UC Berkeley
SEMM Reports Series

Title

Computational Methods for Inverse Finite Elastostatics

Permalink

https://escholarship.org/uc/item/8th4p876

Authors

Govindjee, Sanjay
Mihalic, Paul

Publication Date

1995-02-01

eScholarship.org Powered by the California Digital Library
University of California

REPORT NO.
UCB/SEMM-95/01

STRUCTURAL ENGINEERING
MECHANICS AND MATERIALS

COMPUTATIONAL METHODS FOR INVERSE FINITE ELASTOSTATICS

BY

SANJAY GOVINDJEE

AND

PAUL A. MIHALIC

FEBRUARY 1995

DEPARTMENT OF CIVIL ENGINEERING
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA

Computational Methods for Inverse Finite Elastostatics †

SANJAY GOVINDJEE
PAUL A. MIHALIC

Department of Civil Engineering
University of California
Berkeley, CA 94720.

Submitted To: Computer Methods in Applied Mechanics and Engineering

§ Abstract

In the inverse motion problem in finite hyper-elasticity, the classical formulation relies on conservation laws based on Eshelby's energy-momentum tensor. This formulation is shown to be lacking in several regards for a particular class of inverse motion problems where the deformed configuration and Cauchy traction are given and the undeformed configuration must be calculated. It is shown that for finite element calculations a simple re-examination of the equilibrium equations provides a more suitable finite element formulation. This formulation is also shown to involve only minor changes to existing elements designed for forward motion calculations. Examples illustrating the method in simple and complex situations involving a Neo-Hookean material are presented.

† Research sponsored in part by the Engineering Foundation Grant No. RI-A-94-02.

§1. Introduction

A problem in the design of elastomeric parts is determining the reference/undeformed shape of a part such that under a given load the part will have a desired deformed shape. This problem is a type of inverse problem. However, in contrast to many inverse problems, it is not ill-posed in the sense of Hadamard's definition. For a brief synopsis of the mathematical issues surrounding inverse problems the reader is referred to LAMM [1993]. Here, it is merely noted that (in contrast to most ill-posed problems) the present inverse deformation problem does *not* involve the determination of boundary data for a given interior solution.

A method for formulating such inverse deformation problems in finite elasticity was proposed by SHIELD [1967]. This method relies on exploiting a duality in the equations of finite hyper-elasticity when the role of the spatial and reference coordinates are interchanged. Later CHADWICK [1975] exploited this same duality to formulate Shield's equilibrium equations in terms of Eshelby's energy-momentum tensor, ESHELBY [1956, 1975]. The use of these equations in the finite element analysis of inverse motion problems, however, appears not to have been performed to date. In this paper, it is shown that this classical energy-momentum formulation is *not* the most expedient when dealing with inverse motion problems as defined above. Rather, a simple re-parameterization of the standard equilibrium equations leads to a more efficient and straightforward technique for solving such problems via the finite element method. Additionally, it is found that the implementation of this new formulation involves only minor changes to elements designed for computing forward motion problems.

The paper will be divided into three main sections and two appendices as follows: Section 2 summarizes the problem description and points out some of the difficulties with the energy-momentum formulation; Section 3 presents the finite element formulation of the proposed method; in Section 4 a set of examples are presented to illustrate the method; Appendix A presents a comprehensive set of duality relations for elastostatics; and Appendix B presents and briefly discusses the weak form equations necessary to solve an inverse deformation problem using the energy-momentum formulation.

§2. Problem Description

In the standard problem of elastostatics one considers a stress-free reference placement of a body $B \subset \mathbb{R}^3$ and its motion $\boldsymbol{\phi} : B \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$. Points in the reference placement are denoted by $\boldsymbol{X} \in B$ and map to points $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{X})$ in the deformed configuration $S = \boldsymbol{\phi}(B)$. The motion $\boldsymbol{\phi}(\boldsymbol{X})$ is usually considered the primary unknown when the body is subjected to known boundary displacements and tractions; $\boldsymbol{\phi}$ is determined by solving the equilibrium equations. The problem of interest here is the inverse of this problem where the deformed configuration and boundary conditions are known and the reference configuration and inverse motion $\boldsymbol{\varphi}(\boldsymbol{x}) = \boldsymbol{\phi}^{-1}(\boldsymbol{x})$ are the desired unknowns.

Such problems are typically formulated by exploiting a set of duality relations that allow one to formulate the inverse problem in a form that appears similar to a standard elastostatic problem. These relations, first noted by SHIELD [1967], and later notably expanded by OGDEN [1975A,B] and CHADWICK [1975] follow from the assumption of a homogeneous body whose constitutive response is governed by a stored energy function per unit reference volume $W : \mathrm{Lin}_+ \longrightarrow \mathbb{R}$, where Lin_+ denotes the space of second order tensors with positive determinant. In the standard problem, defining the deformation gradient

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{X}} \tag{2.1}$$

leads to the relation for the first Piola-Kirchhoff stress tensor

$$\boldsymbol{P} = \frac{\partial W(\boldsymbol{F})}{\partial \boldsymbol{F}} . \tag{2.2}$$

The specification of the standard problem is completed by requiring the equilibrium equations to be satisfied for given boundary conditions. Thus, for all $\boldsymbol{X} \in B$

$$\mathrm{DIV}[\boldsymbol{P}] + \widehat{\boldsymbol{B}} = \boldsymbol{O} \qquad \text{and} \qquad \boldsymbol{F}\boldsymbol{P}^T = \boldsymbol{P}\boldsymbol{F}^T , \tag{2.3}$$

for all $\boldsymbol{X} \in \partial B_t$

$$\boldsymbol{P}\boldsymbol{N} = \bar{\boldsymbol{T}} , \tag{2.4}$$

and for all $\boldsymbol{X} \in \partial B_\phi$

$$\boldsymbol{\phi} = \bar{\boldsymbol{\phi}} , \tag{2.5}$$

where $\mathrm{DIV}[\cdot]$ is the divergence operator with respect to $\boldsymbol{X}$, $\widehat{\boldsymbol{B}}$ is a given body force per unit reference volume, $\bar{\boldsymbol{T}}$ is a given traction function per unit reference area, $\bar{\boldsymbol{\phi}}$ is a given surface motion, $\partial B_t \cap \partial B_\phi = \emptyset$, and $\partial B_t \cup \partial B_\phi = \partial B$ the boundary of B. Equations (2.1)–(2.5) constitute a complete set of equations that may be solved for the unknown $\boldsymbol{\phi}$.

If the body is assumed to be homogeneous and body forces absent, then the inverse problem can be formulated in a completely analogous fashion. Start with a stored energy function per unit deformed volume $w : \mathrm{Lin}_+ \longrightarrow \mathbb{R}$ defined as

$$w = W/J , \tag{2.6}$$

where $J = \det[\boldsymbol{F}]$. Next, define the inverse deformation gradient

$$\boldsymbol{f} = \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}} . \tag{2.7}$$

And note that $\boldsymbol{f} = \boldsymbol{F}^{-1} \circ \boldsymbol{\varphi}$, where $\circ$ is the composition symbol; also, for later convenience define $j = \det[\boldsymbol{f}] = 1/(J \circ \boldsymbol{\varphi})$. Next, define

$$\boldsymbol{p} = \frac{\partial w(\boldsymbol{f})}{\partial \boldsymbol{f}} , \tag{2.8}$$

where $\boldsymbol{p}$ is a two-point version of Eshelby's energy-momentum tensor which may be expressed as

$$\boldsymbol{p} = j \left[(W\boldsymbol{I} - \boldsymbol{F}^T\boldsymbol{P}) \circ \boldsymbol{\varphi} \right] \boldsymbol{f}^{-T} , \tag{2.9}$$

where $\boldsymbol{\Sigma} = W\boldsymbol{I} - \boldsymbol{F}^T\boldsymbol{P}$ is Eshelby's energy-momentum tensor in essentially CHADWICK'S notation and $\boldsymbol{I}$ is the second order identity tensor; ESHELBY [1975, §5] denotes $\boldsymbol{\Sigma}$ as $\boldsymbol{P}^*$ and CHADWICK [1975]† denotes it $\boldsymbol{\Sigma}^T$. If one assumes a smooth motion ($\phi \in C^2(B)$) and positive Jacobian ($J > 0$), then static equilibrium is satisfied if and only if for all $\boldsymbol{x} \in S$

$$\mathrm{div}[\boldsymbol{p}] = \boldsymbol{o} \qquad \text{and} \qquad \boldsymbol{f}\boldsymbol{p}^T = \boldsymbol{p}\boldsymbol{f}^T\,, \tag{2.10}$$

where div[·] is the divergence operator with respect to $\boldsymbol{x}$. To complete the statement of the inverse problem, boundary conditions need to be given. In analogy with (2.4) and (2.5), these would read: for all $\boldsymbol{x} \in \partial S_t$

$$\boldsymbol{p}\boldsymbol{n} = \bar{\boldsymbol{t}}_{em}\,, \tag{2.11}$$

and for all $\boldsymbol{x} \in \partial S_\varphi$

$$\boldsymbol{\varphi} = \bar{\boldsymbol{\varphi}}\,, \tag{2.12}$$

where $\bar{\boldsymbol{t}}_{em}$ is a given energy-momentum traction, $\bar{\boldsymbol{\varphi}}$ is a given inverse motion, $\partial S_t \cap \partial S_\varphi = \emptyset$, and $\partial S_t \cup \partial S_\varphi = \partial S$ the deformed body's surface.

Remark 2.1.
Equations (2.7)–(2.12) constitute a complete set of equations that may be used to compute an inverse motion $\boldsymbol{\varphi}$. Further, because their form is identical to the standard equations of elastostatics, existing computational methods can be directly exploited. □

Remark 2.2.
A comprehensive list of duality relations is given in Appendix A. □

Remark 2.3.
There are several obstacles to the practical application of of Equations (2.7)–(2.12). One, the boundary condition (2.11) is not of interest here since the energy-momentum traction is usually unknown. For a given traction $\bar{\boldsymbol{t}}$ per unit deformed area the correct boundary condition is

$$w\boldsymbol{n} - \boldsymbol{f}^T\boldsymbol{p}\boldsymbol{n} = \bar{\boldsymbol{t}}\,, \tag{2.13}$$

where $\boldsymbol{n}$ is the boundary normal in the deformed configuration; this relation follows directly from Eq. $(A.3)_3$. This type of boundary condition, though formally tractable, greatly complicates numerical computations; see Appendix B. Two, the formulation is only valid for homogeneous bodies without body forces. And three, symmetry of the duals to the Cauchy stress, $\boldsymbol{\sigma}$, and the second Piola-Kirchhoff stress, $\boldsymbol{S}$, $\boldsymbol{\Sigma}$ and $\boldsymbol{s}$ respectively, only hold in the case of isotropy. □

† The presence of the transpose in Chadwick's notation merely reflects a difference in the convention of which leg of the stress tensor corresponds to the section normal and which leg corresponds to the traction direction.

Remark 2.4.
In Appendix B, the weak form of this classical formulation of the inverse problem is given for completeness and its primary detractions are briefly discussed. In what follows, however, these relations are not used. Rather, a simple re-examination of the standard problem leads to a simpler formulation of the inverse problem which is devoid of the difficulties stated in Remark **2.3**. □

2.1. Re-parameterization.
An efficient formulation for the inverse problem relies on the following trivial observation: Equations (2.3) can be considered as equations in $\boldsymbol{f}$ and hence as equations in $\boldsymbol{\varphi}$ the inverse motion. This follows from noting that $\boldsymbol{F} = \boldsymbol{f}^{-1} \circ \boldsymbol{\phi}$. In this regard, it proves convenient to rewrite Equations (2.2)-(2.5) in terms of the Cauchy stress $\boldsymbol{\sigma}$ and the known deformed configuration. Thus, the equilibrium equations and boundary conditions become: for all $\boldsymbol{x} \in S$

$$\operatorname{div}[\boldsymbol{\sigma}] + \widehat{\boldsymbol{b}} = \boldsymbol{o} \qquad \text{and} \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T , \tag{2.14}$$

for all $\boldsymbol{x} \in \partial S_t$

$$\boldsymbol{\sigma n} = \bar{\boldsymbol{t}} , \tag{2.15}$$

and for all $\boldsymbol{x} \in \partial S_\varphi$

$$\boldsymbol{\varphi} = \bar{\boldsymbol{\varphi}} , \tag{2.16}$$

where $\widehat{\boldsymbol{b}}$ is a given body force per unit deformed volume. The constitutive relation may be expressed in terms of $\boldsymbol{f}$ as

$$\boldsymbol{\sigma} = j\boldsymbol{P}(\boldsymbol{f}^{-1})\boldsymbol{f}^{-T} . \tag{2.17}$$

§3. Finite Element Formulation

The strong form of the boundary value problem given in the previous section can also be written in the standard weak form

$$G(\boldsymbol{\varphi};\boldsymbol{\eta}) = \int_S \operatorname{grad}[\boldsymbol{\eta}] : \boldsymbol{\sigma} - \int_S \boldsymbol{\eta} \cdot \widehat{\boldsymbol{b}} - \int_{\partial S_t} \boldsymbol{\eta} \cdot \bar{\boldsymbol{t}} = 0 , \tag{3.1}$$

where $\boldsymbol{\eta} : S \longrightarrow \mathbb{R}^3$ represents an arbitrary admissible variation in a suitably chosen function space‡, grad[·] is the gradient operator with respect to $\boldsymbol{x}$, a single (·) between two vectors denotes the standard dot product in $\mathbb{R}^3$ and (:) denotes the following inner product between second order tensors, $\boldsymbol{A} : \boldsymbol{B} = \operatorname{tr}[\boldsymbol{A}^T\boldsymbol{B}]$. Note in particular, that S is a known domain; thus linearization of (3.1) is most easily performed on the Eulerian weak form.

‡ Typically, the configuration space of solutions, $\mathcal{C} = \{\boldsymbol{\varphi} : S \to \mathbb{R}^3 \mid j = \det[\boldsymbol{f}] > 0 \text{ and } \boldsymbol{\varphi}|_{\partial S_\varphi} = \bar{\boldsymbol{\varphi}}\} \subset [W^{1,p}(S)]^3$, where $W^{k,p}$ is the space of k-times weakly differentiable functions, $W^k(S)$, belonging to the classical Lebesgue space $L^p(S)$. The choice of p is dictated by the growth characteristic of the strain energy function; see CIARLET [1988]. The standard linear space of admissible variations, $T_\varphi\mathcal{C}$, is the tangent space to $\mathcal{C}$ at a point $\boldsymbol{\varphi} \in \mathcal{C}$ and is a subset of $[W^{1,p}(S)]^3$ as well.

3.1. Linearization.

Without loss of generality, we consider the special case of isotropy in what follows. Thus, the Cauchy stress may be expressed using the representation theorem for isotropic tensor functions (see *e.g.* GURTIN [1981, §37]) as solely a function of $\boldsymbol{b}$ the left Cauchy-Green strain tensor or equivalently in terms of its inverse $\boldsymbol{c} = \boldsymbol{b}^{-1} = \boldsymbol{f}^T\boldsymbol{f}$. Then, the linearization of (3.1) about a given point(iterate) $\boldsymbol{\varphi}^{(k)}$ in the configuration space in the direction $\boldsymbol{v} : S \longrightarrow \mathbb{R}^3$ is given by

$$\mathcal{L}G(\boldsymbol{\varphi}^{(k)};\boldsymbol{\eta})[\boldsymbol{v}] = G(\boldsymbol{\varphi}^{(k)};\boldsymbol{\eta}) + D_1G(\boldsymbol{\varphi}^{(k)};\boldsymbol{\eta})[\boldsymbol{v}]\,, \tag{3.2}$$

where

$$D_1G(\boldsymbol{\varphi}^{(k)};\boldsymbol{\eta})[\boldsymbol{v}] = \left.\frac{d}{d\theta}\right|_{\theta=0} G(\boldsymbol{\varphi}^{(k)} + \theta\boldsymbol{v};\boldsymbol{\eta})\,. \tag{3.3}$$

For the case being considered

$$D_1G(\boldsymbol{\varphi}^{(k)};\boldsymbol{\eta})[\boldsymbol{v}] = \int_S 2\,\mathrm{sym}\,[\mathrm{grad}[\boldsymbol{\eta}]] : \frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{c}} : \mathrm{sym}\left[\boldsymbol{f}^T\mathrm{grad}[\boldsymbol{v}]\right]\,, \tag{3.4}$$

where $\mathrm{sym}[\cdot] = \frac{1}{2}((\cdot) + (\cdot)^T)$ is the symmetry operator. Note that there is no geometric stiffness in this representation. However, the material tangent operator $\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{c}}$ only has minor symmetries in general; hence any expected computational savings from this observation are voided. Further, note, this lack of symmetry is consistent with the traditional inverse formulation in terms of energy-momentum; see Appendix B.

Remark 3.1.

The weak form above can be approximated by defining finite element sub-spaces to the configuration space and the space of admissible variations. In general, for realistic nonlinearly elastic materials, incompressibility must be considered. However, for the sake of simplicity, such problems are not explicitly dealt with here; see *e.g.* SIMO AND TAYLOR [1991] for a discussion on techniques appropriate to such problem classes. In what follows only the standard Galerkin method will be employed. □

3.2. 2-D Matrix Formulations.

The preceding expressions can be converted to standard matrix form except for the tangent term. Considering the problem on the element level, write

$$\mathrm{sym}\,[\mathrm{grad}[\boldsymbol{\eta}]] = \underline{B}\underline{\eta}\,, \tag{3.5}$$

where $\underline{B} = [\underline{B}_1, \underline{B}_2, \cdots, \underline{B}_{nen}]$ is the discrete gradient operator,

$$\underline{B}_A = \begin{bmatrix} N_{A,1} & 0 \\ 0 & N_{A,2} \\ N_{A,2} & N_{A,1} \end{bmatrix}\,, \tag{3.6}$$

and *nen* is the number of element nodes. In the above, N_A are shape functions and $\underline{\eta} = (\eta_1^1, \eta_2^1, \eta_1^2, \eta_2^2, \cdots, \eta_1^{nen}, \eta_2^{nen},)^T$, where superscripts denote node number and subscripts denote coordinate direction. Similarly,

$$\operatorname{sym}\left[\boldsymbol{f}^T \operatorname{grad}[\boldsymbol{v}]\right] = \underline{B}\underline{f}^T \underline{v}\,, \tag{3.7}$$

where $\underline{f}$ is the block diagonal matrix

$$\underline{f} = \operatorname{diag}[\boldsymbol{f}, \boldsymbol{f}, \cdots, \boldsymbol{f}]_{2nen \times 2nen} \tag{3.8}$$

and $\underline{v}$ is ordered in the same fashion as $\underline{\eta}$. Lastly, the tangent operator is mapped to a 3×3 matrix $\underline{D}$ as:

$$\underline{D} = \begin{bmatrix} \partial\sigma_{11}/\partial c_{11} & \partial\sigma_{11}/\partial c_{22} & \partial\sigma_{11}/\partial c_{12} \\ \partial\sigma_{22}/\partial c_{11} & \partial\sigma_{22}/\partial c_{22} & \partial\sigma_{22}/\partial c_{12} \\ \partial\sigma_{12}/\partial c_{11} & \partial\sigma_{12}/\partial c_{22} & \partial\sigma_{12}/\partial c_{12} \end{bmatrix}. \tag{3.9}$$

Note, this matrix is not symmetric. Thus the tangent term is unsymmetric and on the element level becomes

$$\underline{k}^e = \int \underline{B}^T \underline{D} \underline{B} \underline{f}^T\,, \tag{3.10}$$

where the domain of integration is a given element.

Remark 3.2.

It is seen from (3.10) that the modifications necessary to convert a standard elastostatic element for inverse calculations are relatively minor in this formulation when compared to the energy-momentum formulation. □

§4. Illustrations: Neo-Hookean Material

This section presents a set of example problems which illustrate the capability of this formulation to correctly solve the inverse motion problem. All problems are 2D plane strain and use a compressible Neo-Hookean constitutive relationship with the following strain energy function:

$$W = \frac{\mu}{2}(\operatorname{tr}[\boldsymbol{C}] - 3) - \mu \ln(J) + \frac{\lambda}{2}(\ln(J))^2\,, \tag{4.1}$$

where $\boldsymbol{C}$ is the right Cauchy-Green strain tensor and μ and λ are constitutive parameters that reduce to the Lamé parameters in the small strain limit. Given (4.1), the Cauchy stress

$$\boldsymbol{\sigma} = j\mu(\boldsymbol{c}^{-1} - \boldsymbol{I}) - \lambda j \ln(j)\boldsymbol{I} \tag{4.2}$$

and the tangent operator

$$\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{c}} = j\mu(\tfrac{1}{2}\boldsymbol{c}^{-1} \otimes \boldsymbol{c}^{-1} - \mathbb{I}_{c^{-1}}) - j\tfrac{1}{2}(\mu + \lambda(1 + \ln(J)))\boldsymbol{I} \otimes \boldsymbol{c}^{-1}\,, \tag{4.3}$$

where indicially

$$\mathbb{I}_{c^{-1}} \rightarrow \mathbb{I}_{c^{-1}}^{ijkl} = \tfrac{1}{2}(c_{ik}^{-1}c_{jl}^{-1} + c_{il}^{-1}c_{jk}^{-1}) \tag{4.4}$$

and $\otimes$ denotes the tensor outer product.

4.1. Simple extension. In this example the method is compared to a problem with an analytic solution with a uniform stress state. A rectangular (deformed) domain of width 1.17115 and height 0.96011 is loaded with a uniform (Cauchy) traction of 200 along the vertical surfaces; the material properties are $\mu = 500$ and $\lambda = 1000/3$ (all properties are assumed to be given in consistent units). This corresponds to a material with a small strain Poisson ratio of 0.2. The problem is to determine the undeformed configuration. The exact solution (to the precision given for the geometry) is a 1 by 1 square. Using a uniform 4 element mesh, the problem is solved in a single load step with 5 Newton-Raphson iterations required to quadratically reduce the residual norm by 16 orders of magnitude. The calculated inverse motion produces a 1 by 1 square (to the precision given) and a uniform lateral Cauchy stress field of 200 as expected from the exact answer.

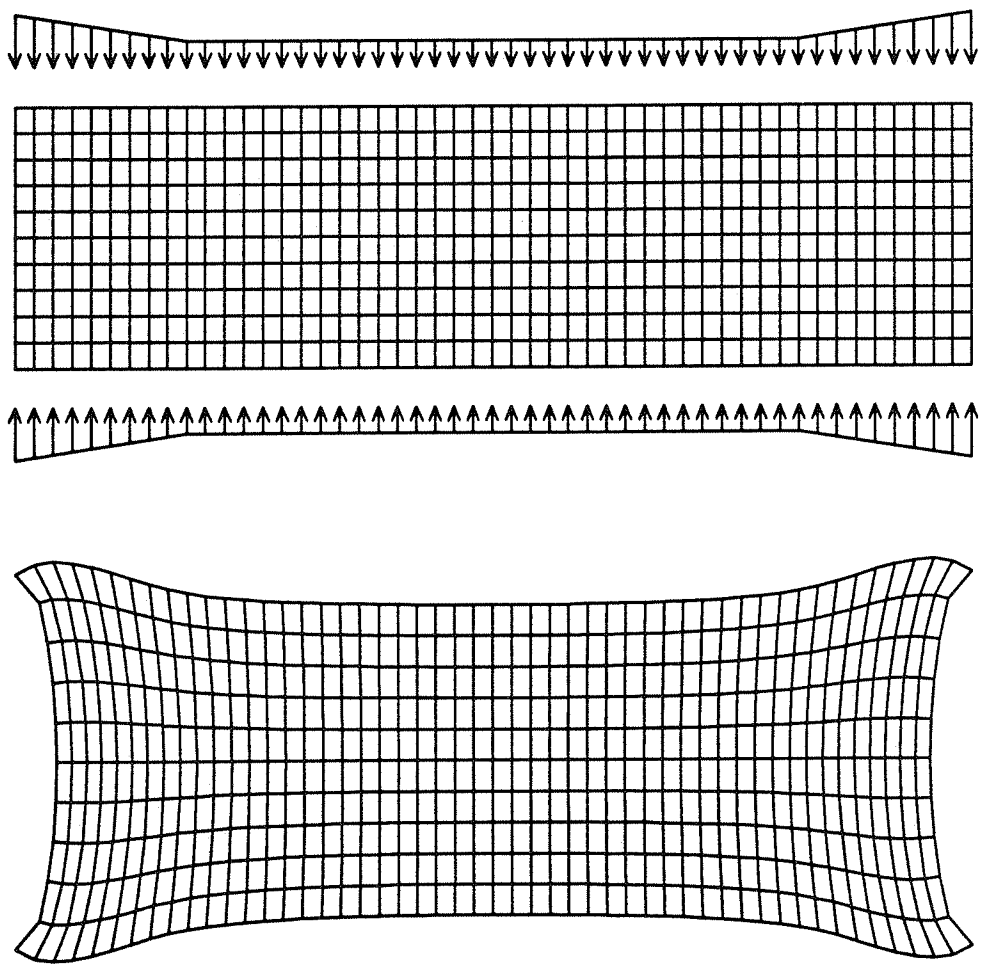

FIGURE 4.1. Top: Deformed gasket cross-section with loading, Bottom: Undeformed gasket cross-section.

4.2. Gasket shape. In this example we consider the determination of the "to be manufactured" shape of a gasket. The design specifications require the gasket to have a (deformed) 1.5 cm $\times$ 0.4 cm rectangular cross-section and to provide a linearly increasing clamping traction near the gasket edges to help prevent the gasket from being compromised.

The initial mesh is shown in the top of Fig. **4.1**. Material properties are $\lambda = 7.143$ MPa and $\mu = 1.786$ MPa. The lateral edges are traction free and the top and bottom faces are subjected to a constant normal traction of 2 MPa over the center 70% of the gasket. On the outer 15% of both sides the traction increases linearly to a value of 4 MPa. The lateral motion on the top and bottom faces is assumed to be zero. By symmetry, only 1/4 of the shown geometry was actually modeled. Shown in the lower half of **4.1** is the calculated undeformed shape of the gasket. This is the shape one would need to manufacture to obtain a gasket with the given performance specifications. The analysis was performed in a single load step with 6 Newton-Raphson iterations; this reduced the residual norm by over 9 orders of magnitude.

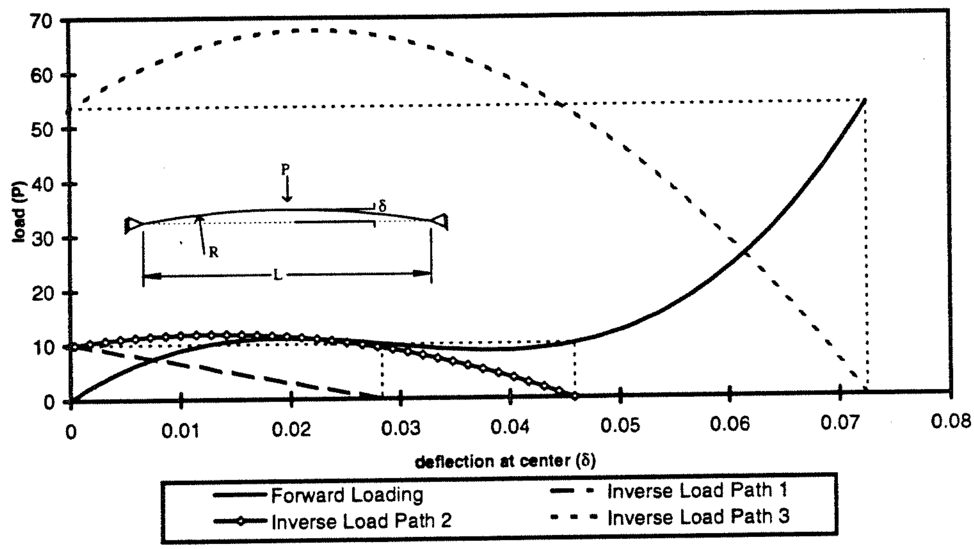

FIGURE 4.2. Snap-through buckling forward and inverse paths.

4.3. Snap-through buckling. In this example, we consider a problem that has multiple solutions. A thin concave slab is pin supported on each end and the geometry is discretized into 20 elements angularly and 4 elements through the thickness; see Fig. **4.2**. Dimensions of the geometry are as follows: $R = 1.0$ is the radius of curvature, $L = 0.2203$ is the distance between the supports, and $t = 0.005$ is the thickness of the slab. The material properties are $\mu = 500$ and $\lambda = 1000/3$, where all parameters are assumed to be given in consistent units. A point load is applied in the center of the mesh with a magnitude large enough to cause snap-through buckling. To follow the load through the bifurcation point an arclength method is used; see *e.g.* CRISFIELD [1983] and references therein. Analysis of the forward problem leads to the solid response curve shown in Fig. **4.2**.

For this problem three inverse calculations were performed from different deformed configurations. The initial deformed configurations were taken from the output of the forward calculation. The first corresponds to a center deflection of 0.0283059, the second a center deflection of 0.0458880, and the third a center deflection of 0.0725270. The load in the inverse problem is incremented from zero and is increased until it reaches the desired load value which is known from the forward calculation. The results show that the mesh deforms to the correct displacement (known from the forward problem) when the corresponding load is applied to the inverse problem. In two cases, load paths 2 and 3 in Fig. **4.2**, the desired load is reached for multiple configurations. This merely reflects the non-unique nature of non-linear elasticity problems of this type. Physically it says that there are multiple undeformed configurations that will produce the same deformed configuration for given boundary conditions. Each solution of course leads to a different stress field. Further note that the inverse paths should not be interpreted as unloading paths.

§5. Closure

This paper has presented an alternative formulation to the energy-momentum formulation for the calculation of inverse motion problems. The energy-momentum formulation has been shown to have several major drawbacks:

i. It is restricted to homogeneous bodies.
ii. It is restricted to problems without body forces.
iii. It results in a weak form that requires deformation gradient information on the boundary or C^1 continuity in finite element methods.

The alternative formulation based simply on equilibrium considerations has been shown to provide a way around these restrictions in a fashion that fits well within a standard finite element program architecture. In particular, elements designed for computing forward motion problems in hyper-elasticity can be easily converted to inverse motion elements.

§ Acknowledgments

The authors would like to acknowledge Prof. R.L. Taylor of the University of California at Berkeley for several helpful discussions and the late Prof. J.C. Simo of Stanford University for suggesting the idea of using an energy-momentum formulation to solve inverse motion problems.

§ References

CHADWICK, P. "Applications of an Energy-Momentum Tensor in Elastostatics," *J. Elasticity* **5**, 249-258 (1975).

CIARLET, P. *Mathematical Elasticity. Vol I: Three Dimensional Elasticity, Studies in Mathematics and its Applications.* North–Holland Press, Amsterdam, (1988).

CRISFIELD, M.A. "An Arc-Length Method Including Line Searches and Accelerations," *Int. J. Num. Meth. Eng.* **19**, 1269-1289 (1983).

ESHELBY, J.D. "The Continuum Theory of Lattice Defects," in *Solid State Physics, Advances in Research and Applications, Vol III,* Eds: F. Seitz and D. Turnbull, Academic Press, New York, 79-144 (1956).

ESHELBY, J.D. "The Elastic Energy-Momentum Tensor," *J. Elasticity* **5**, 321-335 (1975).

GURTIN, M.E. *An Introduction to Continuum Mechanics*, Academic Press, Orlando, (1981).

LAMM, P.K. "Inverse Problems and Ill-Posedness," in *Inverse Problems in Engineering: Theory and Practice,* Eds: N. Zabaras, K.A. Woodbury, & M. Raynaud, ASME Press, New York, 1-10 (1993).

OGDEN, R.W. "A Note on Duality in Finite Elasticity," *J. Elasticity* **5**, 83-88 (1975A).

OGDEN, R.W. "Dual Reciprocal States in Finite Elasticity," *J. Elasticity* **5**, 149-153 (1975B).

SHIELD, R.T. "Inverse Deformation Results in Finite Elasticity," *Z. Agnew. Math. Phys.* **18**, 381-389 (1967).

SIMO, J.C. & TAYLOR, R.L. "Quasi-incompressible Finite Elasticity in Principal Stretches: Continuum Basis and Numerical Algorithms," *Comp. Meth. Appl. Mech. Engng.* **85**, 273-310 (1991).

§Appendix A: Duality Relations in Elastostatics

Through the introduction of the stored energy function per unit undeformed volume W and its counterpart (SHIELD [1967]) per unit deformed volume w [see Equation (2.6)], a set of duality relations between forward and inverse motion can be developed. Without expounding on the details of the derivations, the following dual relations are noted:

Kinematics

$$\begin{array}{cc} \textbf{Forward Motion} & \textbf{Inverse Motion} \\ \boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{X}) & \boldsymbol{X} = \boldsymbol{\varphi}(\boldsymbol{x}) \\ \boldsymbol{F} = \dfrac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{X}} & \boldsymbol{f} = \dfrac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}} \\ \boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} & \boldsymbol{c} = \boldsymbol{f}^T\boldsymbol{f} \\ \boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T & \boldsymbol{B} = \boldsymbol{f}\boldsymbol{f}^T \end{array} \tag{A.1}$$

In the above, $\boldsymbol{\phi}$ is the forward motion mapping points $\boldsymbol{X}$ in the reference configuration to points $\boldsymbol{x}$ in the deformed configuration. $\boldsymbol{\varphi}$ is the point mapping for the inverse motion. $\boldsymbol{F}$ is the deformation gradient and $\boldsymbol{f}$ is the inverse deformation gradient. $\boldsymbol{C}$ is the right Cauchy-Green strain tensor and $\boldsymbol{c}$ is its counterpart in the inverse motion; note that $\boldsymbol{c} = \boldsymbol{b}^{-1}$ where $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T$ is the left Cauchy-Green strain tensor whose dual is $\boldsymbol{B} = \boldsymbol{C}^{-1}$. Note that frame invariance requires the energy $W(\boldsymbol{F}) = \hat{W}(\boldsymbol{C})$; however, for the energy in terms of the inverse motion, the requirement is that $w(\boldsymbol{f}) = \hat{w}(\boldsymbol{B})$. Worthy of note also is that isotropy requires $W(\boldsymbol{F}) = \bar{W}(\boldsymbol{b})$ and $w(\boldsymbol{f}) = \bar{w}(\boldsymbol{c})$.

Constitutive Relations

$$\begin{array}{ll} \textbf{Forward Motion} & \textbf{Inverse Motion} \\ \boldsymbol{P} = \dfrac{\partial W}{\partial \boldsymbol{F}} & \boldsymbol{p} = \dfrac{\partial w}{\partial \boldsymbol{f}} \\ \boldsymbol{S} = 2\dfrac{\partial W}{\partial \boldsymbol{C}} & \boldsymbol{s} = 2\dfrac{\partial w}{\partial \boldsymbol{c}} \text{ (isotropy only)} \\ \mathbb{C} = 4\dfrac{\partial^2 W}{\partial \boldsymbol{C}\partial \boldsymbol{C}} & \mathbb{K} = 4\dfrac{\partial^2 w}{\partial \boldsymbol{c}\partial \boldsymbol{c}} \text{ (isotropy only)} \end{array} \tag{A.2}$$

In the above, $\boldsymbol{P}$ is the first Piola-Kirchhoff stress tensor and $\boldsymbol{p}$ a two-point energy momentum tensor is its dual. $\boldsymbol{S}$ is the second Piola-Kirchoff stress tensor and $\boldsymbol{s}$ is its energy-momentum dual. The given form of $\boldsymbol{s}$ is symmetric but only holds in the case of isotropy; a more generally valid expression is given below in the stress tensors table. $\mathbb{C}$ is the fourth order material tangent and $\mathbb{K}$ is its energy-momentum dual in the case of isotropy.

Stress Measures

$$\begin{array}{ll} \textbf{Forward Motion} & \textbf{Inverse Motion} \\ \boldsymbol{P} = \dfrac{\partial W}{\partial \boldsymbol{F}} & \boldsymbol{p} = j(W\boldsymbol{I} - \boldsymbol{F}^T\boldsymbol{P})\boldsymbol{F}^T \\ \boldsymbol{\sigma} = \dfrac{1}{J}\boldsymbol{P}\boldsymbol{F}^T & \boldsymbol{\Sigma} = \dfrac{1}{j}\boldsymbol{p}\boldsymbol{f}^T = W\boldsymbol{I} - \boldsymbol{C}\boldsymbol{S} \\ \boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{P} & \boldsymbol{s} = \boldsymbol{f}^{-1}\boldsymbol{p} = w\boldsymbol{b} - \boldsymbol{b}\boldsymbol{\sigma} \end{array} \tag{A.3}$$

In the above, $J = \det[\boldsymbol{F}]$ is the Jacobian of the forward motion and $j = \det[\boldsymbol{f}]$ is its counterpart in the inverse motion. $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\boldsymbol{\Sigma}$, Eshelby's energy-momentum tensor, is its dual (see (2.9) and subsequent comments on different energy-momentum tensor conventions). Note that the symmetry of $\boldsymbol{\Sigma}$ and $\boldsymbol{s}$ requires that $\boldsymbol{C}$ and $\boldsymbol{S}$ commute and that $\boldsymbol{b}$ and $\boldsymbol{\sigma}$ commute; this in general happens only in the case of isotropy.

Conservation Laws

Forward Motion	Inverse Motion
$\mathrm{div}[\boldsymbol{\sigma}] = \boldsymbol{o}$	$\mathrm{DIV}[\boldsymbol{\Sigma}] = \boldsymbol{O}$
$\mathrm{DIV}[\boldsymbol{P}] = \boldsymbol{o}$	$\mathrm{div}[\boldsymbol{p}] = \boldsymbol{O}$
$\mathrm{DIV}[\boldsymbol{F}\boldsymbol{S}] = \boldsymbol{o}$	$\mathrm{div}[\boldsymbol{f}\boldsymbol{s}] = \boldsymbol{O}$

(A.4)

In the above, $\mathrm{DIV}[\cdot]$ is the divergence operator with respect to the reference coordinates $\boldsymbol{X}$ and $\mathrm{div}[\cdot]$ is the divergence operator with respect to $\boldsymbol{x}$. Further note that $\mathrm{DIV}[\boldsymbol{P}] = \boldsymbol{o}$ if and only if $\mathrm{DIV}[\boldsymbol{\Sigma}] = \boldsymbol{O}$ for smooth motions (i.e. $\boldsymbol{\phi} \in C^2(B)$) with positive Jacobian. This is seen to be true, since under the assumption of smoothness $\mathrm{DIV}[\boldsymbol{\Sigma}] = -\boldsymbol{F}^T\mathrm{DIV}[\boldsymbol{P}]$.

§Appendix B: Weak Form For Energy-Momentum Formulation

In this appendix, the weak form of the inverse problem in terms of the energy-momentum tensor is presented and briefly discussed in the context of isotropic materials without body forces. Begin by taking the dot product of Equation $(\mathrm{A.4})_3$ by an admissible variation $\boldsymbol{\eta} : S \longrightarrow \mathbb{R}^3$ and integrating over the known domain S:

$$\int_S \boldsymbol{\eta} \cdot \mathrm{div}[\boldsymbol{f}\boldsymbol{s}] = 0 \,. \tag{B.1}$$

Following the standard procedure, integrate Eq. (B.1) by parts and note that $\boldsymbol{\eta}|_{\partial S_\varphi} = 0$ to give

$$\int_S \mathrm{grad}[\boldsymbol{\eta}] : (\boldsymbol{f}\boldsymbol{s}) - \int_{\partial S_t} \boldsymbol{\eta} \cdot \boldsymbol{f}\boldsymbol{s}\boldsymbol{n} = 0 \tag{B.2}$$

Making use of Eq. (2.13) yields the weak form expression

$$G(\varphi;\boldsymbol{\eta}) = \int_S \mathrm{grad}[\boldsymbol{\eta}] : (\boldsymbol{f}\boldsymbol{s}) - \int_{\partial S_t} \boldsymbol{\eta} \cdot w\boldsymbol{f}^{-T}\boldsymbol{n} + \int_{\partial S_t} \boldsymbol{\eta} \cdot \boldsymbol{f}^{-T}\bar{\boldsymbol{t}} = 0 \tag{B.3}$$

which must hold true for all admissible $\boldsymbol{\eta}$.

Remark B.1.

This expression has an immediate drawback: The presence of dependencies on $\boldsymbol{f}$ in the boundary integrals implies the need for gradient information on the boundary of elements. Typical shape function interpolations do not, however, provide this data with any accuracy. Note this dependency appears not just in the $\boldsymbol{f}^{-T}$ terms but also in $w(\boldsymbol{f})$. □

Remark B.2.

A non-standard weighted residual method can be used to circumvent the difficulties noted in the previous remark at the expense of the integrability requirements on the finite element function spaces. By taking the dot product of Eq. (A.4)$_3$ with $\boldsymbol{\eta}\boldsymbol{f}^T$, one obtains

$$\int_S \operatorname{grad}[\boldsymbol{\eta}\boldsymbol{f}^T] : (\boldsymbol{f}\boldsymbol{s}) - \int_{\partial S_t} \boldsymbol{\eta}\cdot w\boldsymbol{n} + \int_{\partial S_t} \boldsymbol{\eta}\cdot\bar{\boldsymbol{t}} = 0\,. \tag{B.4}$$

The boundary term that involves $w(\boldsymbol{f})$ can then be converted, using the divergence theorem, to a domain integral to remove all need for gradient information on the boundary. The detraction, of course, is the higher order continuity conditions that can not be handled by standard methods in multidimensions. □

For completeness, we give the directional derivative of Eq. (B.3) below:

$$\begin{aligned} D_1G(\boldsymbol{\varphi};\boldsymbol{\eta})[\boldsymbol{v}] = &\int_{\boldsymbol{\varphi}(S)} \operatorname{Grad}[\boldsymbol{\eta}] : (\operatorname{Grad}[\boldsymbol{v}]\boldsymbol{\Sigma}) + \int_{\boldsymbol{\varphi}(S)} \operatorname{Grad}[\boldsymbol{\eta}] : \boldsymbol{\varphi}_*(\mathbb{K}) : \operatorname{Grad}[\boldsymbol{v}] \\ &- \int_{\boldsymbol{\varphi}(\partial S_t)} \boldsymbol{\eta}\cdot\boldsymbol{N}\boldsymbol{\Sigma} : \operatorname{Grad}[\boldsymbol{v}] - W\boldsymbol{N}\cdot\operatorname{Grad}[\boldsymbol{v}]\boldsymbol{\eta} \\ &- \int_{\boldsymbol{\varphi}(\partial S_t)} (\boldsymbol{f}^{-T}\bar{\boldsymbol{t}})\cdot\operatorname{Grad}[\boldsymbol{v}]\boldsymbol{\eta}\,, \end{aligned} \tag{B.5}$$

where $\operatorname{Grad}[\cdot]$ is the gradient operator with respect to $\boldsymbol{X}$, $\boldsymbol{N}$ is the boundary normal in the undeformed configuration, and $\boldsymbol{\varphi}_*(\mathbb{K}) = \frac{1}{j}\boldsymbol{f}\boldsymbol{f}\boldsymbol{f}\boldsymbol{f}\mathbb{K}$. For a general isotropic material

$$\begin{aligned} \boldsymbol{\varphi}_*(\mathbb{K}) = &(4J^2W_3 - 2W)\mathbb{I} - 4W_2\mathbb{I}_C + 2(W_1 + \iota_1 W_2)\widehat{\mathbb{I}}(\boldsymbol{C},\boldsymbol{I}) \\ &- 2W_2\widehat{\mathbb{I}}(\boldsymbol{C}^2,\boldsymbol{I}) + (W + 4J^3W_{33})\boldsymbol{I}\otimes\boldsymbol{I} \\ &+ 4(W_{11} + W_2 + 2\iota_1 W_{21} + \iota_1^2 W_{22})\boldsymbol{C}\otimes\boldsymbol{C} + 4W_{22}\boldsymbol{C}^2\otimes\boldsymbol{C}^2 \\ &+ (4J^2(W_{13} + \iota_1 W_{23}) - 2(W_1 + \iota_1 W_2))(\boldsymbol{I}\otimes\boldsymbol{C} + \boldsymbol{C}\otimes\boldsymbol{I}) \\ &+ (2W_2 - 4J^2W_{32})(\boldsymbol{I}\otimes\boldsymbol{C}^2 + \boldsymbol{C}^2\otimes\boldsymbol{I}) \\ &- 4(W_{12} + \iota_1 W_{22})(\boldsymbol{C}\otimes\boldsymbol{C}^2 + \boldsymbol{C}^2\otimes\boldsymbol{C}) \end{aligned} \tag{B.6}$$

In the above, $W_y = \partial W/\partial\iota_y$ and $W_{xy} = \partial^2 W/\partial\iota_x\partial\iota_y$ where $\iota_1 = \operatorname{tr}[\boldsymbol{b}]$, $\iota_2 = \frac{1}{2}(\iota_1^2 - \operatorname{tr}[\boldsymbol{b}^2])$, and $\iota_3 = \det[\boldsymbol{b}]$ (the three invariants of $\boldsymbol{b}$). Indicially, the operators

$$\mathbb{I} \rightarrow \mathbb{I}_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,, \tag{$B.7$}$$

$$\mathbb{I}_C \rightarrow \mathbb{I}_C^{ijkl} = \tfrac{1}{2}(C_{ik}C_{jl} + C_{il}C_{jk})\,, \tag{$B.8$}$$

and

$$\widehat{\mathbb{I}}(\boldsymbol{A},\boldsymbol{B}) \rightarrow \widehat{\mathbb{I}}(\boldsymbol{A},\boldsymbol{B})_{ijkl} = \frac{1}{4}(A_{ik}B_{jl} + A_{il}B_{kj} + A_{jl}B_{ik} + A_{jk}B_{il})\,. \tag{B.7}$$

Note that Eq. (B.6) has both minor and major symmetries but overall (B.5) is unsymmetric.