Computer Programs for Bending Analysis of Elastic Plastic Circular Plates

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Report to
National Aeronautics and Space Administration
NASA Research Grant No. NsG 274 S-2

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STRUCTURAL ENGINEERING LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY CALIFORNIA

Structures and Materials Research
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ABSTRACT

A description of computer programs for the bending analysis of the elastic-plastic circular plates with arbitrary axisymmetric loading and support condition is presented. Complete listings of the main routines together with the concise flow charts of the programs are included. The programs are prepared for IBM 7094 digital computer. Both single precision and double precision Fortran IV language have been employed. An example is presented to aid in the application of the programs. To attain an insight into some of the details of the programs the reader is referred to Ref. [1].*

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* Numbers in brackets designate the references.

I. INTRODUCTION

The computer programs presented in this report have been developed for the elastic-plastic bending analysis of circular plates with axisymmetrical load and support conditions. Incremental theory of plasticity has been used in the analysis. The deformations are assumed to be small, Kirchhoff's hypothesis is adopted and shear deformation is neglected. Finite element type of solution using the direct stiffness method of structural analysis has been employed. The reader is referred to Ref. [1] for a complete account of the theoretical formulation of the problem. A brief description of the basic idea of the method of solution to the extent which is essential to follow and use the computer programs follows.

For the purposes of the analysis, a plate is divided into a number of ring elements as shown in Fig. 1. The positive direction of increments of nodal stress resultants and displacements are indicated in this figure. For a circular plate the central element is a disc as shown in Fig. 2. Each element is further subdivided into a number of layers along the depth of the plate, see Fig. 3. The latter are the smallest subdivisions whose load history is followed in the proposed incremental method of analysis.

The plate is assumed to be initially free from residual stresses. The first increment of load is so chosen that yielding just starts in one of the layers of the plate. Thereafter the load is applied in small increments. For each load increment after the displacements are found, the increments in curvature, strain and stress are calculated. This determines the new state of stress for which the corresponding material properties are found. In the computer program an average value for material property is determined for each

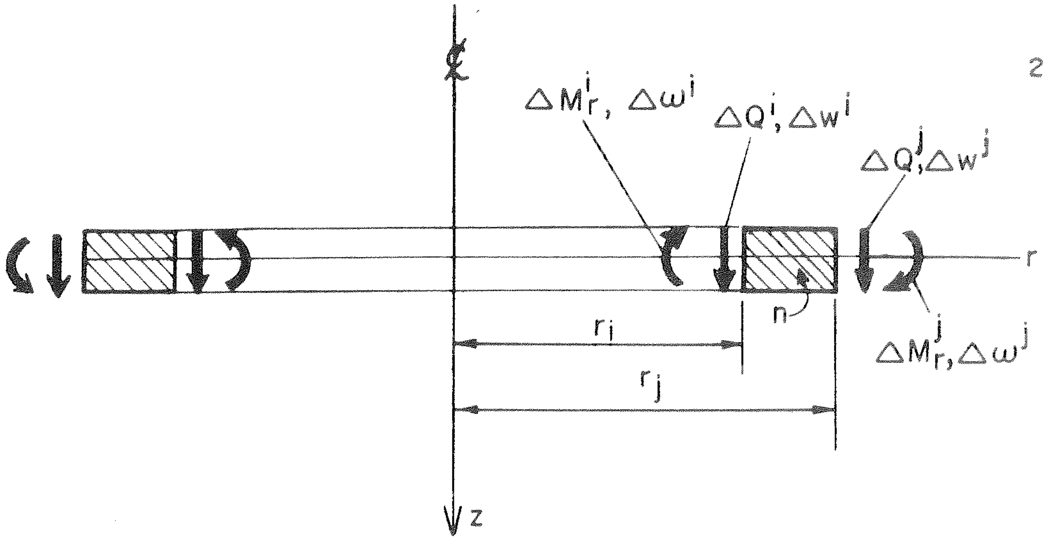


FIG. 1

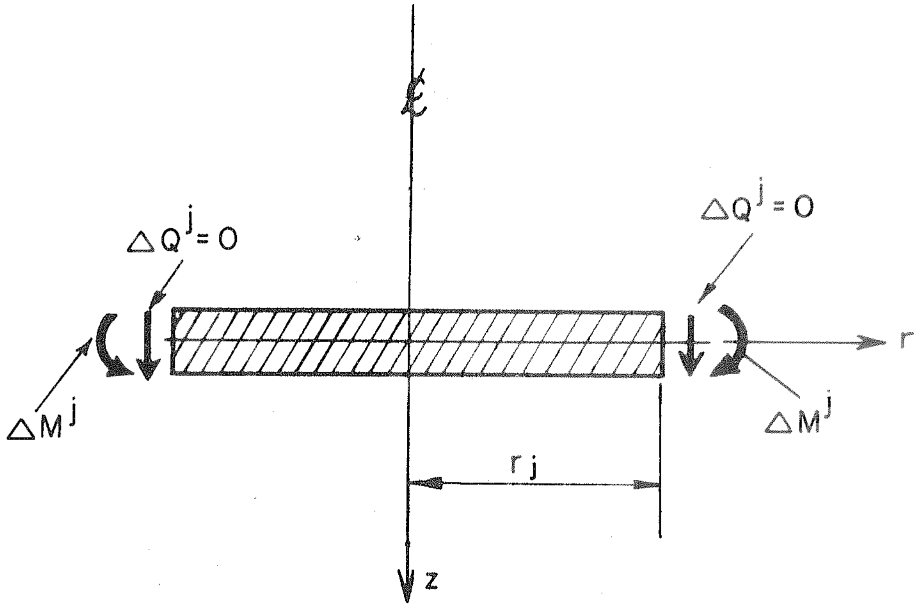


FIG. 2

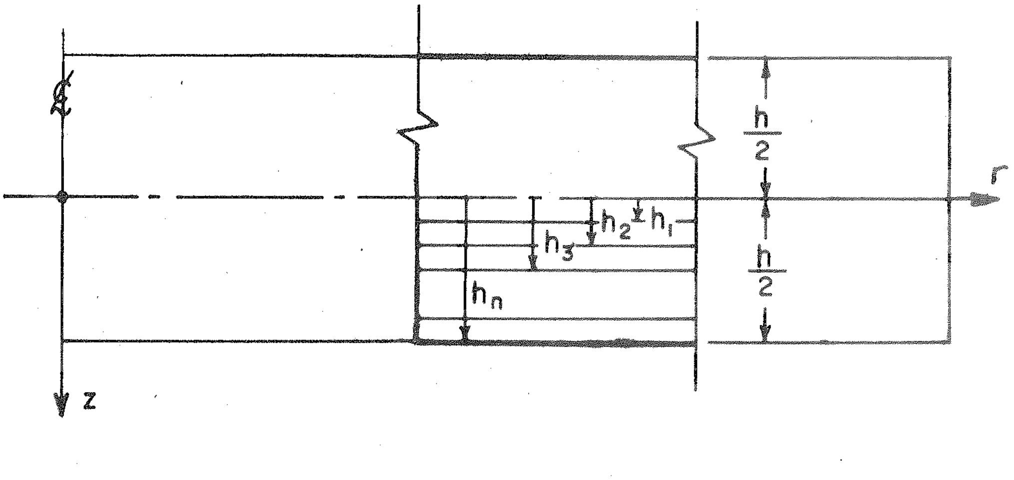


FIG. 3

increment of load. The details of this procedure are presented in Section VI of this report.

Both tributary area and consistent equivalent nodal ring forces [2] have been used. A comparison of the results using the two approaches shows that the difference is not very significant for small size elements.

II. DESCRIPTION OF COMPUTER PROGRAMS

Two sets of computer programs have been prepared in Fortran IV language for the solution of clamped and simply supported plates subject to axisymmetric loading and boundary conditions. The programs have been written for IBM 7094 digital computer. One set of the programs is for elastic perfectly plastic material and the other for hardening material. Although the idea of solution is essentially the same in both cases, there are differences in details due to inherent sources of numerical inaccuracies. This question is discussed in detail in Ref. [1].

In order to achieve the required accuracy, the program for hardening material is written in double precision language in which 16 significant digits are utilized in the analysis. In the program for elastic-perfectly plastic material the elements of matrices $[BV]^{-1}$ and [SK] have been expanded in terms of (1-DR) to treat separately the factors causing ill conditioning.

Each set of programs consists of essentially two parts. There is a main routine in which most of the data are read and displacements and stress resultants at nodal rings are calculated and printed out. In the other part, depending on the type of the material, one or two subroutines are used to calculate the material properties and the quantities associated with them.

The number of elements and layers which can be used varies for the two sets of programs as explained below. The capacity of the programs can be extended by utilizing tapes. Although the present programs have been written for simply supported and clamped circular plates, with slight modifications ring plates with the same type of boundary conditions can be easily treated.

A brief description of the function of each routine follows.

2.1 The Computer Program for Elastic-Perfectly Plastic Material

Presently a 30 element 60 layer plate can be handled. By decreasing the number of layers more elements can be used. Nodal ring loads in this program are tributary. Single Precision Fortran IV language is used.

A. (EPPACP) Elastic-Perfectly Plastic Analysis of Circular Plates

This routine is for the matrix solution of the elastic-perfectly plastic bending of plates. The input data is read in this routine and the nodal ring displacements and stress resultants are calculated and printed out.

B. (AMAFUN) Formation of Average Material Property Functions

The function of this subroutine is to form the average material property function for each increment of external load. These functions are then used in routine EPPACP to determine the displacements and stress resultants.

C. (IMAFUN) Formation of Initial Material Property Functions

The displacements obtained in routine EPPACP are used to find the curvatures, strains, state of stress and magnitude of initial material property functions for the next increment of load.

D. (MATINV) Matrix Inversion

This subroutine is used in EPPACP for inversion of matrix [BK]. In the present form matrix [BK] is nonsymmetric. If it is necessary to save computer storage places, it can be easily symmetrized.

2.2 The Computer Program for Hardening Materials

The routines are all in Double Precision Fortran IV language. The number of elements and layers which is presently handled in this program are 20 and 40, respectively. Other combinations of elements and layers can be used.

A brief description of the routines in this program is as follows:

A. (EPACP) Elastic-Plastic Analysis of Circular Plates

This routine is for the matrix solution of the plate made of hardening material. Most of the input data is read in this routine. Nodal ring displacements and stress resultants are calculated and printed out.

B. (ENRIL) Equivalent Nodal Ring Load

This subroutine is used to calculate the consistent nodal ring forces assuming linear variation of loads between the nodal rings. If it is desired to use tributary nodal ring forces or isolated concentrated ring loads, this subroutine should be deleted and (EPACP) modified accordingly.

C. (MATINV) Matrix Inversion

This subroutine is used in routine EPACP for the inversion of band matrix [BK]. Matrix [BK] is nonsymmetric.

D. (INV) Matrix Inversion

This subroutine is for the inversion of matrix [BV]. Matrix [BV] is nonsymmetric.

E. (MATFUN) Formation of Material Property Functions

In this subroutine both the average material property functions in each step of load and the initial functions for the next increment of load are computed.

F. (INTER) Interpolation

This subroutine is used to perform linear interpolation on effective stress-tangent modulus curve for hardening materials.

III. INPUT DATA ARRANGEMENT FOR ELASTIC-PERFECTLY PLASTIC PROGRAM

The order of data cards is as follows:

Sequence No.	No. of Cards	Description	Format
1*	1	Number of load systems, NP	I2
2	1	Title card containing 72 alphanumeric characters	
3	1	Number of layers, NL Number of elements, NE Number of load increments in one load system, NLL Number indicating the type of boundary conditions, NBC**	4I4
4	Batch of cards	Thickness of elements, H	8F9.5
5	Batch of cards	Radii of elements, R	8F9.5
6	1	Poisson's ratio, U	F9.5
		Modulus of elasticity, E Yield stress, TY	2E12.6
7***	Batch of cards	Nodal ring load increments, PI	6E12.6

* Up to 99 load systems can be analyzed if desired.

** NBC = 0 for simply supported plate, otherwise for clamped plate.

*** The nodal ring loads are positive in the direction of z axis, see Fig. 1. Except for the first increment of load the rest of load increments are used twice, see Section VI. The nodal ring loads in this program are obtained by tributary area method. It has been found that allocating the distributed load half-way between the neighboring elements in the nodal ring load leads to satisfactory results.

IV. INPUT DATA ARRANGEMENT FOR HARDENING MATERIAL PROGRAM

Sequence No.	No. of Cards	Description	Format
1*	1	Number of load systems, NP	I2
2	1	Title card containing 72 alphanumeric characters	
3	1	Number of layers, NL Number of elements, NE Number of load increments in one load system, NLL Number indicating the type of boundary condition, NBC**	4I4
4	1	Thickness of plate, H	D12.6
5	Batch of cards	Radii of elements, R	6D12.6
6	1	Modulus of Elasticity, E Poisson's ratio, U	D12.6 D8.3
7	1	Number of data points in effective stress-tangent modulus diagram, ND	I4
8	Batch of cards	Stress values an effective stress-tan. mod. diagram, SD	6D12.3
9	Batch of cards	Corresponding tan. moduli on the stress-tan. modulus diagram, ED	6D12.3
10***	Batch of cards	Load increments, P	4D15.5

* Up to 99 load systems can be analyzed if desired.

** NBC = 0 for simply supported plate, otherwise for clamped plate.

*** The nodal ring loads are positive in the direction of z axis, see Fig. 1. Except for the first increment of load the rest of load increments are used twice, see Section VI. The loads in this program are the amplitude of the distributed loads at nodal rings and also at the center of the plate, see Fig. 4. These are read in subroutine ENRIL and the corresponding consistent nodal ring forces are calculated. In case tributary area load distribution is to be used subroutine ENRIL is removed and appropriate read statement is inserted in its place in routine EPACP.

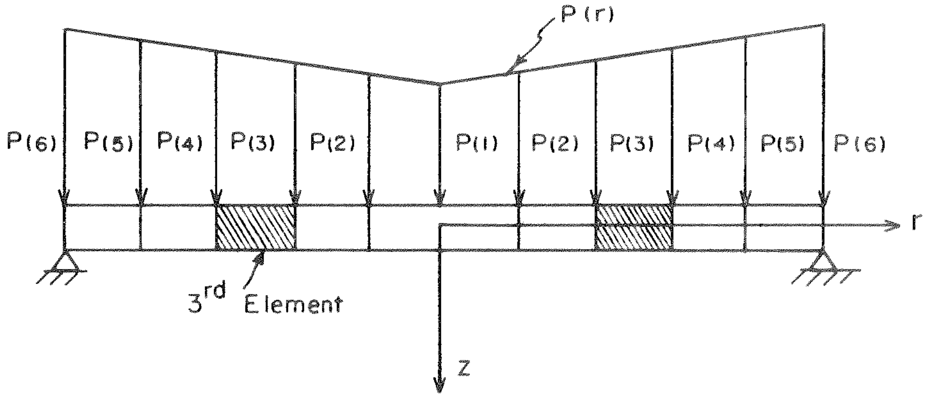


FIG. 4

V. ILLUSTRATIVE EXAMPLE

The following example serves to illustrate the order of data presentation and also that of output arrangement of the computer program for hardening materials. Essentially the same order of output arrangement is also used for the elastic-perfectly plastic program.

A 0.75 x 16 in. simply supported plate is subjected to a uniformly distributed load, see Fig. 5. The tangent modulus-stress diagram of the plate material is shown in Fig. 6. This corresponds to the uniaxial stress-strain diagram in Fig. 7. The Poisson's ratio is assumed to be 0.33. The plate is divided into 16 elements which are further subdivided into 40 layers along the plate thickness. The first load increment is chosen to be 120 psi to cause inelasticity just to start in the central element.

Notice that only part of the computer results for this example are presented in the following pages.

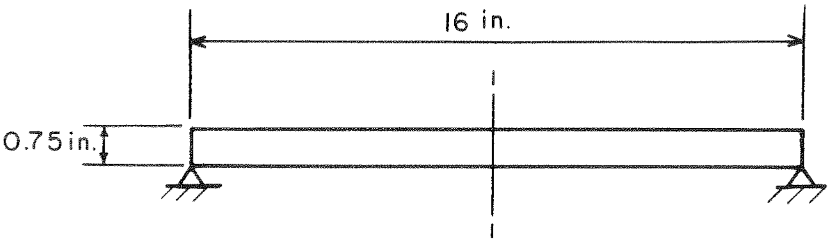


FIG. 5

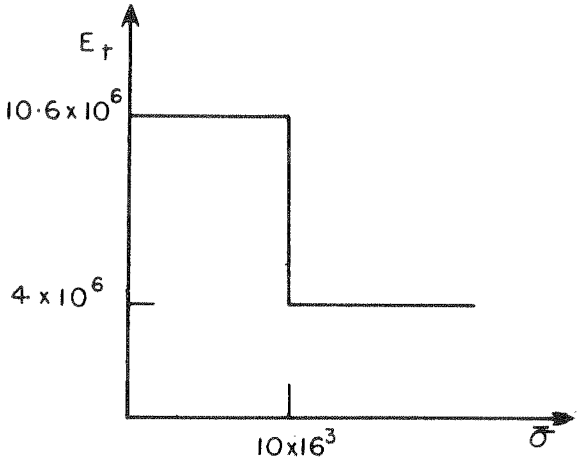


FIG. 6
TANGENT MODULUS - STRESS DIAGRAM

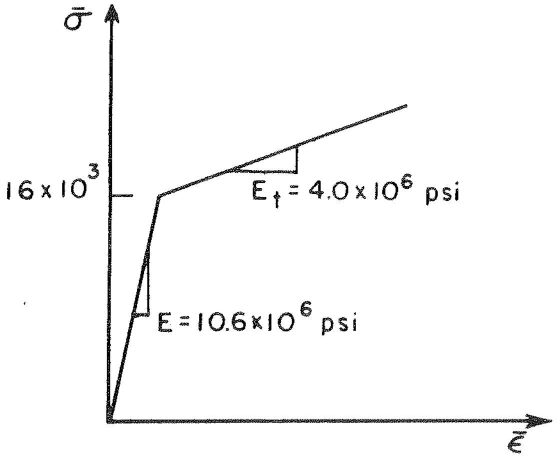


FIG. 7
UNIAXIAL STRESS - STRAIN DIAGRAM

```
$DATA
 1
       S.S. PLATE  UNIFORM LOAD  BILINEAR MATERIAL  (LB-IN)
  40  16   3   0
 0.75     D00
 0.500    D00 1.00     D00 1.50     D00 2.00     D00 2.50     D00 3.00     D00
 3.50     D00 4.00     D00 4.50     D00 5.00     D00 5.50     D00 6.00     D00
 6.50     D00 7.00     D00 7.50     D00 8.00     D00
10.600000D06 .330D00
   4
    0.000D00    16.000D03   16.000D03   40.000D03
   10.600D06    10.600D06    4.000D06    4.000D06
   120.000   D00    120.000   D00    120.000   D00    120.000   D00
   120.000   D00    120.000   D00    120.000   D00    120.000   D00
   120.000   D00    120.000   D00    120.000   D00    120.000   D00
   120.000   D00    120.000   D00    120.000   D00    120.000   D00
   120.000   D00
    20.000   D00     20.000   D00     20.000   D00     20.000   D00
    20.000   D00     20.000   D00     20.000   D00     20.000   D00
    20.000   D00     20.000   D00     20.000   D00     20.000   D00
    20.000   D00     20.000   D00     20.000   D00     20.000   D00
    20.000   D00
    20.000   D00     20.000   D00     20.000   D00     20.000   D00
    20.000   D00     20.000   D00     20.000   D00     20.000   D00
    20.000   D00     20.000   D00     20.000   D00     20.000   D00
    20.000   D00     20.000   D00     20.000   D00     20.000   D00
    20.000   D00
```

```
   S.S. PLATE  UNIFORM LOAD  BILINEAR MATERIAL  (LB-IN)


 LOAD SET NUMBER          1

 NUMBER OF LAYERS =  40  NUMBER OF ELEMENTS =  16  NUMBER OF LOADING INCREMENTS =   3


 THICKNESS OF PLATE = 0.750000D 00

 RADII OF NODAL RINGS
   0.500000D 00 0.100000D 01 0.150000D 01 0.200000D 01 0.250000D 01 0.300000D 01 0.350000D 01 0.400000D 01
   0.450000D 01 0.500000D 01 0.550000D 01 0.600000D 01 0.650000D 01 0.700000D 01 0.750000D 01 0.800000D 01


 MOD. OF ELASTICITY = 0.10600000D 08     POISSON RATIO = 0.33000D 00



 NUMBER OF DATA POINTS IN EQUIVALENT-STRESS TAN. MODULUS TABLE =   4

 STRESS = 0.00000000D-38 0.16000000D 05 0.16000000D 05 0.40000000D 05


 TAN. MODULUS =   0.106000D 08   0.106000D 08   0.400000D 07   0.400000D 07


  LOADING STEP =   1


MAGNITUDE OF DISTRIBUTED LOAD AT NODAL RINGS

   0.12000000D 03 0.12000000D 03 0.12000000D 03 0.12000000D 03 0.12000000D 03 0.12000000D 03 0.12000000D 03
   0.12000000D 03 0.12000000D 03 0.12000000D 03 0.12000000D 03 0.12000000D 03 0.12000000D 03 0.12000000D 03
   0.12000000D 03 0.12000000D 03 0.12000000D 03


                         THE FOLLOWING ARE INCREMENTAL VALUES

NODAL RING          APPLIED LOAD                    DISPLACEMENTS
              VERTICAL        MOMENT             SLOPE          DEFLECTION

     1      0.670096D 02    -0.787383D 00     -0.143242D-02     0.730957D-01
     2      0.599968D 02     0.501296D 00     -0.285140D-02     0.720239D-01
     3      0.599995D 02     0.333696D 00     -0.424347D-02     0.702488D-01
     4      0.599998D 02     0.250151D 00     -0.559519D-02     0.677872D-01
     5      0.599999D 02     0.200077D 00     -0.689311D-02     0.646626D-01
```

```
       6     0.600000D 02      0.166711D 00     -0.812377D-02       0.609053D-01
       7     0.600000D 02      0.142885D 00     -0.927373D-02       0.565522D-01
       8     0.600000D 02      0.125019D 00     -0.103295D-01       0.516472D-01
       9     0.600000D 02      0.111124D 00     -0.112777D-01       0.462406D-01
      10     0.600000D 02      0.100010D 00     -0.121049D-01       0.403896D-01
      11     0.600000D 02      0.909163D-01     -0.127975D-01       0.341582D-01
      12     0.600000D 02      0.833389D-01     -0.133422D-01       0.276168D-01
      13     0.600000D 02      0.769274D-01     -0.137255D-01       0.208428D-01
      14     0.600000D 02      0.714320D-01     -0.139339D-01       0.139204D-01
      15     0.600000D 02      0.666695D-01     -0.139540D-01       0.694029D-02
      16     0.295625D 02     -0.246859D 01     -0.137724D-01       0.000000D-38

 ELEMENT      MOMENT 1          MOMENT 2             SHEAR 1            SHEAR 2

       1                       -0.159344D 04                           -0.000000D-38
       2     0.159265D 04     -0.157320D 04        0.670096D 02       -0.335048D 02
       3     0.157370D 04     -0.154207D 04        0.935016D 02       -0.623344D 02
       4     0.154240D 04     -0.149840D 04        0.122334D 03       -0.917504D 02
       5     0.149865D 04     -0.144224D 04        0.151750D 03       -0.121400D 03
       6     0.144244D 04     -0.137357D 04        0.181400D 03       -0.151167D 03
       7     0.137374D 04     -0.129242D 04        0.211167D 03       -0.181000D 03
       8     0.129256D 04     -0.119877D 04        0.241000D 03       -0.210875D 03
       9     0.119889D 04     -0.109263D 04        0.270875D 03       -0.240778D 03
      10     0.109274D 04     -0.974006D 03        0.300778D 03       -0.270700D 03
      11     0.974106D 03     -0.842892D 03        0.330700D 03       -0.300636D 03
      12     0.842983D 03     -0.699289D 03        0.360636D 03       -0.330583D 03
      13     0.699373D 03     -0.543199D 03        0.390583D 03       -0.360538D 03
      14     0.543276D 03     -0.374620D 03        0.420538D 03       -0.390500D 03
      15     0.374692D 03     -0.193554D 03        0.450500D 03       -0.420467D 03
      16     0.193621D 03      0.363798D-11        0.480467D 03       -0.450438D 03

                            THE FOLLOWING ARE TOTAL VALUES

NODAL RING             APPLIED LOAD                        DISPLACEMENTS
               VERTICAL          MOMENT               SLOPE          DEFLECTION

       1     0.670096D 02     -0.787383D 00     -0.143242D-02       0.730957D-01
       2     0.599968D 02      0.501296D 00     -0.285140D-02       0.720239D-01
       3     0.599995D 02      0.333696D 00     -0.424347D-02       0.702488D-01
       4     0.599998D 02      0.250151D 00     -0.559519D-02       0.677872D-01
       5     0.599999D 02      0.200077D 00     -0.689311D-02       0.646626D-01
       6     0.600000D 02      0.166711D 00     -0.812377D-02       0.609053D-01
       7     0.600000D 02      0.142885D 00     -0.927373D-02       0.565522D-01
       8     0.600000D 02      0.125019D 00     -0.103295D-01       0.516472D-01
       9     0.600000D 02      0.111124D 00     -0.112777D-01       0.462406D-01
      10     0.600000D 02      0.100010D 00     -0.121049D-01       0.403896D-01
      11     0.600000D 02      0.909163D-01     -0.127975D-01       0.341582D-01
      12     0.600000D 02      0.833389D-01     -0.133422D-01       0.276168D-01
      13     0.600000D 02      0.769274D-01     -0.137255D-01       0.208428D-01
      14     0.600000D 02      0.714320D-01     -0.139339D-01       0.139204D-01
      15     0.600000D 02      0.666695D-01     -0.139540D-01       0.694029D-02
      16     0.295625D 02     -0.246859D 01     -0.137724D-01       0.000000D-38
```

```
ELEMENT    MOMENT 1        MOMENT 2          SHEAR 1          SHEAR 2

   1                    -0.159344D 04                    -0.000000D-38
   2     0.159265D 04   -0.157320D 04    0.670096D 02    -0.335048D 02
   3     0.157370D 04   -0.154207D 04    0.935016D 02    -0.623344D 02
   4     0.154240D 04   -0.149840D 04    0.122334D 03    -0.917504D 02
   5     0.149865D 04   -0.144224D 04    0.151750D 03    -0.121400D 03
   6     0.144244D 04   -0.137357D 04    0.181400D 03    -0.151167D 03
   7     0.137374D 04   -0.129242D 04    0.211167D 03    -0.181000D 03
   8     0.129256D 04   -0.119877D 04    0.241000D 03    -0.210875D 03
   9     0.119889D 04   -0.109263D 04    0.270875D 03    -0.240778D 03
  10     0.109274D 04   -0.974006D 03    0.300778D 03    -0.270700D 03
  11     0.974106D 03   -0.842892D 03    0.330700D 03    -0.300636D 03
  12     0.842983D 03   -0.699289D 03    0.360636D 03    -0.330583D 03
  13     0.699373D 03   -0.543199D 03    0.390583D 03    -0.360538D 03
  14     0.543276D 03   -0.374620D 03    0.420538D 03    -0.390500D 03
  15     0.374692D 03   -0.193554D 03    0.450500D 03    -0.420467D 03
  16     0.193621D 03    0.363798D-11    0.480467D 03    -0.450438D 03


ELEMENT  LAYER         RADIAL  STRESS                 TANGENTIAL  STRESS           EQUIVALENT     EQUIVALENT
              INCREMENTAL          TOTAL          INCREMENTAL          TOTAL           STRAIN          STRESS

   1       1   0.424917D 03   0.424917D 03   0.424917D 03   0.424917D 03   0.400865D-04   0.424917D 03
   1       2   0.127475D 04   0.127475D 04   0.127475D 04   0.127475D 04   0.120259D-03   0.127475D 04
   1       3   0.212458D 04   0.212458D 04   0.212458D 04   0.212458D 04   0.200432D-03   0.212458D 04
   1       4   0.297442D 04   0.297442D 04   0.297442D 04   0.297442D 04   0.280605D-03   0.297442D 04
   1       5   0.382425D 04   0.382425D 04   0.382425D 04   0.382425D 04   0.360778D-03   0.382425D 04
   1       6   0.467408D 04   0.467408D 04   0.467408D 04   0.467408D 04   0.440951D-03   0.467408D 04
   1       7   0.552392D 04   0.552392D 04   0.552392D 04   0.552392D 04   0.521124D-03   0.552392D 04
   1       8   0.637375D 04   0.637375D 04   0.637375D 04   0.637375D 04   0.601297D-03   0.637375D 04
   1       9   0.722358D 04   0.722358D 04   0.722358D 04   0.722358D 04   0.681470D-03   0.722358D 04
   1      10   0.807342D 04   0.807342D 04   0.807342D 04   0.807342D 04   0.761643D-03   0.807342D 04
   1      11   0.892325D 04   0.892325D 04   0.892325D 04   0.892325D 04   0.841816D-03   0.892325D 04
   1      12   0.977308D 04   0.977308D 04   0.977308D 04   0.977308D 04   0.921989D-03   0.977308D 04
   1      13   0.106229D 05   0.106229D 05   0.106229D 05   0.106229D 05   0.100216D-02   0.106229D 05
   1      14   0.114728D 05   0.114728D 05   0.114728D 05   0.114728D 05   0.108234D-02   0.114728D 05
   1      15   0.123226D 05   0.123226D 05   0.123226D 05   0.123226D 05   0.116251D-02   0.123226D 05
   1      16   0.131724D 05   0.131724D 05   0.131724D 05   0.131724D 05   0.124268D-02   0.131724D 05
   1      17   0.140223D 05   0.140223D 05   0.140223D 05   0.140223D 05   0.132285D-02   0.140223D 05
   1      18   0.148721D 05   0.148721D 05   0.148721D 05   0.148721D 05   0.140303D-02   0.148721D 05
   1      19   0.157219D 05   0.157219D 05   0.157219D 05   0.157219D 05   0.148320D-02   0.157219D 05
   1      20   0.165718D 05   0.165718D 05   0.165718D 05   0.165718D 05   0.156337D-02   0.165718D 05
   2       1   0.421505D 03   0.421505D 03   0.423231D 03   0.423231D 03   0.398463D-04   0.422371D 03
   2       2   0.126452D 04   0.126452D 04   0.126969D 04   0.126969D 04   0.119539D-03   0.126711D 04
   2       3   0.210753D 04   0.210753D 04   0.211615D 04   0.211615D 04   0.199231D-03   0.211185D 04
   2       4   0.295054D 04   0.295054D 04   0.296261D 04   0.296261D 04   0.278924D-03   0.295659D 04
   2       5   0.379355D 04   0.379355D 04   0.380908D 04   0.380908D 04   0.358617D-03   0.380134D 04
   2       6   0.463656D 04   0.463656D 04   0.465554D 04   0.465554D 04   0.438309D-03   0.464608D 04
   2       7   0.547957D 04   0.547957D 04   0.550200D 04   0.550200D 04   0.518002D-03   0.549082D 04
   2       8   0.632258D 04   0.632258D 04   0.634846D 04   0.634846D 04   0.597694D-03   0.633556D 04
   2       9   0.716559D 04   0.716559D 04   0.719492D 04   0.719492D 04   0.677387D-03   0.718030D 04
   2      10   0.800860D 04   0.800860D 04   0.804138D 04   0.804138D 04   0.757079D-03   0.802504D 04
   2      11   0.885161D 04   0.885161D 04   0.888784D 04   0.888784D 04   0.836772D-03   0.886978D 04
   2      12   0.969462D 04   0.969462D 04   0.973431D 04   0.973431D 04   0.916465D-03   0.971452D 04
```

etc.

```
  16      13     0.630804D 03     0.630804D 03     0.466057D 04     0.466057D 04     0.413149D-03     0.437938D 04
  16      14     0.681268D 03     0.681268D 03     0.503342D 04     0.503342D 04     0.446201D-03     0.472973D 04
  16      15     0.731733D 03     0.731733D 03     0.540627D 04     0.540627D 04     0.479253D-03     0.508008D 04
  16      16     0.782197D 03     0.782197D 03     0.577911D 04     0.577911D 04     0.512305D-03     0.543043D 04
  16      17     0.832661D 03     0.832661D 03     0.615196D 04     0.615196D 04     0.545356D-03     0.578078D 04
  16      18     0.883125D 03     0.883125D 03     0.652480D 04     0.652480D 04     0.578408D-03     0.613113D 04
  16      19     0.933590D 03     0.933590D 03     0.689765D 04     0.689765D 04     0.611460D-03     0.648148D 04
  16      20     0.984054D 03     0.984054D 03     0.727049D 04     0.727049D 04     0.644512D-03     0.683183D 04

                         THE FCLLOWING ARE TOTAL VALUES


 ELEMENT     TANGENTIAL MOMENT     RADIAL CURVATURE     TANGENTIAL CURVATURE

     1          0.159344D 04        -0.286485D-02          -0.286485D-02
     2          0.158175D 04        -0.283612D-02          -0.285921D-02
     3          0.156312D 04        -0.278229D-02          -0.284130D-02
     4          0.153701D 04        -0.270158D-02          -0.281440D-02
     5          0.150344D 04        -0.259397D-02          -0.277854D-02
     6          0.146240D 04        -0.245947D-02          -0.273370D-02
     7          0.141390D 04        -0.229806D-02          -0.267990D-02
     8          0.135794D 04        -0.210975D-02          -0.261713D-02
     9          0.129451D 04        -0.189454D-02          -0.254540D-02
    10          0.122361D 04        -0.165243D-02          -0.246469D-02
    11          0.114526D 04        -0.138342D-02          -0.237502D-02
    12          0.105944D 04        -0.108751D-02          -0.227638D-02
    13          0.966162D 03        -0.764691D-03          -0.216878D-02
    14          0.865420D 03        -0.414977D-03          -0.205221D-02
    15          0.757214D 03        -0.383607D-04          -0.192667D-02
    16          0.641546D 03         0.365156D-03          -0.179216D-02


  LOADING STEP =    2


 MAGNITUDE OF DISTRIBUTED LOAD AT NODAL RINGS

   0.20000000D 02 0.20000000D 02 0.20000000D 02 0.20000000D 02 0.20000000D 02 0.20000000D 02 0.20000000D 02
   0.20000000D 02 0.20000000D 02 0.20000000D 02 0.20000000D 02 0.20000000D 02 0.20000000D 02 0.20000000D 02
   0.20000000D 02 0.20000000D 02 0.20000000D 02


  LOADING STEP =    3


 MAGNITUDE OF DISTRIBUTED LOAD AT NODAL RINGS

   0.20000000D 02 0.20000000D 02 0.20000000D 02 0.20000000D 02 0.20000000D 02 0.20000000D 02 0.20000000D 02
   0.20000000D 02 0.20000000D 02 0.20000000D 02 0.20000000D 02 0.20000000D 02 0.20000000D 02 0.20000000D 02
   0.20000000D 02 0.20000000D 02 0.20000000D 02
```

VI. CONSIDERATION OF VARIATION OF MATERIAL PROPERTY FUNCTION WITHIN AN INCREMENT OF APPLIED LOAD

Considering the plate load and the state of stress as the independent and dependent variables respectively, we can write

$$d\tau = F(\tau)\ dp \tag{1}$$

where $F(\tau)$ represents a function that transforms the external loads into internal stresses and is expressed as a function of the state of stress. Equation (1) is solved numerically by replacing dp and $d\tau$ by finite increments Δp and $\Delta\tau$

$$\Delta\tau = F(\tau)\ \Delta p \tag{2}$$

To solve Equation (2) with reasonable accuracy, Euler's modified method [3] is used where the order of error is $O(\Delta p)^3$. To apply this procedure, first a temporary step of loading is made for which the known initial elastic-plastic modulus matrix is used to calculate the elements of a new modulus matrix. The average of these is then used with the load increment to find the increments of displacements, strains, stresses and the elements of the initial modulus matrix for the next increment of load.

Figure (8) is the flow diagram of the substeps taken in the computer program to complete the (k+1) th step of calculation represented by Equation (2) for hardening material. The procedure is basically the same for elastic-perfectly plastic material.

Whenever unloading from a plastic state takes place in a layer, elastic properties are used.

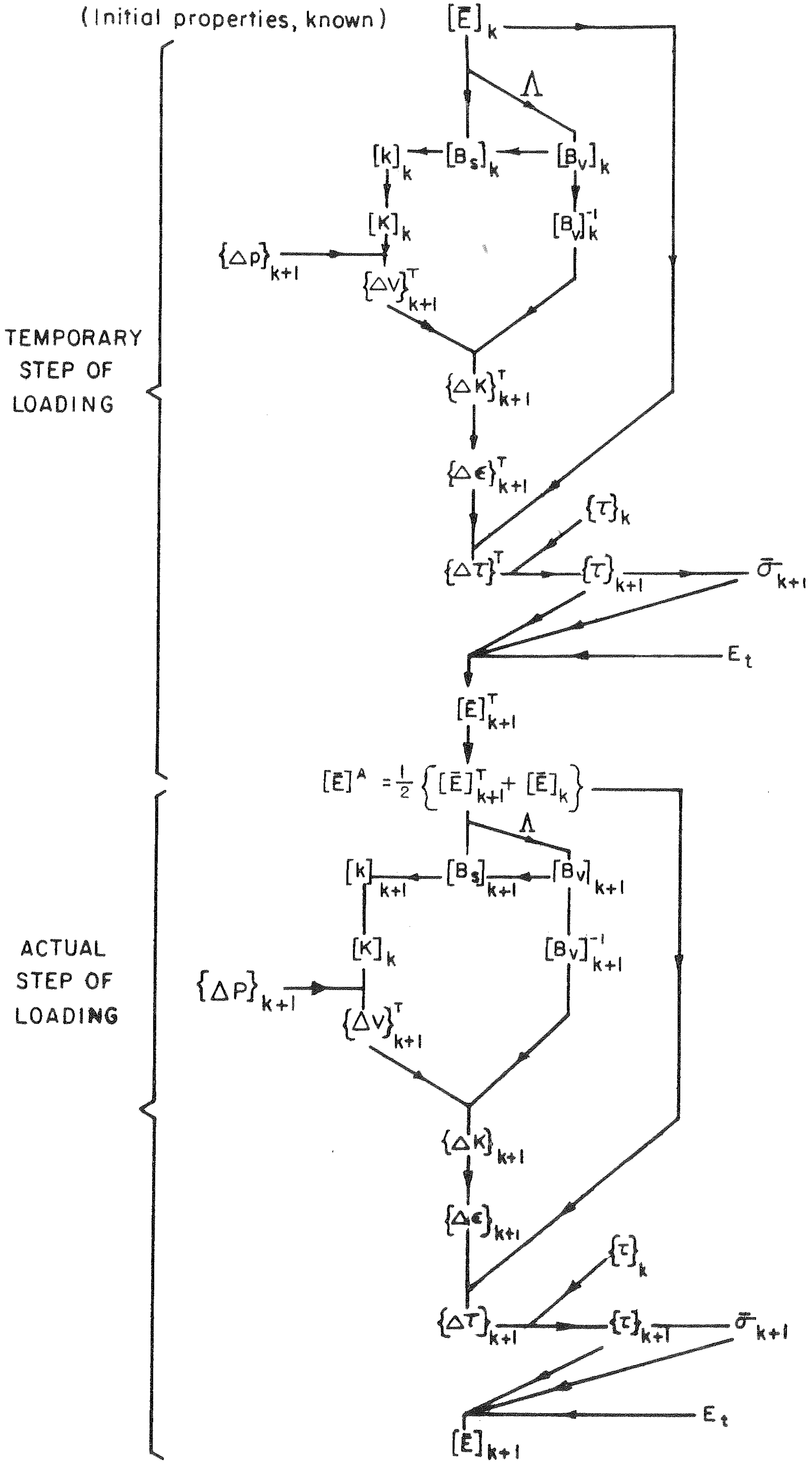

(Initial properties, known)
$[\bar{E}]_k$
Λ
$[k]_k$
$[B_s]_k$
$[B_v]_k$
$[K]_k$
$[B_v]^{-1}_k$
$\{\Delta p\}_{k+1}$
$\{\Delta V\}^T_{k+1}$
TEMPORARY STEP OF LOADING
$\{\Delta K\}^T_{k+1}$
$\{\Delta \epsilon\}^T_{k+1}$
$\{\tau\}_k$
$\{\Delta \tau\}^T$
$\{\tau\}_{k+1}$
$\bar{\sigma}_{k+1}$
E_t
$[\bar{E}]^T_{k+1}$
$[\bar{E}]^A = \frac{1}{2}\left\{[\bar{E}]^T_{k+1} + [\bar{E}]_k\right\}$
Λ
$[k]_{k+1}$
$[B_s]_{k+1}$
$[B_v]_{k+1}$
ACTUAL STEP OF LOADING
$[K]_k$
$[B_v]^{-1}_{k+1}$
$\{\Delta P\}_{k+1}$
$\{\Delta V\}^T_{k+1}$
$\{\Delta K\}_{k+1}$
$\{\Delta \epsilon\}_{k+1}$
$\{\tau\}_k$
$\{\Delta \tau\}_{k+1}$
$\{\tau\}_{k+1}$
$\bar{\sigma}_{k+1}$
E_t
$[\bar{E}]_{k+1}$

FIG. 8

VII. REMARKS

Each set of the two programs in the present form exceeds slightly the core storage of IBM 7094 digital computer. Therefore an overlay link structure is used. For details concerning the method of application of overlay structures the reader may consult Ref. [4].

The execution time of the programs depends mostly on the number of elements in the plate and the number of load increments. The number of layers in the elements does not affect the time consumption appreciably. The double precision program for hardening material uses about 12 seconds for each load increment for a plate with 16 elements and 40 layers. The time used in the single precision program for elastic-perfectly plastic material for the same number of elements and layers is 7 seconds. These estimates are for IBM 7094 digital computer.

To achieve a deeper insight of the manipulations in the computer programs the reader may consult Chapter II of Ref. [1].

VIII. GLOSSARY OF FORTRAN VARIABLE NAMES

A complete list of the variables appearing in the computer programs is presented below. The variables whose definition is evident from the programs themselves are left out. The variables with a parenthesis following them are vector quantities. The commas in the parenthesis indicate two and three dimensional arrays.

A(,) = Generalized coordinates 1x4

BK(,) = Stiffness matrix of the plate NE2xNE2

BS(,) = Force transformation matrix 4x4

BV(,) = Displacement transformation matrix 4x4

BV1(,) = Temporary matrix used for BV 4x4

C(,) = Increments of element curvature 2xNE

D(,,) = Flexural rigidity matrix of elements 2x2xNE

DR( ) = Ratio $[D(2,2,I)/D(1,1,I)]^{1/2}$ NE

E = Modulus of elasticity

ED( ) = Tangent modulus in the data for uniaxial stress strain curve. Dimension is optional.

F(,) = Yield function of layers NLxNE

FI = A ratio, which if unity, indicates the layer is elastic. Also yield function.

H( ) = Thickness of elements NE

I,II,J,JJ,K,L = Indices

IEXTRA = An integer variable which when becomes other than unity will cause the computer to stop executing and will indicate that loading has caused the effective stresses to increase beyond the available data in the input uniaxial stress-tangent modulus curve.

N = Integer variable

N1 = Integer variable indicating the maximum dimension of [BK] after it is modified for boundary conditions.

NBC = An integer indicating the type of boundary conditions specified in input data. For simply supported plate NBC = o, for clamped plate NBC ≠ o

ND = Number of data points in tangent modulus-stress diagram

NE = Number of elements

NK = Integer variable

NL, ANL = Number of layers in an element

NLL = Number of load increments in one load system

NP = Number of load systems

P( ) = Applied load (amplitude of external load distribution at nodal rings); also a temporary vector for PI and VI NE2

PI( ) = Equivalent nodal ring load either tributary or consistent NE2

PT( ) = Total equivalent nodal ring force vector NE2

Q(,) = Total internal stress resultant per unit length at nodal rings. This includes shear forces and radial moments only. 4xNE

QI(,) = Increment of internal stress resultant per unit length at nodal rings 4xNE

QTI( ) = Increment of nodal ring tangential moment per unit length NE

QT( ) = Total nodal ring tangential moment per unit length NE

R( ) = Radii of elements NE

RA = Average radius of a ring element

S1(,,),S2(,,) = Elements of elastic plastic compliance matrix 2xNLxNE

SB = Total effective strain

SD( ) = Stresses in input tangent modulus - stress diagram

SEI( ) = Elastic strain increment 2

SK(,,) = Stiffness matrices of elements 4x4xNE

SPI(,,) = Plastic strain increment 2xNLxNE

STB(,) = Increment of effective plastic strain NLxNE

STI(,,) = Strain increments of the layers of elements 2xNLxNE

T(,,) = Total stresses in layers of elements 2xNLxNE

TBS1(,);TBS2(,)= Effective stresses raised to second power NLxNE

TCR(,) = Total curvatures at nodal rings 2xNE

TI(,,) = Stress increments of the layers of elements 2xNLxNE

U = Poisson's ratio

V(,) = Total displacements at nodal rings 4xNE

VI(,) = Incremental displacements at nodal rings 4xNE

Y = A variable used in the flexural rigidity matrices of elements.

REFERENCES

1. Khojasteh-Bakht, M., Yaghmai, S. and Popov, E. P., "A Bending Analysis of Elastic Plastic Circular Plates," Report to NASA, Struct. Eng. Lab. SESM 66-4, Dept. of Civil Eng., Univ. of California, Berkeley, April 1966.

2. Archer, J. S., "Consistent Matrix Formulations for Structural Analysis Using Finite-Element Techniques," AIAA Jour. Vol. 3, No. 10, October 1965.

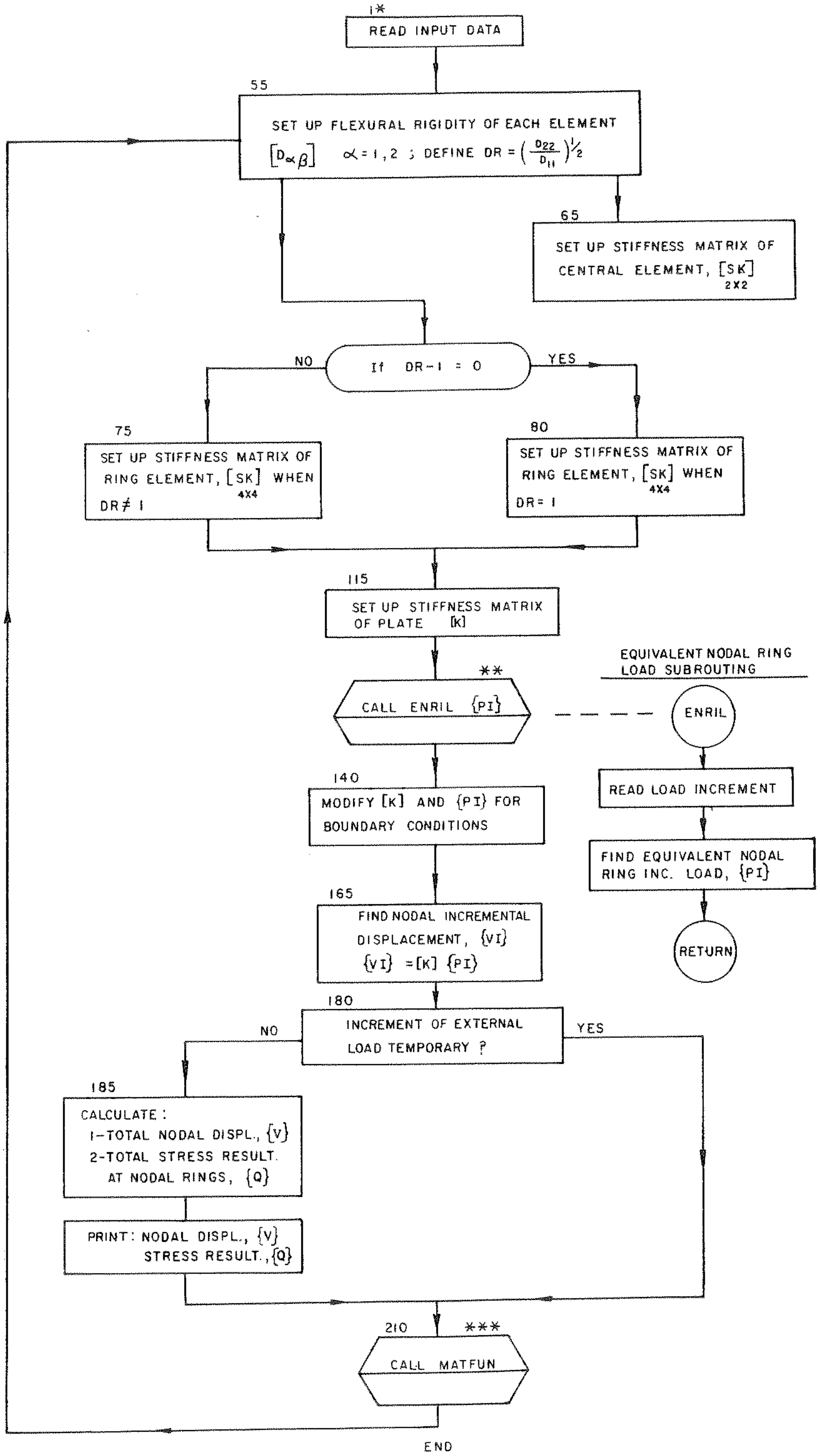
3. Levy, H. and Baggott, E. A., "Numerical Solutions of Differential Equations," Dover 1950.

4. IBM 7090/7094 IBSYS Operating Systems, Version 13, IBJOB Processor, IBM Systems Reference Library File No. 7090-27 form C28-6389-2.

CONCISE FLOW CHARTS OF COMPUTER PROGRAMS

ELASTIC- PLASTIC ANALYSIS OF A CIRCULAR PLATE WITH AXIS-SYMMETRIC LOADING

(EPACP)



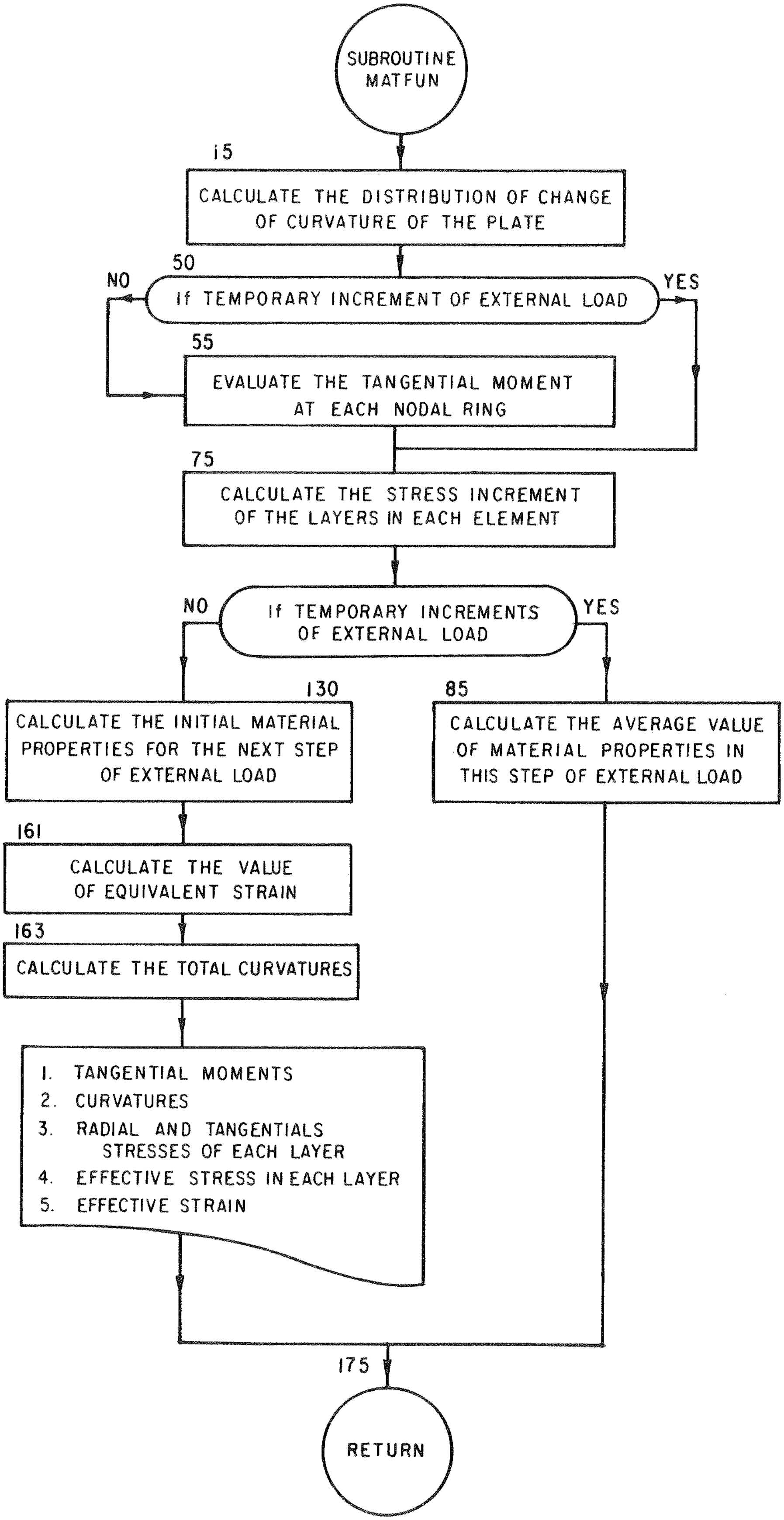
* THE NUMBERS REFER TO FORTRAN NUMBERS STATEMENTS IN THE PROGRAM.

** IN EPPACP, THE LOAD DISTRIBUTION ON NODAL RINGS IS TRIBUTARY HENCE ENRIL IS NOT USED

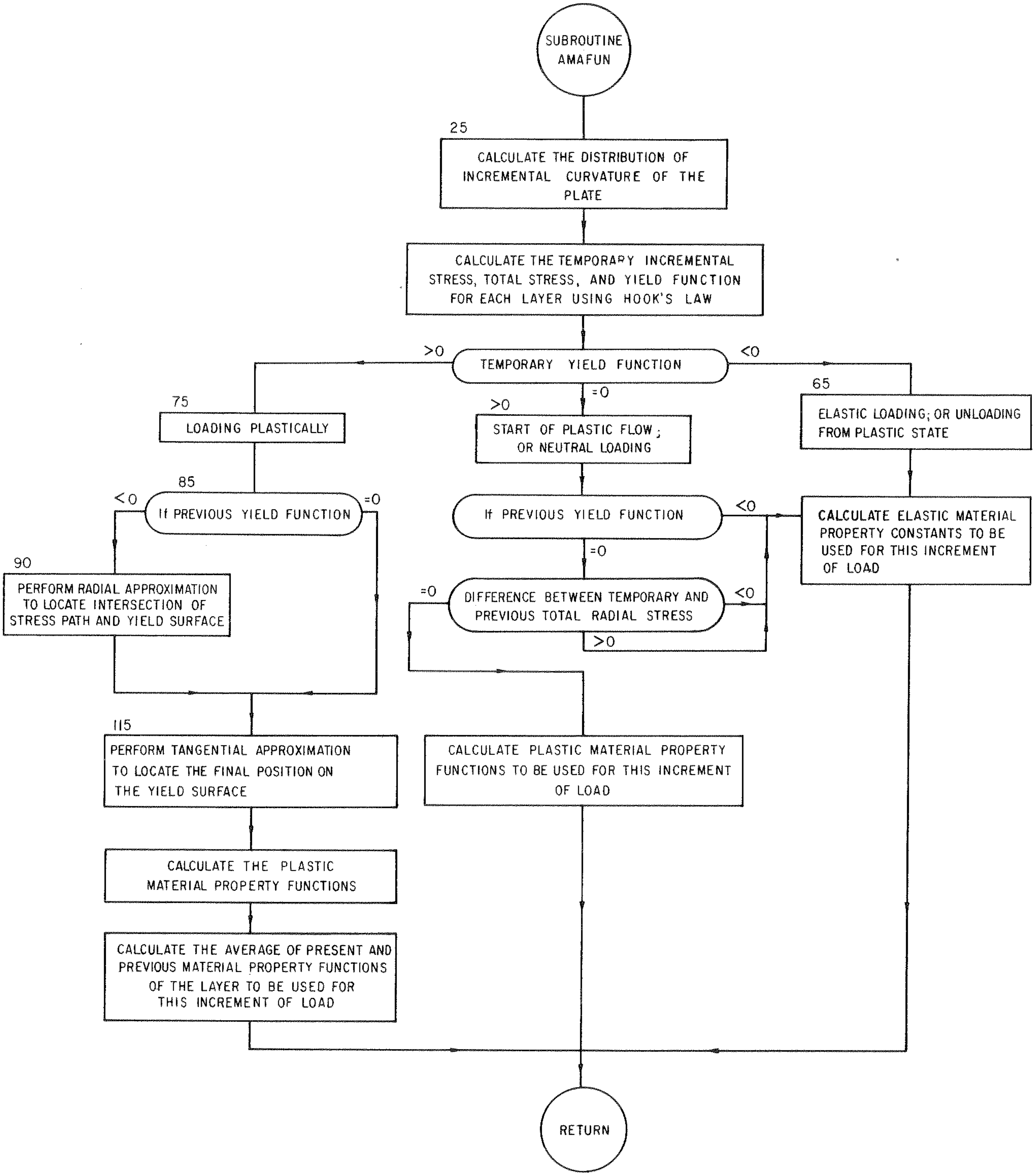
*** IN EPPACAP THERE ARE TWO SUBROUTINES FOR CONSTRUCTION OF MATERIAL PROPERTY FUNCTIONS, AMAFUN AND IMAFUN. WHENEVER THE INCREMENT OF EXTERNAL LOADING IS TEMPORARY AMAFUN IS USED AND IF NOT IMAFUN.

SUBROUTINE FOR THE FORMATION OF MATERIAL PROPERTY FUNCTIONS

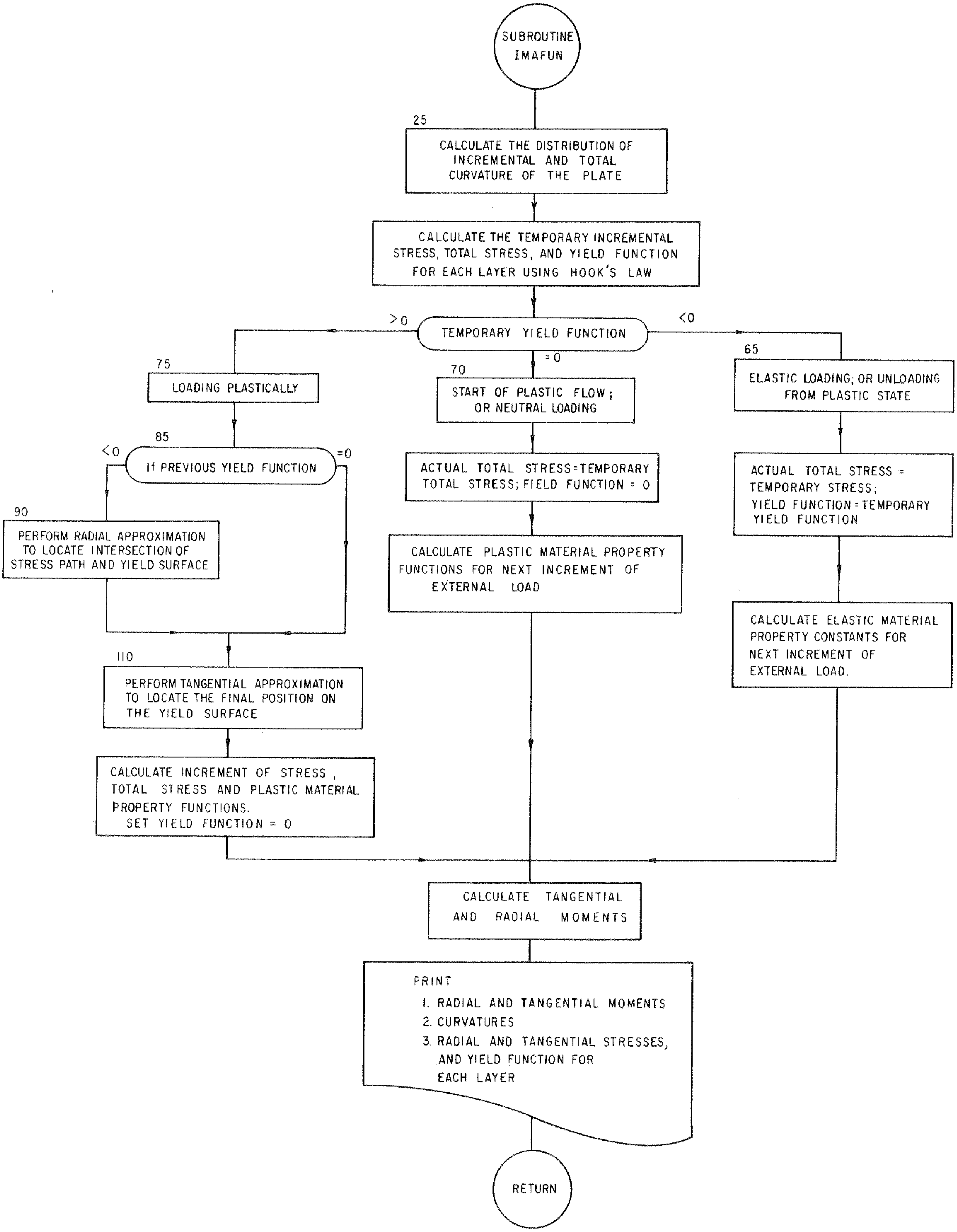
(FOR HARDENING MATERIALS)



SUBROUTINE FOR THE CALCULATION OF AVERAGE MATERIAL PROPERTY FUNCTIONS
(FOR ELASTIC-PERFECTLY PLASTIC MATERIAL)



SUBROUTINE FOR THE CALCULATION OF INITIAL MATERIAL PROPERTY FUNCTIONS OF THE NEXT INCREMENT OF LOAD (FOR ELASTIC PERFECTLY PLASTIC MATERIAL)



LISTING OF COMPUTER PROGRAMS

```
C      ELASTIC PERFECTLY PLASTIC ANALYSIS OF CIRCULAR PLATES WITH
C                        AXI-SYMMETRIC LOAD
C
C
       DIMENSION S1(2,30,30),S2(2,30,30),D(2,2,30),BV(4,4,30),DR(30),
      1SK(4,4,30),BK(60,60),P(60),PI(60),PT(60),VI(4,30),V(4,30),
      2QI(4,30),Q(4,30),R(30),  F(30,30) ,T(2,30,30) ,H(30),RA(30),
      3TCR(2,30)
C      READ DATA
C
       READ 1,NP
       DO 215  N=1,NP
       READ 2
       READ 3,  NL,NE,NLL,NBC
       READ 4,(H(I),I=1,NE)
       READ 5,(R(I),I=1,NE)
       READ 6, U,E,TY
     1 FORMAT (I2)
     2 FORMAT(72H
      1                            )
     3 FORMAT (4I4)
     4 FORMAT (8F9.5)
     5 FORMAT (8F9.5)
     6 FORMAT (F9.5,2E12.6)
C
C      PRINT DATA
C
       PRINT 15
       PRINT 2
       PRINT 16,N,NL,NE,NLL
       PRINT 17,(H(I),I=1,NE)
       PRINT 18
       PRINT 19, (R(I),I=1,NE)
    15 FORMAT(1H1)
    16 FORMAT(////2X,22HLOAD SET NUMBER        I4//(2X,18HNUMBER OF LAYERS
      1 =I4,2X,20HNUMBER OF ELEMENTS =I4,2X,30HNUMBER OF LOADING INCREMEN
      2TS =I4)///)
    17 FORMAT((2X,21HTHICKNESS OF ELEMENTS/(4X,10F9.5)///)
    18 FORMAT (2X,20HRADII OF NODAL RINGS)
    19 FORMAT (4X,12F9.5)
       PRINT 20,E,U,TY
    20 FORMAT(///2X,20HMOD. OF ELASTICITY =E15.8,5X,15HPOISSON RATIO =E12
      1.5,5X,14HYIELD STRESS = E15.8/////)
       NL = NL/2
       DO 30 I=1,NE
       DO 30 J=1,NL
       S1(1,J,I)=E/(1.0-U**2)
       S1(2,J,I)=U*S1(1,J,I)
       S2(1,J,I)= S1(2,J,I)
```

```
      S2(2,J,I) = S1(1,J,I)
      F(J,I) = -TY**2
      DO 30 K=1,2
   30 T(K,J,I)=0.0
      DO 35 I=1,NE
      DO 35 J=1,4
      V(J,I)=0.0
   35 Q(J,I)=0.0
      DO 40 I=1,NE
      DO 40 K=1,2
   40 TCR(K,I) = 0.0
      NE2 = 2*NE
      DO 45  I = 1,NE2
   45 PT(I) = 0.0
      RA(1) = R(1)
      DO 50 I=2,NE
   50 RA(I) = 0.5 *(R(I) + R(I-1))
      DO 215 L=1,NLL
      L1 =L/2
      LD = L-2*L1
      DO 85  I=1,NE
C
C     SET UP THE FLEXURAL RIGIDITIES OF THE ELEMENTS
C
      DO 55 II=1,2
      DO 55 JJ=1,2
   55 D(II,JJ,I)=0.0
      ANL = NL
      DO 60 J=1,NL
      AJ = J
      Y = 2.0/3.0*(H(I)/(2.0*ANL))**3*(3.0*AJ**2-3.0*AJ+1.0)
      D(1,1,I)= D(1,1,I) + S1(1,J,I)*Y
      D(1,2,I)= D(1,2,I) + S1(2,J,I)*Y
      D(2,1,I)=D(1,2,I)
   60 D(2,2,I) = D(2,2,I)+S2(2,J,I)*Y
      DR(I) = SQRT ( D(2,2,I)/D(1,1,I))
      IF (I-1)   65,65,70
C
C     SET UP THE STIFFNESS MATRIX OF THE CENTRAL ELEMENT
C
   65 BV(1,1,1) = 1.0
      BV(1,2,1) =-0.5*R(1)
      BV(2,1,1) = 0.0
      BV(2,2,1) = 0.5/R(1)
      SK(1,1,1) = 0.0
      SK(1,2,1) = 0.0
      SK(2,1,1)=0.0
      SK(2,2,1) = (+D(1,1,1)+D(1,2,1))/R(1)
      GO TO 85
   70 IF (DR(I)-1.0)  75,80,75
C
C     SET UP THE STIFFNESS MATRIX OF A RING IN CASE DR = 1
C
   80 DL1 =    (4.0*ALOG(R(I)/R(I-1))**2-(R(I)/R(I-1)-R(I-1)/R(I))**2)
```

```
       DL = 1.0/DL1
      BV(1,1,I) = DL*(2.0*ALOG(R(I)/R(I-1))*(2.*ALOG(R(I))-1.)-(R(I)/R(I
     1-1))**2+1.)
      BV(1,2,I) = DL*(-2.0*R(I-1)*ALOG(R(I))*ALOG(R(I)/R(I-1)) + R(I)*
     1ALOG(R(I-1))*(R(I)/R(I-1)-R(I-1)/R(I)))
      BV(1,3,I) =-DL*(2.0*ALOG(R(I)/R(I-1))*(2.*ALOG(R(I-1))-1.)+(R(I-1)
     1/R(I))**2-1.0)
      BV(1,4,I) = DL*(2.*R(I)*ALOG(R(I-1))*ALOG(R(I)/R(I-1))-R(I-1)*ALOG
     1(R(I))*(R(I)/R(I-1)-R(I-1)/R(I)))
      BV(2,1,I) = DL*(R(I)/R(I-1)*(2.*ALOG(R(I))+1.)-R(I-1)/R(I)*(2.*
     1ALOG(R(I-1))+1.))/(R(I-1)*R(I))
      BV(2,2,I) = DL*(2.*ALOG(R(I))*ALOG(R(I)/R(I-1))-ALOG(R(I-1))*(1.-
     1(R(I-1)/R(I))**2))/R(I-1)
      BV(2,3,I) = -BV(2,1,I)
      BV(2,4,I) = DL*(-2.*ALOG(R(I-1))*ALOG(R(I)/R(I-1))+ALOG( R(I))*((R
     1(I)/R(I-1))**2-1.))/R(I)
      BV(3,1,I) = -DL*4.*ALOG(R(I)/R(I-1))
      BV(3,2,I) = DL*R(I-1)*(2.*ALOG(R(I)/R(I-1))-(R(I)/R(I-1))**2+1.)
      BV(3,3,I) = -BV(3,1,I)
      BV(3,4,I) = DL*R(I)*(-2.*ALOG(R(I)/R(I-1))+1.-(R(I-1)/R(I))**2)
      BV(4,1,I) = DL*2.*(R(I-1)/R(I)-R(I)/R(I-1))/(R(I-1)*R(I))
      BV(4,2,I) = DL*(1.-(R(I-1)/R(I))**2-2.*ALOG(R(I)/R(I-1)))/R(I-1)
      BV(4,3,I) = -BV(4,1,I)
      BV(4,4,I) = DL*(1.-(R(I)/R(I-1))**2+2.*ALOG(R(I)/R(I-1)))/R(I)
       DL = DL1
      SK(1,1,I) = 8.*D(1,1,I)*(R(I-1)/R(I)-R(I)/R(I-1))/(DL*R(I-1)**2
     1*R(I))
      SK(1,2,I) = 4.*D(1,1,I)*(R(I)/R(I-1)-R(I-1)/R(I)-2.*R(I)/R(I-1)*
     1ALOG(R(I)/R(I-1)))/(DL*R(I-1)*R(I))
      SK(1,3,I) = -SK(1,1,I)
      SK(1,4,I) = 4.*D(1,1,I)*(1.-(R(I)/R(I-1))**2+2.*ALOG(R(I)/R(I-1)))
     1/(DL*R(I-1)*R(I))
      SK(2,1,I) = SK(1,2,I)
      SK(2,2,I) = -(D(1,1,I)+D(1,2,I))/R(I-1)+2.*D(1,1,I)*((R(I-1)/R(I)
     1)**2-(R(I)/R(I-1))**2+4.*ALOG(R(I)/R(I-1)))/(DL*R(I-1))
      SK(2,3,I) = -SK(1,2,I)
      SK(2,4,I) = 4.*D(1,1,I)*((R(I)/R(I-1))**2-1.-ALOG(R(I)/R(I-1))*
     1((R(I)/R(I-1))**2+1.))/(DL*R(I))
      SK(3,1,I) = R(I-1)/R(I)*SK(1,3,I)
      SK(3,2,I) = R(I-1)/R(I)*SK(2,3,I)
      SK(3,3,I) = R(I-1)/R(I)*SK( 1,1,I)
      SK(3,4,I) = -R(I-1)/R(I)*SK(1,4,I)
      SK(4,1,I) =-SK(3,4,I)
      SK(4,2,I) = R(I-1)/R(I)*SK(2,4,I)
      SK(4,3,I) = SK(3,4,I)
      SK(4,4,I) = (D(1,1,I)+D(1,2,I))/R(I) + 2.*D(1,1,I)*((R(I-1)/R(I))*
     1 *2-(R(I)/R(I-1))**2+4.*ALOG(R(I)/R(I-1)))/(DL*R(I))
      GO TO 85
C
C     SET UP THE STIFFNESS MATRIX OF A RING IN CASE   DR  IS NOT UNITY
C
   75 DL = (R(I)/R(I-1))**(1.+DR(I))+(R(I-1)/R(I))**(1.+DR(I))-2.-(1.+
     1DR(I))**2*ALOG(R(I)/R(I-1))**2*((1.-DR(I))**8*ALOG(R(I)/R(I-1))**8
     2/1814400. + (1.-DR(I))**6*ALOG(R(I)/R(I-1))**6/20160. + (1.-DR(I))
```

```
     3**4*ALOG(R(I)/R(I-1))**4/360. + (1.-DR(I))**2*ALOG(R(I)/R(I-1))**2
     4/12. + 1. )
      X = (1.-DR(I))*ALOG(R(I)/R(I-1))
      FAC = 1./362880.
      BETA = X * FAC
      DO 77 N1=1,7
      FAC = -FAC * FLOAT(10-N1)
   77 BETA = (BETA + FAC ) * X
      BETA = ALOG(R(I)/R(I-1))*(BETA+1.)
      BV(1,1,I) = 2.*(R(I)**(-1.-DR(I))-R(I-1)**(-1.-DR(I)))/((1.-DR(I))
     1*DL)
      BV(1,2,I) = -1./R(I-1)**DR(I)*((R(I-1)/R(I))**(1.+DR(I))-(R(I)/
     1R(I-1))**(1.-DR(I))+2.*(R(I)/R(I-1))**(1.-DR(I))*BETA)/((1.-DR(I))
     2*DL)
      BV(1,3,I) = -BV(1,1,I)
      BV(1,4,I) = -1./R(I)**DR(I)*((R(I)/R(I-1))**(1.+DR(I))-(R(I-1)/
     1R(I))**(1.-DR(I))-2.*BETA)/((1.-DR(I))*DL)
      BV(2,1,I) = 2.*(1.+DR(I))/R(I-1)**(1.-DR(I))*BETA/((1.-DR(I))*DL)
      BV(2,2,I) = R(I-1)**DR(I)*((R(I)/R(I-1))**(1.+DR(I))-(R(I-1)/
     1 R(I))**(1.-DR(I))-2.*BETA)/((1.-DR(I))*DL)
      BV(2,3,I) = -BV(2,1,I)
      BV(2,4,I) = R(I)**DR(I)*((R(I-1)/R(I))**(1.+DR(I))-(R(I)/R(I-1))
     1**(1.-DR(I))+2.*(R(I)/R(I-1))**(1.-DR(I))*BETA)/((1.-DR(I))*DL)
      BV(3,1,I) = (1.+DR(I))/(R(I-1)*R(I))*((R(I)/R(I-1))**DR(I)-(R(I-1)
     1/R(I))**DR(I))/((1.-DR(I))*DL)
      BV(3,2,I) = 1./R(I-1)*((R(I-1)/R(I))**(1.+DR(I))-1.+(1.+DR(I))*
     1BETA)/((1.-DR(I))*DL)
      BV(3,3,I) = -BV(3,1,I)
      BV(3,4,I) = 1./R(I)*((R(I)/R(I-1))**(1.+DR(  I))-1.-(1.+DR(I))*(
     1R(I)/R(I-1))**(1.-DR(I))*BETA)/((1.-DR(I))*DL)
      BV(4,1,I) = ((1.-DR(I))*((R(I)/R(I-1))**(1.+DR(I))-1.)-(1.+DR(I))
     1**2*(R(I)/R(I-1))**(1.-DR(I))*BETA)/((1.-DR(I))*DL)
      BV(4,2,I) = R(I-1)*(1.-(R(I)/R(I-1))**(1.+DR(I))+(1.+DR(I))*(R(I)/
     1R(I-1))**(1.-DR(I))*BETA)/((1.-DR(I))*DL)
      BV(4,3,I) = ((1.-DR(I))*((R(I-1)/R(I))**(1.+DR(I))-1.)+(1.+DR(I))
     1**2*BETA)/((1.-DR(I))*DL)
      BV(4,4,I) = -R(I)*((R(I-1)/R(I))**(1.+DR(I))-1.+ (1.+DR(I))*BETA)
     1/((1.-DR(I))*DL)
      SK(1,1,I) = 2.*(1.+DR(I))**2*D(1,1,I)/(R(I-1)**2*R(I))*((R(I)/
     1R(I-1))**DR(I)-(R(I-1)/R(I))**DR(I))/DL
      SK(1,2,I) = 2.*(1.+DR(I))*D(1,1,I)/R(I-1)**2*((R(I-1)/R(I))**
     1(1.+DR(I))-1.+(1.+DR(I))*BETA)/DL
      SK(1,3,I) = -SK(1,1,I)
      SK(1,4,I) = 2.*(1.+DR(I))*D(1,1,I)/(R(I-1)*R(I))*((R(I)/R(I-1))
     1**(1.+DR(I))-1.-(1.+DR(I))*(R(I)/R(I-1))**(1.-DR(I))*BETA)/DL
      SK(2,1,I) = SK(1,2,I)
      SK(2,2,I) = -(D(1,1,I)+D(1,2,I))/R(I-1)+(1.+DR(I))*D(1,1,I)/R(I-1)
     1*((R(I)/R(I-1))**(1.+DR(I))-(R(I-1)/R(I))**(1.+DR(I))-(1.+DR(I))
     2*((R(I)/R(I-1))**(1.-DR(I))+1.)*BETA)/DL
      SK(2,3,I) = -SK(2,1,I)
      SK(2,4,I) = (1.+DR(I))*D(1,1,I)*R(I)/R(I-1)**2*(((R(I-1)/R(I))**
     1(1.-DR(I))+1.)*((R(I-1)/R(I))**(1.+DR(I))-1.)+(1.+DR(I))*((R(I-1)
     2/R(I))**(1.+DR(I))+1.)*BETA)/DL
      SK(3,1,I) = SK(1,3,I)*R(I-1)/R(I)
```

```
      SK(3,2,I) = SK(2,3,I)*R(I-1)/R(I)
      SK(3,3,I) = -SK(3,1,I)
      SK(3,4,I) = -2.*(1.+DR(I))*D(1,1,I)/R(I)**2*((R(I)/R(I-1))**(1.+
     1DR(I))-1.-(1.+DR(I))*(R(I)/R(I-1))**(1.-DR(I))*BETA)/DL
      SK(4,1,I) = SK(1,4,I)*R(I-1)/R(I)
      SK(4,2,I) = SK(2,4,I)*R(I-1 )/R(I)
      SK(4,3,I) = -SK(4,1,I)
      SK(4,4,I) = (D(1,1,I)+D(1,2,I))/R(I)+(1.+DR(I))*D(1,1,I)/R(I)*((
     1R(I)/R(I-1))**(1.+DR(I))-(R(I-1)/R(I))**(1.+DR(I))-(1.+DR(I))*((
     2R(I)/R(I-1))**(1.-DR(I))+1.)*BETA)/DL
   85 CONTINUE
C
C     SET UP THE PLATE STIFFNESS MATRIX
C
      DO  110  I=1,NE2
      DO  110  J=1,NE2
  110 BK(I,J) = 0.0
      BK(1,1) = SK(1,1,1)
      BK(1,2) = SK(1,2,1)
      BK(2,1) = SK(2,1,1)
      BK(2,2) = SK(2,2,1)
      DO  115   I=2,NE
      DO  115   II=1,4
      M = 2*(I-2) + II
      DO   115   JJ = 1,4
      K=2*(I-2)+JJ
  115 BK(M,K)= BK(M,K)+SK(II,JJ,I)
C
C     READ THE INCREMENTAL LOAD
C
      READ 7,(PI(I), I=1,NE2)
    7 FORMAT (6E12.6)
      DO 120  I=1,NE2
  120 P(I) = PI(I)
      IF (LD-1)  135,125,135
  125 DO 130 I=1,NE2
  130 PT(I) = PT(I) +PI(I)
C
C     MODIFY  BK  AND  P  FOR BOUNDARY CONDITIONS
C
  135 IF(NBC)  140,140,150
  140 N1=NE2-1
      DO 145  K=1,4
      NK=NE2-K+1
      BK(N1,NK) = BK(NE2,NK)
  145 BK(NK,N1) = BK(NK,NE2)
      P(N1) = 0.0
      GO TO 155
  150 N1=NE2-2
  155 CALL MATINV (BK,N1,P,1)
      VI(1,1) = P(1)
      VI(2,1) = P(2)
  160 NE1 = NE-1
      DO 165  I=2,NE1
```

```
      DO 165    J=1,4
      K=2*(I-2) +J
  165 VI(J,I) = P(K)
      VI(1,NE) = VI(3,NE1)
      VI(2,NE) = VI(4,NE1)
      IF (NBC)  170,170,175
  170 VI(3,NE) = 0.0
      VI(4,NE) = P(N1)
      GO TO 180
  175 VI(3,NE) = 0.0
      VI(4,NE) = 0.0
  180 IF (LD-1)  210,185,210
  185 V(1,1)=V(1,1)+VI(1,1)
      V(2,1) = V(2,1)+VI(2,1)
      DO 190  I=2,NE
      DO 190  J=1,4
  190 V(J,I) = V(J,I)+VI(J,I)
      DO 195  I=1,2
      QI(I,1) = 0.0
      DO 195  J=1,2
      QI(I,1)=SK(I,J,1)*VI(J,1)+QI(I,1)
  195 Q(I,1) = Q(I,1)+QI(I,1)
      DO  205  I=2,NE
      DO  205  J=1,4
      QI(J,I) = 0.0
      DO 200 K=1,4
  200 QI(J,I) =QI(J,I)+SK(J,K,I)*VI(K,I)
  205  Q(J,I) = Q(J,I)+QI(J,I)
      PRINT 250,L
      PRINT 251
      PRINT 252
      PRINT 253
      I=1
      PRINT  254,I,PI(1),PI(2),VI(2,1),VI(1,1)
      DO 255  I=2,NE
      I2 = 2*I-1
      PRINT 254, I,PI(I2),PI(I2+1),VI(4,I),VI(3,I)
  255 CONTINUE
      PRINT 256
      I=1
      PRINT 257, I,QI(2,I),QI(1,I)
      PRINT 254, (I,QI(2,I),QI(4,I),QI(1,I),QI(3,I) ,I=2,NE)
      PRINT 258
      PRINT 252
      PRINT 253
      I =1
      PRINT 254,  I,PT(1),PT(2),V(2,1),V(1,1)
      DO  259  I=2,NE
      I2=2*I-1
      PRINT 254, I,PT(I2),PT(I2+1),V(4,I),V(3,I)
  259 CONTINUE
      PRINT 256
       I =1
      PRINT 257,I,Q(2,I),Q(1,I)
```

```
      PRINT 254, (I,Q(2,I),Q(4,I),Q(1,I),Q(3,I) ,I=2,NE)
      CALL IMAFUN (S1,S2,BV,DR,VI,RA, F ,T,NE,NL,H,TY,E,U,TCR)
      GO TO 215
  210 CALL AMAFUN (S1,S2,BV,DR,VI,RA,F,T,NE,NL,H,TY,E,U)
  215 CONTINUE
C
  250 FORMAT (///2X,15H LOADING STEP = I4//)
  251 FORMAT  (//25X,36HTHE FOLLOWING ARE INCREMENTAL VALUES//)
  252 FORMAT ( 10HNODAL RING,10X, 12HAPPLIED LOAD,19X,13HDISPLACEMENTS)
  253 FORMAT  (14X, 8HVERTICAL,8X,6HMOMENT,12X,5HSLOPE,8X, 10HDEFLECTION
     1//)
  254 FORMAT  (3X,I4,2X,D15.6,2X,D15.6,2X,D15.6,2X,D15.6)
  256 FORMAT(//2X,7HELEMENT, 4X,8HMOMENT 1, 8X,8HMOMENT 2,10X,7HSHEAR 1,
     1 9X,7HSHEAR 2//)
  257 FORMAT  (3X,I4,19X,D15.6,19X,D15.6)
  258 FORMAT (//25X,30HTHE FOLLOWING ARE TOTAL VALUES//)
  220 STOP
      END
```

```
C       SUBROUTINE FOR THE FORMATION OF AVERAGE MATERIAL PROPERTY
C                                       FUNCTIONS
C
C
        SUBROUTINE AMAFUN   (S1,S2,BV,DR,VI,RA, F  ,T,NE,NL,H,TY,E,U)
        DIMENSION S1(2,30,30),S2(2,30,30),BV(4,4,30),DR(30),VI(4,30),
      1 C(2,30),R(30),F(30,30),TM(2,30),RA(30),STI(2,30,30),TI(2,30,30),
      2 T(2,30,30),A(4,30),H(30)
C
C
        ANL = NL
        DO 125   I=1,NE
C
C       CALCULATE THE CURVATURES
C
        AI = H(I)/(4.0*ANL)
        IF  (I-1)  20,20,30
     20 DO 25   J=1,2
        A(J,1)=0.0
        DO  25   K=1,2
     25 A(J,1)=BV(J,K,1)*VI(K,1)+A(J,I)
        C(1,1)=2.0*A(2,1)
        C(2,1)=2.0*A(2,1)
        GO TO 50
     30 DO 35   J=1,4
        A(J,I)=0.0
        DO  35   K=1,4
     35 A(J,I)=BV(J,K,I)*VI(K,I)+A(J,I)
        IF (DR(I)-1.0) 40,45,40
     40 C(1,I)=A(1,I)*DR(I)*(1.0+DR(I))*RA(I)**(DR(I)-1.0)-A(2,I)*DR(I)*(1
      1.0-DR(I))/(RA(I)**(1.0+DR(I)))+2.0*A(3,I)
        C(2,I)=A(1,I)*(1.0+DR(I))*RA(I)**(DR(I)-1.0)+A(2,I)*(1.0-DR(I))/
      1(RA(I)**(DR(I)+1.0))+2.0*A(3,I)
        GO TO 50
     45 C(1,I) = 2.0*A(2,I)-A(3,I)/(RA(I)**2)+A(4,I)*(2.0*ALOG(RA(I))+3.0)
        C(2,I) = 2.0*A(2,I)+A(3,I)/RA(I)**2+A(4,I)*(2.0*ALOG(RA(I))+1.0)
     50 DO 125   J=1,NL
C
C       CALCULATE THE STRESS INCREMENT OF EACH LAYER
C
        AB=2*J-1
        Z=AB*AI
        STI(1,J,I)  =  -Z*C(1,I)
        STI(2,J,I)  =  -Z*C(2,I)
        TI(1,J,I)  =  0.0
        TI(2,J,I)  =  0.0
        TI1  =  E/(1.-U**2)*(STI(1,J,I)+U*STI(2,J,I))
        TI2  =  E/(1.-U**2)*(STI(2,J,I)+U*STI(1,J,I))
        TE1  =  T(1,J,I)  +  TI1
        TE2  =  T(2,J,I)  +  TI2
        TE3  =  T(1,J,I)
        TE4  =  T(2,J,I)
          F1     =  TE1**2  -  TE1*TE2  +  TE2**2  -  TY**2
        IF  (F1)  65,70,75
```

```
      DEL = E/(TD1**2+TD2**2+2.*U*TD1*TD2)
      TIT1 = DEL*TD2*(TD2*STI(1,J,I)-TD1*STI(2,J,I))
      TIT2 =-DEL*TD1*(TD2*STI(1,J,I)-TD1*STI(2,J,I))
      TT1 = TE3 + TIT1
      TT2 = TE4 + TIT2
      G = TT1**2 - TT1*TT2 + TT2**2
      GAMA = TY/SQRT(G)
      TA1 = GAMA*TT1
      TA2 = GAMA*TT2
      TDT1 =(TA1-0.5*TA2)*2./3.
      TDT2 =(TA2-0.5*TA1)*2./3.
      ALFA = E/(TDT1**2+TDT2**2+2.*U*TDT1*TDT2)
      S1(1,J,I) =(ALFA * TDT2**2+S1(1,J,I))/2.
        S1(2,J,I)=(-ALFA * TDT1*TDT2+S1(2,J,I))/2.
      S2(1,J,I) = S1(2,J,I)
      S2(2,J,I) =(ALFA*TDT1**2+S2(2,J,I))/2.
  125 CONTINUE
      RETURN
      END
```

```
C       SUBROUTINE FOR FORMATION OF INITIAL MATERIAL PROPERTY FUNCTIONS
C                   OF THE NEXT STEP OF LOAD
C
C
        SUBROUTINE IMAFUN  (S1,S2,BV,DR,VI,RA, F ,T,NE,NL,H,TY,E,U,TCR)
        DIMENSION S1(2,30,30),S2(2,30,30),BV(4,4,30),DR(30),VI(4,30),
      1 C(2,30),R(30),F(30,30),TM(2,30),RA(30),STI(2,30,30),TI(2,30,30),
      2 T(2,30,30),A(4,30),H(30),TCR(2,30)
C
C
        PRINT 1
        PRINT 2
        ANL = NL
        DO 125  I=1,NE
C
C       CALCULATE THE CURVATURES
C
        AI = H(I)/(4.0*ANL)
        IF  (I-1)  20,20,30
     20 DO 25  J=1,2
        A(J,1)=0.0
        DO  25   K=1,2
     25 A(J,1)=BV(J,K,1)*VI(K,1)+A(J,I)
        C(1,1)=2.0*A(2,1)
        C(2,1)=2.0*A(2,1)
        GO TO 50
     30 DO 35  J=1,4
        A(J,I)=0.0
        DO  35   K=1,4
     35 A(J,I)=BV(J,K,I)*VI(K,I)+A(J,I)
        IF (DR(I)-1.0) 40,45,40
     40 C(1,I)=A(1,I)*DR(I)*(1.0+DR(I))*RA(I)**(DR(I)-1.0)-A(2,I)*DR(I)*(1
      1.0-DR(I))/(RA(I)**(1.0+DR(I)))+2.0*A(3,I)
        C(2,I)=A(1,I)*(1.0+DR(I))*RA(I)**(DR(I)-1.0)+A(2,I)*(1.0-DR(I))/
      1(RA(I)**(DR(I)+1.0))+2.0*A(3,I)
        GO TO 50
     45 C(1,I) = 2.0*A(2,I)-A(3,I)/(RA(I)**2)+A(4,I)*(2.0*ALOG(RA(I))+3.0)
        C(2,I) = 2.0*A(2,I)+A(3,I)/RA(I)**2+A(4,I)*(2.0*ALOG(RA(I))+1.0)
     50 DO 55 K=1,2
        TCR(K,I) = TCR(K,I)+C(K,I)
     55 TM(K,I) = 0.0
        DO 125  J=1,NL
C
C       CALCULATE THE STRESS INCREMENT OF EACH LAYER
C
        AB=2*J-1
        Z=AB*AI
        STI(1,J,I) = -Z*C(1,I)
        STI(2,J,I) = -Z*C(2,I)
        TI(1,J,I) = 0.0
        TI(2,J,I) = 0.0
        TI1 = E/(1.-U**2)*(STI(1,J,I)+U*STI(2,J,I))
        TI2 = E/(1.-U**2)*(STI(2,J,I)+U*STI(1,J,I))
        TE1 = T(1,J,I) + TI1
```

```
      TE2 = T(2,J,I) + TI2
        F1     = TE1**2 - TE1*TE2 + TE2**2 - TY**2
      IF (F1)  65,70,75
C
C     EITHER LOADING ELASTICALLY OR UNLOADING PLASTICALLY
C
   65 T(1,J,I) = TE1
      T(2,J,I) = TE2
      F(J,I) = F1
      TI(1,J,I) = TI1
      TI(2,J,I) = TI2
      S1(1,J,I) = E/(1.-U**2)
      S1(2,J,I) = U*S1(1,J,I)
      S2(1,J,I) = S1(2,J,I)
      S2(2,J,I) = S1(1,J,I)
      GO TO 115
C
C     EITHER STARTING OF PLASTIC FLOW OR NEUTRAL LOADING
C
   70 T(1,J,I) = TE1
      T(2,J,I) = TE2
      F(J,I) = F1
      TI(1,J,I) = TI1
      TI(2,J,I) = TI2
      TD1 = (TE1-0.50*TE2)*2./3.
      TD2 = (TE2-0.50*TE1)*2./3.
      ALFA = E/(TD1**2 + TD2**2 + 2.*U*TD1*TD2)
      S1(1,J,I) = ALFA * TD2**2
      S1(2,J,I) = -ALFA * TD1*TD2
      S2(1,J,I) = S1(2,J,I)
      S2(2,J,I) = ALFA *TD1**2
      GO TO 115
C
C     LOADING PLASTICALLY
C
   75 IF(I-1) 85,80,85
   80 TI(1,J,I) = TY - T(1,J,I)
      TI(2,J,I) = TY - T(2,J,I)
      T(1,J,I) = TY
      T(2,J,I) = TY
      F(J,I) = 0.0
      S1(1,J,I) = .50 * E/(1. +U)
      S1(2,J,I) = -S1(1,J,I)
      S2(1,J,I) = S1(2,J,I)
      S2(2,J,I) = S1(1,J,I)
      GO TO 115
   85 IF (F(J,I))  90,110,90
   90 FA = ABS(F(J,I))
      FA1 = ABS(F1)
      IF(FA1-FA) 95,100,100
   95 G = TE1**2 - TE1*TE2 + TE2**2
      BETA = TY/SQRT(G)
      TI(1,J,I) = BETA*TE1 - T(1,J,I)
      TI(2,J,I) = BETA*TE2 - T(2,J,I)
```

```
      T(1,J,I) = BETA * TE1
      T(2,J,I) = BETA * TE2
      GO TO 110
  100 G = T(1,J,I)**2 - T(1,J,I)*T(2,J,I) + T(2,J,I)**2
      BETA = TY/SQRT(G)
      DO 105 K=1,2
      TI(K,J,I) =(BETA-1.)*T(K,J,I)
  105 T(K,J,I) = BETA * T(K,J,I)
  110 STE1 = (TI(1,J,I)-U*TI(2,J,I))/E
      STE2 = (-U*TI(1,J,I)+TI(2,J,I))/E
      STI(1,J,I) = STI(1,J,I) - STE1
      STI(2,J,I) = STI(2,J,I) - STE2
      TD1 = (T(1,J,I)-0.50*T(2,J,I))*2./3.
      TD2 = (T(2,J,I)-0.50*T(1,J,I))*2./3.
      DEL = E/(TD1**2+TD2**2+2.*U*TD1*TD2)
      TIT1 = DEL*TD2*(TD2*STI(1,J,I)-TD1*STI(2,J,I))
      TIT2 =-DEL*TD1*(TD2*STI(1,J,I)-TD1*STI(2,J,I))
      TT1 = T(1,J,I) + TIT1
      TT2 = T(2,J,I) + TIT2
      G = TT1**2 - TT1*TT2 + TT2**2
      GAMA = TY/SQRT(G)
      TA1 = GAMA*TT1
      TA2 = GAMA*TT2
      TI(1,J,I) = TA1 - T(1,J,I) + TI(1,J,I)
      TI(2,J,I) = TA2 - T(2,J,I) + TI(2,J,I)
      T(1,J,I) = TA1
      T(2,J,I) = TA2
      F(J,I) = 0.0
      TDT1 = (T(1,J,I) - 0.50*T(2,J,I))*2./3.
      TDT2 = (T(2,J,I) - 0.50*T(1,J,I))*2./3.
      ALFA = E/(TDT1**2+TDT2**2+2.*U*TDT1*TDT2)
      S1(1,J,I) = ALFA * TDT2**2
      S1(2,J,I) =-ALFA * TDT1*TDT2
      S2(1,J,I) = S1(2,J,I)
      S2(2,J,I) = ALFA*TDT1**2
C
C     CALCULATE THE TANGENTIAL AND RADIAL MOMENTS
C
  115 DO 120 K=1,2
  120 TM(K,I) = TM(K,I) + 4.*AI *Z * T(K,J,I)
      PRINT 3,  I,J,TI(1,J,I),T(1,J,I),TI(2,J,I),T(2,J,I),F(J,I)
  125 CONTINUE
      PRINT 4
      PRINT 5
      PRINT  6,   (I,RA(I),(TM(K,I),K=1,2), (TCR(K,I),K=1,2),I=1,NE)
    1 FORMAT (///2X,7HELEMENT,2X,5HLAYER,10X,13HRADIAL STRESS,19X,17HTAN
     1GENTIAL STRESS,10X,14HYIELD FUNCTION)
    2 FORMAT (18X,11HINCREMENTAL,8X,5HTOTAL,11X,11HINCREMENTAL,9X,5HTOTA
     1L//)
    3 FORMAT (2X,I4,4X,I4,2X,E15.6,2X,E15.6,2X,E15.6,2X,E15.6,2X,E15.6)
    4 FORMAT (//25X,30HTHE FOLLOWING ARE TOTAL VALUES//)
    5 FORMAT (//5X,7HELEMENT,9X,6HRADIUS,6X,13HRADIAL MOMENT,7X,17HTANGE
     1NTIAL MOMENT,6X,14HRAD. CURVATURE,9X,14HTAN. CURVATURE//)
    6 FORMAT (7X,I2,9X,F8.3,5X,E15.7,7X,E15.7,7X,E15.7,11X,E15.7/)
```

```
RETURN
END
```

```
C     ELASTIC PLASTIC ANALYSIS OF A CIRCULAR PLATE WITH AXI-SYMMETRIC
C                                LOAD
C
C
      DIMENSION S1(2,20,20),S2(2,20,20),D(2,2,20),BS(4,4),BV(4,4,20),
     1DR(20),SK(4,4,20),BK(40,40),PI(40),PT(40),P(40),VI(4,20),V(4,20),
     2 QI(4,20),Q(4,20),R(20),TBS1(20,20),T(2,20,20),MIV(4),SIV(4),
     3BV1(4,4),SD(30),ED(30),BM(4,4),QT(20),STB(20,20),TCR(2,20)
      DOUBLE PRECISION H,R,E,U,SD,ED,S1,S2,TBS1,T,D,ANL,AJ,Y,DR,BV,SK,
     1 BS,BV1, TCR,SIV  ,BK,P,PI,PT,VI,V,QI,Q,BM,QT,STB
      CALL DPIO
C
C     READ DATA
C
      READ 1,NP
      DO 215  N=1,NP
      READ 2
      READ 3,  NL,NE,NLL,NBC
      READ 4,H
      READ 5,(R(I),I=1,NE)
      READ 6 ,E,U
      READ 7,ND,(SD(I),I=1,ND)
      READ 8, (ED(I),I=1,ND)
    1 FORMAT (I2)
    2 FORMAT(72H
     1                         )
    3 FORMAT (4I4)
    4 FORMAT (D12.6)
    5 FORMAT (6D12.6)
    6 FORMAT( D12.6,D8.3)
    7 FORMAT (I4/(6D12.3))
    8 FORMAT (6D12.3)
C
C     PRINT DATA
C
      PRINT 15
      PRINT 2
      PRINT 16,N,NL,NE,NLL
      PRINT 17,H
      PRINT 18
      PRINT 19, (R(I),I=1,NE)
      PRINT 20,E,U
      PRINT 21,ND,(SD(I),I=1,ND)
      PRINT 22,(ED(I), I=1,ND)
   15 FORMAT(1H1)
   16 FORMAT(////2X,22HLOAD SET NUMBER       I4//(2X,18HNUMBER OF LAYERS
     1 =I4,2X,20HNUMBER OF ELEMENTS =I4,2X,30HNUMBER OF LOADING INCREMEN
     2TS =I4)///)
   17 FORMAT (2X,20HTHICKNESS OF PLATE =D13.6//)
   18 FORMAT (2X,20HRADII OF NODAL RINGS)
   19 FORMAT  (3X,8D13.6)
   20 FORMAT(///2X,20HMOD. OF ELASTICITY =D15.8,5X,15HPOISSON RATIO =D12
     1.5//)
   21 FORMAT(///2X,63HNUMBER OF DATA POINTS IN EQUIVALENT-STRESS TAN. MO
```

```
     1DULUS TABLE =I4//(2X, 8HSTRESS = 7D15.8))
   22 FORMAT (//(2X,14HTAN. MCDULUS = 6D15.6))
      NL = NL/2
      DO 30 I=1,NE
      DO 30 J=1,NL
      S1(1,J,I)=E/(1.0-U**2)
      S1(2,J,I)=U*S1(1,J,I)
      S2(1,J,I)= S1(2,J,I)
      S2(2,J,I) = S1(1,J,I)
      STB(J,I) = 0.0
   30 TBS1(J,I) = 0.0
      DO 35 I=1,NE
      QT(I) = 0.0
      DO 35 J=1,4
      V(J,I)=0.0
   35 Q(J,I)=0.0
      DO 45 I=1,NE
      DO 40 K=1,2
   40 TCR(K,I) = 0.0
      DO 45 J=1,NL
      DO 45 K=1,2
   45 T(K,J,I)=0.0
      NE2 = 2*NE
      DO 50  I = 1,NE2
   50 PT(I) = 0.0
      DO 215 L=1,NLL
      L1 =L/2
      LD = L-2*L1
      DO 105 I=1,NE
C
C     SET UP THE FLEXURAL RIGIDITY OF AN ELEMENT
C
      DO 55 II=1,2
      DO 55 JJ=1,2
   55 D(II,JJ,I)=0.0
      ANL = NL
      DO 60 J=1,NL
      AJ = J
      Y = 2.0/3.0*(H/(2.0*ANL))**3*(3.0*AJ**2-3.0* AJ+1.0)
      D(1,1,I)= D(1,1,I) + S1(1,J,I)*Y
      D(1,2,I)= D(1,2,I) + S1(2,J,I)*Y
      D(2,1,I)=D(1,2,I)
   60 D(2,2,I) = D(2,2,I)+S2(2,J,I)*Y
      DR(I) =DSQRT ( D(2,2,I)/D(1,1,I))
      IF (I-1)  65,65,70
C
C     SET UP THE STIFFNESS MATRIX FOR THE CENTRAL ELEMENT
C
   65 BV(1,1,1) = 1.0
      BV(1,2,1) =-0.5*R(1)
      BV(2,1,1) = 0.0
      BV(2,2,1) = 0.5/R(1)
      SK(1,1,1) = 0.0
      SK(1,2,1) = 0.0
```

```
      SK(2,1,1)=0.0
      SK(2,2,1) = (+D(1,1,1)+D(1,2,1))/R(1)
      GO TO 105
C
C     SET UP THE STIFFNESS MATRIX FOR THE RING ELEMENT WHEN DR IS NOT =1
C
   70 IF(DR(I) -1.0)    75,80,75
   75 BS(1,1) = 0.0
      BS(1,2)=0.0
      BS(1,3)=2.0*(D(1,1,I)-D(2,2,I))/R(I-1)
      BS(1,4)=0.0
      BS(2,1)=-(1.0+DR(I))*(DR(I)*D(1,1,I)+D(1,2,I))*R(I-1)**(DR(I)-1.0)
      BS(2,2) = -(1.0-DR(I))*(-DR(I)*D(1,1,I)+D(1,2,I))/R(I-1)**(
     11.0+DR(I))
      BS(2,3)=-2.0*(D(1,1,I)+D(1,2,I))
      BS(2,4)=0.0
      BS(3,1)=0.0
      BS(3,2)=0.0
      BS(3,3) = -2.0*(D(1,1,I)-D(2,2,I))/R(I)
      BS(3,4)=0.0
      BS(4,1)= (1.0+DR(I))*(DR(I)*D(1,1,I)+D(1,2,I))*R(I   )**(DR(I)-1.0)
      BS(4,2)= (1.0-DR(I))*(-DR(I)*D(1,1,I)+D(1,2,I))/R(I   )**(1.0+
     1DR(I))
      BS(4,3) =  2.0*(D(1,1,I)+D(1,2,I))
      BS(4,4)=0.0
            BV(1,1,I)=R(I-1)**(1.0+DR(I))
      BV(1,2,I)=R(I-1)**(1.0-DR(I))
      BV(1,3,I)=R(I-1)**2
      BV(1,4,I)=1.0
      BV(2,1,I) = (1.0+DR(I))*R(I-1)**DR(I)
      BV(2,2,I)=(1.0-DR(I))/R(I-1)**DR(I)
      BV(2,3,I)=2.0*R(I-1)
      BV(2,4,I)=0.0
      BV(3,1,I)=R(I   )**(1.0+DR(I))
      BV(3,2,I)=R(I   )**(1.0-DR(I))
      BV(3,3,I)=R(I   )**2
      BV(3,4,I)=1.0
      BV(4,1,I)=(1.0+DR(I))*R(I   )**DR(I)
            BV(4,2,I)=(1.0-DR(I))/R(I   )**DR(I)
      BV(4,3,I)=2.0*R(I   )
      BV(4,4,I)=0.0
      GO TO 85
C
C     SET UP THE STIFFNESS MATRIX FOR THE RING ELEMENT WHEN DR=1
C
   80 BS(1,1) = 0.0
      BS(1,2)=0.0
      BS(1,3)=0.0
      BS(1,4)=4.0*D(1,1,I)/R(I-1)
      BS(2,1)=0.0
      BS(2,2)=-2.0*(D(1,1,I)+D(1,2,I))
      BS(2,3)=-(-D(1,1,I)+D(1,2,I))/R(I-1)**2
      BS(2,4)=-D(1,1,I)*(2.0*DLOG(R(I-1))+3.0)-D(1,2,I)*
     1(2.0*DLOG(R(I-1))+1.0)
```

```
      BS(3,1)=0.0
      BS(3,2)=0.0
      BS(3,3)=0.0
      BS(3,4) =-4.0*D(1,1,I)/R(I)
      BS(4,1)=0.0
      BS(4,2)= 2.0*(D(1,1,I)+D(1,2,I))
      BS(4,3)= (-D(1,1,I)+D(1,2,I))/R(I)**2
      BS(4,4)= D(1,1,I)*(2.0*DLOG(R(I   ))+3.0)+D(1,2,I)*(2.0*DLOG(R(I
     1))+1.0)
      BV(1,1,I)=1.0
      BV(1,2,I)=R(I-1)**2
      BV(1,3,I)=DLOG(R(I-1))
      BV(1,4,I)=R(I-1)**2*DLOG(R(I-1))
      BV(2,1,I)=0.0
      BV(2,2,I)=2.0*R(I-1)
      BV(2,3,I)=1.0/R(I-1)
      BV(2,4,I)=R(I-1)*(2.0*DLOG(R(I-1))+1.0)
      BV(3,1,I)=1.0
      BV(3,2,I)=R(I   )**2
      BV(3,3,I)=DLOG(R(I   ))
      BV(3,4,I)=R(I   )**2*DLOG(R(I   ))
      BV(4,1,I)=0.0
      BV(4,2,I)=2.0*R(I   )
      BV(4,3,I)=1.0/R(I   )
      BV(4,4,I) = R(I)*(2.0*DLOG(R(I))+1.0)
   85 DO 90 J=1,4
      DO 90 K=1,4
   90 BV1(J,K)=BV(J,K,I)
      CALL INVERT (BV1,4,4,MIV ,SIV )
      DO 95  J=1,4
      DO 95  K=1,4
   95 BV(J,K,I)=BV1(J,K)
      DO 100  II=1,4
      DO 100  JJ=1,4
      SK(II,JJ,I)=0.0
      DO 100  KK=1,4
  100 SK(II,JJ,I)=SK(II,JJ,I)+BS(II,KK)*BV(KK,JJ,I)
  105 CONTINUE
C
C     SET UP THE PLATE STIFFNESS MATRIX
C
      DO  110  I=1,NE2
      DO  110  J=1,NE2
  110 BK(I,J) = 0.0
      BK(1,1) = SK(1,1,1)
      BK(1,2) = SK(1,2,1)
      BK(2,1) = SK(2,1,1)
      BK(2,2) = SK(2,2,1)
      DO  115   I=2,NE
      DO  115   II=1,4
      M = 2*(I-2) + II
      DO   115   JJ = 1,4
      K=2*(I-2)+JJ
  115 BK(M,K)= BK(M,K)+SK(II,JJ,I)
```

```
C
C     READ THE INCREMENTAL LOAD
C
      PRINT 250, L
      CALL ENRIL(NE,NE2,DR,R,BV,PI,P)
      DO 120  I=1,NE2
  120 P(I) = PI(I)
      IF (LD-1)    135,125,135
  125 DO 130  I=1,NE2
  130 PT(I) = PT(I) +PI(I)
C
C     MODIFY  BK  AND  P  FOR BOUNDARY CONDITIONS
C
  135 IF (NBC)  140,140,150
  140 N1=NE2-1
      DO 145  K=1,4
      NK=NE2-K+1
      BK(N1,NK) = BK(NE2,NK)
  145 BK(NK,N1) = BK(NK,NE2)
      P(N1) = 0.0
      GO TO 155
  150 N1=NE2-2
C
C     FIND NODAL DISPLACEMENTS AND STRESS RESULTANTS
C
  155 CALL MATINV (BK,N1,P,1)
      VI(1,1) = P(1)
      VI(2,1) = P(2)
  160 NE1 = NE-1
      DO 165  I=2,NE1
      DO 165   J=1,4
      K=2*(I-2) +J
  165 VI(J,I) = P(K)
      VI(1,NE) = VI(3,NE1)
      VI(2,NE) = VI(4,NE1)
      IF (NBC)  170,170,175
  170 VI(3,NE) = 0.0
      VI(4,NE) = P(N1)
      GO TO 180
  175 VI(3,NE) = 0.0
      VI(4,NE) = 0.0
  180 IF (LD-1)  210,185,210
  185 V(1,1)=V(1,1)+VI(1,1)
      V(2,1) = V(2,1)+VI(2,1)
      DO 190  I=2,NE
      DO 190  J=1,4
  190 V(J,I) = V(J,I)+VI(J,I)
      DO 195  I=1,2
      QI(I,1) = 0.0
      DO 195  J=1,2
      QI(I,1)=SK(I,J,1)*VI(J,1)+QI(I,1)
  195 Q(I,1) = Q(I,1)+QI(I,1)
      DO  205  I=2,NE
      DO  205  J=1,4
```

```
      QI(J,I) = 0.0
      DO 200 K=1,4
  200 QI(J,I) =QI(J,I)+SK(J,K,I)*VI(K,I)
  205  Q(J,I) = Q(J,I)+QI(J,I)
      PRINT 251
      PRINT 252
      PRINT 253
      I=1
      PRINT  254,I,PI(1),PI(2),VI(2,1),VI(1,1)
      DO 255  I=2,NE
      I2 = 2*I-1
      PRINT 254, I,PI(I2),PI(I2+1),VI(4,I),VI(3,I)
  255 CONTINUE
      PRINT 256
      I=1
      PRINT 257, I,QI(2,I),QI(1,I)
      PRINT 254, (I,QI(2,I),QI(4,I),QI(1,I),QI(3,I) ,I=2,NE)
      PRINT 258
      PRINT 252
      PRINT 253
      I =1
      PRINT 254,  I,PT(1),PT(2),V(2,1),V(1,1)
      DO  259  I=2,NE
      I2=2*I-1
      PRINT 254, I,PT(I2),PT(I2+1),V(4,I),V(3,I)
  259 CONTINUE
      PRINT 256
       I =1
      PRINT 257,I,Q(2,I),Q(1,I)
      PRINT 254, (I,Q(2,I),Q(4,I),Q(1,I),Q(3,I) ,I=2,NE)
  210 CALL MATFUN (S1,S2,BV,DR,VI,R,TBS1,T,NE,NL,H,ND,ED,SD,E,U,IEXTRA,
     1D,QT,STB,LD,TCR)
      IF (IEXTRA-1)  220,215,220
  215 CONTINUE
  250 FORMAT (///2X,15H LOADING STEP = I4//)
  251 FORMAT (////25X,36HTHE FOLLOWING ARE INCREMENTAL VALUES//)
  252 FORMAT ( 10HNODAL RING,10X, 12HAPPLIED LOAD,19X,13HDISPLACEMENTS)
  253 FORMAT  (14X, 8HVERTICAL,8X,6HMOMENT,12X,5HSLOPE,8X, 10HDEFLECTION
     1//)
  254 FORMAT  (3X,I4,2X,D15.6,2X,D15.6,2X,D15.6,2X,D15.6)
  256 FORMAT(//2X,7HELEMENT, 4X,8HMOMENT 1, 8X,8HMOMENT 2,10X,7HSHEAR 1,
     1 9X,7HSHEAR 2//)
  257 FORMAT  (3X,I4,19X,D15.6,19X,D15.6)
  258 FORMAT (//25X,30HTHE FOLLOWING ARE TOTAL VALUES//)
  220 STOP
      END
```

```
C     SUBROUTINE FOR TRANSFORMATION OF DISTRIBUTED LOAD
C                TO NODAL RING LOAD
C
C
      SUBROUTINE ENRIL(NE,NE2,DR,R,BV,PI,P)
      DIMENSION R(20),BV(4,4,20),PI(40),S(4),PG(4),DR(20),P(40)
      DOUBLE PRECISION  DR,R,BV,PI,S,P,PG
      DO  1  I=1,NE2
    1 PI(I) = 0.0
      NE1 = NE+1
      READ 2,(P(I),I=1,NE1)
    2 FORMAT (4D15.5)
      PRINT 10
   10 FORMAT (//44HMAGNITUDE OF DISTRIBUTED LOAD AT NODAL RINGS//)
      PRINT 11,(P(I),I=1,NE1)
   11  FORMAT (3X,7D15.8)
C
C     FIRST ELEMENT
C
      PG(1) = (2.*P(2)+P(1))/6.*R(1)
      PG(2) = (4.*P(2)+P(1))/20.*R(1)**3
      DO  3  J=1,2
      DO  3  K=1,2
    3 PI(J) = BV(K,J,1)*PG(K)+PI(J)
C
C     OTHER ELEMENTS
C
      DO  8  I=2,NE
      IF  (DR(I) - 1.0)    5,4,5
    4 PG(1) = P(I)/2.*(R(I)**2-R(I-1)**2)+(P(I+1)-P(I))/3.*(R(I)**2+
     1 R(I)*R(I-1)+R(I-1)**2)
      PG(2) = P(I)/4.*(R(I)**4-R(I-1)**4)+(P(I+1)-P(I))/5.*(R(I)**4+
     1 R(I)**3 *R(I-1)+R(I)**2*R(I-1)**2+R(I)*R(I-1)**3+R(I-1)**4)
      PG(3) = P(I)/2.*(R(I)**2*DLOG(R(I))-R(I-1)**2*DLOG(R(I-1))+
     1 0.5*(-R(I)**2+R(I-1)**2))+(P(I+1)-P(I))/3.*((DLOG(R(I))-1./3.)*
     2 (R(I)**2+R(I)*R(I-1)+R(I-1)**2)+R(I-1)**3*DLOG(R(I)/R(I-1))/
     3 (R(I)-R(I-1)))
      PG(4) = P(I)/4.*(R(I)**4*DLOG(R(I))-R(I-1)**4*DLOG(R(I-1))+
     1 0.25 *(-R(I)**4+R(I-1)**4))+(P(I+1)-P(I))/5.*((DLOG(R(I))-.2)*
     2 (R(I)**4+R(I)**3*R(I-1)+R(I)**2*R(I-1)**2+R(I)*R(I-1)**3+R(I-1)
     3 **4)+R(I-1)**5*DLOG(R(I)/R(I-1))/(R(I)-R(I-1)))
      GO TO 6
    5  PG(1) = P(I)/(3.+DR(I))*(R(I)**(3.+DR(I))-R(I-1)**(3.+DR(I)))+
     1 (P(I+1)-P(I))/(4.+DR(I))*(R(I)**(4.+DR(I))-R(I-1)**(4.+DR(I)))/
     2 (R(I)-R(I-1))
      PG(2) = P(I)/(3.-DR(I))*(R(I)**(3.-DR(I))-R(I-1)**(3.-DR(I)))+
     1 (P(I+1)-P(I))/(4.-DR(I))*(R(I)**(4.-DR(I))-R(I-1)**(4.-DR(I)))/
     2 (R(I)-R(I-1))
      PG(3) = P(I)/4.*(R(I)**4-R(I-1)**4)+(P(I+1)-P(I))/5.*(R(I)**4+
     1 R(I)**3*R(I-1)+R(I)**2*R(I-1)**2+R(I)*R(I-1)**3+R(I-1)**4)
      PG(4) = P(I)/2.*(R(I)**2-R(I-1)**2)+(P(I+1)-P(I))/3.* (R(I)**2+
     2 R(I)*R(I-1)+R(I-1)**2)
    6 DO  7  J=1,4
      S(J) = 0.0
```

```
      DO   7   K=1,4
    7 S(J) = BV(K,J,I)*PG(K)+S(J)
      I2 = 2*I
      PI(I2-3)= PI(I2-3)+S(1)/R(I-1)
      PI(I2-2)= PI(I2-2)+S(2)/R(I-1)
      PI(I2-1)= PI(I2-1)+S(3)/R(I)
      PI(I2) = PI(I2)+S(4)/R(I)
    8 CONTINUE
       RETURN
      END
```

```
      GO TO 70
   65 QTI(I) = -(1.0+DR(I))*(D(2,2,I)+DR(I)*D(1,2,I))*R(I)**(DR(I)-1.0)
     1*A(1,I)+(1.0-DR(I))*(DR(I)*D(1,2,I)-D(2,2,I))/R(I)**(1.0+  DR(I))
     2 *A(2,I)-2.0*(D(1,2,I)+D(2,2,I))*A(3,I)
   70 QT(I) = QT(I) +QTI(I)
   75 DO 80    J=1,NL
C
C     CALCULATE THE STRESS INCREMENT OF EACH LAYER
C
      AB=2*J-1
      Z=AB*AI
      STI(1,J,I) = -Z*C(1,I)
      STI(2,J,I) = -Z*C(2,I)
      TI(1,J,I) = 0.0
      TI(2,J,I) = 0.0
      DO 80   K= 1,2
      TI(1,J,I) = S1(K,J,I)*STI(K,J,I)+TI(1,J,I)
   80 TI(2,J,I) = S2(K,J,I)*STI(K,J,I)+TI(2,J,I)
      IF (LD-1)  85,130,85
C
C     CALCULATE THE AVERAGE MATERIAL PROPERTY IN THIS STEP OF LOADING
C
   85 DO 125  J=1,NL
      DO 90  K=1,2
      TEMP(K) = T(K,J,I)
   90 T(K,J,I) = T(K,J,I)  +TI(K,J,I)
      TBS2(J,I) = T(1,J,I)**2-T(1,J,I)*T(2,J,I)+T(2,J,I)**2
      TB = DSQRT(TBS2(J,I))
      IF (TBS2(J,I)-TBS1(J,I))   95,105,105
   95 IEXTRA = 1
  100 S1(1,J,I) = E/(1.0-U**2)
      S1(2,J,I) = U*S1(1,J,I)
      S2(1,J,I) = S1(2,J,I)
      S2(2,J,I) = S1(1,J,I)
      GO TO 120
  105 CALL INTER(ED,SD,ND,TB,ET,IEXTRA)
      IF  (IEXTRA -1)   175,110,175
  110  F1 = E/ET
      IF (F1-1.0)  100,100,115
  115 F = F1-1.0
      DELTA =(1.0 - U**2+F*((1.25-U)*(T(1,J,I)**2+T(2,J,I)**2) - (2.0
     1  -2.5*U)*T(1,J,I)*T(2,J,I))/TBS2(J,I))/E
      S1(1,J,I) =((1.0+(T(2,J,I)-0.5*T(1,J,I))**2*F/TBS2(J,I))/DELTA+
     1 S1(1,J,I))/2.
      S1(2,J,I) =((U-(T(1,J,I)-0.5*T(2,J,I))*(T(2,J,I)-0.5*T(1,J,I))*F/
     1 TBS2(J,I))/DELTA+S1(2,J,I))/2.
      S2(1,J,I) = S1(2,J,I)
      S2(2,J,I) =((1.0+(T(1,J,I)-0.5*T(2,J,I))**2*F/TBS2(J,I))/DELTA +
     1 S2(2,J,I))/2.
  120 DO 125  K=1,2
  125 T(K,J,I) = TEMP(K)
      GO TO 165
C
C     CALCULATE THE INITIAL MATERIAL PROPERTIES OF THE NEXT
```

```
C                 STEP OF LOADING
C
  130 DO   162   J=1,NL
      DO 135   K=1,2
  135 T(K,J,I) = T(K,J,I)+TI(K,J,I)
      TBS2(J,I) = T(1,J,I)**2-T(1,J,I)*T(2,J,I)+T(2,J,I)**2
      TB = DSQRT(TBS2(J,I))
      IF (TBS2(J,I)-TBS1(J,I))     140,150,150
  140 IEXTRA = 1
  145 S1(1,J,I) = E/(1.0-U**2)
      S1(2,J,I) = U*S1(1,J,I)
      S2(1,J,I) = S1(2,J,I)
      S2(2,J,I) = S1(1,J,I)
      GO TO 161
  150 CALL INTER(ED,SD,ND,TB,ET,IEXTRA)
      IF  (IEXTRA -1)    175,155,175
  155 F1 = E/ET
      IF(F1-1.0)    145,145,160
  160 TBS1(J,I) = TBS2(J,I)
      F = F1-1.0
      DELTA =(1.0 - U**2+F*((1.25-U)*(T(1,J,I)**2+T(2,J,I)**2) - (2.0
     1  -2.5*U)*T(1,J,I)*T(2,J,I))/TBS2(J,I))/E
      S1(1,J,I) = (1.0+(T(2,J,I)-0.5*T(1,J,I))**2*F/TBS2(J,I))/DELTA
      S1(2,J,I) = (U-(T(1,J,I)-0.5*T(2,J,I))*(T(2,J,I)-0.5*T(1,J,I))*F/
     1 TBS2(J,I))/DELTA
      S2(1,J,I) = S1(2,J,I)
      S2(2,J,I) = (1.0+(T(1,J,I)-0.5*T(2,J,I))**2*F/TBS2(J,I))/DELTA
C
C     CALCULATE THE EQUIVALENT STRAIN
C
  161 SEI(1)      =   (TI(1,J,I)-U*TI(2,J,I))/E
      SEI(2)      =  (-U*TI(1,J,I)+TI(2,J,I))/E
      SPI(1,J,I) = STI(1,J,I)-SEI(1)
      SPI(2,J,I) = STI(2,J,I)-SEI(2)
      STB(J,I) = STB(J,I)+DSQRT(4./3.*(SPI(1,J,I)**2+SPI(2,J,I)**2 +
     1 SPI(1,J,I)*SPI(2,J,I)))
      SB = STB(J,I)+TB/E
  162 PRINT 3,   I,J,TI(1,J,I),T(1,J,I),TI(2,J,I),T(2,J,I),SB,TB
C
C     CALCULATE THE TOTAL CURVATURES
C
      DO 163 K=1,2
  163 TCR(K,I) = TCR(K,I) + C(K,I)
  165 CONTINUE
      IF (LD-1) 175,170,175
  170 PRINT 4
      PRINT 5
      PRINT6,(I,QT(I),TCR(1,I),TCR(2,I),I=1,NE)
    1 FORMAT (///2X,7HELEMENT,2X,5HLAYER,10X,13HRADIAL STRESS,19X,17HTAN
     1GENTIAL STRESS,12X,10HEQUIVALENT,6X,10HEQUIVALENT)
    2 FORMAT (18X,11HINCREMENTAL,8X,5HTOTAL,11X,11HINCREMENTAL,9X,5HTOTA
     1L11X,6HSTRAIN,11X,6HSTRESS//)
    3 FORMAT (2X,I4,4X,I4,2X,D15.6,2X,D15.6,2X,D15.6,2X,D15.6,2X,D15.6,
     12X,D15.6)
```

```
C     SUBROUTINE FOR INTERPOLATION
C
C
      SUBROUTINE  INTER (Y,X,N,A,B,IEXTRA)
      DIMENSION Y(1),X(1)
      DOUBLE PRECISION  Y,X,A,B,D,D1
      DO  30  I=1,N
      IF (A-X(I))   20,15,25
   15 B=Y(I)
      IEXTRA = 1
      GO TO 35
  20  D = A-X(I-1)
      D1 = X(I)-X(I-1)
      B= Y(I-1) + D/D1*(Y(I)-Y(I-1))
      IEXTRA = 1
      GO TO 35
   25  IF (A-X(N))   30,15,40
   30  CONTINUE
   35  RETURN
   40 IEXTRA =2
      PRINT  45
   45 FORMAT  (//60HTHE TABLE OF EQUIVALENT STRESS TAN. MODULUS MUST BE
     1ENLARGED)
      GO TO 35
      END
```