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Control of cyber-physical systems using incremental properties of physical systems

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Majid Zamani

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Abstract of the Dissertation

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Cyber-Physical Systems (CPSs) are complex systems resulting from intricate interaction of digital computational devices with the physical systems. With the recent dazzling advances in computational devices, CPSs have become ubiquitous in modern technology. The increasing presence of CPSs on one hand and the incapability of current methods to analyze them on the other hand, impel the development of novel approaches for analysis and design. In CPSs, embedded computers have the responsibility of monitoring and controlling the physical plants using feedback loops using which physical plants affect computations and vice versa. In these closed-loop fashions, controllers implemented in software are termed embedded control software. Increasing use of embedded control software in life critical applications, such as aircraft flight control systems and automotive engine control systems, demands lots of efforts on software verifications and validations which are very costly. On the other hand, by changing the center of gravity from verification to design, it is possible to synthesize correct-by-design embedded control software while providing formal guarantees of correctness. The foundation of this proposed approach relies on some technical results showing how to construct equivalent finite state models for differential equation models describ-

ing physical plant. These finite state models are simpler descriptions of physical plant in which each state of the finite model represents a collection or aggregate of states in the physical plant. Similar finite state models are used in software and hardware modeling, which enable the composition of such models with the finite models of the physical systems. The results of this composition are finite models capturing the behavior of the physical systems interacting with the digital computation devices. Once such models are available, the methodologies and tools developed in computer science for verification and control synthesis purposes can be easily employed to physical systems, via these models. In the first part of this thesis I take an important step in my quest to synthesize correct-by-design embedded control software for CPSs by constructing finite state models for control systems. I propose a novel technique to compute bisimilar finite state models of incrementally stable nonlinear control systems. I show on practical examples that the finite state models computed by my procedure can be several orders of magnitude smaller than existing approaches. Moreover, I propose another technique to compute (not necessarily bisimilar) finite state models of any nonlinear control system as long as I am interested in its behavior in a compact set. In the second part of this thesis I will show some incremental properties under which nonlinear control systems admit finite state models. I propose some analysis tools to check those properties. Moreover, I provide some design techniques providing controllers enforcing those incremental properties for some special classes of nonlinear control systems.

The dissertation of Majid Zamani is approved.

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CHAPTER 1

Introduction

1.1 Motivation

Embedded control systems result from the integration of computational parts and control processes: on-board computers control physical processes through feedback loops. Although embedded control systems have become ubiquitous in modern technology, their designs are based on ad-hoc solutions trying to make the connection between the classical techniques of control theory and embedded systems engineering. As the number of embedded control systems is expected to grow significantly in the near future, it becomes necessary to develop specific approaches, allowing the system design to be more reliable and efficient.

Most embedded control systems are of hybrid nature: discrete dynamics modeling computing parts including hardware and software and continuous dynamics modeling control systems. The ability to handle the interaction between continuous and discrete dynamics is a prerequisite of a rigorous formal framework for the design and verification of embedded control systems. Verification and control of embedded systems generally require the development of specific techniques combining ideas of the theories of discrete and continuous systems. The complexity induced by the interaction between discrete and continuous dynamics often makes it difficult to obtain analytical results. For this reason, the verification and synthesis of controllers for embedded systems is often addressed by methods of abstraction in which continuous dynamics are approximated by discrete symbolic models. My work in this thesis lies in two different directions. First, I analyze the use of symbolic models for the synthesis of correct-by-design embedded controllers. Second, I analyze and provide the assumptions under which nonlinear control systems admit symbolic models.

The first part of the thesis is devoted to the study of symbolic abstractions in control. Symbolic abstractions result from replacing aggregates or collections of continuous states of a control system by discrete states. I study the construction of symbolic models for some classes of control systems described by differential equations. Similar models are used in software and hardware modeling. Therefore, it enables us to compose such models with the symbolic abstraction of the continuous dynamics. The result of this composition are symbolic models capturing the behavior of the complete embedded control systems. By having specifications in the form of finite symbolic models, the synthesis of controllers for the (finite-state) symbolic abstraction can be reduced to a fixed-point computation. Finally, the resulting controllers can be refined into hybrid-controllers that can be deployed on the actual implementation. The controllers, obtained by this mentioned design flow, are guaranteed to satisfy the provided specification. This justifies the name of correct-by-design synthesis. We can divide abstraction methods for the synthesis of controllers into three stages. First, a discrete approximation of a control system, called a symbolic model, is constructed. This stage is the focus of the first part of my thesis. Then, by having symbolic models with a finite number of states or symbols, we are able to use algorithmic techniques, developed for discrete-event systems to automatically synthesize controllers enforcing control and software requirements using a fixed-point computation. Finally, the controller obtained is refined in order to control the original control system.

The second part of the thesis is devoted to the study of incremental properties of nonlinear control systems. Stability is arguably one of the core concepts upon which our understanding of dynamical and control systems has been built. The related notion of incremental stability, however, has received much less attention until recently, when it was successfully used as a tool for the analysis and design of intrinsic observers, output regulation of nonlinear control systems, frequency estimators, synchronization of coupled identical dynamical systems, symbolic models for nonlinear control systems, bio-molecular systems, nonlinear analog circuits, systems over finite alphabets, global synchronization in networks of cyclic feedback systems, and piecewise affine systems with actuator and sensor faults. However, most of the existing controller design techniques provide controllers enforcing stability rather than incremental stability. Hence, there is a growing need to extend existing methods or develop new ones for the purpose of designing incrementally stabilizing controllers. In the second part of my thesis, I provide some analysis and design approaches providing controllers rendering some classes of control systems incrementally stable.

1.2 Outline of the thesis

This thesis is divided in 5 chapters, the first of which is the current introduction. Chapter 2 presents basic notions from control theory that will be frequently used throughout this thesis. Chapter 3 studies the suitability of symbolic models for the synthesis of correct-by-design embedded controllers. Chapter 4 is devoted to study of incremental properties of nonlinear control systems. Finally, in Chapter 5, a brief suggestions for future research is provided.

For clarity of exposition, both Chapter 3 and Chapter 4 follow a common structure. Both chapters start with an introduction including: a description of the problem addressed, a brief literature review, and a statement of the contributions made. Following the introduction, a section establishing the preliminaries specific to the chapter is included. The developed techniques are detailed in subsequent sections, followed by a section illustrating their efficiency on examples.

The chapters are concluded with a discussion section.

CHAPTER 2

Some notions of mathematical control theory

2.1 Notation

The identity map on a set A is denoted by 1_A. If A is a subset of B we denote by $\imath_A : A \hookrightarrow B$ or simply by $\imath$ the natural inclusion map taking any $a \in A$ to $\imath(a) = a \in B$. The symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the set of natural, nonnegative integer, integer, real, positive, and nonnegative real numbers, respectively.

The symbols I_m, $0_{m\times n}$, and 0_m denote the identity and zero matrices on $\mathbb{R}^{m\times m}$ and $\mathbb{R}^{m\times n}$ and the zero vector in $\mathbb{R}^m$, respectively. Given a vector $x \in \mathbb{R}^n$, we denote by x_i the i–th element of x, by $\|x\|$ the infinity norm of x, and by $\|x\|_2$ the Euclidean norm of x; we recall that $\|x\| = \max\{|x_1|, |x_2|, ..., |x_n|\}$, and $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$, where $|x_i|$ denotes the absolute value of x_i.

Given a measurable function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty := \text{(ess)}\sup\{\|f(t)\|, t \geq 0\}$, $\|f\|_{2,\infty} := \text{(ess)}\sup\{\|f(t)\|_2, t \geq 0\}$ and $\|f\|_{[0,\tau)} := \text{(ess)}\sup\{\|f(t)\|, t \in [0,\tau)\}$; f is essentially bounded if $\|f\|_\infty < \infty$. For a given time $\tau \in \mathbb{R}^+$, define f_τ so that $f_\tau(t) = f(t)$, for any $t \in [0,\tau)$, and $f_\tau(t) = 0$ elsewhere; f is said to be locally essentially bounded if for any $\tau \in \mathbb{R}^+$, f_τ is essentially bounded. A function $f : \mathbb{R}^n \to \mathbb{R}_0^+$ is called radially unbounded if $f(x) \to +\infty$ as $\|x\| \to +\infty$. A function f is said to be smooth if it is an infinitely differentiable function of its arguments.

A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}_\infty$ with respect to r and, for each fixed nonzero r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a global diffeomorphism, and $X : \mathbb{R}^n \to \mathbb{R}^n$ be a map, we denote by $\phi_* X$ the map defined by $(\phi_* X)(y) = \frac{\partial \phi}{\partial x}\big|_{x=\phi^{-1}(y)} X \circ \phi^{-1}(y)$. Let now $G : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ be a smooth map. The notation $\phi^* G : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ denotes the smooth map $(\phi^* G)(x) = (\frac{\partial \phi}{\partial x})^T G(\phi(x))(\frac{\partial \phi}{\partial x})$. A Riemannian metric $G : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is a smooth map on $\mathbb{R}^n$ such that, for any $x \in \mathbb{R}^n$, $G(x)$ is a symmetric positive definite matrix [Lee03]. For any $x \in \mathbb{R}^n$ and smooth functions $I, J : \mathbb{R}^n \to \mathbb{R}^n$, one can define the scalar function $\langle I, J \rangle_G$ as $I^T(x)G(x)J(x)$. We will still use the notation $\langle I, J \rangle_G$ to denote $I^T G J$ even if G does not represent any Riemannian metric. A function $\mathbf{d} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ is a metric on $\mathbb{R}^n$ if for any $x, y, z \in \mathbb{R}^n$, the following three conditions are satisfied: i) $\mathbf{d}(x, y) = 0$ if and only if $x = y$; ii) $\mathbf{d}(x, y) = \mathbf{d}(y, x)$; and iii) $\mathbf{d}(x, z) \leq \mathbf{d}(x, y) + \mathbf{d}(y, z)$. We use the pair $(\mathbb{R}^n, \mathbf{d})$ to denote a metric space $\mathbb{R}^n$ equipped with the metric $\mathbf{d}$. We use the notation $\mathbf{d}_G$ to denote the Riemannian distance function provided by the Riemannian metric G [Lee03]. We refer to the proof of Lemma 4.3.12 in the paper for the definition of $\mathbf{d}_G$. For a set $\mathcal{A} \subseteq \mathbb{R}^n$, a metric $\mathbf{d}$, and any $x \in \mathbb{R}^n$, we abuse the notation by using $\mathbf{d}(x, \mathcal{A})$ to denote the point-to-set distance, defined by $\mathbf{d}(x, \mathcal{A}) = \inf_{y \in \mathcal{A}} \mathbf{d}(x, y)$. Given measurable functions $f : \mathbb{R}_0^+ \to \mathbb{R}^n$ and $g : \mathbb{R}_0^+ \to \mathbb{R}^n$, we define $\mathbf{d}(f, g)_\infty :=$ (ess)sup$\{\mathbf{d}(f(t), g(t)), t \geq 0\}$ and $\mathbf{d}(f, g)_{[0,\tau)} :=$ (ess)sup$\{\mathbf{d}(f(t), g(t)), t \in [0, \tau)\}$.

Given a matrix $M = \{m_{ij}\} \in \mathbb{R}^{n \times m}$, the infinity norm of M is $\|M\| = \max_{1 \leq i \leq m} \sum_{j=1}^n |m_{ij}|$. The closed ball centered at $x \in \mathbb{R}^n$ with radius ε is defined by $\mathcal{B}_\varepsilon(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq \varepsilon\}$. A set $B \subseteq \mathbb{R}^n$ is called a *box* if $B = \prod_{i=1}^n [c_i, d_i]$, where $c_i, d_i \in \mathbb{R}$ with $c_i < d_i$ for each $i \in \{1, \ldots, n\}$. The *span* of a box B is defined as $span(B) = \min\{|d_i - c_i| \mid i = 1, \ldots, n\}$. For a box B and $\eta \leq span(B)$, define the η-approximation $[B]_\eta = \{b \in B \mid b_i = k_i\eta$ for some $k_i \in \mathbb{Z}, i = 1, \ldots, n\}$. Note that $[B]_\eta \neq \varnothing$ for any $\eta \leq span(B)$. Geometrically, for any $\eta \in \mathbb{R}^+$ with $\eta \leq span(B)$ and $\lambda \geq \eta$ the collection of sets $\{\mathcal{B}_\lambda(p)\}_{p \in [B]_\eta}$ is a finite covering of B, i.e., $B \subseteq \bigcup_{p \in [B]_\eta} \mathcal{B}_\lambda(p)$. By defining $[\mathbb{R}^n]_\eta = \{a \in \mathbb{R}^n \mid a_i = k_i\eta, k_i \in \mathbb{Z}, i = 1, \cdots, n\}$, the set $\bigcup_{p \in [\mathbb{R}^n]_\eta} \mathcal{B}_\lambda(p)$ is a countable covering of $\mathbb{R}^n$ for any $\eta \in \mathbb{R}^+$ and $\lambda \geq \eta/2$. We extend the notions of span and approximation to finite unions of boxes as follows. Let $A = \bigcup_{j=1}^{M} A_j$, where each A_j is a box. Define $span(A) = \min\{span(A_j) \mid j = 1, \ldots, M\}$, and for any $\eta \leq span(A)$, define $[A]_\eta = \bigcup_{j=1}^{M} [A_j]_\eta$. We identify a relation $R \subseteq A \times B$ with the map $R : A \to 2^B$ defined by $b \in R(a)$ iff $(a, b) \in R$. Given a relation $R \subseteq A \times B$, R^{-1} denotes the inverse relation defined by $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$.

2.2 Control systems

The class of control systems that we consider in this thesis is formalized in the following definition.

Definition 2.2.1. *A control system is a quadruple:*

$$\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f),$$

where:

- $\mathbb{R}^n$ *is the state space;*
- $\mathsf{U} \subseteq \mathbb{R}^m$ *is the input set;*
- $\mathcal{U}$ *is the set of all measurable and locally essentially bounded functions of time from intervals of the form* $]a, b[\subseteq \mathbb{R}$ *to* U *with* $a < 0$, $b > 0$;
- $f : \mathbb{R}^n \times \mathsf{U} \to \mathbb{R}^n$ *is a continuous map satisfying the following Lipschitz assumption: for every compact set* $Q \subset \mathbb{R}^n$, *there exists a constant* $Z \in \mathbb{R}^+$

such that $\|f(x,u) - f(y,u)\| \leq Z\|x - y\|$ *for all* $x, y \in Q$ *and all* $u \in \mathsf{U}$.

A curve $\xi :]a,b[\to \mathbb{R}^n$ is said to be a *trajectory* of Σ if there exists $\upsilon \in \mathcal{U}$ satisfying:

$$\dot{\xi}(t) = f\left(\xi(t), \upsilon(t)\right), \tag{2.2.1}$$

for almost all $t \in]a,b[$. We also write $\xi_{x\upsilon}(\tau)$ to denote the point reached at time τ under the input υ from initial condition $x = \xi_{x\upsilon}(0)$; this point is uniquely determined, since the assumptions on f ensure existence and uniqueness of trajectories [Son98]. Although we have defined trajectories over open domains, we shall refer to trajectories $\xi_{x\upsilon} : [0,\tau] \to \mathbb{R}^n$ and input curves $\upsilon : [0,\tau[\to \mathsf{U}$ defined on domains $[0,\tau]$, $[0,\tau[$, $\tau \in \mathbb{R}^+$, with the understanding of the existence of a trajectory $\xi'_{x\upsilon'} :]a,b[\to \mathbb{R}^n$ and input curve $\upsilon' :]a,b[\to \mathsf{U}$ such that $a < 0$, $b > \tau$, $\xi_{x\upsilon} = \xi'_{x\upsilon'}|_{[0,\tau]}$ and $\upsilon = \upsilon'|_{[0,\tau[}$. Note that by continuity of ξ, we have that $\xi_{x\upsilon}(\tau)$ is uniquely defined as the left limit of $\xi_{x\upsilon}(t)$ with $t \to \tau$.

A control system Σ is said to be forward complete if every trajectory can be extended to an interval of the form $]a,\infty[$. Sufficient and necessary conditions for a system to be forward complete can be found in [AS99]. A control system Σ is said to be smooth if f is smooth.

2.3 Incremental stability

We start by introducing the following definitions which were inspired by the notions of incremental global asymptotic stability (δ-GAS) and incremental input-to-state stability (δ-ISS) presented in [Ang02].

Definition 2.3.1. *A control system* Σ *is incrementally globally asymptotically stable* ($\delta_\exists$*-GAS) if it is forward complete and there exist a metric* $\mathbf{d}$ *and a* $\mathcal{KL}$ *function* β *such that for any* $t \in \mathbb{R}_0^+$, *any* $x, x' \in \mathbb{R}^n$ *and any* $\upsilon \in \mathcal{U}$ *the following* *condition is satisfied:*

$$\mathbf{d}\left(\xi_{x\upsilon}(t),\xi_{x'\upsilon}(t)\right)\leq\beta\left(\mathbf{d}\left(x,x'\right),t\right). \tag{2.3.1}$$

While δ-GAS, as defined in [Ang02], requires the metric $\mathbf{d}$ to be the Euclidean metric, Definition 2.3.1 only requires the existence of a metric; hence, the existential quantifier in the acronym $\delta_\exists$-GAS. The condition (2.3.1), when the metric $\mathbf{d}$ is the Euclidean metric, is as follows:

$$\left\|\xi_{x\upsilon}(t)-\xi_{x'\upsilon}(t)\right\|\leq\beta\left(\left\|x-x'\right\|,t\right). \tag{2.3.2}$$

We note that while δ-GAS is not generally invariant under changes of coordinates, $\delta_\exists$-GAS is. If $\phi:\mathbb{R}^n\rightarrow\mathbb{R}^n$ is a global diffeomorphism, inequality (2.3.1) transforms under changes of coordinate ϕ to:

$$\mathbf{d}'\left(\phi\circ\xi_{x\upsilon}(t),\phi\circ\xi_{x'\upsilon}(t)\right)\leq\beta\left(\mathbf{d}'\left(\phi(x),\phi(x')\right),t\right),$$

where $\mathbf{d}'(y,y')=\mathbf{d}(\phi^{-1}(y),\phi^{-1}(y'))$. Nevertheless, when the origin is an equilibrium point for Σ and the map $\psi:\mathbb{R}^n\rightarrow\mathbb{R}_0^+$, defined by $\psi(x)=\mathbf{d}(x,0)$, is continuous[1] and radially unbounded[2], both $\delta_\exists$-GAS and δ-GAS imply global asymptotic stability (GAS).

Definition 2.3.2. *A control system Σ is incrementally input-to-state stable ($\delta_\exists$-ISS) if it is forward complete and there exist a metric $\mathbf{d}$, a $\mathcal{KL}$ function β, and a $\mathcal{K}_\infty$ function γ such that for any $t\in\mathbb{R}_0^+$, any $x,x'\in\mathbb{R}^n$, and any υ, $\upsilon'\in\mathcal{U}$ the following condition is satisfied:*

$$\mathbf{d}\left(\xi_{x\upsilon}(t),\xi_{x'\upsilon'}(t)\right)\leq\beta\left(\mathbf{d}\left(x,x'\right),t\right)+\gamma\left(\left\|\upsilon-\upsilon'\right\|_\infty\right). \tag{2.3.3}$$

By observing (2.3.1) and (2.3.3), it is readily seen that $\delta_\exists$-ISS implies $\delta_\exists$-GAS while the converse is not true in general. Moreover, whenever the metric $\mathbf{d}$ is the

---

[1]Here, continuity is understood with respect to the Euclidean metric.

[2]Under the stated assumptions it can be shown that $\underline{\alpha}(\|x\|)\leq\psi(x)\leq\overline{\alpha}(\|x\|)$ for some $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$.

Euclidean metric, $\delta_\exists$-ISS becomes δ-ISS as defined in [Ang02] as the following:

$$\|\xi_{xv}(t) - \xi_{x'v'}(t)\| \leq \beta\left(\|x - x'\|, t\right) + \gamma\left(\|v - v'\|_\infty\right). \tag{2.3.4}$$

We note that while δ-ISS is not generally invariant under changes of coordinates, $\delta_\exists$-ISS is. Once again, although $\delta_\exists$-ISS is not equivalent to δ-ISS, both notions imply input-to-state stability whenever the origin is an equilibrium point for Σ and the map $\psi : \mathbb{R}^n \to \mathbb{R}_0^+$, defined by $\psi(x) = \mathbf{d}(x, 0)$, is continuous[3] and radially unbounded.

Remark 2.3.3. *For linear control systems, the functions β and γ in Definition 2.3.2, when metric is the Euclidean one, can be explicitly computed as follows. It can be readily verified that any linear control system:*

$$\dot{\xi} = A\xi + Bv, \quad \xi(t) \in \mathbb{R}^n, \; v(t) \in \mathsf{U} \subseteq \mathbb{R}^m, \tag{2.3.5}$$

is δ-ISS if and only if A is globally asymptotically stable, i.e., every eigenvalue of A has strictly negative real part. Then, the functions β and γ can be chosen as:

$$\beta(r, t) = \left\|\mathsf{e}^{At}\right\| r; \quad \gamma(r) = \left(\|B\| \int_0^\infty \left\|\mathsf{e}^{As}\right\| ds\right) r, \tag{2.3.6}$$

where $\|\mathsf{e}^{At}\|$ denotes the infinity norm of e^{At}.

In Chapter 4, we provide Lyapunov characterizations and contraction metric descriptions for incremental stability.

2.4 Incremental forward completeness

We now describe a weaker concept that is satisfied even in the absence of stability.

Definition 2.4.1. *A control system Σ is incrementally forward complete (δ-FC) if it is forward complete and there exist continuous functions $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$*

[3] Here, continuity is understood with respect to the Euclidean metric.

and $\gamma : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ *such that for every* $s \in \mathbb{R}^+$*, the functions* $\beta(\cdot, s)$ *and* $\gamma(\cdot, s)$ *belong to class* $\mathcal{K}_\infty$*, and for any* $x, x' \in \mathbb{R}^n$*, any* $t \in \mathbb{R}^+$*, and any* $\upsilon, \upsilon' \in \mathcal{U}$*, the following condition is satisfied:*

$$\|\xi_{x\upsilon}(t) - \xi_{x'\upsilon'}(t)\| \leq \beta(\|x - x'\|, t) + \gamma(\|\upsilon - \upsilon'\|_\infty, t). \tag{2.4.1}$$

Incremental forward completeness[4] requires the distance between two arbitrary trajectories to be bounded by the sum of two terms capturing the mismatch between the initial conditions and the mismatch between the inputs as shown in (2.4.1).

As an example, for a linear control system (not necessarily stable):

$$\dot{\xi} = A\xi + B\upsilon, \;\; \xi(t) \in \mathbb{R}^n, \; \upsilon(t) \in \mathsf{U} \subseteq \mathbb{R}^m,$$

the functions β and γ can be chosen as:

$$\beta(r, t) = \left\|\mathsf{e}^{At}\right\| r; \;\; \gamma(r, t) = \left(\int_0^t \left\|\mathsf{e}^{As}B\right\| ds\right) r, \tag{2.4.2}$$

where $\|\mathsf{e}^{At}\|$ denotes the infinity norm of e^{At}. From (2.3.4) and (2.4.1), we can immediately see that δ-ISS implies δ-FC. However, the converse is not true, in general, since the function β in (2.4.1) is not required to be a decreasing function of t and the function γ in (2.4.1) is allowed to depend on t while this is not the case in (2.3.4).

[4]We note that δ-FC implies uniform continuity of the map $\phi_t : \mathbb{R}^n \times \mathcal{U} \rightarrow \mathbb{R}^n$ defined by $\phi_t(x, \upsilon) = \xi_{x\upsilon}(t)$ for any fixed $t \in \mathbb{R}_0^+$. Here, uniform continuity is understood with respect to the topology induced by the infinity norm on $\mathbb{R}^n$, the sup norm on $\mathcal{U}$, and the product topology on $\mathbb{R}^n \times \mathcal{U}$.

CHAPTER 3

Symbolic models for nonlinear control systems

3.1 Introduction

Many cyber-physical systems involve the complex interplay between continuous controlled dynamical systems and discrete controllers. Correctness requirements for these systems involve temporal specifications about the evolution of the dynamics, which are not easily amenable to classical continuous controller synthesis techniques. As a result, in recent years, a lot of research has focused on *symbolic models* of systems involving both continuous and discrete components (so called *hybrid systems*). A symbolic model is a discrete approximation of the continuous system such that controllers designed for the approximation can be refined to controllers for the original system. Symbolic models are interesting because they allow the algorithmic machinery for controller synthesis of discrete systems w.r.t. temporal specifications [EJ91, Tho95, MPS95] to be used to automatically synthesize controllers for hybrid systems.

3.1.1 Previous work

In the past years several different abstraction techniques have been developed to assist in the synthesis of controllers enforcing complex specifications. This work is concerned with symbolic abstractions resulting from replacing aggregates or collections of states of a control system by symbols. When a symbolic abstraction with a finite number of states or symbols is available, the synthesis of the controllers can be reduced to a fixed-point computation over the finite-state abstraction [Tab09]. Moreover, by leveraging computational tools developed for discrete-event systems [KG95, CL99] and games on automata [dAHM01, MNA03, AVW03], one can synthesize controllers satisfying specifications difficult to enforce with conventional control design methods. Examples of such specification classes include logic specifications expressed in linear temporal logic or automata on infinite strings.

The quest for symbolic abstractions has a long history including results on timed automata [AD90], rectangular hybrid automata [HKPV98], and o-minimal hybrid systems [LPS00, BM05]. Early results for classes of control systems were based on dynamical consistency properties [CW98], natural invariants of the control system [KASL00], l-complete approximations [MRO02], and quantized inputs and states [FJL02, BMP02]. Recent results include work on piecewise-affine and multi-affine systems [HCS06, BH06], set-oriented discretization approach for discrete-time nonlinear optimal control problem [Jun04], abstractions based on an elegant use of convexity of reachable sets for sufficiently small time [Rei09], and the use of incremental input-to-state stability [PGT08, PT09, PPDT10, GPT09].

3.1.2 Contributions

We focus here on digital control systems, in which there is a sampling time τ and the control action is chosen from a compact set and held constant for τ time units.

First contribution: current approaches to building the symbolic model, such as [PGT08, PT09, PPDT10, GPT09], proceed as follows. First, they choose discretizations of the state and input sets. Then, they use either the incremental stability assumption or incremental Lyapunov functions to show that if the discretizations are sufficiently small, and the sampling time τ is sufficiently big, then the resulting discrete abstraction is ε-approximate bisimilar to the original system. If the sampling time, which is usually not under the control of the verification engineer, is not sufficiently large, the technique will not apply. Even if the method applies, the resulting state space is often prohibitively large. This is usually the case for symbolic models built using conservative Lyapunov functions [GPT09].

We show a construction of approximately bisimilar models for digital control systems that improves upon known algorithms. The insight in our construction is to consider a number of sampling steps instead of only one step. That is, we dilate the quantum of time of the control system and observe the system only every k steps, for some parameter k. Then, instead of requiring that the sampling time is sufficiently big, we only require that the number of steps is chosen sufficiently large, so that the technique is always applicable. Further, we demonstrate experimentally in Section 3.4 that our technique can give symbolic models that require a much coarser discretization of the state and input sets, resulting in symbolic models with many fewer states, while guaranteeing ε-approximate bisimulation with the original system.

Second contribution: our next results improve upon most of the existing techniques in two directions: i) by being applicable to larger classes of control systems; ii) by not requiring the exact computation of reachable sets which is a hard task in general. In the first direction, our technique improves upon the results in [BMP02, HCS06, BH06] by being applicable to systems not restricted to nonholonomic chained-form, piecewise-affine, and multi-affine systems, respectively, and upon the results in [PGT08, PT09, PPDT10, GPT09] by not requiring any stability assumption. In the second direction, our technique improves upon the results in [MRO02, FJL02] by not requiring the exact computation of reachable sets. The results in [Jun04] offer a discretization tailored to optimal control while our discretization is independent of the control objective. In [Rei09] a different abstraction technique is proposed that is also applicable to a wide class of control systems and does not require the exact computation of reachable sets. Such technique provides tight over-approximations of reachable sets based on convexity but requires small sampling times. Other efficient techniques are available in the literature for computing over-approximations of reachable sets. For example, [Jun00, DJ02, SP90] provide tight over-approximations of reachable sets, not necessarily convex, at the cost of a higher computational complexity than [Rei09]. In contrast to [Rei09, SP90], our technique imposes no restrictions on the choice of the sampling time but provides less tight over-approximations of the set of reachable states.

We show that symbolic models exist if the control systems satisfy an *incremental forward completeness* assumption which is an incremental version of forward completeness. The second contribution of this chapter is to establish that:

For every nonlinear digital control system satisfying the incremental forward completeness assumption, one can construct a symbolic model that is alternatingly approximately simulated[1] *by the control system and that approximately simulates*[2] *the control system.* Although these results are of theoretical nature, we also provide a simple way of constructing symbolic models which can be improved by using tighter over-approximations of reachable sets such as those described in [Rei09, Jun00, DJ02].

These relationships are weaker than the approximate bisimulation relationships, established in the first part of this chapter and in [PGT08, PT09, PPDT10, GPT09], in the sense that they are only sufficient but not necessary to guarantee that any controller synthesized for the symbolic model can be refined to a controller enforcing the desired specifications on the original control system. In other words, any controller synthesized for the abstraction can be converted into a controller enforcing the specification on the original control system. However, failing to find a controller enforcing the specification on the symbolic model does not prevent the existence of a controller for the original control system. Hence, control

[1] *As defined in Definition 3.2.5.*
[2] *As defined in Definition 3.2.3.*

designers are confronted with the choice between the following two alternatives when using approximate abstractions:

1. design a controller rendering the original control system incrementally input-to-state stable using the results in Chapter 4 and then apply the abstraction techniques proposed in the first part of this chapter or in [PGT08, PT09, PPDT10, GPT09];

2. or construct an abstraction using the results presented in the second part of this chapter.

Since most of the existing controller design techniques provide controllers enforcing stability rather than incremental stability, the second alternative provides a concrete approach to symbolic control design for unstable control systems. In Section 3.4, we illustrate the results through a simple unstable example in which a vehicle is requested to reach a target set while avoiding a number of obstacles.

3.2 Preliminaries

We will use notion of systems to describe both control systems as well as their symbolic models. A more detailed exposition of the notion of system that we now introduce can be found in [Tab09].

Definition 3.2.1. *[Tab09] A system S is a quintuple:*

$$S = (X, U, \longrightarrow, Y, H),$$

consisting of:

- *A set of states X;*

- *A set of inputs U;*

- *A transition relation* $\longrightarrow \subseteq X \times U \times X$*;*

- *An output set* Y*;*

- *An output function* $H : X \to Y$*.*

System S is said to be:

- *metric*, if the output set Y is equipped with a metric $\mathbf{d} : Y \times Y \to \mathbb{R}_0^+$;

- *countable*, if X is a countable set;

- *finite*, if X is a finite set.

A transition $(x, u, x') \in \longrightarrow$ is denoted by $x \xrightarrow{u} x'$. For a transition $x \xrightarrow{u} x'$, state x' is called a u-successor, or simply successor, of state x. We denote by $\mathbf{Post}_u(x)$ the set of u-successors of a state x and by $U(x)$ the set of inputs $u \in U$ for which $\mathbf{Post}_u(x)$ is nonempty. We shall abuse the notation and denote by $\mathbf{Post}_u(Z)$ the set $\mathbf{Post}_u(Z) = \bigcup_{x \in Z} \mathbf{Post}_u(x)$. A system is deterministic if for any state $x \in X$ and any input u, there exists at most one u-successor (there may be none). A system is called nondeterministic if it is not deterministic. Hence, for a nondeterministic system it is possible for a state to have two (or possibly more) distinct u-successors.

Definition 3.2.2. *[Tab09] For a system* $S = (X, U, \longrightarrow, Y, H)$ *and given any state* $x_0 \in X$*, a finite state run generated from* x_0 *is a finite sequence of transitions:*

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} x_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{n-2}} x_{n-1} \xrightarrow{u_{n-1}} x_n,$$

such that $x_i \xrightarrow{u_i} x_{i+1}$ *for all* $0 \leq i < n$*, where* $i \in \mathbb{N}_0$*. In some cases, a finite state run can be extended to an infinite state run. An infinite state run generated from* x_0 *is an infinite sequence:*

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} x_2 \xrightarrow{u_2} x_3 \xrightarrow{u_3} \cdots$$

such that $x_i \xrightarrow{u_i} x_{i+1}$ *for all* $i \in \mathbb{N}_0$*.*

3.2.1 System relations

We first consider simulation and bisimulation relations that are useful when analyzing or synthesizing controllers for deterministic systems. Intuitively, a bisimulation relation between a pair of systems S_a and S_b is a relation between the corresponding sets of states showing how a transition in S_a can be transformed into a transition in S_b and vice versa. A simulation relation is a one-sided version of a bisimulation relation.

Definition 3.2.3. *Let $S_a = (X_a, U_a, \underset{a}{\longrightarrow}, Y_a, H_a)$ and $S_b = (X_b, U_b, \underset{b}{\longrightarrow}, Y_b, H_b)$ be metric systems with the same output sets $Y_a = Y_b$ and metric $\mathbf{d}$, and consider a precision $\varepsilon \in \mathbb{R}^+$. A relation $R \subseteq X_a \times X_b$ is said to be an ε-approximate simulation relation from S_a to S_b, if the following three conditions are satisfied:*

(i) for every $x_a \in X_a$, there exists $x_b \in X_b$ with $(x_a, x_b) \in R$;

(ii) for every $(x_a, x_b) \in R$ we have $\mathbf{d}(H_a(x_a), H_b(x_b)) \leq \varepsilon$;

(iii) for every $(x_a, x_b) \in R$ we have that: $x_a \xrightarrow[a]{u_a} x'_a$ in S_a implies the existence of $x_b \xrightarrow[b]{u_b} x'_b$ in S_b satisfying $(x'_a, x'_b) \in R$.

System S_a is ε-approximately simulated by S_b or S_b ε-approximately simulates S_a, denoted by $S_a \preceq^{\varepsilon}_{\mathcal{S}} S_b$, if there exists an ε-approximate simulation relation from S_a to S_b.

Note that in [Tab09], the condition (i) of Definition 3.2.3 relates states in a set of initial states. In this chapter we assume that all the states are initial states for simplicity of presentation. Hence, the condition (i) is written for all the states in X. The same simplification is used in the forthcoming relations.

Note also that when $\varepsilon = 0$, the condition (ii) in the above definition is changed to $(x_a, x_b) \in R$ if and only if $H_a(x_a) = H_b(x_b)$ and R becomes an exact simulation relation [Mil89, Par81].

Symmetrizing the notion of simulation we obtain the notion of bisimulation, which we report hereafter.

Definition 3.2.4. *Let* $S_a = (X_a, U_a, \underset{a}{\longrightarrow}, Y_a, H_a)$ *and* $S_b = (X_b, U_b, \underset{b}{\longrightarrow}, Y_b, H_b)$ *be metric systems with the same output sets* $Y_a = Y_b$ *and metric* $\mathbf{d}$*, and consider a precision* $\varepsilon \in \mathbb{R}^+$*. A relation* $R \subseteq X_a \times X_b$ *is said to be an* ε*-approximate bisimulation relation between* S_a *and* S_b*, if the following two conditions are satisfied:*

(i) R *is an* ε*-approximate simulation relation from* S_a *to* S_b*;*

(ii) R^{-1} *is an* ε*-approximate simulation relation from* S_b *to* S_a*.*

System S_a is ε-approximate bisimilar to S_b, denoted by $S_a \cong^{\varepsilon}_{\mathcal{S}} S_b$, if there exists an ε-approximate bisimulation relation R between S_a and S_b.

Similarly, a 0-approximate bisimulation relation R is an exact bisimulation relation [Mil89, Par81].

For nondeterministic systems we need to consider relationships that explicitly capture the adversarial nature of nondeterminism. It was illustrated in [PT09] that the preceding notions of simulation and bisimulation are not appropriate for symbolic control design on nondeterministic systems. In the following, we report the notions of alternating approximate simulation and bisimulation which, as illustrated in [PT09], are appropriate for nondeterministic systems.

Definition 3.2.5. *Let* $S_a = (X_a, U_a, \underset{a}{\longrightarrow}, Y_a, H_a)$ *and* $S_b = (X_b, U_b, \underset{b}{\longrightarrow}, Y_b, H_b)$ *be metric systems with the same output sets* $Y_a = Y_b$ *and metric* $\mathbf{d}$*, and consider a precision* $\varepsilon \in \mathbb{R}^+$*. A relation* $R \subseteq X_a \times X_b$ *is said to be an alternating* ε*-approximate simulation relation from* S_a *to* S_b *if the following three conditions are satisfied:*

(i) for every $x_a \in X_a$*, there exists* $x_b \in X_b$ *with* $(x_a, x_b) \in R$*;*

(ii) *for every* $(x_a, x_b) \in R$ *we have* $\mathbf{d}(H_a(x_a), H_b(x_b)) \leq \varepsilon$*;*

(iii) *for every* $(x_a, x_b) \in R$ *and for every* $u_a \in U_a(x_a)$ *there exists* $u_b \in U_b(x_b)$ *such that for every* $x'_b \in \mathbf{Post}_{u_b}(x_b)$ *there exists* $x'_a \in \mathbf{Post}_{u_a}(x_a)$ *satisfying* $(x'_b, x'_a) \in R$*.*

System S_a is alternatingly ε-approximately simulated by S_b or S_b alternatingly ε-approximately simulates S_a, denoted by $S_a \preceq^{\varepsilon}_{\mathcal{AS}} S_b$, if there exists an alternating ε-approximate simulation relation from S_a to S_b.

Note also that when $\varepsilon = 0$, the condition (ii) in the above definition is changed to $(x_a, x_b) \in R$ if and only if $H_a(x_a) = H_b(x_b)$ and R becomes an exact alternating simulation relation [AHKV98].

Symmetrizing the notion of alternating simulation one obtains the notion of alternating bisimulation.

Definition 3.2.6. *Let* $S_a = (X_a, U_a, \underset{a}{\longrightarrow}, Y_a, H_a)$ *and* $S_b = (X_b, U_b, \underset{b}{\longrightarrow},$ $Y_b, H_b)$ *be metric systems with the same output sets* $Y_a = Y_b$ *and metric* $\mathbf{d}$*, and consider a precision* $\varepsilon \in \mathbb{R}^+$*. A relation* $R \subseteq X_a \times X_b$ *is said to be an alternating* ε*-approximate bisimulation relation between* S_a *and* S_b*, if the following two conditions are satisfied:*

(i) R *is an alternating* ε*-approximate simulation relation from* S_a *to* S_b*;*

(ii) R^{-1} *is an alternating* ε*-approximate simulation relation from* S_b *to* S_a*.*

System S_a is alternating ε-approximate bisimilar to S_b, denoted by $S_a \cong^{\varepsilon}_{\mathcal{AS}} S_b$, if there exists an alternating ε-approximate bisimulation relation R between S_a and S_b.

Similarly, an alternating 0-approximate bisimulation relation R is an exact alternating bisimulation relation [AHKV98].

It is readily seen from the above definitions that the notions of approximate (bi)simulation and of alternating approximate (bi)simulation coincide when the systems involved are deterministic. It is shown in [Tab09] that for the nondeterministic case, these two notions are not comparable in the sense that approximate (bi)simulation does not imply alternating approximate (bi)simulation neither does alternating approximate (bi)simulation imply approximate (bi)simulation.

The importance of the preceding notions lies in enabling the transfer of controllers designed for the symbolic models to controllers acting on the original control systems. More details about these notions and how the refinement of controllers is performed can be found in [Tab09].

3.3 Existence of symbolic models

This section contains the main contributions of this chapter. The results in this section rely on additional assumptions on U and $\mathcal{U}$ that we now describe. Such assumptions are not required for the definitions and results in other sections and chapters. We restrict attention to control systems $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ with input set which is assumed to be a finite union of boxes containing the origin. We further restrict attention to digital control systems, where input curves belong to $\mathcal{U}_\tau$ containing only curves, constant in any duration $\tau \in \mathbb{R}^+$, i.e.

$$\mathcal{U}_\tau = \Big\{ \upsilon : \mathbb{R}_0^+ \rightarrow \mathsf{U} \mid \upsilon(t) = \upsilon((k-1)\tau), t \in [(k-1)\tau, k\tau[, k \in \mathbb{N} \Big\},$$

where τ is the sampling time.

3.3.1 First result

We show that a δ-ISS digital control system Σ admits a countable symbolic abstraction. First, we consider a metric system $S_{k\tau}(\Sigma)$ which relates two states if the second is reached from the first in $k\tau$ time for a parameter $k \in \mathbb{N}$.

Given a constant $k \in \mathbb{N}$ and a digital control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f)$, define the metric system:

$$S_{k\tau}(\Sigma) = (X_{k\tau}, U_{k\tau}, \underset{k\tau}{\longrightarrow}, Y_{k\tau}, H_{k\tau}), \tag{3.3.1}$$

consisting of:

- $X_{k\tau} = \mathbb{R}^n$;
- $U_{k\tau} = \mathcal{U}_\tau$;
- $x_{k\tau} \xrightarrow[k\tau]{\upsilon_{k\tau}} x'_{k\tau}$ if there exists a trajectory $\xi_{x_{k\tau}\upsilon_{k\tau}} : [0, k\tau] \to \mathbb{R}^n$ of Σ satisfying $\xi_{x_{k\tau}\upsilon_{k\tau}}(k\tau) = x'_{k\tau}$;
- $Y_{k\tau} = \mathbb{R}^n$;
- $H_{k\tau} = 1_{\mathbb{R}^n}$.

Although the metric system $S_{k\tau}(\Sigma)$ relates states of Σ that are k sampling steps apart, this system is not less accurate than $S_\tau(\Sigma)$ relating states of Σ one sampling time apart in the sense that for any initialized run with k transitions in the latter there is one transition in the former and vice versa.

Now, assume that Σ is δ-ISS. Consider a triple $\mathsf{q} = (\eta, \mu, k)$, where $\eta \in \mathbb{R}^+$ is the *state space quantization* which determines a discretization of the state space, $\mu \in \mathbb{R}^+$ is the *input set quantization* which determines a discretization of the inputs, and $k \in \mathbb{N}$ is a design parameter. Given Σ and q, consider the following metric system:

$$S_{\mathsf{q}}(\Sigma) = (X_{\mathsf{q}}, U_{\mathsf{q}}, \underset{\mathsf{q}}{\longrightarrow}, Y_{\mathsf{q}}, H_{\mathsf{q}}), \tag{3.3.2}$$

consisting of:

- $X_{\mathsf{q}} = [\mathbb{R}^n]_\eta$;
- $U_{\mathsf{q}} = \prod_{i=1}^{k} [\mathsf{U}]_\mu$;

- $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{u_{\mathsf{q}}} x'_{\mathsf{q}}$ if there is a trajectory $\xi_{x_{\mathsf{q}}u_{\mathsf{q}}} : [0, k\tau] \to \mathbb{R}^n$ such that $\|\xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(k\tau) - x'_{\mathsf{q}}\| \leq \eta/2$;

- $Y_{\mathsf{q}} = \mathbb{R}^n$;

- $H_{\mathsf{q}} = \imath : X_{\mathsf{q}} \hookrightarrow Y_{\mathsf{q}}$.

We have abused notation by identifying $v \in [\mathsf{U}]_\mu$ with the constant input curve with domain $[0, \tau[$ and value v and identifying $u \in \prod_{i=1}^{k} [\mathsf{U}]_\mu$ with the concatenation of k control inputs $v^i \in [\mathsf{U}]_\mu$ (i.e., $u(t) = v^i$ for any $t \in [(i-1)\tau, i\tau[$) for $i = 1, \ldots, k$. The set of states of $S_{\mathsf{q}}(\Sigma)$ is countable in general, and finite when the set of states of Σ is restricted to a compact set.

The transition relation of $S_{\mathsf{q}}(\Sigma)$ is well defined in the sense that for every $x_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ and every $u \in \prod_{i=1}^{k} [\mathsf{U}]_\mu$ there always exists $x'_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ such that $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{u_{\mathsf{q}}} x'_{\mathsf{q}}$. This can be seen by noting that by definition of $[\mathbb{R}^n]_\eta$, for any $\widehat{x} \in \mathbb{R}^n$ there always exists a state $\widehat{x}' \in [\mathbb{R}^n]_\eta$ such that $\|\widehat{x} - \widehat{x}'\| \leq \eta/2$. Hence, for $\xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(k\tau)$ there always exists a state $x'_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ satisfying $\|\xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(k\tau) - x'_{\mathsf{q}}\| \leq \eta/2$.

We can now present the first result of this chapter.

Theorem 3.3.1. *Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f)$ be a δ-ISS digital control system, with sampling time τ, and let functions β and γ satisfy* (2.3.4)*. For any $\varepsilon \in \mathbb{R}^+$ and any triple $\mathsf{q} = (\eta, \mu, k)$ of quantization parameters, we have $S_{\mathsf{q}}(\Sigma) \cong_{\mathcal{S}}^{\varepsilon} S_{k\tau}(\Sigma)$ if $\mu \leq span(\mathsf{U})$ and*

$$\beta(\varepsilon, k\tau) + \gamma(\mu) + \eta/2 \leq \varepsilon. \tag{3.3.3}$$

Before giving the proof, we point out that if Σ is δ-ISS, there always exists a triple $\mathsf{q} = (\eta, \mu, k)$ satisfying condition (3.3.3). Since β is a $\mathcal{KL}$ function, there exists sufficiently large $k \in \mathbb{N}$ such that $\beta(\varepsilon, k\tau) < \varepsilon/2$; for this value of k, by choosing sufficiently small values of η and μ, condition (3.3.3) can be fulfilled.

Proof: We start by proving $S_{k\tau}(\Sigma) \preceq_{\mathcal{S}}^{\varepsilon} S_{\mathsf{q}}(\Sigma)$. Consider the relation $R \subseteq X_{k\tau} \times X_{\mathsf{q}}$ defined by $(x_{k\tau}, x_{\mathsf{q}}) \in R$ if and only if $\|H_{k\tau}(x_{k\tau}) - H_{\mathsf{q}}(x_{\mathsf{q}})\| = \|x_{k\tau} - x_{\mathsf{q}}\| \leq \varepsilon$. Since $X_{k\tau} \subseteq \bigcup_{p \in [\mathbb{R}^n]_\eta} \mathcal{B}_{\eta/2}(p)$ and by (3.3.3), for every $x_{k\tau} \in X_{k\tau}$ there always exists $x_{\mathsf{q}} \in X_{\mathsf{q}}$ such that:

$$\|x_{k\tau} - x_{\mathsf{q}}\| \leq \eta/2 \leq \varepsilon. \tag{3.3.4}$$

Hence, $(x_{k\tau}, x_{\mathsf{q}}) \in R$ and condition (i) in Definition 3.2.3 is satisfied. Now consider any $(x_{k\tau}, x_{\mathsf{q}}) \in R$. Condition (ii) in Definition 3.2.3 is satisfied by the definition of R. Let us now show that condition (iii) in Definition 3.2.3 holds.

Consider any $\upsilon_{k\tau} \in U_{k\tau}$ of duration $k\tau$. Choose an input $u_{\mathsf{q}} \in U_{\mathsf{q}}$ satisfying:

$$\|\upsilon_{k\tau}|_{[(l-1)\tau),l\tau[} - u_{\mathsf{q}}|_{[(l-1)\tau),l\tau[}\|_\infty = \|\upsilon_{k\tau}((l-1)\tau) - u_{\mathsf{q}}((l-1)\tau)\| \leq \mu, \tag{3.3.5}$$

for any $l = 1, \ldots, k$. Note that the existence of such u_{q} is guaranteed by the special shape of U and by the inequality $\mu \leq \mathit{span}(\mathsf{U})$ which guarantees that $\mathsf{U} \subseteq \bigcup_{p \in [\mathsf{U}]_\mu} \mathcal{B}_\mu(p)$. Now, we have:

$$\|\upsilon_{k\tau} - u_{\mathsf{q}}\|_\infty = \max_{l=1,\ldots,k} \left\| \upsilon_{k\tau}\big|_{[(l-1)\tau),l\tau[} - u_{\mathsf{q}}\big|_{[(l-1)\tau),l\tau[} \right\|_\infty \leq \mu. \tag{3.3.6}$$

Consider the unique transition $x_{k\tau} \xrightarrow[k\tau]{\upsilon_{k\tau}} x'_{k\tau} = \xi_{x_{k\tau}\upsilon_{k\tau}}(k\tau)$ in $S_{k\tau}(\Sigma)$. It follows from the δ-ISS assumption on Σ and (3.3.6) that the distance between $x'_{k\tau}$ and $\xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(k\tau)$ is bounded as:

$$\|x'_{k\tau} - \xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(k\tau)\| \leq \beta\left(\|x_{k\tau} - x_{\mathsf{q}}\|, k\tau\right) + \gamma\left(\|\upsilon_{k\tau} - u_{\mathsf{q}}\|_\infty\right) \leq \beta\left(\varepsilon, k\tau\right) + \gamma\left(\mu\right). \tag{3.3.7}$$

Since $X_{k\tau} \subseteq \bigcup_{p \in [\mathbb{R}^n]_\eta} \mathcal{B}_{\eta/2}(p)$, there exists $x'_{\mathsf{q}} \in X_{\mathsf{q}}$ such that:

$$\|\xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(k\tau) - x'_{\mathsf{q}}\| \leq \eta/2, \tag{3.3.8}$$

which, by the definition of $S_{\mathsf{q}}(\Sigma)$, implies the existence of $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{u_{\mathsf{q}}} x'_{\mathsf{q}}$ in $S_{\mathsf{q}}(\Sigma)$.

Using the inequalities (3.3.3), (3.3.7), (3.3.8), and triangle inequality, we obtain:

$$\begin{aligned}\|x'_{k\tau} - x'_{\mathsf{q}}\| &\leq \|x'_{k\tau} - \xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(k\tau) + \xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(k\tau) - x'_{\mathsf{q}}\| \\ &\leq \|x'_{k\tau} - \xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(k\tau)\| + \|\xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(k\tau) - x'_{\mathsf{q}}\| \\ &\leq \beta(\varepsilon, k\tau) + \gamma\left(\mu\right) + \eta/2 \leq \varepsilon.\end{aligned}$$

Therefore, we conclude $(x'_{k\tau}, x'_{\mathsf{q}}) \in R$ and condition (iii) in Definition 3.2.3 holds.

Now we prove $S_{\mathsf{q}}(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} S_{k\tau}(\Sigma)$ implying that R^{-1} is a suitable ε-approximate simulation relation. Consider the relation $R \subseteq X_{k\tau} \times X_{\mathsf{q}}$, defined in the first part of the proof. For every $x_{\mathsf{q}} \in X_{\mathsf{q}}$, by choosing $x_{k\tau} = x_{\mathsf{q}}$, we have $(x_{k\tau,}x_{\mathsf{q}}) \in R$ and condition (i) in Definition 3.2.3 is satisfied. Now consider any $(x_{k\tau}, x_{\mathsf{q}}) \in R$. Condition (ii) in Definition 3.2.3 is satisfied by the definition of R. Let us now show that condition (iii) in Definition 3.2.3 holds. Consider any $u_{\mathsf{q}} \in U_{\mathsf{q}}$. Choose the input $\upsilon_{k\tau} = u_{\mathsf{q}}$ and consider the unique $x'_{k\tau} = \xi_{x_{k\tau}\upsilon_{k\tau}}(k\tau)$ in $S_{k\tau}(\Sigma)$. Using δ-ISS assumption for Σ, we bound the distance between $x'_{k\tau}$ and $\xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(k\tau)$ as:

$$\|x'_{k\tau} - \xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(k\tau)\| \leq \beta\left(\|x_{k\tau} - x_{\mathsf{q}}\|, k\tau\right) \leq \beta(\varepsilon, k\tau). \tag{3.3.9}$$

Using the definition of $S_{\mathsf{q}}(\Sigma)$, the inequalities (3.3.3), (3.3.9), and the triangle inequality, we obtain:

$$\begin{aligned}\|x'_{k\tau} - x'_{\mathsf{q}}\| \leq& \|x'_{k\tau} - \xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(k\tau) + \xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(k\tau) - x'_{\mathsf{q}}\| \\ \leq& \|x'_{k\tau} - \xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(k\tau)\| + \|\xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(k\tau) - x'_{\mathsf{q}}\| \\ \leq& \beta(\varepsilon, k\tau) + \eta/2 \leq \varepsilon.\end{aligned}$$

Therefore, we conclude that $(x'_{k\tau}, x'_{\mathsf{q}}) \in R$ and condition (iii) in Definition 3.2.3 holds. □

Remark 3.3.2. *Although we assume the set* U *is infinite, Theorem 3.3.1 still holds when the set* U *is finite, with the following modifications. First, the system* Σ *is required to satisfy the property (2.3.4) for* $\upsilon = \upsilon'$*. Second, take* $U_{\mathsf{q}} = \prod_{i=1}^{k} \mathsf{U}$ *in the definition of* $S_{\mathsf{q}}(\Sigma)$*. Finally, in the condition (3.3.3), set* $\mu = 0$*.*

A concern that arises when using $S_{\mathsf{q}}(\Sigma)$ is the inter-samples behavior: can a specification be violated for $t \in]0, \; k\tau[$ even though it is satisfied at $t = 0$ and $t = k\tau$? This concern arises already in existing approaches to compute discrete abstractions [PGT08, PT09, GPT09] (setting $k = 1$).

In the absence of any bounds on inter-samples behaviors, the results of controller synthesis on $S_{\mathsf{q}}(\Sigma)$ can be interpreted in the following way. If there is no controller satisfying a safety or co-Büchi specification on $S_{\mathsf{q}}(\Sigma)$, respectively, then we can conclude that there is no controller satisfying the same safety or co-Büchi specification on Σ, respectively. Dually, if there is a controller satisfying a reachability or Büchi specification on $S_{\mathsf{q}}(\Sigma)$, respectively, then we can conclude that the refinement of that controller satisfies the same reachability or Büchi specification on Σ, respectively.

In practice, the parameter τ is chosen to be sufficiently small, and if $k \in \mathbb{N}$ is also small, the specification is directly verified against $S_{\mathsf{q}}(\Sigma)$ ignoring inter-samples behaviors. If it is important to include the effects of inter-samples behaviors, e.g., when τ or k are large, there is a naive way to solve the inter-samples behaviors, especially in terms of synthesizing a controller. In the process of constructing abstract transition system $S_{\mathsf{q}}(\Sigma)$, every transition can be labeled not only with the input but also with the sequence of the states visited at times $\tau, 2\tau, \ldots, (k-1)\tau$. Now, one can find a symbolic controller for the constructed abstract transition system with the knowledge of what is happening in the inert-samples. By doing this, we shrink the inter-samples behaviors in only one sample time $]0, \tau[$.

Furthermore, one can over-approximate the reachable states between two sample points using techniques incorporating zonotopes [Gir05, Tab09] or support functions [GG10, FGD+11]. We illustrate the bounding technique using zonotopes. A transition $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{u_{\mathsf{q}}} x'_{\mathsf{q}}$ in $S_{\mathsf{q}}(\Sigma)$ implies the existence of a trajectory $\xi_{x_{\mathsf{q}} u_{\mathsf{q}}}$ of Σ satisfying $\|\xi_{x_{\mathsf{q}} u_{\mathsf{q}}}(k\tau) - x'_{\mathsf{q}}\| \leq \eta/2$. We can thus enclose x_{q} in a zonotope Z_1, enclose $\mathcal{B}_{\eta/2}(x'_{\mathsf{q}})$ in a different zonotope Z_2, and use results in [Gir05] (see also Proposition 7.31 in [Tab09]) for a given $u_{\mathsf{q}} \in U_{\mathsf{q}}$ to obtain another zonotope $Z_{k\tau}(x_{\mathsf{q}}, u_{\mathsf{q}}, x'_{\mathsf{q}})$ containing all the states $\xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(t)$ for $t \in [0,\ k\tau]$.

Fix an ε and q such that $S_{\mathsf{q}}(\Sigma) \cong^{\varepsilon}_{\mathcal{S}} S_{k\tau}(\Sigma)$. Let Z be the smallest zonotope enclosing Z_1 and Z_2. Let $\varepsilon_0(x_{\mathsf{q}}, u_{\mathsf{q}}, x'_{\mathsf{q}})$ be an upper bound on the Hausdorff distance between Z and $Z_{k\tau}(x_{\mathsf{q}}, u_{\mathsf{q}}, x'_{\mathsf{q}})$, and let ε_0 be the supremum over all choices of $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{u_{\mathsf{q}}} x'_{\mathsf{q}}$. Then, if $S_{\mathsf{q}}(\Sigma) \cong^{\varepsilon-\varepsilon_0}_{\mathcal{S}} S_{k\tau}(\Sigma)$, then we know that any trajectory of $S_{\mathsf{q}}(\Sigma)$ is at most ε away from a trajectory of $S(\Sigma)$. If $\varepsilon_0 > \varepsilon$, then one needs to reduce the original precision ε and compute a new q, and iterate. This represents the tradeoff between choosing larger k's and bounding the deviations of inter-samples behaviors: choosing a larger k makes satisfying (3.3.3) easier, but can make ε_0 larger.

Remark 3.3.3. *For linear control systems in (2.3.5) and safety (or co-Büchi) specifications, we can compute bounds on inter-samples behaviors in the following way. Assume $W \in \mathbb{R}^n$ is a compact and convex polyhedron with h vertices $x^1, \ldots, x^h$. Assume Σ is a globally asymptotically stable linear control system, defined in (2.3.5) and $S_{\mathsf{q}}(\Sigma)$ is its symbolic abstraction. Assume there exists a symbolic controller on $S_{\mathsf{q}}(\Sigma)$ satisfying*[3] *$\Box W$. What can we say about the existence of a controller on $S_\tau(\Sigma)$? It can be readily verified that there exists a controller satisfying $\Box\widehat{W}$ on $S_\tau(\Sigma)$, where $\widehat{W}$ is the polyhedron with vertices $\widehat{x}^1, \ldots, \widehat{x}^k$, defined by:*

$$\widehat{x}^i = \mathsf{e}^{Al^*\tau}x^i + A^{-1}\left(\mathsf{e}^{A\tau} - I_n\right)\left[\mathsf{e}^{A(l^*-1)\tau}Bu_1^* + \ldots + \mathsf{e}^{A\tau}Bu_{l^*-1}^* + Bu_{l^*}^*\right], \quad (3.3.10)$$

where l^* and u^* are computed by:

$$(l^*, u_1^*, \ldots, u_{l^*}^*) = \arg \max_{l=1,\ldots,k-1} \min_{u_1,\ldots,u_l \in \mathsf{U}} \|\widetilde{x}^i\|_W, \quad (3.3.11)$$

where the symbol $\|\cdot\|_W$ denotes the point-to-set distance, namely, $\|x\|_W = \min_{w\in W}\|x - w\|$, and

$$\widetilde{x}^i = \mathsf{e}^{Al\tau}x^i + A^{-1}\left(\mathsf{e}^{A\tau} - I_n\right)\left[\mathsf{e}^{A(l-1)\tau}Bu_1 + \ldots + \mathsf{e}^{A\tau}Bu_{l-1} + Bu_l\right]. \quad (3.3.12)$$

[3] *Note that the semantics of LTL would be defined over the output behaviors of $S_{\mathsf{q}}(\Sigma)$.*

If $\widehat{W} \subseteq W$, then no new states are introduced through inter-samples behaviors. A similar analysis can be performed for co-Büchi objectives.

3.3.1.1 Comparison with previous techniques

We now compare our result (Theorem 3.3.1) with existing results on computing ε-approximate bisimilar discrete abstractions for δ-ISS digital control systems.

The construction in Pola, Girard, and Tabuada [PGT08] essentially fixes $k = 1$. That is, it computes the metric system $S_\tau(\Sigma)$ and shows that $S_{\mathsf{q}}(\Sigma)$ is ε-approximate bisimilar to it if $\mu \leq span(\mathsf{U})$ and $\beta(\varepsilon, \tau) + \gamma(\mu) + \eta/2 \leq \varepsilon$. This inequality may not hold for a choice of τ and in that case, the technique fails to construct an ε-approximate bisimilar abstraction. In contrast, we are guaranteed that for every given ε and τ, we can choose parameters η, μ, and k such that $S_{\mathsf{q}}(\Sigma)$ is ε-approximate bisimilar to $S_{k\tau}(\Sigma)$.

Next, we compare with the construction in Girard, Pola, and Tabuada [GPT09]. First, we need the notion of δ-ISS Lyapunov functions, explained in more details in Chapter 4.

Definition 3.3.4. *[Ang02] Fix a control system Σ. A smooth function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ is called a* δ-ISS Lyapunov function *for Σ if there exist $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, and σ, and a positive real $\kappa \in \mathbb{R}^+$ such that:*

(i) for any $x, x' \in \mathbb{R}^n$,
$\underline{\alpha}(\|x - x'\|) \leq V(x, x') \leq \overline{\alpha}(\|x - x'\|)$*;*

(ii) for any $x, x' \in \mathbb{R}^n$ and for any $u, u' \in \mathsf{U}$,
$\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u') \leq -\kappa V(x, x') + \sigma(\|u - u'\|)$.

The results in [GPT09] additionally assume:

$$|V(x, y) - V(x, z)| \leq \widehat{\gamma}(\|y - z\|), \tag{3.3.13}$$

for any $x, y, z \in \mathbb{R}^n$, and some $\mathcal{K}_\infty$ function $\widehat{\gamma}$. As explained in [GPT09], this assumption is not restrictive provided V is smooth and we are interested in the dynamics of Σ on a compact subset of $\mathbb{R}^n$, which is often the case in practice. The main result of [GPT09] is as follows.

Theorem 3.3.5. *[GPT09] Let Σ be a δ-ISS digital control system admitting a δ-ISS Lyapunov function V. For any $\varepsilon \in \mathbb{R}^+$ and parameters $\mathsf{q} = (\eta, \mu, 1)$, we have $S_{\mathsf{q}}(\Sigma) \cong^{\varepsilon}_{\mathcal{S}} S_\tau(\Sigma)$ if $\mu \leq span(\mathsf{U})$ and*

$$\eta/2 \leq \min\left\{\widehat{\gamma}^{-1}\left(\left(1 - e^{-\kappa\tau}\right)\underline{\alpha}(\varepsilon) - \frac{1}{\kappa}\sigma(\mu)\right), \overline{\alpha}^{-1}\left(\underline{\alpha}(\varepsilon)\right)\right\}, \tag{3.3.14}$$

For a given sampling time $\tau \in \mathbb{R}^+$, there always exist $\eta, \mu \in \mathbb{R}^+$ satisfying the condition (3.3.14). However, it can be readily verified that if the sampling time τ is very small, the right hand side of the inequality (3.3.14) is very small as well. Therefore, the upper bound on η will be very small, resulting in a large symbolic abstraction. On the other hand, we can always choose $k \in \mathbb{N}$ in (3.3.3) appropriately, to control the size of the symbolic model, justifying advantage of our proposed approach in comparison with the approach in [GPT09]. In Section 3.4, we demonstrate experimentally that our approach can result in discrete abstractions with orders of magnitude fewer states than the abstractions using Theorem 3.3.5.

3.3.2 Second result

Here, we show that a δ-FC digital control system, suitably restricted to a compact set, admits a finite abstraction.

We consider a δ-FC digital control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f)$, and a triple $\mathsf{q} = (\eta, \mu, \theta)$ of quantization parameters, where $\eta \in \mathbb{R}^+$ is the state space quantization, $\mu \in \mathbb{R}^+$ is the input set quantization, and $\theta \in \mathbb{R}^+$ is a design parameter. Given Σ

and q, define the system:

$$S_{\mathsf{q}}(\Sigma) = (X_{\mathsf{q}}, U_{\mathsf{q}}, \underset{\mathsf{q}}{\longrightarrow}, Y_{\mathsf{q}}, H_{\mathsf{q}}), \tag{3.3.15}$$

consisting of:

- $X_{\mathsf{q}} = [\mathbb{R}^n]_\eta$;
- $U_{\mathsf{q}} = [\mathsf{U}]_\mu$;
- $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{u_{\mathsf{q}}} x'_{\mathsf{q}}$ if $\|\xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(\tau) - x'_{\mathsf{q}}\| \leq \beta(\theta, \tau) + \gamma(\mu, \tau) + \frac{\eta}{2}$;
- $Y_{\mathsf{q}} = \mathbb{R}^n$;
- $H_{\mathsf{q}} = \imath : X_{\mathsf{q}} \hookrightarrow Y_{\mathsf{q}}$,

where β and γ are the functions appearing in (2.4.1). In the definition of the transition relation, and in the remainder of the chapter, we abuse notation by identifying u_{q} with the constant input curve with domain $[0, \tau[$ and value u_{q}.

The transition relation of $S_{\mathsf{q}}(\Sigma)$ is well defined in the sense that for every $x_{\mathsf{q}} \in X_{\mathsf{q}}$ and every $u_{\mathsf{q}} \in U_{\mathsf{q}}$ there always exists $x'_{\mathsf{q}} \in X_{\mathsf{q}}$ such that $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{u_{\mathsf{q}}} x'_{\mathsf{q}}$. This can be seen by noting that by definition of X_{q}, for any $x \in \mathbb{R}^n$ there always exists a state $x'_{\mathsf{q}} \in X_{\mathsf{q}}$ such that $\|x - x'_{\mathsf{q}}\| \leq \eta/2$. Hence, for $x = \xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(\tau)$ there always exists a state $x'_{\mathsf{q}} \in X_{\mathsf{q}}$ satisfying $\|\xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(\tau) - x'_{\mathsf{q}}\| \leq \frac{\eta}{2} \leq \beta(\theta, \tau) + \gamma(\mu, \tau) + \frac{\eta}{2}$.

We stress that while system $S_\tau(\Sigma)$ is not countable, system $S_{\mathsf{q}}(\Sigma)$ is so and it becomes finite when the state set of the control system Σ is restricted to a compact set. We can now state one of the main results of this chapter which relates δ-FC to existence of symbolic models.

Theorem 3.3.6. *Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_\tau, f)$ be a δ-FC digital control system. For any desired precision $\varepsilon \in \mathbb{R}^+$, and any triple $\mathsf{q} = (\eta, \mu, \theta)$ of quantization parameters satisfying $\mu \leq span(\mathsf{U})$ and $\eta \leq 2\varepsilon \leq 2\theta$, we have $S_{\mathsf{q}}(\Sigma) \preceq^{\varepsilon}_{\mathcal{AS}} S_\tau(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} S_{\mathsf{q}}(\Sigma)$.*

Proof: We start by proving $S_\tau(\Sigma) \preceq_{\mathcal{S}}^{\varepsilon} S_{\mathsf{q}}(\Sigma)$. Consider the relation $R \subseteq X_\tau \times X_{\mathsf{q}}$ defined by $(x_\tau, x_{\mathsf{q}}) \in R$ if and only if $\|H_\tau(x_\tau) - H_{\mathsf{q}}(x_{\mathsf{q}})\| = \|x_\tau - x_{\mathsf{q}}\| \leq \varepsilon$. Since $X_\tau \subseteq \bigcup_{p\in[\mathbb{R}^n]_\eta} \mathcal{B}_{\eta/2}(p)$, for every $x_\tau \in X_\tau$ there exists $x_{\mathsf{q}} \in X_{\mathsf{q}}$ such that:

$$\|x_\tau - x_{\mathsf{q}}\| \leq \frac{\eta}{2} \leq \varepsilon. \tag{3.3.16}$$

Hence, $(x_\tau, x_{\mathsf{q}}) \in R$ and condition (i) in Definition 3.2.3 is satisfied. Now consider any $(x_\tau, x_{\mathsf{q}}) \in R$. Condition (ii) in Definition 3.2.3 is satisfied by the definition of R. Let us now show that condition (iii) in Definition 3.2.3 holds.

Consider any $\upsilon_\tau \in U_\tau$, defined on $[0, \tau[$. Choose an input $u_{\mathsf{q}} \in U_{\mathsf{q}}$ satisfying:

$$\|\upsilon_\tau - u_{\mathsf{q}}\|_\infty = \|\upsilon_\tau(0) - u_{\mathsf{q}}(0)\| \leq \mu. \tag{3.3.17}$$

Note that the existence of such u_{q} is guaranteed by the special form of U and by the inequality $\mu \leq span(\mathsf{U})$ which guarantees that $\mathsf{U} \subseteq \bigcup_{p\in[\mathsf{U}]_\mu} \mathcal{B}_\mu(p)$. Consider the unique transition $x_\tau \xrightarrow[\tau]{\upsilon_\tau} x'_\tau = \xi_{x_\tau \upsilon_\tau}(\tau)$ in $S_\tau(\Sigma)$. It follows from the δ-FC assumption that the distance between x'_τ and $\xi_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau)$ is bounded as:

$$\|x'_\tau - \xi_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau)\| \leq \beta(\varepsilon, \tau) + \gamma\left(\mu, \tau\right). \tag{3.3.18}$$

Since $X_\tau \subseteq \bigcup_{p\in[\mathbb{R}^n]_\eta} \mathcal{B}_{\eta/2}(p)$, there exists $x'_{\mathsf{q}} \in X_{\mathsf{q}}$ such that:

$$\|x'_\tau - x'_{\mathsf{q}}\| \leq \frac{\eta}{2}. \tag{3.3.19}$$

Using the inequalities $\varepsilon \leq \theta$, (3.3.18), (3.3.19), and the triangle inequality, we obtain:

$$\begin{aligned}\|\xi_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau) - x'_{\mathsf{q}}\| &= \|\xi_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau) - x'_\tau + x'_\tau - x'_{\mathsf{q}}\| \leq \|\xi_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau) - x'_\tau\| + \|x'_\tau - x'_{\mathsf{q}}\| \\ &\leq \beta(\varepsilon, \tau) + \gamma\left(\mu, \tau\right) + \frac{\eta}{2} \leq \beta(\theta, \tau) + \gamma\left(\mu, \tau\right) + \frac{\eta}{2},\end{aligned}$$

which implies the existence of $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{u_{\mathsf{q}}} x'_{\mathsf{q}}$ in $S_{\mathsf{q}}(\Sigma)$ by the definition of $S_{\mathsf{q}}(\Sigma)$. Therefore, from inequality (3.3.19) and $\frac{\eta}{2} \leq \varepsilon$, we conclude $(x'_\tau, x'_{\mathsf{q}}) \in R$ and condition (iii) in Definition 3.2.3 holds.

Now we prove $S_{\mathsf{q}}(\Sigma) \preceq^{\varepsilon}_{\mathcal{AS}} S_{\tau}(\Sigma)$. Consider the relation $R \subseteq X_{\tau} \times X_{\mathsf{q}}$. For every $x_{\mathsf{q}} \in X_{\mathsf{q}}$, by choosing $x_{\tau} = x_{\mathsf{q}}$, we have $(x_{\tau}, x_{\mathsf{q}}) \in R$ and condition (i) in Definition 3.2.5 is satisfied. Now consider any $(x_{\tau}, x_{\mathsf{q}}) \in R$. Condition (ii) in Definition 3.2.5 is satisfied by the definition of R. Let us now show that condition (iii) in Definition 3.2.5 holds. Consider any $u_{\mathsf{q}} \in U_{\mathsf{q}}$. Choose the input $\upsilon_{\tau} = u_{\mathsf{q}}$ and consider the unique $x'_{\tau} = \xi_{x_{\tau}\upsilon_{\tau}}(\tau) \in \mathbf{Post}_{\upsilon_{\tau}}(x_{\tau})$ in $S_{\tau}(\Sigma)$. From the δ-FC assumption, the distance between x'_{τ} and $\xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(\tau)$ is bounded as:

$$\|x'_{\tau} - \xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(\tau)\| \leq \beta(\varepsilon, \tau). \tag{3.3.20}$$

Since $X_{\tau} \subseteq \bigcup_{p \in [\mathbb{R}^n]_{\eta}} \mathcal{B}_{\eta/2}(p)$, there exists $x'_{\mathsf{q}} \in X_{\mathsf{q}}$ such that:

$$\|x'_{\tau} - x'_{\mathsf{q}}\| \leq \frac{\eta}{2}. \tag{3.3.21}$$

Using the inequalities, $\varepsilon \leq \theta$, (3.3.20), (3.3.21), and the triangle inequality, we obtain:

$$\begin{aligned}
\|\xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(\tau) - x'_{\mathsf{q}}\| &= \|\xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(\tau) - x'_{\tau} + x'_{\tau} - x'_{\mathsf{q}}\| \\
&\leq \|\xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(\tau) - x'_{\tau}\| + \|x'_{\tau} - x'_{\mathsf{q}}\| \leq \beta(\varepsilon, \tau) + \frac{\eta}{2} \\
&\leq \beta(\theta, \tau) + \gamma(\mu, \tau) + \frac{\eta}{2},
\end{aligned}$$

which implies the existence of $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{u_{\mathsf{q}}} x'_{\mathsf{q}}$ in $S_{\mathsf{q}}(\Sigma)$ by definition of $S_{\mathsf{q}}(\Sigma)$. Therefore, from inequality (3.3.21) and $\frac{\eta}{2} \leq \varepsilon$, we can conclude that $(x'_{\tau}, x'_{\mathsf{q}}) \in R$ and condition (iii) in Definition 3.2.5 holds, which completes the proof.

Remark 3.3.7. *Although the set* U *is infinite, the results in Theorem 3.3.6 still hold when the set* U *is finite with the following modifications: the system* Σ *is required to satisfy the property (2.4.1) for* $\upsilon = \upsilon'$ *and in the definition of* $S_{\mathsf{q}}(\Sigma)$ *in (3.3.15),* $U_{\mathsf{q}} = \mathsf{U}$ *and* $\gamma(\mu, \tau) = 0$.

Remark 3.3.8. *The transition relation defined in (3.3.15) can also be written as:*

$$x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{u_{\mathsf{q}}} x'_{\mathsf{q}} \quad \textit{if} \quad \mathcal{B}_{\frac{\eta}{2}}(x'_{\mathsf{q}}) \cap \mathcal{B}_{\beta(\theta,\tau)+\gamma(\mu,\tau)}(\xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(\tau)) \neq \varnothing. \tag{3.3.22}$$

This shows that we place a transition from x_{q} to any point x'_{q} for which the ball $\mathcal{B}_{\frac{\eta}{2}}(x'_{\mathsf{q}})$ intersects the over-approximation of $\mathbf{Post}_{u_{\mathsf{q}}}(\mathcal{B}_{\varepsilon}(x_{\mathsf{q}}))$ given by $\mathcal{B}_{\beta(\theta,\tau)+\gamma(\mu,\tau)}(\xi_{x_{\mathsf{q}}u_{\mathsf{q}}}(\tau))$. It is not difficult to see that the conclusion of Theorem 3.3.6 remains valid if we use any over-approximation of the set $\mathbf{Post}_{u_{\mathsf{q}}}(\mathcal{B}_{\varepsilon}(x_{\mathsf{q}}))$.

The result in Theorem 3.3.6 is only a sufficient result in the sense that if one fails to find a controller enforcing the desired specification on the symbolic model $S_{\mathsf{q}}(\Sigma)$, it can not be concluded the non-existence of a controller enforcing the desired specification on $S_{\tau}(\Sigma)$. On the other hand, if $S_{\tau}(\Sigma)$ is related to $S_{\mathsf{q}}(\Sigma)$ by an alternating approximate bisimulation, not finding a controller for $S_{\mathsf{q}}(\Sigma)$ would imply the non-existence of a controller for $S_{\tau}(\Sigma)$. The existence of a countable approximately bisimilar system to $S_{\tau}(\Sigma)$ was shown in [PGT08] under a δ-ISS assumption. It is therefore natural to ask the extent to which the results in [PGT08] are recovered when δ-FC is replaced by δ-ISS. The next result gives an answer to this question.

Theorem 3.3.9. *Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}_{\tau}, f)$ be a δ-FC digital control system. For any desired precision $\varepsilon \in \mathbb{R}^+$, and any triple $\mathsf{q} = (\eta, \mu, \theta)$ of quantization parameters satisfying $\mu \leq span(\mathsf{U})$, $\eta \leq 2\varepsilon \leq 2\theta$, and*

$$\beta(\varepsilon,\tau) + 2\gamma(\mu,\tau) + \beta(\theta,\tau) + \frac{\eta}{2} \leq \varepsilon, \tag{3.3.23}$$

we have $S_{\tau}(\Sigma) \cong^{\varepsilon}_{\mathcal{AS}} S_{\mathsf{q}}(\Sigma)$.

Although the condition $\eta \leq 2\varepsilon$ follows from (3.3.23), we decided to include it in the statement of Theorem 3.3.9 so that its assumptions can be easily compared with the assumptions in Theorem 3.3.6. Moreover, under a δ-ISS assumption and if τ is sufficiently large, we can always find a quantization vector q satisfying (3.3.23).

Proof: We prove $S_{\tau}(\Sigma) \preceq^{\varepsilon}_{\mathcal{AS}} S_{\mathsf{q}}(\Sigma)$. Consider the relation $R \subseteq X_{\tau} \times X_{\mathsf{q}}$ defined by $(x_{\tau}, x_{\mathsf{q}}) \in R$ if and only if

$$\|H_{\tau}(x_{\tau}) - H_{\mathsf{q}}(x_{\mathsf{q}})\| = \|x_{\tau} - x_{\mathsf{q}}\| \leq \varepsilon.$$

Since $X_\tau \subseteq \bigcup_{q\in[\mathbb{R}^n]_\eta} \mathcal{B}_{\eta/2}(q)$, for every $x_\tau \in X_\tau$ there exists $x_{\mathsf{q}} \in X_{\mathsf{q}}$ such that:

$$\|x_\tau - x_{\mathsf{q}}\| \leq \frac{\eta}{2} \leq \varepsilon. \tag{3.3.24}$$

Hence, $(x_\tau, x_{\mathsf{q}}) \in R$ and condition (i) in Definition 3.2.5 is satisfied. Consider now any $(x_\tau, x_{\mathsf{q}}) \in R$. Condition (ii) in Definition 3.2.5 is satisfied by the definition of R. Let us now show that condition (iii) in Definition 3.2.5 holds.

Consider any $\upsilon_\tau \in U_\tau$, defined on $[0, \tau[$, and choose an input $u_{\mathsf{q}} \in U_{\mathsf{q}}$ satisfying:

$$\|\upsilon_\tau - u_{\mathsf{q}}\|_\infty = \|\upsilon_\tau(0) - u_{\mathsf{q}}(0)\| \leq \mu. \tag{3.3.25}$$

Note that existence of such u_{q} is guaranteed by the special form of U and by the inequality $\mu \leq span(\mathsf{U})$ which guarantees that $\mathsf{U} \subseteq \bigcup_{p\in[\mathsf{U}]_\mu} \mathcal{B}_\mu(p)$. Consider the unique $x'_\tau = \xi_{x_\tau \upsilon_\tau}(\tau) \in \mathbf{Post}_{\upsilon_\tau}(x_\tau)$ in $S_\tau(\Sigma)$. It follows from the δ-FC assumption that the distance between x'_τ and $\xi_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau)$ is bounded as:

$$\|x'_\tau - \xi_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau)\| \leq \beta(\varepsilon, \tau) + \gamma(\mu, \tau). \tag{3.3.26}$$

For all $x'_{\mathsf{q}} \in \mathbf{Post}_{u_{\mathsf{q}}}(x_{\mathsf{q}})$, and based on the definition of the symbolic model, we have:

$$\|\xi_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau) - x'_{\mathsf{q}}\| \leq \beta(\theta, \tau) + \gamma(\mu, \tau) + \frac{\eta}{2}. \tag{3.3.27}$$

Using the inequalities (3.3.23), (3.3.26), (3.3.27), and the triangle inequality, we obtain:

$$\begin{aligned}
\|x'_\tau - x'_{\mathsf{q}}\| &= \|x'_\tau - \xi_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau) + \xi_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau) - x'_{\mathsf{q}}\| \\
&\leq \|x'_\tau - \xi_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau)\| + \|\xi_{x_{\mathsf{q}} u_{\mathsf{q}}}(\tau) - x'_{\mathsf{q}}\| \\
&\leq \beta(\varepsilon, \tau) + \gamma(\mu, \tau) + \beta(\theta, \tau) + \gamma(\mu, \tau) + \frac{\eta}{2} \leq \varepsilon.
\end{aligned}$$

Hence $(x'_\tau, x'_{\mathsf{q}}) \in R$ and condition (iii) in Definition 3.2.5 holds and we have $S_\tau(\Sigma) \preceq^\varepsilon_{\mathcal{AS}} S_{\mathsf{q}}(\Sigma)$.

The proof of the other direction: $S_{\mathsf{q}}(\Sigma) \preceq^\varepsilon_{\mathcal{AS}} S_\tau(\Sigma)$, follows from Theorem 3.3.6.

Since the relationships established in Theorem 3.3.9 in this chapter and Theorem 5.1 in [PGT08] are different: approximate bisimulation in [PGT08] and alternating approximate bisimulation in this chapter, and since the construction in (3.3.15) applies to nondeterministic systems but the construction provided in [PGT08] applies to deterministic systems, one can not compare these two theorems directly. However, we can verify that the assumptions in Theorem 3.3.9 are stronger in the sense that any quadruple $(\tau, \eta, \mu, \theta)$ satisfying (3.3.23) also satisfies the condition $\beta(\varepsilon, \tau) + \gamma(\mu) + \frac{\eta}{2} \leq \varepsilon$ in Theorem 5.1 in [PGT08] while the converse is not true.

The symbolic model $S_{\mathsf{q}}(\Sigma)$ in (3.3.15) has a countably infinite set of states. In order to construct a finite symbolic model we note that in practical applications the physical variables are restricted to a compact set. Velocities, temperatures, pressures, and other physical quantities cannot become arbitrarily large without violating the operational envelop defined by the control problem being solved. By making use of this fact, we can directly compute a finite abstraction $S_{\mathsf{q}D}(\Sigma)$ of $S_\tau(\Sigma)$ capturing the behavior of $S_\tau(\Sigma)$ within a given set D, which is assumed to be a finite union of boxes containing the origin. By having the extra condition $\eta \leq span(D)$, we define the system

$$S_{\mathsf{q}D}(\Sigma) = (X_{\mathsf{q}D}, U_{\mathsf{q}D}, \underset{\mathsf{q}D}{\longrightarrow}, Y_{\mathsf{q}D}, H_{\mathsf{q}D}),$$

where

- $X_{\mathsf{q}D} = [D]_\eta$;
- $U_{\mathsf{q}D} = U_{\mathsf{q}}$
- $x_{\mathsf{q}D} \overset{u_{\mathsf{q}D}}{\underset{\mathsf{q}D}{\longrightarrow}} x'_{\mathsf{q}D}$ if $\|\xi_{x_{\mathsf{q}D}u_{\mathsf{q}D}}(\tau) - x'_{\mathsf{q}D}\| \leq \beta(\theta, \tau) + \gamma(\mu, \tau) + \eta$ and any $x'_{\mathsf{q}} \in \mathbf{Post}_{u_{\mathsf{q}D}}(x_{\mathsf{q}D})$ in $S_{\mathsf{q}}(\Sigma)$ belongs to $X_{\mathsf{q}D}$;
- $Y_{\mathsf{q}D} = Y_{\mathsf{q}}$;

- $H_{\mathsf{q}D} = H_{\mathsf{q}}$.

Note that $S_{\mathsf{q}D}(\Sigma)$ is a finite system because D is a compact set. Moreover, the relation $R \subseteq X_{\mathsf{q}D} \times X_{\mathsf{q}}$ defined by $(x_{\mathsf{q}D}, x_{\mathsf{q}}) \in R$ if $x_{\mathsf{q}D} = x_{\mathsf{q}}$ is an alternating 0-approximate simulation relation from $S_{\mathsf{q}D}(\Sigma)$ to $S_{\mathsf{q}}(\Sigma)$. By combining $S_{\mathsf{q}D}(\Sigma) \preceq^{0}_{\mathcal{AS}} S_{\mathsf{q}}(\Sigma)$ with $S_{\mathsf{q}}(\Sigma) \preceq^{\varepsilon}_{\mathcal{AS}} S_{\tau}(\Sigma)$ we conclude[4] $S_{\mathsf{q}D}(\Sigma) \preceq^{\varepsilon}_{\mathcal{AS}} S_{\tau}(\Sigma)$. Hence, any controller synthesized for the finite model $S_{\mathsf{q}D}(\Sigma)$ can be refined to a controller enforcing the same specification on $S_{\tau}(\Sigma)$. Detailed information on how to construct refinements can be found in [Tab09].

3.4 Examples

We now experimentally demonstrate the effectiveness of our contributions. In the examples below, all constants and variables use SI units. Moreover, the computation of the abstractions $S_{\mathsf{q}}(\Sigma)$ was performed using the tool Pessoa [Pes09] on a laptop with CPU Intel Core 2 Duo @ 2.4GHz. Controllers enforcing the specifications were found by using standard algorithms from game theory, see e.g. [MPS95, Tab09], as implemented in Pessoa. We assume that control inputs are piecewise constant of duration τ

In the examples, we assume that $\mathcal{U}_{\tau}$ is finite. In the first three examples, we assume that $\mathcal{U}_{\tau}$ contains curves taking values in $[\mathsf{U}]_{0.5}$ and in the last example, taking values in $[\mathsf{U}]_{0.3}$. Hence, as explained in Remarks 3.3.2 and 3.3.7, $\mu = 0$ in the conditions (3.3.3) and (3.3.14) and in the definition of $S_{\mathsf{q}}(\Sigma)$ in (3.3.15), $\gamma(\mu, \tau) = 0$.

Table 3.1 summarizes the experimental results for the first three examples.

[4] It is shown in [Tab09] that the composition of two alternating simulation relations is still an alternating simulation relation.

Control Systems	τ	ε	Parameters of $S_{\mathsf{q}}(\Sigma)$ μ=0.5			Time		
			[PGT08] η	[GPT09] η	Our approach (η, k)	[PGT08]	[GPT09]	Our approach
DC motor	0.02	1	N/A	0.012	(0.5, 2)	N/A	∞	1.42s
Robot	0.002	0.075	0.0027	0.0022	(0.01, 4)	∞	∞	45.29s
Pendulum	0.02	0.25	N/A	0.0007	(0.1, 2)	N/A	∞	33s

Table 3.1: Parameters of $S_{\mathsf{q}}(\Sigma)$ and overall required time for constructing $S_{\mathsf{q}}(\Sigma)$ and synthesizing controllers. The notation N/A means not applicable. We use the notation ∞ to indicate that the size of $S_{\mathsf{q}}(\Sigma)$ is too large for Pessoa to finish constructing the abstraction.

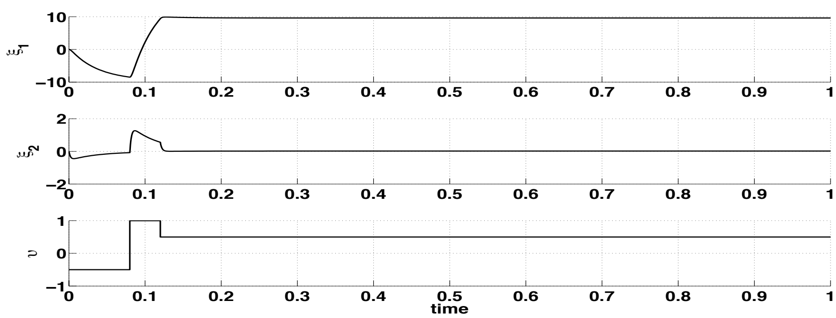

Figure 3.1: Upper and central panels: evolution of ξ_1 and ξ_2 with initial condition $(0,\ 0)$. Lower panel: input signal.

3.4.1 DC motor

Model: Consider a linear DC motor (from [CMU]) described by:

$$\Sigma : \begin{cases} \dot{\xi}_1 = -\frac{b}{J}\xi_1 + \frac{K}{J}\xi_2, \\ \dot{\xi}_2 = -\frac{K}{L}\xi_1 - \frac{R}{L}\xi_2 + \frac{1}{L}\upsilon, \end{cases} \tag{3.4.1}$$

where ξ_1 is the angular velocity of the motor, ξ_2 is the current through the inductor, υ is the source voltage, $b = 10^{-4}$ is the damping ratio of the mechanical system, $J = 10^{-4}$ is the moment of inertia of the rotor, $K = 5 \times 10^{-2}$ is the electromotive force constant, $L = 2 \times 10^{-3}$ is the electric inductance, and $R = 1$ is the electric resistance. Using Remark 2.3.3, it is readily seen that Σ is δ-ISS.

Abstraction: We assume that $\mathsf{U} = [-1,\ 1]$. We work on the subset $D = [-10,\ 10]\times[-10,\ 10]$ of the state space of Σ. For a sampling time $\tau = 0.02$, the function β in (2.3.4) is given by $\beta(\varepsilon, \tau) = 1.26\varepsilon$. Hence, the results in [PGT08] cannot be applied because the condition (3.3.3) of Theorem 3.3.1 cannot be fulfilled when $k = 1$. On the other hand, by choosing $k = 2$, we have $\beta(\varepsilon, k\tau) = 0.73\varepsilon$ implying that the condition (3.3.3) of Theorem 3.3.1 can be fulfilled. For a precision $\varepsilon = 1$, we construct a symbolic model $S_{\mathsf{q}}(\Sigma)$. The parameters of $S_{\mathsf{q}}(\Sigma)$ based on the results in this paper as well as the construction in [GPT09] are given in Table 3.1. The proposed state space quantization parameter in [GPT09] is roughly 42 times smaller than our quantization parameter. Since Σ is a 2 dimensional system, the size of our abstraction is 42^2 times smaller than the one in [GPT09].

Example control problem: Consider the objective to design a controller forcing the trajectories of Σ to reach and stay within $W = [9,\ 10] \times [-1,\ 1]$ thereafter while always remaining within $Z = [-10,\ 10] \times [-1,\ 1]$, that is, the LTL specification $\Diamond\Box W \wedge \Box Z$. Using the result in Remark 3.3.3, we compute $\widehat{Z}$ and $\widehat{W}$ and note that in this case, $\widehat{Z} \subseteq Z$ and $\widehat{W} \subseteq W$. Hence, the existence of a symbolic controller on $S_{\mathsf{q}}(\Sigma)$ satisfying $\Diamond\Box W \wedge \Box Z$ implies the existence of a controller satisfying $\Diamond\Box W \wedge \Box Z$ on Σ. In Figure 3.1, we show the closed-loop trajectory stemming from the initial condition $(0,\ 0)$ as well as the evolution of the input signal. It is readily seen that the specifications are satisfied in the sense that trajectories of Σ reach and stay within $W^{\varepsilon} = [9-\varepsilon,\ 10] \times [-1-\varepsilon,\ 1+\varepsilon]$ thereafter while always remaining within $Z^{\varepsilon} = [-10,\ 10] \times [-1-\varepsilon,\ 1+\varepsilon]$.

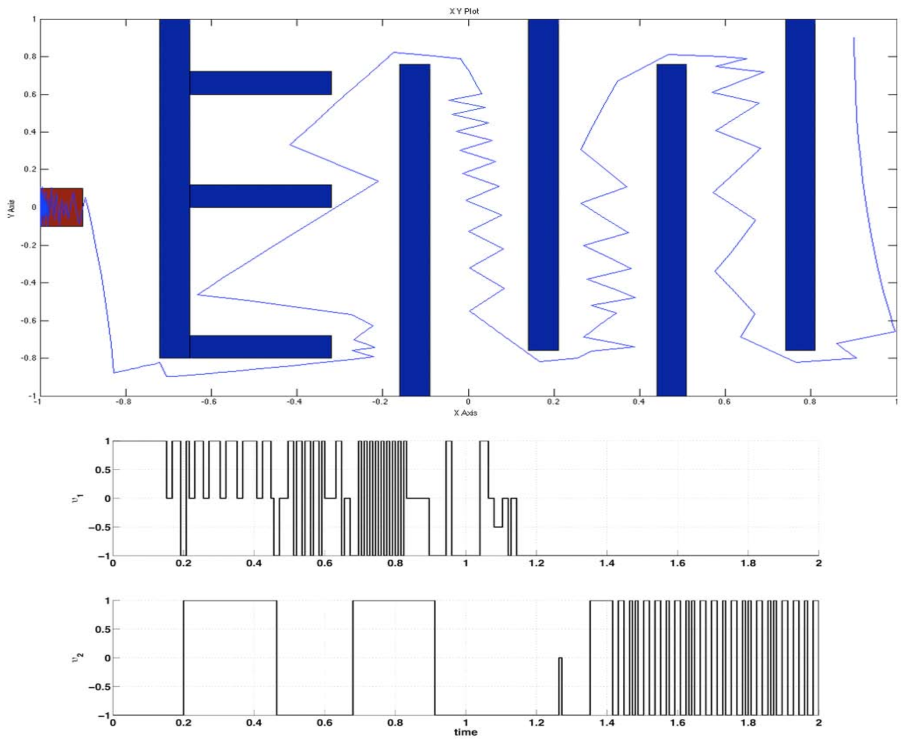

Figure 3.2: Evolution of the robot and the input signals with initial condition (0.9, 0.9).

3.4.2 Motion planing

Model: Consider a linear model of a robot described by:

$$\Sigma : \begin{cases} \dot{\xi}_1 = -10\xi_1 - \xi_2 + 10\upsilon_1, \\ \dot{\xi}_2 = -\xi_1 - 10\xi_2 + 10\upsilon_2. \end{cases} \tag{3.4.2}$$

The position of the robot is given by the pair (ξ_1, ξ_2). The pair (υ_1, υ_2) are the control inputs, expressing the velocity of the wheels. Using Remark 2.3.3, it is readily seen that Σ is δ-ISS.

Abstraction: We assume that $(\upsilon_1, \upsilon_2) \in \mathsf{U} = [-1,\ 1] \times [-1,\ 1]$. We work on the subset $D = [-1,\ 1] \times [-1,\ 1]$ of the state space of Σ. For a sampling time $\tau = 0.002$, the function β in (2.3.4) is given by $\beta(\varepsilon, \tau) = 0.982\varepsilon$. Hence, the results in [PGT08] can be applied. On the other hand, by choosing $k = 4$, we have $\beta(\varepsilon, k\tau) =$

0.93ε implying that the condition (3.3.3) of Theorem 3.3.1 can also be fulfilled. For a precision $\varepsilon = 0.075$, we construct a symbolic model $S_{\mathsf{q}}(\Sigma)$. The parameters of $S_{\mathsf{q}}(\Sigma)$ based on the results in this paper and those from [PGT08, GPT09] are given in Table 3.1. The state space quantization parameters in [PGT08, GPT09] are roughly four times smaller than our η. Therefore, the size of our abstraction is roughly 4^2 times smaller than the ones in [PGT08, GPT09].

Example control problem: Consider the problem of designing a controller navigating the robot to reach the target set $W = [-1,\ -0.9] \times [-0.1,\ 0.1]$, indicated with a target box in the far left hand side in Figure 3.2, while avoiding the obstacles, indicated as rectangular boxes in Figure 3.2, and then remain indefinitely inside W. If we denote by ϕ and ψ the predicates representing the target and obstacles, respectively, this specification can also be expressed by the LTL formula $\Diamond\Box\phi \wedge \Box\neg\psi$. If we express the non-obstacle area in Figure 3.2 as the union of l polyhedra Z_i, for $i = 1, \ldots, l$, then using the result in Remark 3.3.3, we compute $\widehat{Z}_i$ and $\widehat{W}$, and note that for this example, $\widehat{Z}_i \subseteq Z_i$, for each $i = 1, \ldots, l$, and $\widehat{W} \subseteq W$. Hence, a symbolic controller on $S_{\mathsf{q}}(\Sigma)$ satisfying $\Diamond\Box\phi \wedge \Box\neg\psi$ implies there exists a controller satisfying the specification on Σ. In Figure 3.2, we show the closed-loop trajectory stemming from the initial condition $(0.9,\ 0.9)$ and the evolution of the input signals. It is readily seen that the specification is satisfied.

3.4.3 Pendulum with resource constraints

Model: Consider a nonlinear model of a pendulum on a cart (from [PGT08]) described by:

$$\Sigma : \begin{cases} \dot{\xi}_1 = \xi_2, \\ \dot{\xi}_2 = -\frac{g}{l}\sin(\xi_1) - \frac{h}{m}\xi_2 + \upsilon, \end{cases} \tag{3.4.3}$$

where ξ_1 and ξ_2 are the angular position and velocity of the point mass, υ is the torque applied to the cart, $g = 9.8$ is acceleration due to gravity, $l = 5$ is the length of the rod, $m = 0.5$ is the mass, and $h = 3$ is the coefficient of friction. As shown in [PGT08], Σ is δ-ISS.

Abstraction: We assume that $\mathsf{U} = [-1.5,\ 1.5]$. We work on the subset $D = [-1,\ 1] \times [-1,\ 1]$ of the state space of Σ. As shown in [PGT08], the function β in (2.3.4) is given by $\beta(\varepsilon, \tau) = 6.17\mathrm{e}^{-2.08\tau}\varepsilon$, so for a sampling time $\tau = 0.5$, we have $\beta(\varepsilon, \tau) = 2.18\varepsilon$. Hence, the results in [PGT08] cannot be applied because the condition (3.3.3) of Theorem 3.3.1 cannot be fulfilled when $k = 1$. On the other hand, by choosing $k = 2$, we have $\beta(\varepsilon, k\tau) = 0.77\varepsilon$, so the condition (3.3.3) of Theorem 3.3.1 is fulfilled. For a precision $\varepsilon = 0.25$, we construct a symbolic model $S_{\mathsf{q}}(\Sigma)$. The parameters of $S_{\mathsf{q}}(\Sigma)$ based on the results in this paper and [GPT09] are given in Table 3.1. The state space quantization parameter in [GPT09] is roughly 147 times smaller than our quantization parameter. Therefore, the size of the symbolic model computed by our algorithm is roughly $147^2 \sim 2 \times 10^4$ times smaller than the one in [GPT09].

Example control problem: Suppose our objective is to design a controller forcing the trajectories of the system to reach the target set $W = [-0.7,\ -0.6] \times [-1,\ 1]$ and to remain indefinitely inside W. Furthermore, to add a discrete component to the problem, we assume that the controller is implemented on a microprocessor, executing other tasks in addition to the control task. We consider a schedule with epochs of four time slots in which the first two slots are allocated to the control task and the rest of them to other tasks. The expression time slot refer to a time interval of the form $[k'\tau, (k'+1)\tau[$ with $k' \in \mathbb{N}$ and where τ is the sampling time. Therefore, the microprocessor schedule is given by:

$$|\mathsf{aauu}|\mathsf{aauu}|\mathsf{aauu}|\mathsf{aauu}|\mathsf{aauu}|\mathsf{aauu}|\mathsf{aauu}|\ldots,$$

where a denotes a slot available for the control task and u denotes a slot allotted to other tasks. The symbol | separates each epoch of four time slots. The schedulability constraint on the microprocessor can be represented by the finite system in Figure 3.3. Initial states of the finite system are distinguished by being the

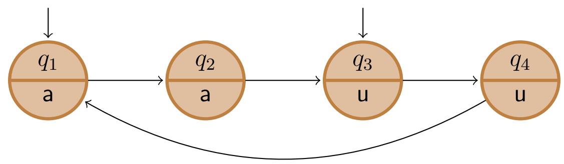

Figure 3.3: Finite system describing the schedulability constraints. The lower part of the states are labeled with the outputs a and u denoting availability and unavailability of the microprocessor, respectively.

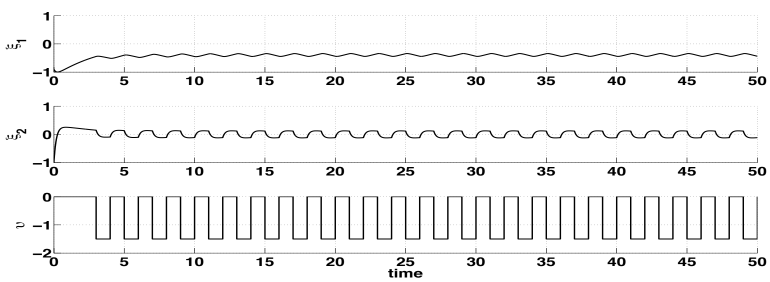

Figure 3.4: Upper and central panels: evolution of ξ_1 and ξ_2 with initial condition $(-0.9, \ -1)$. Lower panel: input signal.

target of a sourceless arrow.

In Figure 3.4, we show the closed-loop trajectory stemming from the initial condition $(-0.9, \ -1)$, and the evolution of the input signal, where the finite system initialized from state q_3.

3.4.4 Vehicle navigation

We borrowed this example from [AM08]. In this model, the motion of the front and rear pairs of wheels are approximated by a single front wheel and a single rear wheel. We consider the following model for the vehicle:

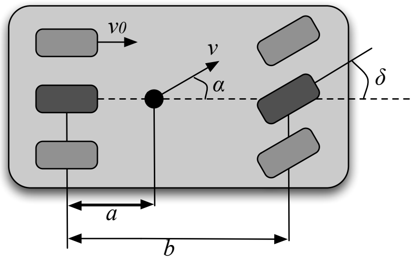

Figure 3.5: Vehicle top view.

$$\Sigma : \begin{cases} \dot{x} = v_0 \frac{\cos(\alpha+\theta)}{\cos(\alpha)}, \\ \dot{y} = v_0 \frac{\sin(\alpha+\theta)}{\cos(\alpha)}, \\ \dot{\theta} = \frac{v_0}{b} \tan(\delta), \end{cases} \tag{3.4.4}$$

where $\alpha = \arctan\left(\frac{a\tan(\delta)}{b}\right)$. The position of the vehicle is given by the pair (x, y), and the orientation of the vehicle is given by θ. The pair (v_0, δ) are the control inputs, expressing the velocity of the rear wheel and the steering angle, respectively. It is readily seen that Σ is not δ-ISS. Hence, the results in the first part of this chapter and in [PGT08, PT09, GPT09] cannot be applied to this system. We assume that $a = 0.5$, $b = 1$, and $(v_0, \delta) \in \mathsf{U} = [-1,\ 1] \times [-1,\ 1]$. Since control inputs are piecewise constant of duration τ, it can be readily checked that for any $t \in [0, \tau]$, we get:

$$\begin{aligned} x(t) =& \frac{b\Big[\sin\big(\alpha + \frac{v_0}{b}\tan(\delta)t + \theta(0)\big) - \sin(\alpha + \theta(0))\Big]}{\cos(\alpha)\tan(\delta)} + x(0), \\ y(t) =& \frac{-b\Big[\cos\big(\alpha + \frac{v_0}{b}\tan(\delta)t + \theta(0)\big) - \cos(\alpha + \theta(0))\Big]}{\cos(\alpha)\tan(\delta)} + y(0), \\ \theta(t) =& \frac{v_0}{b}\tan(\delta)t + \theta(0), \end{aligned}$$

if $\tan(\delta) \neq 0$, and $x(t) = v_0\cos(\theta(0))t + x(0)$, $y(t) = v_0\sin(\theta(0))t + y(0)$, and $\theta(t) = \theta(0)$, if $\tan(\delta) = 0$. It can be verified that for the given U, the function β is given by $\beta(r, t) = (1{+}1.267t)r$. We work on the subset $D = [0,\ 10] \times [0,\ 10] \times [-\pi,\ \pi]$ of the state space of Σ. Our objective is to design a controller navigating the vehicle to reach the target set $W = [9,\ 9.5] \times [0,\ 0.5]$, indicated with a red box in

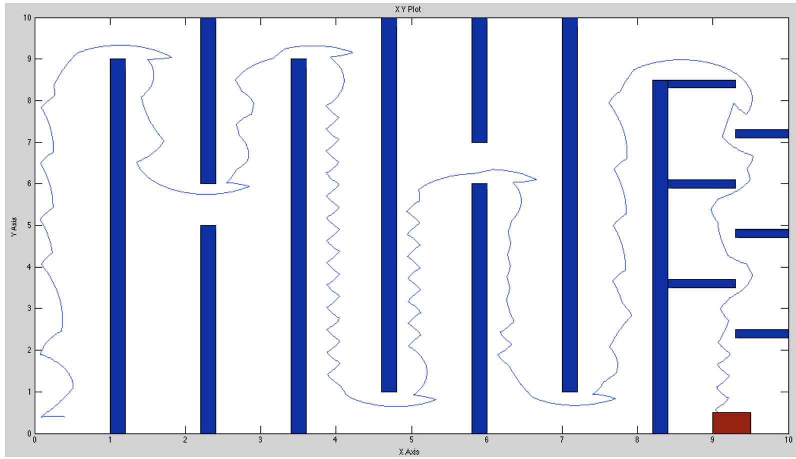

Figure 3.6: Evolution of the vehicle with initial condition $(0.4,\ 0.4,\ 0)$.

Figure 3.6, while avoiding the obstacles, indicated as blue boxes in Figure 3.6, and remain indefinitely inside W. For a precision $\varepsilon = 0.2$, we construct a symbolic model $S_{\mathsf{q}D}(\Sigma)$ by choosing $\theta = 0.2$, $\eta = 0.2$, and $\tau = 0.3$ so that the assumptions of Theorem 3.3.6 are satisfied. The resulting number of states, inputs, and transitions were 91035, 49, and 34020088, respectively. The consumed CPU time for computing the abstraction and synthesizing the controller were 13509.38 and 535.987 seconds, respectively. In Figure 3.6, we show the closed-loop trajectory stemming from the initial condition $(0.4,\ 0.4,\ 0)$. It is readily seen that the specification is satisfied. In Figure 3.7, we show the evolution of the input signals.

3.5 Discussion

In the first part of this chapter, we presented a construction of symbolic models for δ-ISS digital control systems without any restriction on the sampling time and without using Lyapunov functions. The preceding experiments demonstrate that our construction improves upon previous techniques by providing approximate transition systems that are orders of magnitude smaller than previous constructions, enabling automatic controller synthesis for systems that were beyond the

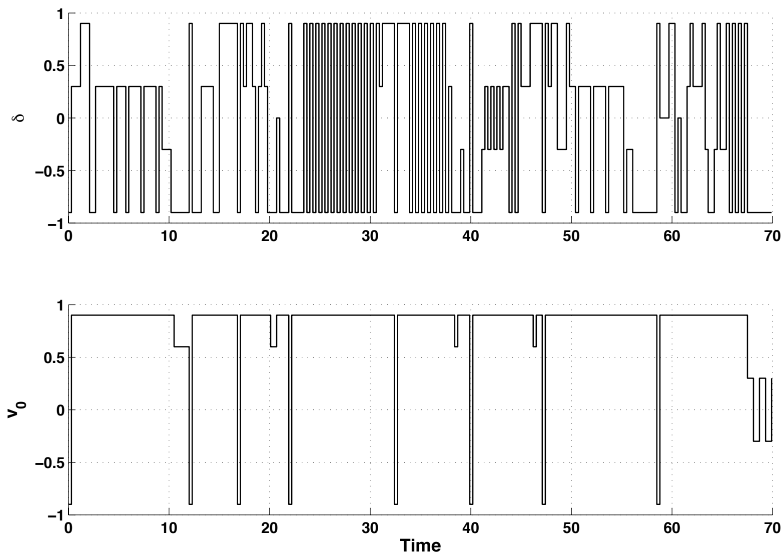

Figure 3.7: Evolution of the input signals.

capacity of current tools. In the second part of the chapter we showed that any δ-FC digital control system, suitably restricted to a compact subset of states, admits a finite (not necessarily bisimilar) symbolic model. Our results improve upon the existing work by being applicable to a large class of control systems and by not requiring the exact computation of reachable sets or the convexity of reachable sets. The symbolic models constructed according to the results presented in this chapter can be used to synthesize controllers enforcing complex specifications given in several different formalisms such as temporal logics or automata on infinite strings. The synthesis of such controllers is well understood and can be performed using simple fixed-point computations as described in [Tab09].

CHAPTER 4

Incremental properties of nonlinear control systems

4.1 Introduction

Stability is a property of dynamical systems comparing trajectories with an equilibrium point or with a particular trajectory. Incremental stability is a stronger property comparing arbitrary trajectories with themselves, rather than with an equilibrium point or with a particular trajectory. It is well-known that for linear systems incremental stability is equivalent to global asymptotic stability. For nonlinear systems, incremental stability is a stronger property requiring separate concepts and techniques for its study.

The notion of incremental stability has a long history that can be traced back to the work of Zames in the 60's, as described in [Zam96]. In [Zam63], incremental stability is introduced and studied under the input-output setting where control systems are regarded as operators mapping input signals to output signals. Incremental stability then arises naturally by considering the Lipschitz constant of the operator. A modern treatment of incremental stability, based on Lyapunov methods, appeared only recently in [Ang02] where incremental global asymptotic stability (δ-GAS) and incremental input-to-state stability (δ-ISS) were defined in a state-space setting. The notion of δ-GAS was defined by requiring the Euclidean distance between two arbitrary system trajectories to converge[1] to zero. Lyapunov

[1] A suitable "small overshoot" requirement is also included in the definition.

characterizations of δ-GAS and δ-ISS were also given in [Ang02]. There are two other stability properties related to incremental stability that have an equally long, if not longer, history.

The first is the notion of convergent system which, according to [PPvdWN04], was introduced in the 60's by B. P. Demidovich in [Dem61, Dem67]. A system is convergent if all the system trajectories converge[1] to a trajectory, bounded on the whole time axis. Furthermore, Demidovich also introduced a sufficient condition for a system to be convergent, called the Demidovich's condition in [PvdWN05]. Since incremental stability requires every trajectory to converge to every other trajectory, an incrementally stable system is also a convergent system whenever a trajectory, bounded on the whole time axis, exists.

The second stability property is contractivity and was introduced in the control community by Lohmiller and Slotine in [LS98] although it had been studied before in the mathematical community [Jou05]. Rather than comparing trajectories, the notion of contracting system is infinitesimal and requires the decrease of a suitable quantity, defined through a Riemannian metric, along trajectories. The definition of contracting system can be seen as a generalization of the Demidovich's condition. Note that while the Demidovich's condition was introduced as a sufficient condition for a system to be convergent, its generalization in [LS98] was directly used as the definition of contracting system.

Although both contractivity as well as convergence are coordinate independent properties, this is no longer the case with incremental stability. In this chapter, however, we work with a variation of incremental stability that is coordinate invariant. This is achieved by no longer insisting on the distance between trajectories being measured by the Euclidean metric.

4.1.1 Previous work

The number of applications of incremental stability has increased in the past years. Examples include building explicit bounds on the region of attraction in phase-locking in the Kuramoto system [FCPL10], modeling of nonlinear analog circuits [BML$^+$10], robustness analysis of systems over finite alphabets [TMD08], global synchronization in networks of cyclic feedback systems [HSSG12], control reconfiguration of piecewise affine systems with actuator and sensor faults [RHvdWL11], intrinsic observer design [AR03], consensus problems in complex networks [WS05], output regulation of nonlinear systems [PvdWN05], design of frequency estimators [SK08], synchronization of coupled identical dynamical systems [RdBS09], construction of symbolic models for nonlinear control systems [PGT08, PT09, GPT09], and the analysis of bio-molecular systems [RdB09]. Our motivation comes from symbolic control where incremental stability was identified as a key property enabling the construction of finite bisimilar abstractions of nonlinear control systems, as shown in Chapter 3 and in [PGT08, PT09, GPT09]. Hence, there is a growing need for design methods providing controllers enforcing incremental stability since most of the existing design methods guarantee stability rather than incremental stability.

Related works include controller design for convergence of Lur'e-type systems [PvdWN05, PvdWN07] and a class of piecewise affine systems [vdWP08] through the solution of linear matrix inequalities (LMIs). In contrast, the current work in this chapter does not require the solution of LMIs and the existence of controllers is always guaranteed for the class of systems under consideration. The quest for backstepping design approaches for incremental stability has received increasing attention recently. Recently obtained results include backstepping design approaches rendering parametric-strict-feedback[2] form sys-

[2]See [KKK95] for a definition.

tems incrementally globally asymptotically stable[3] using the notion of contraction metrics in [JL02, SK09, SK08]. The results in [PvdWN05] offer a backstepping design approach rendering a larger class of control systems than those in parametric-strict-feedback form input-to-state convergent, rather than incrementally input-to-state stable. We will build upon these results in [PvdWN05] and extend those in the scope of incremental stability. The notion of (input-to-state) convergence requires existence of a trajectory which is bounded on the whole time axis which is not the case in incremental input-to-state stability. The notion of input-to-state convergence can not be applied to the results in Chapter 3 and in [PGT08, GPT09, PT09], which require the global asymptotic stability of all trajectories rather than that of to particular trajectory.

4.1.2 Contributions

Angeli [Ang02] proposed the notions of incremental Lyapunov function and incremental input-to-state Lyapunov function, and used these notions to prove characterizations of δ-GAS and δ-ISS. Both proposed notions of Lyapunov functions in [Ang02] are not coordinate independent, in general. In this chapter, we propose new notions of incremental Lyapunov functions and incremental input-to-state Lyapunov functions that are coordinate invariant. Moreover, we use these new notions of Lyapunov functions to characterize notions of $\delta_\exists$-GAS and $\delta_\exists$-ISS. Since the proposed notions of Lyapunov functions in this chapter are coordinate invariant, they potentiate the development of synthesis tools for incremental stability.

In addition to incremental Lyapunov functions, the $\delta_\exists$-GAS and $\delta_\exists$-ISS conditions can be checked by resorting to contraction metrics. The interested reader may consult [LS98] for more detailed information about the notion of contraction metrics. The description of $\delta_\exists$-GAS in terms of existence of a contraction metric was given in [AR03]. In this chapter, we provide the description of $\delta_\exists$-ISS in terms

[3] Understood in the sense of Definition 2.3.1.

of existence of a contraction metric.

Finally, we propose a backstopping design approach providing controllers rendering some classes of control systems $\delta_\exists$-GAS and $\delta_\exists$-ISS. Our techniques improve upon most of the existing backstepping techniques in three directions:

1) by providing controllers enforcing not only incremental global asymptotic stability but also incremental input-to-state stability;

2) by being applicable to larger classes of (non-smooth) control systems;

3) by providing a recursive way of constructing not only contraction metrics but also incremental Lyapunov functions.

In the first direction, our technique extends the results in [JL02, SK09, SK08], where only controllers enforcing incremental global asymptotic stability are designed. In the second direction, our technique improves the results in [JL02, SK09, SK08], which are only applicable to smooth parametric-strict-feedback form systems. In the third direction, our technique extends the results in [JL02, SK09, SK08], where the authors only provide a recursive way of constructing contraction metrics, and the results in [PvdWN05], where the authors do not provide a way to construct Lyapunov functions characterizing the input-to-state convergence property induced by the controller. Note that the explicit availability of incremental Lyapunov functions is necessary in many applications. Examples include the construction of symbolic models for nonlinear control systems [GPT09, Gir05, CGG11], robust test generation of hybrid systems [JFA+07], the approximation of stochastic hybrid systems [JP09], and source-code model checking for nonlinear dynamical systems [KDL+08]. Note that incremental Lyapunov functions can be used as bisimulation functions, recognized as a key tool for the analysis in [JFA+07, JFA+07, KDL+08].

4.2 Preliminaries

In this chapter, we denote by $\|x\|$ the Euclidean norm of x, for any $x \in \mathbb{R}^n$, rather than using $\|x\|_2$.

4.2.1 Incremental Lyapunov function

We start by introducing the following definition which was inspired by the notions of incremental global asymptotic stability (δ-GAS) Lyapunov function and incremental input-to-state stability (δ-ISS) Lyapunov function presented in [Ang02].

Definition 4.2.1. *Consider a control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ and a smooth function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$. Function V is called a $\delta_\exists$-GAS Lyapunov function for Σ, if there exist a metric $\mathbf{d}$, $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, and $\kappa \in \mathbb{R}^+$ such that:*

(i) for any $x, x' \in \mathbb{R}^n$,
$\underline{\alpha}(\mathbf{d}(x,x')) \leq V(x,x') \leq \overline{\alpha}(\mathbf{d}(x,x'))$;

(ii) for any $x, x' \in \mathbb{R}^n$ and any $u \in \mathsf{U}$,
$\frac{\partial V}{\partial x} f(x,u) + \frac{\partial V}{\partial x'} f(x',u) \leq -\kappa V(x,x')$.

Function V is called a $\delta_\exists$-ISS Lyapunov function for Σ, if there exist a metric $\mathbf{d}$, $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, σ, and $\kappa \in \mathbb{R}^+$ satisfying conditions (i) and:

(iii) for any $x, x' \in \mathbb{R}^n$ and for any $u, u' \in \mathsf{U}$,
$\frac{\partial V}{\partial x} f(x,u) + \frac{\partial V}{\partial x'} f(x',u') \leq -\kappa V(x,x') + \sigma(\|u-u'\|)$.

Remark 4.2.2. *Condition (iii) of Definition 4.2.1 can be replaced by:*

$$\frac{\partial V}{\partial x} f(x,u) + \frac{\partial V}{\partial x'} f(x',u') \leq -\rho(\mathbf{d}(x,x')) + \sigma(\|u-u'\|),$$

where ρ is a $\mathcal{K}_\infty$ function. It is known that there is no loss of generality in considering $\rho(\|x-y\|) = \kappa V(x,y)$, by appropriately modifying the δ-ISS Lyapunov function V (see Lemma 11 in [PW96]).

While δ-GAS and δ-ISS Lyapunov functions, as defined in [Ang02], require the metric $\mathbf{d}$ to be the Euclidean metric, Definition 4.2.1 only requires the existence of a metric. We note that while δ-GAS and δ-ISS Lyapunov functions are not invariant under changes of coordinates in general, $\delta_\exists$-GAS and $\delta_\exists$-ISS Lyapunov functions are.

In the next lemma, we show that $\delta_\exists$-GAS and $\delta_\exists$-ISS Lyapunov functions, defined in Definition 4.2.1, are invariant under changes of coordinates.

Lemma 4.2.3. *Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ be a control system and let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a global diffeomorphism. If the function V is a $\delta_\exists$-GAS (resp. $\delta_\exists$-ISS) Lyapunov function for Σ, then the function $\widehat{V}(y, y') = V\left(\phi^{-1}(y), \phi^{-1}(y')\right)$, for any $y, y' \in \mathbb{R}^n$, is a $\delta_\exists$-GAS (resp. $\delta_\exists$-ISS) Lyapunov function for $\Sigma' = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, \phi_* f)$.*

Proof: For simplifying the proof, we abuse the notation and use $V \circ \phi^{-1}$ to denote $V\left(\phi^{-1}, \phi^{-1}\right)$. Inequalities (i) in Definition 4.2.1, transforms under ϕ to:

$$\underline{\alpha}\left(\mathbf{d}\left(\phi^{-1}(y), \phi^{-1}(y')\right)\right) \leq V\left(\phi^{-1}(y), \phi^{-1}(y')\right) \leq \overline{\alpha}\left(\mathbf{d}\left(\phi^{-1}(y), \phi^{-1}(y')\right)\right).$$

Therefore, function $V \circ \phi^{-1}$ satisfies the inequalities (i) in Definition 4.2.1 by the metric $\mathbf{d}'(y, y') = \mathbf{d}\left(\phi^{-1}(y), \phi^{-1}(y')\right)$. Let us now show that condition (ii) in Definition 4.2.1 holds for $V \circ \phi^{-1}$. Using $\frac{\partial \phi^{-1}}{\partial y}\frac{\partial \phi}{\partial x}\left(\phi^{-1}(y)\right) = I_n$, we obtain:

$$\begin{aligned}
&\frac{\partial\left(V \circ \phi^{-1}\right)}{\partial y}(\phi_* f)(y, u) + \frac{\partial\left(V \circ \phi^{-1}\right)}{\partial y'}(\phi_* f)(y', u) \\
&= \frac{\partial V}{\partial x}\Big|_{x=\phi^{-1}(y)}\frac{\partial \phi^{-1}}{\partial y}\left(\phi_* f\right)(y, u) + \frac{\partial V}{\partial x'}\Big|_{x'=\phi^{-1}(y')}\frac{\partial \phi^{-1}}{\partial y'}\left(\phi_* f\right)(y', u) \qquad (4.2.1)\\
&= \frac{\partial V}{\partial x}\Big|_{x=\phi^{-1}(y)} f\left(\phi^{-1}(y), u\right) + \frac{\partial V}{\partial x'}\Big|_{x'=\phi^{-1}(y')} f\left(\phi^{-1}(y'), u\right) \\
&\leq -\kappa V\left(\phi^{-1}(y), \phi^{-1}(y')\right),
\end{aligned}$$

which completes the proof. Similarly, it can be shown that $V \circ \phi^{-1}$ satisfies the condition (iii) in Definition 4.2.1 for Σ' if V satisfies it for Σ.

Now we introduce the following definition which was inspired by the notion of

δ-ISS Lyapunov function in Definition 4.2.1, when the metric d is the Euclidean metric.

Definition 4.2.4. *Consider a control system* $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ *and a smooth function* $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$. *Function* V *is called a* δ*-FC Lyapunov function for* Σ, *if there exist* $\mathcal{K}_\infty$ *functions* $\underline{\alpha}$, $\overline{\alpha}$, σ, *and* $\kappa \in \mathbb{R}$ *such that:*

(i) for any $x, x' \in \mathbb{R}^n$,
$\underline{\alpha}(\|x - x'\|) \leq V(x, x') \leq \overline{\alpha}(\|x - x'\|)$;

(ii) for any $x, x' \in \mathbb{R}^n$ *and for any* $u, u' \in \mathsf{U}$,
$\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u') \leq \kappa V(x, x') + \sigma(\|u - u'\|)$.

Note that κ in the condition (ii) in Definition 4.2.4 can be both negative and positive.

4.2.2 Variational systems

The variational system associated with a smooth control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$, when we only have state variations, is given by the differential equation:

$$\frac{d}{dt}(\delta\xi) = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\xi \\ u=\upsilon}} \delta\xi, \tag{4.2.2}$$

for any $\upsilon \in \mathcal{U}$ and where $\delta\xi$ is the variation[4] of a trajectory ξ of Σ. More details about the variational system can be found in [CS87].

Similarly, the variational system associated with a smooth control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$, when we have state and input variations, is given by the differential equation:

$$\frac{d}{dt}(\delta\xi) = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\xi \\ u=\upsilon}} \delta\xi + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\xi \\ u=\upsilon}} \delta\upsilon, \tag{4.2.3}$$

where $\delta\xi$ and $\delta\upsilon$ are variations of a state trajectory ξ and an input curve υ of Σ, respectively.

[4]The variation $\delta\xi$ can be formally defined by considering a family of trajectories $\xi_{x\upsilon}(t, \epsilon)$ parametrized by $\epsilon \in \mathbb{R}$. The variation of the state is then $\delta\xi = \frac{\partial \xi_{x\upsilon}}{\partial \epsilon}$.

4.2.3 Contraction (expansion) metric

The following definition is adapted from [LS98]:

Definition 4.2.5. *Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ be a smooth control system on $\mathbb{R}^n$ equipped with a Riemannian metric G. The metric G is said to be a contraction (resp. expansion) metric, with respect to states, for system Σ if there exists some $\widehat{\lambda} \in \mathbb{R}^+$ (resp. $\widehat{\lambda} < 0$) such that:*

$$\langle X, X \rangle_F \leq -\widehat{\lambda} \langle X, X \rangle_G \tag{4.2.4}$$

for $F(x,u) = \left(\frac{\partial f}{\partial x}\right)^T G(x) + G(x)\frac{\partial f}{\partial x} + \frac{\partial G}{\partial x} f(x,u)$, any $u \in \mathsf{U}$, $X \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$, or equivalently:

$$X^T \left(\left(\frac{\partial f}{\partial x} \right)^T G(x) + G(x) \frac{\partial f}{\partial x} + \frac{\partial G}{\partial x} f(x,u) \right) X \leq -\widehat{\lambda} X^T G(x) X, \tag{4.2.5}$$

where the constant $\left|\widehat{\lambda}\right|$ is called contraction (resp. expansion) rate.

When the metric G is constant, the condition (4.2.4) or (4.2.5), for $\widehat{\lambda} \in \mathbb{R}^+$, is known as the Demidovich's condition [PvdWN05]. It is shown in [PvdWN05] that such condition implies incremental stability and the convergent system property.

Note that the inequality (4.2.4) or (4.2.5) implies:

$$\frac{d}{dt} \langle \delta\xi, \delta\xi \rangle_G \leq -\widehat{\lambda} \langle \delta\xi, \delta\xi \rangle_G, \tag{4.2.6}$$

when we only have state variations and $\delta\xi$ is the variation of a state trajectory ξ of Σ.

The following definition is a generalization of Definition 4.2.5, when we have state and input variations.

Definition 4.2.6. *Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ be a smooth control system on $\mathbb{R}^n$ equipped with a Riemannian metric G. The metric G is said to be a contraction (resp. expansion) metric, with respect to states and inputs, for system Σ if there exists* *some $\widehat{\lambda} \in \mathbb{R}^+$ (resp. $\widehat{\lambda} < 0$) and $\alpha \in \mathbb{R}_0^+$ such that:*

$$\langle X, X\rangle_F + 2\left\langle \frac{\partial f}{\partial u} Y, X \right\rangle_G \leq -\widehat{\lambda}\langle X, X\rangle_G + \alpha \langle X, X\rangle_G^{\frac{1}{2}} \langle Y, Y\rangle_{I_m}^{\frac{1}{2}} \tag{4.2.7}$$

for $F(x,u) = \left(\frac{\partial f}{\partial x}\right)^T G(x) + G(x)\frac{\partial f}{\partial x} + \frac{\partial G}{\partial x} f(x,u)$, any $X \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $u \in \mathsf{U}$, and $Y \in \mathbb{R}^m$, or equivalently:

$$\begin{aligned} X^T \left(\left(\tfrac{\partial f}{\partial x}\right)^T G(x) + G(x)\tfrac{\partial f}{\partial x} + \tfrac{\partial G}{\partial x} f(x,u)\right) X + 2Y^T \left(\tfrac{\partial f}{\partial u}\right)^T G(x) X \\ \leq -\widehat{\lambda} X^T G(x) X + \alpha (X^T G(x) X)^{\frac{1}{2}} (Y^T Y)^{\frac{1}{2}}, \end{aligned} \tag{4.2.8}$$

where the constant $\left|\widehat{\lambda}\right|$ is called contraction (resp. expansion) rate.

Note that the inequality (4.2.7) or (4.2.8) implies:

$$\frac{d}{dt}\langle \delta\xi, \delta\xi\rangle_G \leq -\widehat{\lambda}\langle \delta\xi, \delta\xi\rangle_G + \alpha\langle \delta\xi, \delta\xi\rangle_G^{\frac{1}{2}} \langle \delta\upsilon, \delta\upsilon\rangle_{I_m}^{\frac{1}{2}}, \tag{4.2.9}$$

when we have state and input variations and where $\delta\xi$ and $\delta\upsilon$ are variations of a state trajectory ξ and an input curve υ of Σ.

4.3 Chracterizations of incremental stability

4.3.1 Lyapunov characterizations of incremental stability

The following theorem, provided in [Ang02], characterizes δ-GAS and δ-ISS in terms of existence of a δ-GAS and a δ-ISS Lyapunov function, respectively.

Theorem 4.3.1 ([Ang02])**.** *Consider a control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$. Then:*

- *If U is compact then Σ is δ-GAS if and only if it admits a δ-GAS Lyapunov function;*
- *If U is closed, convex, compact, contains the origin and $f(0,0) = 0$, then Σ is δ-ISS if and only if it admits a δ-ISS Lyapunov function.*

To provide characterizations of $\delta_\exists$-ISS (resp. $\delta_\exists$-GAS) in terms of the existence of $\delta_\exists$-ISS (resp. $\delta_\exists$-GAS) Lyapunov functions, we need the following technical results.

Here, we introduce the following definition which was inspired by the notion of uniform global asymptotic stability with respect to sets, presented in [LSW96].

Definition 4.3.2. *A control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ is uniformly globally asymptotically stable (U$_\exists$GAS) with respect to a set $\mathcal{A} \subseteq \mathbb{R}^n$ if it is forward complete and there exist a metric $\mathbf{d}$, and a $\mathcal{KL}$ function β such that for any $t \in \mathbb{R}_0^+$, any $x \in \mathbb{R}^n$ and any $\upsilon \in \mathcal{U}$, the following condition is satisfied:*

$$\mathbf{d}(\xi_{x\upsilon}(t), \mathcal{A}) \leq \beta(\mathbf{d}(x, \mathcal{A}), t). \tag{4.3.1}$$

We now introduce the following definition which was inspired by the notion of uniform global asymptotic stability (UGAS) Lyapunov functions in [LSW96].

Definition 4.3.3. *Consider a control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$, a set $\mathcal{A}$, and a smooth function $V : \mathbb{R}^n \to \mathbb{R}_0^+$. Function V is called a U$_\exists$GAS Lyapunov function, with respect to $\mathcal{A}$, for Σ, if there exist a metric $\mathbf{d}$, $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, and $\kappa \in \mathbb{R}^+$ such that:*

(i) for any $x \in \mathbb{R}^n$,
$\underline{\alpha}(\mathbf{d}(x, \mathcal{A})) \leq V(x) \leq \overline{\alpha}(\mathbf{d}(x, \mathcal{A}))$;

(ii) for any $x \in \mathbb{R}^n$ and any $u \in \mathsf{U}$,
$\frac{\partial V}{\partial x} f(x, u) \leq -\kappa V(x)$.

The following theorem characterizes U$_\exists$GAS in terms of the existence of a U$_\exists$GAS Lyapunov function.

Theorem 4.3.4. *Consider a control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ and a set $\mathcal{A}$. If U is compact and $\mathbf{d}$ is a metric such that the function $\psi(\cdot) = \mathbf{d}(\cdot, y)$ is continuous[5] for any $y \in \mathbb{R}^n$ then the following statements are equivalent:*

[5] *Here, continuity is understood with respect to the Euclidean metric.*

(1) Σ is forward complete and there exists a $U_{\exists}GAS$ Lyapunov function with respect to $\mathcal{A}$, equipped with the metric $\mathbf{d}$.

(2) Σ is $U_{\exists}GAS$ with respect to $\mathcal{A}$, equipped with the metric $\mathbf{d}$.

Proof: First we show that the function $\phi(\cdot) = \mathbf{d}(\cdot, \mathcal{A})$ is a continuous function with respect to the Euclidean metric. Assume $\{x_n\}_{n=1}^{\infty}$ is a converging sequence in $\mathbb{R}^n$ with respect to the Euclidean metric, implying that $\|x_n - x^*\| \to 0$ as $n \to \infty$ for some $x^* \in \mathbb{R}^n$. By the triangle inequality, we have:

$$\mathbf{d}\left(x^*, y\right) \leq \mathbf{d}\left(x^*, x_n\right) + \mathbf{d}\left(y, x_n\right), \tag{4.3.2}$$

for any $n \in \mathbb{N}$ and any $y \in \mathcal{A}$. Using inequality (4.3.2), we obtain:

$$\begin{aligned} \phi\left(x^*\right) = \mathbf{d}(x^*, \mathcal{A}) = \inf_{y \in \mathcal{A}} \mathbf{d}\left(x^*, y\right) &\leq \inf_{y \in \mathcal{A}} \left\{\mathbf{d}\left(x^*, x_n\right) + \mathbf{d}\left(y, x_n\right)\right\} \\ &= \mathbf{d}\left(x^*, x_n\right) + \inf_{y \in \mathcal{A}} \mathbf{d}\left(y, x_n\right) = \mathbf{d}\left(x^*, x_n\right) + \phi\left(x_n\right), \end{aligned} \tag{4.3.3}$$

for any $n \in \mathbb{N}$. Using inequality (4.3.3) and the continuity assumption on $\mathbf{d}$, implying that $\lim_{n\to\infty} \mathbf{d}\left(x^*, x_n\right) = 0$, we obtain for any $n \in \mathbb{N}$:

$$\phi\left(x^*\right) \leq \inf_{m \geq n} \left\{\mathbf{d}\left(x^*, x_m\right) + \phi\left(x_m\right)\right\} \Rightarrow \phi\left(x^*\right) \leq \lim_{n\to\infty} \inf \phi\left(x_n\right), \tag{4.3.4}$$

where limit inferior exists because a lower bounded sequence of real numbers always admit a greatest lower bound [RRA09]. By doing the same analysis, we have:

$$\phi\left(x^*\right) \geq \lim_{n\to\infty} \sup \phi\left(x_n\right), \tag{4.3.5}$$

where limit superior exists because an upper bounded sequence of real numbers always admit a least upper bound [RRA09]. Using inequalities (4.3.4) and (4.3.5), one obtains:

$$\phi\left(x^*\right) = \lim_{n\to\infty} \phi\left(x_n\right), \tag{4.3.6}$$

implying that ϕ is a continuous function with respect to the Euclidean metric. Since $\phi(\cdot) = \mathbf{d}(\cdot, \mathcal{A})$ is a continuous, positive semidefinite function, by choosing $\omega_1(\cdot) = \omega_2(\cdot) = \mathbf{d}(\cdot, \mathcal{A})$ in Theorem 1 in [TNK98], the proof completes.

Before showing the main results, we need the following technical lemma, inspired by Lemma 2.3 in [Ang02].

Lemma 4.3.5. *Consider a control system* $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$*. If* Σ *is* $\delta_\exists$*-GAS, then the control system* $\widehat{\Sigma} = (\mathbb{R}^{2n}, \mathsf{U}, \mathcal{U}, \widehat{f})$*, where* $\widehat{f}(\zeta, \upsilon) = \left[f(\xi_1, \upsilon)^T, f(\xi_2, \upsilon)^T\right]^T$*, and* $\zeta = \left[\xi_1^T, \xi_2^T\right]^T$*, is* $U_\exists$*GAS with respect to the diagonal set* Δ*, defined by:*

$$\Delta = \left\{z \in \mathbb{R}^{2n} | \exists x \in \mathbb{R}^n : z = \left[x^T, x^T\right]^T\right\}. \tag{4.3.7}$$

Proof: Since Σ is $\delta_\exists$-GAS, there exists a metric $\mathbf{d} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ such that property (2.3.1) is satisfied. Now we define a new metric $\widehat{\mathbf{d}} : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}_0^+$ by:

$$\widehat{\mathbf{d}}(z, z') = \mathbf{d}(x_1, x_1') + \mathbf{d}(x_2, x_2'), \tag{4.3.8}$$

for any $z = \left[{x_1}^T, {x_2}^T\right]^T$, $z' = \left[{x_1'}^T, {x_2'}^T\right]^T \in \mathbb{R}^{2n}$. It can be readily checked that $\widehat{\mathbf{d}}$ satisfies all three conditions of a metric. Now we show that $\widehat{\mathbf{d}}(z, \Delta)$, for any $z = \left[x_1^T, x_2^T\right]^T \in \mathbb{R}^{2n}$, is proportional to $\mathbf{d}(x_1, x_2)$ which will be exploited later in the proof. We have:

$$\begin{aligned}
\widehat{\mathbf{d}}(z, \Delta) = \inf_{z' \in \Delta} \widehat{\mathbf{d}}(z, z') = \inf_{x' \in \mathbb{R}^n} \widehat{\mathbf{d}}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x' \\ x' \end{bmatrix}\right) \\
= \inf_{x' \in \mathbb{R}^n} \{\mathbf{d}(x_1, x') + \mathbf{d}(x_2, x')\} \leq \inf_{x' = x_1} \{\mathbf{d}(x_1, x') + \mathbf{d}(x_2, x')\} \\
= \mathbf{d}(x_1, x_1) + \mathbf{d}(x_1, x_2) = \mathbf{d}(x_1, x_2).
\end{aligned} \tag{4.3.9}$$

Since $\mathbf{d}$ is a metric, by using the triangle inequality, we have: $\mathbf{d}(x_1, x_2) \leq \mathbf{d}(x_1, x') + \mathbf{d}(x_2, x')$ for any $x' \in \mathbb{R}^n$, implying that:

$$\mathbf{d}(x_1, x_2) \leq \inf_{x' \in \mathbb{R}^n} \{\mathbf{d}(x_1, x') + \mathbf{d}(x_2, x')\} = \widehat{\mathbf{d}}(z, \Delta). \tag{4.3.10}$$

Hence, using (4.3.9) and (4.3.10), one obtains:

$$\mathbf{d}(x_1, x_2) \leq \widehat{\mathbf{d}}(z, \Delta) \leq \mathbf{d}(x_1, x_2) \Rightarrow \mathbf{d}(x_1, x_2) = \widehat{\mathbf{d}}(z, \Delta). \tag{4.3.11}$$

Using equality (4.3.11) and property (2.3.1), we have:

$$\widehat{\mathbf{d}}\left(\zeta_{zv}(t),\Delta\right)=\widehat{\mathbf{d}}\left(\begin{bmatrix}\xi_{x_1 v}(t)\\ \xi_{x_2 v}(t)\end{bmatrix},\Delta\right)=\mathbf{d}\left(\xi_{x_1 v}(t),\xi_{x_2 v}(t)\right) \qquad (4.3.12)$$
$$\leq \beta\left(\mathbf{d}\left(x_1,x_2\right),t\right)=\beta\left(\widehat{\mathbf{d}}\left(z,\Delta\right),t\right),$$

for any $t\in\mathbb{R}_0^+$, and any $\upsilon\in\mathcal{U}$, where $\zeta_{zv}=\left[\xi_{x_1 v}^T,\xi_{x_2 v}^T\right]^T$, and $z=\left[x_1^T,x_2^T\right]^T$. Hence, $\widehat{\Sigma}$ is U$_\exists$GAS with respect to Δ.

We can now provide characterization of $\delta_\exists$-GAS in terms of existence of a $\delta_\exists$-GAS Lyapunov function.

Theorem 4.3.6. *Consider a control system $\Sigma=(\mathbb{R}^n,\mathsf{U},\mathcal{U},f)$. If U is compact and $\mathbf{d}$ is a metric such that the function $\psi(\cdot)=\mathbf{d}(\cdot,y)$ is continuous*[6] *for any $y\in\mathbb{R}^n$ then the following statements are equivalent:*

(1) Σ is forward complete and there exists a $\delta_\exists$-GAS Lyapunov function, equipped with the metric $\mathbf{d}$.

(2) Σ is $\delta_\exists$-GAS, equipped with the metric $\mathbf{d}$.

Proof: First we show the proof from (1) to (2), even in the absence of the compactness assumption on U and the continuity assumption on $\mathbf{d}$. The proof is inspired by the work in [AS99]. By using property (i) in Definition 4.2.1, we obtain:

$$\mathbf{d}\left(\xi_{xv}(t),\xi_{x'v}(t)\right)\leq\underline{\alpha}^{-1}\left(V\left(\xi_{xv}(t),\xi_{x'v}(t)\right)\right), \qquad (4.3.13)$$

for any $t\in\mathbb{R}_0^+$, any $x,x'\in\mathbb{R}^n$, and any $\upsilon\in\mathcal{U}$. By using property (ii) and the comparison lemma [Kha96], one gets:

$$V(\xi_{xv}(t),\xi_{x'v}(t))\leq\mathsf{e}^{-\kappa t}V(\xi_{xv}(0),\xi_{x'v}(0)), \qquad (4.3.14)$$

[6] *Here, continuity is understood with respect to the Euclidean metric.*

for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$, and any $\upsilon \in \mathcal{U}$. By combining inequalities (4.3.13) and (4.3.14), one gets:

$$\mathbf{d}\left(\xi_{x\upsilon}(t), \xi_{x'\upsilon}(t)\right) \leq \underline{\alpha}^{-1}\left(\mathsf{e}^{-\kappa t} V(x, x')\right).$$

Moreover, using $V(x, x') \leq \overline{\alpha}(\mathbf{d}\,(x, x'))$, one obtains:

$$\mathbf{d}(\xi_{x\upsilon}(t), \xi_{x'\upsilon'}(t)) \leq \underline{\alpha}^{-1}\left(\mathsf{e}^{-\kappa t}\overline{\alpha}(\mathbf{d}(x, x'))\right).$$

Therefore, by defining function β as:

$$\beta(\mathbf{d}(x, x'), t) = \underline{\alpha}^{-1}\left(\mathsf{e}^{-\kappa t}\overline{\alpha}\left(\mathbf{d}(x, x')\right)\right) \tag{4.3.15}$$

the condition (2.3.1) is satisfied. Hence, the system Σ is $\delta_\exists$-GAS.

We now prove that (2) implies (1). Since Σ is $\delta_\exists$-GAS, using Lemma 4.3.5, we conclude that the control system $\widehat{\Sigma}$, defined in Lemma 4.3.5, is $\mathrm{U}_\exists$GAS with respect to the diagonal set Δ. Since $\psi(\cdot) = \mathbf{d}(\cdot, y)$ is continuous[7] for any $y \in \mathbb{R}^n$, it can be easily verified that $\widehat{\psi}(\cdot) = \widehat{\mathbf{d}}(\cdot, z')$ is also continuous[7] for any $z' \in \mathbb{R}^{2n}$, where the metric $\widehat{\mathbf{d}}$ was defined in Lemma 4.3.5. Using Theorem 4.3.4, we conclude that there exists a $\mathrm{U}_\exists$GAS Lyapunov function $V : \mathbb{R}^{2n} \to \mathbb{R}_0^+$, with respect to Δ, for $\widehat{\Sigma}$. Thanks to the special form of $\widehat{\Sigma}$, using the equality (4.3.11), and slightly abusing notation, the function V satisfies:

(i) $\underline{\alpha}\left(\widehat{\mathbf{d}}\left(\begin{bmatrix} x \\ x' \end{bmatrix}, \Delta\right)\right) \leq V\left(\begin{bmatrix} x \\ x' \end{bmatrix}\right) \leq \overline{\alpha}\left(\widehat{\mathbf{d}}\left(\begin{bmatrix} x \\ x' \end{bmatrix}, \Delta\right)\right)$
$\Rightarrow \underline{\alpha}(\mathbf{d}(x, x')) \leq V(x, x') \leq \overline{\alpha}(\mathbf{d}(x, x'))$;

(ii) $\left[\frac{\partial V}{\partial x}\ \frac{\partial V}{\partial x'}\right]\begin{bmatrix} f(x, u) \\ f(x', u) \end{bmatrix} \leq -\kappa V\left(\begin{bmatrix} x \\ x' \end{bmatrix}\right)$
$\Rightarrow \frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u) \leq -\kappa V(x, x')$,

for any $x, x' \in \mathbb{R}^n$, any $u \in \mathsf{U}$, some $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}$ and some $\kappa \in \mathbb{R}^+$. Hence, V is a $\delta_\exists$-GAS Lyapunov function for Σ. This completes the proof.

[7]Here, continuity is understood with respect to the Euclidean metric.

Before providing characterization of $\delta_\exists$-ISS in terms of existence of a $\delta_\exists$-ISS Lyapunov function, we need the following technical lemma, inspired by Proposition 5.3 in [Ang02]. By following similar steps as in [Ang02], we need to define the proximal point function $\mathsf{sat}_\mathsf{U} : \mathbb{R}^m \to \mathsf{U}$, defined by:

$$\mathsf{sat}_\mathsf{U}(u) = \arg\min_{u' \in \mathsf{U}} \|u' - u\| . \tag{4.3.16}$$

As explained in [Ang02], by assuming U is closed and convex and since $\|\cdot\| : \mathbb{R}^m \to \mathbb{R}_0^+$ is a proper and convex function, the definition (4.3.16) is well-defined and the minimizer of $\|u' - u\|$ with $u' \in \mathsf{U}$ is unique. Moreover, by convexity of U, we have:

$$\|\mathsf{sat}_\mathsf{U}(u_1) - \mathsf{sat}_\mathsf{U}(u_2)\| \leq \|u_1 - u_2\|, \quad \forall u_1, u_2 \in \mathbb{R}^m. \tag{4.3.17}$$

Lemma 4.3.7. *Consider a control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$, where U is closed and convex. If Σ is $\delta_\exists$-ISS, equipped with a metric $\mathbf{d}$ such that $\psi(\cdot) = \mathbf{d}(\cdot, y)$ is continuous*[8] *for any $y \in \mathbb{R}^n$, then there exists a $\mathcal{K}_\infty$ function ρ such that the control system $\widehat{\Sigma} = (\mathbb{R}^{2n}, \mathsf{D}, \mathcal{D}, \widehat{f})$*[9] *is $U_\exists$GAS with respect to the diagonal set Δ, where:*

$$\widehat{f}(\zeta, \omega) = \begin{bmatrix} f(\xi_1, \mathsf{sat}_\mathsf{U}(\omega_1 + \rho(\mathbf{d}(\xi_1, \xi_2))\omega_2)) \\ f(\xi_2, \mathsf{sat}_\mathsf{U}(\omega_1 - \rho(\mathbf{d}(\xi_1, \xi_2))\omega_2)) \end{bmatrix}, \tag{4.3.18}$$

$\zeta = \left[\xi_1^T, \xi_2^T\right]^T$, $\mathsf{D} = \mathsf{U} \times \mathcal{B}_1(0_m)$, *and* $\omega = \left[\omega_1^T, \omega_2^T\right]^T$.

Proof: The proof was inspired by the proof of Proposition 5.3 in [Ang02]. We include the complete details to ensure that the interested reader can assess the essential differences caused by using the arbitrary metric $\mathbf{d}$ rather than the Euclidean metric. Since Σ is $\delta_\exists$-ISS, equipped with the metric $\mathbf{d}$, there exist some $\mathcal{KL}$ function β and $\mathcal{K}_\infty$ function γ such that:

$$\mathbf{d}(\xi_{xv}(t), \xi_{x'v'}(t)) \leq \max\{\beta(\mathbf{d}(x, x'), t), \gamma(\|v - v'\|_\infty)\}. \tag{4.3.19}$$

[8] *Here, continuity is understood with respect to the Euclidean metric.*

[9] *The set $\mathcal{D}$ contains all measurable and locally essentially bounded functions of time from intervals of the form $]a, b[\subseteq \mathbb{R}$ to D with $a < 0$ and $b > 0$.*

Note that inequality (4.3.19) is a straightforward consequence of inequality (2.3.3) in Definition 2.3.2 (see Remark 2.5 in [Son89]). Using the results in Lemma 4.3.5 and the proposed metric $\widehat{\mathbf{d}}$ in (4.3.8), we have that $\mathbf{d}(x, x') = \widehat{\mathbf{d}}(z, \Delta)$, where $z = \left[x^T, x'^T\right]^T$. Without loss of generality we can assume $\alpha(r) = \beta(r, 0) > r$ for any $r \in \mathbb{R}^+$. Let ρ be a $\mathcal{K}_\infty$ function satisfying $\rho(r) \leq \frac{1}{2}\gamma^{-1} \circ (\alpha^{-1}(r)/4)$ (note that $\gamma, \alpha \in \mathcal{K}_\infty$). Now we show that

$$\gamma\left(\left\|2\omega_2(t)\rho\left(\widehat{\mathbf{d}}(\zeta_{z\omega}(t), \Delta)\right)\right\|\right) \leq \widehat{\mathbf{d}}(z, \Delta)/2, \tag{4.3.20}$$

for any $t \in \mathbb{R}_0^+$, any $z \in \mathbb{R}^{2n}$, and any $\omega = [\omega_1,\ \omega_2]^T \in \mathcal{D}$. Since γ is a $\mathcal{K}_\infty$ function and $\omega_2(t) \in \mathcal{B}_1(0_m)$, it is enough to show

$$\gamma\left(2\rho\left(\widehat{\mathbf{d}}(\zeta_{z\omega}(t), \Delta)\right)\right) \leq \widehat{\mathbf{d}}(z, \Delta)/2. \tag{4.3.21}$$

Since

$$\gamma\left(2\rho\left(\widehat{\mathbf{d}}(\zeta_{z\omega}(0), \Delta)\right)\right) = \gamma\left(2\rho\left(\widehat{\mathbf{d}}(z, \Delta)\right)\right) \leq \alpha^{-1}\left(\widehat{\mathbf{d}}(z, \Delta)\right)/4 < \widehat{\mathbf{d}}(z, \Delta)/4, \tag{4.3.22}$$

and $\varphi(\cdot) = \widehat{\mathbf{d}}(\cdot, \Delta)$ is a continuous[10] function (see proof of Theorem 4.3.4), then for all $t \in \mathbb{R}_0^+$ small enough, we have $\gamma\left(2\rho\left(\widehat{\mathbf{d}}(\zeta_{z\omega}(t), \Delta)\right)\right) \leq \widehat{\mathbf{d}}(z, \Delta)/4$. Now, let

$$t_1 = \inf\left\{t > 0 \mid \gamma\left(2\rho\left(\widehat{\mathbf{d}}(\zeta_{z\omega}(t), \Delta)\right)\right) > \widehat{\mathbf{d}}(z, \Delta)/2\right\}. \tag{4.3.23}$$

Clearly $t_1 > 0$. We will show that $t_1 = \infty$. Now, assume by contradiction that $t_1 < \infty$. Therefore, the inequality (4.3.21) holds for all $t \in [0, t_1)$. Hence, for all $t \in [0, t_1)$, one obtains:

$$\begin{aligned}\gamma\left(\left\|2\omega_2(t)\rho\left(\widehat{\mathbf{d}}(\zeta_{z\omega}(t), \Delta)\right)\right\|\right) &\leq \gamma\left(2\rho\left(\widehat{\mathbf{d}}(\zeta_{z\omega}(t), \Delta)\right)\right) \\ &\leq \widehat{\mathbf{d}}(z, \Delta)/2 < \alpha\left(\widehat{\mathbf{d}}(z, \Delta)\right)/2.\end{aligned} \tag{4.3.24}$$

[10]Here, continuity is understood with respect to the Euclidean metric.

Let υ and υ' be defined as:

$$\upsilon(t) = \mathsf{sat}_{\mathsf{U}}\left(\omega_1(t) + \rho\left(\widehat{\mathbf{d}}(\zeta_{z\omega}(t), \Delta)\right)\omega_2(t)\right),$$
$$\upsilon'(t) = \mathsf{sat}_{\mathsf{U}}\left(\omega_1(t) - \rho\left(\widehat{\mathbf{d}}(\zeta_{z\omega}(t), \Delta)\right)\omega_2(t)\right).$$

By using (4.3.17), we obtain:

$$\|\upsilon(t) - \upsilon'(t)\| \leq \left\|2\omega_2(t)\rho\left(\widehat{\mathbf{d}}(\zeta_{z\omega}(t), \Delta)\right)\right\|.$$

Using (4.3.19) and (4.3.24), we have:

$$\begin{aligned}\widehat{\mathbf{d}}(\zeta_{z\omega}(t), \Delta) = \mathbf{d}\left(\xi_{x\upsilon}(t), \xi_{x'\upsilon'}(t)\right) \leq \beta\left(\mathbf{d}(x, x'), 0\right) &= \beta\left(\widehat{\mathbf{d}}(z, \Delta), 0\right) \\ &= \alpha\left(\widehat{\mathbf{d}}(z, \Delta)\right),\end{aligned} \tag{4.3.25}$$

for any $t \in [0, t_1)$, any $\omega \in \mathcal{D}$, and any $z = \left[x^T, x'^T\right]^T \in \mathbb{R}^{2n}$. Using $\rho(r) \leq \frac{1}{2}\gamma^{-1} \circ (\alpha^{-1}(r)/4)$, the inequality (4.3.25) implies that

$$\gamma\left(2\rho\left(\widehat{\mathbf{d}}(\zeta_{z\omega}(t), \Delta)\right)\right) \leq \widehat{\mathbf{d}}(z, \Delta)/4,$$

for any $t \in [0, t_1)$, contradicting the definition of t_1. Therefore, $t_1 = \infty$ and inequality (4.3.20) is proved for all $t \in \mathbb{R}_0^+$. Therefore, using (4.3.19) and (4.3.20), we obtain:

$$\begin{aligned}\widehat{\mathbf{d}}(\zeta_{z\omega}(t), \Delta) &= \mathbf{d}\left(\xi_{x\upsilon}(t), \xi_{x'\upsilon'}(t)\right) \\ &\leq \max\left\{\beta\left(\mathbf{d}(x, x'), t\right), \gamma\left(\|\upsilon - \upsilon'\|_\infty\right)\right\} \\ &\leq \max\left\{\beta\left(\mathbf{d}(x, x'), t\right), \gamma\left(\left\|2\omega_2\rho\left(\widehat{\mathbf{d}}(\zeta_{z\omega}, \Delta)\right)\right\|_\infty\right)\right\} \\ &\leq \max\left\{\beta\left(\widehat{\mathbf{d}}(z, \Delta), t\right), \widehat{\mathbf{d}}(z, \Delta)/2\right\},\end{aligned} \tag{4.3.26}$$

for any $z = \left[x^T, x'^T\right]^T \in \mathbb{R}^{2n}$, any $\omega \in \mathcal{D}$, and any $t \in \mathbb{R}_0^+$. Since β is a $\mathcal{KL}$ function, it can be readily seen that for each $r > 0$ if $\widehat{\mathbf{d}}(z, \Delta) \leq r$, then there exists some $T_r \geq 0$ such that for any $t \geq T_r$, $\beta\left(\widehat{\mathbf{d}}(z, \Delta), t\right) \leq r/2$ and, hence, $\widehat{\mathbf{d}}(\zeta_{z\omega}(t), \Delta) \leq r/2$. Now we show that the set Δ is a global attractor for the control system $\widehat{\Sigma}$. For any $\varepsilon \in \mathbb{R}^+$, let k be a positive integer such that $2^{-k}r < \varepsilon$.

Let $r_1 = r$ and $r_i = r_{i-1}/2$ for $i \geq 2$, and let $\tau = T_{r_1} + T_{r_2} + \cdots + T_{r_k}$. Then, for $t \geq \tau$, we have $\widehat{\mathbf{d}}(\zeta_{z\omega}(t), \Delta) \leq 2^{-k} r < \varepsilon$ for all $\widehat{\mathbf{d}}(z, \Delta) \leq r$, all $\omega \in \mathcal{D}$, and all $t \geq \tau$. Therefore, it can be concluded that the set Δ is a uniform global attractor for the control system $\widehat{\Sigma}$. Furthermore, since $\widehat{\mathbf{d}}(\zeta_{z\omega}(t), \Delta) \leq \beta\left(\widehat{\mathbf{d}}(z, \Delta), 0\right)$ for all $t \in \mathbb{R}_0^+$, all $z \in \mathbb{R}^{2n}$, and all $\omega \in \mathcal{D}$, the control system $\widehat{\Sigma}$ is uniformly globally stable and as showed in [TNK98], it is $\mathrm{U}_\exists$GAS.

Finally, the next theorem provide a characterization of $\delta_\exists$-ISS in terms of the existence of a $\delta_\exists$-ISS Lyapunov function.

Theorem 4.3.8. *Consider a control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$. If U is compact and convex and $\mathbf{d}$ is a metric such that the function $\psi(\cdot) = \mathbf{d}(\cdot, y)$ is continuous*[11] *for any $y \in \mathbb{R}^n$ then the following statements are equivalent:*

(1) Σ is forward complete and there exists a $\delta_\exists$-ISS Lyapunov function, equipped with metric $\mathbf{d}$.

(2) Σ is $\delta_\exists$-ISS, equipped with metric $\mathbf{d}$.

Proof: First we show the proof from (1) to (2), even in the absence of the compactness and convexity assumptions on U and the continuity assumption on $\mathbf{d}$. The proof is inspired by the work in [AS99]. By using property (i) in Definition 4.2.1, we obtain:

$$\mathbf{d}\left(\xi_{x\upsilon}(t), \xi_{x'\upsilon'}(t)\right) \leq \underline{\alpha}^{-1}\left(V\left(\xi_{x\upsilon}(t), \xi_{x'\upsilon'}(t)\right)\right), \tag{4.3.27}$$

for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$, and any $\upsilon, \upsilon' \in \mathbb{R}^n$. By using property (iii) and the comparison lemma [Kha96], one gets:

$$V(\xi_{x\upsilon}(t), \xi_{x'\upsilon'}(t)) \leq \mathsf{e}^{-\kappa t} V(\xi_{x\upsilon}(0), \xi_{x'\upsilon'}(0)) + \mathsf{e}^{-\kappa t} * \sigma(\|\upsilon(t) - \upsilon'(t)\|), \tag{4.3.28}$$

[11] *Here, continuity is understood with respect to the Euclidean metric.*

for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$, and any $\upsilon, \upsilon' \in \mathbb{R}^n$, where $*$ denotes the convolution integral[12]. By combining inequalities (4.3.27) and (4.3.28), one gets:

$$\begin{aligned}\mathbf{d}\left(\xi_{x\upsilon}(t), \xi_{x'\upsilon'}(t)\right) &\leq \underline{\alpha}^{-1}\left(\mathsf{e}^{-\kappa t}V(x,x') + \mathsf{e}^{-\kappa t} * \sigma(\|\upsilon(t) - \upsilon'(t)\|)\right) \\ &\leq \underline{\alpha}^{-1}\left(\mathsf{e}^{-\kappa t}V(x,x') + \frac{1-\mathsf{e}^{-\kappa t}}{\kappa}\sigma(\|\upsilon - \upsilon'\|_\infty)\right) \\ &\leq \underline{\alpha}^{-1}\left(\mathsf{e}^{-\kappa t}V(x,x') + \frac{1}{\kappa}\sigma(\|\upsilon - \upsilon'\|_\infty)\right) = \gamma(\rho, \phi),\end{aligned}$$

where $\gamma(\rho,\phi) = \underline{\alpha}^{-1}(\rho + \phi)$, $\rho = \mathsf{e}^{-\kappa t}V(x,x')$, and $\phi = \frac{1}{\kappa}\sigma\left(\|\upsilon - \upsilon'\|_\infty\right)$. Since γ is nondecreasing in each variable, we have:

$$\mathbf{d}\left(\xi_{x\upsilon}(t), \xi_{x'\upsilon'}(t)\right) \leq h\left(\mathsf{e}^{-\kappa t}V(x,x')\right) + h\left(\frac{1}{\kappa}\sigma\left(\|\upsilon - \upsilon'\|_\infty\right)\right),$$

where $h(r) = \gamma(r,r) = \underline{\alpha}^{-1}(2r)$ and $h : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a $\mathcal{K}_\infty$ function. Moreover, using $V(x,x') \leq \overline{\alpha}(\mathbf{d}\left(x,x'\right))$, one obtains:

$$\mathbf{d}(\xi_{x\upsilon}(t), \xi_{x'\upsilon'}(t)) \leq \underline{\alpha}^{-1}\left(2\mathsf{e}^{-\kappa t}\overline{\alpha}(\mathbf{d}(x,x'))\right) + \underline{\alpha}^{-1}\left(\frac{2}{\kappa}\sigma(\|\upsilon - \upsilon'\|_\infty)\right).$$

Therefore, by defining functions β and γ as:

$$\begin{aligned}\beta(\mathbf{d}(x,x'), t) =& \underline{\alpha}^{-1}\left(2\mathsf{e}^{-\kappa t}\overline{\alpha}\left(\mathbf{d}(x,x')\right)\right) \\ \gamma\left(\|\upsilon - \upsilon'\|_\infty\right) =& \underline{\alpha}^{-1}\left(\frac{2}{\kappa}\sigma(\|\upsilon - \upsilon'\|_\infty)\right),\end{aligned}$$

the condition (2.3.3) is satisfied. Hence, the system Σ is $\delta_\exists$-ISS.

We now prove that (2) implies (1). As we proved in Lemma 4.3.7, since Σ is $\delta_\exists$-ISS, it implies that the control system $\widehat{\Sigma}$, defined in Lemma 4.3.7, is U$_\exists$GAS with respect to Δ. Since $\psi(\cdot) = \mathbf{d}(\cdot, y)$ is continuous[13] for any $y \in \mathbb{R}^n$, it can be easily verified that $\widehat{\psi}(\cdot) = \widehat{\mathbf{d}}(\cdot, z')$ is continuous[13] for any $z' \in \mathbb{R}^{2n}$, where the metric $\widehat{\mathbf{d}}$ was defined in the proof of Lemma 4.3.5. Using Theorem 4.3.4, we conclude that there exists a U$_\exists$GAS Lyapunov function V, with respect to Δ, for $\widehat{\Sigma}$. By using the special form of $\widehat{\Sigma}$, defined in Lemma 4.3.7, the equality (4.3.11), and slightly abusing notation the function V satisfies:

[12] $\mathsf{e}^{-\kappa t} * \sigma(\|\upsilon(t) - \upsilon'(t)\|) = \int_0^t \mathsf{e}^{-\kappa(t-\tau)}\sigma(\|\upsilon(\tau) - \upsilon'(\tau)\|)d\tau$.

[13] Here, continuity is understood with respect to the Euclidean metric.

(i) $\underline{\alpha}\left(\widehat{\mathbf{d}}\left(\begin{bmatrix} x \\ x' \end{bmatrix}, \Delta\right)\right) \leq V\left(\begin{bmatrix} x \\ x' \end{bmatrix}\right) \leq \overline{\alpha}\left(\widehat{\mathbf{d}}\left(\begin{bmatrix} x \\ x' \end{bmatrix}, \Delta\right)\right)$
$\Rightarrow \underline{\alpha}(\mathbf{d}(x,x')) \leq V(x,x') \leq \overline{\alpha}(\mathbf{d}(x,x'));$

for any $x, x' \in \mathbb{R}^n$, some $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}$ and

(ii)

$$\begin{bmatrix} \frac{\partial V}{\partial x} & \frac{\partial V}{\partial x'} \end{bmatrix} \begin{bmatrix} f(x, \mathsf{sat}_\mathsf{U}(d_1 + \rho(\mathbf{d}(x,x'))d_2)) \\ f(x', \mathsf{sat}_\mathsf{U}(d_1 - \rho(\mathbf{d}(x,x'))d_2)) \end{bmatrix} \leq -\kappa V\left(\begin{bmatrix} x \\ x' \end{bmatrix}\right) \tag{4.3.29}$$
$$\Rightarrow \frac{\partial V}{\partial x} f(x, \mathsf{sat}_\mathsf{U}(d_1 + \rho(\mathbf{d}(x,x'))d_2)) +$$
$$\frac{\partial V}{\partial x'} f(x', \mathsf{sat}_\mathsf{U}(d_1 - \rho(\mathbf{d}(x,x'))d_2)) \leq -\kappa V(x,x'),$$

for some $\kappa \in \mathbb{R}^+$, any $x, x' \in \mathbb{R}^n$, and any $\left[d_1^T, d_2^T\right]^T \in \mathsf{D}$. By choosing $d_1 = (u+u')/2$ and $d_2 = (u-u')/(2\rho(\mathbf{d}(x,x')))$ for any $u, u' \in \mathsf{U}$, it can be readily checked that $\left[d_1^T, d_2^T\right]^T \in \mathsf{U} \times \mathcal{B}_1(0_m)$, whenever $2\rho(\mathbf{d}(x,x')) \geq \|u-u'\|$. Hence, using (4.3.29), we have that the following implication holds:

$$\text{if } \varphi(\mathbf{d}(x,x')) \geq \|u-u'\|, \text{ then } \frac{\partial V}{\partial x} f(x,u) + \frac{\partial V}{\partial x'} f(x',u') \leq -\kappa V(x,x'), \tag{4.3.30}$$

where $\varphi(r) = 2\rho(r)$. As showen in Remark 2.4 in [Son89], there is no loss of generality in modifying inequalities (4.3.30) to

$$\frac{\partial V}{\partial x} f(x,u) + \frac{\partial V}{\partial x'} f(x',u') \leq -\widehat{\kappa} V(x,x') + \sigma(\|u-u'\|), \tag{4.3.31}$$

for some $\mathcal{K}_\infty$ function σ and some $\widehat{\kappa} \in \mathbb{R}^+$, which completes the proof.

The following theorem describes δ-FC in terms of existence of a δ-FC Lyapunov function.

Theorem 4.3.9. *A control system Σ is δ-FC if it admits a δ-FC Lyapunov function.*

Proof: The proof is very similar to the first part of the proof of Theorem 4.3.8. By using property (i) in Definition 4.2.4, we obtain:

$$\|\xi_{xv}(t) - \xi_{x'v'}(t)\| \leq \underline{\alpha}^{-1}\left(V\left(\xi_{xv}(t), \xi_{x'v'}(t)\right)\right), \tag{4.3.32}$$

for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$, and any $v, v' \in \mathsf{U}$. By using the property (ii) and the comparison lemma [Kha96], one gets:

$$V(\xi_{xv}(t), \xi_{x'v'}(t)) \leq \mathsf{e}^{\kappa t} V(\xi_{xv}(0), \xi_{x'v'}(0)) + \mathsf{e}^{\kappa t} * \sigma(\|v(t) - v'(t)\|) \tag{4.3.33}$$

for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$, and any $v, v' \in \mathsf{U}$, where $*$ denotes the convolution integral. By combining inequalities (4.3.32) and (4.3.33), one gets:

$$\begin{aligned}\|\xi_{xv}(t) - \xi_{x'v'}(t)\| \leq& \underline{\alpha}^{-1}\left(\mathsf{e}^{\kappa t} V(x, x') + \mathsf{e}^{\kappa t} * \sigma(\|v - v'\|)\right) \\ \leq& \underline{\alpha}^{-1}\left(\mathsf{e}^{\kappa t} V(x, x') + \frac{\mathsf{e}^{\kappa t} - 1}{\kappa}\sigma(\|v - v'\|_\infty)\right) = \gamma(\rho, \phi)\end{aligned}$$

where $\gamma(\rho, \phi) = \underline{\alpha}^{-1}(\rho + \phi)$, $\rho = \mathsf{e}^{\kappa t} V(x, x')$, and $\phi = \frac{\mathsf{e}^{\kappa t} - 1}{\kappa}\sigma\left(\|v - v'\|_\infty\right)$. Since γ is nondecreasing in each variable, we have:

$$\|\xi_{xv}(t) - \xi_{x'v'}(t)\| \leq h\left(\mathsf{e}^{\kappa t} V(x, x')\right) + h\left(\frac{\mathsf{e}^{\kappa t} - 1}{\kappa}\sigma\left(\|v - v'\|_\infty\right)\right)$$

where $h(r) = \gamma(r, r) = \underline{\alpha}^{-1}(2r)$ and $h : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a $\mathcal{K}_\infty$ function. Moreover, using $V(x, x') \leq \overline{\alpha}(\|x - x'\|)$, one obtains:

$$\|\xi_{xv}(t) - \xi_{x'v'}(t)\| \leq \underline{\alpha}^{-1}\left(2\mathsf{e}^{\kappa t}\overline{\alpha}(\|x - x'\|)\right) + \underline{\alpha}^{-1}\left(2\frac{\mathsf{e}^{\kappa t} - 1}{\kappa}\sigma(\|v - v'\|_\infty)\right).$$

Therefore, by defining functions β and γ as:

$$\begin{aligned}\beta(\|x - x'\|, t) =& \underline{\alpha}^{-1}\left(2\mathsf{e}^{\kappa t}\overline{\alpha}\left(\|x - x'\|\right)\right) \\ \gamma(\|v - v'\|_\infty, t) =& \underline{\alpha}^{-1}\left(2\frac{\mathsf{e}^{\kappa t} - 1}{\kappa}\sigma(\|v - v'\|_\infty)\right),\end{aligned}$$

the condition (2.4.1) is satisfied. Hence, the system Σ is δ-FC.

4.3.2 Contraction (expansion) description of incremental stability

Note that for all definitions and results in this subsection we require function f to be continuously differentiable which was not the case in characterizations of incremental stability using incremental Lyapunov functions.

The following theorem shows that the inequality (4.2.4) implies $\delta_\exists$-GAS.

Theorem 4.3.10. *Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ be a smooth control system on $\mathbb{R}^n$ equipped with a Riemannian metric G. If G is a contraction metric, with respect to states, for Σ and if $(\mathbb{R}^n, \mathbf{d}_G)$ is a complete metric space*[14]*, then Σ is $\delta_\exists$-GAS.*

Different variations of this result appeared in [LS98] and [AR03]; see [AR03] for a concise proof and the proof of Theorem 4.3.11 for a more detailed treatment including the completeness assumption. It is also shown in the proof of Theorem 4.3.11 that the additional assumption $\underline{\omega}\langle X, X\rangle_{I_n} \leq \langle X, X\rangle_G \leq \overline{\omega}\langle X, X\rangle_{I_n}$ for $\underline{\omega}, \overline{\omega} \in \mathbb{R}^+$ leads to the stronger conclusion that Σ is in fact δ-GAS.

The following theorem shows that the inequality (4.2.7) implies $\delta_\exists$-ISS.

Theorem 4.3.11. *Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ be a smooth control system on $\mathbb{R}^n$ equipped with a Riemannian metric G and U be a convex set. If the metric G is a contraction (resp. expansion) metric, with respect to states and inputs, for system Σ and $(\mathbb{R}^n, \mathbf{d}_G)$ is a complete metric space, then Σ is $\delta_\exists$-ISS (resp. δ-FC).*

Similarly to contraction metrics with respect to states, it will be shown that Σ is δ-ISS when the inequalities $\underline{\omega}\langle X, X\rangle_{I_n} \leq \langle X, X\rangle_G \leq \overline{\omega}\langle X, X\rangle_{I_n}$ are satisfied for $\underline{\omega}, \overline{\omega} \in \mathbb{R}^+$.

Proof: Since $(\mathbb{R}^n, \mathbf{d}_G)$ is a complete metric space, using the Hopf-Rinow theorem [Pet97], we conclude that $\mathbb{R}^n$ with respect to the metric G is geodesically complete. The rest of the proof is inspired by the proof of Theorem 2 in [AR03].

[14] *A metric space $(\mathbb{R}^n, \mathbf{d})$ is said to be complete if every Cauchy sequence of points in $\mathbb{R}^n$ has a limit that is also in $\mathbb{R}^n$.*

Consider two points x and x' in $\mathbb{R}^n$ and a geodesic $\chi : [0,1] \to \mathbb{R}^n$ joining $x = \chi(0)$ and $x' = \chi(1)$. The geodesic distance between the points x and x' is given by:

$$\mathbf{d}_G(x, x') = \int_0^1 \sqrt{\left(\frac{d\chi(s)}{ds}\right)^T G(\chi(s))\frac{d\chi(s)}{ds}} ds. \tag{4.3.34}$$

Consider the straight line $\widehat{\chi}_t(s) = (1-s)\upsilon(t) + s\upsilon'(t)$, for fixed $t \in \mathbb{R}_0^+$, fixed $\upsilon, \upsilon' \in \mathcal{U}$, and for any $s \in [0,1]$. The curve $\widehat{\chi}_t$ is a geodesic, with respect to the Euclidean metric, on the convex subset $\mathsf{U} \subseteq \mathbb{R}^m$ joining $\upsilon(t) = \widehat{\chi}_t(0)$ and $\upsilon'(t) = \widehat{\chi}_t(1)$. Consider also the input curve υ_s defined by $\upsilon_s(t) = \widehat{\chi}_t(s)$. Let $l(t)$ be the length of the curve $\xi_{\chi(s)\upsilon_s}(t)$ parametrized by s and with respect to the metric G, i.e.:

$$l(t) = \int_0^1 \sqrt{\delta\xi^T G(\xi_{\chi(s)\upsilon_s}(t))\delta\xi} ds, \quad \text{with} \quad \delta\xi = \frac{\partial}{\partial s}\xi_{\chi(s)\upsilon_s}(t). \tag{4.3.35}$$

In the rest of the proof, we drop the argument of the metric G for the sake of simplicity. By taking the derivative of (4.3.35) with respect to time, we obtain:

$$\begin{aligned}\frac{d}{dt}l(t) &= \int_0^1 \frac{\frac{d}{dt}\left(\delta\xi^T G\delta\xi\right)}{2\sqrt{\delta\xi^T G\delta\xi}} ds \\ &= \int_0^1 \frac{\delta\xi^T \left(\left(\frac{\partial f}{\partial x}\right)^T G + \frac{\partial G}{\partial x} f + G\frac{\partial f}{\partial x}\right)\delta\xi + 2\delta\upsilon^T \left(\frac{\partial f}{\partial u}\right)^T G\delta\xi}{2\sqrt{\delta\xi^T G\delta\xi}} ds, \quad \text{with} \quad \delta\upsilon = \frac{\partial}{\partial s}\upsilon_s(t).\end{aligned}$$

Since G is a contraction (expansion) metric, with respect to states and inputs, with $\widehat{\lambda}$ and α the constants introduced in Definition 4.2.6, the following inequality holds:

$$\frac{d}{dt}l(t) \leq -\frac{\widehat{\lambda}}{2}l(t) + \frac{\alpha}{2}\int_0^1 \sqrt{\delta\upsilon^T\delta\upsilon} ds = -\frac{\widehat{\lambda}}{2}l(t) + \frac{\alpha}{2}\|\upsilon(t) - \upsilon'(t)\|. \tag{4.3.36}$$

Using (4.3.36), and comparison principle [Kha96], we obtain:

$$\begin{aligned} l(t) \leq& \mathsf{e}^{-\frac{\widehat{\lambda}}{2}t} l(0) + \frac{\alpha}{2}\mathsf{e}^{-\frac{\widehat{\lambda}}{2}t} * \|\upsilon(t) - \upsilon'(t)\| \\ \leq& \mathsf{e}^{-\frac{\widehat{\lambda}}{2}t} l(0) + \frac{\alpha}{\widehat{\lambda}}\left(1 - \mathsf{e}^{-\frac{\widehat{\lambda}}{2}t}\right)\|\upsilon - \upsilon'\|_\infty, \end{aligned} \tag{4.3.37}$$

where $*$ denotes the convolution integral. From (4.3.34) and (4.3.35), it can be seen that $l(0) = \mathbf{d}_G(x, x')$. However, for $t \in \mathbb{R}^+$, $l(t)$ is not necessarily the Riemannian distance function, determined by G, because $\xi_{\chi(s)\upsilon_s}(t)$ is not necessarily a geodesic, implying that it is always bigger than or equal to the Riemannian distance function[15]: $\mathbf{d}_G(\xi_{x\upsilon}(t), \xi_{x'\upsilon'}(t)) \leq l(t)$, and, hence, the following inequality holds:

$$\mathbf{d}_G\left(\xi_{x\upsilon}(t), \xi_{x'\upsilon'}(t)\right) \leq \mathsf{e}^{-\frac{\widehat{\lambda}}{2}t}\mathbf{d}_G(x, x') + \frac{\alpha}{\widehat{\lambda}}\left(1 - \mathsf{e}^{-\frac{\widehat{\lambda}}{2}t}\right)\|\upsilon - \upsilon'\|_\infty, \tag{4.3.38}$$

which, in turn, implies that Σ is $\delta_\exists$-ISS when $\widehat{\lambda} \in \mathbb{R}^+$ and δ-FC when $\widehat{\lambda} \in \mathbb{R}$. The proof for the case that G is a contraction (expansion) metric, with respect to states, can be readily verified by just enforcing $\delta\upsilon(t) = 0$ and $\upsilon(t) = \upsilon'(t)$ for any $t \in \mathbb{R}_0^+$.

Using the inequalities $\underline{\omega}\langle X, X\rangle_{I_n} \leq \langle X, X\rangle_G \leq \overline{\omega}\langle X, X\rangle_{I_n}$, it can be readily checked that:

$$\begin{aligned}\sqrt{\underline{\omega}}\|\xi_{x\upsilon}(t) - \xi_{x'\upsilon'}(t)\| \leq& \mathbf{d}_G\left(\xi_{x\upsilon}(t), \xi_{x'\upsilon'}(t)\right),\\ \mathbf{d}_G(x, x') \leq& \sqrt{\overline{\omega}}\|x - x'\|,\end{aligned}$$

Hence, the condition (4.3.38) reduces to:

$$\|\xi_{x\upsilon}(t) - \xi_{x'\upsilon'}(t)\| \leq \sqrt{\frac{\overline{\omega}}{\underline{\omega}}}\mathsf{e}^{-\frac{\widehat{\lambda}}{2}t}\|x - x'\| + \frac{1}{\sqrt{\underline{\omega}}}\frac{\alpha}{\widehat{\lambda}}\left(1 - \mathsf{e}^{-\frac{\widehat{\lambda}}{2}t}\right)\|\upsilon - \upsilon'\|_\infty, \tag{4.3.39}$$

which is the δ-ISS condition in (2.3.4) for $\widehat{\lambda} \in \mathbb{R}^+$ and δ-FC condition in (2.4.1) for $\widehat{\lambda} \in \mathbb{R}$.

Since completeness of the metric space $(\mathbb{R}^n, \mathbf{d}_G)$ is crucial to the previous proof, the following lemma provides a sufficient condition on the metric G guaranteeing that $(\mathbb{R}^n, \mathbf{d}_G)$ is a complete metric space.

Lemma 4.3.12. *The Riemannian manifold $\mathbb{R}^n$ equipped with a Riemannian metric G, satisfying*[16] $\underline{\omega}\|y\|^2 \leq y^TG(x)y$ *for any $x, y \in \mathbb{R}^n$ and for some positive constant $\underline{\omega}$, is complete as a metric space, with respect to $\mathbf{d}_G$.*

[15]Note that given a Riemannian metric G, the Riemannian distance function is the smallest distance, determined by G.

[16]*This condition is nothing more than uniform positive definiteness of G.*

Proof: The proof was suggested to us by C. Manolescu. First, for each pair of points $x, y \in \mathbb{R}^n$, we define the path space:

$$\Omega(x,y) = \{\chi : [0,1] \to \mathbb{R}^n \mid \chi \text{ is piecewise smooth, } \chi(0) = x, \text{ and } \chi(1) = y\}.$$

Recall that a function $\chi : [a,b] \to \mathbb{R}^n$ is piecewise smooth if χ is continuous and if there exists a partitioning $a = a_1 < a_2 < \cdots < a_k = b$ of $[a,b]$ such that $\chi|_{(a_i,a_{i+1})}$ is smooth for $i = 1, \cdots, k-1$. We can then define the Riemannian distance function $\mathbf{d}_G(x,y)$ between points $x, y \in \mathbb{R}^n$ as

$$\mathbf{d}_G(x,y) = \inf_{\chi \in \Omega(x,y)} \int_0^1 \sqrt{\left(\frac{d\chi(s)}{ds}\right)^T G(\chi(s)) \frac{d\chi(s)}{ds}} ds.$$

It follows immediately that $\mathbf{d}_G$ is a metric on $\mathbb{R}^n$. The Riemannian manifold $\mathbb{R}^n$ is a complete metric space, equipped with the metric $\mathbf{d}_G$, if every Cauchy sequence[17] of points in $\mathbb{R}^n$ has a limit in $\mathbb{R}^n$. Assume $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}^n$, equipped with the metric $\mathbf{d}_G$. By using the assumption on G, we have

$$\begin{aligned}\mathbf{d}_G(x_n, x_m) &= \inf_{\chi \in \Omega(x_n,x_m)} \int_0^1 \sqrt{\left(\frac{d\chi(s)}{ds}\right)^T G(\chi(s)) \frac{d\chi(s)}{ds}} ds \qquad (4.3.40) \\ &\geq \sqrt{\underline{\omega}} \inf_{\chi \in \Omega(x_n,x_m)} \int_0^1 \sqrt{\left(\frac{d\chi(s)}{ds}\right)^T \frac{d\chi(s)}{ds}} ds = \sqrt{\underline{\omega}} \|x_n - x_m\|.\end{aligned}$$

It is readily seen from the inequality (4.3.40) that the sequence $\{x_n\}_{n=1}^\infty$ is also a Cauchy sequence in $\mathbb{R}^n$ with respect to the Euclidean metric. Since the Riemannian manifold $\mathbb{R}^n$ with respect to the Euclidean metric is a complete metric space, the sequence $\{x_n\}_{n=1}^\infty$ converges to a point, named x^*, in $\mathbb{R}^n$. By picking a convex compact subset $D \subset \mathbb{R}^n$, containing x^*, and using Lemma 8.18 in [Lee03], we have: $\overline{\omega}\|y\|^2 \geq y^T G(x) y$ for any $y \in \mathbb{R}^n$, $x \in D$, and some positive constant $\overline{\omega}$. Since the sequence $\{x_n\}_{n=1}^\infty$ converges to $x^* \in D$, there exists some integer N such that the sequence $\{x_n\}_{n=N}^\infty$ remains forever inside D. Hence, we have:

$$\sqrt{\underline{\omega}}\|x_n - x^*\| \leq \mathbf{d}_G(x_n, x^*) \leq \sqrt{\overline{\omega}}\|x_n - x^*\|, \qquad (4.3.41)$$

[17] A sequence $\{x_n\}_{n=1}^\infty$ in a metric space X, equipped with a metric $\mathbf{d}$, is a Cauchy sequence if $\lim_{n,m\to\infty} \mathbf{d}(x_n, x_m) = 0$.

for $n > N$. Therefore, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to $x^* \in \mathbb{R}^n$, equipped with the metric $\mathbf{d}_G$. Therefore, $\mathbb{R}^n$ with respect to the metric $\mathbf{d}_G$ is a complete metric space.

Resuming, in this section we have provided a characterization of $\delta_\exists$-GAS and $\delta_\exists$-ISS in terms of the existence of $\delta_\exists$-GAS and $\delta_\exists$-ISS Lyapunov functions and we have provided sufficient conditions for $\delta_\exists$-GAS and $\delta_\exists$-ISS in terms of the existence of a contraction metric. Based on these results, in the next section, we propose a backstepping controller design procedure, providing controllers rendering control systems incrementally stable. Additionally, we will provide incremental Lyapunov functions and contraction metrics for smooth control systems.

4.4 Backstepping Design Procedure

The backsteping method proposed here is inspired by the backstepping method, described in [PvdWN05]. Here, we will extend this approach to render the closed-loop system $\delta_\exists$-ISS and to construct $\delta_\exists$-ISS Lyapunov functions and contraction metrics, with respect to states and inputs. Consider the following subclass of control systems:

$$\Sigma : \begin{cases} \dot{\eta} = f(\eta, \zeta), \\ \dot{\zeta} = \upsilon, \end{cases} \tag{4.4.1}$$

where $x = \left[y^T, z^T\right]^T \in \mathbb{R}^{n_\eta + n_\zeta}$ is the state of Σ, y and z are initial conditions for η, ζ-subsystems, respectively, and υ is the control input.

In support of the main result of this section (Theorem 4.4.2), we need the following technical result.

Lemma 4.4.1. *Consider the following interconnected control system*

$$\Sigma : \begin{cases} \dot{\eta} = f(\eta, \zeta, \upsilon), \\ \dot{\zeta} = g(\zeta, \upsilon). \end{cases} \tag{4.4.2}$$

Let the η-subsystem be $\delta_{\exists}$-ISS with respect to ζ, υ and let the ζ-subsystem be $\delta_{\exists}$-ISS with respect to υ for some metrics $\mathbf{d}_\eta$ and $\mathbf{d}_\zeta$, respectively such that the solutions ${\eta_{y\zeta\upsilon}}$[18] *and $\zeta_{z\upsilon}$ satisfy the following inequalities:*

$$\mathbf{d}_\eta\left(\eta_{y\zeta\upsilon}(t), \eta_{y'\zeta'\upsilon'}(t)\right) \leq \beta_\eta\left(\mathbf{d}_\eta\left(y, y'\right), t\right) + \gamma_\zeta\left(\mathbf{d}_\zeta(\zeta, \zeta')_\infty\right) + \gamma_\upsilon\left(\|\upsilon - \upsilon'\|_\infty\right), \tag{4.4.3}$$

$$\mathbf{d}_\zeta\left(\zeta_{z\upsilon}(t), \zeta_{z'\upsilon'}(t)\right) \leq \beta_\zeta\left(\mathbf{d}_\zeta\left(z, z'\right), t\right) + \widetilde{\gamma}_\upsilon\left(\|\upsilon - \upsilon'\|_\infty\right), \tag{4.4.4}$$

where y, y' and z, z' are the initial conditions for the η, ζ-subsystems, respectively. Then, the interconnected control system Σ in (4.4.2) is $\delta_{\exists}$-ISS with respect to υ.

Proof: The proof was inspired by the proof of Proposition 4.7 in [Ang02]. The essential differences lie in the choice of the metric for the overall system Σ using the metrics for η, ζ-subsystems. We provide the proof so that it can be easily compared with the proof in [Ang02]. Using (4.4.3), (4.4.4) and triangular

[18] *Notation $\eta_{y\zeta\upsilon}$ denotes a trajectory of η-subsystem under the inputs ζ and υ from initial condition $y \in \mathbb{R}^{n_\eta}$.*

inequality, the following chain of inequalities hold:

$$
\begin{aligned}
&\mathbf{d}_\eta\left(\eta_{y\zeta\upsilon}(t), \eta_{y'\zeta'\upsilon'}(t)\right)\\
&\leq \beta_\eta\left(\mathbf{d}_\eta\left(\eta_{y\zeta\upsilon}(t/2), \eta_{y'\zeta'\upsilon'}(t/2)\right), t/2\right) + \gamma_\zeta\left(\mathbf{d}_\zeta\left(\zeta,\zeta'\right)_{[t/2,\infty)}\right) + \gamma_\upsilon\left(\|\upsilon-\upsilon'\|_{[t/2,\infty)}\right)\\
&\leq \beta_\eta\left(\beta_\eta\left(\mathbf{d}_\eta(y,y'), t/2\right) + \gamma_\zeta\left(\mathbf{d}_\zeta\left(\zeta,\zeta'\right)_\infty\right) + \gamma_\upsilon\left(\|\upsilon-\upsilon'\|_\infty\right), t/2\right)\\
&\;+ \gamma_\zeta\left(\mathbf{d}_\zeta\left(\zeta,\zeta'\right)_{[t/2,\infty)}\right) + \gamma_\upsilon\left(\|\upsilon-\upsilon'\|_{[t/2,\infty)}\right)\\
&\leq \beta_\eta\left(3\beta_\eta(\mathbf{d}_\eta(y,y'), t/2), t/2\right) + \beta_\eta\left(3\gamma_\zeta\left(\mathbf{d}_\zeta\left(\zeta,\zeta'\right)_\infty\right), t/2\right)\\
&\;+ \beta_\eta\left(3\gamma_\upsilon\left(\|\upsilon-\upsilon'\|_\infty\right), 0\right) + \gamma_\zeta\left(\mathbf{d}_\zeta\left(\zeta,\zeta'\right)_{[t/2,\infty)}\right) + \gamma_\upsilon\left(\|\upsilon-\upsilon'\|_{[t/2,\infty)}\right)\\
&\leq \beta_\eta\left(3\beta_\eta(\mathbf{d}_\eta(y,y'), t/2), t/2\right) + \beta_\eta\left(3\gamma_\zeta\left(\beta_\zeta\left(\mathbf{d}_\zeta\left(z,z'\right), 0\right) + \widetilde{\gamma}_\upsilon\left(\|\upsilon-\upsilon'\|_\infty\right)\right), t/2\right)\\
&\;+ \beta_\eta\left(3\gamma_\upsilon\left(\|\upsilon-\upsilon'\|_\infty\right), 0\right) + \gamma_\zeta\left(\mathbf{d}_\zeta\left(\zeta,\zeta'\right)_{[t/2,\infty)}\right) + \gamma_\upsilon\left(\|\upsilon-\upsilon'\|_{[t/2,\infty)}\right)\\
&\leq \beta_\eta\left(3\beta_\eta(\mathbf{d}_\eta(y,y'), t/2), t/2\right) + \beta_\eta\left(3\gamma_\zeta\left(2\beta_\zeta\left(\mathbf{d}_\zeta\left(z,z'\right), 0\right)\right), t/2\right)\\
&\;+ \beta_\eta\left(3\gamma_\zeta\left(2\widetilde{\gamma}_\upsilon\left(\|\upsilon-\upsilon'\|_\infty\right)\right), t/2\right) + \beta_\eta\left(3\gamma_\upsilon\left(\|\upsilon-\upsilon'\|_\infty\right), 0\right)\\
&\;+ \gamma_\zeta\left(\mathbf{d}_\zeta\left(\zeta,\zeta'\right)_{[t/2,\infty)}\right) + \gamma_\upsilon\left(\|\upsilon-\upsilon'\|_{[t/2,\infty)}\right)\\
&\leq \beta_\eta\left(3\beta_\eta(\mathbf{d}_\eta(y,y'), t/2), t/2\right) + \beta_\eta\left(3\gamma_\zeta\left(2\beta_\zeta\left(\mathbf{d}_\zeta\left(z,z'\right), 0\right)\right), t/2\right)\\
&\;+ \beta_\eta\left(3\gamma_\zeta\left(2\widetilde{\gamma}_\upsilon\left(\|\upsilon-\upsilon'\|_\infty\right)\right), 0\right) + \beta_\eta\left(3\gamma_\upsilon\left(\|\upsilon-\upsilon'\|_\infty\right), 0\right)\\
&\;+ \gamma_\zeta\left(\beta_\zeta\left(\mathbf{d}_\zeta\left(z,z'\right), t/2\right) + \widetilde{\gamma}_\upsilon\left(\|\upsilon-\upsilon'\|_\infty\right)\right) + \gamma_\upsilon\left(\|\upsilon-\upsilon'\|_{[t/2,\infty)}\right)\\
&\leq \beta_\eta\left(3\beta_\eta(\mathbf{d}_\eta(y,y'), t/2), t/2\right) + \beta_\eta\left(3\gamma_\zeta\left(2\beta_\zeta\left(\mathbf{d}_\zeta\left(z,z'\right), 0\right)\right), t/2\right)\\
&\;+ \beta_\eta\left(3\gamma_\zeta\left(2\widetilde{\gamma}_\upsilon\left(\|\upsilon-\upsilon'\|_\infty\right)\right), 0\right) + \beta_\eta\left(3\gamma_\upsilon\left(\|\upsilon-\upsilon'\|_\infty\right), 0\right)\\
&\;+ \gamma_\zeta\left(2\beta_\zeta\left(\mathbf{d}_\zeta\left(z,z'\right), t/2\right)\right) + \gamma_\zeta\left(2\widetilde{\gamma}_\upsilon\left(\|\upsilon-\upsilon'\|_\infty\right)\right) + \gamma_\upsilon\left(\|\upsilon-\upsilon'\|_\infty\right)\\
&\leq \widehat{\beta}\left(\mathbf{d}_\eta(y,y'), t\right) + \widetilde{\beta}\left(\mathbf{d}_\zeta(z,z'), t\right) + \widehat{\gamma}\left(\|\upsilon-\upsilon'\|_\infty\right),
\end{aligned}
\tag{4.4.5}
$$

where $\widehat{\gamma} \in \mathcal{K}_\infty$, and $\widehat{\beta}, \widetilde{\beta} \in \mathcal{KL}$ are defined as following:

$$
\begin{aligned}
\widehat{\gamma}(r) &= \beta_\eta\left(3\gamma_\zeta\left(2\widetilde{\gamma}_\upsilon(r)\right), 0\right) + \beta_\eta\left(3\gamma_\upsilon(r), 0\right) + \gamma_\zeta\left(2\widetilde{\gamma}_\upsilon(r)\right) + \gamma_\upsilon(r),\\
\widehat{\beta}(r,t) &= \beta_\eta\left(3\beta_\eta\left(r, t/2\right), t/2\right),\\
\widetilde{\beta}(r,t) &= \beta_\eta\left(3\gamma_\zeta\left(2\beta_\zeta(r,0)\right), t/2\right) + \gamma_\zeta\left(2\beta_\zeta(r,t/2)\right).
\end{aligned}
$$

Now we define a new metric $\mathbf{d} : \mathbb{R}^{n_\eta+n_\zeta} \times \mathbb{R}^{n_\eta+n_\zeta} \rightarrow \mathbb{R}_0^+$ by:

$$\mathbf{d}(x, x') = \mathbf{d}_\eta(y, y') + \mathbf{d}_\zeta(z, z'), \tag{4.4.6}$$

for any $x = \left[y^T, z^T\right]^T \in \mathbb{R}^{n_\eta+n_\zeta}$ and $x' = \left[y'^T, z'^T\right]^T \in \mathbb{R}^{n_\eta+n_\zeta}$. It can be readily checked that $\mathbf{d}$ satisfies all three conditions of a metric. By defining $\xi_{x\upsilon} = \left[\eta_{y\zeta\upsilon}^T, \zeta_{z\upsilon}^T\right]^T$, using inequalities (4.4.4) and (4.4.5), and for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^{n_\eta+n_\zeta}$, and any $\upsilon, \upsilon' \in \mathcal{U}$, we obtain:

$$\begin{aligned}
\mathbf{d}\left(\xi_{x\upsilon}(t), \xi_{x'\upsilon'}(t)\right) &= \mathbf{d}_\eta\left(\eta_{y\zeta\upsilon}(t), \eta_{y'\zeta'\upsilon'}(t)\right) + \mathbf{d}_\zeta\left(\zeta_{z\upsilon}(t), \zeta_{z'\upsilon'}(t)\right) \\
&\le \widehat{\beta}\left(\mathbf{d}_\eta(y, y'), t\right) + \widetilde{\beta}\left(\mathbf{d}_\zeta(z, z'), t\right) + \widehat{\gamma}\left(\|\upsilon - \upsilon'\|_\infty\right) \\
&\quad + \beta_\zeta\left(\mathbf{d}_\zeta\left(z, z'\right), t\right) + \widetilde{\gamma}_\upsilon\left(\|\upsilon - \upsilon'\|_\infty\right) \\
&\le \widehat{\beta}\left(\mathbf{d}_\eta(y, y') + \mathbf{d}_\zeta(z, z'), t\right) + \widetilde{\beta}\left(\mathbf{d}_\eta(y, y') + \mathbf{d}_\zeta(z, z'), t\right) \\
&\quad + \beta_\zeta\left(\mathbf{d}_\eta(y, y') + \mathbf{d}_\zeta\left(z, z'\right), t\right) + \widehat{\gamma}\left(\|\upsilon - \upsilon'\|_\infty\right) + \widetilde{\gamma}_\upsilon\left(\|\upsilon - \upsilon'\|_\infty\right) \\
&\le \beta\left(\mathbf{d}(x, x'), t\right) + \gamma\left(\|\upsilon - \upsilon'\|_\infty\right),
\end{aligned} \tag{4.4.7}$$

where $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ are defined as following:

$$\begin{aligned}
\beta(r, t) &= \widehat{\beta}(r, t) + \widetilde{\beta}(r, t) + \beta_\zeta(r, t), \\
\gamma(r) &= \widehat{\gamma}(r) + \widetilde{\gamma}_\upsilon(r).
\end{aligned}$$

Hence, the overall system Σ of the form (4.4.2) is $\delta_\exists$-ISS with respect to υ.

We can now state the main result, on a backstepping controller design approach for the control system Σ in (4.4.1), rendering the resulting closed-loop system $\delta_\exists$-ISS.

Theorem 4.4.2. *Consider the control system Σ of the form (4.4.1). Suppose there exists a continuously differentiable function $\psi : \mathbb{R}^{n_\eta} \rightarrow \mathbb{R}^{n_\zeta}$ such that the control system*

$$\Sigma_\eta : \dot{\eta} = f(\eta, \psi(\eta) + \widetilde{\upsilon}) \tag{4.4.8}$$

is $\delta_\exists$-ISS with respect to the input $\widetilde{\upsilon}$. Then for any $\lambda \in \mathbb{R}^+$, the state feedback control law:

$$\upsilon = k(\eta, \zeta, \widehat{\upsilon}) = -\lambda(\zeta - \psi(\eta)) + \frac{\partial \psi}{\partial y}(\eta) f(\eta, \zeta) + \widehat{\upsilon} \tag{4.4.9}$$

renders the control system Σ $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$.

Proof: Consider the following coordinate transformation:

$$\chi = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \phi(\xi) = \begin{bmatrix} \eta \\ \zeta - \psi(\eta) \end{bmatrix}, \tag{4.4.10}$$

where $\xi = \left[\eta^T, \zeta^T\right]^T$. In the new coordinate χ, we obtain the following dynamics:

$$\widehat{\Sigma} : \begin{cases} \dot{\chi}_1 = f\left(\chi_1, \psi(\chi_1) + \chi_2\right), \\ \dot{\chi}_2 = \upsilon - \frac{\partial \psi}{\partial y}(\chi_1) f\left(\chi_1, \psi(\chi_1) + \chi_2\right). \end{cases} \tag{4.4.11}$$

The proposed control law (4.4.9), given in the new coordinate χ by

$$\upsilon = k(\chi_1, \chi_2 + \psi(\chi_1), \widehat{\upsilon}) = -\lambda\chi_2 + \frac{\partial \psi}{\partial y}(\chi_1) f\left(\chi_1, \psi(\chi_1) + \chi_2\right) + \widehat{\upsilon}, \tag{4.4.12}$$

transforms the control system $\widehat{\Sigma}$ into:

$$\widetilde{\Sigma} : \begin{cases} \dot{\chi}_1 = f(\chi_1, \psi(\chi_1) + \chi_2), \\ \dot{\chi}_2 = -\lambda\chi_2 + \widehat{\upsilon}. \end{cases} \tag{4.4.13}$$

Due to the choice of ψ, the χ_1-subsystem of $\widetilde{\Sigma}$ is $\delta_\exists$-ISS with respect to χ_2. It can be easily verified that the χ_2-subsystem is input-to-state stable with respect to the input $\widehat{\upsilon}$. Since any ISS linear control system is also δ-ISS [Ang02], χ_2-subsystem is also δ-ISS[19] with respect to $\widehat{\upsilon}$. Therefore, using Lemma 4.4.1, we conclude that the control system $\widetilde{\Sigma}$ is $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$. Since, $\delta_\exists$-ISS property is coordinate invariant, we conclude that the original control system Σ in (4.4.1) equipped with the state feedback control law in (4.4.9) is $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$ which completes the proof.

[19] We recall that δ-ISS property is equivalent to $\delta_\exists$-ISS property when the metric is the Euclidean one.

Remark 4.4.3. *The $\delta_\exists$-ISS property of system Σ_η in (4.4.8) can be stablished, for example, using the approaches provided in [PvdWN05, vdWP08] for some relevant classes of control systems (such as piece-wise affine systems and Lur'e-type systems).*

Remark 4.4.4. *The result of Theorem 4.4.2 can be extended to the case that we have arbitrary number of integrators:*

$$\Sigma : \begin{cases} \dot{\eta} = f(\eta, \zeta_1), \\ \dot{\zeta}_1 = \zeta_2, \\ \quad \vdots \\ \dot{\zeta}_k = \upsilon. \end{cases} \tag{4.4.14}$$

Note that in this case, the functions f and ψ must be sufficiently differentiable.

Although the proposed approach in Theorem 4.4.2 provides a controller rendering the control system Σ of the form (4.4.1) $\delta_\exists$-ISS, it does not provide a way of constructing $\delta_\exists$-ISS Lyapunov functions or contraction metrics. In the next lemmas, we show how to construct incremental Lyapunov functions and contraction metrics for the resulting closed-loop system, recursively. Note that the constructed incremental Lyapunov functions can be used as a necessary tool in the analysis in [GPT09, Gir05, JFA$^+$07, KDL$^+$08].

Lemma 4.4.5. *Consider the control system Σ of the form (4.4.1). Suppose there exists a continuously differentiable function $\psi : \mathbb{R}^{n_\eta} \to \mathbb{R}^{n_\zeta}$ such that the smooth function $\widehat{V} : \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_\eta} \to \mathbb{R}_0^+$ is a $\delta_\exists$-ISS Lyapunov function for the control system:*

$$\Sigma_\eta : \dot{\eta} = f(\eta, \psi(\eta) + \widetilde{\upsilon}), \tag{4.4.15}$$

and with respect to the control input $\widetilde{\upsilon}$. Assume that $\widehat{V}$ satisfies condition (iii) in Definition 4.2.1 for some $\kappa \in \mathbb{R}^+$ and some $\sigma \in \mathcal{K}_\infty$, satisfying $\sigma(r) \leq \widehat{\kappa} r^2$ for some $\widehat{\kappa} \in \mathbb{R}^+$ and any $r \in \mathbb{R}_0^+$. Then the function $\widetilde{V} : \mathbb{R}^{n_\eta + n_\zeta} \times \mathbb{R}^{n_\eta + n_\zeta} \to \mathbb{R}_0^+$, *defined as:*

$$\widetilde{V}(x,x') = \widehat{V}(y,y') + \|(z-\psi(y)) - (z'-\psi(y'))\|^2,$$

where $x = \left[y^T, z^T\right]^T$ *and* $x' = \left[y'^T, z'^T\right]^T$, *is a* $\delta_\exists$*-ISS Lyapunov function for* Σ *as in (4.4.1) equipped with the state feedback control law (4.4.9) for all* $\lambda \geq \frac{\kappa+\widehat{\kappa}+1}{2}$.

Proof: As explained in the proof of Theorem 4.4.2, using the proposed state feedback control law (4.4.9) and the coordinate transformation ϕ in (4.4.10), the control system Σ of the form (4.4.1) is transformed to the control system $\widetilde{\Sigma}$ in (4.4.13). Now we show that

$$V(\widehat{x},\widehat{x}') = \widehat{V}(\widehat{x}_1,\widehat{x}_1') + (\widehat{x}_2 - \widehat{x}_2')^T(\widehat{x}_2 - \widehat{x}_2'),$$

is a $\delta_\exists$-ISS Lyapunov function for $\widetilde{\Sigma}$, where $\widehat{x} = \left[\widehat{x}_1^T, \widehat{x}_2^T\right]^T$ and $\widehat{x}' = \left[\widehat{x}_1'^T, \widehat{x}_2'^T\right]^T$ are the states of $\widetilde{\Sigma}$ and $\widehat{x}_1, \widehat{x}_1'$, and $\widehat{x}_2, \widehat{x}_2'$ are the states of χ_1, χ_2-subsystems, respectively. Since $\widehat{V}$ is a $\delta_\exists$-ISS Lyapunov function for χ_1-subsystem when χ_2 is the input, it satisfies condition (i) in Definition 4.2.1 using a metric $\mathbf{d}$ as follows:

$$\underline{\alpha}(\mathbf{d}(\widehat{x}_1,\widehat{x}_1')) \leq \widehat{V}(\widehat{x}_1,\widehat{x}_1') \leq \overline{\alpha}(\mathbf{d}(\widehat{x}_1,\widehat{x}_1')),$$

for some $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$. Now we define a new metric $\widehat{\mathbf{d}} : \mathbb{R}^{n_\eta+n_\zeta} \times \mathbb{R}^{n_\eta+n_\zeta} \rightarrow \mathbb{R}_0^+$ by

$$\widehat{\mathbf{d}}(\widehat{x},\widehat{x}') = \mathbf{d}(\widehat{x}_1,\widehat{x}_1') + \|\widehat{x}_2 - \widehat{x}_2'\|.$$

It can be readily checked that $\widehat{\mathbf{d}}$ satisfies all three conditions of a metric. Using metric $\widehat{\mathbf{d}}$, function V satisfies condition (i) in Definition 4.2.1 as follows:

$$\underline{\mu}\left(\widehat{\mathbf{d}}(\widehat{x},\widehat{x}')\right) \leq V(\widehat{x},\widehat{x}') \leq \overline{\mu}\left(\widehat{\mathbf{d}}(\widehat{x},\widehat{x}')\right),$$

where $\underline{\mu}, \overline{\mu} \in \mathcal{K}_\infty$, $\underline{\mu}\left(\widehat{\mathbf{d}}(\widehat{x},\widehat{x}')\right) = \underline{\alpha}(\mathbf{d}(\widehat{x}_1,\widehat{x}_1')) + \|\widehat{x}_2 - \widehat{x}_2'\|^2$, and $\overline{\mu}\left(\widehat{\mathbf{d}}(\widehat{x},\widehat{x}')\right) = \overline{\alpha}(\mathbf{d}(\widehat{x}_1,\widehat{x}_1')) + \|\widehat{x}_2 - \widehat{x}_2'\|^2$. Now we show that V satisfies condition (iii) in Definition 4.2.1 for $\widetilde{\Sigma}$. Since $\widehat{V}$ is a $\delta_\exists$-ISS Lyapunov function for χ_1-subsystem when χ_2 is the input, $\lambda \geq \frac{\kappa+\widehat{\kappa}+1}{2}$, $\sigma(r) \leq \widehat{\kappa} r^2$, and using the Cauchy Schwarz inequality, we have:

$$\begin{aligned}
&\frac{\partial V}{\partial \widehat{x}}\left[f(\widehat{x}_1,\psi(\widehat{x}_1)+\widehat{x}_2)^T,-\lambda\widehat{x}_2^T+\widehat{u}^T\right]^T \qquad (4.4.16)\\
&+\frac{\partial V}{\partial \widehat{x}'}\left[f(\widehat{x}_1',\psi(\widehat{x}_1')+\widehat{x}_2')^T,-\lambda\widehat{x}_2'^T+\widehat{u}'^T\right]^T \leq\\
&\frac{\partial \widehat{V}}{\partial \widehat{x}_1}f(\widehat{x}_1,\psi(\widehat{x}_1)+\widehat{x}_2)+\frac{\partial \widehat{V}}{\partial \widehat{x}_1'}f(\widehat{x}_1',\psi(\widehat{x}_1')+\widehat{x}_2')\\
&+2(\widehat{x}_2-\widehat{x}_2')^T(-\lambda\widehat{x}_2+\widehat{u})-2(\widehat{x}_2-\widehat{x}_2')^T(-\lambda\widehat{x}_2'+\widehat{u}')\leq\\
&-\kappa\widehat{V}(\widehat{x}_1,\widehat{x}_1')+\sigma(\|\widehat{x}_2-\widehat{x}_2'\|)-2\lambda\|\widehat{x}_2-\widehat{x}_2'\|^2+2(\widehat{x}_2-\widehat{x}_2')^T(\widehat{u}-\widehat{u}')\leq\\
&-\kappa\widehat{V}(\widehat{x}_1,\widehat{x}_1')+\widehat{\kappa}\|\widehat{x}_2-\widehat{x}_2'\|^2-2\lambda\|\widehat{x}_2-\widehat{x}_2'\|^2+2\|\widehat{x}_2-\widehat{x}_2'\|\|\widehat{u}-\widehat{u}'\|\leq\\
&-\kappa\widehat{V}(\widehat{x}_1,\widehat{x}_1')+\widehat{\kappa}\|\widehat{x}_2-\widehat{x}_2'\|^2-2\lambda\|\widehat{x}_2-\widehat{x}_2'\|^2+\|\widehat{x}_2-\widehat{x}_2'\|^2+\|\widehat{u}-\widehat{u}'\|^2\leq\\
&-\kappa V(\widehat{x},\widehat{x}')+\|\widehat{u}-\widehat{u}'\|^2.
\end{aligned}$$

The latter inequality implies that V is a $\delta_\exists$-ISS Lyapunov function for $\widetilde{\Sigma}$. Since $\delta_\exists$-ISS Lyapunov functions are coordinate-invariant, as shown in Lemma 4.2.3, we conclude that the function $\widetilde{V}:\mathbb{R}^{n_\eta+n_\zeta}\times\mathbb{R}^{n_\eta+n_\zeta}\rightarrow\mathbb{R}_0^+$, defined by:

$$\widetilde{V}(x,x')=V(\phi(x),\phi(x'))=\widehat{V}(y,y')+\|(z-\psi(y))-(z'-\psi(y'))\|^2,$$

is a $\delta_\exists$-ISS Lyapunov function for Σ, as in (4.4.1) equipped with the state feedback control law in (4.4.9).

Remark 4.4.6. *One can use the LMI based results in [PvdWN05, PvdWN07, vdWP08] to find a quadratic $\delta_\exists$-ISS Lyapunov function for system Σ_η in (4.4.15).*

The next results provide a backstepping design approach for parametric-strict-feedback and strict-feedback form control systems by recursively applying the results proposed in Lemma 4.4.5.

Consider the class of control systems $\Sigma=(\mathbb{R}^n,\mathsf{U},\mathcal{U},f)$ with f of the parametric-strict-feedback form [KKK95]:

$$
\begin{aligned}
f_1(x,u) &= h_1(x_1) + b_1 x_2, \\
f_2(x,u) &= h_2(x_1,x_2) + b_2 x_3, \\
&\vdots \\
f_{n-1}(x,u) &= h_{n-1}(x_1,\cdots,x_{n-1}) + b_{n-1}x_n, \\
f_n(x,u) &= h_n(x) + g(x)u,
\end{aligned}
\tag{4.4.17}
$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathsf{U} \subset \mathbb{R}$ is the control input. The functions $h_i : \mathbb{R}^i \to \mathbb{R}$, for $i = 1, \ldots, n$, and $g : \mathbb{R}^n \to \mathbb{R}$ are smooth, $g(x) \neq 0$ over the domain of interest, and $b_i \in \mathbb{R}$, for $i = 1, \ldots, n$, are nonzero constants.

We can now state the results, describing a backstepping controller for control system of the form (4.4.17).

Theorem 4.4.7. *For any control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ with f of the form (4.4.17) and for any $\lambda \in \mathbb{R}^+$, the state feedback control law:*

$$
k(x,\widehat{u}) = \frac{1}{g(x)}\Big[k_n(x) - h_n(x)\Big] + \frac{1}{g(x)}\widehat{u}, \tag{4.4.18}
$$

where

$$
\begin{aligned}
k_l(x) = -b_{l-1}\left(x_{l-1} - \phi_{l-2}(x)\right) - \lambda\left(x_l - \phi_{l-1}(x)\right) \\
+\frac{\partial \phi_{l-1}}{\partial x} f(x,k(x)),\ \text{for } l = 1,\cdots,n, \\
\phi_l(x) = \frac{1}{b_l}\Big[k_l(x) - h_l(x)\Big],\ \text{for } l = 1,\cdots,n-1, \\
\phi_{-1}(x) = \phi_0(x) = 0\ \forall x \in \mathbb{R}^n,\ b_0 = 0,\ \text{and } x_0 = 0,
\end{aligned}
\tag{4.4.19}
$$

renders the control system Σ $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$ and the function

$$
\widehat{V}(x,x') = \sqrt{\sum_{l=0}^{n-1}\left[\left(x_{l+1} - \phi_l(x)\right) - \left(x'_{l+1} - \phi_l(x')\right)\right]^2},
$$

is a $\delta_\exists$-ISS Lyapunov function for Σ.

Proof: The proposed control law (4.4.18) transforms a control system $\Sigma = (\mathbb{R}^n, \mathbb{R}, \mathcal{U}, f)$ with f of the form (4.4.17) into:

$$\begin{aligned} f_1(x, k(x, \widehat{u})) &= h_1(x_1) + b_1 x_2, \\ f_2(x, k(x, \widehat{u})) &= h_2(x_1, x_2) + b_2 x_3, \\ &\vdots \\ f_{n-1}(x, k(x, \widehat{u})) &= h_{n-1}(x_1, \cdots, x_{n-1}) + b_{n-1} x_n, \\ f_n(x, k(x, \widehat{u})) &= k_n(x) + \widehat{u}. \end{aligned} \tag{4.4.20}$$

The coordinate transformation $z = \psi(x)$, where $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is the smooth map (with smooth inverse) defined by:

$$z = \psi(x) = \begin{bmatrix} x_1 \\ x_2 - \phi_1(x) \\ x_3 - \phi_2(x) \\ \vdots \\ x_n - \phi_{n-1}(x) \end{bmatrix}, \tag{4.4.21}$$

transforms the control system Σ with f of the form (4.4.20) into:

$$\Sigma' : \left\{ \dot{\zeta} = A\zeta + B\widehat{v} \right., \tag{4.4.22}$$

where

$$A = \begin{bmatrix} -\lambda & b_1 & 0 & 0 & \cdots & 0 \\ -b_1 & -\lambda & b_2 & 0 & \cdots & 0 \\ 0 & -b_2 & -\lambda & b_3 & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & & \cdots & 0 & -b_{n-1} & -\lambda \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{4.4.23}$$

It can be easily checked that the function:

$$V(z, z') = \sqrt{(z - z')^T (z - z')},$$

satisfies

$$\frac{\partial V}{\partial z}(Az + B\widehat{u}) + \frac{\partial V}{\partial z'}(Az' + B\widehat{u}') \leq -\lambda V(z, z') + \|\widehat{u} - \widehat{u}'\|. \tag{4.4.24}$$

Hence the function V satisfies conditions (i) and (iii) in Definition 4.2.1 implying that it is a $\delta_\exists$-ISS Lyapunov function for Σ'. Using Theorem 4.3.8, we obtain that Σ' is $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$. Using Lemma 4.2.3, we conclude that the function:

$$\widehat{V}(x,x') = V\left(\psi(x),\psi(x')\right) = \\ \sqrt{\left(\psi(x)-\psi(x')\right)^T\left(\psi(x)-\psi(x')\right)} = \\ \sqrt{\sum_{l=0}^{n-1}\left[\left(x_{l+1}-\phi_l(x)\right)-\left(x'_{l+1}-\phi_l(x')\right)\right]^2},$$

is a $\delta_\exists$-ISS Lyapunov function for Σ. Therefore, using Theorem 4.3.8, we obtain that Σ is $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$. The $\delta_\exists$-ISS condition (2.3.3), as shown in Theorem 4.3.8, is given by:

$$\mathbf{d}\left(\xi_{x\widehat{\upsilon}}(t),\xi_{x'\widehat{\upsilon}'}(t)\right) \leq 2\mathsf{e}^{-\lambda t}\mathbf{d}(x,x') + \frac{2}{\lambda}\|\widehat{\upsilon}-\widehat{\upsilon}'\|_\infty,$$

where $\mathbf{d}(x,x') = \|\psi(x)-\psi(x')\|$, for any $x,x'\in\mathbb{R}^n$.

Remark 4.4.8. *It can be readily seen that the state feedback control law (4.4.18) renders the control system Σ $\delta_\exists$-GAS and the function*

$$\widehat{V}(x,x') = \sqrt{\sum_{l=0}^{n-1}\left[\left(x_{l+1}-\phi_l(x)\right)-\left(x'_{l+1}-\phi_l(x')\right)\right]^2},$$

is a $\delta_\exists$-GAS Lyapunov function for Σ.

Now, we extend the result in Theorem 4.4.7 to the class of control systems $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ with f of the strict-feedback form [KKK95]:

$$\begin{aligned} f_1(x,u) &= h_1(x_1)+g_1(x_1)x_2, \\ f_2(x,u) &= h_2(x_1,x_2)+g_2(x_1,x_2)x_3, \\ &\vdots \\ f_{n-1}(x,u) &= h_{n-1}(x_1,\cdots,x_{n-1})+g_{n-1}(x_1,\cdots,x_{n-1})x_n, \\ f_n(x,u) &= h_n(x)+g_n(x)u, \end{aligned} \tag{4.4.25}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathsf{U} \subseteq \mathbb{R}$ is the control input. The functions $h_i : \mathbb{R}^i \to \mathbb{R}$, and $g_i : \mathbb{R}^i \to \mathbb{R}$, for $i = 1, \ldots, n$, are smooth, and $g_i(x_1, \cdots, x_i) \neq 0$ over the domain of interest.

Theorem 4.4.9. *Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ be a control system where f is of the form (4.4.25). The state feedback control law $u = k(\varphi(x), \widehat{u})$, where k was defined in (4.4.18) and $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is the smooth map (with smooth inverse) defined by:*

$$\varphi(x) = \begin{bmatrix} x_1 \\ g_1(x_1)x_2 \\ g_1(x_1)g_2(x_1, x_2)x_3 \\ \vdots \\ \prod_{i=1}^{n-1} g_i(x_1, \cdots, x_i)x_n \end{bmatrix}, \tag{4.4.26}$$

renders control system Σ $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$ and the function

$$\widetilde{V}(x, x') = \sqrt{(\psi \circ \varphi(x) - \psi \circ \varphi(x'))^T (\psi \circ \varphi(x) - \psi \circ \varphi(x'))},$$

where ψ was defined in (4.4.21), is a $\delta_\exists$-ISS Lyapunov function for Σ.

Proof: It can be readily verified that the coordinate transformation $\eta = \varphi(\xi)$ transforms the control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ with f of the form (4.4.25) to the control system $\Sigma' = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f')$, where $f' = \varphi_* f$ has the following form:

$$\begin{aligned} f'_1(y, u) &= h'_1(y_1) + y_2, \\ f'_2(y, u) &= h'_2(y_1, y_2) + y_3, \\ &\vdots \\ f'_{n-1}(y, u) &= h'_{n-1}(y_1, \cdots, y_{n-1}) + y_n, \\ f'_n(y, u) &= h'_n(y) + g'(y)u, \end{aligned} \tag{4.4.27}$$

where $h'_i : \mathbb{R}^i \to \mathbb{R}$, for $i = 1, \cdots, n$, are smooth functions, $g' = \prod_{i=1}^{i=n} g_i$, and $y \in \mathbb{R}^n$ is the state of Σ'. As proved in Theorem 4.4.7, the state feedback control law k, defined in (4.4.18), makes the function:

$$\widehat{V}(y, y') = \sqrt{(\psi(y) - \psi(y'))^T (\psi(y) - \psi(y'))}, \tag{4.4.28}$$

a $\delta_\exists$-ISS Lyapunov function, for the control system Σ'. As proved in Lemma 4.2.3, the function:

$$\widetilde{V}(x,x') = \widehat{V}(\varphi(x),\varphi(x')) = \sqrt{(\psi\circ\varphi(x)-\psi\circ\varphi(x'))^T(\psi\circ\varphi(x)-\psi\circ\varphi(x'))},$$

is a $\delta_\exists$-ISS Lyapunov function , for the control system Σ, equipped with the state feedback control law $k(\varphi(x),\widehat{u})$. Therefore, the state feedback control law $k(\varphi(x),\widehat{u})$ makes the control system Σ $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$. The $\delta_\exists$-ISS condition (2.3.3), as shown in Theorem 4.3.8, is given by:

$$\mathbf{d}\left(\xi_{x\widehat{\upsilon}}(t),\xi_{x'\widehat{\upsilon}'}(t)\right) \leq 2\mathsf{e}^{-\lambda t}\mathbf{d}(x,x') + \frac{2}{\lambda}\|\widehat{\upsilon}-\widehat{\upsilon}'\|_\infty,$$

where $\mathbf{d}(x,x') = \|\psi\circ\varphi(x)-\psi\circ\varphi(x')\|$, for any $x,x'\in\mathbb{R}^n$.

Remark 4.4.10. *It can be readily seen that the state feedback control law $k(\varphi(x),\widehat{u})$, where k was defined in (4.4.18), renders the control system Σ $\delta_\exists$-GAS and the function*

$$\widetilde{V}(x,x') = \sqrt{(\psi\circ\varphi(x)-\psi\circ\varphi(x'))^T(\psi\circ\varphi(x)-\psi\circ\varphi(x'))},$$

is a $\delta_\exists$-GAS Lyapunov function for Σ.

The next lemma shows how to construct contraction metrics for the closed-loop system resulting from the backstepping controller synthesis technique in Theorem 4.4.2.

Lemma 4.4.11. *Consider the control system Σ of the form (4.4.1) and assume that function f is smooth. Suppose there exists a continuously differentiable function $\psi:\mathbb{R}^{n_\eta}\rightarrow\mathbb{R}^{n_\zeta}$ such that the metric $\widehat{G}:\mathbb{R}^{n_\eta}\rightarrow\mathbb{R}^{n_\eta\times n_\eta}$ is a contraction metric, with respect to states and inputs, for the control system*

$$\Sigma_\eta : \dot{\eta} = f(\eta,\psi(\eta)+\widetilde{\upsilon}), \tag{4.4.29}$$

satisfying the condition (4.2.7) for some $\widehat{\lambda}\in\mathbb{R}^+$ and $\alpha\in\mathbb{R}_0^+$. Then

$$\widetilde{G}(x) = \begin{bmatrix} \widehat{G}(y) + \left(\frac{\partial\psi}{\partial y}\right)^T\frac{\partial\psi}{\partial y} & -\left(\frac{\partial\psi}{\partial y}\right)^T \\ -\frac{\partial\psi}{\partial y} & I_{n_\zeta} \end{bmatrix},$$

where $x = \left[y^T, z^T\right]^T$*, is a contraction metric, with respect to states and inputs, for* Σ *as in (4.4.1) equipped with the state feedback control law in (4.4.9) for all* $\lambda > \frac{\alpha^2}{8\widehat{\lambda}}$*.*

Proof: As explained in the proof of Theorem 4.4.2, using the proposed control law (4.4.9) and the coordinate transformation ϕ in (4.4.10), the control system Σ of the form (4.4.1) is transformed to the control system $\widetilde{\Sigma}$ in (4.4.13). Now we show that the metric

$$G(\widehat{x}) = \begin{bmatrix} \widehat{G}(\widehat{x}_1) & 0_{n_\eta \times n_\zeta} \\ 0_{n_\zeta \times n_\eta} & I_{n_\zeta} \end{bmatrix},$$

is a contraction metric, with respect to states and inputs, for $\widetilde{\Sigma}$, where $\widehat{x} = \left[\widehat{x}_1^T, \widehat{x}_2^T\right]^T$ is the state of $\widetilde{\Sigma}$, and $\widehat{x}_1$, and $\widehat{x}_2$ are states of χ_1, χ_2-subsystems, respectively. It can be easily seen that G is positive definite because $\widehat{G}$ is positive definite since it is a contraction metric for Σ_η. Now we show that G satisfies the condition (4.2.7) for the control system $\widetilde{\Sigma}$. Since $\widehat{G}$ is a contraction metric, with respect to states and inputs, for χ_1-subsystem when χ_2 is the input, we have:

$$\begin{aligned} &\widehat{X}_1^T \left(\left(\frac{\partial f}{\partial \widehat{x}_1} \right)^T \widehat{G}(\widehat{x}_1) + \widehat{G}(\widehat{x}_1) \frac{\partial f}{\partial \widehat{x}_1} + \frac{\partial \widehat{G}}{\partial \widehat{x}_1} f(\widehat{x}_1, \psi(\widehat{x}_1) + \widehat{x}_2) \right) \widehat{X}_1 \\ &+ 2\widehat{X}_2^T \left(\frac{\partial f}{\partial \widehat{x}_2} \right)^T \widehat{G}(\widehat{x}_1) \widehat{X}_1 \leq -\widehat{\lambda} \widehat{X}_1^T \widehat{G}(\widehat{x}_1) \widehat{X}_1 + \alpha \left(\widehat{X}_1^T \widehat{G}(\widehat{x}_1) \widehat{X}_1 \right)^{\frac{1}{2}} \left(\widehat{X}_2^T \widehat{X}_2 \right)^{\frac{1}{2}}, \end{aligned} \tag{4.4.30}$$

for any $\widehat{X}_1, \widehat{x}_1 \in \mathbb{R}^{n_\eta}$, $\widehat{X}_2, \widehat{x}_2 \in \mathbb{R}^{n_\zeta}$, some $\widehat{\lambda} \in \mathbb{R}^+$, and some $\alpha \in \mathbb{R}_0^+$. By choosing

$\lambda > \frac{\alpha^2}{8\widehat{\lambda}}$, using (4.4.30), and the Cauchy Schwarz inequality we obtain:

$$\left[\widehat{X}_1^T \ \widehat{X}_2^T\right]\left(\left(\frac{\partial\left[f, -\lambda\widehat{x}_2+\widehat{u}\right]^T}{\partial\widehat{x}}\right)^T G(\widehat{x}) + G(\widehat{x})\frac{\partial\left[f, -\lambda\widehat{x}_2+\widehat{u}\right]^T}{\partial\widehat{x}} +\right.$$

$$\left.\frac{\partial G}{\partial\widehat{x}}\begin{bmatrix} f(\widehat{x}_1, \psi(\widehat{x}_1)+\widehat{x}_2) \\ -\lambda\widehat{x}_2+\widehat{u}\end{bmatrix}\right)\begin{bmatrix}\widehat{X}_1 \\ \widehat{X}_2\end{bmatrix} + 2Y^T\begin{bmatrix}0_{n_\eta\times n_\zeta} \\ I_{n_\zeta}\end{bmatrix}^T G(\widehat{x})\begin{bmatrix}\widehat{X}_1 \\ \widehat{X}_2\end{bmatrix} =$$

$$\left[\widehat{X}_1^T \ \widehat{X}_2^T\right]\begin{bmatrix} \left(\frac{\partial f}{\partial\widehat{x}_1}\right)^T\widehat{G}(\widehat{x}_1) + \widehat{G}(\widehat{x}_1)\frac{\partial f}{\partial\widehat{x}_1} + \frac{\partial\widehat{G}}{\partial\widehat{x}_1}f & \widehat{G}(\widehat{x}_1)\frac{\partial f}{\partial\widehat{x}_2} \\ \left(\frac{\partial f}{\partial\widehat{x}_2}\right)^T\widehat{G}(\widehat{x}_1) & -2\lambda I_{n_\zeta}\end{bmatrix}\begin{bmatrix}\widehat{X}_1 \\ \widehat{X}_2\end{bmatrix} + 2Y^T\widehat{X}_2 \leq$$

$$-\widehat{\lambda}\left\langle\widehat{X}_1, \widehat{X}_1\right\rangle_{\widehat{G}} + \alpha\left\langle\widehat{X}_1, \widehat{X}_1\right\rangle_{\widehat{G}}^{\frac{1}{2}}\left\langle\widehat{X}_2, \widehat{X}_2\right\rangle_{I_{n_\zeta}}^{\frac{1}{2}} - 2\lambda\widehat{X}_2^T\widehat{X}_2 + 2Y^T\widehat{X}_2 \leq$$

$$-\widetilde{\lambda}\left\langle\widehat{X}_1, \widehat{X}_1\right\rangle_{\widehat{G}} - \widetilde{\lambda}\widehat{X}_2^T\widehat{X}_2 + 2\sqrt{Y^TY}\sqrt{\widehat{X}_2^T\widehat{X}_2 + \left\langle\widehat{X}_1, \widehat{X}_1\right\rangle_{\widehat{G}}} \leq$$

$$-\widetilde{\lambda}\left\langle\widehat{X}, \widehat{X}\right\rangle_G + 2\left\langle\widehat{X}, \widehat{X}\right\rangle_G^{\frac{1}{2}}\langle Y, Y\rangle_{I_{n_\zeta}}^{\frac{1}{2}},$$

for any $\widehat{X} = \left[\widehat{X}_1^T, \ \widehat{X}_2^T\right]^T \in \mathbb{R}^{n_\eta+n_\zeta}$, any $\widehat{x} = \left[\widehat{x}_1^T, \ \widehat{x}_2^T\right]^T \in \mathbb{R}^{n_\eta+n_\zeta}$, any $Y \in \mathbb{R}^{n_\zeta}$, and some $\widetilde{\lambda} \in \mathbb{R}^+$. Hence, G is a contraction metric, with respect to states and inputs, for $\widetilde{\Sigma}$. Since a contraction metric, with respect to states and inputs, is coordinate invariant, as shown in Lemma 4.4.19, we conclude that $\widetilde{G} = \phi^*G$ is a contraction metric, with respect to states and inputs, for Σ as in (4.4.1) equipped with the state feedback control law in (4.4.9). This completes the proof.

The next results provide a backstepping design approach for parametric-strict-feedback and strict-feedback form control systems by recursively applying the results proposed in Lemma 4.4.11.

Theorem 4.4.12. *For any control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ with f of the form (4.4.17) and for any $\lambda \in \mathbb{R}^+$, the state feedback control law:*

$$k(x, \widehat{u}) \;\; = \;\; \frac{1}{g(x)}\left[k_n(x) - h_n(x)\right] + \frac{1}{g(x)}\widehat{u}, \qquad (4.4.31)$$

where

$$\begin{aligned}
k_l(x,\widehat{u}) =& - b_{l-1}\left(x_{l-1} - \phi_{l-2}(x)\right) - \frac{\lambda}{2}\left(x_l - \phi_{l-1}(x)\right)\\
&+ \frac{\partial \phi_{l-1}}{\partial x} f(x, k(x,\widehat{u})),\ \textit{for } l = 1, \cdots, n,\\
\phi_l(x) =& \frac{1}{b_l}\left[k_l(x) - h_l(x)\right],\ \textit{for } l = 1, \cdots, n-1,\\
\phi_{-1}(x) =& \phi_0(x) = 0\ \forall x \in \mathbb{R}^n,\ b_0 = 0,\ \textit{and } x_0 = 0,
\end{aligned}$$

renders the control system Σ $\delta_\exists$*-GAS.*

Proof: Consider the following system:

$$\Sigma_l : \begin{cases} \dot{\eta}_l = F_l(\eta_l) + B_l \xi_l, \\ \dot{\xi}_l = k_l(\eta_l, \xi_l), \end{cases} \tag{4.4.32}$$

where $\eta_l = [\xi_1, \cdots, \xi_{l-1}]^T,\ B_l = [0, \cdots, 0, b_{l-1}]^T \in \mathbb{R}^{l-1},\ z_l = [y_l^T,\ x_l]^T \in \mathbb{R}^l$ is the state of Σ_l, $F_l(y_l) = [f_1(x,u), \cdots, f_{l-2}(x,u), h_{l-1}(x_1, \cdots, x_{l-1})]^T$, and $y_l = [x_1, \cdots, x_{l-1}]^T$. By using induction on l, we show that the metric G_l, defined by:

$$G_l(y_l) = \begin{bmatrix} G_{l-1}(y_{l-1}) + \left(\frac{\partial \phi_{l-1}}{\partial y_l}\right)^T \frac{\partial \phi_{l-1}}{\partial y_l} & -\left(\frac{\partial \phi_{l-1}}{\partial y_l}\right)^T \\ -\frac{\partial \phi_{l-1}}{\partial y_l} & 1 \end{bmatrix}, \tag{4.4.33}$$

is a contraction metric, with respect to states, for the system (4.4.32) with contraction rate λ. For $l = 1$, it can be easily checked that $G_1(y_1) = 1$ is a contraction metric, with respect to states, with the contraction rate λ for the scalar system:

$$\Sigma_1 : \dot{\xi}_1 = k_1(\xi_1) = -\frac{\lambda}{2}\xi_1.$$

Assume that the metric G_{k-1} is a contraction metric, with respect to states, for the system Σ_{k-1}, for some $2 \leq k \leq n$, and with contraction rate λ. This implies:

$$\begin{aligned}
&[Y^T\ X]\Bigg(\left(\frac{\partial(F_k + B_k\phi_{k-1})}{\partial y_k}\right)^T G_{k-1}(y_{k-1}) + G_{k-1}(y_{k-1})\frac{\partial(F_k + B_k\phi_{k-1})}{\partial y_k}\\
&+ \frac{\partial G_{k-1}}{\partial y_k}\left(F_k + B_k\phi_{k-1}\right)\Bigg)\begin{bmatrix} Y \\ X \end{bmatrix} \leq -\lambda\left[Y^T\ X\right] G_{k-1}(y_{k-1})\begin{bmatrix} Y \\ X \end{bmatrix},
\end{aligned} \tag{4.4.34}$$

for any $Y \in \mathbb{R}^{k-2}$, and $X \in \mathbb{R}$. Since the metric G_{k-1} is only a function of $y_{k-1} = [x_1, \cdots, x_{k-2}]^T$, and the vector B_k has zero entries except for the last entry, it can be easily shown that $\frac{\partial G_{k-1}}{\partial y_k} B_k = 0_{k-1}$, and the inequality (4.4.34) reduces to:

$$[Y^T \; X] \left(\left(\frac{\partial (F_k + B_k \phi_{k-1})}{\partial y_k} \right)^T G_{k-1}(y_{k-1}) + G_{k-1}(y_{k-1}) \frac{\partial (F_k + B_k \phi_{k-1})}{\partial y_k} + \frac{\partial G_{k-1}}{\partial y_k} F_k \right) \begin{bmatrix} Y \\ X \end{bmatrix} \leq -\lambda \, [Y^T \; X] \, G_{k-1}(y_{k-1}) \begin{bmatrix} Y \\ X \end{bmatrix}. \tag{4.4.35}$$

Now, we show that:

$$G_k(y_k) = \begin{bmatrix} G_{k-1}(y_{k-1}) + \left(\frac{\partial \phi_{k-1}}{\partial y_k}\right)^T \frac{\partial \phi_{k-1}}{\partial y_k} & -\left(\frac{\partial \phi_{k-1}}{\partial y_k}\right)^T \\ -\frac{\partial \phi_{k-1}}{\partial y_k} & 1 \end{bmatrix}, \tag{4.4.36}$$

is a contraction metric, with respect to states, for the system Σ_k. For any nonzero vector $\left[Y^T, \; X\right]^T \in \mathbb{R}^k$, we have:

$$[Y^T \; X] \, G_k(y_k) \begin{bmatrix} Y \\ X \end{bmatrix} =$$

$$[Y^T \; X] \begin{bmatrix} G_{k-1}(y_{k-1}) + \left(\frac{\partial \phi_{k-1}}{\partial y_k}\right)^T \frac{\partial \phi_{k-1}}{\partial y_k} & -\left(\frac{\partial \phi_{k-1}}{\partial y_k}\right)^T \\ -\frac{\partial \phi_{k-1}}{\partial y_k} & 1 \end{bmatrix} \begin{bmatrix} Y \\ X \end{bmatrix} =$$

$$Y^T G_{k-1}(y_{k-1}) Y + \left(\frac{\partial \phi_{k-1}}{\partial y_k} Y - X \right)^2. \tag{4.4.37}$$

If $Y \in \mathbb{R}^{k-1}$ is the zero vector, X must be nonzero implying that the equation (4.4.37) is equal to X^2 which is positive. On the other hand, if $Y \in \mathbb{R}^{k-1}$ is nonzero, $Y^T G_{k-1}(y_{k-1}) Y$ is a positive scalar because G_{k-1} is a Riemannian metric. Hence, G_k is positive definite. Using the inequality (4.4.35), the long algebraic manipulations in (4.4.38) show that G_k satisfies (4.2.5) with the contraction rate λ. Hence, the metric G_k is a contraction metric, with respect to states, for the system Σ_k. Therefore, for any $l \leq n$, the metric G_l is a contraction metric, with respect to states, for the system (4.4.32) and with the contraction rate λ.

$$
\begin{aligned}
&\begin{bmatrix} Y^T & X \end{bmatrix} \left(\left(\frac{\partial \left[F_k^T + B_k^T x_k \;\; k_k(x,\widehat{u}) \right]^T}{\partial z_k} \right)^T G_k(y_k) + G_k(y_k) \frac{\partial \left[F_k^T + B_k^T x_k \;\; k_k(x,\widehat{u}) \right]^T}{\partial z_k} + \dot{G}_k(y_k) \right) \begin{bmatrix} Y \\ X \end{bmatrix} = \qquad (4.4.38) \\
&\begin{bmatrix} Y^T & X \end{bmatrix} \left(\begin{bmatrix} \frac{\partial F_k}{\partial y_k} & B_k \\ (F_k + B_k x_k)^T \frac{\partial^2 \phi_{k-1}}{\partial {y_k}^2} + \frac{\partial \phi_{k-1}}{\partial y_k} \frac{\partial F_k}{\partial y_k} + \frac{\lambda}{2} \frac{\partial \phi_{k-1}}{\partial y_k} - B_k^T G_{k-1}(y_{k-1}) & -\frac{\lambda}{2} + \frac{\partial \phi_{k-1}}{\partial y_k} B_k \end{bmatrix}^T \cdot \right. \\
&\begin{bmatrix} G_{k-1}(y_{k-1}) + \left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T \frac{\partial \phi_{k-1}}{\partial y_k} & -\left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T \\ -\frac{\partial \phi_{k-1}}{\partial y_k} & 1 \end{bmatrix} + \begin{bmatrix} G_{k-1}(y_{k-1}) + \left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T \frac{\partial \phi_{k-1}}{\partial y_k} & -\left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T \\ -\frac{\partial \phi_{k-1}}{\partial y_k} & 1 \end{bmatrix} \cdot \\
&\begin{bmatrix} \frac{\partial F_k}{\partial y_k} & B_k \\ (F_k + B_k x_k)^T \frac{\partial^2 \phi_{k-1}}{\partial {y_k}^2} + \frac{\partial \phi_{k-1}}{\partial y_k} \frac{\partial F_k}{\partial y_k} + \frac{\lambda}{2} \frac{\partial \phi_{k-1}}{\partial y_k} - B_k^T G_{k-1}(y_{k-1}) & -\frac{\lambda}{2} + \frac{\partial \phi_{k-1}}{\partial y_k} B_k \end{bmatrix} + \\
&\left. \begin{bmatrix} \frac{\partial G_{k-1}}{\partial y_k} (F_k + B_k x_k) + \frac{\partial^2 \phi_{k-1}}{\partial {y_k}^2} (F_k + B_k x_k) \frac{\partial \phi_{k-1}}{\partial y_k} + \left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T (F_k + B_k x_k)^T \frac{\partial^2 \phi_{k-1}}{\partial {y_k}^2} & -\frac{\partial^2 \phi_{k-1}}{\partial {y_k}^2} (F_k + B_k x_k) \\ -(F_k + B_k x_k)^T \frac{\partial^2 \phi_{k-1}}{\partial {y_k}^2} & 0 \end{bmatrix} \right) \cdot \\
&\begin{bmatrix} Y \\ X \end{bmatrix} = \begin{bmatrix} Y^T & X \end{bmatrix} \cdot \\
&\begin{bmatrix} \left(\left(\frac{\partial (F_k + B_k \phi_{k-1})}{\partial y_k} \right)^T G_{k-1}(y_{k-1}) + G_{k-1}(y_{k-1}) \frac{\partial (F_k + B_k \phi_{k-1})}{\partial y_k} + \frac{\partial G_{k-1}}{\partial y_k} F_k \right) - \lambda \left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T \frac{\partial \phi_{k-1}}{\partial y_k} & \lambda \left(\frac{\partial \phi_{k-1}}{\partial y_k} \right)^T \\ \lambda \frac{\partial \phi_{k-1}}{\partial y_k} & -\lambda \end{bmatrix} \cdot \\
&\begin{bmatrix} Y \\ X \end{bmatrix} \leq -\lambda \begin{bmatrix} Y^T & X \end{bmatrix} G_k(y_k) \begin{bmatrix} Y \\ X \end{bmatrix}.
\end{aligned}
$$

The proposed control law (4.4.31), transforms a control system of the form (4.4.17) into:

$$
\Sigma_n : \begin{cases} \dot{\eta}_n = F_n(\eta_n) + B_n \xi_n, \\ \dot{\xi}_n = k_n(\eta_n, \xi_n) + \widehat{\upsilon}. \end{cases} \qquad (4.4.39)
$$

It can be easily checked that $\widehat{\upsilon}$ does not appear in the variation of Σ_n when we only have state variations. Since the metric G_n is not a function of the n-th state, its derivative with respect to time does not include $\widehat{\upsilon}$. Hence, we can apply the induction results to Σ_n to conclude that the metric G_n is a contraction metric, with respect to states, for Σ_n and with the contraction rate λ. Moreover, it can be readily seen that $G_n = \psi^* I_n$, where ψ was defined in (4.4.21). Note that $\mathbf{d}_{I_n}$ is just the Euclidean metric and we know that $(\mathbb{R}^n, \mathbf{d}_{I_n})$ is a complete metric space. Moreover, since $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry[20], $(\mathbb{R}^n, \mathbf{d}_{G_n})$ is also a complete metric space [Lee03]. By using Theorem 4.3.10, we conclude that a control system of the

[20] Suppose M and $\widetilde{M}$ are Riemannian manifolds with Riemannian metrics G and $\widetilde{G}$, respectively. A smooth map $\psi : M \to \widetilde{M}$ is called an isometry if it is a diffeomorphism satisfying $G = \psi^* \widetilde{G}$.

form (4.4.17), equipped with the state feedback control law (4.4.31), is $\delta_\exists$-GAS. The $\delta_\exists$-GAS condition (2.3.1), as shown in [AR03], is given by:

$$\mathbf{d}_{G_n}\left(\xi_{xv}(t), \xi_{x'v}(t)\right) \leq \mathsf{e}^{-\frac{\lambda}{2}t}\mathbf{d}_{G_n}(x, x'). \tag{4.4.40}$$

Remark 4.4.13. *The contraction metric $G_n(y_n)$, with respect to states, for the control system (4.4.17), equipped with the state feedback control law (4.4.31), is given by:*

$$\begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} [1] + \left(\frac{\partial \phi_1}{\partial y_2}\right)^T \frac{\partial \phi_1}{\partial y_2} & -\left(\frac{\partial \phi_1}{\partial y_2}\right)^T \\ -\frac{\partial \phi_1}{\partial y_2} & 1 \end{bmatrix} + \left(\frac{\partial \phi_2}{\partial y_3}\right)^T \frac{\partial \phi_2}{\partial y_3} & -\left(\frac{\partial \phi_2}{\partial y_3}\right)^T \\ -\frac{\partial \phi_2}{\partial y_3} & 1 \end{bmatrix} + \cdots \\ \vdots \end{bmatrix} + \left(\frac{\partial \phi_{n-1}}{\partial y_n}\right)^T \frac{\partial \phi_{n-1}}{\partial y_n} & -\left(\frac{\partial \phi_{n-1}}{\partial y_n}\right)^T \\ -\frac{\partial \phi_{n-1}}{\partial y_n} & 1 \end{bmatrix}, \tag{4.4.41}$$

where $y_l = [x_1, \cdots, x_{l-1}]^T$, for $l = 2, \cdots, n$, and the contraction rate is λ.

Remark 4.4.14. *It can be checked that the function*

$$V(x) = \frac{1}{2}\sum_{l=0}^{n-1}\left(x_{l+1} - \phi_l(x)\right)^2,$$

is a Lyapunov function [Kha96] for the control system (4.4.17), equipped with the state feedback control law (4.4.31) when $\widehat{u} = 0$. Moreover, the Hessian of $V(x)$ is equal to the contraction metric G_n, with respect to states, defined in (4.4.41).

In the next theorem, we show that control law (4.4.31) also enforces $\delta_\exists$-ISS.

Theorem 4.4.15. *For any control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ with f of the form (4.4.17) and for any $\lambda \in \mathbb{R}^+$, the state feedback control law:*

$$k(x, \widehat{u}) \;=\; \frac{1}{g(x)}\Big[k_n(x) - h_n(x)\Big] + \frac{1}{g(x)}\widehat{u}, \tag{4.4.42}$$

where

$$\begin{aligned} k_l(x, \widehat{u}) =& -b_{l-1}\left(x_{l-1} - \phi_{l-2}(x)\right) - \frac{\lambda}{2}\left(x_l - \phi_{l-1}(x)\right) \\ &+ \frac{\partial \phi_{l-1}}{\partial x} f(x, k(x, \widehat{u})), \;\; \textit{for } l = 1, \cdots, n, \\ \phi_l(x) =& \frac{1}{b_l}\Big[k_l(x) - h_l(x)\Big], \;\; \textit{for } l = 1, \cdots, n-1, \\ \phi_{-1}(x) =& \phi_0(x) = 0 \;\; \forall x \in \mathbb{R}^n, \;\; b_0 = 0, \;\; \textit{and } x_0 = 0, \end{aligned}$$

renders the control system Σ $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$.

Proof: Consider the following system:

$$\Sigma_l: \ \dot{\eta}_l = F_l(\eta_l) + B_l\phi_{l-1}(\eta_l), \text{ for } l = 2, \cdots, n, \tag{4.4.43}$$

where $\eta_l = [\xi_1, \cdots, \xi_{l-1}]^T$, $B_l = [0, \cdots, 0, b_{l-1}]^T \in \mathbb{R}^{l-1}$, $y_l^T = [x_1, \cdots, x_{l-1}]^T$ is the state of Σ_l, and $F_l(\eta_l) = [f_1(\xi, \upsilon), \cdots, f_{l-2}(\xi, \upsilon), h_{l-1}(\xi_1, \cdots, \xi_{l-1})]^T$. As proved in Theorem 4.4.12, the metric G_l in (4.4.33) is a contraction metric, with respect to states, for the system Σ_l and with the contraction rate λ. The proposed control law (4.4.42), transforms a control system of the form (4.4.17) into:

$$\Sigma: \begin{cases} \dot{\eta}_n = F_n(\eta_n) + B_n\xi_n, \\ \dot{\xi}_n = k_n(\eta_n, \xi_n) + \widehat{\upsilon}. \end{cases} \tag{4.4.44}$$

Now, we show that:

$$G_n(y_n) = \begin{bmatrix} G_{n-1}(y_{n-1}) + \left(\frac{\partial\phi_{n-1}}{\partial y_n}\right)^T \frac{\partial\phi_{n-1}}{\partial y_n} & -\left(\frac{\partial\phi_{n-1}}{\partial y_n}\right)^T \\ -\frac{\partial\phi_{n-1}}{\partial y_n} & 1 \end{bmatrix}, \tag{4.4.45}$$

is a contraction metric, with respect to states and inputs, for the control system (4.4.44). For $n = 1$, it can be easily checked that $G_1(y_1) = 1$ is a contraction metric, with respect to states and inputs, with the contraction rate λ, satisfying (4.2.8) with $\alpha = 2$ for the scalar control system:

$$\Sigma: \dot{\xi}_1 = k_1(\xi_1) + \widehat{\upsilon} = -\frac{\lambda}{2}\xi_1 + \widehat{\upsilon}.$$

As proved in Theorem 4.4.12, $G_n(y_n)$ is positive definite. Using the inequality (4.4.35) for $k = n$, long algebraic manipulations in (4.4.46) show that G_n satisfies (4.2.8) with the contraction rate λ and $\alpha = 2$. Hence, the metric G_n is a contraction metric, with respect to states and inputs, for the control system (4.4.44). As explained in the proof of Theorem 4.4.12, we know that $(\mathbb{R}^n, \mathbf{d}_{G_n})$ is a complete metric space. By using Theorem 4.3.11, we conclude that a control system of the form (4.4.17), equipped with the state feedback control law (4.4.42), is $\delta_\exists$-ISS with respect to $\widehat{\upsilon}$. The $\delta_\exists$-ISS condition (2.3.3), as shown in Theorem 4.3.11, is given

$$
\left[Y^T \;\; X\right] \left(\left(\frac{\partial \left[F_n^T + B_n^T x_n \;\; k(y_n, x_n) + \widehat{u}\right]^T}{\partial x} \right)^T G_n(y_n) + G_n(y_n) \frac{\partial \left[F_n^T + B_n^T x_n \;\; k(y_n, x_n) + \widehat{u}\right]^T}{\partial x} + \dot{G}_n(y_n) \right) \begin{bmatrix} Y \\ X \end{bmatrix} +
$$

$$
2U \left(\frac{\partial \left[F_n^T + B_n^T x_n \;\; k(y_n, x_n) + \widehat{u}\right]^T}{\partial \widehat{u}} \right)^T G_n(y_n) \begin{bmatrix} Y \\ X \end{bmatrix} = \left[Y^T \;\; X\right] \cdot \tag{4.4.46}
$$

$$
\left(\begin{bmatrix} \frac{\partial F_n}{\partial y_n} & B_n \\ (F_n + B_n x_n)^T \frac{\partial^2 \phi_{n-1}}{\partial {y_n}^2} + \frac{\partial \phi_{n-1}}{\partial y_n} \frac{\partial F_n}{\partial y_n} + \frac{\lambda}{2} \frac{\partial \phi_{n-1}}{\partial y_n} - B_n^T G_{n-1}(y_{n-1}) & -\frac{\lambda}{2} + \frac{\partial \phi_{n-1}}{\partial y_n} B_n \end{bmatrix}^T \cdot \right.
$$

$$
\begin{bmatrix} G_{n-1}(y_{n-1}) + \left(\frac{\partial \phi_{n-1}}{\partial y_n}\right)^T \frac{\partial \phi_{n-1}}{\partial y_n} & -\left(\frac{\partial \phi_{n-1}}{\partial y_n}\right)^T \\ -\frac{\partial \phi_{n-1}}{\partial y_n} & 1 \end{bmatrix} + \begin{bmatrix} G_{n-1}(y_{n-1}) + \left(\frac{\partial \phi_{n-1}}{\partial y_n}\right)^T \frac{\partial \phi_{n-1}}{\partial y_n} & -\left(\frac{\partial \phi_{n-1}}{\partial y_n}\right)^T \\ -\frac{\partial \phi_{n-1}}{\partial y_n} & 1 \end{bmatrix} \cdot
$$

$$
\begin{bmatrix} \frac{\partial F_n}{\partial y_n} & B_n \\ (F_n + B_n x_n)^T \frac{\partial^2 \phi_{n-1}}{\partial {y_n}^2} + \frac{\partial \phi_{n-1}}{\partial y_n} \frac{\partial F_n}{\partial y_n} + \frac{\lambda}{2} \frac{\partial \phi_{n-1}}{\partial y_n} - B_n^T G_{n-1}(y_{n-1}) & -\frac{\lambda}{2} + \frac{\partial \phi_{n-1}}{\partial y_n} B_n \end{bmatrix} +
$$

$$
\left. \begin{bmatrix} \frac{\partial G_{n-1}}{\partial y_n} (F_n + B_n x_n) + \frac{\partial^2 \phi_{n-1}}{\partial {y_n}^2} (F_n + B_n x_n) \frac{\partial \phi_{n-1}}{\partial y_n} + \left(\frac{\partial \phi_{n-1}}{\partial y_n}\right)^T (F_n + B_n x_n)^T \frac{\partial^2 \phi_{n-1}}{\partial {y_n}^2} & -\frac{\partial^2 \phi_{n-1}}{\partial {y_n}^2} (F_n + B_n x_n) \\ -(F_n + B_n x_n)^T \frac{\partial^2 \phi_{n-1}}{\partial {y_n}^2} & 0 \end{bmatrix} \right) \cdot
$$

$$
\begin{bmatrix} Y \\ X \end{bmatrix} + 2U \left(-\frac{\partial \phi_{n-1}}{\partial y_n} Y + X \right) \leq \left[Y^T \;\; X\right] \cdot
$$

$$
\begin{bmatrix} \left(\left(\frac{\partial (F_n + B_n \phi_{n-1})}{\partial y_n} \right)^T G_{n-1}(y_{n-1}) + G_{n-1}(y_{n-1}) \frac{\partial (F_n + B_n \phi_{n-1})}{\partial y_n} + \frac{\partial G_{n-1}}{\partial y_n} F_n \right) - \lambda \left(\frac{\partial \phi_{n-1}}{\partial y_n}\right)^T \frac{\partial \phi_{n-1}}{\partial y_n} & \lambda \left(\frac{\partial \phi_{n-1}}{\partial y_n}\right)^T \\ \lambda \frac{\partial \phi_{n-1}}{\partial y_n} & -\lambda \end{bmatrix} \cdot
$$

$$
\begin{bmatrix} Y \\ X \end{bmatrix} + 2 \sqrt{\left[Y^T \;\; X\right] G_n(y_n) \begin{bmatrix} Y \\ X \end{bmatrix}} \sqrt{U^2} \leq -\lambda \left[Y^T \;\; X\right] G_n(y_n) \begin{bmatrix} Y \\ X \end{bmatrix} + 2 \sqrt{\left[Y^T \;\; X\right] G_n(y_n) \begin{bmatrix} Y \\ X \end{bmatrix}} \sqrt{U^2}.
$$

by:

$$
\begin{aligned}
\mathbf{d}_{G_n}\left(\xi_{x\widehat{\upsilon}}(t), \xi_{x'\widehat{\upsilon}'}(t)\right) &\leq \mathsf{e}^{-\frac{\lambda}{2}t} \mathbf{d}_{G_n}(x, x') + \frac{2}{\lambda}\left(1 - \mathsf{e}^{-\frac{\lambda}{2}t}\right) \|\widehat{\upsilon} - \widehat{\upsilon}'\|_\infty \\
&\leq \mathsf{e}^{-\frac{\lambda}{2}t} \mathbf{d}_{G_n}(x, x') + \frac{2}{\lambda} \|\widehat{\upsilon} - \widehat{\upsilon}'\|_\infty.
\end{aligned}
$$

Remark 4.4.16. *The contraction metric, with respect to states and inputs, for a control system of the form (4.4.17), equipped with the state feedback control law (4.4.42), is given by (4.4.41).*

Remark 4.4.17. *It can be shown that the function*

$$
V(x) = \frac{1}{2} \sum_{l=0}^{n-1} \left(x_{l+1} - \phi_l(x)\right)^2,
$$

is an input-to-state stability Lyapunov function [Kha96] with respect to $\widehat{\upsilon}$ for a control system of the form (4.4.17), equipped with the state feedback control law (4.4.42).

Now, we extend the results in Theorems 4.4.12 and 4.4.15 to the class of control systems $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ with f of the strict-feedback form in (4.4.25).

In order to extend Theorems 4.4.12, and 4.4.15 to control systems of the form (4.4.25), we need the following technical lemmas.

Lemma 4.4.18. *Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ be a control system and let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a global diffeomorphism. If the metric G is a contraction metric, with respect to states, for $\Sigma' = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, \phi_* f)$ and with contraction rate $\lambda \in \mathbb{R}^+$, then the metric $\phi^* G$ is a contraction metric, with respect to states, for the system Σ and with the contraction rate λ.*

Proof: Since G is a contraction metric, with respect to states, for the system Σ' and with the contraction rate λ, using the inequality (4.2.6), we have:

$$\frac{d}{dt}\langle \delta\eta, \delta\eta \rangle_G \leq -\lambda \langle \delta\eta, \delta\eta \rangle_G, \tag{4.4.47}$$

where $\delta\eta$ is variation of the state trajectory of Σ'. Since G is a metric and $\Theta(x) = \frac{\partial \phi}{\partial x}(x)$ is an invertible matrix[21], it is readily seen that $(\phi^* G)(x)$ is a positive definite matrix. We now show that the metric $\phi^* G$ is a contraction metric, with respect to states, for the system Σ. For the coordinate transformation $\eta = \phi(\xi)$, we have:

$$\delta\eta = \Theta(\xi)\delta\xi. \tag{4.4.48}$$

By taking the derivative of (4.4.48) with respect to time, we obtain:

$$\frac{d}{dt}\delta\eta = \dot{\Theta}(\xi)\delta\xi + \Theta(\xi)\frac{d}{dt}\delta\xi. \tag{4.4.49}$$

[21]For any smooth map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with a smooth inverse, it is easy to show that $\frac{\partial \phi}{\partial x}(x)$ is an invertible matrix for any $x \in \mathbb{R}^n$.

Using (4.4.47), (4.4.48), and (4.4.49), we obtain:

$$\begin{aligned}\frac{d}{dt}\langle\delta\eta,\delta\eta\rangle_G &= \left(\frac{d}{dt}\delta\eta\right)^T G\delta\eta + \delta\eta^T G\frac{d}{dt}\delta\eta + \delta\eta^T\dot{G}\delta\eta \\ &= \left(\dot{\Theta}\delta\xi + \Theta\frac{d}{dt}\delta\xi\right)^T G\Theta\delta\xi + (\Theta\delta\xi)^T G\left(\dot{\Theta}\delta\xi + \Theta\frac{d}{dt}\delta\xi\right) + \delta\eta^T\dot{G}\delta\eta \\ &= \left(\frac{d}{dt}\delta\xi\right)^T \phi^*G\delta\xi + \delta\xi^T\phi^*G\frac{d}{dt}\delta\xi + \delta\xi\frac{d}{dt}\left(\phi^*G\right)\delta\xi = \frac{d}{dt}\langle\delta\xi,\delta\xi\rangle_{\phi^*G} \\ &\leq -\lambda\langle\delta\eta,\delta\eta\rangle_G = -\lambda\delta\eta^T G\delta\eta = -\lambda\left(\Theta\delta\xi\right)^T G\Theta\delta\xi = -\lambda\delta\xi^T\phi^*G\delta\xi \\ &= -\lambda\langle\delta\xi,\delta\xi\rangle_{\phi^*G}.\end{aligned}$$

Hence, the metric ϕ^*G is a contraction metric, with respect to states, for the system Σ and with the contraction rate λ.

In the next lemma, we extend the results of Lemma 4.4.18 to contraction with respect to states and inputs.

Lemma 4.4.19. *Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ be a control system and let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a global diffeomorphism. If the metric G is a contraction metric, with respect to states and inputs, satisfying (4.2.8) with contraction rate $\lambda \in \mathbb{R}^+$, and $\alpha \in \mathbb{R}_0^+$, for $\Sigma' = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, \phi_* f)$, then the metric ϕ^*G is a contraction metric, with respect to states and inputs, satisfying (4.2.8) with the contraction rate λ, and the constant α for the system Σ.*

Proof: Since G is a contraction metric, with respect to states and inputs, for the control system Σ', satisfying (4.2.8) with the contraction rate λ, and $\alpha \in \mathbb{R}_0^+$, using the inequality (4.2.9), we have:

$$\frac{d}{dt}\langle\delta\eta,\delta\eta\rangle_G \leq -\lambda\langle\delta\eta,\delta\eta\rangle_G + \alpha\langle\delta\eta,\delta\eta\rangle_G^{\frac{1}{2}}\langle\delta\upsilon,\delta\upsilon\rangle_{I_m}^{\frac{1}{2}}. \tag{4.4.50}$$

Using (4.4.48), (4.4.49), (4.4.50), and the results of Lemma 4.4.18, we obtain:

$$\begin{aligned}\frac{d}{dt}\langle\delta\eta,\delta\eta\rangle_G =& \frac{d}{dt}\langle\delta\xi,\delta\xi\rangle_{\phi^*G}\\ \leq& -\lambda\langle\delta\eta,\delta\eta\rangle_G + \alpha\langle\delta\eta,\delta\eta\rangle_G^{\frac{1}{2}}\langle\delta\upsilon,\delta\upsilon\rangle_{I_m}^{\frac{1}{2}}\\ =& -\lambda\langle\delta\xi,\delta\xi\rangle_{\phi^*G} + \alpha\left(\left(\Theta\delta\xi\right)^T G\Theta\delta\xi\right)^{\frac{1}{2}}\langle\delta\upsilon,\delta\upsilon\rangle_{I_m}^{\frac{1}{2}}\\ =& -\lambda\langle\delta\xi,\delta\xi\rangle_{\phi^*G} + \alpha\langle\delta\xi,\delta\xi\rangle_{\phi^*G}^{\frac{1}{2}}\langle\delta\upsilon,\delta\upsilon\rangle_{I_m}^{\frac{1}{2}}.\end{aligned}$$

Hence, the metric ϕ^*G is a contraction metric, with respect to states and inputs, satisfying (4.2.8) with the contraction rate λ, and the constant α, for Σ.

We can now state the result for a control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ with f of the form (4.4.25).

Theorem 4.4.20. *Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ be a control system where f is of the form (4.4.25). The state feedback control law $u = k(\varphi(x), \widehat{u})$, where k was defined in (4.4.31) and $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is the smooth map (with smooth inverse), defined in (4.4.26), renders control system Σ $\delta_\exists$-GAS.*

Proof: The coordinate transformation $\eta = \varphi(\xi)$ transforms the control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ with f of the form (4.4.25) to the control system $\Sigma' = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f')$, where $f' = \varphi_* f$. It can be easily checked that f' has the following form:

$$\begin{aligned} f_1'(y,u) &= h_1'(y_1) + y_2,\\ f_2'(y,u) &= h_2'(y_1,y_2) + y_3,\\ &\vdots \\ f_{n-1}'(y,u) &= h_{n-1}'(y_1,\cdots,y_{n-1}) + y_n,\\ f_n'(y,u) &= h_n'(y) + g'(y)u, \end{aligned} \tag{4.4.51}$$

where $h_i' : \mathbb{R}^i \to \mathbb{R}$, for $i = 1, \cdots, n$, are smooth functions, $g' = \prod_{i=1}^{i=n} g_i$, and $y \in \mathbb{R}^n$ is the state of Σ'. As proved in Theorem 4.4.12, the state feedback control law k, defined in (4.4.31), makes the metric G_n, defined in (4.4.41), a contraction metric, with respect to states, for the control system Σ' and with the contraction rate λ. As proved in Lemma 4.4.18, the metric $\varphi^* G_n$ is a contraction metric, with respect to states, with the contraction rate λ, for the control system Σ, equipped with the state feedback control law $k(\varphi(x), \widehat{u})$. Since $(\mathbb{R}^n, \mathbf{d}_{G_n})$ is a complete metric space and φ is an isometry, $(\mathbb{R}^n, \mathbf{d}_{\varphi^* G_n})$ is also a complete metric space [Lee03]. Therefore, the state feedback control law $k(\varphi(x), \widehat{u})$ makes the control system Σ $\delta_\exists$-GAS.

The $\delta_\exists$-ISS version of Theorem 4.4.20 is given by the following result.

Theorem 4.4.21. *Let $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ be a control system where f is of the form (4.4.25). The state feedback control law $u = k(\varphi(x), \widehat{u})$, where k and φ were defined in (4.4.42) and (4.4.26), respectively, renders control system Σ $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$.*

Proof: By following the same steps as in the proof of Theorem 4.4.20, and using Lemma 4.4.19, we obtain that the state feedback control law $u = k(\varphi(x), \widehat{u})$ makes the metric $\varphi^* G_n$ a contraction metric, with respect to states and inputs, for Σ and with the contraction rate λ. Hence, the control system Σ, equipped with the state feedback control law $k(\varphi(x), \widehat{u})$, is $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$.

Remark 4.4.22. *Although we only discussed single input control systems, extensions to multi input control systems are straightforward using the techniques in [KKK95].*

4.5 Examples

We illustrate the results in this chapter on several examples.

4.5.1 Non-smooth control system

We illustrate the results in Theorem 4.4.2 and Lemma 4.4.5 on the following non-smooth control system:

$$\Sigma : \begin{cases} \dot{\eta}_1 = -\eta_1 + |\eta_3|, \\ \dot{\eta}_2 = -\eta_2 + \mathsf{sat}(\eta_1)\mathsf{sat}(\eta_3), \\ \dot{\eta}_3 = \eta_3 + \eta_1\eta_2\eta_3 + \eta_3^3 + \zeta_1, \\ \dot{\zeta}_1 = \zeta_1 + \zeta_1^3 + \eta_3^2 + \upsilon, \end{cases} \tag{4.5.1}$$

where $\mathsf{sat} : \mathbb{R} \to \mathbb{R}$ is the saturation function, defined by:

$$\mathsf{sat}(x) = \begin{cases} -1 & \text{if } x < -1, \\ x & \text{if } |x| \le 1, \\ 1 & \text{if } x > 1. \end{cases} \tag{4.5.2}$$

It can be readily verified that Σ is unstable at $(0, 0, 0, 0)$, implying that Σ is not $\delta_\exists$-ISS. It can be checked that the results in [JL02, SK09, SK08, PvdWN05] can not be applied to design controllers that render the system Σ $\delta_\exists$-ISS. By introducing the feedback transformation $\widehat{\upsilon} = \zeta_1 + \zeta_1^3 + \eta_3^2 + \upsilon$, the control system Σ is transformed into:

$$\widehat{\Sigma} : \begin{cases} \dot{\eta}_1 = -\eta_1 + |\eta_3|, \\ \dot{\eta}_2 = -\eta_2 + \mathsf{sat}(\eta_1)\mathsf{sat}(\eta_3), \\ \dot{\eta}_3 = \eta_3 + \eta_1\eta_2\eta_3 + \eta_3^3 + \zeta_1, \\ \dot{\zeta}_1 = \widehat{\upsilon}. \end{cases} \tag{4.5.3}$$

Now by choosing $\psi(\eta) = -4\eta_3 - \eta_1\eta_2\eta_3 - \eta_3^3$, and substituting $\psi(\eta) + \widetilde{\upsilon}$ instead of ζ_1, we obtain the following η-subsystem:

$$\widehat{\Sigma}_\eta : \begin{cases} \dot{\eta}_1 = -\eta_1 + |\eta_3|, \\ \dot{\eta}_2 = -\eta_2 + \mathsf{sat}(\eta_1)\mathsf{sat}(\eta_3), \\ \dot{\eta}_3 = \eta_3 + \eta_1\eta_2\eta_3 + \eta_3^3 + \psi(\eta) + \widetilde{\upsilon} = -3\eta_3 + \widetilde{\upsilon}. \end{cases} \tag{4.5.4}$$

It remains to show that $\widehat{\Sigma}_\eta$ is $\delta_\exists$-ISS with respect to $\widetilde{\upsilon}$. First we show that the control system:

$$\widehat{\Sigma}_1 : \dot{\eta}_1 = -\eta_1 + |\eta_3|,$$

is $\delta_\exists$-ISS with respect to the input η_3. By choosing the function $V_1(y_1, y_1') = (y_1 - y_1')^2$, where y_1 and y_1' are states of $\widehat{\Sigma}_1$, and using the Cauchy Schwarz inequality, we have that:

$$\begin{aligned} \frac{\partial V_1}{\partial y_1}(-y_1 + |y_3|) + \frac{\partial V_1}{\partial y_1'}(-y_1' + |y_3'|) &\leq -2(y_1 - y_1')^2 + 2|y_1 - y_1'||y_3 - y_3'| \\ &\leq -V_1(y_1, y_1') + |y_3 - y_3'|^2, \end{aligned}$$

showing that V_1 is a $\delta_\exists$-ISS Lyapunov function for $\widehat{\Sigma}_1$ and, hence, $\widehat{\Sigma}_1$ is $\delta_\exists$-ISS with respect to the input η_3. Now we show that the control system:

$$\widehat{\Sigma}_2 : \dot{\eta}_2 = -\eta_2 + \mathsf{sat}(\eta_1)\mathsf{sat}(\eta_3),$$

is $\delta_\exists$-ISS with respect to inputs η_1 and η_3. Similarly, by choosing the function $V_2(y_2, y_2') = (y_2 - y_2')^2$, where y_2 and y_2' are states of $\widehat{\Sigma}_2$, and using the Cauchy Schwarz inequality, we have that:

$$\begin{aligned} &\frac{\partial V_2}{\partial y_2}(-y_2 + \mathsf{sat}(y_1)\mathsf{sat}(y_3)) + \frac{\partial V_2}{\partial y_2'}(-y_2' + \mathsf{sat}(y_1')\mathsf{sat}(y_3')) \leq \\ &-2(y_2 - y_2')^2 + 2|y_2 - y_2'||\mathsf{sat}(y_1)\mathsf{sat}(y_3) - \mathsf{sat}(y_1')\mathsf{sat}(y_3')| = \\ &-2(y_2 - y_2')^2 + 2|y_2 - y_2'||\mathsf{sat}(y_1)\mathsf{sat}(y_3) - \mathsf{sat}(y_1)\mathsf{sat}(y_3') + \mathsf{sat}(y_1)\mathsf{sat}(y_3') - \mathsf{sat}(y_1')\mathsf{sat}(y_3')| \leq \\ &-2(y_2 - y_2')^2 + 2|y_2 - y_2'|\left(|\mathsf{sat}(y_1)||\mathsf{sat}(y_3) - \mathsf{sat}(y_3')| + |\mathsf{sat}(y_3')||\mathsf{sat}(y_1) - \mathsf{sat}(y_1')|\right) \leq \\ &-2(y_2 - y_2')^2 + 2|y_2 - y_2'|\left(|y_3 - y_3'| + |y_1 - y_1'|\right) \leq \\ &-V_2(y_2, y_2') + \left(|y_1 - y_1'| + |y_3 - y_3'|\right)^2, \end{aligned}$$

showing that V_2 is a $\delta_\exists$-ISS Lyapunov function for $\widehat{\Sigma}_2$ and, hence, $\widehat{\Sigma}_2$ is $\delta_\exists$-ISS with respect to inputs η_1 and η_3. Similarly, it can be shown that the control system:

$$\widehat{\Sigma}_3 : \dot{\eta}_3 = -3\eta_3 + \widetilde{\upsilon},$$

is $\delta_\exists$-ISS with respect to $\widetilde{\upsilon}$. By virtue of Lemma 4.4.1, we conclude that the control system $\widehat{\Sigma}_\eta$ is $\delta_\exists$-ISS with respect to $\widetilde{\upsilon}$. By using the results in Theorem

4.4.2 for the control system $\widehat{\Sigma}$, we conclude that the state feedback control law:

$$
\begin{aligned}
\widehat{\upsilon} = k(\eta, \zeta, \bar{\upsilon}) =& -\lambda(\zeta_1 - \psi(\eta)) + \frac{\partial \psi}{\partial y}\dot{\eta} + \bar{\upsilon} \\
=& -\lambda\left(\zeta_1 + 4\eta_3 + \eta_1\eta_2\eta_3 + \eta_3^3\right) \\
&+ \left(\eta_3 + \eta_1\eta_2\eta_3 + \eta_3^3 + \zeta_1\right)\left(-4 - \eta_1\eta_2 - 3\eta_3^2\right) \\
&- \eta_2\eta_3\left(-\eta_1 + |\eta_3|\right) - \eta_1\eta_3\left(-\eta_2 + \mathsf{sat}(\eta_1)\mathsf{sat}(\eta_3)\right) + \bar{\upsilon},
\end{aligned} \tag{4.5.5}
$$

makes the control system $\widehat{\Sigma}$ $\delta_\exists$-ISS with respect to input $\bar{\upsilon}$, for any $\lambda \in \mathbb{R}^+$. Therefore, the state feedback control law

$$
\upsilon = \widehat{k}(\eta, \zeta, \bar{\upsilon}) = k(\eta, \zeta, \bar{\upsilon}) - \zeta_1 - \zeta_1^3 - \eta_3^2, \tag{4.5.6}
$$

makes the control system Σ $\delta_\exists$-ISS with respect to input $\bar{\upsilon}$.

Let us now aim for finding a $\delta_\exists$-ISS Lyapunov function $V : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}_0^+$ for the control system Σ equipped with the state feedback control law $\widehat{k}$. First we show that the function $\widehat{V}(y, y') = (y_1 - y_1')^2 + (y_2 - y_2')^2 + (y_3 - y_3')^2$ is a $\delta_\exists$-ISS Lyapunov function for the control system $\widehat{\Sigma}_\eta$. It can be easily seen that the function $\widehat{V}$ satisfies the condition (i) in Definition 4.2.1. Using the Cauchy

Schwarz inequality, we have that:

$$
\begin{aligned}
&\frac{\partial \widehat{V}}{\partial y} f(y, \psi(y) + \widetilde{u}) + \frac{\partial \widehat{V}}{\partial y'} f(y', \psi(y') + \widetilde{u}') = \\
&2\left(y_1 - y_1'\right)\left(-y_1 + |y_3| + y_1' - |y_3'|\right) + \\
&2\left(y_2 - y_2'\right)\left(-y_2 + \mathsf{sat}(y_1)\mathsf{sat}(y_3) + y_2' - \mathsf{sat}(y_1')\mathsf{sat}(y_3')\right) + \\
&2\left(y_3 - y_3'\right)\left(-3y_3 + \widetilde{u} + 3y_3' - \widetilde{u}\right) \leq \\
&-2\left(y_1 - y_1'\right)^2 - 2\left(y_2 - y_2'\right)^2 - 6\left(y_3 - y_3'\right)^2 + \\
&2|y_1 - y_1'||y_3 - y_3'| + 2|y_2 - y_2'|\left(|y_1 - y_1'| + |y_3 - y_3'|\right) + 2|y_3 - y_3'||\widetilde{u} - \widetilde{u}'| \leq \\
&-\frac{1}{2}\left(y_1 - y_1'\right)^2 - \frac{1}{2}\left(y_2 - y_2'\right)^2 - 2\left(y_3 - y_3'\right)^2 + \\
&-\left(\frac{1}{\sqrt{2}}|y_1 - y_1'| - \sqrt{2}|y_3 - y_3'|\right)^2 - \left(\frac{1}{\sqrt{2}}|y_2 - y_2'| - \sqrt{2}|y_3 - y_3'|\right)^2 + \\
&-\left(|y_1 - y_1'| - |y_2 - y_2'|\right)^2 + 2|y_3 - y_3'||\widetilde{u} - \widetilde{u}'| \leq \\
&-\frac{1}{2}\widehat{V}(y, y') + |\widetilde{u} - \widetilde{u}'|^2.
\end{aligned}
$$

Therefore, the function $\widehat{V}$ satisfies the condition (iii) in Definition 4.2.1. Hence $\widehat{V}$ is a $\delta_\exists$-ISS Lyapunov function for $\widehat{\Sigma}_\eta$. Using Lemma 4.4.5, we conclude that the function:

$$
\begin{aligned}
V(x, x') =& \widehat{V}(y, y') + |\left(z_1 - \psi(y)\right) - \left(z_1' - \psi(y')\right)|^2 \qquad (4.5.7) \\
=& (y_1 - y_1')^2 + (y_2 - y_2')^2 \\
&+ (y_3 - y_3')^2 + \left((z_1 + 4y_3 + y_1 y_2 y_3 + y_3^3) - (z_1' + 4y_3' + y_1' y_2' y_3' + y_3'^3)\right)^2,
\end{aligned}
$$

where $x = [y_1, y_2, y_3, z_1]^T$ is the state of Σ, is a $\delta_\exists$-ISS Lyapunov function for the control system Σ equipped with the state feedback control law $\widehat{k}$ in (4.5.6) with $\lambda > \frac{5}{4}$.

We simulate the closed-loop system with $\lambda = 5$ and $\bar{\upsilon}(t) = 10\sin(2t)$, for any $t \in \mathbb{R}_0^+$. In Figure 4.1, we show the closed-loop trajectories stemming from the initial conditions $(1, 1, 1, 1)$ and $(-1, -1, -1, -1)$, respectively.

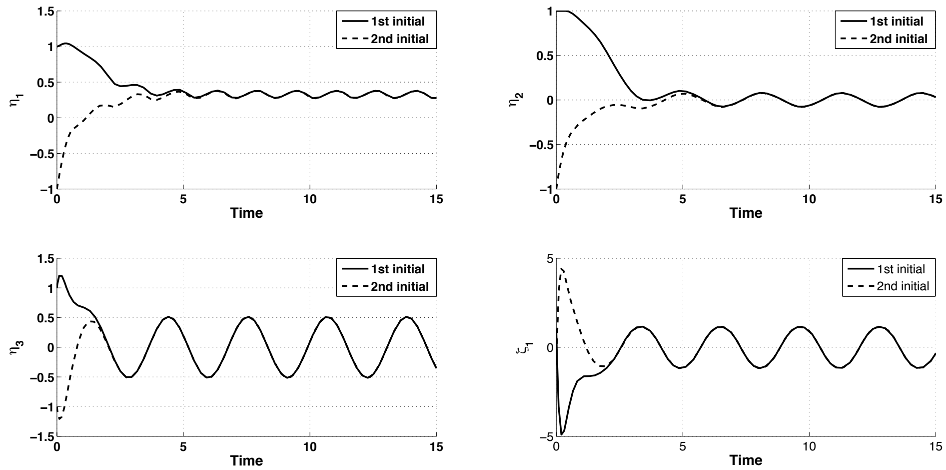

Figure 4.1: Evolution of η_1, η_2, η_3, and ζ_1 with the input $\bar{v}(t) = 10\sin(2t)$ and initial conditions $(1,\ 1,\ 1,\ 1)$, and $(-1,\ -1,\ -1,\ -1)$, respectively.

4.5.2 Single-machine infinite-bus electrical power system

Here, we illustrate the results in Theorem 4.4.9. Consider a single-machine infinite-bus electrical power system with static VAR compensator [SL10]. The control system $\Sigma = (\mathbb{R}^3, \mathsf{U}, \mathcal{U}, f)$ with f of the form:

$$\begin{aligned} f_1(x,u) =& x_2, \\ f_2(x,u) =& -\frac{\omega_0}{H} E'_q V_s y_{svc0} \sin(x_1+\delta_0) - \frac{D}{H} x_2 + \frac{\omega_0}{H} P_m - \frac{\omega_0}{H} E'_q V_s \sin(x_1+\delta_0) x_3, \\ f_3(x,u) =& -\frac{1}{T_{svc}} x_3 + \frac{1}{T_{svc}} u, \end{aligned} \tag{4.5.8}$$

models a single-machine infinite-bus (SIMB) electrical power system with static VAR compensator (SVC). In the mentioned model, x_1 is the deviation of the generator rotor angle, x_2 is the relative speed of the rotor of the generator, x_3 is the deviation of the susceptance of the overall system, δ_0 is the operating point of the generator rotor angle, ω_0 is the operating point of the speed of the generator rotor, H is the inertia constant, P_m is the mechanical power on the generator shaft, D is the damping coefficient, E'_q is the inner generator voltage, V_s is the infinite bus voltage, y_{svc0} is the operating point of the susceptance of the overall system, T_{svc} is the time constant of SVC regulator, and u is the input of SVC regulator. We assume that $\sin(x_1+\delta_0)$ is nonzero over the domain of the interest.

The control system (4.5.8) is of the form (4.4.25). The coordinate transformation (4.4.26), given by:

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \varphi(\xi) = \begin{bmatrix} \xi_1 \\ \xi_2 \\ -\frac{\omega_0}{H} E_q' V_s \sin(\delta_0+\xi_1)\xi_3 \end{bmatrix}, \tag{4.5.9}$$

transforms the control system $\Sigma = (\mathbb{R}^3, \mathsf{U}, \mathcal{U}, f)$ to the control system $\Sigma' = (\mathbb{R}^3, \mathsf{U}, \mathcal{U}, f')$ with $f' = \varphi_* f$ of the form:

$$\begin{aligned} f_1'(y,u) =& h_1'(y_1) + y_2 = y_2, \\ f_2'(y,u) =& h_2'(y_1,y_2) + y_3 = -\frac{D}{H}y_2 + \frac{\omega_0}{H}P_m - \frac{\omega_0}{H}E_q'V_s y_{svc0}\sin(y_1+\delta_0) + y_3, \\ f_3'(y,u) =& h_3'(y) + g'(y)u = y_2\cot(y_1+\delta_0)y_3 - \frac{1}{T_{svc}}y_3 - \frac{\omega_0}{HT_{svc}}E_q'V_s\sin(y_1+\delta_0)u. \end{aligned} \tag{4.5.10}$$

By using the results in Theorem 4.4.7 for a control system of the form (4.5.10) and for $\lambda = 1$, we have:

$$\begin{aligned} \phi_1(y_1) =& -y_1, \\ \phi_2(y_1,y_2) =& -2y_1 + \frac{\omega_0}{H}E_q'V_s y_{svc0}\sin(y_1+\delta_0) - \frac{\omega_0}{H}P_m + \left(\frac{D}{H} - 2\right)y_2, \\ k_3(y) =& -3y_1 + \left(\frac{D}{H} - 3\right)y_3 \\ &+ \left(\frac{D}{H} - 3\right)\frac{\omega_0}{H}P_m + \left(3 - \frac{D}{H}\right)\frac{\omega_0}{H}E_q'V_s y_{svc0}\sin(y_1+\delta_0) \\ &+ \frac{\omega_0}{H}E_q'V_s y_{svc0}\cos(y_1+\delta_0)y_2 + \left(3\frac{D}{H} - \frac{D^2}{H^2} - 5\right)y_2. \end{aligned}$$

Therefore, the state feedback control law:

$$
\begin{aligned}
k(y,\widehat{u}) =& \frac{1}{g'(y)}\left[k_3(y) - h_3'(y)\right] + \frac{1}{g'(y)}\widehat{u} \qquad (4.5.11)\\
=& -\frac{HT_{svc}}{\omega_0 E_q' V_s \sin(\delta_0 + y_1)}\Bigg[-3y_1 + \left(\frac{D}{H} - 3 + \frac{1}{T_{svc}}\right)y_3\\
&+\left(3-\frac{D}{H}\right)\frac{\omega_0}{H}E_q'V_s y_{svc0}\sin(y_1+\delta_0) + \frac{\omega_0}{H}E_q'V_s y_{svc0}\cos(y_1+\delta_0)y_2\\
&+\left(3\frac{D}{H} - \frac{D^2}{H^2} - 5\right)y_2 + \left(\frac{D}{H} - 3\right)\frac{\omega_0}{H}P_m - y_2\cot(y_1+\delta_0)y_3\Bigg]\\
&-\frac{HT_{svc}\widehat{u}}{\omega_0 E_q' V_s \sin(\delta_0 + y_1)},
\end{aligned}
$$

makes the control system Σ' $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$. The corresponding $\delta_\exists$-ISS Lyapunov function for the control system (4.5.10) is given by:

$$
\begin{aligned}
\widehat{V}(y,y') =& \Bigg[(y_1 - y_1')^2 + ((y_1 - y_1') + (y_2 - y_2'))^2 + \Bigg[(y_3 - y_3') + \left(2 - \frac{D}{H}\right)(y_2 - y_2')\\
&+ 2(y_1 - y_1') - \frac{\omega_0}{H}E_q'V_s y_{svc0}(\sin(y_1+\delta_0) - \sin(y_1'+\delta_0))\Bigg]^2\Bigg]^{\frac{1}{2}}.
\end{aligned}
$$

By using Theorem 4.4.9, the state feedback control law (4.5.11), and the coordinate transformation (4.5.9), we obtain the state feedback control law $k(\varphi(x),\widehat{u})$ making Σ $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$. The corresponding $\delta_\exists$-ISS Lyapunov function for the control system Σ is given by:

$$
\begin{aligned}
\widetilde{V}(x,x') =& \Bigg[(x_1 - x_1')^2 + ((x_1 - x_1') + (x_2 - x_2'))^2 + \Bigg[-\frac{\omega_0}{H}E_q'V_s\Bigg(\sin(\delta_0 + x_1)x_3\\
&-\sin(\delta_0 + x_1')x_3'\Bigg) + \left(2 - \frac{D}{H}\right)(x_2 - x_2') + 2(x_1 - x_1')\\
&-\frac{\omega_0}{H}E_q'V_s y_{svc0}(\sin(x_1+\delta_0) - \sin(x_1'+\delta_0))\Bigg]^2\Bigg]^{\frac{1}{2}}.
\end{aligned}
$$

4.5.3 Synchronous generator

We illustrate the results in Theorem 4.4.21 on a synchronous generator. Consider a synchronous generator [RGHS01] connected through a transmission line to an

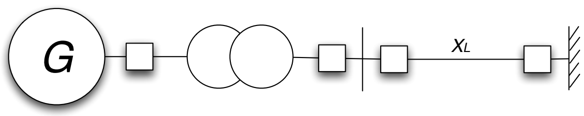

Figure 4.2: Generator connected through a transmission line to an infinite bus.

infinite bus, as shown in Figure 4.2. The control system $\Sigma = (\mathbb{R}^3, \mathsf{U}, \mathcal{U}, f)$ with f of the form:

$$\begin{aligned} f_1(x,u) &= h_1(x_1) + g_1(x_1)x_2, \\ f_2(x,u) &= h_2(x_1,x_2) + g_2(x_1,x_2)x_3, \\ f_3(x,u) &= h_3(x) + g_3(x)u, \end{aligned} \tag{4.5.12}$$

where

$$\begin{aligned} h_1(x_1) =&0, \\ g_1(x_1) =&1, \\ h_2(x_1,x_2) =& -Ex_2 + FP_{m0} + V_s G e_{q0} \sin(\delta_0 + x_1), \\ g_2(x_1,x_2) =& V_s G \sin(\delta_0 + x_1), \\ h_3(x) =& -Ix_3 + JV_s \sin(\delta_0 + x_1)x_2 - Ie_{q0}, \\ g_3(x) =& IK_c, \end{aligned}$$

models a synchronous generator connected to an infinite bus. In the aforementioned model, x_1 is the deviation of the power angle, x_2 is the relative speed of the rotor of the generator, x_3 is the deviation of the quadrature axis voltage of the generator, δ_0 is the operating point of the power angle, P_{m0} is the operating point of the mechanical input power, e_{q0} is the operating point of the quadrature axis voltage of the generator, V_s is the infinite bus voltage, K_c is the gain of the excitation amplifier, and u is the input of the silicon-controlled rectifier amplifier of the generator. Other parameters in (4.5.12) are given by: $E = \frac{D}{2H}$, $I = \frac{1}{T'}$, $F = \frac{\omega_0}{2H}$, $G = -\frac{\omega_0}{2H}\frac{1}{X_{qs}}$, and $J = \frac{X_q - X_d'}{X_{ds}'}$, where D is the per-unit damping constant, H is the inertia constant, ω_0 is the synchronous generator speed,

$T' = \frac{X'_{ds}}{X_{qs}} T'_{d0}$, $X_{qs} = X_T + \frac{1}{2}X_L + X_q$, $X'_{ds} = X_T + \frac{1}{2}X_L + X'_d$, T'_{d0} is the direct axis transient short-circuit time constant, X_T is the reactance of the transformer, X_q is the quadrature axis reactance, X'_d is the direct axis transient reactance and X_L is the reactance of the transmission line. We assume that $\sin(\delta_0 + x_1)$ is nonzero over the domain of the interest.

The control system (4.5.12) is of the form (4.4.25). The coordinate transformation (4.4.26), given by:

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \varphi(\xi) = \begin{bmatrix} \xi_1 \\ \xi_2 \\ V_s G \sin(\delta_0 + \xi_1)\xi_3 \end{bmatrix}, \tag{4.5.13}$$

transforms the control system $\Sigma = (\mathbb{R}^3, \mathsf{U}, \mathcal{U}, f)$ to the control system $\Sigma' = (\mathbb{R}^3, \mathsf{U}, \mathcal{U}, f')$ with $f' = \varphi_* f$ of the form:

$$\begin{aligned} f'_1(y,u) &= h'_1(y_1) + y_2, \\ f'_2(y,u) &= h'_2(y_1, y_2) + y_3, \\ f'_3(y,u) &= h'_3(y) + g'_3(y)u, \end{aligned} \tag{4.5.14}$$

where

$$\begin{aligned} h'_1(y_1) =& 0, \\ h'_2(y_1, y_2) =& - E y_2 + F P_{m0} + V_s G e_{q0} \sin(\delta_0 + y_1), \\ h'_3(y) =& - I V_s G e_{q0} \sin(\delta_0 + y_1) + J V_s^2 G \sin^2(\delta_0 + y_1) y_2 - I y_3 + \cot(\delta_0 + y_1) y_2 y_3, \\ g'_3(y) =& I K_c V_s G \sin(\delta_0 + y_1). \end{aligned}$$

By using the results in Theorem 4.4.15 for the control system of the form (4.5.14) and for $\lambda = 2$, we have:

$$\begin{aligned} \phi_1(\eta_1) =& - \eta_1, \\ \phi_2(\eta_1, \eta_2) =& - 2\eta_1 + (E-2)\,\eta_2 - F P_{m0} - V_s G e_{q0} \sin(\delta_0 + \eta_1), \\ k_3(\eta) =& \left(-5 + 3E - E^2\right)\eta_2 - 3\eta_1 + (E-3)\,\eta_3 + (E-3)\,F P_{m0} \\ &+ (E-3)\,V_s G e_{q0} \sin(\delta_0 + \eta_1) - V_s G e_{q0} \cos(\delta_0 + \eta_1)\eta_2. \end{aligned}$$

Therefore, the state feedback control law:

$$\begin{aligned} k(\eta,\widehat{\upsilon}) &= \frac{1}{g_3'(\eta)}\left[k_3(\eta)-h_3'(\eta)\right]+\frac{1}{g_3'(\eta)}\widehat{\upsilon} \qquad (4.5.15)\\ &= \frac{1}{IK_cV_sG\sin(\delta_0+\eta_1)}\Big[\left(-5+3E-E^2\right)\eta_2-3\eta_1+\left(E-3+I\right)\eta_3\\ &\quad +\left(E-3\right)FP_{m0}+\left(E-3+I\right)V_sGe_{q0}\sin(\delta_0+\eta_1)\\ &\quad -V_sGe_{q0}\cos(\delta_0+\eta_1)\eta_2-JV_s^2G\sin^2(\delta_0+\eta_1)\eta_2\\ &\quad -\cot(\delta_0+\eta_1)\eta_2\eta_3\Big]+\frac{\widehat{\upsilon}}{IK_cV_sG\sin(\delta_0+\eta_1)}, \end{aligned}$$

makes the control system Σ' δ-ISS with respect to the input $\widehat{\upsilon}$. The corresponding contraction metric for the control system (4.5.14) is given by:

$$G(y)=\begin{bmatrix} \begin{bmatrix} 1+\left(\frac{\partial\phi_1}{\partial y_1}\right)^T\frac{\partial\phi_1}{\partial y_1} & -\left(\frac{\partial\phi_1}{\partial y_1}\right)^T\\ -\frac{\partial\phi_1}{\partial y_1} & 1 \end{bmatrix}+\left(\frac{\partial\phi_2}{\partial z_2}\right)^T\frac{\partial\phi_2}{\partial z_2} & -\left(\frac{\partial\phi_2}{\partial z_2}\right)^T\\ -\frac{\partial\phi_2}{\partial z_2} & 1 \end{bmatrix}=$$

$$\begin{bmatrix} 2+\left(2+V_sGe_{q0}\cos(\delta_0+y_1)\right)^2 & -2E+5-\left(E-2\right)V_sGe_{q0}\cos(\delta_0+y_1) & 2+V_sGe_{q0}\cos(\delta_0+y_1)\\ -2E+5-\left(E-2\right)V_sGe_{q0}\cos(\delta_0+y_1) & (E-2)^2+1 & 2-E\\ 2+V_sGe_{q0}\cos(\delta_0+y_1) & 2-E & 1 \end{bmatrix},$$

where $z_2^T=[y_1,\ y_2]^T$. By using Theorem 4.4.21, the state feedback control law (4.5.15), and the coordinate transformation (4.5.13), we have the state feedback control law $k(\varphi(x),\widehat{u})$ making Σ δ-ISS with respect to the input $\widehat{\upsilon}$. The corresponding contraction metric for the control system Σ is given by:

$$\left(\varphi^*G\right)(x)=\Theta^T(x)G(\varphi(x))\Theta(x),$$

where

$$\Theta(x)=\frac{\partial\varphi}{\partial x}(x)=\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ V_sG\cos(\delta_0+x_1)x_3 & 0 & V_sG\sin(\delta_0+x_1) \end{bmatrix}. \qquad (4.5.16)$$

Since the map φ does not transform the first and the second coordinate and the metric G is only function of the first coordinate, we have $G(\varphi(x))=G(x)$.

4.6 Discussion

In this chapter we developed Lyapunov characterizations and contraction metric description of $\delta_\exists$-GAS and $\delta_\exists$-ISS. Moreover, we developed a backstepping procedure to design controllers enforcing incremental input-to-state stability (or contraction properties) for the resulting closed-loop system. The proposed approach in this paper generalizes the work in [JL02, SK09, SK08] by being applicable to larger classes of control systems and the work in [PvdWN05] by enforcing incremental input-to-state stability rather than input-to-state convergence. Moreover, in contrast to the proposed backstepping design approach in [PvdWN05], here we provided a way of constructing incremental Lyapunov functions, which are known to be a key tool in the analysis provided in [GPT09, Gir05, JFA+07, KDL+08].

CHAPTER 5

Conclusion and future contributions

In this thesis I have studied control problems for embedded control systems using incremental properties of control systems. First, in Chapter 3, I addressed constructions of equivalent finite symbolic abstractions for some classes of nonlinear control systems. These abstractions, with finite number of states, simplify the design of controllers satisfying specifications, difficult to enforce with conventional control design methods, such as logic specifications expressed in linear temporal logic or automata on infinite strings. Similar finite abstractions are used in software and hardware modeling, which enables the composition of such abstractions with the finite abstraction of the nonlinear control system. The result of this composition are finite abstractions capturing the behavior of the nonlinear control system interact with the digital computation devices. Once such abstractions are available, the methodologies and tools developed in computer science for verification and control synthesis purposes can be easily employed to control systems, via these abstractions. In Chapter 4, I studied the incremental properties under which control systems admit finite symbolic abstractions. I also provided the characterizations (descriptions) of those incremental properties in terms of existence of incremental Lyapunov functions (contraction metrics). Furthermore, I provided some analysis and design approaches providing controllers enforcing those incremental properties on nonlinear controlled systems. Rather than applications of those properties in the synthesis of finite abstraction, they can also be beneficial in global synchronization in networks of cyclic feedback systems

[HSSG12], control reconfiguration of piecewise affine systems with actuator and sensor faults [RHvdWL11], intrinsic observer design [AR03], consensus problems in complex networks [WS05], output regulation of nonlinear systems [PvdWN05], design of frequency estimators [SK08], synchronization of coupled identical dynamical systems [RdBS09], the analysis of bio-molecular systems [RdB09], and so on.

Switched control systems are a class of hybrid control systems often appearing in embedded control applications. One can use the results in [GPT09] to construct finite abstractions for the switched control systems. However, the extension to the general class of hybrid systems is still open and further research is required. The main difficulty lies in deducing, from the entrance of a single trajectory in a guard set, the entrance of the surrounding trajectories in the same guard set. The exception of switched systems is easy to clarify because for this class of hybrid systems the guards match with the invariant sets.

The current available symbolic abstraction methods only apply to small dimensional systems or restricted classes of dynamics. To use symbolic models for systems of larger size, it is necessary to obtain abstractions with reduced complexity. The first approach to be explored is the use of adaptive multi-scale abstractions. The existing results include the use of nonuniform grid with fixed sampling time [TI09] and the use of uniform grid with varying sampling time [CGG11]. Hence, one possible extension is by using nonuniform grid and varying sampling time. The second approach to deal with reduced complexity of symbolic models is the use of compositional methods for systems defined as an interconnection of subsystems. The initial results in [KS10] provide compositional analysis for linear control systems. The idea is to compute symbolic models for each subsystem, to synthesize local controllers at the subsystem level and to ensure coordination at the global level to control the original overall system. The third approach is by integrating the design of controllers with the construction of symbolic models, such as the results in [PBD12] for incrementally input-to-state stable nonlinear control systems.

While construction of finite abstractions have been recently studied extensively for deterministic nonlinear control systems, they received much less attentions for stochastic control systems. The existing results on stochastic control systems include constructions of finite bisimilar abstractions for probabilistically incrementally globally asymptotically stable dynamical systems [Aba09], for stochastic hybrid dynamical systems with some given ergodic properties [ADD11], and for any stochastic dynamical systems, restricted in some finite time-horizon, [AP11] and constructions of infinite bisimilar abstractions for some classes of stochastic control systems [JP09]. Although the works in [Aba09, ADD11, AP11] provide finite bisimilar abstractions, they only deal with dynamical systems rather than control systems. Hence, they can only be applied for verification purposes rather than synthesis of controllers enforcing complex specifications. To the best of our knowledge, there is no work on the construction of finite bisimilar abstraction for stochastic control systems that are left for future research.

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