, \nabla^{0,1}_{\geq 1}\right]_{FN}\right)
\end{pmatrix}
$$

$$
= \begin{pmatrix}
-\deg_{m-2}\left(\left[\left[\nabla^{1,0}_{\geq 0}, \nabla^{1,0}_{\geq 0}\right]_{FN}, \nabla^{1,0}_{\geq 0}\right]_{FN}\right) \\
------------------------------ \\
-\deg_{m-1}\left(\left[\left[\nabla^{1,0}_{\geq 0}, \nabla^{1,0}_{\geq 0}\right]_{FN}, \nabla^{0,1}_{\geq 1}\right]_{FN} + 2\left[\left[\nabla^{1,0}_{\geq 0}, \nabla^{0,1}_{\geq 1}\right]_{FN}, \nabla^{1,0}_{\geq 0}\right]_{FN}\right) \\
------------------------------ \\
-\deg_{m}\left(\left[\left[\nabla^{0,1}_{\geq 1}, \nabla^{0,1}_{\geq 1}\right]_{FN}, \nabla^{1,0}_{\geq 0}\right]_{FN} + 2\left[\left[\nabla^{1,0}_{\geq 0}, \nabla^{0,1}_{\geq 1}\right]_{FN}, \nabla^{0,1}_{\geq 1}\right]_{FN}\right) \\
------------------------------ \\
-\deg_{m+1}\left(\left[\left[\nabla^{0,1}_{\geq 1}, \nabla^{0,1}_{\geq 1}\right]_{FN}, \nabla^{0,1}_{\geq 1}\right]_{FN}\right)
\end{pmatrix}
$$

Recall from eq 1.4 that for $K, L$ arbitrary vector-valued 1 forms one has

$$
\begin{aligned}
[K, [K, K]_{FN}]_{FN} &= 0 \\
[[K, K]_{FN}, L]_{FN} &= -2\,[[K, L]_{FN}, K]_{FN}
\end{aligned}
$$

Thus one finds

$$\mathrm{d}_B \Lambda_m = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus $\Lambda_m$ is a closed vector-valued 2 form. By theorem 4.1 $\Lambda_m$ is exact. Thus

$$\Lambda_m = \mathrm{d}_B \begin{pmatrix} Id_M \otimes \lambda_m^{1,0} \\ Id_M \otimes \lambda_{m+1}^{0,1} \end{pmatrix}$$

One may then choose

$$\begin{pmatrix} A_{E\,m} \\ B_{E\,m+1} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \lambda_m^{1,0} \\ \lambda_{m+1}^{0,1} \end{pmatrix}$$

which then achieves $\rho_m = 0$. Since all relevant values of $m$ have been checked, $[\nabla, \nabla]_{FN} = \theta + \phi + \tau$ and hence the connection $\nabla$ is flat. One is free to choose a vanishing cross-term: $\nu = 0$. Note also that no further restrictions were placed on $\theta$ and $\phi$.

## B.2 Constructing the Gelfand-Fuks class

The goal is to rewrite $\Lambda_m$ to look more like $GF \circ C(\cdot, \cdot)$. One wishes to show that the difference between the two is exact. If one have a quantization in $\mathfrak{g}_k$, then $\rho_E = 0$ to all degrees, but only up to a maximum power of $\hbar$ of $\hbar^k$. In extending the quantization, one does not impose $\rho_E = 0$ to higher degrees, but rather to higher power of $\hbar$ in each and every degree up to $\infty$.

Rewrite the connection in the form

$$\begin{aligned} \nabla &= Id_M \otimes d_{dR} + d_{dR} \otimes Id_E + S \\ S &= Id_M \otimes S_E + S_M \otimes Id_E \end{aligned} \tag{B.16}$$

be the connection of eq 3.19. Then

$$\begin{aligned} S_E &= \tilde{A}_{E\,0} + \sum_{i=0}^{\infty} A_{E\,i} + \sum_{i=1}^{\infty} B_{E\,i} \\ S_M &= A_{-1} + \tilde{A}_{M\,0} + \sum_{i=0}^{\infty} A_{M\,i} + \sum_{i=1}^{\infty} B_{M\,i} \\ \\ \tilde{A}_{E\,0} &:= \text{"}\nabla_{WE} - d_{dR}^{1,0}\text{"} \\ \tilde{A}_{M\,0} &:= \text{"}\nabla_{ME} - d_{dR}^{1,0}\text{"} \end{aligned} \tag{B.17}$$

The Gelfand Fuks map one will employ is constructed from the $S$

$$\begin{aligned} GF &: \left(C^{\bullet}_{Lie}\left(\mathfrak{g}_\ell,\, \mathfrak{sp}\left(2n,\kappa\right) \times \mathfrak{gl}\left(e,\kappa\right);\, End\left(E\right)\right),\, d_{Lie}\right) \to \left(\Omega^{\bullet}\left(M;\, End\left(E\right),\, \nabla\right)\right) \\ GF\left(\eta\right)&\left(X_1, \cdots, X_j\right) = \eta\left(S\left(X_1\right), \cdots, S\left(X_j\right)\right) \end{aligned} \tag{B.18}$$

where

$$\Omega^j\left(M;\, End\left(E\right)\right) = \left\{ \beta \in \left(\Omega^j\left(\mathcal{P}_E\right) \otimes End\left(E\right)\right)^{G_0} \middle| \, i_X\beta = 0 \;\forall X \in \mathfrak{sp}\left(2n,\kappa\right) \times \mathfrak{gl}\left(e,\kappa\right) \right\} \tag{B.19}$$

and the $\{X_1, \cdots, X_j\}$ are vector fields on $\mathcal{P}_E$. The $GF$ of eq B.18 can be used to map the class $C$ of eq A.9 to de Rham cohomology on $M$, eq B.19.

Thus

$$GF\left(C\right)\left(a,b\right) = C\left(S\left(a\right), S\left(b\right)\right) \tag{B.20}$$

$$
\begin{gathered}
C\left(S\left(a\right),S\left(b\right)\right)\cong\\
\{0\}\ltimes\pi_{k+1,\ell}\left(\left\{\left(S_E\left(B_a\right)\,\exp\left(\hbar\sum_{i=1}^{n}\overleftarrow{\frac{\partial}{\partial y_i}}\overrightarrow{\frac{\partial}{\partial x_i}}\right)\,S_E\left(B_b\right)\right)\right.\right.\\
-\left(S_E\left(B_b\right)\,\exp\left(\hbar\sum_{i=1}^{n}\overleftarrow{\frac{\partial}{\partial y_i}}\overrightarrow{\frac{\partial}{\partial x_i}}\right)\,S_E\left(B_a\right)\right)\\
-\tfrac{1}{\hbar}\left(I_eS_M\left(f_b\right)\exp\left(\hbar\sum_{i=1}^{n}\overleftarrow{\frac{\partial}{\partial y_i}}\overrightarrow{\frac{\partial}{\partial x_i}}\right)S_E\left(B_a\right)-S_E\left(B_a\right)\exp\left(\hbar\sum_{i=1}^{n}\overleftarrow{\frac{\partial}{\partial y_i}}\overrightarrow{\frac{\partial}{\partial x_i}}\right)I_eS_M\left(f_b\right)\right)\\
\left.\left.+\tfrac{1}{\hbar}\left(I_eS_M\left(f_a\right)\exp\left(\hbar\sum_{i=1}^{n}\overleftarrow{\frac{\partial}{\partial y_i}}\overrightarrow{\frac{\partial}{\partial x_i}}\right)S_E\left(B_b\right)-S_E\left(B_b\right)\exp\left(\hbar\sum_{i=1}^{n}\overleftarrow{\frac{\partial}{\partial y_i}}\overrightarrow{\frac{\partial}{\partial x_i}}\right)I_eS_M\left(f_a\right)\right)\right\}_{RYO}\right)
\end{gathered}
\tag{B.21}
$$

Alternatively

$$
\begin{aligned}
C\left(a,b\right) &= \Phi_V\left[\Phi_H a\,,\,\Phi_H b\right]\\
&= \left[\Phi_H a\,,\,\Phi_H b\right]-\Phi_H\left[\Phi_H a\,,\,\Phi_H b\right]
\end{aligned}
$$

$$
C\left(S\left(a\right),S\left(b\right)\right) = \Phi_V\left[\Phi_H S\left(a\right)\,,\,\Phi_H S\left(b\right)\right]
$$

and

$$
\sigma_{k+1,\ell}\left(\pi_{k+1,\ell}\left(B\right)\right)\cong\left\{\begin{array}{ccc}
\pi_{k+1,\ell}\left(B\right) & if & \ell<2k+2\\
\pi_{k+1,2k+1}\left(B\right)+\frac{1}{e}I_e\mathrm{Tr}\left(\pi_{k+1,2k+2}\left(B\right)\right) & if & \ell\geq 2k+2
\end{array}\right\}
\tag{B.22}
$$

Note also that

$$
\begin{aligned}
[\nabla,\nabla]_{FN} = \quad & 2\mathrm{d}_B \begin{pmatrix} \nabla^{1,0}_{M\geq 0}\otimes Id_E + Id_M \otimes \nabla^{1,0}_{E} \\ \nabla^{0,1}_{M\geq 1}\otimes Id_E + Id_M \otimes \nabla^{0,1}_{E\geq 1} \end{pmatrix} \\
& + \left[\nabla^{1,0}_{\geq 0} + \nabla^{0,1}_{\geq 1}, \nabla^{1,0}_{\geq 0} + \nabla^{0,1}_{\geq 1}\right]_{FN} - \tfrac{i}{\hbar}\omega \\
= \quad & 2\mathrm{d}_B \begin{pmatrix} S^{1,0}_{M\geq 0}\otimes Id_E + Id_M \otimes S^{1,0}_{E} \\ S^{0,1}_{M\geq 1}\otimes Id_E + Id_M \otimes S^{0,1}_{E\geq 1} \end{pmatrix} \\
& + 2\mathrm{d}_B \begin{pmatrix} \nabla^{1,0}_{WM}\otimes Id_E + Id_M \otimes \nabla^{1,0}_{WE} \\ 0 \end{pmatrix} \\
& + \left[\nabla^{1,0}_{\geq 0} + \nabla^{0,1}_{\geq 1}, \nabla^{1,0}_{\geq 0} + \nabla^{0,1}_{\geq 1}\right]_{FN} - \tfrac{i}{\hbar}\omega
\end{aligned}
$$

$$
\begin{aligned}
\rho = \quad & [\nabla,\nabla]_{FN} - \begin{pmatrix} \theta \\ \phi \\ \tau \end{pmatrix} \\
= \quad & 2\mathrm{d}_B \begin{pmatrix} S^{1,0}_{M\geq 0}\otimes Id_E + Id_M \otimes S^{1,0}_{E} \\ S^{0,1}_{M\geq 1}\otimes Id_E + Id_M \otimes S^{0,1}_{E\geq 1} \end{pmatrix} \\
& + 2\mathrm{d}_B \begin{pmatrix} \nabla^{1,0}_{WM}\otimes Id_E + Id_M \otimes \nabla^{1,0}_{WE} \\ 0 \end{pmatrix} \\
& + \left[\nabla^{1,0}_{\geq 0} + \nabla^{0,1}_{\geq 1}, \nabla^{1,0}_{\geq 0} + \nabla^{0,1}_{\geq 1}\right]_{FN} - \begin{pmatrix} \theta_{\geq 0} \\ \phi \\ \tau \end{pmatrix}
\end{aligned}
\tag{B.23}
$$

$$
\begin{aligned}
= \quad & 2\mathrm{d}_B \begin{pmatrix} Id_M \otimes S_E^{1,0} \\ Id_M \otimes S_{E\geq 1}^{0,1} \end{pmatrix} + 2\mathrm{d}_B \begin{pmatrix} \nabla_{M\geq 0}^{1,0} \otimes Id_E + Id_M \otimes \nabla_{WE}^{1,0} \\ S_{M\geq 1}^{0,1} \otimes Id_E \end{pmatrix} \\
& + \left[Id_M \otimes d_{dR}^{1,0} + d_{dR}^{1,0} \otimes Id_E + S, Id_M \otimes d_{dR}^{1,0} + d_{dR}^{1,0} \otimes Id_E + S\right]_{FN} \\
& - \begin{pmatrix} \theta_{\geq 0} \\ \phi \\ \tau \end{pmatrix} \\
= \quad & 2\mathrm{d}_B \begin{pmatrix} Id_M \otimes S_E^{1,0} \\ Id_M \otimes S_{E\geq 1}^{0,1} \end{pmatrix} + \Lambda
\end{aligned}
$$

$$
\begin{aligned}
\Lambda := \quad & 2\mathrm{d}_B \begin{pmatrix} \nabla_{M\geq 0}^{1,0} \otimes Id_E + Id_M \otimes \nabla_{WE}^{1,0} \\ S_{M\geq 1}^{0,1} \otimes Id_E \end{pmatrix} \\
& + \left[Id_M \otimes d_{dR}^{1,0} + d_{dR}^{1,0} \otimes Id_E + S \, , Id_M \otimes d_{dR}^{1,0} + d_{dR}^{1,0} \otimes Id_E + S\right]_{FN} \\
& - \begin{pmatrix} \theta_{\geq 0} \\ \phi \\ \tau \end{pmatrix} \\
= \quad & [S, S]_{FN} - \begin{pmatrix} \theta_{\geq 0} \\ \phi \\ \tau \end{pmatrix} + (\text{exact term})
\end{aligned}
$$

The goal is to show that

$$
\sigma_{k,\ell}\left(\pi_{k+1,\ell}\left(GF \circ C\right)\right) = \gamma \left[S, S\right]_{FN} + (\text{exact term}) \tag{B.24}
$$

for some $\gamma \in \kappa$.

To facilitate this proof, break the connection form $S$ into pieces, according to the projection operators $\Phi_H$ and $\Phi_V$

$$\Phi_V : \left(\frac{Der_{-1}(\mathcal{D}_n)}{Der_0(\mathcal{D}_n)}\right) \ltimes \left(\frac{gl_r(\mathcal{D}_n)}{\hbar^{k+1}gl_r(\mathcal{D}_n)} \oplus \frac{\hbar^{k+1}gl_r(\mathcal{D}_n)}{\hbar^{\ell+1}gl_r(\mathcal{D}_n)}\right) \to \{0\} \ltimes \left(\{0\} \oplus \frac{\hbar^{k+1}gl_r(\mathcal{D}_n)}{\hbar^{\ell+1}gl_r(\mathcal{D}_n)}\right)$$
$$\Phi_H := Id - \Phi_V \tag{B.25}$$
$$\ell > k$$

$$\begin{aligned} S = & \quad (\Phi_H + \Phi_V)\, S\, (\Phi_H + \Phi_V) \\ & \Phi_H S \Phi_H + \Phi_V S \Phi_H + \Phi_V S \Phi_V \\ = & \quad S_{HH} + S_{VH} + S_{VV} \end{aligned} \tag{B.26}$$

$$\Phi_H S \Phi_V \equiv 0$$

Note that $\hbar$ is in the center of the Lie algebra, and $S$ contains only non-negative powers of $\hbar$, thus $(\Phi_H S \Phi_V)(a) = 0\ \forall a \in \mathfrak{g}$. Additionally, the Lie bracket can not decrease the power of $\hbar$.

$$(GF \circ C)(a, b) \;=\; C(S(a), S(b)) \;=\; \Phi_V\left[S_{HH}(a)\ ,\ S_{HH}(b)\right] \tag{B.27}$$

Let one assume that quantization exists up to and including order $\hbar^k$ for $\nabla$, and quantization exists to all order in $\hbar$ for $\nabla_M$ (via the Fedosov construction). One wishes to extend the quantization to order $\hbar^\ell$ for some $\ell > k$. Thus by assumption $\rho = 0$ to order $\hbar^k$. The existence of quantization from order $\hbar^{k+1}$ to order $\hbar^\ell$ would require that it is possible to extend the solution of

$$\rho = 2\mathrm{d}_B \begin{pmatrix} Id_M \otimes S_E^{1,0} \\ Id_M \otimes S_{E\geq 1}^{0,1} \end{pmatrix} + \Lambda = 0 \tag{B.28}$$

from order $\hbar^{k+1}$ to order $\hbar^\ell$, for all degrees. By assumption, it is already true that

$$\Phi_H \rho = \Phi_H \left( 2\mathrm{d}_B \begin{pmatrix} Id_M \otimes S_E^{1,0} \\ Id_M \otimes S_{E\geq 1}^{0,1} \end{pmatrix} + \Lambda \right) = 0 \tag{B.29}$$

Thus the goal is to show that

$$\sigma_{k,\ell}\left(\Phi_V\left[S_{HH}\left(a\right)\,,S_{HH}\left(b\right)\right]\right) =$$
$$\gamma\Phi_V\left[S_{HH}+S_{VH}+S_{VV}\,,S_{HH}+S_{VH}+S_{VV}\right]_{FN}\left(a,b\right)+\left(\text{exact term}\right)$$

$$\begin{aligned}\Phi_V\left[S_{HH}+S_{VH}+S_{VV}\,,S_{HH}+S_{VH}+S_{VV}\right]_{FN}\left(a,b\right) =& \\ \Phi_V\left[S_{HH}\,,S_{HH}\right]_{FN}\left(a,b\right)& \\ +2\Phi_V\left[S_{HH}\,,S_{VH}+S_{VV}\right]_{FN}\left(a,b\right)& \\ +\Phi_V\left[S_{VH}\,,S_{VH}+2S_{VV}\right]_{FN}\left(a,b\right)& \\ +\Phi_V\left[S_{VV}\,,S_{VV}\right]_{FN}\left(a,b\right)+\left(\text{exact term}\right)&\end{aligned} \tag{B.30}$$

$$\gamma\in\kappa$$

(recall that $\hbar\notin\kappa$). Since the LHS of eq B.30 is independent of $S_{VH}$ and $S_{VV}$, one must show that terms involving these components are exact.

From appendix C, eq C.5 one has

$$U,W\in\Omega^1\left(M,\left(\mathcal{P}\times\mathfrak{gl}\left(e,\mathcal{D}_n\right)\right)/G\right)$$

$$\left[W\left(a\right),U\left(b\right)\right]+\left[U\left(a\right),W\left(b\right)\right]=U\left(\left(\delta_{Lie}W\right)\left(a,b\right)\right)+\left(\text{exact term}\right) \tag{B.31}$$

$$\left[U,W\right]_{FN}\left(a,b\right)=-\left[U\left(a\right),W\left(b\right)\right]-\left[W\left(a\right),U\left(b\right)\right]+\left(\text{exact term}\right)$$

In particular, note that if $U\left(\left(\delta_{Lie}W\right)\right)=0$, then $[U,W]_{FN}$ is exact. Thus, by eq B.31

$$\left[S_{HH}\,,S_{VH}+S_{VV}\right]_{FN}(a,b)=(\text{exact term})$$

$$\left[S_{VH}\,,S_{VH}+2S_{VV}\right]_{FN}(a,b)=(\text{exact term}) \tag{B.32}$$

$$\Phi_V\left[S_{HH}\,,S_{HH}\right]_{FN}(a,b)=-2\Phi_V\left[S_{HH}(a)\;,S_{HH}(b)\right]+(\text{exact term})$$

Thus

$$\begin{aligned}\Phi_V\left[S_{HH}+S_{VH}+S_{VV}\,,S_{HH}+S_{VH}+S_{VV}\right]_{FN}(a,b)=\\ -2\Phi_V\left[S_{HH}(a)\;,S_{HH}(b)\right]\\ +\Phi_V\left[S_{VV}\,,S_{VV}\right]_{FN}(a,b)+(\text{exact term})\end{aligned} \tag{B.33}$$

To exhibit the $\sigma_{k,\ell}$ on the RHS of eq B.33 (and hence eliminate the $\Phi_V\left[S_{VV}\,,S_{VV}\right]_{FN}$ term), consider the quantity

$$\begin{aligned}&\left[\Phi_V W\Phi_H\,,\Phi_V W\Phi_H\right]_{FN}\\ =\;&-2\left[\Phi_V W\Phi_H\,,\Phi_V W\Phi_H\right]+(\text{exact term})\\ =\;&-\Phi_V W\Phi_H\left(\left(\delta_{Lie}\Phi_V W\Phi_H\right)(a,b)\right)+(\text{exact term})\\ =\;&-\Phi_V W\Phi_H\left(\left[\left(\Phi_V W\Phi_H\right)(a)\,,b\right]+\left[a,\left(\Phi_V W\Phi_H\right)(b)\right]-\Phi_V W\Phi_H\left[a,b\right](a,b)\right)+(\text{exact term})\\ =\;&-\Phi_V W\Phi_H\left(\left[\left(\Phi_V W\Phi_H\right)(a)\,,b\right]+\left[a,\left(\Phi_V W\Phi_H\right)(b)\right]\right)+(\text{exact term})\end{aligned} \tag{B.34}$$

where $W$ is an arbitrary Lie algebra-valued one form. The expression $\left(\Phi_V W\Phi_H\right)(a)$ will produce only terms of power of $\hbar$ greater than $k$. However, since the Lie bracket can not reduce the power of $\hbar$ of a term, the output of the bracket will be terms that all have a power of $\hbar$ greater than $k$. The $\Phi_H$ will kill all such terms. Thus
$\Phi_H\left(\left[\left(\Phi_V W\Phi_H\right)(a)\,,b\right]+\left[a,\left(\Phi_V W\Phi_H\right)(b)\right]\right)=0$, leaving the result

$$\left[\Phi_V W\Phi_H\,,\Phi_V W\Phi_H\right]_{FN}=(\text{exact term}) \tag{B.35}$$

Note that due to the $\Phi_V$ in both arguments of the bracket of the LHS of eq B.35, the minimum power of $\hbar$ of the output of the bracket is $2k+2$. Thus the $End\,(E)$ component of eq B.35 is an arbitrary element of $\hbar^{2k+2}\mathfrak{sl}\,(e,\mathcal{A}_n) + \hbar^{2k+3}\mathfrak{gl}\,(e,\mathcal{D}_n)$. Since this is exact, it may freely be added to eq B.33 without changing its class. This has the effect of quotienting by $\hbar^{2k+2}\mathfrak{sl}\,(e,\mathcal{A}_n) + \hbar^{2k+3}\mathfrak{gl}\,(e,\mathcal{D}_n)$, yielding

$$\begin{aligned} &\Phi_V\left[S_{HH}+S_{VH}+S_{VV}\,,S_{HH}+S_{VH}+S_{VV}\right]_{FN}(a,b) \\ &= -2\sigma_{k,\ell}\left(\Phi_V\left[S_{HH}\left(a\right)\,,S_{HH}\left(b\right)\right]+(\text{exact term})\right) \end{aligned} \tag{B.36}$$

Thus, one has at last

$$\begin{aligned} &\Phi_V\left[S_{HH}+S_{VH}+S_{VV}\,,S_{HH}+S_{VH}+S_{VV}\right]_{FN}(a,b) \\ = \quad & -2\left(GF\circ C\right)(a,b)+(\text{exact term}) \end{aligned} \tag{B.37}$$

or

$$\Lambda = -2\sigma_{k,\ell}\left(GF\circ C\right) - \begin{pmatrix} \theta_{\geq 0} \\ \phi \\ \tau \end{pmatrix} + (\text{exact term}) \tag{B.38}$$

# Appendix C

# Proof that Certain Expressions are Exact

In this appendix it will be proven that certain expressions involving vector-valued forms are exact with respect to $\delta_{Lie}$ of eq 1.2. Consider the general structure of the FN and Lie brackets:

$$U, W \in \Omega^1 \left(M, \left(\mathcal{P}_0 \times \mathfrak{gl}\left(e, \mathcal{D}_n\right)\right)/G_0\right) \tag{C.1}$$

$$
\begin{aligned}
(\delta_{Lie} W)(a,b) &= [W(a), b] + [a, W(b)] - W([a,b]) \\
(\delta_{Lie}(UW))(a,b) &= [UW(a), b] + [a, UW(b)] - UW([a,b]) \\
U((\delta_{Lie} W))(a,b) &= U((\delta_{Lie} W)(a,b)) \\
&= U([W(a), b] + [a, W(b)]) - UW([a,b]) \\
(\delta_{Lie} U)(a, W(b)) &= [U(a), W(b)] + [a, UW(b)] - U([a, W(b)]) \\
(\delta_{Lie} U)(W(a), b) &= [UW(a), b] + [W(a), U(b)] - U([W(a), b])
\end{aligned}
\tag{C.2}
$$

where $[-,-]$ is the Lie Algebra bracket. Combining these one finds

$$
\begin{aligned}
& (\delta_{Lie} U)(W(a), b) + (\delta_{Lie} U)(a, W(b)) - (\delta_{Lie}(UW))(a,b) + U((\delta_{Lie} W))(a,b) \\
= \; & [UW(a), b] + [W(a), U(b)] - U([W(a), b]) + [U(a), W(b)] \\
& + [a, UW(b)] - U([a, W(b)]) - [UW(a), b] - [a, UW(b)] \\
& + UW([a,b]) + U([W(a), b] + [a, W(b)]) - UW([a,b]) \\
= \; & {\color{red}[UW(a), b]} + [W(a), U(b)] - {\color{blue}U([W(a), b])} + [U(a), W(b)] \\
& + {\color{green}[a, UW(b)]} - {\color{blue}U([a, W(b)])} - {\color{red}[UW(a), b]} - {\color{green}[a, UW(b)]} \\
& + {\color{magenta}UW([a,b])} + {\color{blue}U([W(a), b] + [a, W(b)])} - {\color{magenta}UW([a,b])} \\
= \; & [W(a), U(b)] + [U(a), W(b)]
\end{aligned}
\tag{C.3}
$$

$$[U,W]_{FN}(a,b) - ((\delta_{Lie}U)(W(a),b) + (\delta_{Lie}U)(a,W(b)) + (\delta_{Lie}W)(U(a),b)$$
$$+ (\delta_{Lie}W)(a,U(b)) - (\delta_{Lie}(UW))(a,b) - (\delta_{Lie}(WU))(a,b))$$

$$\begin{aligned}
= \quad & [U(a),W(b)] + [W(a),U(b)] + (UW+WU)[a,b] \\
& -U([W(a),b] + [a,W(b)]) - W([U(a),b] + [a,U(b)]) \\
& -[UW(a),b] - [W(a),U(b)] + U([W(a),b]) \\
& -[U(a),W(b)] - [a,UW(b)] + U([a,W(b)]) \\
& -[WU(a),b] - [U(a),W(b)] + W([U(a),b]) \\
& -[W(a),U(b)] - [a,WU(b)] + W([a,U(b)]) \\
& [UW(a),b] + [a,UW(b)] - UW([a,b]) \\
& +[WU(a),b] + [a,WU(b)] - WU([a,b])
\end{aligned} \tag{C.4}$$

$$\begin{aligned}
= \quad & [U(a),W(b)] + [W(a),U(b)] + (UW+WU)[a,b] \\
& -U([W(a),b] + [a,W(b)]) - W([U(a),b] + [a,U(b)]) \\
& -[UW(a),b] - [W(a),U(b)] + U([W(a),b]) \\
& -[U(a),W(b)] - [a,UW(b)] + U([a,W(b)]) \\
& -[WU(a),b] - [U(a),W(b)] + W([U(a),b]) \\
& -[W(a),U(b)] - [a,WU(b)] + W([a,U(b)]) \\
& +[UW(a),b] + [a,UW(b)] - UW([a,b]) + \\
& [WU(a),b] + [a,WU(b)] - WU([a,b])
\end{aligned}$$

$$= \quad -[U(a),W(b)] - [W(a),U(b)]$$

Thus it is true that

$$\begin{aligned}
&[W(a),U(b)] + [U(a),W(b)] = U((\delta_{Lie}W)(a,b)) + (\text{exact term}) \\
&[U,W]_{FN}(a,b) = -[U(a),W(b)] - [W(a),U(b)] + (\text{exact term})
\end{aligned} \tag{C.5}$$