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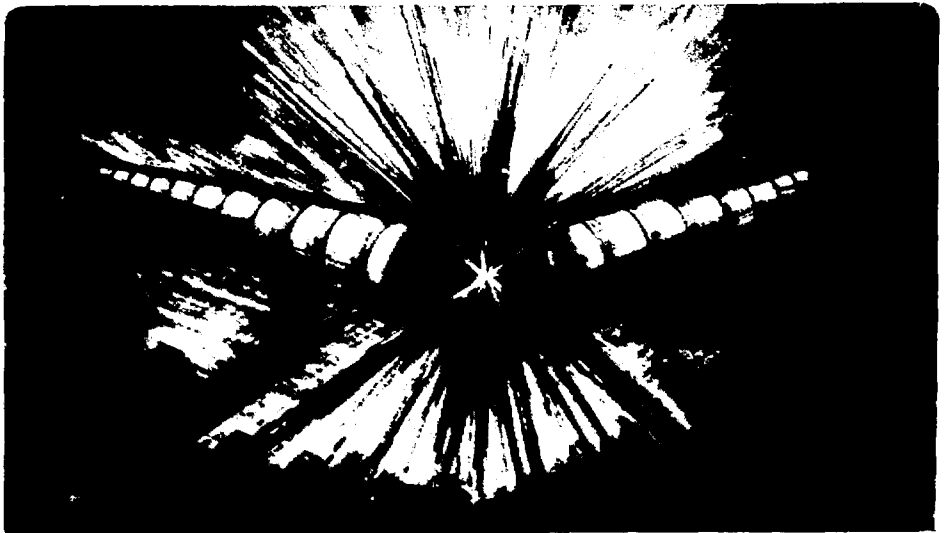
Accelerator & Fusion
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MASTER

TRANSVERSE-LONGITUDINAL COUPLING IN INTENSE BEAMS

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Summary

The coupling between transverse and longitudinal perturbations is studied self-consistently by considering a beam of K-V distribution. The analysis is carried out within the context of linearized Vlasov-Maxwell equations and electrostatic approximation. The perturbation is assumed to be azimuthally symmetric but axially non-uniform ($k_z \neq 0$). It is shown that the coupling affects both the longitudinal and transverse modes significantly in the high density and low frequency region. Two new classes of longitudinal modes are found which would not exist if the transverse motions of particles are neglected. The effect of resistive wall impedance on beam stability is also studied. It is found that the longitudinal impedance can cause the transverse modes also to be weakly unstable.

Introduction

For a stability analysis of a continuous beam in a circular accelerator or storage ring, longitudinal and transverse effects are considered separately, a procedure which is valid because space charge forces are relatively weak and the characteristic frequencies differ by orders of magnitude. In present day r.f. linacs, space charge effects are large and the frequencies in all degrees of freedom are comparable, but deterioration of beam quality is dominated by the effects of mismatches and non-linear coupling in the first few drift tubes. A new situation arises in the use of heavy ions to initiate inertial confinement fusion. In an induction linac and in the final transport lines to a reactor, space charge forces are large and all frequencies are of the order of the plasma frequency, so that one would expect a coupling of longitudinal and transverse effects. The purpose of the paper is to present a first attempt to explore this regime.

Model

We consider an infinitely long continuous beam, circular in cross-section with a constant linear external transverse focusing force. The stationary configuration is characterized by a K-V distribution in transverse coordinates and a velocity distribution in the longitudinal direction. The beam is in a circular pipe with arbitrary wall impedance but the analysis of the linearized Vlasov equation is restricted to azimuthally symmetric modes and electrostatic perturbed fields.

Taking all perturbed quantities to vary as $e^{i(\omega t - kz)}$, Vlasov's equation is solved by integration over the unperturbed orbits and Poisson's equation becomes an integro-differential equation for the electrostatic potential within the beam radius:

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial V}{\partial r} - k^2 V = \frac{k^2 \omega_p^2}{2\pi} \int_0^\infty dt\, t e^{-i\Omega t} \int_0^{2\pi} d\theta V(\rho)$$

$$-i \frac{\Omega \omega_p^2}{\pi v^2} \int_0^\infty dt\, e^{-i\Omega t} \int_0^{2\pi} d\theta \left[\sin^2 v t - \frac{r}{2\sqrt{a^2 - r^2}} \cos\theta \sin 2vt \right] \frac{dV(\rho)}{d\rho^2} + \quad (1)$$

$$\frac{2\omega_p^2 a^2}{v} \delta(r^2 - a^2) \int_0^\infty dt\, e^{-i\Omega t} \sin 2vt \; \frac{dV(R)}{dR^2},$$

where $\rho^2 = \frac{a^2}{2}(1 - \cos 2vt) + r^2 \cos 2vt - r\sqrt{a^2 - r^2} \times \cos\theta \sin 2vt,$

$R^2 = a^2 \cos^2 vt,$

$a =$ beam radius,

$\Omega = \omega - k v_0,$

$\omega_p^2 = 4\pi n_0 z^2 e^2/m \sim$ plasma frequency,

$v^2 = v_0^2 - \omega_p^2/2 \sim$ betatron frequency,

$v_0^2 \sim$ betatron frequency at zero intensity,

and the longitudinal velocity distribution is taken to be a δ-function at the beam velocity, v_0. The solution of (1) is to be matched at the beam radius to the free space solution

$$V(r) \sim K_0(kr)I_0(kb) - I_0(kr)K_0(kb) -$$

$$i \frac{\omega Z}{ck}[I_0'(kb)K_0(kr) - K_0'(kb)I_0(kr)] \quad (2)$$

where b is the pipe radius and Z the wall impedance in units of Z_0. For HIF, $v^2 \ll v_0^2$ and since the bunches are in fact not very long and would almost fill the pipe, values of ka up to unity or greater are of interest.

Equation (1), with $k = 0$, is the same as the one investigated by Gluckstern[1] for stability of transverse modes. In that case, the

$$\text{solutions of (1) are } V = P_n\left(1 - \frac{2r^2}{a^2}\right) + P_{n+1}\left(1 - \frac{2r^2}{a^2}\right),$$

where P_n is the nth Legendre polynomial. Eqn. (1) then reduces to an (n+1) degree polynomial, the roots of which are the desired eigenvalues. With $k \neq 0$, the equation is not soluble with a finite polynomial in r^2, but an expansion in Legendre polynomials of the same argument is suggested in order to see how these identified modes interact with longitudinal modes and with each other.

After considerable manipulation, one arrives at a recursion relation:

$$W_jA_{j+1} + (U_j + W_{j-1} + W_j)A_j + W_{j-1}A_{j-1} = 0$$

$$j = 1,2 \ldots. \tag{3}$$

where A_j is the coefficient of $P_j(1-2r^2/a^2) + P_{j-1}(1 - 2r^2/a^2)$ in the expansion ($j\neq0$), and A_0 is the value of the potential at $r = a$. The matching condition at $r = a$ gives:

$$[kaV_0'/V_0 - W_0]A_0 = W_0A_1 \tag{4}$$

where V_0 is the potential for $r \geq a$, given by Eqn. (2).

The other symbols in (3) and (4) are defined by:

$$U_j = 4j+1 \frac{\omega_p^2}{\nu^2} \int_0^\infty dt \, e^{-i\Omega t}[P_{j-1}(\cos 2\nu t) - P_j(\cos 2\nu t)],$$

$$W_j = \frac{k^2a^2}{2(2j+1)} [1 + \omega_p^2 \int_0^\infty dt \, t e^{-i\Omega t}P_j(\cos 2\nu t)].$$

The frequencies of the modes found by Gluckstern are given by $U_j = 0$.

Results

The eigen-frequencies are given by setting the infinite determinant of the coefficients of A_j in equations (3) and (4) equal to zero; they are approximated for the lower modes by truncating the determinant at some order. Before describing the detailed results, two limiting cases should be mentioned. If the betatron frequency is allowed to approach infinity, we should have an accurate description of purely longitudinal phenomena. For ka << 1, one finds the well-known relation,

$$\Omega^2 = \frac{k^2a^2}{4}\omega_p^2 \left(\frac{1}{2} + \ln \frac{b^2}{a^2} - \frac{2iv_{0z}}{kcb}\right)^* \tag{5}$$

but also an infinite set of frequencies,

$$\Omega^2 = \frac{k^2a^2}{4} \omega_p^2 \frac{\bar{P}_n}{2n(n+1)} \tag{6}$$

which describe a longitudinal motion internal to the beam with a perturbed potential equal to zero at the edge of the beam. Here $\bar{P}_n$ denotes the average value in time of $P_n(\cos 2\nu t)$. Also there appears another class of modes with frequencies $2n\nu$ as k approaches zero. These correspond at k = 0 to a configuration in which the density remains constant in time, and there is no perturbed field, but a distribution in longitudinal velocity is correlated with transverse amplitude and phase in such a way that the average axial velocity at fixed radius varies sinusoidally in time at multiples of twice the betatron frequency. These two sets of modes have not been recognized before; for finite ν and k they can also interact with more familiar ones to cause potential instabilities.

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The factor, 1/2, in the bracket corresponds to an average of E_z over the beam area in the usual longitudinal analysis. For finite ν, it becomes

$$\frac{1}{2}\left(\frac{4\nu^2}{4\nu^2 + \omega_p^2}\right).$$

Out of the sequence of an infinite number of roots of the dispersion relation (4), the first ten roots are shown in Fig. 1 as functions of normalized betatron frequency ν/ν_0 for values of ka = 1.0 and b/a = 1.5. The transverse modes, except for the lower T_2 and at the crossing of T_1 and T_3, are not significantly affected by the longitudinal perturbations. The growth rate of the well known instability in the confluent region of the two lower T_3 modes remains roughly the same as for k = 0, but the merging point is shifted from $\nu/\nu_0 = .375$ for ka = 0 to $\nu/\nu_0 = .39$ for ka = 1.0.

Pronounced longitudinal-transverse coupling appears in the high density and low frequency region, which is more clearly depicted in fig. 2. The first direct impact between longitudinal and transverse modes happens when the frequency of T_2 approaches the frequency of the L_1 mode, which occurs at $\nu/\nu_0 = .44$ in Fig. 2. In that region, the frequencies of L_1 and T_2 modes are complex conjugate pairs; the maximum growth rate is around $.08\nu_0$. A more severe instability occurs when T_2 and L_2 merge in the region from $\nu/\nu_0 = .32$ to $\nu/\nu_0 = .2$. The maximum growth rate in this region is roughly $.2\nu_0$.

In addition to the mixing of T modes and L modes, a mild instability of growth rate about $10^{-3}\nu_0$ is found when T_3 crosses T_1 at $\nu/\nu_0 = .48$. The two upper longitudinal modes ℓ_1 and ℓ_3 appear to remain stable with increasing values of ka.

The effect of a resistive wall impedance on these ten roots has been examined for the value of Z_w/ck ranging from 0 to .3 which range corresponds to conditions anticipated in a linear induction accelerator. It is found that the wall impedance affects modes L_1 and T_1 most. The familiar mode, L_1, is unstable in the familiar way and T_1, the envelope oscillation mode is weakly unstable, with growth rate of $5 \times 10^{-4}\nu_0$ at $Z_w/ck = 0.2$.

Conclusions

Growth rates of ~ $0.1\nu_0$ are very substantial, corresponding to an e-folding distance along an accelerator of about ten magnet periods for a $60°$ focusing lattice. Although the model used in this work is a simple one, the results suggest the need to watch for such effects in a three-dimensional computer simulation, which is probably needed to investigate the problem thoroughly. A qualitative picture is at least established; only modes which involve an average axial field over the beam or surface motion are influenced by the wall impedance, but at the same time there are new modes involving internal motion which must be taken into consideration.

Reference

1. R.L. Gluckstern, Proc. 1970 Proton Linac Conference (NAL), p. 811.
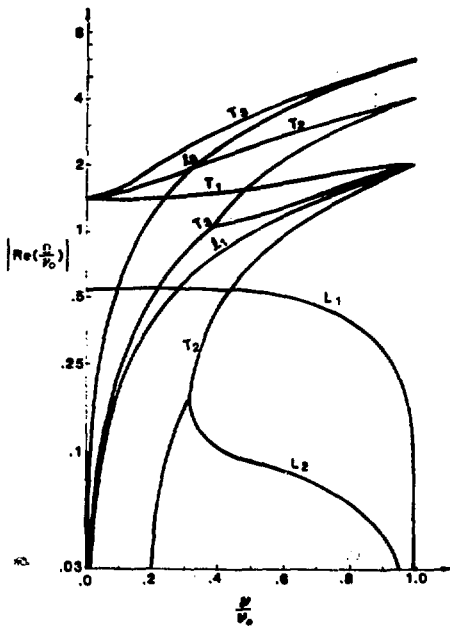
Fig. 1. Real part of the first ten roots of dispersion relation, shown as functions of betatron frequency. Quantities are normalized to the zero current betatron frequency. The transverse modes T_n correspond to

Gluckstern's axi-symmetric modes.[1] ℓ_1 and ℓ_3 are the longitudinal modes due to the correlation between the longitudinal velocity and the transverse motion. L_1 is the ordinary longitudinal mode. L_2 is the lowest mode indicated by Eqn. (6).
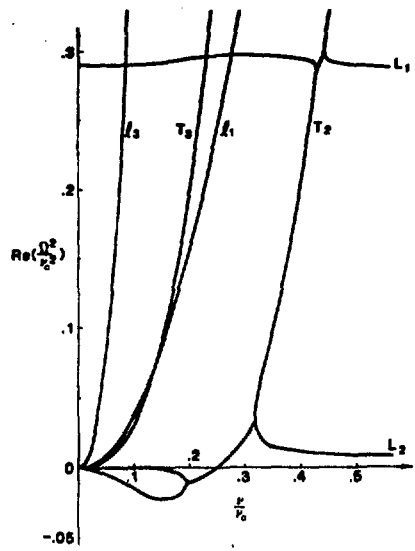

Fig. 2. Real part of the square of the Doppler shifted frequency ($\Omega = \omega - k v_0$) versus betatron frequency in the high density-low frequency domain. Quantities are normalized to v_0^2 and v_0 respectively.