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On the Quasiperiodic Structure of the Complex Dimensions of Self-Similar Fractal Strings

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I dedicate this thesis to my beautiful wife Lilit, for which none of this would have been possible.

ABSTRACT OF THE DISSERTATION

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by

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In crystallography, it was an axiom that any material with a diffraction pattern consisting of bright spots must have an atomic structure that is a repetition of a unit cell consisting of finitely many atoms, i.e., having translational symmetry in three linearly independent spacial directions. The discovery of quasicrystals in 1982 by Dan Shechtman, started an investigation into the question of exactly which atomic structure is necessary to produce a pure point diffraction pattern. In this thesis, we give a special case of the Poisson summation formula, and use it to address a question (in the simpler lattice case) by Lapidus and van Frankenhuijsen on whether or not the complex dimensions of a nonlattice self-similar fractal string can be understood as a mathematical analog for a quasicrystal, in the context of a mathematical idealization of diffraction developed by A. Hof. Also, we provide an implementation of the LLL algorithm to give a numerical exploration of the quasiperiodic structure in the nonlattice case.

Contents

List of Figures

Chapter 1

Introduction

1.1 Overview

In crystallography, without any evidence to the contrary, it was an axiom that only those materials with an atomic structure having periodic translations in all three spacial dimensions could produce a pure point diffraction pattern. In 1982 however, a material scientist by the name of Dan Shechtman, while on sabbatical at the National Institute of Standards and Technology (now called the National Bureau of Standards and Publications), made a startling discovery that changed the world of crystallography forever. Using a rapid cooling technique, Shechtman processed an alloy of aluminum and manganese with an atomic structure that produced a pure point diffraction pattern, but whose symmetry was exactly that of an icosahedron, which includes rotational symmetries other than one, two, three, four, or six. By a mathematical law called the *crystallographic restriction*, translational symmetry and rotational symmetry of order 5 or greater than 6 are incompatible. So, while this material was scattering x-rays like periodic crystals, it could not have a peri-

odic atomic structure. These materials, which have been produced in other labs since their discovery, and are now known to exist in nature, are called *quasicrystals*. (See [SBGC84]). The existence of these materials means that there is still a question as to exactly what interatomic arrangements are necessary and sufficient to produce discrete diffraction patterns, and so it has given rise to the theory of 'mathematical quasicrystals'. One might say that this theory has existed since the early 17th century with Kepler, where he discussed a tiling built with pentagons, pentagrams, decagons. He also called them "monsters". (See [Sen96]). According to Moody in [Moo00], pure point diffractivity and aperiodicity are among the properties considered representative of the concept of quasiperiodicity in general. There is not a unifying definition for mathematical quasicrystal, and we do not attempt to establish one in this thesis. However there are many examples that are widely considered to be working models. According to Lagarias in [Lag00], the basic mathematical model is an infinite discrete subset of n-dimensional Euclidean space that is highly ordered such that, under a certain mathematical analog of diffraction, it has a pure point diffraction pattern. These are called *Model sets*, and they are widely considered to be mathematical quasicrystals. In this thesis, the condition of having a pure point diffraction pattern is relaxed, and we consider a more general candidate, which is a slight extension of a much larger class of models named after the Soviet/Russian mathematician Boris Delone.

Definition 1.1.1 *A Delone set Λ in $\mathbb{R}^n$ is a set with the properties:*

(1) Uniformly Discreteness. There exists $r > 0$ such that each ball of radius r contains at most one element of Λ.

(2) Relatively Denseness. There exists $R > 0$ such that each ball of radius R contains at

least one element of Λ.

Intuitively, a Delone set is an extensive discrete subset of $\mathbb{R}^n$ with a global minimum and maximum distance between all points. Lapidus and van Frankenhuijsen give a candidate for a more general model.

Problem 1 *Is there a natural way in which the quasiperiodic pattern of the set of complex dimensions of a nonlattice self-similar string can be understood in terms of suitable* (*generalized*) *quasicrystal, or of an associated quasiperiodic tiling?*

In short, a *self-similar fractal string* is a bounded open subset of the real line with fractal boundary. The set of *complex dimensions* of a self-similar fractal string is defined as a discrete subset of the complex plane which in some sense, describes the geometry of the boundary. These structures were developed and studied extensively by Lapidus and van Frankenhuijsen in their book [LvF06]. The theory of complex dimensions is motivated from earlier publications by Lapidus, Pomerance, and Maier. From 1991 to 1993, Lapidus and Pomerance produced works which established connections between complex dimensions and the theory of Riemann zeta functions (see [Lap91], [Lap93], and [LP93]). In [LM95], Lapidus and Maier use the intuition coming from the notion of complex dimensions to reformulate the Riemann hypothesis. The notion of complex dimensions was precisely defined and the corresponding rigorous theory of complex dimensions was fully developed by Lapidus and van Frankenhuijsen in the books [LvF00] and [LvF06].

There is an important dichotomy in the set of all self-similar fractal strings, in which any self-similar fractal string is either *lattice* or *nonlattice*, depending on the *scaling ratios* with which a self-similar fractal string is constructed. In the lattice case, the complex

dimensions lie periodically on finitely many vertical lines, and on each line they are separated by a positive real number p called the *oscillatory period* of the string. In the nonlattice case, the structure is nonperiodic. The complex dimensions in this case cannot be obtained explicitly, but are instead approximated via an explicit procedure developed by Lapidus and van Frankenhuijsen, by the complex dimensions of lattice self-similar fractal strings. Such approximating lattice self-similar fractal strings are called *lattice approximations*.

According to Theorem 2.2.3, computing many lattice approximations for a given nonlattice Dirichlet polynomial requires a practical method for obtaining simultaneous Diophantine approximations. The LLL algorithm, which is a lattice basis reduction algorithm, was introduced in [LLL82] as the first polynomial time algorithm to compute the roots of polynomials with rational coefficients. In their paper, the authors also described how the algorithm can be used to compute simultaneous Diophantine approximations. In this chapter, we describe, citing the original paper [LLL82], and [Bre11], exactly how the LLL algorithm can be used to obtain simultaneous Diophantine approximations. We refer the reader to the appendix for a general overview of the treatment of lattice theory and basis reduction from both [LLL82] and [Bre11]. Including, a complete description of the LLL algorithm, and some important facts about its complexity. Also in the appendix, we give an implementation of the LLL algorithm to compute simultaneous Diophantine approximations.

Through the approximation procedure developed Lapidus and van Frankenhuijsen, which we call *lattice approximation*, one sees that the complex dimensions lack translational symmetry, and that they posses a structure which Lapidus and van Frankenhuijsen have

called quasiperiodic. This is described further in Chapter 3. For a complete description of the structure of complex dimensions for self-similar fractal strings, we refer to Chapters two and three in [LvF06]. In Chapter 2, we describe some of the structure which is most relevant to this thesis.

To address Problem 1, we work towards computing the diffraction pattern of the complex dimensions in the nonlattice case. The recipe for the mathematical diffraction pattern of an infinite discrete subset $X \subset \mathbb{R}^n$ mentioned above, which was developed by A. Hof in [Hof95], and in his Ph.D. thesis, begins with writing an analytical representation of X called the *Dirac comb* of X:

$$\mu = \sum_{x \in X} \delta_x,$$

where δ_x is the Dirac delta distribution. The next step is to compute the *autocorrelation measure* γ from μ, which is a distribution that quantifies the frequency of interpoint distance vectors per unit volume. Finally, the *diffraction measure* is defined as the Fourier transform $\hat{\gamma}$ of the autocorrelation measure. This is a standard way to compute diffraction patterns of infinite discrete subsets of $\mathbb{R}^n$, and it will be used in this thesis. In [Lag00], Lagarias computes the diffraction pattern of an infinite discrete subset whose structure is similar to the structure of the complex dimensions in the lattice case.

Definition 1.1.2 (Lagarias, [Lag00]) *An ideal crystal (or perfect crystal) in $\mathbb{R}^n$ is any set Λ that consists of a finite number of translates of a full rank lattice L in $\mathbb{R}^n$. That is, $\Lambda = L + F$, where F is a finite set.*

Theorem 1.1.3 (Lagarias [Lag00]) *An ideal crystal $\Lambda = L + F$, in which L is a full rank lattice in $\mathbb{R}^n$ and F is a finite set, has a unique autocorrelation measure given by the*

tempered distribution

$$\gamma = \frac{1}{|\det(L)|} \sum_{a \in F} \sum_{b \in F} \left(\sum_{x \in L} \delta_{x+(a-b)} \right).$$

Its Fourier transform $\hat{\gamma}$ *is given by the tempered distribution*

$$\hat{\gamma} = \frac{1}{|\det(L)|^2} \sum_{y \in L^*} \left(\sum_{a \in F} \sum_{b \in F} e^{2\pi i (a-b) y} \right) \delta_y.$$

Like an ideal crystal in the plane, the complex dimensions in the lattice case resemble an ideal crystal. This is because they are composed of finitely many translations in the plane of a lattice L, except that L is not of full rank, i.e.,

$$L = \text{span}\{b\},$$

where $\{b\}$ is a basis for $\mathbb{R}^2$. In Chapter 4, following the work of Lagarias on diffraction by ideal crystals, but using a slightly modified recipe for Hof's formulation of mathematical diffraction, the diffraction measure of the complex dimensions of a lattice self-similar fractal string is computed. Via lattice approximation, this is a step forward in the direction of obtaining a diffraction measure in the nonlattice case, which is planned work to be addressed in Chapter 5.

1.2 Main results

In this thesis, Problem 1 is addressed using a measure theoretic idealization of kinematic diffraction. The formula for both the autocorrelation measure, and the diffraction measure for any lattice self-similar fractal string is given. We note that the complex roots in the lattice case may be realized as a degenerate analog of ideal crystals in the plane, which have a pure point diffraction pattern as shown by Lagarias. The degeneracy of the

complex roots in the lattice case manifests itself in the diffraction measure as an integral over $\mathbb{R}$. In [BMP00], the authors present an infinite discrete subset of Euclidean space, that is uniformly discrete, but not relatively dense, and that has a pure point diffraction pattern. The complex roots in the lattice case, while 'close' to a Delone set like the structure presented in the latter paper, the diffraction pattern is not purely discrete. Therefore, our results support the need to relax the diffractivity condition in the work of developing mathematical models for quasicrystals. I am also interested in exploring the quasiperiodic pattern of the set of complex dimensions in the nonlattice case. The work done in this thesis to investigate this structure is based on lattice approximation. However, the procedure required a practical way of obtaining simultaneous Diophantine approximations. Lapidus and van Frankenhuijsen have suggested to use a lattice basis reduction algorithm, called LLL, developed by Lenstra, Lenstra Jr., and Lovász in [LLL82], which can be used to compute simultaneous Diophantine approximations.

Through lattice approximation, the complex dimensions in the nonlattice case are approximated by infinite discrete subsets of the plane, that are slightly more general than ideal crystals, but only because of a degeneracy in one component: The complex dimensions in the lattice case lie on finitely many vertical lines, where on each line they are separated by a positive number, they can be expressed as finitely many translations of a 1-dimensional lattice in the plane, which is not of full rank. In Theorem 4.2.3, the autocorrelation measure for any ideal crystal is given, and then the diffraction measure is obtained by applying the classical Poisson summation formula for full rank lattices. By exploiting the existing periodic structure of the complex dimensions in the lattice case, and

by considering a more suitable average shape, the autocorrelation measure for any lattice self-similar fractal string is computed. Then, by developing an extension of the Poisson summation formula to the setting of locally compact abelian groups, we get the diffraction measure too.

It was briefly mentioned above that the set of complex dimensions of a lattice self-similar fractal string lie on finitely many vertical lines, separated by a positive number p (see [LvF00] or [LvF06]). To be more precise, for a given self-similar fractal string with oscillatory period p, there exist complex numbers

$$\omega_1, \ldots, \omega_u$$

such that its set of complex dimensions are given by the set

$$X = \{\omega_j + inp : \ n \in \mathbb{Z}, j = 1, \ldots, u\}. \tag{1.1}$$

For $1 \leq j, k \leq u$, define $\omega_{j,k} := \omega_j - \omega_k$, and let

$$H_{j,k} = \{\omega_{j,k} + inp : n \in \mathbb{Z}\}. \tag{1.2}$$

As a convention, we use H to denote $H_{j,k}$ when $j = k$.

Lemma 1.2.1 (Lapidus, van Frankenhuijsen, Voskanian) *Let $\mathcal{L}$ be a lattice self-similar fractal string with oscillatory period p. If u is the number of vertical lines describing the set of complex dimensions of $\mathcal{L}$, $1 \leq j, k \leq u$ and $\phi \in \mathscr{S}(\mathbb{R}^2)$, where $\mathscr{S}(\mathbb{R}^2)$ denotes the set of all Schwartz functions on $\mathbb{R}^2$, then*

$$\sum_{a \in H_{j,k}} \phi(a) = \frac{1}{p} \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \hat{\phi}\left(\xi, np^{-1}\right) e^{2\pi i(\xi \operatorname{Re}(\omega_{j,k}) + np^{-1} \operatorname{Im}(\omega_{j,k}))} \, d\xi.$$

Consequently,

$$\sum_{a \in H} \phi(a) = \frac{1}{p} \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \phi\left(\xi, np^{-1}\right) \, d\xi.$$

Corollary 1.2.2 (Lapidus, van Frankenhuijsen, Voskanian)

$$\sum_{a \in H_{j,k}} \hat{\phi}(a) = \frac{1}{p} \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \phi\left(\xi, np^{-1}\right) e^{2\pi i(\xi(\mathrm{Re}(\omega_{j,k})) + np^{-1}(\mathrm{Im}(\omega_{j,k})))} \, d\xi.$$

Theorem 1.2.3 (Lapidus, van Frankenhuijsen, Voskanian) *Let $\mathcal{L}$ be a lattice self-similar fractal string with oscillatory period p, and let X be its set of complex dimensions, as given in (1.2), so that the difference set $X - X$ is given by*

$$X - X = \bigcup_{1 \leq j,k \leq u} H_{j,k},$$

where the sets $H_{j,k}$ are given by (1.2).

Then, X has a unique autocorrelation measure γ, given by the tempered distribution

$$\gamma = \sum_{a \in H} \frac{u}{2bp} \delta_a + \sum_{\substack{\text{distinct } H_{j,k} \\ j \neq k}} f(j,k) \left(\sum_{a \in H_{j,k}} \frac{1}{2bp} \delta_a \right),$$

where $f(j,k)$ is the multiplicity of the set $H_{j,k}$. Moreover, its Fourier transform is the tempered distribution

$$\hat{\gamma} = \frac{1}{2bp^2} \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \omega\left(\xi, np^{-1}\right) \delta_{(\xi, np^{-1})} \, d\xi,$$

where

$$\omega\left(\xi, np^{-1}\right) := u + \sum_{\substack{\textit{distinct } H_{j,k} \\ j \neq k}} f_{j,k} e^{2\pi i(\xi(\mathrm{Re}(\omega_{j,k})) - np^{-1}(\mathrm{Im}(\omega_{j,k})))}.$$

Chapter 2

Background and Motivations

The discovery of quasicrystals in the early 1980s, by Dan Shechtman led to the development of mathematical models for these materials so that we may better understand this new metallic phase. For a given atomic structure, each individual atom serves as an obstacle for incoming electromagnetic waves, e.g., x-rays. The net effect is that each atom becomes a point source of radiation. The outgoing waves interfere with one another, resulting in an image called a diffraction pattern. Quasicrystals have an atomic structure with enough 'order' to scatter waves in such a way that the diffraction pattern consists of bright spots, but at the same time, the pattern lacks translational symmetry. The question still remains: exactly how much order in the atomic makeup of a physical structure is necessary to produce a diffraction pattern consisting of bright spots. My contribution to this effort is through addressing (only in the lattice case) Problem 1 by Lapidus and van Frankenhuijsen, which asks if the quasiperiodic pattern of the complex dimensions of a nonlattice self-similar fractal string can be understood in terms of a mathematical

quasicrystal. In this chapter, we tell the story of the discovery of quasicrystals, which is the main motivation for the work done in this thesis, and then give a complete overview of Problem 1 in the context of a mathematical idealization of diffraction.

2.1 Quasicrystals and mathematical diffraction

We begin with an introduction to the phenomenon of diffraction, and then give a brief recount of Shechtman's discovery. Then, we give a brief introduction to the theory of mathematical quasicrystals, and describe a certain mathematical idealization of diffraction, developed by A. Hof in [Hof95].

2.1.1 The discovery of quasicrystals and forbidden symmetry

The discovery in 1982 of a nonperiodic material with a pure point diffraction pattern by Dan Shechtman changed crystallography in a significant way, fueling an investigation into what long range order actually means. That is, exactly how much order in the atomic structure of a material is necessary to produce a pure point diffraction pattern. Not everyone in the crystallographic community was immediately happy to dispense with the definition of a crystal. From the day Shechtman published [SBGC84] on his findings on quasicrystals in 1984, he experienced hostility. Crystals are modeled as discrete structures, generated by a list of linearly independent finite translations. Because discreteness requires that spacing between lattice points have a lower bound, there can only be finitely many rotational symmetries. The crystallographic restriction says that not all finite groups are compatible with a discrete lattice; in the plane or space. Because of the rotational sym-

metries present in the diffraction pattern observed by Shechtman, he knew that the atomic structure of the material he had processed could not be generated by independent finite translations. The following lemma and theorem are from [BG13]. We refer to the reference for the proof of Lemma 2.1.1, but provide part of the proof for Theorem 2.1.2.

Lemma 2.1.1 *Consider a lattice $\Gamma \subset \mathbb{R}^d$. If $R \in O(d)$ satisfies $R\Gamma \subset \Gamma$, one has $R\Gamma = \Gamma$. Then the corresponding characteristic polynomial $P(\lambda) = \det(R - \lambda I)$ has integer coefficients only, so that $P(\lambda) \in \mathbb{Z}[\lambda]$.*

Theorem 2.1.2 (The crystallographic restriction) *A lattice $\Gamma \subset \mathbb{R}^d$ with $d = 2$ or $d = 3$ can have n-fold rotational symmetry at most for $n = \{1, 2, 3, 4, 6\}$.*

Proof. For $d = 2$, a rotation by an angle θ is given by the matrix

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

with characteristic polynomial

$$f(t) = \det R_\theta - tI) = t^2 - 2\cos(\theta)t + 1,$$

where I is the identity map. By Lemma 2.1.1, $R_\theta\Gamma = \Gamma$, which means that $2\cos(\theta) \in \mathbb{Z}$. Then, $|\cos(\theta)| \in \{0, 1/2, 1\}$, which gives the claim for $n = 2$. See [BG13] for a proof of the claim when $n = 3$. □

Scientists around the world had quickly replicated the discovery and, in 1992, the International Union of Crystallography accepted that quasicrystals exist, and altered their definition of crystal: "a substance in which the constituent atoms, molecules or ions

are packed in a regularly ordered, repeating three-dimensional pattern". In 2011, Dan Shechtman was awarded the Nobel Prize in Chemistry for the discovery of quasicrystals.

Not only have quasicrystals introduced a new extension of solid state physics, but there has been an immense amount of theoretical and experimental research effort to determine the atomic structure of such materials, and to find mechanisms that explain how they form. They have given rise to a mathematical theory motivated by the desire to model these new structures, which are somewhere in between periodic and amorphous. The basic model is an infinite discrete subset of n-dimensional Euclidean space, which, when under a certain mathematical analog of diffraction which is described in the next section, has a pure point diffraction pattern. A well known construction of such a structure is via the cut and project method. These are called *cut-and-project sets*, or *model sets*, and under mild conditions every cut-and-project set is a Delone set. Even though there does not yet exist a unifying definition for mathematical quasicrystals, cut-and-project sets are widely considered to be representative. Cut and project sets are defined as follows. Let $1 \leq d < k$ be integers, and let E be a d-dimensional subspace of $\mathbb{R}^k$, and $F \subset \mathbb{R}^k$ a subspace complementary to E. The subspace E is called the *physical space*, the subspace F is called the *internal space*, and $\mathbb{R}^k$ is called the *total space*. Write π for the projection onto E with respect to the decomposition

$$\mathbb{R}^k = E + F.$$

Choose a set $W \subset F$, and define

$$S = W + E.$$

The set W is called the window, and S is called the strip. For each $s \in \mathbb{T}^k$, we define the

cut-and-project set $Y_s \subset E$ by

$$Y_s = \pi(S \cap \mathbb{Z}^k + s).$$

A more general construction involves replacing $\mathbb{Z}^k$ with any full rank lattice in $\mathbb{R}^k$, which is defined as the integer span of a basis for $\mathbb{R}^k$. See the appendix for an introduction to basic lattice theory. Typically, a cut-and-project set Y in $\mathbb{R}^n$ will be *aperiodic*, meaning that there does not exists a ball B centered at the origin for which Y B has translational symmetry in n linearly independent directions. Under slightly mild conditions, any cut-and-project set is a Delone set. We refer to Lemma 3.2.2 from [Hay16] for the specifics, and for a proof that, under these assumptions, any cut-and-project has a pure point mathematical diffraction pattern. This is why a cut-and-project set is a good model for a mathematical quasicrystal. There are more general models for quasicrystals, which do not require the presence of a pure point diffraction pattern.

As will be shown in chapters 3 and 4, the set of complex dimensions of any lattice self-similar fractal string consists of an aperiodic infinite discrete subset of the complex plane, in the sense described above, and, posses an interesting diffraction pattern. Moreover, the set of complex dimensions do not form a Delone set, but actually a more general ideal crystal, in which the lattice Λ in 1.1.2 is not of full rank, that is, a subset of $\mathbb{R}^n$ that is the integer span of $k < d$ linearly independent vectors of $\mathbb{R}^d$.

2.1.2 Mathematical diffraction

Since its discovery in the early 1900s, x-ray diffraction has been used to determine the atomic structure of solid materials. Diffraction is the scattering that takes place when

waves meet an obstacle. In 1912, Max von Laue showed that for a solid material, its atomic structure acts as a large finite collection of obstacles for x-rays. Each atom becomes a point source of radiation. The interference pattern from the reflected x-rays is called the x-ray diffraction pattern of the material. By studying this pattern, we can determine its atomic structure. Today, x-ray diffraction is a well understood technique for studying the atomic structures of solid materials. In this section, we present a mathematical idealization of diffraction developed by A. Hof in [Hof95]. In Chapter 4, we use this formalization to compute the diffraction pattern of the complex dimensions of a lattice self-similar fractal string. In Chapter 5, I give some ideas for future work, including our strategy for obtaining the diffraction pattern in the nonlattice case, which addresses Problem 1.

We begin by deriving an important diagram, which may be regarded as a description of kinematic diffraction. For any potential function $\rho \in L^1(\mathbb{R}^n)$, we define $\tilde{\rho}$ as the mapping $x \to \tilde{\rho}(x) := \overline{\rho(-x)}$. Then,

$$\begin{aligned}
\hat{\tilde{\rho}}(\xi) &= \int \tilde{\rho}(x) e^{-2\pi i \xi x} \, dx \\
&= \int \overline{\rho(-x) e^{2\pi i \xi x}} \, dx \\
&= \overline{\int \rho(x) e^{-2\pi i \xi x} \, dx} \\
&= \overline{\hat{\rho}(\xi)}.
\end{aligned}$$

As a result of the *Fourier convolution theorem*, see Theorem IX.3 in [RS81], and from the relation $\hat{\tilde{\rho}}(\xi) = \overline{\hat{\rho}(\xi)}$ established above, we get

$$\begin{aligned}
\widehat{\rho * \tilde{\rho}} &= \hat{\rho}\overline{\hat{\rho}} \\
&= |\hat{\rho}|^2.
\end{aligned}$$

And so for any integrable function ρ, there is an associated commutative diagram

$$\begin{array}{ccc} \rho & \xrightarrow{\quad * \quad} & \rho * \tilde{\rho} \\ \downarrow{\scriptstyle \mathscr{F}} & & \downarrow{\scriptstyle \mathscr{F}} \\ \hat{\rho} & \xrightarrow{\quad |\cdot| \quad} & |\hat{\rho}|^2 \end{array} \tag{2.1}$$

which is commonly referred to as a *Wiener diagram*. Kinematic diffraction is the diagonal map $\rho \mapsto |\hat{\rho}|^2$ in (2.1). That is, the intensities of the diffraction pattern are described by the modulus squared of the Fourier coefficients of the potential ρ associated to the solid material. The theory of mathematical diffraction presented in this section requires formulating the analog of (2.1) for *translation bounded measures*. The following derivation from [Hof95] describes a recipe for mathematical diffraction, which is used in Chapter 4 to compute the diffraction of the complex dimensions of a lattice self-similar fractal string.

The basic mathematical object is a discrete set $X \subset \mathbb{R}^n$, which is interpreted as an infinite limit of the atomic configuration of some hypothetical physical structure. Assume that X is a countable subset of n-dimensional Euclidean space that is uniformly locally finite, i.e., for every compact set $K \subset \mathbb{R}^n$ there exists a constant a_K such that the number of points of X in the set $K + x$, given by

$$\{y \in \mathbb{R}^d : y - x \in K\},$$

is bounded by a_K, uniformly in x. The following definition provides an analytic representation for the set X.

Definition 2.1.3 *A continuous linear functional μ on the space $\mathscr{C}_c(\mathbb{R}^n)$ of compactly supported complex valued functions on $\mathbb{R}^n$ is called a measure. By continuity, we mean that for*

any compact subset $K \subset \mathbb{R}^n$, there is a constant c_K such that

$$|\mu(f)| \leq c_K \sup_{x \in \mathbb{R}^n} |f(x)|,$$

for all complex bounded functions with support in K.

Definition 2.1.4 *A positive measure is a measure μ such that if $f \in \mathscr{C}_c(\mathbb{R}^n)$ with $f \geq 0$, then $\mu(f) \geq 0$.*

Theorem 2.1.5 (Riesz-Markov representation theorem) *There is a one-to-one correspondence between positive measures and positive Radon measures; for each positive measure ψ, there is a unique Borel measure μ such that $\psi(f) = \int f \, d\mu$, for all $f \in \mathscr{C}_c(\mathbb{R}^n)$.*

The starting point for obtaining the mathematical diffraction of the set X is to replace it by a measure called the *Dirac comb of X* given by

$$\mu := \sum_{x \in X} \delta_x,$$

where $\delta_x \in \mathscr{D}'(\mathbb{R}^d)$ is the Dirac distribution at x, defined by the property

$$\langle \delta_x, \phi \rangle = \phi(x)$$

for all test functions $\phi \in \mathscr{D}(\mathbb{R}^n)$. Following the Wiener diagram (2.1), we proceed by computing the *autocorrelation measure*: for every positive number L, let C_L denote the closed cube of side L centered around the origin. The restriction of a measure μ to C_L is denoted by μ_L. Since μ_L has compact support,

$$\gamma^L := L^{-n} \mu_L * \tilde{\mu}_L$$

is well-defined. The space of measures is given a topology in which a sequence of measure $\{\mu_n\}$ converges to μ if $\mu_n(f) \to \mu_(f)$ for all $f \in \mathscr{C}_c(\mathbb{R}^n)$.

Definition 2 *Every limit point of γ^L as $L \to \infty$ is called an autocorrelation of μ, and any autocorrelation is a measure.*

In our setting, computing the autocorrelation measure is much simpler. Let A be the set $\{x - y : x, y \in X\}$ of "interatomic vectors". We will assume that A is locally finite. Note that if $a \in A$, then $-a \in A$ too. For $a \in A$ and positive L, let $N_L(a)$ be the number of occurrences of a in the cube C_L:

$$N_L(a) := \{x \in X : x \in C_L \text{ and } x - a \in X \cap C_L\}.$$

Assume that for all $a \in A$ the limit

$$n_a := \lim_{L\to\infty} L^{-d} N_L(a) \tag{2.2}$$

exists and is positive. Now, we have

$$\mu_L * \tilde{\mu}_L = \sum_{x,y\in X\cap C_L} = \sum_{a\in A} N_L(a)\delta_a.$$

Next, choose $N > 0$ and a complex valued continuous function f on $\mathbb{R}^n$ with support in C_N, so that $f(a) = 0$ if $a \in C_N$. Then

$$\gamma^L(f) = L^{-n} \sum_{a\in Aa\in C_N} N_L(a)\delta_a(f).$$

Since the above sum is finite, (2.2) gives

$$\lim_{L\to\infty} \gamma^L(f) = \sum_{a\in A} n_a\delta_a(f).$$

Since N and f were chosen arbitrarily, we get that μ has a unique autocorrelation measure γ given by

$$\gamma := \sum_{a\in A} n_a\delta_a.$$

In Chapter 4, working in the setting described above, we compute the autocorrelation measure of the complex dimensions of a lattice self-similar fractal string. From standard references on diffraction [Cow75], diffraction by an infinite system is described by the Fourier transform of the autocorrelation measure, $\hat{\gamma}$, which is also consistent with the Wiener diagram. We also develop an extended version of the Poisson summation formula to compute the Fourier transform to describe the diffraction pattern.

2.2 Fractal strings and complex dimensions

Lapidus and van Frankenhuijsen present a way to study the Minkowski measurability, and more generally, the oscillations in the geometry and the spectrum of fractal strings, through the notion of *complex dimensions* (see [LvF06] and [LRŽ17] in particular.) This dissertation concerns the structure of the complex dimensions of a *nonlattice self-similar fractal string*, defined in Section 2.2.1. Lapidus and van Frankenhuijsen have shown that the complex dimensions of self-similar fractal strings arise as the complex roots of a class of exponential polynomials called *Dirichlet polynomials*, which is defined in Section 2.2.2. In Section 2.2.3, we state a theorem by Lapidus and van Frankenhuijsen that provides a method to approximate the complex dimensions of a nonlattice self-similar fractal string; in Chapter 3, we explain how this approximation procedure shows that the complex dimensions have a quasiperiodic structure, as is discussed in detail in Chapter 3 of [LvF06]. Problem 1 from [LvF06], stated by Lapidus and van Frankenhuijsen, asks if the quasiperiodic pattern of the complex dimensions of a nonlattice self-similar fractal string can be regarded as a mathematical quasicrystal.

2.2.1 Self-similar fractal strings and complex dimensions

Definition 2.2.1 *A fractal string $\mathcal{L}$ is a bounded open subset Ω of the real line.*

Any open subset Ω of the real line is a union of at most countably many disjoint open intervals, which we will call *the intervals of* $\mathcal{L}$. See Theorem 1.3 in [SS09]. To any fractal string $\mathcal{L}$, we will associate a sequence of positive numbers

$$\ell_1, \ell_2, \ell_3, \dots$$

corresponding to the lengths of the intervals of $\mathcal{L}$, listed according to multiplicity. Since Ω has finite volume, we know that

$$\sum_{j=1}^{\infty} \ell_j < \infty,$$

so that the lengths can be ordered nonincreasingly, i.e.,

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq 0.$$

Definition 2.2.2 *For a fractal string $\mathcal{L}$ with bounded open set Ω, we define the volume of the inner tubular neighborhood of $\partial\Omega$ with radius ϵ, denoted by $V(\epsilon)$, as*

$$V(\epsilon) = \mathrm{vol}_1\{x \in \Omega \ : \ d(x, \partial\Omega) < \epsilon\}.$$

Example 2.2.3 *The complement in $[0,1]$ of the usual Cantor set is a fractal string with lengths,*

$$\frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \dots$$

The volume of the inner tubular neighborhood of $\partial\Omega$ with radius $0 < \epsilon \leq 1/2$ is given by

$$\begin{aligned} V(\epsilon) &= 2\epsilon \cdot (2^n - 1) + \sum_{k=n}^{\infty} 2^k 3^{-(k+1)} \\ &= 2\epsilon \cdot (2^n - 1) + \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k \\ &= 2\epsilon \cdot 2^n + \left(\frac{2}{3}\right)^n - 2\epsilon, \end{aligned}$$

where n is such that $3^{-n} \geq 2\epsilon > 3^{-(n+1)}$.

Definition 2.2.4 *The dimension of a fractal string $\mathcal{L}$ with associated bounded open set Ω, denoted $D = D_{\mathcal{L}}$, is defined as the inner Minkowski dimension of Ω, i.e.,*

$$D = D_{\mathcal{L}} = \inf\{\alpha \geq 0 \ : \ V(\epsilon) = O(\epsilon^{1-\alpha}) \text{ as } \epsilon \to 0^+\}.$$

Let $\mathcal{L}$ be a fractal string with dimension D. Because the infinite sum $\sum_{j=1}^{\infty} \ell_j^{\sigma}$ converges when $\sigma = 1$, it follows from standard references that the (generalized) Dirichlet series

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s$$

defines a holomorphic function for $\mathrm{Re}(s) > 1$ (see [Ser73]).

Definition 2.2.5 *The abscissa of convergence of the series*

$$\sum_{j=1}^{\infty} \ell_j^s$$

is defined by

$$\sigma = \inf\{\alpha \in \mathbb{R} : \sum_{j=1}^{\infty} \ell_j^{\alpha} < \infty\}.$$

Theorem 2.2.6 by Lapidus and van Frankenhuijsen in [LvF06], stated below, establishes an important connection between σ and D, and leads to the following definition.

Theorem 2.2.6 (Lapidus, van Frankenhuijsen, [LvF06]) *Suppose that $\mathcal{L}$ has infinitely many lengths. Then the abscissa of convergence of the geometric zeta function of $\mathcal{L}$ coincides with D, the Minkowski dimension of ∂L.*

Definition 2.2.7 *Let $\mathcal{L}$ be a fractal string. If the geometric zeta function $\zeta_{\mathcal{L}}$ has a meromorphic extension to all of $\mathbb{C}$, we call*

$$\mathcal{D} = \{\omega \in \mathbb{C} \ : \ \zeta \text{ has a pole at } \omega\}$$

the set of complex dimensions of $\mathcal{L}$.

We are now in a position to define the central object from Problem 1. Let $I = [a, b]$. We construct a *self-similar fractal string* $\mathcal{L}$ as follows. Let

$$\Phi_1, \Phi_2, \ldots, \Phi_N$$

be N *contraction similitudes* mapping I into I, i.e.,

$$|\Phi_j(x) - \Phi_j(y)| = r_j|x - y|,$$

for all $x, y \in I$, with respective scaling ratios

$$1 > r_1 \geq r_2 \geq \cdots \geq r_N > 0.$$

Assume that the images of I under the contraction similitudes do not overlap, except possibly at the endpoints, and that

$$\sum_{j=1}^{N} r_j < 1.$$

Subdivide I into the pieces $\Phi_j(I)$. See Figure 2.1. The remaining pieces in between are called the *first intervals* of length $\ell_k = g_k L$, for $k = 1, \ldots, K$, where the scaling factors

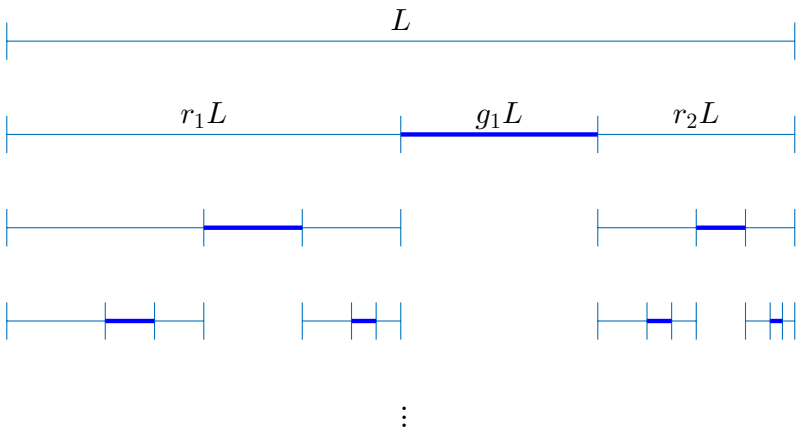


Figure 2.1: The construction of a self-similar string with two scaling ratios r_1, r_2 and one gap g_1.

$g_1, \ldots, g_K$ satisfy

$$1 > g_1 \geq \cdots \geq g_K > 0,$$

and

$$\sum_{j=1}^{N} r_j + \sum_{k=1}^{K} g_k = 1.$$

We refer to the quantities $g_1, \ldots, g_K$ as the *gaps* of the self-similar fractal string. Now, repeat this process with each of the remaining intervals $\Phi_j(I)$. The limiting object will be the self-similar fractal string $\mathcal{L}$, consisting of intervals of length ℓ_n given by

$$r_{\nu_1} \cdots r_{\nu_q} g_k L,$$

for $k = 1, \ldots, K$ and all choices of $q \in \mathbb{N}$ and $\nu_1, \ldots, \nu_q \in \{1, \ldots, N\}$.

Theorem 2.2.8 (Lapidus, van Frankenhuijsen, [LvF06]) *Let $\mathcal{L}$ be a self-similar fractal string. Then the geometric zeta function of $\mathcal{L}$ has a meromorphic continuation to the*

whole complex plane, given by

$$\zeta_{\mathcal{L}}(s) = \frac{L^s \sum_{k=1}^{K} g_k^s}{1 - \sum_{j=1}^{N} r_j^s}.$$

Here, $L = \zeta_{\mathcal{L}}(1)$ is the total length of $\mathcal{L}$, which is also the length of I, the initial interval from which $\mathcal{L}$ is constructed.

Corollary 2.2.9 *Let $\mathcal{L}$ be a self-similar fractal string constructed with a single gap size $g_1 = \cdots = g_K$. Then the complex dimensions is the set of solutions of the equation*

$$\sum_{j=1}^{N} r_j^{\omega} = 1, \quad \omega \in \mathbb{C}, \tag{2.3}$$

with the same multiplicity.

In general, when the gaps have different sizes, the complex dimensions are among the set of solutions of (2.3), and each complex dimension has a multiplicity of at most that of the corresponding solutions.

Remark 2.2.10 *The length L of the initial interval I of a self-similar fractal string may be normalized so that the first length equals 1 by choosing*

$$L = g_1^{-1},$$

where g_1 is the largest gap. This does not affect the complex dimensions.

In the following well-known example from [LvF06], we show that the Cantor string in 2.2.3 is actually a self-similar fractal string, defined by two contraction similitudes.

Example 2.2.11

$$\Phi_1(x) = \frac{1}{3}x, \quad \textit{and} \quad \Phi_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

These are two contraction similitudes mapping $[0,1]$ *to* $[0,1]$*, with scaling ratios*

$$r_1 = r_2 = \frac{1}{3},$$

and first interval $g_1 = 1/3$*. The self-similar string with total length* 1 *and these scaling ratios is called the Cantor string. It consists of lengths* 3^{-n} *with multiplicity* 2^n*,* $n \geq 0$*. The geometric zeta function of this fractal string is*

$$\zeta(s) = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}.$$

By Corollary 2.2.9, the complex dimensions are exactly the roots of the equation

$$3^x \cdot e^{iy \log 3} = 2.$$

We get that

$$x = \log_3 2, \quad y = \frac{2\pi}{\log 3} n \ (n \in \mathbb{Z})$$

and the complex dimensions are given by the set

$$\{\log_3 2 + inp : n \in \mathbb{Z}\},$$

where $p = \frac{2\pi}{\log 3}$*. See Figure 2.2.*

2.2.2 An introduction to Dirichlet polynomials

Problem 1 by Lapidus and van Frankenhuijsen concerns the quasiperiodic structure of the complex dimensions of a nonlattice self-similar fractal string. In the previous section, we saw that the complex dimensions of a self-similar fractal string are among the roots of an associated exponential polynomial. These polynomials, defined below, are called *Dirichlet*

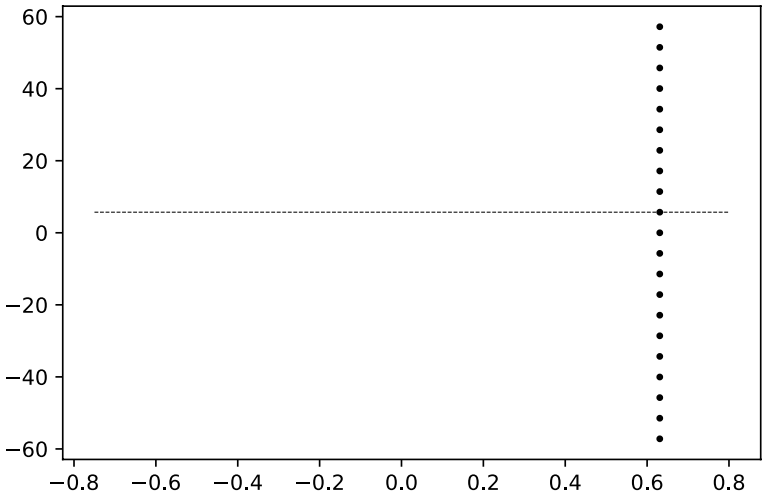


Figure 2.2: The complex dimensions of the Cantor string.

polynomials. There is a dichotomy in the set of all Dirichlet polynomials, in which every one is either *lattice* or *nonlattice*. This induces the same dichotomy in the set of all self-similar fractal strings, i.e., a self-similar fractal string is lattice if its corresponding Dirichlet polynomial is lattice, and nonlattice otherwise. Therefore, we focus on a more general problem:

Problem 3 *Can the quasiperiodic pattern of the set of complex roots of a nonlattice Dirichlet polynomial with integer multiplicities be understood in terms of a suitable (generalized) quasicrystal, or quasiperoidic tiling?*

Definition 4 *For $N+1$ scaling ratios $r_0 > r_1 > \cdots > r_N > 0$ and multiplicities $m_j \in \mathbb{C}$ for $1 \leq j \leq N$, an exponential polynomial in the complex variable s of the form*

$$f(s) = \sum_{j=0}^{N} m_j r_j^s \tag{2.4}$$

is called a Dirichlet polynomial.

For the purposes of studying the complex roots of a Dirichlet polynomial, it is assumed without any loss of generality that $r_0 = 1$ and $m_0 = -1$ so that (2.4) reduces to

$$f(s) = 1 - \sum_{j=1}^{N} m_j r_j^s, \tag{2.5}$$

and that $1 > r_1 > \cdots > r_N > 0$.

Definition 5 *Let $f(s)$ be a Dirichlet polynomial. If the rank of the additive group*

$$G = \sum_{j=1}^{N} \log(r_j)\mathbb{Z}$$

is equal to 1, $f(s)$ is said to be lattice. Otherwise, it is called nonlattice

Remark 2.2.12 *It is straightforward to check that a Dirichlet polynomial $f(s)$ is lattice if and only if there exists a positive real number r, which we call the generator of f, and integers $k_1, \ldots, k_N$ such that $r_j = r^{k_j}$ for $j = 1, \ldots, N$. Furthermore, if for a Dirichlet polynomial $f(s)$, we define $w_j = -\log r_j$ for $1 \leq j \leq N$, then $f(s)$ is nonlattice if and only if w_j/w_1 is irrational for some $1 < j \leq N$.*

The structure of the complex roots of a given Dirichlet polynomial is completely described by Theorem 3.6 in [LvF06]. For instance, the set of complex roots of any Dirichlet polynomial is contained in the strip

$$R := \{z \in \mathbb{C} : D_\ell \leq \operatorname{Re} s \leq D\},$$

where D_ℓ and D are the unique real numbers satisfying the equations

$$1 + \sum_{j=1}^{N-1} |m_j| r_j^{D_\ell} = |m_N| r_N^{D_\ell} \quad \text{and} \quad \sum_{j=1}^{N} |m_j| r_j^{D} = 1,$$

respectively, and the inequality $-\infty < D_\ell \leq D \leq 1$. If the multiplicities are positive integers, and $N \geq 2$ or $m_1 > 1$, as is the case with the Dirichlet polynomial associated to any lattice self-similar fractal string, then the complex roots are symmetric about the real axis, the number D defined above is positive, and it is the only real root. We also note that D is simple.

In the lattice case, the complex roots of $f(s)$ lie periodically on finitely many vertical lines, and on each line they are separated by the positive number

$$p = \frac{2\pi}{\log r}$$

called the *oscillatory period of* $f(s)$. More precisely, as described by Lapidus and van Frankenhuijsen in [LvF06], given a lattice Dirichlet polynomial $f(s)$ of the form (2.5) with oscillatory period p, there exist complex numbers

$$\omega_1, \ldots, \omega_u$$

such that the complex roots of $f(s)$ are given by the set

$$X = \{\omega_j + inp : \ n \in \mathbb{Z}, j = 1, \ldots, u\}.$$

For each $1 \leq j \leq u$, the set $\{\omega_j + inp : \ n \in \mathbb{Z}\}$ will be denoted by H_j. The following example illustrates the relatively simple task of computing the complex roots of a lattice Dirichlet polynomial.

Example 2.2.13 *Consider the lattice Dirichlet polynomial*

$$f(s) = 1 - 2^{-s} - 4^{-s}$$

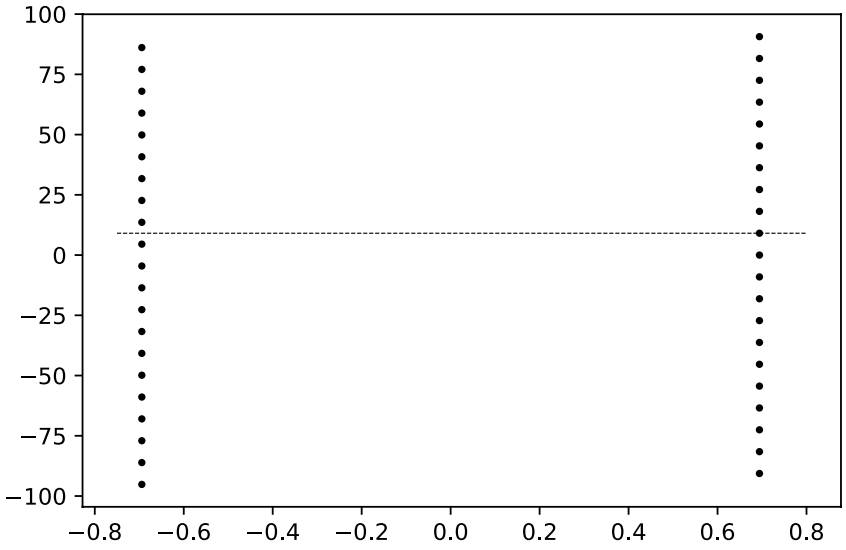


Figure 2.3: Ten positive and negative periods of the set of complex roots of $f(s) = 1 - 2^{-s} - 4^{-s}$, with $p \approx 9.064$.

with scaling ratios $r_1 = 1/2$ *and* $r_2 = 1/4$, *and multiplicities* $m_1 = m_2 = 1$, *which is associated to a lattice self-similar fractal string known as the Fibonacci string. Because* $f(s)$ *is lattice with generator* $r = 1/2$, *we can express it as the polynomial*

$$f(z) = 1 - z - z^2$$

in the complex variable $z = 2^{-s}$. *Then, by a direct computation, the complex roots of* $f(s)$ *are given by the set*

$$X = \{D + inp \ : \ n \in \mathbb{Z}\} \cup \{-D + i\,(n + 1/2)\,p \ : \ n \in \mathbb{Z}\}\,,$$

where $D = \log_2 \phi$ *and* $\phi = (1 + \sqrt{5})/2$. *See Fig. 2.3 for a plot consisting of several periods.*

2.2.3 Lattice approximation

The process for computing the complex roots of a lattice Dirichlet polynomial, as described in Example 2.2.13 cannot be applied in the nonlattice case. While this is not ideal, there is an upshot to this discussion that connects fractal geometry and aperiodic order. Lapidus and van Frankenhuijsen in [LvF06] have established a theorem which gives a procedure to approximate the set of complex roots in the nonlattice case by the sets of complex roots of lattice Dirichlet polynomials. In Chapter 3, we show how this approximation procedure, which we refer to as the *lattice approximation procedure* (or *lattice approximation*), reveals the quasiperiodic pattern in the set of complex roots for a nonlattice Dirichlet polynomial.

Suppose that we are given the nonlattice Dirichlet polynomial

$$f(s) = 1 - \sum_{j=1}^{N} r_j^s \tag{2.6}$$

with positive weights $0 < w_1 < \cdots < w_N$, where

$$\frac{w_j}{w_1} = \log_{r_1} r_j$$

for $j = 1, \ldots, N$. Then, for at least one $j = 2, \ldots, N$, we know that w_j/w_1 is irrational. See Remark 2.2.12. The lattice approximation procedure is based on Dirichlet's Theorem on simultaneous Diophantine approximation, and has a very natural implementation. Before stating the theorem which provides this method of approximation, we give a short preliminary description with $f(s)$ above, which will help to understand the statement of the

theorem. To begin, we attempt to rewrite (2.6) as if it were a lattice Dirichlet polynomial:

$$\begin{aligned} f(s) &= 1 - r_1^s - r_2^s - \cdots - r_N^s \\ &= 1 - (r_1^s)^1 - (r_1^s)^{\log_{r_1} r_2} - \cdots - (r_1^s)^{\log_{r_1} r_N} \end{aligned}$$

Note that unlike 2.2.13, the powers of the complex variable r_1^s are not all rational. By replacing the numbers $\log_{r_1} r_1 = 1$, and $\log_{r_1} r_2 \ldots, \log_{r_1} r_N$ with specific rational approximations q/q, and $k_2/q, \ldots, k_N/q$, respectively, we obtain the lattice Dirichlet polynomial

$$\begin{aligned} \tilde{f}(s) &= 1 - (r_1^s)^{q/q} - (r_1^s)^{k_2/q} - \cdots - (r_1^s)^{k_N/q} \\ &= 1 - (z)^q - (z)^{k_2} - \cdots - (z)^{k_N}, \end{aligned}$$

where $z = r_1^{s/q}$. Note that $\tilde{f}(s)$ has scaling ratios

$$(r_1^{1/q})^q, (r_1^{1/q})^{k_2}, \ldots, (r_1^{1/q})^{k_N},$$

generator $\tilde{r} = r_1^{1/q}$, and oscillatory period $\tilde{p} = -q\frac{2\pi}{\log r_1}$. Theorem 2.2.18 on lattice approximation, stated below, says that for any order of accuracy desired, the set of complex roots of $\tilde{f}(s)$ approximate the set of complex roots of $f(s)$ in an open ball centered at the origin. This is essentially how lattice approximation works.

We conclude this section by stating Dirichlet's theorem on simultaneous Diophantine approximation, which drives the lattice approximation procedure, and then finally stating the theorem on lattice approximation. Theorems 2.2.14, 2.2.15, and Corollary 2.2.16 stated below, which are from [Sch80], provide some of the basics in the theoretical foundations behind approximating real numbers by rational numbers. We give the proof of Theorem 2.2.14, which uses the *pigeon hole principle* stating that if n items are put into m

containers with $n > m$, then at least one container must contain more than one item. The proof of Theorem 2.2.15 theorem is omitted.

Theorem 2.2.14 *Let $\alpha \in \mathbb{R}$, and let $Q > 1$. Then, there exist integers p and q with $1 \leq q < Q$ such that*

$$|q\alpha - p| \leq \frac{1}{Q}.$$

Proof. First we assume that Q is an integer. The $Q + 1$ numbers

$$0, 1, \{\alpha\}, \{2\alpha\}, \ldots, \{(Q-1)\alpha\} \tag{2.7}$$

lie in the interval $[0, 1]$, and we divide this interval into the Q subintervals

$$\left[0, \frac{1}{Q}\right), \quad \left[\frac{1}{Q}, \frac{2}{Q}\right), \quad \ldots, \quad \left[\frac{Q-2}{Q}, \frac{Q-1}{Q}\right), \quad \left[\frac{Q-1}{Q}, 1\right].$$

At least one of these intervals contains two or more of the $Q+1$ numbers (2.7). Then, there exist integers r_1, r_2, s_1, s_2 with $0 \leq r_1, r_2 < Q$ and $r_1 \neq r_2$ such that

$$|(r_1\alpha - s_1) - (r_2\alpha - s_2)| = |(r_1 - r_2)\alpha - (s_1 - s_2)| \leq \frac{1}{Q}.$$

If $r_1 > r_2$, then we put $q = r_1 - r_2$, and $p = s_1 - s_2$. Then, $1 \leq q < Q$ and

$$|q\alpha - p| \leq \frac{1}{Q}.$$

This proves the theorem when Q is an integer. If Q is not an integer, the statement is true for $Q' = [Q] + 1$. That is, there exist integers p, q such that $1 \leq q < Q$ and

$$|q\alpha - p| \leq \frac{1}{Q'} \leq \frac{1}{Q}.$$

Therefore the statement is true for Q. □

Theorem 2.2.15 *Suppose that $\alpha_1, \ldots, \alpha_n$ are n real numbers and that $Q > 1$ is an integer. Then there exist integers $q, p_1, \ldots, p_n$ with $1 \leq q < Q^n$ and*

$$|\alpha_i q - p_i| \leq \frac{1}{Q}$$

for $1 \leq i \leq n$. In particular, when $n = 2$, we get the statement in 2.2.14.

The following corollary guarantees the existence of infinitely many lattice approximations for a given nonlattice Dirichlet polynomials. However, there is still the question of which method is best for obtaining them. When there is only one irrational number to approximate, the method of continued fractions works well. Generally however, we will need to obtain simultaneous Diophantine approximations. Since there is not a perfect analog of the method of continued fractions for obtaining simultaneous Diophantine approximations, we rely on another tool. In [LvF06], the authors suggest using a lattice basis reduction algorithm, known as the LLL algorithm, to compute simultaneous Diophantine approximations. In Chapter 3, we show exactly how this can be done.

Corollary 2.2.16 *Suppose that at least one of the numbers $\alpha_1, \ldots, \alpha_n$ is irrational. Then there are infinitely many n-tuples $p_1/q, \ldots, p_n/q$ with*

$$\left|\alpha_i - \frac{p_i}{q}\right| < \frac{1}{q^{1+1/n}}$$

for $1 \leq i \leq n$.

The following lemma from [LvF06] is just a restatement of Theorem 2.2.15, which is better suited for Theorem 2.2.18 below.

Lemma 2.2.17 *Let $f(s)$ be a Dirichlet polynomial with real numbers $w_1, \ldots, w_N$, and $N \geq 2$. Then for every integer $Q > 1$, there exist integers q and $k_1, \ldots, k_N$ such that*

$$1 \leq q < Q^{N-1} \quad \text{and} \quad |qw_j - k_j w_1| \leq w_1 Q^{-1}$$

for all $j = 1, \ldots, N$.

Theorem 2.2.18 (Lapidus, van Frankenhuijsen, [LvF06]) *Let $f(s)$ be a nonlattice Dirichlet polynomial of the form (2.5) with scaling ratios $1 > r_1 > \cdots > r_N > 0$ and multiplicities $m_1, \ldots, m_N$. Let $Q > 1$, and let q and k_j be as in Lemma 2.2.17. Then, the Dirichlet polynomial*

$$\tilde{f}(s) = 1 - \sum_{j=1}^{N} m_j \left(r_1^{k_j/q}\right)^s$$

is lattice with generator $r_1^{1/q}$ and oscillatory period $p = -q\frac{2\pi}{\log r_1}$. And for every $\epsilon > 0$,

$$|\tilde{f}(s) - f(s)| < \epsilon$$

whenever $|s| < \epsilon CQp$, where

$$C = \frac{\sum_{j=1}^{N} |m_j|}{2\pi} \left(\frac{\sum_{j=0}^{N} |m_j|}{\min\{1, |m_N|\}}\right)^{\frac{-2w_N}{\min\{w_1, w_N - w_{N-1}\}}}.$$

The next chapter is devoted to the exploration of the quasiperiodic structure of the complex dimensions of a nonlattice self-similar fractal string. In Section 2.2.1, we saw that it is enough to study the set of complex roots of a nonlattice Dirichlet polynomial with integer multiplicities. This is because the complex dimensions of any self-similar fractal string are among the set of complex roots of an associated Dirichlet polynomial with integer multiplicities; the Dirichlet polynomial appears in the denominator of the geometric zeta function of the string (see Theorem 2.2.8). The set of complex roots of a lattice Dirichlet

polynomial have a periodic structure. We refer the reader to [LvF06] for more information on the general structure of the set of complex dimensions of a Dirichlet polynomial. In the lattice case, the set of complex roots are computed by rewriting the Dirichlet polynomial as a traditional polynomial in a complex variable. The roots can then be computed using software to approximate the roots, and then by taking logarithms, the desired set of complex roots are obtained.

In the nonlattice case, the set of complex roots cannot be computed explicitly. Rather, they are approximated by the sets of complex roots of lattice Dirichlet polynomials, via an approximation procedure developed by Lapidus and van Frankenhuijsen in [LvF06].

Chapter 3

Exploring the Quasiperiodic Structure Through Diophantine Approximation and Lattice Basis Reduction

In this chapter, we will follow the suggestion from Lapidus and van Frankenhuijsen, and use the LLL algorithm to describe the quasiperiodic structure of the set of complex roots of a nonlattice Dirichlet polynomial. In Chapter 4, using a suitably modified version of the recipe for mathematical diffraction developed by Hof in [Hof95], we compute the diffraction for any lattice self-similar fractal string. In the future, I intend to continue working on Problem 1. Specifically, I would like to compute the diffraction for a nonlattice self-similar fractal string. In Chapter 5, we give some ideas on this work.

3.1 Obtaining simultaneous Diophantine approximations with the LLL algorithm

The lattice approximation procedure is a tool for exploring the quasiperiodic structure of a nonlattice Dirichlet polynomial, but it does require a practical method for obtaining many good simultaneous Diophantine approximations. In [LLL82], the authors present the LLL algorithm, which is a lattice basis reduction algorithm. The following two theorems from [LLL82] make the LLL algorithm a tool for computing simultaneous Diophantine approximations. An implementation using C++ is given in Section A.2.

Theorem 3.1.1 *Let $x_1, x_2, \ldots, x_n$ be a reduced basis for a lattice L, $x_1^*, x_2^*, \ldots, x_n^*$ be its Gram-Schmidt orthogonalization, and let $d(L)$ denote the determinant of L. Then,*

$$|x_j|^2 \leq 2^{i-1} \cdot |x_i^*|^2 \text{ for } 1 \leq j \leq i \leq n,$$

$$d(L) \leq \prod_{i=1}^{n} |x_i| \leq 2^{n(n-1)/4} \cdot d(L),$$

and

$$|x_1| \leq 2^{(n-1)/4} \cdot d(L)^{1/n}.$$

Theorem 3.1.2 *There exists a polynomial-time algorithm that, given a positive integer n and rational numbers $x_1, x_2, \ldots, x_n$, and ϵ satisfying $0 < \epsilon < 1$, which finds integers $p_1, p_2, \ldots, p_n, q$ for which*

$$\left| \frac{p_i}{q} - \alpha_i \right| \leq \frac{\epsilon}{q} \qquad \text{for} \qquad 1 \leq i \leq n,$$

$$1 \leq q \leq 2^{\frac{n(n+1)}{4}} \epsilon^{-n}.$$

Proof. Let L be the lattice or rank $n+1$ spanned by the columns of the $(n+1)\times(n+1)$ matrix

$$\begin{pmatrix} -1 & 0 & \cdots & 0 & -x_1 \\ 0 & -1 & \cdots & 0 & -x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & -x_n \\ 0 & 0 & \cdots & 0 & 2^{\frac{-n(n+1)}{4}}\epsilon^{n+1} \end{pmatrix}$$

By Theorem A.1.12, we obtain, in polynomial time, a reduced basis $y_1, y_2, \ldots, y_{n+1}$. Then by Theorem 3.1.1,

$$|y_1| \leq 2^{\frac{n}{4}} d(L)^{\frac{1}{n+1}} = \epsilon.$$

Since $y_1 \in L$, there exist integers $q, p_1, p_2, \ldots, p_n$ such that

$$y_1 = \left(p_1 - qx_1, \ldots, p_n - qx_n, q2^{\frac{-n(n+1)}{4}}\epsilon^{n+1}\right)^T.$$

If $q = 0$, at least one of the p_i must be nonzero. So, $y_1 = (p_1, p_2, \ldots, p_n, 0)$ satisfies $|y_1| \geq 1$, which is a contradiction. Hence, $q \neq 0$. If $q < 0$, replace y_1 with $-y_1$. Since $q2^{\frac{-n(n+1)}{4}}\epsilon^{n+1} \leq |y_1| < \epsilon$, we have that

$$q < 2^{\frac{n(n+1)}{4}}\epsilon^{-n}.$$

And from the other components of y_1, we obtain

$$\left|x_i - \frac{p_i}{q}\right| < \frac{\epsilon}{q}, \qquad 0 \leq q < 2^{\frac{n(n+1)}{4}}\epsilon^{-n}$$

as required. □

Let us suppose that we want to obtain a simultaneous Diophantine approximation for the set of irrational numbers

$$S = \{\log 2, \log 3, \log 5\}.$$

One can use Theorem A.1.12 to do this. First, we use the continued fraction algorithm to generate a convergent for each of the numbers in S:

$$\begin{array}{lll} \log 2 & 1504/1369 & \text{error} : -1.62\text{e}-07 \\ \log 3 & 1143/1649 & \text{error} : 1.81\text{e}-07 \\ \log 5 & 1603/996 & \text{error} : -1.61\text{e}-07 \end{array}$$

Let $\epsilon = 1/10$, and write the matrix

$$\begin{pmatrix} -1 & 0 & \cdots & 0 & 1504/1369 \\ 0 & -1 & \cdots & 0 & 1143/1649 \\ 0 & 0 & \cdots & -1 & 1603/996 \\ 0 & 0 & \cdots & 0 & 2^{-3}\epsilon^4 \end{pmatrix}$$

We then process this matrix with the LLL algorithm, and upon termination, obtain three rational approximations, with common denominator $466 < 8000$:

$$\begin{array}{ll} 323/466 & \text{error} : 1.41\text{e}-05 < 1/4660 \\ 512/466 & \text{error} : 1.00\text{e}-04 < 1/4660 \\ 750/466 & \text{error} : 4.14\text{e}-06 < 1/4660 \end{array}$$

3.2 The quasiperiodic pattern in the nonlattice case

We now have the tools to apply the lattice approximation procedure. In this section, through lattice approximation, we describe the quasiperiodic pattern of the complex dimensions of a nonlattice self-similar fractal string. Let us begin with a relatively simple

example. Consider the nonlattice Dirichlet polynomial

$$f(s) = 1 - 2^{-s} - 3^{-s}$$

from Section 2.2.3, with scaling ratios $r_1 = 1/2,\ r_2 = 1/3$, multiplicities $m_1 = m_2 = 1$, and period $p = 2\pi/\log 2$. In [LvF06], Lapidus and van Frankenhuijsen call this the 2-3-polynomial. We apply lattice approximation, and begin by writing

$$f(s) = 1 - (2^{-s})^1 - (2^{-s})^{\log_2 3}. \tag{3.1}$$

By Lemma 2.2.17 with $N = 2$, and since $\log_2 3$ is irrational, there exists infinitely many pairs of integers k and $1 \leq q < Q$, such that

$$|\log_2 3 - k/q| \leq (qQ)^{-1}.$$

In this case, we can use the continued fraction algorithm to obtain many approximations that satisfy Lemma 2.2.17. See Figure 3.1. Also, see Section A.2 for an implementation of the continued fraction algorithm in `C`.

We proceed by choosing a convergent to replace $\log_2 3$ in (3.1), which gives a lattice approximation. Say for example we chose the convergent k/q. Then, we have the lattice approximation

$$\tilde{f}(s) = 1 - z^k - z^q,$$

where $z = 2^{-s/q}$. To understand the quality of this approximation, we work backwards and calculate a value of Q such that $1 \leq q < Q$, and

$$\left|\log_2 3 - \frac{k}{q}\right| \leq (qQ)^{-1}.$$

$\approx \log_2 3$	Error
1/1	5.84963e-01
2/1	4.15037e-01
3/2	8.49625e-02
8/5	1.50375e-02
19/12	1.62917e-03
65/41	4.03353e-04
84/53	5.68403e-05
485/306	4.81954e-06
1054/665	9.47061e-08
24727/15601	1.68254e-09
50508/31867	3.28901e-10
125743/79335	6.66425e-11
176251/111202	4.67077e-11
301994/190537	4.88498e-13

Table 3.1: Several convergents to $\log_2 3$, with errors.

Then, according to Theorem 2.2.18, for any $\epsilon > 0$, the set of complex roots of $\tilde{f}(s)$ approximate the set of complex roots of $f(s)$, in the open ball

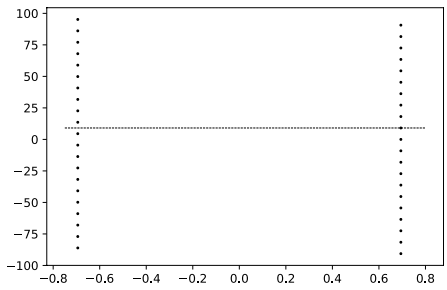
$$\{z \in \mathbb{C} : |z| < \epsilon CQp\}.$$

In Figure 3.1, we plot several periods of the set of complex roots for lattice approximations corresponding to the convergents 2/1, 8/5, 65/41, and 1054/665 to the irrational number $\log_2 3$, with $\epsilon = 1/10$. The plot in Figure 3.1(a) corresponding to 2/1 and $Q = 2$, is not considered to be a very good approximation. In fact, the radius of the ball for which this approximation is valid, up to order ϵ, is roughly 0.0008. What this means is that the unique complex root of $\tilde{f}$ on the real axis is the only one guaranteed by Theorem 2.2.3 to approximates a root of f. To obtain a lattice approximation that is valid for a ball centered at the origin with bigger radius, one chooses a larger value for Q in Lemma 2.2.17.

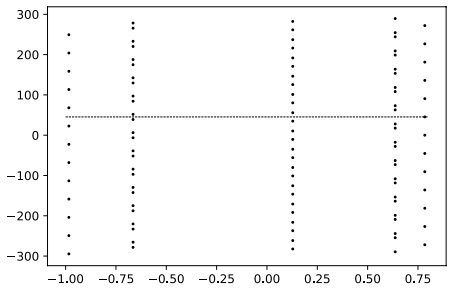
The plot in Figure 3.1(b) corresponding to 8/5 and $Q = 12$, is valid in a ball of radius approximately equal to 0.043. The plots in Figures 3.1(b) and (c) correspond to 65/41 and 1054/665, and have radii of approximations roughly equal to 1.78 and 7233.63, respectively. In Figures 3.2(a) and (b), plots of the set of complex roots for two lattice approximations corresponding to 65/41 and 84/53 are given, respectively. In the shaded region, there is very little difference in the location of the roots. This is consistent with the fact that the lattice approximation corresponding to 84/53 is valid in the ball centered at the origin of radius approximately 12.29.

A given lattice approximation $\tilde{f}$, with common denominator q from the simultaneous Diophantine approximation from which $\tilde{f}$ is determined, has its oscillatory period p directly proportional to q. Also, the radius of the ball for which $\tilde{f}$ is valid is directly proportional to qQ. It follows that the number of periods for which a lattice approximation in Theorem 2.2.18 remains valid tends to infinity as the value of Q in Lemma 2.2.17 tends to infinity. The set of complex roots in the nonlattice case are therefore periodic like in the lattice case, for many periods, except that eventually a new pattern emerges from the next lattice approximation valid for an even larger region. What is interesting is that the set of complex roots of each lattice approximation must incorporate the structure from lattice approximations valid in smaller regions. Lapidus and van Frankenhuijsen call this the quasiperiodic structure of the complex roots in the nonlattice case. We refer to Chapter 3 of [LvF06] for more on the structure of the set of complex roots of nonlattice Dirichlet polynomials and the quasiperiodic structure.

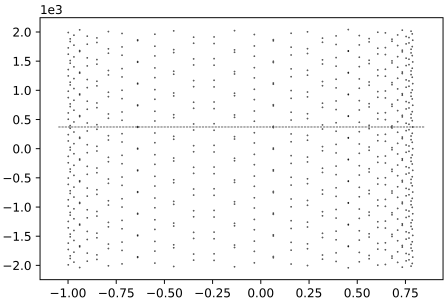
Typically, a lattice approximation that is valid for many periods will be sparse



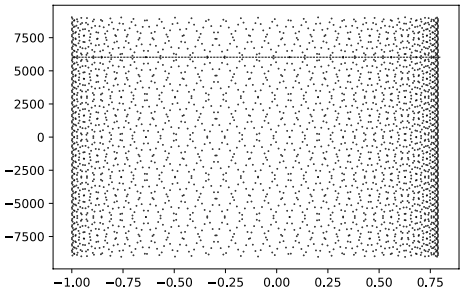
(a) Lattice approximation corresponding to $2/1$, $Q = 2$, and oscillatory period $p \approx 9.06$.



(b) Lattice approximation corresponding to $8/5$, $Q = 12$, and oscillatory period $p \approx 45.32$.

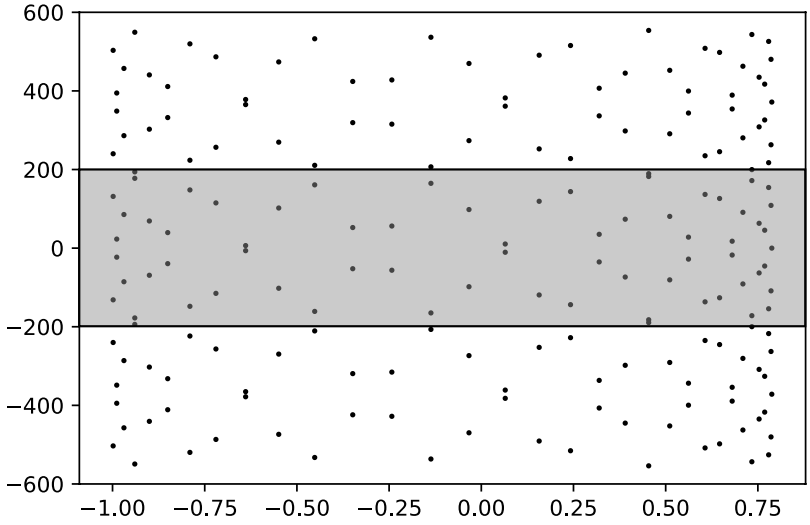


(c) Lattice approximation corresponding to $65/41$, $Q = 2470$, and oscillatory period $p \approx 371.65$.

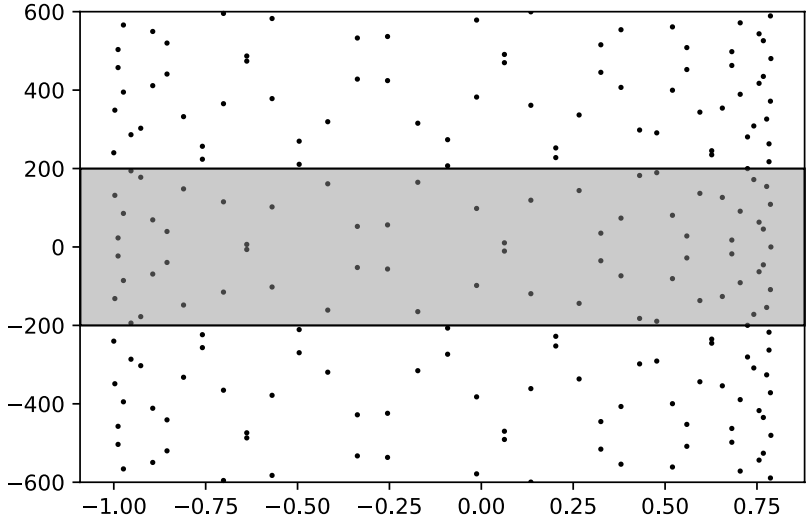


(d) Lattice approximation corresponding to $1054/665$, $Q = 15000$, and oscillatory period $p \approx 6028.03$.

Figure 3.1: Four lattice approximations, with $\epsilon = 1/10$, to the 2-3-polynomial, with radii of approximations roughly 0.0008, 0.043, 1.78, and 7233.63. The dotted horizontal lines intersect the imaginary axis at one period.



(a) A plot of the set of complex roots of the lattice approximation to the 2-3-polynomial corresponding to 65/41, with $Q = 60$, $p \approx 371.65$, and $\epsilon CQp \approx 1.78$



(b) A plot of the set of complex roots of the lattice approximation to the 2-3-polynomial corresponding to 84/53, with $Q = 320$, $p \approx 480.43$, and $\epsilon CQp \approx 12.29$.

Figure 3.2: Lattice approximations from 65/41 (left) and 84/53 (right).

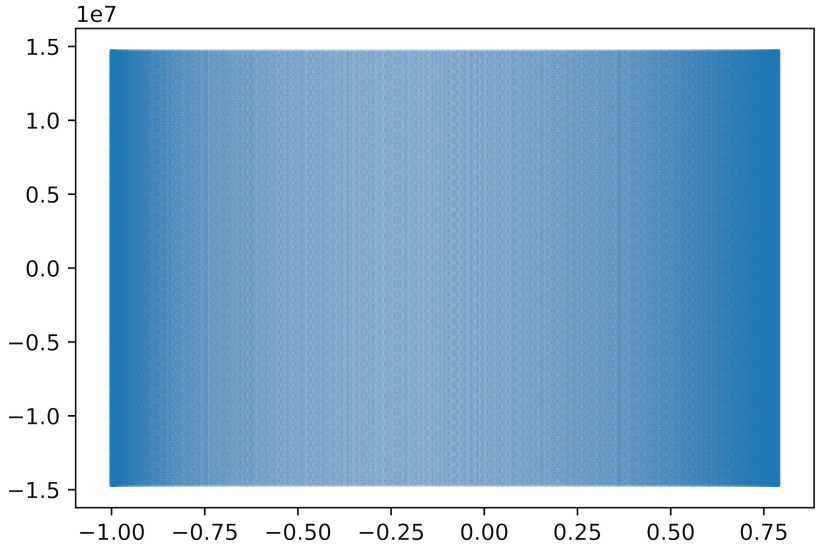


Figure 3.3: A plot of the set of complex roots of the lattice approximation to the 2-3-polynomial corresponding to 125743/79335, with $Q = 100000$, $p \approx 719149.58$, and $\epsilon CQp \approx 5753196.64$.

with very large degree. Previously, programming languages like MATLAB and MATHEMATICA had enough functionality to compute, and plot, the roots of reasonably modest lattice approximations (modest meaning having degree up to 1000). However, plotting better and better approximations required more practical tools. Software developed by NUMPI called mpslove, which can take advantage of sparsity (see [BR14]), has been very useful. One of the goals for the future is to develop practical methods for obtaining better numerical approximations, which will also help to address a conjecture of Lapidus and van Frankenhuijsen on the structure of the closure of the projection of the complex dimensions in the nonlattice case onto the real axis. For the main purpose of this thesis, it is enough to know that the set of complex dimensions of any nonlattice self-similar fractal string are quasiperiodic in the sense of Lapidus and van Frankenhuijsen. In the next chapter, for a given nonlattice

self-similar fractal string, we give a formula for the diffraction pattern of the set of complex dimensions for any lattice approximation.

Chapter 4

Diffraction by the set of complex dimensions of a lattice self-similar fractal string

This chapter is devoted to new results on diffraction concerning Problem 1 by Lapidus and van Frankenhuijsen, stated in [LvF06]. We saw in Chapter 2 that the complex dimensions of a nonlattice self-similar fractal string are approximated by the complex roots of lattice self-similar fractal strings. By exploiting the periodic structure in the lattice case, and by using a slight modification in the recipe for mathematical diffraction described in Chapter 2, it is possible to compute the diffraction of the complex dimensions of a lattice self-similar fractal string.

4.1 The autocorrelation measure of a lattice self-similar fractal string

Calculating the mathematical diffraction pattern in the lattice case begins with an object which serves as an analytical expression for the complex dimensions. This object is given by

$$\mu = \sum_{x \in X} \delta_x,$$

where δ_x is the Dirac distribution at x. We will generally regard μ as a tempered distribution. Note that in Chapter 2, μ is regarded as a continuous linear function on the space $\mathscr{C}_c(\mathbb{R}^n)$. The recipe for the autocorrelation is still valid when extending to Schwartz functions. The next step is to compute the autocorrelation of μ, which in our case will be another tempered distribution. In the next section, we give the Fourier transform of the autocorrelation, which ultimately describes the diffraction pattern. In the theory of kinematic diffraction, the intensities of the diffraction pattern of an object are given by the Fourier transform of that object.

Let $f(s)$ be a lattice Dirichlet polynomial with integer multiplicities, and denote its set of complex roots by X. Then there exists a positive integer u and complex numbers $\omega_1, \ldots, \omega_u \in \mathbb{C}$ such that

$$X = \bigcup_{j=1}^{u} H_j, \quad H_j = \{(\mathrm{Re}(\omega_j), \mathrm{Im}(\omega_j) + np) : n \in \mathbb{Z}\}.$$

The first step in obtaining the diffraction pattern is to compute the autocorrelation, which is governed by the set of all interactions in X, i.e. the set of all difference vectors

$$X - X = \bigcup_{1 \leq j,k \leq u} H_{j,k},$$

where

$$H_{j,k} = \{(\mathrm{Re}(\omega_{j,k}), \mathrm{Im}(\omega_{j,k}) + pn) : n \in \mathbb{Z}\},$$

and $\omega_{j,k} := \omega_j - \omega_k$. We use H to denote $H_{j,k}$ when $j = k$, and note that H is the same set for all $1 \leq j \leq u$. For a positive number L, let R_L denote the rectangle given by

$$R_L = \{z \in \mathbb{C} : |\mathrm{Re}(z)| \leq b,\ |\mathrm{Im}(z)| \leq L\},$$

and $b = \max\{|D|, |D_l|\}$. For each $a \in X - X$, let

$$N_L(a) = \#\{(x, y) : x, y \in X \cap R_L, a = x - y\},$$

that is, the number of occurrences of the interaction a in the rectangle R_L. If for all $a \in A$, the limit

$$n_a := \lim_{L \to \infty} \frac{N_L(a)}{2bL}$$

exists and $n_a > 0$, then X has a unique autocorrelation

$$\gamma_X := \sum_{a \in X - X} n_a \delta_a. \tag{4.1}$$

Therefore, to actually compute the autocorrelation, we are left with computing n_a for all $a \in A$. This is done in the following theorem:

Theorem 4.1.1 (Lapidus, van Frankenhuijsen, Voskanian) *The set of complex roots X of a lattice Dirichlet polynomial $f(s)$ with oscillatory period p, and positive integer multiplicities, has the autocorrelation measure*

$$\gamma_X = \sum_{a \in H} \frac{u}{2bp} \delta_a + \sum_{\substack{\text{distinct } H_{j,k} \\ j \neq k}} f(j,k) \left(\sum_{a \in H_{j,k}} \frac{1}{2bp} \delta_a \right), \tag{4.2}$$

where $H = H\ f(j,k)$ denotes the multiplicity of the set $H_{j,k}$.

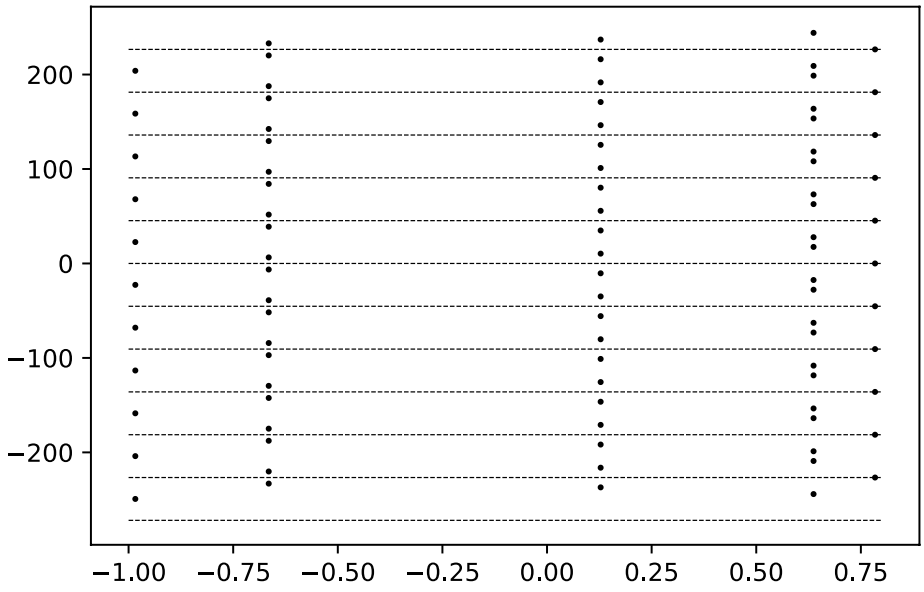


Figure 4.1: A plot of the set of complex dimensions of the lattice approximation to the 2-3-polynomial corresponding to 8/5. The dotted lines tile the plot into repeating cells, which help to count the number of any interaction per unit volume.

Proof. If $a \in H$, then $a = (0, pn)$ for some $n \in \mathbb{Z}$. We exploit the periodic structure of X and count interactions in rectangles that have height equal to copies of $2L$. That is, given $L > 0$, we divide by $2p$ and then round up. The advantage of dividing by $2p$ is that it makes it easier to count frequencies (see Figure 4.1). This way, we get

$$\limsup_{L\to\infty} \frac{N_L(ipn)}{2bL} \leq \limsup_{L\to\infty} \frac{2u\left\lceil \frac{L}{2p} \right\rceil - (nu-1)}{4\left\lceil \frac{L}{2p} \right\rceil pb} = \frac{u}{2pb}.$$

And then by taking a lower estimate, we get

$$\liminf_{L\to\infty} \frac{N_L(ipn)}{2bL} \geq \liminf_{L\to\infty} \frac{2u\left\lfloor \frac{L}{2p} \right\rfloor - (nu-1)}{4\left\lfloor \frac{L}{2p} \right\rfloor pb} = \frac{u}{2pb}.$$

Hence,

$$n_a = \frac{u}{2pb}.$$

By a similar argument, for all $a \in H_{j,k}$ with $j \neq k$, we get

$$n_a = \frac{f(j,k)}{2pb}.$$

We get the autocorrelation measure by substituting n_a into (4.1). $\square$

Now having obtained the autocorrelation measure for the complex dimensions of a lattice self-similar fractal string, the next step is to compute its Fourier transform. The key tool for doing this is an extension of the classical Poisson summation formula to the setting of locally compact abelian groups. To conclude this section, we state and prove the classical version.

Theorem 4.1.2 (Poisson summation formula) *Let $\phi : \mathbb{R}^d \to \mathbb{C}$ be a Schwartz function. Then*

$$\sum_{x \in \mathbb{Z}^d} \hat{\phi}(x) = \sum_{x \in \mathbb{Z}^d} \phi(x). \tag{4.3}$$

Proof. For all $x \in \mathbb{R}^d$,

$$|\phi(x)| \leq \frac{C}{(1+|x|)^{d+1}}.$$

By comparison,

$$\sum_{x \in \mathbb{Z}^d} |\phi(x)| < \infty.$$

Since $\hat{\phi} \in \mathcal{S}(\mathbb{R}^d)$, we have shown that both sums converge absolutely, and thus converge. Now define the function $F : \mathbb{T}^d \to \mathbb{C}$ by

$$F(x) = \sum_{n \in \mathbb{Z}^d} \phi(x+n).$$

Since $F \in C(\mathbb{R}^d)$, $\mathbb{Z}^d$-periodic, and (d+1)-times continuously differentiable, F has a uniformly converging series of the form

$$F(x) = \sum_{m \in \mathbb{Z}^d} c_m e^{2\pi i m \cdot x},$$

where

$$c_m = \int_{\mathbb{T}^d} F(\tau) e^{-2\pi i m \cdot \tau} \, d\tau.$$

Then,

$$\begin{aligned} F(x) &= \sum_{m \in \mathbb{Z}^d} \left(\int_{\mathbb{T}^d} \sum_{n \in \mathbb{Z}^d} \phi(\tau + n) e^{-2\pi i m \cdot \tau} \, d\tau \right) e^{2\pi i m \cdot x} \\ &= \sum_{m \in \mathbb{Z}^d} \left(\sum_{n \in \mathbb{Z}^d} \int_{\mathbb{T}^d} \phi(\tau + n) e^{-2\pi i m \cdot \tau} \, d\tau \right) e^{2\pi i m \cdot x} \\ &= \sum_{m \in \mathbb{Z}^d} \left(\int_{\mathbb{R}^d} \phi(\tau) e^{-2\pi i m \cdot \tau} \, d\tau \right) e^{2\pi i m \cdot x} \\ &= \sum_{m \in \mathbb{Z}^d} \hat{\phi}(m) e^{2\pi i m \cdot x}. \end{aligned}$$

By picking $x = 0$ in the above formula, we get

$$F(0) = \sum_{n \in \mathbb{Z}^d} \phi(n) = \sum_{n \in \mathbb{Z}^d} \hat{\phi}(n)$$

as required. □

4.2 The diffraction measure of a lattice self-similar fractal string

The following lemma and corollary give a suitable extension of the classical Poisson summation formula (4.3) from Theorem 4.1.2. This extension is used to compute the Fourier

transform of the autocorrelation (4.2) of the set of complex dimensions of any lattice self-similar fractal string. In the following theorem, H again denotes $H_{j,k}$ when $j = k$.

Lemma 4.2.1 (Lapidus, van Frankenhuijsen, Voskanian) *Let $\mathcal{L}$ be a lattice self-similar fractal string with oscillatory period p. If u is the number of vertical lines describing the set of complex dimensions of $\mathcal{L}$, $1 \leq j, k \leq u$ and $\phi \in \mathscr{S}(\mathbb{R}^2)$, where $\mathscr{S}(\mathbb{R}^2)$ denotes the set of all Schwartz functions on $\mathbb{R}^2$, then*

$$\sum_{a \in H_{j,k}} \phi(a) = \frac{1}{p} \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \hat{\phi}\left(\xi, np^{-1}\right) e^{2\pi i(\xi \operatorname{Re}(\omega_{j,k}) + np^{-1} \operatorname{Im}(\omega_{j,k}))} \, d\xi.$$

Consequently,

$$\sum_{a \in H} \phi(a) = \frac{1}{p} \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \phi\left(\xi, np^{-1}\right) \, d\xi.$$

Proof. Let $1 \leq j, k \leq u$. For each $a \in H_{j,k}$, there exists an integer n such that

$$a = (\operatorname{Re}(\omega_{j,k}), \operatorname{Im}(\omega_{j,k}) + np).$$

Then,

$$\sum_{a \in H_{j,k}} \phi(a) = \sum_{n \in \mathbb{Z}} \phi(\operatorname{Re}(\omega_{j,k}), \operatorname{Im}(\omega_{j,k}) + np). \tag{4.4}$$

Since $\phi \in \mathscr{S}(\mathbb{R}^2)$, it follows that both sides of equation (4.4) converge absolutely, and that the function $F : \mathbb{R}^2/H \to \mathbb{C}$ defined by

$$F(x, t) = \sum_{n \in \mathbb{Z}} \phi(x + \operatorname{Re}(\omega_{j,k}), \operatorname{Im}(\omega_{j,k}) + t + np)$$

is an element of $\mathscr{S}(\mathbb{R}^2/H)$. Note that F is periodic modulo $p\mathbb{Z}$ in the second coordinate. Applying Fourier inversion in the setting of locally compact abelian groups, the Fourier series expansion of F is given by

$$F(x, t) = \int_{\mathbb{R}} \sum_{m \in \mathbb{Z}} \hat{F}(\xi, m) e^{2\pi i(x\xi + \frac{m}{p} t)} \, d\xi,$$

where

$$\hat{F}(\xi, m) = \int_{\mathbb{R}} \left[\frac{1}{p} \int_0^p F(\tau, s) e^{-\frac{2\pi i m s}{p}} \, ds \right] e^{-2\pi i \xi \tau} \, d\tau.$$

Then,

$$\begin{aligned}
&F(x,t) \\
&\quad = \int_{\mathbb{R}} \sum_{m \in \mathbb{Z}} \left(\int_{\mathbb{R}} \frac{1}{p} \int_0^p \sum_{n \in \mathbb{Z}} \phi(\mathrm{Re}(\omega_{j,k}) + \tau, \mathrm{Im}(\omega_{j,k}) + s + np) e^{\frac{-2\pi i m s}{p}} \, ds \; e^{-2\pi i \xi \tau} \, d\tau \right) \\
&\qquad\qquad e^{2\pi i (x\xi + \frac{m}{p} t)} \, d\xi \\
&\quad = \int_{\mathbb{R}} \sum_{m \in \mathbb{Z}} \left(\int_{\mathbb{R}} \frac{1}{p} \sum_{n \in \mathbb{Z}} \int_0^p \phi(\mathrm{Re}(\omega_{j,k}) + \tau, \mathrm{Im}(\omega_{j,k}) + s + np) e^{\frac{-2\pi i m (s+np)}{p}} e^{-2\pi i \xi \tau} \, ds \, d\tau \right) \\
&\qquad\qquad e^{2\pi i (x\xi + \frac{m}{p} t)} \, d\xi \\
&\quad = \int_{\mathbb{R}} \sum_{m \in \mathbb{Z}} \left(\frac{1}{p} \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(\mathrm{Re}(\omega_{j,k}) + \tau, \mathrm{Im}(\omega_{j,k}) + s) e^{-2\pi i (\xi\tau + \frac{m}{p} s)} \, ds \, d\tau \right) e^{2\pi i (x\xi + \frac{m}{p} t)} \, d\xi \\
&\quad = \int_{\mathbb{R}} \sum_{m \in \mathbb{Z}} \left(\frac{1}{p} \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(\mathrm{Re}(\omega_{j,k}) + \tau, \mathrm{Im}(\omega_{j,k}) + s) e^{-2\pi i (\xi\tau + \frac{m}{p} s)} \, d\tau \, ds \right) e^{2\pi i (x\xi + \frac{m}{p} t)} \, d\xi.
\end{aligned}$$

Because ϕ is a Schwartz function, all of the sums and integrals in the above calculation are absolutely convergent, which justifies the interchanges between them. Now, using the identity

$$e^{-2\pi i (\xi (\mathrm{Re}(\omega_{j,k}) + \tau) + \frac{m}{p} (\mathrm{Im}(\omega_{j,k}) + s))} = e^{-2\pi i (\xi\tau + \frac{m}{p} s)} e^{-2\pi i (\xi \, \mathrm{Re}(\omega_{j,k}) + \frac{m}{p} \, \mathrm{Im}(\omega_{j,k}))},$$

we get

$$\begin{aligned}
F(x,t) &= \frac{1}{p} \int_{\mathbb{R}} \sum_{m \in \mathbb{Z}} \left(\int_{\mathbb{R}^2} \phi(\mathrm{Re}(\omega_{j,k}) + \tau, \mathrm{Im}(\omega_{j,k}) + s) e^{-2\pi i (\xi (\mathrm{Re}(\omega_{j,k}) + \tau) + \frac{m}{p} (\mathrm{Im}(\omega_{j,k}) + s))} \, d\tau \, ds \right) \\
&\quad \cdot e^{2\pi i (\xi \, \mathrm{Re}(\omega_{j,k}) + \frac{m}{p} \, \mathrm{Im}(\omega_{j,k}))} e^{2\pi i (x\xi + \frac{m}{p} t)} \, d\xi \\
&= \frac{1}{p} \int_{\mathbb{R}} \sum_{m \in \mathbb{Z}} \left(\int_{\mathbb{R}^2} \phi(\tau, s) e^{-2\pi i (\xi\tau + \frac{m}{p} s)} \, d\tau \, ds \right) e^{2\pi i (\xi \, \mathrm{Re}(\omega_{j,k}) + \frac{m}{p} \, \mathrm{Im}(\omega_{j,k}))} e^{2\pi i (x\xi + \frac{m}{p} t)} \, d\xi \\
&= \frac{1}{p} \int_{\mathbb{R}} \sum_{m \in \mathbb{Z}} \hat{\phi}(\xi, \frac{m}{p}) e^{2\pi i (\xi (\mathrm{Re}(\omega_{j,k}) + x) + \frac{m}{p} (\mathrm{Im}(\omega_{j,k}) + t))} \, d\xi.
\end{aligned}$$

By picking $x = 0$ and $t = 0$ in the formula above, we get

$$\sum_{a \in H_{j,k}} \phi(a) = \frac{1}{p} \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \hat{\phi}(\xi, \frac{n}{p}) e^{2\pi i (\xi(\mathrm{Re}(\omega_{j,k})) + \frac{n}{p}(\mathrm{Im}(\omega_{j,k})))} \, d\xi.$$

□

Corollary 4.2.2 (Lapidus, van Frankenhuijsen, Voskanian)

$$\sum_{a \in H_{j,k}} \hat{\phi}(a) = \frac{1}{p} \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \phi\left(\xi, np^{-1}\right) e^{2\pi i (\xi(\mathrm{Re}(\omega_{j,k})) + np^{-1}(\mathrm{Im}(\omega_{j,k})))} \, d\xi.$$

Proof. If $\phi \in \mathscr{S}(\mathbb{R}^2)$ then $\hat{\hat{\phi}}(x) = \phi(-x)$, and

$$\sum_{n \in \mathbb{Z}} \phi(-n) = \sum_{n \in \mathbb{Z}} \phi(n).$$

□

Applying Corollary 4.2.2 to the autocorrelation associated to a lattice self-similar fractal string, we obtain the following main theoretical result of this thesis.

Theorem 4.2.3 (Lapidus, van Frankenhuijsen, Voskanian) *Let $\mathcal{L}$ be a lattice self-similar fractal string with oscillatory period p, and let X be its set of complex dimensions, as given in (1.2), so that the difference set $X - X$ is given by*

$$X - X = \bigcup_{1 \le j,k \le u} H_{j,k},$$

where the sets $H_{j,k}$ are given by (1.2).

Then the autocorrelation measure γ is the tempered distribution

$$\gamma = \sum_{a \in H} \frac{u}{2bp} \delta_a + \sum_{\substack{\text{distinct } H_{j,k} \\ j \neq k}} f(j,k) \left(\sum_{a \in H_{j,k}} \frac{1}{2bp} \delta_a \right),$$

where $f(j,k)$ is the multiplicity of the set $H_{j,k}$. Moreover, its Fourier transform is the tempered distribution

$$\hat{\gamma} = \frac{1}{2bp^2} \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} \omega\left(\xi, np^{-1}\right) \delta_{(\xi, np^{-1})} \, d\xi,$$

where

$$\omega\left(\xi, np^{-1}\right) := u + \sum_{\substack{\text{distinct } H_{j,k} \\ j \neq k}} f_{j,k} e^{2\pi i (\xi(\mathrm{Re}(\omega_{j,k})) - np^{-1}(\mathrm{Im}(\omega_{j,k})))}.$$

Chapter 5

Conclusion and Future Plans

In summary, having computed a diffraction measure for the set of complex dimensions of any lattice self-similar fractal string, we can now obtain an infinite sequence of diffraction measures corresponding to infinitely many lattice approximations to the set of complex dimensions of a given nonlattice self-similar fractal string. This is a half step forward in addressing Problem 1, which is the central focus of this thesis. That being said, the full step will be to write down a formula for the diffraction measure of the final product. That is, to compute a diffraction measure in the nonlattice case. This will be the topic of a future paper. In addition to having addressed Problem 1, we note that Theorem 4.2.3 extends Theorem 1.1.3 by Lagarias on diffraction by ideal crystals. This is because the set of complex dimensions in the lattice case consist of finitely many translations in the plane of the group H, which may also be regarded as a *degenerate lattice of rank* 1 in $\mathbb{R}^2$ (see A.1.1).

All of the work leading up to this point has naturally led to explorations in other existing, and possibly new extensions of the classical Poisson summation formula. For example, in [Lin11], the author gives an extension of the Poisson summation formula to a semi-direct product. Another idea is to extend the Poisson summation formula to the setting of degenerate lattices in $\mathbb{R}^n$. Using such an extension, it could be possible to compute a diffraction pattern for any infinite discrete subset of $\mathbb{R}^n$ consisting of finitely many translations in $\mathbb{R}^n$ of a degenerate lattice. One might call such a structure a *degenerate ideal crystal* in $\mathbb{R}^n$.

In Chapter 3, we saw that to obtain many increasingly good lattice approximations numerically, one requires practical tools for obtaining good simultaneous Diophantine approximations with relatively small denominators, and for approximating the roots of sparse polynomials of very large degree. By harnessing enough resources to do computations on a very large scale, and by being careful about precision, I will explore more of the quasiperiodic structure in the nonlattice case. In particular, I will address conjectures concerning the notion of *dimension of fractality* for nonlattice Dirichlet polynomials.

5.1 Diffraction in the nonlattice case

The next step we are taking in addressing problem 1 on quasicrystals and complex dimensions, stated by Lapidus and van Frankenhuijsen, is to compute the diffraction pattern for the complex dimensions of a nonlattice self-similar fractal string. We briefly recall the setting:

Let $f(s)$ be a nonlattice Dirichlet polynomial, and let $\epsilon > 0$ be given. By Theorem

2.2.18, and by Dirichlet's theorem on simultaneous Diophantine approximation, we can obtain a sequence of lattice approximations $\{f_j\}_{j=1}^{\infty}$, such that for each j, the set of complex roots of f_j approximate, up to order ϵ, the roots of f in the ball of radius ϵCQp_j centered at the origin, where p_j is the oscillatory period of f_j. The sequence $\{f_j\}_{j=1}^{\infty}$ can be chosen so that $\epsilon CQp_{j+1} > \epsilon CQp_{j+1}$ for all j. By Theorem 4.2.3, we can write down a formula for the diffraction measure of the set of complex dimensions of each lattice approximation f_j:

$$\hat{\gamma}(\phi) = \frac{1}{2bp_j^2} \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}} \omega\left(\xi, np_j^{-1}\right) \delta_{\left(\xi, np_j^{-1}\right)}\, d\xi,$$

where

$$\omega\left(\xi, \frac{n}{p_j}\right) := u + \sum_{\substack{\text{distinct } H_{j,k} \\ j\neq k}} f(j,k) e^{2\pi i(\xi(\mathrm{Re}(\omega_{j,k})) - np_j^{-1}(\mathrm{Im}(\omega_{j,k})))}.$$

It is not immediately clear what this formula will look like in the limit. However, there is something we can do to better understand what is going on. Let us break down the work of computing the Fourier transform of the tempered distribution

$$\gamma = \sum_{a\in H} \frac{u}{2bp}\delta_a + \sum_{\substack{\text{distinct } H_{j,k} \\ j\neq k}} f(j,k) \left(\sum_{a\in H_{j,k}} \frac{1}{2bp}\delta_a \right).$$

Since the Fourier transform is a linear operator, we apply it to both

$$\sum_{a\in H} \frac{u}{2bp}\delta_a$$

and

$$\sum_{\substack{\text{distinct } H_{j,k} \\ j\neq k}} f(j,k) \left(\sum_{a\in H_{j,k}} \frac{1}{2bp}\delta_a \right)$$

separately, and then add the results. Consider the Fourier transform of the summation over

the group H, which is given by

$$\begin{aligned}\frac{u}{2bp}\sum_{a\in H}\hat{\phi}(a) &= \frac{u}{2bp}\left[\frac{1}{p}\int_{\mathbb{R}}\sum_{n\in\mathbb{Z}}\phi(\xi, np^{-1})\ d\xi\right] \\ &= \frac{u}{2bp}\left[\int_{\mathbb{R}}\lim_{N\to\infty}\sum_{n=1}^{N}\phi(\xi,(n-1)p^{-1})+\phi(\xi,-np^{-1})\ d\xi\right],\end{aligned}$$

for all $\phi \in \mathscr{S}(\mathbb{R}^2)$. For a function $\phi \in \mathscr{S}(\mathbb{R}^2)$, a real number ξ, and an integer $N \geq 1$, we define the function

$$f_\xi : [-Np^{-1}, (N-1)p^{-1}] \to \mathbb{R},$$

given by

$$f_\xi(\tau) = \phi(\xi, (\tau-1)p^{-1}) + \phi(\xi, -\tau p^{-1}).$$

Note that this function is continuous. Let

$$P = \left\{[-Np^{-1}, (-N+1)p^{-1}], \ldots, [(N-2)p^{-1}, (N-1)p^{-1}]\right\}$$

be a partition of the interval $[-Np^{-1}, (N-1)p^{-1}]$, where

$$a = -Np^{-1} < (-N+1)p^{-1} < \cdots < (N-2)p^{-1} < (N-1)p^{-1} = b.$$

Then,

$$\begin{aligned}\frac{u}{2bp}\sum_{a\in H}\hat{\phi}(a) &= \frac{u}{2bp}\left[\int_{\mathbb{R}}\lim_{N\to\infty}\sum_{n=1}^{N}\phi(\xi,(n-1)p^{-1})+\phi(\xi,-np^{-1})\ d\xi\right] \\ &= \frac{u}{2bp}\left[\int_{\mathbb{R}}\lim_{N\to\infty}\sum_{n=1}^{N}f_\xi(n)\frac{1}{p}\ d\xi\right],\end{aligned}$$

and for each fixed N and ξ, the term

$$\sum_{n=1}^{N} f_\xi(n)\frac{1}{p}$$

is a Riemann sum of f_ξ over the interval $[-Np^{-1}, (N-1)p^{-1}]$ with partition P.

5.2 Generic nonlattice self-similar fractal strings, and the dimensions of fractality.

In Chapter 2, the lattice nonlattice dichotomy in the set of all Dirichlet polynomials was defined. In Remark 2.2.12, an equivalent definition was given. There is another characterization from [LvF06] of this dichotomy, stated in the language of groups: Let f be a Dirichlet polynomial with scaling ratios $r_1, \ldots, r_N$. Then, $f(s)$ is lattice if and only if the associated additive group

$$G := \sum_{j=1}^{N} \log r_j \mathbb{Z}$$

has rank 1. If f is a nonlattice Dirichlet polynomial whose associated additive group G has full rank, i.e., if the numbers $\log r_1, \ldots, \log r_N$ are all rationally independent, we call f *generic*. Otherwise, it is called *nongeneric*. A nonlattice self-similar fractal string is nongeneric if the associated Dirichlet polynomial is nongeneric, and generic otherwise. The following definition from Remark 3.54 in [LvF06], [DS14], and [LvF03] is the basis for a planned project on the complex dimensions in the nongeneric case.

Definition 5.2.1 *Let f be a Dirichlet polynomial with positive integer multiplicities. The set of dimensions of fractality is given by*

$$R_f := \overline{\{\operatorname{Re} z : f(z) = 0\}}.$$

The dimensions of fractality of a self-similar fractal string $\mathcal{L}$ is defined to be the set of dimensions of fractality of the associated Dirichlet polynomial, which has positive integer multiplicities. In the first edition of [LvF06], Lapidus and van Frankenhuijsen conjecture that the dimensions of fractality of a nonlattice self-similar fractal string is a bounded

connected interval $[\sigma_\ell, D]$, where σ_ℓ is some real number, and D is the Minkowski dimension of the fractal string. Moreover, they conjecture that in the generic nonlattice case, $\sigma_\ell = D_\ell$, which was eventually proved in [PSV13]. As future work, I will investigate the set of dimensions of fractality in the nongeneric case. Specifically, Lapidus and van Frankenhuijsen have also conjectured in [LvF06] that the set of dimensions of fractality in the nongeneric case is equal to a finite number of closed intervals, and certain cases are proved by Dubon and Sepulcre in [DS14].

To demonstrate this observation in the nongeneric nonlattice case, we give the following example, were a lattice approximation for a nongeneric Dirichlet polynomial is computed. A plot is given in Figure 5.1.

Example 5.2.2 *Consider the nongeneric Dirichlet polynomial*

$$f(s) = 1 - 2^{-s} - 3^{-s} - 4^{-s} - 5^{-s} - 6^{-s} - 7^{-s},$$

which is re expressed as

$$f(s) = 1 - (2^{-s})^1 - (2^{-s})^{\log_2 3} - (2^{-s})^2 - (2^{-s})^{\log_2 5} - (2^{-s})^{\log_2 6} - (2^{-s})^{\log_2 7}.$$

Now, we obtain a simultaneous Diophantine approximation for the numbers

$$\log_2 3, \quad \log_2 5, \quad \log_2 6, \quad \log_2 7.$$

Using the method of continued fractions, we compute convergents, and then process the basis matrix for the lattice determined by those convergents with the LLL algorithm, as in Section 3.2. This process gives us the simultaneous Diophantine approximation

$$29092/18355, \quad 42619/18355, \quad 47447/18355, \quad 51529/18355$$

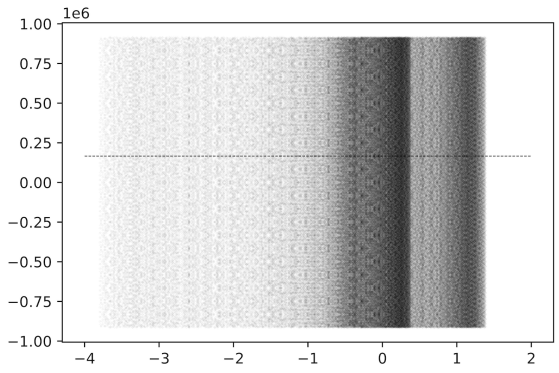


Figure 5.1: Lattice approximation to the nongeneric Dirichlet polynomial in Example 5.2.2, with $\epsilon = 1/10$, and period $p \approx 166382.94$. The dotted horizontal line intersects the imaginary axis at p.

for the real numbers $\log_2 3$, $\log_2 5$, $\log_2 6$, $\log_2 7$, *respectively, with* $Q = 70$, *and common denominator* $q = 18355$ *satisfying* $1 \leq q < (70)^5$. *By Theorem 2.2.18, for* $\epsilon = 1/10$, *the set of complex roots of the lattice Dirichlet polynomial*

$$\tilde{f}(s) = 1 - (2^{-s/18355})^{18355} - (2^{-s/18355})^{29092} - (2^{-s/18355})^{36710} \\ - (2^{-s/18355})^{42619} - (2^{-s/18355})^{47447} - (2^{-s/18355})^{51529},$$

with period $q\frac{2\pi}{\log 2}$, *approximate the set of complex roots of* $f(s)$ *in the ball* $\{z \in \mathbb{C} : |z| < \rho\}$, *where* $\rho = (1/10)C(70)(p)(q)$.

In Figure 5.1, we give a plot of the set of complex roots of the lattice approximation in Example 5.2.2. Notice the two rectangular regions where the complex roots seem to be concentrated. Compare this approximation to the one for the generic 2-3-polynomial in Figure 3.3.

Chapter A

Appendix

A.1 The implementation of LLL for lattice approximation

To conduct numerical experiments in exploring the quasiperiodic structure of the complex dimensions of a nonlattice Dirichlet polynomial, we require a practical method for obtaining many good simultaneous Diophantine approximations for a finite list of real numbers. Lapidus and Van Frankenhuijsen in [LvF06] have suggested to use a lattice basis reduction algorithm known as the LLL algorithm, which was originally introduced in [LLL82] as the first polynomial time algorithm to factor polynomials with rational coefficients. In this Appendix, we give some preliminaries on lattice theory and basis reduction from the book [Bre11], which is a self contained treatment of Lattice basis reduction, the LLL algorithm, and much more. Then, we relay the exposition of LLL, along with its complexity from [Bre11] and [LLL82]. To conclude, an implementation of the LLL algorithm (in C++) to compute simultaneous Diophantine approximations is given.

A.1.1 The basics of lattice theory

Lattices in n-dimensional Euclidean space

A set $L \subset \mathbb{R}^n$ is called a *lattice* if it is a discrete subgroup of $\mathbb{R}^n$, such that the factor group $\mathbb{R}^n/L$ is compact. The following definition from [Bre11] is equivalent, and includes the notion of the determinant of a lattice.

Definition A.1.1 ([Bre11]) *Let n be a positive integer, and let $x_1, x_2, \ldots, x_n$ be a basis of $\mathbb{R}^n$. The lattice with dimension n and basis $x_1, x_2, \ldots, x_n$ is the set L of all linear combinations of the basis vectors with integer coefficients:*

$$L = \sum_{i=1}^{n} x_i \mathbb{Z} = \left\{ \sum_{i=1}^{n} a_i x_i : a_1, a_2, \ldots, a_n \in \mathbb{Z} \right\}.$$

The basis vectors $x_1, x_2, \ldots, x_n$ are said to generate or span the lattice. For $i = 1, 2, \ldots, n$ we write the row vector

$$x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$$

and form the $n \times n$ matrix X given by $(X)_{ij} = x_{ij}$. The determinant of the lattice L with basis $x_1, x_2, \ldots, x_n$ is given by

$$\det(L) = |\det(X)|.$$

Let n be a positive integer. If $k < n$ and $x_1, \ldots, x_k$ is a linearly independent set of vectors in $\mathbb{R}^n$, then

$$L = \sum_{i=1}^{k} x_i \mathbb{Z}$$

will be called a *degenerate lattice of rank k* in $\mathbb{R}^n$. A lattice as defined in Definition A.1.1 is sometimes called a lattice of full rank. The concept of lattice basis and lattice determinant

is generalized to any set of k linearly independent vectors in $\mathbb{R}^n$ (see Definition 1.15 in [Bre11]).

Theorem A.1.2 ([Bre11]) *Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$, be two bases for the same lattice $L \subset \mathbb{R}^n$. Let X (respectively Y) be the $n \times n$ matrix with x_i (respectively y_i) in row i for $i = 1, 2, \ldots, n$. Then*

$$Y = CX$$

for some $n \times n$ matrix C with integer entries and determinant ± 1.

Proof. Every y_i belongs to the lattice with basis $x_1, \ldots, x_n$, and every x_i belongs to the lattice with basis $y_1, \ldots, y_n$. It follows that

$$x_i = \sum_{j=1}^{n} b_{ij} y_j, \quad y_i = \sum_{j=1}^{n} c_{ij} x_j \quad (i = 1, \ldots, n),$$

where B and C are $n \times n$ matrices with integer entries given by $(B)_{ij} = b_{ij}$ and $(C)_{ij} = c_{ij}$, respectively. Writing these two equations in matrix form gives $X = BY$ and $Y = CX$, and hence $X = B(CX)$ and $Y = C(BY)$. Since the matrices X and Y are invertible, we get

$$BC = CB = I$$

by cancellation. Then,

$$\det(BC) = \det(B)\det(C)$$

$$= 1$$

so that because the matrices B and C have integer entries, $\det(B) = \det(C) = \pm 1$. □

Corollary A.1.3 ([Bre11]) *The determinant of a lattice does not depend on the basis.*

Proof. Let $L \subset R^n$ be a lattice with bases $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. Then,

$$\begin{aligned} |\det(Y)| &= |\det(CX)| \\ &= |\det(C)\det(X)| \\ &= |\pm \det(X)| \\ &= |\det(X)|. \end{aligned}$$

□

The LLL algorithm takes as input a basis matrix for a lattice, and then return a reduced basis (see Definition A.1.8). Specifically, the LLL algorithm performs finitely many row operations to the input matrix. After each operation, the result is another basis for the lattice. These are called unimodular row operations, and they form the basis of the LLL algorithm.

Definition A.1.4 ([Bre11]) *A unimodular row operation on a matrix is one of the following elementary row operations:*

(*i*) *multiply any row by 1;*

(*ii*) *interchange any two rows;*

(*iii*) *add an integral multiple of any row to any other row.*

Definition A.1.5 ([Bre11]) *An $n \times n$ matrix with integer entries and determinant ± 1 will be called unimodular.*

Let $L \subset \mathbb{R}^n$ be a lattice with basis $x_1, \ldots, x_n$ with basis matrix X, and let and let C be an $n \times n$ unimodular matrix. Since the rows of X are linearly independent, we know

that $\det(X) \neq 0$. Then, denoting $Y = CX$,

$$|\det(Y)| = |\det(CX)| = |\det(X)| \neq 0,$$

which means that the rows of Y are linearly independent. And since there are n of them, they form a basis for $\mathbb{R}^n$. The rows of Y are given by

$$\begin{aligned}
y_1 &= \left(\sum_{k=1}^{n} c_{1,k} x_{k,1}, \sum_{k=1}^{n} c_{1,k} x_{k,2}, \ldots, \sum_{k=1}^{n} c_{1,k} x_{k,n} \right) \\
y_2 &= \left(\sum_{k=1}^{n} c_{2,k} x_{k,1}, \sum_{k=1}^{n} c_{2,k} x_{k,2}, \ldots, \sum_{k=1}^{n} c_{2,k} x_{k,n} \right) \\
&\vdots \\
y_n &= \left(\sum_{k=1}^{n} c_{n,k} x_{k,1}, \sum_{k=1}^{n} c_{n,k} x_{k,2}, \ldots, \sum_{k=1}^{n} c_{n,k} x_{k,n} \right).
\end{aligned}$$

It follows that the integer span of the vectors $x_1, \ldots, x_n$ includes the integer span of the vectors $y_1, \ldots, y_n$. Because C is a unimodular matrix with nonzero determinant, its inverse C^{-1} exists, and it has integer entries. Therefore, we can write the equation $X = C^{-1}Y$, and use the same reasoning to conclude that the bases $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ generate the same lattice.

It is easy to produce a basis with longer vectors, as one can easily determine unimodular matrices, with large integer entries, by performing finitely many *unimodular row operations* to the identity matrix I_n. The difficulty is getting shorter vectors. In general, a basis $x_1, \ldots, x_n$ for a lattice $L \subset \mathbb{R}^n$ might consist of vectors that are large in magnitude. The problem of lattice basis reduction is finding another basis for the same lattice with shorter vectors. The LLL algorithm takes as input a basis matrix X for a lattice L, and then multiplies it by a unimodular matrix, several times, until a basis with shorter vectors is achieved.

α-reduced bases for a lattice

Given a lattice $L \subset \mathbb{R}^n$, we would like to determine what is called an *α-reduced* basis for the lattice L. Lattice basis reduction algorithms are usually used in modern number theoretical applications, but as we have already mentioned, they can be used to obtain simultaneous Diophantine approximations. We now define the notion of an *α-reduced basis*, and then state an important fact about a lattice with such a basis.

Before we can define an α-reduced basis, we must first recall the *Gram-Schmidt orthogonalization process*, which takes a basis $x_1, \ldots, x_n$ for $\mathbb{R}^n$ and constructs an orthogonal basis.

Definition A.1.6 *Let $x_1, x_2, \ldots, x_n$ be an ordered basis for $\mathbb{R}^n$, and let $x_1^* = x_1$. For $1 < i \leq n$, define*

$$x_i^* = x_i - \sum_{j=1}^{i-1} \mu_{i,j} x_j^*,$$

where for $1 \leq j < i \leq n$,

$$\mu_{i,j} = \frac{x_i \cdot x_j^*}{x_j^* \cdot x_j^*} = \frac{x_i \cdot x_j^*}{|x_j^*|^2}.$$

The vectors $x_1^*, \ldots, x_n^*$ are called the Gram-Schmidt orthogonalization, and the numbers $\mu_{i,j}$ are called the Gram-Schmidt coefficients.

The following theorem stated in [LLL82] gives some fundamental properties needed to understand the analysis of the LLL algorithm. For a proof, see [LLL82] or [Bre11].

Theorem A.1.7 ([LLL82]) *Let $x_1, \ldots, x_n$ be a basis of $\mathbb{R}^n$ and let $x_1^*, \ldots, x_n^*$ be its Gram-Schmidt orthogonalization. Let X (respectively X^*) be the $n \times n$ matrix in which row i is the vector x_i (respectively x_i^*) for $1 \leq i \leq n$. We have:*

(a) $x_i^* \cdot x_j^* = 0$ *for* $1 \leq i < j \leq n$.

(b) *span*$\{x_1^*, \ldots, x_k^*)\}$ = *span*$\{x_1, \ldots, x_k)\}$ *for* $1 \leq k \leq n$.

(c) *For* $1 \leq k \leq n$, *the vector* x_k^* *is the projection of* x_k *onto the orthogonal complement of span*$\{x_1, \ldots, x_k\}$.

(d) $|x_k^*| \leq |x_k|$ *for* $1 \leq k \leq n$.

(e) $\det(X^*) = \det(X)$.

In the previous section, we said that the difficulty is finding a suitable finite sequence of unimodular row operations to obtain a basis of short vectors for a given lattice. We now define the notion of such a desirable basis, which is called an α-*reduced basis* with *reduction parameter* $1/4 < \alpha < 1$. The standard value of the parameter is $\alpha = 3/4$.

Definition A.1.8 ([Bre11]) *Let* $x_1, x_2, \ldots, x_n$ *be an ordered basis for the lattice* $L \subset \mathbb{R}^n$, *and let* $x_1^*, x_2^*, \ldots, x_n^*$ *be its Gram-Schmidt orthogonalization (GSO), with Gram-Schmidt coefficients* $\mu_{i,j}$ *for* $1 \leq j < i \leq n$.

Write $X = MX^*$ *where* X *(respectively* X^**) is the matrix with* x_i *(respectively* x_i^**) as row* i, *and* M, *called the matrix of Gram-Schmidt coefficients, is defined by* $(M)_{i,j} = \mu_{i,j}$. *The basis* $x_1, x_2, \ldots, x_n$ *is called* α*-reduced if it satisfies*

(i) $|\mu_{i,j}| \leq 1/2, \quad$ *for* $1 \leq j < i \leq n$,

(ii) $|x_i^* + \mu_{i,i-1} x_{i-1}^*|^2 \geq \alpha |x_{i-1}^*|^2, \quad$ *for* $1 < i \leq n$.

Using part (b) of Theorem A.1.7 we have

$$\begin{aligned}|x_i^* + \mu_{i,i-1}x_{i-1}^*|^2 &= \langle x_i^* + \mu_{i,i-1}x_{i-1}^*, x_i^* + \mu_{i,i-1}x_{i-1}^*\rangle \\ &= \langle x_i^*, x_i^*\rangle + 2\langle x_i^*, \mu_{i,i-1}x_{i-1}^*\rangle + \mu_{i,i-1}^2\langle x_{i-1}^*, x_{i-1}^*\rangle \\ &= |x_i^*|^2 + \mu_{i,i-1}^2|x_{i-1}^*|^2.\end{aligned}$$

It follows that the inequality in condition (i) in Definition A.1.8 becomes

$$|x_i^*|^2 \geq (\alpha - \mu_{i,i-1}^2)|x_{i,i-1}^*|^2. \tag{A.1}$$

The following theorem is key to understanding how the LLL algorithm can be used to find short vectors in a lattice. See [Bre11] for a proof.

Theorem A.1.9 (The LLL Theorem, [Bre11]) *If $x_1, \ldots, x_n$ is an α-reduced basis of the lattice $L \subset \mathbb{R}^n$, and $y \in L$ is any nonzero lattice vector, then*

$$|x_1| \leq \beta^{(n-1)/2}|y|.$$

In particular, the first vector in the α-reduced basis is no longer than $\beta^{(n-1)/2}$ times the shortest nonzero vector in L, where $\beta = \frac{4}{4\alpha-1}$.

A.1.2 The analysis of the LLL algorithm

Let $L \subset \mathbb{R}^n$ be a lattice with ordered basis given by $X = \{x_1, \ldots, x_n\}$. We now describe the LLL algorithm, which transforms X into an α-reduced basis with reduction parameter $1/4 < \alpha < 1$. This exposition follows the one from [Bre11], and the original paper [LLL82]. We refer to [Bre11] for an extensive treatment of LLL, including examples, applications, projects, and references to further developments of the LLL algorithm.

The first step of the algorithm is to compute the GSO vectors

$$x_1^*, x_2^*, \ldots, x_n^*,$$

and coefficients

$$\mu_{i,j} = \frac{x_i \cdot x_j^*}{|x_j^*|^2}$$

for $1 \leq j < i \leq n$. Say that the input basis X is not α-reduced for a given reduction parameter $1/4 < \alpha < 1$. Then, several changes are made to X in such a way that the resulting set of vectors after each change form another basis for the lattice L. This process continues until a reduced basis is achieved. Each change to the basis comes from calling one of two subroutines that make up the LLL algorithm. In [Bre11], one of these is called reduce, and the other is called exchange.

For $k = 2, 3, \ldots, n$, the reduce procedure checks whether or not the Gram-Schmidt coefficient $\mu_{k,k-1}$ satisfies condition (i) in Definition A.1.8. If if does, then the reduce procedure does nothing. Otherwise, x_k is replaced with $x_k - rx_{k-1}$, where $r = \lfloor \mu_{k,k-1} \rceil$, $\mu_{k,k-1}$ is replaced with $\mu_{k,k-1} - r$, and for $j = 1, 2, \ldots, k-2$, and $\mu_{k,k-1} - r\mu_{k,j}$. After these changes, a new basis and GSO is obtained with $\mu k, k-1 \leq 1/2$. For $k = 2, 3, \ldots, n$, the exchange procedure checks if the Gram-Schmidt vector x_k^* satisfies condition (ii) in Definition A.1.8. If it does, exchange does nothing. Otherwise, the basis vectors x_k and x_{k-1} are swapped. Then, the Gram-Schmidt vectors and coefficients are modified so that a new basis and Gram-Schmidt orthogonalization is obtained with the length of x_{k-1}^* being less than $3/4$ times its old length.

The pseudocode for the subroutines reduce and exchange from [Bre11] and [LLL82] are given below, along with an implementation at the end of this appendix.

REDUCE SUBROUTINE

Input: $\mu_{k,k-1}$

if $|\mu_{k,k-1}| > 1/2$

$r \leftarrow \lfloor \mu_{k,k-1} \rceil$

$\mu_{k,k-1} \leftarrow \mu_{k,k-1} - r$

$b_k \leftarrow b_k - r b_{k-1}$

for $j = 1, 2, \ldots, k-2$

$\mu_{k,j} \leftarrow \mu_{k,j} - r\mu_{k-1,j}$

end if

EXCHANGE SUBROUTINE

Input: b_k

$b_k \leftrightarrow b_{k-1}$ (exchange b_{k-1} and b_k)

$\mu_{k,k-1}$, $\mu_{k-1,j}$ for $j = 1, 2, \ldots, k-2$. **for** $i = 1$ to n **do**

Set $b_i \leftarrow x_i$

end for

Compute the Gram-Schmidt orthogonalization of $b_1, b_2, \ldots, b_n$.

Set $k \leftarrow 2$

while $k \leq n$

Call $reduce(k, k-1)$

if $|b_k^* + \mu_{k,k-1} b_{k-1}^*|^2 < \frac{3}{4}|b_{k-1}^*|^2$

Call $exchange(k)$

Set $k \leftarrow k-1$

else

for $\ell = k-2, \ldots, 2, 1$

Call $reduce(k, \ell)$

end for

Set $k \leftarrow k + 1$

end if

end while

We now describe the LLL algorithm: Starting with $k = 2$, the first step is to call reduce and ensure that $|\mu_{k,k-1}| \leq 1/2$. Then, before the remaining coefficients are checked, exchange is called and the algorithm enters one of two cases: In the first case,

$$|b_k^* + \mu_{k,k-1} b_{k-1}^*|^2 < \frac{3}{4} |b_{k-1}^*|^2,$$

and the current basis and Gram-Schmidt orthogonalization are updated as described above. After this change, k is replaced by $k - 1$, and the algorithm returns to the first step. The other case is when

$$|b_k^* + \mu_{k,k-1} b_{k-1}^*|^2 \geq \frac{3}{4} |b_{k-1}^*|^2,$$

or $k = 1$. In this case, for $j = 1, 2, \ldots, k - 2$, reduced is called so that the remaining Gram-Schmidt coefficients $\mu_{k,1}, \mu_{k,2}, \ldots, \mu_{k,k-2}$ satisfy condition (i) in Definition A.1.8. Then, k is replaced with $k + 1$, and the algorithm returns to step 1. The number of times the algorithm passes through case 1 is bounded above. Also, the number of times the algorithm can pass through case 2 is at most $n-1$ more times than the number of times the algorithm passes through case 1. Therefore, the LLL algorithm terminates, and upon termination, ensures a reduced basis for the lattice L. See [LLL82] for a more complete explanation. The pseudocode for the entire LLL algorithm is given below.

Input: A basis $x_1, x_2, \ldots, x_n$ for the lattice $L \subset \mathbb{R}^n$.

Output: A reduced basis for L.

for $i = 1$ to n **do**

Set $b_i \leftarrow x_i$

end for

Compute the Gram-Schmidt orthogonalization of $b_1, b_2, \ldots, b_n$.

Set $k \leftarrow 2$

while $k \leq n$

Call $reduce(k, k-1)$

if $|b_k^* + \mu_{k,k-1} b_{k-1}^*|^2 < \frac{3}{4}|b_{k-1}^*|^2$

Call $exchange(k)$

Set $k \leftarrow k - 1$

else

for $\ell = k-2, \ldots, 2, 1$

Call $reduce(k, \ell)$

end for

Set $k \leftarrow k + 1$

end if

end while

The complexity of LLL

To study the complexity of the LLL algorithm, it is important to keep track of how the Gram-Schmidt orthogonalization (GSO) changes during the course of the algorithm. Specifically, we need to determine how the basis and coefficients change during the reduction in step (4)(b)(i), and during the exchange in step (4)(b)(iii).

Suppose that we process a lattice with basis $y_1, \ldots, y_n$ with the LLL algorithm, and that we are at the beginning of some iteration with $2 \leq k < n$. We continue on to Step 4(a), which performs the reduction reduce$(k, k-1)$. After this call, the GSO coefficient $\mu_{k,k-1}$ satisfies condition (i) in Definition A.1.8. Next, we check that γ_k^* satisfies inequality (A.1). If it does, we enter Step 4(b)(i), which updates the remaining Gram-Schmidt coefficients $\mu_{k,1}, \mu_{k,2}, \ldots, \mu_{k,k-2}$. Consider one call to reduce in this step, and suppose that a reduction is actually performed, i.e., suppose that $|\mu_{k,\ell}| > 1/2$. After the call, we have

$$y_k = y_k - \lceil \mu_{k,\ell} \rfloor y_\ell, \qquad \mu_{k,j} = \mu_{k,j} - \lceil \mu_{k,\ell} \rfloor \mu_{\ell,j} \ (1 \leq j \leq \ell - 1),$$

and

$$\mu_{k,\ell} = \mu_{k,\ell} - \lceil \mu_{k,\ell} \rfloor.$$

Then, k is incremented in Step(4)(b)(ii), we repeat. The following lemma says that after one call to reduce in Step(4)(b)(i), the Gram-Schmidt orthogonalization does not change. This is good news, as the Gram-Schmidt process is not only costly, but also unstable. Lemma A.1.11 that follows, which concerns the exchange step, partly ensures that the algorithm will terminate after a finite number of steps.

Lemma A.1.10 ([Bre11]) *Consider one call to reduce in Step (4)(b)(i) with given k and ℓ. Let $Y = MY^*$ and $Z = NZ^*$ be the matrix equations for the Gram-Schmidt orthogonalization before and after the call to reduce(k, ℓ), respectively. Write $\nu = \lceil \mu_{k,\ell} \rfloor$, and let $E = I_n - \nu E_{k,\ell}$ be the elementary matrix which represents subtracting ν times row ℓ from row k (thus $(E)_{i,i} = 1$ for $1 \leq i \leq n$, $(E)_{k,\ell} = \nu$, and $(E)_{i,j} = 0$ otherwise). Then,*

$$Z = EY, \quad N = EM, \quad Z^* = Y^*.$$

In particular, the Gram-Schmidt orthogonal basis does not change. Before the call to reduce(k, ℓ) we have

$$|\mu_{k,j}| \leq 1/2 \quad (\ell < j < k).$$

After the execution of the loop on ℓ in Step (4)(b)(i) we have

$$|\mu_{k,j}| \leq 1/2 \quad (1 \leq j < k).$$

Lemma A.1.11 ([Bre11]) *Consider the call to exchange with a given value of k in Step (4)(b)(iii). Let $Y = MY^*$ and $Z = NZ^*$ be the matrix equations for the GSO before and after the exchange. We have*

$$z_i^* = y_i^* \quad (i \neq k-1, k), \qquad |z_{k-1}^*|^2 < \alpha |y_{k-1}^*|^2, \qquad |z_k^*| \leq |y_{k-1}^*|.$$

Theorem A.1.12 ([Bre11]) *Let $L \subset \mathbb{Z}$ be a lattice with basis $b_1, b_2, \ldots, b_n$, and let $B \in \mathbb{R}$, $B \geq 2$, be such that $|b_i|^2 \leq B$ for $1 \leq i \leq n$. Then the number of arithmetic operations needed by the basis reduction algorithm is $O(n^4 \log B)$, and the integers on which these operations are performed each have binary length $O(n \log B)$.*

Remark (1.37) in [LLL82] states that, apart from some minor changes, the analysis of the algorithm remains valid if $b_i \cdot b_j \in \mathbb{Z}$ for $1 \leq i, j \leq n$.

The following example shows just how the algorithm works in a relatively small application. Notice that k is increased and decreased several times until it is equal to 4, which is when the algorithm terminates.

Example 6 *Let L be an n-dimensional lattice spanned by the basis given by the vectors*

$$x_1 = (1, 1, 1), \quad x_2 = (-1, 0, 2), \quad x_3 = (3, 5, 6).$$

The basis is processed by the LLL algorithm after 5 updates:

$k = 2$

$$|\mu_{2,1}| \leq 1/2$$

$$\gamma_2^* \geq (3/4 - \mu_{2,1}^2)\gamma_1^*$$

$k = 3$

$$|\mu_{3,2}| > 1/2$$

$reduce(3, 2)$

$$X = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & 5 \\ 1 & 2 & 4 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 1/3 & 13/3 \\ 0 & 1 & -1/14 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma^* = \begin{pmatrix} 3 \\ 14/3 \\ 9/14 \end{pmatrix}$$

$$\gamma_3^* < (3/4 - \mu_{3,2}^2)\gamma_2^*$$

$exchange(3)$

$$B = \begin{pmatrix} 1 & 4 & -1 \\ 1 & 5 & 0 \\ 1 & 4 & 2 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 13/3 & 1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma^* = \begin{pmatrix} 3 \\ 2/3 \\ 9/2 \end{pmatrix}$$

$k = 2$

$$|\mu_{2,1}| > 1/2$$

$reduce(2, 1)$

$$B = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 1/3 & 1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma^* = \begin{pmatrix} 3 \\ 2/3 \\ 9/2 \end{pmatrix}$$

$$\gamma_2^* < (3/4 - \mu_{2,1}^2)\gamma_1^*$$

$exchange(2)$

$$B = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma^* = \begin{pmatrix} 1 \\ 2 \\ 41/9 \end{pmatrix}$$

$k = 2$

$$|\mu_{2,1}| > 1/2$$

$reduce(2, 1)$

$$B = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma^* = \begin{pmatrix} 1 \\ 2 \\ 41/9 \end{pmatrix}$$

$$\gamma_2^* \geq (3/4 - \mu_{2,1}^2)\gamma_1^*$$

$k = 3$

$$|\mu_{3,2}| \leq 1/2$$

$$\gamma_3^* \geq (3/4 - \mu_{3,2}^2)\gamma_2^*$$

$$|\mu_{3,1}| \leq \frac{1}{2}$$

$k = 4$

Algorithm terminates

A.2 Source code

The following programs were written to generate lattice approximations. The first is an implementation of the continued fraction algorithm on C, and the second is an implementation of the LLL algorithm on C++ to compute simultaneous Diophantine approximations.

```c
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
/* Declares sin(), cos(), etc. */

/*
Continued fraction algorithm: Edward Voskanian
To compile: cc -o CF continuedFraction.c -lm
To execute: ./CF [enter a real number] [enter the convergent you want]

Note: If you want better convergent, you must enter a better approximation.
For example: To obtain the 3rd convergent for Pi, 3.14 is not enough
 decimal places.
*/

int main(int argc,char* argv[]) {

    // Variable definitions and declarations.
    double alpha = atof(argv[1]);
    double a = floor(alpha);
    int n = atoi(argv[2]);
    int hold_p,hold_q;
    int p_1 = 1,p_2 = 0,q_1 = 0,q_2 = 1;
    int p,q;
    int i;

    // Main for loop that generates the nth convergent.
    for (i = 0;i < n;i++) {
        hold_p = p_1;
        hold_q = q_1;
        p = a*p_1 + p_2;
        q = a*q_1 + q_2;
        p_1 = p;
        q_1 = q;
        p_2 = hold_p;
        q_2 = hold_q;
        alpha = 1.0 / (alpha - a);
        a = floor(alpha);
        printf("Convergent %d: %d / %d\n",i + 1,p,q);
    }

    return 0;
```

```
}

/*

 Edwards-MBP:Coding edwardvoskanian$ cc -o CF continuedFraction.c -lm
 Edwards-MBP:Coding edwardvoskanian$ ./CF 3.14159265358979323846264643 7
 Convergent 1: 3 / 1
 Convergent 2: 22 / 7
 Convergent 3: 333 / 106
 Convergent 4: 355 / 113
 Convergent 5: 103993 / 33102
 Convergent 6: 104348 / 33215
 Convergent 7: 208341 / 66317
 Edwards-MBP:Coding edwardvoskanian$

*/
```

```
#include <iostream>
#include <math.h>
using namespace std;

void gramSchmidt(double ** Y,double ** Y_star,double ** M,double
                 * gamma_star);
void reduce(int,int,double ** Y,double ** M,double ** C);
void exchange(int,double ** Y,double ** M,double * gamma_star,double ** C);

//GLOBAL VARIABLES
int n = 5;
double epsilon = 1.0;
const double ALPHA = 0.75;

int main() {
    int i, j, l = 0;
    double ** X = new double * [n];
    for (i = 0; i < n; i++) {
        X[i] = new double[n];
        for (j = 0; j < n; j++) {
            X[i][j] = 0.0;
        }
    }
    for (i = 1; i < n; i++) {
        X[i][i] = -1.0;
    }
    double ** Y = new double * [n];
    for (i = 0; i < n; i++) {
        Y[i] = new double[n];
        for (j = 0; j < n; j++) {
            Y[i][j] = 0.0;
        }
    }
    double ** Y_star = new double * [n];
    for (i = 0; i < n; i++) {
        Y_star[i] = new double[n];
        for (j = 0; j < n; j++) {
```

```
38              Y[i][j] = 0.0;
39          }
40      }
41      double ** M = new double * [n];
42      for (i = 0; i < n; i++) {
43          M[i] = new double[n];
44          for (j = 0; j < n; j++) {
45              M[i][j] = 0.0;
46          }
47      }
48      for (i = 0; i < n; i++) {
49          M[i][i] = 1.0;
50      }
51      double * gamma_star = new double[n];
52      double ** C = new double * [n];
53      for (i = 0; i < n; i++) {
54          C[i] = new double[n];
55          for (j = 0; j < n; j++) {
56              C[i][j] = 0.0;
57          }
58      }
59      for (i = 0; i < n; i++) {
60          C[i][i] = 1.0;
61      }
62
63      // FIRST, WE FILL THE MATRIX TO BE PROCESSED BY THE LLL ALGORITHM
64
65      //    MATRIX INPUT
66      cout << "Input epsilon" << endl;
67      cin >> epsilon;
68      X[0][0] = pow(2.0,-(n - 1.0) * n / 4.0) * pow(epsilon,n);
69      for (i = 1; i < n; i++) {
70          cin >> X[0][i];
71      }
72
73      // NOW WE PROCESS X
74
75      //    STEP 1:MAKE A COPY OF X
76      for (i = 0;i < n;i++) {
77          Y[i] = new double[n];
78          for (j = 0;j < n;j++) {
79              Y[i][j] = X[i][j];
80          }
81      }
82      //    STEP 2: COMPUTE THE GRAM-SCHMIDT ORTHOGONALIZATION
83      gramSchmidt(Y,Y_star,M,gamma_star);
84      //    STEP 3: START BY PROCESSING Y_2
85      int k = 1;
86      //    STEP 4: PERFORM THE BASIS REDUCTION
87      while (k < n) {
88          reduce(k,k - 1,Y,M,C);
89          if (gamma_star[k] >= ( ALPHA - ( M[k][k - 1] * M[k][k - 1] ) )
90              * gamma_star[k - 1]) {
91              for (l = k - 2;l >= 0;l--) {
92                  reduce(k,l,Y,M,C);
93              }
94              k = k + 1;
```

```
95          }
96          else {
97              exchange(k,Y,M,gamma_star,C);
98              if (k > 1) {
99                  k = k - 1;
100             }
101         }
102     }
103
104     // DISPLAY THE RESULTS WITH ERRORS
105
106     //    DISPLAY RESULTS
107     cout << endl;
108     cout << endl;
109     for (i = 0; i < n; i++) {
110         for (j = 0; j < n; j++) {
111             cout << C[i][j] << " ";
112         }
113         cout << endl;
114     }
115     cout << endl;
116     cout << endl;
117
118     double denominator = C[0][0];
119     for (i = 1; i < n; i++) {
120         double numerator = C[0][i];
121         cout << numerator << "/" << C[0][0] << "
    error: " << fabs(numerator/denominator - X[0][i]) << endl;
122     }
123     cout << endl;
124     cout << endl;
125
126     return 0;
127 }
128
129 // HERE WE DEFINE THE GRAM-SCHMIDT, REDUCE, AND EXCHANGE FUNCTIONS
130
131 void gramSchmidt(double ** Y,double ** Y_star,double ** M,double
132                  * gamma_star) {
133
134     // Local variables
135     int i,j,k = 0;
136     double sum = 0.0;
137
138     for (i = 0; i < n; i++) {
139         for (j = 0; j < n; j++) {
140             Y_star[i][j] = Y[i][j];
141         }
142         for (j = 0; j < i; j++) {
143             sum = 0.0;
144             for (k = 0; k < n; k++) {
145                 sum = sum + Y[i][k] * Y_star[j][k];
146             }
147             M[i][j] = sum / gamma_star[j];
148             for (k = 0; k < n; k++) {
149                 Y_star[i][k] = Y_star[i][k] - M[i][j] * Y_star[j][k];
150             }
```

```
151             }
152             for (j = 0;j < n;j++) {
153                 gamma_star[i] = gamma_star[i] + Y_star[i][j] * Y_star[i][j];
154             }
155         }
156 
157         return;
158     }
159 
160     void reduce(int k,int l,double ** Y,double ** M,double ** C) {
161 
162         /* Local variables */
163         int i,j = 0;
164 
165         if (fabs(M[k][l]) >= 0.5) {
166             for (i = 0;i < n;i++) {
167                 Y[k][i] = Y[k][i] - ( round(M[k][l]) * Y[l][i] );
168                 C[k][i] = C[k][i] - ( round(M[k][l]) * C[l][i] );
169             }
170             for (j = 0;j < l;j++) {
171                 M[k][j] = M[k][j] - ( round(M[k][l]) * M[l][j] );
172             }
173             M[k][l] = M[k][l] - round(M[k][l]);
174         }
175 
176         return;
177     }
178 
179     void exchange(int k,double ** Y,double ** M,double * gamma_star,double ** C) {
180 
181         /* Local variables */
182         int i = 0;
183         double temp = 0.0;
184         double temp_1 = 0.0;
185         double temp_2 = 0.0;
186         double nu, delta, xsi = 0.0;
187 
188         /* exchange y_k-1 and y_k */
189         for (i = 0;i < n;i++) {
190             temp_1 = Y[k - 1][i];
191             temp_2 = C[k - 1][i];
192             Y[k - 1][i] = Y[k][i];
193             C[k - 1][i] = C[k][i];
194             Y[k][i] = temp_1;
195             C[k][i] = temp_2;
196         }
197 
198         nu = M[k][k - 1];
199         delta = gamma_star[k] + (nu * nu) * gamma_star[k - 1];
200 
201         M[k][k - 1] = nu * gamma_star[k - 1] / delta;
202         gamma_star[k] = gamma_star[k] * gamma_star[k - 1] / delta;
203         gamma_star[k - 1] = delta;
204 
205         /* exchange mu(k-1,t) and mu(k,t) */
206         for (i = 0;i < k - 1;i++) {
207             temp = M[k - 1][i];
```

```
208             M[k - 1][i] = M[k][i];
209             M[k][i] = temp;
210         }
211 
212         for (i = k + 1;i < n;i++) {
213             xsi = M[i][k];
214             M[i][k] = M[i][k - 1] - ( nu * M[i][k] );
215             M[i][k - 1] = ( M[k][k - 1] * M[i][k] ) + xsi;
216         }
217 
218         return;
219     }
220 
221     // USAGE EXAMPLE
222 
223     //(base) Edwards-MacBook-Pro:LLL edwardvoskanian$ g++ -o main main.cpp
224     //(base) Edwards-MacBook-Pro:LLL edwardvoskanian$ ./main
225     //Input epsilon
226     //.1
227     //0.69314718055994530
228     //1.098612288668
229     //1.6094379124341003
230     //
231     //
232     //466 323 512 750
233     //-2819 -1954 -3097 -4537
234     //-2079 -1441 -2284 -3346
235     //-2809 -1947 -3086 -4521
236     //
237     //
238     //323/466     error: 1.41333e-05
239     //512/466     error: 0.000100158
240     //750/466     error: 4.14765e-06
241     //
242     //
243     //(base) Edwards-MacBook-Pro:LLL edwardvoskanian$
```

Bibliography

[BG13] Michael Baake and Uwe Grimm. *Aperiodic Order*, volume 1. Cambridge University Press, New York, 2013.

[BMP00] Michael Baake, Robert Vaughan Moody, and Peter A. B. Pleasants. Diffraction from visible lattice points and kth power free integers. *Discrete Mathematics*, 221(1):3–42, 2000.

[BR14] Dario A. Bini and Leonardo Robol. Solving secular and polynomial equations: A multiprecision algorithm. *Journal of Computational and Applied Mathematics*, 272:276–292, 2014.

[Bre11] Murray R. Bremner. *Lattice Basis Reduction: An Introduction to the LLL Algorithm and Its Applications*. Taylor & Francis, Boca Raton, 2011.

[Cow75] John Maxwell Cowley. *Diffraction Physics*. Elsevier Publication Co., New York, 1975.

[DS14] Eric Dubon and Juan Matías Sepulcre. On the complex dimensions of non-lattice fractal strings in connection with dirichlet polynomials. *Experimental Mathematics*, 23:13–24, 2014.

[Hay16] Alan Haynes. Notes on diophantine approximation and aperiodic order. pages 38–41, 79–96, 2016. https://www.math.uh.edu/ haynes/files/Aperiodic-Notes.pdf.

[Hof95] A. Hof. On diffraction by aperiodic structures. *Communications in mathematical physics*, 169(1):25–43, 1995.

[Lag00] Jeffery C. Lagarias. Mathematical quasicrystals and the problem of diffraction. *Directions in Mathematical Quasicrystals*, 13:61–93, 2000.

[Lap91] Michel L. Lapidus. Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl-Berry conjecture. *Transactions of the American Mathematical Society*, 325(2):465–529, 1991.

[Lap93] Michel L. Lapidus. Vibrations of fractal drums, the Riemann hypothesis, waves in fractal media and the Weyl-Berry conjecture. *Pitman Research Notes in Mathematics Series*, pages 126–209, 1993.

[Lin11] Jason Michael Lind. *An extension of the poisson summation formula to a semi-direct product*. PhD thesis, Carleton University, 2011.

[LLL82] Arjen Klaas Lenstra, Hendrik Willem Lenstra, and László Lovász. Factoring polynomials with rational coefficients. *Mathematische Annalen*, 261(4):515–534, 1982.

[LM95] Michel L. Lapidus and Helmut Maier. The Riemann hypothesis and inverse spectral problems for fractal strings. *Journal of the London Mathematical Society*, 52(1):15–34, 1995.

[LP93] Michel L. Lapidus and Carl Pomerance. The Riemann zeta-function and the one-dimensional Weyl-Berry conjecture for fractal drums. *Proceedings of the London Mathematical Society*, 3(1):41–69, 1993.

[LRŽ17] Michel L. Lapidus, Goran Radunović, and Darko Žubrinić. *Fractal Zeta Functions and Fractal Drums: Higher-Dimensional Theory of Complex Dimensions*. Springer Monographs in Mathematics, Springer, New York, 2017.

[LvF00] Michel L. Lapidus and Machiel van Frankenhuijsen. *Fractal Geometry and Number Theory: Complex dimensions of fractal strings and zeros of zeta functions*. Springer Science & Business Media, Birkhäuser, Boston, 2000.

[LvF03] Michel L. Lapidus and Machiel van Frankenhuijsen. Complex dimensions of self-similar fractal strings and Diophantine approximation. *Experimental Mathematics*, 12(1):41–69, 2003.

[LvF06] Michel L. Lapidus and Machiel van Frankenhuijsen. *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings (2nd rev. and enl. ed., 2013)*. Springer Monographs in Mathematics, Springer, New York, 2006.

[Moo00] Robert Vaughan Moody. Model sets: A survey. In *From Quasicrystals to More Complex Systems*. Springer, 2000.

[PSV13] Gaspar Mora Porta, Juan Matías Sepulcre, and Tomás Vidal. On the existence of exponential polynomials with prefixed gaps. *Bulletin of the London Mathematical Society*, 45(6):1148–1162, 2013.

[RS81] Michael Reed and Barry Simon. *I: Functional Analysis. Methods of Modern Mathematical Physics*. Elsevier Science, 1981.

[SBGC84] Dan Shechtman, Ilan Blech, Denis Gratias, and John W. Cahn. Metallic phase with long-range orientational order and no translational symmetry. *Physical Review Letters*, 20(20):19–51, 1984.

[Sch80] Wolfgang M. Schmidt. *Diophantine Approximation*, volume 785. Springer-Verlag, New York, 1980.

[Sen96] Marjorie Senechal. *Quasicrystals and Geometry.* Cambridge University Press, New York, 1996.

[Ser73] Jean-Pierre Serre. *A Course in Arithmetic,* English translation. Springer-Verlag, Berlin, 1973.

[SS09] Elias M. Stein and Rami Shakarchi. *Real Analysis: Measure Theory, Integration, and Hilbert Spaces.* Princeton University Press, New Jersey, 2009.