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Publication Date
1972-12-01

Peer reviewed

COMMITTEE ON ECONOMETRICS AND MATHEMATICAL ECONOMICS

Working Paper IP-175

SMOOTH PREFERENCES

AND DIFFERENTIABLE DEMAND FUNCTIONS

by

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December 1972

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Smooth Preferences

and Differentiable Demand Functions*

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There is a whole variety of problems in equilibrium theory (on unique-
ness, stability, continuity of equilibria, for example) the study of which,
to be tractable and fruitful, has to be placed in a smooth, differentiable
environment. The common situation then is that one wants to keep the theory
grounded on the formalization of an agent (consumer, from now on) as a pref-
erence relation but that what is technically needed is the differentiability
of the demand functions. As it turns out (and is well known) this is not
implied by the natural smoothness hypothesis on preferences. The reason
should be clear to economists, at least since Samuelson's Foundations: not
every maximum is regular (i.e., the second order conditions are not nec-
essary); an example has recently been given by D. Katzner [13]. Still, one
intuitively feels, and much theoretical work has proceeded on this assumption,
that, given the smoothness of preferences, the absence of differentiability
of demand is exceptional, that is to say that, granted smoothness, we can
safely expect that any practical situation we may confront will be "well

behaved." Following the suggestions of G. Debreu in [3] (page 31), where

the problem treated in this paper is raised, we will provide a formulation

and proof of this contention in the framework of a standard mathematical

approach which, starting in the study of G. Debreu [2] on the finiteness of

the set of price-equilibrium, has been increasingly used in economics for

the formalization of notions of this kind (H. and E. Dierker [7], F. Delbaen

[5], S. Smale [19]). In a few words, what is done is to endow the space of

smooth (convex and monotone) preferences with a C^1-type metric with respect

to which it becomes a complete metric space and to show then, that the

"good" set is residual (hence of the second category) in it, i.e. a countable

intersection of open, dense sets (hence, by Baire Category Theorem, dense).

An application (which may serve as a test of the usefulness of the result

as an avenue for recovering the preference framework in some of the work

referred to above) is mentioned.

The analysis reported here is closely related to the subject matter

studied in [15]. There it is considered a space of continuous preferences

endowed with the usual, C^0-type, closed convergence topology and it is

shown that every preference relation can be approximated by a C^∞ one.

However, if the preference relation is itself smooth there is no guaranty

that the sequence approximates in the stronger (C^1) topology we adopt here.

So a proof is given in the present context but we want to point out that

the question turns out to be, in this case, an altogether simpler one.

The problem so far examined has a dual; the demand function

$h(p^1,p^2,w) = \dfrac{w}{p^1+p^2}$ is C^∞ and generable by preferences $\left(u(x^1,x^2) = \min\{x^1,x^2\}\right)$

but not by smooth ones. In analogy to what has been said in the preceding

paragraphs, it makes sense to ask for the position of the set of demand

functions generated by smooth preferences in a space of differentiable

demand functions satisfying rationality requirements (i.e., the symmetry and

negative semidefiniteness of the Slutsky matrices). The theorem parellel to

the one for preferences is then proved, partly as an application of it an

integrability-type result is obtained.

Section I is devoted to preliminaries, the theorems are stated and

discussed in Section II, Section III contains the proofs.

I. Definitions and Preliminaries

The consumption set is $P = \{x \in \mathbb{R}^{\ell} : x \gg 0\}$ $\underline{1/}$. The set of monotone,

continuous, convex preference relations on P is denoted $\mathcal{R}$. For every

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[1] Some notational conventions and definitions: superscripts denote
components of vectors and subscripts denote vectors. $x \gg y$ means $x^i > y^i$
for every i; $x > y$ means $x^i \geq y^i$ for every i, and $x \neq y$; $x \geq y$ means $x > y$
or $x = y$. τx is the transpose of the (row) vector x. No distinction is
made between a finite-dimensional Euclidean space and its dual, so
$xy = \sum_i x^i y^i$, $x,y \in \mathbb{R}^n$. A function $f : A \to \mathbb{R}^n$, $A \subset \mathbb{R}^m$ is $C^r (r \geq 1)$ if it can
be extended in a neighborhood of A to a function for which every component
has continuous partial derivatives up to the r-th order; it is C^{∞} if it is
C^r for every $r \geq 1$. The derivative of a C^1 mapping $f : A \to \mathbb{R}^n$ at $x \in A$ is
denoted $Df(x)$; $D_{x^i}f(x)$, $D_{x'}f(x',x'')$ have the obvious meaning; if $f : A \to \mathbb{R}$
$A \subset \mathbb{R}^m$ is C^2, $Hf(x)$ designates the Hessian at $x \in A$. Let $f : A \to \mathbb{R}^n$,
$A \subset \mathbb{R}^m$, then $\|f\|_A^o = \sup_{x \ A} \|f(x)\|$ and, if f is C^1, $\|f\|_A^1 = \|f\|_A^o + \sum_{i=1}^{m} \|D_{x^i}f(x)\|_A^o$
For every set $A \subset \mathbb{R}^n$, $\bar{A}$, ∂A, Int A denote, respectively, the closure,
boundary and interior. The (n-1) unit circle is S^n, and the ε-ball, B_{ε}^n;
$S_+^n = \{x \in S^n : x \gg 0\}$. Intervals are designated $[a;b]$, $[a;\infty]$, etc.,
$I = [0;1]$.

$(R,x) \in \mathcal{R} \times P$ we let $U(R,x) = \{y \in P : (y,x) \in R\}$[2], and for every $R \in \mathcal{R}$ the correspondence $g_R : P \to S^\ell$ is defined by $g_R(x) = \{p \in S^\ell : py \geq px$ for every $y \in U(R,x)\}$. Clearly, $g_R(P) \subset \overline{P}$.

A preference relation $R \in \mathcal{R}$ will be said to be of _class_ $C^r (r = 2,\ldots,\infty)$ if g_R is a C^{r-1} function or, equivalently, if it is representable by a C^r utility function with no critical points.[3] We refer to G. Debreu [3] for a proof of this fact and for a thorough discussion of those, related and subsequent concepts; one has:

If R, R' are of class C^r $(r > 1)$ and $g_R = g_{R'}$, then $R = R'$. $\qquad$ (1.1)

Let $\mathcal{R}^r = \{R \in \mathcal{R} : R$ is $C^r\}$, $r = 2,\ldots,\infty$.

A preference relation $R \in \mathcal{R}^2$ is _regular_ at $x \in P$ if $M(R,x) \neq 0$, where

$$M(R,x) = \begin{vmatrix} Dg_R & {}^Tg_R \\ g_R & 0 \end{vmatrix}.$$

Let $u : P \to \mathbb{R}$ be a C^2 utility function with no critical points representing R. Then R is regular at $x \in P$ if and only if the restriction to the hyperplane $\{y \in \mathbb{R}^\ell : Du(x)y = 0\}$ of the form determined by $Hu(x)$ is negative definite. Let $\mathcal{R}^{(r)}(K) = \{R \in \mathcal{R}^r : R$ is regular at every $x \in K\}$, $K \subset P$, $r = 2,\ldots,\infty$; $\mathcal{R}^{(r)} = \mathcal{R}^{(r)}(P)$. A $R \in \mathcal{R}^{(r)}$ will be called regular.

Denote by $\mathcal{R}_b$ (and, similarly, $\mathcal{R}_b^r$, $\mathcal{R}_b^{(r)}$,...) the subset of $\mathcal{R}$ whose elements R fulfill the boundary condition:

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[2]A preference relation R is a subset of PxP which is _reflexive_ (i.e., $(x,x) \in R$ for every $x \in P$), _total_ (i.e., $(x,y) \in R$ or $(y,x) \in R$ for every x,y P, $x \neq y$), and _transitive_ (i.e., if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$, for every $x,y,z \in P$). It is _continuous_ if it is closed (rel. to PxP); it is _convex_ (resp. _monotone_) if, for every $x \in P$, (R,x) is convex (resp. contains $x + P$).

[3]In general we keep the notation in line with G. Debreu [3].

For every $x \in P$, $\overline{U(R,x)} \subset P$. $\qquad$ (1.2)

Let $V = P \times (0; \infty)$. Then, for every $R \in \mathcal{R}_b$, the demand correspondence $h^R : V \to P$ is well defined.[4/] The interest of regularity comes from the fact that if $R \in \mathcal{R}^r$ $(r > 1)$ is regular at $x = h^R(p,w)$, then h^R is a C^{r-1} function at (p,w) (see D. Katzner [13] or G. Debreu [3]).

We want to endow $\mathcal{R}^2$ with a C^1-type topology. We do this by giving to $\mathcal{R}^2$ the topology induced by the uniform convergence on compact sets of the g_R's functions and their first partial derivatives (see G. Debreu [3], p. 31); in other words, we say that $\langle R_n \rangle \overset{1}{\to} R$, R_n, $R \in \mathcal{R}^2$ if and only if $\|g_{R_n} - g_R\|_K^1 \to 0$ for every compact $K \subset P$. Actually, under the natural identification of R with g_R, $\mathcal{R}^2$ becomes a complete separable metric space (see 1.1)).

The C^o-type topology on $\mathcal{R}$ which has proved most fruitful (see W. Hildenbrand [10]) is the one induced by the closed convergence of sets.[5/] The adequacy of the topology we are introducing on $\mathcal{R}^2$ rest in the fact that it genuinely refines the closed convergence one, that is to say (letting Lim denote closed limit):

If $< R_n > \overset{1}{\to} R$ $(R_n, R \in \mathcal{R}^2)$, then $\text{Lim } R_n = R$. $\qquad$ (1.3)

---

[4] From now on if we write h^R, $R \in \mathcal{R}$, we are assuming that this is well defined (i.e. $h^R(p,w) \neq \emptyset$, for every $(p,w) \in V$).

[5] The fact that we are taking P instead of $\overline{P}$ as the consumption set does not affect the definition of the closed convergence topology. The basic theorems carry over unaltered; for example, the demand correspondence is continuous on $\mathcal{R}_b \times V$ and, in general, whenever it is defined and compact valued.

<u>Proof</u>:   We can assume that Lim R_n = T (F. Hausdorff [9], p. 169) where

T is closed (rel. PxP).   Every x ϵ P is a boundary point of the nonempty,

convex set U(T,x) = {y ϵ T : (y,x) ϵ T}.   Therefore a correspondence

g_T : P $\rightarrow$ S^ℓ can be defined analogously to the g_R's above.   It is immediate

that $\langle R_n \rangle \xrightarrow{1}$ R implies $g_R \subset g_T$.   Suppose that, for some x ϵ P,

$g_T(x) \not\subset g_R(x)$; then a simple argument (using the fact that the set on which

g_T is singlevalued is dense in $\partial U(T,x)$) yields the existence of a sequence

$\langle x_n \rangle \rightarrow$ x with $g_T(x_n)$ singlevalued (i.e., $g_T(x_n) = g_R(x_n)$) and

$\langle g_T(x_n) \rangle \rightarrow p \neq g_R(x)$ contradicting the continuity of g_R.

Hence $g_T = g_R$.   We should have T = R.   Otherwise we could find a y ϵ P

such that $\partial U(T,y)$ and $\partial U(R,y)$ would be two distinct (maximal) connected C^1

integral manifolds through y of the $(\ell-1)$ C^1 distribution determined by g_R,

which is impossible (see G. Debreu [3] and F. Warner [20], p. 42).

For every C^1 f: V $\rightarrow$ P and (p,w) ϵ V define the (substitution term or

Slutsky) matrix Sf(p,w) = Df(p,w) + ^{T}f(p,w) D_wf(p,w).   Let $\varkappa^r$ (r=2,...,∞)

be the set of C^{r-1} functions h : V $\rightarrow$ P which, for every (p,w) ϵ V, satisfy:

$$ph(p,w) = w \quad \text{and} \quad h(\lambda p, \lambda w) = h(p,w), \quad \lambda > 0 . \tag{1.4}$$

Sh(p,w) is symmetric and negative semidefinite. $\tag{1.5}$

A h ϵ $\varkappa^r$ (r > 1) is <u>regular</u> at (p,w) ϵ V if rkSh(p,w) = $\ell-1$ or,

equivalently, if every $(\ell-1)$ principal minor of Sh(p,w) is nonvanishing.

Let $\varkappa^{(r)}(J) = \{h \epsilon \varkappa^r$: h is regular at every (p,w) ϵ J$\}$ J $\subset$ V,

r = w,...,∞; $\varkappa^{(r)} = \varkappa^{(r)}(V)$.   A h $\epsilon \varkappa^{(r)}$ will be called regular.

Define $\mathscr{H}^r_b$(resp. $\mathscr{H}^{(r)}_b$, $r > 1$, to be the subsets of $\mathscr{H}^r$ (resp. $\mathscr{H}^{(r)}_b$) whose elements h satisfy the boundary condition:

If $\langle p_n, w_n \rangle \to (p,w)$, $(p_n, w_n) \in V$, $0 \neq p \notin P$, $w > 0$,

then $\lim \| h(p_n, w_n) \| = \infty$. $\qquad$ (1.6)

Let $h \in \mathscr{H}^r$ $(r > 1)$ and suppose that $h | S^\ell_+ \times (0;\infty)$ possesses a C^{r-1} inverse. The composition of this inverse with the projection on S^ℓ yields a (C^{r-1}) function $g : h(V) \to S^\ell$ satisfying (because of the symmetry part of (1.5); see P. A. Samuelson [18] or L. Hurwicz [11]) the (local) integrability condition at every point of its domain. Suppose now that $h(V) = P$. Then, since $g(P) \subset P$, a global integrability result of G. Debreu ([3], p. 16) $\underline{6/}$ applies, and we can conclude the existence of a C^2 function $u : P \to \mathbb{R}$ such that $g(x) = \frac{Du(x)}{\|Du(x)\|}$, $x \in P$. Moreover (negative semidifiniteness part of (1.5)) u is quasi concave. Therefore $g = g_R$ for some $R \in \mathbb{R}^r$. In addition, $h^R = h$ implying $R \in \mathbb{R}^{(r)}$. The interest of regularity comes from the fact:

If $h \in \mathscr{H}^r$ $(r > 1)$ is regular and satisfies the boundary

condition (1.6), i.e., $h \in \mathscr{H}^{(r)}_b$, then $h(V) = P$ and

$h | S^\ell_+ \times (0;\infty)$ is invertible. $\qquad$ (1.7)

<u>Proof</u>:$\underline{7/}$ Let $P_1 = (0;\infty)^{\ell-1}$ and take a arbitrary $x \in P$. Define the "normalized excess demand" function $d_x : P_1 \to P_1 + \{-x^2, \ldots, -x^\ell\}$ by

---

^{6}See, also, S. N. Afriat [1], p. 213.

^{7}It seems to have been taken for granted that if a (normalized) demand function h has everywhere symmetric and negative definite Slutsky matrix, then it is invertible (see, for example, P. A. Samuelson [18], p. 377, last paragraph). Of course, by the Inverse Function Theorem, this is true locally but the similar <u>global</u> assertion needs proof. It would follow from the fact

$d_x(p') = (h^2(p,px)-x^2,\ldots,h^\ell(p,px)-x^\ell)$, where $p = (1,p'^2,\ldots,p'^\ell)$. Because

of (1.4) the proof will be concluded if it is shown that the equation

$d_x(p') = 0$ has a unique, regular (i.e., if $dx(\bar{p}') = 0$, then $|Dd_x(\bar{p}')| \neq 0$)

solution. Given that h satisfies (1.6), it is known that this will be the

case (see E. Dierker [6] or A. Mas-Colell [14]) if the following condition

is fulfilled: If $d_x(\bar{p}') = d_x(\bar{\bar{p}}') = 0$, then $\text{sign}|Dd_x(\bar{p}')| = \text{sign}|Dd_x(\bar{\bar{p}}')| \neq 0$.

But this is true since if $d_x(\bar{p}') = 0$, then $|Dd_x(\bar{p}')|$ equals the south-west

$(\ell-1)$ principal minor of $Sh(p,px)$. This ends the proof.

Summarizing:

> If $h \in \mathcal{N}_b^{(r)}$ $(r = 2,\ldots,\infty)$, then $h = h^R$ for some $R \in \mathcal{R}^{(r)}$,
>
> i.e., if h is a regular C^r demand function satisfying (1.4),
>
> (1.5), and (1.6), then it can be generated by a C^r, regular
>
> preference relation on P. (1.8)

We endow $\mathcal{N}^2$ with the topology of uniform convergence on compact sets

of the functions and their first partial derivative, i.e., $\langle h_n \rangle \xrightarrow{1} h$,

---

that, under the above hypothesis, $h^{-1}(x)$ is always a convex set. However, every demonstration of this statement that we know of (one is subsumed in Lemma 4 of L. Hurwicz and H. Uzawa [1], p. 126; for another, see S.A. Afriat [1], p. 222) uses "income-compensation" functions obtained through the integration of the direct demand functions (which, paradoxically, is what is supposed to be dispensable if the function is invertible). Since we are purposely trying to avoid this specific integration step (see Remark 3 in Section II) our proof of (1.7) will proceed, with the help of the boundary condition, by assimilating the problem to one of existence of a unique competitive equilibrium for a one-consumer economy and applying, then, existence and uniqueness results relying on fixed-point index theorems. So, Frobenius is left out but Poincaré is brought in; this is no limitation to our line of attack, but it would, indeed, be nice if an elementary proof could be worked out.

$h_n, h \in \mathcal{H}^2$, if and only if $\|h_n - h\|_J^2 \to 0$ for every compact $J \subset V$. Analogously to $\mathcal{H}^2$, $\mathcal{H}^2$ becomes a complete separable metric space.

II. Theorems

Theorem 1: $\underline{\mathbb{R}_b^{(\infty)}}$ <u>is a dense subset of</u> $\mathbb{R}^2$, <u>and for every compact</u> $K \subset P$, $\mathbb{R}^{(2)}(K)$ <u>is open</u>; i.e., every C^2 preference relation on P can be approximated by a C^∞ one generating a C^∞ demand function; moreover, the set of C^2 preference relations which are regular on a given compact set is C^1-open.

Remark 1: An immediate consequence of Theorem 1 is that $\mathbb{R}^{(2)}$ is a Baire (i.e., residual) subset of $\mathbb{R}^2$ (a complete metric space). Therefore, defining a consumer as a preferences (in $\mathbb{R}^2$)-endowments pair, it is a corollary of Theorem 1 in G. Debreu [2] that the subset of the space of pure exchange n-participants economies (with the product topology) which elements have a discreet and stable price equilibrium set is of the second category. This is a result along the lines of a theorem of S. Smale [19] (see G. Debreu [3], p. 31); it has less scope than the latter since Smale does not assume convexity, but for precisely this reason it cannot be deduced from it.

Remark 2: The generalization of Theorem 1 to a space of nonsaturated preferences can be worked out rather straighforwardly. With the obvious changes in the statements the C^1 topology can be replaced throughout by a C^r topology ($r \geq 1$) without any change in the proofs.

Theorem 2 is the dual of Theorem 1:

Theorem 2: $\mathscr{N}^{(\infty)}_b$ <u>is a dense subset of</u> $\mathscr{N}^2$, <u>and for every compact</u>
$J \subset V$, $\mathscr{N}^{(2)}(J)$ <u>is open.</u> <u>Moreover, every</u> $h \in \mathscr{N}^2$ <u>is the demand function</u>
<u>generated by an upper-semicontinuous, convex, monotone preference</u>
<u>relation R.</u> <u>If</u> $h \in \mathscr{N}^2_b$, <u>then R is continuous.</u>

Remark 3:   The theorem can be proved in two alternative manners. One
is to exploit the duality structure and to reduce the proof to the solution
of a problem which is formally identical to the one solved by Theorem 1
(essentially, direct utility functions would be substituted by indirect ones).
The other, which is perhaps more instructive and is the one we follow,
shows that Theorem 2 is, in substance, a corollary of Theorem 1 (or
vice versa); that is to say, there are not two independent (although formally
similar) theorems but very much the same one.

The second part of the theorem is an integrability result for direct
demand functions very related to the theorems of L. Hurwicz and H. Uzawa [12]
although not covered by them.[8] It is, in essence, the same one that
was obtained recently by S. N. Afriat [1]. However, the proof is different
(see last paragraph): we do not integrate direct demand functions, the tradi-
tional "Antonelli" approach (integration of the indirect demand functions) is
retained, and the theorem is proved by the combination of an approximation
argument (relying on the first part of the theorem and Theorem 1) with some
revealed preference results in [16].

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[8]They require a certain income Lipschitz condition (not implied by the
demand function being C^1) to hold (Condition (E), p. 117 in [12]) in order to
get global integrability. But, as has been shown in G. Debreu [3], the
positivity of prices and commodity bundles can be exploited for this purpose.
One should emphasize that the main thrust of the work of L. Hurwicz and
H. Uzawa is to go beyond the C^1 framework which we certainly do not do.

III. Proof of the Theorems

1. Proof of Theorem 1

For every compact $K \subset P$, $\mathcal{R}^{(2)}(K)$ is open. $\qquad$ (3.1)

Proof: It follows immediately from the compactness of K and the continuity of $M(R,x)$ on $\mathcal{R}^2 \times P$.

If $v : P \to \mathbb{R}$ is a C^r $(r > 1)$ quasi-concave, increasing[9] function, then $R(v) \in \mathcal{R}^r$ will denote the preference relation that it represents.

$\mathcal{R}^{(r)}$ is dense in $\mathcal{R}^r$, $r = 2,\ldots,\infty$. $\qquad$ (3.2)

Proof: Consider a fix $1 < r \le \infty$. Let $R \in \mathcal{R}^r$ and $K \subset P$ be a compact set. Take $u : P \to \mathbb{R}$ to be a C^r function representing R. Note that (letting $e_n = \left(\frac{1}{n},\ldots,\frac{1}{n} \right)$) if we define $u'_n : P \to \mathbb{R}$ $(0 < n < \infty)$ by $u'_n(x) = u(x + x^1 e_n)$, then u'_n is a C^r quasi-concave function satisfying: $D_{x^1} u'_n(x) \gg 0$ for every $x \in P$. Obviously, $\langle R(u'_n) \rangle \overset{1}{\to} R$. Therefore we can assume, for our purposes, that

$$D_{x^1} u(x) \gg 0 \text{ for every } x \in P. \qquad (3.3)$$

Let $P_1 = (0;\infty)^{\ell-1}$ and denote the generic element of P_1 by $y = (y^2,\ldots,y^\ell)$. Take a concave function $f : \bar{P}_1 \to \mathbb{R}$ of class C^r bounded above by ℓ and having strictly positive gradient and nonsingular Hessian at every $y \in P_1$ (for example, $f(y) = \ell - \sum_{i=2}^{\ell} \frac{1}{y^i+1}$).

---

[9] I.e., $Dv(x) > 0$ for every $x \in P$.

Roughly speaking, we are going to form a new preference relation by substracting vertically a fraction of the function f from every indifference curve (see Fig. 1). For every $0 < n \leq \infty$ define the C^r function $u_n : P \to \mathbb{R}$ by

$$u_n(x) = u(x^1 + \frac{1}{n} f(x^2,\ldots,x^\ell),x^2,\ldots,x^\ell).$$

By (3.3):

For every $x \in P$ and $0 < n \leq \infty$, $D_x u_n(x) >> 0$. $\qquad\qquad$ (3.4)

Since u is quasi concave and increasing, and f is concave, u_n is quasi concave. Therefore $R(u_n) \in \mathbb{R}^r$. Moreover, $\langle R(u_n) \rangle \overset{1}{\to} R$. We want to show that for every $n < \infty$, $R(u_n)$ is regular.

For every $0 < n \leq \infty$, $\bar{x} \in P$, it follows by the I.F.T. and (3.4) that the equation $u_n(x) = u_n(\bar{x})$ can be solved so as to express x^1 as a C^{r+1} function of $(x^2,\ldots,x^\ell)$ in a neighborhood of $(\bar{x}^2,\ldots,\bar{x}^\ell)$. Let $\Phi_n^{\bar{x}}$ be this function.

Take any $x' \in P$ and $0 < n < \infty$, and regard them as fixed. Define $x'' = (x'^1 + \frac{1}{n} f(x'^2,\ldots,x'^\ell), x'^2,\ldots,x'^\ell)$. It is immediately seen (Fig. 1) that, in a neighborhood of $(x'^2,\ldots,x'^\ell)$:

$$\Phi_n^{x'}(x^2,\ldots,x^\ell) = \Phi_\infty^{x''}(x^2,\ldots,x^\ell) - \frac{1}{n} f(x^2,\ldots,x^\ell)$$

and

$$u(\Phi_\infty^{x''}(x^2,\ldots,x^\ell),x^2,\ldots,x^\ell) = u(x'').$$

Thus

$$H\Phi_n^{x'}(x'^2,\ldots,x'^\ell) = H\Phi_\infty^{x''}(x'^2,\ldots,x'^\ell) - \frac{1}{n} Hf(x'^2,\ldots,x'^\ell)$$

is positive definite which implies (actually, it is equivalent) that $R(u_n)$ is regular at x' . (Let $zHu_n(x')^T z = 0$ for some $z \in \mathbb{R}^\ell$, $z \neq 0$, $Du_n(x')z = 0$,
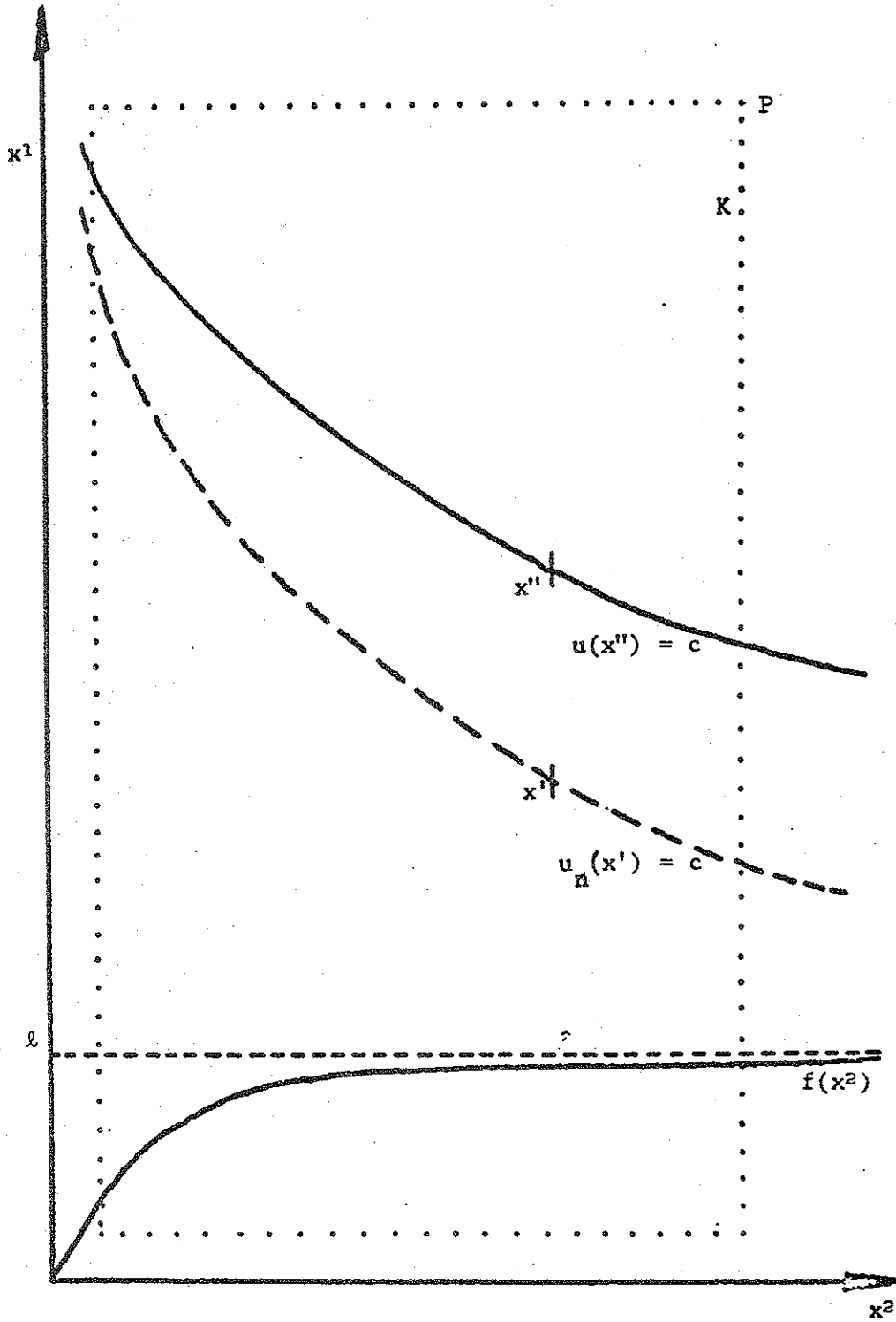
Figure 1

then if $\bar{z} = (z^2, \ldots, z^\ell)$, a straighforward calculation yields

$\bar{z} H \Phi_n^{x'}(x'^2, \ldots, x'^\ell)^T \bar{z} = 0$; of course, $\bar{z} \neq 0$). This ends the proof.

$$\mathbb{R}^\infty \text{ is dense in } \mathbb{R}^{(2)} \tag{3.5}$$

Proof: Let $R \in \mathbb{R}^{(2)}$ and $K, K', K'' \subset P$ be compact, convex sets with $K \subset \text{Int } K'$, $K' \subset \text{Int } K''$. By definition, R is representable by a quasi-concave C^2 function $u : P \to \mathbb{R}$ having no critical points and such that $h^{R(u)}$ is C^1. It has been noted by R. Aumann[10] that those hypothesis on u met the sufficiency conditions of a theorem of W. Fenchel ([8], see Chap. III, Sec. 8, and the historical note on p. 143) for the existence of a strictly increasing C^2 function $f: \mathbb{R} \to \mathbb{R}$, making $f \circ u | K''$ concave.[11] Therefore, there is a C^2, concave function $v' : K'' \to \mathbb{R}$, representing R on K''. We can let $v : \mathbb{R}^\ell \to \mathbb{R}$ be a C^2 concave, increasing function such that $v | K' = v' | K'$ (for example, $v = \inf_{x \in K'} v_x$ where $v_x : \mathbb{R}^\ell \to \mathbb{R}$ is given by $v_x(y) = v'(x) + Dv'(x)y$).

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[10] I owe this reference to Professor G. Debreu.

[11] Actually, this is not difficult to prove: let $L = \{(s,x) \in S^\ell \times K'' : D_u(x)s = 0\}$ and for every $(s,x) \in S^\ell \times K''$, define $\gamma(s,x) = sHu(x)^T s$, $\lambda(s,x) = s^T Du(x)D(x)^T s \geq 0$. By the regularity hypothesis, continuity and compactness there is an open set $A \subset S^\ell x K''$ such that $L \subset 0$ and $\gamma(s,x) < 0$ for every $(s,x) \in A$. Let $T = (S^\ell x K'') \sim A$. Clearly, $\lambda(s,x) > 0$ for every $(s,x) \in T$. Therefore, $r = \sup_{(s,x) \in T} \frac{\gamma(s,x)}{\lambda(s,x)}$ is finite, and if we let $n > \max\{0,r\}$, then for every $(s,x) \in S^\ell x K''$ the expression $\gamma(s,x) + n\lambda(s,x)$ is negative. Take $f : R \to R$ to be $f(t) = -e^{-nt}$ (see de Finetti [14]). Then $Df(t) > 0$ and $\frac{D^2 f(t)}{Df(t)} = n$, $t \in R$. Hence, for every $x \in K''$, $H(f \circ u)(x) = Df(u(x))(Hu(x) + n^T Du(x)Du(x))$ is negative definite which implies that $f \circ u | K''$ is concave.

If $\langle v_n \rangle$, $v_n : P \to \mathbb{R}$ is a sequence of C^∞ concave, increasing functions converging to v, uniformly on K, in the first and second partial derivatives, then $\|g_{R(v_n)} - g_R\|_K^1 \to 0$. Hence our problem reduces to find such a sequence. This is standard: let $\Psi_n : \mathbb{R}^\ell \to \mathbb{R}$ be a C^∞ function such that $\Psi_n(x) = 0$ if $x \notin B_{1/n}^\ell$, $\Psi_n(x) > 0$ if $x \in B_{1/n}^\ell$ and $\int_{\mathbb{R}^\ell} \Psi_n = 1$. Define $v_n : \mathbb{R}^\ell \to R$ by $v_n(x) = \int_{\mathbb{R}^\ell} v(t)\Psi_n(x+t)dt$ (i.e., $v_n = v * \Psi_n$). Then v_n is C^∞ and $\langle v_n \rangle$ converges to v, uniformly on K, in the first and second partial derivatives (see J. Munkres [17], Lemma 4.1 and its proof, pp. 39-40). Note that, with a change of variable, $v_n(x) = \int_{\mathbb{R}^\ell} v(s-x)\Psi_n(s)ds$, $x \in P$. From this and the nonnegativity of Ψ_n the concavity and increasingness of v_n follow.

$$\mathcal{R}_b^{(\infty)} \text{ is dense in } \mathcal{R}^{(\infty)} . \tag{3.6}$$

<u>Proof</u>: Let $R \in \mathcal{R}^{(\infty)}$ and $K \subseteq P$ be a compact, convex set. We can assume that (see proof of (3.5)) there is a C^∞, concave function $u : \mathbb{R}^\ell \to \mathbb{R}$ such that $u|K$ represents R on K. For every $n > 1$ define a concave, increasing, C^∞ function $u_n : P \to R$ by $u_n(x) = u(x) + \frac{1}{n}\sum_{i=1}^{\ell} \log x^i$. Obviously, $\|g_{R(u_n)} - g_R\|_K^1 \to 0$, and it is easily seen that for every u, $R(u_n)$ is regular and satisfies the boundary condition, i.e., $R(u_n) \in \mathcal{R}_b^{(\infty)}$.

The density of $\mathcal{R}_b^{(\infty)}$ in $\mathcal{R}^2$ follows by combining (3.2), (3.5), and (3.6). Hence the proof of the theorem is completed.

2. Proof of Theorem 2:

For every compact $J \subset V$, $\varkappa^{(2)}(J)$ is open. (3.7)

Proof: Straightforward.

$\varkappa_b^{(\infty)}$ is a dense subset of $\varkappa^2$. (3.8)

Proof: For every $s > 0$, define $P_s = P + \{(s,\ldots,s)\}$, $A_s = \text{Int}(B_s^{\ell} \cap P)$ and $V_s = A \times (\frac{1}{s} ; \infty)$. Denote by $\varkappa_s^r$ (and, similarly, $\varkappa_s^{(r)}$) the set of functions $h : V_s \to P$ which satisfy at every point of its domain the conditions defining a C^r (resp. regular) demand function ((1.4), (1.5)).

Let $h \in \varkappa^2$ and $J \subset V$ be compact. Take an N such that $J \subset V_N$ and $h(J) \subset P_N$.

Now we claim:

There is a sequence $\langle h_n \rangle$, $n > N$, $h_n \in \varkappa_n^{(2)}$ such that $\|h_n - h\|_J^1 \to 0$; $P_n \subset h_n(V_n)$ and $h \,|\, h^{-1}(P_n) \cap S^{\ell} \times (0;\infty)$

has a C^1 inverse. (3.9)

In order to prove (3.9), let $f : P \to \mathbb{R}$ be a C^{∞}, concave linear homogeneous function such that:

i) $f|S_+^{\ell}$ is bounded;

ii) for every $p \in P$, $Df(p) \gg 0$ and $\text{rkHf}(p) = \ell - 1$;

iii) if $\langle p_n \rangle \to p$, $p \in P$, $p^i = 0$, $p \neq 0$, then $\lim D_i f(p_n) = \infty$.

We could take $f(p) = (\sum_{i=1}^{\ell} (p^i)^{1/2})^2$.

Let $0 < \delta < \min\left\{\dfrac{1}{n}, \dfrac{1/N}{\displaystyle\sup_{p\in A_n} f(p)}\right\}$ and define $h_n : V_n \to P(n > N)$ by

$h_n(p,w) = h\left(p, w - \dfrac{\delta}{n}f(p)\right) + \dfrac{\delta}{n}Df(p)$. Easy calculations yield:

   i) for every (p,w), $(\lambda p, \lambda w) \in Vn$, $\lambda > 0$, $ph_n(p,w) = w$ and

   $h_n(\lambda p, \lambda w) = h_n(p,w)$;

   ii) for every $(p,w) \in Vn$, $Sh_n(p,w) = Sh(p,w) + \dfrac{1}{n}Hf(p)$. Therefore

   $rkSh_n(p,w) = \ell-1$;

   iii) $\|h_n - h\|_J^1 \to 0$;

Moreover:

   iv) $P_n \subset h_n(V_n)$ and $h_n|h_n^{-1}(P_n) \cap S^\ell x(0;\infty)$ has a C^1 inverse.

To see this for every $x \in P_n$, define $d_x : P \to P + \{-x\}$ by

$d_x(p) = h_n\left(\dfrac{p}{\|p\|}, \dfrac{p}{\|p\|}x\right)$. Since $\dfrac{p}{\|p\|}x \geq \dfrac{1}{n}$ for every p, $x \in P$, this is

always possible. Because of the boundary condition on f, d_x satisfies: if

$<p_n> \to p$, $0 \neq p \not\in P$, $p_n \in P$, then $\lim \|d_x(p_n)\| = \infty$. The conclusion now

follows as in (1.7).

   Let $h' \in \mathcal{N}_s^{(2)}$ ($s > 0$) be such that $J \subset V_s$, $h'(J) \subset P_s \subset h'(V_s)$,

   and $h'|h'^{-1}(P_s) \cap S^\ell x(0;\infty)$ has a C^1 inverse, then there is a

   sequence $\langle h_n \rangle$, $h_n \in \mathcal{N}_b^{(\infty)}$, such that $\|h_n - h'\|_J^1 \to 0$.   (3.10)

To prove (3.10) let $K \subset P_s$ be a compact set with $h'(J) \subset$ Int K. Denote

by $g : P_s \to S^\ell$ the projection on S^ℓ of the inverse of $h'|\overline{h'^{-1}(P_s) \cap S^\ell x(0;\infty)}$.

Applying, mutatis mutandis, the reasoning leading to (1.8) and some of the

arguments in the proof of (3.5) we immediately find an $R' \in \mathcal{R}^2$ with

$g_{R'}|K = g|K$ (implying $h^{R'}|J = h'|J$). Take $\langle R_n \rangle \xrightarrow{1} R'$, $R_n \in \mathcal{R}_b^{(\infty)}$ (Theorem 1). Clearly $h^{R_n} \in \mathcal{U}_b^{(\infty)}$ for every n. Note that $R_n \in \mathcal{R}^{(2)}(K)$, for n greater than a large enough N'.

$$M(R_n, x) \text{ is bounded away from zero for } x \in K, n > N'. \qquad (3.11)$$

Moreoever ((1.3) and the continuity of the demand correspondence, see footnote 5 and [15] pg 16):

$$\|h^{R_n} - h'\|_J^0 \to 0. \qquad (3.12)$$

Applying the I.F.T., (3.11) and (3.12) yield the convergence (uniform on J) of the partial derivatives of h^{R_n} to the partial derivatives of h'. So we have $\|h^{R_n} - h'\|_J^1 \to 0$ finishing the proof of (3.10).

The proof of (3.8) follows combining (3.9) and (3.10). Note that if $\langle h_n \rangle$ is the sequence which existence is asserted in (3.9) the fact that $h(J) \subset P_N$ does eventually imply $h_n(J) \subset P_n$.

If $h \in \mathcal{U}^2$ then h can be generated by an upper semi-continuous monotone, convex preference relation R, (i). If h $h \in \mathcal{U}_b^2$ then R is continuous (ii). $\qquad (3.13)$

Proof: It has been seen in the proof of (3.8) that h is the limit of a sequence of demand functions satisfying the Strong Axiom of Revealed preference (i.e., generated by preferences). Hence it satisfies likewise this Axiom (A. Mas-Colell [16], Proposition 2). Therefore i) and ii) follow, respectively, from Propositions 1 and 5 in [16].

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