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The Importance of Using Probabilistic Effective Hydraulic Conductivity in Darcy's Law and Groundwater Flow Calculations

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Abstract

The effective saturated hydraulic conductivity is a parameter that relates the average groundwater specific discharge to the average hydraulic gradient. This paper outlines a procedure to calculate the effective saturated hydraulic conductivity in local-scale groundwater flow. Examples are presented in this work.

Introduction

Let q_w (dimensions of length over time) denote the component of the specific discharge in the direction w, where w can be the x or y coordinal directions on a horizontal plane, or the z direction on a vertical plane (perpendicular to the plane containing x and y). Also, let j_w (dimensionless) be the component of the hydraulic gradient in the direction w. Darcy's law expresses the relation between q_w, K, and j_w as shown in the next equation:

$$q_w = -K\, j_w \qquad (1)$$

K, j_w, and q_w are random variables. K is a tensor whose principal directions are assumed aligned with the coordinal axes x, y, z. Therefore, an operational analog of equation (1) expressed in terms of measurable variables is needed. This can be achieved by taking the expected value on both sides of equation (1) and expressing Darcy's law in terms of the average specific discharge (Q_w), the average hydraulic gradient (J_w), and the effective saturated hydraulic conductivity along direction w (K_{ew}, a deterministic or non-random

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entity, which does not equal the expected value of saturated hydraulic conductivity) in the following manner:

$$Q_w = -K_{ew}\, J_w \qquad w = x,\ y,\ \text{or } z \text{ coordinal direction} \qquad (2).$$

This paper presents a methodology to calculate the effective saturated hydraulic conductivity K_{ew}. When the effective saturated hydraulic conductivity is independent of the choice of coordinal direction in isotropic aquifers it is denoted by K_e.

Calculation of the effective hydraulic conductivity when the saturated hydraulic conductivity has arbitrary pdf and variance.

This section presents equations and methods to calculate the effective saturated hydraulic conductivity for K with arbitrary pdf and either axisymmetric or isotropic covariance.

Axisymmetric covariance case. Dagan (1989, p. 193-201) reported an approach to obtain the effective saturated hydraulic conductivities in the horizontal plane (K_{eh}) and in the vertical direction (K_{ez}) when the hydraulic conductivity has an axisymmetric covariance. These results are applicable for arbitrary pdf of the saturated hydraulic conductivity and circumvents the assumption of very small log-conductivity variance (i.e., that $\sigma_Y^2 \leq 0.01$). The horizontal effective saturated hydraulic conductivity is as follows (Dagan, 1989, p. 198):

$$K_{eh} = \frac{1}{2}\left[\int_{\text{all } x} \frac{f_K(x)\, dx}{(x - K_{eh})\eta + 2K_{eh}}\right]^{-1} \qquad (3)$$

in which $f_K(x)$ is the pdf of the hydraulic conductivity (lognormal, gamma, exponential, log-gamma, beta, for example) whose domain is represented by "all x" values in equation (3). In the case of a lognormal pdf, for instance, the integration on the right-hand side of equation (3) is over the interval $x \geq 0$ (represented by "all x"). Other terms in equation (8.1) are (Dagan, 1989, p. 192):

$$\eta = \frac{\kappa^2}{1-\kappa^2}\left[\frac{1}{\kappa\sqrt{1-\kappa^2}} \tan^{-1}\left(\sqrt{\frac{1}{\kappa^2} - 1}\right) - 1\right] \qquad (4)$$

in which the inverse tangent function ($\tan^{-1}$) is expressed in radians, and (Dagan, 1989, p. 196):

$$\kappa = \frac{I_{Kz}}{I_{Kh}} \sqrt{\frac{K_{eh}}{K_{ez}}} \tag{5}$$

I_{Kh} and I_{Kz} are the horizontal and vertical integral scales of the saturated hydraulic conductivity, respectively.

The vertical effective saturated hydraulic conductivity is (Dagan, 1989, p. 198):

$$K_{ez} = \left[\int_{\text{all } x} \frac{f_K(x)\, dx}{x + \eta \cdot (K_{ez} - x)} \right]^{-1} \tag{6}$$

Equations (3) and (6) are coupled integral equations. This is so because the factor κ in equation (3) contains the ratio K_{eh} / K_{ez}, which appears in both equations via the term η (see equation (4)). Therefore, equations (3) and (6) must be solved jointly and iteratively to obtain the horizontal and vertical effective saturated hydraulic conductivities. The equations for axisymmetric covariances presented in this section and for isotropic covariance in section 8.2 do not constrain the variance of log-conductivity and are applicable to arbitrary pdf of saturated hydraulic conductivity.

Isotropic covariance case. If the covariance of saturated hydraulic conductivity is isotropic, then $K_{eh} = K_{ez}$ and equations (3) and (6) converge to the following effective saturated hydraulic conductivity (K_e) (Dagan, 1989, p. 199):

$$K_e = \frac{1}{3} \left[\int_{\text{all } x} \frac{f_K(x)\, dx}{x + 2K_e} \right]^{-1} \tag{7}$$

Calculation of the effective saturated hydraulic conductivity in which K is log-gamma distributed with isotropic covariance

The log-gamma pdf was identified as a suitable model for the saturated hydraulic conductivity in Loáiciga et al. (2006) The following equation is used to calculate the log-gamma pdf ($f_K(x)$) (ASCE, 2008):

$$f_K(x) = \frac{\left(\dfrac{\ln(x) - \theta_Y}{\beta_Y}\right)^{\alpha_Y} \left|\ln(x) - \theta_Y\right|^{-1} e^{-\left(\frac{\ln(x)-\theta_Y}{\beta_Y}\right)}}{x\,\Gamma(\alpha_Y)} \tag{8}$$

where x represents the value of the saturated hydraulic conductivity at which the log-gamma pdf is calculated; α_Y, β_Y, and θ_Y are the shape, scale, and upper or lower bound parameters of the log-gamma pdf, respectively. The log-gamma pdf may have either a lower bound:

$$x \geq e^{\theta_Y} \quad \text{if } \beta_Y > 0 \tag{9}$$

or it may have a lower bound (equal to zero) and an upper bound simultaneously, as follows:

$$0 < x \leq e^{\theta_Y} \quad \text{if } \beta_Y < 0 \tag{10}$$

$\Gamma(\alpha_Y)$ in equation (8) denotes the gamma function evaluated at α_Y. The gamma function can be calculated using commercially available spreadsheets and numerical software.

The log-gamma pdf approaches the lognormal pdf when the coefficient of skew of logconductivity tends to zero (Loáiciga et al., 2006, equations (52)-(53)). Therefore, the lognormal pdf is a special case of the log-gamma pdf. Applications will be presented during the oral lecture of this paper.

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