UNIVERSITY OF CALIFORNIA,
IRVINE

Geometry and Energy: Global and Local Perspectives

DISSERTATION

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for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

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DEDICATION

To my family whose love I anchor to along this passionate and challenging journey.

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VITA

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ABSTRACT OF THE DISSERTATION

Geometry and Energy: Global and Local Perspectives

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This thesis includes the analysis on initial data sets with singularities which helps identify sufficient conditions on the singularity guaranteeing the positivity of mass, characterising the dominant energy condition on polyhedra, and showing the relation between boundary energy and interior energy.

The main contribution of this thesis is to provide both global and local perspectives of the relation between geometry and physics. First, we show a spacetime positive mass theorem with corners. Then, by putting Gromov's dihedral rigidity conjecture and fill-in conjecture into the context of general relativity, we can use the aforementioned theorem to provide partial solutions to these conjectures by constructing suitable extensions for compact initial data sets.

Chapter 1

Introduction

Riemannian geometry is the study of manifolds in terms of their curvature. The Gauss-Bonnet Theorem is an elegant theorem which connects the topology of a 2 dimensional manifold Σ to the curvature of its interior and boundary. If $\partial\Sigma$ consists of l piecewise smooth components, we have,

$$\int_{\Sigma} K + \int_{\partial\Sigma} \kappa = 2\pi\chi(\Sigma) - \sum_{i=1}^{l}(\pi - \alpha_i), \tag{1.0.1}$$

where α_i is the diherdral angle between 2 smooth components. On the other hand, if $\partial\Sigma$ is smooth, we have,

$$\int_{\Sigma} K + \int_{\partial\Sigma} \kappa = 2\pi\chi(\Sigma). \tag{1.0.2}$$

From this, we can see the relation among the manifold's interior curvature, boundary's mean (exterior) curvature and dihedral angles. Gromov ([25, 26]) has proposed a lot of ideas and questions around the interaction of the boundary geometry and the interior curvature, including his dihedral rigidity conjecture and fill-in conjecture. With the Hamiltonian formulation of gravitation ([29]), we will see that Gromov's questions arise naturally from the physical perspective of energy.

1.1 General relativity, ADM mass and quasilocal mass

Now we shall discuss the deep connection between geometry and relativity. General relativity is a theory described by the Einstein equation for a spacetime $(\bar{M}, \bar{g})$,

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}_{\bar{g}}\bar{g}_{\mu\nu} = T_{\mu\nu}. \tag{1.1.1}$$

The Einstein tensor on the left (also denoted by $G_{\mu\nu}$) and the stress-energy-momentum tensor on the right respectively represent the geometry and the physical content of the spacetime.

Definition 1.1.1. *The constraint equations are defined on M by the Gauss equations and the Codazzi equations as if M sits in a spacetime satisfying* (1.1.1). *The mass density μ and the current density J by*

$$\mu := T_{00} = \frac{1}{2}(R_g + (tr_g k)^2 - |k|_g^2), \qquad J_i := T_{0i} = div_g(k - (tr_g k)g)_i = div_g \pi_i.$$

Consider a 3-tuple (M, g, k), where g is a Riemannian metric and k is a symmetric $(0,2)$-tensor. The Cauchy problem is to construct a Lorentzian manifold $(\overline{M}^{n+1}, \overline{g})$ such that

1. $\bar{g}$ satisfies the Einstein Equation

$$G_{\mu\nu}(:= \overline{R}_{\mu\nu} - \frac{1}{2}\overline{R}_{\overline{g}}\overline{g}_{\mu\nu}) = T_{\mu\nu},$$

   where $T_{00}|_M = \mu$ and $T_{0i}|_M = J_i$.

2. there exists an isometric embedding $(M, g) \hookrightarrow (\bar{M}, \bar{g})$,

3. k is the second fundamental form of M with respect to $\bar{M}$.

Hence, we call (M, g, k) an initial data set. On the other hand, solving the Cauchy problem allows us to regard (M, g, k) as a spacelike slice in a spacetime.

Definition 1.1.2. *(M, g, k) is said to satisfy the dominant energy condition (DEC) if*

$$\mu \geq |J|_g.$$

Define the conjugate momentum tensor by $\pi = k - (tr_g k)g$. (M, g, k) is said to satisfy the boundary dominant energy condition (BDEC) if on ∂M

$$H \geq |\pi(\cdot, \nu)|_g,$$

where the mean curvature H is computed with respect to the unit outward normal ν.

Definition 1.1.3. *We say (M, g, k) is asymptotically flat (AF) if there exists a compact set $\mathcal{C} \subset M$ such that $M \setminus \mathcal{C} = \coprod_{i=1}^{k} N_i$, where each end $N_i = \mathbb{R}^n \setminus B_{r_i}$ through a coordinate diffeomorphism in which*

$$g_{ij} = \delta_{ij} + O^2(|x|^{-q}),$$

and

$$k_{ij} = O^1(|x|^{-q-1}),$$

where $q > \frac{n-2}{2}$, $\mu, J \in L^1(M)$ and for a function f on M, $f = O^m(|x|^{-p})$ means $\sum_{|l|=0}^{m} ||x|^{p+|l|}\partial^l f|$ is bounded near the infinity.

There have been proposals of geometric invariants which represent the total energy of an isolated gravitational system ([3]). After those quantities were proposed, proving their meaningfulness sparked a lot of interesting problems on both geometry and physics, for example, the positive mass theorem which was first proved by Schoen and Yau ([57]). If $k \equiv 0$, we can see the two energy conditions are purely geometric: non negative scalar curvature and non negative mean curvature. The positive mass theorem and its proof are thus useful in tackling geometry problems.

Definition 1.1.4. *For each end of an asymptotically flat initial data set, the ADM energy-momentum vector (E, P) and the ADM mass $\mathfrak{m}$ [3] are given by*

$$E = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} (g_{ij,i} - g_{ii,j})\nu^j,$$

$$P_i := \frac{1}{(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{|x|=r} \pi_{ij}\nu^j, \qquad i = 1, 2, ..., n$$

and

$$\mathfrak{m} = \sqrt{E^2 - |P|^2},$$

where the outward unit normal ν and surface integral are with respect to the Euclidean metric. Moreover, ω_{n-1} denotes the area of $\mathbb{S}^{n-1} \subset (\mathbb{R}^n, g_{Euc})$.

Theorem 1.1.1 (Spacetime positive mass theorem)**.** *For $3 \leq n \leq 7$, let (M^n, g, k) be an asymptotically flat initial data set that satisfies the dominant energy condition. Then*

$$E \geq |P|.$$

We refer readers to [57, 58, 20] and [70] ([53]) for its proof. This theorem shows that the ADM mass is a reasonable measure of the total energy of an isolated gravitational system since it is non-negative whenever the energy of the mater is pointwise non-negative.

If a physical system is not isolated or cannot be viewed from infinity where asymptotic symmetry exists, e.g. compact initial data sets with boundary, the ADM mass is not well defined.

Different notions of quasilocal mass have been suggested ([12, 6, 34, 44, 69]). To show these masses are positive, a general approach is to construct an asymptotically flat extension and study the relation between the ADM mass of such extensions and the proposed mass. The resulting initial data set would inevitably have singularities along the boundary (corners). Hence, the positive mass theorem with corners is important and has been extensively studied ([35], [36], [40], [46], [47], [59], [60]). In this thesis, we will establish a spacetime positive mass theorem with corners which leads to a new notion of quasilocal mass and certain relativistic formulations of local geometric phenomena.

Theorem 1.1.2 ([68])**.** *Let M^3 be a complete non-compact smooth manifold and $\tilde{\Sigma} \subset M$ be a piecewise smooth surface. Assume the metric g and the symmetric (0,2)-tensor k on M satisfy the following:*

1. *(g, k) is asymptotically flat,*
2. *g and k are smooth up to each component of $\tilde{\Sigma}$,*
3. *g is Lipschitz,*
4. *k need not be continuous across $\tilde{\Sigma}$.*

Let $\mathcal{E}$ be an asymptotically flat end of M. Assume there exists $\mathcal{S}$, a finite (possibly empty) disjoint union of connected weakly trapped surfaces which do not intersect $\tilde{\Sigma}$, such that $H_2(M_{ext}, \mathcal{S}, \mathbb{Z}) = 0$, where M_{ext} is the exterior region of M containing $\mathcal{E}$ with $\partial M_{ext} = \mathcal{S}$. Denote $\tilde{\Sigma} \cap M_{ext}$ by Σ. Then for $\mathcal{E}$, there exists a spacetime harmonic function u such that

$$\begin{aligned} 16\pi(E - |P|) \geq & \int_{M_{ext}\setminus\Sigma} \left(\frac{|\overline{\nabla}\nabla u|^2}{|\nabla u|} + 2(\mu|\nabla u| + \langle J, \nabla u\rangle) \right) \\ & + 2\int_{\Sigma} (H_- - H_+)|\nabla u| - 2\int_{\Sigma} (\pi_- - \pi_+)(\nabla u, \nu), \end{aligned} \tag{1.1.2}$$

where $\pi_\pm$ and $H_\pm$ respectively denote the conjugate momentum tensors of $k_\pm$ and the mean curvatures of $g_\pm$ on Σ with respect to ν, the unit normal pointing into the infinity of $\mathcal{E}$. In *particular, if the dominant energy condition holds on $M_{ext} \setminus \Sigma$ and*

$$(H_- - H_+) - |\omega_- - \omega_+| \geq 0$$

on Σ, then we have

$$E \geq |P|,$$

where $\omega_\pm := \pi_\pm(\cdot, \nu)$.

Corollary 1.1.1 ([68])**.** *Assume the dominant energy condition holds on $M_{ext} \setminus \Sigma$ and*

$$(H_- - H_+) - |\omega_- - \omega_+| \geq 0$$

on Σ. If $E = |P|$, then M is diffeomorphic to $\mathbb{R}^3$. If $E = |P| = 0$, then (M, g, k) arises from an isometric embedding into Minkowski space as the graph of a linear combination of spacetime harmonic functions.

To suggest a meaningful quasilocal quantity, we adopt the following approach.

1.1.1 Hamiltonian formulation (Hamilton-Jacobi analysis)

([3],[55],[12],[29]) Let (Ω^n, g, k) be a compact initial data set with boundary Σ. A spacetime $(N^{n+1}, \bar{g})$ with boundary $\bar{\Sigma}$ can be constructed by infinitesimally deforming the initial data set (Ω, g, k, Σ) in a transversal, timelike direction $\partial_t = V\vec{n} + W^i\partial_i$ which satisfies $\bar{\nabla}_{\partial_t} t = 1$, where V is the lapse function, $\vec{n}$ is the timelike unit normal of Ω in N and W is the shift vector. Further assume that Ω meets $\bar{\Sigma}$ orthogonally. The purely gravitational contribution $\mathcal{H}_{grav}$ to the total Hamiltonian at the slice Ω is given by

$$c(n)\mathcal{H}_{grav}(V, W) = \int_\Omega (\mu V + \langle J, W \rangle) - \int_\Sigma (HV - \pi(\nu, W)), \tag{1.1.3}$$

where H is the mean curvature of Σ with respect to the outward normal of Ω and π is the conjugate momentum tensor. From this, we can expect that the contribution to the boundary geometry is from the mean curvature H and the 1-form $\pi(\nu, \cdot)$.

Corollary 1.1.2 ([68]). *Let (Ω^3, g, k) be a compact initial data set satisfying the dominant energy condition. Assume there exists $\mathcal{S}$, a finite (possibly empty) disjoint union of connected weakly trapped surfaces, such that $H_2(\Omega_{ext}, \mathcal{S}, \mathbb{Z}) = 0$, where Ω_{ext} denotes the portion of Ω outside $\mathcal{S}$. Suppose $\Sigma = \partial\Omega$ is a smooth surface with finitely many components with Gaussian curvature $\kappa > 0$ and mean curvature H with respect to the outward normal ν. Denote the mean curvature of an isometric embedding of Σ into $\mathbb{R}^3$ with respect to the outward normal by H_0. If $H > |\omega|$, where $\omega = \pi(\cdot, \nu)$, then*

$$\mathcal{W}(\Sigma) := \frac{1}{8\pi} \int_\Sigma H_0 - (H - |\omega|) \geq 0.$$

If $\mathcal{W}(\Sigma) = 0$, then Σ is connected, Ω is diffeomorphic to a domain in $\mathbb{R}^3$ and can be isometrically embedded into Minkowski space.

1.2 Gromov's conjectures

There are also several interesting implications in geometry by Theorem 1.1.2. Gromov ([25] Section 2.2) proposed the following conjecture to study the geometry of scalar curvature with a lower bound and to define non-negative scalar curvature for C^0 metric.

Conjecture 1.2.1 (The dihedral rigidity conjecture). *Suppose (M, g) is a Riemannian polyhedron with nonnegative scalar curvature and weakly mean convex faces. Suppose that the dihedral angles of (M, g) are not larger than the (constant) dihedral angle between corresponding faces of the model Euclidean polyhedron (M, g_{Euc}). Then (M, g) is isometric to a flat Euclidean polyhedron.*

Definition 1.2.1 (cf. [37] Definition 1.1, [38] Definition 1.4, 1.5 and [39] Definition 2.1, 2.2)**.** *Let $P \subset \mathbb{R}^n$ be a polyhedron. A compact manifold (M^n, g) with non-empty boundary is said to be of type P if M admits a Lipschitz diffeomorphism $\Psi : M \to P$ such that Ψ^{-1} is smooth when restricted to the interior, the faces and the edges of P.*

Li has made major progress on this problem; in particular, the following results are obtained.

Theorem 1.2.1 ([37],[38])**.** *Let $2 \leq n \leq 7$, P^n be a Euclidean prism with dihedral angles at most $\pi/2$, and if $n = 3$, P^3 can be an arbitrary simplex in $\mathbb{R}^3$. Assume M^n is a Riemannian polyhedron of type P. Then Conjecture 1.2.1 holds for M. Precisely, if g is a $C^{2,\alpha}$ metric on M such that*

1. *The scalar curvature of g is nonnegative;*

2. *Each face of M is weakly mean convex;*

3. *The dihedral angles between adjacent faces of (M, g) are everywhere less than or equal to the corresponding (constant) dihedral angles of (P, g_{Euc}).*

Then (M, g) is isometric to a Euclidean polyhedron.

There is also a polyhedral comparison result for hyperbolic polyhedra ([39]). From the perspective of relativity, the aforementioned results give a comparison of a given polyhedral initial data set to standard ones $(\mathbb{R}^n, g_{Euc}, 0)$ and $(\mathbb{H}^n, g_{\mathbb{H}}, g_{\mathbb{H}})$ in Minkowski space.

Section 1.1.1 provides us a model to formulate Conjecture 1.2.1 in terms of initial data sets and energy conditions. In particular, we have obtained the following generalisation.

Theorem 1.2.2 ([67])**.** *Let (M^3, g, k) be an initial data set of cube type which simultaneously satisfies:*

1. *the dominant energy condition,*
2. *the boundary dominant energy condition,*
3. *everywhere the dihedral angle between two faces of M is less than or equal to $\pi/2$.*

Then, (M, g, k) can be isometrically embedded into Minkowski space with boundary isometric to the boundary of a Euclidean rectangular prism.

As a corollary we obtain the following result.

Corollary 1.2.1 ([67], cf. [25])**.** *Let (M^3, g, k) be an initial data set of cube type. Then (M, g, k) cannot simultaneously satisfy:*

1. *the dominant energy condition,*
2. *the boundary dominant energy condition,*
3. *all dihedral angles of M are acute.*

It is shown in [25] Section 4.9 that there exists a mean convex cubical domain with negative scalar curvature and strictly acute dihedral angles. Hence, Corollary 1.2.1 can be seen as a precise local characterization of the dominant energy condition. Moreover, we will see its connection with the spacetime positive mass theorem.

For geometry on compact manifolds with boundary, Gromov proposed the following conjecture ([26] Sect 3.12.2 III., IV.).

Conjecture 1.2.2 (The fill-in conjecture)**.** *Let (M, g) be a compact Riemannian manifold with scalar curvature $R \geq \sigma$. Then there exists Λ depending only on σ and the intrinsic geometry of $(\partial M, g|_{T(\partial M)})$ such that*

$$\int_{\partial M} H \leq \Lambda, \tag{1.2.1}$$

where H is the mean curvature of the boundary ∂M in (M, g) with respect to the outward unit normal vector.

Definition 1.2.2 ([7] Definition 2, [62])**.** *For $n \geq 3$, a tuple $(\Sigma^{n-1}, \gamma, H)$ is called a Bartnik data set D_B, where (Σ, γ) is an oriented closed null-cobordant manifold with H being a smooth function. A compact manifold (Ω^n, g, k) is called a fill-in of D_B if there is an isometry $\phi : (\Sigma^{n-1}, \gamma) \to (\partial\Omega, g|_{\partial\Omega})$ such that $\phi^* H_g = H$, where H_g is the mean curvature of $\partial\Omega$ to g with respect to the outward unit normal ν.*

In [62] and [61], there was a partial affirmative answer given by the parabolic method to construct an asymptotically flat extension done in [59]. It is shown that if the mean curvature is too large and an NNSC (non-negative scalar curvature) fill-in to a Bartnik data set exists, then there is a contradiction to the positive mass theorem with corners. Heuristically from the perspective of energy (1.1.3), we can see that if the boundary energy is too large, then the gravitational contribution must be negative. Therefore, it again provides us a direction in which Conjecture 1.2.2 can be formulated in terms of relativity. In this thesis, a physical formulation of Conjectures 1.2.1 and 1.2.2 will be given along with partial solutions.

Theorem 1.2.3. *Let $D_{SB} := (\Sigma^2, \gamma, \alpha, H, \beta)$ be a spacetime Bartnik data set where Σ^2 can be embedded into $\mathbb{R}^3$ and γ is smooth. There exists a constant $C_0 = C_0(\Sigma, \gamma) > 0$ such that if*

$$H - f \geq C_0,$$

where $f := \sqrt{(\operatorname{tr}_\Sigma \alpha)^2 + |\beta|^2_\gamma}$, then D_{SB} cannot admit a fill-in satisfying both of the following:

1. *there exists $\mathcal{S}$, a finite (possibly empty) disjoint union of connected weakly trapped surfaces, such that $H_2(\Omega_{ext}, \mathcal{S}, \mathbb{Z}) = 0$, where Ω_{ext} denotes the portion of Ω outside $\mathcal{S}$,*

2. *the dominant energy condition.*

This thesis is structured as follows. In Chapter 2, a review of curvatures and an introduction of different techniques for scalar curvature geometry are given. In Chapter 3, we will prove Theorem 1.1.2 and Corollary 1.1.1. Then, a new notion of quasilocal mass is discussed in Chapter 4. In Chapter 5 and 6, we will give a physical perspective on Gromov's conjectures and provide partial solutions. Finally, in Chapter 7, the existence and regularity of spacetime harmonic functions will be discussed in detail.

Chapter 2

Preliminaries

2.1 Review of curvatures

Let (M^n, g) be a Riemannian manifold. We define the Riemann curvature tensor and the Ricci curvature tensor as follows,

$$\begin{aligned} R(X,Y,Z,W) &= \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W \rangle, \\ Ric(X,W) &= \sum_{i=1}^{n} R(X, e_i, e_i, W), \end{aligned} \tag{2.1.1}$$

where $\{e_i\}$ is an orthonormal basis of TM.

Moreover, the scalar curvature is defined as

$$R = tr_g Ric. \tag{2.1.2}$$

Geometrically, the scalar curvature R reflects volume. Let $p \in (M, g)$, for small $\varepsilon > 0$,

$$\frac{Vol_g(B_\varepsilon(p))}{Vol_{g_{Euc}}(B_\varepsilon(0))} = 1 - \frac{R(p)}{6(n+2)}\varepsilon^2 + O(\varepsilon^4).$$

Let Σ^{n-1} be a 2-sided closed hypersurface in (M^n, g). We can then define the second fundamental form and the mean curvature as follows. Let $X, Y \in T\Sigma$,

$$\begin{aligned} h(X,Y) =& \langle \nabla_X \nu, Y \rangle, \\ H =& tr_\Sigma h, \end{aligned} \tag{2.1.3}$$

where ν is a unit normal of Σ with respect to g. Under this convention, the mean curvature of $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is $n-1$ with respect to the outward normal.

Geometrically, the mean curvature H reflects the change of area. Let $\{\Sigma_t\}$ be a smooth family of surfaces such that it is a variation along the vector field $\phi\nu$ and $\Sigma_0 = \Sigma$, then

$$\frac{d}{dt}|_{t=0} Area(\Sigma_t) = \int_\Sigma \phi H. \tag{2.1.4}$$

These geometric interpretations together with the Gauss-Bonnet theorem and the Hamiltonian formulation (1.1.3) give us insight to formulate and prove various phenomena in physics and geometry.

2.2 Approaches on scalar curvature geometry

For scalar curvature geometry, there have been 2 fundamental approaches. The first one is stable minimal surfaces ([56]) (stable prescribed mean curvature surfaces [24]). By Schoen-

Yau rearrangement of the Gauss equation on the stability operator, an area-minimising hypersurface Σ reveals the relation between its own scalar curvatures and that of the ambient manifold (M, g). For all ϕ in some suitable (weighted) spaces, one has

$$\int_{\Sigma} |\nabla\phi|^2 + \frac{1}{2} R^{\Sigma} \phi^2 \geq \frac{1}{2} \int_{\Sigma} \left(R^M + |A|^2\right) \phi^2. \tag{2.2.1}$$

By the conformal Laplacian and an induction on dimension, one ultimately concerns the Gauss curvature of a 2 dimensional surface and hence the Gauss-Bonnet theorem can be applied.

The second important technique is spinors ([42], [28, 27]). By Lichnerowicz formula, one get the following integral identity. Let $\psi \in S(M)$, the space of spinors of M, we have

$$\int_M -|\mathcal{D}\psi|^2 + |\nabla\psi|^2 + \frac{1}{4} R^M |\psi|^2 = \int_{\partial M} \langle \nu \cdot \mathcal{D}^{\Sigma}\psi, \psi\rangle + \frac{1}{2} H |\psi|^2, \tag{2.2.2}$$

where $\mathcal{D}$ and $\mathcal{D}^{\Sigma}$ is the Dirac operator and the induced boundary Dirac operator respectively. Both of these techniques have a corresponding version for initial data sets, which are stable marginally trapped surfaces [17, 18, 19, 20] and the Dirac-Witten operator [70, 53] for spinors.

Stern ([65]) recently suggested a level set method which is based on harmonic 1-form, and later in some subsequent works on harmonic functions (e.g. [9]).

Unlike the minimal surface technique, its application is only on 3 manifolds with some topological restrictions $H_2(M, \mathbb{Z}) = 0$ or $H_2(M, \partial M, \mathbb{Z}) = 0$. It gives an integral which reveals information from the scalar curvature like spinors, giving an alternative proof of the Riemannian positive mass theorem ([9]), which is reminiscent to the harmonic function approach by Bartnik [5] and Jezierski-Kijowski [33]. Yet, since this integral arises from a solution to PDE instead of a section of spinors, by well established existence theories of PDE solutions with various boundary conditions, one can get some results not yet discovered by the spinorial approach which are the main contributions of this thesis. Correspondingly, Hirsch, Kazaras and Khuri [30] formulated spacetime harmonic functions as inspired by the Dirac-Witten operator which treats an initial data set as if it already sits in a spacetime.

If we consider (M, g, k) as a spacelike slice of a spacetime $(\overline{M}, \overline{g})$, then for a smooth function $\tilde{u}$ on $\overline{M}$, for $X, Y \in TM$, the spacetime Hessian $\overline{\nabla\nabla}\tilde{u}(X,Y) = \nabla\nabla\tilde{u}(X,Y) + k(X,Y)\vec{n}(\tilde{u})$, where $\vec{n}$ is the timelike unit normal of M in $\overline{M}$. And if $\overline{\nabla}\tilde{u}$ is null, we have $\overline{\nabla\nabla}\tilde{u}(X,Y) = \nabla\nabla\tilde{u}(X,Y) + |\nabla\tilde{u}|k(X,Y)$.

Definition 2.2.1. *A function u on M is called spacetime harmonic if*

$$\overline{\Delta}u := tr_g\overline{\nabla\nabla}u = \Delta u + (tr_g k)|\nabla u| = 0.$$

On an initial data set with boundary, to apply the spacetime harmonic function u, one first establishes the following integral formula.

$$\begin{aligned}&\int_\Omega \frac{1}{2}\frac{|\overline{\nabla\nabla}u|^2}{|\nabla u|} + \mu|\nabla u| + \langle J, \nabla u\rangle\, dV \\ \leq &\int_{\partial_{\neq 0}\Omega} \partial_\nu|\nabla u|\, d\sigma + \int_{\partial\Omega} k(\nabla u, \nu) d\sigma + \frac{1}{2}\int_{\underline{u}}^{\overline{u}}\int_{\Sigma_t} R_{\Sigma_t} dA dt,\end{aligned} \tag{2.2.3}$$

where $\partial_{\neq 0}\Omega = \{x \in \partial\Omega \mid |\nabla u| \neq 0\}$, $\Sigma_t = \{u = t\}$, ν is the outward unit normal, $\overline{u}$ and $\underline{u}$ denote the maximum and the minimum of u respectively.

By interpreting spacetime harmonic functions as solution to a prescribed mean curvature surface (or, second fundamental form) equation, it echos with the stable minimal hypersurface approach. Assigning the boundary conditions or the asymptotics of a spacetime harmonic function, one can reveal not only the interaction of interior energy and boundary energy, but also various interesting geometry phenomena.

Chapter 3

Spacetime Positive Mass Theorem with Corners

3.1 The set up of Theorem 1.1.2

$\tilde{\Sigma} \subset M$ we consider is a piecewise smooth surface with (possibly empty) piecewise smooth boundary. Since we can fill in $\partial\tilde{\Sigma}$ by a surface in M, hereafter, it is assumed that $\tilde{\Sigma}$ is some open sets' boundary consisting of piecewise smooth surfaces whose boundaries are piecewise smooth curves and vertices, where the dihedral angles between faces are bounded from below by a positive constant. For example, $\tilde{\Sigma}$ can be the boundary of balls, cylinders, polyhedra and cones in $\mathbb{R}^3$. In this setting, motivated by Hamiltonian formulation (see Section 1.1.1), Theorem 1.1.2 provides partial results on dihedral rigidity for initial data sets in [68].

Denote the designated end by $\mathcal{E}$. Let Σ_i be a connected component of Σ. Let ν denote the normal on faces of Σ_i pointing toward $\mathcal{E}$. A neigbourhood of Σ_i in M on the same side to which ν is pointing is denoted by U_+ while the one on the opposite by U_-. The metrics on $U_\pm$ induced by g are denoted by $g_\pm$ and their mean curvatures on Σ_i with respect to ν are denoted by $H_\pm$. Similarly, we can define $k_\pm$ and $\pi_\pm$ on Σ.

The regularity assumptions of (g, k) in Theorem 1.1.2 naturally arise from the fill-in and extension problems (e.g. [6], [47], [59], [62], [61]). For example, let (M_1, g_1), (M_2, g_2) be two Riemannian manifolds with smooth boundary, where ∂M_1 is isometric to ∂M_2. As mentioned in Section 3 of [47], one can respectively identify the Gauss tubular neighbourhoods of ∂M_1 in M_1 and ∂M_2 in M_2 with $U_1 = \partial M_1 \times (-2\varepsilon, 0]$ and $U_2 = \partial M_2 \times [0, 2\varepsilon)$ for some $\varepsilon > 0$ by Fermi coordinates (x, t). Then, $g_1 \cup g_2$ would be a continuous metric on the glued manifold $M_1 \cup M_2$ under this chart. The smooth structure might be altered but the topology remains the same.

For a smooth closed hypersurface $S \subset M$, we say S is a weakly outer trapped surface if on S, the outer null expansion

$$\theta_+ = H + tr_S k \leq 0,$$

and a marginally outer trapped surface ($MOTS$) if

$$\theta_+ = 0;$$

correspondingly, a weakly inner trapped surface if the inner null expansion

$$\theta_- = H - tr_S k \leq 0,$$

and a marginally inner trapped surface ($MITS$) if

$$\theta_- = 0,$$

where H is computed with respect to the normal pointing to the infinity of the designated end $\mathcal{E}$. A surface is weakly trapped if it is either weakly outer trapped or weakly inner trapped.

If M contains more than one ends, by the decay rate of g and k, we know large coordinate spheres in all the ends other than $\mathcal{E}$ satisfy $\theta^+ < 0$. Therefore, we can assume that M_{ext} has one end $\mathcal{E}$ only.

Since $H_2(M_{ext}, \mathcal{S}, \mathbb{Z}) = 0$, we can compartment M_{ext} into different components as follows,

$$M_{ext} = M_0 \cup_{i=1}^{l} K_i \cup_{j=1}^{m} \Omega_j,$$

for some $l, m \geq 0$, where

1. M_0 is the component containing $\mathcal{E}$ with the boundary composed of components of $\mathcal{S}$ and components of Σ,
2. K_i is compact with the boundary composed only of components of Σ,
3. Ω_j is compact with the boundary composed of components of $\mathcal{S}$ and a component of Σ.
4. (g, k) is smooth on each of the components.

We are going to construct an asymptotically flat initial data set to show the significance the conditions on corners stated in Theorem 1.1.2.

3.1.1 Hyperbolic space patched with negative mass Schwarzschild

Let us consider a rotationally symmetric data set of the form $g = u(r)dr^2 + r^2 g_{\mathbb{S}^2}$, where $g_{\mathbb{S}^2}$ is the standard metric on $\mathbb{S}^2$. Let $u = \frac{1}{1+r^2}$ for $0 \leq r \leq 1$ and $u = \frac{1}{1-\frac{2m}{r}}$ for $r \geq 1$, take $m = -\frac{1}{2}$ so that u is continuous at $r = 1$.

Note that the metric g_- for $r < 1$ is the hyperbolic metric while g_+ for $r > 1$ is the negative mass Schwarzschild metric. This metric $g = (g_-, g_+)$ is then Lipschitz across $\Sigma = \{r = 1\}$ with $H_- = H_+$ on Σ. If we take either $k_- = g_-$ or $-g_-$ for $r < 1$ and $k_+ = 0$ for $r > 1$, then away from Σ we see that (g, k) satisfies the vacuum constraint equations, $\mu = |J| = 0$. Moreover, $H_- - H_+ - |\omega_- - \omega_+| < 0$ on Σ.

For this initial data set, $E = m = \frac{-1}{2}$ and $|P| = 0$. By the definition of ADM energy-momentum vector, we can see that under different choices of k, $E - |P|$ is still of the same sign. This tells us the jump of expansions $\theta_\pm = H \pm tr_\Sigma k$ would not be a sufficient condition for the spacetime positive mass theorem with corners in general. The example also shows that the negativity of $E - |P|$ can be expected from the conditions on the corner.

3.2 Regular level set topology

The existence of regularity of a spacetime harmonic coordinate is stated as below and whose proof will be postponed to Appendix 7.1.

Proposition 3.2.1. *For the asymptotically flat coordinate x^1, for any $\phi \in C^\infty(\mathcal{S})$, there exists $u \in W^{2,p}_{loc}(M_{ext}) \cap W^{3,p}_{loc}(M_{ext} \setminus \Sigma)$ such that*

1. $\Delta u + K|\nabla u| = 0$ *on* M_{ext}*,*
2. $u = \phi$ *on* $\mathcal{S}$*,*
3. $u - x^1 = O^2(|x|^{1-q})$ *as* $|x| \to \infty$*,*
4. $u|_\Sigma$ *is* C^2 *on faces of* Σ*.*

In this section, we would first discuss the regular level set as a whole in M_{ext}. Then we would further study the intersection of the regular level set with the corner Σ. This is essential for analysis in Section 3.3 and 3.4 when we study the boundary terms of the integral formula (Lemma 3.3.1). We would denote a level set $\{u = t\}$ by Σ_t.

3.2.1 Structure of regular level sets in M_{ext}

Denote each component of $\mathcal{S}$ by $\partial_i M$, $i = 1, 2, ...n$. Let $u_{\vec{c}}$, where $\vec{c} = (c^1, c^2, ..., c^n)$ is a constant vector, be a spacetime harmonic function such that

1. $\Delta u_{\vec{c}} + K|\nabla u_{\vec{c}}| = 0$ in M_{ext},
2. $u_{\vec{c}} = c^i$ on $\partial_i M$,
3. $u_{\vec{c}} = v + O^2(|x|^{1-2q})$ as $x \to \infty$.

We are going to show the following 2 conclusions from [30] are still valid for the solution we have obtained which is of slightly lower regularity.

Lemma 3.2.1. *([30] Lemma 5.1) Let $a_i \in \{-1, 1\}$ for $i = 1, 2, ..., n$. There exists a constant $\vec{c}$ such that for each i, there exists $y_i \in \partial_i M$ with $|\nabla u_{\vec{c}}(y_i)| = 0$, and $(-1)^{a_i}(\partial_\nu u_{\vec{c}}) \geq 0$ on $\partial_i M$, where ν is the unit normal pointing out of M_{ext}.*

Theorem 3.2.1. *([30] Theorem 5.2) Let $\vec{c}$ be the constant obtained from Lemma 3.2.1, then all regular level sets of $u_{\vec{c}}$ are connected and non-compact with a single end modeled on $\mathbb{R}^2 \setminus B_1$. Hence, a regular level set would have Euler characteristic ≤ 1.*

It suffices to show that $u_{\vec{c}}$ is continuously differentiable in $\vec{c}$, in the sense of Section 5 in [30], which is as follows.

Lemma 3.2.2. $\Psi : \mathbb{R}^n \to C^{1,\alpha}(M_{ext})$ *is a C^1 map, where $\Psi(\vec{c}) := u_{\vec{c}} - v$ and v is defined as in Section 7.1.*

Proof. For simplicity, say $n = 1$. Now, we have 2 spacetime harmonic functions u_t and u_s, define $w := u_t - u_s = \Psi(t) - \Psi(s)$, we can see w solve the following Dirichlet problem,

1. $\Delta w - K \left(\frac{\nabla u_t + \nabla u_s}{|\nabla u_t| + |\nabla u_s|} \right) \cdot \nabla w = 0$ in M_{ext},

2. $w = t - s$ on ∂M,

3. $w = O^2(|x|^{1-2q})$ as $|x| \to \infty$.

Let $R >> 1$, denote the part of M_{ext} enclosed by coordinate sphere $S_R = \{|x| = R\}$ by M_R. Let ϕ_R be a function satisfying the boundary conditions $\phi_R = t - s$ on ∂M and $\phi_R = w = O(R^{1-2q})$ at S_R. We can extend ϕ_R into M_R such that $||\phi_R||_{C^0} = |t - s|, |\partial^k \phi_R| \leq \frac{C}{R^k}$, $k = 1, 2$. Then by Theorem 8.33 in [23], we have

$$||w||_{C^{1,\alpha}(M_R)} \leq C \left(||w||_{C^0(M_R)} + ||\phi_R||_{C^{1,\alpha}(M_R)} \right).$$

Note that the coefficient on the zeroth order term is zero and hence maximum principle ([23] Theorem 9.1) can be applied. Then we know $||w||_{C^0(M_R)} = ||w||_{C^0(\partial M_R)} = |t-s|$. Therefore, we have,

$$||w||_{C^{1,\alpha}(M_R)} \leq C\left(|t-s| + \frac{C}{R}\right).$$

Take $R \to \infty$, we have.

$$||w||_{C^{1,\alpha}(M_{ext})} \leq C(|t-s|).$$

Therefore,

$$\frac{\Psi(t) - \Psi(s)}{t-s}$$

converges subsequently as $t \to s$. Hence, Ψ is differentiable in c.

Further note that $\partial_c v = 0$, define $u_c^{'} = \partial_c u_c = \partial_c \Psi$, then we have (equations (5.3) and (5.4) in [30]),

1. $\Delta u_c^{'} + K\frac{\nabla u_c}{|\nabla u_c|} \cdot \nabla u_c^{'} = 0$ in M_{ext},
2. $u_c^{'} = 1$ on ∂M,
3. $u_c^{'} = O(|x|^{1-2q})$ as $|x| \to \infty$.

Note that, for all c, $u_c^{'}$ are bounded by 1 in L^∞ by maximum principle and satisfy a PDE with uniformly bounded coefficients. Therefore, they have uniform $W^{2,p}_{loc}$ bound. In particular, $||\nabla u_c^{'}||_{L^p_{loc}}$ are uniformly bounded.

Fix t, for all s, define $\overline{w}_s := u_t^{'} - u_s^{'}$, we have

$$\begin{aligned} L(\overline{w}_s) :=& \Delta(\overline{w}_s) + K\frac{\nabla u_t}{|\nabla u_t|}\nabla(\overline{w}_s) \\ =& f_s \\ :=& K\left(\frac{\nabla u_s}{|\nabla u_s|} - \frac{\nabla u_t}{|\nabla u_t|}\right)\nabla u_s^{'}. \end{aligned} \tag{3.2.1}$$

For all s, $\overline{w}_s = 0$ on ∂M, $||\overline{w}_s||_{L^\infty(M_{ext})} \leq 2$, while the equation above is with uniformly bounded coefficients. Therefore, $||\overline{w}_s||_{W^{2,p}_{loc}}$ are uniformly bounded. Also note that, $\overline{w}_s = O(|x|^{1-2q})$ and $f_s \to 0$ in L^p_{loc} as $s \to t$. Then as $s \to t$, there is a diagonal subsequence convergent to $\overline{w}$ satisfying

1. $L(\overline{w}) = 0$ in M_{ext},
2. $\overline{w} = 0$ on ∂M,
3. $\overline{w} = O(|x|^{1-2q})$ as $|x| \to \infty$.

By maximum principle, $\overline{w} \equiv 0$. Therefore, $c \mapsto \partial_c \Psi$ is continuous. The same argument can be extended to multiple boundary components correspondingly. $\square$

Note that $|\nabla u| = \frac{\nabla u}{|\nabla u|} \cdot \nabla u$. Hence, the maximum principle still applies. Moreover, $u_{\vec{c}}$ is C^2 around $\mathcal{S}$, therefore Hopf lemma also applies on each $\partial_i M$. Therefore, we can follow Section 5 in [30] to conclude Lemma 3.2.1 and Theorem 3.2.1.

3.2.2 Intersection of Σ_t and Σ

Recall from Section 3, we have $M_{ext} = M_0 \cup_{i=1}^{l} K_i \cup_{j=1}^{m} \Omega_j$. Notate $\cup_{i=1}^{l} K_i \cup_{j=1}^{m} \Omega_j$, faces and edges of Σ respectively by $\tilde{\Omega}$, F and γ. From Lemma 3.2.1, we know that $u|_{\tilde{\Omega}\setminus\Sigma}$ and $u|_{M_0\setminus\Sigma}$ are $W^{3,p}_{loc}$, $u|_F$ is C^2 and $u|_\gamma$ is C^1. By [16] (cf. [21]), this is sufficient to conclude Sard's

Theorem on these 4 functions. Let a and b be the infimum and the supremum of $u|_{\tilde{\Omega}}$. In particular for $u|_\Sigma$, a.e. $t \in [a,b]$, $\tau_t = \{u|_\Sigma = t\}$ is a closed piecewise embedded curve and since Σ is compact, we know τ_t is of finitely many components. Since $u \in C^{1,\alpha}(M_{ext})$, we can see that a.e. $t \in [a,b]$, the level set Σ_t intersects Σ transversely along some closed piecewise embedded curves.

3.3 Boundary formulae

With spacetime harmonic functions, we can study ADM energy and momentum by the following integral formula.

Lemma 3.3.1. *(cf. [68] Lemma 3.1, [30] Proposition 3.2) Let (Ω, g, k) be a compact initial data set with $\Sigma := \partial\Omega$. Then, for any spacetime harmonic function u which is $C^{1,\alpha}(\bar{\Omega}) \cap C^{2,\alpha}_{loc}(\bar{\Omega} \setminus \bar{\mathcal{E}})$, where $\mathcal{E}$ denotes the edge components of Σ,*

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\frac{|\overline{\nabla\nabla} u|^2}{|\nabla u|} + \mu|\nabla u| + \langle J, \nabla u\rangle \, dV \\
\leq &\int_{\partial_{\neq 0}\Omega} \partial_\nu |\nabla u| \, d\sigma + \int_{\partial\Omega} k(\nabla u, \nu) d\sigma + \frac{1}{2}\int_{\underline{u}}^{\overline{u}} \int_{\Sigma_t} R_{\Sigma_t} dA dt,
\end{aligned} \tag{3.3.1}$$

where $\partial_{\neq 0}\Omega = \{x \in \partial\Omega \mid |\nabla u| \neq 0\}$, $\Sigma_t = \{u = t\}$, ν is the outward unit normal, $\overline{u}$ and $\underline{u}$ denote the maximum and the minimum of u respectively.

Proof. We here assume that $|\nabla u| \neq 0$ for the simplicity of presentation. For the full generality, one should first consider $\sqrt{|\nabla u|^2 + \delta^2}$ for $\delta > 0$ and then take limit as $\delta \to 0$ (see [65],[10],[30],[31] *Remark* 3.3).

It suffices to verify the divergence theorem such that the following holds.

$$\int_{\partial\Omega} \partial_\nu |\nabla u| = \int_\Omega \Delta |\nabla u|. \tag{3.3.2}$$

Let $\{\Omega_r\}_{r>0}$ be an exhaustion of Ω with vertices and edges of Ω being smoothed out, where r is the parameter of radius of spherical cap around the vertices and rounded-off cylinders along the edges. The functions are regular enough on Ω_r so that the divergence theorem can be applied.

$$\int_{\partial\Omega_r} \partial_{\nu_r}|\nabla u| = \int_{\Omega_r} \Delta|\nabla u|. \tag{3.3.3}$$

From a remark in the proof of Theorem 1.4 in [37], elliptic estimates with scaling are important in showing integrability. Let $p \in \bar{\mathcal{E}}$, w.l.o.g., identified as 0 in a local coordinate chart. From the fact that $u \in C^{1,\alpha}(\bar{\Omega})$ and Schauder estimates with scaling (e.g. [64], [23] Corollary 6.3) applied on u in a (conic) annulus $A(r)$ around p, where $r > 0$ is small, we have $|\nabla\nabla u|_{C^0(A(r))} \leq Cr^{\alpha-1}$. Thus, $|\nabla\nabla u|$ is integrable on $\partial\Omega$ and Ω. Moreover, $r|\nabla\nabla u| \to 0$ as $r \to 0$. Therefore,

$$\int_{\partial\Omega_r} \partial_{\nu_r}|\nabla u| \to \int_{\partial\Omega} \partial_{\nu}|\nabla u|. \tag{3.3.4}$$

On the other hand, $\Delta u = -K|\nabla u|$, first note that by Lemma 3.1 in [30], we have

$$\Delta|\nabla u| \geq -C(||g||_{C^2} + ||k||_{C^1})|\nabla u|. \tag{3.3.5}$$

In particular,

$$(\Delta|\nabla u|)_- \leq C(||g||_{C^2} + ||k||_{C^1})|\nabla u|, \tag{3.3.6}$$

i.e. $(\Delta|\nabla u|)_-$ is integrable on Ω.

By (3.3.3) and integrability of $(\Delta|\nabla u|)_-$, we have on Ω_r,

$$\int_{\Omega_r}(\Delta|\nabla u|)_+ = \int_{\partial\Omega_r}\partial_{\nu_r}|\nabla u| + \int_{\Omega_r}(\Delta|\nabla u|)_-. \tag{3.3.7}$$

Hence, although u is not necessarily C^2 on $\bar{\mathcal{E}}$, we can conclude that (3.3.2) holds by (3.3.4) and monotone convergence theorem as $r \to 0$. Moreover, integrability of the integrands in (3.3.1) follows from the elliptic estimates aforementioned. $\square$

We will express the boundary terms of Lemma 3.3.1 explicitly for spacetime harmonic functions on manifolds with boundary. Note that we have to make use of the fact that $\Delta u = -K|\nabla u|$ instead of 0 in [31].

Lemma 3.3.2. *(cf. [31] Proposition 2.2) Let (Ω, g, k) be a compact initial data set with $\Sigma := \partial\Omega$. Then, for any spacetime harmonic function u which is $C^{1,\alpha}(\bar{\Omega}) \cap C^{2,\alpha}_{loc}(\bar{\Omega} \setminus \bar{\mathcal{E}})$, where $\mathcal{E}$ denotes the edge components of Σ,*

$$\begin{aligned}
&\int_{\Sigma_{\neq 0}}\partial_\nu|\nabla u|\,d\sigma + \int_\Sigma k(\nabla u,\nu)\,d\sigma\\
={}&\int_\Sigma \pi(\nabla u,\nu) - H|\nabla u| \ d\sigma + \int_{\underline{u}}^{\overline{u}}\int_{\tau_t}\kappa\,ds\,dt\\
&+\int_{\Sigma_{\neq 0}} -\frac{\nu(u)}{|\nabla u|}\Delta_\Sigma\eta + \frac{(\nabla_\Sigma\eta)(\nu(u))}{|\nabla u|} \ d\sigma\\
&+\int_{\Sigma_{\neq 0}\cap\{\nabla_\Sigma\eta\neq 0\}} -\frac{\nu(u)}{|\nabla u|}\langle\nabla_{\tau_t'}\tau_t',\nabla_\Sigma\eta\rangle\,d\sigma,
\end{aligned} \tag{3.3.8}$$

where $\eta = u|_\Sigma$, $\Sigma_{\neq 0} = \{x \in \Sigma \mid |\nabla u| \neq 0\}$, $\Sigma_t = \{u = t\}$, $\tau_t = \Sigma_{\neq 0} \cap \Sigma_t \cap \{\nabla_\Sigma\eta \neq 0\}$, H is computed with respect to the outward unit normal ν, $\overline{u}$ and $\underline{u}$ are the maximum and the minimum of u respectively.

Proof. As discussed in Section 3.2.2, a.e. $t \in [\underline{u}, \overline{u}]$, t is regular value of u and Σ_t intersects transversely with Σ on τ_t which is a closed piecewise embedded curve of finitely many components. Then, we can consider

$$\begin{aligned}&\int_{\Sigma_{\neq 0}} \partial_\nu |\nabla u|\, d\sigma \\ =&\int_{\Sigma_{\neq 0}} \partial_\nu |\nabla u|\, d\sigma - \int_{\underline{u}}^{\overline{u}} \left(\int_{\tau_t} \kappa\, ds\right) dt + \int_{\underline{u}}^{\overline{u}} \left(\int_{\tau_t} \kappa\, ds\right) dt.\end{aligned} \tag{3.3.9}$$

We are going to to express $\partial_\nu|\nabla u|$ and κ explicitly. First, for $\partial_\nu|\nabla u|$, we have

$$\begin{aligned}\partial_\nu|\nabla u| =&\frac{\nabla\nabla u(\nabla u,\nu)}{|\nabla u|} \\ =&\frac{\nu(u)}{|\nabla u|}\nabla\nabla u(\nu,\nu) + \frac{1}{|\nabla u|}\nabla\nabla u(\nabla_\Sigma \eta,\nu)\end{aligned} \tag{3.3.10}$$

Using $\Delta_\Omega u = -K|\nabla u|$, we have

$$\nabla\nabla u(\nu,\nu) = \Delta_\Omega u - H\nu(u) - \Delta_\Sigma \eta = -K|\nabla u| - H\nu(u) - \Delta_\Sigma\eta. \tag{3.3.11}$$

We also have,

$$\begin{aligned}\nabla\nabla u(\nabla_\Sigma\eta,\nu) =&(\nabla_\Sigma\eta)(\nu(u)) - (\nabla_{\nabla_\Sigma\eta}\nu)(u) \\ =&(\nabla_\Sigma\eta)(\nu(u)) - \langle \nabla_{\nabla_\Sigma\eta}\nu, \nabla_\Sigma\eta\rangle + \nu(u)\langle\nabla_{\nabla_\Sigma\eta}\nu,\nu\rangle \\ =&(\nabla_\Sigma\eta)(\nu(u)) - \Pi(\nabla_\Sigma\eta,\nabla_\Sigma\eta),\end{aligned} \tag{3.3.12}$$

where Π denotes the second fundamental form on Σ with respect to ν.

Thus, we have,

$$\begin{aligned}\partial_\nu|\nabla u| = & -\frac{\Pi(\nabla_\Sigma\eta,\nabla_\Sigma\eta)}{|\nabla u|} + \frac{(\nabla_\Sigma\eta)(\nu(u))}{|\nabla u|} \\ & - K\nu(u) - H\frac{|\nu(u)|^2}{|\nabla u|} - \frac{\nu(u)}{|\nabla u|}\Delta_\Sigma\eta.\end{aligned} \tag{3.3.13}$$

And in particular, if $\nabla_\Sigma \eta = 0$,

$$\partial_\nu |\nabla u| = -K\nu(u) - H|\nabla u| - \frac{\nu(u)}{|\nabla u|}\Delta_\Sigma \eta. \tag{3.3.14}$$

Then, we have to study the geodesic curvature κ. At a point $x \in \tau_t$, in particular, we know $\nabla_\Sigma \eta \neq 0$, we have the following geometric vectors:

- ν, the outward unit normal to $\partial\Omega$;
- $\nabla_\Sigma \eta$, the gradient of $u = \eta$ on $\partial\Omega$, which is perpendicular to τ_t in Σ. We let $n_t = \frac{1}{|\nabla_\Sigma \eta|}\nabla_\Sigma \eta$;
- τ_t', the unit tangent vector to the curve τ_t;
- ν_t, the outward unit normal to τ_t with respect to Σ_t; and
- $n = \frac{1}{|\nabla u|}\nabla u$, the normal direction to Σ_t along which u increases.

Both

$$\{\nu, n_t\} \text{ and } \{\nu_t, n\}$$

are orthonormal basis for the normal bundle ${\tau_t'}^{\perp}$. We can express n in the basis of ν and n_t,

$$\begin{aligned} n =& \langle n, \nu\rangle \nu + \langle n, n_t\rangle n_t \\ =& \frac{\nu(u)}{|\nabla u|}\nu + \frac{|\nabla_\Sigma \eta|}{|\nabla u|} n_t \end{aligned} \tag{3.3.15}$$

Let $\theta \in [0, \pi]$ be the angle between ν and n, then we have

$$\cos\theta = \frac{\nu(u)}{|\nabla u|}, \tag{3.3.16}$$

and

$$\sin\theta = \frac{|\nabla_\Sigma \eta|}{|\nabla u|}. \tag{3.3.17}$$

On the other hand, consider the geodesic curvature κ, by definition,

$$\kappa = \langle \nabla_{\tau_t'} \nu_t, \tau_t' \rangle, \tag{3.3.18}$$

and

$$\nu_t = \sin\theta\, \nu - \cos\theta\, n_t. \tag{3.3.19}$$

Thus,

$$\begin{aligned} -\kappa =& \langle \nabla_{\tau_t'} \tau_t', \nu_t \rangle \\ =& \ \langle \nabla_{\tau_t'} \tau_t',\ \sin\theta\, \nu - \cos\theta\, n_t \rangle \\ =& \ -\sin\theta\, \Pi(\tau_t', \tau_t') - \cos\theta\, \langle \nabla_{\tau_t'} \tau_t', n_t \rangle \end{aligned} \tag{3.3.20}$$

Therefore, on Σ, by co-area formula, (3.3.16) and (3.3.17), we have,

$$\begin{aligned} & \int_{\underline{u}}^{\overline{u}} \left(-\int_{\tau_t} \kappa\, ds \right) dt \\ =& \int_{\Sigma_{\neq 0} \cap \{\nabla_\Sigma \eta \neq 0\}} -|\nabla_\Sigma \eta| \sin\theta\, \Pi(\tau_t', \tau_t')\ d\sigma \\ & - \int_{\Sigma_{\neq 0} \cap \{\nabla_\Sigma \eta \neq 0\}} |\nabla_\Sigma \eta| \cos\theta\, \langle \nabla_{\tau_t'} \tau_t', n_t \rangle\ d\sigma \\ =& \int_{\Sigma_{\neq 0} \cap \{\nabla_\Sigma \eta \neq 0\}} -\frac{|\nabla_\Sigma \eta|^2}{|\nabla u|} \Pi(\tau_t', \tau_t')\ d\sigma - \int_{\Sigma_{\neq 0} \cap \{\nabla_\Sigma \eta \neq 0\}} \frac{\nu(u)}{|\nabla u|} \langle \nabla_{\tau_t'} \tau_t', \nabla_\Sigma \eta \rangle\ d\sigma. \end{aligned} \tag{3.3.21}$$

Together with (3.3.13), (3.3.14), we have

$$
\begin{aligned}
&\int_{\Sigma_{\neq 0}} \partial_\nu |\nabla u|\, d\sigma + \int_{\underline{u}}^{\overline{u}} \left(- \int_{\tau_t} \kappa\, ds \right) dt \\
= &\int_{\Sigma_{\neq 0} \cap \{\nabla_\Sigma \eta = 0\}} -K\nu(u) - H|\nabla u| - \frac{\nu(u)}{|\nabla u|} \Delta_\Sigma \eta \;\; d\sigma \\
&+ \int_{\Sigma_{\neq 0} \cap \{\nabla_\Sigma \eta \neq 0\}} -K\nu(u) - H \frac{|\nu(u)|^2}{|\nabla u|} \\
&+ \int_{\Sigma_{\neq 0} \cap \{\nabla_\Sigma \eta \neq 0\}} -\frac{|\nabla_\Sigma \eta|^2}{|\nabla u|} \Pi(n_t, n_t) - \frac{|\nabla_\Sigma \eta|^2}{|\nabla u|} \Pi(\tau_t', \tau_t') \;\; d\sigma \\
&+ \int_{\Sigma_{\neq 0} \cap \{\nabla_\Sigma \eta \neq 0\}} -\frac{\nu(u)}{|\nabla u|} \langle \nabla_{\tau_t'} \tau_t', \nabla_\Sigma \eta \rangle - \frac{\nu(u)}{|\nabla u|} \Delta_\Sigma \eta + \frac{(\nabla_\Sigma \eta)(\nu(u))}{|\nabla u|} \;\; d\sigma \\
= &\int_\Sigma -Kg(\nabla u, \nu) - H|\nabla u| \;\; d\sigma \\
&+ \int_{\Sigma_{\neq 0}} -\frac{\nu(u)}{|\nabla u|} \Delta_\Sigma \eta + \frac{(\nabla_\Sigma \eta)(\nu(u))}{|\nabla u|} \;\; d\sigma \\
&+ \int_{\Sigma_{\neq 0} \cap \{\nabla_\Sigma \eta \neq 0\}} -\frac{\nu(u)}{|\nabla u|} \langle \nabla_{\tau_t'} \tau_t', \nabla_\Sigma \eta \rangle \, d\sigma.
\end{aligned}
\tag{3.3.22}
$$

By the definition of the conjugate momentum tensor π, we can conclude the lemma. $\square$

3.4 Proof of Theorem 1.1.2

Recall the decomposition $M_{ext} = M_0 \cup_{i=1}^{l} K_i \cup_{j=1}^{m} \Omega_j$ in Section 3. We may first assume that $M_{ext} = M_0 \cup \Omega$. Without loss of generality, we can rotate the coordinates prior to solving for the spacetime harmonic coordinate so that $P = (-|P|, 0, 0)$. Under the new coordinate system $x = (u, x^2, x^3)$, where u is the spacetime harmonic coordinate obtained in Lemma 3.2.1. Let $L >> 1$, define the following,

1. $T_L = \{x \in M_0 \mid |u| \leq L, (x^2)^2 + (x^3)^2 = L^2\}$,

2. $D_L^{\pm} = \{x \in M_0 \mid u = \pm L, (x^2)^2 + (x^3)^2 \leq L^2\}$,

3. $C_L = T_L \cup D_L^+ \cup D_L^-$.

We would then label u by x^1. Let M_L be the portion of M_0 bounded by C_L and the corner Σ. Since $L >> 1$, we can assume $\mathcal{S} \subset M_L \cup \Omega$. We would use the following notations.

- $\Sigma_t^L = \Sigma_t \cap M_L$,
- $\Sigma_t' = \Sigma_t \cap \Omega$,
- $\tau_t^L = \Sigma_t \cap C_L$,
- $\tau_t = \Sigma_t \cap \Sigma$,
- N, the outward unit normal on ∂M_L,
- ν_L, the unit normal vector on C_L pointing to the infinity of $\mathcal{E}$,
- ν, the unit normal vector on Σ pointing to the infinity of $\mathcal{E}$,
- $\nu_{\mathcal{S}}$, the unit normal vector on $\mathcal{S}$ pointing out of M_{ext},
- $\mathcal{S}^L$, the subcollection of $\mathcal{S}$ which are in M_0,
- $\mathcal{S}'$, the subcollection of $\mathcal{S}$ which are in Ω,
- $\eta = u|_\Sigma$,
- $\nabla^\pm$ and $|\cdot|_\pm$, the connections and norms with respect to $g_\pm$,
- $A_{\neq 0} = \{x \in A \mid |\nabla u| \neq 0\}$ for any $A \subset M$.

From Lemma 3.2.1, we can choose $\vec{c}$ such that u on weakly outer trapped and weakly inner trapped components of $\mathcal{S}$, we would have $\partial_{\nu_{\mathcal{S}}} u \leq 0$ and $\partial_{\nu_{\mathcal{S}}} u \geq 0$ respectively. Furthermore, $\mathcal{S}$ has empty intersection with regular level sets. From Section 3.2.2, we know τ_t are closed piecewise embedded curves for a.e. t. Apply Lemma 3.3.1 on M_L and Lemma 3.3.2 on Σ, we have,

$$
\begin{aligned}
&\int_{M_L} \frac{1}{2}\left(\frac{|\overline{\nabla}\nabla u|^2}{|\nabla u|} + 2\left(\mu|\nabla u| + \langle J, \nabla u\rangle\right)\right) dV \\
\leq &\int_{\partial_{\neq 0} M_L} \partial_N |\nabla u|\, d\sigma + \int_{\partial M_L} k(\nabla u, N)\, d\sigma + \frac{1}{2}\int_{-L}^{L}\int_{\Sigma_t^L} R_{\Sigma_t^L}\, dA dt \\
= &\int_{\mathcal{S}^L_{\neq 0}} \partial_{\nu_{\mathcal{S}}}|\nabla u|\, d\sigma + \int_{\mathcal{S}^L} k(\nabla u, \nu_{\mathcal{S}})\, d\sigma \\
&+ \int_{\partial C_L} \partial_{\nu_L}|\nabla u|\, d\sigma + \int_{-L}^{L}\left(-\int_{\tau_t^L}\kappa\, ds\right) dt + \int_{\partial C_L} k(\nabla u, \nu_L)\, d\sigma \\
&- \int_{\Sigma} \pi_+(\nabla u, \nu) - H_+|\nabla u| \;\; d\sigma \\
&+ \int_{\Sigma_{\neq 0}} \frac{\nu(u)}{|\nabla u|}\Delta_\Sigma \eta - \frac{(\nabla_\Sigma \eta)(\nu(u))}{|\nabla u|} \;\; d\sigma + \int_{\Sigma_{\neq 0}\cap\{\nabla_\Sigma \eta \neq 0\}} \frac{\nu(u)}{|\nabla u|}\langle \nabla_{\tau_t'}\tau_t', \nabla_\Sigma \eta\rangle\, d\sigma \\
&+ \frac{1}{2}\int_{-L}^{L}\int_{\Sigma_t} R_{\Sigma_t} dA\, dt + \int_{-L}^{L}\int_{-\tau_t}\kappa\, ds\, dt + \int_{-L}^{L}\int_{\tau_t^L}\kappa\, ds\, dt.
\end{aligned} \tag{3.4.1}
$$

On the other hand, by the asymptotics of u and maximum principle, τ_t^L is a circle. By computations in Section 6 of [9] and Section 6 of [30], we get,

$$
\begin{aligned}
&\int_{\partial C_L} \partial_{\nu_L}|\nabla u|\, d\sigma + \int_{-L}^{L}\left(-\int_{\tau_t^L}\kappa\, ds\right) dt + \int_{\partial C_L} k(\nabla u, \nu_L)\, d\sigma \\
= &-4\pi L + \frac{1}{2}\int_{C_L}\left(g_{ij,i} - g_{ii,j}\right)\nu_L^j\, dA + \int_{C_L}\pi_{1j}\nu_L^j\, dA + O(L^{1-2q}) + O(L^{-q}).
\end{aligned} \tag{3.4.2}
$$

Similarly, apply Lemma 3.3.1 on Ω and Lemma 3.3.2 on Σ, we have,

$$
\begin{aligned}
&\int_{\Omega} \frac{1}{2}\left(\frac{|\overline{\nabla\nabla} u|^2}{|\nabla u|} + 2\left(\mu|\nabla u| + \langle J, \nabla u\rangle\right)\right) dV \\
\leq &\int_{\mathcal{S}'_{\neq 0}} \partial_{\nu_{\mathcal{S}}}|\nabla u|\, d\sigma + \int_{\mathcal{S}'} k(\nabla u, \nu_{\mathcal{S}})\, d\sigma + 2\pi \int_{-L}^{L} \chi(\Sigma'_t)\, dt \\
&+ \int_{\Sigma} \pi_-(\nabla u, \nu) - H_-|\nabla u| \; d\sigma \\
&+ \int_{\Sigma_{\neq 0}} -\frac{\nu(u)}{|\nabla u|}\Delta_{\Sigma}\eta + \frac{(\nabla_{\Sigma}\eta)(\nu(u))}{|\nabla u|} \; d\sigma \\
&+ \int_{\Sigma_{\neq 0}\cap\{\nabla_{\Sigma}\eta\neq 0\}} -\frac{\nu(u)}{|\nabla u|}\langle \nabla_{\tau'_t}\tau'_t, \nabla_{\Sigma}\eta\rangle\, d\sigma \\
&+ \frac{1}{2}\int_{-L}^{L}\int_{\Sigma_t} R_{\Sigma_t} dA\, dt + \int_{-L}^{L}\int_{\tau_t} \kappa\, ds\, dt..
\end{aligned}
\tag{3.4.3}
$$

By (3.3.14), on $\mathcal{S}$, with the corresponding choice of sign of normal derivatives as aforementioned,

$$
\begin{aligned}
&\int_{\mathcal{S}_{\neq 0}} \partial_{\nu_{\mathcal{S}}}|\nabla u| + k(\nabla u, \nu_{\mathcal{S}})\, d\sigma \\
=&\sum_{i=1}^{n} \int_{\partial_i M_{\neq 0}} H|\partial_{\nu_{\mathcal{S}}} u| - tr_{\partial_i M} k(\partial_{\nu_{\mathcal{S}}} u)\, d\sigma \\
\leq & \, 0,
\end{aligned}
\tag{3.4.4}
$$

where $\partial_i M$ are the components of $\mathcal{S}$ and H is computed with respect to $-\nu_{\mathcal{S}}$.

Note that u is C^1 across Σ, $\nu(u) + (-\nu)(u)$ is constantly zero on Σ. Furthermore, g is continuous, in case $\pm\tau_t$ have some turning angles, they are of the opposite signs. Moreover, by Theorem 3.2.1, we know that Σ_t has a single end modeled on $\mathbb{R}^2 \setminus B_1$. Therefore, for $L >> 1$, a.e. $t \in [a, b]$, $1 \geq \chi(\Sigma_t) = \chi(\Sigma_t^L) + \chi(\Sigma'_t)$. Summing equations (3.4.1) and (3.4.3), and applying Gauss-Bonnet Theorem, we have,

$$\begin{aligned}
&\int_{M_L \cup \Omega} \frac{1}{2}\left(\frac{|\overline{\nabla}\nabla u|^2}{|\nabla u|} + 2\left(\mu|\nabla u| + \langle J, \nabla u\rangle\right)\right) dV \\
\leq\, & 2\pi \int_{-L}^{L} \chi(\Sigma_t)\, dt - 4\pi L + \frac{1}{2}\int_{C_L} \left(g_{ij,i} - g_{ii,j}\right) \nu_L^j \, dA + \int_{C_L} \pi_{1j}\nu_L^j \, dA \\
&- \int_{\Sigma} \pi_+(\nabla u, \nu) - H_+|\nabla u| \;\, d\sigma + \int_{\Sigma} \pi_-(\nabla u, \nu) - H_-|\nabla u| \;\, d\sigma \\
&+ O(L^{1-2q}) + O(L^{-q}) \\
\leq\, & \frac{1}{2}\int_{C_L} \left(g_{ij,i} - g_{ii,j}\right) \nu_L^j \, dA + \int_{C_L} \pi_{1j}\nu_L^j \, dA \\
&+ \int_{\Sigma} \left(H_+ - H_-\right)|\nabla u| \;\, d\sigma + \int_{\Sigma} (\pi_- - \pi_+)(\nabla u, \nu) \;\, d\sigma \\
&+ O(L^{1-2q}) + O(L^{-q}).
\end{aligned} \tag{3.4.5}$$

By Proposition 4.1 in [5], as $L \to \infty$, we have

$$\frac{1}{2}\int_{C_L} \left(g_{ij,i} - g_{ii,j}\right) \nu_L^j \, dA + \int_{C_L} \pi_{1j}\nu_L^j \, dA \to 8\pi\left(E + P_1\right) = 8\pi\left(E - |P|\right).$$

For the general case, we can apply the same idea onto each component M_0, K_i and Ω_j and sum up the integrals. Theorem 1.1.2 is therefore proved.

As we can see from the proof above, particularly the term $\pi_+(\nabla u, \nu)$, in general we get

Corollary 3.4.1. *For $\mathbf{a} \in \mathbb{S}^2 \subset \mathbb{R}^3$, if a spacetime harmonic function $u_{\mathbf{a}}$ is asymptotic to $a^i x^i$, then*

$$\begin{aligned}
16\pi(E + \langle \mathbf{a}, P\rangle) \geq & \int_{M_{ext}\setminus\Sigma} \left(\frac{|\overline{\nabla}\nabla u_{\mathbf{a}}|^2}{|\nabla u_{\mathbf{a}}|} + 2(\mu|\nabla u_{\mathbf{a}}| + \langle J, \nabla u_{\mathbf{a}}\rangle)\right) \\
&+ 2\int_{\Sigma}(H_- - H_+)|\nabla u_{\mathbf{a}}| - 2\int_{\Sigma}(\pi_- - \pi_+)(\nabla u_{\mathbf{a}}, \nu).
\end{aligned} \tag{3.4.6}$$

3.5 Proof of Corollary 1.1.1

We can follow Section 7 in [30] with slight modifications to conclude the equality case. Some details are provided to explain how to deal with discontinuity of k and u being only $C^{1,\alpha}$ across Σ.

3.5.1 $E = |P|$ case

Under the assumptions of Corollary 1.1.1, from the inequality of Theorem 1.1.2, if $E = |P|$, we have $\overline{\nabla\nabla} u = \nabla\nabla u + |\nabla u| k = 0$. Then, by Kato's inequality, we get

$$|\nabla|\nabla u|| \leq |\nabla\nabla u| \leq |k||\nabla u|.$$

Therefore, by standard ODE technique, there exists a constant $C > 0$ such that $|\nabla u| \geq C$ on M_0. Since $u \in C^{1,\alpha}_{loc}$, $|\nabla u| \geq C$ on Σ. Hence, use the same technique again within the remaining compact portions of M_{ext}, we can conclude that $|\nabla u| \geq \tilde{C} > 0$ for some $\tilde{C}$ on M_{ext}. This is inconsistent with the choice of normal derivatives of $u_{\vec{c}}$ on $\mathcal{S}$ by Lemma 3.2.1. Hence, $\mathcal{S}$ is empty.

Let $\tau >> 1$, $\Sigma_\tau = \{u = \tau\}$ is an asymptotically flat complete plane and since $|\nabla u|$ does not vanish, we can see along the level set flow, the topology does not change. Therefore, M is diffeomorphic to $\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$. Moreover, we can see that on each level set Σ_t, $\frac{\nabla\nabla u}{|\nabla u|}|_{T\Sigma_t} + k|_{T\Sigma_t} = h_t + k|_{T\Sigma_t} = 0$ (spacetime totally geodesic), where h_t is the second fundamental form of Σ_t with respect to $\frac{\nabla u}{|\nabla u|}$. M is thus foliated by stable MOTS. Then, by Theorem 1 (2) of [13], we know that each Σ_t has vanishing Gauss curvature and hence is isometric to $\mathbb{R}^2$. Thus, together with asymptotic flatness, the metric can be expressed as

$g(u, x^2, x^3) = \frac{1}{|\nabla u|^2} du^2 + \delta_{ij} dx^i dx^j$.

3.5.2 Isometric embedding into Minkowski space for the case $E = |P| = 0$.

As $\mathcal{S}$ is empty, we now have $M = M_0 \cup_{i=1}^{l} K_i$. Here, for notation simplicity, we denote $\cup_{i=1}^{l} K_i$ by $\tilde{K}$.

For $M \cong \mathbb{R}^3$, let (x^1, x^2, x^3) be a global coordinate system which coincides with the asymptotically flat coordinate on $M \setminus \mathcal{C}$. And we can, by Lemma 3.2.1, construct a spacetime harmonic function $u(a_1, a_2, a_3)$ which is asymptotic to $a_i x^i$, where $\sum_{i=1}^{3} (a_i)^2 = 1$. Then as in Theorem 7.3 in [30], we can define a lapse function α and a shift vector β by

$$\alpha = \left|\nabla u \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)\right| + \left|\nabla u \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right| - \left|\nabla u \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right|, \tag{3.5.1}$$

and

$$\beta = \nabla u \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) + \nabla u \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) - \nabla u \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right). \tag{3.5.2}$$

Then, we can define a stationary spacetime, $(\overline{M} = \mathbb{R} \times M, \overline{g})$ where

$$\overline{g} = -(\alpha^2 - |\beta|^2) dt^2 + 2\beta_i dx^i dt + g, \tag{3.5.3}$$

where the Killing vector is

$$\partial_t = \alpha \vec{n} + \beta, \tag{3.5.4}$$

where $\vec{n}$ is the unit normal to the hypersurface constant t-slice. We can see that (M, g) is isometric to a constant time slice in $\overline{M}$ under such construction. First, notice that α and β

are differentiable on $M \setminus \Sigma$ and continuous across Σ. From equations (7.9) to (7.11) in [30], it is shown that $\alpha^2 - |\beta|^2$ is constant in $M \setminus \tilde{K}$ and $\tilde{K} \setminus \Sigma$ respectively. By continuity, we have $\alpha^2 - |\beta|^2$ is a constant on M. Since $\alpha^2 - |\beta|^2 \to 1$ as $r \to \infty$, we have $\alpha^2 - |\beta|^2 \equiv 1$. we thus have,

$$\begin{aligned} \overline{g} = -dt^2 + 2\beta_i dx^i dt + g &= -\left(dt - \beta_i dx^i\right)^2 + \left(g_{ij} + \beta_i\beta_j\right)dx^i dx^j \\ &= -\left(dt - d\Psi\right)^2 + \left(g + d\Psi\right)^2, \end{aligned} \tag{3.5.5}$$

where $\Psi = u\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) + u\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) - u\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Notice that β is exact since $\beta = \nabla\Psi$. Then, on $M \setminus \Sigma$, we have, $a, b, c = 0, 1, 2, 3$, where $\partial_0 = \partial_t$.

$$\overline{\Gamma}^a_{it} = \frac{1}{2}\overline{g}^{ac}\left(\overline{g}_{ic,t} + \overline{g}_{tc,i} - \overline{g}_{it,c}\right) = \overline{g}^{ac}\left(\partial_i\beta_c - \partial_c\beta_i\right) = 0.$$

On the other hand, since $E = |P| = 0$, for all $|\vec{a}| = 1$, we have $\nabla\nabla u(a_1, a_2, a_3) = -|\nabla u(a_1, a_2, a_3)|k$, and hence

$$\nabla_i\beta_j = -\alpha k_{ij}.$$

With these, we can show k is the corresponding 2nd fundamental from of M with respect to this embedding since on $M \setminus \Sigma$,

$$\langle \overline{\nabla}_i \vec{n}, \partial_j \rangle = \alpha^{-1}\langle \overline{\nabla}_i(\partial_t - \beta), \partial_j \rangle = \alpha^{-1}\overline{\Gamma}^b_{it}\overline{g}_{bj} + \alpha^{-1}\nabla_i\beta_j = k_{ij}.$$

Therefore, (M, g, k) arises as a constant time slice in $(\overline{M}, \overline{g})$.

For $l = 1$, 2 and 3, construct vector fields X_l on M as follows,

$$X_l = \nabla u_l + |\nabla u_l|\vec{n},$$

where

$$u_1 = u(1,0,0), \;\; u_2 = u(0,1,0), \;\; u_3 = u(0,0,1),$$

i.e. the spacetime harmonic coordinates corresponding to the original asymptotically flat coordinates (x^1, x^2, x^3). These vector fields are differentiable on $M \setminus \Sigma$ and continuous across Σ. Extend these vector fields trivially along ∂_t to $\overline{M}$. Then, as shown in equations (7.13), (7.18) to (7.21) in [30], we know that on $\mathbb{R} \times (M \setminus \tilde{K})$ and $\mathbb{R} \times (\tilde{K} \setminus \Sigma)$, these vector fields and ∂_t are covariantly constant. And hence by continuity, the metric on these vector fields is constant on $\overline{M}$. They are linearly independent at the asymptotic end and thus linearly independent on $\overline{M}$. Therefore, $(\overline{M}, \overline{g})$ is flat. Further, by a change of coordinate, $\overline{t} = t - \Psi(x)$ and $\overline{x} = x$, we have

$$\overline{g} = -d\overline{t}^2 + (g + d\Psi^2).$$

From this construction, we can see (M, g, k) can be expressed as a graph $\overline{t} = -\Psi(\overline{x})$. Also note that $(\mathbb{R}^3, g + d\Psi^2)$ is asymptotically flat and therefore complete. Furthermore, it is a constant $\overline{t}$ slice in this splitting of $\overline{g}$ and hence it is flat and isometric to Euclidean space. Therefore, we have $\overline{M}$ is isometric to Minkowski space.

Chapter 4

A new quasilocal mass $\mathcal{W}$

In [59], the Riemannian positive mass theorem with Lipschitz metric along corners is used to prove positivity of the Brown York mass. This motivates us to consider if Theorem 1.1.2 can provide an insight into some quasilocal quantities. In particular, we have the following.

4.1 Positivity of $\mathcal{W}(\Sigma)$ for 3 dimensional initial data sets

Corollary 4.1.1. *Let (Ω^3, g, k) be a compact initial data set satisfying the dominant energy condition. Assume there exists $\mathcal{S}$, a finite (possibly empty) disjoint union of connected weakly trapped surfaces, such that $H_2(\Omega_{ext}, \mathcal{S}, \mathbb{Z}) = 0$, where Ω_{ext} denotes the portion of Ω outside $\mathcal{S}$. Suppose $\Sigma = \partial\Omega$ is a smooth surface of finitely many components with Gaussian curvature $\kappa > 0$ and mean curvature H with respect to the outward normal ν. Denote the mean curvature of isometric embedding of Σ into $\mathbb{R}^3$ with respect to the outward normal by H_0. If*

$H > |\omega|$, where $\omega = \pi(\cdot, \nu)$, then

$$\mathcal{W}(\Sigma) := \frac{1}{8\pi}\int_\Sigma H_0 - (H - |\omega|) \geq 0.^1$$

If $\mathcal{W}(\Sigma) = 0$, then Σ is connected, Ω is diffeomorphic to a domain in $\mathbb{R}^3$ and can be isometrically embedded into Minkowski space.

Proof. We would follow Bartnik-Shi-Tam construction of quasi-spherical metric ([6], [59]) for each component Σ_i of Σ. First, define on Σ,

$$u = \frac{H_0}{H - |\omega|}. \tag{4.1.1}$$

As Nirenberg ([49]) and independently, Pogorelov ([54]) have solved Weyl's isometric embedding problem, we know that by its positive Gauss curvature, Σ_i can be isometrically embedded into $\mathbb{R}^3$. We notate the image of isometric embedding of Σ_i into $\mathbb{R}^3$ by Σ_{0i}, and the unbounded region of $\mathbb{R}^3$ outside of Σ_{0i} by $M_i = \Sigma_{0i} \times [0, \infty)$ which stands for a foliation by unit normal flow. Then as in [59], we can construct an asymptotically flat metric $g_i = u(r)^2 dr^2 + g_r$ with zero scalar curvature on M_i ([59] Theorem 2.1(b)), where $u(0) = u$ and g_r stands for the metric induced on $\Sigma_r = \Sigma_{0i} \times \{t = r\}$ by the Euclidean metric on $\mathbb{R}^3$. Since the Gauss curvature of Σ_r is positive, we have (Lemma 4.2 in [59]),

$$\begin{aligned} 8\pi\frac{d}{dr}Q(\Sigma_r) :=& \frac{d}{dr}\int_{\Sigma_r} H_0(r)\left(1 - \frac{1}{u(r)}\right) d\sigma_r \\ =& -\frac{1}{2}\int_{\Sigma_r} R_{\Sigma_r} u^{-1}(1-u)^2 \leq 0, \end{aligned} \tag{4.1.2}$$

where $H_0(r)$ is the mean curvature of Σ_r with respect to the Euclidean metric of $\mathbb{R}^3$. More-

[1] One can compare $\mathcal{W}(\Sigma)$ to the expression of the physical Hamiltonian in equation (2.14) in [29].

over, by Theorem 2.1 (c) in [59], we have

$$\lim_{r\to\infty} Q(\Sigma_r) = E(g_i).$$

Therefore to prove the inequality, i.e. $\mathcal{W}(\Sigma_i) = Q(\Sigma_{0i}) \geq 0$, it suffices to show $E(g_i) \geq 0$.

Consider the glued initial data set $\tilde{M} = \Omega \cup^n_{\Sigma_i, i=1} M_{+i}$, with metric $\tilde{g} = (g, \{g_i\}^n_{i=1})$ and symmetric 2 tensor $\tilde{k} = (k, \{k_i = 0\}^n_{i=1})$, and correspondingly $\pi_i = 0$ for $i = 1, 2, ..., n$. By the construction above, we know that $\tilde{g}$ is Lipschitz across Σ and the dominant energy condition is satisfied on $\tilde{M} \setminus \Sigma$. And by equation (1.6) in [59], we have on Σ_i, the mean curvature H_i of g_i with respect to the outward normal is $H - |\omega|$. Therefore, on Σ_i, we have

$$H - H_i - |\omega - 0| = H - (H - |\omega|) - |\omega| = 0.$$

Fix l, for g_l, as discussed in Section 3, for each of other extensions, a large coordinate sphere can act as a weakly trapped surface with respect to g_l. These spheres together with $\mathcal{S}$ are the boundary of $\tilde{M}_{ext}$. Hence, by Theorem 1.1.2, we have $0 \leq E(g_l) - |P_l| = E(g_l)$. As $\mathcal{W}(\Sigma) = \sum_{i=1}^n \mathcal{W}(\Sigma_i)$, we can conclude the positivity. And by Corollary 1.1.1, we can conclude the equality case. □

4.2 Positivity of $\mathcal{W}(\Sigma)$ on spin compact initial data sets

As we can see from the proof based on [59] above, if we consider spin condition as in [63] and [36], we can arrive at the following conclusion.

Corollary 4.2.1. *(cf. [59] Theorem 4.1) For $n \geq 3$, let (Ω^n, g, k) be a compact initial data set in a spacetime N^{n+1} satisfying the dominant energy condition. Assume that $\Sigma := \partial\Omega$*

has finitely many components. Let H denote the mean curvature of Σ with respect to the outward normal ν. Suppose Ω is spin and Σ can be isometrically embedded into $\mathbb{R}^n$ as a strictly convex closed hypersurface. Denote the mean curvature of isometric embedding of Σ into $\mathbb{R}^n$ with respect to the outward normal by H_0. If $H > |\omega|$, where $\omega = \pi(\cdot, \nu)$, then

$$\mathcal{W}(\Sigma) := \frac{1}{8\pi}\int_{\Sigma} H_0 - (H - |\omega|) \geq 0,$$

and equality implies that Σ is connected and N is a flat spacetime along Ω.

Chapter 5

Dihedral Rigidity of Initial Data Sets

We now outline the proof. First, we consider a solution u to the following mixed boundary value problem.

Lemma 5.0.1. *Given an initial data set (M^3, g, k) of type P, where all dihedral angles are everywhere smaller than π, then there exists a non-negative spacetime harmonic function $u \in C^{0,\alpha}(M) \cap C^{1,\alpha}_{loc}(M \setminus (\bar{T} \cup \bar{B})) \cap C^{2,\alpha}_{loc}(M \setminus \bar{\mathcal{E}}) \cap W^{3,p}_{loc}(\mathring{M})$ such that*

1. *$G_0(u) := \Delta u + K|\nabla u| = 0$ in $\mathring{M}$,*

2. *$u = 0$ on B and $u = 1$ on T,*

3. *$\partial_\nu u = 0$ on F,*

where $K := tr_g k$, ν denotes the outward unit normal of ∂M; T, B, F and $\mathcal{E}$ denote the top, the bottom, the side faces and the edges of M respectively.

The proof of Lemma 5.0.1 is based on Leray-Schauder fixed point theorem and a reflection technique. It will be shown in Appendix 7.3. Then, under the assumptions of Theorem 1.2.2, we can see that M is smoothly foliated by level sets of u. In particular, we show M is foliated by stable free boundary MOTS. We then apply the results of [1] Section 5 to study each level set using certain integral formulae for spacetime harmonic functions ([30],[31],[14],[68]). Then, the flow generated by $\frac{\nabla u}{|\nabla u|^2}$ on M is studied. Finally, we can conclude the proof using the geometric assumptions on M.

5.1 Integral Formula

The following integral formula links the interior energy condition and the boundary behaviour of an initial data set.

Lemma 5.1.1. *([68], cf. [30] Proposition 3.2) Let (M^3, g, k) be an initial data set of type P, where all dihedral angles are everywhere smaller than π. Further assume that the dihedral angles between T and F and those of B and F are everywhere less than or equal to $\pi/2$. Then, for a spacetime harmonic function u in Lemma 5.0.1,*

$$\begin{aligned} &\int_M \frac{1}{2}\frac{|\overline{\nabla\nabla} u|^2}{|\nabla u|} + \mu|\nabla u| + \langle J, \nabla u\rangle \, dV - \frac{1}{2}\int_0^1 \int_{\Sigma_t} R_{\Sigma_t} dA dt \\ \leq &\int_{\partial_{\neq 0} M} \partial_\nu |\nabla u| \, d\sigma + \int_{\partial M} k(\nabla u, \nu) d\sigma, \end{aligned} \tag{5.1.1}$$

where $\partial_{\neq 0} M = \{x \in \partial M \mid |\nabla u| \neq 0\}$, $\Sigma_t = \{u = t\}$ and ν is the outward unit normal on ∂M.

Proof. We here assume that $|\nabla u| \neq 0$ for the simplicity of presentation. For the full generality, one should first consider $\sqrt{|\nabla u|^2 + \delta^2}$ for $\delta > 0$ and then take limit as $\delta \to 0$ (see [65],[10],[9],[30],[31] *Remark* 3.3).

It suffices to verify the divergence theorem such that the following holds since the remaining would be the same as in the references aforementioned (application of Bochner formula, Gauss equation and coarea formula).

$$\int_{\partial M} \partial_\nu |\nabla u| = \int_M \Delta |\nabla u|. \tag{5.1.2}$$

From a remark in the proof of Theorem 1.4 in [37], elliptic estimates with scaling (e.g. [64], [23] Corollary 6.3) are essential for determining integrability. Let $\{M_r\}_{r>0}$ be an exhaustion of M with vertices and edges of M being smoothed out, where r is the parameter of radius of spherical cap around the vertices and rounded-off cylinders along the edges. The functions are regular enough on M_r so that the divergence theorem can be applied.

$$\int_{\partial M_r} \partial_{\nu_r} |\nabla u| = \int_{M_r} \Delta |\nabla u|. \tag{5.1.3}$$

1. <u>L.H.S. of (3.3.2).</u> To show that $\int_{\partial M} \partial_\nu |\nabla u|$ is well-defined, we consider the following.

Proposition 5.1.1. *(cf. [31] Proposition 2.2, [14],[68]) Let Σ be a face of a type P initial data set (M^3, g, k), for $|\nabla u| > 0$,*

1. *if $u = constant$ on Σ,*

$$\partial_\nu |\nabla u| = -K\nu(u) - H|\nabla u|; \tag{5.1.4}$$

2. *if $\partial_\nu u = 0$ on Σ,*

$$\partial_\nu |\nabla u| = -\,|\nabla u| \Pi\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right), \tag{5.1.5}$$

where Π is the second fundamental form of Σ with respect to the outward normal ν.

Proof. Let $\eta = u|_\Sigma$,

$$\begin{aligned}\partial_\nu |\nabla u| =& \frac{\nabla\nabla u(\nabla u, \nu)}{|\nabla u|} \\ =& \frac{\nu(u)}{|\nabla u|}\nabla\nabla u(\nu,\nu) + \frac{1}{|\nabla u|}\nabla\nabla u(\nabla_\Sigma \eta, \nu)\end{aligned} \tag{5.1.6}$$

Using $\Delta_M u = -K|\nabla u|$, we have

$$\nabla\nabla u(\nu,\nu) = \Delta_M u - H\nu(u) - \Delta_\Sigma \eta = -K|\nabla u| - H\nu(u) - \Delta_\Sigma \eta. \tag{5.1.7}$$

We also have,

$$\begin{aligned}\nabla\nabla u(\nabla_\Sigma \eta, \nu) =& (\nabla_\Sigma \eta)(\nu(u)) - (\nabla_{\nabla_\Sigma \eta}\nu)(u) \\ =& (\nabla_\Sigma \eta)(\nu(u)) - \langle \nabla_{\nabla_\Sigma \eta}\nu, \nabla_\Sigma \eta\rangle + \nu(u)\langle \nabla_{\nabla_\Sigma \eta}\nu, \nu\rangle \\ =& (\nabla_\Sigma \eta)(\nu(u)) - \Pi(\nabla_\Sigma \eta, \nabla_\Sigma \eta),\end{aligned} \tag{5.1.8}$$

Thus, we have,

$$\begin{aligned}\partial_\nu |\nabla u| =& -\frac{\Pi(\nabla_\Sigma \eta, \nabla_\Sigma \eta)}{|\nabla u|} + \frac{(\nabla_\Sigma \eta)(\nu(u))}{|\nabla u|} \\ & - K\nu(u) - H\frac{|\nu(u)|^2}{|\nabla u|} - \frac{\nu(u)}{|\nabla u|}\Delta_\Sigma \eta.\end{aligned} \tag{5.1.9}$$

Hence, if $u = constant$ on Σ,

$$\partial_\nu |\nabla u| = -K\nu(u) - H|\nabla u|. \tag{5.1.10}$$

If $\partial_\nu u = 0$ on Σ,

$$\begin{aligned}\partial_\nu |\nabla u| = & -\frac{\Pi(\nabla_\Sigma \eta, \nabla_\Sigma \eta)}{|\nabla u|} \\ = & -|\nabla u| \Pi(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}).\end{aligned} \tag{5.1.11}$$

$\square$

From Proposition 5.1.1 above, we have on ∂M,

$$\int_{\partial M} \partial_\nu |\nabla u| = \int_F -|\nabla u| \Pi(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}) + \int_{T \cup B} -K\nu(u) - H|\nabla u|. \tag{5.1.12}$$

In particular, on ∂M,

$$|\partial_\nu |\nabla u|| \leq C(||g||_{C^1} + ||k||_{C^0})|\nabla u|. \tag{5.1.13}$$

Therefore, for the well-definedness of $\int_{\partial M} \partial_\nu |\nabla u|$, it suffices to check if $|\nabla u|$ is integrable on ∂M. Let $p \in \bar{\mathcal{E}}$, w.l.o.g., identified as 0 in a local coordinate chart. From the fact that $u \in C^{0,\alpha}(M)$, apply $W^{2,p}$ estimate followed by Sobolev embedding onto $w_r(x) := u(rx) - u(0)$, where $r > 0$ fixed, in a (conic) annulus $A(1)$ around p, we have

$$|\nabla u|_{C^0(A(r))} \leq C|\nabla \nabla u|_{L^p(A(r))} \leq Cr^{\alpha - 1}. \tag{5.1.14}$$

$|\nabla u|$ is therefore integrable on ∂M and also on M.

2. $\underline{\int_{\partial M_r} \partial_{\nu_r} |\nabla u| \to \int_{\partial M} \partial_\nu |\nabla u|}$. First, let's consider the convergence along the horizontal edges by a blow-up argument.

Proposition 5.1.2. *Let W be a compact neighbourhood along the interior of an horizontal edge E_H and $r(p) = dist(p, E_H)$. We have $r|\nabla\nabla u| \to 0$ uniformly in W as $r \to 0$.*

Proof. Assume on the contrary that $r|\nabla\nabla u|$ does not converge to 0 uniformly in W as $r \to 0$. Hence, there exists $\varepsilon_0 > 0$ and a sequence $\{p_i\}$ in W with

$$L_i|\nabla\nabla u|(p_i) \geq \varepsilon_0, \tag{5.1.15}$$

where $L_i = r(p_i)$ and $L_i \to 0$.

Denote the point on E_H closest to p_i by q_i. Let p denote a subsequential limit of p_i and hence also the limit of q_i. W.l.o.g., we still denote the subsequence by $\{p_i\}$.

Then, on a ball (intersecting with a wedge) denoted by B_1, define a sequence of functions u_i by scaling around each q_i, that is,

$$u_i(x) := \frac{u(q_i + L_i x)}{L_i}. \tag{5.1.16}$$

Check that for each i,

1. $u_i(0) = 0$,
2. $\partial u_i(x) = (\partial u)(q_i + L_i x)$, and
3. $\partial\partial u_i(x) = L_i \partial\partial u(q_i + L_i x)$.

Thus,

$$\Delta_i u_i(x) = L_i \Delta u(q_i + L_i x) = -L_i K(q_i + L_i x)|\nabla u(q_i + L_i x)| \tag{5.1.17}$$

since u is spacetime harmonic, where Δ_i denotes the Laplace-Beltrami operator with respect to $g_i := \frac{1}{L_i^2}\phi_i^* g$, where $\phi_i : B_1 \to (\mathcal{N}_{L_i}(q_i) \subset M, g)$. Note that, $g_i \to g(p)$.

For regularity of u on W, one reflects the domain along the corresponding side face, then by [66] and [41], we know that u is uniformly Lipschitz on W.

Then for u_i, since u is uniformly Lipschitz, $u_i \to v$ in some $C^{0,\alpha}$ norm, while v itself is still a Lipschitz function. Moreover, R.H.S of (5.1.17) $\to 0$ as $L_i \to 0$.

Therefore, we have v satisfying $\Delta_{g_{Euc}} v = 0$ and mixed boundary condition on a model wedge with angle θ. Furthermore, from (5.1.15) there exists a point y with distance 1 away from p such that,

$$|\partial\partial v(y)| \geq \varepsilon_0. \tag{5.1.18}$$

There are 2 cases. First, if θ is less than $\pi/2$, then in M, p lies on a segment where the dihedral angle is less than $\pi/2$. Hence, there exists an open neighbourhood U of p which

1. sits along a segment where the dihedral angle is less than $\pi/2$,
2. contains a compact set V containing p_i for all large i, and hence p.

By [4] Proposition (Satz) 3.1 (and (V1*)), we know that $u \in C^{1,\alpha}(V)$, then by weighted Schauder estimates on u in an annulus $A(r)$ around p, we get $r|\nabla\nabla u| \leq C(V)r^{1+(1+\alpha)-2} \leq C(V)r^{\alpha} \to 0$ as $r \to 0$. This contradicts (5.1.15).

While for the second case, if $\theta = \pi/2$, then $\partial\partial v$ should vanish as v should be linear by boundary Harnack inequality. This contradicts (5.1.18). □

Meanwhile, for the vertical edges of F along which u is $C^{1,\alpha}_{loc}$, when weighted Schauder estimates are applied on u in $A(r) \subset \tilde{\Omega}$, a compact neghbourhood along the segment, we have $|\nabla\nabla u|_{C^0(A(r))} \leq C(\tilde{\Omega})r^{\alpha-1}$. From this, the integrability of Gauss curvature and geodesic curvature for each level set also follows. With Gauss-Bonnet Theorem, it will be useful to show Lemma 5.1.2 which connects energy conditions with dihedral angles.

We then consider for p being a vertex, by weighted Schauder estimates in $A(r)$ around p, we have

$$|\nabla\nabla u|_{C^0(A(r))} \leq Cr^{\alpha-2}. \tag{5.1.19}$$

Note that as ∂M_r approaches the vertices, the difference in the area is of order r^2. To sum up, as $r \to 0$,

$$\int_{\partial M_r} \partial_{\nu_r}|\nabla u| \to \int_{\partial M} \partial_\nu |\nabla u|. \tag{5.1.20}$$

3. <u>R.H.S. of (5.1.2).</u>

$$\begin{aligned} &\Delta|\nabla u| \\ =&\frac{1}{|\nabla u|}(|\nabla\nabla u|^2 + \langle \nabla u, \nabla \Delta u\rangle - |\nabla|\nabla u||^2 + |\nabla u|^2 Ric(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|})). \end{aligned} \tag{5.1.21}$$

As $\Delta u = -K|\nabla u|$, note that by Lemma 3.1 in [30], we have

$$\Delta|\nabla u| \geq - C(||g||_{C^2} + ||k||_{C^1})|\nabla u|. \tag{5.1.22}$$

In particular,

$$(\Delta|\nabla u|)_- \leq C(||g||_{C^2} + ||k||_{C^1})|\nabla u|, \tag{5.1.23}$$

i.e. $(\Delta|\nabla u|)_-$ is integrable on M.

4. Conclusion. By (3.3.3) and integrability of $(\Delta|\nabla u|)_-$, on M_r,

$$\int_{M_r} (\Delta|\nabla u|)_+ = \int_{\partial M_r} \partial_{\nu_r}|\nabla u| + \int_{M_r} (\Delta|\nabla u|)_-. \tag{5.1.24}$$

We can thus by (3.3.4) and monotone convergence theorem conclude that as $r \to 0$,

$$\int_{\partial M} \partial_\nu |\nabla u| = \int_M \Delta|\nabla u|. \tag{5.1.25}$$

□

Then by Lemma 5.0.1, Lemma 5.1.1 and Proposition 5.1.1, we can conclude the following which links energy conditions to dihedral angles.

Lemma 5.1.2. *Let (M^3, g, k) be an initial data set of type P where the dihedral angles are everywhere less than π. Further assume that the dihedral angles between T and F and those of B and F are everywhere less than or equal to $\pi/2$. Let u be a spacetime harmonic function* *in Lemma 5.0.1, we have*

$$\begin{aligned}
&\int_M \frac{1}{2}\frac{|\overline{\nabla}\nabla u|^2}{|\nabla u|} + \mu|\nabla u| + \langle J, \nabla u\rangle \, dV \\
&+ \int_{\partial M} H|\nabla u| - \pi(\nabla u, \nu)\, d\sigma \\
\leq &\int_0^1 \int_{\Sigma_t} \frac{1}{2} R_{\Sigma_t} dA\, dt + \int_0^1 \int_{\partial\Sigma_t} \kappa \, d\tau\, dt \\
= &\int_0^1 \left(2\pi\chi(\Sigma_t) - \sum_{j=1}^{q}(\pi - \alpha_j)\right) dt.
\end{aligned} \tag{5.1.26}$$

where $\Sigma_t = \{u^{-1}(t)\}$, *q denotes the number of sides of* P_0, α_j *is the dihedral angle between the edges of the level sets and H is the mean curvature computed with respect to* ν, *the outward unit normal of M.*

Proof. First, we know that each regular level set of u must reach the side faces by maximum principle. Moreover, since $u \in W^{3,p}_{loc}(\mathring{M})$, $C^{2,\alpha}_{loc}$ on each face and $C^{1,\alpha}_{loc}$ around the vertical edges, Sard's theorem is applicable ([16]). Together with the topology of a prism, each component of a level set of regular values is homeomorphic to P_0. Furthermore, by homogeneous Neumann condition, the dihedral angle of the boundary of the level sets is the same as the dihedral angle between corresponding side faces.

We are going to show that the boundary terms of (5.1.1) actually reveal the boundary energy condition. For T and B, on which $|\nabla u|$ is nowhere vanishing by maximum principle and u is a constant, by Proposition 5.1.1, we have

$$\begin{aligned}
&\int_{T\cup B} \partial_\nu |\nabla u| + k(\nabla u, \nu)\, d\sigma \\
= &\int_{T\cup B} -H|\nabla u| - Kg(\nabla u, \nu) + k(\nabla u, \nu)\, d\sigma \\
= &\int_{T\cup B} -H|\nabla u| + \pi(\nabla u, \nu)\, d\sigma.
\end{aligned} \tag{5.1.27}$$

Then, on F, $\partial_\nu u = 0$. From Proposition 5.1.1 and coarea formula, we have

$$
\begin{aligned}
&\int_F \partial_\nu |\nabla u| + k(\nabla u, \nu)\, d\sigma \\
=& \int_0^1 \int_{\partial \Sigma_t} -\Pi\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) + k\left(\frac{\nabla u}{|\nabla u|}, \nu\right) d\tau\, dt \\
=& \int_F -H|\nabla u| + k(\nabla^F u, \nu)\, d\sigma + \int_0^1 \int_{\partial \Sigma_t} \kappa\, d\tau\, dt \\
=& \int_F -H|\nabla u| + \pi(\nabla u, \nu)\, d\sigma + \int_0^1 \int_{\partial \Sigma_t} \kappa\, d\tau\, dt.
\end{aligned}
\tag{5.1.28}
$$

The proof is then concluded by Lemma 5.1.1 and Gauss-Bonnet theorem. □

5.2 Comparison Theorem and Dihedral Rigidity

In this section, we are going to use Lemma 5.1.2 to show the relation between energy conditions and the geometry and dihedral rigidity of a polyhedral initial data set.

5.2.1 General spacetime case

First, we consider the case of a type P initial data set in general.

Lemma 5.2.1. *Let (M^3, g, k) be an initial data set of type P, where P_0 is a convex q-gon, which simultaneously satisfies:*

1. *the dominant energy condition,*
2. *$H \geq -tr_T k$ on T, $H \geq tr_B k$ on B,*
3. *$H \geq |\pi^T(\cdot, \nu)|$ on F, where the superscript ^{T} means the projection onto the tangent bundle of the corresponding domain and H denotes the mean curvature computed with respect to ν, the outward unit normal of M,*

where T, B, and F denote the top, the bottom and the side faces of M respectively.

Assume that the dihedral angles are everywhere less than π; moreover, the dihedral angles between T and F and those between B and F are everywhere less than or equal to $\pi/2$. Let u be a spacetime harmonic function in Lemma 5.0.1 and $\Sigma_t = \{u^{-1}(t)\}$, where $t \in [0,1]$. Then, the following holds.

1. *Let $\{E_j\}_{j=1}^{q}$ denote the vertical edges of F and $\theta_j := \sup_{E_j} \alpha_j$, where α_j denotes the dihedral angle between F on E_j. Then,*

$$\sum_{j=1}^{q} \theta_j \geq (q-2)\pi.$$

   In particular, the dihedral angles of M cannot be everywhere less than those of P.

2. *If the dihedral angles of M are further assumed to be everywhere less than or equal to those of P, then*
   - *(a) $\mu - |J| = 0$ on M.*
   - *(b) The dihedral angles between T and F and those between B and F are everywhere $\pi/2$.*
   - *(c) M is smoothly foliated by $\Sigma_t, t \in [0,1]$. On each Σ_t, the following properties are satisfied.*
     - *i. $h_{\Sigma_t} + k_{\Sigma_t} = 0$, where h_{Σ_t} denotes the second fundamental from of Σ_t with respect to $\frac{\nabla u}{|\nabla u|}$. In particular, Σ_t is a free boundary (stable) totally spacetime geodesic MOTS.*
     - *ii. $\mu + \langle J, \frac{\nabla u}{|\nabla u|} \rangle = 0$.*
     - *iii. The dihedral angles of Σ_t are all equal to those of P_0, $R_{\Sigma_t} = 0$ and $\kappa_{\partial \Sigma_t} = 0$. Therefore, each level set is isometric to P_0 up to scaling.*

(d) On ∂M,

i. $H = -tr_T k$ on T, $H = tr_B k$ on B,

ii. $H = |\pi^T(\cdot, \nu)|$ on F.

Proof. First, by maximum principle, $\partial_\nu u < 0$ on B while $\partial_\nu u > 0$ on T. By Lemma 5.1.2, we have

$$
\begin{aligned}
&\int_M \frac{1}{2}\frac{|\overline{\nabla\nabla} u|^2}{|\nabla u|} + (\mu - |J|)|\nabla u|\, dV \\
&+ \int_T (H + tr_T k)|\nabla u|\, d\sigma + \int_B (H - tr_B k)|\nabla u|\, d\sigma \\
&+ \int_F (H - |\pi^T(\cdot,\nu)|)|\nabla u| \; d\sigma \\
\leq &\int_0^1 \int_{\Sigma_t} \frac{1}{2}\frac{|\overline{\nabla\nabla} u|^2}{|\nabla u|^2} + \mu + \langle J, \frac{\nabla u}{|\nabla u|}\rangle \, dA\, dt \\
&+ \int_T (H + tr_T k)|\nabla u|\, d\sigma + \int_B (H - tr_B k)|\nabla u|\, d\sigma \\
&+ \int_F (H|\nabla u| - \pi(\nabla u, \nu)\, d\sigma \\
\leq &\int_0^1 \int_{\Sigma_t} \frac{1}{2} R_{\Sigma_t} dA\, dt + \int_0^1 \int_{\partial \Sigma_t} \kappa \, d\tau\, dt \\
= &\int_0^1 \left(2\pi\chi(\Sigma_t) - \sum_{j=1}^q (\pi - \alpha_j) \right) dt.
\end{aligned}
\tag{5.2.1}
$$

By the dominant energy condition and the assumptions on ∂M, we know that

$$
\sum_{j=1}^q \theta_j \geq (q-2)\pi. \tag{5.2.2}
$$

Hence, the dihedral angles of M cannot be everywhere less than those of P.

If it is further assumed that the dihedral angles of M are everywhere less than or equal to those of P, then the dihedral angles of Σ_t are all equal to those of P_0 by (5.2.2). Then on

∂M, we have

$$H = -tr_T k \text{ on } T, H = tr_B k \text{ on } B \text{ and } H = |\pi^T(\cdot,\nu)| \text{ on } F. \tag{5.2.3}$$

Moreover, on M,

$$\mu - |J| = 0, \tag{5.2.4}$$

and

$$\bar{\nabla}\bar{\nabla} u = \nabla\nabla u + |\nabla u| k = 0. \tag{5.2.5}$$

Note that u is not a constant function, hence ∇u is non-vanishing somewhere. Furthermore, by Kato's inequailty, on M,

$$|\nabla|\nabla u|| \leq |\nabla\nabla u| \leq |k||\nabla u|. \tag{5.2.6}$$

Then by ODE technique, we know that ∇u is nowhere vanishing and hence each $t \in [0,1]$ is a regular value. By compactness of M, there exists $c_1, c_2 \in \mathbb{R}$ such that

$$c_2 \geq |\nabla u| \geq c_1 > 0. \tag{5.2.7}$$

Hence, M is a smooth foliation of regular level sets. Moreover, assume on the contrary that there exists p on the horizontal edges where T or B meet F such that the dihedral angle is less than $\pi/2$, then by [4] Proposition (Satz) 3.1 (and its remark on P.343), u is $C^{1,\alpha}$ around p. Since the dihedral angle is less than $\pi/2$, $\nabla u(p) = 0$. A contradiction arises.

Then we recall that on each level set Σ_t, for $X, Y \in T\Sigma_t$, by (5.2.5) we have

$$h(X,Y) = \frac{\nabla\nabla u}{|\nabla u|}(X,Y) = -k(X,Y), \tag{5.2.8}$$

where h denotes the second fundamental form for Σ_t with respect to $\frac{\nabla u}{|\nabla u|}$. From (5.2.8), M is a smooth foliation of totally spacetime geodesic MOTS.

5.2.2 Stability of free boundary MOTS

To further study the geometry of each level set and M, we have to verify and use the fact that Σ_t is a stable free boundary MOTS.

Proposition 5.2.1. *([20] Proposition 2) For a 2-sided MOTS Σ, let $\phi \in C^\infty(\Sigma)$ and N be a continuous unit normal vector field on Σ, we have*

$$\begin{aligned} &\delta_{\phi N}(H + tr_\Sigma k) \\ =& -\Delta_\Sigma \phi + 2\langle W_\Sigma, \nabla^\Sigma \phi\rangle \\ &+ (div_\Sigma W_\Sigma - |W_\Sigma|^2 + \frac{1}{2}R_\Sigma - \mu - J(N) - \frac{1}{2}|k_\Sigma + h_\Sigma|^2)\phi, \end{aligned} \tag{5.2.9}$$

where h_Σ and H respectively denote the second fundamental form and mean curvature of Σ with respect to N and $W_\Sigma \in T\Sigma$ is dual to $k(N,\cdot)|_{T\Sigma}$.

A definition of stable capillary MOTS is proposed in [1], and here we state the free boundary case only.

Definition 5.2.1. *([1] Definition 5.1, cf. [2] Definition 2) A free boundary MOTS $\Sigma \subset M$ is* stable *with respect to the variation vector field $X = \varphi N$, where N is a continuous unit normal vector field on Σ, if and only if there exists a non-negative function $\varphi \in C^2(\Sigma)$, $\varphi \not\equiv 0$ satisfying Robin boundary condition $\frac{\partial \varphi}{\partial \nu} = \Pi(N,N)\varphi$ such that $\delta_X(H + tr_\Sigma k) \geq 0$, where Π*

is the second fundamental form of ∂M with respect to the outward normal ν. Moreover, it is called strictly stably outermost *with respect to the direction X if, moreover, $\delta_X(H+tr_\Sigma k) \neq 0$ somewhere on Σ.*

On Σ_t, let $N := \frac{\nabla u}{|\nabla u|}$. By (5.2.7), $\frac{1}{|\nabla u|}$ is well defined on M. Then, we can consider the flow generated by $\frac{\nabla u}{|\nabla u|^2}$. For M being foliated by level sets of u and each Σ_t is a free boundary MOTS, we have

$$\delta_{\frac{1}{|\nabla u|}N}(H_{\Sigma_t} + tr_{\Sigma_t} k) = 0. \tag{5.2.10}$$

While on $\partial\Sigma_t$,

$$\begin{aligned} \partial_\nu\left(\frac{1}{|\nabla u|}\right) &= -\frac{\nabla_\nu\langle\nabla u, \nabla u\rangle}{2|\nabla u|^3} \\ &= \frac{-\langle\nabla_{\nabla u}\nabla u, \nu\rangle}{|\nabla u|^3} \\ &= \frac{\Pi(N,N)}{|\nabla u|}. \end{aligned} \tag{5.2.11}$$

Therefore, we can conclude that Σ_t is a stable free boundary MOTS.

Following [1] Lemma 5.4 (equation (5.16) in [1], equation (2.9) in [22]), the stability of Σ_t yields to a positive-semidefinite bilinear form G by integrating $\frac{w^2}{|\nabla u|}\delta_{|\nabla u|N}(H + tr_{\Sigma_t} k)$ over Σ_t,

$$G(w,w) := \int_{\Sigma_t} \left(|\nabla^{\Sigma_t} w|^2 + Qw^2\right) dA - \int_{\partial\Sigma_t} \left(\Pi(N,N) - \langle W_{\Sigma_t}, \nu\rangle\right) w^2 d\tau \geq 0 \tag{5.2.12}$$

for all $w \in C^\infty(\Sigma_t)$, where $Q := \frac{1}{2}R_\Sigma - \mu - J(N) - \frac{1}{2}|k_\Sigma + h_\Sigma|^2$. Moreover, if $G(1,1) = 0$, then

$$Q = 0 \ , \ W_{\Sigma_t} = \nabla^{\Sigma_t} \log \frac{1}{|\nabla u|} \ \text{ and } \ \Pi(N, N) = \langle W_{\Sigma_t}, \nu \rangle. \tag{5.2.13}$$

Now, by (5.2.1), (5.2.5) and (5.2.8), we get that

$$\begin{aligned} 0 =& \int_{\Sigma_t} \frac{|\overline{\nabla\nabla} u|^2}{|\nabla u|^2} - \frac{1}{2} R_{\Sigma_t} + \mu + \langle J, N \rangle \, dA + \int_{\partial\Sigma_t} \Pi(N, N) - \pi(\nu, N) \, d\tau \\ =& \int_{\Sigma_t} - \frac{1}{2} R_{\Sigma_t} + \mu + \langle J, N \rangle + \frac{1}{2} |k_{\Sigma_t} + h_{\Sigma_t}|^2 \, dA + \int_{\partial\Sigma_t} \Pi(N, N) - \langle W_{\Sigma_t}, \nu \rangle \, d\tau \\ =& - G(1, 1). \end{aligned} \tag{5.2.14}$$

Then, by (5.2.13), we have

$$R_{\Sigma_t} = 0. \tag{5.2.15}$$

Let τ_t' denote the unit tangent vector of $\partial\Sigma_t$. From (5.2.3), we can see that on F

$$H = |\pi^T(\nu, \cdot)| = \sqrt{|\pi(\nu, N)|^2 + |\pi(\nu, \tau_t')|^2}, \tag{5.2.16}$$

hence,

$$H \geq |\pi(\nu, N)| \geq 0. \tag{5.2.17}$$

Furthermore, by (5.2.5), on F,

$$\begin{aligned}\Pi(N,N) =&\langle \nabla_{\frac{\nabla u}{|\nabla u|}}\nu, \frac{\nabla u}{|\nabla u|}\rangle \\ =&\frac{-\nabla\nabla u(\nabla u,\nu)}{|\nabla u|^2} \\ =&k(N,\nu) \\ =&\pi(N,\nu).\end{aligned} \tag{5.2.18}$$

As a result,

$$\kappa_{\partial\Sigma_t} = \Pi(\tau_t', \tau_t') = H - \Pi(N,N) \geq 0. \tag{5.2.19}$$

By (5.2.15) and Gauss-Bonnet Theorem, we can conclude that

$$\kappa_{\partial\Sigma_t} = 0. \tag{5.2.20}$$

Therefore, Σ_t is isometric to P_0 up to scaling. Then g can be expressed as $\frac{1}{|\nabla u|^2}dt^2 + f(t)g_{Euc}$ for some function f which depends on t only. This split form will be useful in Corollary 5.2.2 for showing dihedral rigidity of parabolic prisms. $\square$

Remark 5.2.1. Following the observation by P.-F. Yip [71], (cf. [32]), for dimension 3, a foliation by totally spacetime geodesic MOTS implies isometric embedding into Minkowski space by Gauss Codazzi equation and the fundamental theorem of hypersurface. In particular, we have the following corollary.

Corollary 5.2.1. *Let (M^3, g, k) be an initial data set of type P, where P_0 is a convex q-gon, which simultaneously satisfies:*

1. *the dominant energy condition,*
2. *$H \geq -tr_T k$ on T, $H \geq tr_B k$ on B,*

3. *$H \geq |\pi^T(\cdot, \nu)|$ on F, where the superscript ^{T} means the projection onto the tangent bundle of the corresponding domain and H denotes the mean curvature computed with respect to ν, the outward unit normal of M,*

where T, B, and F denote the top, the bottom and the side faces of M respectively.

Assume that the dihedral angles are everywhere less than π; moreover, the dihedral angles between T and F and those between B and F are everywhere less than or equal to $\pi/2$. If the dihedral angles of M are further assumed to be everywhere less than or equal to those of P, then (M, g, k) can be isometrically embedded into Minkowski space.

For P being a rectangular prism (cube), we can further deduce the geometry of the boundary from its symmetry.

Theorem 5.2.1. *Let (M^3, g, k) be a type P initial data set, where P_0 is a rectangle, which simultaneously satisfies:*

1. *the dominant energy condition,*
2. *the boundary dominant energy condition,*
3. *everywhere the dihedral angle between two faces of M is less than or equal to $\pi/2$.*

Let $\Sigma_t = \{u^{-1}(t)\}$, where $t \in [0, 1]$ and u be a spacetime harmonic function solving the mixed boundary problem in Lemma 5.0.1. Then,

1. *M is smoothly foliated by $\Sigma_t, t \in [0, 1]$. On each Σ_t, the following properties are satisfied.*
   (a) *$h_{\Sigma_t} + k_{\Sigma_t} = 0$, where h_{Σ_t} denotes the second fundamental from of Σ_t with respect to $\frac{\nabla u}{|\nabla u|}$. In particular, Σ_t is a free boundary (stable) totally spacetime geodesic MOTS.*

(b) $\mu + \langle J, \frac{\nabla u}{|\nabla u|} \rangle = 0$.

(c) *The 4 dihedral angles of the edges are all equal to* $\pi/2$, $\kappa_{\partial \Sigma_t} = 0$ *and* $R_{\Sigma_t} = 0$. *Hence, each level set is isometric to a Euclidean rectangle.*

2. $\mu = |J| = 0$ *on* M.

3. *On* ∂M,

(a) $R_{\partial M} = 0$, $\Pi = k|_{T(\partial M)} = 0$. *where* Π *is the second fundamental form of* ∂M *with respect to the outward normal* ν. *Consequently,* $H_{\partial M} = tr_{\partial M} k = |\pi^T(\nu, \cdot)| = 0$, *where the superscript* ^{T} *means the projection onto the tangent bundle of the corresponding domain. In particular,* ∂M *is isometric to the boundary of a Euclidean rectangular prism.*

(b) $(\nabla u)|_{\partial M}$ *is a parallel vector field, i.e.* $\nabla_X \nabla u \equiv 0$ *for* $X \in T(\partial M)$.

4. (M, g, k) *can be isometrically embedded into Minkowski space.*

Proof. Based on Lemma 5.2.1, we are going to study the geometry of M further by its symmetry. We get that the dihedral angles are everywhere $\pi/2$ as in the proof of Lemma 5.2.1. Then, by Proposition 7.2.1, $u \in C^{2,\alpha}(M) \cap W^{3,p}_{loc}(\mathring{M})$. Furthermore, $u \in C^{3,\alpha}_{loc}(\mathring{M})$ by (5.2.7).

From (5.2.8), on T and B, we respectively have $\Pi = h = -k|_{T(\partial M)}$ and $\Pi = -h = k|_{T(\partial M)}$. Therefore, when we reverse the identification of T and B and solve for another spacetime harmonic function, we get that on both T and B,

$$\Pi = k|_{T(\partial M)} = 0. \tag{5.2.21}$$

And since we can choose T, B and F freely, we actually get that (5.2.21) holds on all 6 faces. In particular, the geodesic curvature of all the edges of M vanishes. Moreover, $R_{\partial M} = 0$

since each face of ∂M is a stable free boundary MOTS. Hence, we can further conclude that ∂M is isometric to the boundary of a Euclidean rectangular prism $\tilde{P} = [0, a_1] \times [0, a_2] \times [0, a_3]$ for some $a_1, a_2, a_3 > 0$.

Moreover, by the boundary dominant energy condition, we can conclude that on ∂M,

$$H_{\partial M} = tr_{\partial M} k = |\pi^T(\cdot, \nu)| = |k^T(\cdot, \nu)| = 0. \tag{5.2.22}$$

Let $X, Y \in T(\partial M)$, by (5.2.5), (5.2.21) and (5.2.22), we get that on ∂M,

$$\begin{aligned} \nabla_X \nabla_Y u =& - |\nabla u| k(X, Y) \\ =&0, \end{aligned} \tag{5.2.23}$$

$$\begin{aligned} \nabla_X \nabla_\nu u =& - |\nabla u| k(X, \nu) \\ =&0. \end{aligned} \tag{5.2.24}$$

Hence, we can conclude that

$$\nabla u \text{ is a parallel vector field on } \partial M. \tag{5.2.25}$$

Moreover, if we solve for u^i, $1 \leq i \leq 3$, with the corresponding choice of $B_i \subset \{x^i = 0\}$ and $T_i \subset \{x^i = a_i\}$ with $u^i = a_i$ on T_i. It is straight forward to check that (u^1, u^2, u^3) is a coordinate system on a neighbourhood of ∂M with corresponding vector fields $\tilde{\partial}_i := \frac{\nabla u^i}{|\nabla u^i|^2}$.

While on each Σ_t,

$$\mu + \langle J, \frac{\nabla u}{|\nabla u|}\rangle = 0. \tag{5.2.26}$$

Since we can choose another orientation of cubes, then we have another spacetime harmonic function w such that on M,

$$\langle J, \frac{\nabla u}{|\nabla u|}\rangle = \langle J, \frac{\nabla w}{|\nabla w|}\rangle = -|J|, \tag{5.2.27}$$

Note that $\frac{\nabla u}{|\nabla u|}$ and $\frac{\nabla w}{|\nabla w|}$ must be different somewhere and hence nowhere equal on M by the following lemma and ODE technique.

Lemma 5.2.2. *([68] Lemma 8.1) Let $X = \nabla u/|\nabla u|$ and let $Y = \nabla\widetilde{u}/|\nabla\widetilde{u}|$ where u and $\widetilde{u}$ are spacetime harmonic functions, then*

$$|\nabla(|X - Y|^2)| \leq 2|k||X - Y|^2.$$

Proof.

$$\begin{aligned}\nabla X =&\nabla(\frac{\nabla u}{|\nabla u|})\\ =&\frac{\nabla\nabla u}{|\nabla u|} - \frac{1}{|\nabla u|^2}\frac{\nabla\nabla u(\nabla u, \cdot)}{|\nabla u|}\nabla u\\ =&-k + k(X, \cdot)X,\end{aligned} \tag{5.2.28}$$

i.e. in local coordinates, $\nabla_i X^j = -k_i^j + k_{mi}X^m X^j$. Similarly,

$$\nabla Y = -k + k(Y, \cdot)Y. \tag{5.2.29}$$

Hence,

$$\begin{aligned}\nabla(|X-Y|^2) =& -2\langle \nabla X, Y\rangle - 2\langle X, \nabla Y\rangle \\ =&2(k(Y,\cdot) - k(X,\cdot)\langle X, Y\rangle + k(X,\cdot) - k(Y,\cdot)\langle X, Y\rangle) \\ =&2(1-\langle X, Y\rangle)k(X+Y,\cdot) \\ =&|X-Y|^2 k(X+Y,\cdot).\end{aligned} \tag{5.2.30}$$

And

$$\begin{aligned}|\nabla(|X-Y|^2)| \leq& |X-Y|^2|k|(|X|+|Y|) \\ \leq& 2|X-Y|^2||k|.\end{aligned} \tag{5.2.31}$$

□

As a result, on M,

$$\mu = |J| = 0. \tag{5.2.32}$$

Consider the initial data set $(\mathbb{R}^3 \setminus \tilde{P}, g_{Euc}, 0)$, by (5.2.21), we can identify ∂M and $\partial\tilde{P}$ to form an initial data set with corners ∂M,

$$(M_1, g_1, k_1) = (M \cup (\mathbb{R}^3 \setminus \tilde{P}), g \cup g_{Euc}, k \cup 0). \tag{5.2.33}$$

Note that ∂M is isometric to $\partial\tilde{P}$ and the dihedral angle is everywhere $\pi/2$. Then, one can take Fermi coordinates or $\{\tilde{\partial}_i\}_{i=1}^3$ as aforementioned on ∂M so that under this chart, g_1 is Lipschitz and k_1 is L^∞ on M_1 while smooth up to ∂M and $\partial(\mathbb{R}^3 \setminus \tilde{P}) = \partial\tilde{P}$ respectively. We see that M_1 is $\mathbb{R}^3$ topologically and satisfies $E = |P| = 0$. By (5.2.4) and (5.2.22), we can apply Corollary 1.1 in [68] or Section VI of [63]. Therefore, (M_1, g_1, k_1), in particular (M, g, k), can be isometrically embedded into Minkowski space (as a graph of a linear combination of spacetime harmonic functions). $\square$

5.2.3 $k = g$ hyperbolic space

For the special case $k = g$, we can conclude the dihedral rigidity for general prisms.

Definition 5.2.2. *Let $(\mathbb{H}^3, g_{\mathbb{H}})$ be the hyperbolic space with sectional curvature -1. Fix the coordinate system (x^1, x^2, x^3) such that $g_{\mathbb{H}}$ takes the form*

$$g_{\mathbb{H}} = (dx^1)^2 + e^{2x^1}\big((dx^2)^2 + (dx^3)^2\big). \tag{5.2.34}$$

Corollary 5.2.2. *(cf. [39] Theorem 2.4) Let (M^3, g, g) be an initial data set of type P which simultaneously satisfies:*

1. *the dominant energy condition,*
2. *$H \geq \pi^{\perp}(\nu, \cdot)$ on T,*
3. *$H \geq -\pi^{\perp}(\nu, \cdot)$ on B,*
4. *$H \geq |\pi^{T}(\nu, \cdot)|$ on F,*
5. *everywhere the dihedral angles between two faces of M is less than or equal to those of P,*

where T and B are identified with the face lying on $\{x^1 = 0\}$ and $\{x^1 = 1\}$ respectively[1], H is computed with respect to ν, the unit outward normal of M. Then (M, g, g) is isometric to a parabolic prism in $(\mathbb{H}^3, g_{\mathbb{H}})$.

[1]Note that our identification of "top" and "bottom" faces is the reverse of [39].

Proof. Let u be the spacetime harmonic function in Lemma 5.0.1. By (5.2.1) and $k = g$, we have $H = |\pi^T(\nu, \cdot)| = 0$ on F. Moreover, by (5.2.5), on F,

$$\begin{aligned}
\Pi(N,N) =& \langle \nabla_{\frac{\nabla u}{|\nabla u|}} \nu, \frac{\nabla u}{|\nabla u|} \rangle \\
=& \frac{-\nabla\nabla u(\nabla u, \nu)}{|\nabla u|^2} \\
=& g(N, \nu) \\
=& 0,
\end{aligned} \tag{5.2.35}$$

where Π denotes the second fundamental form of ∂M with respect to the outward normal ν. On the other hand, on each Σ_t, let $X \in T\Sigma_t$

$$\begin{aligned}
& \nabla_X |\nabla u| \\
=& \frac{\nabla\nabla u(\nabla u, X)}{|\nabla u|} \\
=& -|\nabla u| g(N, X) \\
=& 0.
\end{aligned} \tag{5.2.36}$$

From the proof of Lemma 5.2.1, we can first conclude the following. Let $\Sigma_t = \{u^{-1}(t)\}$, then,

1. M is smoothly foliated by $\Sigma_t, t \in [0,1]$. On each Σ_t, the following properties are satisfied.

   (a) $h_{\Sigma_t} + g_{\Sigma_t} = 0$, where h_{Σ_t} denotes the second fundamental from of Σ_t with respect to $N = \frac{\nabla u}{|\nabla u|}$. In particular, Σ_t is a free boundary stable totally spacetime geodesic MOTS (horosphere).

   (b) The dihedral angles of the edges are all equal to those of P_0, $\kappa_{\partial\Sigma_t} = 0$ and $R_{\Sigma_t} = 0$, in particular, each level set is isometric to P_0 up to scaling.

(c) $|\nabla u||_{\Sigma_t}$ is constant.

2. $R^M = -6$ on M.

3. The dihedral angles between T and F and those between B and F are everywhere $\pi/2$.

4. On ∂M,

   (a) $H = -2$ on T and $H = 2$ on B.

   (b) $H = |\pi^T(\nu, \cdot)| = 0$ on F.

   (c) $\Pi = 0$ on F.

Similar to the proof of Theorem 5.2.1, we are going to consider the flow generated by $\frac{\nabla u}{|\nabla u|^2}$. Since $|\nabla u|$ is constant on each level set Σ_t, we can make a change of coordinate to express g in the following form on M,

$$g = ds^2 + f(s)\delta_{ij}dx^i dx^j, \tag{5.2.37}$$

where f is a function depending on s only. Then, we consider on each level set with respect to the ∂_s direction,

$$-2 = H(s) = \frac{1}{f(s)}\delta^{ij}\frac{1}{2}\partial_s(f(s)\delta_{ij}) = \frac{\partial_s f(s)}{f(s)}. \tag{5.2.38}$$

We have

$$g = ds^2 + e^{-2s+C}\delta_{ij}dx^i dx^j. \tag{5.2.39}$$

Note that since ∂_s is pointing in the decreasing x^1 direction, after a change of direction, we can see that (M, g, g) is isometric to a parabolic prism in $(\mathbb{H}^3, g_{\mathbb{H}})$. $\square$

5.3 Application to the spacetime positive mass theorem

(cf. [38] Section 5) In this section, we observe that if a general version of Lemma 5.2.1 holds, then we can prove the spacetime positive mass theorem (Theorem 1.1.1) with Lohkamp's construction of $(\mu - |J|_g) > 0$-island (Section 2 in [45]) which is based on PDE analysis for the density theorem in [20]. First, here is proposed a general version of Lemma 5.2.1.

Conjecture 5.3.1. *Let $n \geq 3$, P^n be a Euclidean prism ($P_0 \times [0,1]^{n-2}$) and (Ω^n, g, k) be an initial data set admitting a degree one map onto P. Further assume that (Ω, g, k) simultaneously satisfies:*

1. *the dominant energy condition,*
2. *the boundary dominant energy condition,*
3. *the dihedral angles of Ω are everywhere less than or equal to those of P.*

Then, on Ω,

$$\mu - |J|_g = 0.$$

Definition 5.3.1. *([45] Definition 2.8) An asymptotically flat initial data set (M^n, g, k) is called a $(\mu - |J|_g) > 0$-island if there exists a non-empty open set $U \subset M$ with compact closure such that*

1. *$\mu - |J|_g > 0$ on U, and*
2. *$(M \setminus U, g, k) \equiv (\mathbb{R}^n \setminus B_r(0), g_{Euc}, 0)$.*

Proposition 5.3.1. *Conjecture 5.3.1 implies the spacetime positive mass theorem.*

Proof. Let (M^n, g, k) be an asymptotically flat initial data set satisfying the dominant energy condition. Assume on the contrary that, w.l.o.g. by Christodoulou and O'Murchadha's boost argument ([15]), $E < 0 \leq |P|$. From [45] Section 2 which based on the PDE system analysis for the density theorem in [20], one can construct a $(\tilde{\mu} - |\tilde{J}|_{\tilde{g}}) > 0$-island $(\tilde{M}, \tilde{g}, \tilde{k})$.

Then, consider a large scaling of P^n such that ∂P can be isometrically embedded into $(\tilde{M} \setminus \tilde{U}, \tilde{g}, \tilde{k})$ and encloses $\tilde{U}$. Let $\tilde{\Omega}$ denote the region in $\tilde{M}$ bounded by ∂P. $(\tilde{\Omega}, \tilde{g}, \tilde{k})$ clearly satisfies the assumptions of Conjecture 5.3.1, where a degree one map is taking $\tilde{\Omega} \setminus \tilde{U}$ to $P \setminus \{0\}$ and $\tilde{U}$ to $\{0\}$. Therefore, particularly, $\tilde{\mu} - |\tilde{J}|_{\tilde{g}} = 0$ on $\tilde{U}$. A contradiction arises. $\square$

In particular, for the 3 dimensional case, Theorem 1.1.1 can be proved by considering a large cube as follows. Assume on the contrary that the spacetime positive mass theorem does not hold, as aforementioned, then there exists a $(\tilde{\mu} - |\tilde{J}|_{\tilde{g}}) > 0$-island $(\tilde{M}, \tilde{g}, \tilde{k})$. Now, the construction of the generalised exterior region ([30] Section 2) can be carried out on $(\tilde{M}, \tilde{g}, \tilde{k})$. In particular, there exists $(\hat{M}, \hat{g}, \hat{k})$ with boundary $\partial\hat{M}$ composed of MOTS and MITS such that $H_2(\hat{M}, \partial\hat{M}, \mathbb{Z}) = 0$. Moreover, there exists a non-empty open set $\hat{U} \subset \hat{M}$ with compact closure such that

1. $\hat{\mu} - |\hat{J}|_{\hat{g}} > 0$ on $\hat{U}$,
2. $(\hat{M} \setminus \hat{U}, \hat{g}, \hat{k}) = (\tilde{M} \setminus \tilde{U}, \tilde{g}, \tilde{k}) \equiv (\mathbb{R}^n \setminus B_r(0), g_{Euc}, 0)$.

Then, let P^3 be a cube. As aforementioned, consider a large scaling of P such that ∂P can be isometrically embedded into $(\hat{M} \setminus \hat{U}, \hat{g}, \hat{k})$ and encloses $\hat{U}$. Let $\hat{\Omega}$ denote the region in $\hat{M}$ bounded by ∂P. Hence, we have $H_2(\hat{\Omega}, \partial\hat{M}, \mathbb{Z}) = 0$. The arguments in Appendix 7.2 can be modified correspondingly to show the following lemma.

Lemma 5.3.1. *Let $\vec{c} \in \mathbb{R}^m$, where m denotes the number of components of $\partial \hat{M}$. Then there exists a non-negative spacetime harmonic function $u_{\vec{c}} \in C^{2,\alpha}(\hat{\Omega}) \cap W^{3,p}_{loc}(int\,\hat{\Omega})$ such that*

1. $\hat{\Delta} u_{\vec{c}} + \hat{K}|\hat{\nabla} u_{\vec{c}}| = 0$ *in* $int\,\hat{\Omega}$,
2. $u_{\vec{c}} = c_i$ *on* $\partial_i \hat{M}$ *for* $i = 1, ...m$, *where* $\partial_i \hat{M}$ *denotes the i-th component of* $\partial \hat{M}$,
3. $u_{\vec{c}} = 0$ *on* B *and* $u_{\vec{c}} = 1$ *on* T,
4. $\partial_\nu u_{\vec{c}} = 0$ *on* F,

where $\hat{K} := tr_{\hat{g}} \hat{k}$, T, B, F *and* $\mathcal{E}$ *denote the top, the bottom, the side faces and the edges of* $\partial \hat{\Omega} \cap (\hat{M} \setminus \hat{U})$ *respectively.*

Furthermore, note that Lemma 3.2.1 can be applied to $\hat{\Omega}$ as we reduce the mixed boundary value problem into a Dirichlet problem in Appendix 7.2.

Then, as in the proof of Theorem 5.2.1, we get,

$$\begin{aligned}
0 \leq & \int_{\hat{\Omega}} \frac{1}{2} \frac{|\hat{\nabla}\hat{\nabla} u_{\vec{c}}|^2}{|\hat{\nabla} u_{\vec{c}}|} + (\hat{\mu} - |\hat{J}|_{\hat{g}})|\hat{\nabla} u_{\vec{c}}|\, dV + \int_{T\cup B\cup F} (\hat{H} - |\hat{\pi}(\cdot,\nu)|)|\hat{\nabla} u_{\vec{c}}|\, d\sigma \\
\leq & \int_0^1 \int_{\Sigma_t} \frac{1}{2} R_{\Sigma_t} dA\, dt + \int_0^1 \int_{\partial\Sigma_t} \kappa\, d\tau\, dt + \sum_{i=1}^m \int_{\partial_i \hat{M}} \hat{H}|\partial_\nu u_{\vec{c}}| - tr_{\partial_i \hat{M}} \hat{k}(\partial_\nu u_{\vec{c}})\, d\sigma \\
\leq & \sum_{i=1}^m \int_{\partial_i \hat{M}} \hat{H}|\partial_\nu u_{\vec{c}}| - tr_{\partial_i \hat{M}} \hat{k}(\partial_\nu u_{\vec{c}})\, d\sigma \\
= & 0.
\end{aligned} \tag{5.3.1}$$

The last equality follows from Lemma 3.2.1 that we can choose $\vec{c}$ such that $u_{\vec{c}}$ on MOTS and MITS components of $\partial \hat{M}$, we would have $\partial_\nu u_{\vec{c}} \leq 0$ and $\partial_\nu u_{\vec{c}} \geq 0$ respectively. Therefore, $\hat{\mu} - |\hat{J}|_{\hat{g}} = 0$ in $\hat{\Omega}$, a contradiction arises.

Chapter 6

Non-existence of DEC fill-ins

First, in [62], the construction of a scalar flat and asymptotically flat metric and decreasing total mean curvature difference along the radial direction proved in [59] are the main inputs to show that under certain assumptions, if an NNSC fill-in, i.e. (Ω, g) with $R_g \geq 0$, of Bartnik data (Σ, γ, H) exists, then there is a contradiction to the Riemannian positive mass theorem with corners ([59], [47]). Heuristically from the perspective of energy, by (1.1.3), we can see that if the boundary energy is too large, then the gravitation contribution must be negative. Motivated by this, with Theorem 1.1.2 and [63], we can obtain similar results regarding the energy condition of the fill-in of a spacetime Bartnik data set. In other words, a partial answer to the spacetime version of Conjecture 1.2.2 can be obtained.

Definition 6.0.1. *(cf. [7] Definition 2, [62]) For $n \geq 3$, a tuple $(\Sigma^{n-1}, \gamma, \alpha, H, \beta)$ is called a spacetime Bartnik data set, where (Σ, γ, α) is an oriented closed null-cobordant initial data set with $\alpha \in C^{1,\alpha}$, while H and β are respectively a smooth function and a $C^{1,\alpha}$ 1-form on Σ. A compact initial data set (Ω^n, g, k) is called a fill-in of D_{SB} if there is an isometry $\phi : (\Sigma^{n-1}, \gamma) \to (\partial\Omega, g|_{\partial\Omega})$ such that*

1. *$\phi^* H_g = H$, where H_g is the mean curvature of $\partial\Omega$ to g with respect to the outward unit*

normal ν*,*

2. $\phi^* tr_{\partial\Omega} k = tr_\Sigma \alpha$*, and*

3. $\phi^*(k(\nu, \cdot)) = \beta$*.*

We can see from the above definition that on Σ,

$$\phi^*(|\omega|_g) = \phi^*(|\pi(\nu, \cdot)|_g) = \sqrt{(\mathrm{tr}_\Sigma\, \alpha)^2 + |\beta|^2_\gamma}.$$

Definition 6.0.2. *A fill-in* (Ω^n, g, k) *is said to satisfy a topological assumption* (T) *if either one of the following holds:*

1. *for* $n = 3$*, there exists* $\mathcal{S}$*, a finite (possibly empty) disjoint union of connected weakly trapped surfaces, such that* $H_2(\Omega_{ext}, \mathcal{S}, \mathbb{Z}) = 0$*, where* Ω_{ext} *denotes the portion of* Ω *outside* $\mathcal{S}$*.*

2. *for* $n \geq 3$*,* Ω *is spin.*

The condition (T) allows applications of Theorem 1.1.2 or [63] Section VI. Following in this section, $n \geq 3$. For notation simplicity, the isometric embedding ϕ is omitted after gluing and identification if without ambiguity. Moreover, γ_{std} denotes the standard metric on $\mathbb{S}^{n-1}$ induced from the Euclidean space.

Theorem 6.0.1. *(cf. [62] Theorem 1.3) Let* $D_{SB} := (\mathbb{S}^{n-1}, \gamma, \alpha, H, \beta)$ *be a spacetime Bartnik data set. If* γ *is isotopic to* γ_{std} *in* $\mathcal{M}^q_{psc}(\mathbb{S}^{n-1}) := \{\eta : C^q \text{ metrics on } \mathbb{S}^{n-1} \text{ with } R_\eta > 0\}$*, where* $q \geq 5$*, then there exists a constant* $h_0 = h_0(n, \gamma) > 0$ *such that if*

$$H - f > 0 \;\; and \;\; \int_{\mathbb{S}^{n-1}} H - f \, d\mu_\gamma > h_0,$$

where $f := \sqrt{(\mathrm{tr}_\Sigma\, \alpha)^2 + |\beta|^2_\gamma}$*, then* D_{SB} *cannot admit a fill-in satisfying* (T) *and the dominant energy condition.*

Proof. Notice that, since γ is isotopic to γ_{std} in $\mathcal{M}^q_{psc}$ (as mentioned in [62], by [52] Proposition 2.1 and its proof, the path γ_t can be assumed to be smooth), [62] Lemma 2.1 shows that we can construct an asymptotically flat exterior extension $(M_+ = \mathbb{S}^{n-1}\times[1,\infty), \bar{g} = dr^2+\bar{\gamma}_r)$ of $(\Sigma^n, \gamma = \bar{\gamma}_1)$, which is exactly Euclidean (i.e. $\gamma_r = r^{n-1}\gamma_{std}$), for $r \geq s_0 = s_0(\gamma_t, \varepsilon)$ for any $\varepsilon > 0$. The choice of ε is to be determined.

In the proof of [62] Lemma 2.1, we can see that $R_{\bar{g}}$ is bounded by some constants depending on $\varepsilon > 0$ to be determined and γ_t. For this M_+, since $R_{\bar{g}}$ is bounded, the argument for the solvability of the initial value problem ([62] P.14 equation (7)) for u such that $g_+ = u^2(r)dr^2+\bar{\gamma}_r$ is scalar flat and asymptotically flat in [59] is also applicable . The initial value $u(1) > 0$ is to be determined. Let $\bar{H}_r$ and H^+_r denote the mean curvature for $\Sigma_r := \mathbb{S}^{n-1}\times\{r\}$ in $\bar{g}$ and g_+ respectively.

By equation (1.6) in [59], and further note that $(\Sigma\times[s_o,\infty), \bar{g})$ is Euclidean, then Lemma 4.2 and Lemma 2.10 in [59] are applicable. Together with the fact that, Σ_{s_0} is a standard sphere in $\bar{g}$, we have

$$\begin{aligned} c(n)E_{ADM}(g_+) \leq & \int_{\Sigma_{s_0}} \bar{H}_{s_0} - H^+_{s_0} d\mu_{\bar{\gamma}_{s_0}} \\ \leq & n(n-1)|\mathbb{S}^{n-1}|s_0^{n-2} - \int_{\Sigma_{s_0}} H^+_{s_0} d\mu_{\bar{\gamma}_{s_0}} \end{aligned} \tag{6.0.1}$$

What remains is to link the quantity $\int_{\Sigma_{s_0}} H^+_{s_0}\, d\mu_{\bar{\gamma}_{s_0}}$ to $\int_{\mathbb{S}^{n-1}} H-\omega\, d\mu_\gamma$. Following [62] Section 3.1, we thus have

$$c(n)E_{ADM}(g_+) \leq n(n-1)|\mathbb{S}^{n-1}|s_0^{n-2} - s_0^{\frac{(n-2)(1-\varepsilon)}{2}} \int_{\mathbb{S}^{n-1}} H^+_1 d\mu_\gamma. \tag{6.0.2}$$

Up to this step, it is the same as in the proof of Theorem 1.3 in [62] as the construction for the extension only concerns the metric γ.

Again, by equation (1.6) in [59], we have $H_1^+ = \frac{\bar{H}_1}{u(1)}$. Note that $\bar{H}_1 > 0$ by Lemma 2.1 equation (2) of [62] and by fixing a choice of $\varepsilon << 1$. If we choose the initial value $u(1) := \frac{\bar{H}_1}{H-f}$, then we have,

$$c(n)E_{ADM}(g_+) \leq n(n-1)|\mathbb{S}^{n-1}|s_0^{n-2} - s_0^{\frac{(n-2)(1-\varepsilon)}{2}} \int_{\mathbb{S}^{n-1}} H - f \, d\mu_\gamma. \tag{6.0.3}$$

Thus, if we choose $h_0 = h_0(n,\gamma) = n(n-1)|\mathbb{S}^{n-1}|s_0^{n-2-\frac{(n-2)(1-\varepsilon)}{2}}$, then $E_{ADM}(g_+) < 0$. Let (Ω, g, k) be a fill-in which satisfies the assumptions of the proposition. Then, (Ω, g, k) with $(M_+, g_+, 0)$, altogether is an initial data set with a corner $\mathbb{S}^{n-1}$ on which

$$H_g - H_1^+ - |\pi(\nu,\cdot)|_g = H - (H-f) - f = 0,$$

and M_+ satisfies the dominant energy condition by construction. If (Ω, g, k) furthermore simultaneously satisfies (T) and the dominant energy condition, by Theorem 1.1.2 or [63] Section VI., $E_{ADM}(g_+) \geq 0$, contradiction arises. □

Theorem 6.0.2. *(cf. [62] Theorem 1.4) Let $D_{SB} := (\mathbb{S}^{n-1}, \gamma, \alpha, H, \beta)$ be a spacetime Bartnik data set. If $\gamma \in \mathcal{M}^n_{c,d} := \{\eta : C^\infty$ metrics on $\mathbb{S}^{n-1}$ with*
$|Rm_\eta| \leq c$, $diam(\eta) \leq d$, $vol(\eta) = vol(\gamma_{std})\}$, then there exists a constant $C_0(n,c,d) > 0$ such that if

$$H - f \geq C_0,$$

where $f := \sqrt{(\mathrm{tr}_\Sigma \alpha)^2 + |\beta|^2_\gamma}$, then D_{SB} cannot admit a fill-in satisfying (T) and the dominant *energy condition.*

Proof. The construction of an asymptotically flat extension with a corner where mean curvatures match is done in [62] Lemma 2.4 followed by Lemma 2.1. The solvability of the initial value problem for u is again by [59], where $u(1) = \frac{\bar{H}_1(n,c)}{C_0}$ while C_0 is to be determined. If $C_0 > 0$ is sufficiently big, depending on the curvature of the extension constructed in [62] Lemma 2.4 which depends on n, c and d, then $0 < u < 1$ on $M_+ = \mathbb{S}^{n-1} \times [1, \infty)$. Moreover, $E_{ADM}(g_+ = u(r)^2 + \bar{\gamma}_r) < 0$.

Assume on the contrary that there exists a fill-in of D_{SB}, (Ω, g, k) which satisfies the assumptions of the proposition, (T) and the dominant energy condition. Gluing Ω and M_+, we have got an asymptotically flat initial data sets with 2 disjoint corners. For the corner in M_+, as mentioned, mean curvatures match by the construction in [62] Lemma 2.4. For the corner $\partial M_+ = \Sigma_1$, we have $H - C_0 - f \geq 0$. By Theorem 1.1.2 or Section VI. in [63], $E_{ADM}(g_+) \geq 0$, contradiction arises. $\square$

While in [61], the parabolic method to extend metric in [59] is used to construct a PSC ($R_g > 0$) collar, which combined with he Riemannian positive mass theorem with corners can show non-existence of NNSC fill-ins. In the same spirit as above, we can arrive at the following conclusion.

Theorem 6.0.3. *(cf. [61] Theorem 1.2) Let $D_{SB} := (\Sigma^{n-1}, \gamma, \alpha, H, \beta)$ be a spacetime Bartnik data set where Σ^{n-1} can be smoothly embedded into $\mathbb{R}^n$ and γ is smooth. There exists a constant $C_0 = C_0(\Sigma, \gamma) > 0$ such that if*

$$H - f \geq C_0,$$

where $f := \sqrt{(\operatorname{tr}_\Sigma \alpha)^2 + |\beta|^2_\gamma}$, then D_{SB} cannot admit a fill-in satisfying (T) and the dominant *energy condition.*

Proof. Let $F : \Sigma^{n-1} \hookrightarrow \mathbb{R}^n$ be an embedding. For $\lambda > 0$, λF is also an embedding. There exists a $\lambda_0 > 0$ such that $\gamma_1 := \lambda_0^2 F^*(g_{Euc}) > \gamma$, where g_{Euc} is the Euclidean metric on $\mathbb{R}^n$. Let $\tilde{h}$ denote the mean curvature of γ_1 with respect to the outward normal in $\mathbb{R}^n$. Denote the unbounded region of $\mathbb{R}^n$ outside $\lambda_0 F(\Sigma)$ by M_+.

By [61] Lemma 2.1, we know that there exists a cobordism $(\Sigma \times [0,1], \hat{g})$ and $h_0, h_1 \in C^\infty(\Sigma)$ such that

1. $\hat{g}|_{\Sigma\times\{0\}} = \gamma$ and $\hat{g}|_{\Sigma\times\{1\}} = \gamma_1$,
2. With respect to $\hat{g}$ and the outward normal, the mean curvature of $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ are respectively h_0 and h_1,
3. $h_1 > \tilde{h}$ and
4. $R_{\hat{g}} > 0$.

Pick $C_0 = \max(-h_0)$. Let (Ω, g, k) be a fill-in satisfying the assumption of the proposition. Then glue $\partial\Omega$ to $\Sigma \times [0,1]$ along $\Sigma \times \{0\}$ and further glue $\Sigma \times [0,1]$ along $\Sigma \times \{1\}$ to M_+. Altogether, we have a manifold with a flat end and hence with $E_{ADM} = 0$. Across the corners $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$, we respectively have $H - (-h_0) - f \geq 0$ and $h_1 - \tilde{h} > 0$. If (Ω, g, k) satisfies both (T) and the dominant energy condition, by Theorem 1.1.2 or [63] Section VI., $E_{ADM} > 0$, contradiction arises. □

Remark 6.0.1. For a charged initial data set $(M, g, \mathcal{E})$ with corners, we can consider charged harmonic functions ([8] Section 8) and the quantity associated with the divergence-free electric field $2\langle \mathcal{E}_\pm, \nu \rangle$ like $\pi_\pm(\nu, \cdot)$, in particular, $\mathcal{E}$ are only required to be L^∞ across the hypersurface where the corner of g occurs. The observations in this note are applicable and the corresponding results can be obtained.

Chapter 7

Existence and regularity of spacetime harmonic functions

7.1 Spacetime harmonic coordinates on asymptotically flat initial data sets with Lipschitz singularities

For notation simplicity, write $tr_g k$ by K. Let $\mathcal{S}$ be a finite (possibly empty) disjoint union of connected weakly trapped surfaces which do not intersect $\tilde{\Sigma}$ such that $H_2(M_{ext}, \mathcal{S}, \mathbb{Z}) = 0$. Let $\Sigma = \tilde{\Sigma} \cap M_{ext}$. Following the strategy of Section 4 in [30], we prove the following proposition.

Proposition 7.1.1. *For the asymptotically flat coordinate x^1, for any $\phi \in C^\infty(\mathcal{S})$, there exists $u \in W^{2,p}_{loc}(M_{ext}) \cap W^{3,p}_{loc}(M_{ext} \setminus \Sigma)$ such that*

1. $\Delta u + K|\nabla u| = 0$ *on* M_{ext},

2. $u = \phi$ *on* $\mathcal{S}$,

3. *$u - x^1 = O^2(|x|^{1-q})$ as $|x| \to \infty$,*

4. *$u|_\Sigma$ is C^2 on faces of Σ.*

Remark 7.1.1. In Section 3.2, we would discuss that ϕ can be chosen to achieve suitable signs on the normal derivative of u on $\mathcal{S}$.

Proof. By slightly generalising Proposition 2.2 and Theorem 3.1 in [5], we can have a function $v \in W^{2,p}_{loc}(M)$, where $p > 3$, such that

1. $\Delta v = -K$ on M_{ext},

2. $v = 0$ on $\mathcal{S}$,

3. $v = x^1 + O^2(|x|^{1-q})$ as $|x| \to \infty$.

Note that by elliptic regularity, v is smooth on $M_{ext} \setminus \Sigma$ and $C^{1,\alpha}$ across Σ by Sobolev embedding. We also define a compactly supported smooth function v_0 such that $v_0 = \phi$ on $\mathcal{S}$. Define $\tilde{v} = v + v_0$.

Let $r >> 1$ and M_r denote the region of M_{ext} enclosed by the coordinate sphere $S_r = \{|x| = r\}$. Consider the following localised Dirichlet problem,

1. $\Delta u^r + K|\nabla u^r| = 0$ in M_r,

2. $u^r = \phi$ on $\mathcal{S}$,

3. $u^r = \tilde{v}$ on S_r.

Let $w^r = u^r - \tilde{v}$. It is then equivalent to seek existence of w^r which solves,

1. $\Delta w^r = -K\left(\frac{\nabla(w^r+2\tilde{v})}{|\nabla(w^r+\tilde{v})|+|\nabla\tilde{v}|}\right)\cdot\nabla w^r - \Delta\tilde{v} - K|\nabla\tilde{v}|$ in M_r,

2. $w^r = 0$ on $\mathcal{S}$,

3. $w^r = 0$ on S_r.

Construct a map $\mathcal{F} : C_0^{1,\alpha}(M_r)\times[0,1]\to C_0^{1,\alpha}(M_r)$ by

$$\mathcal{F}(w,\sigma) = \sigma\Delta^{-1}F(w), \tag{7.1.1}$$

where $F : C_0^{1,\alpha}(M_r)\to L^p(M_r)$ is defined by

$$F(w) = -K\left(\frac{\nabla(w+2\tilde{v})}{|\nabla(w+\tilde{v})|+|\nabla\tilde{v}|}\right)\cdot\nabla w - \Delta\tilde{v} - K|\nabla\tilde{v}|.$$

In particular, we can see $w^r = \mathcal{F}(w^r,1)$. Consider the following composition (cf. equation (4.10) in [30]),

$$C_0^{1,\alpha}(M_r)\xrightarrow{F} L^p(M_r)\xrightarrow{\Delta^{-1}} W^{2,p}(M_r)\cap W_0^{1,p}(M_r)\xrightarrow{\iota} C_0^{1,\alpha}(M_r). \tag{7.1.2}$$

Note that F and Δ^{-1} are bounded and the inclusion is compact by Sobolev embedding. Let $w_\sigma = \mathcal{F}(w_\sigma,\sigma)$, we have

$$\Delta w_\sigma + \sigma K\left(\frac{\nabla(w_\sigma+2\tilde{v})}{|\nabla(w_\sigma+\tilde{v})|+|\nabla\tilde{v}|}\right)\cdot\nabla w_\sigma = -\sigma\Delta\tilde{v} - \sigma K|\nabla\tilde{v}|. \tag{7.1.3}$$

Since the zeroth order term coefficient vanishes, maximum principle (Theorem 9.1 in [23]) is applicable, we can have a uniform $W^{2,p}(M_r)$ apriori estimate for all w_σ by Theorem 9.11 and 9.13 in [23]. Thus, w_σ is uniformly bounded in $C^{1,\alpha}(M_r)$ by Sobolev embedding.

By Leray Schauder fixed point theorem ([23] Theorem 11.6), we can seek existence of w^r. And by the barrier function of order $O(|x|^{1-2q})$ constructed in Section 4.2 in [30] and maximum principle, we can obtain a uniform $W^{2,p}_{loc}$ bound for all w^r. Hence, w_r is uniformly $C^{1,\alpha}_{loc}$ bounded. Away from Σ, we can see that $\Delta w^r \in C^{0,\alpha}$, and hence $w^r \in C^{2,\alpha}_{loc}(M_{ext} \setminus \Sigma)$, uniformly bounded .

Hence, by taking a diagonal subsequence as $r \to \infty$, we have a spacetime harmonic function $u := \lim_{r\to\infty} w^r + \tilde{v} = \tilde{v} + O^2(|x|^{1-2q}) = x^1 + O^2(|x|^{1-q})$, $u \in C^{1,\alpha}_{loc}(M_{ext}) \cap C^{2,\alpha}_{loc}(M_{ext} \setminus \Sigma)$. Furthermore, since $|\nabla u| \in W^{1,p}_{loc}(M_{ext})$ by Kato's inequality, we have $u \in W^{3,p}_{loc}(M_{ext} \setminus \Sigma)$ by Theorem 9.19 in [23].

Then, we are going to consider the regularity of u nearby Σ. Let $\hat{\Sigma}$ be a smooth surface component of Σ. Let $p \in \hat{\Sigma}$ and V be a neighborhood of p in M which does not intersect $\Sigma \backslash \hat{\Sigma}$. Apply Fermi coordinate along $\hat{\Sigma}$, $(x^1, x^2, t) \in \Sigma \times (-\varepsilon, \varepsilon)$, considering the difference quotients along ∂_1 direction, let $\phi^h = \Delta^h u$, where $\Delta^h f(x_1, x_2, t) := \dfrac{f((x_1 + h, x_2, t) - f((x_1, x_2, t))}{h}$ for a function f. Since u is spacetime harmonic, we have

$$\begin{aligned} & g^{ij}\phi^h_{ij} - g^{ij}\Gamma^k_{ij}\phi^h_k \\ =& f^h := -\Delta^h g * \widetilde{\partial^2 u} + \Delta^h (g * \Gamma) * \widetilde{\partial u} - (\Delta^h K)\,\widetilde{|\nabla u|} - K \Delta^h |\nabla u|, \end{aligned} \tag{7.1.4}$$

where $*$ denotes multiplication with indices suppressed and $\tilde{\phi}(x_1, x_2, t) = \phi(x_1 + h, x_2, t)$ for functions on V. Observe that the $g \in C^{0,1}(V)$, $\Gamma \in L^\infty(V)$ and $u \in W^{2,p}(V)$. While along ∂_1 direction, except on a $\mathcal{H}^3$-measure zero set $\Sigma \cap V$, g and k are also smooth. Moreover, $|\nabla u| \in W^{1,p}(V)$. Hence, its difference quotient is uniformly bounded in $L^p(V)$. Therefore, f^h is uniformly bounded in $L^p(V)$. By Theorem 9.11 in [23], we know for any $U \subset\subset V$, $||\phi^h||_{W^{2,p}(U)}$ and hence $||\phi^h||_{C^{1,\alpha}(U)}$ is uniformly bounded. Therefore, we have $\phi := \lim_{h\to 0} \phi^h = \partial_1 u \in C^{1,\alpha}(U)$ by [23] Lemma 7.24.

By varying the direction which is tangential to $\hat{\Sigma}$ and the neighbourhood for difference quotients, we can see the same argument applies. Therefore, $u|_\Sigma$ is C^2 on faces of Σ. $\square$

7.2 Spacetime harmonic functions on regular cubes

In this section, we discuss the existence of solutions to the PDE in Lemma 5.0.1 when the dihedral angles are $\pi/2$ everywhere. This illustrates the ideas of reducing a mixed boundary problem to a Dirichlet problem.

Proposition 7.2.1. *Given $([0,1]^3, g, k)$, where all dihedral angles are $\pi/2$, there exists a non-negative spacetime harmonic function $u \in C^{2,\alpha}([0,1]^3) \cap W^{3,p}_{loc}((0,1)^3)$ such that*

1. $G_0(u) := \Delta u + K|\nabla u| = 0$ *in* $(0,1)^3$,
2. $u = 0$ *on* B *and* $u = 1$ *on* T,
3. $\partial_\nu u = 0$ *on* F,

where $K = tr_g k$, T, B and F denote the top, the bottom and the side faces of the cube respectively and ν is the outward unit normal of $\partial[0,1]^3$.

7.2.1 Invertibility of Linear operators

First, we are going to show the solvability of certain linear mixed boundary value problems, which will be used in Section 7.2.5 to prove Proposition 7.2.1 above.

Definition 7.2.1. *Let $\mathcal{B} := \{w \in C^{2,\alpha}([0,1]^3) \,|\, \partial_\nu w = 0, \exists\, C_1, C_2 \in \mathbb{R}$ s.t. $w = C_1$ on T and $w = C_2$ on $B\}$, which is a Banach space with $C^{2,\alpha}([0,1]^3)$ norm.*

Definition 7.2.2. *Let $\mathcal{B}_0 = \{w \in \mathcal{B} \,|\, w = 0$ on T and $B\}$, which is also a Banach space with $C^{2,\alpha}([0,1]^3)$ norm.*

The following lemma is implied by the proof of Section 3 in [14]. And here we provide an alternative proof by reflection (cf. [38] Appendix B, [50]) which reduces the mixed boundary problem to a Dirichlet boundary problem. This approach can further be utilised when we study the mixed boundary problem on general prisms in Appendix 7.3.

Lemma 7.2.1. *Given* $([0,1]^3, g)$*, where all dihedral angles are* $\pi/2$*. If* X *is a vector field of regularity* $C^{0,\alpha}([0,1]^3)$*, then the operator* $L : \mathcal{B}_0 \to C^{0,\alpha}([0,1]^3)$ *defined by*

$$L(u) = \Delta(u) + \langle X, \nabla u \rangle$$

is invertible.

Proof. Consider

1. $L(u) = \Delta(u) + \langle X, \nabla u \rangle = f$ in $(0,1)^3$, where $f \in C^{0,\alpha}([0,1]^3)$,
2. $u = 0$ on B and $u = 0$ on T,
3. $\partial_\nu u = 0$ on F,

First, say B and T are identified with $\{x^3 = 0 \,|\, (x^1, x^2) \in [0,1]^2\}$ and $\{x^3 = 1 \,|\, (x^1, x^2) \in [0,1]^2\}$. We can make an even isometric reflection along the one of the side faces F. Then, make another even reflection along one of the longer faces. Without loss of generality, the quadruple cube is obtained by reflecting along $\{x^1 = 1\}$, then $\{x^2 = 1\}$, identified by $[0,2]^2 \times [0,1]$ and denoted by Q.

Correspondingly, under the coordinate charts (general Fermi coordinate) introduced in [38] Lemma 2.2, the metric components g^{ij}, the Christoffel symbols Γ_{ij}^k, X^i and f are evenly reflected twice as in [38] Appendix B. They then would be denoted by $\tilde{g}$, $\tilde{\Gamma}$, $\tilde{X}$ and $\tilde{f}$ respectively. Note that, since the dihedral angle is everywhere $\pi/2$, on the edges and vertices where doubling takes place, $\tilde{g}$ is still a well-defined Lipschitz metric on Q. Identifying and gluing the faces of Q lying on $\{x^1 = 0\}$ and $\{x^1 = 2\}$, then the faces lying on $\{x^2 = 0\}$ and

$\{x^2 = 2\}$, we have obtained $T^2 \times [0,1] = S^1 \times S^1 \times [0,1]$. We can see that the component functions of $\tilde{g} \in C^{0,1}(T^2 \times [0,1])$ while $\tilde{f} \in C^{0,\alpha}(T^2 \times [0,1])$ and $\tilde{\Gamma}, \tilde{X} \in L^\infty(T^2 \times [0,1])$.

Then we consider the following PDE,

1. $\tilde{\Delta} u + \tilde{g}(\tilde{X}, \tilde{\nabla} u) = \tilde{f}$ in $T^2 \times (0,1)$,

2. $u = 0$ on $T^2 \times \{0\}$ and $u = 0$ on $T^2 \times \{1\}$.

By standard elliptic theory ([23] Theorem 9.15, Theorem 9.13 and Lemma 9.16), there exists a unique strong solution $v \in W^{2,p}(T^2 \times [0,1])$, $p > 3$, hence $C^{1,\alpha}(T^2 \times [0,1])$. In order to show that this v when restricted to one of the cubes solves the mixed boundary problem. It suffices to show that v is actually periodically evenly reflected. Back to Q, define a new functions $\hat{v}$ and $\hat{f}$ by reflection as follows

$$\begin{aligned} \hat{v}(x^1, x^2, x^3) &= v(2 - x^1, x^2, x^3), \\ \hat{f}(x^1, x^2, x^3) &= \tilde{f}(2 - x^1, x^2, x^3), \end{aligned} \tag{7.2.1}$$

note that $\hat{f} = \tilde{f}$.

Now, we consider the following PDE,

1. $\tilde{\Delta} u + \tilde{g}(\tilde{X}, \tilde{\nabla} u) = \hat{f}$ in $T^2 \times (0,1)$,

2. $u = 0$ on $T^2 \times \{0\}$ and $u = 0$ on $T^2 \times \{1\}$.

By the symmetry of $\tilde{f}$, $v \in W^{2,p}(T^2 \times [0,1])$ is the unique solution. On the other hand, by the symmetry of coefficients, obviously $\hat{v}$ is also a solution. Hence, $v = \hat{v}$. Hence v is even along the plane $\{x^1 = 1\}$. Similarly, we can conclude that v is symmetric along the other side faces. Therefore, we can conclude that $v|_{[0,1]^3} \in W^{2,p}([0,1]^3)$ is a strong solution to the mixed boundary problem. Moreover, since $\partial_\nu v = 0$ on F, v is actually C^2 in $T^2 \times (0,1)$. For its regularity up to the boundary, in particular across the closed side faces and edges, let Ω be a neighbourhood in $T^2 \times [0,1]$ near the boundary. For ν on F can be extended as a Fermi coordinate system, consider

1. $\tilde{g}^{ij}v_{ij} + \tilde{g}^{ij}\tilde{\Gamma}^a_{ij}\partial_a v + \tilde{X}^a\partial_a v = \tilde{f} - \tilde{g}^{ij}\tilde{\Gamma}^\nu_{ij}\partial_\nu v - \tilde{X}^\nu\partial_\nu v$ in Ω,
2. $v = 0$ on $T^2 \times \{0\}$,

where $i, j \in \{1,2,3\}$ while $a, b \in \{1,2\}$ stands for the component perpendicular to ν. Note that $\partial_\nu v \in C^1(T^2 \times (0,1)) \cap C^{0,\alpha}(T^2 \times [0,1])$ and vanishes along the side faces, hence the right hand side is $C^{0,\alpha}(\bar{\Omega})$. Similarly, we can deal with the edges. Therefore, by [23] Lemma 6.18, we know that $v \in C^{2,\alpha}(T^2 \times [0,1])$. As a result, $L : \mathcal{B}_0 \to C^{0,\alpha}([0,1]^3)$ is invertible. $\square$

7.2.2 Regularised operators

After showing invertibility of linear operators in Lemma 7.2.1, we are going to show that the mixed boundary problem in Proposition 7.2.1 is solvable by implicit function theorem. Since the operator there is not linearisable, we first have to consider the following regularised operator.

Let $\delta \in (0,1)$, let G_δ be a regularised operator defined by $G_\delta(u) := \Delta u + K\sqrt{\delta^2 + |\nabla u|^2} - \delta K$. We are going to consider the following regularised PDE.

1. $G_\delta(u) = 0$ in $(0,1)^3$,
2. $u = 0$ on B and $u = 1$ on T,
3. $\partial_\nu u = 0$ on F,

where $K = tr_g k$ and T, B, F denotes the top, the bottom and the side faces of a cube respectively.

7.2.3 Aprori estimates

Let $u \in \mathcal{B}$ be a solution to the PDE above. As in Section 7.2.1, by reflection and the fact that

$$\Delta u = -K\sqrt{\delta^2 + |\nabla u|^2} + \delta K,$$

together with interpolation inequality ([23] Lemma 6.35), we get the following estimate

$$||u||_{C^{2,\alpha}([0,1]^3)} \leq C(||u||_{C^0([0,1]^3)} + \delta||K||_{C^{0,\alpha}([0,1]^3)}). \tag{7.2.2}$$

where C depends on metric g. Moreover, by [23] Theorem 9.1, we have $||u||_{C^0([0,1]^3)} \leq C(1 + ||K||_{C^0([0,1]^3)})$. Altogether, we have

$$||u||_{C^{2,\alpha}([0,1]^3)} \leq C(1 + ||K||_{C^{0,\alpha}([0,1]^3)}), \tag{7.2.3}$$

which is independent of δ.

7.2.4 Uniqueness of solutions

Proposition 7.2.2. *The solution to the regularised PDE is unique.*

Proof. If u and v are solutions to the regularised PDE, then we have

1. $\Delta(u - v) + K\frac{\nabla(u+v)}{\sqrt{\delta^2+|\nabla u|^2}+\sqrt{\delta^2+|\nabla v|^2}} \cdot \nabla(u - v) = 0$ in $(0, 1)^3$,
2. $u - v = 0$ on B and $u - v = 0$ on T,

3. $\partial_\nu(u-v)=0$ on F,

Then by maximum principle, $u=v$. □

7.2.5 Linearisation of the regularised operator

Let $\phi \in \mathcal{B}$ with $\phi = 0$ on B and $\phi = 1$ on T. For a fixed $\delta \in (0,1)$, we consider a mapping $T_\delta : \mathcal{B}_0 \times [0,1] \to C^{0,\alpha}([0,1]^3)$ defined by

$$T_\delta[u,t] = G_\delta(u+t\phi).$$

And for each t, let $T_\delta^{(1)}|_{[u,t]} : \mathcal{B}_0 \to C^{0,\alpha}([0,1]^3)$ denote its linearisation in the parameter of $\mathcal{B}_0$.

Let $A = \{t \in [0,1] \,|\, \exists\, w \in \mathcal{B}_0 \text{ such that}\, T_\delta[w,t] = 0.\}$. Equivalently, if $t \in A$, there exists $u \in \mathcal{B}$ solving the following mixed boundary problem,

1. $G_\delta(u) = 0$ in $(0,1)^3$,
2. $u = 0$ on B and $u = t$ on T,
3. $\partial_\nu u = 0$ on F.

A is non-empty obviously since $T_\delta[0,0] = 0$. We first show that A is open as follows. Let $\bar{t} \in A$, i.e. there exists $\bar{u} \in \mathcal{B}_0$ such that

$$T_\delta[\bar{u},\bar{t}] = 0.$$

Consider the linearisation

$$T_\delta^{(1)}|_{[\bar{u},\bar{t}]}(v) = \Delta v + K \frac{\nabla(\bar{u} + \bar{t}\phi)}{\sqrt{\delta^2 + |\nabla(\bar{u} + \bar{t}\phi)|^2}} \cdot \nabla v,$$

which is invertible by Lemma 7.2.1. By implicit function theorem ([23] Theorem 17.6), we know that there exists a neighbourhood $\mathcal{N}$ of $\bar{t}$ in $[0, 1]$ such that for all $t \in \mathcal{N}$, there exists $u_{\delta,t} \in \mathcal{B}_0$ such that $T_\delta[u_{\delta,t}, t] = 0$.

By the estimate (7.2.3) which is independent of $\delta \in (0, 1)$ and $t \in [0, 1]$, we have that A is closed and hence $A = [0, 1]$. Therefore, for all $\delta \in (0, 1)$, there exists $u_\delta \in \mathcal{B}$ solving the PDE,

1. $G_\delta(u_\delta) = 0$,
2. $u_\delta = 0$ on B and $u_\delta = 1$ on T,
3. $\partial_\nu u_\delta = 0$ on F.

Again, by the uniform estimate (7.2.3), we have $u_\delta \xrightarrow[\delta \to 0]{C^{2,\beta}([0,1^3])} u \in \mathcal{B}$ for all $0 < \beta < \alpha$, satisfying

1. $\Delta u + K|\nabla u| = 0$,
2. $u = 0$ on B and $u = 1$ on T,
3. $\partial_\nu u = 0$ on F.

Furthermore, by Kato's inequailty and [23] Theorem 9.19, we get $u \in W^{3,p}_{loc}((0, 1)^3)$ and by maximum principle, $u \geq 0$.

7.3 Spacetime harmonic functions on prisms

To prove Lemma 5.0.1, Leray-Schauder fixed point theorem is applied for existence and reflection is still an essential tool for regularity. Generally, for P_0 being a q-gon, we can locally around each vertical edge perform a reflection twice for regularity estimates. (For example, see [50] and [51] which apply a bi-Lipschitz mapping locally onto the boundary where Neumann conditions are imposed to get it straightened followed by a reflection.) Therefore, after identifying the side faces where Neumann conditions are imposed, we can apply standard results in [23] for estimates on Dirichlet problems.

However, since the angles are no longer necessarily $\pi/2$ or constant, though local bi-Lipschitz mappings are applied through identification with reflection is carried out twice as in Appendix 7.2, the coefficients could be discontinuous, yet still uniformly bounded across the edges and vertices. Correspondingly, we need to apply weak solution theory instead and consider different Banach spaces. $G_0(u)$ is then expressed as

$$div(\nabla u) + K|\nabla u|$$

whose structure would be preserved under bi-Lipschitz transformation.

For a type P initial data set (M, g, k), we define the following.

Definition 7.3.1. *Let* $\mathring{M} := int\, M$*,* $\tilde{\mathcal{H}} := \{w \in W^{1,2}(\mathring{M}) \,|\, \exists\, C_1, C_2 \in \mathbb{R}$ *s.t.* $w = C_1$ *on* T *and* $w = C_2$ *on* $B\}$*, which is a Banach space with* $W^{1,2}(\mathring{M})$ *norm.*

Definition 7.3.2. *Let* $\tilde{\mathcal{H}}_0 = \{w \in W^{1,2}(\mathring{M}) \,|\, w = 0$ *on* T *and* $B\}$*, which is also a Banach space with* $W^{1,2}(\mathring{M})$ *norm.*

7.3.1 Invertibility of Linear operators

We first have to show that weak (generalised) solutions to the following linear PDE exist,

1. $div(\nabla u) + \langle X, \nabla u\rangle = \tilde{f}$, where X is a bounded vector field and $\tilde{f} \in \tilde{\mathcal{H}}_0^*$,
2. $u = 0$ on B and $u = 1$ on T,
3. $\partial_\nu u = 0$ on F.

It is then equivalent to consider the following.

Lemma 7.3.1. *Given a type P initial data set (M, g, k). If X is a bounded vector field, then the operator $\tilde{L} : \tilde{\mathcal{H}}_0 \to \tilde{\mathcal{H}}_0^*$ defined by*

$$\tilde{L}(u)(v) = \int_M \langle \nabla u, \nabla v\rangle - \langle X, \nabla u\rangle v$$

for $v \in \tilde{\mathcal{H}}_0$, is invertible.

Proof. First, an alternative weak maximum principle where we only need to consider $\sup_{T\cup B} u_+$ or $\inf_{T\cup B} u_-$ follows from the proof of [23] Theorem 8.1.

Define a bilinear functional $\mathcal{L} : \tilde{\mathcal{H}}_0 \times \tilde{\mathcal{H}}_0 \to \mathbb{R}$ by

$$\mathcal{L}(u, v) = \int_M \langle \nabla u, \nabla v\rangle - \langle X, \nabla u\rangle v$$

for $(u, v) \in \tilde{\mathcal{H}}_0 \times \tilde{\mathcal{H}}_0$. As in the proof of [23] Theorem 8.3, we can see there exists a sufficiently large $\lambda > 0$ such that the bilinear functional $\mathcal{L}_\lambda$ defined by

$$\mathcal{L}_\lambda(u, v) = \int_M \langle \nabla u, \nabla v\rangle - \langle X, \nabla u\rangle v + \lambda uv$$

for $(u, v) \in \tilde{\mathcal{H}}_0 \times \tilde{\mathcal{H}}_0$ is coercive. Then follow the argument of [23] Theorem 8.3, by Lax-Milgram theorem and Fredholm alternative we can conclude that for all $\tilde{f} \in \tilde{\mathcal{H}}_0^*$, there exists a unique $u \in \tilde{\mathcal{H}}_0$ such that

$$\mathcal{L}(u, v) = \tilde{f}(v)$$

for all $v \in \tilde{\mathcal{H}}_0$. By [48] Ch. IV Section 1.1, u is by definition the weak solution satisfying the boundary conditions: $u = 0$ on $T \cup B$ and $\partial_\nu u = 0$ on F. An alternative way is to first reduce the mixed boundary problem to the Dirchlet problem by bi-Lipschitz map followed by a reflection as in Appendix 7.2. After the existence of solution is established as above, we can see the solution satisfies $\partial_\nu u = 0$ on F. □

7.3.2 Existence of solutions

(cf. [30] Section 4, [68] Section 3) By Lemma 7.3.1, there exists $v \in \tilde{\mathcal{H}}$ such that

1. $div(\nabla v) = 0$ in $\mathring{M}$,
2. $v = 0$ on B,
3. $v = 1$ on T,
4. $\partial_\nu v = 0$ on F.

Let $u \in \tilde{\mathcal{H}}$ denote a solution to the mixed boundary problem in Lemma 5.0.1, it is then equivalent to seek existence of $w := u - v \in \tilde{\mathcal{H}}_0$ which satisfies

$$div(\nabla w) = -K \left(\frac{\nabla(w + 2v)}{|\nabla(w + v)| + |\nabla v|} \right) \cdot \nabla w - K|\nabla v|$$

in $\mathring{M}$.

Construct a map $\mathcal{F} : \tilde{\mathcal{H}}_0 \times [0,1] \to \tilde{\mathcal{H}}_0$ by

$$\mathcal{F}(\phi,\sigma) = \sigma\Delta^{-1}\tilde{I}G(\phi), \tag{7.3.1}$$

where $G : \tilde{\mathcal{H}}_0 \to L^2(\mathring{M})$ and $\tilde{I} : L^2(\mathring{M}) \to \tilde{\mathcal{H}}_0^*$ are respectively defined as follows. For all $\phi \in \tilde{\mathcal{H}}_0$,

$$G(\phi) = -\,K\left(\frac{\nabla(\phi+2v)}{|\nabla(\phi+v)|+|\nabla v|}\right)\cdot\nabla\phi - K|\nabla v|. \tag{7.3.2}$$

And for all $(\xi,\psi) \in L^2(\mathring{M}) \times \tilde{\mathcal{H}}_0$,

$$\tilde{I}(\xi)(\psi) = \int_M \xi\psi. \tag{7.3.3}$$

In particular, we can see $w = \mathcal{F}(w,1)$. Consider the following composition (cf. equation (4.10) in [30], equation (3.2) in [68]),

$$\tilde{\mathcal{H}}_0 \xrightarrow{G} L^2(\mathring{M}) \xrightarrow{\tilde{I}} \tilde{\mathcal{H}}_0^* \xrightarrow{\Delta^{-1}} \tilde{\mathcal{H}}_0. \tag{7.3.4}$$

Note that G and Δ^{-1}, by Lemma 7.3.1, are bounded while $\tilde{I}$ is compact by Sobolev embedding and Schauder theorem ([11] Theorem 6.4). Let $w_\sigma = \mathcal{F}(w_\sigma,\sigma)$. To apply Leray-Schauder fixed point theorem ([23] Theorem 11.6), it remains to obtain a uniform apriori $W^{1,2}$ estimate for w_σ. Consider

$$\Delta w_\sigma + \sigma K\left(\frac{\nabla(w_\sigma+2v)}{|\nabla(w_\sigma+v)|+|\nabla v|}\right)\cdot\nabla w_\sigma = -\sigma K|\nabla v|. \tag{7.3.5}$$

By Corollary 8.7 in [23] and the definition of w_σ, we know that

$$||w_\sigma||_{W^{1,2}(\mathring{M})} \le C(||w_\sigma||_{L^2(\mathring{M})} + ||v||_{W^{1,2}(\mathring{M})}) \le C(||u||_{L^2(\mathring{M})} + ||v||_{W^{1,2}(\mathring{M})}).$$

Following the proof Theorem 8.1 in [23], we get that $u \geq 0$ and an apriori estimate $||u||_{L^\infty(M)} \leq 1$. Thus, w_σ is uniformly bounded in $\tilde{\mathcal{H}}_0$. And thus $u = v + w \in \tilde{\mathcal{H}}_0$ exists. Again, by reducing the mixed boundary problem to the Dirchlet problem as aforementioned, we can see the solution satisfies $\partial_\nu u = 0$ on F.

7.3.3 Regularity of solutions

Regarding regularity, since the structural inequality in [23] Section 8.5 is satisfied, with reflection as aforementioned, by [23] Theorem 8.22 and 8.29, $u \in C^{0,\alpha}(M) \cap W^{1,2}(\mathring{M})$. Then by elliptic regularity theory as in Appendix 7.2, we get that u is $C^{2,\alpha}$ away from the edges. In particular, for each compact $\Omega \subset M \setminus \bar{\mathcal{E}}$, where $\mathcal{E}$ denotes the edges, by interior estimates and boundary estimates as in Section 7.2.3, $||u||_{C^{2,\alpha}(\Omega)}$ is bounded since $||u||_{C^0(M)} \leq 1$. Furthermore, by [43] Theorem 4.1, since the dihedral angles are assumed to be less than π everywhere, one can see that u is $C^{1,\alpha}$ up to the vertical edges away from $\bar{T}$ and $\bar{B}$. Therefore, a classical solution $u \in C^{0,\alpha}(M) \cap C^{2,\alpha}_{loc}(M \setminus \bar{\mathcal{E}}) \cap C^{1,\alpha}_{loc}(M \setminus (\bar{T} \cup \bar{B}))$ to the mixed boundary problem in Lemma 5.0.1 is obtained. Moreover, $u \in W^{3,p}_{loc}(\mathring{M})$ by Kato's inequality and [23] Theorem 9.19.

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