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Abstract

In infinite-date models the received definitions of the payoffs of finite portfolio strategies imply discontinuous valuation. Accordingly, in the absence of trading restrictions, arbitrage results when infinite trading strategies are admitted. We propose an alternative that is free of these problems. The alternative produces a cleaner, if more abstract, treatment of equilibrium in financial models in infinite-date settings. We consider the bearing of the revised treatment on the theory of rational speculative bubbles and equivalent martingale measures.

1 Introduction[1]

In finite settings there is no ambiguity about the definition of portfolio payoffs: at any event the payoff of a given portfolio strategy equals the value of the portfolio chosen at the immediately preceding event minus the value of the portfolio chosen at that event. In infinite settings, however, this definition of payoffs is sometimes inapplicable[2], and when it is applicable its appropriateness is not as clear. For example, under the definition just presented the payoff of a portfolio strategy that is self-financing at every date—that is, where the payoff is zero at every date—is zero. Where did the value represented by the initial cost of the portfolio go?

In continuous-time models where there is no terminal date the received definition of the portfolio payoff is precisely analogous to the finite-time definition just stated. In particular, the definition in the continuous-time case has in common with the definition in the discrete-time case the property that the payoff of a self-financing

[1] I have received helpful comments from Mark Fisher, Christian Gilles, Zari Rachev and Jan Werner. I am indebted to seminar participants at University of Karlsruhe, University of Trento and University of Rome for comments.

[2] As seen below, when there is an infinite number of dates and also a terminal date, events at the terminal date may have no immediate predecessors.

portfolio strategy is zero. In continuous-time models where there is a terminal date (the usual case) the payoff of any portfolio strategy at the terminal date is defined via a stochastic integral when that integral is well-defined, and via a pointwise limit when the integral is undefined.

The latter specification also has problems of interpretation, as exemplified by the doubling strategy, a portfolio trading strategy that apparently produces an arbitrage. The existence of such arbitrages has led many analysts to conclude that models with infinite time have equilibria only in the presence of trading restrictions, so that arbitrage either is ruled out or is limited in scale. These trading restrictions greatly complicate the analysis of such models, as we will see below.

We propose an alternative definition of the payoffs of infinite portfolio strategies, one that does not result in arbitrage even in the absence of trading restrictions. Accordingly, there is no need to introduce trading restrictions, although one can still do so if desired. For this reason the alternative definition produces a much cleaner, although somewhat more abstract, treatment of equilibrium in infinite-date models than the received definition.

We also show how the alternative definition affects the analysis of rational bubbles. Finally, we demonstrate that the revised definition allows a concise demonstration of when equivalent martingale measures do and do not exist.

2 Payoffs as Pointwise Limits

Assume that time is infinite and countable[3], and therefore that it can be indexed by the natural numbers. Under this specification the setting assumes an *infinite horizon* if there exists no terminal date, or a *finite horizon* if there exists a terminal date $\overline{T}$. In the infinite horizon case time is interpreted as $0, 1, 2, 3, ...,$; in the finite horizon case it may be interpreted as $0, 1/2, 3/4, 7/8, ..., 1$, for example, in which case the horizon occurs at date $\overline{T} = 1$. The latter specification is analogous to the continuous-time setting, in which there exists an infinite number of trading dates, but (usually) there also exists a last date.

Uncertainty is represented by a probability space $(\Omega, \mathcal{F}, \mu)$, where $\mathcal{F}$ is a σ-algebra on Ω and μ is a probability measure on $\mathcal{F}$, and an event tree with a finite number of events (subsets of Ω) at each date. There is a single event ξ_0 at date 0. Agents trade n securities at each node of the event tree, where n is finite. In this setting *finite portfolio strategies* are representable as finitely nonzero sequences $\Theta = \{\theta(\xi)\}$, where $\theta(\xi)$ is an n-tuple in which the i-th element represents the holding of security i at event ξ.[4]

[3] We choose a discrete-time setting so that the theory corresponds directly to an extended discrete-time example, presented in the last section. The treatment in continuous time is broadly similar to that of discrete time.

[4] We will generally use capital letters to denote stochastic processes and lower-case letters to denote random variables, n-tuples or scalars.

The payoff $X = \{x_\Theta(\xi)\}$ of a finite portfolio strategy is defined at all events (other than the initial event and, in the finite-horizon case, events at $\overline{T}$) by

$$x_\Theta(\xi) = \pi(\xi)\left[\theta(\xi^-) - \theta(\xi)\right] \tag{1}$$

when the securities pay no dividends. Here ξ^- is the event that immediately precedes ξ and $\pi(\xi)$ is the n-tuple of security prices at event ξ. In the finite-horizon case, payoffs at $\overline{T}$ are given by (1) with $\theta(\xi) = 0$. The *initial cost* of Θ is $\pi(\xi_0)\theta(\xi_0)$, where ξ_0 is the root node.

Portfolio strategies representable by infinitely nonzero sequences are *infinite portfolio strategies.* The question is how to define their payoffs. Established practice is different in the case of an infinite horizon vs. a finite horizon. Because in the infinite-horizon case every event other than ξ_0 has an immediate predecessor, (1) can be interpreted as applying to infinite as well as finite portfolio strategies (Santos and Woodford [34], Huang and Werner [23], for example), and the usual practice is to do so. This specification has the implication that the payoff of any self-financing infinite portfolio strategy (a portfolio strategy for which $x_\Theta(\xi) = 0$ at every event ξ) is zero.

In the case of a finite horizon, (1) does not define the payout at $\overline{T}$, the events at $\overline{T}$ not having immediate predecessors. In the continuous-time literature the practice is to specify a portfolio's payoff at $\overline{T}$ either via a stochastic integral or, if that is undefined, as the pointwise limit of the portfolio's value at t as t approaches $\overline{T}$ (Duffie [9], Appendix D). We will provisionally carry over the latter specification to the discrete-time case.

In order to attain a symmetric treatment of the infinite-horizon and finite-horizon cases we will modify the specification of payoffs in the infinite-horizon case so as to render the treatment the same as in the finite-horizon case. We do this by appending a date called ∞ in the infinite-horizon case and defining the payoffs of infinite portfolio strategies so as to include payoffs at ∞. This component of the payoff is defined to equal the limit of the portfolio value at t as t approaches ∞, exactly as in the finite-horizon case (with ∞ in the infinite-horizon case replacing $\overline{T}$ in the finite-horizon case).

To assure the existence of the payoff under the alternative definition it is necessary to provide some additional structure. Let L^p be the set of adapted stochastic processes X for which $\|X\|_p$, defined by

$$\|X\|_p = \left[E \sum_t |x_t|^p\right]^{1/p}, \tag{2}$$

is finite. Here x_t is the date-t component of X; that is, the random variable that takes values $x(\xi_t)$ at the events ξ_t occurring at date t.

In the absence of arbitrage in finite portfolio strategies, there exists a *state price deflator* M, a strictly positive adapted process such that the initial cost $\pi(\xi_0)\theta(\xi_0)$

of any finite portfolio strategy Θ equals the expectation of its summed payoff valued using M:

$$\pi(\xi_0)\theta(\xi_0) = E\left[\sum_t x_t m_t\right], \tag{3}$$

where m_t is the date-t component of M.

Following Fisher and Gilles [13], suppose that $M \in \mathbf{L}^p$ for some p, and consider Θ^q, the set of infinitely nonzero portfolio strategies Θ that are mapped by (1) to elements of $\mathbf{L}^q$, where

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{4}$$

Since $M \in \mathbf{L}^p$ and $X \in \mathbf{L}^q$, it follows that $Y = MX \in \mathbf{L}^1$, implying that the image X of $\Theta \in \Theta^q$ under (1) has finite value under (3).

Here Y can be interpreted as resulting from applying a numeraire change to X, so as to measure in units of date-0 value. Since, from (3),

$$\pi(\xi_0)\theta(\xi_0) = E\left[\sum_t y_t\right], \tag{5}$$

where y_t is the date-t component of Y, we have that the date-0 value of any positive Y equals its norm. Note that we are not characterizing X or Y as the payoff of Θ, except when Θ is a finite portfolio strategy.

Let v_t be the date-t value of portfolio strategy $\Theta \in \Theta^q$, measured in units of date-0 value (i.e., $v_t = \pi_t\theta_t m_t$), and (in this section) let v_∞ be the pointwise limit of v_t as t approaches ∞, assuming the limit exists (the limit always exists for portfolio strategies with nonnegative payoffs at each date t, since v_t is positive and finite at each t, and decreases with t).

Payoffs of infinite portfolio strategies Θ are denoted (Y, v_∞) in the infinite horizon case, and $(Y, v_{\overline{T}})$ in the finite-horizon case, whenever the limit exists.[5]

2.1 Local Martingales and Martingales

Define L^p as the space of random variables v with finite norm, where the norm is defined by

$$\|v\|_p = \left[E\,|x_t|^p\right]^{1/p}. \tag{6}$$

A stochastic process $V = \{v_t\}$ defined on $(0, 1, 2, ..., \infty)$ is a *local martingale* if

$$v_t \in L^1 \tag{7}$$

[5] In order to situate payoffs of finite portfolio strategies as a special case of payoffs of infinite portfolio strategies, one would replace $X = \{x_\theta(\xi)\}$, where $x_\theta(\xi)$ is defined by (1), by $(X, 0)$, and similarly with Y.

and

$$v_t = E_t(v_\tau) \tag{8}$$

for $t, \tau \in \{0, 1, 2, ...\}$ and $\tau \geq t$. It is a *martingale* if in addition

$$v_t = E_t(v_\infty). \tag{9}$$

A *self-financing infinite portfolio strategy* is an infinite portfolio strategy for which $Y = 0$, so that its payoff measured in units of date-0 value equals $(0, v_\infty)$ for some random variable v_∞ (hereafter the exposition assumes an infinite-horizon setting for convenience). For a self-financing portfolio strategy, $V = \{v_t\}$ is a local martingale. Therefore the martingale convergence theorem (see, for example, Chung and Williams [6]) guarantees the existence of $v_\infty = \lim_{t\to\infty}(v_t)$ as an element of L^1.

3 Discontinuous Valuation

The fact that V is a local martingale implies that the random variables v_t have common expectation. As the martingale convergence theorem makes clear, it does not necessarily follow that v_∞ has the same expectation as the v_t. Therefore V may not be a martingale. However, under the additional restriction that V is uniformly integrable (or if any of a variety of other conditions is satisfied) then $E(v_\infty) = E(v_t)$ for all t, so that V is a martingale. In that case $E(v_\infty)$, the date-0 value of the limiting payoff, equals the initial cost of the infinite portfolio strategy with payoff $(0, v_\infty)$.

However, the uniform integrability condition may not be satisfied. If it is not, the v_t may converge (pointwise almost everywhere) to the payoff of a finite-horizon portfolio strategy with different initial cost, since when uniform integrability fails the expectation of the limit is not necessarily equal to the limit of the expectations. In that case, valuation of finite portfolio strategies is discontinuous in the topology of pointwise convergence.

Discontinuous valuation of finite portfolio strategies implies existence of arbitrage when infinite portfolio strategies are included. To see this, suppose that V converges pointwise to v_∞ almost everywhere, where v_∞ is the payoff of a finite portfolio strategy, but $E(v_t) \neq E(v_\infty)$. Now consider the finite portfolio strategy with date-t payoff $v_t - v_\infty$, and construct the associated infinite portfolio strategy by letting t go to infinity. This portfolio strategy has zero payoff since v_t converges to v_∞, but since $E(v_t) \neq E(v_\infty)$, its initial cost is nonzero. Therefore the law of one price fails, and this portfolio strategy or its negative is an arbitrage.

3.1 Example

The doubling strategy and the "suicide strategy" (as Harrison and Pliska [20] called the negative of the doubling strategy) are familiar examples of the failure of the law of one price that results from discontinuous valuation. Suppose that at each date t an agent can buy without cost θ_t shares of a gamble each share of which produces a payoff of $+1$ or -1 with equal probability. The trading strategy is to keep playing, doubling the bet at each stage, until the first win occurs, at which time the agent stops betting. After t rounds the value v_t of the doubling strategy is

$$v_t = \begin{cases} 1 & \text{with probability } 1 - 2^{-t} \\ 1 - 2^t & \text{with probability } 2^{-t}. \end{cases} \tag{10}$$

The expectation of this payoff is, of course, zero, as it must be because the initial cost of the portfolio strategy is zero and the gamble in each round is fair. However, the payoff converges pointwise to 1 almost surely, an arbitrage. Note here that V is not uniformly integrable.

A uniform lower bound on wealth rules out the doubling strategy (Dybvig and Huang [11]), but permits a suicide strategy, in which an agent gambles until his first loss rather than his first win. This strategy produces a loss almost everywhere.

3.2 Critique

Defining the payoffs of infinite-horizon portfolio strategies as pointwise limits makes mathematical sense, but it is not clear that doing so makes economic sense. In the case of the doubling strategy, the prospect of not winning after t rounds contributes $2^{-t} - 1$ (the product of $1 - 2^t$ and 2^{-t}) to the expected payoff of the finite doubling strategy. This converges to -1. Despite this, the prospect of never winning contributes nothing to the expected payoff of the infinite-horizon doubling strategy. Thus the fact that the probability of never winning goes to zero trumps the fact that the payoff in the event of not winning goes to $-\infty$, this despite the fact that the product of the two converges to -1.

Agents are modeled as believing that the opportunity to gamble with each other forever enables each of them to produce wealth out of nothing. Is it obvious that people are that irrational?

In a curious passage discussing the doubling strategy, Delbaen and Schachermayer ([8], p. 465), observed that the player who

> doubles his bet until the first time he wins .... has an almost sure win .... However, his accumulated losses are not bounded below. Everybody, especially the casino boss, knows that this is a very risky way of winning 1 ECU.

They then argue from this that the doubling strategy has to be ruled out via trading restrictions. What is curious about the passage is that under the received mathematical treatment—defining the limiting payoff via pointwise convergence—the strategy, like any arbitrage, is modeled as in fact having no risk of loss, and it is attractive to risk-averse investors precisely for that reason. Rather than motivating imposition of trading restrictions, Delbaen and Schachermayer, apparently without realizing it, are in fact expressing doubts about the received mathematical treatment, which represents as riskless the payoff of a gambling strategy that, according to them, is far from riskless.

When the analysis involves an infinite number of states rather than dates, most economists understand that notions of convergence are very much at issue. For example, issues similar to those discussed here arise in the Arbitrage Pricing Theory of Ross [31], [32]: analysts assume absence of "approximate arbitrage" so as to avoid introducing arbitrage when defining payoffs of diversified portfolios—usually implicitly, it is true—as limits of payoffs of finite portfolios. They (sometimes) state explicitly that by absence of approximate arbitrage they mean continuity in the mean-square norm. Nevertheless, when the commodity index set represents time rather than states of nature, "convergence" always seems to mean "pointwise convergence". This is so despite the inconvenient fact that, under this specification, valuation necessarily involves discontinuity, which economists find difficult to motivate in other contexts.

4 Trading Restrictions

In the absence of trading restrictions, existence of arbitrage implies nonexistence of optimal portfolios (when agents prefer more to less), so it cannot occur in equilibrium . A variety of trading restrictions may be used to create an environment in which agents have optimal portfolios.[6] These restrictions complicate the analysis of valuation to different degrees and in different ways.

4.1 Portfolio Restrictions Implying Linear Valuation

In the simplest case the assumed restrictions leave the portfolio space as a linear space, but one on which valuation of payoffs of feasible finite portfolio strategies is (pointwise) continuous, so that inclusion of infinite portfolio strategies does not induce arbitrage. When the portfolio space is a linear space, there are two implications: (1) there can be no arbitrage in equilibrium, and (2) valuation is linear, just as in the finite-date case in the absence of trading restrictions.

[6] See Huang and Werner [23] for a general discussion of trading restrictions in discrete-time sequential equilibrium.

With regard to (1), absence of arbitrage again follows from the fact that when portfolios lie in a linear space any arbitrage could be operated on an arbitrary scale, implying nonexistence of optimal portfolio strategies. With regard to (2), valuation is linear because if there existed payoffs x and y such that the price of $x+y$ exceeded the price of x plus the price of y, there would exist an obvious arbitrage: sell $x+y$ short and buy x and y separately. The reverse transaction is indicated if the price of $x+y$ were less than the price of x plus the price of y. A similar arbitrage would be profitable if the price of α units of x equaled anything other than α times the price of x.

The reasoning in the preceding paragraph required that the set of feasible portfolios be a linear space, but does not require that the dimension of the portfolio or payoff spaces be finite. Therefore when the set of feasible portfolios is a linear space we have linear valuation even in the infinite case. However, it does not follow from this that the price of every security can be expressed as a linear function of its dividends. The reason is that linearity applies only to payoffs of feasible portfolio strategies. Depending on the trading restrictions, the portfolio strategy of buying a security and holding it forever may not be in the feasible set. If not, there is no reason to presume that the price of that security will necessarily depend linearly on its dividends.

4.1.1 Example

An example of a portfolio restriction that leaves the portfolio space as a linear space is that feasible portfolio strategies must have zero present value of asymptotic indebtedness. Again, doubling strategies are infeasible under this restriction. Further, under this restriction the portfolio choice set is a linear space, so there exists a linear functional that values payoffs. Magill and Quinzii [27], [28] showed that this specification allows the existence of bubbles in equilibrium that cannot be arbitraged away. In the case of money, the portfolio strategy of selling money short and rolling over the position forever is infeasible since it involves nonzero asymptotic indebtedness.

4.2 Portfolio Restrictions Implying Nonlinear Valuation

The trading restrictions that are imposed to assure continuous valuation may be such that the set of feasible portfolios is not a linear space. For example, suppose that a lower bound is imposed on security holdings; a prohibition on short sales is the obvious example. Even in that case valuation may be linear. This is the case if, for example, the assumed trading restrictions are never binding at equilibrium portfolio strategies.[7]

[7] Dybvig and Huang [11] showed that a lower bound on wealth eliminates arbitrage in a setting that assumed linear valuation via a risk-neutral probability measure (that is, a setting in which portfolio values equal simple expected payoffs). Valuation via a risk-neutral probability measure

However, in general matters are more complicated. When the set of feasible portfolios is not a linear space, discontinuous valuation can occur in equilibrium as long as the resulting arbitrages on infinite-horizon portfolio strategies cannot be operated on an arbitrary scale (Huang and Werner [23]).[8] Besides possibly implying discontinuous valuation, portfolio restrictions that result in a portfolio set that is not a linear space complicate the analysis of valuation. First, the law of one price may fail, as long as the implied arbitrage can be operated only on a limited scale. For example, under short sale restrictions a zero-payoff asset may have a positive price in equilibrium, since the trading restrictions may imply that the resulting arbitrage can be operated only on a finite scale.

When different portfolio strategies produce the same payoff at different initial costs, the question arises what exactly one means by the value of a payoff. As in the finite case (LeRoy and Werner [24], Ch. 7), the value of a payoff is most usefully defined as the cost of the portfolio that produces that payoff at minimum cost, since under increasing preferences that is the portfolio that utility-maximizing agents will choose. So defined, the portfolio valuation functional may or may not be linear (Huang and Werner [23]). Further, valuation may not even be positive. For example, if portfolio constraints allow operation of a Ponzi scheme on a limited scale, the zero payoff can be obtained at negative initial price, which is inconsistent with positive valuation. For further discussion of valuation in sequential models see Luttmer [26], Santos and Woodford [34] and Huang [21].

5 Payoffs as Weak* Limits

The difficulties discussed above—discontinuous valuation of finite portfolio strategies, implying the existence of arbitrage when payoffs of infinite portfolio strategies are modeled as limits of payoffs of finite portfolio strategies—are all consequences of working with pointwise limits. The principal argument in favor of pointwise convergence is that it is tractable, although why that argument appears decisive as regards time but not uncertainty is unclear, as discussed in Subsection 3.2. In the remainder of this paper the date-infinity payoffs v_∞ of infinite portfolio strategies will be taken as weak* limits of nets of portfolio values v_t, instead of as pointwise limits. We will see that making this change has many attractive consequences.[9]

generally obtains only if the restriction on wealth is not binding at equilibrium portfolio strategies.

[8] See LeRoy and Werner [24], p. 36 for a finite-state example of a model in which equilibrium asset prices imply existence of a limited arbitrage.

[9] The remainder of this paper presumes some knowledge of functional analysis. Much of the relevant material is summarized in Gilles [15], Gilles and LeRoy [17], [18] and Fisher and Gilles [13] in a context very close to that of the present paper. For more extended and more rigorous discussion, see Royden [33], Aliprantis and Border [1], Dunford and Schwartz [10] and Bhaskara Rao and Bhaskara Rao [30].

In the discussion to follow, whenever there exists an isometric isomorphism between two function spaces, we will not distinguish between the two. For example, the space of norm-continuous

The values v_t of self-financing infinite portfolio strategies lie in L^1, so it is natural to consider replacing pointwise convergence with convergence in the L^1 norm. However, with this change there is no assurance that the limiting payoff exists. For example, in the case of the doubling strategy considered in Section 3.1, $\{v_t\}$ diverges in the L^1 norm, implying the unsatisfactory outcome that the doubling strategy is not a well-defined portfolio strategy.

Suppose instead that we embed L^1 in its second topological dual, the space $ba(\Omega, \mathcal{F}, \mu)$ of signed charges on $\mathcal{F}$. Assuming that the net $\{v_t\}$ directed by t is bounded, Alaoglu's theorem guarantees the existence of convergent subnets in ba. We will take the limit points v_∞ of such nets as the date-∞ payoffs of infinite portfolio strategies.

The Yosida-Hewitt theorem guarantees that the charge v_∞ is the sum of a measure component and a pure charge component. The former can be identified with an element of L^1, by the Radon-Nikodym theorem. Further, it coincides with the pointwise limit discussed above (this follows from Fisher and Gilles' [13] Theorem A.5). Thus the present treatment differs from that of Section 2 in including the pure charge component of the limiting payoff. For example, in the context of the doubling strategy, the limiting payoff consists of the measure 1 plus a pure charge centered on the event that the agent loses forever.

5.1 Continuous Valuation

Weak* convergence of v_t to v_∞ means that $E(xv_t)$ converges to the integral of xv_∞ for any $x \in L^\infty$ (see Gilles and LeRoy [18], appendix, for a minimally technical discussion of integration with respect to a charge).[10] Since the functional 1 is an element of L^∞, we have that the values $E(v_t)$ of the liquidating payoffs v_t converge to the integral of v_∞, so valuation is continuous. In the context of the doubling strategy the measure component of the limiting payoff has value 1 and the pure charge component has value -1. Thus the date-0 value of the limiting payoff—the sum of the values of the measure component and the pure charge component—equals the initial cost (zero) of the portfolio strategy.

Continuous valuation of finite portfolio strategies has the attractive feature that allowing for infinite portfolio strategies does not produce arbitrage: if there existed an infinite portfolio strategy with positive payoff and negative initial cost, continuity

functions on L^1 is isometrically isomorphic to a space of charges. Instead of adopting notation that distinguishes between the two and stating the isometry, we will simply identify charges and continuous functions, thereby economizing on verbiage and notation. The sources cited provide the rigorous treatment.

[10] Note that we cannot generally write the integral of v_∞ as $E(v_\infty)$. This is so because v_∞, as a charge, is not a random variable, so its expectation is not defined. It is true that in the special case when v_∞ has a zero pure charge component it can be identified with a measure (by the Yosida-Hewitt theorem), and therefore (by the Radon-Nikodym theorem) with a random variable in L^1, so in that case there is no difficulty with identifying the date-0 value of v_∞ with $E(v_\infty)$.

implies that there would also exist finite portfolio strategies with the same properties, but this was ruled out in assuming the existence of a state price deflator. For example, the doubling strategy is not an arbitrage when the limiting payoff is defined as a weak* limit because the pure charge component of v_∞ is negative (and, as noted in the preceding paragraph, has value equal in absolute magnitude to that of the measure component).

Nonexistence of arbitrage implies that there is no need to impose portfolio restrictions although, of course, one can still do so if desired. Thus the canonical model of finite-time-and-state finance, which specifies unrestricted portfolio strategies, has a direct counterpart in the infinite case. In contrast, we saw that under pointwise convergence trading restrictions must be imposed to prevent arbitrage, and the features of the model depend critically on what restrictions are adopted (as emphasized by Huang and Werner [23]). Therefore under pointwise convergence there can be no canonical model.

In the absence of trading restrictions, portfolio choice sets are linear spaces. Since with payoffs of infinite portfolio strategies defined as weak* limits there is no need for trading restrictions, it follows that valuation is linear in the canonical version. In contrast, we observed in Section 4 above that when payoffs of infinite portfolio strategies are defined as pointwise limits, the fact that portfolio restrictions must be imposed implies that in such settings valuation may or may not be linear depending on the restrictions imposed and whether or not they are binding in equilibrium.

Under most specification of preferences agents do not value consumption at infinity or $\overline{T}$, implying that optimal portfolios will have $v_\infty = 0$ (Gilles and LeRoy [17] is an exception). In such cases the most important consequence of the specification of weak* convergence in place of pointwise convergence is that it allows greater latitude in specifying portfolio restrictions, as discussed in the preceding paragraph.

6 Bubbles

The preceding discussion has direct implications for the analysis of (rational) bubbles. First, we define the *fundamental* of any portfolio payoff (Y, v_∞) as its stochastic process component Y. The remaining component v_∞ is the bubble. Correspondingly, the *fundamental value* of (Y, v_∞) is the date-0 value of the fundamental which, from (5), equals $\sum_t E(y_t)$. The *bubble value* equals $\lim_{t\to\infty} E(v_t)$.

The definitions of the fundamental and bubble components of portfolio strategies and the associated values have obvious counterparts for securities: a security's fundamental is the fundamental of the portfolio strategy consisting of buying one share of that security and holding it forever. The definitions of fundamental value, bubble and bubble value are similar.

The definitions just presented agree with general usage in earlier discussions of bubbles, except that in most discussions no distinction is drawn between bubble and bubble value, and similarly for fundamental and fundamental value. Suppression of this distinction in the earlier studies is justified because these studies generally imposed trading restrictions, implying that asset values are not generally related linearly to payoffs. Here, however, we do not impose trading restrictions, implying that valuation is linear. Therefore portfolio values always have counterparts in portfolio payoffs, and it is necessary to adopt terminology that distinguishes between portfolio values and portfolio payoffs.

Several recent discussions of bubbles adopt a framework similar to that set out here, but define bubbles differently. For example, Loewenstein and Willard [25] implicitly and Fisher and Gilles [13] explicitly identified bubbles with the pure charge component of v_∞. In contrast, Clark [7] identified bubbles with the measure component of v_∞. Both of these alternative definitions entail a departure from received usage, which identifies fundamental value via the present-value relation (that is, as the value attributable to payoffs at finite dates). Adhering to received usage, as we have done, implies identifying bubbles with v_∞, rather than either its measure component or its pure charge component separately. This is purely a semantic matter, but it is difficult to see any rationale here for deviating from the received definitions.

Gilles [15] was first to point out the connection between bubbles and charges. Gilles and LeRoy [16], [17], [18], [19] continued work along these lines. All of these papers, however, worked exclusively with valuation operators and payoffs spaces, so that there was no representation of portfolio strategies. Specifically, in Gilles [15] and Gilles and LeRoy [17] the payoff space was L^∞ and valuation operators lay in ba, while the reverse specification was adopted in Gilles and LeRoy [18]. In the present analysis we follow Fisher and Gilles [13] in representing bubbles as a component of portfolio payoffs measured in units of date-0 value (this was implied in working with $Y = MX$ rather than X). Therefore bubbles are not traceable to either the valuation operator or the payoff space separately.

The present framework for the analysis of bubbles is superior to that of the earlier papers. First, it makes explicit the connection between the analysis of bubbles and the characterization of portfolio payoffs (via pointwise or weak* limits). Second, it situates bubbles as consequences of portfolio strategies that push their payoffs into the infinite future, which has no direct connection either with the specification of payoff spaces or of price systems. Making explicit the connection between bubbles and portfolio strategies allows integration of the analysis of bubbles into the mainstream finance literature (for an example in the finance literature of the lack of such integration, see Sethi [36], where it is pointed out that the present-value relation may fail; no connection is made there with the economics literature on bubbles).

6.1 Equilibrium Bubbles in Finite-Agent Settings

In recent years a literature has arisen identifying conditions under which bubbles can and cannot exist in equilibrium. Here we consider the extent to which this analysis applies when the date-$\mathbf{1}$ components of portfolio payoffs are defined as weak* limits.

Our discussion is necessarily informal: as noted above, a major difference between the two settings is that under pointwise limits portfolio restrictions must be imposed if arbitrage is to be eliminated, whereas no such restrictions are needed under weak* limits rather tha pointwise limits. Further, as observed above, the analysis of bubbles under pointwise limits depends on what restrictions are imposed, implying that there can exist no canonical analysis of bubbles under pointwise limits. The discussion below, which does not specify portfolio restrictions, must therefore be regarded as identifying lines of argument rather than providing rigorous analysis.

It is clear that bubbles always exist in the sense that agents can always construct self-financing portfolio strategies, and the payoffs of these are bubbles, as we have seen. However, such portfolios are not optimal under standard characterizations of preferences. The question is under what conditions agents' optimal portfolios will or can have payoffs with bubbles.

The answer to this question is different in finite-agent and infinite-agent settings. In the early literature on bubbles it was stated flatly that bubbles cannot exist in finite-agent settings (for example, Tirole [38], Blanchard and Fischer [5]). It is known now that this conclusion requires qualification. Most obviously, equilibria involving payoff bubbles can occur when the representative agent's endowment contains a pure charge component and agents derive utility from consumption in the infinite future. However, specifications like this appear, and are, contrived and unrealistic. Assumptions implying that security or portfolio payoffs in the very distant future are unimportant serve to eliminate bubbles. For example, if payoffs are bounded and agents discount the future, then bubble values equal zero (Bewley [4], Magill and Quinzii [28], Santos and Woodford [34], Huang and Werner [22], Stokey, Lucas and Prescott [37]). This is so because in a finite-agent setting a security can have a bubble only if some agent's optimal portfolio strategy has a bubble, and the stated assumption rules that out.

There exist several other lines of argument that exclude bubbles in some settings:

1. *Transversality condition.* Several analysts (Obstfeld and Rogoff [29], for example) have pointed out that an optimal portfolio strategy can have a bubble only if a transversality condition is violated. This condition has nothing to do with how infinite portfolio payoffs are defined, so it applies under both pointwise and weak* limits if it applies at all. However, the argument does not apply universally: some optimization problems do not have necessary transversality conditions, for example.

2. *Arbitrage.* Some analysts have argued that for a security to have a bubble would imply existence of an arbitrage opportunity. This is so because in the presence of a positive bubble an agent could sell short the asset with a bubble, assuming portfolio restrictions permit this, and hold long an infinite portfolio consisting of the dividends on the security. This argument presumes pointwise convergence, since the payoff on a short position in the security and a long

position in its first n dividend payments converges to zero pointwise. However, this payoff does not weak* converge to zero, so the argument does not carry over to the present setting.

3. *Walras' Law.* For a security in positive net supply to have a bubble would imply a violation of Walras' Law since agents with a positive endowment of that security would receive a positive wealth transfer that is not completely offset by negative wealth transfers on the part of agents with negative endowments of that security (Santos and Woodford [34], Huang and Werner [22]). This argument, being unrelated to how payoffs of infinite portfolio strategies are modeled, carries over to weak* convergence whenever it applies under pointwise convergence.

These arguments have led some analysts (Santos and Woodford [34], for example) to conclude that there is a strong presumption against the existence of bubbles in finite-agent settings.

6.2 Overlapping Generations

The principal setting for the analysis of bubbles is overlapping generations models (Tirole [39] and Wallace [40], the latter in the context of monetary economics). The reason is that overlapping generations models specify an infinite number of heterogeneous agents, implying existence of equilibria in which aggregate endowments have infinite value. In such equilibria Walras' Law cannot be invoked to show that securities in positive net supply do not have bubbles.[11] Generally there exists a continuum of equilibria indexed by bubble values (Wallace [40]).

In overlapping generations models as customarily specified, trading restrictions prevent agents from arbitraging bubbles away, although the trading restrictions are not always specified explicitly (Tirole [39], for example). Generally it is presumed that agents can trade only when they are alive, implying that no agent can sell a bubble short and roll the position over forever, the arbitrage that would otherwise exploit a positive-valued bubble.

As pointed out above, when payoffs of infinite portfolio strategies are specified as in this paper, absence of arbitrage in finite portfolio strategies implies absence of arbitrage, so no trading restrictions need be imposed. It is instructive to think about overlapping generations models that do not have trading restrictions. First,

[11] The role of Walras' Law, or lack thereof, in proving nonexistence of bubbles in overlapping generations models is similar to its role in proving that equilibrium allocations are Pareto optimal. Specifically, in finite settings the proof of Pareto optimality involves the assertion that any allocation that is Pareto-superior to the equilibrium allocation must have value greater than that of the equilibrium allocation, and therefore is infeasible, by Walras' law. If the aggregate endowment has infinite value, this argument fails, and there generally do exist Pareto-superior allocations that are feasible.

in equilibria that have bubbles, these bubbles cannot be arbitraged away: in the present setting the portfolio strategy consisting of selling one dollar short and rolling over the position forever has a payoff of $(0, -1)$, as opposed to zero as in the usual setting. The former is not an arbitrage.

Similarly, nonzero valuation of securities that have a zero dividend is consistent with linear valuation: it is portfolio payoffs that are valued linearly in the absence of trading restrictions, not (necessarily) securities' dividends. Recall that the fundamental and bubble on a security were identified with the fundamental and bubble on the portfolio strategy consisting of buying that security and rolling over the position forever. This portfolio strategy has payoff $(0, 1)$ in the case of money, implying a fundamental value of zero and a bubble value of 1 even though money has a zero dividend.

7 Equivalent Martingale Measures

The analysis just presented allow a simple characterization of when equivalent martingale measures exist: the equivalent martingale measure corresponding to any numeraire choice exists if the bubble associated with that numeraire has a zero pure charge component, and only then.

Setting up this result requires some preparation. In finite economies there exists a correspondence between probability measures and numeraire choices[12]: for any probability measure ν that is equivalent to the original measure μ there exists a positive-valued self-financing portfolio strategy with value Z (and $z_0 = 1$) such that V/Z is a ν-martingale, and conversely, where V is the value of any self-financing portfolio strategy. If the money market account is chosen as the numeraire portfolio strategy Z, then ν is the *risk-neutral measure*. Dybvig and Ross [12] called this result the "Fundamental Theorem of Finance".

In finite economies the proof is elementary. We have already seen that if v_t is the date-t value of any self-financing portfolio strategy, then in a finite setting $m_t v_t$ is a μ-martingale:

$$m_t v_t = E_t(m_T v_T), \tag{11}$$

where T is the terminal date. Now let Z be the value of a positive-valued self-financing portfolio strategy with initial value 1 that is to be taken as numeraire. Dividing (11) by $m_t z_t$ results in

$$\frac{v_t}{z_t} = E_t\left(\frac{m_T z_T v_T}{m_t z_t z_T}\right) = E_t^{\nu}\left(\frac{v_T}{z_T}\right), \tag{12}$$

[12] For an excellent discussion emphasizing the analytical benefits of an appropriate numeraire choice, see Geman, El Karoui and Rochet [14].

if ν is the measure that has $m_T z_T$ as its Radon-Nikodym derivative. Therefore V/Z is a martingale under ν. Further, the change-of-measure process $m_t z_t = E_t(m_T z_T)$ is a martingale under μ.[13]

It is known from a large recent literature (for example, Schachermayer [35], Delbaen and Schachermayer [8], Back and Pliska [2], Gilles and LeRoy [19] and the papers cited in these) that neither the necessity nor the sufficiency of this condition extends to infinite settings, at least in the absence of qualification. We give a simple and intuitive account of why the Fundamental Theorem of Finance may fail in an infinite setting.

In infinite time MZ is a local martingale, but it is not necesarily a martingale. If it is not a martingale, then $m_t z_t$ does not converge to a random variable with expectation 1. This is true under both pointwise and weak* convergence: in the former case $m_t z_t$ may converge to a random variable with expectation not equal to 1, while in the latter case $m_t z_t$ may converge to a charge that has no representation as a random variable (because it has a nonzero pure charge component). In neither case does MZ define the Radon-Nikodym derivative associated with a change of measure, implying that no equivalent martingale measure is associated with Z.

In particular, there exists a risk-neutral measure if and only if the bubble associated with the money-market account has a zero pure charge component.

7.1 Example

The preceding analysis is illustrated in an example. Consider a representative agent model in which the agent maximizes

$$\sum_{t=0}^{\infty} 2^{-t} E\left[\ln(c_t)\right]. \tag{13}$$

The agent's endowment equals 2^t, $t = 0, 1, 2, ...$ if the state[14] is high at t or at any date prior to t, and 1 otherwise. The transitions between the high and low states are governed by a 2×2 transition matrix with 1/2 in each position, so that the states are independent and equally likely. The state at date 0, the initial date, is L.

It is easily checked that event prices at date t—the prices of one unit of consumption contingent upon a particular sequence of H's and L's up to date t—equal 2^{-3t} if the state is high at t or at any date prior to t, and 2^{-2t} otherwise. Accordingly, the state price deflator M—the ratio of event prices to probabilities—is given by

[13] Baxter and Rennie [3] has a very accessible and intuitive discussion of changes of measure in the discrete-time case.

[14] Here we are not using the term "state" in its dynamic programming sense, since the level of the aggregate endowment at any date is not a sufficient statistic for the past history of the economy up to that date.

$$m_t = \begin{cases} 2^{-2t} & \text{if } \tau \le t, \\ 2^{-t} & \text{otherwise} \end{cases}, \tag{14}$$

where τ is the date of the first high endowment realization. Since the endowment realization after τ equals 2^t regardless of the state, it is clear that all uncertainty is resolved at τ. The event that $\tau \le t$ has probability $1 - 2^{-t}$ for any t.

Note here that, because of the presence of the discount factor in the utility function (13), the state price deflator declines with t. However, it declines an order of magnitude faster when $\tau \le t$ than otherwise, and this is what gives rise to the distinctive features of the example.

Let r_t be the gross one-period interest rate from $t - 1$ to t. It is given by

$$r_t = \begin{cases} 4 & \text{if } \tau < t, \\ 4(1 + 2^{-t})^{-1} & \text{otherwise} \end{cases}. \tag{15}$$

The date-t value b_t of the money-market account B, equal to the cumulated value of one unit of consumption invested at the one-period interest rate and rolled over at each date, is

$$b_t = \prod_{i=1} r_i = \begin{cases} 2^{2t} \prod_{i=1}^{\tau}(1 + 2^{-i})^{-1} & \text{if } \tau \le t, \\ 2^{2t} \prod_{i=1}^{t}(1 + 2^{-i})^{-1} & \text{otherwise} \end{cases}. \tag{16}$$

Define W as the product of M and B. From (14) and (16), W is given by

$$w_t \equiv m_t b_t = \begin{cases} \prod_{i=1}^{\tau}(1 + 2^{-i})^{-1} & \text{if } \tau \le t, \\ 2^t \prod_{i=1}^{t}(1 + 2^{-i})^{-1} & \text{otherwise.} \end{cases} \tag{17}$$

The process w is a local martingale.[15] However, it is not a martingale. To see this, note that the event that t consecutive realizations of L occur, which has probability 2^{-t}, contributes $\prod_{i=1}^{t}(1+2^{-i})^{-1}$ to $E(w_t)$. This term converges to 0.4194, which therefore equals the date-0 value of the pure charge component of the bubble w_∞. Therefore the expectation of the measure component of w_∞ equals 0.5806 ($= 1 - 0.4194$). The portfolio strategy w is seen to be a suicide strategy.

The foregoing discussion establishes that, despite the absence of arbitrage, there does not exist a risk-neutral probability measure in this example.

As a local martingale, w_t equals the date-t value of a self-financing portfolio strategy. The fundamental value of W is zero and its bubble value is 1. However, suppose that instead of rolling over the portfolio, the investment is terminated as

[15] For example, the date-0 value of m is 1, its date-1 values are $m_H = 2/3$; $m_L = 4/3$; its date-2 values are $m_{HH} = m_{HL} = 2/3$; $m_{LH} = 8/15$ and $m_{LL} = 32/15$.

soon as the first high state occurs. In that case the fundamental value of W is 0.5806 and its bubble value is 0.4194, agreeing with Loewenstein and Willard's [25] terminology. For this portfolio strategy the bubble is a pure charge centered on the event $(L, L, L, ...)$.

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