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MODEL REPRESENTATION OF AGING VISCOELASTIC MATERIALS

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Summary

A study is made of the behavior of mechanical models made of elements (springs and dashpots) whose properties (stiffness and viscosity) vary in time. A particular distinction is made between elastic and hypoelastic behavior of aging springs. A further distinction is shown between models described by differential and by integro-differential equations, and a classification is made of the types of differential equations that may be obtained. Several examples of model behavior are studied in detail.

1. Introduction

While it is well established that the integral, or Boltzmann-Volterra, representation of the behavior of linear viscoelastic continua is more general than the differential-equation representation[1], the latter is still frequently used; and though the principles of thermodynamics have been used in establishing it[2], the starting point is usually an analysis of the behavior of spring-dashpot models[3],[4]. The response of a model of any degree of complexity can be determined by superposition of the responses of the elements comprising it, starting from the equations governing the behavior of the basic elements - dashpot and spring - which are, respectively,

$$\sigma = \mu \dot{\varepsilon} \tag{1}$$

and

$$\sigma = E \varepsilon, \tag{2}$$

where σ and ε denote stress and strain, which we shall use as the mechanical variables in place of force and extension. A superposition of the stresses given by eqs. (1) and (2) (parallel combination) results in the Voigt element. The Maxwell element consists of a spring and a dashpot in series; the superposition, however, is not of strains, but of strain rates, and eq. (2) must be used in its differentiated form

$$\dot{\sigma} = E \dot{\varepsilon}. \tag{2'}$$

Provided E is constant, eq. (2') is equivalent to (2) if it is supplemented by the usual initial condition of zero strain at zero stress (definition of initial state). For more complex models, the basic equations need to be differentiated several times, but the correspondingly more complicated initial conditions first proposed by the author (quoted in Ref. 5) may be obviated by letting stress and strain vanish smoothly as $t \to -\infty$.

On the other hand, no study appears to have been made of the behavior of mechanical models whose properties (viscosity, stiffness) vary in time, i.e. models representing aging viscoelastic materials. Hansen[6)] implicitly used such a model in studying the creep of concrete, but on the apparent assumption that the equation of the model was analogous to that of the non-aging model. Virtually all workers in the rheology of concrete have used the Volterra representation[7),8),9)], and the differential-equation approach has only recently begun to be explored[10)], though both have long been used in systems theory[11)]. The same may be largely said of work in thermoviscoelasticity[12)], which is, in fact, a branch of the study of aging viscoelastic media (or "general viscoelasticity"[13)]) (aging induced by temperature variation).

2. Elastic and Hypoelastic Bodies

Let us consider a linear spring with stiffness varying in time. In this case, the two forms of Hooke's law corresponding respectively to (2) and (2'), namely

$$\sigma = E(t)\,\varepsilon \tag{3}$$

and

$$\dot{\sigma} = E(t)\,\dot{\varepsilon} \tag{4}$$

This may be immediately generalized to an aging medium by replacing $S(t-\tau)$ with the more general $R(t,\tau)$:

$$W_p = \int_{-\infty}^{t} R(t,\tau)\dot{\varepsilon}(\tau)[\varepsilon(t)-\varepsilon(\tau)]\,d\tau. \tag{9}$$

The mechanical work done on a general viscoelastic body may be given by

$$\begin{aligned} W &= \int_{-\infty}^{t} \sigma(\tau)\dot{\varepsilon}(\tau)\,d\tau \\ &= \int_{-\infty}^{t} \dot{\sigma}(\tau)[\varepsilon(t)-\varepsilon(\tau)]\,d\tau, \end{aligned} \tag{10}$$

the two forms being equivalent if stress and strain vanish smoothly as $t\to-\infty$ For the elastic and hypoelastic spring, respectively, we have

$$W = \int_{-\infty}^{t} E(\tau)\varepsilon(\tau)\dot{\varepsilon}(\tau)\,d\tau \tag{11}$$

and

$$W = \int_{-\infty}^{t} E(\tau)\dot{\varepsilon}(\tau)[\varepsilon(t)-\varepsilon(\tau)]\,d\tau. \tag{12}$$

When thus see that the work done on a hypoelastic spring equals its Volterra potential energy. Hence, an aging hypoelastic medium is nondissipative in the Volterra sense. Indeed, the rate of dissipation is

$$D = \sigma\dot{\varepsilon} - \dot{W}_p = -\int_{-\infty}^{t} \frac{\partial R}{\partial t}(t,\tau)\dot{\varepsilon}(\tau)[\varepsilon(t)-\varepsilon(\tau)]\,d\tau, \tag{13}$$

and this vanishes for an arbitrary strain history if and only if

$$\frac{\partial R}{\partial t} = 0$$

i.e., if the body is hypoelastic. The Volterra dissipation is therefore a measure of energy loss due to relaxation, regardless of whether this takes place because of viscous mechanisms or of aging.

3. Models with Dashpots

An aging viscous body (dashpot) may be unequivocally characterized by

$$\sigma = \mu(t)\dot{\varepsilon}; \tag{14}$$

this follows from the definition of viscosity.[17] Models may be constructed with dashpots and either hypoelastic or elastic springs (or both).

A Maxwell element with a hypoelastic and an elastic spring is described, respectively, by

$$\mathring{\varepsilon} = \frac{\dot{\sigma}}{E(t)} + \frac{\sigma}{\mu(t)}, \tag{15}$$

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E(t)} + \left[\frac{1}{\mu(t)} + \frac{d}{dt}\left(\frac{1}{E(t)}\right)\right]\sigma. \tag{16}$$

Both equations (15) and (16) may be written in the standard form

$$\mathring{\varepsilon} = q_0(t)\dot{\sigma} + q_1(t)\sigma \tag{17}$$

with $q_0(t), q_1(t)$ independent functions of time. Consequently, a body obeying a constitutive law given by (17) may be represented by a Maxwell model with either an elastic or a hypoelastic spring.

A Voigt element with an elastic spring is described by

$$\sigma = E(t)\varepsilon + \mu(t)\dot{\varepsilon}. \tag{18}$$

With a hypoelastic spring, on the other hand, we have

$$\mathring{\sigma} = [E(t) + \dot{\mu}(t)]\dot{\varepsilon} + \mu(t)\ddot{\varepsilon} \tag{19}$$

Equation (18) and (19) are <u>not</u> equivalent. Indeed, the body described by (18) has the property that in a creep test, i.e., under a stress σ_0 applied at time τ and maintained thereafter, the ultimate strain is $\sigma_0 / E(\infty)$, independent of the loading time τ. This property (which is not in general possessed by actual aging materials, such as concrete) is, in fact, common to all bodies described by differential equations in which the lowest order of strain derivative is zero (except, possibly, for differential equations with singular behavior at infinity). Equation (19), on the other hand, leads to an ultimate strain equal to $\sigma_0 / E(\tau)$.

Let us turn, now, to a model consisting of two springs and a dashpot. In the classical theory of viscoelasticity it is well known that such a model may be made by placing a Maxwell element in parallel with, or a Voigt element in series with, a spring, both combinations leading to the same differential equation (the "standard solid" equation). With aging models, however, the number of possible combinations is greatly increased. Let us denote an elastic and a hypoelastic spring by E and H, respectively; a Voigt element with an elastic and a hypoelastic spring by V_e and V_h; and a Maxwell element by M (we have already seen that either kind of spring may be used in the latter).

Further, let $-$ and $||$ denote series and parellel connections, respectively. Then we may have

(a) $V_e - E$
(b) $V_h - E$
(c) $V_h - H$
(d) $V_e - H$
(e) $M \,||\, E$
(f) $M \,||\, H$

Let us consider case (a), i.e. a Voigt model with an elastic spring of stiffness

$E(t)$ and a dashpot of viscosity $\mu(t)$, in series with an elastic spring of stiffness $E'(t)$. Let ε_1 and ε_2 denote the strain of the spring and of the Voigt element, respectively, then

$$\varepsilon = \varepsilon_1 + \varepsilon_2$$
$$\varepsilon_1 = \sigma / E'(t)$$
$$E(t)\varepsilon_2 + \mu(t)\dot{\varepsilon}_2 = \sigma.$$

Eliminating ε_1 and ε_2, we obtain

$$E\varepsilon + \mu\dot{\varepsilon} = \left[1 + \frac{E}{E'} + \mu\frac{d}{dt}\left(\frac{1}{E'}\right)\right]\sigma + \frac{\mu}{E'}\dot{\sigma}. \qquad \text{(a)}$$

Since this equation contains the three independent functions $\mu(t)$, $E(t)$ and $E'(t)$, it is the most general first-order differential constitutive equation, and may be written in the form

$$\dot{\varepsilon} + p_1(t)\varepsilon = q_0(t)\dot{\sigma} + q_1(t)\sigma \qquad (20)$$

with $p_1(t)$, $q_0(t)$ and $q_1(t)$ independent.

A similar synthesis of models (b),(c) and (d) leads, respectively, to

$$(E+\dot{\mu})\dot{\varepsilon} + \mu\ddot{\varepsilon} = \dot{\sigma} + (E+\dot{\mu})\frac{d}{dt}\left(\frac{\sigma}{E'}\right) + \mu\frac{d^2}{dt^2}\left(\frac{\sigma}{E'}\right), \qquad \text{(b)}$$

$$(E+\dot{\mu})\dot{\varepsilon} + \mu\ddot{\varepsilon} = \left[1 + \mu\frac{d}{dt}\left(\frac{1}{E'}\right)\right]\dot{\sigma} + \frac{\mu}{E'}\ddot{\sigma}, \qquad \text{(c)}$$

$$\dot{\varepsilon} + \frac{d}{dt}\left(\frac{\mu}{E}\dot{\varepsilon}\right) = \frac{d}{dt}\left(\frac{\sigma}{E} + \frac{\mu\dot{\sigma}}{EE'}\right) + \frac{\dot{\sigma}}{E'}. \qquad \text{(d)}$$

In dealing with models (e) and (f) we shall use, for simplicity, the Maxwell model with a hypoelastic spring (stiffness E). We then have, respectively,

$$\left(\frac{E'}{\mu}+\frac{\dot{E}'}{E}\right)\varepsilon+\left(1+\frac{E'}{E}\right)\dot{\varepsilon}=\frac{\sigma}{\mu}+\frac{\dot{\sigma}}{E} \qquad \text{(e)}$$

and

$$E'\dot{\varepsilon}+\frac{d}{dt}\left[\left(1+\frac{E'}{E}\right)\mu\dot{\varepsilon}\right]=\dot{\sigma}+\frac{d}{dt}\left(\frac{\mu}{E}\dot{\sigma}\right). \qquad \text{(f)}$$

Equation (e) is likewise of the form (20). Equations (c) and (f) may be written in the form

$$\ddot{\varepsilon}+p_1(t)\dot{\varepsilon}=q_0(t)\ddot{\sigma}+q_1(t)\dot{\sigma} \qquad (21)$$

with $p_1(t)$, $q_0(t)$ and $q_1(t)$, again, independent.

Equations (b) and (d) are of the form

$$\ddot{\varepsilon}+p_1(t)\dot{\varepsilon}=q_0(t)\ddot{\sigma}+q_1(t)\dot{\sigma}+q_2(t)\sigma \qquad (22)$$

of the four functions $p_1(t), \ldots q_2(t)$, only three are independent: there exists a differential equation among them which is, in general, different for models (b) and (d). In both cases, however, $q_2(t)$ vanishes as $t\to\infty$ if the other functions are bounded.

Let us come, next, to the model composed of two springs and two dashpots which, in rheology, is known as a Burgers body; it may be composed of a standard solid and a dashpot in series or of two Maxwell elements in parallel. The governing differential equation has the form

$$\ddot{\varepsilon}+a_1\dot{\varepsilon}=b_0\ddot{\sigma}+b_1\dot{\sigma}+b_2\sigma, \qquad (23)$$

where $a_1, \ldots b_2$ are positive constants.

If, now, we build a model composed of two aging Maxwell elements in parallel (as before, we need not distinguish between elastic and hypoelastic springs), we find

$$\sigma = \sigma_1 + \sigma_2,$$
$$\dot{\varepsilon} = \frac{\dot{\sigma}_1}{E_1(t)} + \frac{\sigma_1}{\mu_1(t)},$$
$$\dot{\varepsilon} = \frac{\dot{\sigma}_2}{E_2(t)} + \frac{\sigma_2}{\mu_2(t)}.$$

It is immediately apparent that σ_1 and σ_2 cannot be eliminated from the above equations without leading to an integro-differential equation in σ and ε.

Turning, then, to the alternative construction of the aging Burgers model, we see that we can place a dashpot in series with any one of the three standard solid models described, respectively, by equations (20), (21) and (22). It is remarkable that each choice leads to a differential equation of the form

$$\ddot{\varepsilon} + p_1(t)\dot{\varepsilon} = q_0(t)\ddot{\sigma} + q_1(t)\dot{\sigma} + q_2(t)\sigma, \qquad (24)$$

with $p_1(t), \ldots q_2(t)$ independent functions of time. Equation (24) is the generalization of (23) with variable coefficients, and a recent study[10)] has shown it to be a valuable means of describing the behavior of actual aging materials, such as concrete.

4. Generalizations

For non-aging bodies, a differential constitutive equation of the form

$$\left(\frac{d^n}{dt^n} + a_1\frac{d^{n-1}}{dt^{n-1}} + \ldots\ldots + a_n\right)\varepsilon = \left(b_0\frac{d^n}{dt^n} + \ldots\ldots + b_n\right)\sigma \qquad (25)$$

with $b_n \neq 0$, will always be obtained from spring-dashpot models, and, in particular, from a generalized Maxwell or Voigt representation[3]. Equation (25), in fact, represents four standard types of behavior, corresponding to (a) $a_n = 0 = b_0$, (b) $a_n \neq 0 = b_0$, (c) $a_n = 0 \neq b_0$, (d) $a_n \neq 0 \neq b_0$. In terms of a generalized Maxwell representation, for example, the corresponding model consists (a) n-1 Maxwell bodies in parallel with a dashpot, (b) n-1 Maxwell bodies in parallel with a Voigt body (c) n Maxwell bodies, (d) n Maxwell bodies and a spring.

For aging bodies, a general differential constitutive equation may be written in the form

$$\left[\frac{d^n}{dt^n} + p_1(t)\frac{d^{n-1}}{dt^{n-1}} + \dots + p_n(t)\right]\varepsilon = \left[q_0(t)\frac{d^n}{dt^n} + \dots + q_n(t)\right]\sigma. \quad (26)$$

We have seen, however, that the remarks applied to (25) do not hold for (26). In particular, we may have $q_n(t) = 0$ (as in (21)). Furthermore, a generalized model having two or more Maxwell elements in parallel, or Voigt elements in series, will not, in general, lead to a differential equation at all, but to an integro-differential equation. Let us call such a model "bad". A "good" model (i.e. one leading to a differential equation of the form (26)) can be constructed only by placing a simple element (spring or dashpot) in series or in parallel with a "good" model. (The resulting "good" model may, of course, be degenerate.) For example, the differential equation (25) of type (a) with n = 3 will be obtained from any one of the six models shown in the Figure, but the rheologist would in all likelihood use only models I and II, the generalized Maxwell and Voigt models, respectively. If, however, the models shown have time-variable elements, then, in general, models I, II and III are "bad", while IV, V and VI are "good".

Of course, the differential equation obtained from a "good" model will depend on whether the springs in it are elastic or hypoelastic. The distinction is immaterial only in the case of models in which every force path goes through a dashpot (e.g. the dashpot, Maxwell element, Burgers model), which lead to differential equations of the form (26) with $p_n(t) = 0 \neq q_n(t)$, this last function being independent of the others. In all other models there exists a force path going entirely through springs (only one such path if the model is nondegenerate in the usual sense); then, depending on whether the springs in the path are (a) all elastic, (b) all hypoelastic, or (c) mixed, we have (a) $p_n(t) \neq 0 \neq q_n(t)$, (b) $p_n(t) = 0 = q_n(t)$, or (c) $p_n(t) = 0 \neq q_n(t)$. In case (a) $q_n(t)$ is an independent function. In case (c) it is not, and, in particular, it vanishes as $t \to \infty$ if all spring compliances remain finite, so that singular behavior of the differential equation may be expected.

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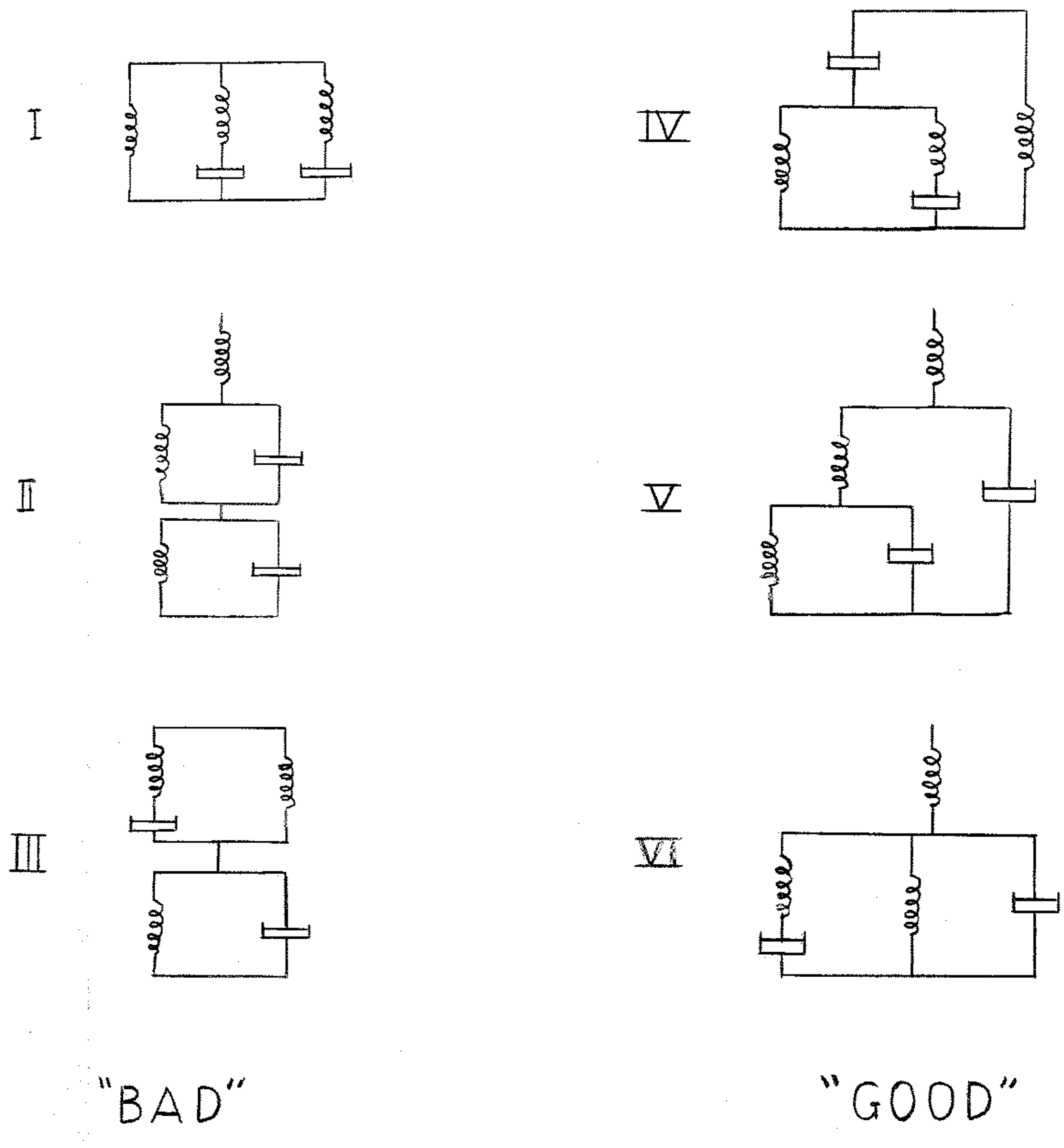

FIGURE 1