Albanese and Picard 1-Motives in Positive Characteristic

by

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Abstract

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The goal of this Thesis is to develop the theory of Picard and Albanese 1-motives attached to a variety X defined over a perfect field of positive characteristic, and to relate these 1-motives to the étale cohomology groups of X. This should be viewed as a generalization of the classical theory of Picard and Albanese varieties attached to a smooth and proper variety X. Moreover, giving such a theory allows us to relate the dimension- and codimension-one étale cohomology groups in the most natural ('motivic') way possible; in particular, independence-of-ℓ type results in dimension- and codimension-one are automatic once one has developed such a theory.

In the case of a base field of characteristic zero, the corresponding 1-motives have been constructed and studied in previous work of Barbieri-Viale and Srinivas. When one deals with a positive-characteristic base field, new difficulties arise due to the fact that resolution of singularities in positive characteristic is still an open problem. This forces us to introduce new methods, especially a strong form of de Jong's results that allows us to resolve (in a weak sense) an arbitrary separated finite type k-scheme by a smooth Deligne-Mumford stack. A large part of this thesis is devoted to preliminary results on divisors and cycle class maps for Deligne-Mumford stacks that we need when applying the methods of Barbieri-Viale and Srinivas with stacks rather than schemes.

In the end, we manage to construct the Picard 1-motives of an arbitrary separated finite type k-scheme with no additional assumptions, and prove various functoriality and compatibility properties for these 1-motives. The situation with the Albanese 1-motives is more complicated; over an arbitrary perfect field, we only manage to show that the Albanese 1-motives of X exist after possibly base extending X via a finite field extension K/k. We show, however, that in the case that k is a finite field or an algebraically closed field, no such field extension is necessary. The case of a finite field uses a method for descending 1-motives along an extension of finite fields when the 1-motive is only given up to isogeny. This method may be of some independent interest.

We conclude the thesis with a brief Chapter indicating how to use this theory to prove some new independence-of-ℓ results in dimension- and codimension-one.

To my parents

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Chapter 1

Introduction

1.1 Picard and Albanese for smooth projective varieties

Our goal in this thesis is to generalize the theory of Picard and Albanese varieties for smooth projective varieties in positive characteristic to arbitrary separated finite type k-schemes. To make sense of this, we will first briefly review the theory of Picard and Albanese varieties in the smooth and projective case, and then indicate what properties of this theory we would like to generalize to the singular/non-proper case.

1.1.1. Let X be a smooth and projective variety over an algebraically closed field k (of any characteristic). In the 1950's the work of multiple people (in particular Weil, Chow, and Matsusaka [Mat52]) established the theory of the Picard and Albanese varieties associated to X. This had been known in the case $k = \mathbb{C}$ far earlier, but only in the 1950's were algebraic constructions given. This theory was then incorporated by Grothendieck into the more general theory of Picard functors of schemes [FGA, Exp. V-VI]. For an excellent review of the complicated history of these ideas, see the Introduction of [Kle06]. Here we will only review the modern, functorial definitions of the Picard and Albanese varieties. The article [Kle06] is also an excellent place for proofs of the following facts.

1.1.2. Fix a closed point $x \in X$, and also (by abuse of notation) let $x : \operatorname{Spec} k \hookrightarrow X$ denote the corresponding closed immersion. The Picard variety of X, which we denote by $\mathbf{P}(X)$, is characterized by the following universal property: $\mathbf{P}(X)$ is an abelian variety (in particular a smooth projective variety) such that for any other (connected) smooth projective variety Y over k, giving a map $f : Y \to \mathbf{P}(X)$ is the same as giving the following data: (1) a line bundle $\mathscr{L}$ on $X \times Y$, and (2) an *x-rigidification* of $\mathscr{L}$, i.e., an isomorphism $\alpha_{\mathscr{L}} : x_Y^* \mathscr{L} \xrightarrow{\sim} \mathcal{O}_Y$, where $x_Y : Y \to X \times Y$ is the section determined by x. This property uniquely characterizes $\mathbf{P}(X)$ once we have fixed the *Poincaré bundle* $\mathcal{P}$ on $X \times \mathbf{P}(X)$, together with an x-rigidification $\alpha_{\mathcal{P}} : x_{\mathbf{P}(X)}^* \mathcal{P} \xrightarrow{\sim} \mathcal{O}_{\mathbf{P}(X)}$. Here $x_{\mathbf{P}(X)} : \mathbf{P}(X) \to X \times \mathbf{P}(X)$ is the section determined by x. Then for any map $f : Y \to \mathbf{P}(X)$, the corresponding line bundle $\mathscr{L}$ on $X \times Y$ is obtained

by pulling back $\mathcal{P}$ along $1 \times f : X \times Y \to X \times \mathbf{P}(X)$, and the section $\alpha_{\mathscr{L}}$ is obtained by pulling back $\alpha_{\mathcal{P}}$ along f.

1.1.3. The Albanese variety of X, denoted $\mathbf{A}(X)$ or $\mathrm{Alb}(X)$, is characterized by the following universal property: again, fix a closed point $x \in X$. Then $\mathbf{A}(X)$ is an abelian variety with the property that for any other abelian variety V over k, giving a homomorphism of abelian varieties from $\mathbf{A}(X)$ to V is equivalent to giving a map $f : X \to V$ such that $f(x)$ is the identity element of V. This property uniquely characterizes $\mathbf{A}(X)$ once we have fixed the canonical map $A : X \to \mathbf{A}(X)$ such that $A(x) = 0$. In the case where X is a curve, this map is known as the Abel map; in the general case, one calls it the *Albanese map*.

1.1.4. The Albanese and Picard varieties have the key property that they are *dual* to one another in the sense of abelian varieties, i.e., $\mathbf{A}(X)$ is the Picard variety of $\mathbf{P}(X)$ and vice versa [Kle06, 9.5.25].

1.1.5. In [FGA], Grothendieck generalized this theory by introducing the Picard functor of an arbitrary scheme X over a base S. To define this functor, first consider the 'naïve' Picard functor $(Sch/S)^{op} \longrightarrow \mathrm{Set}$ sending Y to $\mathrm{Pic}(X \times_S Y)$. This functor is not representable as it is not a sheaf, even for the Zariski topology. So we define the *Picard functor* of X/S, also denoted by $\mathbf{Pic}_{X/S}$ to be the *sheafification* of the 'naïve' Picard functor for the fppf topology. Then we have the following key result:

Theorem. *[FGA, p. 236-12] Let X be a proper, separated finite type k-scheme, where k is a field. Then the Picard functor $\mathbf{Pic}_{X/k}$ is representable by a locally finite type k-group scheme.*

1.1.6. If X is smooth and proper over an algebraically closed field k, the classically defined Picard and Albanese varieties of X can be written in terms of $\mathbf{Pic}_{X/k}$; namely, $\mathbf{P}(X) = \mathbf{Pic}_{X/k}^{0,red}$ is the reduction of the connected component of $\mathbf{Pic}_{X/k}$ containing the identity, and $\mathbf{A}(X) = \mathbf{P}(X)^{\vee}$ is the dual abelian variety of $\mathbf{P}(X)$.

1.1.7. More generally, whenever X is a smooth and proper variety over any *perfect* field k we define the Picard and Albanese varieties of X by the formulas $\mathbf{P}(X) := \mathbf{Pic}_{X/k}^{0,red}$ and $\mathbf{A}(X) := \mathbf{P}(X)^{\vee}$. Note that these are in fact abelian varieties over k. If X has a k-point, then these abelian varieties can be given the same functorial description as in 1.1.2-1.1.3.

Remark 1.1.8. For a general (non-perfect) field k, one cannot define the Picard and Albanese varieties of a smooth proper X over k (at least not by this formula). This is because over non-perfect fields, the reduction of a group scheme is not necessarily a group scheme, and so one does not know that the formula $\mathbf{Pic}_{X/k}^{0,red}$ defines a group scheme, let alone an abelian variety. See [Mil12, 6.5].

Relation to Weil cohomologies

1.1.9. We refer the reader to [And04, 3.3-3.4] for the definition of a Weil cohomology. These are cohomology theories defined on the category of smooth projective varieties over a field k. A Weil cohomology theory is a functor $(SmProj/k)^{op} \to \mathcal{C}$ to an abelian category $\mathcal{C}$ that can be vaguely described as 'linear algebraic' in nature. Rather than try to make this notion precise, we simply give the primary examples of Weil cohomologies below. We will only consider these theories in this thesis, and in fact after this introductory chapter we will focus solely on ℓ-adic étale cohomology.

1.1.10. If char $k = 0$, we have the following Weil cohomology theories:

- Fix an algebraic closure $k \hookrightarrow \overline{k}$. Then for each prime number ℓ, we have the ℓ-adic étale cohomology groups $H_\ell^*(X) := H_{et}^*(X_{\overline{k}}, \mathbb{Z}_\ell)$. These are objects in $\text{Rep}_{\mathbb{Z}_\ell}(\text{Gal}(\overline{k}/k))$, which is defined as the category of finitely generated modules over $\mathbb{Z}_\ell$ given a continuous representation of $\text{Gal}(\overline{k}/k)$. Therefore ℓ-adic étale cohomology defines a contravariant functor $H_\ell^i(-) : (SmProj/k)^{op} \to \text{Rep}_{\mathbb{Z}_\ell}(\text{Gal}(\overline{k}/k))$.

- Algebraic de Rham cohomology $H_{dR}^*(X)$, defined as the hypercohomology of the de Rham complex $\Omega_{X/k}^*$. Each group $H_{dR}^i(X)$ is a finite-dimensional vector space over k with a filtration coming from the (degenerate) spectral sequence $H^q(X, \Omega_{X/k}^p) \Rightarrow H_{dR}^{p+q}(X)$. In other words, $H_{dR}^i(X)$ is an object of the category $FVS(k)$ of vector spaces over k endowed with a finite increasing filtration, and de Rham cohomology defines a contravariant functor $H_{dR}^i(-) : (SmProj/k)^{op} \to FVS(k)$.

- If $k \subseteq \mathbb{C}$, we have the singular cohomology $H_{sing}^*(X) := H_{sing}^*(X(\mathbb{C}), \mathbb{Z})$. Each group $H_{sing}^i(X)$ is a pure Hodge structure of weight i; i.e., $H_{sing}^i(X)$ is a finitely generated $\mathbb{Z}$-module such that $H_{sing}^i(X) \otimes \mathbb{C}$ has a Hodge decomposition

$$H_{sing}^i(X) \otimes \mathbb{C} = \bigoplus_{p+q=i} H^{pq}(X),$$

where $H^{pq}(X)$ is a $\mathbb{C}$-vector space satisfying $\overline{H^{pq}(X)} = H^{qp}(X)$. This defines a functor $H_{sing}^i(-) : (SmProj/k)^{op} \to HS$, where HS denotes the category of pure Hodge structures.

1.1.11. If char $k = p > 0$, then we have the following theories:

- For each prime $\ell \neq p$, and fixing an algebraic closure $k \hookrightarrow \overline{k}$, we have ℓ-adic étale cohomology defined as above. Namely, for a smooth projective variety X over k, we have finitely generated $\mathbb{Z}_\ell$-modules $H_\ell^*(X) := H_{et}^*(X_{\overline{k}}, \mathbb{Z}_\ell)$ endowed with a continuous action of $\text{Gal}(\overline{k}/k)$.

- If k is perfect, we have the crystalline cohomology H_{cris}^*. For a smooth projective variety X/k, $H_{cris}^*(X) := H_{cris}^*(X/W(k))$ is a finitely generated module over the ring

of Witt vectors $W(k)$. If $\sigma : W(k) \to W(k)$ is the lift of the Frobenius endomorphism of k, then each group $H^i_{cris}(X)$ is endowed with a σ-linear endomorphism F which becomes a bijection when tensored with the fraction field $K := W(k)[1/p]$. This gives $H^i_{cris}(X)$ the structure of an *F-isocrystal* over $W(k)$. We let $FCrys(k)$ denote the category of F-isocrystals over $W(k)$, so that crystalline cohomology defines a functor $H^i_{crys}(-) : (SmProj/k)^{op} \to FCrys(k)$.

Tate Twists

1.1.12. In each of the categories $\mathrm{Rep}_{\mathbb{Z}_\ell}(\mathrm{Gal}(\overline{k}/k))$, $FVS(k)$, HS, $FCrys(k)$ that is a target for a Weil cohomology described above, there is special object called the *Tate object*. This object is necessary for describing the relation between the Picard and Albanese varieties of X and the cohomology of X. The Tate object is described as follows in each of the preceding cohomology theories:

- For ℓ-adic étale cohomology over k with a fixed embedding $k \hookrightarrow \overline{k}$, the Tate object is denoted by $\mathbb{Z}_\ell(1)$ and is defined as $\varprojlim_n \mu_{\ell^n}$ with $\mathrm{Gal}(\overline{k}/k)$ acting through the ℓ-adic cyclotomic representation. Note that upon forgetting the Galois action, there is a non-canonical isomorphism $\mathbb{Z}_\ell(1) \cong \mathbb{Z}_\ell$ obtained by choosing a compatible system of primitive ℓ^n-th roots of unity.

- For algebraic de Rham cohomology over k, the Tate object is denoted $k(1)$, and is equal to k as a vector space, with increasing filtration given by $F^{<-2} = 0$, $F^{\leq -2} = k$.

- For singular cohomology, the Tate object $\mathbb{Z}(1)$ is given by $2\pi i \mathbb{Z}$, with Hodge structure given by placing $2\pi i \mathbb{Z} \otimes \mathbb{C}$ in bidegree $(-1, -1)$.

- For crystalline cohomology the Tate object is $W(k)$ with σ-linear endomorphism $F : W(k) \to W(k)$ given by $\frac{1}{p}\sigma$.

For each cohomology group $H^*_{(-)}(X)$ (where $(-)$ signifies any of the above Weil cohomologies) and any integer $n \in \mathbb{Z}$, we define $H^*_{(-)}(X)(n) := H^*_{(-)}(X) \otimes 1(1)^{\otimes n}$, where $1(1)$ signifies the Tate object in any of the above Weil cohomology theories.

Relation between $\mathbf{P}(X)$ and $H^1_{(-)}(X)(1)$

1.1.13. Fix a smooth projective variety X over a perfect field k. For each of the above Weil cohomology theories, the cohomology group $H^1_{(-)}(X)(1)$ can be reconstructed from the Picard variety $\mathbf{P}(X)$ in a canonical way. We briefly review these constructions below:

- For ℓ-adic étale cohomology, we have a canonical $\mathrm{Gal}(\overline{k}/k)$-equivariant isomorphism

$$H^1_\ell(X)(1) \cong T_\ell \mathbf{P}(X) := \varprojlim_n \mathbf{P}(X)(\overline{k})[\ell^n]$$

(where for an abelian group A, $A[m]$ denotes the m-torsion in A). This is proved via the Kummer exact sequence of étale sheaves

$$0 \to \mu_{\ell^n} \longrightarrow \mathbb{G}_{m,X} \xrightarrow{\cdot \ell^n} \mathbb{G}_{m,X} \to 0$$

and taking the inverse limit. As usual, $T_\ell A$ denotes the Tate module of A for any abelian variety A.

- For de Rham cohomology over a field k of characteristic 0, we have a canonical isomorphism
$$H^1_{dR}(X)(1) \cong \mathbf{Lie}(\mathbf{P}(X)^\natural),$$
where $\mathbf{P}(X)^\natural$ is the universal extension of $\mathbf{P}(X)$ by a vector group (see [MaMe74, I-4]). Here by $\mathbf{Lie}(-)$ we mean the functor taking any group scheme to its associated Lie algebra (see [Mil12, Ch. XI]). For any abelian variety A, we let $T_{dR}(A) := \mathbf{Lie}(A^\natural)$. This follows the notation of [Del74, Sect. 10].

- For singular cohomology with $k \subseteq \mathbb{C}$, using the exponential sequence
$$0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathcal{O}_X \xrightarrow{exp} \mathcal{O}_X^* \longrightarrow 0$$
we have a canonical isomorphism
$$H^1_{sing}(X)(1) \cong \mathrm{Ker}(\mathbf{Lie}(\mathbf{P}(X)) \xrightarrow{exp} \mathbf{P}(X)).$$
Moreover, the Hodge filtration is defined by setting $Fil^{-1} \subset H^1_{sing}(X)(1)$ to be
$$Fil^{-1} := \mathrm{Ker}(H^1_{sing}(X)(1) \otimes \mathbb{C} \longrightarrow \mathbf{Lie}(\mathbf{P}(X))).$$
Following [Del74, Sect. 10], for any abelian variety A we let $T_\mathbb{Z} A := \mathrm{Ker}(\mathbf{Lie}(A) \to A)$.

- For crystalline cohomology over a perfect field of positive characteristic, we have that $H^1_{crys}(X)(1)$ is canonically isomorphic to the Dieudonné module attached the Barsotti-Tate group of $\mathbf{P}(X)$ [Ill79, II.3.11.2]. For any abelian variety A, we let $T_{crys}A$ be the Dieudonné module attached to the Barsotti-Tate group of A.

Relation between $\mathbf{A}(X)$ and $H^{2d-1}_{(-)}(X)(d)$

Again let X be smooth and projective over a perfect field k. In each of the above cohomology theories, we have *Poincaré duality*, which leads to isomorphisms

$$H^1_{(-)}(X)^\vee \xrightarrow{\sim} H^{2d-1}_{(-)}(X)(d)/(torsion).$$

(Note that if we replace 1 and $2d-1$ by i and $2d-i$ we would have to mod out by torsion on both sides, but $H^1_{(-)}(X)$ is already torsion-free for each cohomology theory as follows from the description of $H^1_{(-)}(X)$ in terms of $\mathbf{P}(X)$ above).

On the other hand, for any abelian variety A and its dual $A^\vee$, Deligne [Del74, Sect. 10] constructs canonical perfect pairings

$$T_{(-)}(A) \times T_{(-)}(A^\vee) \to 1(1)$$

for the de Rham, Hodge, and ℓ-adic realizations (recall that $1(1)$ denotes the Tate object in any Weil cohomology). In [ABV03, Sect. 2] there is constructed the corresponding pairing for the crystalline realization. In each Weil cohomology the pairing is ultimately induced by the Poincaré bundle on $A \times A^\vee$; for example, the pairing $T_\ell A \times T_\ell A^\vee \to \mathbb{Z}_\ell(1)$ is precisely the classical Weil pairing. These perfect pairings lead to canonical isomorphisms

$$T_{(-)}(A^\vee) \xrightarrow{\sim} (T_{(-)}(A)(-1))^\vee.$$

Combining these isomorphisms with the Poincaré duality isomorphisms above, along with the fact that $\mathbf{A}(X) = \mathbf{P}(X)^\vee$, we see that we have natural isomorphisms (for each Weil cohomology)

$$T_{(-)}(\mathbf{A}(X)) \xrightarrow{\sim} H^{2d-1}_{(-)}(X)(d)/(torsion).$$

1.2 Generalizing to arbitrary separated finite type k-schemes

1.2.1. Now we turn to the category of arbitrary separated finite type k-schemes. Our goal is to find a theory of Picard and Albanese geometric objects associated to X which bear the same relation to the cohomology groups of X as the classical Picard and Albanese varieties do for smooth and projective varieties. First note that the Weil cohomologies discussed in the previous section for smooth projective varieties generalize to arbitrary separated finite type k-schemes as follows:

- ℓ-adic étale cohomology generalizes immediately to separated finite type k-schemes. Each group $H^i_\ell(X)$ comes with a continuous action of $\mathrm{Gal}(\overline{k}/k)$ as in the smooth projective case.

- Singular cohomology also generalizes immediately to separated finite type k-schemes when $k \subseteq \mathbb{C}$. Each group $H^i_{sing}(X)$ comes with a *mixed Hodge structure*, i.e., an increasing filtration $W^\bullet$ such that each quotient W^i/W^{i-1} is a pure Hodge structure of weight i.

- de Rham cohomology does not generalize immediately to arbitrary separated finite type k-schemes. However, by using the comparison theorem $H^i_{dR}(X) \cong H^i_{sing}(X)$ for X smooth and $k \subseteq \mathbb{C}$, together with the cohomological descent isomorphism $H^i_{sing}(X) \cong H^i_{sing}(X_\bullet)$ for any proper hypercover $X_\bullet \to X$ [Del74], we conclude that $H^i_{dR}(X_\bullet) := H^i(X_\bullet, \Omega^\bullet_{X_\bullet})$ is independent of the proper hypercover $X_\bullet \to X$ if $X_\bullet$ has smooth terms. Therefore we can define $H^*_{dR}(X)$ to be $H^*_{dR}(X_\bullet)$ for any proper hypercover $X_\bullet \to X$.

- Crystalline cohomology is not known to generalize to arbitrary separated finite type k-schemes if one wishes the resulting groups $H^i_{crys}(X)$ to be finitely generated $W(k)$-modules. However, if one only asks for $H^i_{crys}(X)$ to a finite-dimensional vector space over the fraction field $K(W(k))$, then rigid cohomology gives a good generalization of crystalline cohomology to arbitrary separated finite type k-schemes [Ked06].

1.2.2. To motivate the introduction of the category of 1-motives, we start with the following naïve question:

Question. Fix a separated finite type k-scheme X of dimension d, where k is a perfect field. Do there exist abelian varieties $\mathbf{P}(X)$, $\mathbf{A}(X)$ such that for all primes $\ell \neq \operatorname{char} k$ we have natural isomorphisms $T_\ell \mathbf{P}(X) \cong H^1_\ell(X)(1)$ and $T_\ell \mathbf{A}(X) \cong H^{2d-1}_\ell(X)(d)$? More generally, do $\mathbf{P}(X)$, and $\mathbf{A}(X)$ have natural relations to the other Weil cohomology groups $H^1_{(-)}(X)(1)$, $H^{2d-1}_{(-)}(X)(d)$?

The answer to this question is immediately seen to be negative, since (for example) the cohomology group $H^1_\ell(X)(1)$ can be one-dimensional (e.g., for $X = \mathbb{G}_m$ or X a rational nodal curve), whereas $T_\ell A$ is always even-dimensional for any abelian variety A. However, by looking more carefully at what goes wrong one quickly arrives at a replacement for the category of abelian varieties wherein a theory of Picard and Albanese objects can be reasonably searched for.

1.2.3. First consider $X = \mathbb{G}_m$. Let $\overline{X} = \mathbb{P}^1$, and let $D = \mathbb{P}^1 - \mathbb{G}_m = \{0, \infty\}$. The long exact sequence of relative cohomology for the triple $(X, \overline{X}, D)$ leads to a short exact sequence

$$0 \longrightarrow H^1(\mathbb{P}^1, \mathbb{Z}_\ell(1)) \longrightarrow H^1(\mathbb{G}_m, \mathbb{Z}_\ell(1)) \longrightarrow \operatorname{Ker}(H^2_D(\mathbb{P}^1, \mathbb{Z}_\ell(1)) \to H^2(\mathbb{P}^1, \mathbb{Z}_\ell(1))) \longrightarrow 0.$$

Here the left-hand term is 0, while the right-hand term identifies with

$$\operatorname{Ker}(\deg : \operatorname{Div}_D(\mathbb{P}^1) \to \mathbb{Z}) \otimes \mathbb{Z}_\ell,$$

where $\operatorname{Div}_D(X) \cong \mathbb{Z}^2$ is the free abelian group on the divisors of $\mathbb{P}^1$ supported on $\{0, \infty\}$ and *deg* sends each divisor to its degree. If we let $K := \operatorname{Ker}(deg : \operatorname{Div}_D(\mathbb{P}^1) \to \mathbb{Z})$ then we have a natural isomorphism $H^1_\ell(\mathbb{G}_m)(1) \cong K \otimes \mathbb{Z}_\ell$, indicating that $H^1_\ell(\mathbb{G}_m)(1)$ can be reconstructed from the free finitely generated abelian group K.

1.2.4. Next let X be a proper rational nodal curve. Since X is proper, the Kummer exact sequence yields an isomorphism

$$H^1_\ell(X)(1) \cong T_\ell \mathbf{Pic}^{0,red}_{X/k}.$$

It is well-known that $\mathbf{Pic}^{0,red}_{X/k}$ is canonically isomorphic to the *torus* $\mathbb{G}_m$.

In [Del74], Deligne defined a category encompassing these examples as well as the category of abelian varieties:

Definition 1.2.5. [Del74, Définition 10.1.1] Let k be a field. The category of (free) 1-motives over k, denoted 1-Mot$_k$, is the category of 2-term complexes

$$[L \xrightarrow{u} G]$$

of commutative group schemes over k, where

- L is an étale-locally constant sheaf such that $L(k^s)$ is a free finitely generated abelian group, and

- G is a semi-abelian variety over k; i.e., an extension $0 \to T \to G \to A \to 0$ of an abelian variety A by a torus T.

The morphisms in 1-Mot$_k$ are the morphisms of complexes of sheaves.

1.2.6. The category of 1-motives comes equipped with *realization functors* that relate the category of 1-motives to the various cohomology theories described above:

- For each prime $\ell \neq$ char k, there exist ℓ-adic realization functors T_ℓ : 1-Mot$_k \to$ Rep$_{\mathbb{Z}_\ell}(\mathrm{Gal}(\overline{k}/k))$ generalizing the Tate module functor on abelian varieties [Del74, 10.1.5]. These functors are discussed in more detail in Chapter 5.

- If $k \subseteq \mathbb{C}$, there is a Hodge realization functor $T_\mathbb{Z}$ from the category of 1-motives to the category of mixed Hodge structures [Del74, 10.1.3].

- There is a de Rham realization functor T_{dR} from the category of 1-motives to the category of filtered vector spaces [Del74, 10.1.3].

- If k is perfect of positive characteristic, there is a functor T_{crys} from the category of 1-motives to the category of *filtered F-isocrystals* [ABV03, Sect. 1].

If the 1-motive M is simply an abelian variety A, then these functors are given by the same formulae as in 1.1.13. One should think of these functors as giving the *first homology group* of a 1-motive in the various cohomology theories.

1.2.7. Deligne conjectured in [Del74, 10.4.1] that certain mixed Hodge structures associated to a separated finite type scheme over $\mathbb{C}$ arise naturally from 1-motives; in particular, he conjectured that the mixed Hodge structures $H^1_{sing}(X)(1)$ and $H^{2d-1}_{sing}(X)(d)$/torsion occur as the Hodge realizations of 1-motives $M^1(X)$ and $M^{2d-1}(X)$, respectively, defined purely algebraically (here $d = \dim(X)$ as usual). This special case of Deligne's conjecture was solved in [BVS01], and more general cases were studied in [BRS03] and [BVK12].

1.2.8. For étale cohomology, one can make the following conjecture, which is an ℓ-adic analogue of this special case of Deligne's conjectures on 1-motives that also encompasses compactly supported cohomology. Note that we restrict ourselves to the case where k is perfect; as noted in [Ram04, p. 3], it is not clear that this conjecture should be true for non-perfect fields.

Conjecture 1.2.9. *Fix a perfect field k of characteristic $p \geq 0$, and let Sch/k denote the category of separated finite type k-schemes. Then there exist canonically defined functors*

$$M^1(-), \ M_c^1(-) : (Sch/k)^{op} \longrightarrow 1\text{-}Mot_k,$$

with the property that for all primes $\ell \neq p$,

$$T_\ell M^1(X) \cong H^1(X_{\bar{k}}, \mathbb{Z}_\ell(1)) \ \ \text{and}$$
$$T_\ell M_c^1(X) \cong H_c^1(X_{\bar{k}}, \mathbb{Z}_\ell(1))$$

functorially in X. Here the subscript c indicates compactly supported cohomology; we only ask that $M_c^1(-)$ be contravariantly functorial for proper morphisms.

In addition, let $Sch_d/k \subset Sch/k$ be the full subcategory of d-dimensional separated finite type k-schemes. Then there exist canonically defined functors

$$M^{2d-1}(-), \ M_c^{2d-1}(-) : (Sch_d/k)^{op} \longrightarrow 1\text{-}Mot_k,$$

with the property that for all primes $\ell \neq p$,

$$T_\ell M^{2d-1}(X) \cong H^{2d-1}(X_{\bar{k}}, \mathbb{Z}_\ell(d))/\text{torsion} \ \ \text{and}$$
$$T_\ell M_c^{2d-1}(X) \cong H_c^{2d-1}(X_{\bar{k}}, \mathbb{Z}_\ell(d))/\text{torsion}$$

functorially in X. Again, we only ask that $M_c^{2d-1}(-)$ be contravariantly functorial for proper morphisms.

1.2.10. We call $M^1(X)$ and $M_c^1(X)$ (if they exist) the *Picard* and *compactly-supported Picard* 1-motives of X, and $M^{2d-1}(X)$ and $M_c^{2d-1}(X)$ the *Albanese* and *compactly-supported Albanese* 1-motives of X. If they exist, they give our desired generalization of the theory of Picard and Albanese varieties to arbitrary separated finite type k-schemes, at least in so far as this theory relates to ℓ-adic étale cohomology.

Notation 1.2.11. Throughout the rest of this thesis, the subscript $_{(c)}$, as in $H_{(c)}^i(X)$, refers to both an object and its compactly supported variant. For example, '$H_{(c)}^i(X)$' is shorthand for '$H^i(X)$ and $H_c^i(X)$'.

Remark 1.2.12. One would also like the 1-motives $M_{(c)}^1(X)$, $M_{(c)}^{2d-1}(X)$ to have Hodge, de Rham, and/or crystalline realizations that are compatible with the corresponding ohomology groups of X. In the case of the Hodge and de Rham realizations this has been studied in great detail in [BVS01] as discussed below. We hope to study a crystalline version of the above conjecture in future work; the crystalline realization of $M^1(X)$ has been studied in [ABV03]. For this thesis, however, *we will focus on ℓ-adic étale cohomology.*

1.2.13. In [BVS01], Barbieri-Viale and Srinivas solve Conjecture 1.2.9 in the case where char $k = 0$, for non-compactly supported cohomology. In that paper, they construct one-motives $\mathrm{Pic}^+(X)$ and $\mathrm{Alb}^+(X)$ (in our notation these correspond to $M^1(X)$ and $M^{2d-1}(X)$, respectively) for X a separated finite type k-scheme, and show that these 1-motives have the correct ℓ-adic, Hodge and de Rham realizations. This gives an essentially complete development of the theory of Picard and Albanese 1-motives in characteristic 0, except that compactly supported variants are not considered.

In addition, the papers [ABV03] and [Ram04] provide (independently) definitions of $M^1(X)$ for k perfect of positive characteristic. But the full conjecture above, most importantly defining the Albanese 1-motives $M^{2d-1}_{(c)}(X)$ in positive characteristic, has not been dealt with to our knowledge. In this thesis we will partially resolve this conjecture. The precise statement is Theorem 1.2.17 below.

1.2.14. In this thesis we investigate the possibility of defining 1-motives $M^1_{(c)}(X)$, $M^{2d-1}_{(c)}(X)$ over a perfect field of positive characteristic. In Chapter 6 we provide definitions of the Picard 1-motives $M^1(X)$ and $M^1_c(X)$. As indicated above, the definition of $M^1(X)$ has previously appeared in [ABV03] and [Ram04]. We generalize this by defining a 1-motive $M^1_{D,E}(\overline{X})$ associated to any triple $(\overline{X}, D, E)$ consisting of a proper scheme $\overline{X}$ and two disjoint closed subschemes $D, E \subset \overline{X}$. The 1-motive $M^1_{D,E}(\overline{X})$ realizes the relative cohomology group $H^1(\overline{X}_{\overline{k}} - E_{\overline{k}}, D_{\overline{k}}, \mathbb{Z}_\ell(1))$, which we also write as $H^1_{D,E}(X, \mathbb{Z}_\ell(1))$. Then to define $M^1(X)$ and $M^1_c(X)$ we choose a compactification $X \hookrightarrow \overline{X}$ with closed complement D, and define $M^1(X) := M^1_{\emptyset,D}(\overline{X})$ and $M^1_c(X) = M^1_{D,\emptyset}(\overline{X})$. Of course, we show that these definitions are independent of the choice of compactification.

Remark 1.2.15. Because we do not deal with crystalline realizations in this paper, we only show that $M^1_{D,E}(X)$ is well-defined as an object of 1-$\mathrm{Mot}_k[1/p]$, the category of 1-motives up to p-isogeny (see Chap. 5). The crystalline realization of $M^1(X)$ is discussed in [ABV03]; we hope to consider the crystalline realization of $M^1_{D,E}(\overline{X})$ in future work.

1.2.16. Defining the Albanese 1-motives $M^{2d-1}(X)$ and $M^{2d-1}_c(X)$ turns out to be harder, and most of Chapters 2-4 are devoted to preliminary facts on Picard groups and divisors for Deligne-Mumford stacks that we will use in our construction. Even so, the 1-motives $M^{2d-1}_{(c)}(X)$ might only be defined over a finite extension K/k, and they are only well-defined up to *isogeny* of 1-motives; the difficulty is due to lack of resolution of singularities as will be discussed below. Using a theory of isogeny descent developed in Chapter 9, we show that if k is finite then the 1-motives $M^{2d-1}_{(c)}(X)$ can be defined over k. We can therefore state our main result as follows:

Theorem 1.2.17. *(Combining Theorems 6.4.9, 7.4.10, 8.3.7, 9.4.1 in text) Let k be a perfect field of characteristic $p \geq 0$, and let Sch/k be the category of separated finite type k-schemes. Then there exist canonically defined functors*

$$M^1(-), M^1_c(-) : (Sch/k)^{op} \longrightarrow \text{1-}Mot_k[1/p]$$

with the property that

$$T_\ell M^1(X) \cong H^1(X_{\overline{k}}, \mathbb{Z}_\ell(1)) \quad \text{and}$$
$$T_\ell M_c^1(X) \cong H_c^1(X_{\overline{k}}, \mathbb{Z}_\ell(1))$$

functorially in X, for $\ell \neq p$. (These isomorphisms are well-defined after choosing an embedding $k \hookrightarrow \overline{k}$.) We only require that $M_c^1(-)$ be contravariantly functorial for proper morphisms.

Next assume k is either a finite field or an algebraically closed field (of characteristic p), and let Sch_d/k be the full subcategory of d-dimensional separated finite type k-schemes. Then there exist canonically defined functors

$$M^{2d-1}(-), M_c^{2d-1}(-) : (Sch_d/k)^{op} \longrightarrow \textit{1-Mot}_k \otimes \mathbb{Q}$$

such that

$$T_\ell M^{2d-1}(X) \otimes \mathbb{Q} \cong H^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell(d)) \quad \text{and}$$
$$T_\ell M_c^{2d-1}(X) \otimes \mathbb{Q} \cong H_c^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell(d)).$$

functorially in X, for $\ell \neq p$. We only require that $M_c^{2d-1}(-)$ be contravariantly functorial for proper morphisms.

Here 1-Mot$_k \otimes \mathbb{Q}$ denotes the category of 1-motives up to isogeny; see Chapter 5. Also note the restriction to finite or algebraically closed fields in the second half of the theorem; for a general perfect field k, we can only show that the 1-motives $M_{(c)}^{2d-1}(X)$ exist after a finite extension of the base field.

1.3 Description of the 1-motives defined in this thesis

In the next few sections we describe the 1-motives constructed in this paper; the details are to be found in Chapters 6-9.

Description of the 1-motive $M_{D,E}^1(\overline{X})$

1.3.1. To define the 1-motive $M_{D,E}^1(\overline{X})$, the basic technique is to replace $\overline{X}$ by an appropriate simplicial scheme using the results of [dJng96]. Therefore, first consider a simplicial scheme $\overline{X}_\bullet$ with each $\overline{X}_n$ proper and smooth. Assume that $U_\bullet := \overline{X}_\bullet - (D_\bullet \cup E_\bullet)$ is dense in each component of each $\overline{X}_n$ (we want to exclude the case where $D_\bullet$ or $E_\bullet$ has a component equal to a component of $\overline{X}_\bullet$).

1.3.2. Let $p_\bullet : \overline{X}_\bullet \to \operatorname{Spec} k$ be the structure morphism, and $i_\bullet : D_\bullet \hookrightarrow \overline{X}_\bullet$ the inclusion. Then consider the sheaf

$$\mathbf{Pic}_{\overline{X}_\bullet, D_\bullet} := R^1(p_\bullet)_*(\ker(\mathbb{G}_{m,\overline{X}_\bullet} \to (i_\bullet)_* \mathbb{G}_{m,D_\bullet}))$$

on $(Sch/k)_{fppf}$. $\mathbf{Pic}_{\overline{X}_\bullet, D_\bullet}$ classifies isomorphism classes of pairs $(\mathscr{L}^\bullet, \sigma)$, where $\mathscr{L}^\bullet$ is a line bundle on $\overline{X}_\bullet$ and σ is an isomorphism of $\mathscr{L}^\bullet|_{D_\bullet}$ with $\mathcal{O}_{D_\bullet}$. A straightforward reduction (6.1.5) from well-known representability results shows that $\mathbf{Pic}_{\overline{X}_\bullet, D_\bullet}$ is representable by a locally finite type commutative k-group scheme. Moreover, let $\mathbf{Pic}^{0,red}_{\overline{X}_\bullet, D_\bullet}$ denote the reduction of the connected component of the identity of $\mathbf{Pic}_{\overline{X}_\bullet, D_\bullet}$. Then (loc. cit.) $\mathbf{Pic}^{0,red}_{\overline{X}_\bullet, D_\bullet}$ is a semi-abelian variety.

1.3.3. Next, for any closed subscheme C of a scheme X, let $\mathrm{Div}_C(X)$ be the finitely generated abelian group of Weil divisors on X with support contained in C. Then for the simplicial closed subscheme $E_\bullet$ of $\overline{X}_\bullet$, we define

$$\mathrm{Div}_{E_\bullet}(\overline{X}_\bullet) := \mathrm{Ker}(p_1^* - p_2^* : \mathrm{Div}_{E_0}(\overline{X}_0) \to \mathrm{Div}_{E_1}(\overline{X}_1)),$$

where $p_1, p_2 : \overline{X}_1 \to \overline{X}_0$ are the simplicial structure maps from $\overline{X}_1$ to $\overline{X}_0$. Because $E_\bullet$ and $D_\bullet$ are disjoint, there is a natural map $\mathrm{Div}_{E_\bullet}(\overline{X}_\bullet) \to \mathrm{Pic}(\overline{X}_\bullet, D_\bullet)$ (defined in Paragraph 6.1.6). We define $\mathrm{Div}^0_{E_\bullet}(\overline{X}_\bullet)$ to be the subgroup mapping into $\mathrm{Pic}^0(\overline{X}_\bullet, D_\bullet)$. More generally, we can define an étale k-group scheme $\mathbf{Div}^0_{E_\bullet}(\overline{X}_\bullet)$ with $\overline{k}$-points $\mathrm{Div}^0_{E_{\overline{k},\bullet}}(\overline{X}_{\overline{k},\bullet})$ and $Gal(\overline{k}/k)$ acting in the obvious way. Then there is a natural map of group schemes

$$\mathbf{Div}^0_{E_\bullet}(\overline{X}_\bullet) \to \mathbf{Pic}^{0,red}_{\overline{X}_\bullet, D_\bullet}.$$

Definition 1.3.4. With notation as above, we define a 1-motive $M^1_{D_\bullet, E_\bullet}(\overline{X}_\bullet)$ by the formula

$$M^1_{D_\bullet, E_\bullet}(\overline{X}_\bullet) := [\mathbf{Div}^0_{E_\bullet}(\overline{X}_\bullet) \to \mathbf{Pic}^{0,red}_{\overline{X}_\bullet, D_\bullet}].$$

1.3.5. Now consider a proper separated finite type k-scheme $\overline{X}$, and disjoint closed subschemes D and E of $\overline{X}$ such that $U := \overline{X} - (D \cup E)$ is everywhere dense in $\overline{X}$. Choose a proper hypercover $\pi_\bullet : \overline{X}_\bullet \to \overline{X}$ such that each $\overline{X}_n$ is proper smooth, and let $D'_\bullet := \pi_\bullet^{-1}(D)$ and $E'_\bullet := \pi_\bullet^{-1}(E)$. A technical issue arises from the fact that $D'_\bullet$ and $E'_\bullet$ have components equal to components of $\overline{X}_\bullet$; i.e., $U_\bullet := \overline{X}_\bullet - (D'_\bullet \cup E'_\bullet)$ is not everywhere dense in $\overline{X}_\bullet$. Therefore we consider the closure $\overline{U}_\bullet$ of $U_\bullet$ in $\overline{X}_\bullet$. Let $D_\bullet = D'_\bullet \cap \overline{U}_\bullet$ and $E_\bullet = E'_\bullet \cap \overline{U}_\bullet$. Then to the triple $(\overline{U}_\bullet, D_\bullet, E_\bullet)$ we can define an associated 1-motive $M^1_{D_\bullet, E_\bullet}(\overline{U}_\bullet)$ by the procedure defined above, and we let
$$M^1_{D,E}(\overline{X}) := M^1_{D_\bullet, E_\bullet}(\overline{U}_\bullet).$$

The following Proposition summarizes the main properties of the 1-motive $M^1_{D,E}(\overline{X})$:

Proposition 1.3.6. *(Summary of results of Chapter 6) The 1-motive $M^1_{D,E}(\overline{X})$ is, up to a canonical p-isogeny, independent of the choice of simplicial hypercover $\overline{X}_\bullet \to \overline{X}$. There is a natural isomorphism (for $\ell \neq p$)*

$$\alpha : T_\ell M^1_{D,E}(\overline{X}) \cong H^1_{D,E}(\overline{X}, \mathbb{Z}_\ell(1)).$$

Given a morphism of triples $f : (\overline{X}, D, E) \to (\overline{Y}, A, B)$ such that $f^{-1}(A) = D$ and $f^{-1}(B) \subseteq E$, there is an induced morphism of 1-motives

$$f^* : M^1_{A,B}(\overline{Y}) \to M^1_{D,E}(\overline{X}).$$

1.3.7. As special cases of the above construction, consider a separated finite type k-scheme X. Choose a compactification $X \hookrightarrow \overline{X}$, with closed complement $C \subset \overline{X}$. Then we define

$$M^1(X) := M^1_{\emptyset,C}(\overline{X}) = [\mathbf{Div}^0_{C_\bullet}(\overline{X}_\bullet) \to \mathbf{Pic}^{0,red}_{\overline{X}_\bullet}]$$

and

$$M^1_c(X) := M^1_{C,\emptyset}(\overline{X}) = [0 \to \mathbf{Pic}^{0,red}_{\overline{X}_\bullet,C_\bullet}].$$

We show in Chapter 6 that $M^1(X)$ and $M^1_c(X)$ are independent of the choice of compactification, and that $M^1(X)$ is contravariantly functorial for arbitrary morphisms, while $M^1_c(X)$ is contravariantly functorial for proper morphisms.

Definition of the 1-motives $M^{2d-1}(X)$ and $M^{2d-1}_c(X)$

1.3.8. We next define the 1-motives $M^{2d-1}(X)$ and $M^{2d-1}_c(X)$. If the field k has characteristic 0, then $M^{2d-1}(X)$ corresponds to the 1-motive $\mathrm{Alb}^+(X)$ constructed in [BVS01]. We would like to follow the approach of that paper as much as possible to construct the corresponding 1-motives in positive characteristic, but their construction uses resolution of singularities (in the strong form) throughout, and so we must make additional assumptions. Choose a compactification $X \hookrightarrow \overline{X}$. Then by [dJng96, 7.3], *after a finite extension of the base field k* there exists a sequence of maps

$$\overline{\mathfrak{X}} \xrightarrow{\ p\ } \overline{X}'' \xrightarrow{\ q\ } \overline{X}' \xrightarrow{\ r\ } \overline{X},$$

satisfying the following conditions:

1. r is purely inseparable and surjective, therefore a universal homeomorphism;

2. q is proper and birational;

3. $\overline{\mathfrak{X}}$ is a smooth proper Deligne-Mumford stack (in fact a global quotient $[U/G]$ of a smooth proper k-scheme U by a finite group G) and p identifies $\overline{X}''$ with the coarse moduli space of $\overline{\mathfrak{X}}$.

We call the morphism $\overline{\mathfrak{X}} \to \overline{X}$ a *weak resolution* of $\overline{X}$.

1.3.9. Assume that there exists a weak resolution of $\overline{X}$ (which may require a finite extension of the base field). We let $\overline{\pi} : \overline{\mathfrak{X}} \to \overline{X}$ be the composition $r \circ q \circ p$, and $\pi : \mathfrak{X} \to X$ the restriction to X where $\mathfrak{X} = \overline{\pi}^{-1}(X)$. Let $C = \overline{X} - X$ and $\mathcal{C} = \overline{\mathfrak{X}} - \mathfrak{X}$. The key property of $\overline{\pi}$ we will use is that there exists an open dense subset $U \subset X$ with the property that $\pi^{-1}(U) \to U$

induces an isomorphism $\mathbb{Q}_{\ell,U} \xrightarrow{\sim} R\pi_* \mathbb{Q}_{\ell,\pi^{-1}(U)}$ in $D_c^b(U)$; see Proposition 7.1.3. We say that $\pi^{-1}(U) \to U$ is a $\mathbb{Q}_\ell$-*cohomological isomorphism*. Choose such an open subset U, and let $Z = X - U$, $\mathcal{Z} := \mathfrak{X} - \pi^{-1}(U)$. Finally, let $\overline{Z}$ (resp. $\overline{\mathcal{Z}}$) be the closure of Z in $\overline{X}$ (resp. of $\mathcal{Z}$ in $\overline{\mathfrak{X}}$). In summary, we have diagrams

$$
\begin{array}{ccccc}
\mathfrak{X} & \overset{\alpha'}{\hookrightarrow} & \overline{\mathfrak{X}} & \overset{\beta'}{\hookleftarrow} & \mathcal{C} \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\overline{\pi}} & & \downarrow{\scriptstyle\overline{\pi}|_{\mathcal{C}}} \\
X & \overset{\alpha}{\hookrightarrow} & \overline{X} & \overset{\beta}{\hookleftarrow} & C
\end{array}
$$

and

$$
\begin{array}{ccccc}
\mathcal{U} & \overset{j'}{\hookrightarrow} & \mathfrak{X} & \overset{i'}{\hookleftarrow} & \mathcal{Z} \\
\downarrow{\scriptstyle\pi|_U} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi|_Z} \\
U & \overset{j}{\hookrightarrow} & X & \overset{i}{\hookleftarrow} & Z.
\end{array}
$$

1.3.10. We can define sheaves $\mathbf{Pic}_{\overline{\mathfrak{X}}}$ and $\mathbf{Pic}_{\overline{\mathfrak{X}},\mathcal{C}}$ by the same formulas as in the case of schemes, so that $\mathbf{Pic}_{\overline{\mathfrak{X}}}$ classifies isomorphism classes of line bundles on $\overline{\mathfrak{X}}$ and $\mathbf{Pic}_{\overline{\mathfrak{X}},\mathcal{C}}$ classifies isomorphism classes of pairs $(\mathscr{L}, \sigma)$ where $\mathscr{L}$ is a line bundle on $\overline{\mathfrak{X}}$ and σ is an isomorphism of $\mathscr{L}|_{\mathcal{C}}$ with $\mathcal{O}_{\mathcal{C}}$. The sheaves $\mathbf{Pic}_{\overline{\mathfrak{X}}}$ and $\mathbf{Pic}_{\overline{\mathfrak{X}},\mathcal{C}}$ are both representable by locally finite type commutative k-group schemes (6.6.4). Moreover, $\mathbf{Pic}_{\overline{\mathfrak{X}}}^{0,red}$ and $\mathbf{Pic}_{\overline{\mathfrak{X}},\mathcal{C}}^{0,red}$ are both semi-abelian varieties.

1.3.11. To define the 1-motive $M_c^{2d-1}(X)$, first consider a divisor $D \in \mathrm{Div}_{\mathcal{C} \cup \overline{\mathcal{Z}}}(\overline{\mathfrak{X}})$. Such a divisor can be uniquely written as $D = D_1 + D_2$, with D_1 supported on $\mathcal{C}$ and D_2 supported on $\overline{\mathcal{Z}}$. Then we let $\mathrm{Div}_{\mathcal{C} \cup \overline{\mathcal{Z}}/\overline{Z}}^0(\overline{\mathfrak{X}})$ be the group of divisors $D = D_1 + D_2$ supported on $\mathcal{C} \cup \overline{\mathcal{Z}}$ such that

1. D maps to 0 in $NS(\overline{\mathfrak{X}})$, and

2. D_1 maps to 0 under the proper pushforward $\overline{\pi}_* : \mathrm{Div}_{\overline{\mathcal{Z}}}(\overline{\mathfrak{X}}) \to \mathrm{Div}_{\overline{Z}}(\overline{X})$.

Let $\mathbf{Div}_{\mathcal{C} \cup \overline{\mathcal{Z}}/\overline{Z}}^0(\overline{\mathfrak{X}})$ be the étale k-group scheme with $\overline{k}$-points $\mathrm{Div}_{\mathcal{C}_{\overline{k}} \cup \overline{\mathcal{Z}}_{\overline{k}}/\overline{Z}_{\overline{k}}}^0(\overline{\mathfrak{X}}_{\overline{k}})$ and natural Galois action. We then define

$$
M_c^{2d-1}(X) := [\mathbf{Div}_{\mathcal{C} \cup \overline{\mathcal{Z}}/\overline{Z}}^0(\overline{\mathfrak{X}}) \to \mathbf{Pic}_{\overline{\mathfrak{X}}}^{0,red}]^\vee
$$

where the superscript $^\vee$ indicates taking the Cartier dual (see Chapter 5 for background on Cartier duality for 1-motives).

1.3.12. To define the 1-motive $M^{2d-1}(X)$, let $\mathrm{Div}_{\mathcal{Z}}(\overline{\mathfrak{X}})$ be the group of divisors on $\overline{\mathfrak{X}}$ supported on $\mathcal{Z}$ (note that these are equal to the divisors supported on $\overline{\mathcal{Z}}$ which are disjoint from $\mathcal{C}$). Since these divisors are disjoint from $\mathcal{C}$, there is a natural map $\mathrm{Div}_{\mathcal{Z}}(\overline{\mathfrak{X}}) \to \mathbf{Pic}_{\overline{\mathfrak{X}},\mathcal{C}}^{0,red}$. We let $\mathrm{Div}_{\mathcal{Z}/Z}^0(\overline{\mathfrak{X}})$ be the subgroup of $\mathrm{Div}_{\mathcal{Z}}(\overline{\mathfrak{X}})$ of divisors D such that

1. D maps to 0 in $NS(\overline{\mathfrak{X}}, \mathcal{C})$, and

2. D maps to 0 under the proper pushforward $\pi_* : \mathrm{Div}_{\mathcal{Z}}(\overline{\mathfrak{X}}) \to \mathrm{Div}_Z(\overline{X})$.

If $\mathbf{Div}^0_{\mathcal{Z}/Z}(\overline{\mathfrak{X}})$ is the associated group scheme, we define

$$M^{2d-1}(X) = [\mathbf{Div}^0_{\mathcal{Z}/Z}(\overline{\mathfrak{X}}) \to \mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}},\mathcal{C}}]^\vee.$$

Chapters 7 and 8 go into more detail on the construction of $M^{2d-1}(X)$ and $M^{2d-1}_c(X)$, and show that they have the correct ℓ-adic realizations.

Defining $M^{2d-1}_{(c)}(X)$ when X is a scheme over a finite field

1.3.13. Because [dJng96, 7.3] is only valid after a possible finite extension of the base field, the theory of Paragraphs 1.3.8-1.3.12 only yields 1-motives defined over k if k is algebraically closed. However, we can use the functoriality of the 1-motives $M^{2d-1}_{(c)}(X)$, together with a type of descent that works for *isogeny* 1-motives, to guarantee that for a *finite* base field k, the 1-motives $M^{2d-1}_{(c)}(X)$ are always defined over k. We will briefly overview our isogeny descent result below. Note first that for any 1-motive $M = [L \to G]$ over a finite field $\mathbb{F}_q$, there exists an absolute $\mathbb{F}_q$-linear Frobenius endomorphism $Fr_M : M \to M$; namely take $Fr_M : [L \to G] \longrightarrow [L \to G]$ to be the $\mathbb{F}_q$-linear Frobenius endomorphisms on L and on G.

Definition 1.3.14. Let $k = \mathbb{F}_q$, and choose an algebraic closure $k \hookrightarrow \overline{k}$. Let M be a 1-motive over $\overline{k}$. A *descent isogeny* $g : M \to M$ is an isogeny such that there exists a 1-motive $\overline{M}$ defined over a finite field extension $k \hookrightarrow K$ and an isomorphism $\overline{M} \times_K \overline{k} \cong M$ inducing a commutative diagram

$$
\begin{array}{ccc}
\overline{M} \times_K \overline{k} & \overset{\sim}{\longrightarrow} & M \\
{\scriptstyle Fr_{\overline{M}}}\downarrow & & \downarrow{\scriptstyle g^n} \\
\overline{M} \times_K \overline{k} & \overset{\sim}{\longrightarrow} & M
\end{array}
$$

where $Fr_{\overline{M}}$ is the K-linear Frobenius endomorphism of $\overline{M}$.

Some of the motivation behind this definition is explained in 9.1.8.

Our main result on isogeny descent is then the following:

Theorem 1.3.15. *(Theorem 9.3.4 in text) Let $k = \mathbb{F}_q$, and fix an algebraic closure $k \hookrightarrow \overline{k}$. Let $\mathscr{D}_{k \hookrightarrow \overline{k}} \otimes \mathbb{Q}$ be the category of pairs (M, g) where M is an isogeny 1-motive over $\overline{k}$, and $g : M \to M$ is a descent isogeny relative to k. Consider the natural pullback functor*

$$p^* : \textit{1-Mot}_k \otimes \mathbb{Q} \longrightarrow \mathscr{D}_{k \hookrightarrow \overline{k}} \otimes \mathbb{Q}$$

sending a 1-motive N to the pair $(N \times_k \overline{k}, Fr_N \times_k \overline{k})$ where Fr_N is the k-linear Frobenius endomorphism of N. Then p^* is an equivalence of categories. Moreover, p^* has a natural quasi-inverse p_*.

This theorem is proved in Chapter 9.

1.3.16. To apply this result, consider a d-dimensional separated finite type k-scheme X, where $k = \mathbb{F}_q$. By the results of the Chapters 7-8 for algebraically closed fields, there exist Albanese 1-motives $M_{(c)}^{2d-1}(X_{\overline{k}})$ associated to the base change $X_{\overline{k}}$. Let $Fr : X \to X$ be the k-linear Frobenius endomorphism of X, and let $\overline{Fr} : X_{\overline{k}} \to X_{\overline{k}}$ be the base change to $\overline{k}$. As explained in Chapter 9, the functoriality of the construction of $M_{(c)}^{2d-1}(X_{\overline{k}})$ leads to an isogeny descent endomorphism $\tilde{F} : M_{(c)}^{2d-1}(X_{\overline{k}}) \to M_{(c)}^{2d-1}(X_{\overline{k}})$. By the preceding theorem, we get a 1-motive over k which we denote by $M_{(c)}^{2d-1}(X)$. We show in Chapter 9 that this 1-motive has a natural Galois-equivariant realization isomorphism $V_\ell M_{(c)}^{2d-1}(X) \cong H_{(c)}^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell(d))$, and that it is functorial in X (for proper morphisms in the case of the compactly supported variant). This shows the existence of Albanese 1-motives for finite fields and completes the proof of Theorem 1.2.17.

1.4 Other results

In order to define $M^{2d-1}(X)$ and $M_c^{2d-1}(X)$ we need some results on cycle classes for Deligne-Mumford stacks that aren't in the literature. Some of these results may be of independent interest and are highlighted below.

In Chapter 2 we extend the theory of 1-motivic sheaves [BVK12, App. C] to show that the Picard sheaf of a smooth Artin stack is 1-motivic. See Chapter 2 for the definition of a 1-motivic sheaf; beyond Chapter 2 we will only use the following corollary:

Corollary 1.4.1. *(of Proposition 2.2.8) Let $\mathfrak{X}$ be a smooth Artin stack of finite type over an algebraically closed field k having quasi-compact separated diagonal. Then there exists a divisible group $\mathrm{Pic}^0(\mathfrak{X})$, a finitely generated group $NS(\mathfrak{X})$, and a sequence*

$$0 \to \mathrm{Pic}^0(\mathfrak{X}) \to \mathrm{Pic}(\mathfrak{X}) \to NS(\mathfrak{X}) \to 0$$

which becomes exact after inverting $p := \mathrm{char}\ k$.

Presumably this is true even without inverting char k, although we don't know how to prove it.

In Chapter 3 we review the theory of Weil and Cartier divisors on a Deligne-Mumford stack $\mathfrak{X}$. In addition, we prove the following:

Proposition 1.4.2. *(Proposition 3.2.4 in text) Let $\mathfrak{X}$ be a geometrically reduced, separated finite type Deligne-Mumford stack over a field k. Then the quotient $\mathrm{Pic}(\mathfrak{X})/\mathrm{CaCl}(\mathfrak{X})$ is a finite group.*

This has the following corollary:

Corollary 1.4.3. *(Corollary 3.2.7 in text) Let $\mathfrak{X}$ be a smooth proper Deligne-Mumford stack over an algebraically closed field k. Then every element of $\mathrm{Pic}^0(\mathfrak{X})$ is represented by a Weil divisor.*

Chapter 4 develops the theory of cycle class maps $CH^d(\mathfrak{X}) \to H^{2d}(\mathfrak{X}, \mathbb{Z}_\ell(d))$ for a smooth separated Deligne-Mumford stack $\mathfrak{X}$. We proceed in the same manner as the article [SGA4h, Cycle], and the results of that article are used repeatedly in Chapter 4.

In Chapter 5 we review the necessary background on 1-motives. All of the results in this chapter can be found in several other sources, for example [BVK12, App. C]. In Chapters 6 through 9 we define the 1-motives $M^1_{(c)}(X)$ and $M^{2d-1}_{(c)}(X)$ as discussed above.

We conclude the thesis with the short Chapter 10, which gives the following easy consequence of our work on 1-motives:

Proposition 1.4.4. *(Proposition 10.1.2 in text) Let X be a d-dimensional separated finite type k-scheme where $k = \bar{k}$. Let $f : X \to X$ be an endomorphism of X. Define the polynomials*

$$P^i_\ell(f, t) := \det(1 - tf \mid H^i(X, \mathbb{Q}_\ell)) \quad \text{and}$$
$$P^i_{\ell,c}(f, t) := \det(1 - tf \mid H^i_c(X, \mathbb{Q}_\ell)).$$

(for $P^i_{\ell,c}(f, t)$ we assume f is proper). Then for $i = 0, 1, 2d - 1, 2d$, these polynomials have integer coefficients independent of ℓ.

Probably the only new case here is $i = 2d - 1$, although the other cases aren't clearly stated in the literature. Of course, when X is smooth and proper, this proposition is known for all i due to the work of [Del80] and [KM74].

Using known results on trace formulas ([Fuj97, 5.4.5] and [Ols2, Thm 1.1]), we get the following corollary in the case of surfaces over a finite field $\mathbb{F}_q$:

Corollary 1.4.5. *(Corollary 10.1.5 in text) Let X_0 be a 2-dimensional separated finite type $\mathbb{F}_q$-scheme, and let $X = X_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. If $f : X \to X$ is any proper endomorphism, then for all values of i, the polynomial $P^i_{\ell,c}(f, t)$ has rational coefficients independent of ℓ. If $f : X \to X$ is any quasi-finite endomorphism, then the polynomial $P^i_\ell(f, t)$ has rational coefficients independent of ℓ for all i.*

In particular, if $f = F$ is the Frobenius endomorphism, then $P^i_\ell(F, t)$ and $P^i_{\ell,c}(F, t)$ have rational coefficients independent of ℓ (and hence integer coefficients since the roots of these polynomials are algebraic integers [Ill06, 4.2]).

Chapter 2

Preliminaries on Picard Functors of Smooth Stacks

2.1 Representability of Picard Functors on Proper Stacks

2.1.1. Let $\mathfrak{X}$ be an Artin stack of finite type over a field k, with quasi-compact and separated diagonal. Let $\pi : \mathfrak{X} \to \operatorname{Spec} k$ be the structure morphism, and let $\mathbb{G}_{m,\mathfrak{X}}$ be the sheaf on $(Sch/\mathfrak{X})_{fppf}$ sending T to $\Gamma(T, \mathcal{O}_T^{\times})$. Recall that the Picard functor $\mathbf{Pic}_{\mathfrak{X}/k} \in Sh(Sch/k)_{fppf}$ is defined to be the sheaf $R^1\pi_*(\mathbb{G}_{m,\mathfrak{X}})$, or equivalently, the fppf-sheafification of the functor

$$Y \mapsto \operatorname{Pic}(\mathfrak{X} \times_k Y).$$

We then have the following representability results in case $\mathfrak{X}$ is proper, due to Brochard [Bro09], [Bro12]:

Theorem 2.1.2. *Let $\mathfrak{X}$ be a proper Artin stack over the field k. Then the following hold:*

1. *The sheaf $\mathbf{Pic}_{\mathfrak{X}/k}$ is representable by a locally finite type commutative group scheme over k [Bro12, 2.3.7].*

2. *If $\mathbf{Pic}^0_{\mathfrak{X}/k}$ denotes the connected component containing the identity of $\mathbf{Pic}_{\mathfrak{X}/k}$, then we have an exact sequence of group schemes*

$$0 \to \mathbf{Pic}^0_{\mathfrak{X}/k} \to \mathbf{Pic}_{\mathfrak{X}/k} \to \mathbf{NS}_{\mathfrak{X}/k} \to 0,$$

   defining the group scheme $\mathbf{NS}_{\mathfrak{X}/k}$. Furthermore, $\mathbf{NS}_{\mathfrak{X}/k}$ is an étale group scheme, and $\mathbf{NS}_{\mathfrak{X}/k}(\overline{k})$ is a finitely generated abelian group for any algebraic closure $k \hookrightarrow \overline{k}$ [Bro12, 3.4.1].

3. *Assume in addition that $\mathfrak{X}$ is smooth, and k is a perfect field. Then the reduced group scheme $\mathbf{Pic}^{0,red}_{\mathfrak{X}/k}$ is an abelian variety [Bro09, 4.2.2].*

2.1.3. We write $\text{Pic}^0(\mathfrak{X})$ and $NS(\mathfrak{X})$ for the k-points of $\mathbf{Pic}^0_{\mathfrak{X}/k}$ and $\mathbf{NS}_{\mathfrak{X}/k}$, respectively (note however that if k is not algebraically closed then an element of $\text{Pic}^0(\mathfrak{X})$ might not be given by a line bundle on $\mathfrak{X}$).

2.2 1-Motivic Sheaves

2.2.1. For our application to 1-motives, we need to make sense of the groups $\text{Pic}^0(\mathfrak{X})$ and $NS(\mathfrak{X})$ when $\mathfrak{X}$ is smooth, but not necessarily proper. In this case we cannot expect the Picard sheaf $\mathbf{Pic}_{\mathfrak{X}/k}$ to be representable, but it satisfies a weak form of representability which is sufficient for our purposes. This can be summarized in the statement that $\mathbf{Pic}_{\mathfrak{X}/k}$ is a *1-motivic sheaf*, at least when k is perfect and after inverting the characteristic $p := \text{char k}$ in Hom-groups. We will prove that $\mathbf{Pic}_{\mathfrak{X}/k}$ is 1-motivic for any smooth Artin stack $\mathfrak{X}$ over k with quasi-compact separated diagonal. The argument is a straightforward generalization of [BVK12, 3.4]; we include the details for the convenience of the reader. Recall the following definitions, following [BVK12, Sect. 3]:

Definition 2.2.2. Let k be a perfect field, and $(Sm/k)_{et}$ the category of smooth separated k-schemes, with the étale topology. We denote by $Sh(Sm/k)_{et}[1/p]$ the category of sheaves of abelian groups on this site, with $\text{Hom}(\mathscr{F}, \mathscr{G})$ replaced by $\text{Hom}(\mathscr{F}, \mathscr{G})[1/p]$. We say that $\mathscr{F} \in Sh(Sm/k)_{et}[1/p]$ is *discrete* if it is locally constant for the étale topology, and $\mathscr{F}(\bar{k})$ is a finitely generated abelian group.

Definition 2.2.3. [BVK12, 3.2.1] Let $\mathscr{F} \in Sh(Sm/k)_{et}[1/p]$ be a sheaf of abelian groups as above. We say that $\mathscr{F}$ is a *1-motivic sheaf* if there exists a semi-abelian variety G and a morphism $f : G \to \mathscr{F}$ of sheaves on $(Sm/k)_{et}$, such that $\ker f$ and $\text{coker} f$ are discrete sheaves. We denote by $Shv_1(k)[1/p] \subset Sh(Sm/k)_{et}[1/p]$ the full subcategory of 1-motivic sheaves.

Remark 2.2.4. In the theory of this chapter we will invert p in all Hom-groups. Most of our propositions are false as stated if we do not do this. If we wish to avoid inverting p in Hom-groups, we probably must use a finer topology like the fppf topology (see Remark 2.2.7). But this seemingly forces us to replace Sm/k with the bigger category Sch/k, which breaks down the proof that $\mathbf{Pic}_{\mathfrak{X}/k}$ is 1-motivic. See [Ber12] for some discussion of this point.

Example 2.2.5. Let X be a smooth variety over a perfect field k, and suppose that X embeds into a smooth proper variety $\overline{X}$ with complement $D := \overline{X} - X$. Then for every smooth scheme U/k, we have an exact sequence

$$\text{Div}_{D \times U}(\overline{X} \times U) \to \text{Pic}(\overline{X} \times U) \to \text{Pic}(X \times U) \to 0.$$

Here $\text{Div}_{D \times U}(\overline{X} \times U)$ denotes the free abelian group of Weil divisors on $\overline{X} \times U$ supported on $D \times U$. Let $\mathbf{Pic}_{X/k}$ be the lisse-étale Picard sheaf of X/k, that is, the sheafification in $(Sm/k)_{et}$ of the functor

$$U \mapsto \text{Pic}(X \times U).$$

Then the above exact sequence shows that we have an exact sequence in $(Sm/k)_{et}$

$$\mathbf{Div}_D(\overline{X}) \to \mathbf{Pic}_{\overline{X}/k} \to \mathbf{Pic}_{X/k} \to 0,$$

where $\mathbf{Div}_D(\overline{X})$ is the étale-locally constant sheaf of divisors on $\overline{X}$ supported on D. By Proposition 2.2.6(c) below, we conclude that $\mathbf{Pic}_{X/k}$ is 1-motivic.

We have the following key facts about 1-motivic sheaves:

Proposition 2.2.6.

(a) *Given a 1-motivic sheaf $\mathscr{F}$, there exists a unique (up to isomorphism) semi-abelian variety G together with a map $b : G \to \mathscr{F}$ such that $\ker b$ is torsion-free. We say that such a pair (G, b) is normalized.*

(b) *Given 1-motivic sheaves $\mathscr{F}_1$, $\mathscr{F}_2$ and normalized morphisms $b_i : G_i \to \mathscr{F}_i$ for $i = 1, 2$. Then for any morphism of sheaves $f : \mathscr{F}_1 \to \mathscr{F}_2$, there exists a unique morphism of group schemes $\varphi_f : G_1 \to G_2$ making the diagram*

$$\begin{array}{ccc} G_1 & \xrightarrow{\ b_1\ } & \mathscr{F}_1 \\ {\scriptstyle \varphi_f}\downarrow & & \downarrow{\scriptstyle f} \\ G_2 & \xrightarrow{\ b_2\ } & \mathscr{F}_2 \end{array}$$

commute.

(c) *The full subcategory $Shv_1(k)[1/p] \subset Sh(Sm/k)_{et}[1/p]$ is stable under kernels, cokernels, and extensions.*

Proof. This is [BVK12, 3.2.3 and 3.3.1]. □

Remark 2.2.7. The above proposition is false if we do not invert p. For example, over a field k of characteristic $p > 0$ consider the pth power morphism $F : \mathbb{G}_m \to \mathbb{G}_m$, where we view $\mathbb{G}_m$ as a sheaf on $(Sm/k)_{et}$. Then $\mathrm{coker}\, F$ is not 1-motivic. To see this, first note that $\mathrm{coker}\, F$ is non-zero: if it were zero, then F would have zero kernel and cokernel as a morphism of sheaves on $(Sm/k)_{et}$, implying that F is an isomorphism by the Yoneda lemma. So $\mathrm{coker}\, F$ is non-zero; on the other hand, $(\mathrm{coker}\, F)(\overline{k}) = 0$ since F is an epimorphism of fppf sheaves. This is impossible for a 1-motivic sheaf.

We now state and prove our main fact for this chapter:

Proposition 2.2.8. *Let $\mathfrak{X}$ be a smooth Artin stack over a perfect field k, with quasi-compact separated diagonal. Then the restriction of the sheaf $\mathbf{Pic}_{\mathfrak{X}/k}$ to $(Sm/k)_{et}$ (which we also denote by $\mathbf{Pic}_{\mathfrak{X}/k}$) is 1-motivic.*

Remark 2.2.9. In the case where X is a scheme, this is [BVK12, 3.4.1]. For the reader's convenience, we include the proof of this special case in our argument below.

Proof. (of Proposition 2.2.8) We prove Proposition 2.2.8 by an increasingly general sequence of lemmas.

Lemma 2.2.10. *Let X be a proper smooth scheme over a field k, and let $\pi : X \to \operatorname{Spec} k$ be the structure morphism. Let $\mathbb{G}_{m,X}$ denote the representable sheaf on $(Sch/X)_{fppf}$ sending Y to $\Gamma(Y, \mathcal{O}_Y)^\times$. Then the sheaf $R^0\pi_*\mathbb{G}_{m,X}$ on $(Sch/k)_{fppf}$ is representable by a torus.*

Proof. Because the statement is étale-local on k, we may assume that π has a section $s : \operatorname{Spec} k \to X$. By [EGAIII, 7.7.6], the sheaf $R^0\pi_*\mathbb{G}_a$ sending T to $\Gamma(X_T, \mathcal{O}_{X_T})$ is representable by $\operatorname{Spec} V$, where V is a finite-dimensional k-vector space. More specifically, we have a functorial bijection

$$\Gamma(X_T, \mathcal{O}_{X_T}) \longrightarrow \operatorname{Hom}_k(V, \Gamma(T, \mathcal{O}_T)).$$

Using the fact that π has a section, applying $T = k$ in the above bijection yields

$$V \cong \Gamma(X, \mathcal{O}_X)^\vee \cong k^n,$$

where $n = |\pi_0(X)|$. Then $R^0\pi_*\mathbb{G}_m$ is representable by $\mathbb{G}_m^n$. $\square$

Lemma 2.2.11. *Let X be a proper smooth scheme over a perfect field k, and let $\pi : X \to \operatorname{Spec} k$ be the structure morphism. Then the sheaves $R^0\pi_*\mathbb{G}_{m,X}$ and $R^1\pi_*\mathbb{G}_{m,X}$ are 1-motivic.*

Proof. This is immediate from Lemma 2.2.10 and Theorem 2.1.2. $\square$

Lemma 2.2.12. *Let X be a smooth separated scheme of finite type over the perfect field k, such that there exists an open immersion $j : X \hookrightarrow \overline{X}$ with $\overline{X}$ smooth and proper. Then (letting $\pi : X \to \operatorname{Spec} k$ be the structure morphism) the sheaves $R^0\pi_*\mathbb{G}_{m,X}$ and $R^1\pi_*\mathbb{G}_{m,X}$ are 1-motivic.*

Proof. Let $i : D \hookrightarrow \overline{X}$ be the complement of U in $\overline{X}$. Then we have an exact sequence of sheaves

$$0 \longrightarrow R^0\overline{\pi}_*\mathbb{G}_{m,\overline{X}} \longrightarrow R^0\pi_*\mathbb{G}_{m,X} \longrightarrow \mathbf{Div}_D(\overline{X}) \longrightarrow \mathbf{Pic}_{\overline{X}/k} \longrightarrow R^1\pi_*\mathbb{G}_{m,X} \to 0,$$

using the fact that $\overline{X}$ is smooth. Here $\overline{\pi} : \overline{X} \to \operatorname{Spec} k$ is the structure morphism, and $\mathbf{Div}_D(\overline{X})$ is the locally constant sheaf of Weil divisors on $\overline{X}$ supported on D. By Proposition 2.2.6(c), $R^0\pi_*\mathbb{G}_{m,X}$ and $R^1\pi_*\mathbb{G}_{m,X} := \mathbf{Pic}_{X/k}$ are 1-motivic. $\square$

Lemma 2.2.13. *Let X be a smooth separated algebraic space of finite type over k. Then $R^0\pi_*\mathbb{G}_{m,X}$ and $R^1\pi_*\mathbb{G}_{m,X}$ are 1-motivic sheaves.*

Proof. By [CLO12], we can choose a compactification $X \hookrightarrow \overline{X}$ and then by [dJng96, 3.1], we can find an alteration $\overline{X}_0 \to \overline{X}$ which is generically étale, with $\overline{X}_0$ smooth proper. That is, we have a commutative diagram

$$
\begin{array}{ccccc}
U_0 & \hookrightarrow & X_0 & \hookrightarrow & \overline{X}_0 \\
\downarrow & & \downarrow & & \downarrow \\
U & \hookrightarrow & X & \hookrightarrow & \overline{X},
\end{array}
$$

where $U_0 \to U$ is finite étale. We can then extend $\overline{X}_0 \to \overline{X}$ to a proper hypercover $\overline{X}_\bullet \to \overline{X}$ with each $\overline{X}_i \to \overline{X}$ generically étale, and by possibly further restricting U, we can arrange that the restriction $U_\bullet \to U$ of this simplicial scheme to U has the property that $U_i \to U$ is finite étale for $i \in \{0,1,2\}$. Let $\pi_p : U_p \to U$ be the projection. Then in the spectral sequence

$$
R^q \pi_{p*}\left(\pi_p^* \mathbb{G}_{m,U} \right) \Rightarrow R^{p+q}\pi_* \mathbb{G}_{m,U}, \tag{2.2.13.1}
$$

we have $\pi_p^* \mathbb{G}_{m,U} = \mathbb{G}_{m,U_p}$ for $p = 0,1,2$. For all q, let $\mathscr{K}^{q\bullet}$ be the complex with terms

$$
\mathscr{K}^{qp} = R^q \pi_{p*}\mathbb{G}_{m,U_p}.
$$

By Lemma 2.2.12, $\mathscr{K}^{qp}$ is 1-motivic for $q = 0,1$ and all p. Moreover, the spectral sequence 2.2.13.1 yields

$$
R^0 \pi_* \mathbb{G}_{m,U} \cong \mathscr{H}^0(\mathscr{K}^{0\bullet})
$$

and an exact sequence

$$
0 \longrightarrow \mathscr{H}^1(\mathscr{K}^{0\bullet}) \longrightarrow R^1\pi_*\mathbb{G}_{m,U} \longrightarrow \mathscr{H}^0(\mathscr{K}^{1\bullet}) \longrightarrow \mathscr{H}^2(\mathscr{K}^{0\bullet}).
$$

By Proposition 2.2.6(c), the homology sheaves $\mathscr{H}^p(\mathscr{K}^{q,\bullet})$ are 1-motivic for p arbitrary and $q = 0,1$. Another application of Proposition 2.2.6(c) shows that $R^1\pi_*\mathbb{G}_{m,U}$ is 1-motivic.

Now let $i : D \hookrightarrow X$ be the inclusion of the complement $D = X - U$, and let π_X, π_U, π_D be the structure morphisms to $\operatorname{Spec} k$. Then we have an exact sequence of sheaves on $(Sm/k)_{et}$

$$
0 \to R^0\pi_{X*}\mathbb{G}_{m,X} \to R^0\pi_{U*}\mathbb{G}_{m,U} \to \mathbf{Div}_D(X) \to R^1\pi_{X*}\mathbb{G}_{m,X} \to R^1\pi_{U*}\mathbb{G}_{m,U} \to 0,
$$

where $\mathbf{Div}_D(X)$ is the étale group scheme of Weil divisors on X supported on D. Here the exactness on the right is because X and U are smooth. Therefore Proposition 2.2.6 shows that $R^i\pi_{X*}\mathbb{G}_{m,X}$ is 1-motivic for $i = 0,1$, as was to be shown. $\qquad\square$

We can now complete the proof of Proposition 2.2.8. Let $\mathfrak{X}$ be a smooth Artin stack of finite type over k, with separated quasi-compact diagonal. Choose a smooth cover $U \to \mathfrak{X}$, and take the corresponding Cech simplicial cover $U_\bullet \to \mathfrak{X}$. We then have a spectral sequence

$$
R^q \pi_{p*}\mathbb{G}_{m,U_p} \Rightarrow R^{p+q}\pi_*\mathbb{G}_{m,\mathfrak{X}}.
$$

By the previous lemma, $R^q\pi_{p*}\mathbb{G}_{m,U_p}$ is 1-motivic for $q = 0,1$ and p arbitrary. Therefore Proposition 2.2.6(c) shows that $R^i\pi_*\mathbb{G}_{m,\mathfrak{X}}$ is 1-motivic for $i = 0,1$ by the same argument as in Lemma 2.2.13. In particular, $\mathbf{Pic}_{\mathfrak{X}/k}$ is 1-motivic. $\qquad\square$

2.2.14. We can now use Propositions 2.2.6 and 2.2.8 to define sheaves $\mathbf{Pic}^{0,red}_{\mathfrak{X}/k}$ and $\mathbf{NS}_{\mathfrak{X}/k}$ for any smooth Artin stack $\mathfrak{X}/k$. Namely, by Proposition 2.2.6(a) there exists a semiabelian variety mapping to $\mathbf{Pic}_{\mathfrak{X}/k}$ with discrete cokernel. We let $\mathbf{Pic}^{0,red}_{\mathfrak{X}/k}$ be the image of this mapping, and $\mathbf{NS}_{\mathfrak{X}/k}$ be its cokernel.

Remark 2.2.15. If $\mathfrak{X}$ is proper and smooth over a perfect field k, then these definitions agree with the definitions in Proposition 2.1.2. Note that for any group scheme G, $\mathrm{Hom}(Y, G) = \mathrm{Hom}(Y, G_{red})$ for any smooth scheme Y, so the sheaves on $(Sm/k)_{et}$ defined by G and by G^{red} agree.

2.2.16. There is another subtlety in positive characteristic p, namely that we inverted p in all Hom-groups in our study of 1-motivic sheaves. Therefore the abelian groups $\mathrm{Pic}^0(\mathfrak{X}) := \mathbf{Pic}^{0,red}_{\mathfrak{X}/k}(k)$ and $NS(\mathfrak{X}) := \mathbf{NS}_{\mathfrak{X}/k}(k)$ are only well-defined up to a kernel and cokernel annihilated by a power of p. We will only deal with these groups through their ℓ-adic avatars $T_\ell \mathrm{Pic}^0(\mathfrak{X})$ and $\varprojlim_n NS(\mathfrak{X})/\ell^n NS(\mathfrak{X})$, however (for $\ell \neq p$), and the following simple proposition shows that these are well-defined:

Proposition 2.2.17. *Let A and B be abelian groups, and assume that we have an exact sequence*

$$0 \to K \to A \to B \to C \to 0$$

with K and C annihilated by some power p^r. Then for any $\ell \neq p$ and integer n, $A[\ell^n] \cong B[\ell^n]$ and $A/\ell^n A \cong B/\ell^n B$.

Proof. Let I be the image of A in B, so that we have exact sequences

$$0 \to K \to A \to I \to 0 \quad \text{and}$$
$$0 \to I \to B \to C \to 0.$$

Because $K[\ell^n] = K/\ell^n K = 0$ (and similarly for C), applying the functor of ℓ^n-torsion to both sequences yields $A[\ell^n] \cong I[\ell^n] \cong B[\ell^n]$ and $A/\ell^n A \cong I/\ell^n I \cong B/\ell^n B$. $\qquad\square$

Therefore the groups $\mathrm{Pic}^0(\mathfrak{X})[\ell^n]$ and $NS(\mathfrak{X})/\ell^n NS(\mathfrak{X})$ are well-defined.

Chapter 3

Weil Divisors and Cartier Divisors on Deligne-Mumford Stacks

3.1 Weil Divisors

3.1.1. Let $\mathfrak{X}$ be a d-dimensional separated Deligne-Mumford stack of finite type over a field k, and assume that the connected components of $\mathfrak{X}$ are equidimensional of dimension d. We recall the definition of the Chow groups $A^*(\mathfrak{X})$ [Vis89, 3.4]. For each i between 0 and d, define a presheaf $\mathscr{Z}^i$ on $\mathfrak{X}_{et}$ by setting

$$\mathscr{Z}^i(X \to \mathfrak{X}) := Z^i(X),$$

where $Z^i(X)$ is the free abelian group on the integral closed subschemes of X of codimension i, and the transition maps are induced by flat pullback of cycles. The presheaf $\mathscr{Z}^i$ is in fact a sheaf [Gil84, 4.2]. Also define a sheaf $\mathscr{W}^i$ on $\mathfrak{X}_{et}$ by setting

$$\mathscr{W}^i(X \to \mathfrak{X}) := \bigoplus_V k(V)^\times,$$

where the direct sum is over all subvarieties V of X of codimension $i-1$ and $k(V)^\times$ is the group of invertible rational functions on V. Since rational equivalence is preserved under flat pullback, we have a morphism of sheaves $\mathscr{W}^i \to \mathscr{Z}^i$ defined by taking the associated divisor of a rational function. We define

$$
\begin{aligned}
Z^i(\mathfrak{X}) &:= \Gamma(\mathfrak{X}, \mathscr{Z}^i), \\
W^i(\mathfrak{X}) &:= \Gamma(\mathfrak{X}, \mathscr{W}^i), \quad \text{and} \\
A^i(\mathfrak{X}) &:= Z^i(\mathfrak{X})/W^i(\mathfrak{X}).
\end{aligned}
$$

We define the group Div $\mathfrak{X}$ of Weil divisors on $\mathfrak{X}$ to be the group $Z^1(\mathfrak{X})$. We define the Weil class group Cl $\mathfrak{X}$ to be $A^1(\mathfrak{X})$.

3.2 Cartier Divisors

3.2.1. Next we define the notion of Cartier divisor on the stack $\mathfrak{X}$. For simplicity we will assume that the stack $\mathfrak{X}$ is geometrically reduced over k. Then we can define the sheaf $\mathscr{K}$ of rational maps on $\mathfrak{X}_{et}$ by setting

$$\mathscr{K}(X \to \mathfrak{X}) := \varinjlim_{U \subseteq X \text{ open dense}} \mathrm{Hom}(U, \mathbb{A}^1_k).$$

We define $\mathscr{K}^*$ to be the subsheaf of invertible elements of $\mathscr{K}$ under multiplication. We then define

$$\mathrm{Ca}\ \mathfrak{X} := \Gamma(\mathfrak{X}, \mathscr{K}^*/\mathbb{G}_m),$$

the group of Cartier divisors on $\mathfrak{X}$, where $\mathbb{G}_m$ is the usual sheaf of invertible sections on $\mathfrak{X}_{et}$. Then define the Cartier class group

$$\mathrm{CaCl}\ \mathfrak{X} := \Gamma(\mathfrak{X}, \mathscr{K}^*/\mathbb{G}_m)/\Gamma(\mathfrak{X}, \mathscr{K}^*).$$

Proposition 3.2.2. *Let $\mathfrak{X}$ be a smooth separated Deligne-Mumford stack of finite type over k. Then there is a canonical isomorphism $\mathrm{Ca}\ \mathfrak{X} \cong \mathrm{Div}\ \mathfrak{X}$. Moreover, this isomorphism induces an isomorphism $\mathrm{CaCl}\ \mathfrak{X} \cong \mathrm{Cl}\ \mathfrak{X}$.*

Proof. We first note that we have a commuting diagram

$$\begin{array}{ccc} \mathscr{K}^* & \longrightarrow & \mathscr{K}^*/\mathbb{G}_m \\ \cong \downarrow & & \cong \downarrow \\ \mathscr{W}^1 & \longrightarrow & \mathscr{Z}^1 \end{array} \qquad (3.2.2.1)$$

of sheaves on $\mathfrak{X}_{et}$, where the vertical maps are isomorphisms. To see that we have such a diagram, notice that any etale $X \in (Et/\mathfrak{X})$ is smooth, and so taking sections of the above square at X, we require a commuting diagram

$$\begin{array}{ccc} k(X)^\times & \longrightarrow & \mathrm{Ca}\ X \\ \cong \downarrow & & \cong \downarrow \\ k(X)^\times & \longrightarrow & \mathrm{Div}\ X. \end{array}$$

The fact that this diagram is commutative (and that the right-hand vertical arrow is an isomorphism) is a standard consequence of X being smooth. Therefore we have the commuting diagram (3.2.2.1). Taking global sections of the right hand map in (3.2.2.1) gives us an isomorphism $\mathrm{Ca}\ \mathfrak{X} \cong \mathrm{Div}\ \mathfrak{X}$, while taking cokernels of the maps on global sections induced by the horizontal arrows gives us $\mathrm{CaCl}\ \mathfrak{X} \cong \mathrm{Cl}\ \mathfrak{X}$. $\qquad \square$

3.2.3. From the exact sequence

$$0 \to \mathbb{G}_m \to \mathscr{K}^*_\mathfrak{X} \to \mathscr{K}^*_\mathfrak{X}/\mathbb{G}_m \to 0$$

of sheaves on $\mathfrak{X}_{et}$, we get an injection $\mathrm{CaCl}\ \mathfrak{X} \hookrightarrow \mathrm{Pic}\ \mathfrak{X}$. Unlike the case of schemes, however, we cannot expect this map to be an isomorphism, even when $\mathfrak{X}$ is smooth. For example, if $\mathfrak{X} = BG$ where G is a finite group, then $\mathrm{CaCl}\ BG = 0$ while

$$\mathrm{Pic}\ BG = \mathrm{Hom}(G, \mathbb{G}_m) \neq 0.$$

The best we can do is the following:

Proposition 3.2.4. *Let $\mathfrak{X}$ be a geometrically reduced, separated Deligne-Mumford stack of finite type over a field k. Then the quotient*

$$H := \mathrm{Pic}\ \mathfrak{X}/\mathrm{CaCl}\ \mathfrak{X}$$

is a finite group.

Proof. Taking cohomology in the exact sequence

$$0 \to \mathbb{G}_{m,\mathfrak{X}} \to \mathscr{K}_{\mathfrak{X}}^* \to \mathscr{K}_{\mathfrak{X}}^*/\mathbb{G}_{m,\mathfrak{X}} \to 0,$$

we have an exact sequence of groups

$$0 \to \mathrm{CaCl}\ \mathfrak{X} \to \mathrm{Pic}\ \mathfrak{X} \to H^1(\mathfrak{X}, \mathscr{K}^*),$$

so it suffices to show that $H^1(\mathfrak{X}, \mathscr{K}^*)$ is finite. Let X be the coarse moduli space of $\mathfrak{X}$, let $\xi_1, ..., \xi_n$ be the generic points of X, and let $\xi = \xi_1 \coprod ... \coprod \xi_n$ be their disjoint union. Finally, let $\mathscr{G}_i = \xi_i \times_X \mathfrak{X}$ and $\mathscr{G} = \xi \times_X \mathfrak{X}$ be the fiber products, and $\iota_i : \mathscr{G}_i \hookrightarrow \mathfrak{X}$, $\iota : \mathscr{G} \hookrightarrow \mathfrak{X}$ the resulting maps.

Lemma 3.2.5. *With notation as above, we have an equality*

$$\iota_* \mathbb{G}_m = \mathscr{K}^*$$

of sheaves on $\mathfrak{X}_{et}$.

Proof. (of lemma) Let $V \to \mathfrak{X}$ be étale. Then

$$\mathscr{K}^*(V \to \mathfrak{X}) = \varinjlim_{U \subset V \mathrm{dense}} \mathrm{Hom}(U, \mathbb{G}_m).$$

On the other hand,

$$\begin{aligned}
(\iota_* \mathbb{G}_m)(V \to \mathfrak{X}) &= \mathrm{Hom}(V \times_{\mathfrak{X}} \mathscr{G}, \mathbb{G}_m) \\
&= \mathrm{Hom}(V \times_X \xi, \mathbb{G}_m) \\
&= \mathscr{K}^*(V \to \mathfrak{X}).
\end{aligned}$$

The last equality is because V is quasi-finite over X with open image, and hence the fiber over ξ consists of the generic points of V. $\square$

Continuing with the proof of Proposition 3.2.4, from the spectral sequence

$$H^p(\mathfrak{X}, R^q \iota_* \mathbb{G}_m) \Rightarrow H^{p+q}(\mathscr{G}, \mathbb{G}_m),$$

we get an inclusion $H^1(\mathfrak{X}, \iota_* \mathbb{G}_m) \hookrightarrow \mathrm{Pic}(\mathscr{G})$. This reduces us to showing that $\mathrm{Pic}(\mathscr{G})$ is finite. Moreover, since $\mathscr{G} = \amalg_i \mathscr{G}_i$, it suffices to show that $\mathrm{Pic}(\mathscr{G}_i)$ is finite for each i. Set $\mathscr{H} := \mathscr{G}_i$ for any i, and $\zeta := \xi_i$. Then $\mathscr{H} \to \zeta$ is an fppf-gerbe: Since $\zeta \to X$ is flat, $\mathscr{H} \to \zeta$ is the coarse moduli space of $\mathscr{H}$ (since the property of being a coarse moduli space is stable under flat base change), so the topological space of $\mathscr{H}$ has just one point. By [LMB00, Thm 11.5], $\mathscr{H}$ must be an fppf-gerbe over ζ. Moreover, $\mathscr{H} \to \zeta$ is banded by a finite group G since $\mathscr{H}$ is a Deligne-Mumford stack. Therefore the following lemma will complete the proof of Proposition 3.2.4. $\square$

Lemma 3.2.6. *Let $\mathscr{H}$ be a Deligne-Mumford stack such that $\pi : \mathscr{H} \to \zeta$ is an fppf-gerbe, with ζ the spectrum of a field. Then $\mathrm{Pic}(\mathscr{H})$ is finite.*

Proof. Let $\zeta = \mathrm{Spec}\, F$, and let $F \hookrightarrow L$ be a finite field extension such that $\mathscr{H} \times_\zeta \mathrm{Spec}\, L$ is isomorphic to BG, where G is a finite group. Then $\mathrm{Pic}(\mathscr{H} \times_\zeta \mathrm{Spec}\, L) = \mathrm{Pic}(BG) = \mathrm{Hom}(G, \mathbb{G}_m)$ is a finite group. Now consider the Picard functor $\mathbf{Pic}_{\mathscr{H}/\zeta}$. The spectral sequence

$$H^p_{fppf}(\zeta, R^q \pi_* \mathbb{G}_{m, \mathscr{H}}) \Rightarrow H^{p+q}(\mathscr{H}, \mathbb{G}_m)$$

yields an exact sequence

$$0 \to H^1_{fppf}(\zeta, \pi_* \mathbb{G}_{m, \mathscr{H}}) \to \mathrm{Pic}(\mathscr{H}) \to \mathbf{Pic}_{\mathscr{H}/\zeta}(\zeta).$$

Note that we have $\pi_* \mathbb{G}_{m, \mathscr{H}} = \mathbb{G}_{m, \zeta}$ as sheaves in ζ_{fppf}. This is because for any $Y \to \zeta$, the stack $Y \times_\zeta \mathscr{H}$ has Y as coarse moduli space (since $Y \to \zeta$ is flat) and therefore $\mathrm{Hom}(X \times_\zeta \mathscr{H}, \mathbb{G}_m) = \mathrm{Hom}(X, \mathbb{G}_m)$. It is well-known that $H^1_{fppf}(\zeta, \mathbb{G}_m) = \mathrm{Pic}(\zeta) = 0$, and we conclude that $\mathrm{Pic}(\mathscr{H})$ injects into $\mathbf{Pic}_{\mathscr{H}/\zeta}(\zeta)$. Finally, by faithfully flat descent, $\mathbf{Pic}_{\mathscr{H}/\zeta}(\zeta)$ injects into $\mathbf{Pic}_{\mathscr{H}/\zeta}(\mathrm{Spec}\, L)$ which is finite since it equals $\mathrm{Pic}(\mathscr{H} \times_\zeta \mathrm{Spec}\, L) = \mathrm{Pic}(BG)$. $\square$

Corollary 3.2.7. *Let $\mathfrak{X}$ be a smooth proper Deligne-Mumford stack over an algebraically closed field k. Then every element of $\mathrm{Pic}^0(\mathfrak{X})$ is represented by a Weil divisor.*

Proof. Temporarily, let $\mathrm{Cl}^0(\mathfrak{X}) \subset \mathrm{Pic}^0(\mathfrak{X})$ denote the subgroup of the Weil class group which maps to 0 in $NS(\mathfrak{X})$. Then we have an injection

$$\mathrm{Pic}^0(\mathfrak{X})/\mathrm{Cl}^0(\mathfrak{X}) \hookrightarrow \mathrm{Pic}(\mathfrak{X})/\mathrm{Cl}(\mathfrak{X}),$$

so $\mathrm{Pic}^0(\mathfrak{X})/\mathrm{Cl}^0(\mathfrak{X})$ is finite. On the other hand, it is a quotient of the divisible group $\mathrm{Pic}^0(\mathfrak{X})$, hence divisible. Therefore it is trivial. $\square$

Chapter 4

Cycle class map on smooth Deligne-Mumford stacks

4.1 Cycle map on non-compactly supported cohomology

4.1.1. Let $\mathfrak{X}$ be a smooth Deligne-Mumford stack of pure dimension N over a field k. In this chapter we review the definition of the cycle class map

$$cl : A^d(\mathfrak{X}) \to H^{2d}(\mathfrak{X}_{\overline{k}}, \mathbb{Q}_\ell(d)),$$

where ℓ is different from $p = \mathrm{char}(k)$. For the definition of a cycle class map for singular Deligne-Mumford stacks and more general coefficient rings, see [Ols2, Sect. 3].

4.1.2. For ease of notation we assume $k = \overline{k}$; it will be clear from our construction that the cycle class map we produce will be invariant under Galois action. Let $\mathfrak{X}$ be a purely N-dimensional Deligne-Mumford stack over k, let $D \in A^d(\mathfrak{X})$ be a cycle, and let $e = N - d$ be the dimension of D. Write $D = \sum a_i D_i$ as a sum of integral cycles D_i, and let $U_i \subset D_i$ be the smooth locus.

Lemma 4.1.3. *For each D_i, there is a canonical trace map*

$$Tr_i : H_c^{2e}(D_i, \mathbb{Q}_\ell(e)) \xrightarrow{\sim} \mathbb{Q}_\ell$$

Before starting the proof of Lemma 4.1.3 we note the following fact, which will be used repeatedly in this paper:

Lemma 4.1.4. *Let $\mathfrak{X}$ be a separated finite type Deligne-Mumford stack, and let $\pi : \mathfrak{X} \to X$ be its coarse moduli space. Then the natural map $\mathbb{Q}_{\ell,X} \to R\pi_*\mathbb{Q}_{\ell,\mathfrak{X}}$ is an isomorphism in $D_c^b(X, \mathbb{Q}_\ell)$.*

Proof. Combining Theorem 5.1 and Corollary 5.8 of [Ols1], we have that $R\pi_*\mathbb{Q}_\ell$ is acyclic in non-zero degrees, so we only need to show that $R^0\pi_*\mathbb{Q}_{\ell,\mathfrak{X}} = \mathbb{Q}_{\ell,X}$ which follows easily from the fact that the topological spaces of $\mathfrak{X}$ and X are homeomorphic. $\square$

We now return to Lemma 4.1.3:

Proof. (of Lemma 4.1.3) Let U_i be the non-empty smooth locus of D_i, and $j : U_i \hookrightarrow D_i$ the inclusion, and $k : Z_i \hookrightarrow D_i$ the inclusion of $Z_i := D_i - U_i$ into D_i. Then from the short exact sequence

$$0 \to j_!\mathbb{Q}_{\ell,U_i} \to \mathbb{Q}_{\ell,D_i} \to i_*\mathbb{Q}_{\ell,Z_i} \to 0,$$

we get a long exact sequence

$$... \to H_c^{2e-1}(Z_i, \mathbb{Q}_\ell(e)) \to H_c^{2e}(U_i, \mathbb{Q}_\ell(e)) \to H_c^{2e}(D_i, \mathbb{Q}_\ell(e)) \to H_c^{2e}(Z_i, \mathbb{Q}_\ell(e)) \to ...$$

But $H_c^{2e-1}(Z_i, \mathbb{Q}_\ell(e)) = H_c^{2e}(Z_i, \mathbb{Q}_\ell(e)) = 0$ because $\dim(Z_i) < e$ (use Lemma 4.1.4 and the fact that the statement is true for algebraic spaces.) Now Poincaré duality on the smooth stack U_i [LO08, 4.4.1] gives the required map Tr_i. $\square$

An easy extension of the above lemma shows that the trace maps Tr_i induce a canonical isomorphism

$$\mathrm{Hom}(H_c^{2e}(D, \mathbb{Q}_\ell(e)), \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{Hom}(\oplus_i H_c^{2e}(U_i, \mathbb{Q}_\ell(e)), \mathbb{Q}_\ell) \xrightarrow{\sim} \mathbb{Q}_\ell^{I(D)}, \qquad (4.1.4.1)$$

where $I(D) = \{D_1, ..., D_r\}$ is the set of irreducible components of D. Let $\alpha : D \hookrightarrow \mathfrak{X}$ be the inclusion. Let $\mathscr{D}_\mathfrak{X}$ and $\mathscr{D}_k$ be the Verdier dualities on $\mathfrak{X}$ and k, respectively. Then we have

$$\mathscr{D}_k R\Gamma_c \alpha^*\mathbb{Q}_\ell(e) = R\Gamma R\alpha^!\mathscr{D}_\mathfrak{X}(\mathbb{Q}_\ell(e)) = R\Gamma R\alpha^!\mathbb{Q}_\ell[2N](N-e),$$

where we used the fact that $\mathfrak{X}$ is smooth in the right-hand equality. This induces a canonical isomorphism

$$\mathrm{Hom}(H_c^{2e}(D, \mathbb{Q}_\ell(e)), \mathbb{Q}_\ell) \xrightarrow{\sim} H_D^{2d}(\mathfrak{X}, \mathbb{Q}_\ell(d)) \qquad (4.1.4.2)$$

(recall that $d = N - e$).

Definition 4.1.5. In the above notation, we obtain from 4.1.4.1 a canonical element $[D] \in \mathrm{Hom}(H_c^{2e}(D, \mathbb{Q}_\ell(e)), \mathbb{Q}_\ell)$ corresponding to $\sum a_i D_i \in \mathbb{Q}_\ell^{I(D)}$. Then we set the localized cycle class of D, denoted $cl(D)$, to be the image of $[D]$ in $H_D^{2d}(\mathfrak{X}, \mathbb{Q}_\ell(d))$ under 4.1.4.2.

4.1.6. We also write $cl(D)$ for the image of the localized cycle class under the map

$$H_D^{2d}(\mathfrak{X}, \mathbb{Q}_\ell(d)) \to H^{2d}(\mathfrak{X}, \mathbb{Q}_\ell(d))$$

(we call $cl(D)$ the global cycle class of D). When we need to distinguish between the local cycle class and global cycle class of D, we will denote these by $cl_{loc}(D)$ and $cl_{gl}(D)$, respectively. It is standard in the case of schemes that the resulting map

$$cl : Z^d(\mathfrak{X}) \to H^{2d}(\mathfrak{X}, \mathbb{Q}_\ell(d))$$

passes to the Chow group $A^d(\mathfrak{X})$, and is compatible with the contravariant functoriality for $A^d(-)$ and $H^{2d}(-, \mathbb{Q}_\ell(d))$. The same proof works for stacks.

4.1.7. Now let $D \subset \mathfrak{X}$ be any reduced closed subscheme, and $\alpha : D \hookrightarrow \mathfrak{X}$ the inclusion. Let $\dim(D) = e$ and $\dim(\mathfrak{X}) = N$. We can use the above cycle class map to give a cycle-theoretic description of the Poincaré dual of the restriction map

$$\alpha^* : H_c^{2e}(\mathfrak{X}, \mathbb{Q}_\ell(e)) \to H_c^{2e}(D, \mathbb{Q}_\ell(e)).$$

Let $D = \cup_i D_i$ be the decomposition of D into its irreducible components, and let $I_e(D)$ be the set of e-dimensional irreducible components of D.

Proposition 4.1.8. *With notation as above, we have a commutative square*

$$
\begin{array}{ccc}
H_c^{2e}(D, \mathbb{Q}_\ell(e))^\vee & \xrightarrow{(\alpha^*)^\vee} & H_c^{2e}(\mathfrak{X}, \mathbb{Q}_\ell(e))^\vee \\
\sim\downarrow & & \sim\downarrow \\
\mathbb{Q}_\ell^{I_e(d)} & \longrightarrow & H^{2N-2e}(\mathfrak{X}, \mathbb{Q}_\ell(N-e))
\end{array}
$$

where the vertical arrows are the isomorphisms induced by Poincaré duality, and the lower arrow sends $\sum a_i[D_i]$ to $\sum a_i cl(D_i)$.

Proof. This is immediate from the above description of the cycle class map. $\qquad\square$

4.1.9. In the case of divisors, the cycle class map can be described as follows. Let $\mathfrak{X}$ be a smooth Deligne-Mumford stack over k, and D a closed subscheme of $\mathfrak{X}$; let $i : D \hookrightarrow \mathfrak{X}$ be the inclusion, and $j : U \hookrightarrow \mathfrak{X}$ the inclusion of the open complement $U = \mathfrak{X} - D$. Recall that there is a natural bijection between $H_D^1(\mathfrak{X}, \mathbb{G}_m)$ and the group of Cartier divisors supported on D. To see this, first define

$$\mathcal{H}_D^j(\mathfrak{X}, \mathbb{G}_m) := i_* R^j i^! \mathbb{G}_m.$$

Then we have an exact sequence on sheaves on $\mathfrak{X}_{et}$

$$0 \to \mathcal{H}_D^0(\mathfrak{X}, \mathbb{G}_m) \to \mathbb{G}_m \to j_* \mathbb{G}_{m,U} \to \mathcal{H}_D^1(\mathfrak{X}, \mathbb{G}_m) \to 0,$$

so that

$$\mathcal{H}_D^1(\mathfrak{X}, \mathbb{G}_m) \cong \mathrm{coker}(\mathbb{G}_m \to j_* \mathbb{G}_{m,U}).$$

By its definition, giving a global section of the latter sheaf is the same as giving a Cartier divisor supported on D. Moreover, it is clear that $\mathcal{H}_D^0(\mathfrak{X}, \mathbb{G}_m) = 0$. Therefore the spectral sequence

$$H^p(\mathfrak{X}, \mathcal{H}_D^q(\mathfrak{X}, \mathbb{G}_m)) \Rightarrow H_D^{p+q}(\mathfrak{X}, \mathbb{G}_m)$$

shows that

$$H_D^1(\mathfrak{X}, \mathbb{G}_m) \cong H^0(\mathfrak{X}, \mathcal{H}_D^1(\mathfrak{X}, \mathbb{G}_m)),$$

and hence that $H_D^1(\mathfrak{X}, \mathbb{G}_m)$ is isomorphic to the group of Cartier divisors supported on D.

Now suppose D is a Cartier divisor on $\mathfrak{X}$. Then we have a canonical class $cl'(D) \in H_D^1(\mathfrak{X}, \mathbb{G}_m)$ corresponding to D. For any integer n prime to $p = \operatorname{char} k$, we produce a class $cl''(D) \in H_D^2(\mathfrak{X}, \mathbb{Q}_\ell(1))$ using the Kummer exact sequence of sheaves

$$0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

to induce a map $H_D^1(\mathfrak{X}, \mathbb{G}_m) \to H_D^2(\mathfrak{X}, \mu_n)$ and taking the limit over $n = \ell^m$.

Proposition 4.1.10. *The class $cl''(D) \in H_D^2(\mathfrak{X}, \mathbb{Q}_\ell(1))$ agrees with the class $cl(D)$ defined earlier.*

Proof. In the case of schemes this is [SGA4h, Cycle, 2.3.6], and one easily reduces to this case since the definition of $cl(D) \in H_D^2(\mathfrak{X}, \mathbb{Q}_\ell(1))$ is compatible with étale localization. $\square$

Using this fact, and finite generation of the Néron-Severi group (2.2.14) we get the following:

Proposition 4.1.11. *Let $\mathfrak{X}$ be a smooth stack over $k = \bar{k}$ as above, and choose a prime $\ell \neq \operatorname{char} k$. Then the cycle class map*

$$cl : \operatorname{Pic}(\mathfrak{X}) \to H^2(\mathfrak{X}, \mathbb{Q}_\ell(1))$$

factors as

$$\operatorname{Pic}(\mathfrak{X}) \to NS(\mathfrak{X}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \hookrightarrow H^2(\mathfrak{X}, \mathbb{Q}_\ell(1)),$$

where the map $\operatorname{Pic}(\mathfrak{X}) \to NS(\mathfrak{X}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ is the obvious one, and the map on the right is an injection.

Proof. From the previous proposition we get that $cl : A^1(\mathfrak{X}) \to H^2(\mathfrak{X}, \mathbb{Q}_\ell(1))$ factors through the injection (induced by the Kummer exact sequence)

$$\left(\varprojlim_n \frac{\operatorname{Pic}(\mathfrak{X})}{\ell^n \operatorname{Pic}(\mathfrak{X})} \right) \otimes \mathbb{Q} \hookrightarrow H^2(\mathfrak{X}, \mathbb{Q}_\ell(1)).$$

By Lemma 2.2.14 we have an exact sequence

$$0 \to \operatorname{Pic}^0(\mathfrak{X}) \to \operatorname{Pic}(\mathfrak{X}) \to NS(\mathfrak{X}) \to 0$$

with $\operatorname{Pic}^0(\mathfrak{X})$ divisible (at least up to p-torsion), and therefore

$$\frac{\operatorname{Pic}(\mathfrak{X})}{\ell^n \operatorname{Pic}(\mathfrak{X})} = \frac{NS(\mathfrak{X})}{\ell^n NS(\mathfrak{X})}.$$

Therefore the map cl factors as

$$\operatorname{Pic}(\mathfrak{X}) \to \varprojlim_n \frac{NS(\mathfrak{X})}{\ell^n NS(\mathfrak{X})} \otimes \mathbb{Q} \hookrightarrow H^2(\mathfrak{X}, \mathbb{Q}_\ell(1)).$$

Therefore we are reduced to showing that

$$\varprojlim_n \frac{NS(\mathfrak{X})}{\ell^n NS(\mathfrak{X})} \cong NS(\mathfrak{X}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell,$$

which is clear since $NS(\mathfrak{X})$ is finitely generated. $\square$

4.2 A variant of the cycle class map for compactly supported cohomology

Let $\overline{\mathfrak{X}}$ be a smooth proper Deligne-Mumford stack over a field $k = \overline{k}$, and let $i : D \hookrightarrow \overline{\mathfrak{X}}$ be a reduced closed subscheme of $\overline{\mathfrak{X}}$. Let $\mathfrak{X} = \overline{\mathfrak{X}} - D$, and let $j : \mathfrak{X} \hookrightarrow \overline{\mathfrak{X}}$ be the inclusion. Finally, let $\mathrm{Div}_{\mathfrak{X}}(\overline{\mathfrak{X}})$ be the free abelian group of divisors on $\overline{\mathfrak{X}}$ whose support is contained in $\mathfrak{X}$, i.e., disjoint from D. In this section we describe a cycle class map to compactly supported cohomology

$$cl_c : \mathrm{Div}_{\mathfrak{X}}(\overline{\mathfrak{X}}) \longrightarrow H_c^2(\mathfrak{X}, \mathbb{Q}_\ell(1)) := H^2(\overline{\mathfrak{X}}, j_!\mathbb{Q}_\ell(1)) \qquad (4.2.0.1)$$

and prove some compatibilities regarding this cycle class map.

First definition of 4.2.0.1. First recall the following well-known proposition:

Proposition 4.2.1. *Let $i : D \hookrightarrow \overline{\mathfrak{X}}$ be the inclusion, and define*

$$\mathbb{G}_{m,\overline{\mathfrak{X}},D} := \mathrm{Ker}(\mathbb{G}_{m,\overline{\mathfrak{X}}} \to i_*\mathbb{G}_{m,D}).$$

Then $H^1(\overline{\mathfrak{X}}, \mathbb{G}_{m,\overline{\mathfrak{X}},D})$ is in bijection with isomorphism classes of pairs $(\mathscr{L}, \sigma)$, where $\mathscr{L}$ is a line bundle on $\overline{\mathfrak{X}}$ and $\sigma : \mathscr{L}|_D \xrightarrow{\sim} \mathcal{O}_D$ is a trivialization of $\mathscr{L}$ on D.

Proof. The statement when $\overline{\mathfrak{X}}$ is a scheme is well-known (see, e.g., [BVS01, App. A]) and the same proof applies to Deligne-Mumford stacks. $\qquad\qquad\square$

We denote this group (following standard notation) by $\mathrm{Pic}(\overline{\mathfrak{X}}, D)$.

Given $E \in \mathrm{Div}_{\mathfrak{X}}(\overline{\mathfrak{X}})$, there is a natural class

$$cl_c'(E) \in \mathrm{Pic}(\overline{\mathfrak{X}}, D)$$

consisting of the pair $(\mathcal{O}(E), s)$ where $\mathcal{O}(E)$ is the line bundle of the divisor E and $s : \mathcal{O}_X \to \mathcal{O}(E)$ is the canonical meromorphic section of E, which is an isomorphism restricted to D since $D \cap E = \emptyset$. To define a class in $H_c^2(\mathfrak{X}, \mu_n)$ note that we have an exact sequence of sheaves

$$0 \longrightarrow j_!\mu_{n,\mathfrak{X}} \longrightarrow \mathbb{G}_{m,\overline{\mathfrak{X}},D} \xrightarrow{\cdot n} \mathbb{G}_{m,\overline{\mathfrak{X}},D} \longrightarrow 0.$$

Then we write

$$cl_{c,1}(E) \in H_c^2(\mathfrak{X}, \mathbb{Q}_\ell(1))$$

for the image of $cl_c'(E) \in H^1(\overline{\mathfrak{X}}, \mathbb{G}_{m,\overline{\mathfrak{X}},D})$ under the boundary map, taking the limit over $n = \ell^m$.

Second definition of 4.2.0.1. Given $E \in \mathrm{Div}_{\mathfrak{X}}(\overline{\mathfrak{X}})$, let $v : E \hookrightarrow \overline{\mathfrak{X}}$ be the inclusion. Consider the canonical map in $D_c^b(\overline{\mathfrak{X}})$

$$f : v_*Rv^!\mathbb{Q}_{\ell,\overline{\mathfrak{X}}}(1) \longrightarrow \mathbb{Q}_{\ell,\overline{\mathfrak{X}}}(1).$$

By definition, this map sends the local cycle class $cl_{loc}(E) \in H_E^2(\overline{\mathfrak{X}}, \mathbb{Q}_\ell(1))$ to the global cycle class $cl_{gl}(E) \in H^2(\overline{\mathfrak{X}}, \mathbb{Q}_\ell(1))$. Notice, moreover, that the composition

$$v_* Rv^! \mathbb{Q}_{\ell,\overline{\mathfrak{X}}}(1) \xrightarrow{f} \mathbb{Q}_{\ell,\overline{\mathfrak{X}}}(1) \longrightarrow i_* \mathbb{Q}_{\ell,D}(1)$$

is zero, since D and E are disjoint. Therefore f factors through a unique map

$$\tilde{f} : v_* Rv^! \mathbb{Q}_{\ell,\overline{\mathfrak{X}}}(1) \longrightarrow j_! \mathbb{Q}_{\ell,\mathfrak{x}} = \mathrm{Ker}(\mathbb{Q}_{\ell,\overline{\mathfrak{X}}}(1) \to i_* \mathbb{Q}_{\ell,D}(1))$$

($\tilde{f}$ is unique because any two maps f_1, f_2 define a map

$$f_1 - f_2 : v_* Rv^! \mathbb{Q}_{\ell,\overline{\mathfrak{X}}}(1) \to i_* \mathbb{Q}_{\ell,D}[-1]$$

which must be the zero map, since $v_* Rv^! \mathbb{Q}_{\ell,\overline{\mathfrak{X}}}(1)$ is supported on E and $i_* \mathbb{Q}_{\ell,D}(1)$ is supported on D).

Taking global sections of the map $\tilde{f}$ induces a map

$$H_E^2(\overline{\mathfrak{X}}, \mathbb{Q}_\ell(1)) \longrightarrow H_c^2(\mathfrak{X}, \mathbb{Q}_\ell(1))$$

and we define $cl_{c,2}(E) \in H_c^2(\mathfrak{X}, \mathbb{Q}_\ell(1))$ to be the image of the local cycle class $cl_{loc}(E)$ under this map.

Proposition 4.2.2. *The two cycle classes defined above agree, i.e., $cl_{c,1}(E) = cl_{c,2}(E)$.*

Proof. We can follow the same steps as in the definition of $cl_{c,2}(E)$ to show that the canonical map

$$g : v_* Rv^! \mathbb{G}_{m,\overline{\mathfrak{X}}} \longrightarrow \mathbb{G}_{m,\overline{\mathfrak{X}}}$$

factors uniquely through a map

$$\tilde{g} : v_* Rv^! \mathbb{G}_{m,\overline{\mathfrak{X}}} \longrightarrow \mathbb{G}_{m,\overline{\mathfrak{X}},D} = \mathrm{Ker}(\mathbb{G}_{m,\overline{\mathfrak{X}}} \to \mathbb{G}_{m,D}).$$

Taking global sections of $\tilde{g}$ yields a map

$$\tilde{g} : H_E^1(\overline{\mathfrak{X}}, \mathbb{G}_m) \to H^1(\overline{\mathfrak{X}}, \mathbb{G}_{m,\overline{\mathfrak{X}},D}) = \mathrm{Pic}(\overline{\mathfrak{X}}, D).$$

In terms of Cech cohomology, this map is described as follows: let $\mathcal{O}(E)$ be the line bundle of E. Viewing E as a Cartier divisor, we get a transition function $\alpha \in \mathbb{G}_m(U \to \overline{\mathfrak{X}})$ (where U is some étale cover of $\overline{\mathfrak{X}}$) defining $\mathcal{O}(E)$ such that α does not vanish along D. Therefore α also defines a transition function for an element of $H^1(\overline{\mathfrak{X}}, \mathbb{G}_{m,\overline{\mathfrak{X}},D})$, and this transition function is precisely the image of $cl(E)$ under $\tilde{g}$.

From this description it is clear that $\tilde{g}(cl_{loc}(E)) = cl_c'(E) \in \mathrm{Pic}(\overline{\mathfrak{X}}, D)$, where $cl_c'(E)$ is the class defined earlier. Using the Kummer exact sequence, we conclude that the two classes

$$cl_{c,1}(E), cl_{c,2}(E) \in H_c^2(\mathfrak{X}, \mathbb{Q}_\ell(1))$$

are the same. $\square$

We write $cl_c(E)$ for the element $cl_{c,1}(E) = cl_{c,2}(E)$.

Corollary 4.2.3. *Let $\overline{\mathfrak{X}}$, $\mathfrak{X}$, and D be as above. Then the cycle class map*

$$cl_c : \operatorname{Div}_{\mathfrak{X}}(\overline{\mathfrak{X}}) \longrightarrow H_c^2(\mathfrak{X}, \mathbb{Q}_\ell(1))$$

factors as

$$\operatorname{Div}_{\mathfrak{X}}(\overline{\mathfrak{X}}) \to NS(\overline{\mathfrak{X}}, D) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \hookrightarrow H_c^2(\mathfrak{X}, \mathbb{Q}_\ell(1))$$

where the map $\operatorname{Div}_{\mathfrak{X}}(\overline{\mathfrak{X}}) \to NS(\overline{\mathfrak{X}}, D)$ is the natural map and the map on the right is an injection.

Proof. The proof of Proposition 4.1.11 works here as well, using the fact that $\operatorname{Pic}^0(\overline{\mathfrak{X}}, D)$ is divisible and $NS(\overline{\mathfrak{X}}, D)$ is finitely generated (see Proposition 6.6.4). $\square$

Chapter 5

Review of 1-motives

5.1   The category of 1-motives

In this chapter we review the theory of 1-motives as introduced in [Del74, §10]. We work over a perfect base field k of characteristic $p \geq 0$, and choose an algebraic closure $k \hookrightarrow \bar{k}$. All of this material (and much more) appears in [BVK12, App. C].

Definition 5.1.1. A 1-motive over k is a 2-term complex

$$[L \to G]$$

of abelian sheaves on $(Sch/k)_{fppf}$, where

- L is an étale-locally constant sheaf, and $L(\bar{k})$ is a finitely generated free abelian group.

- G is (represented by) a semi-abelian variety; that is, there is an extension

$$0 \to T \to G \to A \to 0,$$

where T is a torus and A is an abelian variety.

Our convention is that L is placed in degree -1 and G is placed in degree 0. Since k is perfect, L is fully described by the free abelian group $L(\bar{k})$ together with its action by $Gal(\bar{k}/k)$.

5.1.2. If $M = [L \to G]$ and $M' = [L' \to G']$ are 1-motives, then a morphism of 1-motives $F = (f, g) : M \to M'$ is a commuting diagram

$$
\begin{array}{ccc}
L & \xrightarrow{f} & L' \\
\downarrow & & \downarrow \\
G & \xrightarrow{g} & G'.
\end{array}
$$

5.1.3. Let R be a subring of $\mathbb{Q}$ containing $\mathbb{Z}$. We will always take either $R = \mathbb{Z}[p^{-1}]$ or $R = \mathbb{Q}$. The category of *R-isogeny* 1-motives over k, denoted 1-Mot$_k \otimes R$ has the same objects as 1-Mot$_k$, and for 1-motives M, M', we set

$$\text{Hom}_{1\text{-Mot}_k \otimes R}(M, M') := \text{Hom}_{1\text{-Mot}_k}(M, M') \otimes R.$$

We also write 1-Mot$_k[p^{-1}]$ when $R = \mathbb{Z}[p^{-1}]$.

Proposition 5.1.4. *The category of $\mathbb{Q}$-isogeny 1-motives over k is abelian.*

Proof. This is [BVK12, C.7.3], but we give a more elementary argument here. First notice that 1-Mot$_k \otimes \mathbb{Q}$ is clearly additive. Let $M = [L \xrightarrow{\alpha} G]$ and $M' = [L' \xrightarrow{\alpha'} G']$ be 1-motives, and let $F : M \to M'$ be a morphism of 1-motives given by the diagram

$$
\begin{array}{ccc}
L & \xrightarrow{\;\;f\;\;} & L' \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\
G & \xrightarrow{\;\;g\;\;} & G'.
\end{array}
$$

We first describe the kernel of F. Let $\text{Ker}^0(g)$ be the reduction of the connected component of the identity of $\text{Ker}(g)$, and let

$$\text{Ker}^0(f) := \text{Ker}(f) \cap \alpha^{-1}(\text{Ker}^0(g)).$$

Then we set

$$K := [\text{Ker}^0(f) \to \text{Ker}^0(g)],$$

and claim that K is the kernel of $F : M \to M'$. Let $M'' = [L'' \to G'']$ be another 1-motive, and let

$$
\begin{array}{ccc}
L'' & \xrightarrow{\;\;u\;\;} & L \\
\downarrow & & \downarrow \\
G'' & \xrightarrow{\;\;v\;\;} & G
\end{array}
$$

be a morphism such that the composition with F is 0 in 1-Mot$_k \otimes \mathbb{Q}$. Then for some $n \in \mathbb{N}$, $n(F \circ (u,v)) = 0$ in 1-Mot$_k$, and hence $f \circ nu = g \circ nv = 0$. Therefore nu and nv factor through $\text{Ker}(f)$ and $\text{Ker}(g)$, respectively. Since $\text{Ker}(f)/\text{Ker}^0(f)$ and $\text{Ker}(g)/\text{Ker}^0(g)$ are finite groups, we get that for some $m \in \mathbb{N}$, mnu and mnv factor through $\text{Ker}^0(f)$ and $\text{Ker}^0(g)$ respectively. Then

$$\frac{1}{mn}(mnu, mnv) : [L'' \to G''] \to [\text{Ker}^0(f) \to \text{Ker}^0(g)]$$

is a morphism in 1-Mot$_k \otimes \mathbb{Q}$ factoring $(u, v) : [L'' \to G''] \to [L \to G]$.

Now we describe the cokernel of F. Let T be the torsion subgroup of $\text{coker}(f)$. We set

$$C := [\text{coker}(f)/T \to \text{coker}(g)/\alpha'(T)].$$

A check similar to that for K shows that C is the cokernel of F, and that the axioms of an abelian category are satisfied. $\qquad\square$

Remark 5.1.5. The category of 1-motives over k is not abelian, nor is the category of p-isogeny 1-motives where $p = \mathrm{char}\ k$. However, there is an abelian category ${}^t\text{1-Mot}_k[p^{-1}]$ of torsion 1-motives over k which is abelian (described in [BVK12, App. C]), and a fully faithful functor

$$\text{1-Mot}_k[p^{-1}] \hookrightarrow {}^t\text{1-Mot}_k[p^{-1}]$$

[BVK12, C.5.3]. This provides $\text{1-Mot}_k[p^{-1}]$ with an exact structure with respect to which the ℓ-adic realization functors described below are exact [BVK12, C.6.2].

5.2 Realization functors

5.2.1. Let ℓ be a prime distinct from the characteristic of k. We review the ℓ-adic realization functor from 1-motives over k to ℓ-adic representations of $\mathrm{Gal}(\overline{k}/k)$. Let $M = [L \xrightarrow{\alpha} G]$ be a 1-motive. For any integer n prime to char k, we set M/n to be the cone of $\cdot n : M \to M$. We then set $T_{\mathbb{Z}/n}(M) = H^{-1}(M/n)$. More concretely, the $\overline{k}$-points of $T_{\mathbb{Z}/n}(M)$ can be written

$$T_{\mathbb{Z}/n}(M) = \frac{\{(x, g) \in L \times G(\overline{k}) \mid u(x) = -mg\}}{\{(mx, -u(x)) \mid x \in L\}}.$$

From the exact triangle

$$G \to M \to L \to G[1],$$

we get an exact triangle

$$G/n \to M/n \to L/n \to G/n[1].$$

Here G/n and L/n are defined as cones of multiplication by n, in the same manner as M/n was defined. Taking cohomology sheaves, we get an exact sequence

$$0 \to {}_nG \to T_{\mathbb{Z}/n}(M) \to L/nL \to 0. \tag{5.2.1.1}$$

Therefore $T_{\mathbb{Z}/n}(M)$ is an etale sheaf, and can be fully described by its $\overline{k}$-points together with the action by $\mathrm{Gal}(\overline{k}/k)$. Now set

$$T_\ell M := \varprojlim_n T_{\mathbb{Z}/\ell^n}(M).$$

Because the collection $({}_{\ell^n}G)_n$ satisfies the Mittag-Leffler condition, we get an exact sequence

$$0 \to T_\ell G \to T_\ell M \to \varprojlim_n L/\ell^n L \to 0$$

where (as usual) $T_\ell G$ is the Tate module of the $\overline{k}$-points of G. Finally, we set

$$V_\ell M := T_\ell M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Therefore we have defined functors

$$\hat{T}_p := \prod_{\ell \neq p} T_\ell : \quad \text{1-Mot}_k[p^{-1}] \longrightarrow \prod_{\ell \neq p} \text{Rep}_{\mathbb{Z}_\ell}(\text{Gal}(\overline{k}/k))$$

and

$$V_\ell : \text{1-Mot}_k \otimes \mathbb{Q} \to \text{Rep}_{\mathbb{Q}_\ell}(\text{Gal}(\overline{k}/k)).$$

Proposition 5.2.2. *The functors $\hat{T}_p$ and V_ℓ defined above are exact, faithful, and reflect isomorphisms.*

Proof. We show the statement for V_ℓ; the statement for $\hat{T}_p$ is [ABV03, A.1.1]. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of 1-motives for $\mathbb{Q}$-isogeny. Letting $M = [L \to G]$, $M' = [L' \to G']$, $M'' = [L'' \to G''']$, we have sequences

$$0 \to G' \to G \to G'' \to 0 \quad \text{and}$$
$$0 \to L' \to L \to L'' \to 0$$

which are exact up to isogeny. Therefore, we have exact sequences

$$0 \to V_\ell G' \to V_\ell G \to V_\ell G'' \to 0 \quad \text{and}$$
$$0 \to V_\ell L' \to V_\ell L \to V_\ell L'' \to 0.$$

Using the functoriality of V_ℓ, we get a commuting diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_\ell G' & \longrightarrow & V_\ell M' & \longrightarrow & V_\ell L' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_\ell G & \longrightarrow & V_\ell M & \longrightarrow & V_\ell L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_\ell G'' & \longrightarrow & V_\ell M'' & \longrightarrow & V_\ell L'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

with exact rows, and such that the left and right columns are exact. By standard homological algebra, this implies that the middle column is also exact, proving that V_ℓ is an exact functor.

To show that V_ℓ is faithful, it suffices to show: given a morphism $F = M \to M'$ of 1-motives, if $V_\ell F = 0$ then $F = 0$. Suppose that F is given by a commuting diagram

$$
\begin{array}{ccc}
L & \xrightarrow{f} & L' \\
\downarrow & & \downarrow \\
G & \xrightarrow{g} & G'.
\end{array}
$$

Then we get that $V_\ell f = V_\ell g = 0$. This immediately implies that $f = 0$ since $L(\bar{k})$ and $L'(\bar{k})$ are finitely generated free abelian groups. To show that $g = 0$, note that by exactness of V_ℓ we have

$$V_\ell(\mathrm{Ker}(g)) = \mathrm{Ker}(V_\ell g) = V_\ell G.$$

This implies that $\mathrm{Ker}(g)$ is a closed subvariety of G of equal dimension to G. Therefore $\mathrm{Ker}(g) = G$, i.e., $g = 0$. The fact that V_ℓ reflects isomorphisms follows formally from being exact and faithful. □

5.3 Cartier duals of 1-motives

In [Del74, 10.2] there is defined a notion of Cartier duality for 1-motives. We briefly recall this definition below.

Suppose given a 1-motive $M = [L \to G]$, which we write as a commutative diagram

$$
\begin{array}{c}
L \\
\downarrow f \\
0 \longrightarrow T \xrightarrow{\ g\ } G \xrightarrow{\ h\ } A \longrightarrow 0.
\end{array}
$$

The Cartier dual 1-motive is then defined by a diagram

$$
\begin{array}{c}
T^\vee \\
\downarrow g^\vee \\
0 \longrightarrow L^\vee \xrightarrow{\ f^\vee\ } G^u \xrightarrow{\ h^\vee\ } A^\vee \longrightarrow 0.
\end{array}
$$

where $T^\vee$, $L^\vee$ and $A^\vee$ are the usual duals, while G^u is defined as follows: consider the 1-motive $M/W_{-2}M = [L \to A]$. We have an exact sequence of 1-motives

$$0 \to A \to M/W_{-2}M \to L[1] \to 0.$$

Applying $\mathscr{R}\mathscr{H}om(-, \mathbb{G}_m)$ to this sequence of complexes of $fppf$-sheaves and taking cohomology yields an exact sequence

$$0 \to L^\vee \to \mathscr{E}xt^1(M/W_{-2}M, \mathbb{G}_m) \to A^\vee \to 0.$$

Here we have used the fact that $A^\vee \cong \mathscr{E}xt^1(A, \mathbb{G}_m)$ [MaMe74, Chap. 1]. We define G^u to be the group scheme $\mathscr{E}xt^1(M/W_{-2}M, \mathbb{G}_m)$.

It remains to define the map $g^\vee : T^\vee \to G^u$. By the definition of G^u, this means that for every $x \in T^\vee$, we must give

1. an extension $\tilde{x}$ of A by $\mathbb{G}_m$, and

2. a trivialization of the pullback of $\tilde{x}$ via $h \circ f : L \to A$.

Since $T^\vee = \mathrm{Hom}(T, \mathbb{G}_m)$, we can let $\tilde{x}$ be the pushforward extension $x_*G \in \mathscr{E}xt^1(A, \mathbb{G}_m)$ (where $x \in \mathrm{Hom}(T, \mathbb{G}_m)$), defining part (1) of our desired map $g^\vee : T^\vee \to G^u$. The trivialization of part (2) is determined by the fact that $h \circ f : L \to A$ lifts (trivially) to $f : L \to G$.

5.3.1. The key property of Cartier duals we will use is the following:

Proposition 5.3.2. *Let $M = [L \to G]$ be a 1-motive and $M^\vee$ its Cartier dual. Then for every n prime to the characteristic of k there is a canonical perfect pairing*

$$T_{\mathbb{Z}/n}M \otimes T_{\mathbb{Z}/n}M^\vee \to \mathbb{Z}/n(1)$$

which is functorial in M.

Proof. This is [Del74, 10.2.5]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Chapter 6

Construction of $M^1_{D,E}(\overline{X})$

6.1 Definition of the 1-motive $M^1_{D_\bullet,E_\bullet}(\overline{X}_\bullet)$

6.1.1. Fix a perfect field k of characteristic $p \geq 0$ and an algebraic closure $k \hookrightarrow \overline{k}$. Let $\overline{X}$ be a proper reduced k-scheme with two reduced closed subschemes D and E such that $D \cap E = \emptyset$. Let $X = \overline{X} - E$, $\tilde{U} = \overline{X} - D$ and $U = X - D = \tilde{U} - E$. We assume further that the complement U is dense everywhere in $\overline{X}$, so that D and E everywhere have codimension ≥ 1. We label our various maps as follows:

$$
\begin{array}{ccccc}
U & \stackrel{\tilde{j}}{\hookrightarrow} & X & \stackrel{\tilde{i}}{\hookleftarrow} & D \\
\tilde{u} \downarrow & & u \downarrow & & \| \\
\tilde{U} & \stackrel{j}{\hookrightarrow} & \overline{X} & \stackrel{i}{\hookleftarrow} & D \\
\tilde{v} \uparrow & & v \uparrow & & \\
E & =\!\!=\!\!= & E & &
\end{array}
$$

Here in each row and column, the term in the middle is the union of the outer two terms. We then define

$$ H^i_{D,E}(\overline{X}, \mathscr{F}) := H^i(X_{\overline{k}}, \tilde{j}_! \mathscr{F}) $$

for an étale sheaf $\mathscr{F}$ on U_{et}. In this Chapter we define a 1-motive $M^1_{D,E}(\overline{X})$ such that we have natural isomorphisms for $\ell \neq p$

$$ T_\ell M^1_{D,E}(\overline{X}) \xrightarrow{\sim} H^1_{D,E}(\overline{X}, \mathbb{Z}_\ell(1)), $$

functorial in the triple $(\overline{X}, D, E)$. We also give such a definition in case $\overline{X}$ is a smooth proper Deligne-Mumford stack, as this case will be necessary in Chapters 7 and 8.

6.1.2. First, however, we will work in the simplicial setting, as our basic technique for defining $M^1_{D,E}(\overline{X})$ is to replace $\overline{X}$ (via [dJng96]) by an appropriate proper hypercover by a

simplicial scheme. Therefore let $\overline{X}_\bullet$ a simplicial scheme such that each $\overline{X}_n$ is proper and smooth, and let $D_\bullet$ and $E_\bullet$ be closed subschemes. By a *closed subscheme $C_\bullet$* of a simplicial scheme $\overline{X}_\bullet$ we mean a simplicial scheme $C_\bullet$ and a map of simplicial schemes $i_\bullet : C_\bullet \to \overline{X}_\bullet$, such that each $i_n : C_n \to \overline{X}_n$ is a closed immersion and for each simplicial map p in the category Δ defined by a non-decreasing map $\{0, ..., m\} \to \{0, ..., n\}$, the corresponding diagram

$$
\begin{array}{ccc}
C_n & \xrightarrow{i_n} & X_n \\
p_C \downarrow & & p_X \downarrow \\
C_m & \xrightarrow{i_m} & X_m
\end{array}
$$

is set-theoretically cartesian (but not necessarily scheme-theoretically cartesian). This guarantees that the collection of open complements $V_n := \overline{X}_n - C_n$ forms a simplicial scheme $V_\bullet$.

Returning to our two closed simplicial subschemes $D_\bullet$ and $E_\bullet$ of $\overline{X}_\bullet$, we let $X_\bullet = \overline{X}_\bullet - E_\bullet$, $\tilde{U}_\bullet = \overline{X}_\bullet - D_\bullet$ and $U_\bullet = \overline{X}_\bullet - (D_\bullet \cup E_\bullet)$, and label our maps in the same way as before (with a subscript for simplicial index):

$$
\begin{array}{ccccc}
U_\bullet & \xhookrightarrow{\tilde{j}_\bullet} & X_\bullet & \xhookleftarrow{\tilde{i}_\bullet} & D_\bullet \\
\tilde{u}_\bullet \downarrow & & u_\bullet \downarrow & & \| \\
\tilde{U}_\bullet & \xhookrightarrow{j_\bullet} & \overline{X}_\bullet & \xhookleftarrow{i_\bullet} & D_\bullet \\
\tilde{v}_\bullet \uparrow & & v_\bullet \uparrow & & \\
E_\bullet & = & E_\bullet & &
\end{array}
$$

We assume that $U_\bullet$ is dense in $X_\bullet$; i.e., for each n, U_n is dense in $\overline{X}_n$.

Review of the Relative Picard scheme

6.1.3. We review some facts on the relative Picard group $\mathrm{Pic}(\overline{X}_\bullet, D_\bullet)$ of the pair $(\overline{X}_\bullet, D_\bullet)$. This is defined as the group of isomorphism classes of pairs $(\mathscr{L}^\bullet, \sigma : \mathscr{L}^\bullet|_{D_\bullet} \xrightarrow{\sim} \mathcal{O}_{D_\bullet})$, where $\mathscr{L}^\bullet$ is an invertible sheaf on $\overline{X}_\bullet$ and σ is a trivialization of $\mathscr{L}^\bullet$ restricted to $D_\bullet$. Recall [BVS01, Prop. 4.1] that an invertible sheaf $\mathscr{L}^\bullet$ on $\overline{X}_\bullet$ corresponds to an invertible sheaf $\mathscr{L}^0$ on $\overline{X}_0$, together with an isomorphism $\alpha : \pi_1^* \mathscr{L}^0 \xrightarrow{\sim} \pi_2^* \mathscr{L}^0$ where $\pi_1, \pi_2 : \overline{X}_1 \to \overline{X}_0$ are the simplicial face maps, such that α satisfies a cocycle condition relative to the three pullbacks to $\overline{X}_2$.

Proposition 6.1.4. *Define the fppf simplicial sheaf* $\mathbb{G}_{m,\overline{X}_\bullet,D_\bullet} := \mathrm{Ker}(\mathbb{G}_{m,\overline{X}_\bullet} \to \mathbb{G}_{m,D_\bullet})$. *Then there is a natural isomorphism* $\mathrm{Pic}(\overline{X}_\bullet, D_\bullet) \xrightarrow{\sim} H^1(\overline{X}_\bullet, \mathbb{G}_{m,\overline{X}_\bullet,D_\bullet})$.

Proof. Note that the case of an ordinary scheme $\overline{X}$ is [SV96, 2.1]. We handle the simplicial case via a similar method.

Consider a pair $(\mathscr{L}^\bullet, \sigma) \in \mathrm{Pic}(\overline{X}_\bullet, D_\bullet)$. From this pair we can define a $\mathbb{G}_{m,\overline{X}_\bullet,D_\bullet}$-torsor by the rule sending $(U \to \overline{X}_n) \in (Sch/X_\bullet)_{fppf}$ to the group of isomorphisms $\alpha : \mathscr{L}^n|_U \cong \mathcal{O}_U$ inducing a commutative diagram

$$
\begin{array}{ccc}
\mathscr{L}^n|_{D_n} & \xrightarrow{\ \sigma\ } & \mathcal{O}_{D_n} \\
{\scriptstyle \alpha} \downarrow & & \downarrow {\scriptstyle id} \\
\mathcal{O}_{D_n} & \xrightarrow{\ id\ } & \mathcal{O}_{D_n}.
\end{array}
$$

(see [Con01, Ch. 6] for a desctiption of the site $(Sch/\overline{X}_\bullet)_{fppf}$ along with a proof that the topos of sheaves on this site is equivalent to the category of simplicial sheaves $\mathscr{F}^\bullet$ on $\overline{X}_\bullet$.)

By general nonsense [StProj, Tag 03AG], $H^1(\overline{X}_\bullet, \mathbb{G}_{m,\overline{X}_\bullet,D_\bullet})$ is in bijection with isomorphism classes of $\mathbb{G}_{m,\overline{X}_\bullet,D_\bullet}$-torsors. Therefore we have defined a map $f : \mathrm{Pic}(\overline{X}_\bullet, D_\bullet) \to H^1(\overline{X}_\bullet, \mathbb{G}_{m,\overline{X}_\bullet,D_\bullet})$. It is clear that this map fits into a commutative diagram

$$
\begin{array}{ccccccccc}
\Gamma(\overline{X}_\bullet, \mathbb{G}_m) & \longrightarrow & \Gamma(D_\bullet, \mathbb{G}_m) & \longrightarrow & \mathrm{Pic}(\overline{X}_\bullet, D_\bullet) & \longrightarrow & \mathrm{Pic}(\overline{X}_\bullet) & \longrightarrow & \mathrm{Pic}(D_\bullet) \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle \sim}\downarrow & & {\scriptstyle \sim}\downarrow \\
H^0(\overline{X}_\bullet, \mathbb{G}_m) & \longrightarrow & H^0(D_\bullet, \mathbb{G}_m) & \longrightarrow & H^1(\overline{X}_\bullet, \mathbb{G}_{m,\overline{X}_\bullet,D_\bullet}) & \longrightarrow & H^1(\overline{X}_\bullet, \mathbb{G}_m) & \longrightarrow & H^1(D_\bullet, \mathbb{G}_m).
\end{array}
$$

The two right-hand vertical maps are isomorphisms by [StProj, Tag 040D]. Therefore by the five lemma, f is an isomorphism as well. $\qquad\square$

We can also define an associated sheaf $\mathbf{Pic}_{\overline{X}_\bullet, D_\bullet}$, defined as the fppf-sheafification of the functor on $(Sch/k)_{fppf}$,

$$
Y \mapsto \mathrm{Pic}(\overline{X}_\bullet \times Y, D_\bullet \times Y).
$$

Equivalently, let $\mathbb{G}_{m,\overline{X}_\bullet,D_\bullet} = \ker(\mathbb{G}_{m,\overline{X}_\bullet} \to (i_\bullet)_* \mathbb{G}_{m,D_\bullet})$, and let $\overline{p}_\bullet : \overline{X} \to \mathrm{Spec}\, k$ and $p_{D,\bullet} : D \to \mathrm{Spec}\, k$ be the structure maps. Then we have

$$
\mathbf{Pic}_{\overline{X}_\bullet, D_\bullet} = R^1(\overline{p}_\bullet)_* \mathbb{G}_{m,\overline{X}_\bullet,D_\bullet}.
$$

Proposition 6.1.5. *The sheaf $\mathbf{Pic}_{\overline{X}_\bullet, D_\bullet}$ defined above is representable. Moreover, let $\mathbf{Pic}^0_{\overline{X}_\bullet, D_\bullet}$ denote the connected component of the identity of this group scheme. Then the reduction $\mathbf{Pic}^{0,red}_{\overline{X}_\bullet, D_\bullet}$ is a semiabelian variety, and we have an exact sequence*

$$
0 \to \mathbf{Pic}^{0,red}_{\overline{X}_\bullet, D_\bullet} \to \mathbf{Pic}^{red}_{\overline{X}_\bullet, D_\bullet} \to \mathbf{NS}_{\overline{X}_\bullet, D_\bullet} \to 0,
$$

where $\mathbf{NS}_{\overline{X}_\bullet, D_\bullet}$ is an étale group scheme over k whose $\overline{k}$-points form a finitely generated abelian group.

Proof. We start with the exact sequence of fppf sheaves on Sch/k

$$
R^0(\overline{p}_\bullet)_* \mathbb{G}_{m,\overline{X}_\bullet} \xrightarrow{\ a\ } R^0(p_{D,\bullet})_* \mathbb{G}_{m,D_\bullet} \to \mathbf{Pic}_{\overline{X}_\bullet, D_\bullet} \to \mathbf{Pic}_{\overline{X}_\bullet} \xrightarrow{\ b\ } \mathbf{Pic}_{D_\bullet}.
$$

The two sheaves on the right are representable by [BVS01, Appendix A.2], and moreover (by the same reference) the reduction $\mathbf{Pic}_{\overline{X}_\bullet}^{0,red}$ of the connected component of the identity is a semiabelian variety. It is clear that the the two left-hand sheaves are representable by tori. This gives us a short exact sequence

$$0 \to \operatorname{coker}(a) \to \mathbf{Pic}_{\overline{X}_\bullet, D_\bullet} \to \ker(b) \to 0$$

where $\operatorname{coker}(a)$ is a torus and $\ker(b)$ is (after taking the reduced part) an extension of a finitely generated étale group scheme by a semiabelian variety. This implies the proposition statement. □

6.1.6. Now consider the closed subscheme $E_\bullet$. Let

$$\operatorname{Div}_{E_\bullet}(\overline{X}_\bullet) := \ker(p_1^* - p_2^* : \operatorname{Div}_{E_0}(\overline{X}_0) \to \operatorname{Div}_{E_1}(\overline{X}_1)),$$

where $p_1, p_2 : \overline{X}_1 \to \overline{X}_0$ are the simplicial projections. Let $\operatorname{Div}_{E_i}(\overline{X}_i)$ be the group of Weil divisors on $\overline{X}_i$ supported on E_i (for any i). Because $D_n \cap E_n = \emptyset$ in each $\overline{X}_n$, there is a well-defined cycle class map

$$cl : \operatorname{Div}_{E_\bullet}(\overline{X}_\bullet) \longrightarrow \operatorname{Pic}(\overline{X}_\bullet, D_\bullet).$$

Concretely, given a divisor A supported on E_0, we have the associated line bundle $\mathcal{O}(A)$ on X_0 and meromorphic section $s : \mathcal{O}_{X_0} \to \mathcal{O}(A)$. Since $D_0 \cap E_0 = \emptyset$, this induces an isomorphism $\mathcal{O}_{D_0} \xrightarrow{\sim} \mathcal{O}(A)|_{D_0}$. Moreover, since $p_1^* A = p_2^* A$ as divisors, the meromorphic sections of $\mathcal{O}(p_1^* A)$ and $\mathcal{O}(p_2^* A)$ yield a canonical isomorphism $\rho : p_1^* \mathcal{O}(A) \xrightarrow{\sim} p_2^* \mathcal{O}(A)$ verifying a cocycle condition, so $\mathcal{O}(A)$ defines a line bundle on $X_\bullet$ with a trivialization on $D_\bullet$.

Let $\mathbf{Div}_{E_\bullet}(\overline{X}_\bullet)$ be the étale group scheme naturally associated with $\operatorname{Div}_{E_\bullet}(\overline{X}_\bullet)$; i.e., the $\overline{k}$-points are given by $\operatorname{Div}_{E_{\overline{k},\bullet}}(\overline{X}_{\overline{k},\bullet})$ and the Galois action is the natural one. Alternatively, $\mathbf{Div}_{E_\bullet}(\overline{X}_\bullet)$ can be defined as the sheafification of the functor

$$Y \mapsto \operatorname{Div}_{E_\bullet \times Y}(\overline{X}_\bullet \times Y)$$

for any étale k-scheme Y. Then the cycle class map cl above naturally extends to a map of group schemes

$$cl : \mathbf{Div}_{E_\bullet}(\overline{X}_\bullet) \to \mathbf{Pic}_{\overline{X}_\bullet, D_\bullet}^{red}.$$

(This simply means that the cycle class map defined above is Galois-equivariant). With these preliminaries in hand, we can make the following definition:

Definition 6.1.7. Let $(\overline{X}_\bullet, D_\bullet, E_\bullet)$ be as above. Consider the composition

$$\overline{cl} : \mathbf{Div}_{E_\bullet}(\overline{X}_\bullet) \xrightarrow{cl} \mathbf{Pic}_{\overline{X}_\bullet, D_\bullet}^{red} \to \mathbf{NS}_{\overline{X}_\bullet, D_\bullet}$$

sending any divisor to its class in the Neron-Severi group. Then let

$$\mathbf{Div}_{E_\bullet}^0(\overline{X}_\bullet) := \operatorname{Ker}(\overline{cl}).$$

The cycle class map induces a natural map $cl^0 : \mathbf{Div}_{E_\bullet}^0(\overline{X}_\bullet) \to \mathbf{Pic}_{\overline{X}_\bullet, D_\bullet}^{0,red}$. We then define

$$M_{D_\bullet, E_\bullet}^1(\overline{X}_\bullet) := [\mathbf{Div}_{E_\bullet}^0(\overline{X}_\bullet) \xrightarrow{cl^0} \mathbf{Pic}_{\overline{X}_\bullet, D_\bullet}^{0,red}].$$

6.2 ℓ-adic realization

6.2.1. Consider a triple $(\overline{X}_\bullet, D_\bullet, E_\bullet)$ and define $U_\bullet$, $X_\bullet$, etc. as in 6.1.2. For an étale sheaf $\mathscr{F}^\bullet$ on $U_\bullet$ we define

$$H_{D_\bullet, E_\bullet}^i(\overline{X}_\bullet, \mathscr{F}^\bullet) := H^i(\overline{X}_\bullet, \tilde{j}_{\bullet,!}\mathscr{F}^\bullet).$$

Here the extension-by-zero functor $\tilde{j}_{\bullet,!}$ is defined by the same rule as for schemes; i.e., $(\tilde{j}_{\bullet,!}\mathscr{F}^\bullet)^n = \tilde{j}_{n,!}\mathscr{F}^n$. Our goal in this section is to show that there is a natural Galois-equivariant isomorphism (for $\ell \neq p$)

$$T_\ell M_{D_\bullet, E_\bullet}^1(\overline{X}_\bullet) \xrightarrow{\sim} H_{D_\bullet, E_\bullet}^1(\overline{X}_\bullet, \mathbb{Z}_\ell(1)).$$

6.2.2. First off, recall that we have a commutative diagram of inclusions

$$
\begin{array}{ccc}
U_\bullet & \overset{\tilde{j}_\bullet}{\hookrightarrow} & X_\bullet \\
\tilde{u}_\bullet \downarrow & & \downarrow u_\bullet \\
\tilde{U}_\bullet & \overset{j_\bullet}{\hookrightarrow} & \overline{X}_\bullet.
\end{array}
$$

Because $D_\bullet$ and $E_\bullet$ are disjoint, we have

$$Ru_{\bullet,*}\tilde{j}_{\bullet,!} \cong j_{\bullet,!}R\tilde{u}_{\bullet,*}$$

as functors on $D^b(U_\bullet)$. Let $\tilde{v}_\bullet : E_\bullet \hookrightarrow \tilde{U}_\bullet$ be the inclusion, and consider the exact triangle in $D_c^b(\tilde{U}_\bullet)$

$$\tilde{v}_{\bullet*}R\tilde{v}_\bullet^!\mu_{n,\tilde{U}_\bullet} \to \mu_{n,\tilde{U}_\bullet} \to R\tilde{u}_{\bullet*}\mu_{n,U_\bullet} \to \tilde{v}_{\bullet*}R\tilde{v}_\bullet^!\mu_{n,\tilde{U}_\bullet}[1]$$

(where n is prime to the characteristic p). Applying $j_{\bullet,!}$ to this triangle gives a triangle (note that $v_{\bullet,*} = j_{\bullet,!} \circ \tilde{v}_{\bullet,*}$)

$$v_{\bullet,*}Rv_\bullet^!\mu_{n,\overline{X}_\bullet} \to j_{\bullet,!}\mu_{n,\tilde{U}_\bullet} \to j_{\bullet,!}R\tilde{u}_{\bullet*}\mu_{n,U_\bullet} \to v_{\bullet,*}Rv_\bullet^!\mu_{n,\overline{X}_\bullet}[1]. \qquad (6.2.2.1)$$

Taking cohomology of this triangle, and using the fact that $Ru_{\bullet*}\tilde{j}_{\bullet,!} \cong j_{\bullet,!}R\tilde{u}_{\bullet*}$, we get an exact sequence

$$0 \to H_c^1(\tilde{U}_\bullet, \mu_n) \to H_{D_\bullet, E_\bullet}^1(\overline{X}_\bullet, \mu_n) \to H_{E_\bullet}^2(\overline{X}_\bullet, \mu_n) \to H_c^2(\tilde{U}_\bullet, \mu_n). \qquad (6.2.2.2)$$

The following propositions give motivic interpretations of the groups appearing in this sequence:

Proposition 6.2.3. *We have a natural isomorphism of groups*

$$H_c^1(\tilde{U}_\bullet, \mu_n) \longleftrightarrow \mathbf{Pic}^0(\overline{X}_\bullet, D_\bullet)[n].$$

Proof. Let $\mathbb{G}_{m,\overline{X}_\bullet,D_\bullet} := \mathrm{Ker}(\mathbb{G}_{m,\overline{X}_\bullet} \to i_{\bullet,*}\mathbb{G}_{m,D_\bullet})$. Then we have a commutative diagram of sheaves on $\overline{X}_\bullet$, where the rows and columns are exact:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & j_{\bullet,!}\mu_{n,\tilde{U}_\bullet} & \longrightarrow & \mathbb{G}_{m,\overline{X}_\bullet,D_\bullet} & \xrightarrow{\cdot n} & \mathbb{G}_{m,\overline{X}_\bullet,D_\bullet} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mu_{n,\overline{X}_\bullet} & \longrightarrow & \mathbb{G}_{m,\overline{X}_\bullet} & \xrightarrow{\cdot n} & \mathbb{G}_{m,\overline{X}_\bullet} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & i_{\bullet,*}\mu_{n,D_\bullet} & \longrightarrow & i_{\bullet,*}\mathbb{G}_{m,D_\bullet} & \xrightarrow{\cdot n} & i_{\bullet,*}\mathbb{G}_{m,D_\bullet} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

Taking cohomology along the top row, we have an exact sequence

$$H^0_c(\tilde{U}_\bullet, \mathbb{G}_m) \xrightarrow{\cdot n} H^0_c(\tilde{U}_\bullet, \mathbb{G}_m) \longrightarrow H^1_c(\tilde{U}_\bullet, \mu_n) \longrightarrow \mathrm{Pic}(\overline{X}_\bullet, D_\bullet) \xrightarrow{\cdot n} \mathrm{Pic}(\overline{X}_\bullet, D_\bullet). \quad (6.2.3.1)$$

But $H^0_c(\tilde{U}_\bullet, \mathbb{G}_m) = \mathrm{Ker}(H^0(X_\bullet, \mathbb{G}_m) \to H^0(D_\bullet, \mathbb{G}_m))$ is a torus; since the $\overline{k}$-points of a torus are divisible, we have

$$H^1_c(\tilde{U}_\bullet, \mu_n) = \mathrm{Pic}(\overline{X}_\bullet, D_\bullet)[n] = \mathbf{Pic}^0(\overline{X}_\bullet, D_\bullet)[n]$$

as desired. $\qquad\square$

Notice that if we extend the long exact sequence 6.6.8.1 a little, we have an injection

$$\mathrm{Pic}(\overline{X}_\bullet, D_\bullet)/n\mathrm{Pic}(\overline{X}_\bullet, D_\bullet) \hookrightarrow H^2_c(\tilde{U}_\bullet, \mu_n). \quad (6.2.3.2)$$

Because $\mathrm{Pic}^0(\overline{X}_\bullet, D_\bullet)$ is divisible, we have

$$\mathrm{Pic}(\overline{X}_\bullet, D_\bullet)/n\mathrm{Pic}(\overline{X}_\bullet, D_\bullet) \cong NS(\overline{X}_\bullet, D_\bullet)/nNS(\overline{X}_\bullet, D_\bullet) = NS(\overline{X}_\bullet, D_\bullet) \otimes \mathbb{Z}/(n).$$

Proposition 6.2.4. *We have a canonical isomorphism*

$$H^2_{E_\bullet}(\overline{X}_\bullet, \mu_n) \cong \mathrm{Div}_{E_\bullet}(\overline{X}_\bullet) \otimes \mathbb{Z}/(n),$$

and the map $H^2_{E_\bullet}(\overline{X}_\bullet, \mu_n) \to H^2_c(\tilde{U}_\bullet, \mu_n)$ *obtained from sequence 6.6.7.2 factors as*

$$\mathrm{Div}_{E_\bullet}(\overline{X}_\bullet) \otimes \mathbb{Z}/(n) \to NS(\overline{X}_\bullet, D_\bullet) \otimes \mathbb{Z}/(n) \hookrightarrow H^2_c(\tilde{U}_\bullet, \mu_n),$$

where the map $\mathrm{Div}_{E_\bullet}(\overline{X}_\bullet) \to NS(\overline{X}_\bullet, D_\bullet)$ *is the cycle class map of 6.1.6, and the injection* $NS(\overline{X}_\bullet, D_\bullet) \otimes \mathbb{Z}/(n) \hookrightarrow H^2_c(\tilde{U}_\bullet, \mu_n)$ *is induced by 6.2.3.2.*

Proof. The statement that $H_E^2(\overline{X}, \mu_n) = \mathrm{Div}_E(\overline{X}) \otimes \mathbb{Z}/(n)$ for a closed subscheme E (of codimension ≥ 1) of a proper smooth scheme X over k is well known [SGA4h, Cycle 2.1.4]. The case of simplicial schemes follows by considering the spectral sequence $H_{E_p}^q(\overline{X}_p, \mu_n) \Rightarrow H_{E_\bullet}^{p+q}(\overline{X}_\bullet, \mu_n)$. The claim regarding the map $H_{E_\bullet}^2(\overline{X}_\bullet, \mu_n) \to H_c^2(\tilde{U}_\bullet, \mu_n)$ is a simplicial variant of 4.2.2 and 4.2.3: first consider the canonical map in $D_c^b(\overline{X}_\bullet)$

$$\alpha : v_{\bullet,*} v_\bullet^! \mathbb{G}_{m, \overline{X}_\bullet} \longrightarrow \mathbb{G}_{m, \overline{X}_\bullet}.$$

Because $D_\bullet$ and $E_\bullet$ are disjoint, the composition

$$v_{\bullet,*} v_\bullet^! \mathbb{G}_{m, \overline{X}_\bullet} \xrightarrow{\alpha} \mathbb{G}_{m, \overline{X}_\bullet} \longrightarrow i_{\bullet,*} \mathbb{G}_{m, D_\bullet}$$

is the zero map, implying that α factors through a unique map

$$\tilde{\alpha} : v_{\bullet,*} v_\bullet^! \mathbb{G}_{m, \overline{X}_\bullet} \longrightarrow \mathbb{G}_{m, \overline{X}_\bullet, D_\bullet}.$$

Taking global sections induces a map

$$H_{E_\bullet}^1(\overline{X}_\bullet, \mathbb{G}_m) \to \mathrm{Pic}(\overline{X}_\bullet, D_\bullet).$$

The group on the left is canonically isomorphic to $\mathrm{Div}_{E_\bullet}(X_\bullet)$, and we leave it to the reader to check that this map sends a divisor $W_\bullet$ to $(\mathcal{O}(W_\bullet), s)$ where s is the canonical meromorphic section of $\mathcal{O}(W_\bullet)$ (the verification of this fact is the same as in the proof of 4.2.2). Using the Kummer exact sequence, we conclude that the map $H_{E_\bullet}^2(\overline{X}_\bullet, \mu_n) \to H_c^2(\tilde{U}_\bullet, \mu_n)$ can be described as claimed in the proposition statement. $\qquad \square$

6.2.5. Summarizing the last two propositions, we see that we have a diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}^0(\overline{X}_\bullet, D_\bullet)[n] & \longrightarrow & T_{\mathbb{Z}/n}(M_{D_\bullet, E_\bullet}^1(\overline{X}_\bullet)) & \longrightarrow & \mathrm{Div}_{E_\bullet}^0(\overline{X}_\bullet) \otimes \mathbb{Z}/n & \longrightarrow & 0 \\
& & \simeq\downarrow & & & & \simeq\downarrow & & \\
0 & \longrightarrow & H_c^1(\tilde{U}_\bullet, \mu_n) & \longrightarrow & H_{D_\bullet, E_\bullet}^1(\overline{X}_\bullet, \mu_n) & \longrightarrow & \mathrm{Ker}(H_{E_\bullet}^2(\overline{X}_\bullet, \mu_n) \to H_c^2(\tilde{U}_\bullet, \mu_n)) & \longrightarrow & 0
\end{array}
$$

where the left-hand and right-hand maps are isomorphisms (the upper row is the exact sequence 5.2.1.1). Therefore we need only define a map

$$f : T_{\mathbb{Z}/n}(M_{D_\bullet, E_\bullet}^1(\overline{X}_\bullet)) \longrightarrow H_{D_\bullet, E_\bullet}^1(\overline{X}_\bullet, \mu_n)$$

fitting into the middle of the diagram, and by the five lemma it will be an isomorphism. By definition, $T_{\mathbb{Z}/n}(M_{D_\bullet, E_\bullet}^1(\overline{X}_\bullet))$ consists of data $(C_\bullet, \mathscr{L}^\bullet, \varphi)$, where

1. $C_\bullet \in \mathrm{Div}_{E_\bullet}^0(\overline{X}_\bullet)$,

2. $\mathscr{L}^\bullet$ is a line bundle on $\overline{X}_\bullet$, and

3. $\varphi : \mathcal{O}_{D_\bullet} \xrightarrow{\sim} \mathscr{L}^\bullet|_{D_\bullet}$ is an isomorphism. We also require that

4. There exists at least one isomorphism $\eta : (\mathscr{L}^\bullet)^{\otimes n} \xrightarrow{\sim} \mathcal{O}(-C_\bullet)$ identifying $\varphi^{\otimes n}$ with the canonical meromorphic section of $\mathcal{O}(-C_\bullet)$ restricted to $D_\bullet$ (which is an isomorphism since $C_\bullet$ is disjoint from $D_\bullet$).

We then mod out by elements of the form $(-nC_\bullet, \mathcal{O}(C_\bullet), s)$ where $s : \mathcal{O}_{\overline{X}_\bullet} \to \mathcal{O}(C_\bullet)$ is the canonical meromorphic section.

6.2.6. By the general machinery of sites [StProj, tag 03AJ] the group $H^1_{D_\bullet, E_\bullet}(\overline{X}_\bullet, \mu_n) = H^1(X_\bullet, \tilde{j}_{\bullet,!}\mu_n)$ is in bijection with isomorphism classes of $\tilde{j}_{\bullet,!}\mu_n$-torsors on $X_\bullet$. Our map $f : T_{\mathbb{Z}/n} M^1_{D_\bullet, E_\bullet}(\overline{X}_\bullet) \to H^1_{D_\bullet, E_\bullet}(\overline{X}_\bullet, \mu_n)$ is defined as follows: given an object $(C_\bullet, \mathscr{L}^\bullet, \varphi) \in T_{\mathbb{Z}/n} M^1_{D_\bullet, E_\bullet}(\overline{X}_\bullet)$, choose an isomorphism η as in bullet point (4) above. Since $C_\bullet$ is disjoint from $X_\bullet = \overline{X}_\bullet - E_\bullet$, η defines a trivialization of $(\mathscr{L}^\bullet|_{X_\bullet})^{\otimes n}$ carrying $\varphi^{\otimes n}$ to the identity morphism of $\mathcal{O}_{D_\bullet}$. We then set $f(C_\bullet, \mathscr{L}^\bullet, \varphi)$ to be the $\tilde{j}_{\bullet,!}\mu_n$-torsor of local isomorphisms $\mathcal{O}_{X_\bullet} \xrightarrow{\sim} \mathscr{L}^\bullet|_{X_\bullet}$ compatible with η on nth tensor products and reducing to φ on $D_\bullet$.

6.2.7. We still must show that this map is well-defined; i.e., independent of the choice of η. Suppose we chose a different isomorphism $\eta' : (\mathscr{L}^\bullet)^{\otimes n} \xrightarrow{\sim} \mathcal{O}(-C_\bullet)$. Then η and η' differ by an element

$$\alpha \in H^0(\overline{X}_\bullet, \tilde{j}_{\bullet,!}\mathbb{G}_m) = \text{Ker}(H^0(\overline{X}_\bullet, \mathbb{G}_m) \to H^0(D_\bullet, \mathbb{G}_m)).$$

This group is a torus, which implies that we can choose an nth root $\sqrt[n]{\alpha}$ (since n is prime to char k). Then if $\psi : \mathcal{O}_{X_\bullet} \xrightarrow{\sim} \mathscr{L}^\bullet|_{X_\bullet}$ is an isomorphism compatible with η, then $\sqrt[n]{\alpha}\psi$ is an isomorphism which is compatible with η'. Therefore multiplication by $\sqrt[n]{\alpha}$ defines an isomorphism between the $j_{\bullet,!}\mu_n$-torsors defined by choosing η or η'. Moreover, it is clear that elements of the form $(-nC_\bullet, \mathcal{O}(C_\bullet), can)$ map to the trivial torsor, which shows that $f : T_{\mathbb{Z}/n}(M^1_{D_\bullet, E_\bullet}(\overline{X}_\bullet)) \to H^1_{D_\bullet, E_\bullet}(\overline{X}_\bullet, \mu_n)$ is well-defined.

6.2.8. It remains to show that f fits into the diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \text{Pic}^0(\overline{X}_\bullet, D_\bullet)[n] & \longrightarrow & T_{\mathbb{Z}/n}(M^1_{D_\bullet, E_\bullet}(\overline{X}_\bullet)) & \longrightarrow & \text{Div}^0_{E_\bullet}(\overline{X}_\bullet) \otimes \mathbb{Z}/n & \longrightarrow & 0 \\
& & \downarrow & & f\downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1_c(\tilde{U}_\bullet, \mu_n) & \longrightarrow & H^1_{D_\bullet, E_\bullet}(\overline{X}_\bullet, \mu_n) & \longrightarrow & \text{Ker}(H^2_{E_\bullet}(\overline{X}_\bullet, \mu_n) \to H^2_c(\tilde{U}_\bullet, \mu_n)) & \longrightarrow & 0.
\end{array}
$$

To check that we have such a diagram, we must show that $\text{Pic}^0(\overline{X}_\bullet, D_\bullet)[n]$ is precisely the inverse image of $H^1_c(\tilde{U}_\bullet, \mu_n)$ under the mapping f. As a subgroup of $T_{\mathbb{Z}/n}(M^1_{D_\bullet, E_\bullet}(\overline{X}_\bullet))$, $\text{Pic}^0(\overline{X}_\bullet, D_\bullet)[n]$ consists of the elements of the form $(0, \mathscr{L}^\bullet, \varphi)$, i.e., the divisor is empty. This means that $(\mathscr{L}^{\otimes n}, \varphi^{\otimes n})$ is isomorphic to $(\mathcal{O}_{\overline{X}_\bullet}, 1)$ (rather than simply isomorphic when restricted to $X_\bullet$). This is precisely the condition for an element to factor through the subgroup $H^1_c(\tilde{U}_\bullet, \mu_n)$ of $H^1_{D_\bullet, E_\bullet}(\overline{X}_\bullet, \mu_n)$. Therefore we have the diagram above, and by Propositions 6.6.8 and 6.6.9, the left-hand and right-hand vertical arrows are isomorphisms. This implies that f is an isomorphism, so we have proved the following:

Proposition 6.2.9. *For any triple $(\overline{X}_\bullet, D_\bullet, E_\bullet)$ as above, there is a natural isomorphism*

$$T_\ell M^1_{D_\bullet, E_\bullet}(\overline{X}_\bullet) \xrightarrow{\sim} H^1_{D_\bullet, E_\bullet}(\overline{X}_\bullet, \mathbb{Z}_\ell(1)).$$

6.3 Functoriality

6.3.1. Let $(\overline{X}_\bullet, D_\bullet, E_\bullet)$ and $(\overline{Y}_\bullet, A_\bullet, B_\bullet)$ be triples consisting of a proper simplicial scheme $\overline{X}_\bullet$ (resp. $\overline{Y}_\bullet$), and two disjoint reduced closed subschemes $D_\bullet$ and $E_\bullet$ (resp. $A_\bullet$ and $B_\bullet$). We let $X_\bullet = \overline{X}_\bullet - E_\bullet$, $U_\bullet = \overline{X}_\bullet - (D_\bullet \cup E_\bullet)$, and $\tilde{U}_\bullet = \overline{X}_\bullet - D_\bullet$. We similarly let $Y_\bullet = \overline{Y}_\bullet - B_\bullet$, $V_\bullet = \overline{Y}_\bullet - (A_\bullet \cup B_\bullet)$, and $\tilde{V}_\bullet = \overline{Y}_\bullet - A_\bullet$. We label the various maps between these spaces as follows:

$$
\begin{array}{ccccc}
U_\bullet & \overset{\tilde{j}_\bullet}{\hookrightarrow} & X_\bullet & \overset{\tilde{i}_\bullet}{\hookleftarrow} & D_\bullet \\
{\scriptstyle\tilde{u}_\bullet}\downarrow & & {\scriptstyle u_\bullet}\downarrow & & \| \\
\tilde{U}_\bullet & \overset{j_\bullet}{\hookrightarrow} & \overline{X}_\bullet & \overset{i_\bullet}{\hookleftarrow} & D_\bullet \\
{\scriptstyle\tilde{v}_\bullet}\uparrow & & {\scriptstyle v_\bullet}\uparrow & & \\
E_\bullet & = & E_\bullet & &
\end{array}
$$

and

$$
\begin{array}{ccccc}
V_\bullet & \overset{\tilde{b}_\bullet}{\hookrightarrow} & Y_\bullet & \overset{\tilde{a}_\bullet}{\hookleftarrow} & A_\bullet \\
{\scriptstyle\tilde{r}_\bullet}\downarrow & & {\scriptstyle r_\bullet}\downarrow & & \| \\
\tilde{V}_\bullet & \overset{b_\bullet}{\hookrightarrow} & \overline{Y}_\bullet & \overset{a_\bullet}{\hookleftarrow} & A_\bullet \\
{\scriptstyle\tilde{s}_\bullet}\uparrow & & {\scriptstyle s_\bullet}\uparrow & & \\
B_\bullet & = & B_\bullet & &
\end{array}
$$

We will describe the contravariant functoriality of the 1-motives $M^1_{D_\bullet, E_\bullet}(\overline{X}_\bullet)$ and $M^1_{A_\bullet, B_\bullet}(\overline{Y}_\bullet)$. For this, consider a map $f : \overline{X}_\bullet \to \overline{Y}_\bullet$ such that

1. $f^{-1}(B_\bullet) \subseteq E_\bullet$, and

2. $f^{-1}(A_\bullet) = D_\bullet$.

Notice that (2) implies that f restricts to a proper map $\tilde{U}_\bullet \to \tilde{V}_\bullet$. In this situation it will turn out that there is a well-defined map $M^1_{A_\bullet, B_\bullet}(\overline{Y}_\bullet) \to M^1_{D_\bullet, E_\bullet}(\overline{X}_\bullet)$.

6.3.2. We get a natural map

$$
\hat{f}^* : \mathbf{Pic}^{0,red}_{\overline{Y}_\bullet, A_\bullet} \to \mathbf{Pic}^{0,red}_{\overline{X}_\bullet, D_\bullet}
$$

by pulling back a line bundle $\mathscr{L}^\bullet$ to $\overline{X}_\bullet$; the trivialization of $\mathscr{L}^\bullet$ on $A_\bullet$ pulls back to a trivialization on $D_\bullet$ because $f^{-1}(A_\bullet) = D_\bullet$. Similarly, because $f^{-1}(B_\bullet) \subseteq E_\bullet$, there is an induced pullback map on divisors

$$
\mathrm{Div}^0_{B_\bullet}(\overline{Y}_\bullet) \to \mathrm{Div}^0_{E_\bullet}(\overline{X}_\bullet).
$$

Putting these maps together, we get a map of 1-motives $\hat{f}^* : M^1_{A_\bullet, B_\bullet}(\overline{Y}_\bullet) \to M^1_{D_\bullet, E_\bullet}(\overline{X}_\bullet)$.

6.3.3. In the situation of 6.3.1, because $f^{-1}(A_\bullet) = D_\bullet$ there is a morphism of pairs $(X_\bullet, D_\bullet) \to (Y_\bullet, A_\bullet)$. Therefore there is an induced morphism of cohomology groups

$$H_{A_\bullet, B_\bullet}^1(\overline{Y}_\bullet, \mathbb{Z}_\ell(1)) = H^1(Y_\bullet, b_{\bullet !}\mathbb{Z}_\ell(1)) \to H^1(X_\bullet, j_{\bullet !}\mathbb{Z}_\ell(1)) = H_{D_\bullet, E_\bullet}^1(\overline{X}_\bullet, \mathbb{Z}_\ell(1)).$$

Using the Kummer exact sequence, one sees that this morphism is induced by pullback of line bundles. Therefore we have the following:

Proposition 6.3.4. *In the notation above, we have a commutative diagram (for $\ell \neq p$)*

$$
\begin{array}{ccc}
T_\ell M_{D_\bullet, E_\bullet}^1(\overline{X}_\bullet) & \xrightarrow{\alpha_{\overline{X}_\bullet}} & H_{D_\bullet, E_\bullet}^1(\overline{X}_\bullet, \mathbb{Z}_\ell(1)) \\
{\scriptstyle T_\ell \hat{f}^*} \downarrow & & \downarrow {\scriptstyle f^*} \\
T_\ell M_{A_\bullet, B_\bullet}^1(\overline{Y}_\bullet) & \xrightarrow{\alpha_{\overline{Y}_\bullet}} & H_{A_\bullet, B_\bullet}^1(\overline{X}_\bullet, \mathbb{Z}_\ell(1)),
\end{array}
$$

where $\alpha_{\overline{X}_\bullet}$ and $\alpha_{\overline{Y}_\bullet}$ are the comparison isomorphisms of Proposition 6.2.9.

6.4 The 1-motive $M_{D,E}^1(\overline{X})$

6.4.1. Now we return to the setting of a triple $(\overline{X}, D, E)$, where $\overline{X}$ is a separated finite type k-scheme and D,E are disjoint closed subschemes of $\overline{X}$, such that the complement $U := \overline{X} - (D \cup E)$ is everywhere dense in $\overline{X}$. Define schemes $\tilde{U}, X$ and maps as in Paragraph 6.1.1. We wish to define the 1-motive $M_{D,E}^1(\overline{X})$. To do so, choose via [dJng96] a proper hypercover $\overline{\pi}_\bullet : \overline{X}_\bullet \to \overline{X}$ such that each $\overline{X}_n$ is proper smooth, and let $D_\bullet' := \overline{\pi}_\bullet^{-1}(D)$ and $E_\bullet' := \overline{\pi}_\bullet^{-1}(E)$. Let $U_\bullet = \overline{X}_\bullet - (D_\bullet' \cup E_\bullet')$; note that $U_\bullet$ is a proper hypercover of U.

Note that $U_\bullet$ is not dense in each component of $\overline{X}_\bullet$ since there are components of $\overline{X}_\bullet$ which are equal to components of $D_\bullet' \cup E_\bullet'$. Therefore let $\overline{U}_\bullet$ be the closure of $U_\bullet$ in $\overline{X}_\bullet$, and define $D_\bullet := D_\bullet' \cap \overline{U}_\bullet$ and $E_\bullet := E_\bullet' \cap \overline{U}_\bullet$. Then $(\overline{U}_\bullet, D_\bullet, E_\bullet)$ is a simplicial triple to which the results of the previous section apply, and we define

$$M_{D,E}^1(\overline{X}) := M_{D_\bullet, E_\bullet}^1(\overline{U}_\bullet).$$

In 6.4.7, we will show that this definition is independent of the choice of hypercover $\overline{X}_\bullet$.

6.4.2. Recall from the beginning of this chapter that we define

$$H_{D,E}^i(\overline{X}, \mathbb{Z}_\ell(1)) := H^1(X_{\overline{k}}, \tilde{j}_!\mathbb{Z}_\ell(1))$$

where $\tilde{j} : U \hookrightarrow X$ is the inclusion. We define $k_\bullet : U_\bullet \hookrightarrow \overline{U}_\bullet - E_\bullet$ to be the inclusion, and define (in accordance with our standard notation)

$$H_{D_\bullet, E_\bullet}^i(\overline{U}_\bullet, \mathbb{Z}_\ell(1)) := H^i(\overline{U}_\bullet - E_\bullet, k_{\bullet !}\mathbb{Z}_\ell(1)).$$

The following proposition justifies the above definition of $M_{D,E}^1(\overline{X})$:

Proposition 6.4.3. *Let $(\overline{X}, D, E)$ be a triple as above. Then there is a canonical isomorphism*

$$H^1_{D,E}(\overline{X}, \mathbb{Z}_\ell(1)) \cong H^1_{D_\bullet, E_\bullet}(\overline{U}_\bullet, \mathbb{Z}_\ell(1)).$$

Proof. Recall that we started with a proper hypercover $\overline{\pi}_\bullet : \overline{X}_\bullet \to \overline{X}$, and we let $D'_\bullet = \overline{\pi}^{-1}_\bullet(D)$ and $E'_\bullet = \overline{\pi}^{-1}_\bullet(E)$. We first claim that we have a canonical isomorphism

$$H^1_{D,E}(\overline{X}, \mathbb{Z}_\ell(1)) \cong H^1_{D'_\bullet, E'_\bullet}(\overline{X}_\bullet, \mathbb{Z}_\ell(1)).$$

To see this, first let $X = \overline{X} - E$ and $X_\bullet = \overline{X}_\bullet - E'_\bullet$, so that $H^1_{D,E}(\overline{X}, \mathbb{Z}_\ell(1)) = H^1(X, \tilde{j}_! \mathbb{Z}_\ell(1))$ and $H^1_{D'_\bullet, E'_\bullet}(\overline{X}_\bullet, \mathbb{Z}_\ell(1)) = H^1(X_\bullet, \tilde{j}_{\bullet,!} \mathbb{Z}_\ell(1))$. Consider the commutative diagram

$$
\begin{array}{ccc}
U_\bullet & \xrightarrow{\tilde{j}_\bullet} & X_\bullet \\
{\scriptstyle \pi_{\bullet,U}} \downarrow & & \downarrow {\scriptstyle \pi_{\bullet,X}} \\
U & \xrightarrow{\tilde{j}} & X.
\end{array}
$$

By cohomological descent (see [Con01, Chap. 7] for an exposition of cohomological descent) we have an isomorphism

$$\tilde{j}_! \mathbb{Z}_\ell(1) \cong (R\pi_{\bullet,X})_* (\pi_{\bullet,X})^* \tilde{j}_! \mathbb{Z}_\ell(1). \tag{6.4.3.1}$$

Moreover, we have a base change isomorphism

$$(\pi_{\bullet,X})^* \tilde{j}_! \mathbb{Z}_\ell(1) \cong \tilde{j}_{\bullet,!} (\pi_{\bullet,U})^* \mathbb{Z}_\ell(1). \tag{6.4.3.2}$$

This isomorphism exists because the functors $(\pi_{\bullet,X})^*$, $(\pi_{\bullet,U})^*$ and $\tilde{j}_{\bullet,!}$ are all defined in the same way as for ordinary schemes; e.g., $((\pi_{\bullet,X})^* \mathscr{F}^\bullet)^n = (\pi_{n,X})^* \mathscr{F}^n$, and similarly for the other functors. Therefore to show the above base change isomorphism, it suffices to do so for each level of the simplicial scheme (i.e., for each $X_n \to X$), where the base change isomorphism is standard [FK88, 8.7].

Combining Equations 6.4.3.1 and 6.4.3.2, we conclude that we have an isomorphism

$$R(\pi_{X,\bullet})_* \tilde{j}_{\bullet,!} \mathbb{Z}_\ell(1) \cong \tilde{j}_! \mathbb{Z}_\ell(1).$$

Applying the derived global sections functor allows us to conclude that

$$H^1(X, \tilde{j}_! \mathbb{Z}_\ell(1)) \cong H^1(X_\bullet, \tilde{j}_{\bullet,!} \mathbb{Z}_\ell(1))$$

as desired.

To complete the proof, we wish to show that we have an isomorphism

$$H^1(X_\bullet, \tilde{j}_! \mathbb{Z}_\ell(1)) \cong H^1(\overline{U}_\bullet - E_\bullet, k_{\bullet,!} \mathbb{Z}_\ell(1)) := H^1_{D_\bullet, E_\bullet}(\overline{U}_\bullet, \mathbb{Z}_\ell(1)).$$

For this, let $k_\bullet : U_\bullet \hookrightarrow \overline{U}_\bullet - E_\bullet$ and $\alpha_\bullet : \overline{U}_\bullet - E_\bullet \to X_\bullet$. Note that we have a commutative diagram

$$
\begin{array}{ccc}
U_\bullet & \overset{k_\bullet}{\hookrightarrow} & \overline{U}_\bullet - E_\bullet \\
 & \underset{\tilde{j}_\bullet}{\searrow} & \big\uparrow{\scriptstyle \alpha_\bullet} \\
 & & X_\bullet
\end{array} ,
$$

where each arrow is a simplicial open immersion. Moreover, $\alpha_\bullet$ is proper: for $X_\bullet = \overline{X}_\bullet - E'_\bullet$, and so the difference between $\overline{U}_\bullet - E_\bullet$ and $X_\bullet$ is that we have removed those components of $D'_\bullet$ that equal components of $\overline{X}_\bullet$. Therefore we have $\tilde{j}_{\bullet,!} = R\alpha_{\bullet,*} \circ k_{\bullet,!}$, and taking derived global sections we conclude that

$$
H^1(X_\bullet, \tilde{j}_{\bullet,!}\mathbb{Z}_\ell(1)) \cong H^1(\overline{U}_\bullet - E_\bullet, k_{\bullet,!}\mathbb{Z}_\ell(1))
$$

as desired. $\qquad\square$

Combining Proposition 6.4.3 with Proposition 6.2.9, we conclude the following:

Proposition 6.4.4. *Let $(\overline{X}, D, E)$ be a triple as above. Choose a hypercover $\overline{X}_\bullet \to \overline{X}$ with $\overline{X}_n$ proper smooth for each n, and use this hypercover to construct the 1-motive $M^1_{D,E}(\overline{X})$ by the procedure described above. Then (for any such choice) we have a natural isomorphism for $\ell \neq p$*

$$
T_\ell M^1_{D,E}(\overline{X}) \cong H^1_{D,E}(\overline{X}, \mathbb{Z}_\ell(1)).
$$

Functoriality of $M^1_{D,E}(\overline{X})$

Now consider two triples $(\overline{X}, D, E)$ and $(\overline{Y}, A, B)$ with $\overline{X}$ and $\overline{Y}$ proper, and D, E (resp. A, B) disjoint closed subschemes. Let $U = \overline{X} - (D \cup E)$ and $V = \overline{Y} - (A \cup B)$, and assume that U (resp. V) is dense in $\overline{X}$ (resp. in $\overline{Y}$). We let $X = \overline{X} - E$ and $\tilde{U} = \overline{X} - D$. We similarly let $Y = \overline{Y} - B$ and $\tilde{V} = \overline{Y} - A$. We label the various maps between these spaces as follows:

$$
\begin{array}{ccccc}
U & \overset{\tilde{j}}{\hookrightarrow} & X & \overset{\tilde{i}}{\hookleftarrow} & D \\
\big\downarrow{\scriptstyle\tilde{u}} & & \big\downarrow{\scriptstyle u} & & \big\| \\
\tilde{U} & \overset{j}{\hookrightarrow} & \overline{X} & \overset{i}{\hookleftarrow} & D \\
\big\uparrow{\scriptstyle\tilde{v}} & & \big\uparrow{\scriptstyle v} & & \\
E & =\!\!=\!\!= & E & &
\end{array}
$$

and

$$
\begin{array}{ccccc}
V & \overset{\tilde{b}}{\hookrightarrow} & Y & \overset{\tilde{a}}{\hookleftarrow} & A \\
\downarrow{\scriptstyle\tilde{r}} & & \downarrow{\scriptstyle r} & & \| \\
\tilde{V} & \overset{b}{\hookrightarrow} & \overline{Y} & \overset{a}{\hookleftarrow} & A \\
\uparrow{\scriptstyle\tilde{s}} & & \uparrow{\scriptstyle s} & & \\
B & = & B. & &
\end{array}
$$

6.4.5. Let $f : \overline{X} \to \overline{Y}$ be a morphism of triples such that

1. $f^{-1}(B) \subseteq E$, and

2. $f^{-1}(A) = D$.

Then there exist hypercovers $\overline{X}_\bullet \to \overline{X}$, $\overline{Y}_\bullet \to Y$, and a map $f_\bullet : \overline{X}_\bullet \to \overline{Y}_\bullet$ such that there is a commutative diagram

$$
\begin{array}{ccc}
\overline{X}_\bullet & \overset{f_\bullet}{\longrightarrow} & \overline{Y}_\bullet \\
\downarrow & & \downarrow \\
\overline{X} & \overset{f}{\longrightarrow} & \overline{Y}.
\end{array}
$$

Let $D'_\bullet$, $E'_\bullet$ $A'_\bullet$, $B'_\bullet$ be the inverse images of D, E, A, and B, respectively. We then have $f_\bullet^{-1}(B'_\bullet) \subseteq E'_\bullet$, and $f_\bullet^{-1}(A'_\bullet) = D'_\bullet$.

Now let $U_\bullet = \overline{X}_\bullet - (D'_\bullet \cup E'_\bullet)$ and $V_\bullet = \overline{Y}_\bullet - (A'_\bullet \cup B'_\bullet)$. Let $\overline{U}_\bullet$ be the closure of $U_\bullet$ in $\overline{X}_\bullet$ and $\overline{V}_\bullet$ the closure of $V_\bullet$ in $\overline{Y}_\bullet$. Let $D_\bullet = \overline{U}_\bullet \cap D'_\bullet$, and define $E_\bullet$, $A_\bullet$, and $B_\bullet$ similarly. Then $f_\bullet$ restricts to a morphism $f_\bullet : \overline{U}_\bullet \to \overline{V}_\bullet$ satisfying

1. $f_\bullet^{-1}(B_\bullet) \subseteq E_\bullet$, and

2. $f_\bullet^{-1}(A_\bullet) = D_\bullet$.

Therefore by our work in Section 6.3 there is an induced morphism $M^1_{A_\bullet,B_\bullet}(\overline{V}_\bullet) \to M^1_{D_\bullet,E_\bullet}(\overline{U}_\bullet)$ compatible with the ℓ-adic realizations. This implies the following, which summarizes our work in this subsection:

Proposition 6.4.6. *Let $f : (\overline{X}, D, E) \to (\overline{Y}, A, B)$ be a morphism of triples such that $f^{-1}(A) = D$ and $f^{-1}(B) \subseteq E$. Then there exist proper hypercovers $\overline{X}_\bullet \to \overline{X}$ and $\overline{Y}_\bullet \to Y$, such that if one defines $M^1_{D,E}(\overline{X})$ and $M^1_{A,B}(\overline{Y})$ via $\overline{X}_\bullet$ and $\overline{Y}_\bullet$ respectively, one has an induced morphism*

$$
f^* : M^1_{A,B}(\overline{Y}) \longrightarrow M^1_{D,E}(\overline{X})
$$

compatible with the ℓ-adic realizations of Proposition 6.4.4, for $\ell \neq p$.

6.4.7. We are now ready to show that the above definition of $M^1_{D,E}(\overline{X})$ is independent of the choice of simplicial cover. Suppose that $\pi_\bullet : \overline{X}_\bullet \to \overline{X}$ and $\rho_\bullet : \overline{X}'_\bullet \to \overline{X}$ are two simplicial covers by proper smooth schemes. Then as explained in [Con01, pp. 26-31], we can choose a third simplicial cover $\tau : \overline{X}''_\bullet \to \overline{X}$ with the property that there are simplicial maps

$$\overline{X}_\bullet \xleftarrow{f_\bullet} \overline{X}''_\bullet \xrightarrow{g_\bullet} \overline{X}'_\bullet$$

lying over the identity on $\overline{X}$.

Proposition 6.4.8. *Let $M^1_{D,E}(\overline{X})$, $M^1_{D,E}(\overline{X})'$, and $M^1_{D,E}(\overline{X})''$ be the 1-motives constructed by using the simplicial covers $\overline{X}_\bullet$, $\overline{X}'_\bullet$, and $\overline{X}''_\bullet$ respectively. Then the induced morphisms of 1-motives*

$$M^1_{D,E}(\overline{X}) \xrightarrow{\hat{f}} M^1_{D,E}(\overline{X})'' \xleftarrow{\hat{g}} M^1_{D,E}(\overline{X})'$$

are isomorphisms in 1-Mot$_k[1/p]$, and the composite isomorphism

$$\hat{g}^{-1} \circ \hat{f} : M^1_{D,E}(\overline{X}) \xrightarrow{\sim} M^1_{D,E}(\overline{X})'$$

is the unique isomorphism in 1-Mot$_k[1/p]$ inducing the identity map $H^1_{D,E}(\overline{X}, \mathbb{Z}_\ell(1)) \to H^1_{D,E}(\overline{X}, \mathbb{Z}_\ell(1))$ for every $\ell \neq p$.

Hence the 1-motive $M^1_{D,E}(\overline{X}) \in$ 1-Mot$_k[1/p]$ is (up to unique isomorphism) independent of the choice of simplicial cover.

Proof. By Proposition 6.3.4, for each $\ell \neq p$, the ℓ-adic realizations $T_\ell \hat{f}$ and $T_\ell \hat{g}$ induce the identity map on $H^1_{D,E}(\overline{X}, \mathbb{Z}_\ell(1))$ (since they lie over the identity on X). Hence by Prop. 5.2.2, the maps f and g are isomorphisms in 1-Mot$_k[1/p]$. It is also clear that f and g are the only isomorphisms of 1-motives in 1-Mot$_k[1/p]$ inducing the identity on $H^1_{D,E}(\overline{X}, \mathbb{Z}_\ell(1))$ for each $\ell \neq p$, because the realization functor $T_\ell(-)$ is faithful. $\qquad\square$

The results of this chapter up to this point can be summarized as follows:

Theorem 6.4.9. *Let $(\overline{X}, D, E)$ be a triple consisting of a proper finite type k-scheme $\overline{X}$, and two disjoint closed subschemes $D, E \subset \overline{X}$ such that $U := \overline{X} - (D \cup E)$ is everywhere dense in $\overline{X}$. Then there exists a 1-motive $M^1_{D,E}(\overline{X})$ defined up to unique p-isogeny, such that there is a natural isomorphism*

$$T_\ell M^1_{D,E}(\overline{X}) \cong H^1_{D,E}(\overline{X}_{\overline{k}}, \mathbb{Z}_\ell(1)) := H^1(X_{\overline{k}}, j_! \mathbb{Z}_\ell(1))$$

for all $\ell \neq p$. $M^1_{D,E}(\overline{X})$ is functorial for morphisms of triples $f : (\overline{X}, D, E) \to (\overline{Y}, A, B)$ such that $f^{-1}(A) = D$ and $f^{-1}(B) \subseteq E$.

6.5   The 1-motives $M^1(X)$ and $M_c^1(X)$

As special cases of the above constuction, for any separated finite type k-scheme X we can define 1-motives $M^1(X)$ and $M_c^1(X)$ that realize the cohomology groups $H^1(X_{\overline{k}}, \mathbb{Z}_\ell(1))$ and $H_c^1(X_{\overline{k}}, \mathbb{Z}_\ell(1))$ respectively. To do this, choose a compactification $j : X \hookrightarrow \overline{X}$, and let $i : C \hookrightarrow \overline{X}$ be the closed complement. Then we set

$$M^1(X) = M_{\emptyset,C}^1(\overline{X}) = [\mathbf{Div}_{C_\bullet}^0(\overline{X}_\bullet) \to \mathbf{Pic}_{\overline{X}_\bullet}^{0,red}]$$

and

$$M_c^1(X) = M_{C,\emptyset}^1(\overline{X}) = [0 \to \mathbf{Pic}_{\overline{X}_\bullet,C_\bullet}^{0,red}],$$

where $\pi_\bullet : \overline{X}_\bullet \to \overline{X}$ is any proper smooth simplicial hypercover of $\overline{X}$, and $C_\bullet = \pi_\bullet^{-1}(C)$.

Proposition 6.5.1. *The above definitions are independent (up to unique isomorphism) of the choice of the compactification $\overline{X}$. They define contravariant functors*

$$M^1(-), M_c^1(-) : Sch/k \longrightarrow \textit{1-Mot}_k[1/p],$$

where $M^1(-)$ is functorial for arbitrary morphisms, while $M_c^1(-)$ is functorial for proper *morphisms.*

Proof. The proof that $M^1(X)$ and $M_c^1(X)$ are independent of the choice of compactification is essentially the same argument as in 6.4.7; any two compactifications are dominated by a third, and the induced morphisms of 1-motives are seen to be isomorphisms by looking at ℓ-adic realizations. It is clear that $M^1(-)$ is functorial for arbitrary morphisms. To check that $M_c^1(-)$ is functorial for proper morphisms, we need to show the following lemma:

Lemma 6.5.2. *Let $f : X \to Y$ be a proper morphism of schemes over k, and choose compactifications $j : X \hookrightarrow \overline{X}$, $v : Y \hookrightarrow \overline{Y}$, and suppose that there exists a morphism $\overline{f} : \overline{X} \to \overline{Y}$ compactifying f. Let $C = \overline{X} - X$, and let $D = \overline{Y} - Y$. Then $\overline{f}^{-1}(D) = C$.*

Proof. (of lemma) It is equivalent to show that $X = \overline{f}^{-1}(Y)$. We may assume that X, Y, $\overline{X}$, and $\overline{Y}$ are connected. Let $Z = \overline{f}^{-1}(Y)$, which is an open subset of $\overline{X}$ containing X. Considering the commutative diagrams

$$
\begin{array}{ccc}
X & \xrightarrow{\ \alpha\ } & Z \\
 & {\scriptstyle k}\searrow & \ \downarrow{\scriptstyle p_1} \\
 & & \overline{X},
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{\ \alpha\ } & Z \\
 & {\scriptstyle f}\searrow & \ \downarrow{\scriptstyle p_2} \\
 & & Y,
\end{array}
$$

we get that α is an open immersion by the first diagram, and that α is proper by the second diagram (since k and p_1 are open immersions, while f and p_2 are proper). Therefore α is an isomorphism onto a connected component of Z. Because X and $\overline{X}$ were assumed to be connected and $X \hookrightarrow \overline{X}$ is dense, it follows that Z is connected as well; therefore $\alpha : X \to Z$ is an isomorphism. $\square$

    This completes the proof of Prop. 6.5.1. $\square$

6.6   The 1-Motive $M^1_{\mathcal{D},\mathcal{E}}(\overline{\mathfrak{X}})$ for $\overline{\mathfrak{X}}$ a smooth proper stack

6.6.1. Let $\overline{\mathfrak{X}}$ be a smooth proper Deligne-Mumford stack, and let $\mathcal{D}$ and $\mathcal{E}$ be disjoint closed subschemes such that $\mathcal{U} := \overline{\mathfrak{X}} - (\mathcal{D} \cup \mathcal{E})$ is dense in $\overline{\mathfrak{X}}$. Let $\mathfrak{X} := \overline{\mathfrak{X}} - \mathcal{E}$, and $j : \mathcal{U} \hookrightarrow \mathfrak{X}$ be the inclusion. Define

$$H^i_{\mathcal{D},\mathcal{E}}(\overline{\mathfrak{X}}, \mathbb{Q}_\ell(1)) := H^i(\mathfrak{X}_{\overline{k}}, j_! \mathbb{Q}_\ell(1))$$

(In this section we work with $\mathbb{Q}_\ell$-coefficients in order to avoid certain technical issues regarding the ℓ-adic cohomology of stacks; see [LO08]).

6.6.2. Our goal in this section is to define a 1-motive $M^1_{\mathcal{D},\mathcal{E}}(\overline{\mathfrak{X}})$ such that for each prime $\ell \neq p$, we have a canonical isomorphism

$$V_\ell M^1_{\mathcal{D},\mathcal{E}}(\overline{\mathfrak{X}}) \cong H^1_{\mathcal{D},\mathcal{E}}(\overline{\mathfrak{X}}, \mathbb{Q}_\ell(1))$$

(recall that $V_\ell(-) := T_\ell(-) \otimes \mathbb{Q}$). The procedure is much that same as defining $M^1_{D,E}(\overline{X})$ in the case of schemes, except that we do not use a cover of $\overline{\mathfrak{X}}$ by a simplicial scheme. There are two reasons for this: (1) this eliminates the arbitrary choice inherent in the definition of $M^1_{D,E}(\overline{X})$, and (2) we will need the direct (non-simplicial) construction in Chapters 7 and 8.

6.6.3. To start the construction, first consider the relative Picard group of the pair $(\overline{\mathfrak{X}}, \mathcal{D})$, defined by the formula

$$\mathrm{Pic}(\overline{\mathfrak{X}}, \mathcal{D}) = H^1(\overline{\mathfrak{X}}, \mathrm{Ker}(\mathbb{G}_{m,\overline{\mathfrak{X}}} \to i_* \mathbb{G}_{m,\mathcal{D}})),$$

where $i : \mathcal{D} \hookrightarrow \overline{\mathfrak{X}}$ is the inclusion. By the same proof as Proposition 6.1.4 the elements of $\mathrm{Pic}(\overline{\mathfrak{X}}, \mathcal{D})$ correspond to pairs $(\mathscr{L}, \varphi)$, where $\mathscr{L}$ is a line bundle on $\overline{\mathfrak{X}}$ and $\varphi : \mathcal{O}_{\mathcal{D}} \xrightarrow{\sim} \mathscr{L}|_{\mathcal{D}}$ is an isomorphism. We have the following analogue of Proposition 6.1.5:

Proposition 6.6.4. *Let $\mathbf{Pic}_{\overline{\mathfrak{X}},\mathcal{D}}$ be the fppf-sheafification of the functor on Sch/k,*

$$Y \mapsto \mathrm{Pic}(\overline{\mathfrak{X}} \times Y, \mathcal{D} \times Y).$$

Then $\mathbf{Pic}_{\overline{\mathfrak{X}},\mathcal{D}}$ is representable. Moreover, let $\mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}},\mathcal{D}}$ be the reduced connected component of the identity. Then $\mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}},\mathcal{D}}$ is a semi-abelian variety, and there is an exact sequence

$$0 \to \mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}},\mathcal{D}} \to \mathbf{Pic}^{red}_{\overline{\mathfrak{X}},\mathcal{D}} \to \mathbf{NS}_{\overline{\mathfrak{X}},\mathcal{D}} \to 0,$$

where $\mathbf{NS}_{\overline{\mathfrak{X}},\mathcal{D}}$ is a finitely generated étale-locally constant k-group scheme.

Proof. The proof is essentially the same as Proposition 6.1.5: we start with the exact sequence of fppf sheaves on Sch/k

$$R^0(\overline{p}_\bullet)_* \mathbb{G}_{m,\overline{\mathfrak{X}}} \xrightarrow{a} R^0(p_\mathcal{D})_* \mathbb{G}_{m,\mathcal{D}} \to \mathbf{Pic}_{\overline{\mathfrak{X}},\mathcal{D}} \to \mathbf{Pic}_{\overline{\mathfrak{X}}} \xrightarrow{b} \mathbf{Pic}_{\mathcal{D}}.$$

The two sheaves on the right are representable by Theorem 2.1.2, and moreover (by the same reference) the reduction $\mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}}}$ of the connected component of the identity is a semiabelian variety. It is clear that the the two left-hand sheaves are representable by tori. This gives us a short exact sequence

$$0 \to \operatorname{coker}(a) \to \mathbf{Pic}_{\overline{\mathfrak{X}},\mathcal{D}} \to \ker(b) \to 0$$

where $\operatorname{coker}(a)$ is a torus and $\ker(b)$ is (after taking the reduced part) an extension of a semi-abelian variety by a finitely generated étale group scheme. This implies the proposition statement. $\qquad\square$

6.6.5. We define the group $\operatorname{Div}_\mathcal{E}(\overline{\mathfrak{X}})$ to be the group of Weil divisors on $\overline{\mathfrak{X}}$ supported on $\mathcal{E}$. Because $\mathcal{E}$ and $\mathcal{D}$ are disjoint, there is a natural map

$$cl : \operatorname{Div}_\mathcal{E}(\overline{\mathfrak{X}}) \longrightarrow \operatorname{Pic}(\overline{\mathfrak{X}}, \mathcal{D}).$$

More generally, if we let $\mathbf{Div}_\mathcal{E}(\overline{\mathfrak{X}})$ be the étale group scheme defined by sheafifying the functor

$$Y \mapsto \operatorname{Div}_{\mathcal{E} \times Y}(\overline{\mathfrak{X}} \times Y)$$

for smooth Y/k (equivalently, as the group scheme with $\bar{k}$-points equal to $\operatorname{Div}_{\mathcal{E}_{\bar{k}}}(\overline{\mathfrak{X}}_{\bar{k}})$ and natural Galois action), we have a cycle map

$$cl : \mathbf{Div}_\mathcal{E}(\overline{\mathfrak{X}}) \longrightarrow \mathbf{Pic}^{red}_{\overline{\mathfrak{X}},\mathcal{D}}.$$

We define $\mathbf{Div}^0_\mathcal{E}(\overline{\mathfrak{X}})$ to be the kernel of the composition

$$\overline{cl} : \mathbf{Div}_\mathcal{E}(\overline{\mathfrak{X}}) \longrightarrow \mathbf{Pic}^{red}_{\overline{\mathfrak{X}},\mathcal{D}} \to \mathbf{NS}_{\overline{\mathfrak{X}},\mathcal{D}}.$$

Then there is a natural map $cl^0 : \mathbf{Div}^0_\mathcal{E}(\overline{\mathfrak{X}}) \to \mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}},\mathcal{D}}$.

Definition 6.6.6. With the pair $(\overline{\mathfrak{X}}, \mathcal{D}, \mathcal{E})$ as above, we define

$$M^1_{D,\mathcal{E}}(\overline{\mathfrak{X}}) := [\mathbf{Div}^0_\mathcal{E}(\overline{\mathfrak{X}}) \overset{cl^0}{\to} \mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}},\mathcal{D}}].$$

Proposition 6.6.7. *With $M^1_{D,\mathcal{E}}(\overline{\mathfrak{X}})$ defined as above, there is a natural isomorphism (for $\ell \neq p$)*

$$V_\ell M^1_{D,\mathcal{E}}(\overline{\mathfrak{X}}) \cong H^1_{D,\mathcal{E}}(\overline{\mathfrak{X}}, \mathbb{Q}_\ell(1)) := H^1(\mathfrak{X}, j_! \mathbb{Q}_\ell(1)),$$

where $\mathfrak{X} := \overline{\mathfrak{X}} - \mathcal{E}$, $\mathcal{U} := \overline{\mathfrak{X}} - (\mathcal{D} \cup \mathcal{E})$, and $j : \mathcal{U} \hookrightarrow \mathcal{X}$ is the inclusion.

Proof. The proof is essentially the same as that of Proposition 6.2.9, replacing $\overline{X}_\bullet$ by $\overline{\mathfrak{X}}$, etc. Therefore we will omit some of the more routine verifications in our proof below.

Define $\tilde{\mathcal{U}} := \overline{\mathfrak{X}} - \mathcal{D}$. Then we have a commutative diagram of inclusions

$$
\begin{array}{ccc}
\mathcal{U} & \overset{\tilde{j}}{\hookrightarrow} & \mathfrak{X} \\
\tilde{u} \downarrow & & \downarrow u \\
\tilde{\mathcal{U}} & \overset{j}{\hookrightarrow} & \overline{\mathfrak{X}}.
\end{array}
$$

Because $\mathcal{D}$ and $\mathcal{E}$ are disjoint, we have

$$
Ru_* \tilde{j}_! \cong j_! R\tilde{u}_*
$$

as functors on $D^b(\mathcal{U})$. Let $\tilde{v} : \mathcal{E} \hookrightarrow \tilde{\mathcal{U}}$ be the inclusion, and consider the exact triangle in $D_c^b(\tilde{\mathcal{U}})$

$$
\tilde{v}_* R\tilde{v}^! \mu_{n,\tilde{\mathcal{U}}} \to \mu_{n,\tilde{\mathcal{U}}} \to R\tilde{u}_* \mu_{n,\mathcal{U}} \to \tilde{v}_* R\tilde{v}^! \mu_{n,\tilde{\mathcal{U}}}[1]
$$

(where n is prime to the characteristic p). Applying $j_!$ to this triangle gives a triangle (note that $v_* = j_! \circ \tilde{v}_*$)

$$
v_* Rv^! \mu_{n,\overline{\mathfrak{X}}} \to j_! \mu_{n,\tilde{\mathcal{U}}} \to j_! R\tilde{u}_* \mu_{n,\mathcal{U}} \to v_* Rv^! \mu_{n,\overline{\mathfrak{X}}}[1]. \tag{6.6.7.1}
$$

Taking cohomology of this triangle, and using the fact that $Ru_* \tilde{j}_! \cong j_! R\tilde{u}_*$, we get an exact sequence

$$
0 \to H_c^1(\tilde{\mathcal{U}}, \mu_n) \to H_{\mathcal{D},\mathcal{E}}^1(\overline{\mathfrak{X}}, \mu_n) \to H_{\mathcal{E}}^2(\overline{\mathfrak{X}}, \mu_n) \to H_c^2(\tilde{\mathcal{U}}, \mu_n). \tag{6.6.7.2}
$$

Proposition 6.6.8. *We have a natural bijection*

$$
H_c^1(\tilde{\mathcal{U}}, \mu_n) \longleftrightarrow \mathbf{Pic}^0(\overline{\mathfrak{X}}, \mathcal{D})[n].
$$

Proof. Let $\mathbb{G}_{m,\overline{\mathfrak{X}},\mathcal{D}} := \mathrm{Ker}(\mathbb{G}_{m,\overline{\mathfrak{X}}} \to i_{\bullet,*}\mathbb{G}_{m,\mathcal{D}})$. Then we have a commutative diagram of sheaves on $\overline{\mathfrak{X}}$, where the rows and columns are exact:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & j_! \mu_{n,\tilde{\mathcal{U}}} & \longrightarrow & \mathbb{G}_{m,\overline{\mathfrak{X}},\mathcal{D}} & \overset{\cdot n}{\longrightarrow} & \mathbb{G}_{m,\overline{\mathfrak{X}},\mathcal{D}} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mu_{n,\overline{\mathfrak{X}}} & \longrightarrow & \mathbb{G}_{m,\overline{\mathfrak{X}}} & \overset{\cdot n}{\longrightarrow} & \mathbb{G}_{m,\overline{\mathfrak{X}}} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & i_* \mu_{n,\mathcal{D}} & \longrightarrow & i_* \mathbb{G}_{m,\mathcal{D}} & \overset{\cdot n}{\longrightarrow} & i_* \mathbb{G}_{m,\mathcal{D}} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

Taking cohomology along the top row, we have an exact sequence

$$
H_c^0(\tilde{\mathcal{U}}, \mathbb{G}_m) \overset{\cdot n}{\longrightarrow} H_c^0(\tilde{\mathcal{U}}, \mathbb{G}_m) \longrightarrow H_c^1(\tilde{\mathcal{U}}, \mu_n) \longrightarrow \mathrm{Pic}(\overline{\mathfrak{X}}, \mathcal{D}) \overset{\cdot n}{\longrightarrow} \mathrm{Pic}(\overline{\mathfrak{X}}, \mathcal{D}). \tag{6.6.8.1}
$$

But $H_c^0(\tilde{\mathcal{U}}, \mathbb{G}_m) = \mathrm{Ker}(H^0(\overline{\mathfrak{X}}, \mathbb{G}_m) \to H^0(\mathcal{D}, \mathbb{G}_m))$ is a torus; since the $\overline{k}$-points of a torus are divisible, we have

$$H_c^1(\tilde{\mathcal{U}}, \mu_n) = \mathrm{Pic}(\overline{\mathfrak{X}}, \mathcal{D})[n] = \mathbf{Pic}^0(\overline{\mathfrak{X}}, \mathcal{D})[n]$$

as desired. □

Proposition 6.6.9. *We have a canonical isomorphism*

$$H_{\mathcal{E}}^2(\overline{\mathfrak{X}}, \mu_n) \cong \mathrm{Div}_{\mathcal{E}}(\overline{\mathfrak{X}}) \otimes \mathbb{Z}/(n),$$

and the map $H_{\mathcal{E}}^2(\overline{\mathfrak{X}}, \mu_n) \to H_c^2(\tilde{\mathcal{U}}, \mu_n)$ on the right-hand side of sequence 6.6.7.2 factors as

$$\mathrm{Div}_{\mathcal{E}}(\overline{\mathfrak{X}}) \otimes \mathbb{Z}/(n) \to NS(\overline{\mathfrak{X}}, \mathcal{D}) \otimes \mathbb{Z}/(n) \hookrightarrow H_c^2(\tilde{\mathcal{U}}, \mu_n),$$

where the map $\mathrm{Div}_{\mathcal{E}}(\overline{\mathfrak{X}}) \to NS(\overline{\mathfrak{X}}, \mathcal{D})$ sends a divisor to its associated cycle class, and the map on the right is an injection.

Proof. This is precisely Corollary 4.2.3. □

6.6.10. Summarizing the last two propositions, we see that we have a diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}^0(\overline{\mathfrak{X}}, \mathcal{D})[n] & \longrightarrow & T_{\mathbb{Z}/n}(M_{\mathcal{D},\mathcal{E}}^1(\overline{\mathfrak{X}})) & \longrightarrow & \mathrm{Div}_{\mathcal{E}}^0(\overline{\mathfrak{X}}) \otimes \mathbb{Z}/n & \longrightarrow & 0 \\
& & \sim\downarrow & & & & \sim\downarrow & & \\
0 & \longrightarrow & H_c^1(\tilde{\mathcal{U}}, \mu_n) & \longrightarrow & H_{\mathcal{D},\mathcal{E}}^1(\overline{\mathfrak{X}}, \mu_n) & \longrightarrow & \mathrm{Ker}(H_{\mathcal{E}}^2(\overline{\mathfrak{X}}, \mu_n) \to H_c^2(\tilde{\mathcal{U}}, \mu_n)) & \longrightarrow & 0
\end{array}
$$
$$(6.6.10.1)$$

where the left-hand and right-hand maps are isomorphisms (the upper row is the exact sequence 5.2.1.1). Therefore we need only define a map

$$f : T_{\mathbb{Z}/n}(M_{\mathcal{D},\mathcal{E}}^1(\overline{\mathfrak{X}})) \longrightarrow H_{\mathcal{D},\mathcal{E}}^1(\overline{\mathfrak{X}}, \mu_n)$$

fitting into the middle of the diagram, and by the five lemma it will be an isomorphism. The definition of this map is the same as in Proposition 6.2.9: namely, recall that $T_{\mathbb{Z}/n}(M_{\mathcal{D},\mathcal{E}}^1(\overline{\mathfrak{X}}))$ consists of data $(\mathcal{C}, \mathscr{L}^\bullet, \varphi)$, where

1. $\mathcal{C} \in \mathrm{Div}_{\mathcal{E}}^0(\overline{\mathfrak{X}})$,

2. $\mathscr{L}^\bullet$ is a line bundle on $\overline{\mathfrak{X}}$, and

3. $\varphi : \mathcal{O}_{\mathcal{D}} \xrightarrow{\sim} \mathscr{L}^\bullet|_{\mathcal{D}}$ is an isomorphism. We also require that

4. There exists at least one isomorphism $\eta : (\mathscr{L}^\bullet)^{\otimes n} \xrightarrow{\sim} \mathcal{O}(-\mathcal{C})$ identifying $\varphi^{\otimes n}$ with the canonical meromorphic section of $\mathcal{O}(-\mathcal{C})$ restricted to $\mathcal{D}$ (which is an isomorphism since $\mathcal{C}$ is disjoint from $\mathcal{D}$).

We then mod out by elements of the form $(-n\mathcal{C}, \mathcal{O}(\mathcal{C}), s)$ where $s : \mathcal{O}_{\overline{\mathfrak{X}}} \to \mathcal{O}(\mathcal{C})$ is the canonical meromorphic section.

Given an object $(\mathcal{C}, \mathscr{L}^\bullet, \varphi) \in T_{\mathbb{Z}/n} M^1_{\mathcal{D}, \mathcal{E}}(\overline{\mathfrak{X}})$, choose an isomorphism η as in bullet point (4) above. Since $\mathcal{C}$ is disjoint from $X_\bullet = \overline{\mathfrak{X}} - \mathcal{E}$, η defines a trivialization of $(\mathscr{L}^\bullet|_{X_\bullet})^{\otimes n}$ carrying $\varphi^{\otimes n}$ to the identity morphism of $\mathcal{O}_{\mathcal{D}}$. We then set $f(\mathcal{C}, \mathscr{L}^\bullet, \varphi)$ to be the $\tilde{\jmath}_{\bullet,!}\mu_n$-torsor of local isomorphisms $\mathcal{O}_{X_\bullet} \xrightarrow{\sim} \mathscr{L}^\bullet|_{X_\bullet}$ compatible with η on nth tensor products and reducing to φ on $\mathcal{D}$. One checks (by the same procedure as in the proof of Proposition 6.2.9) that this map is well-defined, and fits into the diagram 6.6.10.1. Therefore by the five-lemma, it is an isomorphism, which completes the proof of Proposition 6.6.7.

$\square$

Chapter 7

Construction of $M_c^{2d-1}(X)$

7.1 Definition

Let X be a d-dimensional separated finite type k-scheme, where k is perfect. In this section we show that after possibly a finite extension of the base field k, there exists an isogeny 1-motive $M_c^{2d-1}(X)$ realizing the cohomology group $H_c^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell(d))$. Our main tool is the following [dJng96, 7.4]:

Theorem 7.1.1. *Let X be a d-dimensional separated finite type k-scheme. There exists a finite extension K of k, separated finite type K-schemes X'' and X', and a DM stack $\mathfrak{X}$ over K, such that we have a sequence of maps*

$$\mathfrak{X} \xrightarrow{p} X'' \xrightarrow{q} X' \xrightarrow{r} X_K$$

satisfying the following conditions:

  1. r is purely inseparable and surjective, therefore a universal homeomorphism;

  2. q is proper and birational;

  3. $\mathfrak{X}$ is a smooth Deligne-Mumford stack (in fact a global quotient $[U/G]$ of a smooth k-scheme U by a finite group G) and p identifies X'' with the coarse moduli space of $\mathfrak{X}$.

Proof. [dJng96, 7.4] proves this statement for an algebraically closed field k, but if we start over an arbitrary field k, then all of the objects and maps will exist over some finite extension of k since the objects and maps are all of finite presentation over $\overline{k}$. $\qquad\square$

Definition 7.1.2. Let X be a separated finite type k-scheme, and let $\pi : \mathfrak{X} \to X$ be a map from a smooth proper Deligne-Mumford stack $\mathfrak{X}$ which factors as in Theorem 7.1.1. Then we call $\pi : \mathfrak{X} \to X$ a *weak resolution* of X.

The main fact we will use about weak resolutions is the following:

Proposition 7.1.3. *There exists an open dense subscheme $U \subset X$ such that $\pi|_U : \pi^{-1}(U) \to U$ induces an isomorphism*

$$\mathbb{Q}_{\ell,U} \xrightarrow{\sim} R\pi_{U*}\mathbb{Q}_{\ell,\pi^{-1}(U)}$$

in $D_c^b(U, \mathbb{Q}_\ell)$.

Definition 7.1.4. We define a $\mathbb{Q}_\ell$-*cohomological isomorphism* to be any morphism of Deligne-Mumford stacks $\pi : X \to Y$ inducing an isomorphism $\mathbb{Q}_{\ell,Y} \xrightarrow{\sim} R\pi_*\mathbb{Q}_{\ell,X}$.

Therefore Proposition 7.1.3 says that any weak resolution $\mathfrak{X} \to X$ is a $\mathbb{Q}_\ell$-cohomological isomorphism on an open dense subset of X.

Proof. (of Proposition 7.1.3) This follows from the following two facts:

1. The morphism $r : X' \to X$ induces an isomorphism $\mathbb{Z}_{\ell,X} \xrightarrow{\sim} Rr_*\mathbb{Z}_{\ell,X'}$.

2. The morphism $p : \mathfrak{X} \to X''$ induces an isomorphism $\mathbb{Q}_{\ell,X''} \xrightarrow{\sim} Rp_*\mathbb{Q}_{\ell,\mathfrak{X}}$.

The first fact is well-known [FK88, 3.12], while the second is Lemma 4.1.4. $\qquad\square$

7.1.5. We now construct the 1-motive $M_c^{2d-1}(X)$ as follows. Given a separated finite type k-scheme X, choose a compactification $\alpha : X \hookrightarrow \overline{X}$, such that there exists weak resolution $\overline{\pi} : \overline{\mathfrak{X}} \to \overline{X}$. It may be necessary to extend the base field to do this. Let $U \subset X$ be an open dense subscheme of X such that $\pi|_U : \pi^{-1}(U) \to U$ is a $\mathbb{Q}_\ell$-cohomological isomorphism. We write $\mathcal{U}$ for $\overline{\pi}^{-1}(U)$, and let $\mathfrak{X} = \overline{\mathfrak{X}} \times_{\overline{X}} X$, $C = \overline{X} - X$, and $\mathcal{C} = \overline{\mathfrak{X}} - \mathfrak{X}$. Finally, let $Z = X - U$ and $\mathcal{Z} = \mathfrak{X} - \mathcal{U}$. We label our various maps as follows:

$$
\begin{array}{ccccc}
\mathfrak{X} & \xrightarrow{\alpha'} & \overline{\mathfrak{X}} & \xleftarrow{\beta'} & \mathcal{C} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} & & \downarrow{\scriptstyle \pi_C} \\
X & \xrightarrow{\alpha} & \overline{X} & \xleftarrow{\beta} & C
\end{array}
$$

and

$$
\begin{array}{ccccc}
\mathcal{U} & \xrightarrow{j'} & \mathfrak{X} & \xleftarrow{i'} & \mathcal{Z} \\
\downarrow{\scriptstyle \pi_U} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_Z} \\
U & \xrightarrow{j} & X & \xleftarrow{i} & Z.
\end{array}
$$

Here $\overline{\mathfrak{X}} = \mathfrak{X} \cup \mathcal{C}$, $\overline{X} = X \cup C$, etc. Let $\overline{Z}$ be the closure of Z in $\overline{X}$, and $\overline{\mathcal{Z}}$ the closure of $\mathcal{Z}$ in $\overline{\mathfrak{X}}$.

Remark 7.1.6. Note that it is possible to choose $\overline{\mathfrak{X}}$ so that $\mathcal{C}$ is a (reduced) strict normal crossings divisor [dJng96, 7.4]. However, it is not known whether we can arrange that $\mathcal{Z}$ be a strict normal crossings divisor.

7.1.7. Let $\mathbf{Div}_{\mathcal{C}\cup\overline{\mathcal{Z}}}(\mathfrak{X})$ be the étale group scheme of Weil divisors on $\mathfrak{X}$ supported on $\mathcal{C}\cup\overline{\mathcal{Z}}$. Consider the composition

$$\overline{cl} : \mathbf{Div}_{\mathcal{C}\cup\overline{\mathcal{Z}}}(\mathfrak{X}) \xrightarrow{cl} \mathbf{Pic}_{\mathfrak{X}}^{red} \longrightarrow \mathbf{NS}_{\overline{\mathfrak{X}}}$$

and the proper pushforward map

$$\overline{\pi}_* : \mathbf{Div}_{\mathcal{C}\cup\overline{\mathcal{Z}}}(\mathfrak{X}) \longrightarrow \mathbf{Div}_{C\cup\overline{Z}}(\overline{X}).$$

We define

$$\mathbf{Div}_{\mathcal{C}\cup\overline{\mathcal{Z}}/\overline{Z}}^0(\overline{\mathfrak{X}}) := \mathrm{Ker}(\overline{cl}\oplus\overline{\pi}_*) : \mathbf{Div}_{\mathcal{C}\cup\overline{\mathcal{Z}}}(\overline{\mathfrak{X}}) \longrightarrow \mathbf{NS}_{\overline{\mathfrak{X}}}\oplus\mathbf{Div}_{C\cup\overline{Z}}(\overline{X}).$$

More concretely, $\mathbf{Div}_{\mathcal{C}\cup\overline{\mathcal{Z}}/\overline{Z}}^0(\overline{\mathfrak{X}})$ is the étale group scheme of divisors D on $\mathfrak{X}$ supported on $\mathcal{C}\cup\overline{\mathcal{Z}}$ satisfying

1. The cycle class $cl(D) = 0$ in $NS(\overline{\mathfrak{X}})$.

2. Write D as $D = D_1 + D_2$ with D_1 supported on $\mathcal{C}$ and D_2 supported on $\overline{\mathcal{Z}}$. This decomposition is unique since $\mathcal{C}$ and $\overline{\mathcal{Z}}$ have no codimension-1 irreducible components in common. Consider the proper pushforward map

$$\pi_* : \mathrm{Div}_{\overline{\mathcal{Z}}}(\overline{\mathfrak{X}})\otimes\mathbb{Q} \longrightarrow \mathrm{Div}_{\overline{Z}}(\overline{X})\otimes\mathbb{Q}.$$

We then require that $\pi_* D_2 = 0$.

There is a natural map of group schemes

$$\mathbf{Div}_{\mathcal{C}\cup\overline{\mathcal{Z}}/\overline{Z}}^0(\overline{\mathfrak{X}}) \xrightarrow{cl^0} \mathbf{Pic}_{\overline{\mathfrak{X}}}^{0,red}$$

defined by the cycle class map. We can now define the 1-motive $M_c^{2d-1}(X)$:

Definition 7.1.8. Let X be a separated scheme of finite type, of dimension d over k. Assume that there exists a compactification $X \hookrightarrow \overline{X}$ and a resolution $\mathfrak{X} \to \overline{X}$ as above (it may be necessary to take a finite extension of the base field). We define $M_c^{2d-1}(X)$ to be the 1-motive

$$M_c^{2d-1}(X) := [\mathbf{Div}_{\mathcal{C}\cup\overline{\mathcal{Z}}/\overline{Z}}^0(\overline{\mathfrak{X}}) \xrightarrow{cl^0} \mathbf{Pic}_{\overline{\mathfrak{X}}}^{0,red}]^\vee,$$

where $\mathbf{Div}_{\mathcal{C}\cup\overline{\mathcal{Z}}/\overline{Z}}^0$ is defined above, and $(-)^\vee$ indicates taking the Cartier dual of a 1-motive (5.3). We view $M_c^{2d-1}(X)$ as an object of 1-$\mathrm{Mot}_k\otimes\mathbb{Q}$.

This is an abuse of notation as we have not shown that $M_c^{2d-1}(X)$ is independent of the choice of $\mathfrak{X}$. In fact, it is only independent up to $\mathbb{Q}$-isogeny (see 7.4.9), which is why we only ever consider it as an object of 1-$\mathrm{Mot}_k\otimes\mathbb{Q}$.

7.2 ℓ-adic realization

To understand this definition of $M_c^{2d-1}(X)$ (and to show that it is, up to isogeny, independent of the choice of $\overline{X}$ and $\overline{\mathfrak{X}}$), we must discuss the ℓ-adic realization of $M_c^{2d-1}(X)$. In doing so we will assume that $k = \overline{k}$, to reduce clutter in the notation. All of the maps we define below are clearly Galois-equivariant, so the statements (especially the key statement Proposition 7.2.3) are true over an arbitrary perfect field.

Choose a prime $\ell \neq p$. Then in $D_c^b(X, \mathbb{Q}_\ell)$ we have a commuting diagram with exact rows and columns

$$
\begin{array}{ccccccc}
j_! \mathbb{Q}_{\ell,U} & \xrightarrow{\sim} & R\pi_* j'_! \mathbb{Q}_{\ell,\mathcal{U}} & \longrightarrow & 0 & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{Q}_{\ell,X} & \longrightarrow & R\pi_* \mathbb{Q}_{\ell,\mathfrak{X}} & \longrightarrow & A & \longrightarrow & \\
\downarrow & & \downarrow & & \sim\downarrow & & \\
i_* \mathbb{Q}_{\ell,Z} & \longrightarrow & R\pi_* i'_* \mathbb{Q}_{\ell,\mathcal{Z}} & \longrightarrow & i_* i^* A & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & &
\end{array}
$$

Here $A := \operatorname{Cone}(\mathbb{Q}_{\ell,X} \to R\pi_* \mathbb{Q}_{\ell,\mathfrak{X}})$. The upper left-hand arrow is an isomorphism because it equals the composition

$$
j_! \mathbb{Q}_{\ell,U} \xrightarrow{\text{ad}} j_! R\pi_* \mathbb{Q}_{\ell,\mathcal{U}} \xrightarrow{=} R\pi_* j'_! \mathbb{Q}_{\ell,\mathcal{U}},
$$

and the adjoint map $\mathbb{Q}_{\ell,U} \to R\pi_* \mathbb{Q}_{\ell,\mathcal{U}}$ is an isomorphism in $D_c^b(U, \mathbb{Q}_\ell)$ (Lemma 4.1.4).

7.2.1. If we apply $\alpha_!$ to this diagram and take cohomology of the two left-hand vertical columns, we get a commuting diagram with exact rows

$$
\begin{array}{ccccccccc}
H_c^{2d-2}(X, \mathbb{Q}_\ell) & \longrightarrow & H_c^{2d-2}(Z, \mathbb{Q}_\ell) & \longrightarrow & H_c^{2d-1}(U, \mathbb{Q}_\ell) & \longrightarrow & H_c^{2d-1}(X, \mathbb{Q}_\ell) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\sim} & & \downarrow & & \\
H_c^{2d-2}(\mathfrak{X}, \mathbb{Q}_\ell) & \longrightarrow & H_c^{2d-2}(\mathcal{Z}, \mathbb{Q}_\ell) & \longrightarrow & H_c^{2d-1}(\mathcal{U}, \mathbb{Q}_\ell) & \longrightarrow & H_c^{2d-1}(\mathfrak{X}, \mathbb{Q}_\ell) & \longrightarrow & 0.
\end{array}
$$

Let $I_{d-1}(Z)$ (resp. $I_{d-1}(\mathcal{Z})$) be the set of $(d-1)$-dimensional irreducible components of Z (resp. of $\mathcal{Z}$). Twisting in the above diagram by $d-1$, taking duals, and applying Poincaré duality, we get a diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_c^{2d-1}(X, \mathbb{Q}_\ell(d\text{-}1))^\vee & \longrightarrow & H_c^{2d-1}(U, \mathbb{Q}_\ell(d\text{-}1))^\vee & \longrightarrow & \mathbb{Q}_\ell^{I_{d-1}(Z)} & \longrightarrow & H_c^{2d-2}(X, \mathbb{Q}_\ell(d\text{-}1))^\vee \\
& & \uparrow & & \uparrow{\scriptstyle\sim} & & \uparrow & & \uparrow \\
0 & \longrightarrow & H^1(\mathfrak{X}, \mathbb{Q}_\ell(1)) & \longrightarrow & H^1(\mathcal{U}, \mathbb{Q}_\ell(1)) & \longrightarrow & \mathbb{Q}_\ell^{I_{d-1}(\mathcal{Z})} & \longrightarrow & H^2(\mathfrak{X}, \mathbb{Q}_\ell(1)).
\end{array}
$$
(7.2.1.1)

(The isomorphisms $H_c^{2d-2}(\mathcal{Z}, \mathbb{Q}_\ell(d-1))^\vee \cong \mathbb{Q}_\ell^{I_{d-1}(\mathcal{Z})}$ and $H_c^{2d-2}(Z, \mathbb{Q}_\ell(d-1))^\vee \cong \mathbb{Q}_\ell^{I_{d-1}(Z)}$ are induced by sums of trace maps as in 4.1.3). We have

$$
\mathbb{Q}_\ell^{I_{d-1}(Z)} = \operatorname{Div}_Z(X) \otimes \mathbb{Q}_\ell = \operatorname{Div}_{\overline{Z}}(\overline{X}) \otimes \mathbb{Q}_\ell
$$

and

$$\mathbb{Q}_\ell^{I_{d-1}(\mathcal{Z})} = \mathrm{Div}_{\mathcal{Z}}(\mathfrak{X}) \otimes \mathbb{Q}_\ell = \mathrm{Div}_{\overline{\mathcal{Z}}}(\overline{\mathfrak{X}}) \otimes \mathbb{Q}_\ell,$$

and under these identifications the map $\mathbb{Q}_\ell^{I_{d-1}(\mathcal{Z})} \to \mathbb{Q}_\ell^{I_{d-1}(Z)}$ of diagram 7.2.1.1 is induced by proper pushforward $\pi_* : \mathrm{Div}_{\overline{\mathcal{Z}}}(\overline{\mathfrak{X}}) \otimes \mathbb{Q} \to \mathrm{Div}_{\overline{Z}}(\overline{X}) \otimes \mathbb{Q}$, while the map $\mathbb{Q}_\ell^{I_{d-1}(\mathcal{Z})} \to H^2(\mathfrak{X}, \mathbb{Q}_\ell(1))$ is induced by the divisor class map $\mathrm{Div}_{\mathcal{Z}}(\mathfrak{X}) \to NS(\mathfrak{X})$ (Proposition 4.1.10). Then diagram 7.2.1.1 shows that we have

$$H_c^{2d-1}(X, \mathbb{Q}_\ell(d-1))^\vee \cong \mathrm{Ker}(H^1(\mathcal{U}, \mathbb{Q}_\ell(1)) \to \mathbb{Q}_\ell^{I_{d-1}(\mathcal{Z})} \overset{\pi_*}{\to} \mathbb{Q}_\ell^{I_{d-1}(Z)}).$$

7.2.2. Now recall that we defined $\mathcal{U} := \overline{\mathfrak{X}} - (\mathcal{C} \cup \overline{\mathcal{Z}})$. By Proposition 6.6.7 (with $\mathcal{D} := \emptyset$, $\mathcal{E} := \mathcal{C} \cup \mathcal{Z}$), if we set

$$M^1(\mathcal{U}) := [\mathbf{Div}_{\mathcal{C} \cup \overline{\mathcal{Z}}}^0(\overline{\mathfrak{X}}) \to \mathbf{Pic}^0(\overline{\mathfrak{X}})],$$

we have a canonical isomorphism $V_\ell M^1(\mathcal{U}) \cong H^1(\mathcal{U}, \mathbb{Q}_\ell(1))$. Then from Definition 7.1.8 we have

$$M_c^{2d-1}(\mathfrak{X})^\vee = \mathrm{Ker}(M^1(\mathcal{U}) \longrightarrow [\mathbf{Div}_{\overline{Z}}(\overline{X}) \to 0]),$$

where the map is induced by proper pushforward of divisors. Applying the functor $V_\ell(-)$, we get a commuting diagram

$$
\begin{array}{ccccc}
0 \longrightarrow & V_\ell(M_c^{2d-1}(X)^\vee) & \longrightarrow & V_\ell(M^1(\mathcal{U})) & \longrightarrow & \mathrm{Div}_{\overline{Z}}(\overline{X}) \otimes \mathbb{Q}_\ell \\
& \downarrow & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
0 \longrightarrow & H_c^{2d-1}(X, \mathbb{Q}_\ell(d-1))^\vee & \longrightarrow & H^1(\mathcal{U}, \mathbb{Q}_\ell(1)) & \longrightarrow & \mathbb{Q}_\ell^{I_{d-1}(Z)}.
\end{array}
$$

By the five lemma, the map on the left is also an isomorphism. Taking Cartier duals, we have the following:

Proposition 7.2.3. *For X a separated scheme of finite type over k, of dimension d, set $M_c^{2d-1}(X) = [\mathrm{Div}_{\mathscr{C} \cup \overline{\mathcal{Z}}/\overline{Z}}^0(\overline{\mathfrak{X}}) \to \mathrm{Pic}^0(\overline{\mathfrak{X}})]^\vee$. Then we have a canonical isomorphism*

$$V_\ell M_c^{2d-1}(X) \cong H_c^{2d-1}(X, \mathbb{Q}_\ell(d)).$$

7.3 Preliminaries on pushforward of line bundles on Deligne-Mumford Stacks

Our next goal is to prove that the 1-motive $M_c^{2d-1}(X)$ is contravariantly functorial for proper morphisms $X \to Y$. To do this, we need to prove a key proposition regarding pushforward of line bundles on Deligne-Mumford stacks. Because the proof is quite lengthy and the result will be used in both Chapters 7 and 8, we present it in its own section.

Proposition 7.3.1. *Let* $f : X \to X'$ *be a proper, surjective, representable morphism between* d-*dimensional smooth proper Deligne-Mumford stacks over a perfect field* k. *Let* $\partial X \subset X$ *and* $\partial X' \subset X'$ *be reduced strict normal crossings divisors (i.e., the irreducible components of* ∂X, $\partial X'$ *are smooth) such that* $f^{-1}(\partial X')_{red} \subseteq \partial X$. *Then there is pushforward morphism of algebraic groups*

$$f_* : \mathbf{Pic}_{X,\partial X}^{0,red} \to \mathbf{Pic}_{X',\partial X'}^{0,red}$$

satisfying the following conditions:

1. *For appropriate proper maps* f *and* g, $(g \circ f)_* = g_* \circ f_*$,

2. *Let* $\mathbf{Div}_{X-\partial X}^0(X)$ *be the free abelian group of divisors of degree zero on* X *supported on* $X - \partial X$, *and define* $\mathbf{Div}_{X'-\partial X'}^0(X')$ *similarly. Then there is a commutative diagram*

$$
\begin{array}{ccc}
\mathbf{Div}_{X-\partial X}^0(X) & \xrightarrow{cl} & \mathbf{Pic}_{X,\partial X}^{0,red} \\
{\scriptstyle f_*}\downarrow & & {\scriptstyle f_*}\downarrow \\
\mathbf{Div}_{X'-\partial X'}^0(X') & \xrightarrow{cl} & \mathbf{Pic}^{0,red}X', \partial X'
\end{array}
$$

where the left-hand vertical map is the pushforward map on divisors, and the horizontal maps are cycle class maps.

Proof. We remark that this is a refinement and generalization [BVS01, Lemma 6.2] to the case when X and X' are stacks and the base field k has positive characteristic. Because their proof uses resolution of singularities, we use a different approach.

To begin the proof, first note that by considering the obvious restriction functor

$$\mathbf{Pic}_{X,\partial X}^{0,red} \to \mathbf{Pic}_{X,f^{-1}(\partial X')_{red}}^{0,red},$$

we may assume $\partial X = f^{-1}(\partial X')_{red}$. Next let $U' \subset X$ be an open substack of X such that $f^{-1}(U') \to U'$ is finite and flat and such that $Z' := X' - U'$ is of codimension 2. To see that such a U' exists, one easily reduces to the case of schemes since f is representable, and in that case it follows since the dimension of fibers is an upper semi-continuous function [Har77, Ex. 3.22] and f is flat over every codimension-1 point of X'. Let $U = f^{-1}(U')$, $\partial U' = U' \cap \partial X'$, and $\partial U = f^{-1}(\partial U)_{red}$, so we have a commutative diagram of pairs

$$
\begin{array}{ccc}
(U, \partial U) & \hookrightarrow & (X, \partial X) \\
{\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow \\
(U', \partial U') & \hookrightarrow & (X', \partial X')
\end{array}
$$

where $f : (U, \partial U) \to (U', \partial U')$ is finite flat.

Now consider sheaves $\mathbf{Pic}_{U,\partial U}^{red}$ and $\mathbf{Pic}_{U',\partial U'}^{red}$ on $(Sm/k)_{et}$ defined as in the case when U, U' are proper, namely $\mathbf{Pic}_{U,\partial U}^{red}$ is the sheafification of the functor

$$W \mapsto \mathrm{Pic}(U \times W, \partial U \times W)$$

(with a similar definition for $\mathbf{Pic}_{U',\partial U'}^{red}$). As in the case of proper schemes, we have

$$\mathbf{Pic}_{U,\partial U}^{red} = R^1\pi_*(\mathbb{G}_{m,U,\partial U})$$

where $\pi : U \to \operatorname{Spec} k$ is the structure morphism and $\mathbb{G}_{m,U,\partial U} = \operatorname{Ker}(\mathbb{G}_{m,U} \to a_*\mathbb{G}_{m,\partial U})$ and $a : \partial U \hookrightarrow U$ is the inclusion. A similar formula holds for $\mathbf{Pic}_{U,\partial U'}^{red}$. There is a natural map of sheaves

$$N : \mathbf{Pic}_{U,\partial U}^{red} \longrightarrow \mathbf{Pic}_{U',\partial U'}^{red}$$

induced by the norm map on sheaves

$$N : f_*\mathbb{G}_{m,U} \to \mathbb{G}_{m,U'}$$

and then applying $R^1\pi_*$ (see [BVS01, p. 61] for a proof that N restricts to a morphism of subsheaves $f_*\mathbb{G}_{m,U,\partial U} \to \mathbb{G}_{m,U',\partial U'}$). We then have a map

$$f_* : \mathbf{Pic}_{X,\partial X}^{0,red} \xrightarrow{restr} \mathbf{Pic}_{U,\partial U}^{red} \xrightarrow{N} \mathbf{Pic}_{U',\partial U'}^{red}.$$

We claim the following, which defines the required map $f_* : \mathbf{Pic}_{X,\partial X}^{0,red} \to \mathbf{Pic}_{X',\partial X'}^{0,red}$:

Lemma 7.3.2. *The inclusion $U' \hookrightarrow X'$ induces an injection of sheaves $\mathbf{Pic}_{X',\partial X'}^{red} \hookrightarrow \mathbf{Pic}_{U',\partial U'}^{red}$, and the map f_* defined above factors through this subsheaf. Since $\mathbf{Pic}_{X,\partial X}^{0,red}$ is connected, this implies that f_* factors through $\mathbf{Pic}_{X',\partial X'}^{0,red}$.*

Remark 7.3.3. The resulting map f_* is independent of the choice of $U' \subset X'$: first note that any two choices U_1' and U_2', the intersection $U_3' = U_1' \cap U_2'$ also has complement of codimension ≥ 2. Considering the map $\mathbf{Pic}_{X',\partial X'}^{red} \hookrightarrow \mathbf{Pic}_{U_3',\partial U_3'}^{red}$ defined by the above procedure applied to U_3', it is easy to see that the maps $\mathbf{Pic}_{X,\partial X}^{0,red} \to \mathbf{Pic}_{X',\partial X'}^{0,red}$ defined using these three open subsets are the same.

Proof. (of Lemma 7.3.2) It suffices to check this on the level of $\bar{k}$-points, so we may assume $k = \bar{k}$ and consider the maps $\operatorname{Pic}^0(X', \partial X') \to \operatorname{Pic}(U', \partial U')$, etc. Now consider the inclusion $U' \hookrightarrow X'$; it induces a commutative diagram

$$
\begin{array}{ccccccccc}
\mathcal{O}^*(X') & \longrightarrow & \mathcal{O}^*(\partial X') & \longrightarrow & \operatorname{Pic}(X',\partial X') & \longrightarrow & \operatorname{Pic}(X') & \longrightarrow & \operatorname{Pic}(\partial X') \\
\downarrow{\scriptstyle\sim} & & \uparrow & & \downarrow & & \downarrow{\scriptstyle\sim} & & \downarrow \\
\mathcal{O}^*(U') & \longrightarrow & \mathcal{O}^*(\partial U') & \longrightarrow & \operatorname{Pic}(U',\partial U') & \longrightarrow & \operatorname{Pic}(U') & \longrightarrow & \operatorname{Pic}(\partial U').
\end{array}
$$

The map $\operatorname{Pic}(X') \to \operatorname{Pic}(U')$ is an isomorphism because the complement $Z' \subset X'$ has codimension ≥ 2. The five-lemma implies that $\operatorname{Pic}(X', \partial X')$ injects into $\operatorname{Pic}(U', \partial U')$; moreover, if we let $C := \operatorname{Coker}(\operatorname{Pic}(X', \partial X') \to \operatorname{Pic}(U', \partial U'))$, we have an exact sequence

$$0 \to \mathcal{O}^*(\partial U')/\mathcal{O}^*(\partial X') \to C \to \operatorname{Ker}(\operatorname{Pic}(\partial X') \to \operatorname{Pic}(\partial U')).$$

We need to show that if $L = (\mathscr{L}, \sigma : \mathcal{O}_{\partial X} \xrightarrow{\sim} \mathscr{L}|_{\partial X}) \in \mathrm{Pic}^0(X, \partial X)$, then the image of $f_* L$ in C is 0.

First we show that the image of L in $\mathrm{Ker}(\mathrm{Pic}(\partial X') \to \mathrm{Pic}(\partial U'))$ is 0. Let

$$K := \mathrm{Ker}(\mathrm{Pic}(\partial X') \to \mathrm{Pic}(\partial U')).$$

Concretely, the map $\mathrm{Pic}^0(X, \partial X) \to K$ is defined as follows: given $L = (\mathscr{L}, \sigma) \in \mathrm{Pic}^0(X, \partial X)$, we have $(N(\mathscr{L}|_U), \det(\sigma|_U)) \in \mathrm{Pic}(U', \partial U')$. Then there exists a line bundle $M \in \mathrm{Pic}(X')$ with $M|_{U'} = N(\mathscr{L}|_U)$, and the image of L in K is $M|_{\partial X'}$. From this description it is clear that the map $\mathrm{Pic}^0(X, \partial X) \to \mathrm{Pic}(\partial X')$ factors through $\mathrm{Pic}^0(X)$ (i.e., the image in K only depends on the line bundle $\mathscr{L}$ and not on the trivialization σ). We therefore have a factorization

$$\mathrm{Pic}^0(X, \partial X) \longrightarrow A \longrightarrow K,$$

where $A := \mathrm{Image}(\mathrm{Pic}^0(X, \partial X) \to \mathrm{Pic}^0(X))$ is an abelian variety (since $\mathrm{Pic}^0(X)$ is).

We claim that $\mathrm{Ker}(\mathrm{Pic}(\partial X') \to \mathrm{Pic}(\partial U'))$ is a group variety whose connected component of the identity is a torus. From this it will follow that the map $\mathrm{Pic}^0(X, \partial X) \to K$ is zero, since it factors through the abelian variety A. To prove this claim, we first set up some notation: let C_i be the (smooth) irreducible components of $\partial X'$, and for each increasing sequence $i_0 < ... < i_n$, let $C_{i_0...i_n} = C_{i_0} \cap ... \cap C_{i_n}$. Then [Bak10, Lemma 3.2] we have a resolution of sheaves on $(\partial X')_{et}$

$$0 \longrightarrow \mathcal{O}_{\partial X'} \longrightarrow \bigoplus_i \mathcal{O}_{C_i} \longrightarrow \bigoplus_{i<j} \mathcal{O}_{C_{ij}} \longrightarrow ...,$$

where we have abused notation and written $\mathcal{O}_{C_i}$ instead of $\iota_* \mathcal{O}_{C_i}$ for $\iota : C_i \hookrightarrow \partial X$ the closed immersion (the reference (loc. cit) only proves this for schemes, but the statement is local for the étale topology and so immediately follows for stacks). This sequence remains exact when one takes units: to see this, we can work locally. Each morphism of local rings $\mathcal{O}_{C_{i_0...i_n}} \to \mathcal{O}_{C_{i_0...i_{n+1}}}$ is a surjection of local rings (whenever it is non-zero), and for a surjection of local rings $\pi : R \to S$, $r \in R$ is a unit if and only if $\pi(r)$ is a unit. From this the exactness of the above sequence on units is immediate, giving an exact sequence of sheaves of abelian groups

$$0 \longrightarrow \mathbb{G}_{m, \partial X'} \longrightarrow \bigoplus_i \mathbb{G}_{m, C_i} \longrightarrow \bigoplus_{i<j} \mathbb{G}_{m, C_{ij}} \longrightarrow ... \qquad (7.3.3.1)$$

From this resolution we get an exact sequence

$$0 \to T \to \mathrm{Pic}(\partial X') \to \bigoplus_i \mathrm{Pic}(C_i),$$

where

$$T := \frac{\mathrm{Ker}(\oplus_{i<j} \mathcal{O}^*(C_{ij}) \to \oplus_{i<j<k} \mathcal{O}^*(C_{ijk}))}{\mathrm{Image}(\oplus_i \mathcal{O}^*(C_i) \to \oplus_{i<j} \mathcal{O}^*(C_{ij}))}$$

is an extension of a finite abelian group by a torus. Here the maps are induced by restriction.

We can restrict sequence 7.3.3.1 to $\partial U'$ and get an exact sequence

$$0 \to T_U \to \operatorname{Pic}(\partial U') \to \oplus_i \operatorname{Pic}(C_i|_U'),$$

where T_U is defined by the same formula as for $\partial X'$ (replacing C_i by $C_i|_{U'}$). Moreover, the inclusion $\partial U' \hookrightarrow \partial X'$ induces a commutative diagram

$$
\begin{array}{ccccc}
0 \longrightarrow & T & \longrightarrow & \operatorname{Pic}(\partial X') & \longrightarrow & \bigoplus_i \operatorname{Pic}(C_i) \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & T_{U'} & \longrightarrow & \operatorname{Pic}(\partial U') & \longrightarrow & \bigoplus_i \operatorname{Pic}(C_i|_{U'}).
\end{array}
$$

But notice that since C_i is smooth, the kernel of the map on the right is finitely generated. Therefore we have an exact sequence

$$0 \longrightarrow \tilde{T} \longrightarrow K \to F$$

where $\tilde{T} := \operatorname{Ker}(T \to T_U)$ is an extension of a torus by a finite group and F is finitely generated. Therefore the map $\operatorname{Pic}^0(X, \partial X) \to K$ must be zero, since it factors through the abelian variety $\operatorname{Image}(\operatorname{Pic}^0(X, \partial X) \to \operatorname{Pic}^0(X))$, and there are no non-zero maps from an abelian variety to a torus.

We have shown that the map $\operatorname{Pic}^0(X, \partial X) \to C$ factors through $\mathcal{O}^*(\partial U')/\mathcal{O}^*(\partial X')$, where we recall that $C = \operatorname{Coker}(\operatorname{Pic}(X', \partial X') \to \operatorname{Pic}(U', \partial U'))$. We want to show that the resulting map

$$f : \operatorname{Pic}^0(X, \partial X) \to \mathcal{O}^*(\partial U')/\mathcal{O}^*(\partial X') \tag{7.3.3.2}$$

is zero. Recall that we have an exact sequence

$$0 \to \mathcal{O}^*(\partial X)/\mathcal{O}^*(X) \to \operatorname{Pic}^0(X, \partial X) \to \operatorname{Pic}^0(X).$$

To show that the map in 7.3.3.2 is zero, we start by showing that the restriction

$$f|_{\mathcal{O}^*(\partial X)} : \mathcal{O}^*(\partial X)/\mathcal{O}^*(X) \to \mathcal{O}^*(\partial U')/\mathcal{O}^*(\partial X') \tag{7.3.3.3}$$

is zero. To do this, we first explicitly describe this map. An element of $\mathcal{O}^*(\partial X)$ can be given as follows: first label the connected components of ∂X as $C_1, ..., C_n$. Then for each i we let let $a_i \in k^*$ be the unit which is multiplication by a_i on C_i, and the identity on the other connected components. Then we have a corresponding element

$$L := (\mathcal{O}_X, \Pi_i a_i) \in \operatorname{Pic}^0(X, \partial X).$$

We claim that $f_* L \in \operatorname{Pic}(U', \partial U')$ can be described as follows:

Lemma 7.3.4. *Let $D_i \subset \partial U'$ be any connected component, and let $E_1, ..., E_s$ be the connected components of $f^{-1}(D_i)_{red}$, and $d_1, ..., d_s$ the degrees of these connected components under the map $\partial U \times_{\partial U'} D_i \to D_i$, and let $r_1, ..., r_s$ be their ramification degrees (so $e_1 r_1 + ... + e_s r_s = \deg(f)$). For each $j, 1 \leq j \leq s$, let $C_{\phi(j)}$ be the connected component of ∂X containing E_j. Then let*

$$b_i = \prod_{j=1}^{s} (a_{\phi(j)})^{e_j r_j} \in k^*,$$

which we think of as a unit in $\mathcal{O}^*(\partial U')$ which is multiplication by b_i on D_i, and the identity on the other connected components. We then have

$$f_* L = (\mathcal{O}_{U'}, \prod_i b_i).$$

Proof. (of lemma 7.3.4) In general, the map $\mathcal{O}^*(\partial U) \to \mathcal{O}^*(\partial U')$ is obtained by locally lifting an element of $\mathcal{O}^*(\partial U)$ to $\mathcal{O}^*(U)$, and then applying the norm map and restricting to $\partial U'$. However, if we let $\overline{\partial U} = U \times_{U'} \partial U'$ (so that $\partial U = \overline{\partial U}_{red}$), we have a commutative diagram [EGAII, 6.4.8]

$$\begin{array}{ccc}
\mathbb{G}_{m,U} & \longrightarrow & \iota_* \mathbb{G}_{m,\overline{\partial U}} \\
N \downarrow & & N \downarrow \\
\mathbb{G}_{m,U'} & \longrightarrow & \iota'_* \mathbb{G}_{m,\partial U'}
\end{array}$$

where ι, ι' are the inclusions. This implies that we only have to lift a section to $\mathcal{O}^*(\overline{\partial U})$ and apply the norm map there. For the section $\Pi_i a_i$ we are interested in, this can be done globally and the resulting formula for $N(\Pi_i a_i)$ given in the lemma statement is immediate. $\square$

We return to showing that the map in 7.3.3.3 is zero. From the description of $f_* L$ given in Lemma 7.3.4, we see that if D_i and D_j are connected components of ∂U which belong to the same connected component of $\partial X'$, then $b_i = b_j$. This implies that the section $\Pi_i b_i$ extends to an section of $\mathcal{O}^*(\partial X')$, which in turn implies that the image of $f_* L$ under the map 7.3.3.3 is zero.

We have shown that the map f of 7.3.3.2 factors through $\mathrm{Pic}^0(X, \partial X)/\mathcal{O}^*(\partial X)$, which is a subvariety of $\mathrm{Pic}^0(X)$ and hence an abelian variety. But $\mathcal{O}^*(\partial U')/\mathcal{O}^*(\partial X')$ is an extension of finitely generated group by a torus: if $\partial U'$ is smooth then this is clear, while in the general case it follows from the commuting diagram (where the rows are equalizers)

$$\begin{array}{ccc}
\mathcal{O}^*(\partial X') \longrightarrow \bigoplus_i \mathcal{O}^*(\partial X'_i) \rightrightarrows \bigoplus_{i<j} \mathcal{O}^*(\partial X'_{ij}) \\
\downarrow \qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \\
\mathcal{O}^*(\partial U') \longrightarrow \bigoplus_i \mathcal{O}^*(\partial U'_i) \rightrightarrows \bigoplus_{i<j} \mathcal{O}^*(\partial U'_{ij}),
\end{array}$$

where $\partial X'_i$ are the (smooth) irreducible components of ∂X, and similarly for $\partial U'$. This implies that the resulting map $\mathrm{Pic}^0(X, \partial X)/\mathcal{O}^*(\partial X) \to \mathcal{O}^*(\partial U')/\mathcal{O}^*(\partial X')$ is zero (since

there's no non-zero map from an abelian variety to a torus). We have finally shown that the map $\mathrm{Pic}^0(X, \partial X) \to C$ is the zero map, completing the proof of Lemma 7.3.2. $\qquad\square$

This defines the required map $f_* : \mathbf{Pic}_{X,\partial X}^{0,red} \to \mathbf{Pic}_{X',\partial X'}^{0,red}$. It is clear that this map satisfies conditions (1) and (2) of the proposition statement, since we can reduce to the case when f is finite and flat (by the way f_* was defined), where it follows from standard compatibility properties between pushforward of divisors and the norm map. $\qquad\square$

7.4   Functoriality

Let X and Y be separated finite type k-schemes, and let $f : X \to Y$ be a proper morphism. In this section we show (again after a finite extension of the base field k) that there is an induced map on 1-motives $\hat{f}^* : M_c^{2d-1}(Y) \to M_c^{2d-1}(X)$. We start with a preliminary fact on functoriality for the weak resolutions of Theorem 7.1.1:

Proposition 7.4.1. *Let $f : X \to Y$ be a morphism of separated finite type k-schemes, with $k = \bar{k}$. Then there exist weak resolutions (in the sense of Definition 7.1.2) $\pi : \mathfrak{X} \to X$, $\sigma : \mathcal{Y} \to Y$ with $\mathfrak{X}$ and $\mathcal{Y}$ smooth, and a* representable *map $f' : \mathfrak{X} \to \mathcal{Y}$ making the diagram*

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{f'} & \mathcal{Y} \\ {\scriptstyle \pi} \downarrow & & {\scriptstyle \sigma} \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

commute.

Proof. By Theorem 7.1.1, we can choose a weak resolution $\sigma : \mathcal{Y} \to Y$ with $\mathcal{Y} = [V/H]$. Then let

$$\mathfrak{X}_1 = \mathcal{Y} \times_Y X = [(V \times_Y X)/H].$$

By [dJng96, Thm. 7.3], there exists a quotient stack $\mathfrak{X} = [U/G]$ with U smooth and G finite, together with a proper map $\phi : \mathfrak{X} \to \mathfrak{X}_1$ such that the composition $\mathfrak{X} \to \mathfrak{X}_1 \to X$ is a resolution. The induced map $f' : \mathfrak{X} \to \mathcal{Y}$ satisfies all the conditions of Proposition 7.4.1 except representability. To make f' representable, we replace $\mathcal{Y}$ by $\mathcal{Y} \times_k BG$ (note that $\mathcal{Y} \times BG$ has the same coarse moduli space as $\mathcal{Y}$, so it is still a weak resolution of Y). $\qquad\square$

7.4.2. Now let X and Y be separated finite type k-schemes of dimension d, and $f : X \to Y$ a proper morphism. Choose compactifications $k : X \hookrightarrow \overline{X}$, $j : Y \hookrightarrow \overline{Y}$, and a map $\overline{f} : \overline{X} \to \overline{Y}$; then the diagram

$$\begin{array}{ccc} X & \xhookrightarrow{k} & \overline{X} \\ {\scriptstyle f} \downarrow & & \downarrow {\scriptstyle \overline{f}} \\ Y & \xhookrightarrow{j} & \overline{Y} \end{array}$$

is cartesian because f is proper; see Lemma 6.5.2. Using Proposition 7.4.1, make a finite extension of k so that we can choose weak resolutions $\overline{\pi} : \overline{\mathfrak{X}} \to \overline{X}$ and $\overline{\sigma} : \overline{\mathcal{Y}} \to \overline{Y}$ and a representable map $\overline{f}' : \overline{\mathfrak{X}} \to \overline{\mathcal{Y}}$ lying over $\overline{f}$. If we let $\mathfrak{X} = \overline{\mathfrak{X}} \times_{\overline{X}} X$, $\mathcal{Y} = \overline{\mathcal{Y}} \times_{\overline{Y}} Y$, $\mathcal{C} = \overline{\mathfrak{X}} - \mathfrak{X}$ and $\mathcal{D} = \overline{\mathcal{Y}} - \mathcal{Y}$, we then have a diagram

$$
\begin{array}{ccccc}
\mathfrak{X} & \overset{k'}{\hookrightarrow} & \overline{\mathfrak{X}} & \hookleftarrow & \mathcal{C} \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \overline{f}'} & & \downarrow \\
\mathcal{Y} & \overset{j'}{\hookrightarrow} & \overline{\mathcal{Y}} & \hookleftarrow & \mathcal{D}.
\end{array}
$$

Let $V \subset Y$ be an open subset of Y such that $\overline{\sigma}^{-1}(V) \to V$ is a $\mathbb{Q}_\ell$-cohomological isomorphism, and define $\mathcal{V} := \overline{\sigma}^{-1}(V)$. By possibly shrinking V, we can arrange that

$$
\mathfrak{X} \times_{\mathcal{Y}} \mathcal{V} \longrightarrow X \times_Y V
$$

is also a $\mathbb{Q}_\ell$-cohomological isomorphism, since $\mathfrak{X} \to X$ is a $\mathbb{Q}_\ell$-cohomological isomorphism on an open dense subset of X. If we set $\mathcal{U} = \mathfrak{X} \times_{\mathcal{Y}} \mathcal{V}$ and $U = X \times_Y V$, and $\mathcal{Z} = \mathfrak{X} - \mathcal{U}$, $\mathcal{W} = \mathcal{Y} - \mathcal{V}$, we then have a commuting diagram

$$
\begin{array}{ccccc}
\mathcal{U} & \overset{a'}{\hookrightarrow} & \mathfrak{X} & \hookleftarrow & \mathcal{Z} \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f'} & & \downarrow \\
\mathcal{V} & \overset{b'}{\hookrightarrow} & \mathcal{Y} & \hookleftarrow & \mathcal{W}
\end{array}
$$

where $a' : \mathcal{U} \hookrightarrow \mathfrak{X}$ and $b' : \mathcal{V} \hookrightarrow \mathcal{Y}$ are open immersions $\mathcal{Z} \hookrightarrow \mathfrak{X}$ and $\mathcal{W} \hookrightarrow \mathcal{Y}$ are closed immersions, and $\overline{\pi}$ and $\overline{\sigma}$ restrict to $\mathbb{Q}_\ell$-cohomological isomorphisms $\mathcal{U} \to U$ and $\mathcal{V} \to V$ respectively. Let $\overline{\mathcal{Z}}$ and $\overline{\mathcal{W}}$ be the closures of $\mathcal{Z}$ and $\mathcal{W}$ in $\overline{\mathfrak{X}}$ and $\overline{\mathcal{Y}}$, respectively. We then can set

$$
M_c^{2d-1}(X) = [\mathbf{Div}^0_{\mathcal{C} \cup \overline{\mathcal{Z}}/\overline{Z}}(\overline{\mathfrak{X}}) \to \mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}}}]^\vee \quad \text{and}
$$
$$
M_c^{2d-1}(Y) = [\mathbf{Div}^0_{\mathcal{D} \cup \overline{\mathcal{W}}/\overline{W}}(\overline{\mathcal{Y}}) \to \mathbf{Pic}^{0,red}_{\overline{\mathcal{Y}}}]^\vee .
$$

In order to define a morphism of 1-motives $\hat{f}^* : M_c^{2d-1}(Y) \to M_c^{2d-1}(X)$, we would like to define a covariant morphism of 1-motives

$$
\hat{f}_* : [\mathbf{Div}^0_{\mathcal{C} \cup \overline{\mathcal{Z}}/\overline{Z}}(\overline{\mathfrak{X}}) \to \mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}}}] \longrightarrow [\mathbf{Div}^0_{\mathcal{D} \cup \overline{\mathcal{W}}/\overline{W}}(\overline{\mathcal{Y}}) \to \mathbf{Pic}^{0,red}_{\overline{\mathcal{Y}}}]. \tag{7.4.2.1}
$$

7.4.3. We define $\hat{f}_*$ via proper pushforwards. We first want to reduce to working component by component on $\overline{\mathfrak{X}}$ and $\overline{\mathcal{Y}}$. Let $\{\overline{\mathfrak{X}}_i\}_{i=1}^n$ be the connected components of $\overline{\mathfrak{X}}$, and similarly let $\{\overline{\mathcal{Y}}_j\}_{j=1}^m$ be the connected components of $\overline{\mathcal{Y}}$. Then one sees that

$$
[\mathbf{Div}^0_{\mathcal{C} \cup \overline{\mathcal{Z}}/\overline{Z}}(\overline{\mathfrak{X}}) \to \mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}}}] = \bigoplus_i [\mathbf{Div}^0_{\mathcal{C} \cup \overline{\mathcal{Z}}/\overline{Z}}(\overline{\mathfrak{X}}_i) \to \mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}}_i}],
$$

and there is a similar decomposition for $[\mathbf{Div}^0_{\mathcal{D}\cup\mathcal{W}/\overline{W}}(\overline{\mathcal{Y}}) \to \mathbf{Pic}^{0,red}_{\overline{\mathcal{Y}}}]$. Therefore to define the map $\hat{f}_*$ it suffices to define maps

$$\hat{f}_{i*} : [\mathbf{Div}^0_{\mathcal{C}\cup\overline{\mathcal{Z}}/\overline{Z}}(\overline{\mathfrak{X}}_i) \to \mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}}_i}] \longrightarrow [\mathbf{Div}^0_{\mathcal{D}\cup\overline{\mathcal{W}}/\overline{W}}(\overline{\mathcal{Y}}_j) \to \mathbf{Pic}^{0,red}_{\overline{\mathcal{Y}}_j}]$$

for each i, where $\overline{\mathcal{Y}}_j$ is the component mapped into by $\overline{\mathfrak{X}}_i$ under $\overline{f}'$. This reduces us to the case where $\overline{\mathfrak{X}}$ and $\overline{\mathcal{Y}}$ are connected.

7.4.4. In the case where $\overline{\mathfrak{X}}$ and $\overline{\mathcal{Y}}$ are connected, we define $\hat{f}_*$ as follows. If $\dim(\overline{f}'(\overline{\mathfrak{X}})) < d$, we define $\hat{f}_*$ to be the zero map (as it should be if it is to be compatible with pushforward of divisors and line bundles). So for the rest of this section we assume $\dim(\overline{f}'(\overline{\mathfrak{X}})) = d$. Since $\overline{\mathfrak{X}}$ and $\overline{\mathcal{Y}}$ are proper, $\overline{f}'$ is surjective. Therefore, using Proposition 7.3.1 with $X = \overline{\mathfrak{X}}$, $\partial X = \emptyset$, $X' = \overline{\mathcal{Y}}$ and $\partial X' = \emptyset$, we conclude that there is a commutative diagram

$$
\begin{array}{ccc}
\mathbf{Div}^0_{\mathcal{C}\cup\overline{\mathcal{Z}}/\overline{Z}}(\overline{\mathfrak{X}}) & \longrightarrow & \mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}}} \\
\overline{f}'_* \downarrow & & \overline{f}'_* \downarrow \\
\mathbf{Div}^0_{\mathcal{D}\cup\overline{\mathcal{W}}/\overline{W}}(\overline{\mathcal{Y}}) & \longrightarrow & \mathbf{Pic}^{0,red}_{\overline{\mathcal{Y}}},
\end{array}
$$

which defines the required map of 1-motives $\hat{f}_*$. (Note that Proposition 7.3.1 is much easier to prove in the case $\partial X = \partial X' = \emptyset$, but we will need the more general case in Chapter 8.)

7.4.5. Let $\hat{f}^* : M_c^{2d-1}(Y) \to M_c^{2d-1}(X)$ be the morphism of 1-motives obtained by taking the Cartier dual of the map $\hat{f}_*$ of 7.4.2.1. We wish to show that $V_\ell \hat{f}^* : V_\ell M_c^{2d-1}(Y) \to V_\ell M_c^{2d-1}(X)$ agrees with the pullback map on cohomology $f^* : H_c^{2d-1}(Y, \mathbb{Q}_\ell(d)) \to H_c^{2d-1}(X, \mathbb{Q}_\ell(d))$. More precisely, we claim the following:

Proposition 7.4.6. *There is a commutative diagram*

$$
\begin{array}{ccc}
V_\ell M_c^{2d-1}(Y) & \xrightarrow{\alpha_Y} & H_c^{2d-1}(Y, \mathbb{Q}_\ell(d)) \\
V_\ell \hat{f}^* \downarrow & & f^* \downarrow \\
V_\ell M_c^{2d-1}(X) & \xrightarrow{\alpha_X} & H_c^{2d-1}(X, \mathbb{Q}_\ell(d))
\end{array}
$$

where α_X and α_Y are the comparison isomorphisms of Proposition 7.2.3 and f^*, $\hat{f}^*$ are as defined above.

Proof. We continue with the notation of 7.4.2. If we define

$$M^1(\mathcal{U}) := [\mathbf{Div}^0_{\mathcal{C}\cup\overline{\mathcal{Z}}}(\overline{\mathfrak{X}}) \to \mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}}}]$$

and

$$M^1(\mathcal{V}) := [\mathbf{Div}^0_{\mathcal{D}\cup\overline{\mathcal{W}}}(\overline{\mathcal{Y}}) \to \mathbf{Pic}^{0,red}_{\overline{\mathcal{Y}}}]$$

(so we have omitted the requirement that the divisor push forward to 0 in Z, resp. W), then by Proposition 6.6.7 we have natural isomorphisms $V_\ell M^1(\mathcal{U}) \cong H^1(\mathcal{U}, \mathbb{Q}_\ell(1))$ and $V_\ell M^1(\mathcal{V}) \cong H^1(\mathcal{V}, \mathbb{Q}_\ell(1))$. It is clear that we can again define a map

$$\hat{f}_* : M^1(\mathcal{U}) \to M^1(\mathcal{V})$$

by proper pushforward of divisors using Proposition 7.3.1. Since we have commutative diagrams

$$
\begin{array}{ccc}
V_\ell(M_c^{2d-1}(X)^\vee) \lhook\joinrel\longrightarrow V_\ell(M^1(\mathcal{U})) \\
\downarrow{\scriptstyle\alpha_X} \qquad\qquad \downarrow{\scriptstyle\sim} \\
H_c^{2d-1}(X, \mathbb{Q}_\ell(d-1))^\vee \lhook\joinrel\longrightarrow H^1(\mathcal{U}, \mathbb{Q}_\ell(1))
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
V_\ell(M_c^{2d-1}(Y)^\vee) \lhook\joinrel\longrightarrow V_\ell(M^1(\mathcal{V})) \\
\downarrow{\scriptstyle\alpha_Y} \qquad\qquad \downarrow{\scriptstyle\sim} \\
H_c^{2d-1}(Y, \mathbb{Q}_\ell(d-1))^\vee \lhook\joinrel\longrightarrow H^1(\mathcal{V}, \mathbb{Q}_\ell(1))
\end{array}
$$

(see 7.2.2), to show that we have a commuting diagram as in Proposition 7.4.6 it suffices to show the following:

Proposition 7.4.7. *There is a commutative diagram*

$$
\begin{array}{ccc}
V_\ell M^1(\mathcal{U}) & \xrightarrow{\alpha_U} & H^1(\mathcal{U}, \mathbb{Q}_\ell(1)) \\
{\scriptstyle V_\ell \hat{f}_*}\downarrow & & \downarrow{\scriptstyle f_*} \\
V_\ell M^1(\mathcal{V}) & \xrightarrow{\alpha_V} & H^1(\mathcal{V}, \mathbb{Q}_\ell(1))
\end{array}
$$

where α_U and α_V are the comparison isomorphisms of Proposition 6.6.7, and f_* is the push-forward map on cohomology, i.e., the Poincaré dual to the map

$$f^* : H_c^{2d-1}(\mathcal{V}, \mathbb{Q}_\ell(d-1)) \to H_c^{2d-1}(\mathcal{U}, \mathbb{Q}_\ell(d-1)).$$

Proof. First consider the case where $\dim f(\mathcal{U}) < \dim f(\mathcal{V})$. Then the proper pushforward on cohomology is clearly 0 since the map on f^* on H_c^{2d-1} must be 0. On the other hand, the proper pushforward map $V_\ell \hat{f}_* : V_\ell M^1(\mathcal{U}) \to V_\ell M^1(\mathcal{V})$ is also 0 by definition.

Now assume $f : \mathcal{U} \to \mathcal{V}$ is finite and flat. Then the pushforward map on cohomology is induced by a trace map

$$tr_f : f_* f^* \mu_n \to \mu_n$$

[Ols1, Thm 4.1]. This trace map is shown in loc. cit. to be compatible with étale localization, and to agree with the usual trace map in the case when $\mathcal{U}$ and $\mathcal{V}$ are schemes. To check Proposition 7.4.7 we may work étale-locally on $\mathcal{V}$ and hence may assume $\mathcal{V}$ is a scheme; since f is representable, $\mathcal{U}$ is a scheme as well. In this case, the proposition follows because the trace map agrees with the norm map on invertible sections [FK88, p. 136], and it is clear that the norm map on invertible sections induces the proper pushforward on divisors.

In the general case f is generically finite flat (since it is proper and representable and $\dim(\mathcal{U}) = \dim(\mathcal{V})$, $\dim f(\mathcal{U}) = \dim f(\mathcal{V})$). Let $\mathcal{V}' \subset \mathcal{V}$, $\mathcal{U}' := \mathcal{U} \times_\mathcal{V} \mathcal{V}' \subset \mathcal{U}$ be open substacks

such that $f : \mathcal{U}' \to \mathcal{V}'$ is finite flat. Then we have commutative diagrams

$$
\begin{array}{ccc}
H^1(\mathcal{U}, \mathbb{Q}_\ell(1)) \longrightarrow H^1(\mathcal{U}', \mathbb{Q}_\ell(1)) & \quad & V_\ell M^1(\mathcal{U}) \longrightarrow V_\ell M^1(\mathcal{U}') \\
\downarrow{\scriptstyle f_*} \qquad\qquad \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle V_\ell \hat{f}_*} \qquad\qquad \downarrow{\scriptstyle V_\ell \hat{f}_*} \\
H^1(\mathcal{V}, \mathbb{Q}_\ell(1)) \longrightarrow H^1(\mathcal{V}', \mathbb{Q}_\ell(1)), & & V_\ell M^1(\mathcal{V}) \longrightarrow V_\ell M^1(\mathcal{V}')
\end{array}
$$

where the left-hand diagram is induced from the restriction maps $\mathcal{U}' \hookrightarrow \mathcal{U}$ etc., and the right-hand diagram is essentially from the definition of the various 1-motives appearing and of the maps $\hat{f}_*$. Therefore it suffices to prove Proposition 7.4.7 for the map $f : \mathcal{U}' \to \mathcal{V}'$, which is finite flat and hence has already been considered. $\qquad\square$

This completes the proof of Proposition 7.4.6. $\qquad\square$

7.4.8. Finally we return to the question of independence of compactification. For a given $X \in Sch_d/k$, suppose that we choose two compactifications $\overline{X}$, $\overline{X}'$ of X and resolutions $\overline{\mathfrak{X}} \to \overline{X}$, $\overline{\mathfrak{X}}' \to \overline{X}'$, and assume that these are defined over the same base field K. Then we aim to show the following:

Proposition 7.4.9. *Let $M_c^{2d-1}(X)$ and $M_c^{2d-1}(X)'$ be the 1-motives of Definition 7.1.8 constructed using $\overline{\mathfrak{X}}$ and $\overline{\mathfrak{X}}'$ respectively. Then there exists a unique K-linear isogeny of 1-motives*

$$
f : M_c^{2d-1}(X) \to M_c^{2d-1}(X)'
$$

fitting into a diagram

$$
\begin{array}{ccc}
V_\ell M_c^{2d-1}(X) & \xrightarrow{\alpha_X} & H_c^{2d-1}(X, \mathbb{Q}_\ell(d)) \\
{\scriptstyle V_\ell f}\downarrow & & \downarrow{\scriptstyle =} \\
V_\ell M_c^{2d-1}(X)' & \xrightarrow{\alpha_X'} & H_c^{2d-1}(X, \mathbb{Q}_\ell(d))
\end{array}
$$

for all $\ell \neq p$, where α_X and α_X' are the comparison isomorphisms of Proposition 7.2.3.

Proof. First we show that there exists a third weak resolution $\overline{\mathfrak{X}}''$ of $\overline{X}$, with maps $\overline{\mathfrak{X}}'' \to \overline{\mathfrak{X}}'$ and $\overline{\mathfrak{X}}'' \to \overline{\mathfrak{X}}$. Note that we may have to move a bigger extension L/K to do this.

Let $\overline{X}'' = \overline{X} \times_X \overline{X}'$, a third compactification of X which dominates $\overline{X}$ and $\overline{X}'$. Recall that we can write $\overline{\mathfrak{X}}$ and $\overline{\mathfrak{X}}'$ as global quotient stacks, say $\overline{\mathfrak{X}} = [V/G]$ and $\overline{\mathfrak{X}}' = [V'/G']$. Then set

$$
\mathcal{Y} := \overline{\mathfrak{X}} \times_X \overline{\mathfrak{X}}' = [V \times_X V'/G \times G'].
$$

Since this is a global quotient stack, an application of [dJng96, Thm 7.3] gives a smooth proper stack $\overline{\mathfrak{X}}'' \to \mathcal{Y}$ which is purely inseparable on an open dense substack (we may have to move to a bigger extension field L/K to do this). In fact, we can write $\overline{\mathfrak{X}}''$ as a global

quotient stack $\overline{\mathcal{X}}'' = [W/H]$, and then we have commutative diagrams

$$
\begin{array}{ccc}
\overline{\mathfrak{X}}'' \xrightarrow{f_1'} \overline{\mathfrak{X}} \times BH & \qquad & \overline{\mathfrak{X}}'' \xrightarrow{f_2} \overline{\mathfrak{X}}' \times BH \\
\downarrow \qquad \qquad \downarrow & \qquad & \downarrow \qquad \qquad \downarrow \\
\overline{X}'' \xrightarrow{f_1} \overline{X}, & \qquad & \overline{X}'' \xrightarrow{f_2} \overline{X}'
\end{array}
$$

where f_1' and f_2' are representable, and f_1 and f_2 restrict to the identity on X. Note that replacing $\overline{\mathfrak{X}}$ by $\overline{\mathfrak{X}} \times BH$ does not change the 1-motive $M_c^{2d-1}(X)$ constructed from $\overline{\mathfrak{X}}$ (similarly, replacing $\overline{\mathfrak{X}}'$ by $\overline{\mathfrak{X}}' \times BH$ leaves $M_c^{2d-1}(X)'$ unchanged). Let $M_c^{2d-1}(X)''$ be the 1-motive of Definition 7.1.8 constructed from $\overline{\mathfrak{X}}'' \to \overline{X}''$. Then f_1 and f_2 induce morphisms of 1-motives

$$\hat{f}_1^* : M_c^{2d-1}(X) \to M_c^{2d-1}(X)''$$

and

$$\hat{f}_2^* : M_c^{2d-1}(X)' \to M_c^{2d-1}(X)''$$

which induce the identity on $H_c^{2d-1}(X, \mathbb{Q}_\ell)$ when one applies the functor $V_\ell(-)$. Therefore, $\hat{f}_1^*$ and $\hat{f}_2^*$ are isogenies of 1-motives by Proposition 5.2.2, and

$$f := (\hat{f}_2^*)^{-1} \circ \hat{f}_1^* : M_c^{2d-1}(X) \to M_c^{2d-1}(X)'$$

is an isogeny of 1-motives fitting into the commuting diagram of Proposition 7.4.9. It is clear that this isomorphism is uniquely defined since V_ℓ is a faithful functor.

A priori, the morphism $f : M_c^{2d-1}(X) \to M_c^{2d-1}(X)'$ is only L-linear, not K-linear. To show it is K-linear is equivalent to showing that f is invariant under the action of $\mathrm{Gal}(\overline{k}/K)$. But because f induces the identity map on ℓ-adic realizations, we see that $V_\ell f$ is invariant under the action of $\mathrm{Gal}(\overline{k}/K)$ and since the functor $V_\ell(-)$ is faithful, we conclude that the same is true of f. $\qquad \square$

For the convenience of the reader, we summarize the results of this Chapter:

- Given a separated finite type k-scheme X, there exists a 1-motive $M_c^{2d-1}(X)$ defined over a finite extension $k \hookrightarrow K$.

- There is a natural ℓ-adic realization isomorphism $V_\ell M_c^{2d-1}(X) \cong H_c^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell(d))$, equivariant under the action of $\mathrm{Gal}(\overline{k}/K)$.

- The 1-motive $M_c^{2d-1}(X)$ is unique up to K-linear isogeny in the sense that given two 1-motives $M_c^{2d-1}(X)$, $M_c^{2d-1}(X)'$ defined by two different compactifications/weak resolutions, there exists a canonically defined K-linear isogeny between them.

- The 1-motive $M_c^{2d-1}(-)$ is contravariant for proper morphisms, again after extending the base field.

In the case of an algebraically closed base field, we can state our results as follows

Theorem 7.4.10. *Let k be an algebraically closed field, and let Sch_dProp/k be the category of d-dimensional separated finite type k-schemes, with the morphisms in this category being the proper ones. Then there exists a functor*

$$M_c^{2d-1}(-) : (Sch_dProp/k)^{op} \to \text{1-}Mot_k \otimes \mathbb{Q},$$

unique up to canonical isomorphism, such that we have

$$V_\ell M_c^{2d-1}(X) \cong H_c^{2d-1}(X, \mathbb{Q}_\ell(d))$$

for all $\ell \neq p$.

Chapter 8

Construction of $M^{2d-1}(X)$

8.1 Definition

8.1.1. Fix a d-dimensional separated finite type k-scheme X. In this Chapter we show that after a finite extension $k \hookrightarrow K$, there exists a 1-motive $M^{2d-1}(X)$ defined over K, with a $\mathrm{Gal}(\overline{k}/K)$-equivariant isomorphism $V_\ell M^{2d-1}(X) \cong H^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell(d))$. The argument follows along the same lines as in Chapter 7.

8.1.2. We start with the same setup as in (7.1.5): choose a compactification $X \hookrightarrow \overline{X}$, and a weak resolution $\overline{\pi} : \overline{\mathfrak{X}} \to \overline{X}$ (possibly extending the base field), and commutative diagrams

$$
\begin{array}{ccccc}
\mathfrak{X} & \overset{\alpha'}{\hookrightarrow} & \overline{\mathfrak{X}} & \overset{\beta'}{\hookleftarrow} & \mathcal{C} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} & & \downarrow{\scriptstyle \pi_C} \\
X & \overset{\alpha}{\hookrightarrow} & \overline{X} & \overset{\beta}{\hookleftarrow} & C
\end{array}
$$

and

$$
\begin{array}{ccccc}
\mathcal{U} & \overset{j'}{\hookrightarrow} & \mathfrak{X} & \overset{i'}{\hookleftarrow} & \mathcal{Z} \\
\downarrow{\scriptstyle \pi_U} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_Z} \\
U & \overset{j}{\hookrightarrow} & X & \overset{i}{\hookleftarrow} & Z
\end{array}
$$

where $\mathcal{U} \to U$ is a $\mathbb{Q}_\ell$-cohomological isomorphism. Let $\overline{Z}$ (resp. $\overline{\mathcal{Z}}$) be the closure of Z in $\overline{X}$ (resp. of $\mathcal{Z}$ in $\overline{\mathfrak{X}}$).

8.1.3. Consider the relative Picard group of the pair $(\overline{\mathfrak{X}}, \mathcal{C})$, defined by the formula

$$
\mathrm{Pic}(\overline{\mathfrak{X}}, \mathcal{C}) = H^1(\overline{\mathfrak{X}}, \mathrm{Ker}(\mathbb{G}_{m,\overline{\mathfrak{X}}} \to \beta'_* \mathbb{G}_{m,\mathcal{C}})).
$$

The elements of $\mathrm{Pic}(\overline{\mathfrak{X}}, \mathcal{C})$ correspond to pairs $(\mathscr{L}, \varphi)$, where $\mathscr{L}$ is a line bundle on $\overline{\mathfrak{X}}$ and $\varphi : \mathcal{O}_\mathcal{C} \overset{\sim}{\longrightarrow} \mathscr{L}|_\mathcal{C}$ is an isomorphism. By Proposition 6.6.4, the associated group scheme $\mathbf{Pic}_{\overline{\mathfrak{X}}, \mathcal{C}}$ is representable, and we have an exact sequence

$$
0 \to \mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}}, \mathcal{C}} \to \mathbf{Pic}^{red}_{\overline{\mathfrak{X}}, \mathcal{C}} \to \mathbf{NS}_{\overline{\mathfrak{X}}, \mathcal{C}} \to 0
$$

where $\mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}},\mathcal{C}}$ is a semi-abelian variety and $\mathbf{NS}_{\overline{\mathfrak{X}},\mathcal{C}}$ is a finitely generated étale-locally constant group scheme.

8.1.4. Now consider the étale group scheme $\mathbf{Div}_{\mathcal{Z}}(\overline{\mathfrak{X}})$ of divisors on $\overline{\mathfrak{X}}$ supported on $\mathcal{Z}$. This is not the same as $\mathbf{Div}_{\overline{\mathcal{Z}}}(\overline{\mathfrak{X}})$; the $\overline{k}$-points are the free abelian group on the proper components of $\mathcal{Z}_{\overline{k}}$. Since $\mathcal{Z}$ is disjoint from $\mathcal{C}$, there is a cycle class map

$$cl : \mathbf{Div}_{\mathcal{Z}}(\overline{\mathfrak{X}}) \to \mathbf{Pic}^{red}_{\overline{\mathfrak{X}},\mathcal{C}},$$

sending D to $(\mathcal{O}(D), s|_{\mathcal{C}} : \mathcal{O}_C \to \mathcal{O}(D)|_{\mathcal{C}})$ where $s : \mathcal{O}_X \to \mathcal{O}(D)$ is the meromorphic section associated to D. We define $\overline{cl}$ to be the composition

$$\overline{cl} : \mathbf{Div}_{\mathcal{Z}}(\overline{\mathfrak{X}}) \xrightarrow{cl} \mathbf{Pic}^{red}_{\overline{\mathfrak{X}},\mathcal{C}} \to \mathbf{NS}_{\overline{\mathfrak{X}},\mathcal{C}}$$

and define

$$\pi_* : \mathbf{Div}_{\mathcal{Z}}(\overline{\mathfrak{X}}) \to \mathbf{Div}_Z(\overline{X})$$

to be the proper pushforward map on divisors. We then define

$$\mathbf{Div}^0_{\mathcal{Z}/Z}(\overline{\mathfrak{X}}) := \mathrm{Ker}(\overline{cl} \oplus \pi_* : \mathbf{Div}_{\mathcal{Z}}(\overline{\mathfrak{X}}) \to \mathbf{NS}_{\overline{\mathfrak{X}},\mathcal{C}} \oplus \mathbf{Div}_Z(\overline{X})).$$

Alternatively, we can describe $\mathbf{Div}^0_{\mathcal{Z}/Z}(\overline{\mathfrak{X}})$ as the étale group scheme of divisors D supported on $\mathcal{Z}$ such that

1. $cl(D) = 0$ in $NS(\overline{\mathfrak{X}}, \mathcal{C})$, and

2. $\overline{\pi}_*(D) = 0$ under the proper pushforward map $\overline{\pi}_* : \mathrm{Div}_{\mathcal{Z}}(\overline{\mathfrak{X}}) \to \mathrm{Div}_Z(\overline{X})$.

The above cycle class map cl restricts to a map

$$cl^0 : \mathbf{Div}^0_{\mathcal{Z}/Z}(\overline{\mathfrak{X}}) \to \mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}},\mathcal{C}}.$$

Definition 8.1.5. Let X be a separated scheme of finite type, and choose a compactification $X \hookrightarrow \overline{X}$ and weak resolution $\overline{\pi} : \overline{\mathfrak{X}} \to \overline{X}$ as above. Then we define

$$M^{2d-1}(X) := [\mathbf{Div}^0_{\mathcal{Z}/Z}(\overline{\mathfrak{X}}) \to \mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}},\mathcal{C}}]^\vee,$$

where the superscript $^\vee$ indicates taking the Cartier dual of a 1-motive.

We must show that $M^{2d-1}(X)$ is functorial and independent of choice of compactification. First we discuss the ℓ-adic realization of $M^{2d-1}(X)$.

8.2 ℓ-adic realization

Continuing with the notation of (8.1.2), we have a commuting diagram in $D_c^b(X, \mathbb{Q}_\ell)$ with exact rows and columns

$$
\begin{array}{ccccccc}
j_! \mathbb{Q}_{\ell,U} & \xrightarrow{\sim} & R\pi_* j_!' \mathbb{Q}_{\ell,\mathcal{U}} & \longrightarrow & 0 & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{Q}_{\ell,X} & \longrightarrow & R\pi_* \mathbb{Q}_{\ell,\mathfrak{X}} & \longrightarrow & A & \longrightarrow & \\
\downarrow & & \downarrow & & \sim\downarrow & & \\
i_* \mathbb{Q}_{\ell,Z} & \longrightarrow & R\pi_* i_*' \mathbb{Q}_{\ell,\mathcal{Z}} & \longrightarrow & i_* i^* A & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & &
\end{array}
$$

where A is defined to be $\mathrm{cone}(\mathbb{Q}_{\ell,X} \to R\pi_* \mathbb{Q}_{\ell,\mathfrak{X}})$. Taking global sections of the two left-hand vertical columns, we get a diagram with exact rows

$$
\begin{array}{ccccccccc}
H^{2d-2}(X, \mathbb{Q}_\ell) & \longrightarrow & H^{2d-2}(Z, \mathbb{Q}_\ell) & \longrightarrow & H^{2d-1}(X, j_!\mathbb{Q}_\ell) & \longrightarrow & H^{2d-1}(X, \mathbb{Q}_\ell) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow\sim & & \downarrow & & \\
H^{2d-2}(\mathfrak{X}, \mathbb{Q}_\ell) & \longrightarrow & H^{2d-2}(\mathcal{Z}, \mathbb{Q}_\ell) & \longrightarrow & H^{2d-1}(\mathfrak{X}, j_!'\mathbb{Q}_\ell) & \longrightarrow & H^{2d-1}(\mathfrak{X}, \mathbb{Q}_\ell) & \longrightarrow & 0.
\end{array}
$$
$$(8.2.0.1)$$

We will apply Poincaré duality to the terms in this diagram to interpret them in terms of divisors and cycle maps. We start with some preliminary lemmas:

Lemma 8.2.1. *Poincaré duality induces an isomorphism*

$$H^{2d-2}(Z, \mathbb{Q}_\ell(d-1))^\vee \xrightarrow{\sim} \mathrm{Div}_Z(\overline{X}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell,$$

where $\mathrm{Div}_Z(\overline{X})$ is (as usual) the group of Weil divisors on $\overline{X}$ supported on Z (note that Z is not closed in $\overline{X}$). Similarly, we have

$$H^{2d-2}(\mathcal{Z}, \mathbb{Q}_\ell(d-1))^\vee \cong \mathrm{Div}_{\mathcal{Z}}(\overline{\mathfrak{X}}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell.$$

Finally, let

$$(\pi^*)^\vee : H^{2d-2}(\mathcal{Z}, \mathbb{Q}_\ell(d-1))^\vee \to H^{2d-2}(Z, \mathbb{Q}_\ell(d-1))^\vee$$

be the map induced by applying Poincaré duality to the map $\pi^* : H^{2d-2}(Z, \mathbb{Q}_\ell) \to H^{2d-2}(\mathcal{Z}, \mathbb{Q}_\ell)$. Then under the above isomorphisms, $(\pi^*)^\vee$ corresponds to the proper pushforward map on Weil divisors

$$\pi_* : \mathrm{Div}_{\mathcal{Z}}(\overline{\mathfrak{X}}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to \mathrm{Div}_Z(\overline{X}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell.$$

Proof. We prove the statement for Z; the proof for $\mathcal{Z}$ is the same after passing to the coarse moduli space of $\mathcal{Z}$. Choose a weak resolution $Z' \to Z$ in the sense of Definition 7.1.2; this

induces an isomorphism $H^{2d-2}(Z, \mathbb{Q}_\ell(d-1)) \cong H^{2d-2}(Z', \mathbb{Q}_\ell(d-1))$. By Poincaré duality applied on the smooth stack Z', we have that $H^{2d-2}(Z', \mathbb{Q}_\ell(d-1))^\vee$ is free on the proper $(d-1)$-dimensional connected components of Z'. This can be identified with the set of $(d-1)$-dimensional proper irreducible components of Z; hence $H^{2d-2}(Z, \mathbb{Q}_\ell(d-1))^\vee \cong \mathrm{Div}_Z(\overline{X})$ as was to be shown. The fact that $(\pi^*)^\vee$ corresponds to proper pushforward of divisors is then reduced to the case when Z and $\mathcal{Z}$ are smooth, where it is standard. $\square$

Next we give a concrete description of the Poincaré dual of the map

$$H^{2d-2}(\mathfrak{X}, \mathbb{Q}_\ell) \to H^{2d-2}(\mathcal{Z}, \mathbb{Q}_\ell)$$

induced by the inclusion $\mathcal{Z} \hookrightarrow \mathfrak{X}$. By the above lemma, this corresponds to a map

$$g : \mathrm{Div}_{\mathcal{Z}}(\overline{\mathfrak{X}}) \otimes \mathbb{Q}_\ell \longrightarrow H_c^2(\mathfrak{X}, \mathbb{Q}_\ell(1)).$$

Lemma 8.2.2. *The map g above factors as*

$$\mathrm{Div}_{\mathcal{Z}}(\overline{\mathfrak{X}}) \otimes \mathbb{Q}_\ell \to NS(\overline{\mathfrak{X}}, \mathcal{C}) \otimes \mathbb{Q}_\ell \hookrightarrow H_c^2(\mathfrak{X}, \mathbb{Q}_\ell(1)),$$

where the map on the right is an injection and the map $\mathrm{Div}_{\mathcal{Z}}(\overline{\mathfrak{X}}) \to NS(\overline{\mathfrak{X}}, \mathcal{C})$ is the class map defined earlier, sending a divisor $[D]$ to the class of $(\mathcal{O}(D), s : \mathcal{O}_X \to \mathcal{O}(D))$, where s is the canonical meromorphic section of D (restricted to $\mathcal{C}$, which is disjoint from the support of D).

Proof. Another way of describing g is as H_c^2 of the map on complexes

$$i_* R i'^! \mathbb{Q}_{\ell, \mathfrak{X}}(1) \to \mathbb{Q}_{\ell, \mathfrak{X}}(1).$$

Therefore g is precisely the cycle class map 4.2.0.1 for compactly supported cohomology. The lemma then follows from Proposition 4.2.3. $\square$

8.2.3. At this point, we apply Poincaré duality to the diagram 8.2.0.1. Taking into account the previous two lemmas, we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{2d-1}(X, \mathbb{Q}_\ell(d\text{-}1))^\vee & \longrightarrow & H^{2d-1}(X, j_! \mathbb{Q}_\ell(d\text{-}1))^\vee & \longrightarrow & \mathrm{Div}_Z(\overline{X}) \otimes \mathbb{Q}_\ell & \longrightarrow & H^{2d-2}(X, \mathbb{Q}_\ell(d\text{-}1))^\vee \\
& & \uparrow & & \uparrow{\scriptstyle\sim} & & \uparrow & & \uparrow \\
0 & \longrightarrow & H_c^1(\mathfrak{X}, \mathbb{Q}_\ell(1)) & \longrightarrow & H^{2d-1}(\mathfrak{X}, j'_! \mathbb{Q}_\ell(d\text{-}1))^\vee & \longrightarrow & \mathrm{Div}_{\mathcal{Z}}(\overline{\mathfrak{X}}) \otimes \mathbb{Q}_\ell & \longrightarrow & NS(\overline{\mathfrak{X}}, \mathcal{C}) \otimes \mathbb{Q}_\ell.
\end{array}$$
$$(8.2.3.1)$$

We give a concrete interpretation of the group $H^{2d-1}(\mathfrak{X}, j'_! \mathbb{Q}_\ell(d\text{-}1))^\vee$. By Poincaré duality, this group is isomorphic to $H_c^1(\mathfrak{X}, R j'_* \mathbb{Q}_\ell(1))$. We will show the following:
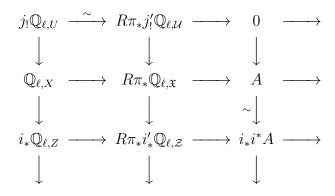
Proposition 8.2.4. *Let M be the 1-motive*

$$M := [\mathbf{Div}_{\mathcal{Z}}^0(\overline{\mathfrak{X}}) \to \mathbf{Pic}^{0,red}(\overline{\mathfrak{X}}, \mathcal{C})].$$

Then there is a canonical isomorphism $V_\ell M \xrightarrow{\sim} H_c^1(\mathfrak{X}, R j'_* \mathbb{Q}_\ell(1))$.

Proof. The argument that follows is very similar to [BVS01, Sect. 2.5]. The main point is to define a map $V_\ell M \to H^1_c(\mathfrak{X}, Rj'_*\mathbb{Q}_\ell(1))$. It will then be easy (using the five lemma) to show that the map is an isomorphism. For any n prime to p, we define a map

$$\varphi_n : T_{\mathbb{Z}/n}(M) \longrightarrow H^1_c(\mathfrak{X}, Rj'_*\mu_n).$$

Recall that $T_{\mathbb{Z}/n}(M)$ is defined as

$$T_{\mathbb{Z}/n}(M) = \frac{\{(\mathscr{L}, a, D) \in \mathrm{Pic}^0(\overline{\mathfrak{X}}, \mathcal{C}) \times \mathrm{Div}^0_{\mathcal{Z}}(\overline{\mathfrak{X}}) | (\mathscr{L}^{\otimes n}, a^{\otimes n}) \cong (\mathcal{O}(-D), s)\}}{\{(\mathcal{O}(D), s, -nD) | D \in \mathrm{Div}^0_{\mathcal{Z}}(\overline{\mathfrak{X}})\}}$$

where

1. $\mathscr{L}$ is a line bundle on $\overline{\mathfrak{X}}$,

2. $a : \mathcal{O}_\mathcal{C} \xrightarrow{\sim} \mathscr{L}|_\mathcal{C}$ is a trivialization of $\mathscr{L}$ on $\mathcal{C}$, and

3. $D \in \mathrm{Div}^0_{\mathcal{Z}}(\overline{\mathfrak{X}})$ is such that the class of $-D$ in $\mathrm{Pic}^0(\overline{\mathfrak{X}}, \mathcal{C})$ is the same as the class of $(\mathscr{L}^{\otimes n}, a^{\otimes n})$.

Suppose given $(\mathscr{L}, a, D) \in T_{\mathbb{Z}/n}(M)$, and let D_{red} be the support of D viewed as a closed subscheme of $\mathfrak{X}$ (it is also closed in $\overline{\mathfrak{X}}$). Note that D_{red} is disjoint from $\mathcal{C}$; let $\tilde{\mathcal{U}} = \mathfrak{X} - D_{red}$ and $\mathcal{U}' = \overline{\mathfrak{X}} - D_{red}$. We then have the following diagram of inclusions:

where along every row and column, the term in the middle is the union of the terms on the ends, and each square is cartesian.

Consider the cohomology group $H^1(\mathcal{U}', \alpha_! \mu_n)$, which by general nonsense [StProj, Tag 03AJ] is in bijection with $\alpha_! \mu_n$-torsors on $\mathcal{U}'$. Given the class $(\mathscr{L}, a, D) \in T_{\mathbb{Z}/n}(M)$ as above, choose an isomorphism

$$\eta : \mathcal{O}(-D) \xrightarrow{\sim} \mathscr{L}^{\otimes n}$$

such that $\eta|_\mathcal{C} : \mathcal{O}(-D)|_\mathcal{C} \cong \mathcal{O}_\mathcal{C} \to \mathscr{L}^{\otimes n}|_\mathcal{C}$ agrees with section $a^{\otimes n} : \mathcal{O}_\mathcal{C} \xrightarrow{\sim} \mathscr{L}^{\otimes n}|_\mathcal{C}$. Such an isomorphism η exists by bullet point (3) above. Notice that η restricts to an isomorphism on $\mathcal{U}'$. Therefore we can define a class

$$\psi_n(\mathscr{L}, a, D) \in H^1(\mathcal{U}', \alpha_! \mu_n)$$

to be the $\alpha_! \mu_n$-torsor of local isomorphisms $\mathcal{O}'_{\mathcal{U}} \xrightarrow{\sim} \mathscr{L}$ which are compatible with η on nth tensor powers and reduce to a on D. By the same argument as in 6.2.7, $\psi_n(\mathscr{L}, a, D)$ does not depend on the choice of η.

Next notice that because $\mathcal{C}$ and D_{red} are disjoint, we have an isomorphism in $D^b_c(\overline{\mathfrak{X}})$

$$Ru'_* \alpha_! \cong \alpha'_! Ru_*,$$

so we have a sequence of maps

$$H^1(\mathcal{U}', \alpha_! \mu_n) \xrightarrow{\sim} H^1_c(\mathfrak{X}, Ru_* \mu_n) \longrightarrow H^1_c(\mathfrak{X}, Ru_* Rg_* \mu_n) = H^1_c(\mathfrak{X}, Rj'_* \mu_n),$$

where we recall that $j' = u \circ g : \mathcal{U} \hookrightarrow \mathfrak{X}$. We let $\varphi_n(\mathscr{L}, a, d) \in H^1_c(\mathfrak{X}, j'_! \mu_n)$ be the image of $\psi_n(\mathscr{L}, a, D)$ under this sequence. Taking the limit over $n = \ell^m$, we get an element

$$\varphi(\mathscr{L}, a, D) \in H^1_c(\mathfrak{X}, \mathbb{Q}_\ell(1)).$$

It is not hard to show that the map φ fits into a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V_\ell \mathrm{Pic}^0(\overline{\mathfrak{X}}, \mathcal{C}) & \longrightarrow & V_\ell M & \longrightarrow & \mathrm{Div}^0_{\mathcal{Z}}(\overline{\mathfrak{X}}) \otimes \mathbb{Q}_\ell & \longrightarrow & 0 \\
& & \downarrow & & {\scriptstyle\varphi}\downarrow & & \| & & \\
0 & \longrightarrow & H^1_c(\mathfrak{X}, \mathbb{Q}_\ell(1)) & \longrightarrow & H^1_c(\mathfrak{X}, Rj'_* \mathbb{Q}_\ell(1)) & \longrightarrow & \mathrm{Div}^0_{\mathcal{Z}}(\overline{\mathfrak{X}}) \otimes \mathbb{Q}_\ell & \longrightarrow & 0,
\end{array}
$$

where the lower row is given by the lower row of 8.2.3.1. Since the left-hand and right-hand vertical arrows are isomorphisms, the five lemma implies that φ is an isomorphism. $\qquad\square$

Applying Proposition 8.2.4 to diagram 8.2.3.1, we get an exact sequence

$$0 \longrightarrow H^{2d-1}(X, \mathbb{Q}_\ell(1))^\vee \longrightarrow V_\ell M \longrightarrow \mathrm{Div}_Z(\overline{X}) \otimes \mathbb{Q}_\ell,$$

where $M = [\mathrm{Div}^0_{\mathcal{Z}}(\overline{\mathfrak{X}}) \to \mathbf{Pic}^0(\overline{\mathfrak{X}}, \mathcal{C})]$.

We leave it to the reader to check that the map $V_\ell M \to \mathrm{Div}_Z(\overline{X}) \otimes \mathbb{Q}_\ell$ is the obvious one, defined by the projection $M \to \mathrm{Div}^0_{\mathcal{Z}}(\overline{\mathfrak{X}})$ followed by the proper pushforward $\mathrm{Div}_{\mathcal{Z}}(\overline{\mathfrak{X}}) \to \mathrm{Div}_Z(\overline{X})$. From this it is clear that we have an isomorphism $V_\ell M^{2d-1}(X)^\vee \cong H^{2d-1}(X, \mathbb{Q}_\ell(1))^\vee$. Dualizing this statement, we have shown the following:

Proposition 8.2.5. *Let $M^{2d-1}(X)$ be defined as above. Then for every $\ell \neq p$, there is a canonical isomorphism*
$$V_\ell M^{2d-1}(X) \xrightarrow{\sim} H^{2d-1}(X, \mathbb{Q}_\ell(d)).$$
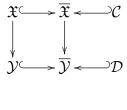
8.3   Functoriality

8.3.1. We can now define the functoriality of $M^{2d-1}(X)$ as follows. Let $f : X \to Y$ be a morphism between d-dimensional separated finite type k-schemes, and choose a compactified
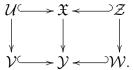
morphism $\overline{f} : \overline{X} \to \overline{Y}$. As in 7.4.1 and 7.4.2, over some finite extension of k we can choose a commutative diagram

$$
\begin{array}{ccc}
\overline{\mathfrak{X}} & \xrightarrow{\ \overline{f}'\ } & \overline{\mathcal{Y}} \\
\pi \downarrow & & \sigma \downarrow \\
\overline{X} & \xrightarrow{\ \overline{f}\ } & \overline{Y}
\end{array}
$$

where $\pi : \overline{\mathfrak{X}} \to \overline{X}$ and $\sigma : \overline{\mathcal{Y}} \to \overline{Y}$ are weak resolutions and $\overline{f}'$ is representable. Let $\mathfrak{X} = X \times_{\overline{X}} \overline{\mathfrak{X}}$ and $\mathcal{Y} := Y \times_{\overline{Y}} \overline{\mathcal{Y}}$. Then we can arrange that $\mathcal{C} := \overline{\mathfrak{X}} - \mathfrak{X}$ and $\mathcal{D} := \overline{\mathcal{Y}} - \mathcal{Y}$ are reduced strict normal crossings divisors; we have $f^{-1}(\mathcal{D})_{red} \subseteq \mathcal{C}$. Let $V \subset Y$ be an open subset of V such that $\mathcal{V} := \sigma^{-1}(V) \to V$ is a $\mathbb{Q}_\ell$-cohomological resolution. Moreover, by shrinking V we can arrange that $\mathfrak{X} \times_{\mathcal{Y}} \mathcal{V} \to X \times_Y V$ is a $\mathbb{Q}_\ell$-cohomological resolution. Set $\mathcal{U} = \mathfrak{X} - \mathfrak{X} \times_{\mathcal{Y}} \mathcal{V}$ and $U = X - X \times_Y V$. Finally, let $Z = X - U$, $\mathcal{Z} = \mathfrak{X} - \mathcal{U}$, $W = Y - V$, $\mathcal{W} = \mathcal{Y} - \mathcal{V}$. We get commuting diagrams

$$
\begin{array}{ccccc}
\mathfrak{X} & \hookrightarrow & \overline{\mathfrak{X}} & \hookleftarrow & \mathcal{C} \\
\downarrow & & \downarrow & & \\
\mathcal{Y} & \hookrightarrow & \overline{\mathcal{Y}} & \hookleftarrow & \mathcal{D}
\end{array}
$$

and

$$
\begin{array}{ccccc}
\mathcal{U} & \hookrightarrow & \mathfrak{X} & \hookleftarrow & \mathcal{Z} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{V} & \hookrightarrow & \mathcal{Y} & \hookleftarrow & \mathcal{W}.
\end{array}
$$

With this notation, to define a map $\hat{f}^* : M^{2d-1}(Y) \to M^{2d-1}(X)$, we want to define a map

$$
\hat{f}_* : [\mathbf{Div}^0_{\mathcal{Z}/Z}(\overline{\mathfrak{X}}) \to \mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}},\mathcal{C}}] \longrightarrow [\mathbf{Div}^0_{\mathcal{W}/W}(\overline{\mathcal{Y}}) \to \mathbf{Pic}^{0,red}_{\overline{\mathcal{Y}},\mathcal{D}}]. \tag{8.3.1.1}
$$

By the same reasoning is in 7.4.3, we can reduce to the case that $\overline{\mathfrak{X}}$ and $\overline{\mathcal{Y}}$ are connected (hence irreducible). In the case that $\dim(\overline{f}'(\overline{\mathfrak{X}})) < d$, we define $\hat{f}_* = 0$. Otherwise, since $\overline{f}'$ is proper it must be surjective. Then we can apply Proposition 7.3.1 with $X = \overline{\mathfrak{X}}$, $\partial X = C$, $X' = \overline{\mathcal{Y}}$ and $\partial X' = \mathcal{D}$ to obtain a commutative diagram

$$
\begin{array}{ccc}
\mathbf{Div}^0_{\mathcal{Z}/Z}(\overline{\mathfrak{X}}) & \longrightarrow & \mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}},\mathcal{C}} \\
\downarrow & & \downarrow \\
\mathbf{Div}^0_{\mathcal{W}/W}(\overline{\mathcal{Y}}) & \longrightarrow & \mathbf{Pic}^{0,red}_{\overline{\mathcal{Y}},\mathcal{D}},
\end{array}
$$

defining the required map $\hat{f}_*$. The dual of $\hat{f}_*$ is our desired map $\hat{f}^* : M^{2d-1}(Y) \to M^{2d-1}(X)$.

8.3.2. Our next goal is to show that this pullback map $\hat{f}^* : M^{2d-1}(Y) \to M^{2d-1}(X)$ is compatible with ℓ-adic realizations; i.e., we claim the following:

Proposition 8.3.3. *In the notation of 8.3.1, we have a commutative diagram*

$$
\begin{array}{ccc}
V_\ell M^{2d-1}(Y) & \xrightarrow{\alpha_Y} & H^{2d-1}(Y, \mathbb{Q}_\ell(d)) \\
\Big\downarrow{\scriptstyle V_\ell \hat{f}^*} & & \Big\downarrow{\scriptstyle f^*} \\
V_\ell M^{2d-1}(X) & \xrightarrow{\alpha_X} & H^{2d-1}(X, \mathbb{Q}_\ell(d)),
\end{array}
$$

where α_Y and α_X are the comparison isomorphisms of 8.2.5.

Proof. Consider the 1-motives

$$
M := [\mathbf{Div}^0_{\mathcal{Z}}(\overline{\mathfrak{X}}) \to \mathbf{Pic}^{0,red}_{\overline{\mathfrak{X}},\mathcal{C}}] \quad \text{and}
$$
$$
N := [\mathbf{Div}^0_{\mathcal{W}}(\overline{\mathcal{Y}}) \to \mathbf{Pic}^{0,red}_{\overline{\mathcal{Y}},\mathcal{D}}].
$$

It is clear that the map $\hat{f}_*$ of 8.3.1.1 extends to a map of 1-motives $\hat{f}_* : M \to N$ defined by the same method. Moreover, in the notation of 8.2.3.1, we have $V_\ell M \cong H^1_c(\mathfrak{X}, Rj'_* \mathbb{Q}_\ell(1))$ and $V_\ell N \cong H^1_c(\mathcal{Y}, Rk'_* \mathbb{Q}_\ell(1))$, where $j' : \mathcal{U} \hookrightarrow \mathfrak{X}$ and $k' : \mathcal{V} \hookrightarrow \mathcal{Y}$ are the inclusions. Since $H^{2d-1}(X, \mathbb{Q}_\ell(d-1))^\vee$ injects into $H^1_c(\mathfrak{X}, Rj'_* \mathbb{Q}_\ell(1))$ and $H^{2d-1}(Y, \mathbb{Q}_\ell(d-1))^\vee$ injects into $H^1_c(\mathcal{Y}, Rk'_* \mathbb{Q}_\ell(1))$, it suffices to show that $V_\ell \hat{f}_* : V_\ell M \to V_\ell N$ is compatible with the proper pushforward $f_* : H^1_c(\mathfrak{X}, \mathbb{Q}_\ell(1)) \to H^1_c(\mathcal{Y}, \mathbb{Q}_\ell(1))$. Note further that f_* and $V_\ell \hat{f}_*$ both induce morphisms of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_c(\mathfrak{X}, \mathbb{Q}_\ell(1)) & \longrightarrow & H^1_c(\mathfrak{X}, Rj'_* \mathbb{Q}_\ell(1)) & \longrightarrow & \mathrm{Div}^0_{\mathcal{Z}}(\overline{\mathfrak{X}}) \otimes \mathbb{Q}_\ell & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle f_*, V_\ell \hat{f}_*} & & \Big\downarrow{\scriptstyle f_*, V_\ell \hat{f}_*} & & \Big\downarrow{\scriptstyle f_*, V_\ell \hat{f}_*} & & \\
0 & \longrightarrow & H^1_c(\mathcal{Y}, \mathbb{Q}_\ell(1)) & \longrightarrow & H^1_c(\mathcal{Y}, Rk'_* \mathbb{Q}_\ell(1)) & \longrightarrow & \mathrm{Div}^0_{\mathcal{W}}(\overline{\mathcal{Y}}) \otimes \mathbb{Q}_\ell & \longrightarrow & 0.
\end{array}
$$

Therefore it suffices to show that the maps induced by f_* and $V_\ell \hat{f}_*$ agree on $H^1_c(\mathfrak{X}, \mathbb{Q}_\ell(1))$ and on $\mathrm{Div}^0_{\mathcal{Z}}(\overline{\mathfrak{X}}) \otimes \mathbb{Q}_\ell$. Since f_* and $V_\ell \hat{f}_*$ are both defined by proper pushforward on $\mathrm{Div}^0_{\mathcal{Z}}(\overline{\mathfrak{X}}) \otimes \mathbb{Q}_\ell$, it is clear that the action of $V_\ell \hat{f}_*$ and f_* on this group agree. Therefore we are left with showing that f_* and $V_\ell \hat{f}_*$ induce the same map on $H^1_c(\mathfrak{X}, \mathbb{Q}_\ell(1))$. We state this as the following lemma, which completes the proof of 8.3.3. $\qquad\square$

Lemma 8.3.4. *In the notation above, we have a commutative diagram*

$$
\begin{array}{ccc}
V_\ell \mathrm{Pic}^0(\overline{\mathfrak{X}}, \mathcal{C}) & \xrightarrow{\sim} & H^1_c(\mathfrak{X}, \mathbb{Q}_\ell(1)) \\
\Big\downarrow{\scriptstyle V_\ell \hat{f}_*} & & \Big\downarrow{\scriptstyle f_*} \\
V_\ell \mathrm{Pic}^0(\overline{\mathcal{Y}}, \mathcal{D}) & \xrightarrow{\sim} & H^1_c(\mathcal{Y}, \mathbb{Q}_\ell(1)),
\end{array}
$$

where the horizontal arrows are the canonical comparison isomorphisms.

Proof. Let $B \subset \overline{\mathcal{Y}}$ be an open substack such that $f^{-1}(B) \to B$ is finite flat, and $\overline{\mathcal{Y}} - B$ has codimension 2. Let $A = f^{-1}(B)$, and let $\alpha : A \cap \mathfrak{X} \hookrightarrow A$ and $\beta : B \cap \mathcal{Y} \hookrightarrow B$ be the inclusions. Then the inclusions $A \hookrightarrow \overline{\mathfrak{X}}$ and $B \hookrightarrow \overline{\mathcal{Y}}$ induce a commutative diagram

$$
\begin{array}{ccc}
H_c^1(\overline{\mathfrak{X}}, \mathbb{Q}_\ell(1)) & \longhookrightarrow & H^1(A, \alpha_! \mathbb{Q}_\ell(1)) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
H_c^1(\overline{\mathcal{Y}}, \mathbb{Q}_\ell(1)) & \longhookrightarrow & H^1(B, \beta_! \mathbb{Q}_\ell(1))
\end{array}
$$

where the horizontal arrows are injections. Since $f : A \to B$ is finite flat, f_* is induced by the trace mapping $Tr : f_* f^* \mathbb{Q}_\ell(1) \to \mathbb{Q}_\ell(1)$. By [FK88, p. 136], we have a commutative diagram of sheaves

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & f_* \alpha_! \mu_{\ell^n} & \longrightarrow & f_* \mathbb{G}_{m,A,\mathcal{C} \cap A} & \xrightarrow{\ell^n} & f_* \mathbb{G}_{m,A,\mathcal{C} \cap A} & \longrightarrow & 0 \\
& & {\scriptstyle Tr}\downarrow & & {\scriptstyle N}\downarrow & & {\scriptstyle N}\downarrow & & \\
0 & \longrightarrow & \beta_! \mu_{\ell^n} & \longrightarrow & \mathbb{G}_{m,B,\mathcal{D} \cap B} & \xrightarrow{\ell^n} & \mathbb{G}_{m,B,\mathcal{D} \cap B} & \longrightarrow & 0.
\end{array}
$$

Here $N : f_* \mathbb{G}_{m,A} \to \mathbb{G}_{m,B}$ is the norm mapping. Taking global sections and then inverse limits induces a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & G & \longrightarrow & H^1(A, \alpha_! \mathbb{Q}_\ell(1)) & \longrightarrow & V_\ell \mathrm{Pic}(A, \mathcal{C} \cap A) & \longrightarrow & 0 \\
& & \downarrow & & {\scriptstyle f_*}\downarrow & & {\scriptstyle N}\downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & H^1(B, \beta_! \mathbb{Q}_\ell(1)) & \longrightarrow & V_\ell \mathrm{Pic}(B, \mathcal{D} \cap B) & \longrightarrow & 0,
\end{array}
$$

where

$$
G := \varprojlim_n \frac{\mathrm{Ker}(\mathcal{O}^*(A) \to \mathcal{O}^*(A \cap \mathcal{C}))}{\ell^n \mathrm{Ker}(\mathcal{O}^*(A) \to \mathcal{O}^*(A \cap \mathcal{C}))}
$$

(note that the corresponding group for B is zero since B is of codimension 2 in the smooth proper Deligne-Mumford stack $\overline{\mathcal{Y}}$). In summary, we have a commuting diagram

$$
\begin{array}{ccccc}
H_c^1(\overline{\mathfrak{X}}, \mathbb{Q}_\ell(1)) & \longhookrightarrow & H^1(A, \alpha_! \mathbb{Q}_\ell(1)) & \longrightarrow & V_\ell \mathrm{Pic}(A, \mathcal{C} \cap A) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle N} \\
H_c^1(\overline{\mathcal{Y}}, \mathbb{Q}_\ell(1)) & \longhookrightarrow & H^1(B, \beta_! \mathbb{Q}_\ell(1)) & \xrightarrow{\sim} & V_\ell \mathrm{Pic}(B, \mathcal{D} \cap B)
\end{array}
$$

showing that the cohomological pushforward f_* is compatible with taking norms of line bundles. On the other hand, the pushforward of 1-motives $\hat{f}_* : \mathrm{Pic}^0(\overline{\mathfrak{X}}, \mathcal{C}) \to \mathrm{Pic}^0(\overline{\mathcal{Y}}, \mathcal{D})$ is defined so that there is a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Pic}^0(\overline{\mathfrak{X}}, \mathcal{C}) & \longrightarrow & \mathrm{Pic}(A, \mathcal{C} \cap A) \\
\downarrow & & \downarrow \\
\mathrm{Pic}^0(\overline{\mathcal{Y}}, \mathcal{D}) & \longhookrightarrow & \mathrm{Pic}(B, \mathcal{D} \cap B).
\end{array}
$$

Applying $V_\ell(-)$ to this diagram, and combining with the diagram above, shows that f_* and $V_\ell \hat{f}_*$ are compatible in the sense of the proposition statement. $\qquad\square$

8.3.5. Finally we discuss independence of the choice of compactification and resolution. The main statement is the following:

Proposition 8.3.6. *Let X be a d-dimensional separated finite type k-scheme, and let $\overline{\mathfrak{X}}$, $\overline{\mathfrak{X}}'$ be weak resolutions of compactifications of X, both defined over the same base field K/k. Let $M^{2d-1}(X)$ and $M^{2d-1}(X)'$ be the 1-motives constructed using $\overline{\mathfrak{X}}$ and $\overline{\mathfrak{X}}'$, respectively. Then there exists a unique K-linear isogeny of 1-motives*

$$f : M^{2d-1}(X) \to M^{2d-1}(X)'$$

fitting into a diagram

$$\begin{array}{ccc} V_\ell M^{2d-1}(X) & \xrightarrow{\alpha_X} & H^{2d-1}(X, \mathbb{Q}_\ell(d)) \\ {\scriptstyle V_\ell f}\downarrow & & \downarrow{\scriptstyle =} \\ V_\ell M^{2d-1}(X)' & \xrightarrow{\alpha_X{}'} & H^{2d-1}(X, \mathbb{Q}_\ell(d)) \end{array}$$

for all $\ell \neq p$, where α_X and α'_X are the comparison isomorphisms of Proposition 8.2.5.

Proof. As in the proof of Proposition 7.4.9, over some larger field L/K we can find a third weak resolution $\overline{\mathfrak{X}}''$, together with maps $f_1 : \overline{\mathfrak{X}}'' \to \overline{\mathfrak{X}}$ and $f_2 : \overline{\mathfrak{X}}'' \to \overline{\mathfrak{X}}'$ lying over the identity on $\overline{X}$. Then f_1 and f_2 induce morphisms of 1-motives

$$\hat{f}_1^* : M_c^{2d-1}(X) \to M_c^{2d-1}(X)''$$

and

$$\hat{f}_2^* : M_c^{2d-1}(X)' \to M_c^{2d-1}(X)''$$

which induce the identity on $H_c^{2d-1}(X, \mathbb{Q}_\ell)$ when one applies the functor $V_\ell(-)$. Therefore, $\hat{f}_1^*$ and $\hat{f}_2^*$ are isogenies of 1-motives by Proposition 5.2.2, and

$$f := (\hat{f}_2^*)^{-1} \circ \hat{f}_1^* : M_c^{2d-1}(X) \to M_c^{2d-1}(X)'$$

is an isogeny of 1-motives fitting into the commuting diagram of Proposition 7.4.9. It is clear that this isomorphism is uniquely defined since V_ℓ is a faithful functor.

A priori, the morphism $f : M_c^{2d-1}(X) \to M_c^{2d-1}(X)'$ is only L-linear, not K-linear. To show it is K-linear is equivalent to showing that f is invariant under the action of $\mathrm{Gal}(\overline{k}/K)$. But because f induces the identity map on ℓ-adic realizations, we see that $V_\ell f$ is invariant under the action of $\mathrm{Gal}(\overline{k}/K)$ and since the functor $V_\ell(-)$ is faithful, we conclude that the same is true of f. $\qquad\square$

We can summarize our results in this Chapter as follows:

- Given a separated finite type k-scheme X, there exists a 1-motive $M^{2d-1}(X)$ defined over a finite extension $k \hookrightarrow K$.

- There is a natural ℓ-adic realization isomorphism $V_\ell M_c^{2d-1}(X) \cong H^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell(d))$, equivariant under the action of $\mathrm{Gal}(\overline{k}/K)$.

- The 1-motive $M^{2d-1}(X)$ is unique up to K-linear isogeny in the sense that given two 1-motives $M^{2d-1}(X)$, $M^{2d-1}(X)'$ defined by two different compactifications/weak resolutions, there exists a canonically defined K-linear isogeny between them.

- The 1-motive $M^{2d-1}(-)$ is contravariant, except that given a morphism $f : X \to Y$, the pullback morphism $f_1^* : M^{2d-1}(Y) \to M^{2d-1}(X)$ may only be L-linear for some finite extension L/k.

The results look simpler when one works over an algebraically closed field:

Theorem 8.3.7. *Let k be an algebraically closed field, and Sch_d/k the category of d-dimensional separated finite type k-schemes. Then there exists a functor*

$$M^{2d-1}(-) : (Sch_d/k)^{op} \to \text{1-Mot}_k \otimes \mathbb{Q},$$

unique up to canonical isomorphism, such that we have a natural isomorphism

$$V_\ell M^{2d-1}(X) \cong H^{2d-1}(X, \mathbb{Q}_\ell(d-1))$$

for all $\ell \neq p$.

Chapter 9

Isogeny Descent and Albanese 1-Motives over Finite Fields

9.1 Motivation

9.1.1. Our goal in this section is to complete the proof of Theorem 1.2.17. What we have left to do is to construct functorial Albanese 1-motives $M^{2d-1}(X)$ and $M_c^{2d-1}(X)$ in the case where X is a separated finite type scheme over $k = \mathbb{F}_q$. The main new tool we need is a method for descending motives defined over $\overline{\mathbb{F}}_q$ to $\mathbb{F}_q$. We explain why in Paragraphs 9.1.2-9.1.3 below.

9.1.2. Suppose given a d-dimensional separated finite type k-scheme X, where $k = \mathbb{F}_q$ is a finite field, and fix an algebraic closure $k \hookrightarrow \overline{k}$. Then by Theorems 7.4.10 and 8.3.7 we have isogeny 1-motives $M_{(c)}^{2d-1}(X_{\overline{k}})$ associated to the base change $X_{\overline{k}}$ (in fact, we only need to pass to some finite extension of k). We would like to find 1-motives $M_{(c)}^{2d-1}(X)$ over k with the following two properties:

1. There exists a canonical isogeny $M_{(c)}^{2d-1}(X) \times_k \overline{k} \cong M_{(c)}^{2d-1}(X_{\overline{k}})$, and therefore $V_\ell M_{(c)}^{2d-1}(X)$ is isomorphic to $V_\ell M_{(c)}^{2d-1}(X_{\overline{k}})$ as a group (i.e., forgetting the Galois action on $V_\ell M_{(c)}^{2d-1}(X)$).

2. Under the realization isomorphism

$$\alpha : V_\ell M_{(c)}^{2d-1}(X) = V_\ell M_{(c)}^{2d-1}(X_{\overline{k}}) \xrightarrow{\sim} H_{(c)}^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell(d)),$$

   appearing in Theorems 7.4.10 and 8.3.7, the Galois actions on $V_\ell M_{(c)}^{2d-1}(X)$ and $H_{(c)}^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell(d))$ coincide.

9.1.3. As a first step towards constructing $M_{(c)}^{2d-1}(X)$, we construct an endomorphism $\tilde{F} : M_{(c)}^{2d-1}(X_{\overline{k}}) \to M_{(c)}^{2d-1}(X_{\overline{k}})$ with the property that under the realization isomorphism α above,

$\tilde{F}$ acts on $V_\ell M_{(c)}^{2d-1}(X_{\overline{k}})$ in the same way as the geometric Frobenius element $\sigma \in \mathrm{Gal}(\overline{k}/k)$ acts on $H_{(c)}^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell(d))$. Here, by the *geometric* Frobenius element $\sigma \in \mathrm{Gal}(\overline{k}/k)$, we mean the *inverse* of the automorphism $x \mapsto x^q$ of $\overline{k}$.

We construct $\tilde{F}$ as follows: begin by defining $F : X \to X$ to be the k-linear absolute Frobenius endomorphism of X. The base change $F_{\overline{k}} : X_{\overline{k}} \to X_{\overline{k}}$ is a $\overline{k}$-linear endomorphism of $X_{\overline{k}}$ which induces the action of $\sigma \in \mathrm{Gal}(\overline{k}/k)$ on $H_{(c)}^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell)$ (which is a Tate twist of the action on $H_{(c)}^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell(d))$). By the functoriality of the 1-motive $M_{(c)}^{2d-1}(X_{\overline{k}})$, we get an induced isogeny

$$\overline{F}_{\overline{k}} : M_{(c)}^{2d-1}(X_{\overline{k}}) \to M_{(c)}^{2d-1}(X_{\overline{k}})$$

which induces the action of σ on $H_{(c)}^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell)$. To obtain an endomorphism inducing the action on $H_{(c)}^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell(d))$, we must "Tate-twist" $\overline{F}_{\overline{k}}$: let $\tilde{F} := \overline{F}_{\overline{k}}/q^d$ (the division by q^d effectively shifts the weights of the Galois action by d, performing the role of a Tate twist). Then $\tilde{F}$ is an isogeny of $M_{(c)}^{2d-1}(X_{\overline{k}})$ such that under the comparison isomorphism

$$\alpha : V_\ell M_{(c)}^{2d-1}(X_{\overline{k}}) \overset{\sim}{\longrightarrow} H_{(c)}^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell(d)),$$

$V_\ell \tilde{F}$ corresponds to the natural action of $\sigma \in \mathrm{Gal}(\overline{k}/k)$ on $H_{(c)}^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell(d))$ as desired.

As noted above, there exist 1-motives $M_{(c)}^{2d-1}(X_K)$ for some finite extension $k \hookrightarrow K$ descending $M_{(c)}^{2d-1}(X_{\overline{k}})$. Our goal is to further descend $M_{(c)}^{2d-1}(X_K)$ to 1-motives defined over k. Let Fr be the K-linear Frobenius endomorphism of $M_{(c)}^{2d-1}(X_K)$. More specifically, if $M = [L \to G]$, then Fr is the morphism of complexes $M \to M$ which is the K-linear Frobenius endomorphism on L and on G. Then by looking at Tate modules, one sees that $\tilde{F}^n = Fr$, where $n = [K : k]$. In other words, $\tilde{F}^n$ is an nth root of the Frobenius endomorphism of M. We would like to find a 1-motive $\tilde{M}$ defined over k such that the k-linear Frobenius endomorphsm of $\tilde{M}$ is given by $\tilde{F}$. We study this problem in a more general context below.

Isogeny descent for 1-motives over $\mathbb{F}_q$

We give two versions of isogeny descent for 1-motives over $\mathbb{F}_q$, one for finite field extensions and one for the extension $\mathbb{F}_q \hookrightarrow \overline{\mathbb{F}}_q$. Write $k = \mathbb{F}_q$, and first consider a degree-n extension $k \hookrightarrow K$. Consider the category $\mathscr{D}_{k \hookrightarrow K} \otimes \mathbb{Q}$ whose objects are pairs (M, g) where M is an isogeny 1-motive over K and $g : M \to M$ is an isogeny such that $g^n = Fr_M$, where Fr_M is the K-linear Frobenius endomorphism of M. There is a natural pullback functor

$$p^* : 1\text{-}\mathrm{Mot}_k \otimes \mathbb{Q} \longrightarrow \mathscr{D}_{k \hookrightarrow K} \otimes \mathbb{Q}$$
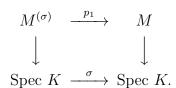
sending a 1-motive N over k to the pair $(N \times_k K, Fr_N \times_k K)$, where $Fr_N : N \to N$ is the k-linear Frobenius endomorphism of N. Our first main theorem is then the following:

Theorem 9.1.4. *(Theorem 9.3.1 in text) The functor p^* defined above is an equivalence of categories. In fact, p^* has a natural quasi-inverse $p_* : \mathscr{D}_{k \hookrightarrow K} \otimes \mathbb{Q} \longrightarrow 1\text{-}Mot_k \otimes \mathbb{Q}$.*

The functor p_* is constructed in Section 9.3 using the Weil restriction functor and a bit of linear algebra. The necessary prerequisites on Weil restriction are reviewed in Section 9.2.

Remark 9.1.5. The theorem above can be thought of as a "geometric" version of Galois descent for the field extension $k \hookrightarrow K$ in the sense that it uses an nth root of the geometric Frobenius endomorphism as descent data rather than directly using an action of $\mathrm{Gal}(K/k)$. Recall that standard descent theory for an extension of finite fields can be stated as follows: let $M = [L \to G]$ be a 1-motive over K (or a group scheme, or abelian sheaf, or...). Here we do not mean a 1-motive up to isogeny, but rather a conventional 1-motive. Let $M^{(\sigma)} = [L^{(\sigma)} \to G^{(\sigma)}]$ be the 1-motive obtained by the fiber product diagram
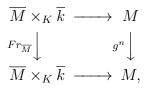
$$
\begin{array}{ccc}
M^{(\sigma)} & \xrightarrow{\ p_1\ } & M \\
\downarrow & & \downarrow \\
\mathrm{Spec}\ K & \xrightarrow{\ \sigma\ } & \mathrm{Spec}\ K.
\end{array}
$$

Suppose given an isomorphism $f : M^\sigma \xrightarrow{\sim} M$ of 1-motives. Then (since $\mathrm{Gal}(K/k)$ is cyclic with generator σ) the pair (M, f) defines a descent datum for the extension $k \hookrightarrow K$, and there exists a unique 1-motive (or group scheme, or abelian sheaf...) $\tilde{M}$ over k that induces the pair (M, f).

It seems a bit difficult to extend the above theory directly to the case where f is an isogeny rather than an isomorphism: for any pair (M, f) arising from a 1-motive over k (i.e., such that $M = \tilde{M} \times_k K$ for some 1-motive $\tilde{M}$ over k), the morphism f is automatically an isomorphism, and it is not clear how to get pairs (M, f) with f an isogeny to be in the essential image of $1\text{-}Mot_k \otimes \mathbb{Q}$ under pullback. This is why we choose to develop our theory of Galois descent as described above. Note that these theories are related as follows: suppose given a pair (M, f) where $f : M^\sigma \to M$ is an isogeny. Let $F_\sigma : M \to M^\sigma$ be the morphism whose composition with the projection $p_1 : M^{(\sigma)} \to M$ in the diagram above yields the k-linear Frobenius $M \to M$. We can then define a K-linear morphism $g = f \circ F_\sigma$, and one easily sees that $g^n = Fr_M$, where Fr_M is the K-linear Frobenius endomorphism of M.

9.1.6. Now we give a version of isogeny descent for the extension $k \hookrightarrow \overline{k}$. First we define what we mean by descent data for a 1-motive over $\overline{k}$:

Definition 9.1.7. Let $M = [L \to G]$ be a 1-motive over $\overline{k}$. A *descent isogeny* $g : M \to M$ relative to k is an isogeny such that there exists a 1-motive $\overline{M}$ over a degree-n extension $k \hookrightarrow K$, such that there is an isogeny $\overline{M} \times_K \overline{k} \xrightarrow{\sim} M$ inducing a commutative diagram

$$
\begin{array}{ccc}
\overline{M} \times_K \overline{k} & \longrightarrow & M \\
{\scriptstyle Fr_{\overline{M}}} \downarrow & & \downarrow {\scriptstyle g^n} \\
\overline{M} \times_K \overline{k} & \longrightarrow & M,
\end{array}
$$

where $Fr_{\overline{M}}$ is the Frobenius endomorphism of $\overline{M}$.

Remark 9.1.8. The choice of a 1-motive over some intermediate field K may seem somewhat strange; it plays the role of enforcing "continuity" of the resulting action of $\mathrm{Gal}(\overline{k}/k)$ on M. More specifically, the isogeny $g : M \to M$ can be seen as given an action on M of the subgroup $\mathbb{Z} \subset \mathrm{Gal}(\overline{k}/k) \cong \hat{\mathbb{Z}}$ generated by σ. But any endomorphism arising via pullback from a 1-motive over k actually extends to an action of the full Galois group $\hat{\mathbb{Z}}$. The extra condition we impose essentially says that the action induced by $g : M \to M$ extends to an action of an open subgroup of $\hat{\mathbb{Z}}$.

With this definition in hand, we can state our second main theorem:

Theorem 9.1.9. *(Theorem 9.3.4 in text) Let $\mathscr{D}_{k \hookrightarrow \overline{k}} \otimes \mathbb{Q}$ be the category of pairs (M, g) where M is an isogeny 1-motive over $\overline{k}$, and $g : M \to M$ is a descent isogeny relative to k. Consider the natural pullback functor*

$$p^* : \textit{1-Mot}_k \otimes \mathbb{Q} \longrightarrow \mathscr{D}_{k \hookrightarrow \overline{k}} \otimes \mathbb{Q}$$

sending a 1-motive N to the pair $(N \times_k \overline{k}, Fr_N \times_k \overline{k})$ where Fr_N is the k-linear Frobenius endomorphism of N. Then p^* is an equivalence of categories. Moreover, p^* has a natural quasi-inverse p_*.

The proof is a straightforward extension of Theorem 9.1.4.

In Section 9.4 we apply Theorem 9.1.9 following the outline indicated above to prove the existence of functorial 1-motives $M^{2d-1}_{(c)}(X)$ attached to a d-dimensional separated finite type k-scheme X, where k is a finite field.

9.2 Weil Restriction of 1-Motives

We start by recalling some facts about Weil restriction of varieties along Galois extensions. A convenient reference is [FrLa06]. Let $k \hookrightarrow K$ be a finite Galois extension, and let X be a separated finite type K-scheme. One way of describing the Weil restriction $W_{K/k}(X)$ is as follows: for each $\tau \in \mathrm{Gal}(K/k)$, let $X^{(\tau)}$ be the variety obtained by applying τ to the equations defining X. More precisely, $X^{(\tau)}$ is defined by the cartesian diagram

$$
\begin{array}{ccc}
X^{(\tau)} & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathrm{Spec}\ k & \overset{\tau}{\longrightarrow} & \mathrm{Spec}\ k.
\end{array}
$$

Then form the product $\prod_{\tau \in \mathrm{Gal}(K/k)} X^{(\tau)}$. The Galois group $\mathrm{Gal}(K/k)$ naturally acts on this product by the following rule: for $g \in \mathrm{Gal}(K/k)$ and $x = (x_\tau)_{\tau \in \mathrm{Gal}(K/k)}$, we set

$$(g.x)_\tau = g(x_{g^{-1}\tau}).$$

Then $W_{K/k}(X)$ is the descent to k of $\prod_{\tau \in \mathrm{Gal}(K/k)} X^{(\tau)}$ under this Galois action. It is a separated finite type k-scheme.

The full Galois group $\mathrm{Gal}(\bar{k}/k)$ acts on the $\bar{k}$-points of $W_{K/k}(X)$ by a similar rule: given $x = (x_\tau) \in \prod_\tau X^\tau(\bar{k})$, and $g \in \mathrm{Gal}(\bar{k}/k)$, let $\bar{g}$ be the image of g in $\mathrm{Gal}(K/k)$. Then we have

$$(g.x)_\tau = g(x_{\bar{g}^{-1}\tau}).$$

9.2.1. We will now specialize to the case where X is a group scheme over K, and $k \hookrightarrow K$ is a degree-n extension where $k = \mathbb{F}_q$. Let $\sigma \in \mathrm{Gal}(\bar{k}/k)$ be the geometric Frobenius element, i.e., the inverse of the automorphism $x \mapsto x^q$. Then the Galois group $\mathrm{Gal}(K/k)$ is of course cyclic, generated by the restriction $\bar{\sigma}$ of σ to K. We write $X^{(i)}$ rather than $X^{(\bar{\sigma}^i)}$, and then an arbitrary element of $W_{K/k}(X)(\bar{k})$ is written as an n-tuple $(x_0, ..., x_{n-1})$, where $x_i \in X^{(i)}$. Then the natural action of σ on $W_{K/k}(X)(\bar{k})$ is given by $\sigma(x_0, ..., x_{n-1}) = (\sigma(x_{n-1}), \sigma(x_0), \sigma(x_1), ..., \sigma(x_{n-2}))$.

9.2.2. It will be useful to have an explicit matrix representation of the action of σ on $V_\ell W_{K/k}(X)$ with respect to a suitable basis. Let $V_0 = (v_{10}, ..., v_{m0})$ be a basis for $V_\ell G$. Then for each i let $\sigma^i|_X : V_\ell X \to V_\ell X^{(i)}$ be the map on Tate modules induced by σ^i. We define

$$V_i := \sigma^i|_X(V_0) = (\sigma^i|_X(v_{10}), ..., \sigma^i|_X(v_{m0})).$$

Then V_i is a basis for $V_\ell X^{(i)}$, and $(V_0, ..., V_{n-1})$ is a basis for $V_\ell W_{K/k}(X)$. We can describe the action of σ on $W_{K/k}(X)$ with respect to this basis as follows: let $M(\sigma^n)$ be the $m \times m$ matrix representation of $\sigma^n|_X : V_\ell X \to V_\ell X^{(n)} = V_\ell X$ with respect to the basis V_0. Then one checks easily that the matrix of the action of σ on $W_{K/k}(X)$ with respect to the basis $(V_0, ..., V_{n-1})$ is given by

$$\begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & M(\sigma^n) \\ I & 0 & 0 & \cdot & 0 & 0 \\ 0 & I & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & I & 0 \end{bmatrix};$$

i.e., a block matrix with $m \times m$ identity matrices below the main diagonal, and $M(\sigma^n)$ in the upper right corner.

9.2.3. Moreover, let $F : X \to X$ be any K-linear endomorphism (or isogeny) of X, and also write F for the induced action on $V_\ell X$. Let $F^{(i)} : X^{(i)} \to X^{(i)}$ be the natural map obtained by conjugation by σ^i. Then $W_{K/k}(F)$ acts on the $\bar{k}$-points of $W_{K/k}(X)$ by the rule $(x_0, ..., x_{n-1}) \mapsto (F(x_0), F^{(1)}(x_1), ..., F^{(n-1)}(x_{n-1}))$. With respect to the basis $(V_0, ..., V_{n-1})$ of $V_\ell W_{K/k}(X)$ described above, one sees that F acts via the block-diagonal matrix

$$\begin{bmatrix} M(F) & 0 & \cdot & \cdot & 0 \\ 0 & M(F) & 0 & \cdot & \cdot \\ \cdot & 0 & M(F) & 0 & \cdot \\ \cdot & \cdot & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & M(F) \end{bmatrix},$$

where $M(F)$ is the matrix representation of $V_\ell F : V_\ell X \to V_\ell X$ with respect to the basis V_0.

9.2.4. Now consider a 1-motive $M = [L \to G]$ over the field K, where as above $k \hookrightarrow K$ is a degree-n extension of finite fields. We define the Weil restriction $W_{K/k}(M)$ to be the 1-motive

$$W_{K/k}(M) = [W_{K/k}(L) \to W_{K/k}(G)].$$

This is indeed a 1-motive since the property of being a 1-motive is étale-local (since the properties of being a lattice and being a semiabelian variety are étale-local), and we have

$$W_{K/k}(M) \times_k K = \prod_{\sigma \in \mathrm{Gal}(K/k)} M^{(\sigma)},$$

where $M^{(\sigma)} = [L^{(\sigma)} \to G^{(\sigma)}]$ is defined as before.

The discussion above on Weil restriction for group schemes holds without change for 1-motives: for each $\sigma^i \in \mathrm{Gal}(\overline{k}/k)$, there is a map $\sigma^i|_M : M \to M^{(i)}$ induced by σ^i. Let $V_0 = \{v_{10}, ..., v_{m0}\}$ be a basis for $V_\ell M$; then for each i, $V_i := \sigma^i|_M(V_0)$ is a basis for $V_\ell M^{(i)}$, and $(V_0, ..., V_{n-1})$ is a basis for $V_\ell W_{K/k}(M)$. With respect to this basis, $\sigma \in \mathrm{Gal}(\overline{k}/k)$ acts by the same matrix as before, namely

$$\begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & M(\sigma^n) \\ I & 0 & 0 & \cdot & 0 & 0 \\ 0 & I & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & I & 0 \end{bmatrix}$$

where $M(\sigma^n)$ is the matrix representation of σ^n acting on $V_\ell M$.

In addition, for any K-linear isogeny of 1-motives $F : M \to M$, $W_{K/k}(F)$ acts on $V_\ell W_{K/k}(M)$ (with respect to this basis) by the matrix

$$\begin{bmatrix} M(F) & 0 & \cdot & \cdot & 0 \\ 0 & M(F) & 0 & \cdot & \cdot \\ \cdot & 0 & M(F) & 0 & \cdot \\ \cdot & \cdot & 0 & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & M(F) \end{bmatrix}.$$

9.3 Isogeny Descent for 1-Motives over Finite Fields

Recall the setup from the beginning of this chapter:

Let $k \hookrightarrow K$ be a degree-n extension where $k = \mathbb{F}_q$. Consider the category $\mathscr{D}_{k \to K} \otimes \mathbb{Q}$ whose objects are pairs (M, g) where M is an isogeny 1-motive over K and $g : M \to M$ is

an isogeny such that $g^n = Fr_M$, where Fr_M is the K-linear Frobenius endomorphism of M. There is a natural functor

$$p^* : \text{1-Mot}_k \otimes \mathbb{Q} \longrightarrow \mathscr{D}_{k \hookrightarrow K} \otimes \mathbb{Q}$$

sending a 1-motive N over k to the pair $(N \times_k K, Fr_N \times_k K)$, where $Fr_N : N \to N$ is the k-linear Frobenius endomorphism. Our main theorem is then the following:

Theorem 9.3.1. *The functor p^* defined above is an equivalence of categories. In fact, p^* has a natural quasi-inverse $p_* : \mathscr{D}_{k \hookrightarrow K} \otimes \mathbb{Q} \longrightarrow \text{1-Mot}_k \otimes \mathbb{Q}$.*

Proof. Suppose given an object (M, g) of $\mathscr{D}_{k \hookrightarrow K} \otimes \mathbb{Q}$. We first define the object $p_*(M, g) \in \text{1-Mot}_k \otimes \mathbb{Q}$. Let $W_{K/k}(M)$ be the Weil restriction of M to k. Let $\pi : W_{K/k}(M) \to W_{K/k}(M)$ be the k-linear Frobenius endomorphism of $W_{K/k}(M)$, and let $W_{K/k}(g) : W_{K/k}(M) \to W_{K/k}(M)$ be the Weil restriction of g. We define

$$p_* M := \text{Ker}(\pi - W_{K/k}(g)),$$

where the kernel is taken in the category of isogeny 1-motives over k (an abelian category; see the Appendix).

We wish to show that there are natural isomorphisms $p^* p_*(M, g) \xrightarrow{\sim} (M, g)$ for $(M, g) \in \mathscr{D}_{k \hookrightarrow K} \otimes \mathbb{Q}$, and $N \xrightarrow{\sim} p_* p^* N$ for $N \in \text{1-Mot}_k \otimes \mathbb{Q}$. We start by giving a natural isomorphism $N \xrightarrow{\sim} p_* p^* N$, as this is easier. Unwinding the definitions, we see that $p_* p^* N$ is given as follows: consider the Weil restriction $W_{K/k}(N_K)$, where $N_K = N \times_k K$. Because $N_K^{(i)} \cong N_K$ for each i, we can write $W_{K/k}(N_K) \times_k K = \prod_{i=0}^{n-1} N_K$. The k-linear Frobenius $\pi : W_{K/k}(N_K) \to W_{K/k}(N_K)$ acts on $\overline{k}$-points by the rule

$$(x_0, ..., x_{n-1}) \mapsto (\pi(x_{n-1}), \pi(x_0), ..., \pi(x_{n-2})),$$

while $W_{K/k}(Fr_N \times_k K)$ acts by the rule

$$(x_0, ..., x_{n-1}) \mapsto (\pi(x_0), ..., \pi(x_{n-1})).$$

From this we see that $\text{Ker}(\pi - W_{K/k}(Fr_N \times_k K)) \times_k K$ is given by the diagonal $N_K \hookrightarrow \prod_{i=0}^{n-1} N_K$. Therefore there is a K-linear isomorphism $N_K \xrightarrow{\sim} \text{Ker}(\pi - W_{K/k}(Fr_n \times_k K)) \times_k K$, and it is clear from the construction of $\text{Ker}(\pi - W_{K/k}(Fr_N \times_k K))$ that the Galois actions on N_K and $\text{Ker}(\pi - W_{K/k}(Fr_N \times_k K))$ are identified by this map. Thefore this map descends to a natural isomorphism $N \xrightarrow{\sim} \text{Ker}(\pi - W_{K/k}(Fr_n \times_k K)) = p_* p^* N$.

Next we give a natural isomorphism $p^* p_*(M, g) \xrightarrow{\sim} (M, g)$ for $(M, g) \in \mathscr{D}_{k \hookrightarrow K} \otimes \mathbb{Q}$. This amounts to defining a natural isogeny $\alpha : p_* M \times_k K \xrightarrow{\sim} M$ fitting into a commuting diagram

$$
\begin{array}{ccc}
p_* M_K & \xrightarrow{\alpha} & M \\
\scriptstyle{F_{p_* M \times_k K}} \downarrow & & \downarrow \scriptstyle{g} \\
p_* M_K & \xrightarrow{\alpha} & M
\end{array}
\qquad (9.3.1.1)
$$

where F_{p_*M} is the k-linear Frobenius endomorphism of p_*M. For ease of notation we have written p_*M_K for $p_*M \times_k K$.

To define α, recall (using the notation of Section 2) that we have $W_{K/k}(M)_K = \prod_{i=0}^{n-1} M^{(i)}$. Let

$$p_0 : W_{K/k}(M)_K \to M$$

be the projection onto the 0th factor. Restricting p_0 to p_*M_K yields a map

$$\alpha : p_*M_K = \mathrm{Ker}(\pi - W_{K/k}(g))_K \xrightarrow{p_0} M.$$

We claim that α is an isogeny that fits into the commutative diagram 9.3.1.1.

To show that α is an isogeny it suffices to fix a prime $\ell \neq p$ and look at the induced map on Tate modules $V_\ell \alpha : V_\ell(p_*M_K) \to V_\ell M$. To understand this map better, we consider the basis $(V_0, ..., V_{n-1})$ for $V_\ell W_{K/k}(M)$ constructed in Paragraphs 9.12-9.14 and use this to compute a basis for $V_\ell(p_*M) = V_\ell \mathrm{Ker}(\pi - W_{K/k}(g))$. Write $M(g)$ for the matrix representation of g on $V_\ell M$ with respect to the basis V_0. Then by our work in the previous section, $\pi - W_{K/k}(g)$ acts on $V_\ell W_{K/k}(M)$ by the matrix

$$\begin{bmatrix} -M(g) & 0 & 0 & \cdot & 0 & M(g)^n \\ I & -M(g) & 0 & \cdot & 0 & 0 \\ 0 & I & -M(g) & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & I & -M(g) \end{bmatrix}.$$

In order to write $M(g)^n$ in the upper right corner, we have used the fact that g^n agrees with the geometric Frobenius action on M.

An easy Gaussian elimination row reduces this to the matrix

$$\begin{bmatrix} -I & 0 & 0 & \cdot & 0 & M(g)^{n-1} \\ 0 & -I & 0 & \cdot & 0 & M(g)^{n-2} \\ 0 & 0 & -I & \cdot & 0 & M(g)^{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & -I & M(g) \\ 0 & 0 & 0 & \cdot & 0 & 0 \end{bmatrix}.$$

After some manipulations, one finds that a basis for $V_\ell \mathrm{Ker}(\pi - W_{K/k}g)$ is given by

$$\{(v, M(g)^{-1}v, M(g)^{-2}v, ..., M(g)^{-(n-1)}v)\}_{v \in V_0}.$$

Since V_0 is a basis for $V_\ell M$, from this immediately follows that $\alpha : p_*M_K \to M$ is an isogeny. Moreover, from the definition of p_*M we have that the k-linear Frobenius endomorphism of p_*M acts via g, which implies that we have the commutative diagram 9.3.1.1. Therefore $p^*p_*M \cong M$ via the isomorphism α. This completes the proof of Theorem 9.3.1. $\square$

Descending 1-motives from $\overline{k}$ to k

In this section we prove a variant of Theorem 9.3.1 for descending 1-motives from $\overline{k}$ to k. To state our theorem, we start by defining the isogenies we would like to use as descent data.

Definition 9.3.2. Let $M = [L \to G]$ be a 1-motive over $\overline{k}$. A *descent isogeny* $g : M \to M$ relative to k is an isogeny such that there exists a 1-motive $\overline{M}$ over a degree-n extension K of k, such that there is an isogeny $\tilde{M} \times_K \overline{k} \overset{\sim}{\to} M$ inducing a commutative diagram

$$
\begin{array}{ccc}
\overline{M} \times_K \overline{k} & \longrightarrow & M \\
{\scriptstyle Fr_{\overline{M}}}\downarrow & & \downarrow{\scriptstyle g^n} \\
\overline{M} \times_K \overline{k} & \longrightarrow & M,
\end{array}
$$

where $Fr_{\overline{M}}$ is the Frobenius endomorphism of $\overline{M}$.

Note that we do not assume g is K-linear. However, it is always possible to find a pair $(K, \overline{M})$ as in Definition 9.3.2 with the additional property that g descends to a K-linear map. We state this as the following lemma.

Lemma 9.3.3. *Given a descent isogeny $g : M \to M$ of the 1-motive M, there exists a degree-n extension $k \hookrightarrow K$ with a 1-motive $\overline{M}$ over K as in Definition 9.3.2, with the additional property that g descends to a morphism $g : \overline{M} \to \overline{M}$.*

Proof. Begin with a 1-motive $\overline{M}$ over K as in Definition 9.3.2. Then there exists a finite extension $K \hookrightarrow K'$ such that g descends to a K'-linear isogeny of $\overline{M} \times_K K'$. Let $n' = [K' : k]$. It is easy to see that we have a commutative diagram

$$
\begin{array}{ccc}
(\overline{M} \times_K K') \times_{K'} \overline{k} & \longrightarrow & M \\
{\scriptstyle Fr_{\overline{M}'}}\downarrow & & \downarrow{\scriptstyle g^{n'}} \\
(\overline{M} \times_K K') \times_{K'} \overline{k} & \longrightarrow & M
\end{array}\ .
$$

Therefore $\overline{M} \times_K K'$ is a 1-motive over K' satisfying the condition of Definition 9.3.2 with the additional property that g descends to an endomorphism of $\overline{M} \times_K K'$. $\qquad\square$

We can now prove the natural generalization of Theorem 9.3.1 to 1-motives over $\overline{k}$.

Theorem 9.3.4. *Let $\mathscr{D}_{k \hookrightarrow \overline{k}} \otimes \mathbb{Q}$ be the category of pairs (M, g) where M is an isogeny 1-motive over $\overline{k}$, and $g : M \to M$ is a descent isogeny relative to k. Consider the natural pullback functor*

$$
p^* : \textit{1-Mot}_k \otimes \mathbb{Q} \longrightarrow \mathscr{D}_{k \hookrightarrow \overline{k}} \otimes \mathbb{Q}
$$

sending a 1-motive N to the pair $(N \times_k \overline{k}, Fr_N \times_k \overline{k})$ where Fr_N is the k-linear Frobenius endomorphism of N. Then p^* is an equivalence of categories. Moreover, p^* has a natural quasi-inverse p_*.

Proof. We start by defining the functor p_*. Let $(M, g) \in \mathscr{D}_{k \hookrightarrow \overline{k}} \otimes \mathbb{Q}$, and let $k \hookrightarrow K$ be a degree-n field extension, and $\overline{M}/K$ a 1-motive such that $\overline{M} \times_K \overline{k} \cong M$ in a way that identifies g^n with $Fr_{\overline{M}}$ as in Definition 9.3.2, and choose K such that g descends to an endomorphism of $\overline{M}$ as in Lemma 9.3.3. We then apply Theorem 3.1 to the pair $(\overline{M}, g)$ to obtain a 1-motive $\tilde{M}$ over k such that there is a canonical commutative diagram

$$
\begin{array}{ccc}
\tilde{M} \times_k K & \xrightarrow{\sim} & \overline{M} \\
{\scriptstyle Fr_{\tilde{M}}} \downarrow & & \downarrow {\scriptstyle g} \\
\tilde{M} \times_k K & \xrightarrow{\sim} & \overline{M}
\end{array}
$$

where $Fr_{\tilde{M}}$ is the k-linear Frobenius (base changed to K). We define $p_*(M, g) := \tilde{M}$.

We need to show that $p_*(M, g)$ is independent (up to unique isogeny) of the choice of intermediate field K appearing in the construction. Suppose that we have a second field extension $k \hookrightarrow K'$, of degree n', and a 1-motive $\overline{M}'$ over K' such that $g^{n'} = Fr_{\overline{M}'}$ as in Definition 9.3.2, such that g descends to a K'-linear endomorphism of $\overline{M}'$. From $\overline{M}'$ we obtain a 1-motive $\tilde{M}'$ over k satisfying a commutative diagram as in 9.3. Combining the diagrams for $\overline{M}$ and $\overline{M}'$ and base changing to $\overline{k}$, we obtain a commutative diagram

$$
\begin{array}{ccccccc}
\tilde{M} \times_k \overline{k} & \xrightarrow{\sim} & \overline{M} \times_K \overline{k} & \xrightarrow{\sim} & \overline{M}' \times_{K'} \overline{k} & \xleftarrow{\sim} & \tilde{M}' \times_k \overline{k} \\
{\scriptstyle Fr_{\tilde{M}}} \downarrow & & {\scriptstyle g} \downarrow & & {\scriptstyle g} \downarrow & & {\scriptstyle Fr_{\tilde{M}'}} \downarrow \\
\tilde{M} \times_k \overline{k} & \xrightarrow{\sim} & \overline{M} \times_K \overline{k} & \xrightarrow{\sim} & \overline{M}' \times_{K'} \overline{k} & \xleftarrow{\sim} & \tilde{M}' \times_k \overline{k}
\end{array}
$$

The inner square comes from the fact that we are given isomorphisms of $\overline{M} \times_K \overline{k}$ and $\overline{M} \times_{K'} \overline{k}$ with M in such a way that g descends to both $\overline{M}$ and $\overline{M}'$. Following the outer rectangle defines a map $f : \tilde{M} \times_k \overline{k} \xrightarrow{\sim} \tilde{M}' \times_k \overline{k}$ that is compatible with the actions of $Fr_{\tilde{M}}$ and $Fr_{\tilde{M}'}$. Hence it descends to an isogeny $\tilde{M} \to \tilde{M}'$ which is uniquely defined. Hence $p_*(M, g) := \tilde{M}$ is well-defined up to canonical isogeny.

The functoriality of $p_* : \mathscr{D}_{k \hookrightarrow \overline{k}} \otimes \mathbb{Q} \to 1\text{-Mot}_k \otimes \mathbb{Q}$ is now easily deduced from Theorem 9.3.1. Namely, suppose given a map $f : (M, g) \to (M', g')$ which amounts to a commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{f} & M' \\
{\scriptstyle g} \downarrow & & \downarrow {\scriptstyle g'} \\
M & \xrightarrow{f} & M'
\end{array} .
$$

We can choose a finite field K together with 1-motives $\overline{M}, \overline{M}'$ over K as in Definition 9.3.2 such that g, f, and g' all descend to K. Then from the functoriality in Theorem 9.3.1 we get a morphism $p_*(M, g) \to p_*(M', g')$.

Having defined the functor p_*, it is obvious from Theorem 9.3.1 that there are canonical isomorphisms $p^* p_*(M, g) \cong (M, g)$ and $N \cong p_* p^* N$ for $(M, g) \in \mathscr{D}_{k \hookrightarrow \overline{k}} \otimes \mathbb{Q}$ and $N \in 1\text{-Mot}_k \otimes \mathbb{Q}$.

$\square$

9.4 Application: Albanese 1-Motives over Finite Fields

In this section we apply Theorem 9.3.4 of the previous section to prove the following:

Theorem 9.4.1. *Let $k = \mathbb{F}_q$, and let Sch_d/k be the category of d-dimensional separated finite type k-schemes. Then there exist functors*

$$M^{2d-1}(-), M_c^{2d-1}(-) : (Sch_d/k)^{op} \longrightarrow 1-Mot_k \otimes \mathbb{Q},$$

unique up to unique isomorphism, such that there are natural isomorphisms (functorial in X)

$$V_\ell M_{(c)}^{2d-1}(X) \cong H_{(c)}^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell(d))$$

for each prime $\ell \neq p$, and these isomorphisms are compatible with the natural actions of $\mathrm{Gal}(\overline{k}/k)$ on each side.

This will complete the proof of our main Theorem 1.2.17.

Proof. Let X be a separated finite type k-scheme, $k = \mathbb{F}_q$. Fix an algebraic closure $k \hookrightarrow \overline{k}$. By our work in Chapters 7 and 8, we have 1-motives $M_{(c)}^{2d-1}(X_{\overline{k}})$ attached to the base change $X_{\overline{k}}$. Our goal is to descend these 1-motives to 1-motives $M_{(c)}^{2d-1}(X)$ defined over k, such that the natural Galois action on $V_\ell M_{(c)}^{2d-1}(X)$ corresponds to the Galois action on $H_{(c)}^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell(d))$.

As in the Introduction, we can define an endomorphism $\tilde{F} : M_{(c)}^{2d-1}(X_{\overline{k}}) \to M_{(c)}^{2d-1}(X_{\overline{k}})$ such that on Tate modules, $\tilde{F}$ induces the action on $H_{(c)}^{2d-1}(X_{\overline{k}}, \mathbb{Q}_\ell(d))$. Moreover, $\tilde{F}$ defines a descent isogeny of $M_{(c)}^{2d-1}(X_{\overline{k}})$. Therefore by Theorem 9.3.4, we get a 1-motive $M_{(c)}^{2d-1}(X)$ over k with the property that the k-linear Frobenius of $M_{(c)}^{2d-1}(X)$ is given by $\tilde{F}$. Moreover, $M_{(c)}^{2d-1}(X)$ is contravariantly functorial in X. This proves Theorem 9.4.1.

$\square$

Chapter 10

Application to Independence of ℓ

10.1 Independence of ℓ in Dimension and Codimension One

10.1.1. Let k be an algebraically closed field and $f : X \to X$ an endomorphism of a separated finite type k-scheme (we are primarily thinking of the case $k = \overline{\mathbb{F}}_q$ and f is the geometric Frobenius endomorphism of a scheme X defined over $\mathbb{F}_q$). Then for any i and $\ell \neq p$ we can define

$$P_\ell^i(f, t) := \det(1 - tf | H^i(X, \mathbb{Q}_\ell))$$

and in case f is proper,

$$P_{\ell,c}^i(f, t) := \det(1 - tf | H_c^i(X, \mathbb{Q}_\ell)).$$

An old conjecture is that these polynomials have integer coefficients independent of ℓ. Based on our work on 1-motives, we can prove the following:

Proposition 10.1.2. *Let $f : X \to X$ be an endomorphism of a separated finite type k-scheme. Then the polynomials $P_\ell^i(f, t)$ have integer coefficients independent of ℓ for $i = 0, 1, 2d - 1, 2d$. If f is proper, then the same holds true for the polynomials $P_{\ell,c}^i(f, t)$ for the same values of i.*

Proof. First we handle the cases $i = 0$ and $i = 2d$. Let $C(X)$ and $PC(X)$ be, respectively, the sets of connected components and proper connected components of X. Also, let $I_d(X)$ and $PI_d(X)$ be, respectively, the sets of d-dimensional irreducible components and d-dimensional proper irreducible components of X. It is clear that we have functorial isomorphisms $H^0(X, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell^{C(X)}$ and $H_c^0(X, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell^{PC(X)}$. This proves the case $i = 0$. By Lemmas 4.1.3 and 8.2.1, we have functorial isomorphisms $H^{2d}(X, \mathbb{Q}_\ell(d)) \cong \mathbb{Q}_\ell^{PI_d(X)}$ and $H_c^{2d}(X, \mathbb{Q}_\ell(d)) \cong \mathbb{Q}_\ell^{I_d(X)}$. This deals with the case $i = 2d$. We have shown that $M_{(c)}^i(X)$ is the realization of a natural 1-motive for $i = 1, 2d - 1$, so we see that the following proposition completes the proof. $\qquad \square$

Proposition 10.1.3. *Let $M = [L \to G]$ be a 1-motive over k, and let $f : M \to M$ be an endomorphism of M. For any $\ell \neq p$ define the polynomial*

$$P_\ell^i(t) := \det(1 - tf|V_\ell M).$$

Then $P_\ell^i(t)$ has integer coefficients independent of ℓ.

Proof. The endomorphism f induces a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V_\ell G & \longrightarrow & V_\ell M & \longrightarrow & V_\ell L & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle f} & & \downarrow {\scriptstyle f} & & \downarrow {\scriptstyle f} & & \\
0 & \longrightarrow & V_\ell G & \longrightarrow & V_\ell M & \longrightarrow & V_\ell L & \longrightarrow & 0,
\end{array}
$$

so it suffices to prove the proposition individually for $V_\ell G$ and $V_\ell L$. Since $V_\ell L = L \otimes \mathbb{Q}_\ell$, the statement is clear for $V_\ell L$. For $V_\ell G$, let T be the torus part of G and A the abelian quotient. Then f induces a diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V_\ell T & \longrightarrow & V_\ell G & \longrightarrow & V_\ell A & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle f} & & \downarrow {\scriptstyle f} & & \downarrow {\scriptstyle f} & & \\
0 & \longrightarrow & V_\ell T & \longrightarrow & V_\ell G & \longrightarrow & V_\ell A & \longrightarrow & 0
\end{array}
$$

So it suffices to prove the proposition individually for a torus T and an abelian variety A, where both cases are well known [Dem72, p. 96]. $\qquad\square$

10.1.4. Now consider the case when X is 2-dimensional. Then for any endomorphism $f : X \to X$, we have proved ℓ-independence for $P_{\ell,(c)}^i(f, t)$ for all i except $i = 2$. But this single remaining value of i can be dealt with by the trace formula (for certain f). We obtain the following:

Corollary 10.1.5. *Let X be a 2-dimensional separated finite type k-scheme. If $f : X \to X$ is any proper endomorphism, then for all values of i, the polynomial $P_{\ell,c}^i(t)$ has rational coefficients independent of ℓ. If $f : X \to X$ is any quasi-finite endomorphism, then the polynomial $P_\ell^i(t)$ has rational coefficients independent of ℓ for all i.*

Proof. The statement for $P_{\ell,c}^i(t)$ follows from the trace formula on compactly supported cohomology, known as Fujiwara's theorem [Fuj97, 5.4.5]. The statement for $P_\ell^i(t)$ follows from a trace formula for quasi-finite morphisms [Ols2, Thm 1.1]. $\qquad\square$

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