Title
Perelman's entropy on ancient solutions to the Ricci flow

Author
Zhang, Yongjia

Publication Date
2019

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA SAN DIEGO

Perelman's entropy on ancient solutions to the Ricci flow

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Yongjia Zhang

Committee in charge:

Professor Bennett Chow, Chair
Professor Lei Ni, Co-Chair
Professor Michael Fogler
Professor Kenneth Intriligator
Professor Bo Li

2019

The dissertation of Yongjia Zhang is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Co-Chair

Chair

University of California San Diego

2019

DEDICATION

A mighty fortress is our God,
    A bulwark never failing:
Our helper He, amid the flood
    Of mortal ills prevailing.
For still our ancient foe
    Doth seek to work his woe;
        His craft and power are great,
        And armed with cruel hate,
On earth is not his equal.

Did we in our own strength confide,
    Our striving would be losing;
Were not the right Man on our side,
    The Man of God's own choosing.
Dost ask who that may be?
    Christ Jesus, it is he;
        Lord Sabaoth is his name,
        From age to age the same,
And He must win the battle.

And though this world, with devils filled,
    Should threaten to undo us,
We will not fear, for God hath willed
    His truth to triumph through us.
The Prince of Darkness grim,—
    We tremble not for him;
        His rage we can endure,
        For lo! His doom is sure,—
One little word shall fell him.

That word above all earthly powers—
    No thanks to them—abideth;
The Spirit and the gifts are ours
    Through him who with us sideth.
Let goods and kindred go,
    This mortal life also:
        The body they may kill:
        God's truth abideth still,
His kingdom is for ever.

—Rev.Dr. Martin Luther

EPIGRAPH

Behold, the fear of the LORD, that is wisdom;
and to depart from evil is understanding.
—Job 28:28, KJV

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

Thanks be to my family—my father Zhenfu, my mother Qionghua, my sister Yi, and my sister Qin—for their love, support, and companionship in all my life.

Thanks be to my advisors, Professor Bennett Chow and Professor Lei Ni, for their professional support.

Thanks be to Reverend Doctor Jonathan Doolittle of Clairemont Lutheran Church, who baptized me into Jesus Christ and nourished me with the words of God.

Thanks be to my friends Doctor Qingjun Luo and his wife Zihui Zhang, my precious brother and sister in Christ, through whom the grace, mercy, and wisdom of God flow to me constantly, in whose fellowship is the foretaste of our eternal life with our common Father in heaven.

Thanks be to my friend Doctor Gang Pang, through whose tongue was preached unto me the good news of Jesus Christ the Lord.

The result of chapter 4 is my own work, currently accepted for publication on Communications in Analysis and Geometry.

The result of chapter 5 is my own work, currently accepted for publication online on Journal für die reine und angewandte Mathematik (Crelle's Journal).

VITA

2013	B. S. in Theoretical and Applied Mechanics, Peking University
2015-2019	Graduate Teaching Assistant, University of California San Diego
2019	Ph. D. in Mathematics, University of California San Diego

PUBLICATIONS

Yongjia Zhang. "Entropy, noncollapsing, and a gap theorem for ancient solutions to the Ricci flow." *Communications in Analysis and Geometry*, to appear.

Yongjia Zhang. "On the equivalence between noncollapsing and bounded entropy for ancient solutions to the Ricci flow." *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, published online, 2018.

Xiaodong Cao, Bennett Chow, and Yongjia Zhang. "Three-dimensional noncompact κ-solutions that are Type I forward and backward." *Proceedings of the American Mathematical Society*, to appear.

Yongjia Zhang. "On three-dimensional Type I κ-solutions to the Ricci flow." *Proceedings of the American Mathematical Society*, published online, 2018.

Yongjia Zhang. "The rigidity of $\mathbb{S}^3 \times \mathbb{R}$ under ancient Ricci flow." *The Journal of Geometric Analysis*, published online, 2018.

Yongjia Zhang. "A note on Perelman's no shrinking breather theorem." *The Journal of Geometric Analysis*, published online, 2018.

Yongjia Zhang. "Compactness theorems for 4-dimensional gradient Ricci solitons." *Pacific Journal of Mathematics*, to appear.

Xiaolong Li and Yongjia Zhang. "Ancient solutions to the Ricci flow in higher dimensions." *arXiv preprint arXiv:1812.04156*, 2018.

Bennett Chow, Michael Freedman, Henry Shin, and Yongjia Zhang. "Curvature growth of some 4-dimensional gradient Ricci soliton singularity models." *arXiv preprint arXiv:1903.09181*, 2019.

ABSTRACT OF THE DISSERTATION

Perelman's entropy on ancient solutions to the Ricci flow

by

Yongjia Zhang
Doctor of Philosophy in Mathematics

University of California San Diego, 2019

Professor Bennett Chow, Chair
Professor Lei Ni, Co-Chair

In this dissertation we will first give an exposition for some topics on Perelman's entropy and some results related to the analysis of the entropy, and then present the content of two papers among the author's publication list, [42] and [41].

Chapter 1 is an introduction to Perelman's entropy and the author's main results. The statement of these main results can be found in section 1.3.

Chapter 2 and chapter 3 are expository materials on Perelman's entropy and its related analytic tools; these results are included because of their importance to our

main theorems.

In chapter 4, we prove some estimates for the *Nash entropy* on ancient solutions and thereby prove a gap theorem for the *asymptotic entropy*.

In chapter 5, we prove an assertion made by Perelman [33], saying that for an ancient solution to the Ricci flow with bounded and nonnegative curvature operator, bounded entropy is equivalent to κ-noncollapsing on all scales. This proof is based on accurate gaussian upper and lower estimates for the *conjugate heat kernel*.

Part I

Introduction and preliminaries

Chapter 1

Introduction and the main results

1.1 The Ricci flow

The Ricci flow was first invented by Richard Hamilton in 1982. In his seminal paper [15], Hamilton considered an n-manifold M^n with a one-parameter family of metrics $\{g(t)\}_{t\in[0,T)}$, evolving by the equation

$$\frac{\partial}{\partial t} g = -2Ric.$$

This evolving equation is called the *Ricci flow equation*, and a manifold with such an evolving metric is called a *Ricci flow*.

The Ricci flow equation is a nonlinear parabolic geometric evolution equation. On the one hand, because the equation is parabolic, if one starts the Ricci flow from a closed manifold with certain good curvature condition, then one may reasonably hope that the Ricci flow would improve the curvature condition and evolve the metric to a standard one. It would then follow that the original manifold is diffeomorphic to a standard manifold. Indeed, there are many results in this fashion. For instance, Hamilton [15] first proved that every closed three-dimensional manifold with positive Ricci curvature is diffeomorphic to a round space form. Then Hamilton [16] also proved that every four-dimensional closed manifold with positive curvature operator is diffeomorphic to a round space form—a result eventually extended to every dimension by Böhm and Wilking [4]. Brendle and Schoen [6] proved that a closed manifold with strict pointwise quarterly pinched sectional curvature is diffeomorphic to a round space form.

On the other hand, the Ricci flow is highly nonlinear. It is therefore reasonable to think that in general cases the Ricci flow does not exist for all time and does develop singularities, for instance the neckpinch [1]. If a singularity happens at finite time, then it is called a *finite-time singularity*. Please refer to section 3.1 in chapter 3 for more details about finite-time singularities.

In [21] Hamilton proved that a specific type of Ricci flows on three-dimensional closed manifolds—called *nonsingular solutions* by Hamilton—always evolve the metric on the manifold to the extent that it becomes a standard space as described in Thurston's geometrization conjecture, or multiple pieces connected with tori. Such phenomenon fits very well in Hamilton's project of proving the geometrization conjecture with the Ricci flow [20].

1.2 Perelman's entropy

One of the main obstacles in Hamilton's program is the possibility of *collapsing*, that is, along the Ricci flow, normalized such that the curvature is uniformly bounded, the injectivity radius might converge to zero. If collapsing ever happens, one cannot use Hamilton's dilation method [20] to analyze the singularity. This obstacle was removed by Perelman [33] using either one of his two monotonicity formulae, *the entropy* and *the reduced volume*. Furthermore, Perelman [34] also built up the rigorous surgery process and completed the proof of Thurston's geometrization conjecture. In this section we will give a brief introduction to Perelman's entropy, since it is the main subject of our study. A nice monograph on Perelman's entropy is [40].

Let M be an n-dimensional smooth manifold. In [33], Perelman defined his $\mathcal{W}$-functional to be an operator on a tuple (g, f, τ), where g is a Riemannian metric, f is a smooth function, and τ is a positive scale

$$\mathcal{W}(g,f,\tau) = \int_M \Big(\tau(|\nabla f|^2 + R) + f - n\Big)(4\pi\tau)^{-\frac{n}{2}} e^{-f} dg. \tag{1.1}$$

He also defined the μ functional

$$\mu(g,\tau) = \inf\left\{\mathcal{W}(g,f,\tau) : \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dg = 1\right\}$$

and the ν functional

$$\nu(g) = \inf\{\mu(g,\tau) : \tau > 0\}.$$

Indeed, $\mu(g,\tau)$ and ν are logarithmic Sobolev constants for the Riemannian manifold (M,g).

Let $(M^n, g(t))_{t\in[0,T]}$ be a complete Ricci flow with bounded curvature. Let $u : M \times [0,T] \to \mathbb{R}$ be a positive solution to the conjugate heat equation coupled with the Ricci flow, that is

$$\frac{\partial}{\partial t} g = -2Ric, \qquad (1.2)$$

$$-\frac{\partial}{\partial t} u = \Delta u - Ru. \qquad (1.3)$$

It will become clear in chapter 2 why (1.3) is called the conjugate heat equation. If we write u as

$$u(x,t) = (4\pi\tau)^{-\frac{n}{2}} e^{-f(x,t)},$$

where $\tau = T - t$, then the quantity $\mathcal{W}(g(t), f(t), \tau(t))$ is monotonically increasing in time t.

Theorem 1.1 (Perelman, 2002)**.**

$$\frac{d}{dt}\mathcal{W}(g,f,\tau) = \int_M 2\tau\left|Ric + \nabla^2 f - \frac{1}{2\tau}g\right|^2 (4\pi\tau)^{-\frac{n}{2}} e^{-f} dg_t \geq 0. \tag{1.4}$$

Consequently, we have

$$\frac{d}{dt}\mu(g(t),\tau) \geq 0.$$

Remark: Indeed Perelman proved the above theorem for closed manifolds, but from his proof it is easy to observe that (1.4) is true so long as the integration by parts holds. In particular, Theorem 1.1 is true for a conjugate heat kernel; see chapter 2 for more details.

Perelman [33] has discovered another monotonicity formula called *the reduced volume*. This is also a very important formula and is frequently implemented in the proof of our main theorems. We will postpone its exposition until chapter 3.

1.3 Perelman's entropy on ancient solutions

In this section we introduce the main results of the author—the gap theorem for Perelman's entropy on ancient solutions to the Ricci flow, and the equivalence between bounded entropy and noncollapsing. These results are all proved for ancient solutions to the Ricci flow. We assume that all the ancient solutions in our discussion have *bounded curvature*.

Definition 1.2 (Ancient solution)**.** *A complete Ricci flow* $(M, g(t))$ *is called an ancient solution if its existence interval extends to* $-\infty$*. For the sake of convenience, we can always shift time and assume* $g(t)$ *exists for* $t \in (-\infty, 0]$*.*

Consider a complete ancient solution $(M, g(t))_{t\in(-\infty,0]}$. Let

$$H(y, s|x, t) = (4\pi(t-s))^{-\frac{n}{2}} e^{f_{(y,s)}(x,t)} \tag{1.5}$$

be the fundamental solution to the conjugate heat equation (1.3), where $s < t \leq 0$. We define the *pointed entropy* to be

$$W_{(x,t)}(s) = \mathcal{W}\big(g(s), f_{(\cdot,s)}(x,t), t-s\big), \tag{1.6}$$

where (x, t) is called the *base point*. The *asymptotic entropy* is defined by

$$\bar{W}(x, t) = \lim_{s\to\infty} W_{(x,t)}(s). \tag{1.7}$$

It is easy to see that the asymptotic entropy is a function of the base point (x, t). Since $W_{(x,t)}(s)$ is monotonically increasing in s, the limit in (1.6) always makes sense. Furthermore, $\bar{W}(x, t)$ is always less than or equal to 0; see Lemma 2.12 and Lemma 2.13(1).

1.3.1 The gap theorem

Forasmuch as the ancient solution is more likely to be unmasked when the time approaches negative infinity, the asymptotic entropy, being determined by the properties of the ancient solution at time negative infinity, is believed to reveal some mysteries of the ancient solution. By Lemma 2.13(6), if the asymptotic entropy is 0, then Perelman's monotonicity formula immediately implies that the ancient solution is a gaussian shrinker (static Euclidean space). We prove a more rigid result for noncollapsed ancient solutions, showing that there must be a fixed gap between 0 and every $\bar{W}(x,t)$ for a nontrivial noncollapsed ancient solution. The following is the main theorem of [42].

Theorem 1.3 (Zhang, 2018)**.** *There exists $\varepsilon > 0$ depending only on the dimension n such that the following holds. Let $(M^n, g(t))_{t\in(-\infty,0]}$ be a complete ancient noncollapsed solution to the Ricci flow such that $\sup\limits_{M\times(-\infty,0]} |Rm| < \infty$. If there exists $(x,t) \in M \times (-\infty, 0]$ such that*

$$\bar{W}(x,t) \geq -\varepsilon.$$

Then $(M, g(t))$ is a gaussian shrinker.

Indeed, Yokota [37] proved a similar theorem for the reduced volume, it is interesting to compare his theorem to Theorem 1.3.

Theorem 1.4 (Yokota, 2009)**.** *There exists $\varepsilon > 0$ depending only on the dimension n such that the following holds. Let $(M^n, g(\tau))_{\tau\in[0,\infty)}$ be a complete ancient solution to* *the Ricci flow with a uniform lower bound for the Ricci tensor, where τ is the backward time. If there exists $(x, \tau) \in M \times [0, \infty)$ such that*

$$\bar{\mathcal{V}}(x, \tau) \geq 1 - \varepsilon$$

Then $(M, g(\tau))$ is a Gaussian shrinker. Here $\bar{\mathcal{V}}(x, \tau)$ is the asymptotic reduced volume defined in (3.32).

1.3.2 The equivalence between bounded entropy and noncollapsing

In Perelman [33], he made the following comment

> ...require that $g_{ij}(t)$ to be κ-noncollapsed on all scales... It is not hard to show that this requirement is equivalent to a uniform bound on the entropy S, defined as in 5.1 using an arbitrary fundamental solution to the conjugate heat equation.

Here he was referring to the pointed entropy defined in (1.6). And the definition of noncollapsing is as follows.

Definition 1.5 (Strong noncollapsing)**.** *A Ricci flow $(M, g(t))_{t\in[0,T]}$ is called strongly κ-noncollapsed on scale ρ, where $\kappa > 0$ and $\rho > 0$, if for any $r \in (0, \rho]$ and $(x, t) \in M \times [0, T]$, $R \leq r^{-2}$ on $B_{g(t)}(x, r)$ implies $\mathrm{Vol}_{g(t)}(B_{g(t)}(x, r)) \geq \kappa r^n$.*

Definition 1.6 (Weak noncollapsing)**.** *A Ricci flow $(M, g(t))_{t\in[0,T]}$ is called weakly κ-noncollapsed on scale ρ, where $\kappa > 0$ and $\rho > 0$, if for any $r \in (0, \rho]$ and $(x, t) \in M \times [r^2, T]$, $|Rm| \leq r^{-2}$ on $B_{g(t)}(x, r) \times [t - r^2, t]$ implies $\mathrm{Vol}_{g(t)}(B_{g(t)}(x, r)) \geq \kappa r^n$.*

In Perelman's statement, the noncollapsing condition is either strong or weak on all scales. Indeed they are equivalent in the case of ancient solutions with nonnegative and bounded curvature operator.

Though he claimed this equivalence to be obvious, we are not aware of a complete proof, until the author proved it in [41]. We present the main theorem below.

Theorem 1.7 (Zhang, 2018)**.** *There exists two positive functions $\kappa_0 = \kappa_0(W,n)$ and $W_0 = W_0(\kappa,n)$, such that the following holds. Let $(M^n, g(t))_{t\in(-\infty,0]}$ be an ancient solution to the Ricci flow such that on each time-slice the curvature operator is bounded and nonnegative. Then*

(1) If $(M^n, g(t))_{t\in(-\infty,0]}$ is κ-noncollapsed on all scales, where $\kappa > 0$, then $\bar{W}(x,t) \geq -W_0(\kappa,n)$ for each $(x,t) \in M \times (-\infty,0]$.

(2) If there exists $W > 0$, such that $\bar{W}(x,t) > -W$ for each $(x,t) \in M \times (-\infty,0]$, then $(M^n, g(t))_{t\in(-\infty,0]}$ is κ-noncollapsed on all scales, where $\kappa = \kappa_0(W,n)$.

Here $\bar{W}(x,t)$ is the asymptotic entropy defined in (1.6).

Remark: Because of the Lemma 2.12 and Lemma 2.13(4), we have that

$$\bar{W}(x,t) \geq -W, \quad \text{for all} \quad (x,t) \in M \times (-\infty,0]$$

is indeed equivalent to

$$|W_{(x,t)}(s)| \leq W, \quad \text{for all} \quad x \in M \text{ and } s < t \leq 0.$$

We also obtain an equality between the asymptotic entropy and the asymptotic reduced volume, which is the same conclusion as in Xu [36] with his Type I condition replaced by the nonnegative curvature operator assumption.

Corollary 1.8. *Let* $(M^n, g(t))_{t\in(-\infty,0]}$ *be an ancient solution to the Ricci flow such that on each time-slice the curvature operator is bounded and nonnegative. Furthermore, assume that* $\inf_{x\in M} \mathrm{Vol}_{g(0)}(B_{g(0)}(x,1)) > 0$. *Then we have*

$$\bar{W}(x,t) = \log \bar{\mathcal{V}}(x,\tau), \tag{1.8}$$

for all $(x,t) \in M \times (-\infty, 0]$ *and* $\tau = -t$. *We allow both sides of (*1.8*) to be* $-\infty$ *and in this case* $(M, g(t))_{t\in(-\infty,0]}$ *is collapsed. Here* $\bar{\mathcal{V}}(x,\tau)$ *is the asymptotic reduced volume defined in (3.32).*

As shown by Carrillo and Ni [8] and Yokota [38], on a *shrinking gradient Ricci soliton* (see section 3.1 for an exposition), the asymptotic reduced volume and the entropy coincide. It turns out that the known cases where (1.8) holds are the cases where the *asymptotic shrinker* (see Theorem 3.14) exists (that is, type I case and nonnegative curvature operator case, and indeed more generally, PIC2 case [5]). The reason, as one may perceive, is because in such cases both the entropy and the reduced volume will limit to corresponding quantities on the asymptotic shrinker, where they coincide in the way of (1.8).

Finally, we prove a nice gaussian upper and lower bound for the conjugate heat kernel on κ-noncollapsed ancient solutions with bounded nonnegative curvature operator.

Corollary 1.9. *There exists $c > 0$ and $C < \infty$ depending only on κ and the dimension n such that the following holds. Let $(M, g(t))_{t\in(-\infty,0]}$ be a κ-noncollapsed ancient solution to the Ricci flow such that on each time-slice the curvature operator is bounded and nonnegative. Let $H(z, s|x, t)$ be the conjugate heat kernel. Then H satisfies*

$$\frac{c}{(t-s)^{\frac{n}{2}}} \exp\left(-\frac{C}{t-s} dist_{g(s)}(z, p(s))^2\right) \leq H(z, s|x, t) \leq \frac{C}{(t-s)^{\frac{n}{2}}} \exp\left(-\frac{c}{t-s} dist_{g(s)}(z, p(s))^2\right), \tag{1.9}$$

for all $s \in (-\infty, t)$, where $p(s)$ is chosen such that $l_{(x,-t)}(p(s), t-s) \leq \frac{n}{2}$. Here l is the reduced distance defined in (3.18).

The above Gaussian bounds (1.9) are somewhat surprising. They imply that the conjugate heat kernel always looks like a Gaussian kernel. But its "center", instead of being fixed at the base point, is moving along with the "center" of the reduced distance. One may imagine a conjugate heat kernel on a Bryant soliton based at the origin, as s evolves to negative infinity, $p(s)$ will, of necessity, be drifted away to infinity, since along those $p(s)$'s one eventually gets a cylinder, which is the only possible asymptotic shrinker for the Bryant soliton. We hope this phenomenon will provide a better understanding of the relationship between two fundamental tools in the Ricci flow, the entropy and the reduced geometry.

Chapter 2

Preliminaries in analysis

Perelman's pointed entropy, as defined in (1.6), is a formula based on the conjugate heat equation and its fundamental solution. Therefore in the study of Perelman's entropy, the analysis of the heat equation on the Ricci flow is not merely inevitable, but also fundamental. In order to have the readers familiarized with these methods and prepared for the proofs of our main theorems, we compose this chapter as an exposition to these analytic preliminaries. The theorems introduced in this chapter are from Qi S Zhang [39], Chau-Tam-Yu [9], and Hein-Naber [23]. The readers are strongly encouraged to read these papers.

2.1 Heat equation on evolving manifold

2.1.1 Parabolic mean value inequality

The classical mean value inequality for the heat equation was proved by using the property of fundamental solutions. In modern analysis, there are many other stronger techniques, proving mean value inequality for more general cases, especially on manifolds. For instances, Li and Yau [25] established there famous differential Harnack inequality, which can be used to prove a mean value inequality for heat the equation on static Riemannian manifolds. In [24], one can also find a parabolic mean value inequality on static manifolds proved by using a local Sobolev inequality and the Moser iteration. In this section we introduce a parabolic mean value inequality on a manifold with evolving Riemannian metric. This result is proved by Chau, Tam, and Yu [9].

Consider a smooth one-parameter family of complete Remannian metrics $\{g(t) : t \in [0,T]\}$ on a n-dimensional manifold M^n, such that

$$\frac{\partial}{\partial t} g_{ij}(t) = 2h_{ij}(t) \tag{2.1}$$

on $M \times [0,T]$, where $h_{ij}(t)$ is a evolving 2-tensor on M. Consider the following heat-type equation

$$\frac{\partial}{\partial t} u - \Delta u + Qu = 0, \tag{2.2}$$

where Δ is the Laplacian operator with respect to the metric $g(t)$ at each time t, and Q is a function on $M \times [0,T]$. We impose the following conditions.

(A) $|h|_{g(t)}$ and $|\nabla h|_{g(t)}$ are uniformly bounded on $M \times [0,T]$.

(B) The sectional curvatures of $g(t)$ are uniformly bounded on $M \times [0,T]$.

(C) $|Q|_{g(t)}$, $|\nabla Q|_{g(t)}$, and $|\Delta Q|_{g(t)}$ are uniformly bounded on $M \times [0,T]$.

Then we have the following mean value inequality for equation (2.2) (Lemma 3.1 in [9]; see also [39]).

Theorem 2.1. *Let u be a positive subsolution of (2.2) on $\Omega \times [0,T]$, where Ω is a domain in M. Moreover, assume $\tilde{g}$ is a metric on M satisfying*

$$\frac{1}{C_0}\tilde{g} \leq g(0) \leq C_0\tilde{g} \text{ on } M \quad \text{and} \quad Ric(\tilde{g}) \geq -K \text{ on } \Omega,$$

where $C_0 > 0$ and $K \geq 0$. Let $\tilde{Q}_r(x,t) := B_{\tilde{g}}(x,r) \times [t-r^2, t]$. Then for any $(x,t) \in \Omega \times (0,T]$ such that $\tilde{Q}_{2r}(x,t) \subset\subset \Omega \times [0,T]$, it holds that

$$\sup_{\tilde{Q}_r(x,t)} u \leq \frac{C\exp(At + B\sqrt{k}r)}{r^2 \operatorname{Vol}(B_{\tilde{g}}(x,r))} \int_{\tilde{Q}_{2r}(x,t)} u d\tilde{g} ds, \tag{2.3}$$

where A depends on the upper bounds of $|Q|$ and $|\operatorname{tr}_{g(t)} h(t)|_{g(t)}$ on $\Omega \times [0,T]$, B depends only on n, and C depends on C_0, n, T, and the upper bound of $|h(t)|_{g(t)}$ on $\Omega \times [0,T]$.

The proof of Theorem 2.1, as in the static metric case mentioned before, is by using a local Sobolev inequality and iteration technique. Since this proof is lengthy and almost standard, we omit it here.

2.1.2 A coarse gaussian bound for heat kernels.

As a consequence of the parabolic mean value inequality in Theorem 2.1 and a Li-Yau type differential Harnack inequality, Chau, Tam, and Yu [9] also proved gaussian upper and lower bounds for the fundamental solutions of equation (2.2). These estimates, of course, are too coarse for out purpose. However, as mentioned in the remark below Theorem 1.1, one need integration by parts at infinity to justify Perelman's monotonicity formula for the entropy. These gaussian upper and lower bounds provide such justification. In this subsection we will summarize these results without giving any proof; one may refer to [9] or [13] for details.

Lemma 2.2 (A rough upper bound)**.** *Let $\tilde{g}$, C_0, A, B, and C be as in Theorem 2.1, where we let $\Omega = M$. There exists D, depending on C_0, A, B, and C, such that the following holds. Let $H(y,s|x,t)$ be the fundamental solution to (2.2) on $M \times [0,T]$. Then we have*

$$H(y,s|x,t) \leq \min\left\{\frac{D}{\mathrm{Vol}_{\tilde{g}}\left(B_{\tilde{g}}\left(x,\sqrt{\frac{t-s}{2}}\right)\right)}, \frac{D}{\mathrm{Vol}_{\tilde{g}}\left(B_{\tilde{g}}\left(y,\sqrt{\frac{t-s}{2}}\right)\right)}\right\},$$

for all x, $y \in M$ and for all $0 \leq s < t \leq T$

Theorem 2.3 (Upper gaussian bound)**.** *Let $g(t)$ be an evolving metric on $M^n \times [0,T]$ as defined in (2.1). There exists a constant C depending only on n, T, the lower bound* *for $Ric(g(0))$, and the upper bound for $|h|_{g(t)}$ on $M \times [0,T]$, such that the following holds. Let $H(y,s|x,t)$ be the fundamental solution to (2.2) on $M \times [0,T]$. Then*

$$H(y,s|x,t) \leq \frac{C}{\sqrt{\mathrm{Vol}_{\tilde{g}}\left(B_{\tilde{g}}\left(x,\sqrt{\frac{t-s}{2}}\right)\right)}\cdot\sqrt{\mathrm{Vol}_{\tilde{g}}\left(B_{\tilde{g}}\left(y,\sqrt{\frac{t-s}{2}}\right)\right)}}\exp\left(-\frac{C dist^2_{\tilde{g}}(x,y)}{t-s}\right)$$

for all x, $y \in M$ and for all $0 \leq s < t \leq T$. Here $\tilde{g}$ is defined as in Theorem 2.1. More specifically, we have

$$H(y,s|x,t) \leq \frac{C}{\mathrm{Vol}_{\tilde{g}}\left(B_{\tilde{g}}\left(x,\sqrt{\frac{t-s}{2}}\right)\right)}\exp\left(-\frac{C dist^2_{\tilde{g}}(x,y)}{t-s}\right) \tag{2.4}$$

and

$$H(y,s|x,t) \leq \frac{C}{\mathrm{Vol}_{\tilde{g}}\left(B_{\tilde{g}}\left(y,\sqrt{\frac{t-s}{2}}\right)\right)}\exp\left(-\frac{C dist^2_{\tilde{g}}(x,y)}{t-s}\right). \tag{2.5}$$

Theorem 2.4 (Lower gaussian bound)**.** *Let $g(t)$ be and evolving metric on $M^n \times [0,T]$ as defined in (2.1). There exists positive constants C_1 and C_2, where C_1 depends on n, T, and the upper bounds of $|Ric(g(0))|$, $|h|_{g(t)}$, $|\nabla h|_{g(t)}$, and $|\Delta h|_{g(t)}$, and C_2 depends on n, T, and the upper bound of $|h|_{g(t)}$, such that the following holds. Let $H(y,s|x,t)$ be the fundamental solution to (2.2) on $M \times [0,T]$. Then*

$$H(y,s|x,t) \geq \frac{C_1}{\sqrt{\mathrm{Vol}_{\tilde{g}}\left(B_{\tilde{g}}\left(x,\sqrt{\frac{t-s}{2}}\right)\right)}\cdot\sqrt{\mathrm{Vol}_{\tilde{g}}\left(B_{\tilde{g}}\left(y,\sqrt{\frac{t-s}{2}}\right)\right)}}\exp\left(-\frac{C_2 dist^2_{\tilde{g}}(x,y)}{t-s}\right)$$

for all $x,\ y \in M$ and for all $0 \le s < t \le T$. Here $\tilde{g}$ is defined as in Theorem 2.1. More specifically, we have

$$H(y,s|x,t) \ge \frac{C_1}{\mathrm{Vol}_{\tilde{g}}\left(B_{\tilde{g}}\left(x, \sqrt{\frac{t-s}{2}}\right)\right)} \exp\left(-\frac{C_2 dist^2_{\tilde{g}}(x,y)}{t-s}\right) \tag{2.6}$$

and

$$H(y,s|x,t) \ge \frac{C_1}{\mathrm{Vol}_{\tilde{g}}\left(B_{\tilde{g}}\left(y, \sqrt{\frac{t-s}{2}}\right)\right)} \exp\left(-\frac{C_2 dist^2_{\tilde{g}}(x,y)}{t-s}\right). \tag{2.7}$$

2.2 Heat equation on the Ricci flow

Now we focus on the heat equation coupled with the Ricci flow on $M \times [0,T]$

$$\begin{cases} \frac{\partial}{\partial t} g &= -2Ric, \\ \frac{\partial}{\partial t} u &= \Delta u, \end{cases} \tag{2.8}$$

where Δ is the Laplacian operator with respect to $g(t)$ at each time t. The heat operator

$$\Box := \frac{\partial}{\partial t} - \Delta$$

has the following conjugate operator

$$\Box^* := -\frac{\partial}{\partial t} - \Delta + R.$$

To see this, let u and v be two smooth functions on $M \times [0, T]$ such that at each time both u and v are compactly supported. Then we can compute

$$\begin{aligned}
\int_M uv dg_t \Big|_{t=0}^{t=T} &= \int_0^T \frac{d}{dt}\Big(\int_M uv dg_t\Big) dt \qquad (2.9)\\
&= \int_0^T \left(\int_M \Big(\frac{\partial u}{\partial t} v + u\frac{\partial v}{\partial t}\Big) dg_t + \int_M uv \frac{\partial}{\partial t}(dg_t)\right) dt\\
&= \int_0^T \left(\int_M \Big(\frac{\partial u}{\partial t} v + u\frac{\partial v}{\partial t} - v\Delta u + u\Delta v\Big) dg_t - \int_M uvR dg_t\right) dt\\
&= \int_0^T \int_M \Big(v\Box u - u\Box^* v\Big) dg_t dt,
\end{aligned}$$

where we have used the fact

$$\frac{\partial}{\partial t}(dg_t) = \frac{1}{2}\left(tr_{g(t)}\Big(\frac{\partial g}{\partial t}\Big)\right) dg_t = -R dg_t.$$

Now we consider the fundamental solution $H(y, s|x, t)$ to (2.8). We will see below that this is also the fundamental solution to the conjugate heat equation, if one fixes (x, t) and let (y, s) be the variable; this argument can be found in [23]. By the definition of the fundamental solution, we obviously have

$$\lim_{t \to s+} H(y, s|x, t) = \delta_y(x), \qquad (2.10)$$

where $\delta_y(x)$ is the Dirac delta functional based at y.

Lemma 2.5. *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete Ricci flow with bounded curvature.*

(1) $H(y,s|x,t)$ is also the fundamental solution to the conjugate heat operator $\Box^* := -\partial_s - \Delta_{g(s)} + R$.

(2) $\int_M H(\cdot,s|x,t)dg_s \equiv 1$ for all $s \in [0,t)$.

Proof. (1) Fix arbitrary (x,t) and $(y,s) \in M \times [0,T]$ such that $s < t$. Let H^* be the fundamental solution of the conjugate operator. Define

$$\begin{aligned} u &= H(y,s|\cdot,\cdot), \\ v &= H^*(\cdot,\cdot|x,t). \end{aligned}$$

Then we have

$$\Box u = 0, \quad \Box^* v = 0,$$

and

$$\lim_{\eta\to s+} u(z,\eta) = \delta_y(z) \quad \text{and} \quad \lim_{\eta\to t-} v(z,\eta) = \delta_x(z). \tag{2.11}$$

Since by Theorem 2.3 and Theorem 2.4 we have that both u and v have gaussian upper and lower bounds. Taking in to account the gradient estimates in Theorem

2.9 (note that one does not need this current lemma to apply Theorem 2.9), we have that the integration by parts in the computation of (2.9) is justified. We have

$$\int_M uv dg_\eta \Big|_{\eta=t_1}^{\eta=t_2} = 0,$$

for all $s < t_1 < t_2 < t$. Taking $t_1 \to s+$, $t_2 \to t-$, and by (2.11), we have

$$H(y, s|x, t) = H^*(y, s|x, t).$$

(2) Let $v(y, s) := H(x, t|y, s)$ be a conjugate heat kernel. As argued in (1), we can do the following computation

$$\begin{aligned} \frac{d}{ds}\int_M v dg_s &= \int_M \frac{\partial v}{\partial s} dg_s + \int_M v \frac{\partial}{\partial s}(dg_s) \\ &= \int_M \Big(-\Delta v + Rv\Big) dg_s - \int_M Rv dg_s \\ &\equiv 0. \end{aligned}$$

Hence we have

$$\int_M v dg_s \equiv \lim_{s \to t-} \int_M v dg_s = 1.$$

□

2.3 Gradient estimates

In this section we introduce some gradient estimates for the heat equation and the conjugate heat equation coupled with Ricci flow. They are important in the proof of the main theorems, especially the Bochner's formula in the following subsection.

2.3.1 A Bochner's formula for the heat equation

We begin with a nice Bochner formula found by Qi S Zhang [39]. Such type of formula was first discovered by Hamilton [18]. Qi S Zhang's adaption shows how the heat equation fits in the Ricci flow.

Theorem 2.6 (Qi S Zhang, 2006). *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete Ricci flow, let $u : M \times [0,T] \to \mathbb{R}_+$ be a positive solution to (2.8). Then*

(1) For any $A > 0$, it holds that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(u \log \frac{A}{u}\right) = \frac{|\nabla u|^2}{u}. \tag{2.12}$$

(2)

$$\left(\frac{\partial}{\partial t} - \Delta\right)\frac{|\nabla u|^2}{u} = -\frac{2}{u}\left|\nabla^2 u - \frac{\nabla u \otimes \nabla u}{u}\right|^2 \tag{2.13}$$

Proof. In the proof we will use $\Box := \partial_t - \Delta$ to denote the heat operator.

(1)

$$\begin{aligned}\square\left(u\log\frac{A}{u}\right) &= (\square u)\log\frac{A}{u}+u\square\log\frac{A}{u}-2\left\langle\nabla u,\nabla\log\frac{A}{u}\right\rangle\\ &= u\left(-\frac{1}{u}\square u-\frac{1}{u^2}|\nabla u|^2\right)+2\frac{|\nabla u|^2}{u}\\ &= \frac{|\nabla u|^2}{u}.\end{aligned}$$

(2)

$$\begin{aligned}\square\frac{|\nabla u|^2}{u} &= \frac{1}{u}\square|\nabla u|^2+|\nabla u|^2\square\frac{1}{u}-2\left\langle\nabla|\nabla u|^2,\nabla\frac{1}{u}\right\rangle\\ &= \frac{1}{u}\left(2Ric(\nabla u,\nabla u)+2\langle\nabla u,\nabla\partial_t u\rangle\right)-\frac{1}{u}\Delta|\nabla u|^2\\ &\quad+|\nabla u|^2\left(-\frac{1}{u^2}\square u-\frac{2}{u^3}|\nabla u|^2\right)+\frac{4}{u^2}\langle\nabla^2 u,\nabla u\otimes\nabla u\rangle\\ &= \frac{1}{u}\left(2Ric(\nabla u,\nabla u)+2\langle\nabla u,\nabla\Delta u\rangle-\Delta|\nabla u|^2\right)\\ &\quad-\frac{2}{u^3}|\nabla u|^4+\frac{4}{u^2}\langle\nabla^2 u,\nabla u\otimes\nabla u\rangle.\end{aligned}$$

Using the classical Bochner's formula

$$\Delta|\nabla u|^2=2|\nabla^2 u|^2+2Ric(\nabla u,\nabla u)+2\langle\nabla u,\nabla\Delta u\rangle$$

we have

$$\begin{aligned}\square\frac{|\nabla u|^2}{u} &= -\frac{2}{u}|\nabla^2 u|^2-\frac{2}{u^3}|\nabla u|^4+\frac{4}{u^2}\langle\nabla^2 u,\nabla u\otimes\nabla u\rangle\\ &= -\frac{2}{u}\left|\nabla^2 u-\frac{\nabla u\otimes\nabla u}{u}\right|^2.\end{aligned}$$

□

We then have a immediate consequence of Theorem 2.6.

Corollary 2.7. *Let $(M, g(t))_{t\in[0,T]}$ be a Ricci flow on a closed manifold. Let $u : M \times [0,T] \to \mathbb{R}_+$ be a positive solution to (2.8) such that $0 < u \leq A$ everywhere, where $A > 0$ is a constant. Then we have*

$$\frac{|\nabla u|^2}{u} \leq \frac{1}{t} u \log \frac{A}{u}$$

on $M \times (0,T]$.

Proof. By Theorem 2.6 we have

$$\Box \left(t\frac{|\nabla u|^2}{u} - u \log \frac{A}{u} \right) = -\frac{2t}{u} \left| \nabla^2 u - \frac{\nabla u \otimes \nabla u}{u} \right|^2 \leq 0.$$

The corollary then follows from the parabolic maximum principle. Note that

$$t\frac{|\nabla u|^2}{u} - u \log \frac{A}{u} \leq 0, \quad \text{when} \quad t = 0.$$

□

2.3.2 Gradient estimates on noncompact Ricci flow

The maximum principle, unfortunately, is in general not true for arbitrary subsolution on noncompact manifolds. As we would like to apply Corollary 2.7 to noncompact Ricci flows, the following justification is needed.

Theorem 2.8. *Let* $(M, g(t))_{t\in[0,T]}$ *be a complete Ricci flow with bounded curvature. Let* $u : M^n \times [0,T] \to \mathbb{R}_+$ *be a positive solution to (2.8) such that* $0 < u \leq A$ *everywhere on* $M \times [0,T]$*. Then we have*

$$\frac{|\nabla u|^2}{u} \leq \frac{1}{t} u \log \frac{A}{u}. \tag{2.14}$$

Proof. Let C_0 be the bound of the Ricci curvature, that is, $|Ric| \leq C_0$ on $M \times [0,T]$. Applying Theorem 2.2 of [2], we have

$$\frac{|\nabla u|}{u}(x,t) \leq B\left(\frac{1}{2\rho} + \frac{1}{\sqrt{t}} + \sqrt{C_0}\right)\left(1 + \log\frac{A}{u}\right), \text{ for all } x \in B_{g(t)}(O,\rho) \text{ and } t \neq 0,$$

where B is a constant depending only on the dimension of M and O is a fixed point on M.

Taking $\rho \to \infty$, we have

$$t\frac{|\nabla u|^2}{u} \leq B^2(1+\sqrt{C_0 t})^2(u^{\frac{1}{2}} + u^{\frac{1}{2}}\log\frac{A}{u})^2 \leq C, \text{ on } M \times [0,T],$$

where C is a constant depending on B, C_0, T and A, and in the last inequality, we have also used the fact that the function $-x^{\frac{1}{2}}\log x$ is bounded from above and below on the interval $(0, A]$. Therefore, the subsolution to the heat equation (see the proof of Corollary 2.7)

$$t\frac{|\nabla u|^2}{u} - u\log\frac{A}{u}$$

is bounded on $M \times [0,T]$ and is nonpositive at time $t = 0$. Hence we may apply the parabolic weak maximum principle (c.f. Theorem 12.10 of [12]) to obtain the conclusion. □

Now we prove growth estimates for the heat kernel and the conjugate heat kernel. These estimates are far from optimal, but they suffice to justify the integration by parts at infinity in the computations involving entropies defined by the conjugate heat kernel.

Theorem 2.9. *Let $M \times [0,T]$ be a complete Ricci flow with bounded curvature and bounded first covariant derivatives for the curvature. Let H be the fundamental solution to (2.8). Then we have*

(1)

$$\left|\nabla_x \log H(y,s|x,t)\right|^2_{g(t)} \le C_1 + C_1 dist_{g(t)}(x,y)^2,$$

where $C_1 < \infty$ depends on the upper bound of $|Rm|$, $\mathrm{Vol}_{g(0)}(B_{g(0)}(y,\sqrt{t-s}))^{-1}$, $(t-s)^{-1}$, n, T.

(2)

$$\left|\nabla_y \log H(y,s|x,t)\right|^2_{g(s)} \leq C_2 + C_2 dist_{g(s)}(x,y)^4,$$

where $C_2 < \infty$ depends on the upper bound of $|Rm|$, the upper bound of $|\nabla Rm|$, $(t-s)^{-1}$, $\mathrm{Vol}_{g(0)}(B_{g(0)}(x,\sqrt{t-s}))^{-1}$, n, T.

Proof. (1) follows from combining Lemma 2.2, Theorem 2.3, Theorem 2.4, and applying Theorem 2.8 on $M \times [\frac{s+t}{2}, t]$.

For (2) we recall Theorem 10 of [14], where they have shown the following: suppose that on $\Omega(2J) = \bigcup_{\tau\in[0,\bar{t}]} B_{g(\tau)}(q, 2J)$ we have the bounds

$$|Ric| \leq K_1 \quad \text{and} \quad |\nabla R| \leq K_2.$$

Let u be a positive solution to the conjugate heat equation with $u \leq A$ on $\Omega(2J)$, then it holds that

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} \quad \leq \quad & \left(1 + \log\frac{A}{u}\right)^2 \left(\frac{1}{\bar{t}-t} + C_1 K_1 + \sqrt{K_2}\right. \\ & \left. + K_2 + \frac{C_1\sqrt{K_2}J\coth(\sqrt{K_2}J) + C_2}{A^2}\right)\end{aligned}$$

on $\Omega(J)$ and for $t \in [0, \bar{t})$, where C_1 and C_2 are constants. Now taking $J \to +\infty$ and argue as in (1), we obtain (2) also. □

Remark One may continue using Shi's method [35] to obtain higher derivative growth estimates. It turns out that all derivatives have at most polynomial growth. Since the heat kernel has rapid decay like e^{-r^2}, and the volume growth is at most e^r due to the Bishop-Gromov comparison theorem, we can then justify all the integration by parts at infinity that appear in this dissertation.

2.4 Hein and Naber's logarithmic Sobolev inequality

In this section we introduce the logarithmic Sobolev inequality of Hein and N-aber, and their gaussian concentration theorem. These results are much more accurate estimates than the theorems in the last subsection, and are of fundamental importance in the proofs of our main theorems. We will present the proofs of all the results in this section, since they are concise, beautiful, and important. These proofs, of course, are due to Hein and Naber [23].

2.4.1 The logarithmic Sobolev inequality and the Poincaré inequality

Consider a complete Ricci flow $(M^n, g(t))_{t\in[0,T]}$ with bounded curvature. Let $H(y, s|x, t)$ be the conjugate heat kernel. For each $0 \le s < t \le T$ and for each $x \in M$, we define the following measure

$$\nu^s_{(x,t)}(A) := \int_A H(\cdot, s|x,t) dg_s, \ A \subset M. \tag{2.15}$$

We readily see from Lemma 2.5(2) that $\nu^s_{(x,t)}$ is a probability measure. Hein and Naber proved the following Poincaré-type inequality and logarithmic Sobolev inequality with respect to the measure $\nu^s_{(x,t)}$.

Theorem 2.10 (Hein and Naber, 2014)**.** *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete Ricci flow with bounded curvature. Let $\nu^s_{(x,t)}$ be the measure defined in (2.15), where $0 \le s < t \le T$ and $x \in M$. Then the following hold.*

(1) If $u \in C_0^\infty(M)$, then

$$\int_M u^2 d\nu^s_{(x,t)} - \left(\int_M u d\nu^s_{(x,t)}\right)^2 \le 2(t-s) \int_M |\nabla u|^2 d\nu^s_{(x,t)}.$$

(2) If $u \in C_0^\infty(M)$ and $u \ge 0$, then

$$\int_M u \log u d\nu^s_{(x,t)} - \left(\int_M u d\nu^s_{(x,t)}\right) \log\left(\int_M u d\nu^s_{(x,t)}\right) \le (t-s) \int_M \frac{|\nabla u|^2}{u} d\nu^s_{(x,t)}.$$

Let us now discuss the proof of Theorem 2.10. First, we observe the following Bochner's formula for the heat equation coupled with Ricci flow; this is a easier version of Theorem 2.6. Let u be a solution to (2.8), then

$$\Box|\nabla u|^2 = \left\langle \nabla \frac{\partial u}{\partial t}, \nabla u \right\rangle + 2Ric(\nabla u, \nabla u) - \Delta|\nabla u|^2 = -2|\nabla^2 u|^2. \tag{2.16}$$

We define the following "homotopy of functions". Let $u : M \to \mathbb{R}$ be a smooth and compactly supported function,

$$P_{st}u := \int_M H(\cdot, s|x, t) u dg_s.$$

The following Lemma is due to straightforward computation.

Lemma. *(1) Let $U : M \times [0, T] \to \mathbb{R}$ be a smooth function. Then*

$$\frac{d}{ds} P_{st} U(\cdot, s) = P_{st}\big((\Box_s U)(\cdot, s)\big),$$

for all $0 \leq s < t \leq T$, where $\Box_s := \partial_s - \Delta_{g(s)}$.

(2) Let $u \in C_0^\infty(M)$ and denote $u_t(x) = P_{st}u(x)$ such that $\Box_t u_t = 0$. Fix ϕ, $\psi : \mathbb{R} \to \mathbb{R}$. Then we have

$$\begin{aligned} U(x,t) := \phi(u_t(x)) &\Rightarrow \Box_t U = -\phi''(u_t)|\nabla u_t|^2_{g(t)}, \\ U(x,t) := \psi(u_t(x))|\nabla u_t|^2_{g(t)} &\Rightarrow \\ \Box_t U &= -2\psi(u_t)|\nabla u_t|^2_{g(t)} - 4\psi'(u_t)\langle \nabla^2 u_t, \nabla u_t \otimes \nabla u_t \rangle \\ &\quad -\psi''(u_t)|\nabla u_t \otimes \nabla u_t|^2_{g(t)}. \end{aligned}$$

From the above lemma, we then have

$$\begin{aligned}\int_M \phi(u)d\nu^s_{(x,t)} - \phi\left(\int_M u d\nu^s_{(x,t)}\right) &= -\int_s^t \frac{d}{d\eta} P_{\eta t}(\phi(P_{s\eta}u))d\eta \\ &= \int_s^t P_{\eta t}\big(\phi''(P_{s\eta}u)|\nabla P_{s\eta}u|^2_{g(\eta)}\big)d\eta.\end{aligned} \tag{2.17}$$

Now we take $\phi(x) = x^2$ and $\phi(x) = x\log x$ and estimate the term

$$\int_s^t P_{\eta t}\big(\phi''(P_{s\eta}u)|\nabla P_{s\eta}u|^2_{g(\eta)}\big)d\eta$$

in (2.17). In the case $\phi(x) = x^2$, we may compute (2.16) and item (1) of the above lemma

$$\begin{aligned}2\int_s^t P_{\eta t}\big(|\nabla P_{s\eta}u|^2_{g(\eta)}\big)d\eta &= 2\int_s^t \Big(-\int_\eta^t \frac{d}{d\tau}P_{\tau t}|\nabla P_{s\tau}u|^2_{g(\tau)}d\tau + |\nabla P_{st}u|^2_{g(t)}\Big)d\eta \\ &= -2\int_s^t\int_\eta^t P_{\tau t}\Box_\tau|\nabla P_{s\tau}u|^2_{g(\tau)}d\tau d\eta + 2(t-s)|\nabla P_{st}u|^2_{g(t)} \\ &= 4\int_s^t\int_\eta^t P_{\tau t}\Box|\nabla^2 P_{s\eta}u|^2_{g(\tau)}d\tau d\eta + 2(t-s)|\nabla P_{st}u|^2_{g(t)} \\ &= 4\int_s^t (\eta - s)P_{\eta t}|\nabla^2 P_{s\eta}u|^2_{g(\eta)}d\eta + 2(t-s)|\nabla P_{st}u|^2_{g(t)}.\end{aligned}$$

On the other hand

$$\begin{aligned}|\nabla P_{st}u|^2_{g(t)} &= P_{st}|\nabla u|^2_{g(s)} + \int_s^t \frac{d}{d\eta}\big(P_{\eta t}|\nabla P_{s\eta}u|^2_{g(\eta)}\big)d\eta \\ &= P_{st}|\nabla u|^2_{g(s)} - 2\int_s^t P_{\eta t}|\nabla^2 P_{s\eta}u|^2_{g(\eta)}d\eta.\end{aligned}$$

In combination we have

$$\begin{aligned} 2\int_s^t P_{\eta t}\big(|\nabla P_{s\eta}u|^2_{g(\eta)}\big)d\eta &= 2(t-s)|\nabla P_{st}u|^2_{g(t)} - 4\int_s^t (t-\eta)P_{\eta t}|\nabla^2 P_{s\eta}u|^2_{g(\eta)}d\eta \\ &\leq 2(t-s)|\nabla P_{st}u|^2_{g(t)}, \end{aligned}$$

this proves Theorem 2.10 (1).

In the case $\phi(x) = x\log x$, that is, $\psi(x) := \phi''(x) = x^{-1}$, item (2) of the above lemma reduces to Theorem 2.6. In other words, any positive solution u to the heat equation satisfies

$$\nabla \frac{|\nabla u|^2}{u} = -\frac{2}{u}\left|\nabla^2 u - \frac{\nabla u \otimes \nabla u}{u}\right|^2 = -2u|\nabla^2 \log u|^2.$$

With this we may perform the following computation as before.

$$
\begin{aligned}
\int_s^t P_{\eta t}\left(\frac{|\nabla P_{s\eta}u|^2_{g(\eta)}}{P_{s\eta}u}\right)d\eta &= \int_s^t\left(-\int_\eta^t \frac{d}{d\tau}P_{\tau t}\Big(\frac{|\nabla P_{s\tau}u|^2_{g(\tau)}}{P_{s\tau}u}\Big)d\tau\right.\\
&\quad\left.+\frac{|\nabla P_{st}u|^2_{g(t)}}{P_{st}u}\right)d\eta\\
&= -\int_s^t\int_\eta^t P_{\tau t}\Box_\tau\frac{|\nabla P_{s\tau}u|^2_{g(\tau)}}{P_{s\tau}u}d\tau d\eta\\
&\quad+(t-s)\frac{|\nabla P_{st}u|^2_{g(t)}}{P_{st}u}\\
&= 2\int_s^t\int_\eta^t P_{\eta t}\Big((P_{s\tau}u)|\nabla^2\log P_{s\tau}u|^2_{g(\tau)}\Big)d\tau d\eta\\
&\quad+(t-s)\frac{|\nabla P_{st}u|^2_{g(t)}}{P_{st}u}\\
&= 2\int_s^t(\eta-s)P_{\eta t}\Big((P_{s\eta}u)|\nabla^2\log P_{s\eta}u|^2_{g(\eta)}\Big)d\eta\\
&\quad+(t-s)\frac{|\nabla P_{st}u|^2_{g(t)}}{P_{st}u}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\frac{|\nabla P_{st}u|^2_{g(t)}}{P_{st}u} &= P_{st}\frac{|\nabla u|^2_{g(s)}}{u}+\int_s^t\frac{d}{d\eta}\left(P_{\eta t}\frac{|\nabla P_{s\eta}u|^2_{g(\eta)}}{P_{s\eta}u}\right)d\eta\\
&= P_{st}\frac{|\nabla u|^2_{g(s)}}{u}-2\int_s^t P_{\eta t}\Big((P_{s\eta}u)|\nabla^2\log P_{s\eta}u|^2_{g(\eta)}\Big)d\eta.
\end{aligned}
$$

In combination we have

$$\begin{aligned}
\int_s^t P_{\eta t}\left(\frac{|\nabla P_{s\eta}u|^2_{g(\eta)}}{P_{s\eta}u}\right)d\eta &= (t-s)P_{st}\frac{|\nabla u|^2_{g(s)}}{u} \\
&\quad -2\int_s^t (t-\eta)P_{\eta t}\Big((P_{s\eta}u)|\nabla^2 \log P_{s\eta}u|^2_{g(\eta)}\Big)d\eta \\
&\le (t-s)P_{st}\frac{|\nabla u|^2_{g(s)}}{u},
\end{aligned}$$

this proves Theorem 2.10 (2).

Furthermore, one may easily see from the proof of Theorem 2.10 that if the equality in Theorem 2.10 (1) or (2) holds, then either u is a constant or $(M, g(t))$ splits as a direct product.

2.4.2 Gaussian concentration

In this subsection we prove Hein and Naber's gaussian concentration for the conjugate heat kernel. This theorem is the cornerstone of the proof of Corollary 1.9.

Theorem 2.11 (Gaussian concentration)**.** *Let $(M, g(t))_{t\in[0,T]}$ be a Ricci flow with bounded curvature. Let $\nu^s_{(x,t)}$ be the probability measure defined in (2.15). Then we have*

$$\nu^s_{(x,t)}(A)\nu^s_{(x,t)}(B) \le \exp\left(-\frac{1}{8(t-s)}dist_{g(s)}(A,B)^2\right),$$

for any A, $B \subset M$. Here the distance is in the usual sense.

Proof. For the sake of convenience, in this proof we will suppress all the indices of $\nu^s_{(x,t)}$. We fix any $F \in C^\infty(M)$ such that

$$\int F d\nu = 0 \quad \text{and} \quad |\nabla F| \leq 1,$$

and define the following Laplacian-type transform

$$U(\lambda) := \frac{1}{\lambda} \log \int e^{\lambda F} d\nu.$$

We observe that

$$\begin{aligned}
\lim_{\lambda \to 0+} U(\lambda) &= \lim_{\lambda \to 0+} \frac{\int F e^{\lambda F} d\nu}{\int e^{\lambda F} d\nu}, \\
\lim_{\lambda \to 0+} \left| \frac{\int F e^{\lambda F} d\nu}{\int e^{\lambda F} d\nu} \right| &\leq \lim_{\lambda \to 0+} \frac{\int |F(e^{\lambda F} - 1)| d\nu}{\int e^{\lambda F} d\nu} = 0,
\end{aligned}$$

where we have used the dominate convergence theorem, where $|F|(e^{|F|} + 1)$, being integrable with respect to ν according to (2.3) and according to the fact $|\nabla F| \leq 1$, is a dominant for $|F(e^{\lambda F} - 1)|$, which converges to 0 pointwise. It follows that

$$U(\lambda) = o(1) \quad \text{as} \quad \lambda \to 0+ . \tag{2.18}$$

On the other hand, we apply Theorem 2.10 (2) to $\dfrac{e^{\lambda F}}{\int e^{\lambda F} d\nu}$ and obtain

$$\begin{aligned} \lambda \frac{\int F e^{\lambda F} d\nu}{\int e^{\lambda F} d\nu} - \log\left(\int e^{\lambda F} d\nu\right) &= \int \frac{e^{\lambda F}}{\int e^{\lambda F} d\nu} \log\left(\frac{e^{\lambda F}}{\int e^{\lambda F} d\nu}\right) d\nu \\ &\leq \lambda^2 (t-s) \frac{\int |\nabla F|^2 e^{\lambda F} d\nu}{\int e^{\lambda F} d\nu} \\ &\leq \lambda^2 (t-s), \end{aligned}$$

or in other words

$$\frac{d}{d\lambda} U(\lambda) \leq (t-s). \tag{2.19}$$

Combining (2.18) and (2.19) we have

$$U(\lambda) \leq (t-s)\lambda \quad \text{for all} \quad \lambda > 0. \tag{2.20}$$

Finally we set

$$F = dist_{g(s)}(\cdot, B) - \int dist_{g(s)}(\cdot, B) d\nu.$$

Note that though F is only Lipschitz with Lipschitz constant 1, since it is in $W^{1,2}(\nu)$, the above argument still holds. We have

$$\begin{aligned}
\int_A\int_B e^{\lambda F(y)-\lambda F(z)}d\nu(z)d\nu(y) &= \int_A\int_B e^{\lambda dist(y,B)-\lambda dist(z,B)}d\nu(z)d\nu(y)\\
&= \int_A\int_B e^{\lambda dist(y,B)}d\nu(z)d\nu(y)\\
&\geq \nu(A)\nu(B)e^{\lambda dist(A,B)}
\end{aligned}$$

and

$$\begin{aligned}
\int_A\int_B e^{\lambda F(y)-\lambda F(z)}d\nu(z)d\nu(y) &= \int_A e^{\lambda F}d\nu\cdot\int_B e^{-\lambda F}d\nu\\
&\leq \int_M e^{\lambda F}d\nu\cdot\int_M e^{-\lambda F}d\nu\\
&\leq e^{2\lambda^2(t-s)}.
\end{aligned}$$

Hence

$$\nu(A)\nu(B)\leq e^{-\lambda dist(A,B)+2\lambda^2(t-s)}\quad\text{for all}\quad \lambda>0.$$

The conclusion then follows from optimizing $\lambda>0$. □

2.5 Nash entropy

The definition of Perelman's entropy is complicated. However, if we observe the integrand of (1.1), we may see that the $\tau(|\nabla f|^2+R)(4\pi\tau)^{-\frac{n}{2}}e^{-f}$ is an "energy-like" term, while the logarithmic term $(f-n)(4\pi\tau)^{-\frac{n}{2}}e^{-f}$ is "entropy-like". Indeed, if we take away the energy part, what remains is still an entropy—the Nash entropy.

As in chapter 1, we let $H(y, s|x, t)$ be the fundamental solution to the heat equation coupled with the Ricci flow (2.8). Let f be defined as in (1.5). Then if we fix (x, t) and let (y, s) be the variables, H is the fundamental solution of the conjugate heat operator $\Box^* := -\partial_s - \Delta_{g(s)} + R$ based at (x, t). We defined the *pointed Nash entropy* based at (x, t) and evaluated at s as

$$N_{(x,t)}(s) = (4\pi(t-s))^{-\frac{n}{2}} \int_M \Big(f_{(\cdot,s)}(x,t) - \frac{n}{2}\Big) e^{-f_{(\cdot,s)}(x,t)} dg_s. \tag{2.21}$$

We next collect some known facts for Perelman's entropy and Nash entropy. Most of the following results in the rest of this section can be found in [23].

Lemma 2.12. *Let $(M, g(t))_{t\in[0,T]}$ is a complete Ricci flow with bounded curvature, then we have*

$$\lim_{s\to t-} W_{(x,t)}(s) = 0 \quad \text{and} \quad \lim_{s\to t-} N_{(x,t)}(s) = 0,$$

for any $(x, t) \in M \times (0, T]$.

Proof. We take an arbitrary positive sequence $\lambda_i \searrow 0$ and consider the scaled sequence $\{(M, g_i(s), (x, 0))_{[-\frac{t}{\lambda_i}, 0]}\}_{i=1}^{\infty}$ such that

$$\begin{aligned} g_i(s) &= \frac{1}{\lambda_i} g(t + s\lambda_i), \\ H_i(y, s|x, 0) &= \frac{1}{(4\pi(-s))^{-\frac{n}{2}}} e^{-f^i_{(y,s)}(x,0)} := \frac{1}{(4\pi(-s))^{-\frac{n}{2}}} e^{-f(y,t+s\lambda_i)(x,t)}. \end{aligned}$$

Apparently H_i is the conjugate heat kernel of g_i. Hence we have

$$\begin{aligned} W_{(x,t)}(t+s\lambda_i) &= W^{(i)}_{(x,0)}(s), \\ N_{(x,t)}(t+s\lambda_i) &= N^{(i)}_{(x,0)}(s) \end{aligned}$$

for all $s \in [-\frac{t}{\lambda_i}, 0)$.

Now we use [19] to extract a subsequence from $\{(M, g_i(s), (x,0))_{[-\frac{t}{\lambda_i},0]}\}_{i=1}^{\infty}$ which converges to $(M_\infty, g_\infty, (x_\infty, 0))_{(-\infty,0]}$. By [26] H_i will also converge to the fundamental solution H_∞ to the conjugate heat equation on (M_∞, g_∞) based at $(x_\infty, 0)$. By Theorem 2.3, Theorem 2.4, and Theorem 2.9, we have uniform estimates for all the integrand of $W^{(i)}$ and $N^{(i)}$. Note also that the heat kernel, being the weight of those integrations, have decay as fast as e^{-r^2}. This them implies that these integrations are uniformly negligible outside large compact sets. Thus a standard argument yields

$$\lim_{i\to\infty} W^{(i)}(s) = W_\infty(s) \quad \text{and} \quad \lim_{i\to\infty} N^{(i)}(s) = N_\infty(s),$$

for all $s \in (-\infty, 0)$, where W_∞ and N_∞ are Perelman's entropy and Nash entropy of (M_∞, g_∞) based at $(x_\infty, 0)$. Finally, it is obvious that (M_∞, g_∞) is the flat Euclidean space, and a direct computation implies that $W_\infty \equiv N_\infty \equiv 0$. Since $\{\lambda_i\}$ is arbitrary, the lemma follows. $\square$

Now let $H(y,s|x,t)$ and $f_{(y,s)}(x,t)$ be defined as in (1.5). We may easily compute

$$\frac{\partial}{\partial s} f_{(\cdot,s)}(x,t) = -\Delta_{g(s)} f_{(\cdot,s)}(x,t) + |\nabla f_{(\cdot,s)}(x,t)|^2_{g(s)} - R_{g(s)} + \frac{n}{2(t-s)}.$$

Hence

$$\begin{aligned}
\frac{d}{ds} N_{(x,t)}(s) &= \int_M \Bigg(\Big(-\Delta f_{(\cdot,s)}(x,t) + |\nabla f_{(\cdot,s)}(x,t)|^2 - R + \frac{n}{2(t-s)} \Big) \\
&\quad \times (4\pi(t-s))^{-\frac{n}{2}} e^{-f_{(\cdot,s)}(x,t)} \\
&\quad - \big(f_{(\cdot,s)}(x,t) - \frac{n}{2}\big)(4\pi(t-s))^{-\frac{n}{2}} \Delta e^{-f_{(\cdot,s)}(x,t)} \Bigg) dg_s \\
&= \frac{1}{t-s} \int_M \Big(-(t-s)\big[|\nabla f_{(\cdot,s)}(x,t)|^2_{g(s)} + R\big] + \frac{n}{2} \Big) \\
&\quad \times (4\pi(t-s))^{-\frac{n}{2}} e^{-f_{(\cdot,s)}(x,t)} dg_s \\
&= \frac{1}{t-s} \big(N_{(x,t)}(s) - W_{(x,t)}(s) \big).
\end{aligned}$$

The above identity can be easily integrated. Taking into account also Lemma 2.12, one obtains the following properties for Perelman's entropy and Nash entropy.

Lemma 2.13. *Let $(M, g(t))_{t\in[0,T]}$ be a complete Ricci flow with bounded curvature, then the following hold for $0 \le s < t \le T$:*

(1) $\frac{d}{ds} N_{(x,t)}(s) = \frac{1}{t-s}(N_{(x,t)}(s) - W_{(x,t)}(s)) \ge 0$

(2) $N_{(x,t)}(s) = \frac{1}{t-s} \int_s^t W_{(x,t)}(\eta) d\eta$

(3) $W_{(x,t)}(s) \le N_{(x,t)}(s) \le 0$

(4) $\frac{d}{ds}W_{(x,t)}(s) = 2(t-s)\int_M \left| Ric + \nabla^2 f_{(\cdot,s)}(x,t) - \frac{g}{2(t-s)} \right|^2_{g(s)} d\nu^s_{(xt)} \geq 0$

(5) $N_{(x,t)}(s) = -\int_s^t 2(t-\eta)(1-\frac{t-\eta}{t-s})$
$\times \int_M \left| Ric + \nabla^2 f_{(\cdot,\eta)}(x,t) - \frac{g}{2(t-\eta)} \right|^2_{g(\eta)} d\nu^\eta_{(x,t)}(z) d\eta$

(6) If there exists a $s < t$ such that $W_{(x,t)}(s) = 0$, then the Ricci flow must be a gaussian shrinker, that is, a static Euclidean space.

In the above Lemma, (4) is merely a reinterpretation of Theorem 1.1, (5) follows from (2) and (4). As for (6), see section 3.2.3.

Similar to (1.7), we can also define *asymptotic Nash entropy* for an ancient solution $(M, g(t))_{t\in(-\infty,0]}$

$$\bar{N}(x,t) := \lim_{s\to-\infty} N_{(x,t)}(s) \tag{2.22}$$

for each $(x,t) \in M \times (-\infty, 0]$. By Lemma 2.13(1), if the ancient solution has bounded curvature, then the asymptotic Nash entropy always exists and is a function of the base point (x,t).

2.6 A Bochner's formula for Perelman's entropy

As before, let us consider the conjugate heat kernel

$$H(y,s|x,t) = \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-f_{(y,s)}(x,t)} \tag{2.23}$$

and the pointed entropy

$$W_{(x,t)}(s) = \int_M \Bigg((t-s)\big(|\nabla f_{(\cdot,s)}(x,t)|^2 + R_{g(s)}\big) + f_{(\cdot,s)}(x,t) - n \Bigg) \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-f_{(\cdot,s)}(x,t)} dg_s.$$

An integration by parts yields

$$W_{(x,t)}(s) = \int_M \Bigg((t-s)\Big(2\Delta f_{(\cdot,s)}(x,t) - |\nabla f_{(\cdot,s)}(x,t)|^2 + R_{g(s)}\Big) + f_{(\cdot,s)}(x,t) - n \Bigg) \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-f_{(\cdot,s)}(x,t)} dg_s. \tag{2.24}$$

Indeed, the integrand of the above formula is a localized version of Perelman's entropy. Perelman [33] proved the following nice formula by direct computation. (see [9] and [31] for a detailed proof.)

Theorem 2.14. *Let $(M, g(t))$ be a Ricci flow and $u := (4\pi\tau)^{-\frac{n}{2}} e^f$ be a positive solution to the conjugate heat equation, where $\tau > 0$ stands for the backward time. Define*

$$v := \Big(\tau(2\Delta f - |\nabla f|^2 + R) + f - n\Big)u.$$

Then we have

$$\begin{aligned}\Box^* v &:= \Big(\frac{\partial}{\partial \tau} - \Delta + R\Big) v \\ &= -2\tau \left| Ric + \nabla^2 f - \frac{1}{2\tau} g \right|^2 u.\end{aligned}$$

Applying the above theorem to the conjugate heat kernel, Perelman also proved the following differential Harnack inequality (see also [9] and [31]).

Corollary 2.15. *Consider a complete Ricci flow with bounded curvature operator. Let H and f be defined as in (2.23). Then the following inequality always hold*

$$2\Delta_{g(s)} f_{(\cdot,s)}(x,t) - |\nabla f_{(\cdot,s)}(x,t)|^2_{g(s)} + R_{g(s)} + \frac{f_{(\cdot,s)}(x,t) - n}{t-s} \leq 0.$$

Chapter 3

On ancient solutions to the Ricci flow

An ancient solution, as defined in Definition 1.2, is a Ricci flow $(M, g(t))_{t\in(-\infty,0]}$ whose existence domain extends to time negative infinity. Ancient solutions are extensively studied by Hamilton [20] and Perelman [33]. As Ricci flow is a parabolic equation, it is reasonable to think that ancient solutions are rigid and have many nice properties. This is exactly the case. In this chapter we give a brief exposition of ancient solutions and collect some results that are used in our main theorems. In the study of ancient solutions, sometimes it is convenient to set $\tau = -t$ as the backward time. If this is the case, we will write the ancient solution as $(M, g(\tau))_{\tau\in[0,\infty)}$. This shall always be clear in the context.

3.1 Ancient solution as singularity model

Since the Ricci flow is a nonlinear parabolic equation, it is not surprising that the Ricci flow starting from an arbitrary closed Riemannian manifold in general does not exist for all time. As Hamilton [20] has proved, a Ricci flow exists so long as the norm of the Riemann curvature tensor is bounded. In other words, if we let $(M, g(t))_{t\in[0,T)}$ be a Ricci flow on a closed manifold, where $[0,T)$ is the maximum existence interval of $g(t)$ and $T<\infty$, then we have

$$\lim_{t\to T-}\sup_{M\times[0,t]}|Rm| = \infty. \tag{3.1}$$

If this happens, we say that $g(t)$ has a *singularity* at T and that T is a *singular time* of $g(t)$. Since T is finite, such type of singularity is also called a *finite-time singularity.* For the *infinite-time singularities* (called Type II(b) and Type III by Hamilton), please refer to [20].

To deal with singularities, Hamilton invented a method of dilation. We consider the most easy case. Let us assume the above scenario happens, that is, $(M, g(t))_{t\in[0,T)}$ is a Ricci flow on a closed manifold such that $T<\infty$ and (3.1) holds. Since $g(t)$ is smooth and M is closed, we can then find a sequence of space time points $\{(x_i,t_i)\}_{i=1}^{\infty}$

such that

$$t_i \nearrow T, \tag{3.2}$$

$$|Rm|(x_i, t_i) = \sup_{M\times[0,t_i]} |Rm| \nearrow \infty. \tag{3.3}$$

For the sake of convenience, we denote $Q_i := |Rm|(x_i, t_i)$, and define $g_i(t) := Q_i g(t_i + tQ_i^{-1})$. Then we obtain a sequence of (pointed) Ricci flows

$$\{(M, g_i(t), (x_i, 0))_{t\in[-t_iQ_i,0]}\}_{i=1}^{\infty}.$$

By (3.3) we have

$$\sup_{M\times[-t_iQ_i,0]} |Rm_{g_i}| = Q_i^{-1} \sup_{M\times[0,t_i]} |Rm| \equiv 1.$$

At the same time we also have

$$|Rm_{g_i}|(x_i, 0) = Q_i^{-1}|Rm|(x_i, t_i) = 1. \tag{3.4}$$

Hence the sequence of Ricci flows $\{(M, g_i(t), (x_i, 0))_{t\in[-t_iQ_i,0]}\}_{i=1}^{\infty}$ has uniformly bounded curvature. Moreover, Perelman's no local collapsing theorem [33] implies that $\mathrm{inj}_{g_i(0)}(x_i)$ is uniformly bounded from below by a constant depending only on $g(0)$, T, and the dimension n. One may find a subsequence converging to a limit Ricci flow.

Since (3.2) and (3.3) imply that $-t_i Q_i \to -\infty$, the limit Ricci flow

$$(M_\infty, g_\infty(t), (x_\infty, 0))_{t\in(-\infty,0]}$$

is an ancient solution. Furthermore, $|Rm_\infty|(x_\infty, 0) = 1$ and hence $g_\infty(t)$ is nonflat.

From the above argument we see that the study of ancient solutions is an essential part of singularity analysis. If an ancient solution arises from a dilation around a finite-time singularity of a Ricci flow on a closed manifold, then we call such an ancient solution a *singularity model*.

3.2 Shrinking gradient Ricci solitons

3.2.1 Definition

A *gradient Ricci soliton* is a tuple (M^n, g, f), where (M^n, g) is a Riemannian manifold and f is a smooth function on M called the *potential function*, satisfying the following equation

$$Ric + \nabla^2 f = \frac{\lambda}{2} g, \tag{3.5}$$

where λ is a constant. When $\lambda > 0$, or $\lambda = 0$, or $\lambda < 0$, the Ricci soliton is called *shrinking*, or *steady*, or *expanding*, respectively. In this paper we only consider shrinking gradient Ricci solitons, or shrinkers for short. For most of the time, we scale the shrinker and normalize in the way that $\lambda = 1$.

The following Lemma is follows from elementary computation.

Lemma 3.1. *Let (M^n, g, f) be a shrinking gradient Ricci soliton satisfying (3.5) with $\lambda = 1$. Then the following identities hold.*

$$|\nabla f|^2 + R - f = 0, \tag{3.6}$$

$$2\Delta f - |\nabla f|^2 + R + f - n = 0, \tag{3.7}$$

$$\nabla R = 2Ric(\nabla f). \tag{3.8}$$

Remark: Note that in (3.6) one can only prove that $|\nabla f|^2 + R - f$ is a constant on each shrinker. However, by adding a constant to f we can always make this constant 0. Henceforth, unless otherwise specified, we always assume the shrinkers in question have the following standard normalization

$$\begin{aligned} Ric + \nabla^2 f &= \frac{1}{2} g, \\ |\nabla f|^2 + R &= f. \end{aligned} \tag{3.9}$$

3.2.2 Properties

We collect some well-known facts of the shrinking gradient Ricci solitons. These results are fundamentally important in the study of Ricci shrinkers. The following growth estimate is due to Cao and Zhou [7]. What we present below is an improvement of Haslhofer and Müller [22].

Theorem 3.2. *Let (M^n, g, f) be a noncompact shrinking gradient Ricci soliton normalized as in (3.9). Let p be a point where f attains its minimum. Then the following holds.*

$$\frac{1}{4}(dist(x,p) - 5n)_+^2 \leq f(x) \leq \frac{1}{4}(dist(x,p) + \sqrt{2n})^2, \tag{3.10}$$

where $u_+ := \max\{u, 0\}$ denotes the positive part of a function.

The following Theorem due to Munteanu [28] and Cao and Zhou [7] shows that Ricci shrinkers has at most Euclidean volume growth. What we present below is an improvement of Munteanu and Wang [29].

Theorem 3.3. *There exists $C < \infty$ depending only on the dimension n, such that under the same assumption of Theorem 3.2 the following holds.*

$$\mathrm{Vol}(B(p,r)) \leq Cr^n \tag{3.11}$$

for all $r > 0$, where p is the minimum point of f .

3.2.3 The canonical form

Ricci shrinker is a self-similar ancient solution to the Ricci flow. Let us consider a Ricci shrinker normalized as in (3.9). Let $t \in (-\infty, 1)$ and $\tau(t) := 1 - t$, we define the one-parameter family of self-diffeomorphisms $\phi_t : M \to M$ by

$$\frac{\partial}{\partial t}\phi_t = \frac{1}{\tau(t)}\nabla_g f \circ \phi(t).$$

If we define

$$g(t) = \tau(t)\phi_t^* g, \tag{3.12}$$

then it is easy to check from the shrinker equation that

$$\frac{\partial}{\partial t}g(t) = -2Ric_{g(t)}.$$

The Ricci flow $(M, g(t))_{t\in(-\infty,1)}$ is called the canonical form of the Ricci shrinker. Along the canonical form, one readily checks the following identities. For the sake of convenience, we write $f \circ \phi_t$ as $f(t)$.

$$\begin{aligned}
Ric_{g(t)} + \nabla^2_{g(t)} f(t) - \frac{1}{2\tau(t)}g(t) &= 0, && (3.13)\\
|\nabla f(t)|^2_{g(t)} + R_{g(t)} - \frac{1}{2\tau(r)}f(t) &= 0, && (3.14)\\
\tau(t)\big(2\Delta_{g(t)} f(t) - |\nabla f(t)|^2_{g(t)} + R_{g(t)}\big) + f(t) - n &= 0. && (3.15)\\
-\frac{\partial}{\partial t}f(t) - \Delta_{g(t)} f(t) + |\nabla f(t)|^2_{g(t)} - R_{g(t)} + \frac{n}{2\tau(t)} &= 0. && (3.16)
\end{aligned}$$

Now if we define

$$u := \frac{1}{(4\pi\tau(t))^{\frac{n}{2}}} e^{-f(t)},$$

then u is a solution to the conjugate heat solution

$$-\frac{\partial}{\partial t}u - \Delta_{g(t)}u + R_{g(t)}u = 0,$$

and (3.13) and Theorem 1.1 implies

$$\frac{d}{dt}\mathcal{W}(g(t), f(t), \tau(t)) = \int_M 2\tau(t)\left|Ric_{g(t)} + \nabla^2 f(t) - \frac{1}{2\tau(t)}g(t)\right|^2_{g(t)} udg_t = 0,$$

where $\mathcal{W}$ is the $\mathcal{W}$-functional defined in (1.1). Note that u is not a fundamental solution to the conjugate heat equation, since its "base time" $t = 1$ is the singular time of $g(t)$ (as we will see shortly).

It follows from the definition of the canonical form (3.12) that $|Rm_{g(t)}| = \tau(t)^{-1}|Rm_{\phi_t^* g}|$, and consequently

$$\sup_M |Rm_{g(t)}| = \frac{1}{\tau(t)} \sup |Rm_g|.$$

Since $\tau(t) \to 0$ as $t \to 1$, unless (M, g) is flat, we have $\sup_M |Rm_{g(t)}| \nearrow \infty$ as $t \to 1-$.

Now lets consider a complete Ricci flow $(M, g(t))_{t\in[0,T]}$ with bounded curvature. Let $0 \leq s < t \leq T$ and $x \in M$ such that $W_{(x,t)}(s) = 0$. Then Lemma 2.13(4) implies that (M, g) is the canonical form of a Ricci shrinker. According to out argument above, the Ricci flow must go singular at t unless it is flat. Since (M, g) has bounded curvature in $[0, T]$, we have that it is a flat shrinker. Moreover a flat shrinker satisfies the equation

$$\nabla^2 f = \frac{1}{2} g.$$

The Morse theory implies that such manifold is diffeomorphic to $\mathbb{R}^n$. Hence it is isometric to $\mathbb{R}^n$. This justifies Lemma 2.13(6).

3.3 Perelman's distance distortion estimate

In this section we divert a bit and prove a distance distortion estimate due to Perelman [33]. This estimate, being valid for any complete Ricci flow (no curvature boundedness assumption needed), proves to be very handy, and we have used it for multiple times in the proofs of our main theorems. A remarkable fact of this distance distortion estimate is that the distortion rate of the distance between two points depends only on the local Ricci upper bound near these two points, and one does not even need to know the curvature bound on a geodesic connecting these two points.

Theorem 3.4 (Perelman, 2002). *Cosider a complete Ricci flow. Let x_0 and x_1 be two fixed points on the manifold. Then for almost every t_0 in the existence interval of this Ricci flow, the following holds. If $dist_{g(t_0)}(x_0, x_1) \geq 2r_0$ and $Ric(x, t_0) \leq (n-1)K$ for* *all* $x \in B_{g(t_0)}(x_0, r_0) \cup B_{g(t_0)}(x_1, r_0)$, *where* $K \geq 0$ *and* $r > 0$. *Then*

$$\frac{d}{dt} dist_{g(t)}(x_0, x_1) \geq -2(n-1)\left(\frac{2}{3}Kr_0 + r_0^{-1}\right) \quad at\ t = t_0. \tag{3.17}$$

Proof. Let $\gamma : [0, L] \to M$ be a normalized shortest geodesic connecting x_0 and x_1 with respect to the metric $g(t_0)$. If $dist_{g(t)}(x_0, x_1)$ is differentiable (with respect to variable t) at $t = t_0$, then we may compute at $t = t_0$

$$\frac{d}{dt} dist_{g(t)}(x_0, x_1) = -\int_0^L Ric(\dot{\gamma}(s), \dot{\gamma}(s)) ds.$$

On the other hand, since γ is a shortest geodesic, the second variation formula implies that, for any Lipschitz function $\phi : [0, L] \to \mathbb{R}$ with $\phi(0) = \phi(L) = 0$, it holds that

$$\int_0^L \Big((n-1)(\dot{\phi}(s))^2 - (\phi(s))^2 Ric(\dot{\gamma}(s), \dot{\gamma}(s))\Big) ds \geq 0.$$

Finally, letting $\phi \equiv 1$ on $[r_0, L - r_0]$ and linear on $[0, r_0]$ and $[L - r_0, L]$, and making use of the condition of Ricci upper bounde in $B_{g(t_0)}(x_0, r_0) \cup B_{g(t_0)}(x_1, r_0)$, we can estimate

$$-\int_0^L Ric(\dot{\gamma}(s), \dot{\gamma(s)}) ds \geq -2(n-1)\left(\frac{2}{3}Kr_0 + r_0^{-1}\right).$$

Therefore (3.17) holds whenever $dist_{g(t)}(x_0, x_1)$ is differentiable. To see that this holds for almost all t_0, please refer to the following remark. □

Remark: If we fix x_0 and x_1, it is not difficult to see that $dist_{g(t)}(x_0, x_1)$ is a locally Lipschitz function in t. Indeed, Fix arbitrary $t_1 < t_2$ and Let A be a local curvature bound of a compact domain containing every geodesic segment connecting x_0 and x_1 in $[t_0, t_1]$, then we have that for any t, $s \in [t_1, t_2]$, it holds that

$$dist_{g(s)}(x_0, x_1)e^{-C(n)A(t_2-t_1)} \leq dist_{g(t)}(x_0, x_1) \leq dist_{g(s)}(x_0, x_1)e^{C(n)A(t_2-t_1)}.$$

It follows that (3.17) is not only true for almost all t_0 but also can be integrated with respect to t.

3.4 Perelman's reduced geometry

In this section we introduce Perelman's reduced geometry. This is one of the two monotonicity formulae discovered by Perelman, it can be seen as parallel to Perelman's entropy. Throughout this section we will consider a backward Ricci flow $(M^n, g(\tau))_{\tau \in [0,T]}$ with

$$\frac{\partial g}{\partial \tau} = 2Ric.$$

3.4.1 Definition

Let $(x_0, \tau_0) \in M \times [0, T]$ be a point fixed in space-time. The reduced distance is a function defined on $M \times (\tau_0, T]$ by

$$l_{(x_0,\tau_0)}(x, \tau) := \frac{1}{2\sqrt{\tau - \tau_0}} \inf_{\gamma} \mathcal{L}(\gamma), \tag{3.18}$$

where the inf is taken among all piecewise smooth $\gamma : [\tau_0, \tau] \to M$ with $\gamma(\tau_0) = x_0$ and $\gamma(\tau) = x$, and $\mathcal{L}$ is the functional

$$\mathcal{L}(\gamma) := \int_{\tau_0}^{\tau} \sqrt{\zeta - \tau_0}\big(R(\gamma(\zeta), \zeta) + |\dot{\gamma}(\zeta)|^2_{g(\zeta)}\big) d\zeta.$$

Note that the subindex in $l_{(x_0,\tau_0)}(x, \tau)$ stands for the *base point*. When the base point is understood, we also omit the subindex. The *reduced volume* is defined by

$$\mathcal{V}_{(x_0,\tau_0)}(\tau) = \int_M (4\pi(\tau - \tau_0))^{-\frac{n}{2}} e^{-l_{(x_0,\tau_0)}} dg_\tau. \tag{3.19}$$

Again, we may omit the subindex (x_0, τ_0) if the base point is understood.

Remark: The minimizer of the $\mathcal{L}$-functional, if it exists, is called a shortest $\mathcal{L}$-geodesic (connecting (x_0, τ_0) and (x, τ)). If the curvature is bounded on every time slice, then the shortest $\mathcal{L}$-geodesic connecting any (x_0, τ_0) and (x, τ) with $\tau_0 < \tau$ always exists, but may not be unique; but it is unique for almost every such pair of space-time points. Please refer to [11] for more details.

3.4.2 Properties

Let us collect some properties of the reduced distance and the reduced volume. Most of these properties are proved by Perelman [33]. More detailed proofs can be found in [27] or [11].

Lemma 3.5. *Let l be the reduced distance function based at $(x_0, 0)$. Then the following hold.*

$$
\begin{aligned}
\frac{\partial}{\partial \tau} l &= R - \frac{1}{\tau} l + \frac{1}{2\tau^{\frac{3}{2}}} K, && (3.20)\\
|\nabla l|^2 &= -R + \frac{l}{\tau} - \frac{1}{\tau^{\frac{3}{2}}} K, && (3.21)\\
\Delta l &\leq -R + \frac{n}{2\tau} - \frac{1}{2\tau^{\frac{3}{2}}} K, && (3.22)
\end{aligned}
$$

where

$$K(x, \tau) = \int_0^\tau \zeta^{\frac{3}{2}} H(\dot{\gamma}(\zeta)) d\zeta,$$

γ is the shortest $\mathcal{L}$-geodesic connecting the base point $(x_0, 0)$ with (x, τ), and H is Hamilton's trace Harnack [17]

$$H(\gamma(\zeta)) = -\frac{\partial R}{\partial \zeta} - \frac{1}{\zeta} R - 2\langle \nabla R, \dot{\gamma}(\zeta) \rangle + 2Ric(\dot{\gamma}(\zeta), \dot{\gamma}(\zeta)). \tag{3.23}$$

Furthermore, if γ is the unique shortest $\mathcal{L}$-geodesic connecting (x_0) and (x, τ), then we *also have*

$$\big(\nabla_{g(\zeta)} l\big)\big(\gamma(\zeta), \zeta\big) = \dot{\gamma}(\zeta), \tag{3.24}$$

for all $\zeta \in (0, \tau]$.

Inequalities (3.20)—(3.22) are proved by an elegant but straightforward second variation argument. One may refer to [33] for the details, though the reader is strongly encouraged to figure them out by himself. A few remarks are to be made below.

Remarks:

(1) For the sake of convenience we let the base time to be 0 in (3.20)—(3.24). If one replaces the base time by an arbitrary τ_0, then every τ and ζ that appeared in (3.20)—(3.24) should correspondingly be replaced by $\tau - \tau_0$ or $\zeta - \tau_0$, respectively.

(2) If the shortest $\mathcal{L}$-geodesic connecting $(x_0, 0)$ and (x, τ) is unique, then (3.20)—(3.22) can be understood in the usual smooth sense at (x, τ).

(3) In the case such shortest $\mathcal{L}$-geodesic is not unique, then (3.20)—(3.22) can be understood in the barrier sense.

(4) For any τ fixed, one may understand (3.20)—(3.22) in the sense of distribution (or signed Radon measure on M). For instance, for any τ and for any $\phi \in C_0(M)$ with ϕ nonnegative and Lipschitz, (3.22) implies

$$-\int_M \langle \nabla_{g(\tau)}\phi, \nabla_{g(\tau)} l\rangle dg_\tau \;\; \leq \;\; \int_M \phi\Big(- R_{g(\tau)} + \frac{n}{2\tau} - \frac{1}{2\tau^{\frac{3}{2}}} K\Big) dg_\tau.$$

Combining (3.20)—(3.22) to cancel the terms containing K, one obtains the following clean formulae.

Lemma 3.6.

$$l_\tau - \Delta l + |\nabla l|^2 - R + \frac{n}{2\tau} \geq 0, \tag{3.25}$$

$$2\Delta l - |\nabla l|^2 + R + \frac{l-n}{\tau} \leq 0, \tag{3.26}$$

$$(4\tau l)_\tau + \Delta(4\tau l) \leq 2n. \tag{3.27}$$

$$2\frac{\partial l}{\partial \tau} + |\nabla l|^2 = R - \frac{l}{\tau}. \tag{3.28}$$

In the case when the curvature is bounded on each time-slice, we may closely examine definition (3.18) of l to observe that

$$l_{(x_0,0)}(x,\tau) \sim C(\tau) dist_{g(\tau)}(x_0,x)^2 \pm C(\tau). \tag{3.29}$$

Indeed, to see this, let $\gamma : [0,\tau]$ be a smooth curve connecting x_0 and x, we may estimate

$$\begin{aligned}\mathcal{L}(\gamma) &= \int_0^\tau \sqrt{\zeta}\big(R(\gamma(\zeta),\zeta) + |\dot\gamma(\zeta)|^2_{g(\zeta)}\big)d\zeta \\ &\geq -C(\tau) + C(\tau)\int_0^\tau |\dot\gamma(\zeta)|^2_{g(\tau)}d\zeta,\end{aligned}$$

where $C(\tau) > 0$ depends on the curvature bound. Since $\int_0^\tau |\dot\gamma(d\zeta)|^2_{g(\tau)}d\zeta$ is the curve energy for the Riemannian manifold $(M, g(\tau))$, it is minimized by the normalized shortest geodesic; this proves the lower estimate. The upper estimate is even easier to prove and is left to readers. It then follows that $4\tau l - 2n\tau$ always attains its minimum. Applying the maximum principle to $4\tau l - 2n\tau$ using (3.27), we have

Lemma 3.7. *For every $\tau > 0$, we have*

$$\inf_M l_{(x_0,0)}(\cdot, \tau) \leq \frac{n}{2}.$$

Now we present Perelman's monotonicity formula for the reduced volume. Clearly (3.25) implies

$$\Box^* \left(\frac{1}{(4\pi\tau)^{-\frac{n}{2}}} e^{-l} \right) := \left(\frac{\partial}{\partial \tau} - \Delta + R \right) \left(\frac{1}{(4\pi\tau)^{-\frac{n}{2}}} e^{-l} \right) \leq 0 \tag{3.30}$$

in the sense of distribution. Integrating (3.30) on M (indeed (3.30) is true in the sense of distribution, but (3.29) implies that e^{-l} has very fast decay at infinity, one may take an exhausting sequence of test functions), we have the following important monotonicity property of Perelman.

Theorem 3.8.

$$\frac{d}{d\tau} \mathcal{V}_{(x_0,0)}(\tau) \leq 0.$$

Remark: In our discussion we assume that the Ricci flow has bounded curvature on each time-slice. However, the condition for Theorem 3.8 can be very weak. For instance, Yokota [37] proved that a lower bound for the Ricci curvature suffices. Note that it is impossible to give an explicit formula for $\frac{d}{d\tau}\mathcal{V}$ in general cases.

(3.30) implies that the integrand $\frac{1}{(4\pi\tau)^{-\frac{n}{2}}}e^{-l}$ of the reduced volume is a sub-solution to the conjugate heat equation. However, if we assume (x,τ) is very close to $(x_0,0)$ so that the metric barely changed, then (3.18) implies

$$\frac{1}{(4\pi\tau)^{-\frac{n}{2}}}e^{-l} \sim \frac{1}{(4\pi\tau)^{-\frac{n}{2}}}e^{-\frac{dist_{g(0)}(x,x_0)^2}{4\tau}}, \quad \text{when} \quad (x,\tau)\sim(x_0,0).$$

Consequently we have

$$\lim_{\tau\to 0+}\frac{1}{(4\pi\tau)^{-\frac{n}{2}}}e^{-l_{(x_0,0)}} = \delta_{x_0}. \tag{3.31}$$

Applying the maximum principle to $\frac{1}{(4\pi\tau)^{-\frac{n}{2}}}e^{-l}$ using (3.30) and (3.31), we have

Lemma 3.9. *Let* $\frac{1}{(4\pi\tau)^{-\frac{n}{2}}}e^{-l_{(x_0,0)}}$ *be the integrand of the reduced volume based at* $(x_0,0)$. *Let* $H_{(x_0,0)}(x,\tau)$ *be the fundamental solution to the conjugate heat equation* $\Box^* u = 0$ *based at* $(x_0,0)$. *It then holds that*

$$\frac{1}{(4\pi\tau)^{-\frac{n}{2}}}e^{-l_{(x_0,0)}} \le H_{(x_0,0)}(x,\tau)$$

for all $x\in M$ *and for all* $\tau>0$. *In particular, we have*

$$\lim_{\tau\to 0+}\mathcal{V}_{(x_0,0)}(\tau)=1 \quad \textit{and} \quad \mathcal{V}_{(x_0,0)}(\tau)\le 1.$$

If there is a $\tau > 0$ *such that* $\mathcal{V}_{(x_0,0)}(\tau) = 1$*, then the Ricci flow is the trivial gaussian shrinker.*

For the last assertion above please refer to [37]. It is also convenient to define the following *asymptotic reduced volume* for a backward ancient solution $(M, g(\tau))_{\tau\in[0,\infty)}$

$$\bar{\mathcal{V}}(x_0, \tau_0) = \lim_{\tau\to\infty} \mathcal{V}_{(x_0,\tau_0)}(\tau). \tag{3.32}$$

3.4.3 Growth estimates on ancient solutions

Let us now consider an ancient solution to the Ricci flow $(M, g(\tau))_{\tau\in[0,\infty)}$, where τ is the backward time, such that each time-slice has bounded nonnegative curvature operator. Recall that when a Ricci flow has bounded nonnegative curvature operator, it is known that Hamilton's trace Harnack [17] holds.

Theorem 3.10 (Hamilton, 1993)**.** *Let* $(M^n, g(t))_{t\in(0,T)}$ *be a complete Ricci flow with bounded nonnegative curvature operator. For any* 1*-vector* V *we have*

$$\frac{\partial}{\partial t}R + \frac{R}{t} + 2\langle\nabla R, V\rangle + 2Ric(V,V) \geq 0.$$

In particular, on an ancient solution with bounded nonnegative curvature operator, we have

$$\frac{\partial}{\partial t}R + 2\langle\nabla R, V\rangle + 2Ric(V,V) \geq 0.$$

Applying Hamilton's trace Harnack to (3.23), we shall see that (3.20)—(3.22) yield much better estimates.

$$\begin{aligned} H(\gamma(\zeta)) &= -\frac{\partial R}{\partial \zeta} + \frac{1}{T-\zeta}R - 2\langle \nabla R, \dot\gamma(\zeta)\rangle + 2Ric(\dot\gamma(\zeta),\dot\gamma(\zeta)) - \left(\frac{1}{\zeta} + \frac{1}{T-\zeta}\right)R \\ &\geq -\frac{T}{\zeta(T-\zeta)}R, \end{aligned}$$

for any $T > \zeta > 0$. In particular, if we take $T = 2\zeta$, the above inequality becomes

$$H(\gamma(\zeta)) \geq -\frac{2}{\zeta}R(\gamma(\zeta),\zeta).$$

Hence

$$\begin{aligned} K(x,\tau) &= \int_0^\tau \zeta^{\frac{3}{2}} H(\dot\gamma(\zeta))d\zeta \\ &\geq -2\int_0^\tau \sqrt{\zeta}R(\gamma(\zeta),\zeta)d\zeta \\ &\geq -4l(x,\tau). \end{aligned}$$

Then, (3.20), (3.21), and (3.28) become the following important estimates on ancient solutions with bounded nonnegative curvature operator.

Lemma 3.11. *On an ancient solution to the Ricci flow with bounded nonnegative* *curvature operator. Let $l = l_{(x_0,0)}$ for arbitrary $x_0 \in M$. Then the following holds*

$$\begin{aligned} |\nabla l|^2 + R &\leq \frac{Cl}{\tau}, && (3.33)\\ \frac{\partial l}{\partial \tau} + \frac{Cl}{\tau} &\geq R, && (3.34)\\ 2\frac{\partial l}{\partial \tau} + \frac{l}{\tau} &\leq R, && (3.35)\end{aligned}$$

where C depends only on the dimension n.

The following lemma shows that l satisfies a much more accurate estimate than (3.29). These estimates play an very important role in the proof of Theorem 1.7 and Corollary 1.9. The proof presented below are originated from [27] (see Lemma 9.25).

Theorem 3.12. *Let l be the reduced distance satisfying the same condition as the previous lemma, then we have*

$$-2l(y,\tau) - C + \frac{c}{\tau} dist^2_{g(\tau)}(x,y) \leq l(x,\tau) \leq 2l(y,\tau) + \frac{C}{\tau} dist^2_{g(\tau)}(x,y), \qquad (3.36)$$

where c, C are constants depending only on the dimension n.

Proof. The second inequality follows immediately from the gradient estimate (3.33). To prove the first inequality, we let $\gamma_1(\zeta)$, $\gamma_2(\zeta)$ be the shortest $\mathcal{L}$-geodesics connecting the base point $(x_0, 0)$ and (x, τ), (y, τ), respectively. We compute

$$
\begin{aligned}
& \frac{d}{d\zeta}(dist_{g(\zeta)}(\gamma_1(\zeta), \gamma_2(\zeta))) \\
= \quad & \langle \nabla r_{\gamma_2(\zeta)}, \dot{\gamma}_1(\zeta) \rangle_{g(\zeta)} + \langle \nabla r_{\gamma_1(\zeta)}, \dot{\gamma}_2(\zeta) \rangle_{g(\zeta)} \\
& + \left(\frac{\partial}{\partial \zeta} dist_\zeta \right) (\gamma_1(\zeta), \gamma_2(\zeta)) \\
\leq \quad & |\nabla l|(\gamma_1(\zeta), \zeta) + |\nabla l|(\gamma_2(\zeta), \zeta) + \left(\frac{\partial}{\partial \zeta} dist_{g(\zeta)} \right) (\gamma_1(\zeta), \gamma_2(\zeta)) \\
\leq \quad & \frac{C}{\zeta^{\frac{1}{2}}} \left(\sqrt{l(\gamma_1(\zeta), \zeta)} + \sqrt{l(\gamma_2(\zeta), \zeta)} \right) + \left(\frac{\partial}{\partial \zeta} dist_{g(\zeta)} \right) (\gamma_1(\zeta), \gamma_2(\zeta)) \\
\leq \quad & \frac{C\tau^{\frac{1}{4}}}{\zeta^{\frac{3}{4}}} \left(\sqrt{l(x, \tau)} + \sqrt{l(y, \tau)} \right) + \left(\frac{\partial}{\partial \zeta} dist_\zeta \right) (\gamma_1(\zeta), \gamma_2(\zeta)).
\end{aligned}
$$

In the previous computation, we have used (3.24) as well as the following obvious estimate

$$
l(\gamma(\zeta), \zeta) = \frac{1}{2\sqrt{\zeta}} \mathcal{L}(\gamma|_{[0,\zeta]}) \leq \frac{1}{2\sqrt{\zeta}} \mathcal{L}(\gamma|_{[0,\tau]}) = \frac{\tau^{\frac{1}{2}}}{\zeta^{\frac{1}{2}}} l(\gamma(\tau), \tau).
$$

Note that $\gamma : [0, \tau] \to M$ may not be the unique shortest $\mathcal{L}$-geodesic connecting $(x_0, 0)$ and (x, τ), but for each $\zeta \in (0, \tau)$, $\gamma|_{[0,\zeta]} : [0, \zeta] \to M$ is the unique shortest $\mathcal{L}$-geodesic connecting $(x_0, 0)$ and $(\gamma(\zeta), \zeta)$.

To estimate the distance distortion term we use a variation of Theorem 3.4, whose proof is almost identical to Theorem 3.4:

$$
\frac{d}{d\zeta} dist_{g(\zeta)}(x, y) \leq (n - 1) \left(\frac{2}{3} K_1 r_1 + \frac{2}{3} K_2 r_2 + r_1^{-1} + r_2^{-1} \right),
$$

if $dist_{g(\zeta)}(x,y) \geq r_1 + r_2$, $Ric \leq (n-1)K_1$ in $B_{g(\zeta)}(x, r_1)$, and $Ric \leq (n-1)K_2$ in $B_{g(\zeta)}(y, r_2)$. Let

$$r_1(\zeta) = \frac{\zeta^{\frac{3}{4}}}{\tau^{\frac{1}{4}}} \frac{1}{(1 + l(x,\tau))^{\frac{1}{2}}} \leq \frac{\zeta^{\frac{3}{4}}}{\tau^{\frac{1}{4}}}, \; r_2(\zeta) = \frac{\zeta^{\frac{3}{4}}}{\tau^{\frac{1}{4}}} \frac{1}{(1 + l(y,\tau))^{\frac{1}{2}}} \leq \frac{\zeta^{\frac{3}{4}}}{\tau^{\frac{1}{4}}}$$

be the radius r_1, r_2 with which we shall apply the above distance distortion estimate. Notice that we have the curvature estimate $R \leq \frac{Cl}{\zeta}$ from (3.33), so the radius that is compatible with this scale should be $(\frac{l}{s})^{-\frac{1}{2}}$. The choice of $r_1(\zeta)$ and $r_2(\zeta)$ originates from here. Now we estimate the curvature in the small balls around $\gamma_1(\zeta)$, $\gamma_2(\zeta)$ using $|\nabla l| \leq \frac{Cl}{\zeta}$ from (3.33):

$$\begin{aligned} \sqrt{l} \;&\leq\; \sqrt{l(\gamma_1(\zeta),\zeta)} + C\zeta^{-\frac{1}{2}} \cdot \frac{\zeta^{\frac{3}{4}}}{\tau^{\frac{1}{4}}} \\ &\leq\; C\frac{\zeta^{\frac{1}{4}}}{\tau^{\frac{1}{4}}}(l(x,\tau)+1)^{\frac{1}{2}}, \;\; \text{in } B_{g(\zeta)}(\gamma_1(\zeta), r_1(\zeta)). \end{aligned}$$

Hence

$$R \leq \frac{C}{\tau^{\frac{1}{2}}\zeta^{\frac{1}{2}}}(1 + l(x,\tau)), \quad \text{in} \quad B_{g(\zeta)}(\gamma_1(\zeta), r_1(\zeta)).$$

Similarly, we have

$$R \leq \frac{C}{\tau^{\frac{1}{2}}\zeta^{\frac{1}{2}}}(1 + l(y,\tau)), \quad \text{in} \quad B_{g(\zeta)}(\gamma_2(\zeta), r_2(\zeta)).$$

By Perelman's distance distortion estimate, we have

$$\begin{aligned}&\left(\frac{\partial}{\partial\zeta}dist_{g(\zeta)}\right)(\gamma_1(\zeta),\gamma_2(\zeta))\\&\leq C\left[(l(x,\tau)+1)^{\frac{1}{2}}+(l(y,\tau)+1)^{\frac{1}{2}}\right]\left(\frac{\zeta^{\frac{1}{4}}}{\tau^{\frac{3}{4}}}+\frac{\tau^{\frac{1}{4}}}{\zeta^{\frac{3}{4}}}\right),\end{aligned}$$

whenever $dist_{g(\zeta)}(\gamma_1(\zeta),\gamma_2(\zeta))\geq 2\frac{\zeta^{\frac{3}{4}}}{\tau^{\frac{1}{4}}}$. Integrating we have

$$dist_{g(\tau)}(x,y)\leq C\tau^{\frac{1}{2}}\left((l(x,\tau)+1)^{\frac{1}{2}}+(l(y,\tau)+1)^{\frac{1}{2}}\right),$$

whence follows the conclusion. □

3.5 The asymptotic shrinker

In the last two sections of the introduction part, we shall consider a special type of ancient solutions—κ-solutions. This specific type of ancient solutions are particularly interesting in dimension three, since all possible singularity models are κ-solutions. κ-solutions was extensively studied by Perelman, and all the results stated in the following two sections can be found in [32]. For a detailed account one may also refer to [27].

Definition 3.13 (κ-solution)**.** *A nonflat ancient solution $(M^n, g(t))_{t\in(-\infty]}$ is called a κ-solution, where $\kappa > 0$, if it is κ-noncollapsed on all scales and each times slice $(M, g(t))$ has bounded and nonnegative curvature operator.*

Perelman's asymptotic shrinker theorem indeed reals the mystery of κ-solutions.

It says that when one moves to negative infinity in time, one should always see a Ricci shrinker. Or in other words, a κ-solution always flows "out of" a shrinking gradient Ricci soliton.

Theorem 3.14. *Let $(M, g(\tau))_{\tau\in[0,\infty)}$ be a κ-solution, where τ stands for the backward time. Let $\tau_k \to \infty$ be an increasing positive sequence. Let l be the reduced distance based an arbitrarily fixed $(x_0, 0) \in M \times \{0\}$. Let $\{x_k\}_{k=1}^{\infty} \subset M$ be such that $\limsup_{k\to\infty} l(x_k, \tau_k) < \infty$. Then the scaled sequence of Ricci flows*

$$\{(M, g_k(\tau), (x_k, 1), l_k)_{\tau\in[\frac{1}{2},1]}\}_{k=1}^{\infty},$$

where $g_k(\tau) := \tau_k^{-1} g(\tau\tau_k)$ and $l_k(\tau) = l(\tau\tau_k)$, subconverges in the pointed Cheeger-Gromov sense to a non-flat Ricci shrinker $(M_\infty, g_\infty(\tau), (p_\infty), l_\infty)_{\tau\in[\frac{1}{2},1]}$. Here l_∞ is the shrinker potential satisfying

$$Ric_\infty + \nabla^2 l_\infty - \frac{1}{2\tau} g_\infty = 0, \tag{3.37}$$

$$2\Delta l_\infty - 2|\nabla l_\infty|^2 + R_\infty + \frac{l_\infty - n}{\tau} = 0 \tag{3.38}$$

on $M_\infty \times [\frac{1}{2}, 1]$. In other words, $(M_\infty, g_\infty(1), l_\infty(1))$ is normalized as in (3.9).

Remark: Even if $\{\tau_k\}_{k=1}^{\infty}$ and $(x_0, 0)$ are arbitrarily given, $\{x_k\}_{k=1}^{\infty}$ always exists due to Lemma 3.7.

Sketch of proof. For the sake of simplicity, we denote $l_k(\cdot, \tau) := l_{(x_0,0)}(\tau\tau_k)$. Let $C < \infty$ be a constant independent of k such that $l_k(x_k, 1) \leq C$. Such C exists by the assumption. Combining (3.33) and (3.34) we have $\frac{\partial l_k}{\partial \tau} \geq -\frac{Cl}{\tau}$. Hence $l_k(x_k, \tau) \leq C$ for all k and for all $l \in [\frac{1}{2}, 1]$. (3.36) implies

$$-C + \frac{c}{\tau} dist^2_{g_k(\tau)}(x_k, x) \leq l_k(x, \tau) \leq C + \frac{C}{\tau} dist^2_{g_k(\tau)}(x_k, x), \tag{3.39}$$

for all $(x, \tau) \in M \times [\frac{1}{2}, 1]$. Combining (3.39) and (3.33)—(3.35), we have:

(1) A uniform growth estimate for $R_{g_k(\tau)}$ on $M \times [\frac{1}{2}, 1]$. Combining with κ-noncollapsing condition, this yields the smooth compactness of the scaled Ricci flows

$$\{(M, g_k(\tau), (x_k, 1))_{\tau \in [\frac{1}{2}, 1]}\}_{k=1}^{\infty}.$$

Let the limit flow be

$$(M_\infty, g_\infty(\tau), (x_\infty, 1))_{\tau \in [\frac{1}{2}, 1]}.$$

(2) Locally uniform C^1 estimates for l_k on $M \times [\frac{1}{2}, 1]$. It follows that $\{l_k\}_{k=1}^{\infty}$ is compact in the C^α_{loc} or weak $(W^{1,2}_{loc})^*$ sense. Let the limit function be

$$l_\infty : M_\infty \times [\frac{1}{2}, 1] \to \mathbb{R}.$$

Moreover, since the uniform estimate (3.39) implies that outside each large compact set the integrations

$$\mathcal{V}_k(\tau) := \mathcal{V}(\tau\tau_k) = \int_M \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-l_k} dg_k.$$

are uniformly negligible. A standard argument yields

$$\mathcal{V}_k(\tau) \to \mathcal{V}_\infty(\tau) := \int_{M_\infty} \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-l_\infty} dg_\infty.$$

Since $\mathcal{V}$ is nonnegative and monotonically decreasing, and since $\tau_k \nearrow \infty$, we have that $\mathcal{V}_\infty$ is a constant. Then we have

$$\begin{aligned} 0 &= \int_{\frac{1}{2}}^{1} \frac{\partial}{\partial\tau} V_\infty(\tau) \qquad (3.40) \\ &= \int_{M_\infty \times [\frac{1}{2},1]} \left(\frac{\partial}{\partial\tau} l_\infty - \Delta l_\infty + |\nabla l_\infty|^2 - R_{g_\infty} + \frac{n}{2\tau} \right) \\ &\quad \times \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-l_\infty} dg_\infty d\tau. \end{aligned}$$

On the other hand (3.25) implies that each

$$\frac{\partial}{\partial\tau} l_k - \Delta l_k + |\nabla l_k|^2 - R_{g_k} + \frac{n}{2\tau}$$

is a nonnegative distribution. Hence

$$\frac{\partial}{\partial\tau} l_\infty - \Delta l_\infty + |\nabla l_\infty|^2 - R_{g_\infty} + \frac{n}{2\tau}$$

is a nonnegative distribution by taking the limit. Together with (3.40) this implies that

$$\frac{\partial}{\partial \tau} l_\infty - \Delta l_\infty + |\nabla l_\infty|^2 - R_{g_\infty} + \frac{n}{2\tau} \equiv 0$$

is a zero distribution, or in other words, l_∞ is a weak solution to the above parabolic equation. It follows immediately that l_∞ is smooth and

$$\begin{aligned} 0 &= \int_{\frac{1}{2}}^{1} \frac{\partial}{\partial \tau} V_\infty(\tau) \\ &= \int_{M_\infty \times [\frac{1}{2},1]} 2\tau \left| Ric_{g_\infty} + \nabla^2 l_\infty - \frac{g_\infty}{2\tau} \right|^2 \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-l_\infty} dg_\infty d\tau. \end{aligned}$$

Consequently $(M_\infty, g_\infty, l_\infty)$ is a Ricci shrinker.

Finally, if $(M_\infty, g_\infty, l_\infty)$ is the flat gaussian shrinker, then one easily checks that $\mathcal{V}_\infty \equiv 1$. By Lemma 3.9, $(M, g(\tau))$ is the flat Euclidean space, contradicting the definition of a κ-solution.

□

3.6 Perelman's κ-compactness theorem

Up to this day, Perelman's κ-compactness theorem [33] is still one of the nicest and most sophisticated techniques in the study of the Ricci flow. Much as we want to give more detailed exposition of this theorem, since it is not highly relied on in the proof of our main theorems (except for the final corollary of this section), and since we are restricted by the length and main purpose of this dissertation, we shall refrain ourselves from giving too much details of this beautiful theorem. Perelman's κ-compactness theorem is essentially a consequence of the following "bounded curvature at bounded distance" theorem.

Theorem 3.15. *There exists a positive function $C(r) = C(r, \kappa, n)$, such that the following holds. Let $(M^n, g(t))_{t\in(-\infty,0]}$ be a κ-solution and $x_0 \in M$. Then $R(x,t) \leq C(dist_{g(t)}(x, x_0))R(x_0, t)$ for all $(x,t) \in (-\infty, 0]$.*

We briefly discuss the geometric intuition of Theorem 3.15. First, Perelman proved that an n-dimensional nonflat κ-solution cannot have volume growth rate comparable to r^n, because this would imply that the κ-solution is "conical" at space infinity, and a limiting argument yields a (local) Ricci flow on a nonflat cone with nonnegative curvature operator, this contradicts Hamilton's strong maximum principle [16]. With this being understood, let us assume that there are x and x_0 such that $\frac{R(x,t)}{R(x_0,t)}$ is very large. Then, if we stand at (x,t) and measure the manifold with scale $R(x,t)^{-\frac{1}{2}}$, the point (x_0, t) is almost at space infinity, hence the volume of discs around (x_0, t) must be very collapsed. This then contradicts the κ-noncollapsing assumption. We can now state the κ-compactness theorem.

Theorem 3.16. *Let $\kappa > 0$ be a fixed real number.*

(1) The space of three-dimensional κ-solution is compact in the smooth Cheeger-Gromov sense up to scaling. More specifically, for any sequence of κ-solutions $\{(M_k, g_k)\}_{k=1}^{\infty}$, if one fixes arbitrary base points on these Ricci flows, and scales the Ricci flows,

such that the scalar curvature at each base point is equal to 1 *for each member in this sequence, then after passing to a subsequence, the Ricci flows converges to a* κ*-solution.*

(2) In dimension higher than three, the conclusion is the same as above, except that the limit Ricci flow is only nonflat and κ*-noncollapsed with nonnegative curvature operator, and the curvature may not be bounded.*

In dimension three, each such limit Ricci flow has a dimension reduction at infinity: along every point towards space infinity, the scaled Ricci flow converges to the standard shrinking cylinder $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{S}^2/\mathbb{Z}_2 \times \mathbb{R}$. Hence unbounded curvature implies the existence of necks with arbitrary small radius—a contradiction to Sharafutdinov retraction. In higer dimensions, the dimension reduction is more complicated and hence one has only a precompactness theorem.

Finally, the following corollary is a direct consequence of Theorem 3.15 and Shi's local derivative estimates [35]. It is very useful in the study κ-solutions. We used this corollary in the proof of Corollary 1.9.

Corollary 3.17. *Let* $(M^n, g(\tau))_{\tau\in[0,\infty)}$ *be a* κ*-solution, where* $\kappa > 0$ *and* τ *is the backward time. Then there exists* $C < \infty$ *depending only on* κ *and the dimension* n*, such that*

$$\left|\frac{\partial R}{\partial \tau}\right| \leq CR^2, \tag{3.41}$$

$$|\nabla R| \leq CR^{\frac{3}{2}}, \tag{3.42}$$

hold on $M \times [0, \infty)$. *More generally, we have*

$$\left| \frac{\partial^l}{\partial \tau^l} \nabla^m Rm \right| \leq \eta R^{1+l+\frac{m}{2}}, \tag{3.43}$$

for any positive integers l *and* m, *where* η *depends on* κ, n, l, *and* m.

Part II

Perelman's entropy on ancient solutions

Chapter 4

Estimates of Nash entropy and the gap theorem

The goal of this chapter is to prove Theorem 1.3. The main technical part is to prove some delicate estimates for the asymptotic Nash entropy $\bar{N}(x,t)$ (see (2.22) for its definition). These estimates indicate that $\bar{N}(x,t)$ is independent of x and is decreasing in t. In fact, we will show that under some appropriate assumptions, it holds that

$$\lim_{s\to-\infty} |\nabla_x N_{(x,t)}(s)|_{g(t)} = 0, \text{ uniformly in } x, \tag{4.1}$$

$$\int_N \bar{N}(\cdot,s)d\nu^s_{(x,t)} \geq \bar{N}(x,t), \text{ for all } s<t\leq 0, \tag{4.2}$$

where $\nu^s_{(x,t)}$ is the probability measure defined in (2.15). The

This chapter is organized as follows. In section 4.1 we will prove a version of

Perelman's no local collapsing Theorem [33]. This version of no local collapsing is what we need to apply, and it does not follow trivially from Perelman's theorem. In section 4.2 we prove the estimates (4.1) and (4.2). This is the main technical part of this chapter. Finally in section 4.3 we prove Theorem 1.3.

4.1 Fundamental noncollapsing theorems

In this section we prove the following theorem.

Theorem 4.1. *For all $\beta > 0$, there exists $\kappa = \kappa(\beta, n) > 0$ such that the following holds. Consider a complete Ricci flow $(M^n, g(t))_{t\in[0,T]}$. Let $(x_0, t_0) \in M \times (0, T]$ and $r_0 \in (0, \sqrt{t_0}]$ be such that $|Rm|(x, t) \le r_0^{-2}$ on $B_{g(t_0)}(x_0, r_0) \times [t_0 - r_0^2, t_0]$. If $N_{(x_0,t_0)}(t_0 - r_0^2) \ge -\beta$, then $\mathrm{Vol}_{g(t_0)}(B_{g(t_0)}(x_0, r_0)) \ge \kappa r_0^n$.*

Indeed, if one compares Theorem 4.1 with Perelman's weak no local collapsing theorem (7.3 in [33]), one readily sees that if one replaces the Nash entropy in Theorem 4.1 by the reduced volume (defined in (3.19)) based at the same point evaluated at the same time, then one obtains a statement that is identical to the following theorem of Perelman.

Theorem 4.2 (Perelman, 2002)**.** *For all $\beta \in (0, 1)$, there exists $\kappa = \kappa(\beta, n) > 0$ such that the following holds. Consider a complete Ricci flow $(M^n, g(t))_{t\in[0,T]}$. Let $(x_0, t_0) \in M \times (0, T]$ and $r_0 \in (0, \sqrt{t_0}]$ be such that $|Rm|(x, t) \le r_0^{-2}$ on $B_{g(t_0)}(x_0, r_0) \times [t_0 - r_0^2, t_0]$. If $\mathcal{V}_{(x_0,t_0)}(t_0 - r_0^2) \ge -\beta$, then $\mathrm{Vol}_{g(t_0)}(B_{g(t_0)}(x_0, r_0)) \ge \kappa r_0^n$. Here $\mathcal{V}_{(x_0,t_0)}$ is the reduced volume based at (x_0, t_0).*

Furthermore, from Theorem 4.1 one may also obtain Perelman's original no local collapsing Theorem for Ricci flows on closed manifold.

Corollary 4.3 (Perelman's noncollapsing theorem)**.** *Let* $(M^n, g(t))_{t\in[0,T)}$ *be a complete Ricci flow with* $\nu[g(0), T] > -\infty$. *Then there exists a positive number* κ *depending only on* n *and* $\nu[g(0), T]$ *such that the following holds. Let* $(x_0, t_0) \in M \times (0, T)$ *and* $0 < r_0 \le \sqrt{t_0}$ *be such that* $|Rm|(x, t) \le r_0^{-2}$ *for all* $(x, t) \in B_{g(t_0)}(x_0, r_0) \times [t_0 - r_0^2, t_0]$. *Then* $\mathrm{Vol}_{g(t_0)}(B_{g(t_0)}(x_0, r_0)) \ge \kappa r_0^n$.

Proof. By Lemma 2.13(1)(3), we have

$$N_{(x_0,t_0)}(t_0 - r_0^2) \ge N_{(x_0,t_0)}(0) \ge W_{(x_0,t_0)}(0) \ge \mu(g(0), t_0) \ge \nu[g(0), T]$$

for all (x_0, t_0) and r_0 satisfying the condition in the statement of the corollary. The conclusion follows from Theorem 4.1. □

4.1.1 On flat space forms—the model of collapsing case

The proof of Theorem 4.1 is an argument by contradiction. Assuming there is a contradicting sequence, then after properly scaling and taking a limit, one obtains a flat (static) ancient solutions to the Ricci flow that is not the Euclidean space. Thus we need to understand what happens to the Nash entropy on a flat non-Euclidean space form (when regarded as a Ricci flow). Indeed in this case the conjugate heat kernel is simply the classical heat kernel since the metric is static and the scalar curvature is zero.

Let us begin with the computation of the Nash entropy on $\mathbb{T}^k$, any flat torus of dimension $k \geq 1$. In fact, all the properties we need here for $\mathbb{T}^k$ are the Ricci-flatness and the finiteness of the volume. Hence the following lemma is true for any Ricci flat closed manifold. Let

$$H(x, y, t) = \frac{1}{(4\pi t)^{\frac{k}{2}}} e^{-f(x,y,t)}$$

be the heat kernel on $\mathbb{T}^k$; H coincides with the conjugate heat kernel coupled with the static backward Ricci flow, if t is regarded as backward time. Then the Nash entropy centered at $(x, 0)$ is defined as

$$\begin{aligned} N(t) &= \int_{\mathbb{T}^k} f(x,y,t)H(x,y,t)dy - \frac{k}{2} \\ &= -\int_{\mathbb{T}^k} H(x,y,t)\log H(x,y,t)dy - \frac{k}{2}\log(4\pi t) - \frac{k}{2}. \end{aligned} \tag{4.3}$$

Lemma 4.4. $\lim_{t\to\infty} N(t) = -\infty.$

Proof. From $H > 0$ and $-H \log H \leq \frac{1}{e}$ for all $t > 0$, it follows that

$$\begin{aligned} N(t) &= -\int_{\mathbb{T}^k} H(x,y,t)\log H(x,y,t)dy - \frac{k}{2}\log(4\pi t) - \frac{k}{2} \\ &\leq \frac{1}{e}\mathrm{Vol}(\mathbb{T}^k) - \frac{k}{2}\log(4\pi t) - \frac{k}{2}, \text{ for all } t > 0. \end{aligned}$$

The right-hand side obviously approaches negative infinity as t approaches infinity. □

We have the following corollary.

Corollary 4.5. *Let $N_{\mathbb{T}^k\times\mathbb{R}^{n-k}}(t)$ be the Nash entropy of the linear heat equation on the flat manifold $\mathbb{T}^k\times\mathbb{R}^{n-k}$, where $k\geq 1$. Then $\lim_{t\to\infty} N_{\mathbb{T}^k\times\mathbb{R}^{n-k}}(t) = -\infty$.*

Proof. It is easy to see that, if

$$\begin{aligned} H_1(x,t) &= \frac{1}{(4\pi t)^{\frac{n_1}{2}}}\exp\left(-f_1(x,t)\right), \\ H_2(y,t) &= \frac{1}{(4\pi t)^{\frac{n_2}{2}}}\exp\left(-f_2(y,t)\right) \end{aligned}$$

are the heat kernels on Riemannian manifolds (M_1,g_1), (M_2,g_2) centered at $(x_0,0)$, $(y_0,0)$, respectively, then $H_1(x,t)H_2(y,t)$ and $N_{(x_0,0)}(t)+N_{(y_0,0)}(t)$ are the heat kernel and Nash entropy of $(M_1\times M_2, g_1+g_2)$ centered at $\big((x_0,y_0),0\big)$, respectively. It follows from Lemma 4.4 that $\lim_{t\to\infty} N_{\mathbb{T}^k\times\mathbb{R}^{n-k}}(t) = -\infty$. □

Now we consider an arbitrary non-Euclidean flat space form. It follows from Bieberbach's theorem (c.f. Theorem 98 of [3]) that such a space form can always be written as $(\mathbb{T}^k\times\mathbb{R}^{n-k})/\Gamma$, where $k\geq 1$, $\mathbb{T}^k$ is a flat torus, and Γ is a finite group. With this known we can prove the following proposition.

Proposition 4.6. *Let (M^n,g) be a non-Euclidean flat space form. Then the Nash entropy $N_{(x,0)}(t)\searrow -\infty$ as $t\nearrow\infty$ for any base point $(x,0)$.*

Proof. Let $(\tilde{M},\tilde{g})\cong\mathbb{T}^k\times\mathbb{R}^{n-k}$ be a finite covering space of (M,g) and let $\pi:(\tilde{M},\tilde{g})\to(M=\tilde{M}/\Gamma,g)$ be the finite Riemannian covering map, so that $|\Gamma|<\infty$. Let $\tilde{x}\in\tilde{M}$

be such that $\pi(\tilde{x}) = x$, we shall show that

$$\tilde{N}_{(\tilde{x},0)}(t) \geq N_{(\pi(\tilde{x}),0)}(t) := N_{(x,0)}(t) \tag{4.4}$$

for all $t > 0$, where $\tilde{N}_{(\tilde{x},0)}(t)$ is the Nash entropy on $(\tilde{M}, \tilde{g})$ based at $(\tilde{x}, 0)$.

It is well known that

$$H(\pi(\tilde{x}), \pi(\tilde{y}), t) = \sum_{h \in \Gamma} \tilde{H}(\tilde{x}, h(\tilde{y}), t),$$

where $\tilde{H}$ is the heat kernel on $(\tilde{M}, \tilde{g})$. We let $\tilde{M}_1 \subset \tilde{M}$ be a fixed fundamental domain of the covering $\tilde{M}$. It follows that

$$\begin{aligned}
N_{(\pi(\tilde{x}),0)}(t) &= -\int_{\tilde{M}_1} \sum_{h\in\Gamma} \tilde{H}(\tilde{x}, h(\tilde{y}), t) \log\left(\sum_{k\in\Gamma} \tilde{H}(\tilde{x}, k(\tilde{y}), t)\right) d\tilde{g}(\tilde{y}) - \frac{n}{2}\log(4\pi t) - \frac{n}{2} \\
&\leq -\int_{\tilde{M}_1} \sum_{h\in\Gamma} \tilde{H}(\tilde{x}, h(\tilde{y}), t) \log(\tilde{H}(\tilde{x}, h(\tilde{y}), t)) d\tilde{g}(\tilde{y}) - \frac{n}{2}\log(4\pi t) - \frac{n}{2} \\
&= -\int_{\tilde{M}} H(\tilde{x}, \tilde{y}, t) \log H(\tilde{x}, \tilde{y}, t) d\tilde{g}(\tilde{y}) - \frac{n}{2}\log(4\pi t) - \frac{n}{2} \\
&= \tilde{N}_{(\tilde{x},0)}(t).
\end{aligned}$$

Combining (4.4) with Corollary 4.6, the proposition follows.

□

Remarks:

(1) Note that in all the statements of this subsection we regard t as forward time and consider the heat operator. But in real application we will let t be the backward

time and consider the conjugate heat operator. Note that this does not make any difference in the static case.

(2) (4.4) works also for solutions to the Ricci flow in place of static Riemannian manifolds, so long as the covering is finite. Note that we have only applied it to the static Ricci flat case.

4.1.2 Nash entropy on a collapsing sequence.

Now we continue with the proof of Theorem 4.1. As described before, we will assume that the conclusion of Theorem 4.1 is not true and take a limit from a scaled contradicting sequence to get into a scenario as in Proposition 4.6. Although we do not know whether the scaled Nash entropies would converge or not in this case, we can observe an important property from Lemma 2.13(5)—seeing its integrand is nonpositive—that with respect to the topology of pointed Cheeger-Gromov-Hamilton convergence, the pointed Nash entropy is *upper semicontinuous*. This suffices our purpose.

Proof of Theorem 4.1. Consider a sequence of counterexample

$$(M_i, g_i(t))_{t\in[0,T_i]},$$

$$(x_i, t_i) \in M_i \times (0, T_i],$$

$$r_i \in (0, \sqrt{t_i}],$$

$$|Rm(g_i)| \le r_i^{-2} \text{ on } B_{g_i(t_i)}(x_i, r_i) \times [t_i - r_i^2, t_i],$$

$$N_{(x_i,t_i)}(t_i - r_i^2) \ge -\beta,$$

but at the same time

$$\mathrm{Vol}(B_{g_i(t_i)}(x_i, r_i))/r_i^n \to 0,$$

or equivalently

$$\mathrm{inj}_{g_i(t_i)}(x_i)/r_i \to 0.$$

Now chose a sequence of scaling factor Q_i, such that

$$\mathrm{inj}_{Q_i g_i(t_i)}(x_i) = 1,$$

hence

$$Q_i r_i^2 \to \infty.$$

Consider the scaled Ricci flows $\bar{g}_i(t) = Q_i g_i(tQ_i^{-1} + t_i)$:

$$|Rm(\bar{g}_i)| \leq Q_i^{-1} r_i^{-2} \text{ on } B_{\bar{g}_i(0)}(x_i, Q_i^{\frac{1}{2}} r_i) \times [-Q_i r_i^2, 0], \tag{4.5}$$

$$\text{inj}_{\bar{g}_i(0)}(x_i) = 1, \tag{4.6}$$

$$N_{(x_i,0)}(t) \geq -\beta \text{ for } t \in [-Q_i r_i^2, 0]. \tag{4.7}$$

Hence by [19], we can extract a subsequence from

$$\{(M_i, \bar{g}_i(t), (x_i, 0))_{t\in[-Q_i r_i^2, 0]}\}_{i=1}^{\infty}$$

converging in the pointed smooth Cheeger-Gromov sense to a flat ancient Ricci flow

$$(M_\infty^n, g_\infty(t), (x_\infty, 0))_{t\in(-\infty,0]}$$

whose injectivity radius at $(x_\infty, 0)$ is 1. Obviously, (M_∞, g_∞) is a non-Euclidean flat space form.

Claim. *$N_{(x_\infty,0)}(t) \geq -\beta$, for all $t < 0$.*

Proof of the claim. Let us fix a $t < 0$ and let $\{\Omega_i \times [A_i, 0]\}_{i=1}^{\infty}$ be a compact space-time exhaustion of the limit flow. We assume without loss of generality that $A_i < t$ for any i. Let $\bar{g}_i$ and $H_i(z, s) = \frac{1}{(4\pi(-s))^{\frac{n}{2}}} \exp\left(-f_i(z, s)\right)$ also denote the pull-back metrics and conjugate heat kernels, respectively, via diffeomorphisms given by the pointed Cheeger-Gromov convergence on $\Omega_i \times [A_i, 0]$, where all the heat kernels are centered at the base points $(x_i, 0)$. According to [26], by passing to a subsequence the conjugate heat kernels centered at the base points will also converge to the heat kernel on the limit manifold centered at $(x_\infty, 0)$, that is, $f_i \to f_\infty$ and $H_i \to H_\infty$ locally smoothly. By (4.7) and Lemma 2.13(5), we have

$$\int_t^0 2s\left(1 - \frac{s}{t}\right) \int_{\Omega_k} \left| Ric_i(\cdot, \eta) + \nabla^2 f_i(\cdot, \eta) + \frac{g_i}{2\eta} \right|^2_{g_i(\eta)} H_i(\cdot, \eta) dg_i(\eta) d\eta \geq -\beta,$$

for all $i \geq 1$ and for all $k \geq 1$. Notice here we have used the fact that the integrands are nonpositive, and hence their contributions to the integration outside $\Omega_k \times [t, 0]$ are also nonpositive. Passing i to the limit and using the locally smooth convergence of the integrand, we have

$$\int_t^0 2s\left(1 - \frac{s}{t}\right) \int_{\Omega_k} \left| Ric_\infty(\cdot, \eta) + \nabla^2 f_\infty(\cdot, \eta) + \frac{g_\infty}{2\eta} \right|^2_{g_\infty(\eta)} H_\infty(\cdot, \eta) dg_\infty(\eta) d\eta \geq -\beta,$$

for all $k \geq 1$. Taking $k \to \infty$ and using Lemma 2.13(5) again for $(M_\infty, g_\infty(t))$ completes the proof. □

Now the above claim is a contradiction to Proposition 4.6; this completes the proof of Theorem 4.1.

□

4.2 Estimates on the asymptotic Nash entropy

In this section, we shall prove the estimates (4.1) and (4.2). We shall make the following assumptions for the ancient solution $(M^n, g(t))_{t\in(-\infty,0]}$ throughout this section.

$$\sup_M |Rm_{g(t)}| \;<\; \infty, \tag{4.8}$$

$$\inf_M \mathrm{Vol}_{g(t)}(B_{g(t)}(\cdot,1)) \;>\; 0, \tag{4.9}$$

for all $t \in (-\infty, 0]$. In our estimates we will consider two cases—a more general case and the case of nonnegative curvature operator. The estimates in these two cases are similar and parallel, but the conditions they require are not quite the same.

We will introduce another notation. Let $H(y,s|x,t)$ be the heat kernel on $(M^n, g(t))_{t\in(-\infty,0]}$, that is, the fundamental solution to (2.8). We define

$$M_{s,t} := \sup_{x\in M, y\in M} H(y,s|x,t).$$

By Lemma 2.2 we have that $M_{s,t}$ depends only on

$$\sup_{M\times[s,t]} |Rm| < \infty, \quad \inf_{M\times[s,t]} \mathrm{Vol}_{g(\cdot)}(B_{g(\cdot)}(\cdot,1)) > 0, \text{ and } t-s>0. \tag{4.10}$$

As we know from chapter 2, since $H(\cdot,\cdot|x,t)$ and $H(y,s|\cdot,\cdot)$ are solutions to the conjugate heat equation $\Box^* u = 0$ and the heat equation $\Box u = 0$, respectively, and since by Theorem 2.3 and Theorem 2.4 $H(\cdot,\cdot|x,t)$ and $H(y,s|\cdot,\cdot)$ always attain their maximums, we can apply the maximum principle and obtain

$$H(\cdot,\eta|x,t) \le M_{s,t}, \quad \text{whenever } \eta \le s, \tag{4.11}$$

$$H(y,s|\cdot,\zeta) \le M_{s,t}, \quad \text{whenever } \zeta \ge t. \tag{4.12}$$

Note that when applying the maximum principle to the conjugate heat equation $\Box^* u = (-\partial_s - \Delta + R)u = 0$, one needs to use a result of Chen [10], saying that the scalar curvature on every ancient solution is nonnegative.

4.2.1 The independence of space

We proceed to prove (4.1). The idea is by direct computation and by using the gradient estimate Theorem 2.8. The key point is that since we are working with an ancient solution, the $\frac{1}{t}$ factor on the right-hand side of (2.14) has a fast rate of decaying.

Lemma 4.7. *Let $(M^n, g(t))_{t\in(-\infty,0]}$ be a complete ancient solution to the Ricci flow.*

(1) If there exists $C_0 > 0$ such that $\sup_{M\times(-\infty,0]} |Rm| \le C_0$ and $\inf_{M\times(-\infty,0]} \mathrm{Vol}_{g(\cdot)}(B_{g(\cdot)}(\cdot,1)) \ge C_0^{-1}$, then there exists a constant $C > 0$ depending only on C_0 such that

$$0 \ge N_{(x,t)}(\eta) \ge -\frac{n}{2}\log\left(4\pi(t-\eta)\right) - C,$$

for all $(x,t) \in M \times (-\infty,0]$ and $\eta < t-1$.

(2) If $(M^n, g(t))_{t\in(-\infty,0]}$ *has nonnegative curvature operator and if there exists* $C_0 > 0$ *such that* $\sup_M R(\cdot,0) \le C_0$ *and* $\inf_M \mathrm{Vol}_{g(0)}(B_{g(0)}(\cdot,1)) \ge C_0^{-1}$ *then there exists a constant* $C > 0$ *depending only on* C_0 *such that*

$$0 \ge N_{(x,t)}(\eta) \ge -\frac{n}{2}\log(4\pi(t-\eta)) - n\log(-\eta + C) - C,$$

for all $(x,t) \in M \times (-\infty, 0]$ *and* $\eta < t - 1$.

Proof. In our argument we let C be a constant depending only on C_0 and which may vary from line to line.

(1) By our assumptions and by Lemma 2.2, since

$$\sup_{M\times[\eta,\eta+1]} |Rm| < C_0, \quad \inf_{M\times[\eta,\eta+1]} \mathrm{Vol}_{g(\cdot)}(B_{g(\cdot)}(\cdot,1)) > C_0^{-1}$$

$$M_{\eta,\eta+1} \le C(C_0), \text{ for all } \eta \le -1, \tag{4.13}$$

Note that the constant $C(C_0)$ on the right-hand side does not depend on η.

From (4.11) we have

$$-\frac{n}{2}\log(4\pi(t-\eta)) - f_{(z,\eta)}(x,t) = \log H(z,\eta|x,t) \le \log M_{\eta,\eta+1}.$$

Integrating this inequality against $d\nu^{\eta}_{(x,t)} = H(\cdot,\eta|x,t)dg_\eta$ and using the definition of the Nash entropy (2.21), we have

$$0 \geq N_{(x,t)}(\eta) \geq -\frac{n}{2} - \log M_{\eta,\eta+1} - \frac{n}{2}\log\left(4\pi(t-\eta)\right) \tag{4.14}$$
$$\geq -C - \frac{n}{2}\log\left(4\pi(t-\eta)\right).$$

Whence follows the conclusion.

(2) The proof is only a slight modification of part (1). We need to estimate $M_{\eta,\eta+1}$ for this case. According to Theorem 3.10, we have $\sup\limits_{M\times(-\infty,0]} |Rm| \leq C(n)C_0$. Applying Theorem 3.4 with $K = C_0$ and $r_0 = \frac{1}{2}$, we obtain that for all $y \in B_{g(0)}(x,1)$, it holds that

$$\frac{d}{ds}dist_{g(s)}(x,y) \geq -C(n)C_0, \text{ for almost all } s < 0 \text{ whenever } dist_{g(s)}(x,y) \geq 1.$$

Integrating the above inequality from η to 0 we have that

$$dist_{g(\eta)}(x,y) \leq dist_{g(0)}(x,y) - C(n)C_0\eta$$
$$\leq 1 - C(n)C_0\eta.$$

Hence there exists $C > 0$ depending only on C_0 such that $B_{g(\eta)}(x, 1 - C\eta) \supseteq B_{g(0)}(x,1)$, for all $\eta < 0$ and $x \in M$. Furthermore, the Ricci flow with nonnegative scalar curvature shrinks the measure. Thus

$$\mathrm{Vol}_{g(\eta)}(B_{g(\eta)}(x, 1 - C\eta)) \geq \mathrm{Vol}_{g(0)}(B_{g(0)}(x, 1)) \geq C_0^{-1}$$

and

$$\mathrm{Vol}_{g(\eta)}(B_{g(\eta)}(x, 1)) \geq \frac{1}{C(1 - C\eta)^n}$$

by the Bishop-Gromov volume comparison theorem. Applying the rough upper bound for fundamental solutions of heat-type equations again (c.f. Lemma 26.17 of [13]), we have

$$M_{\eta,\eta+1} \leq C(1 - C\eta)^n. \tag{4.15}$$

Inserting this inequality into (4.14) completes the proof.

□

The following lemma is inspired by Hein-Naber's argument (see Theorem 4.5 of [23]).

Lemma 4.8. *Let* $(M^n, g(t))_{t\in(-\infty,0]}$ *be a complete ancient solution to the Ricci flow such that either:*

(1) There exists $C_0 > 0$ *such that* $\sup_{M\times(-\infty,0]} |Rm| \leq C_0$ *and* $\inf_{M\times(-\infty,0]} \mathrm{Vol}_{g(0)}(B_{g(\cdot)}(\cdot, 1)) \geq C_0^{-1}$*, or*

(2) M has nonnegative curvature operator and there exists $C_0 > 0$ *such that* $\sup_M R(\cdot, 0) \leq C_0$ *and* $\inf_M \mathrm{Vol}_{g(0)}(B_{g(0)}(\cdot, 1)) \geq C_0^{-1}$.

Then

$$\lim_{\eta \to -\infty} |\nabla_x N_{(x,t)}(\eta)|_{g(t)} = 0$$

for each $(x, t) \in M \times (-\infty, 0]$. *Moreover, the convergence is uniform in* x.

Proof. In the proof we always assume that $\eta < t - 1$. By definition,

$$\nabla_x N_{(x,t)}(\eta) = \int_M [\nabla_x f_{(\cdot,\eta)}(x,t) - f_{(\cdot,\eta)}(x,t) \nabla_x f_{(\cdot,\eta)}(x,t)] d\nu^{\eta}_{(x,t)}$$

and hence

$$|\nabla_x N_{(x,t)}(\eta)|_{g(t)} \leq \|\nabla_x f_{(\cdot,\eta)}(x,t)\|_2 (1 + \|f_{(\cdot,\eta)}(x,t)\|_2),$$

by the Cauchy-Schwarz inequality, where the L^2 norms are taken with respect to $d\nu^{\eta}_{(x,t)}$.

By Theorem 2.8 we have

$$\begin{aligned} |\nabla_x f_{(z,\eta)}(x,t)|^2_{g(t)} &= \frac{|\nabla_x H(z,\eta|x,t)|^2_{g(t)}}{H(z,\eta|x,t)^2} \\ &\leq \frac{1}{t-\eta-1} \log \frac{M_{\eta,\eta+1}}{H(z,\eta|x,t)} \\ &= \frac{1}{t-\eta-1} \Big(\log M_{\eta,\eta+1} + \frac{n}{2} \log\left(4\pi(t-\eta) + f_{(z,\eta)}(x,t)\right) \Big). \end{aligned}$$

Integrating against the measure $d\nu^{\eta}_{(x,t)}(z)$, we have

$$\|\nabla_x f_{(\cdot,\eta)}(x,t)\|_2^2 \leq \frac{1}{t-\eta-1}\left(\log M_{\eta,\eta+1} + \frac{n}{2}\log\left(4\pi(t-\eta)\right) + \frac{n}{2}\right), \tag{4.16}$$

where the last term on the right-hand side is because of the fact $N_{(x,t)}(\eta) \leq 0$.

To estimate $\|f_{(\cdot,\eta)}(x,t)\|_2$, we apply Theorem 2.10(1) to obtain

$$\begin{aligned} \|f_{(\cdot,\eta)}(x,t)\|_2^2 &= \int_M f_{(\cdot,\eta)}(x,t)^2 d\nu^{\eta}_{(x,t)} \\ &\leq \left(\int_M f_{(\cdot,\eta)}(x,t) d\nu^{\eta}_{(x,t)}\right)^2 + 2(t-\eta)\int_M |\nabla_z f_{(z,\eta)}(x,t)|^2_{g(\eta)} d\nu^{\eta}_{(x,t)}(z) \\ &\leq \left(N_{(x,t)}(\eta) + \frac{n}{2}\right)^2 + n, \end{aligned} \tag{4.17}$$

where the last inequality follows from

$$\begin{aligned} & (t-\eta)\int_M |\nabla_z f_{(z,\eta)}(x,t)|^2_{g(\eta)} d\nu^{\eta}_{(x,t)}(z) \\ \leq\ & (t-\eta)\int_M \left(|\nabla_z f_{(z,\eta)}(x,t)|^2_{g(\eta)} + R(z,\eta)\right) d\nu^{\eta}_{(x,t)}(z) \\ \leq\ & \frac{n}{2}, \end{aligned}$$

which in turn is implied by the fact that $W_{(x,t)}(\eta) \leq N_{(x,t)}(\eta)$.

Combining (4.16) and (4.17), we have

$$|\nabla_x N_{(x,t)}(\eta)|^2_{g(t)} \leq \frac{1}{t-\eta-1}\left(\left(N_{(x,t)}(\eta)+\frac{n}{2}\right)^2+n\right) \times \left(\log M_{\eta,\eta+1}+\frac{n}{2}\log\left(4\pi(t-\eta)\right)+\frac{n}{2}\right). \tag{4.18}$$

Applying Lemma 4.7 as well as (4.13) or (4.15) to (4.18), we have:

Case (1).

$$|\nabla_x N_{(x,t)}(\eta)|^2_{g(t)} \leq \frac{1}{t-\eta-1}\left(\frac{n}{2}\log\left(4\pi(t-\eta)\right)+C\right)^3,$$

or

Case (2).

$$|\nabla_x N_{(x,t)}(\eta)|^2_{g(t)} \leq \frac{1}{t-\eta-1}\left(\frac{n}{2}\log\left(4\pi(t-\eta)\right)+C+n\log\left(-\eta+C\right)\right)^3,$$

where C depends only on C_0. In either case, the right-hand side converges to 0 as $\eta \to -\infty$ whence follows the conclusion. □

Proposition 4.9. *Let $(M^n, g(t))_{t\in(-\infty,0]}$ be a complete ancient solution to the Ricci flow such that either:*

(a) there exists $C_0 > 0$ *such that* $\sup_{M\times(-\infty,0]} |Rm| \leq C_0$ *and* $\inf_{M\times(-\infty,0]} \mathrm{Vol}_{g(\cdot)}(B_{g(\cdot)}(\cdot,1)) \geq C_0^{-1}$*, or*

(b) M *has nonnegative curvature operator and there exists* $C_0 > 0$ *such that* $\sup_M R(\cdot,0) \leq C_0$ *and* $\inf_M \mathrm{Vol}_{g(0)}(B_{g(0)}(\cdot,1)) \geq C_0^{-1}$.

Then $\bar{N}(x,t)$ *is independent of* x.

Proof. It suffices to show that if there exists $x \in M$ such that $\bar{N}(x,t) > -\infty$, then for any $y \in M$, $\bar{N}(y,t) = \bar{N}(x,t)$. By Lemma 4.8, in either case, it holds that for all $\varepsilon > 0$, there exists $\eta_0 \ll t$ such that $|\nabla_y N_{(y,t)}(\eta)|_{g(t)} < \varepsilon$ for all $y \in M$ and $\eta < \eta_0$. Hence $|N_{(y,t)}(\eta) - N_{(x,t)}(\eta)| \leq \varepsilon dist_{g(t)}(x,y)$ for all $y \in M$ and $\eta < \eta_0$. Taking $\eta \to -\infty$, we have $|\bar{N}(y,t) - \bar{N}(x,t)| \leq \varepsilon dist_{g(t)}(x,y)$. The proposition follows from taking $\varepsilon \to 0$. □

4.2.2 A mean-value-type inequality

In this subsection we prove (4.2). This is indeed a surprising inequality, and the main technique we use is Hein and Naber's logarithmic Sobolev inequality (Theorem 2.10(2)).

Proposition 4.10. *Let* $(M^n, g(t))_{t\in(-\infty,0]}$ *be a complete ancient solution to the Ricci flow such that either:*

(a) there exists $C_0 > 0$ *such that* $\sup_{M\times(-\infty,0]} |Rm| \leq C_0$ *and* $\inf_{M\times(-\infty,0]} \mathrm{Vol}_{g(\cdot)}(B_{g(\cdot)}(\cdot,1)) \geq C_0^{-1}$*, or*

(b) M has nonnegative curvature operator and there exists $C_0 > 0$ *such that* $\sup_M R(\cdot, 0) \le C_0$ *and* $\inf_M \mathrm{Vol}_{g(0)}(B_{g(0)}(\cdot, 1)) \ge C_0^{-1}$.

Then for all $x, z \in M$ *and* $s < t \le 0$ *it holds that*

$$\bar{N}(x,t) \le \int_M \bar{N}(\cdot, s) d\nu^s_{(x,t)} \equiv \bar{N}(z,s). \tag{4.19}$$

The last equality above is due to Proposition 4.9. Hence $\bar{N}(x,t)$ *is independent of* x *and is monotonically decreasing in* t.

Proof. Let $z \in M$ and fix a $\eta < 0$ such that $\eta + 1 \ll s < t \le 0$. Applying Theorem 2.10(2) to

$$\begin{aligned} u(\cdot) &= H(z, \eta | \cdot, s) > 0, \\ d\nu &= d\nu^s_{(x,t)} = H(\cdot, s | x, t) dg_s, \end{aligned}$$

we have

$$\begin{aligned} & -H(z,\eta|x,t) \log H(z,\eta|x,t) \\ = & -\left(\int_M H(z,\eta|\cdot,s) d\nu^s_{(x,t)}\right) \log \left(\int_M H(z,\eta|\cdot,s) d\nu^s_{(x,t)}\right) \\ \le & -\int_M H(z,\eta|\cdot,s) \log \big(H(z,\eta|\cdot,s)\big) d\nu^s_{(x,t)} \\ & +(t-s) \int_M \frac{|\nabla_y H(z,\eta|y,s)|^2}{H(z,\eta|y,s)} d\nu^s_{(x,t)}(y), \end{aligned}$$

where we have used the fact that

$$\int_M H(z,\eta|\cdot,s)d\nu^s_{(x,t)} = \int_M H(z,\eta|\cdot,s)H(\cdot,s|x,t)dg_s = H(z,\eta|x,t).$$

Integrating against $dg_\eta(z)$ and using definition (2.21), we have

$$\begin{aligned}
& N_{(x,t)}(\eta) + \frac{n}{2} + \frac{n}{2}\log(4\pi(t-\eta)) \\
\leq\; & \int_M N_{(y,s)}(\eta)d\nu^s_{(x,t)}(y) + \frac{n}{2} + \frac{n}{2}\log(4\pi(s-\eta)) \\
& +(t-s)\int_M\int_M \frac{|\nabla_y H(z,\eta|y,s)|^2}{H(z,\eta|y,s)}d\nu^s_{(x,t)}(y)dg_\eta(z),
\end{aligned} \tag{4.20}$$

where the last term needs to be estimated. By Theorem 2.8 we have

$$\frac{|\nabla_y H(z,\eta|y,s)|^2}{H(z,\eta|y,s)} \leq \frac{1}{s-\eta-1}\Big(\log M_{\eta,\eta+1} - \log H(z,\eta|y,s)\Big)H(z,\eta|y,s).$$

Inserting this inequality into (4.20), we have

$$\begin{aligned}
N_{(x,t)}(\eta) \;\leq\; & \int_M N_{(\cdot,s)}(\eta)d\nu^s_{(x,t)} + \frac{n}{2}\log\frac{s-\eta}{t-\eta} + \frac{t-s}{s-\eta-1}\log M_{\eta,\eta+1} \\
& -\frac{t-s}{s-\eta-1}\int_M\int_M H(z,\eta|y,s)\log\big(H(z,\eta|y,s)\big)d\nu^s_{(x,t)}(y)dg_\eta(z) \\
=\; & \left(1+\frac{t-s}{s-\eta-1}\right)\int_M N_{(\cdot,s)}(\eta)d\nu^s_{(x,t)} + \frac{t-s}{s-\eta-1}\log M_{\eta,\eta+1} \\
& +\frac{t-s}{s-\eta-1}\left(\frac{n}{2}\log(4\pi(s-\eta)) + \frac{n}{2}\right) + \frac{n}{2}\log\frac{s-\eta}{t-\eta}.
\end{aligned} \tag{4.21}$$

By (4.13) and (4.15) we may observe that

$$\lim_{\eta\to-\infty}\frac{t-s}{s-\eta-1}\log M_{\eta,\eta+1}=0$$

in either case (a) or case (b). Therefore, by taking $\eta\to-\infty$ in formula (4.21), we have

$$\bar{N}(x,t)\le\lim_{\eta\to-\infty}\int_M N_{(\cdot,s)}(\eta)d\nu^s_{(x,t)}.$$

Notice that $\{N_{(y,s)}(\eta)\}_{\eta\in(-\infty,s-1]}$ is a family of nonpositive functions of y that are monotonic in η; the conclusion follows from Fatou's lemma.

□

To conclude this section, we remark that Proposition 4.9 holds also for the asymptotic entropy, simply because the asymptotic entropy coincides with the asymptotic Nash entropy.

Corollary 4.11. *Let $(M,g(t))_{t\in(-\infty,0]}$ be a complete ancient solution to the Ricci flow with bounded geometry on every compact time interval. Then*

$$\bar{W}(x,0)=\bar{N}(x,0),$$

for all $x\in M$. In particular, Proposition 4.10 holds for the asymptotic entropy $\bar{W}(x,t)$.

Proof. By Lemma 2.13(2)(3)(4), for any $\varepsilon > 0$ and $\eta < 0$, we have

$$\begin{aligned} W_{(x,0)}(\eta) \le N_{(x,0)}(\eta) &= -\frac{1}{\eta}\int_{\eta}^{0} W_{(x,0)}(s)ds \le -\frac{1}{\eta}\int_{\eta}^{\varepsilon\eta} W_{(x,0)}(s)ds \\ &\le -\frac{1}{\eta}\int_{\eta}^{\varepsilon\eta} W_{(x,0)}(\varepsilon\eta)ds = (1-\varepsilon)W_{(x,0)}(\varepsilon\eta). \end{aligned}$$

The conclusion follows from first taking $\eta \to -\infty$ and then $\varepsilon \to 0$. □

4.3 Proof of the gap and noncollapsing theorems

In this section, we present two propositions which imply Theorem 1.3 immediately via Lemma 2.13 that $N_{(x,t)}(\eta) \ge W_{(x,t)}(\eta)$ and Corollary 4.11 that $\bar{N}(x,t) = \bar{W}(x,t)$.

Proposition 4.12. *Let $(M^n, g(t))_{t\in(-\infty,0]}$ be a complete ancient solution to the Ricci flow with nonnegative curvature operator. Assume*

$$\sup_{M\times(-\infty,0]} |Rm| < \infty \quad \textit{and} \quad \inf_{M} \mathrm{Vol}_{g(0)}(B_{g(0)}(\cdot,1)) > 0.$$

Let $\bar{N}$ be the asymptotic Nash entropy defined in (2.22). If there exists a point (x_0, t_0) such that $\bar{W}(x_0,t_0) \equiv \bar{N}(x_0,t_0) \ge -\beta$ for some $\beta < \infty$, then there exists a $\kappa > 0$ such that $(M, g(t))_{t\in(-\infty,t_0]}$ is κ-noncollapsed on all scales, where κ depends only on β and the dimension n.

Proof. Under the assumption of this proposition, we may apply part (2) of Proposition

4.10 to conclude that $\bar{N}(x,t) \geq -\beta$, for all $t \leq t_0$, $x \in M$. Then we may apply Theorem 4.1 to these points. So there exists a $\kappa = \kappa(n,\beta) > 0$ such that for any $r > 0$, if $|Rm| \leq r^{-2}$ on $B_{g(t)}(x,r) \times [t-r^2, t]$, then $\mathrm{Vol}_{g(t)}(B_{g(t)}(x,r)) \geq \kappa r^n$. That is, $(M, g(t))_{t\in(-\infty,t_0]}$ is κ-noncollapsed on all scales. □

The next corollary follows similarly.

Corollary 4.13. *Let $(M^n, g(t))_{t\in(-\infty,0]}$ be a complete ancient solution to the Ricci flow. Assume*

$$\sup_{M\times(-\infty,0]} |Rm| < \infty \quad \text{and} \quad \inf_{M\times(-\infty,0]} \mathrm{Vol}_{g(\cdot)}(B_{g(\cdot)}(\cdot,1)) > 0.$$

If there exists a point (x_0,t_0) such that $\bar{W}(x_0,t_0) \equiv \bar{N}(x_0,t_0) \geq -\beta > -\infty$, then there exists a $\kappa > 0$ such that $(M, g(t))_{t\in(-\infty,t_0]}$ is κ-noncollapsed on all scales, where κ depends only on β and the dimension n.

Proof. The proof is similar to that of the previous proposition, where now part (2) is replaced by part (1) in Proposition 4.10. □

To prove the next proposition, one may apply a similar technique as used by Yokota [37]. He implemented a point picking method on the ancient solution to construct a contradicting sequence, which implies an ε-regularity theorem; his gap theorem follows from that. With all the tools we have developed by far, we are able to prove this theorem in a slightly different—if not significantly easier—way.

Proposition 4.14. *There exists $\varepsilon > 0$ depending only on the dimension n such that the following holds. Let $(M^n, g(t))_{t\in(-\infty,0]}$ be a complete ancient noncollapsed solution to the Ricci flow such that $\sup_{M\times(-\infty,0]} |Rm| < \infty$. If there exists $(x,t) \in M \times (-\infty, 0]$ such that $\bar{N}(x,t) \geq -\varepsilon$, then $(M, g(t))$ is a Gaussian shrinker.*

Proof. Suppose $(M, g(t))$ is flat but not Euclidean. Then the noncollapsing assumption implies maximum volume growth. By Peter Li (see Corollary 16.3 of [24] for instance), M is a finite quotient of Euclidean space. However, any finite group action $\Gamma\times\mathbb{R}^n \to \mathbb{R}^n$ has a fixed point, which is a contradiction since M is smooth. To see this, one may take any $id \neq \gamma \in \Gamma$ and $x \in \mathbb{R}^n$; then $x + \gamma x + ... + \gamma^{|\gamma|-1}x$ is a fixed point, where $|\gamma| > 1$ is the order of γ. So, henceforth we assume $(M, g(t))$ is non-flat.

Suppose the theorem is not true. Then we may find a sequence

$$\{(M_k, g_k(t), (x_k, 0))_{t\in(-\infty,0]}\}_{k=1}^{\infty},$$

such that the $(M_k, g_k(t))$ are non-flat and noncollapsed on all scales,

$$\sup_{M_k\times(-\infty,0]} |Rm_k| < \infty, \quad \text{and} \quad \bar{N}(x_k, 0) \geq -\frac{1}{k}.$$

By the noncollapsing assumption, $\inf_{M_k\times(-\infty,0]} \mathrm{Vol}_{g(\cdot)}(B_{g(\cdot)}(\cdot, 1)) > 0$ for each k, and it follows from Corollary 4.13 that the sequence of ancient solutions are κ-noncollapsed with respect to a uniform κ.

For each k, we pick $(\tilde{x}_k, \tilde{t}_k) \in M_k \times (-\infty, 0]$ such that

$$|Rm_k|(\tilde{x}_k, \tilde{t}_k) \geq \frac{1}{2} \sup_{M_k \times (-\infty,0]} |Rm_k|.$$

By parabolic rescaling the ancient flows centered at $(\tilde{x}_k, \tilde{t}_k)$ by the factors $|Rm_k|(\tilde{x}_k, \tilde{t}_k)$ with time shifts of $\tilde{t}_k$ to 0, we obtain another sequence

$$\{(\tilde{M}_k, \tilde{g}_k(t), (\tilde{x}_k, 0))_{t \in (-\infty,0]}\}_{k=1}^{\infty}$$

which is κ-noncollapsed with respect to a universal κ, satisfying

$$\sup_{\tilde{M}_k \times (-\infty,0]} |\tilde{Rm}_k| < 2, \quad |\tilde{Rm}_k|(\tilde{x}_k, 0) = 1, \text{ and } \bar{N}(\tilde{x}_k, 0) \geq -\frac{1}{k}.$$

Here we have used the invariance of the Nash entropy under parabolic rescaling as well as Proposition 4.10 to conclude that $\bar{N}(\tilde{x}_k, 0) \geq -\frac{1}{k}$.

By [19], we can extract a subsequence of $\{(\tilde{M}_k, \tilde{g}_k(t), (\tilde{x}_k, 0))_{t \in (-\infty,0]}\}_{k=1}^{\infty}$ converging in the pointed Cheeger-Gromov sense to an ancient κ-noncollapsed Ricci flow $(M_\infty, g_\infty(t), (x_\infty, 0))_{t \in (-\infty,0]}$ with $|Rm_\infty|(x_\infty, 0) = 1$, where by [26] the conjugate heat kernels centered at $(\tilde{x}_k, 0)$ converge to the conjugate heat kernel centered at $(x_\infty, 0)$. By the same argument as shown in the proof of 4.1, $N_{(x_\infty,0)}(\eta) \equiv 0$. Moreover, since the Nash entropy is the time average of pointed entropies (see Lemma 2.13), $W_{(x_\infty,0)}(\eta) \equiv 0$. It follows that $(M_\infty, g_\infty(t))$ is a Gaussian shrinker, contradicting $|Rm_\infty|(x_\infty, 0) = 1$. □

The result of chapter 4 is my own work, currently accepted for publication on Communications in Analysis and Geometry.

Chapter 5

Equivalence between bounded entropy and noncollapsing

In this chapter we prove Theorem 1.7, Corollary 1.8, and Corollary 1.9. The main part of this chapter will be devoted to the necessity part in Theorem 1.7. In other words, let us consider an ancient solution to the Ricci flow with bounded nonnegative curvature operator $(M, g(t))_{t\in[0,\infty)}$, such that $g(t)$ is κ-noncollapsed on all scales for some $\kappa > 0$. We proceed to prove that for all $(x, t) \in M \times (-\infty, 0]$ and for all $s < t$, $|W_{(x,t)}(s)|$ is bounded by a number depending only on κ and n.

The main idea of the proof is to apply Theorem 3.14 and the monotonicity formula Lemma 2.13(4). Indeed, because of Lemma 2.13(4) and Lemma 2.12, we need only to show that for all $(x, t) \in M \times (-\infty, 0]$, $\bar{W}(x, t)$ is bounded from below by a number depending only on κ and n. On the other hand, Theorem 3.14 implies that when time approaches negative infinity, the Ricci flow looks like the canonical form of a shrinker. If we can show that $W_{(x,t)}(s)$ also converges to a corresponding quantity on the asymptotic shrinker, and this quantity depends only on n and κ, we would establish the necessity part of Theorem 1.7.

5.1 Notations

5.1.1 Ancient solution and conjugate heat kernel

Now let us simplify the notation for the sake of convenience. Throughout this whole chapter, unless otherwise specified, we will use the following notation. We will use $(M^n, g(\tau))_{\tau\in[0,\infty)}$ to denote the ancient solution in question, such that τ stands for the backward time. That is to say, each slice $(M^n, g(\tau))$ has bounded nonnegative curvature operator, and is κ-noncollapsed on all scales. Let us fix a base point $(x_0, 0)$ for the conjugate heat kernel and the reduced distance—seeing that it is always possible to shift time and make the base time 0. Let

$$u(x,\tau) = \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-f(x,\tau)} := H(x, -\tau|x_0, 0)$$

be the conjugate heat kernel based at $(x_0, 0)$. Or in other words

$$\Box^* u := \left(\frac{\partial}{\partial\tau} - \Delta + R\right) u$$

and

$$\lim_{\tau \to 0+} u(\cdot, \tau) = \delta_{x_0}.$$

Finally, let us use

$$l(x, \tau) := l_{(x_0, 0)}(x, \tau)$$

to denote the reduced distance based at $(x_0, 0)$.

5.1.2 Parabolic scaling

Let us fix an arbitrary sequence of positive numbers $\{\tau_k\}_{k=1}^{\infty}$ such that

$$\tau_k \nearrow \infty, \quad \text{as} \quad k \nearrow \infty.$$

According to Lemma 3.7, we can find a sequence of points $\{p_k\}_{k=1}^{\infty}$ in the manifold M such that

$$0 \leq l(p_k, \tau_k) \leq \frac{n}{2},$$

where the first inequality above is obvious from the definition of l and from the fact that $R \geq 0$ (see Chen [10]). For the sake of convenience we define the following notation for the parabolic scaling:

$$\begin{aligned}
g_k(\tau) &= \tau_k^{-1} g(\tau\tau_k), \\
Rm_k &= Rm_{g_k}, \\
R_k &= R_{g_k}, \\
l_k(p,\tau) &= l(p,\tau\tau_k), \\
u_k(p,\tau) &= \tau_k^{\frac{n}{2}} u(p,\tau\tau_k) = \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-f_k(p,\tau)}, \\
f_k(p,\tau) &= f(p,\tau\tau_k).
\end{aligned}$$

One may easily check by the definition that, u_k is the fundamental solution to the conjugate heat equation, and l_k is reduced distance, both of backward Ricci flow $g_k(\tau)$ and based at $(x_0, 0)$. Because of this property of l_k, we have that (3.33)—(3.36) also hold for each l_k:

$$|\nabla l_k|^2_{g_k} + R_k \leq \frac{Cl_k}{\tau}, \tag{5.1}$$

$$\frac{\partial l_k}{\partial \tau} + \frac{Cl_k}{\tau} \geq R_k, \tag{5.2}$$

$$2\frac{\partial l_k}{\partial \tau} + \frac{l_k}{\tau} \leq R_k, \tag{5.3}$$

on $M \times (0, \infty)$ and for all k, and

$$-2l_k(y,\tau) - C + \frac{c}{\tau} dist^2_{g_k(\tau)}(x,y) \leq l_k(x,\tau) \leq 2l_k(y,\tau) + \frac{C}{\tau} dist^2_{g_k(\tau)}(x,y), \tag{5.4}$$

for all $(x, y) \in M$ and for all $\tau \in (0, \infty)$. The constants c and C in (5.1)—(5.4) depend only on the dimension n.

For most of the time we will use C and c to denote positive estimation constants **depending only on $\kappa > 0$ and the dimension n**, which could be different from line to line. We use the capital letter C to denote those constants that are intuitively large, and the lower case letter c to denote those constants that are intuitively small.

5.2 Estimates of the conjugate heat kernel

5.2.1 Pointwise lower bound

We the following lower bound of u_k is simply a consequence of the fact that the integrand of the reduced volume is a subsolution to the conjugate heat equation.

Lemma 5.1 (The lower bound)**.** *There exists a constant $C < \infty$ depending only on $\kappa > 0$ and the dimension n, such that*

$$f_k(x, \tau) \leq C + C dist_{g_k(\tau)}(p_k, x)^2, \tag{5.5}$$

for all $x \in M$ and $\tau \in [\frac{1}{2}, 2]$. In particular, $u_k(x, \tau) > c$ for all $x \in B_{g_k(\tau)}(p_k, 1)$ and $\tau \in [\frac{1}{2}, 2]$, where $c > 0$ is a constant.

Proof. First, from Lemma 3.9 we have

$$\frac{1}{(4\pi\tau)^{\frac{n}{2}}}e^{-l_k(x,\tau)} \le \frac{1}{(4\pi\tau)^{\frac{n}{2}}}e^{-f_k(x,\tau)},$$

for all $(x,\tau) \in M \times (0,\infty)$, or equivalently

$$f_k(x,\tau) \le l_k(x,\tau), \tag{5.6}$$

for all $(x,\tau) \in M \times (0,\infty)$.

On the other hand, (5.1)—(5.3) implies

$$-\frac{Cl_k}{\tau} \le \frac{\partial l_k}{\partial \tau} \le \frac{Cl_k}{\tau}.$$

Integrating this on $[\frac{1}{2}, 2]$ and using the fact that $l_k(p_k, 1) = l(p_k, \tau_k) \in [0, \frac{n}{2}]$, we have (note that l_k is always nonnegative because of its definition and the fact $R_k \ge 0$)

$$0 \le l_k(p_k, \tau) \le C,$$

for all $\tau \in [\frac{1}{2}, 2]$. Combining this with (5.4) we have

$$-C + c\,dist_{g_k(\tau)}(p_k, x) \le l_k(x,\tau) \le C + C\,dist_{g_k(\tau)}(p_k, x), \tag{5.7}$$

for all $(x,\tau) \in M \times [\frac{1}{2}, 2]$. The Lemma then follows from (5.6) and (5.7).

$\square$

5.2.2 Pointwise upper bound

Next we apply Hein-Naber's gaussian concentration inequality to obtain an integral estimate, and furthermore pointwise estimate of u_k. Note that the result of this subsection is not yet the final form. We will obtain the nice gaussian upper bound for u_k in the next subsection.

Proposition 5.2. *There exists $c > 0$ and $C < \infty$ depending only on κ and the dimension n such that*

$$f_k(x,\tau) \geq -C - C\log\left(\max\{R_k(x,\tau),1\}\right) + c\,dist_{g_k(\tau)}(x,p_k)^2, \tag{5.8}$$

for all $x \in M$ and $\tau \in [\frac{3}{4}, 2]$.

Proof. First of all, we use Theorem 2.11 to obtain an integral estimate for u_k.

Claim 1. *There exists $C < \infty$ depending only on κ and the dimension n such that for all $p \in M$, $\tau \in [\frac{1}{2}, 2]$, and for all $r \in (0,1]$, it holds that*

$$\int_{B_{g_k(\tau)}(x,r)} u_k(\cdot,\tau)dg_k(\tau) \leq C\exp\left(-\frac{1}{32}dist_{g_k(\tau)}(p_k,x)^2\right). \tag{5.9}$$

Proof. Let us apply Theorem 2.11 to the sets $A = B_{g_k(\tau)}(p_k,1)$ and $B = B_{g_k(\tau)}(x,r)$. By Lemma 5.1 we have $u_k(y,\tau) \geq c$ for all $y \in B_{g_k(\tau)}(p_k,1)$. By (5.1) and (5.7) we have

$$R_k(y,\tau) \le C \quad \text{for} \quad y \in B_{g_k(\tau)}(p_k, 1),$$

hence by the noncollapsing assumption we have

$$\mathrm{Vol}_{g_k(\tau)}(B_{g_k(\tau)}(p_k, 1)) > c.$$

Therefore,

$$\nu_\tau(A) = \int_{B_{g_k(\tau)}(p_k,1)} u_k(\cdot,\tau) dg_k(\tau) > c$$

and (5.9) follows immediately. □

Next, we will extend inequality (5.9) to a space-time cube, so as to apply the parabolic mean value inequality.

Claim 2. *There exists $c_2 > 0$ depending only on κ and the dimension n such that for any $(x,\tau) \in M \times [\frac{3}{4}, 2]$ and $r := \min\{1, R_k(x,\tau)^{-\frac{1}{2}}\}$ it holds that.*

$$\int_{\tau-(c_2 r)^2}^{\tau} \int_{B_{g_k(\tau)}(x,c_2 r)} u_k(\cdot, s) dg_k(\tau) ds \le C_2 r^2 \exp\left(-\frac{1}{64} dist_{g_k(\tau)}(p_k, x)^2\right). \quad (5.10)$$

Proof. We fix $\tau \in [\frac{3}{4}, 2]$ and $x \in M$, and let r be as defined in the statement of the claim. By applying (3.41) and (3.42) we can find $c_1 \in (0, \frac{1}{4}]$ and $C_1 < \infty$ such that

$$R_k(y,s) \leq C_1 r^{-2}, \quad \text{for all} \quad (y,s) \in \bigcup_{\bar{\tau} \in [\tau - c_1 r^2, \tau]} B_{g_k(\bar{\tau})}(x,r). \tag{5.11}$$

Moreover, by applying Theorem 3.4 to the points x and p_k with $K = C_1 r^{-2}$ and $r_0 = r$, we have that

$$\frac{d}{ds} dist_{g_k(s)}(p_k, x) \leq 10nC_1 r^{-2},$$

for all $s \in [\tau - c_1 r^2, \tau]$ and whenever $dist_{g_k(s)}(p_k, x) \geq 2r$. Integrating this inequality from s to τ we have

$$dist_{g_k(s)}(p_k, x) \geq dist_{g_k(\tau)}(p_k, x) - 10nC_1\sqrt{c_1}, \tag{5.12}$$

for all $s \in [\tau - c_1 r^2, \tau]$. Integrating (5.9) in time and taking (5.12) into account, we obtain the following integral estimate

$$\int_{\tau - c_1 r^2}^{\tau} \int_{B_{g_k(s)}(x,r)} u_k(\cdot, s) dg_k(s) ds \leq c_1 C r^2 \exp\left(-\frac{1}{64} dist_{g_k(\tau)}(p_k, x)^2\right). \tag{5.13}$$

Because of (5.11) we can apply a standard local distortion argument, and choose $c_2 > 0$ small enough and $C_2 < \infty$ large enough, depending on all the previous estimation constants, such that

$$
\begin{aligned}
B_{g_k(\tau)}(x, c_2 r) \subset B_{g_k(s)}(x, r) \quad &, \quad \text{for all } s \in [\tau - (c_2 r)^2, \tau] \subset [\tau - c_1 r^2, \tau], \\
R_k(y, s) \le C_2 r^{-2} \quad &, \quad \text{for all } (y, s) \in B_{g_k(\tau)}(x, c_2 r) \times [\tau - (c_2 r)^2, \tau], \\
C_2^{-1} g_k(s) \le g_k(\tau) \le C_2 g_k(s) \quad &, \quad \text{in } B_{g_k(\tau)}(x, c_2 r) \times [\tau - (c_2 r)^2, \tau].
\end{aligned}
$$

Given this, we may obtain (5.10) from (5.13). □

Now we are ready to apply the parabolic mean value inequality Theorem 2.1. To keep track of the curvature condition, we perform the parabolic scaling in our scenario. Let $\tilde{u}(s) = u_k(\tau + sr^2)$ and $\tilde{g}(s) = r^{-2} g_k(\tau + sr^2)$, then we have

$$
\begin{aligned}
&\int_{-(c_2)^2}^{0} \int_{B_{\tilde{g}(0)}(x, c_2)} \tilde{u}(\cdot, s) d\tilde{g}(0) ds \\
&= \frac{1}{r^{n+2}} \int_{\tau - (c_2 r)^2}^{\tau} \int_{B_{g_k(\tau)}(x, c_2 r)} u_k(\cdot, s) dg_k(\tau) ds \le \frac{C_2}{r^n} \exp\left(-\frac{1}{64} dist_{g_k(\tau)}(p_k, x)^2 \right),
\end{aligned}
$$

with the curvature bound

$$
\tilde{R}(y, s) \le C_2, \text{ for all } (y, s) \in B_{\tilde{g}(0)}(x, c_2) \times [-(c_2)^2, 0].
$$

On the other hand, the κ-noncollapsing assumption implies

$$
\mathrm{Vol}_{\tilde{g}(0)} \left(B_{\tilde{g}(0)} \left(x, \frac{c_2}{4} \right) \right) \ge c > 0.
$$

Therefore we have

$$
\begin{aligned}
u_k(x,\tau) &= \tilde{u}(x,0) \\
&\leq \frac{C}{(\frac{c_2}{4})^2 \operatorname{Vol}_{\tilde{g}(0)}\left(B_{\tilde{g}(0)}\left(x,\frac{c_2}{4}\right)\right)} \cdot \frac{C_2}{r^n} \exp\left(-\frac{1}{64} dist_{g_k(\tau)}(p_k,x)^2\right) \\
&\leq \frac{C}{r^n} \exp\left(-\frac{1}{64} dist_{g_k(\tau)}(p_k,x)^2\right),
\end{aligned}
$$

Taking into account the definition of $r = \min\{1, R_k(x,\tau)^{-\frac{1}{2}}\}$ we obtain the conclusion.

□

5.2.3 Growth estimates of derivatives and curvatures

Once we have the above pointwise estimate for the conjugate heat kernel, we can apply the curvature growth condition given by (5.1) and (5.7) to deal with the $\log(\max\{R_k(x,\tau),1\})$ term in (5.8), and furthermore obtain a gradient growth estimate by using a localized gradient estimate for the conjugate heat equation.

Proposition 5.3. *The following holds for every element of the sequence of backward Ricci flows* $\{(M, g_k(\tau), (p_k,1))_{\tau\in[\frac{1}{2},2]}\}_{k=1}^{\infty}$.

(1) There exists $C<\infty$ *depending only on* κ *and the dimension* n*, such that*

$$
R_k(x,\tau) \leq C + C dist_{g_k(\tau)}(p_k,x)^2, \tag{5.14}
$$

for all $x \in M$ *and* $\tau \in [\frac{1}{2}, 2]$.

(2) There exists $C < \infty$ and $c > 0$ depending only on κ and the dimension n, such that

$$f_k(x,\tau) \geq -C + c dist_{g_k(\tau)}(p_k,x)^2, \tag{5.15}$$

for all $x \in M$ and $\tau \in [\frac{3}{4}, 2]$.

(3) There exists $C < \infty$ depending only on κ and the dimension n, such that

$$|\nabla f_k(x,\tau)|^2 \leq C + C dist_{g_k(\tau)}(p_k,x)^7, \tag{5.16}$$

for all $x \in M$ and $\tau \in [1,2]$.

In particular, every constant in these estimates is independent of k.

Proof. (1) This follow directly from (5.1) and (5.7).

(2) Inserting (5.14) into (5.8) we have that

$$\begin{aligned} f_k(x,\tau) &\geq -C - C\log\left(C + C dist_{g_k(\tau)}(p_k,x)^2\right) + c dist_{g_k(\tau)}(x,p_k)^2 \\ &\geq -C + \frac{c}{2} dist_{g_k(\tau)}(x,p_k)^2 \\ &\quad + \left(\frac{c}{2} dist_{g_k(\tau)}(x,p_k)^2 - C\log\left(C + C dist_{g_k(\tau)}(p_k,x)^2\right)\right). \end{aligned}$$

The conclusion of (b) follows from the simple fact that the function

$$\phi(s) := \frac{c}{2}s^2 - C\log(C + Cs^2), \;\; s \geq 0$$

is bounded from below, and its lower bound depends only on c and C.

(3) We apply Theorem 10 of [14], where they have shown the following: suppose that on $\Omega(2A) = \bigcup_{\tau\in[0,\bar{t}]} B_{g(\tau)}(q, 2A)$ we have the bounds

$$|Ric| \leq K_1 \text{ and } |\nabla R| \leq K_2.$$

Let u be a positive solution to the conjugate heat equation with $u \leq J$ on $\Omega(2A)$, then it holds that

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} &\leq \left(1 + \log \frac{J}{u}\right)^2 \\ &\quad \times \left(\frac{1}{\tau} + C_1 K_1 + \sqrt{K_2} + K_2 + \frac{C_1\sqrt{K_2}A\coth(\sqrt{K_2}A) + C_2}{A^2}\right)\end{aligned}$$

on $\Omega(A)$ and for $\tau \in (0, \bar{t}\,]$, where C_1 and C_2 are constants. In order to apply this theorem to each u_k on $[\frac{3}{4}, 2]$, we need to check every bound that is needed. We fix an arbitrary $A > 1$ and consider $\Omega(2A) = \bigcup_{\tau\in[\frac{3}{4},2]} B_{g_k(\tau)}(p_k, 2A)$. By (5.15) we have a uniform upper bound J for u_k. By (5.14) and (3.42), we may take $K_1 = C+CA^2$ and $K_2 = C+CA^3$. Moreover, (5.5) provides an upper bound for $\log\frac{J}{u} \leq C+CA^2$. Hence we have

$$\begin{aligned}\frac{|\nabla u_k|^2}{u_k^2}(x,\tau) &\leq C(1+A^2)^2 \\ &\quad \times \left(1 + A^{\frac{3}{2}} + A^2 + A^3 + \frac{A(1+A^{\frac{3}{2}})\coth(CA(1+A^{\frac{3}{2}}))+1}{A^2}\right)\end{aligned}$$

for all $x \in B_{g_k(\tau)}(p_k, A)$ and $\tau \in [1,2]$, whence follows the result.

□

Proof of Corollary 1.9. Notice that in all the estimates above, we may indeed let τ_k be any positive number. Taking $\tau = 2$ in the estimates (5.5) and (5.15), and scaling them back by the factor τ_k, we obtain the conclusion. □

5.3 Proof of the main theorems

We continue using the definitions and notations introduced in the previous section. In Proposition 5.3 we have derived a uniform upper bound for the u_k's on the interval $[1,2]$. Combining (3.42) and (5.14) we obtain uniform growth estimates for all the derivatives of the curvatures on $[1,2]$. By standard parabolic regularity theory, we have locally uniform estimates for all the derivatives of each u_k—and it is easy to see that these growth rates are at most polynomial. By Theorem 3.14, after passing to a subsequence, $\{(M, g_k(\tau), (p_k, 1), l_k(\tau))_{\tau\in[1,2]}\}$ converges in the pointed smooth Cheeger-Gromov sense to the canonical form of a nonnegatively curved asymptotic shrinker $\big(M_\infty, g_\infty(\tau), (p_\infty, 1), l_\infty(\tau)\big)_{\tau\in[1,2]}$. At the same time, because of the uniform growth estimates for all derivatives, $\{u_k\}_{k=1}^\infty$ converges locally smoothly to a solution

$$u_\infty := \frac{1}{(4\pi\tau)^{-\frac{n}{2}}} e^{-f_\infty}$$

to the conjugate heat equation on (M_∞, g_∞), with $f_k \to f_\infty$ locally smoothly. Here we would like to remark that the parabolic regularity theory actually ensures the smooth convergence of $\{u_k\}_{k=1}^\infty$ on any space-time compact subset of $M \times (1,2]$, however, it is not hard to extend all those estimates we obtained in the previous section to an interval larger than $[1,2]$, say $[\frac{7}{8}, 4]$. Therefore, to keep our notations concise, we simply assume that the smooth convergence of $\{u_k\}_{k=1}^\infty$ happens on any compact subset of $M \times [1,2]$. The quadratic upper and lower bounds (5.5) and (5.15) of f_k, and the derivative growth bound (5.16) carry to f_∞ on $M_\infty \times [1,2]$:

$$-C + c\,dist_{g_\infty(\tau)}(p_\infty, x)^2 \leq f_\infty(x,\tau) \leq C + C\,dist_{g_\infty(\tau)}(p_\infty, x)^2, \tag{5.17}$$

$$|\nabla f_\infty(x,\tau)|^2 \leq C + C\,dist_{g_\infty(\tau)}(p_\infty, x)^7, \tag{5.18}$$

for all $x \in M_\infty$ and $\tau \in [1,2]$. In particular, (5.17) implies u_∞ is not a zero solution. Furthermore, we observe the following.

Lemma 5.4. *The asymptotic shrinker $(M_\infty, g_\infty, l_\infty)$ has bounded curvature and constant scalar curvature.*

Proof. By Corollary 4 in Munteanu and Wang [30], we have that the universal cover of the asymptotic shrinker $(M_\infty, g_\infty, l_\infty)$ must either be compact or split as a Euclidean space and a compact symmetric space. Hence, the asymptotic shrinker has bounded curvature and parallel Riemann curvature tensor. □

Remark: Indeed, given (5.17) and the fact that the asymptotic shrinker has bounded curvature, we can use derivative estimates to obtain a result much better than (5.18). However, (5.18) is sufficient for our application.

We denote

$$\begin{aligned} W_k(\tau) &= \int_M \Big(\tau(|\nabla f_k|^2 + R_k) + f_k - n\Big)\frac{1}{(4\pi\tau)^{\frac{n}{2}}}e^{-f_k}dg_k(\tau) \\ &= \int_M \Big(\tau(2\Delta f_k - |\nabla f_k|^2 + R_k) + f_k - n\Big)\frac{1}{(4\pi\tau)^{\frac{n}{2}}}e^{-f_k}dg_k(\tau), \\ W_\infty(\tau) &= \int_{M_\infty} \Big(\tau(|\nabla f_\infty|^2 + R_\infty) + f_\infty - n\Big)\frac{1}{(4\pi\tau)^{\frac{n}{2}}}e^{-f_\infty}dg_\infty(\tau). \\ &= \int_{M_\infty} \Big(\tau(2\Delta f_\infty - |\nabla f_\infty|^2 + R_\infty) + f_\infty - n\Big)\frac{1}{(4\pi\tau)^{\frac{n}{2}}}e^{-f_\infty}dg_\infty(\tau). \end{aligned}$$

The integrand of both $W_k(\tau)$ and $W_\infty(\tau)$ are absolutely integrable, because of Proposition 5.3, (5.17), (5.18), Lemma 5.4, and the Bishop-Gromov comparison theorem.

5.3.1 No loss of entropy

We first show that there is no loss of entropy in the process of obtaining the asymptotic shrinker.

Proposition 5.5. *$W_\infty(\tau) > -\infty$ is a constant, and is equal to $\lim_{\tau\to\infty} W_{x_0}(\tau) = \bar{W}(x_0)$. Moreover, f_∞ is a potential function of the asymptotic Ricci shrinker.*

We split the proof into several lemmas, first we show that $\liminf_{k\to\infty} W_k(\tau) \geq W_\infty(\tau) > -\infty$, and then we show that $\limsup_{k\to\infty} W_k(\tau) \leq W_\infty(\tau)$, finally we show that $W_\infty(\tau)$ is a constant.

Lemma 5.6. *There exists a* $A_0 < \infty$ *independent of* k*, such that*

$$\Big(\tau(|\nabla f_k|^2 + R_k) + f_k - n\Big)(x,\tau) \;\geq\; 0 \tag{5.19}$$

$$\Big(\tau(|\nabla f_\infty|^2 + R_\infty) + f_\infty - n\Big)(x,\tau) \;\geq\; 0 \tag{5.20}$$

whenever $dist_{g_k(\tau)}(p_k, x) \geq A_0$*,* $dist_{g_\infty(\tau)}(p_\infty, x) \geq A_0$*, and for all* $\tau \in [1,2]$*.*

Lemma 5.6 follows immediately from the quadratic growth estimates (5.15) and (5.17) of f_k and f_∞, respectively.

Lemma 5.7. $\liminf\limits_{k\to\infty} W_k(\tau) \geq W_\infty(\tau) > -\infty$ *for all* $\tau \in [1,2]$

Proof. By the local smooth convergence of $\{f_k\}_{k=1}^{\infty}$ and $\{u_k\}_{k=1}^{\infty}$ and by Lemma 5.6, we have that for any $A > A_0$ and $\tau \in [1,2]$, the following holds

$$\begin{aligned}
\liminf_{k\to\infty} W_k(\tau) &\geq \lim_{k\to\infty} \int_{B_{g_k(\tau)}(p_k,A)} \Big(\tau(|\nabla f_k|^2 + R_k) + f_k - n\Big)\frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-f_k} dg_k(\tau) \\
&= \int_{B_{g_\infty(\tau)}(p_\infty,A)} \Big(\tau(|\nabla f_\infty|^2 + R_\infty) + f_\infty - n\Big)\frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-f_\infty} dg_\infty(\tau) \\
&\geq \int_{B_{g_\infty(\tau)}(p_\infty,A_0)} \Big(\tau(|\nabla f_\infty|^2 + R_\infty) + f_\infty - n\Big)\frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-f_\infty} dg_\infty(\tau) \\
&= C(\tau) \\
&> -\infty.
\end{aligned}$$

Here we use $C(\tau)$ to denote a constant depending on τ but independent of A. Taking $A \to \infty$ in the above inequality and applying the dominated convergence theorem, we have $\liminf\limits_{k\to\infty} W_k(\tau) \geq W_\infty(\tau) > -\infty$. □

Lemma 5.8. $\limsup_{k\to\infty} W_k(\tau) \le W_\infty(\tau)$.

Proof. We use the equivalent definition (2.24) of Perelman's entropy. By Perelman's differential Harnack inequality Corollary 2.15, we have

$$\tau(2\Delta f_k - |\nabla f_k|^2 + R_k) + f_k - n \le 0,$$

on $M_k \times [1, 2]$ and for all k, obviously this inequality carries to f_∞ on $M_\infty \times [1, 2]$

$$\tau(2\Delta f_\infty - |\nabla f_\infty|^2 + R_\infty) + f_\infty - n \le 0 \tag{5.21}$$

because of the locally smooth convergence.

Let $A < \infty$ and $0 \le \phi_k \le 1$ be cut-off functions such that $\phi_k = 1$ on $B_{g_k(\tau)}(p_k, A)$, $\phi_k = 0$ outside $B_{g_k(\tau)}(p_k, A + 1)$, $|\nabla\phi_k| \le 2$, and $\phi_k \to \phi_\infty$ uniformly, where ϕ_∞ is a cut-off function on M_∞ with the same properties as ϕ_k's.

$$\begin{aligned}
&\limsup_{k\to\infty} W_k(\tau) \qquad\qquad (5.22)\\
&\le \lim_{k\to\infty} \int_M \phi_k^2 \Big(\tau(2\Delta f_k - |\nabla f_k|^2 + R_k) + f_k - n\Big) \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-f_k} dg_k(\tau)\\
&= \int_{M_\infty} \phi_\infty^2 \Big(\tau(2\Delta f_\infty - |\nabla f_\infty|^2 + R_\infty) + f_\infty - n\Big) \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-f_\infty} dg_\infty(\tau).
\end{aligned}$$

The lemma is proved if we can take the limit as $A \to \infty$ and the last integral in the above formula converges. Hence, we need only to verify the condition that

$$\Big(\tau(2\Delta f_\infty - |\nabla f_\infty|^2 + R_\infty) + f_\infty - n\Big)\frac{1}{(4\pi\tau)^{\frac{n}{2}}}e^{-f_\infty}$$

is absolutely integrable. To see this, we apply Shi's Bernstein type estimate [35] directly to the equation satisfied by ∇f_∞, and use Lemma 5.4 and (5.18). It then follows that each derivative of f_∞ has at most polynomial growth rate. This argument is standard and we omit here. Therefore, we have not only obtained the convergence of the last integral in (5.22) as $A \to \infty$, but also verified the validity of integration by parts at infinity for the entropy formula involving f_∞. $\square$

We have established $\lim_{k\to\infty} W_k(\tau) = W_\infty(\tau)$ for all $\tau \in [1,2]$. It follows from the monotonicity of $W_{x_0}(\tau)$ that for all $\tau \in [1,2]$,

$$\lim_{k\to\infty} W_k(\tau) = \lim_{k\to\infty} W_{x_0}(\tau\tau_k) = \lim_{s\to\infty} W_{x_0}(s),$$

and hence $W_\infty(\tau) \equiv \lim_{s\to\infty} W_{x_0}(s) = \bar{W}(x_0)$ is a constant. Finally, since the monotonicity formula Lemma 2.13 (4) also holds for $W_\infty(\tau)$, we have that f_∞ is a potential function of the asymptotic shrinker. This completes the proof of Proposition 5.5.

5.3.2 Relation between asymptotic entropy and asymptotic reduced volume

Proof of Corollay 1.8. Now we make the assumption as in Corollay 1.8. Let

$$(M, g(\tau))_{\tau\in[0,\infty)}$$

be an ancient solution with bounded nonnegative curvature operator and

$$\inf_M \mathrm{Vol}_{g(0)}(B_{g(0)}(\cdot, 1)) > 0.$$

In the case that $(M, g(\tau))_{\tau\in[0,\infty)}$ is collapsed, both sides of (1.8) are $-\infty$. Here $\bar{W}(x_0) = -\infty$ follows from Proposition 4.12 and $\log \bar{V}(x_0) = -\infty$ follows from 7.3 in Perelman [33] and Lemma 3.1 in Yokota [37]. Hence we only consider the noncollapsed case.

First of all, we establish the fact that there is no integration loss when taking limits of u_k and $(4\pi\tau)^{-\frac{n}{2}} e^{-l_k(x,\tau)}$.

Claim.

$$\int_{M_\infty} u_\infty dg_\infty(\tau) \equiv 1, \tag{5.23}$$

$$\int_{M_\infty} \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-l_\infty} dg_\infty(\tau) \equiv \bar{V}(x_0), \tag{5.24}$$

for all $\tau \in [1, 2]$.

Proof of the claim. To prove the first equality, we apply the quadratic growth estimate of f_k (5.15) and the Bishop-Gromov volume comparison to see that the contribution of u_k to the integral outside a large ball is negligible. To wit, for any $\varepsilon > 0$, there exists $A_0 < \infty$, such that for all $A > A_0$ and for all k, it holds that

$$1 - \varepsilon < \int_{B_{g_k}(p_k, A)} u_k(\cdot, \tau) dg_k(\tau) \le 1.$$

The conclusion follows from first taking $k \to \infty$, then taking $A \to \infty$, and finally taking $\varepsilon \to 0$. The second equality follows from the same argument. This proves Claim 1.

Now we study the Ricci shrinker potentials f_∞ and l_∞ with τ fixed at 1. By Carrillo and Ni [8], we know that $\bar{W}(x_0) = W_\infty(1) = \mu(g_\infty, 1)$, since f_∞ is normalized in the way of (5.23). Moreover, if we define $\tilde{l}_\infty = l_\infty + \log \bar{V}(x_0)$, by (5.24) we have $\int_{M_\infty} (4\pi)^{-\frac{n}{2}} e^{-\tilde{l}_\infty} = 1$ and $\tilde{l}_\infty$ is normalized in the same way as f_∞ in (5.23). Therefore, by applying Carrillo and Ni [8] again we have

$$\begin{aligned} \mu(g_\infty, 1) &= \int_{M_\infty} \left(2\Delta \tilde{l}_\infty - |\nabla \tilde{l}_\infty|^2 + R_\infty + \tilde{l}_\infty - n\right) (4\pi)^{-\frac{n}{2}} e^{-\tilde{l}_\infty} dg_\infty \\ &= \log \bar{V}(x_0), \end{aligned}$$

where in the last equality we have used (3.38) and the definition of $\tilde{l}_\infty$. Hence we have proved the theorem.

□

5.3.3 The lower bound for all possible asymptotic entropies

In Proposition 5.5 we have already proved that on a κ-noncollapsed ancient solution to the Ricci flow with bounded nonnegative curvature operator, the pointed entropy based at a fixed point is bounded in time, since the asymptotic entropy is its lower bound and 0 is its upper bound. We now proceed to show that the set of all possible asymptotic entropies has a lower bound depending only on $\kappa > 0$ and the dimension n, and is independent of the ancient solution and the base point.

Proof of Theorem 1.7 (1). We argue by contradiction. Assume the conclusion is not true for some $\kappa > 0$, then we can find a sequence of κ-noncollapsed ancient solutions with bounded nonnegative curvature operator and a sequence of base points, such that the corresponding asymptotic entropies goes to negative infinity. Let (M_k, g_k, l_k, p_k) be the corresponding asymptotic shrinker, where l_k is the potential function normalized as in (3.9) and $p_k \in M_k$ is the point where l_k attains its minimum. Here we fix $\tau = 1$. It then follows from Corollary 1.8 that

$$\int_{M_k} e^{-l_k} dg_k \to 0, \quad \text{as} \quad k \to \infty. \tag{5.25}$$

By Theorem 3.2 we have that

$$\frac{1}{4}\Big(dist(p_k, x) - 5n\Big)_+^2 \le l_k(x) \le \frac{1}{4}\Big(dist(p_k, x) + \sqrt{2n}\Big)^2, \tag{5.26}$$

where p_k is the minimum point of l_k. It then follows from the identity

$$|\nabla l_k|^2 + R_k = l_k$$

and (5.26) that $R_k(p_k)$ is uniformly bounded independent of k. By Lemma 5.4, since each asymptotic shrinker has constant scalar curvature, we have that the curvature of (M_k, g_k) is uniformly bounded independent of k, and hence so is each derivative of curvature, according to Shi [35]. By applying derivative estimates to

$$\Delta l_k - |\nabla l_k|^2 = \frac{n}{2} - l_k,$$

we can obtain uniform growth bounds for all the derivatives of l_k. Therefore, by passing to a (not relabeled) subsequence, $\{(M_k, g_k, l_k, p_k)\}_{k=1}^{\infty}$ converges to a shrinker in the pointed smooth Cheeger-Gromov sense. Note that this sequence of Ricci shrinkers are κ-noncollapsed on all scales with respect to a uniform $\kappa > 0$ according to our assumption. We denote the limit as $(M_\infty, g_\infty, l_\infty, p_\infty)$, where the potential function l_∞ also satisfies (5.26). It follows from the same argument as the proof of (5.23) and (5.24) that

$$\lim_{k\to\infty} \int_{M_k} e^{-l_k} dg_k = \int_{M_\infty} e^{-l_\infty} dg_\infty > 0.$$

This contradicts (5.25). □

5.3.4 The sufficiency part

Finally, Theorem 1.7(2) follows from Corollary 4.13 and the following proposition.

Proposition 5.9. *Let* $(M, g(\tau))_{\tau\in[0,\infty)}$ *be an ancient solution to the Ricci flow such that on each time-slice the curvature operator is bounded and nonnegative. Assume that* $\inf_{x\in M} \mathrm{Vol}_{g(0)}(B_{g(0)}(x,1)) = 0$. *Then the following holds.*

(1) $(M, g(\tau))_{\tau\in[0,\infty)}$ *is not* κ*-noncollapsed for any* $\kappa > 0$*, and*

(2) There exists a sequence of points $\{x_k\}_{k=1}^{\infty}$*, such that* $\bar{W}(x_k) \to -\infty$.

Proof. (1) follows directly from the definition of κ-noncollapsing, we prove (2). Let $\{x_k\}_{k=1}^{\infty}$ be a sequence of points along which

$$\mathrm{Vol}_{g(0)}(B_{g(0)}(x_k,1)) \to 0. \tag{5.27}$$

We claim that $\bar{W}(x_k) \to -\infty$. Assume by contradiction that there exists $\beta \in (0,\infty)$, such that

$$\bar{W}(x_k) > -\beta \tag{5.28}$$

for all k. Let $A_k > 0$ be a sequence of numbers such that

$$\mathrm{inj}_{A_k g(0)}(x_k) = 1.$$

Then we have $A_k \to \infty$ because of (5.27). We consider the scaled flows

$$\{(M, g_k(\tau), (x_k, 0))_{\tau\in[0,\infty)}\}_{k=1}^{\infty},$$

where $g_k(\tau) = A_k g(A_k^{-1}\tau)$. Let $u_k := (4\pi\tau)^{-\frac{n}{2}} e^{-f_k}$ be conjugate heat kernel coupled with backward Ricci flow g_k based at $(x_k, 0)$. Because of the scaling invariant of the entropy and because of (5.28), we have

$$W_k(\tau) := \int_M \Big(\tau(2\Delta f_k - |\nabla f_k|^2 + R_k) + f_k - n\Big) u_k dg_k(\tau) \geq -\beta,$$

for all $\tau \in [0, \infty)$. By Hamilton [19], after passing to a (not relabeled) subsequence, $\{(M, g_k(\tau), x_k)_{\tau\in[0,\infty)}, \}_{k=1}^{\infty}$ converges in smooth pointed Cheeger-Gromov sense to an ancient Ricci flow $(M, g_\infty(\tau), x_\infty)_{\tau\in[0,\infty)}$. We have that $g_\infty(\tau)$ is flat and static and that

$$\mathrm{inj}_{g_\infty}(x_\infty) = 1.$$

In particular, (M_∞, g_∞) is not the Euclidean space. Furthermore, by Lu [26], we have that u_k converges locally smoothly to the fundamental solution to the heat equation based at $(x_\infty, 0)$. For any $A >> 1$ and any $\tau > 0$, we have

$$
\begin{aligned}
&\int_{B_{g_\infty}(x_\infty,A)} \Big(\tau(2\Delta f_\infty - |\nabla f_\infty|^2) + f_\infty - n\Big) u_\infty dg_\infty(\tau) \\
&= \lim_{k\to\infty} \int_{B_{g_k(\tau)}(x_k,A)} \Big(\tau(2\Delta f_k - |\nabla f_k|^2 + R_k) + f_k - n\Big) u_k dg_k(\tau) \\
&\geq W_k(\tau) \\
&\geq -\beta,
\end{aligned}
$$

where we have used the fact that the integrand in the above formula is nonpositive (see section 9 in [33]). It then follows taking $A \to \infty$ that

$$
\int_{M_\infty} \Big(\tau(2\Delta f_\infty - |\nabla f_\infty|^2) + f_\infty - n\Big) u_\infty dg_\infty(\tau) \geq -\beta > -\infty,
$$

for any $\tau > 0$. This is a contradiction, as we have proved in Proposition 3.3 of [42], the entropy is always unbounded from below on a flat non-Euclidean manifold.

□

The result of chapter 5 is my own work, currently accepted for publication online on Journal für die reine und angewandte Mathematik (Crelle's Journal).

Bibliography

[1] Sigurd Angenent and Dan Knopf. An example of neckpinching for Ricci flow on sn. *Math. Res. Lett*, 11(4):493–518, 2004.

[2] Mihai Bailesteanu, Xiaodong Cao, and Artem Pulemotov. Gradient estimates for the heat equation under the Ricci flow. *Journal of Functional Analysis*, 258(10):3517–3542, 2010.

[3] Marcel Berger. *A Panoramic View of Riemannian Geometry*. Springer Science and Business Media, 2012.

[4] Christoph Böhm and Burkhard Wilking. Manifolds with positive curvature operators are space forms. *Annals of Mathematics*, 167(3):1079–1097, 2008.

[5] Simon Brendle. A generalization of Hamilton's differential Harnack inequality for the Ricci flow. *Journal of Differential Geometry*, 82(1):207–227, 2009.

[6] Simon Brendle and Richard Schoen. Manifolds with 1/4-pinched curvature are space forms. *Journal of the American Mathematical Society*, 22(1):287–307, 2009.

[7] Huai-Dong Cao and Detang Zhou. On complete gradient shrinking Ricci solitons. *Journal of Differential Geometry*, 85(2):175–186, 2010.

[8] José Carrillo and Lei Ni. Sharp logarithmic Sobolev inequalities on gradient solitons and applications. *Communications in Analysis and Geometry*, 17(4):721–753, 2009.

[9] Albert Chau, Luen-Fai Tam, and Chengjie Yu. Pseudolocality for the Ricci flow and applications. *arXiv preprint math/0701153*, 2007.

[10] Bing-Long Chen. Strong uniqueness of the Ricci flow. *Journal of Differential Geometry*, 82(2):363–382, 2009.

[11] Bennett Chow, Sun-Chin Chu, David Glickenstein, Christine Guenther, Jim Isenberg, Tom Ivey, Dan Knopf, Peng Lu, Feng Luo, and Lei Ni. *The Ricci Flow:*

Techniques and Applications: Part I: Geometric Aspects. American Mathematical Society, 2007.

[12] Bennett Chow, Sun-Chin Chu, David Glickenstein, Christine Guenther, Jim Isenberg, Tom Ivey, Dan Knopf, Peng Lu, Feng Luo, and Lei Ni. *The Ricci Flow: Techniques and Applications: Part II: Analytic Aspects*. American Mathematical Society, 2008.

[13] Bennett Chow, Sun-Chin Chu, David Glickenstein, Christine Guenther, Jim Isenberg, Tom Ivey, Dan Knopf, Peng Lu, Feng Luo, and Lei Ni. *The Ricci Flow: Techniques and Applications: Part III: Geometric-Analytic Aspects*. American Mathematical Society, 2010.

[14] Klaus Ecker, Dan Knopf, Lei Ni, and Peter Topping. Local monotonicity and mean value formulas for evolving Riemannian manifolds. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2008(616):89–130, 2008.

[15] Richard Hamilton. Three-manifolds with positive Ricci curvature. *Journal of Differential Geometry*, 17(2):255–306, 1982.

[16] Richard Hamilton. Four-manifolds with positive curvature operator. *Journal of Differential Geometry*, 24(2):153–179, 1986.

[17] Richard Hamilton. The Harnack estimate for the Ricci flow. *Journal of Differential Geometry*, 37(1):225–243, 1993.

[18] Richard Hamilton. A matrix Harnack estimate for the heat equation. *Communications in Analysis and Geometry*, 1(1):113–126, 1993.

[19] Richard Hamilton. A compactness property for solutions of the Ricci flow. *American Journal of Mathematics*, 117(3):545–572, 1995.

[20] Richard Hamilton. Formation of singularities in the Ricci flow. *Surveys in Differential Geometry,* International Press, 2:7–136, 1995.

[21] Richard S Hamilton. Non-singular solutions of the ricci flow on three-manifolds. *Communications in analysis and geometry*, 7(4):695–729, 1999.

[22] Robert Haslhofer and Reto Müller. A compactness theorem for complete Ricci shrinkers. *Geometric and Functional Analysis*, 21(5):1091–1116, 2011.

[23] Hans-Joachim Hein and Aaron Naber. New logarithmic Sobolev inequalities and an ε-regularity theorem for the Ricci flow. *Communications on Pure and Applied Mathematics*, 67(9):1543–1561, 2014.

[24] Peter Li. *Geometric Analysis*. Cambridge University Press, 2012.

[25] Peter Li and Shing Tung Yau. On the parabolic kernel of the Schrödinger operator. *Acta Mathematica*, 156(1):153–201, 1986.

[26] Peng Lu. Convergence of fundamental solutions of linear parabolic equations under Cheeger–Gromov convergence. *Mathematische Annalen*, 353(1):193–217, 2012.

[27] John Morgan and Gang Tian. *Ricci flow and the Poincaré conjecture*, volume 3. American Mathematical Society, 2007.

[28] Ovidiu Munteanu. The volume growth of complete gradient shrinking Ricci solitons. *arXiv preprint arXiv:0904.0798*, 2009.

[29] Ovidiu Munteanu and Jiaping Wang. Geometry of manifolds with densities. *Advances in Mathematics*, 259:269–305, 2014.

[30] Ovidiu Munteanu and Jiaping Wang. Positively curved shrinking Ricci solitons are compact. *Journal of Differential Geometry*, 106(3):499–505, 2017.

[31] Lei Ni. A note on Perelman's Li-Yau-Hamilton inequality. *Communications in Analysis and Geometry*, 14(5):883–905, 2006.

[32] Grisha Perelman. Proof of the soul conjecture of Cheeger and Gromoll. *Journal of Differential Geometry*, 40(1):209–212, 1994.

[33] Grisha Perelman. The entropy formula for the Ricci flow and its geometric applications. *arXiv preprint math/0211159*, 2002.

[34] Grisha Perelman. Ricci flow with surgery on three-manifolds. *arXiv preprint math/0303109*, 2003.

[35] Wan-Xiong Shi. Deforming the metric on complete Riemannian manifolds. *Journal of Differential Geometry*, 30(1):223–301, 1989.

[36] Guoyi Xu. An equation linking W-entropy with reduced volume. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, Published online, 03-31-2015.

[37] Takumi Yokota. Perelman's reduced volume and a gap theorem for the Ricci flow. *Communications in Analysis and Geometry*, 17(2):227–263, 2009.

[38] Takumi Yokota. Addendum to Perelmans reduced volume and a gap theorem for the Ricci flow. *Communications in Analysis and Geometry*, 20(5):949–955, 2012.

[39] Qi S. Zhang. Some gradient estimates for the heat equation on domains and for an equation by Perelman. *International Mathematics Research Notices*, 2006:O92314, 2006.

[40] Qi S Zhang. *Sobolev inequalities, heat kernels under Ricci flow, and the Poincaré conjecture*. CRC Press, 2010.

[41] Yongjia Zhang. On the equivalence between noncollapsing and bounded entropy for ancient solutions to the Ricci flow. *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, published online, 2018.

[42] Yongjia Zhang. Entropy, noncollapsing, and a gap theorem for ancient solutions to the Ricci flow. *Communications in Analysis and Geometry*, to appear.