The Generalized External Order, and Applications to Zonotopal Algebra

by

Bryan R. Gillespie

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Professor Bernd Sturmfels
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Professor David Aldous

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Abstract

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Extrapolating from the work of Las Vergnas on the external active order for matroid bases, and inspired by the structure of Lenz's forward exchange matroids in the theory of zonotopal algebra, we develop the combinatorial theory of the generalized external order. This partial ordering on the independent sets of an ordered matroid is a supersolvable join-distributive lattice which is a refinement of the geometric lattice of flats, and is fundamentally derived from the classical notion of matroid activity. We uniquely classify the lattices which occur as the external order of an ordered matroid, and we explore the intricate structure of the lattice's downward covering relations, as well as its behavior under deletion and contraction of the underlying matroid.

We then apply this theory to improve our understanding of certain constructions in zonotopal algebra. We first explain the fundamental link between zonotopal algebra and the external order by characterizing Lenz's forward exchange matroids in terms of the external order. Next we describe the behavior of Lenz's zonotopal $\mathcal{D}$-basis polynomials under taking directional derivatives, and we use this understanding to provide a new algebraic construction for these polynomials. The construction in particular provides the first known algorithm for computing these polynomials which is computationally tractible for inputs of moderate size. Finally, we provide an explicit construction for the zonotopal $\mathcal{P}$-basis polynomials for the internal and semi-internal settings.

To my parents, for fanning my passions
and giving me a foundation on which to build.

To my wife, for supporting me through this
and filling my days with love and joy.

Contents

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Chapter 1

Introduction

The theory of zonotopal algebra studies the characteristics and applications of certain finite-dimensional polynomial vector spaces, and related annihilating ideals, which are derived from the structure of particular combinatorial and geometric objects formed from a finite ordered list of vectors. The theory lies at an interface between several starkly contrasting mathematical disciplines, especially approximation theory, commutative algebra, and matroid theory, and wide-ranging connections have been found with topics in enumerative combinatorics, representation theory, and discrete geometry.

Zonotopal algebra has at its foundation a collection of ideas in numerical analysis and approximation theory that were developed in the 1980s and early 90s, in works such as [1, 9, 10, 11, 17, 19, 16]. In these works, the *central* zonotopal spaces were discovered in relation to the approximation of multivariate functions by so-called (exponential) box splines, and as the solutions of certain classes of multivariate differential and difference equations.

The central zonotopal spaces lie in the real polynomial ring $\Pi = \mathbb{R}[x_1, \ldots, x_d]$, and consist of two finite-dimensional polynomial vector spaces $\mathcal{D}(X)$ and $\mathcal{P}(X)$, and two related ideals $\mathcal{I}(X)$ and $\mathcal{J}(X)$ which are constructed from the columns of a $d \times n$ matrix X. The $\mathcal{D}$-space, also known as the Dahmen-Michelli space, is the space of polynomials spanned by the local polynomial pieces of the *box spline* associated with the matrix X. It can be realized as the differential kernel or Macaulay inverse system of the $\mathcal{J}$-ideal, an ideal generated by products of linear forms corresponding with the cocircuits of the linear independence matroid of the columns of X.

The $\mathcal{P}$-space is defined using the notion of *matroid activity* of bases of X, and acts as the dual vector space of the $\mathcal{D}$-space under the identification $p \mapsto \langle p, \cdot \rangle$, where $\langle \cdot, \cdot \rangle : \Pi \times \Pi \to \mathbb{R}$ denotes the differential bilinear form given by $\langle p, q \rangle := \big(p(\partial) q\big)\big|_{x=0}$, and where $p(\partial)$ is the differential operator obtained from p by replacing each variable x_i with the operator $\partial/\partial x_i$. This space is realized as the differential kernel of the $\mathcal{I}$-ideal, a *power ideal* generated by certain powers of linear forms corresponding to orthogonal vectors of the hyperplanes spanned by column vectors of X.

The $\mathcal{D}$- and $\mathcal{P}$-spaces in particular each contain a unique interpolating polynomial for any function defined on the vertex set of certain hyperplane arrangements associated with the matrix X, a property known as *correctness*. In addition, the Dahmen-Michelli space has a simple construction in terms of the *least map*, an operator introduced in [11] which plays an important role in interpolation theory. Furthermore, relations dual to these can be developed with respect to generic vertex sets for the *zonotope* of X, which is formed by taking the Minkowski sum of the segments $[0, x] = \{tx : t \in [0, 1]\}$ for the column vectors x of X. It is from this geometric connection (and related combinatorial structure) that the name *zonotopal algebra* is derived.

$$X = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \rightsquigarrow$$

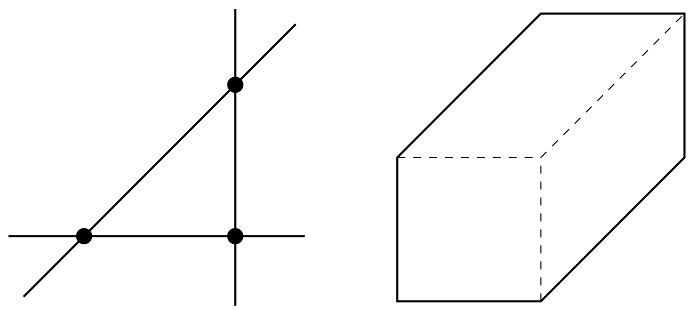


Figure 1.1: A hyperplane arrangement and corresponding dual zonotope of the matrix X.

Particularly in the last decade, the area of zonotopal algebra has witnessed a resurgence of interest, with applications and generalizations of the theory emerging in a variety of new directions. In [38], Sturmfels and Xu study sagbi bases of Cox-Nagata rings, and discuss an algebraic generalization of the zonotopal spaces called zonotopal Cox rings which can be described as subalgebras of particular Cox-Nagata rings. In [33], Luca Moci generalizes some of the results of zonotopal algebra regarding hyperplane arrangements to a discrete setting related to the geometry of toric arrangements. In [12, 13, 14], De Concini, Processi and Vergne explore connections between zonotopal spaces, vector partition functions, and the index theory of transversally elliptic operators. Very recently, Gleb Nenashiv in [34] classifies up to isomorphism the quotient algebras associated with certain zonotopal ideals.

In their seminal work [23], Holtz and Ron produced a surprising generalization of the theory by defining new spaces, the *external* and the *internal* zonotopal spaces, which exhibit characteristics analogous to those of the central spaces, with corresponding interpretations in terms of polynomial interpolation and differential equations. The existence of such a generalization was not expected, but it begged the question of whether the two new cases were isolated instances, or whether a more general theory was yet to be formulated.

In the following years, additional generalizations of the zonotopal spaces were indeed discovered. In [4], Ardila and Postnikov explored combinatorial and geometric structure in power ideals which yielded insight into the nature of the $\mathcal{I}$-ideals and $\mathcal{P}$-spaces. In [24], Holtz, Ron and Xu introduced the *semi-external* and *semi-internal* zonotopal spaces, which give a discrete collection of spaces interpolating between the central and the external and internal zonotopal spaces respectively. In [29], Lenz gave a construction which integrated ideas from both of the preceding works to introduce a hierarchy of spaces which extend beyond the external spaces by raising the powers of generators in the $\mathcal{I}$-ideal, but which also encompass the discrete interpolation exhibited by the semi-external spaces of [24]. In [31], Li and Ron presented a further generalization which is parametrized by certain extensions of the collection of matroid bases associated with the matrix X.

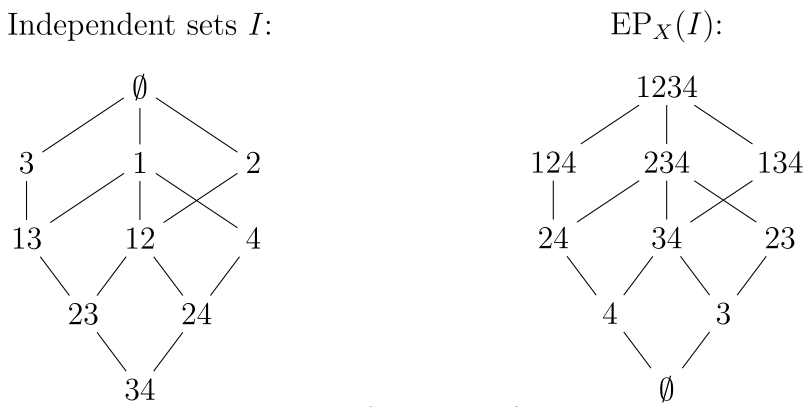
The most sweeping generalization of zonotopal spaces to date was introduced by Lenz in [30]. In this work, Lenz isolated a combinatorial structure, the *forward exchange matroid*, which captures certain matroid properties which are important to the behavior of zonotopal spaces. A collection of *generalized zonotopal spaces* can be defined for each forward exchange matroid, and in his exposition, Lenz shows that these generalized spaces include as particular instances all of the zonotopal spaces mentioned above, with the exception of the internal and semi-internal $\mathcal{P}$-spaces and $\mathcal{I}$-ideals.

This last point in particular has been something of a mystery. Lenz's generalized zonotopal spaces seem to capture the correct structure of the zonotopal $\mathcal{D}$-spaces and $\mathcal{J}$-ideals, as well as key duality properties between the $\mathcal{D}$-spaces and the $\mathcal{P}$-spaces. However, the definition of the generalized $\mathcal{P}$-space fails to capture the behavior of zonotopal spaces in one key respect: the generalized $\mathcal{P}$-space is not always equal to the differential kernel of the generalized $\mathcal{I}$-ideal. Even worse, for some forward exchange matroids the $\mathcal{P}$-space can fail to be closed under taking derivatives, which implies that it is not equal to the differential kernel of *any* homogeneous ideal.

The source of this difficulty can be observed even in the construction of the classical internal $\mathcal{P}$-space by Holtz and Ron in [23]. In their construction, they begin as a first step with the $\mathcal{P}$-space corresponding with Lenz's forward exchange $\mathcal{P}$-space. However, in order to produce a space which satisfies the requisite differential properties, their construction requires a separate abstract, and rather nontrivial, manipulation, which is described in the proof of their Theorem 5.7, and is discussed just prior to their Corollary 5.11. The same complication arises in the semi-internal setting, and more broadly, a fundamental difficulty seems to exist in defining zonotopal spaces where the $\mathcal{P}$- and $\mathcal{D}$-spaces are smaller than the central analogues.

To better understand this difficulty as well as other subtle aspects of the zonotopal spaces, we introduce and develop in this thesis the combinatorial theory of the *generalized external order* of an ordered matroid. If $M = (E, \mathbb{I})$ is a matroid with independent sets $\mathbb{I}$ and ordered ground set E, then the external order $\leq^*_{\text{ext}}$ is a partial ordering on $\mathbb{I}$ which is a refinement of the geometric lattice of flats of M, and which has the structure of a supersolvable join-distributive lattice. It is formed by ordering independent sets I by inclusion of their sets $\mathrm{EP}(I)$ of *externally passive* elements, and these externally passive sets in particular form the feasible sets of an *antimatroid*, a particular class of greedoid. Dually, the complements of these sets give the convex sets of a *convex geometry*, a discrete abstraction of the notion of convexity.

This generalized external order is an extension of a poset introduced by Michel Las Vergnas in [27]. In this work, Las Vergnas defines the external order for the *bases* of M, producing a poset on the bases which is nearly a lattice, and which is extended to a lattice by introducing an extra minimal element. The generalized external order can be realized



$$X = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Figure 1.2: The generalized order $\leq^*_{\text{ext}}$ on independent sets I of the column vectors of a matrix X, and the corresponding externally passive sets $\mathrm{EP}_X(I)$.

by appropriately assembling copies of Las Vergnas's order for each flat in the lattice of flats of M. The resulting natural object exhibits stronger lattice theoretic properties, and in particular provides a new refinement of the classical connection between matroid theory and lattice theory via the geometric lattice of flats.

After defining the generalized external order and characterizing its fundamental properties, we obtain the following classification for the orderings which come about as the external order of some ordered matroid.

Theorem. *A finite lattice L is isomorphic to the external order $\leq^*_{\text{ext}}$ of an ordered matroid if and only if it is join-distributive, matroidal, and S_n EL-shellable.*

Join-distributivity of a lattice is a class intermediate between semimodular and distributive lattices, and in particular such lattices are "locally Boolean" in a certain sense. S_n EL (or "edge lexicographic") shellability of a lattice is equivalent to the well-known notion of *supersolvability*, introduced by Stanley in [37], and in particular implies shellability of the lattice's order complex. Finally, a lattice is called matroidal if the *covering rank* function r_c counting the number of lattice elements covering a given element is decreasing and satisfies the semimodular inequality

$$r_c(x \wedge y) + r_c(x \vee y) \leq r_c(x) + r_c(y)$$

for all pairs of elements x and y.

After defining and classifying the external order, we explore some of its more interesting combinatorial properties. The operations of deletion and contraction on an ordered matroid induce corresponding operations on the external order, and we describe the nontrivial connections that exist between these operations and deletion and contraction operations which have been defined in the theory of antimatroids.

The upward covering relations of an independent set I in the external order are simple to describe, and correspond directly with the elements of I. The downward covering relations, on the other hand, do not exhibit such regular behavior. To better understand these downward covering relations, we introduce a new operator on subsets of an ordered matroid, the *spread operator*. If M is a matroid with ordered ground set E, then for $A \subseteq E$ and $x \in E$, define

$$\mathrm{Spr}_A(x) \coloneqq \mathrm{span}(\{a \in A \,:\, a > x\} \cup x) \setminus \mathrm{span}(\{a \in A \,:\, a > x\}).$$

Thus the spread operator gives the collection of elements which are additionally spanned by adding x to the set of elements from A which lie after it in the ordering of E. The definition is based on a construction used by Lenz in [30], and captures fundamental information about matroid activity. We use it to define a certain *canonical passive exchange* operator $\varepsilon_I : \mathrm{EP}(I) \to \mathbb{I}$ associated with each independent set, and we characterize the order structure of the independent sets obtained from canonical passive exchanges. In particular, it is shown that the independent sets covered by I in the external order are among those obtained by canonical passive exchanges, thus offering a satisfactory description of the downward covering relations.

With regards to the theory of zonotopal algebra, the generalized external order provides a richly structured way to understand Lenz's construction of forward exchange matroids. Specifically, Lenz defined a forward exchange matroid as a matroid M with bases $\mathbb{B}$, along with a collection $\mathbb{B}' \subseteq \mathbb{B}$ satisfying a defining condition called the *forward exchange property*. We express the forward exchange property in elementary terms using the external order, characterizing it as follows:

Theorem. *A collection of bases $\mathbb{B}' \subseteq \mathbb{B}$ satisfies the forward exchange property if and only if it is downward closed in the external order. That is, if $B \in \mathbb{B}'$ and $B' \leq^*_{\mathrm{ext}} B$, then $B' \in \mathbb{B}'$.*

More broadly, the combinatorial structure of the external order provides a concrete combinatorial framing for many of the constructions in zonotopal algebra. We present several applications of this outlook which revolve around forming a better understanding of certain canonical bases of the zonotopal $\mathcal{P}$- and $\mathcal{D}$-spaces in several settings. In the central setting, the spaces $\mathcal{P}(X)$ and $\mathcal{D}(X)$ are spanned respectively by polynomials which we denote P^X_B and D^X_B, which are indexed by the bases of X. These polynomials in particular are closely related to the bases of the zonotopal $\mathcal{P}$- and $\mathcal{D}$-spaces in other zonotopal settings, so a combinatorial understanding of these polynomials provides insight into much of the zonotopal theory.

One difficulty in the theory of zonotopal algebra has been to understand in a combinatorial way the behavior of the $\mathcal{D}$-spaces and their corresponding basis polynomials. We show that the directional derivatives of the $\mathcal{D}$-basis polynomials D^X_B are explicitly described by the local structure of the external order around the basis B. Specifically,

Proposition. *If $B \subseteq X$ is a basis, and $b \in B$, then*

$$\ell_b(\partial) D_B = \sum_{B' = B \setminus b \cup x} [b]^x_{B'} D_{B'},$$

where the sum is over bases B' covered by B in the external order. Here, $\ell_b \in \Pi$ denotes the linear form dual to b, and $[b]^x_{B'}$ denotes the x-coordinate of b represented in terms of the basis B'.

We additionally describe a direct algebraic construction for the polynomials D^X_B, which works by induction on the size of the matrix X using simple linear projection operations in certain polynomial subspaces. In particular, our understanding of the differential properties of the D-polynomials plays a key role in deducing the final simplified form of the projection operation. The construction also produces a practical algorithm for computing the D-polynomials which is the first such which is computationally tractable for matrices of moderate size. (For more details, refer to Section 4.4.)

Returning to the difficulties surrounding the internal and semi-internal $\mathcal{P}$-spaces, we take a first step in understanding the unusual behavior in these settings by presenting a new description for the semi-internal $\mathcal{P}$-basis polynomials in terms of a certain explicit projection operator. In light of the fact that the semi-internal spaces are defined using the notion of *internal* matroid activity, it is not surprising that the arguments involved in this characterization relate to the external order on the dual matroid of the matrix X. We conjecture that similar techniques could be used to provide an improved definition for the generalized zonotopal $\mathcal{P}$-spaces of broader classes of forward exchange matroids, which will satisfy the differential properties exhibited by the other existing spaces.

The upcoming chapters are organized as follows. In Chapter 2, we begin by reviewing necessary background in the areas of matroid theory and zonotopal algebra.

In Chapter 3, we define and characterize the generalized external order, in particular presenting its lattice-theoretic classification in Theorem 3. In Section 3.4 we describe the behavior of the external order under the operations of matroid deletion and contraction, and in Section 3.5 we develop the theory of the spread operator and use it to describe the canonical passive exchanges and downward covering relations in the ordering.

In Chapter 4, we present our applications of the generalized external order to zonotopal algebra. Section 4.1 characterizes the precise relation between the external order and Lenz's forward exchange matroids. In Section 4.2 we describe the canonical $\mathcal{P}$- and $\mathcal{D}$-space basis polynomials, we give a uniform description for the bases of the central, external, internal, semi-external, and semi-internal settings, and we explore further properties exhibited by the P- and D-polynomials. In Section 4.3 we characterize the behavior of the D-polynomials under directional derivatives, and in Section 4.4 we present a recursive algebraic construction of the D-polynomials, which is summarized in its simplest form in Theorem 6. In Section 4.5 we give a construction of the canonical semi-internal $\mathcal{P}$-space basis polynomials in terms of explicit projection operators.

Finally, in Appendix A, we give an overview, usage examples, and implementation details of a new Python software library for interacting with the external order and zonotopal spaces using the SageMath open-source mathematics software system. Section A.1 deals with the `OrderedMatroid` class, which in particular provides a method to construct the external order of a desired ordered matroid. Section A.2 deals with a utility class `PolynomialFreeModule` which simplifies working with the vector space structure of polynomial rings in Sage. Last, Section A.3 discusses classes which compute ideal generators of the $\mathcal{I}$- and $\mathcal{J}$- ideals and canonical bases of the $\mathcal{P}$- and $\mathcal{D}$-spaces for the central, external, and internal zonotopal settings.

Chapter 2

Background

In the following, we review some of the concepts and theory that will be needed as background for the developments in the following chapters, and give additional references for further reading. Section 2.1 reviews basic notions related to matroids, antimatroids, and join-distributive lattices which are necessary for the development of the generalized external order in Chapter 3. Section 2.2 gives an introduction to the theory of zonotopal algebra, and summarizes the constructions of several variants of the zonotopal spaces which will be the subject of study in Chapter 4.

2.1 Matroids and Antimatroids

Throughout Chapter 3 will be studying the relations between several objects in the areas of lattice theory and discrete geometry, for which significant theory has been developed. We provide a brief review of relevant background here, and refer the reader to standard sources for additional details. The material in this section is adapted from the author's exposition in [21], Section 2.

For general matroid notions, Oxley [35] is comprehensive, and for concepts related to matroid activity, Björner [7] gives a concise overview. For the topics of greedoids and antimatroids, our primary references are Björner and Ziegler's survey [8], as well as the book [26] by Korte, Lovász and Schrader. General lattice theory is developed in detail in Stanley [36], Chapter 3, and the literature on join-distributive lattices is discussed in some detail in the introduction of Czédli [15].

Matroids

To begin, we define matroids, a combinatorial object which generalizes both the concept of linear independence of vectors in a vector space, and the concept of cycle-freeness of edge sets in a graph. The basic object of interest is the set system.

Definition 2.1.1. If E is a finite set, a **set system** is a pair $(E, \mathcal{F})$ where $\mathcal{F}$ is a nonempty collection of subsets of E. We will sometimes refer to $\mathcal{F}$ as a set system when we don't need to emphasize the ground set.

A common notation in the study of finite set systems is to use a string of lower-case characters or numbers to refer to a small finite set. For instance, if $a, b \in E$ are elements of a ground set, then the string ab denotes the set $\{a, b\}$. If $A \subseteq E$, then $A \cup ab$ denotes the set $A \cup \{a, b\}$. In practice this notation enhances rather than confounds communication, so we will adopt it in the present work when the meaning is clear from the context.

We can now define matroids in terms of their collections of "independent sets" as follows.

Definition 2.1.2. A set system $M = (E, \mathbb{I})$ is called a **matroid** if

- If $I \in \mathbb{I}$ and $J \subseteq I$, then $J \in \mathbb{I}$; and
- For $I, J \in \mathbb{I}$, if $|I| > |J|$, then there is an element $x \in I$ such that $J \cup x \in \mathbb{I}$.

A set in $\mathbb{I}$ is called an **independent set** of the matroid M.

The first property above is called the **hereditary property** for a set system, and the second is called the matroid **independence exchange axiom**.

The independence axioms for matroids are one of many different equivalent definitions of matroids frequently called "cryptomorphisms". Among the classical cryptomorphisms are axiom systems for bases, circuits, rank functions, closure operators, and the greedy algorithm. A fluent understanding of the definitions of these concepts and the relations between them will be helpful in the remainder of this work, and is explored in detail in [35] Chapter 1.

A pair of constructions which will be used frequently are the basic circuit and basic bond.

Definition 2.1.3. Let $M = (E, \mathbb{I})$ be a matroid, and let B be a basis of M. For $x \notin B$, define $\mathrm{ci}_M(B, x)$ the **basic circuit** of x in B to be the unique circuit contained in $B \cup x$.

Dually, for $b \in B$ define $\mathrm{bo}_M(B, b)$ the **basic cocircuit** or **basic bond** of b in B to be the unique cocircuit contained in $(E \setminus B) \cup b$.

A classical characterization of the basic circuit and basic bond is given by the following lemma.

Lemma 2.1.4. *Let M be a matroid with a basis B, and let $b \in B$ and $x \notin B$. Then the following are equivalent:*

- $b \in \mathrm{ci}(B, x)$
- $x \in \mathrm{bo}(B, b)$
- $B \setminus b \cup x$ *is a basis of* M

For notational convenience, we extend the definition of basic circuits and basic cocircuits in the following way.

Definition 2.1.5. Let M be a matroid, let $I \in \mathbb{I}(M)$, and denote $F = \mathrm{span}(I)$. For $x \in F \backslash I$, define

$$\mathrm{ci}(I, x) = \mathrm{ci}_{M|_F}(I, x),$$

and for $y \in I$, define

$$\mathrm{bo}(I, y) = \mathrm{bo}_{M|_F}(I, y).$$

For elements outside of F, neither of these expressions are defined.

A concept of fundamental importance in the remainder of this work is the notion of matroid activity.

Definition 2.1.6. An **ordered matroid** is a matroid $M = (E, \mathbb{I})$ along with a total order $\leq$ on the ground set E. We will frequently refer to M as an ordered matroid without specifying the order when no ambiguity arises.

Definition 2.1.7. Let $M = (E, \mathbb{I})$ be an ordered matroid, and let B be a basis of M. For $x \in E \setminus B$, we call x **externally active** with respect to B if x is the minimum element of the basic circuit $\mathrm{ci}(B, x)$, and **externally passive** otherwise. For $b \in B$, we call b **internally active** with respect to B if b is the minimum element of the basic cocircuit $\mathrm{bo}(B, b)$, and **internally passive** otherwise.

We denote the sets of externally active and externally passive elements with respect to a basis B by $\mathrm{EA}_M(B)$ and $\mathrm{EP}_M(B)$, and the sets of internally active and internally passive elements by $\mathrm{IA}_M(B)$ and $\mathrm{IP}_M(B)$.

Note in particular that the internal and external activities are dual notions. If M^* is the dual matroid of M, then $\mathrm{EA}_M(B) = \mathrm{IA}_{M^*}(E \setminus B)$, and similarly for the other sets.

Historically, the most important property of the notions of matroid activity is that they generate an important algebraic invariant of matroids called the Tutte polynomial.

Proposition 2.1.8. *Given an ordered matroid M, the Tutte polynomial of M is given by*

$$T_M(x, y) = \sum_{B \in \mathbb{B}(M)} x^{|\mathrm{IA}(B)|} y^{|\mathrm{EA}(B)|},$$

and is independent of the ordering of M.

The Tutte polynomial is what is called the *universal Tutte-Grothendieck invariant* for the class of all matroids, and in particular it encodes a breadth of combinatorial data corresponding to a matroid.

Greedoids

Greedoids are a generalization of matroids which capture the structure necessary for the matroid greedy algorithm to apply. The generalization gives rise to a rich hierarchy of subclasses, including matroids, which are outlined in exquisite detail in [8], Figure 8.5.

Definition 2.1.9. A set sytem $G = (E, \mathcal{F})$ is called a **greedoid** if

- For every non-empty $X \in \mathcal{F}$, there is an $x \in X$ such that $X \setminus x \in F$; and
- For $X, Y \in \mathcal{F}$, if $|X| > |Y|$, then there is an element $x \in X$ such that $Y \cup x \in \mathcal{F}$.

A set in $\mathcal{F}$ is called a **feasible set** of the greedoid E.

The first property above is a weakening of the matroid hereditary property called **accessibility**, and the second property above is exactly the matroid independence exchange axiom, which we sometimes will call the **greedoid exchange axiom** for clarity.

Antimatroids and Convex Geometries

For our discussion, the most important subclass of greedoids aside from matroids is the antimatroids, defined by:

Definition 2.1.10. A set system $(E, \mathcal{F})$ is called an **antimatroid** if

- $\mathcal{F}$ is a greedoid; and
- if $X \subseteq Y$ are sets in $\mathcal{F}$ and $a \in E \setminus Y$ with $X \cup a \in \mathcal{F}$, then $Y \cup a \in \mathcal{F}$.

The second property in this definition is called the **interval property without upper bounds**. Antimatroids as set system of feasible sets can be formulated in a variety of equivalent manners, and we will state for reference several of these which will also be useful.

Proposition 2.1.11 ([8], Proposition 8.2.7)**.** *If $\mathcal{F}$ is a set system, then the following conditions are equivalent.*

- *$\mathcal{F}$ is an antimatroid;*
- *$\mathcal{F}$ is accessible, and closed under taking unions; and*
- *$\emptyset \in \mathcal{F}$, and $\mathcal{F}$ satisfies the exchange axiom that if X, Y are sets in $\mathcal{F}$ such that $X \nsubseteq Y$, then there is an element $x \in X \setminus Y$ such that $Y \cup x \in \mathcal{F}$.*

Independent Sets, Circuits and Cocircuits

As with matroids, the theory of antimatroids admits a number of cryptomorphic definitions, which include a theory of **rooted circuits** and a dual theory of **rooted cocircuits**. For more details, see [8] Section 8.7.C as well as [26] Section 3.3.

Definition 2.1.12. If $(E, \mathcal{F})$ is a set system and $A \subseteq E$, define the **trace** $\mathcal{F}\!:\!A$ by

$$\mathcal{F}\!:\!A \coloneqq \{X \cap A \,:\, X \in \mathcal{F}\}\,.$$

If $\mathcal{F}$ is a greedoid, then $A \subseteq E$ is called **free** or **independent** if $\mathcal{F}\!:\!A = 2^A$. If A is not independent, it is called **dependent**.

Definition 2.1.13. If $(E, \mathcal{F})$ is a set system and $A \in \mathcal{F}$, then the **feasible extensions** of A are the elements of

$$\Gamma(A) := \{x \in E \setminus A \,:\, A \cup x \in \mathcal{F}\}\,.$$

The following lemma relates freeness to feasible extensions, and follows directly from Lemma 3.1 of [26].

Lemma 2.1.14. *If $(E, \mathcal{F})$ is an antimatroid, then $X \subseteq E$ is independent if and only if it is equal to the feasible extensions $\Gamma(A)$ of some feasible set $A \in \mathcal{F}$.*

Of particular note is that the collection of independent sets of an antimatroid is closed under taking subsets, and thus forms a simplicial complex as a set system. We will discuss more properties of independent sets and their relationship with feasible sets of an antimatroid in Section 3.1.

The cryptomorphisms of rooted circuits and rooted cocircuits are presented in terms of rooted sets:

Definition 2.1.15. If A is a set and $a \in A$, then the pair (A, a) is called a **rooted set** with **root** a. In this case, we may equivalently refer to A as a rooted set if the root is clear from context.

Now we can define the circuits of an antimatroid.

Definition 2.1.16. A **circuit** of an antimatroid $(E, \mathcal{F})$ is a minimal dependent subset of E.

In particular, the following holds for circuits of an antimatroid.

Proposition 2.1.17 ([8])**.** *If $(E, \mathcal{F})$ is an antimatroid and $C \subseteq E$, then there is a unique element $a \in C$ such that $\mathcal{F}\!:\!C = 2^C \setminus \{\{a\}\}$. We call the rooted set (C, a) a **rooted circuit** of $\mathcal{F}$.*

Let $\mathcal{C}(\mathcal{F})$ denote the collection of rooted circuits of an antimatroid $\mathcal{F}$. Rooted circuits give a cryptomorphism for antimatroids due to the following fundamental result.

Proposition 2.1.18 ([8], Proposition 8.7.11)**.** *Let $(E, \mathcal{F})$ be an antimatroid and $A \subseteq E$. Then A is feasible if an only if $C \cap A \neq \{a\}$ for every rooted circuit (C, a).*

That is, an antimatroid is fully determined by its collection of rooted circuits. Further, we can axiomatize the rooted families which give rise to an antimatroid.

Proposition 2.1.19 ([8], Theorem 8.7.12)**.** *Let $\mathcal{C}$ be a family of rooted subsets of a finite set E. Then $\mathcal{C}$ is the family of rooted circuits of an antimatroid if an only if the following two axioms are satisfied:*

(CI1) If $(C_1, a) \in \mathcal{C}$, then there is no rooted set $(C_2, a) \in \mathcal{C}$ with $C_2 \subsetneq C_1$.

(CI2) If $(C_1, a_1), (C_2, a_2) \in \mathcal{C}$ *and* $a_1 \in C_2 \setminus a_2$*, then there is a rooted set* $(C_3, a_2) \in \mathcal{C}$ *with* $C_3 \subseteq C_1 \cup C_2 \setminus a_1$.

Björner and Ziegler noted that these axioms bear a curious resemblance to the circuit axioms for matroids, and we will see in Section 3.2 that this resemblance is not superficial.

A second cryptomorphism for antimatroids is their rooted cocircuits, which form a certain type of dual to their rooted circuits.

Definition 2.1.20. If $(E, \mathcal{F})$ is an antimatroid and $F \in \mathcal{F}$, then an element $a \in F$ is called an **endpoint** of F if $F \setminus a \in \mathcal{F}$. If $F \in \mathcal{F}$ has a single endpoint a, then we call F a **cocircuit**, and we call the rooted set (F, a) a **rooted cocircuit** of $\mathcal{F}$. Equivalently, (F, a) is a rooted cocircuit iff $F \in \mathcal{F}$ is minimal containing a. We denote by $\mathcal{C}^*(\mathcal{F})$ the collection of rooted cocircuits of an antimatroid $\mathcal{F}$.

In many places in the literature, antimatroid cocircuits are also called *paths*, but we use the name cocircuit to emphasize their duality with antimatroid circuits. The descriptive power of these rooted sets is exemplified by the following lemma.

Lemma 2.1.21 ([26], Lemma 3.12)**.** *If* $(E, \mathcal{F})$ *is an antimatroid and* $A \subseteq E$*, then* A *is feasible if and only if it is a union of cocircuits. If* A *has* k *endpoints* $\{a_1, \ldots, a_k\}$*, then* A *is a union of* k *cocircuits* $\{A_1, \ldots, A_k\}$*, where the root of each* A_i *is* a_i.

In particular, this shows that the cocircuits of an antimatroid also uniquely determine the feasible sets. As with circuits, there is also an axiomatic characterization of the set systems which form the collection of rooted cocircuits of an antimatroid.

Proposition 2.1.22. *Let* $\mathcal{C}^* \subseteq \{(D, a) : D \subseteq E, a \in D\}$ *be a family of rooted subsets of a finite set* E*. Then* $\mathcal{C}^*$ *is the family of rooted cocircuits of an antimatroid* $(E, \mathcal{F})$ *if an only if the following two axioms are satisfied:*

(CC1) If $(D_1, a) \in \mathcal{C}^*$*, then there is no rooted set* $(D_2, a) \in \mathcal{C}^*$ *with* $D_2 \subsetneq D_1$.

(CC2) If $(D_1, a_1) \in \mathcal{C}^*$ *and* $a_2 \in D_1 \setminus a_1$*, then there is a rooted set* $(D_2, a_2) \in \mathcal{C}^*$ *with* $D_2 \subseteq D_1 \setminus a_1$.

Since rooted circuits and rooted cocircuits suffice to specify an antimatroid, when convenient we will sometimes denote an antimatroid using these rooted set systems, as a pair $(E, \mathcal{C})$ or $(E, \mathcal{C}^*)$.

Finally, we describe the duality which relates the circuits and cocircuits of an antimatroid.

Definition 2.1.23. If E is a finite set and $\mathcal{U}$ is a collection of subsets of E, then $\mathcal{U}$ is called a **clutter** if no set in $\mathcal{U}$ is contained in another. If $\mathcal{U}$ is a clutter, then the **blocker** of $\mathcal{U}$, denoted $\mathcal{B}(\mathcal{U})$ is the collection of minimal subsets

$$\mathcal{B}(\mathcal{U}) := \min \{V \subseteq E : V \cap U \text{ is nonempty for each } U \in \mathcal{U}\}.$$

A basic result of blockers is that the operation of taking blockers is an involution on clutters.

Lemma 2.1.24. *For any clutter $\mathcal{U}$, the blocker $\mathcal{V} = \mathcal{B}(\mathcal{U})$ is a clutter, and $\mathcal{B}(\mathcal{V}) = \mathcal{U}$.*

In particular, this involution provides the essential connection between antimatroid circuits and cocircuits.

Definition 2.1.25. If $\mathcal{A}$ is a collection of rooted subsets of a ground set E and $x \in E$, let $\mathcal{A}_x$ denote the collection of sets $\{A \setminus x \,:\, (A, x) \in \mathcal{A}\}$.

Proposition 2.1.26. *Let $(E, \mathcal{F})$ be an antimatroid with circuits and cocircuits $\mathcal{C}$ and $\mathcal{C}^*$ respectively. Then for each $x \in E$, we have that $\mathcal{C}_x$ and $\mathcal{C}^*_x$ are clutters, and $\mathcal{C}^*_x$ is the blocker of $\mathcal{C}_x$ and vice versa.*

Minors

Finally, we will recall two notions of minors which may be defined respectively for greedoids and for antimatroids. First, we give the standard definitions of deletion and contraction for general greedoids.

Definition 2.1.27. If $G = (E, \mathcal{F})$ is a greedoid and $A \subseteq E$, then the **greedoid deletion** $G \setminus A$ is the set system $(E \setminus A, \mathcal{F} \setminus A)$, where

$$\mathcal{F} \setminus A = \{F \subseteq E \setminus A \,:\, F \in \mathcal{F}\}\,.$$

The **greedoid contraction** $G \,/\, A$ is the set system $(E \setminus A, F \,/\, A)$ where

$$F \,/\, A = \{F \subseteq E \setminus A \,:\, F \cup A \in \mathcal{F}\}\,.$$

A greedoid deletion $G \setminus A$ is always a greedoid, while in general a greedoid contraction $G \,/\, A$ is a greedoid only when A is feasible, as otherwise $\emptyset$ is not included in the resulting set system.

A greedoid minor is a deletion of a contraction of a greedoid. Aside from the limitation that the contracting set is feasible, greedoid minors behave like matroid minors in that the deletion and contraction operations commute with themselves and each other.

We provide these definitions for arbitrary greedoids primarily for background and context. For antimatroids in particular, there is an alternate formulation of minors based on rooted circuits which will be central to the discussion in Section 3.4.

Definition 2.1.28. If $\mathcal{A} = (E, \mathcal{C})$ is an antimatroid with rooted circuits $\mathcal{C}$ and $A \subseteq E$, then the **antimatroid deletion** $\mathcal{A} \setminus A$ is the pair $(E \setminus A, \mathcal{C} \setminus A)$ where

$$\mathcal{C} \setminus A = \{(C, x) \,:\, (C, x) \in \mathcal{C}, C \cap S = \emptyset\}\,.$$

The **antimatroid contraction** $\mathcal{A} \,/\, A$ is the pair $(E \setminus A, \mathcal{C} \,/\, A)$ where

$$\mathcal{C} \,/\, A = \min \{(C \setminus S, x) \,:\, (C, x) \in \mathcal{C}, x \notin S\},$$

and where $\min \mathcal{R}$ for a collection $\mathcal{R}$ of rooted sets denotes the subcollection of those which are (non-strictly) minimal under inclusion as non-rooted sets.

In particular, these deletion and contraction operations produce antimatroids, and also behave like matroid minors.

Proposition 2.1.29 ([18], Propositions 12 and 14)**.** *If $\mathcal{A} = (E, \mathcal{F})$ is an antimatroid and $A \subseteq E$, then $\mathcal{A} \,/\, A$ and $\mathcal{A} \setminus A$ are antimatroids. If $A, B \subseteq E$ are disjoint, then*

- $(\mathcal{A} \setminus A) \setminus B = (\mathcal{A} \setminus B) \setminus A$
- $(\mathcal{A} \setminus A) \,/\, B = (\mathcal{A} \,/\, B) \setminus A$
- $(\mathcal{A} \,/\, A) \,/\, B = (\mathcal{A} \,/\, B) \,/\, A$

An antimatroid minor may then be defined as a deletion of a contraction of an antimatroid. Although not immediately obvious from the circuit definition, these operations may also be characterized in the following way in terms of antimatroid feasible sets.

Proposition 2.1.30. *If $(E, \mathcal{F})$ is an antimatroid and $A \subseteq E$, then*

- *$\mathcal{F} \setminus A$ is given by the trace $\mathcal{F} \colon (E \setminus A)$*
- *$\mathcal{F} \,/\, A$ is given by the greedoid deletion $\mathcal{F}/A = \{F \in \mathcal{F} \,:\, F \cap A = \emptyset\}$*

Antimatroid deletion by a set A can in general be thought of as collapsing the edges of the antimatroid Hasse diagram whose labels for the natural edge labeling (see Definition 3.1.1) are elements of A.

Lattice Theory

We now review necessary background on the class of posets called join-distributive lattices, which fundamentally connect antimatroids with lattice theory. To begin, we briefly review some standard definitions of lattice theory, as discussed in [36], Chapter 3.

Definition 2.1.31. A **partially ordered set** or **poset** is a set P along with a binary relation $\leq$ satisfying

- *Reflexibity:* For all $x \in P$, $x \leq x$
- *Antisymmetry:* For $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$.
- *Transitivity:* For $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

A poset P is called **finite** if the set P is finite, and **locally finite** if any interval

$$[x, y] \coloneqq \{z \in P \,:\, x \leq z \leq y\}$$

is finite. If $x, y \in P$ are elements such that $x < y$ and there is no element z such that $x < z < y$, then we say that y **covers** x, and write $y \gtrdot x$. We will denote by $\mathrm{Cov}(P) \subseteq P \times P$ the set of **covering pairs** of P,

$$\mathrm{Cov}(P) \coloneqq \{(x, y) \,:\, x \lessdot y\} \,.$$

Lattices are a particular type of poset which admits meets and joins of poset elements.

Definition 2.1.32. If $(P, \leq)$ is a poset, then for $x, y \in P$, an element $z \in P$ is called a **least upper bound** of x and y if $z \geq x$ and $z \geq y$ and for any $w \in P$ satisfying $w \geq x$ and $w \geq y$ we have $w \geq z$. As usual, a least upper bound of x and y is unique, and is denoted $x \vee y$, or "x join y". A **greatest lower bound** of x and y is defined similarly, and the unique greatest lower bound of x and y is denoted $x \wedge y$, or "x meet y".

A poset P is called a **lattice** if each pair of elements $x, y \in P$ has a least upper bound and a greatest lower bound.

The meet and join operations of a lattice are commutative, associative, and idempotent. If x, y are lattice elements, we additionally have

- $x \wedge (x \vee y) = x \vee (x \wedge y) = x$.
- $x \wedge y = x$ iff $x \vee y = y$ iff $x \leq y$.

If a lattice L is finite (and nonempty), then we additionally note the existence of a minimal element 0 and a maximal element 1, such that $0 \leq x$ and $x \leq 1$ for every $x \in L$.

The theory of matroids is classically related to lattice theory by considering the collection of flats of a matroid (sets F such that $\mathrm{cl}(F) = F$), which form a type of lattice called a *geometric lattice*.

Definition 2.1.33. A lattice L is called **semimodular** or **upper semimodular** if for all $x, y \in L$, if $x \gtrdot x \wedge y$, then $x \vee y \gtrdot y$.

Definition 2.1.34. If L is a finite lattice, then an **atom** of L is an element $x \in L$ such that $x \gtrdot 0$. We say that L is **atomistic** if every element can be expressed as the join of some collection of atoms.

Definition 2.1.35. A lattice L is called a **geometric lattice** if it is semimodular and atomistic.

If M is a matroid, let $\mathcal{F}(M)$ denote the poset of flats of M ordered by inclusion. We say that M is **simple** if it has no circuits of size 1 or 2. The connection between matroids and geometric lattices is then given by the following.

Proposition 2.1.36. *If M is a matroid, then $\mathcal{F}(M)$ is a geometric lattice, and any geometric lattice is isomorphic to $\mathcal{F}(M)$ for some matroid M. Further, if L is a geometric lattice, then there exists a unique simple matroid M (up to isomorphism) such that $L = \mathcal{F}(M)$.*

We will see as a central result in Chapter 3 that a new class of lattices, the *matroidal join-distributive lattices*, form a a refinement of the geometric lattice of flats of a matroid which encodes the isomorphism class of an *arbitrary* matroid.

The class of join-distributive lattices in particular has been extensively studied in relation to greedoid theory and abstract convexity theory. We now review relevant definitions and structural properties, following the exposition of [15].

Definition 2.1.37. A lattice L is called **meet semidistributive** if it satisfies the meet semidistributive law, that for all $x, y \in L$ and for any $z \in L$, if $x \wedge z = y \wedge z$, then the common value of these meets is $(x \vee y) \wedge z$.

Definition 2.1.38. Given a lattice L, an element $x \in L$ is called meet-irreducible if it is covered by exactly one element of L, and is called join-irreducible if it covers exactly one element of L. We denote the set of meet-irreducibles of L by $\mathrm{MI}(L)$, and the set of join-irreducibles of L by $\mathrm{JI}(L)$.

Definition 2.1.39. Given a lattice L and an element $x \in L$, an **irredundant meet decomposition** of x is a representation $x = \bigwedge Y$ with $Y \subseteq \mathrm{MI}(L)$ such that $x \neq \bigwedge Y'$ for any proper subset Y' of Y. The lattice L is said to have **unique meet-irreducible decompositions** if each $x \in L$ has a unique irredundant meet-decomposition.

Definition 2.1.40. If $x \in L$ is a member of a locally finite lattice, let $j(x)$ denote the join of all elements covering x.

Using this terminology, we can define join-distributive lattices and give several equivalent formulations which will be variously useful for our discussion.

Definition 2.1.41. A finite lattice is called **join distributive** if it is semimodular and meet-semidistributive.

Proposition 2.1.42 ([15], Proposition 2.1)**.** *For a finite lattice L, the following are equivalent.*

1. *L is join-distributive*
2. *L has unique meet-irreducible decompositions*
3. *For each $x \in L$, the interval $[x, j(x)]$ is a boolean lattice*
4. *The length of each maximal chain in L is equal to $|\mathrm{MI}(L)|$.*

The most important property of join-distributive lattices for our purposes is a remarkable correspondence with antimatroids, very similar to the correspondence of Birkhoff's representation theorem for finite distributive lattices.

Definition 2.1.43 ([8])**.** Given a finite join-distributive lattice L, let $\mathcal{F}(L)$ denote the set system which is the image of the map $T : L \to 2^{\mathrm{MI}(L)}$ given by

$$T : x \mapsto \{y \in \mathrm{MI}(L) \,:\, y \ngeq x\}\,.$$

Proposition 2.1.44 ([8], Theorem 8.7.6)**.** *T is a poset isomorphism from L to $\mathcal{F}(L)$ ordered by inclusion, and joins in L correspond to unions in $\mathcal{F}(L)$. $\mathcal{F}(L)$ is an antimatroid with ground set $\mathrm{MI}(L)$, and the poset $\mathcal{F}$ of feasible sets of any antimatroid, ordered by inclusion, forms a join-distributive lattice.*

Figure 2.1 demonstrates the application of this map to produce an antimatroid from a join-distributive lattice.

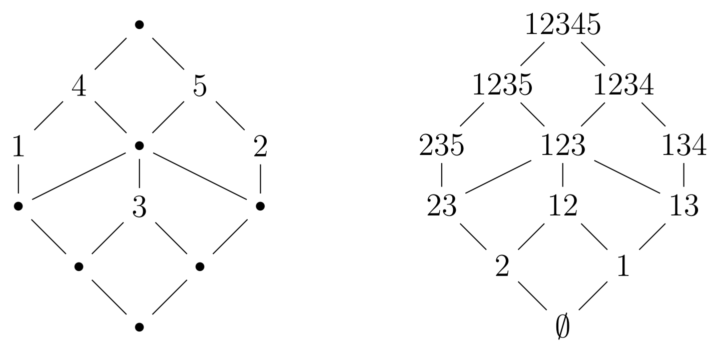


Figure 2.1: The T map applied to a join-distributive lattice with labeled meet irreducibles

The primary consequence of this correspondence is that join-distributive lattices are essentially equivalent to antimatroids: T gives a one-to-one correspondence between join-distributive lattices and antimatroids $\mathcal{F}$ which have no loops, or equivalently, for which the ground set E is covered by the feasible sets of $\mathcal{F}$.

Explicitly, if $\mathcal{F}$ is an antimatroid with ground set $E = \bigcup_{F \in \mathcal{F}} F$, let $L(\mathcal{F})$ denote the join-distributive lattice formed by the feasible sets of $\mathcal{F}$ under set inclusion. Then the elements of E are in bijection with the meet irreducibles of $L(\mathcal{F})$ by the map $x \mapsto S_x$, where $S_x \in \mathcal{F}$ is the unique meet irreducible in $L(\mathcal{F})$ covered by $(S_x \cup x) \in \mathcal{F}$. This bijection of ground sets induces a canonical isomorphism between $\mathcal{F}$ and $T(L(\mathcal{F}))$.

In general, we will allow for antimatroids with loops. This introduces a slight ambiguity in the equivalence between antimatroids and join-distributive lattices, as an antimatroid with loops has the same feasible sets and associated join-distributive lattice as a corresponding antimatroid with loops removed. This should not cause confusion in practice, however, so we will often refer to general antimatroids and join-distributive lattices interchangeably, keeping this subtlety in mind.

2.2 Zonotopal Algebra

We now present definitions and fundamental results in the theory of zonotopal algebra which will necessary for the developments undertaken in Chapter 4. The material in this section is primarily derived from [23], [24] and [30], and Holtz and Ron's exposition in [23] in particular gives an excellent overview of the broader connections of the theory.

One detail important to mention before we proceed is that throughout this section and Chapter 4, we will be adopting an ordering convention that is common in the zonotopal algebra literature, but which is unusual in other settings. Specifically, when discussing matroid activity, we will use definitions that correspond with the standard definitions in matroid theory, but using the *reverse* ordering on the ground set. For instance, we will say that an element x in an ordered matroid is externally active with respect to a basis B if it is *maximal* in the fundamental circuit $\mathrm{ci}(B, x)$. This does not have a substantive impact on any relevant structure of matroid activity, but it does mean that an extra ordering reversal is necessary when referencing e.g. material in Chapter 3 on the external order.

Polynomial Spaces and the Differential Bilinear Form

The central objects of the theory are certain spaces of polynomials which are related by a differential action. We now recall the basic concepts, and we set notation for use later. For the material related to differential kernels, we follow in somewhat simplified form the exposition of [23], Section 2.5.

Throughout this work we will be working with vectors in the real vector space $\mathbb{R}^d$, and will assume as background standard notions and notations of linear algebra, such as vector space spans, subspaces, sums and direct sums, coordinates, projections, inner products, etc. Particular notation that will be used throughout Chapter 4 is for coordinates with respect to a fixed basis: if $B \subseteq \mathbb{R}^d$ is a basis of $\mathbb{R}^d$ and $b \in B$, then for $x \in \mathbb{R}^d$, we write $[x]_B^b$ to denote the b-coordinate of x represented in terms of the basis B.

In the following, we will use Π to denote symmetric algebra $\mathbb{R}[x_1, \ldots, x_d]$ of $\mathbb{R}^d$. If $I \subseteq \Pi$ is an ideal generated by a set Q, we will write $I = \mathrm{Ideal}(Q)$. Additionally denote the **algebraic variety** $\mathrm{Var}(I)$ of I by

$$\mathrm{Var}(I) := \left\{ v \in \mathbb{R}^d \,:\, p(v) = 0 \text{ for all } p \in I \right\}.$$

Linear forms and products of linear forms in Π play an extensive role in zonotopal algebra, so we will adopt the following notation to simplify working with them. If $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$, then denote by ℓ_v the homogeneous linear polynomial

$$\ell_v := \sum_i v_i x_i.$$

If $S \subseteq \mathbb{R}^d$ is a finite collection of vectors, let ℓ_S denote the product

$$\ell_S := \prod_{v \in S} \ell_v.$$

Another construction fundamental to the theory is the differential operator derived from a polynomial, as well as the corresponding differential pairing.

Definition 2.2.1. If $p \in \Pi$ is a real multivariate polynomial, then let $p(\partial)$ denote the differential operator

$$p(\partial) \coloneqq p\left(\partial/\partial x_1, \ldots, \partial/\partial x_d\right).$$

If $q \in \Pi$, additionally define the pairing $\langle p, q\rangle$ by

$$\langle p, q\rangle \coloneqq (p(\partial)q)\big|_{x=0}.$$

The pairing $\langle\cdot, \cdot\rangle$ is a bilinear form on Π, which we will refer to as the **differential bilinear form**. It is symmetric and nondegenerate, and the monomials x^α form an orthogonal basis of Π. Considering the homogeneous linear polynomials as an embedding of $\mathbb{R}^d$ into its symmetric algebra Π, the bilinear form gives a canonical extension of the standard inner product to Π.

A simple but important property of the bilinear form is the fact that multiplication and differentiation by a polynomial are adjoint operators:

Lemma 2.2.2. *If $p_1, p_2, q \in \Pi$, then*

$$\langle qp_1, p_2\rangle = \langle p_1, q(\partial)p_2\rangle.$$

For polynomials which are products of linear forms, repeated applications of the product rule for derivatives allows us to describe the behavior of the differential bilinear form as an explicit sum.

Lemma 2.2.3. *If $p_1 = \prod_{i=1}^k \ell_i^{(1)}$ and $p_2 = \prod_{i=1}^k \ell_i^{(2)}$, where $\ell_i^{(j)} \in \Pi$ are homogeneous linear polynomials, then*

$$\langle p_1, p_2\rangle = \sum_\sigma \left\langle \ell_i^{(1)}, \ell_{\sigma(i)}^{(2)}\right\rangle,$$

where the sum is over all permutations $\sigma \in S_k$.

When a particular linear form has higher multiplicity in a product, the above representation allows us to deduce a simpler sufficient condition for polynomial orthogonality.

Lemma 2.2.4. *Let $p = \prod_{i=1}^k \ell_i$ be a product of linear forms. If $q = \ell^k$ is a power of a single linear form, then $\langle p, q\rangle = 0$ iff at least one of the linear forms ℓ_i are orthogonal to ℓ. If $q = \ell^j q_0$ where $q_0 \in \Pi$ is homogeneous of degree $k - j$, then $\langle p, q\rangle = 0$ if at least $k - j + 1$ of the linear forms ℓ_i are orthogonal to ℓ.*

A more general characterization of the above orthogonality is given in Lemma 2.2.10 in terms of the notion of the *linear support* of a polynomial or collection of polynomials.

Linear Support of Polynomial Subspaces

We now introduce terminology to describe polynomial subrings of $\Pi = \mathbb{R}[x_1, \ldots, x_d]$ identified with linear subspaces of the ambient vector space $\mathbb{R}^d$, and we describe the interactions between such subrings and the differential bilinear form. The material is related to the classical commutative algebra notion of *polynomial apolarity* (see for instance [25]), but our discussion here is self-contained.

Definition 2.2.5. If $U \subseteq \mathbb{R}^d$ is a subspace, let Π_U denote the polynomial subring of Π given by

$$\Pi_U := \mathbb{R}[\ell_u : u \in U].$$

Equivalently, by linear algebra considerations, if $S \subseteq U$ is any spanning set, then

$$\Pi_U = \mathbb{R}[\ell_u : u \in S].$$

These linear polynomial subrings relate to the underlying vector space geometry of $\mathbb{R}^d$ in a straightforward fashion.

Lemma 2.2.6. *If $x \in \Pi$ and $U \subseteq \mathbb{R}^d$ is a subspace, then $x \in \Pi_U$ if and only if $\ell_v(\partial)x = 0$ for each $v \in U^\perp$.*

Proof. Denote $V = U^\perp$, and let B_U, B_V be orthonormal bases of U and V respectively, so that $B = B_U \cup B_V$ is an orthonormal basis of $\mathbb{R}^d$. Then x can be represented uniquely as a sum of monomials in the variables ℓ_b, $b \in B$, and in particular, $x \in \Pi_U$ if and only if this representation does not include any monomials with the linear forms ℓ_v, $v \in B_V$. The result then follows from the product rule applied to the monomials of this representation, and the fact that $\ell_v(\partial)\ell_u = \langle v, u \rangle$ for vectors $u, v \in \mathbb{R}^d$. $\square$

Lemma 2.2.7. *If $U_1, U_2 \subseteq \mathbb{R}^d$ are subspaces, then*

- $\Pi_{U_1} \cap \Pi_{U_2} = \Pi_{U_1 \cap U_2}$
- $\Pi_{U_1} + \Pi_{U_2} = \Pi_{U_1 + U_2}$

Proof. By Lemma 2.2.6, $x \in \Pi_{U_1} \cap \Pi_{U_2}$ if and only if $\ell_v(\partial)x = 0$ for each $v \in U_1^\perp \cup U_2^\perp$, and hence by linearity for each $v \in U_1^\perp + U_2^\perp = (U_1 \cap U_2)^\perp$. This is true if and only if $x \in \Pi_{U_1 \cap U_2}$. The second equality holds similarly. $\square$

We now introduce notation to describe the minimum linear polynomial subring in which a given collection of polynomials can be represented. This notion will be useful for inductive constructions involving lower-dimensional zonotopal spaces, which are only properly defined as zero-dimensional ideals and their kernels in appropriate polynomial subrings.

Definition 2.2.8. If $S \subseteq \Pi$, define the **linear support** of S, denoted $\mathrm{Supp}(S)$, as the minimum subspace $U \subseteq \mathbb{R}^d$ such that $S \subseteq \Pi_U$.

Lemma 2.2.9. *If $S \subseteq \Pi$ and $V = \big\{v \in \mathbb{R}^d \,:\, \ell_v(\partial)x = 0$ for $x \in S\big\}$, then* $\operatorname{Supp}(S) = V^\perp$.

Proof. By Lemma 2.2.6, $U = V^\perp$ satisfies $S \subseteq \Pi_U$. If U' is another subspace such that $S \subseteq \Pi_{U'}$, then again by Lemma 2.2.6, we have $\ell_v(\partial)x = 0$ for $x \in S$ and $v \in V' := (U')^\perp$. In particular, $V' \subseteq V$, which implies that $U' \supseteq U$. This shows that U is minimal with $S \subseteq \Pi_U$, and thus is equal to $\operatorname{Supp}(S)$. □

Distinguishing polynomials by their linear supports in particular is frequently useful for simplifying the differential action of one polynomial on another.

Lemma 2.2.10. *Suppose $p, q_1, q_2 \in \Pi$, and $\operatorname{Supp}(p) \subseteq \operatorname{Supp}(q_1)^\perp$. Then*

$$p(\partial)\big(q_1 q_2\big) = q_1 p(\partial) q_2.$$

Proof. This is trivial when p is a constant. When p is a linear form, this follows by the derivative product rule, noting that $p(\partial)q_1 = 0$ in this case by Lemma 2.2.9. For general p, note that p can be written as a polynomial in the variables $\{\ell_{b_i}\}$ where $b_1, \dots, b_k$ form a basis of $\operatorname{Supp}(p)$. Thus the identity follows for monomials in this representation of p by induction on the degree, and then for p itself by linearity. □

Differential Kernels

Let $I \subseteq \Pi$ be an ideal, and let Π' denote the dual space of linear functionals on Π. The space Π' can be realized as the space of formal power series $\mathbb{R}[[x_1, x_2, \dots, x_d]]$, where a power series q is identified with the differential operation $\langle q, \cdot\rangle$. In particular, with respect to this identification, the exponential map $e_\alpha : t \mapsto \exp(\alpha \cdot t)$ for a fixed vector $\alpha \in \mathbb{R}^d$ plays the important role of the evaluation functional δ_α:

$$\langle e_\alpha, p\rangle = p(\alpha) = \delta_\alpha p.$$

We will be concerned with a particular subset of the annihilator in Π' of the ideal I called the **differential kernel**, or just **kernel**, $\ker I$ of I:

$$\ker I := \{e_\alpha p \,:\, \langle e_\alpha p, q\rangle = 0 \text{ for all } q \in I, \text{ where } \alpha \in \operatorname{Var}(I),\, p \in \Pi\}\,.$$

In general the set $\ker I$ is a proper subset of the annihilator of I, in particular because it consists only of exponential polynomials which in particular are absolutely convergent as power series. However, the collection still has the property that it distinguishes between ideals in Π, so that

$$\big(\ker I\big)^\perp := \{p \in \Pi \,:\, \langle f, p\rangle = 0 \text{ for all } f \in \ker I\} = I.$$

We will in particular be concerned with ideals which are *zero-dimensional* or *Artinian*, meaning their variety $\operatorname{Var}(I)$ is a finite collection of points, or equivalently that the quotient ring Π/I is finite-dimensional as an $\mathbb{R}$-vector space. In this case, $\ker I$ is a finite-dimensional space with dimension equal to that of Π/I.

The most important setting for our purposes is when I is additionally homogeneous, so that $\mathrm{Var}(I) = \{0\}$. In this case, the following gives a refined characterization of the kernel in terms of differential operators.

Lemma 2.2.11. *Let $I \subseteq \Pi$ be a zero-dimensional homogeneous ideal generated by a set Q. Then*

$$\ker I = \{p \in \Pi \,:\, q(\partial)p = 0 \text{ for all } q \in Q\}\,.$$

For non-homogeneous ideals, a rich duality exists between the kernel operator and the *least* and *most* maps, which map a polynomial (or power series for the least map) to respectively its lowest or highest degree homogeneous part. This duality is what enables the fundamental connection of the zonotopal spaces to interpolation theory, but the details are omitted here as they will not be needed for the developments of this dissertation. More information and additional references may be found in [23], Sections 2.4 and 2.5.

Zonotopal Spaces

We now have the tools and terminology needed to define the **zonotopal spaces**. In the following, let X denote a $d \times n$ real matrix, which is frequently thought of as an ordered multiset of column vectors in $\mathbb{R}^d$. We will define several variants of four polynomial spaces, each consisting of two polynomial ideals $\mathcal{I}$ and $\mathcal{J}$, and two finite-dimensional polynomial subspaces $\mathcal{P}$ and $\mathcal{D}$. The ideal $\mathcal{I}$ is in most cases a **power ideal**, i.e. is generated by powers of linear forms, and the ideal $\mathcal{J}$ is generated by products of linear forms which are related to the matroid combinatorics of the columns of X. The spaces $\mathcal{P}$ and $\mathcal{D}$ are differential kernels of the ideals $\mathcal{I}$ and $\mathcal{J}$ respectively, and each have dimension given by the size of a collection of bases of X which depends on the variant in question.

The following definition summarizes some of the main algebraic properties shared by the zonotopal spaces in different settings.

Definition 2.2.12. Let $\mathcal{I}$ and $\mathcal{J}$ be zero-dimensional homogeneous ideals in Π, and let $\mathcal{P}$ and $\mathcal{D}$ be vector subspaces of Π. We say that the collection $(\mathcal{I}, \mathcal{J}, \mathcal{P}, \mathcal{D})$ is **zonotopal** if it satisfies the following properties.

1. $\mathcal{J} \oplus \mathcal{P} = \mathcal{I} \oplus \mathcal{D} = \Pi$
2. $\mathcal{P} = \ker \mathcal{I}$, and $\mathcal{D} = \ker \mathcal{J}$
3. $\mathcal{D}$ acts as the dual to $\mathcal{P}$ under the identification $q \mapsto \langle q, \cdot \rangle$

For a zonotopal collection of spaces, we additionally can conclude that $\mathcal{J} \oplus \mathcal{D} = \mathcal{I} \oplus \mathcal{P} = \Pi$, and that $\mathcal{D}$ and $\mathcal{P}$ have identical Hilbert series. Algebraically, the $\mathcal{J}$ ideal and $\mathcal{P}$ space can be thought of as lying in one polynomial space while the $\mathcal{I}$ ideal and $\mathcal{P}$ space lies in the dual polynomial space under identification using the differential bilinear form. The distinction is not important for our purposes, so we will adopt the convention of [23] and consider all four spaces as lying in the same polynomial ring.

We now give definitions for the variants of zonotopal spaces which will be discussed throughout Chapter 4. We will use the following common notation throughout. Let U denote the column space of X, and let $M(X)$ denote the matroid whose ground set is the columns of X, and whose independent sets are the collections of columns that are linearly independent. We will frequently refer to X and $M(X)$ synonymously, as the matroid structure of X is of central importance in the constructions of zonotopal spaces. Then $\mathbb{B}(X)$ and $\mathbb{I}(X)$ denote the bases and independent sets of X respectively, and we let $\mathcal{F}(X)$ denote the **hyperplanes** (sometimes referred to as facet hyperplanes) of X, which are the flats H with $r(H) = r(X) - 1$.

For each $H \in \mathcal{F}(X)$, let $\eta_X(H)$ be a unit length normal vector of H in the column space U, and let n_H^X denote the linear form $\ell_{\eta(H)}$. Additionally let $m(H)$ denote the cardinality of the cocircuit complementary to H, that is, $m(H) = |X \setminus H|$. Finally, if $\mathcal{S} \subseteq 2^X$, define the **$\mathcal{S}$-long** subsets of X by

$$L(X, \mathcal{S}) \coloneqq \{Y \subseteq X \,:\, Y \cap S \neq \emptyset \text{ for each } S \in \mathcal{S}\},$$

and the **$\mathcal{S}$-short** subsets of X by

$$S(X, \mathcal{S}) \coloneqq \{Y \subseteq X \,:\, Y \text{ is disjoint from some } S \in \mathcal{S}\}.$$

The $\mathbb{B}(X)$-long sets are the dependent sets in the dual matroid of X (the *codependent* sets) while the $\mathbb{B}(X)$-short sets are the independent sets of the dual matroid of X (the *coindependent* sets), so the $\mathcal{S}$-long and short sets can be thought of as certain extensions of these notions.

As a general remark, note that the definitions of zonotopal spaces in many places in the literature assume that the underlying matrix X is of full rank. For our purposes, it will be convenient to remove this restriction and work with *relative* zonotopal spaces. This amounts to taking the standard full-rank construction thinking of X as a full-rank matrix with respect to its column space U. The underlying polynomial space is then the linear polynomial subring Π_U, which should be represented in terms of generating variables corresponding to an orthonormal basis of U to preserve relevant geometric structure. This introduces a slight clash of notation, as the usual definitions of zonotopal spaces are typically degenerate in the case when X is not of full rank. This should not cause confusion in our usage.

The **central zonotopal spaces** are defined using the (central) long and short sets $L(X) = L(X, \mathbb{B}(X))$ and $S(X) = S(X, \mathbb{B}(X))$, and are given by the following.

Definition 2.2.13 ([23] Section 3)**.** The *central* zonotopal spaces are defined by

- $\mathcal{I}(X) \coloneqq \text{Ideal}\left(\left(n_H^X\right)^{m(H)} : H \in \mathcal{F}(X)\right)$
- $\mathcal{J}(X) \coloneqq \text{Ideal}\left(\ell_Y : Y \in L(X)\right) = \text{Ideal}\left(\ell_D : D \text{ is a cocircuit of } X\right)$

- $\mathcal{P}(X) := \operatorname{span}\left(\ell_Y : Y \in S(X)\right)$
- $\mathcal{D}(X) := \{p \in \Pi \,:\, \ell_Y(\partial)\, p = 0,\ \text{for every } Y \in L(X)\} = \ker \mathcal{J}(X)$

To define the **external zonotopal spaces**, let B_{ext} denote an ordered collection of vectors in $\mathbb{R}^d$ which form a basis of the column space of X, which we will call the **extending basis**, and let X' denote the matrix (or ordered collection) obtained by appending the ordered column vectors in B_{ext} to the end of X.

Additionally, for each independent set $I \in \mathbb{I}(X)$, let $\operatorname{ex}(I)$ denote the greedy extension of I to a basis of X' using elements of B_{ext}, and let $\mathbb{B}_+(X)$ denote the set of **external bases**, given by $\mathbb{B}_+(X) := \{\operatorname{ex}(I) \,:\, I \in \mathbb{I}(X)\}$. Finally, let $L_+(X) := L(X', \mathbb{B}_+(X))$ and $S_+(X) := S(X, \mathbb{I}(X)) = 2^X$.

Definition 2.2.14 ([23] Section 4)**.** The *external* zonotopal spaces are defined by

- $\mathcal{I}_+(X) := \operatorname{Ideal}\left(\left(n_H^X\right)^{m(H)+1} : H \in \mathcal{F}(X)\right)$
- $\mathcal{J}_+(X) := \operatorname{Ideal}\left(\ell_Y : Y \in L_+(X)\right)$
- $\mathcal{P}_+(X) := \operatorname{span}\left(\ell_Y : Y \subseteq X\right) = \operatorname{span}\left(\ell_Y : Y \in S_+(X)\right)$
- $\mathcal{D}_+(X) := \{p \in \Pi \,:\, \ell_Y(\partial)\, p = 0,\ \text{for every } Y \in L_+(X)\} = \ker \mathcal{J}_+(X)$

The external spaces are generalized to the **semi-external zonotopal spaces** by introducing an extra parameter, given by a family of independent sets of X. Let $\mathcal{O}$ denote a collection of flats of X which is upward closed with respect to inclusion:

$$\text{If } F \in \mathcal{O} \text{ and } F' \supseteq F, \text{ then } F' \in \mathcal{O}.$$

Now let $\mathbb{I}' \subseteq \mathbb{I}(X)$ denote the collection of independent sets whose span lies in $\mathcal{O}$. Let $\mathbb{B}_+(X, \mathbb{I}')$ denote the set of semi-external bases, given by $\mathbb{B}_+(X, \mathbb{I}') := \{\operatorname{ex}(I) \,:\, I \in \mathbb{I}'\}$, and let $L_+(X, \mathbb{I}') := L(X', \mathbb{B}_+(X, \mathbb{I}'))$ and $S_+(X, \mathbb{I}') := S(X, \mathbb{I}')$. If F is a flat of X, let $F^\perp$ denote the orthogonal complement of $\operatorname{span}(F)$ in the ambient space U, and if Q is a collection of polynomials let $Q^{(j)}$ denote the degree j homogeneous polynomials in Q. Finally, let $\mathcal{O}'$ denote the flats of X not contained in $\mathcal{O}$, given by $\mathcal{O}' = \{\operatorname{span}(I) \,:\, I \in \mathbb{I}(X) \setminus \mathbb{I}'\}$.

Definition 2.2.15 ([24] Section 2.2)**.** The *semi-external* zonotopal spaces are defined by

- $\mathcal{I}_+(X, \mathbb{I}') := \mathcal{I}_+(X) + \operatorname{Ideal}\left\{\left(\Pi_{F^\perp}\right)^{(|X \setminus F|)} \,:\, F \in \mathcal{O}'\right\}$
- $\mathcal{J}_+(X, \mathbb{I}') := \operatorname{Ideal}\left(\ell_Y : Y \in L_+(X, \mathbb{I}')\right)$
- $\mathcal{P}_+(X, \mathbb{I}') := \operatorname{span}\left(\ell_Y : Y \in S_+(X, \mathbb{I}')\right)$
- $\mathcal{D}_+(X, \mathbb{I}') := \{p \in \Pi \,:\, \ell_Y(\partial)\, p = 0,\ \text{for every } Y \in L_+(X, \mathbb{I}')\} = \ker \mathcal{J}_+(X, \mathbb{I}')$

This definition in particular has the property that it interpolates between the central and the external definitions, where the central spaces appear for the choice $\mathbb{I}' = \mathbb{B}(X)$, and the external spaces appear for the choice $\mathbb{I}' = \mathbb{I}(X)$. Interestingly, the semi-external ideal $\mathcal{I}_+(X, \mathbb{I}')$ may not always be a power ideal, in contrast with the central and the external cases. In [24] the authors study additional restrictions on the collection $\mathbb{I}'$ which ensure that the semi-external $\mathcal{I}$-ideal is a power ideal.

The **internal zonotopal spaces** are defined using the notion of *internal activity* of a matroid basis. If $B \subseteq X$ is a basis and $b \in B$, then b is said to be **internally active** with respect to B if it is externally active with respect to $X \setminus B$ considered as a basis in the dual matroid of X. More concretely, b is internally active if it is the maximal element in the fundamental cocircuit $\mathrm{bo}_X(B, b)$. (As a reminder, this is the reverse ordering convention of the most standard definition of this notion.)

We call a basis B an **internal basis** if B has no internally active elements, and we denote the collection of internal bases of X by $\mathbb{B}_-(X)$. Additionally, we define the **barely long** subsets of X by $L_-(X) \coloneqq L(X, \mathbb{B}_-(X))$.

Definition 2.2.16 ([23] Section 5)**.** The *internal* zonotopal spaces are defined by

- $\mathcal{I}_-(X) \coloneqq \mathrm{Ideal}\left(\left(n_H^X\right)^{m(H)-1} : H \in \mathcal{F}(X)\right)$
- $\mathcal{J}_-(X) \coloneqq \mathrm{Ideal}\left(\ell_Y : Y \in L_-(X)\right)$
- $\mathcal{P}_-(X) \coloneqq \bigcap_{x \in X} \mathcal{P}(X \setminus x)$
- $\mathcal{D}_-(X) \coloneqq \{p \in \Pi \,:\, \ell_Y(\partial)\, p = 0, \text{ for every } Y \in L_-(X)\} = \ker \mathcal{J}_-(X)$

The internal spaces are generalized to the **semi-internal zonotopal spaces** by specifying an additional independent set $I_{\mathrm{int}} \subseteq X$ chosen greedily from the elements at the end of X, so that I_{int} is lex maximal among independent sets of X of its rank. More specifically, if $x = \min(I_{\mathrm{int}})$, then we require that for any $y > x$ in $X \setminus I_{\mathrm{int}}$, we have that y is spanned by the elements of I_{int} larger than it.

Additionally define the **semi-internal bases** $\mathbb{B}_-(X, I_{\mathrm{int}})$ and the **semi-internal facet hyperplanes** $\mathcal{F}(X, I_{\mathrm{int}})$ by

$$\begin{aligned} \mathbb{B}_-(X, I_{\mathrm{int}}) &\coloneqq \{B \in \mathbb{B}(X) \,:\, B \text{ has no internally active element in } I_{\mathrm{int}}\} \\ \mathcal{F}(X, I_{\mathrm{int}}) &\coloneqq \{H \in \mathcal{F}(X) \,:\, I_{\mathrm{int}} \nsubseteq H\}\,. \end{aligned}$$

We call a cocircuit D semi-internal if $D = X \setminus H$ for $H \in \mathcal{F}(X, I_{\mathrm{int}})$, and we define the collection $L_-(X, I_{\mathrm{int}})$ of I_{int}-long sets by $L_-(X, I_{\mathrm{int}}) \coloneqq L(X, \mathbb{B}_-(X, I_{\mathrm{int}}))$.

Definition 2.2.17 ([24], Section 3.2)**.** The *semi-internal* zonotopal spaces are defined by

- $\mathcal{I}_-(X, I_{\mathrm{int}}) \coloneqq \mathrm{Ideal}\left(\left\{\eta_H^{m(H)-1} : H \in \mathcal{F}(X, I_{\mathrm{int}})\right\} \cup \left\{\eta_H^{m(H)} : H \in \mathcal{F}(X) \setminus \mathcal{F}(X, I_{\mathrm{int}})\right\}\right)$

- $\mathcal{J}_-(X, I_{\text{int}}) \coloneqq \text{Ideal}\big(\ell_Y : Y \in L_-(X, I_{\text{int}})\big)$
- $\mathcal{P}_-(X, I_{\text{int}}) \coloneqq \bigcap_{x \in I_{\text{int}}} \mathcal{P}(X \setminus x)$
- $\mathcal{D}_-(X, I_{\text{int}}) \coloneqq \{p \in \Pi : \ell_Y(\partial)\, p = 0, \text{ for every } Y \in L_-(X, I_{\text{int}})\} = \ker \mathcal{J}_-(X, I_{\text{int}})$

The semi-internal spaces in particular interpolate between the central and the internal spaces, where the central spaces are recovered by the choice of $I_{\text{int}} = \emptyset$, and the internal spaces are recovered by the choice of I_{int} as the lex maximal basis of X.

Concerning the definition of I_{int}, in [24] it is assumed when defining the semi-internal spaces that I_{int} consists of the *last* $|I_{\text{int}}|$ elements of X, so that I_{int} forms a pure suffix of the columns of X, and the proofs there are conditioned on this assumption. However, the definitions reduce to those for the standard internal spaces whenever I_{int} is a basis, *without* the additional requirement that I_{int} consist of a suffix of X. We thus present definitions without the extra assumption on I_{int}, and note that it is expected that the basic facts and theory of the semi-internal spaces are still valid in this slightly more general context.

One oddity of the definitions for both the internal and the semi-internal spaces is the indirect presentation of the $\mathcal{P}$-spaces as an intersection of central $\mathcal{P}$-spaces corresponding to submatrices of X. In Corollary 4.5.11 we give an alternate characterization of the semi-internal $\mathcal{P}$-space which matches the format of the central and external definitions, using an appropriate notion of short subsets and a new projection operation.

The spaces defined so far constitute what we call the *classical* zonotopal spaces (the central, external, and internal spaces), and the *semi-classical* zonotopal spaces (the semi-external and semi-internal space). A primary characteristic of these spaces is that they each satisfy the collection of basic properties required of zonotopal spaces:

Proposition 2.2.18 ([23, 24])**.** *The central, external, semi-external, internal, and semi-internal zonotopal spaces all form zonotopal collections in the sense of Definition 2.2.12.*

Another essential characteristic which we do not emphasize here is that all of the above spaces can be interpreted as spaces of polynomials *correct* (a notion in interpolation theory) with respect to certain vertex sets derived from hyperplane arrangements and zonotopes underlying the matrix X. These connections are explored in more detail in [23] and [24].

We now define the **forward exchange** or **generalized zonotopal spaces**, defined by Lenz in [30] to broadly generalize the zonotopal spaces defined above, as well as others. The spaces are defined in terms of a combinatorial object called a *forward exchange matroid*.

Definition 2.2.19. Let $M = (E, \mathbb{B})$ denote a matroid with ground set E and bases $\mathbb{B}$. If $\mathbb{B}' \subseteq \mathbb{B}$, we say that $\mathbb{B}'$ satisfies the **forward exchange property** if for any basis $B \in \mathbb{B}'$ and any x externally passive with respect to B we have that the basis B' given by

$$B' = B \setminus \max(\text{ci}(B, x)) \cup x$$

is also contained in $\mathbb{B}'$. If $\mathbb{B}'$ satisfies the forward exchange property, then we call the triple $(E, \mathbb{B}, \mathbb{B}')$ a **forward exchange matroid**.

Additionally, define the *generalized cocircuits* of a forward exchange matroid as follows.

Definition 2.2.20. If $M = (E, \mathbb{B}, \mathbb{B}')$ is a forward exchange matroid, then the **generalized cocircuits** of M, or **$\mathbb{B}'$-cocircuits**, are given by the minimal sets (under inclusion) in the collection $L(E, \mathbb{B}') = \{Y \subseteq E \,:\, Y \cap B \neq \emptyset \text{ for each } B \in \mathbb{B}'\}$.

In Section 4.1 we relate forward exchange matroids and generalized cocircuits to the *generalized external order*, which is explored in depth in Chapter 3.

In the following, let $\mathbb{B}'$ denote a subset of $\mathbb{B}(X)$ which satisfies the forward exchange property. Additionally, recalling that U denotes the column span of X, for $\eta \in U \setminus \{0\}$ let $H(\eta)$ denote the hyperplane orthogonal to η in U, and let $m(\mathbb{B}', \eta) \coloneqq \max_{B \in \mathbb{B}'} |\mathrm{EP}(B) \setminus H(\eta)|$.

Definition 2.2.21 ([30], Section 7)**.** The *forward exchange* zonotopal spaces are defined by

- $\mathcal{I}(X, \mathbb{B}') \coloneqq \mathrm{Ideal}\left(\ell_\eta^{m(\mathbb{B}',\eta)+1} : \eta \in U \setminus \{0\}\right)$
- $\mathcal{J}(X, \mathbb{B}') \coloneqq \mathrm{Ideal}\left(\ell_Y : Y \in L(X, \mathbb{B}')\right) = \mathrm{Ideal}\left(\ell_Y : Y \text{ is a } \mathbb{B}'\text{-cocircuit}\right)$
- $\mathcal{P}(X, \mathbb{B}') \coloneqq \mathrm{span}\left(\ell_Y : Y = \mathrm{EP}(B) \text{ for } B \in \mathbb{B}'\right)$
- $\mathcal{D}(X, \mathbb{B}') \coloneqq \{p \in \Pi \,:\, \ell_Y(\partial)\, p = 0 \text{ for each } \mathbb{B}'\text{-cocircuit}\} = \ker \mathcal{J}(X, \mathbb{B}')$

These definitions in particular give the central, external, and semi-external zonotopal spaces when specialized to $\mathbb{B}(X)$, $\mathbb{B}_+(X)$ and $\mathbb{B}_+(X, \mathbb{I}')$, each collection of which satisfies the forward exchange property. They also give the $\mathcal{J}$-ideal and the $\mathcal{D}$-space for the internal and semi-internal cases when specialized to $\mathbb{B}_-(X)$ and $\mathbb{B}_-(X, I_{\mathrm{int}})$, but notably, not the $\mathcal{I}$-ideal and $\mathcal{P}$-space.

General forward exchange zonotopal spaces satisfy some, but not all, of the properties typically expected of zonotopal spaces, summarized in the following.

Proposition 2.2.22. *If $\mathcal{I}$, $\mathcal{J}$, $\mathcal{P}$ and $\mathcal{D}$ denote the forward exchange zonotopal spaces associated with a forward exchange matroid $M = (X, \mathbb{B}, \mathbb{B}')$, then*

- $\mathcal{J} \oplus \mathcal{P} = \Pi$
- $\mathcal{D} = \ker \mathcal{J}$
- *$\mathcal{D}$ acts as the dual to $\mathcal{P}$ under the identification $q \mapsto \langle q, \cdot\rangle$*

In particular, the forward exchange $\mathcal{P}$-space, which should be the differential kernel of a homogeneous ideal $\mathcal{I}$, is not always even closed under differentiation, a necessary condition for being a differential kernel of this kind. The forward exchange ideal as defined is the only power ideal for which the relation $\mathcal{P} = \ker \mathcal{I}$ could hold, but in general we can only be assured that $\mathcal{P} \subseteq \ker \mathcal{I}$.

Despite these difficulties, many of the properties required for zonotopal spaces are satisfied by these definitions, and we conjecture that a modification of the forward exchange $\mathcal{P}$-space construction may yield a space with improved properties. As a first step in understanding what such a construction might look like, in Section 4.5 we characterize the difference between the semi-internal $\mathcal{P}$-space and the corresponding forward exchange $\mathcal{P}$-space for the semi-internal bases.

Chapter 3

The Generalized External Order

The classical notion of matroid activity plays an important role in understanding fundamental properties of a matroid, including the h-vector of its independence complex and the matroid Tutte polynomial. In 2001, Michel Las Vergnas introduced another structure derived from matroid activity, a collection of partial orders on the bases of a matroid which he called the **active orders** [27]. These orders elegantly connect matroid activity to a system of basis exchange operations, and are closely related to the broken circuit complex and the Orlik-Solomon algebra of a matroid.

In [2] and [3], the combinatorial structure of these active orders arises in relation to the initial ideal of certain projective varieties derived from affine linear spaces. In the theory of zonotopal algebra, the active orders connect to Lenz's **forward exchange matroids** [30], where the bases associated with a forward exchange matroid satisfy axioms which are equivalent to their forming a downward closed set in the external order.

In this chapter we will define a generalization of Las Vergnas's external order which extends the order to the independent sets of a matroid. If M is an ordered matroid, we define for each independent set I a set of externally passive elements, $\mathrm{EP}_M(I)$, using a general definition given in [27]. The external order can be generalized to independent sets by the following.

Definition. If I and J are independent sets of the ordered matroid M, then we define the **generalized external order** $\leq_{\mathrm{ext}}$ by

$$I \leq_{\mathrm{ext}} J \text{ iff } \mathrm{EP}_M(I) \supseteq \mathrm{EP}_M(J).$$

By [27], Proposition 3.1, this is equivalent to Las Vergnas's ordering in the case where I and J are two bases. For a variety of technical reasons, throughout this exposition we will instead work with the *reverse* of this ordering:

$$I \leq^*_{\mathrm{ext}} J \text{ iff } \mathrm{EP}_M(I) \subseteq \mathrm{EP}_M(J).$$

Whenever we refer to the "external order" throughout this work, we will be referring to this reversed order unless otherwise noted. We use distinct notation for these two orders to reduce ambiguity, particularly because there are other contexts in which Las Vergnas's original ordering convention fits more naturally with existing literature.

By associating each independent set with its corresponding set of externally passive elements, we define a set system $\mathcal{F}_{\text{ext}} := \{\text{EP}(I) : I \in \mathbb{I}(M)\}$, and show:

Theorem. *If M is an ordered matroid, then the set system $\mathcal{F}_{\text{ext}}$ of externally passive sets of M is an* antimatroid.

An **antimatroid** is a special class of greedoid which appears particularly in connection with convexity theory. Specifically, associated with any antimatroid is a **convex closure operator**, a closure operator on the ground set which combinatorially abstracts the operation of taking a convex hull, in the same way that a matroid closure operator abstracts the operation of taking a linear span. The convex closure operator on an ordered matroid derived from $\mathcal{F}_{\text{ext}}$ in particular bears a strong similarity to the convex closure operator for *oriented* matroids, which were first explored by Las Vergnas in [28].

In a 1985 survey paper [20], American mathematicians Paul Edelman and Robert Jamison noted:

> The authors have previously referred to these objects by the cacophonous name of 'antimatroids'. We hope there is time to rectify this and that Gresham's Law does not apply to mathematical nomenclature.

In the intervening 30 years, the name nevertheless appears to have become ensconced in the mathematical literature. However, in light of our Theorem 1 and other structural results of antimatroids, the name is perhaps not so poorly chosen, as the generalized external order provides an explicit connection between antimatroids and their combinatorial namesake.

The characterization of $\mathcal{F}_{\text{ext}}$ as an antimatroid further allows us to connect the external order with the large existing literature on lattice theory. The feasible sets of an antimatroid have a highly structured inclusion ordering called a **join-distributive lattice**[1], which is thus inherited by the generalized external order. Moreover, the poset is in fact a refinement of the geometric lattice of flats associated with the matroid M, obtained by suitably combining copies of Las Vergnas's original external order for the different flats of M.

Figure 3.1 compares Las Vergnas's external order with the generalized order for the linear matroid represented by the columns of the matrix

$$X = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

where the numbers 1 through 4 indicate the column number, labeled from left to right.

Las Vergnas's original construction required the inclusion of an additional zero element (the '$*$' in Figure 3.1) in order to form a proper lattice structure. In the generalized order,

[1] In fact, join-distributive lattices are essentially equivalent to antimatroids via a construction similar to that of Birkhoff's representation theorem.

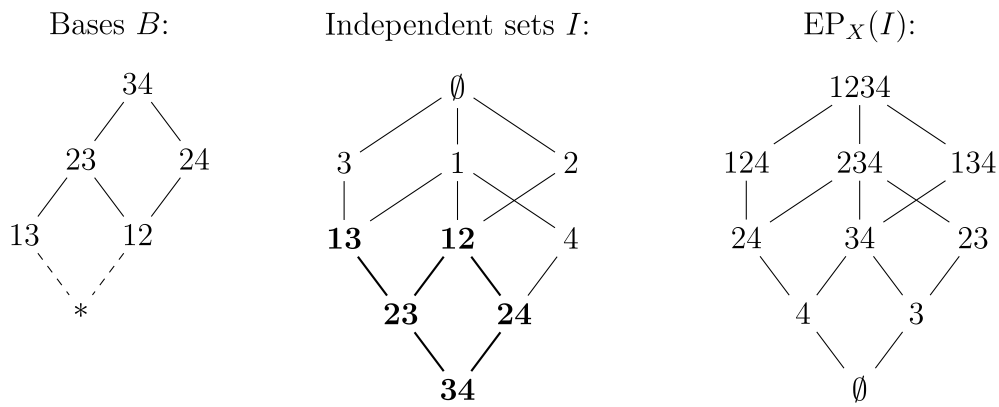


Figure 3.1: Las Vergnas's external order $\leq_{\mathrm{ext}}$ on bases B, the generalized order $\leq_{\mathrm{ext}}^*$ on independent sets I, and the corresponding externally passive sets $\mathrm{EP}_X(I)$. Note that Las Vergnas's order embeds in the generalized order (in bold) in reversed orientation.

bases whose meet in the original order would have been the extra zero element instead are joined at an independent set of lower rank.

The fact that the external order comes from an antimatroid allows us to describe features of the lattice structure combinatorially. In addition, using results of Gordon and McMahon [22] for general greedoids, we are able to further derive the following explicit partition of the boolean lattice.

Proposition. *If M is an ordered matroid with ground set E, then the intervals*

$$[I, I \cup \mathrm{EA}(I)] \text{ for } I \text{ independent}$$

form a partition of the boolean lattice 2^E.

This partition bears a resemblance to the well-known partition of Crapo, described in [7], and in fact it can be shown that this partition is a proper refinement of Crapo's.

Another main purpose of this chapter is to discuss the way in which the external orders fit into the context of antimatroids and join-distributive lattices. To refine our understanding, we characterize a proper subclass of the join-distributive lattices which we call **matroidal join-distributive lattices**.

Definition. Given a lattice L and an element $x \in L$, let $r_c(x)$ denote the number of elements in L which cover x. A join-distributive lattice L is called **matroidal** if r_c is decreasing in L, and it satisfies the semimodular inequality

$$r_c(x \wedge y) + r_c(x \vee y) \leq r_c(x) + r_c(y).$$

For an element x of a join-distributive lattice L, one can associate a set $I(x)$ called the **independent set** corresponding with x. If L is the external order lattice of an ordered matroid M, then the I operator recovers the matroid independent sets of M. Even for an arbitrary join-distributive lattice, the collection of independent sets is closed under taking subsets, and thus forms a simplicial complex. This join-distributive independence complex in fact provides an alternate characterization of matroidal join-distributive lattices.

Theorem. *A join-distributive lattice L is matroidal if and only if its independent sets are those of a matroid.*

In particular, this shows that the external order of an ordered matroid is a matroidal join-distributive lattice. This result goes a long way towards understanding where the external order sits among all join-distributive lattices, but surprisingly, there are matroidal join-distributive lattices which are not an external order. If we denote the class of join-distributive lattices by JD, the class of matroidal join-distributive lattices by MJD, and the class of lattices derived from the external order by EO, then

$$\mathrm{EO} \subsetneq \mathrm{MJD} \subsetneq \mathrm{JD}.$$

Figure 3.2 in Section 3.3 gives an example of a lattice in MJD but not EO, and Figure 2.1 in Section 2.1 gives an example of a lattice in JD but not MJD.

A further refinement is necessary to precisely classify the lattices isomorphic to an external order, and that refinement comes from the notion of **edge lexicographic** or **EL-shellability**. A graded poset P is EL-shellable if its Hasse diagram admits a labeling of its edges by integers which satisfies certain lexicographic comparability conditions on unrefinable chains. EL-shellability of a graded poset implies shellability of its order complex, and the notion has been widely studied for different classes of posets.

The external order is EL-shellable, and in fact it satisfies a stronger property called $\mathbf{S_n}$ **EL-shellability**. We study how S_n EL-shellability relates to antimatroids, and we show that

Theorem. *A finite lattice L is isomorphic to the external order $\leq^*_{\mathrm{ext}}$ of an ordered matroid if and only if it is join-distributive, matroidal, and S_n EL-shellable.*

McNamara introduced S_n EL-shellability in [32] as a way to characterize the *supersolvable* lattices of Stanley [37], and in particular, he proved that the two properties are equivalent. This implies that one may replace "S_n EL-shellable" with "supersolvable" in the above classification of the external order.

The remainder of the document is structured as follows. Section 3.1 develops technical results relating feasible and independent sets of join-distributive lattices. Section 3.2 constructs the generalized external order and explores its structure and connections with greedoid theory, characterizes matroidal join-distributive lattices, and relates them to S_n EL-shellability. Section 3.4 relates the deletion and contraction operations of matroids and antimatroids. Finally, Section 3.5 explores the downward covering relations of the external order in terms of the newly defined **spread** operator.

We note that the majority of the work in this chapter is based on the author's exposition in [21], which has been submitted to the *Electronic Journal of Combinatorics* for publication. The primary exception to this is Section 3.5, which is new material that has not previously been disseminated.

3.1 Feasible and Independent Sets of Join-distributive Lattices

Before embarking on the main new results of this chapter, we develop some theory in the realm of antimatroids and join-distributive lattices which will be useful later. Our aim is to explore the robust connections between the independent sets and the feasible sets of an antimatroid, so we will work in the equivalent context of join-distributive lattices, which provide a more symmetric way to represent these set systems.

To begin, we give some notation to describe covering relations and independent sets in join-distributive lattices.

Definition 3.1.1. Let L be a join-distributive lattice. Recall from Definition 2.1.43 the map $T : L \to 2^{\mathrm{MI}(L)}$ which maps L to its associated antimatroid, and let $e : \mathrm{Cov}(L) \to \mathrm{MI}(L)$ denote the **natural edge labeling**, given by $e : (x, y) \mapsto T(y) \setminus T(x)$. Such set differences are singletons, hence the map is well-defined into $\mathrm{MI}(L)$.

Definition 3.1.2. If $x \in L$ is an element of a join-distributive lattice, let $I(x)$ denote the set of elements

$$I(x) = \{e(x, y) \,:\, y \in L, (x, y) \in \mathrm{Cov}(L)\},$$

and let $J(x)$ denote the set of elements

$$J(x) = \{e(w, x) \,:\, w \in L, (w, x) \in \mathrm{Cov}(L)\}.$$

$I(x)$ is the **independent set** associated to x, and is equal to the independent set of feasible extensions of $T(x)$ in the antimatroid corresponding to L. We adopt the following additional notation.

Definition 3.1.3. If L is a join-distributive lattice,

- Let $\mathcal{F}(L) = (\mathrm{MI}(L), \{T(x) \,:\, x \in L\})$ denote the (loopless) antimatroid associated with L
- Let $\mathbb{I}(L) = \{I(x) \,:\, x \in L\}$ denote collection of independent sets of L
- Let $\mathcal{C}(L)$ denote the collection of rooted circuits of $\mathcal{F}(L)$, which we interchangeably refer to as the rooted circuits of L

Notice that $I(x)$ is disjoint from $T(x)$, and $J(x)$ is a subset of $T(x)$. The meet-irreducible elements $x \in \mathrm{MI}(L)$ are characterized by the condition $|I(x)| = 1$, in which case $I(x) = \{x\}$. The join-irreducible elements $y \in \mathrm{JI}(L)$ are characterized by the condition $|J(x)| = 1$, and in particular correspond with the rooted cocircuits of $\mathcal{F}(L)$.

Of particular importance is the following:

Lemma 3.1.4. *For $x, y \in L$ elements of a join-distributive lattice, $T(x)$ has empty intersection with $I(y)$ if and only if $x \leq y$.*

Proof. If $x \leq y$, then $T(x) \subseteq T(y)$. If $a \in I(y) \cap T(x)$, then a is a member of both $I(y)$ and $T(y)$, contradicting disjointness.

Otherwise, $x \vee y > y$. In particular, there is a covering element y_a for some $a \in I(y)$ such that $T(y_a) = T(y) \cup a$, and $y_a \leq x \vee y$. Thus $a \in T(y_a) \subseteq T(x \vee y) = T(x) \cup T(y)$, so because $a \notin T(y)$ we conclude that $a \in T(x) \cap I(y)$. □

Corollary 3.1.5. *The map $I : L \to \mathbb{I}(L)$ is one-to-one.*

Proof. If $x, y \in L$ satisfy $I(x) = I(y)$, then $T(x) \cap I(y) = T(y) \cap I(x) = \emptyset$, so $x \leq y$ and $y \leq x$. □

In particular, an element of a join-distributive lattice is uniquely identified with its corresponding independent set. In fact, this property characterizes the antimatroids among all greedoids.

Proposition 3.1.6. *A greedoid $(E, \mathcal{F})$ is an antimatroid if and only if the feasible extension operator $\Gamma : A \mapsto \{x \in E \setminus A \,:\, A \cup x \in \mathcal{F}\}$ is one-to-one.*

Proof. The forward direction is just restating Corollary 3.1.5 in the context of antimatroids. So suppose that $\mathcal{F}$ is a greedoid and the map Γ is one-to-one.

To see that $\mathcal{F}$ is an antimatroid, we prove that it satisfies the interval property without upper bounds. As a base case, suppose $A, B \in \mathcal{F}$ with $B = A \cup x$ for some $x \notin A$. If $A \cup y \in \mathcal{F}$ for some $y \notin B$, we want to show that $B \cup y \in \mathcal{F}$ as well.

Suppose this is not the case, so that $B \cup y = A \cup xy \notin \mathcal{F}$. Then we will show that $A \cup x$ and $A \cup y$ are mapped to the same set under Γ. To this end, suppose that $z \in \Gamma(A \cup x)$ for some z, so that $A \cup xz \in \mathcal{F}$.

Then in particular, $|A \cup y| < |A \cup xz|$, so by the greedoid exchange axiom we know there is an element $w \in (A \cup xz) \setminus (A \cup y) = xz$ such that $A \cup yw \in \mathcal{F}$. However, by assumption we know that $A \cup xy \notin \mathcal{F}$, so we must have $w = z$. Then $A \cup yz \in \mathcal{F}$, so $z \in \Gamma(A \cup y)$.

This implies that $\Gamma(A \cup x) \subseteq \Gamma(A \cup y)$. A symmetric argument proves the reverse inclusion, so we see that Γ maps the two sets to the same independent set, a contradiction. We conclude that in this context, $B \cup y = A \cup xy \in \mathcal{F}$.

In general if $A, B \in \mathcal{F}$ with $A \subsetneq B$, then by repeatedly applying the greedoid exchange axiom, there is a sequence of covering sets $A_i \in \mathcal{F}$ with

$$A = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_k = B,$$

where $A_{i+1} = A_i \cup x_i$ for some $x_i \in E$. The interval property without upper bounds follows by inducting on the length of this chain using the previous base case. $\square$

As mentioned previously, the independent sets of an antimatroid are closed under taking subsets, and so form a simplicial complex. In terms of the lattice structure of L, we get a stronger fact, that the inclusion order on the complex embeds in L in the following way. If $A \in \mathbb{I}(L)$, let x_A denote the corresponding lattice element $I^{-1}(A)$.

Lemma 3.1.7. *If J is an independent set of a join-distributive lattice L, and $I \subseteq J$, then I is independent, and $x_I \geq x_J$.*

Proof. If $I \subsetneq J$, there is a lattice element $x_{J'} \gtrdot x_J$ such that $I \subseteq J'$. This follows because if $a \in J \backslash I$, then by definition of independent sets, there is a covering element $x_{J'} \gtrdot x_J$ such that $T(x_{J'}) \setminus T(x_J) = \{a\}$. In particular, because L is join-distributive, the interval $[x_J, j(x_J)]$ is boolean, and so $j(x_{J'}) \geq j(x_J)$. Noting that for any $x \in L$ the relation $I(x) = T(j(x)) \backslash T(x)$ holds, we have

$$J' = T(j(x_{J'})) \setminus T(x_{J'}) \supseteq T(j(x_J)) \setminus (T(x_J) \cup a) = J \setminus a \supseteq I.$$

Since L is of finite length, repeated applications of the above must terminate, producing a saturated chain whose greatest element is x_K for an independent set K satisfying $K \supseteq I$ but not $K \supsetneq I$. Hence $I = K$ is independent, and $x_I \geq x_J$. $\square$

We now state and prove some additional lemmas concerning independent sets of join-distributive lattices which will be useful in later sections.

Lemma 3.1.8. *If $x \leq y$ in a join-distributive lattice L, then $I(x) \subseteq I(y) \cup T(y)$.*

Proof. Suppose that $a \in I(x)$, and $a \notin T(y)$. Then there is an element $x_a \gtrdot x$ such that $T(x_a) = T(x) \cup a$, and by the antimatroid interval property without upper bounds, there must be an element $y_a \in L$ such that $T(y_a) = T(y) \cup a$, and so we have $y_a \gtrdot y$. We conclude that $a \in I(y)$. $\square$

Lemma 3.1.9. *If I, J are independent sets of a join-distributive lattice L, then if $x_I \wedge x_J = x_K$ for K independent, then $K \subseteq I \cup J$.*

Proof. Let $a \in K$, and suppose that $a \notin I \cup J$. Since $x_K \leq x_I, x_J$, we know that $a \in I \cup T(x_I)$ and $a \in J \cup T(x_J)$. Thus since a is in neither I nor J, we can conclude that $a \in T(x_I) \cap T(x_J)$.

However, since $a \in K$, there exists K' independent such that $T(x_{K'}) = T(x_K) \cup a$. Since $T(x_K) \subseteq T(I) \cap T(J)$ and $a \in T(I) \cap T(J)$, we have that $T(x_{K'}) \subseteq T(I) \cap T(J)$. We see now that $x_K \lessdot x_{K'} \leq x_I, x_J$, and this contradicts the claim that x_K is the meet of x_I and x_J. $\square$

If $A \subseteq \mathrm{MI}(L)$, let x_A denote the meet of all elements x_I for $I \subseteq A$ independent. The element $x_A \in L$ is equal to x_K for some independent set K, and by induction on Lemma 3.1.9, we have that $K \subseteq A$. Let $I(A)$ denote this independent set, and note that if A is itself independent, then $I(A) = A$ by Lemma 3.1.7.

Lemma 3.1.10. *If $A, B \subseteq \mathrm{MI}(L)$, then $x_A \vee x_B \leq x_{A\cap B}$, and $x_A \wedge x_B \leq x_{A\cup B}$.*

Proof. For the first inequality, let I be independent with $I \subseteq A \cap B$, and note that $I \subseteq A$ and $I \subseteq B$, so $x_A \leq x_I$ and $x_B \leq x_I$. In particular, $x_A \vee x_B \leq x_I$, so since this holds for arbitrary $I \subseteq A \cap B$, it is also true for the meet of all such elements, hence $x_A \vee x_B \leq x_{A\cap B}$.

For the second inequality, let I be independent with $I \subseteq A \cup B$, and let $I_1 = I \cap A$, and $I_2 = I \cap B$. By Lemma 3.1.7 both I_1 and I_2 are independent, and they satisfy $x_{I_1}, x_{I_2} \geq x_I$. Thus $x_{I_1} \wedge x_{I_2} \geq x_I$, and in fact we will see that $x_{I_1} \wedge x_{I_2} = x_I$.

If K is independent with $x_K = x_{I_1} \wedge x_{I_2}$, then by Lemma 3.1.9, we have that $K \subseteq I_1 \cup I_2 = I$. For $a \in I$, suppose without loss of generality that $a \in I_1$. In particular, $a \notin T(x_{I_1})$, and this implies $a \notin T(x_K)$ because $x_{I_1} \geq x_K$. But by Lemma 3.1.8, since $x_K \geq x_I$, we have that $I \subseteq K \cup T(x_K)$, and so we conclude that $a \in K$. Since $a \in I$ was arbitrary, we thus have $I \subseteq K$, so the two sets are equal.

Finally, note that since $x_{I_1} \wedge x_{I_2} = x_I$ and $I_1 \subseteq A$, $I_2 \subseteq B$, we have that $x_A \wedge x_B \leq x_I$. Since I was chosen arbitrarily in $A \cup B$, we conclude $x_A \wedge x_B \leq x_{A\cup B}$. □

3.2 Definition and Fundamental Properties

In [27], Michel Las Vergnas defined partial orderings on the bases of an ordered matroid which are derived from the notion of matroid activity. His external order, defined in terms of matroid external activity, is the starting point for the coming developments.

Definition 3.2.1 ([27])**.** Let M be an ordered matroid. Then **Las Vergnas's external order** on the set of bases of M is defined by

$$B_1 \leq_{\text{ext}} B_2 \text{ iff } \mathrm{EP}(B_1) \supseteq \mathrm{EP}(B_2).$$

The poset obtained by this definition depends on the ordering associated with M, but has some suggestive properties, summarized in the following.

Proposition 3.2.2. *Let $M = (E, \mathbb{I})$ be an ordered matroid, and let $P = (\mathbb{B}(M), \leq^*_{\text{ext}})$ be the external order on the bases of M. Let L denote the poset P with an additional minimal element $\mathbf{0}$ added to the ground set. Then*

- *P is a graded poset, graded by $|\mathrm{EP}(B)|$*
- *Two bases B_1 and B_2 satisfy a covering relation $B_1 \lessdot B_2$ in P iff $B_2 = B_1 \setminus b \cup a$, where $b \in B_1$, and a is the maximal element of $\mathrm{bo}(B_1, b)$ externally active with respect to B_1. In this case, $\mathrm{EP}(B_2) = \mathrm{EP}(B_1) \cup b$*
- *L is a lattice with combinatorially defined meet and join operators*

A dual order, the **internal order**, can be derived from the external order on the dual ordered matroid M^*, and has analogous properties.

The Generalized External Order

In the same paper, Las Vergnas defined a generalized notion of matroid activity which will be the key to generalizing the external order.

Definition 3.2.3. Let $M = (E, \mathbb{I})$ be an ordered matroid, and let $A \subseteq E$. Then we say that $x \in E$ is **M-active** with respect to A if there is a circuit C of M with $x \in C \subseteq A \cup x$ such that x is the smallest element of C. We denote the set of such M-active elements by $\mathrm{Act}_M(A)$, and define

1. $\mathrm{EA}_M(A) \coloneqq \mathrm{Act}_M(A) \setminus A$
2. $\mathrm{EP}_M(A) \coloneqq (E \setminus A) \setminus \mathrm{EA}_M(A)$
3. $\mathrm{IA}_M(A) \coloneqq \mathrm{Act}_{M^*}(E \setminus A) \cap A$
4. $\mathrm{IP}_M(A) \coloneqq A \setminus \mathrm{IA}_M(A)$

In particular, the above definition reduces to the classical definition of matroid activity when A is chosen to be a basis of M.

One of the primary properties of external activity that allows the construction of the external lattice on bases is the fact that the map

$$B \mapsto \mathrm{EP}(B)$$

is one-to-one. This characteristic fails spectacularly for the generalized definition of external activity. However, when we restrict our attention to independent sets, the situation is better.

Lemma 3.2.4. *Let $M = (E, \mathbb{I})$ be an ordered matroid, and let $I \in \mathbb{I}$. Then if F is the flat spanned by I, we have*

$$\mathrm{Act}_M(I) = \mathrm{Act}_{M|_F}(I).$$

Proof. Suppose that $x \in \mathrm{Act}_{M|_F}(I)$. Then $x \in F$, and there is a circuit C of $M|_F$ such that $x \in C \subseteq I \cup x$ and x is the smallest element of C. However, the circuits of $M|_F$ are just the circuits of M which are contained in F, so in particular we have that C is also a circuit of M, which shows that $x \in \mathrm{Act}_M(I)$.

Now suppose that $x \in \mathrm{Act}_M(I)$. Then there is a circuit C of M such that $x \in C \subseteq I \cup x$ and x is the smallest element of C. In particular, we have that $C \setminus x$ is an independent subset of F.

If $x \notin F$, then we would have $x \notin \mathrm{span}(C \setminus x)$, which would imply that $C = (C \setminus x) \cup x$ is independent, a contradiction. Thus it must be the case that $x \in F$. This means that $C \subseteq I \cup x \subseteq F$, so this implies that C is also a circuit of $M|_F$. Since C still satisfies the conditions required by the definition of activity in $M|_F$, we conclude that $x \in \mathrm{Act}_{M|_F}(I)$. $\square$

In particular, we have the following.

Corollary 3.2.5. *If $M = (E, \mathbb{I})$ is an ordered matroid and $I \in \mathbb{I}$ with $F = \mathrm{span}(I)$, then*

$$\mathrm{EP}_M(I) = \mathrm{EP}_{M|_F}(I) \cup (E \setminus F),$$

and in particular,

$$F = \mathrm{span}(E \setminus \mathrm{EP}_M(I)).$$

Proof. The first equality follows directly from the above lemma, noting that $\mathrm{Act}_M(I) = \mathrm{Act}_{M|_F}(I) \subseteq F$. The second equality follows because

$$I \subseteq E \setminus \mathrm{EP}_M(I) \subseteq F.$$

□

Corollary 3.2.6. *If M is an ordered matroid, then the map $\mathrm{EP}_M : \mathbb{I} \to 2^E$ is one-to-one.*

Proof. From previous theory we know that EP is one-to-one when restricted to the bases of a matroid. Now let I, J be distinct independent sets of M, with $F_I = \mathrm{span}(I)$ and $F_J = \mathrm{span}(J)$. If $F_I \neq F_J$, then by the above lemma,

$$\mathrm{span}(E \setminus \mathrm{EP}_M(I)) = F_I \neq F_J = \mathrm{span}(E \setminus \mathrm{EP}_M(J)).$$

Thus in this case the two passive sets cannot be equal.

If $F_I = F_J$, call this common spanning flat F. Then I and J are distinct bases of the restriction matroid $M|_F$. This gives that $\mathrm{EP}_{M|_F}(I) \neq \mathrm{EP}_{M|_F}(J)$, so

$$\mathrm{EP}_M(I) = \mathrm{EP}_{M|_F}(I) \cup (E \setminus F) \neq \mathrm{EP}_{M|_F}(J) \cup (E \setminus F) = \mathrm{EP}_M(J)$$

because the unions with $(E \setminus F)$ are disjoint unions. □

With this result in mind, we extend Las Vergnas's external order to the independent sets of an ordered matroid.

Definition 3.2.7. Let M be an ordered matroid. Then the **external order** on the independent sets of M is defined by

$$I_1 \leq_{\mathrm{ext}} I_2 \text{ iff } \mathrm{EP}(I_1) \supseteq \mathrm{EP}(I_2).$$

In particular, because EP restricted to the bases of M is the same as the classical definition used by Las Vergnas, the original external order on the bases of M appears as a subposet of this generalization. As noted in the introduction, for technical convenience we will work with the *reverse* of this order,

$$I_1 \leq^*_{\mathrm{ext}} I_2 \text{ iff } \mathrm{EP}(I_1) \subseteq \mathrm{EP}(I_2).$$

Whenever we refer to the external order, we will be referring to the reversed order $\leq^*_{\mathrm{ext}}$ unless otherwise noted.

To understand the properties of the generalized external order, we will relate the notion of matroid external activity to an analogous notion for antimatroids, as follows. We first note that the rooted circuits of an antimatroid can be thought of as minimal obstructions to extending feasible sets.

Lemma 3.2.8. *Let $(E, \mathcal{F})$ be an antimatroid with associated join-distributive lattice L, let $x \in L$, and let $a \in E \setminus T(x)$. Then $a \in I(x)$ if and only if each rooted circuit (C, a) of $\mathcal{F}$ has nonempty intersection with $T(x)$.*

Proof. If $a \in I(x)$, then $T(x) \cup a$ is a feasible set. If a rooted circuit (C, a) is disjoint from $T(x)$, then the intersection of C with $T(x) \cup a$ is equal to the singleton set $\{a\}$. However, this violates the definition from Proposition 2.1.17 of the root of a rooted circuit.

On the other hand, if each rooted circuit (C, a) has nonempty intersection with $T(x)$, then the intersection of C with $T(x) \cup a$ is not equal to the singleton set $\{a\}$, and so by Proposition 2.1.18, we have that $T(x) \cup a \in \mathcal{F}$, so $a \in I(x)$. □

A consequence of this fact is that the rooted circuits of an antimatroid allow us to recover the feasible set associated to a given independent set without reference to any other global structure of the antimatroid.

Lemma 3.2.9. *Let $(E, \mathcal{F})$ be an antimatroid with associated join-distributive lattice L, and let $x \in L$. Then*

$$T(x) = \{a \in E \setminus I(x) \,:\, C \nsubseteq I(x) \cup a \text{ for any } (C, a) \in \mathcal{C}(\mathcal{F})\}\,.$$

Proof. Let $T_0(x)$ denote the set in the right side of the equality, and let a be an arbitrary element in $E \setminus I(x)$. If $a \notin T_0(x)$, then there is a rooted circuit $(C, a) \in \mathcal{C}$ such that $C \subseteq I(x) \cup a$. But then $C \setminus a \subseteq I(x)$, so $C \cap T(x)$ is either $\{a\}$ if $a \in T(x)$ or empty if $a \notin T(x)$. By Proposition 2.1.18, since $T(x) \in \mathcal{F}$, we see that $C \cap T(x) \neq \{a\}$, so we conclude that in this case, $a \notin T(x)$. Thus $T(x) \subseteq T_0(x)$.

Now suppose that $a \in T_0(x)$. If $I(x) \cup a$ is independent, say $I(y) = I(x) \cup a$, then by Lemma 3.1.7 we know that $x \geq y$, so by Lemma 3.1.8, $I(y) \subseteq I(x) \cup T(x)$, and thus $a \in T(x)$.

If $I(x) \cup a$ is not independent, it contains a rooted circuit $(C, b) \in \mathcal{C}$. Since any subset of $I(x)$ is independent and thus not a circuit, we must have that $a \in C$. However, a cannot be the root of C because in this case $C \subseteq I(x) \cup a$ violates the fact that $a \in T_0(x)$. However, if $b \neq a$ then $b \in I(x)$, so by Lemma 3.2.8 we have that $C \cap T(x)$ is nonempty. Since all elements of C aside from a are in $I(x)$ which is disjoint from $T(x)$, we conclude that $a \in T(x)$. Thus $T_0(x) \subseteq T(x)$ as well. □

In light of this lemma, it makes sense to define the external activity in an antimatroid as follows.

Definition 3.2.10. Let $(E, \mathcal{F})$ be an antimatroid with rooted circuits $\mathcal{C}$, and let I be an independent set. Then for $a \in E \setminus I$, we say that a is **externally active** with respect to I if there exists a rooted circuit $(C, a) \in \mathcal{C}$ such that $C \subseteq I \cup a$. Otherwise we say that a is **externally passive**.

We denote the active elements of $\mathcal{F}$ by $\mathrm{EA}_{\mathcal{F}}(I)$, and the passive elements by $\mathrm{EP}_{\mathcal{F}}(I)$, where the subscripts may be omitted if there is no risk of ambiguity. If L is a join-distributive lattice, then $\mathrm{EA}_L(x)$ and $\mathrm{EP}_L(x)$ denote the active and passive elements of $I(x)$ in the associated antimatroid $\mathcal{F}(L)$.

In particular, for $x \in L$ a join-distributive lattice, Lemma 3.2.9 shows that $T(x)$ is the set of externally passive elements of $I(x)$.

We can now connect the external order with the theory of antimatroids.

Proposition 3.2.11. *If M is an ordered matroid, then the collection of rooted sets*

$$\mathcal{C} = \mathcal{C}_{\mathrm{ext}}(M) := \{(C, \min(C)) \,:\, C \in \mathcal{C}(M)\}$$

satisfies the axioms of rooted antimatroid circuits.

Proof. For axiom (CI1), note that if (C_1, a) and (C_2, a) are in $\mathcal{C}$, then C_1 and C_2 are circuits of M, and thus C_1 is not a proper subset of C_2 by properties of matroid circuits.

For axiom (CI2), suppose $(C_1, a_1), (C_2, a_2) \in \mathcal{C}$ with $a_1 \in C_2 \setminus a_2$. By definition of $\mathcal{C}$ we know that $a_1 = \min(C_1)$ and $a_2 = \min(C_2)$, so in particular we know that $a_1 > a_2$, and $a_2 \notin C_1$.

Note that matroid circuits satisfy the following strong elimination axiom: If C_1, C_2 are circuits with $a_1 \in C_1 \cap C_2$ and $a_2 \in C_2 \setminus C_1$, then there is a circuit $C_3 \subseteq (C_1 \cup C_2) \setminus a_1$ which contains a_2.

Applying this elimination axiom to our present circuits, we obtain a matroid circuit $C_3 \subseteq (C_1 \cup C_2) \setminus a_1$ with $a_2 \in C_3$. a_2 is minimal in $C_1 \cup C_2$, so this implies that $a_2 = \min(C_3)$, and $(C_3, a_2) \in \mathcal{C}$. Thus $\mathcal{C}$ satisfies axiom (CI2) as well. $\square$

Proposition 3.2.11 allows us to conclude the following structural characterization of the generalized external order.

Definition 3.2.12. If M is an ordered matroid, let

$$\mathcal{F}_{\mathrm{ext}} = \mathcal{F}_{\mathrm{ext}}(M) := \{\mathrm{EP}_M(I) \,:\, I \in \mathbb{I}(M)\}\,.$$

Theorem 1. *If M is an ordered matroid, then $\mathcal{F}_{\mathrm{ext}}(M)$ is the collection of feasible sets of the antimatroid with rooted circuits $\mathcal{C}_{\mathrm{ext}}(M)$.*

Proof. Denote $M = (E, \mathbb{I})$. By Proposition 3.2.11, $\mathcal{C}_{\mathrm{ext}}(M)$ forms the rooted circuits of an antimatroid $(E, \mathcal{F})$. Let L be the associated join-distributive lattice. By definition of antimatroid circuits as minimal dependent sets, we have that $\mathbb{I}(L) = \mathbb{I}$ so that the sets $I(x), x \in L$ are in correspondence with the matroid independent sets of M.

By Lemma 3.2.9, any element $x \in L$ has

$$T(x) = \mathrm{EP}_L(x) = \{a \in E \setminus I(x) \,:\, C \nsubseteq I(x) \cup a \text{ for any } (C, a) \in \mathcal{C}_{\mathrm{ext}}(M)\}\,.$$

In particular, we can see that $\mathrm{EP}_L(x) = \mathrm{EP}_M(I(x))$ for each $x \in L$, and so the feasible set of $\mathcal{F}$ associated with each independent set $I(x)$ is given by the set of (matroid) externally passive elements of $I(x)$. Thus the feasible sets of $\mathcal{F}$ are exactly the sets in $\mathcal{F}_{\mathrm{ext}}(M)$, as we wished to show. $\square$

A further consequence of this argument is that the independent set associated with each feasible set $\mathrm{EP}_M(I)$ in $\mathcal{F}_{\mathrm{ext}}(M)$ is in fact I. Following from this correspondence with antimatroids, we may apply Proposition 2.1.44 to obtain the following.

Corollary 3.2.13. *If $M = (E, \mathbb{I})$ is an ordered matroid, then the external order $\leq^*_{\mathrm{ext}}$ on $\mathbb{I}$ is a join-distributive lattice. Meet-irreducible sets in the lattice correspond with the non-loops of E, and joins correspond to taking unions of externally passive sets.*

Combinatorial Structure

Using the antimatroid structure of the generalized external order, we are able to prove a variety of properties of the poset, many of which generalize the properties enjoyed by the classical order on matroid bases. In the following, $M = (E, \mathbb{I})$ denotes an ordered matroid.

Lemma 3.2.14. *The following basic properties hold for independent sets and externally passive sets in M.*

1. *If $I, J \in \mathbb{I}$, then $I \leq^*_{\mathrm{ext}} J$ if and only if $\mathrm{EP}(I) \cap J = \emptyset$*
2. *If $I, J \in \mathbb{I}$ and $J \supseteq I$, then $J \leq^*_{\mathrm{ext}} I$*
3. *If $I, J \in \mathbb{I}$, then $I \wedge J \subseteq I \cup J$*

Proof. The three parts are restatements of Lemmas 3.1.4, 3.1.7 and 3.1.9 respectively in the context of the generalized external order. □

Lemma 3.2.15. *If $M = (E, \mathbb{I})$ is an ordered matroid, $I \in \mathbb{I}$, and $a \in E \setminus \mathrm{EP}(I)$, the set $\mathrm{EP}(I) \cup a$ is the set of externally passive elements of some independent set iff $a \in I$.*

Proof. Let L be the join-distributive lattice associated with the antimatroid $\mathcal{F}_{\mathrm{ext}}(M)$, and let $x \in L$ be the element with $I(x) = I$. Then $I(x)$ is the set of feasible extensions of $T(x) = \mathrm{EP}(I)$, so $\mathrm{EP}(I) \cup a$ is feasible in $\mathcal{F}_{\mathrm{ext}}(M)$ iff $a \in I(x) = I$. The result follows because the feasible sets are exactly the sets of externally passive elements. □

We now characterize the covering relations in the external order.

Definition 3.2.16. For an ordered matroid M, if I us independent and $a \in I$, define the **active chain** of a in I to be the set

$$\mathrm{ch}(I, a) = \mathrm{EA}_M(I) \cap \mathrm{bo}(I, a).$$

Proposition 3.2.17. *Let M be an ordered matroid, and let $I \in \mathbb{I}(M)$. Then for each $a \in I$, define the independent set J_a by*

- *If $\mathrm{ch}(I, a)$ is nonempty, $J_a = I \setminus a \cup \max(\mathrm{ch}(I, a))$.*
- *If $\mathrm{ch}(I, a)$ is empty, $J_a = I \setminus a$.*

For each $a \in I$, we have $\mathrm{EP}(J_a) = \mathrm{EP}(I) \cup a$*, and thus the sets J_a are the independent sets covering I in the external order.*

Proof. Let $a \in I$, and denote $F = \mathrm{span}(I)$, $I_0 = I \setminus a$, and $F_0 = \mathrm{span}(I_0)$.

From Lemma 3.2.15 we know that there exists an independent set J such that $\mathrm{EP}(J) = \mathrm{EP}(I) \cup a$. Since $E \setminus F \subseteq \mathrm{EP}(I)$ and $\mathrm{EP}(J) \cap J = \emptyset$, we have that $J \subseteq F$.

Using the antimatroid interval property without upper bounds, with the fact that independent sets are the sets of antimatroid feasible extensions, we know that $I_0 \subseteq J$. Thus since J is independent and contained in F, either $J = I_0$, or $J = I_0 \cup b$ for some $b \in \mathrm{bo}(I, a)$. In the latter case, since $b \in J$, $b \notin \mathrm{EP}(J) = \mathrm{EP}(I) \cup a$, so this implies that b is an element of the active chain $\mathrm{ch}(I, a)$.

If $\mathrm{ch}(I, a)$ is empty, then we must be in the first case above, so $J = I \setminus a = J_a$ as desired.

If $\mathrm{ch}(I, a)$ is nonempty, let c be its maximal element, which in particular is in $F \setminus F_0$, and is not in $\mathrm{EP}(I) \cup a$. On one hand, suppose that $J = I_0$. Then $F \setminus \mathrm{span}(J) = F \setminus F_0 \subseteq \mathrm{EP}(J)$, so $c \in \mathrm{EP}(J)$, and this implies that $\mathrm{EP}(J) \neq \mathrm{EP}(I) \cup a$, a contradiction.

On the other hand, suppose that $J = I_0 \cup c'$ for some $c' \in \mathrm{ch}(I, a)$, $c' < c$. Then because $c \notin F_0$, we must have $\mathrm{ci}(J', c) \nsubseteq I_0 \cup c$, so $c' \in \mathrm{ci}(J', c)$. This implies that c is externally passive since $c' < c$, so again $\mathrm{EP}(J) \neq \mathrm{EP}(I) \cup a$.

Since there is only one remaining possibility for J, we conclude that $J = I \setminus a \cup c = J_a$. □

The downward covering relations are somewhat more complicated to describe in general, but a particular covering always exists.

Lemma 3.2.18. *Let M be an ordered matroid. If I is independent and $x = \min(\mathrm{EP}(I))$, then there is an independent set J such that* $\mathrm{EP}(J) = \mathrm{EP}(I) \setminus x$*.*

Proof. If $x \notin \mathrm{span}(I)$, then let $J = I \cup x$. Then the active chain $\mathrm{ch}(J, x)$ is empty, so from Proposition 3.2.17, $\mathrm{EP}(J) = \mathrm{EP}(I) \setminus x$.

If $x \in \mathrm{span}(I)$, then let $y = \min(\mathrm{ci}(I, x))$, and let $J = I \setminus y \cup x$. Then $\mathrm{ci}(J, y) = \mathrm{ci}(I, x)$, so since $y < x$, we have that y is externally active with respect to J, and in particular is contained in the active chain $\mathrm{ch}(J, x)$.

In fact, we can show that $y = \max(\mathrm{ch}(J, x))$. If this were not the case, then there is an element $z > y$ with $z \in \mathrm{EA}(J) \cap \mathrm{bo}(J, x)$. Then $z \in \mathrm{bo}(J, x) = \mathrm{bo}(I, y)$, which means that $x \in \mathrm{ci}(J, z)$ and $y \in \mathrm{ci}(I, z)$. Since $z \in \mathrm{EA}(J)$, we have $z < x$, and since $z > y$ we have that $z \in \mathrm{EP}(I)$. This contradicts the assumption that x was minimal in $\mathrm{EP}(I)$.

We conclude that $y = \max(\mathrm{ch}(J, x))$, so again by Proposition 3.2.17, we have that $\mathrm{EP}(J) = \mathrm{EP}(I) \setminus x$. □

Corollary 3.2.19. *If $M = (E, \mathbb{I})$ is an ordered matroid and $I, J \in \mathbb{I}$ satisfy $I \leq^*_{\mathrm{ext}} J$, then I is lexicographically greater than or equal to J, where prefixes are considered small.*

Proof. This follows because $\mathrm{ch}(I, x)$ consists only of elements smaller than x, so any covering relation corresponds with either a replacement of an element with a smaller one, or with removal of an element entirely. □

We can give explicit combinatorial formulations for the meet and join of independent sets in the external order.

Lemma 3.2.20. *If $A \subseteq E$, then the lex maximal basis B of $M \setminus A$ satisfies* $\mathrm{EP}(B) \subseteq A$. *If $I >^*_{\mathrm{ext}} B$ for some independent set I, then* $\mathrm{EP}(I) \setminus A$ *is nonempty.*

Proof. Suppose $x \in \mathrm{EP}(B) \setminus A$. Then the element $y = \min(\mathrm{ci}(B, x))$ is an element of B, and the basis $B' = B \setminus y \cup x$ gives a basis in $M \setminus A$ which is lex greater than B, a contradiction. Thus $\mathrm{EP}(B) \subseteq A$.

If $I >^*_{\mathrm{ext}} B$, then there is an independent set $J \leq^*_{\mathrm{ext}} I$ which covers B, so that $\mathrm{EP}(J) = \mathrm{EP}(B) \cup x$ for some $x \in E$. However, such a J exists exactly when $x \in B$, so since $B \subseteq E \setminus A$, we have $x \notin A$. Thus $\mathrm{EP}(I) \setminus A$ is nonempty. □

Proposition 3.2.21. *The minimum element of the external order is the lex maximal basis of M, and the maximum element of the external order is the empty set. If $I, J \in \mathbb{I}$, then meets and joins in the external order are described by*

- *$I \wedge J$ is the lex maximal basis of $M \setminus (\mathrm{EP}(I) \cap \mathrm{EP}(J))$*
- *$I \vee J$ is the lex maximal basis of $M \setminus (\mathrm{EP}(I) \cup \mathrm{EP}(J))$*

Proof. The proof is by repeated application of Lemma 3.2.20. The lex maximal basis B of $M = M \setminus \emptyset$ has $\mathrm{EP}(B) \subseteq \emptyset$, so B is the minimum element in the external order. Likewise, $\mathrm{EP}(\emptyset)$ is the ground set of M minus any loops (which are never externally passive), so $\emptyset$ is the maximum element.

To characterize meets, let K be the lex maximal basis of $M \setminus (\mathrm{EP}(I) \cap \mathrm{EP}(J))$. Then $\mathrm{EP}(K) \subseteq \mathrm{EP}(I) \cap \mathrm{EP}(J)$, so we have that $K \leq^*_{\mathrm{ext}} I \wedge J$. Further, if $K' \geq^*_{\mathrm{ext}} K$, then $\mathrm{EP}(K')$ contains an element outside of $\mathrm{EP}(I) \cap \mathrm{EP}(J)$, which shows that K' is not less than one of I or J. Since $K \leq^*_{\mathrm{ext}} I, J$ and no larger independent set is, we conclude that $K = I \wedge J$.

To characterize joins, let K be the lex maximal basis of $M \setminus (\mathrm{EP}(I) \cup \mathrm{EP}(J))$, so that $\mathrm{EP}(K) \subseteq \mathrm{EP}(I) \cup \mathrm{EP}(J)$. By properties of antimatroids, $\mathrm{EP}(I \vee J) = \mathrm{EP}(I) \cup \mathrm{EP}(J)$, so in particular, we have $K \leq^*_{\mathrm{ext}} I \vee J$. If this relation is not equality however, we note that $\mathrm{EP}(I \vee J)$ contains an element outside of $\mathrm{EP}(I) \cup \mathrm{EP}(J)$, which is a contradiction. Thus we must have equality, so $K = I \vee J$. □

From this we also conclude

Corollary 3.2.22. *I is the lex maximal basis of $M \setminus \mathrm{EP}(I)$ for any independent set I.*

As a further consequence, we obtain the following partition of the boolean lattice into boolean subintervals.

Proposition 3.2.23. *If M is an ordered matroid with ground set E, then the intervals*

$$[I, I \cup \mathrm{EA}(I)] \text{ for } I \text{ independent}$$

form a partition of the boolean lattice 2^E.

This partition resembles the classic partition of Crapo (see for instance [7]), and in fact, it can be shown that this partition is a refinement of Crapo's. Gordon and McMahon [22] mention that the existence of such a partition is implied by their Theorem 2.5 applied to matroid independent sets, and this explicit form can be proved by first generalizing the idea of their Proposition 2.6 to external activity for arbitrary independent sets. Interestingly, an independent proof is obtained by instead applying Theorem 2.5 to the antimatroid $\mathcal{F}_{\mathrm{ext}}(M)$. This gives the interval partition

$$[\mathrm{EP}(I), E \setminus I] \text{ for } I \text{ independent},$$

and the desired interval partition is obtained from this by taking set complements. The details of these proofs are omitted.

Finally, we note that the external order is a refinement of the geometric lattice of flats of the associated matroid.

Proposition 3.2.24. *The natural map from the external order $\leq^*_{\mathrm{ext}}$ on M to the geometric lattice of flats of M given by $I \mapsto \mathrm{span}(I)$ is surjective and monotone decreasing. In particular, the external order on M is a refinement of the geometric lattice of flats of M.*

Proof. Suppose I and J are independent with $I \leq^*_{\mathrm{ext}} J$. In particular, $\mathrm{EP}(I)$ contains all elements outside of $\mathrm{span}(I)$, and by Lemma 3.2.14, we also have $\mathrm{EP}(I) \cap J = \emptyset$. Thus $J \subseteq \mathrm{span}(I)$, so we conclude $\mathrm{span}(J) \subseteq \mathrm{span}(I)$. □

Note in particular that the classical ordering convention $\leq_{\mathrm{ext}}$ which is consistent with Las Vergnas's original definition then gives an order *preserving* surjection onto the geometric lattice of flats of a matroid. This is a significant reason why in some contexts the classical order convention, rather than the reverse, may be more convenient.

3.3 Lattice Theoretic Classification

With the external order identified as a join-distributive lattice, a natural question which arises is to classify the lattices this construction produces. To do so, we will need to incorporate two main ideas.

First, we will define the subclass of *matroidal join-distributive lattices* which characterizes the join-distributive lattices whose independent are those of a matroid. Second, we will identify a property, S_n *EL-shellability*, which ensures a certain order consistency condition for the roots of circuits.

We will see in Theorem 3 that these two lattice-theoretic properties, which are satisfied by the external order, are in fact enough to characterize the lattices isomorphic to the external order of an ordered matroid.

Matroidal Join-distributive Lattices

The most apparent connection between the external order and the underlying ordered matroid is in the equality of the matroid and antimatroid independent sets. We now define the class of matroidal join-distributive lattices to further explore this connection.

Definition 3.3.1. If L is a join-distributive lattice, define the **covering rank function** r_{c} of L by

$$r_{\mathrm{c}} : x \mapsto |I(x)|,$$

counting the number of elements in L which cover x.

Definition 3.3.2. We call a join-distributive lattice L **matroidal** if the covering rank function r_{c} is decreasing, and satisfies the semimodular inequality

$$r_{\mathrm{c}}(x \wedge y) + r_{\mathrm{c}}(x \vee y) \leq r_{\mathrm{c}}(x) + r_{\mathrm{c}}(y).$$

Proposition 3.3.3. *If L is a matroidal join-distributive lattice, then $\mathbb{I}(L)$ is the collection of independent sets of a matroid with ground set* $\mathrm{MI}(L)$.

Proof. For notational convenience, let $\mathbb{I} = \mathbb{I}(L)$ and let $E = \mathrm{MI}(L)$. We will show that the function $r : 2^E \to \mathbb{Z}_{\geq 0}$ defined by

$$r(A) = \max\{|I| \,:\, I \in \mathbb{I}, I \subseteq A\}$$

is a matroid rank function on 2^E whose independent sets are $\mathbb{I}$.

Both the fact that $0 \leq r(A) \leq |A|$ for any subset A and that $r(A) \leq r(B)$ for subsets $A \subseteq B \subseteq E$ are clear from the definition of r. Thus all that remains is to prove the semimodular inequality

$$r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$$

for any subsets $A, B \subseteq E$.

Recall that for arbitrary $A \subseteq E$, we denote by x_A the meet of the elements

$$\mathbb{I}_A = \{x_I \,:\, I \subseteq A \text{ is independent}\},$$

In general, x_A is equal to a minimal element x_I with $I \subseteq A$ independent, and since r_{c} is decreasing in L, covering rank is maximized in $\mathbb{I}_A$ by x_I. This means that I is a maximal size independent subset of A, so we conclude that $r(A) = r_{\mathrm{c}}(x_A)$.

Now for $A, B \subseteq E$, by Lemma 3.1.10 we know $x_A \wedge x_B \leq x_{A\cup B}$ and $x_A \vee x_B \leq x_{A\cap B}$. Thus with the semimodular inequality for r_{c} and because r_{c} is a decreasing function, we have

$$\begin{aligned} r(A \cup B) + r(A \cap B) &= r_{\mathrm{c}}(x_{A\cup B}) + r_{\mathrm{c}}(x_{A\cap B}) \\ &\leq r_{\mathrm{c}}(x_A \wedge x_B) + r_{\mathrm{c}}(x_A \vee x_B) \\ &\leq r_{\mathrm{c}}(x_A) + r_{\mathrm{c}}(x_B) \\ &= r(A) + r(B). \end{aligned}$$

Thus r satisfies the semimodular inequality.

Finally, note that if A is independent, then $r(A) = |A|$, and if A is not independent, then the independent subsets of A are proper, so $r(A) < |A|$. Thus the sets $A \in \mathbb{I}$ are exactly the subsets of E for which $r(A) = |A|$, and so $\mathbb{I}$ is the set of independent sets of the matroid with rank function r. $\square$

With a little more work, we can also prove the converse of this statement: a join-distributive lattice whose independent sets form a matroid is itself matroidal. To this end, a few additional lemmas will be useful.

Definition 3.3.4. Let L be a join distributive lattice whose independent sets are the independent sets of a matroid. Then for $x \in L$, let F_x denote the matroid flat $\mathrm{cl}(T(x)^c)$.

Lemma 3.3.5. *If L is a join-distributive lattice whose independent sets are the independent sets of a matroid M, then for any $x \in L$, the independent set $I(x)$ is a basis of F_x. In particular, $r_c(x) = |I(x)| = r(F_x)$.*

Proof. Since $I(x) \subseteq T(x)^c$, we have $I(x) \subseteq F_x$ for any x, so suppose there is an $x \in L$ such that $I(x)$ doesn't span F_x. In particular, by properties of matroids there is an element $a \in T(x)^c \setminus I(x)$ such that $I(x) \cup a$ is independent in M, and since $\mathbb{I}(L) = \mathbb{I}(M)$, there is an element $y \in L$ with $I(y) = I(x) \cup a$. By Lemma 3.1.7, we have $y < x$, and by Lemma 3.1.8, this means that $I(y) \subseteq I(x) \cup T(x)$. However, this is a contradiction since $a \in T(x)^c \setminus I(x)$. $\square$

Lemma 3.3.6. *Let L be a join-distributive lattice whose independent sets are the independent sets of a matroid M. If $x, y \in L$ satisfy $I(x) \supseteq I(y)$, then the elements of $T(y) \setminus T(x)$ lie outside of F_y.*

Proof. If $x = y$ this is vacuously true, so suppose $x \neq y$. By Lemma 3.1.7, we have $x < y$, so there is a sequence of elements $x = z_0 \lessdot z_1 \lessdot \cdots \lessdot z_k = y$ with edge labels $a_i = e(z_{i-1}, z_i)$. In particular, $T(y) \setminus T(x) = \{a_1, \ldots, a_k\}$.

For each i, $a_i \in I(z_{i-1})$. If a_i were in $I(y)$ for some i, then we would have $a_i \in T(z_i) \subseteq T(y)$, so in particular this contradicts disjointness of $T(y)$ and $I(y)$. By induction using Lemma 3.1.8, we see that $I(z_i) \supseteq I(y)$ for each i. Thus the sets $I(y) \cup a_i \subseteq I(z_{i-1})$ are independent, and $a_i \notin \mathrm{cl}(I(y))$ for each i. The conclusion follows from Lemma 3.3.5. $\square$

Lemma 3.3.7. *Let L be a join-distributive lattice whose independent sets are the independent sets of a matroid M. If $x, y \in L$, then*

- $F_{x \vee y} \subseteq F_x \cap F_y$
- $F_{x \wedge y} = \mathrm{cl}(F_x \cup F_y)$

Proof. For the first relation, note that $T(x \vee y) = T(x) \cup T(y)$, so

$$F_{x\vee y} = \mathrm{cl}((T(x) \cup T(y))^{\mathrm{c}}) = \mathrm{cl}(T(x)^{\mathrm{c}} \cap T(y)^{\mathrm{c}}) \subseteq \mathrm{cl}(T(x)^{\mathrm{c}}) \cap \mathrm{cl}(T(y)^{\mathrm{c}}) = F_x \cap F_y.$$

For the second, begin by noticing that $T(x \wedge y) \subseteq T(x) \cap T(y)$, so

$$F_{x\wedge y} = \mathrm{cl}(T(x \wedge y)^{\mathrm{c}}) \supseteq \mathrm{cl}((T(x) \cap T(y))^{\mathrm{c}}) = \mathrm{cl}(T(x)^{\mathrm{c}} \cup T(y)^{\mathrm{c}}) = \mathrm{cl}(F_x \cup F_y).$$

Let $G_{x\wedge y} := \mathrm{cl}(F_x \cup F_y)$, and suppose the containment $F_{x\wedge y} \supseteq G_{x\wedge y}$ is proper. Then since $I(x \wedge y)$ is a basis for $F_{x\wedge y}$, we have $I(x \wedge y) \setminus G_{x\wedge y}$ is nonempty, containing an element a. Then there exists $z \in L$ with $I(z) = I(x \wedge y) \setminus a$, and by Lemma 3.1.7, we have $z > x \wedge y$.

Since a lies outside of $G_{x\wedge y}$, we know that $I(z) = I(x \wedge y) \setminus a$ has span $F_z \supseteq G_{x\wedge y}$, so in particular F_z contains both F_x and F_y. By Lemma 3.3.6, since $I(x \wedge y) \supseteq I(z)$, we know that $T(z) \setminus T(x \wedge y)$ contains only elements outside of F_z. However, since $F_x, F_y \subseteq F_z$, we have

$$T(z) \setminus T(x \wedge y) \subseteq F_z^{\mathrm{c}} \subseteq F_x^{\mathrm{c}} \cap F_y^{\mathrm{c}} \subseteq T(x) \cap T(y).$$

Noting that $T(x \wedge y) \subseteq T(x) \cap T(y)$, we further conclude that $T(z) \subseteq T(x) \cap T(y)$, and thus $z \leq x \wedge y$. This contradicts $z > x \wedge y$, so we see that the inclusion $F_{x\wedge y} \supseteq G_{x\wedge y}$ must be equality as desired. □

Finally, we can prove the converse to Proposition 3.3.3.

Proposition 3.3.8. *Let L be a join-distributive lattice. If $\mathbb{I}(L)$ is the collection of independent sets of a matroid, then L is matroidal.*

Proof. Suppose that $x \leq y$ in L, so that $T(x) \subseteq T(y)$. Then in particular, $F_x = \mathrm{cl}(T(x)^{\mathrm{c}}) \supseteq \mathrm{cl}(T(y)^{\mathrm{c}}) = F_y$, so

$$r_{\mathrm{c}}(x) = r(F_x) \geq r(F_y) = r_{\mathrm{c}}(y),$$

and thus r_{c} is decreasing. To prove that r_{c} satisfies the semimodular inequality, we appeal to the corresponding inequality for matroid rank functions. Using Lemmas 3.3.5 and 3.3.7, we have

$$\begin{aligned} r_{\mathrm{c}}(x \wedge y) + r_{\mathrm{c}}(x \vee y) &= r(F_{x\wedge y}) + r(F_{x\vee y}) \\ &\leq r(\mathrm{cl}(F_x \cup F_y)) + r(F_x \cap F_y) \\ &= r(F_x \cup F_y) + r(F_x \cap F_y) \\ &\leq r(F_x) + r(F_y) \\ &= r_{\mathrm{c}}(x) + r_{\mathrm{c}}(y). \end{aligned}$$

□

Gathering the above results, we have proven the following.

Theorem 2. *A join-distributive lattice L is matroidal if and only if $\mathbb{I}(L)$ is the collection of independent sets of a matroid.*

It is clear from this result that the generalized external order for an ordered matroid M gives a matroidal join-distributive lattice. A natural question to address, then, is whether *all* matroidal join-distributive lattices arise as the external order for some ordering of their underlying matroid. In fact, this question can be answered in the negative, as the following counterexample demonstrates.

Example. Consider the antimatroid on ground set $E = \{a, b, c, d\}$ whose feasible sets are $\mathcal{F} = \{\emptyset, d, c, bd, cd, ac, abd, bcd, acd, abc\}$. The Hasse diagram for the corresponding join-distributive lattice appears in Figure 3.2.

In particular, the collection of independent sets of this antimatroid is the uniform matroid U_4^2 of rank 2 on 4 elements. Suppose this were the external order with respect to some total ordering $<$ on E. In this case, we observe that

- a is active with respect to $I = bc$, so a is smallest in the basic circuit $\mathrm{ci}(bc, a) = abc$
- b is active with respect to $I = ad$, so b is smallest in the basic circuit $\mathrm{ci}(ad, b) = abd$

But this implies that both $a < b$ and $b < a$, a contradiction. Thus this matroidal join-distributive lattice cannot come from a total ordering on the ground set E.

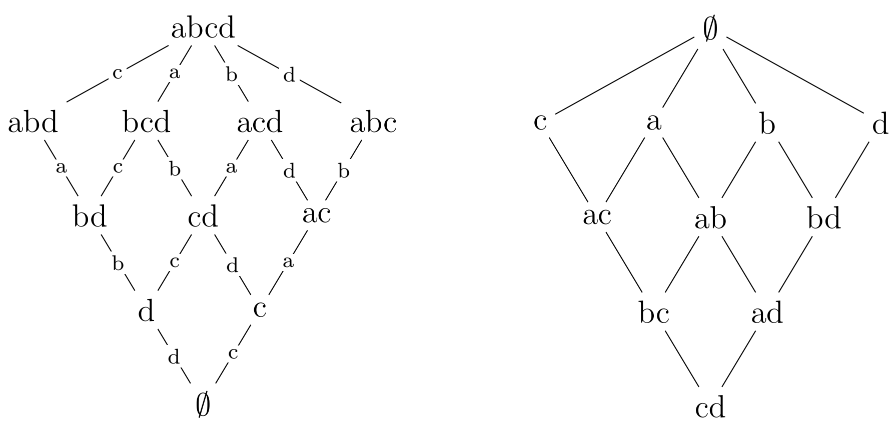


Figure 3.2: Feasible sets of $\mathcal{F}$ with edge labels, and corresponding independent sets

The External Order and $\mathbf{S_n}$ EL-labelings

To bridge the gap between matroidal join-distributive lattices and the external order, we will need one more key notion, a combinatorial construction on a graded poset called an S_n EL-labeling, or snelling.

Definition 3.3.9. If P is a finite poset, then a map $\lambda : \mathrm{Cov}(P) \to \mathbb{Z}$ on the covering pairs of P is called an **edge labeling** of P.

If $\mathfrak{m}$ is an unrefinable chain $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k$ in P, then the sequence

$$\lambda(\mathfrak{m}) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{k-1}, x_k))$$

is called the **label sequence** of $\mathfrak{m}$, and an unrefinable chain $\mathfrak{m}$ is called **increasing** if $\lambda(\mathfrak{m})$ is increasing.

Definition 3.3.10. If P is a finite graded poset, then an edge labeling λ is called an **edge lexicographic** or **EL-labeling** if

- Any interval $[x, y] \subseteq P$ has a unique increasing maximal chain $\mathfrak{m}_0$
- Any other maximal chain in $[x, y]$ has edge labels which are lex greater than the edge labels of $\mathfrak{m}_0$

The existence of an EL-labeling on a poset P in particular implies that the order complex of P is *shellable*, and this is the application for which the notion was introduced by Björner in [6]. In particular, a poset which admits an EL-labeling is called **EL-shellable**.

EL-labelings are not sufficiently rigid to capture the combinatorial property we are trying to isolate, but the following strengthening, first introduced by McNamara in [32], couples well with the set system structure of antimatroids.

Definition 3.3.11. An EL-labeling on a finite graded poset P is called an $\mathbf{S_n}$ **EL-labeling** or **snelling** if the label sequence $\lambda(\mathfrak{m})$ of any maximal chain in P is additionally a permutation of the integers 1 to n. A poset which admits an S_n EL-labeling is called $\mathbf{S_n}$ **EL-shellable**.

We proceed to relate S_n EL-labelings of join-distributive lattices to the following useful property for antimatroid circuits.

Definition 3.3.12. If $(E, \mathcal{F})$ is an antimatroid with rooted circuits $\mathcal{C}$, we say that $\mathcal{F}$ is **confluent** if there is an ordering $\leq$ on the elements of E such that the root of any rooted circuit $C \in \mathcal{C}$ is given by $x = \max_{\leq}(C)$. We call such an ordering a **confluent ordering** for $\mathcal{F}$. Similarly, a join-distributive lattice is called confluent if its corresponding antimatroid is confluent.

This definition captures the essential structure that distinguishes the external order from other matroidal join-distributive lattices. A useful consequence of confluence is that comparable feasible sets in a confluent antimatroid have lex comparable independent sets in the following sense.

Lemma 3.3.13. *In a confluent join-distributive lattice L, if $x, y \in L$ satisfy $x \leq y$, then $I(x) \leq I(y)$ in lex ordering, where prefixes of a word S are considered larger than S.*

Proof. If $x \lessdot y$, then $T(y) = T(x) \cup a$ for some $a \in E = \mathrm{MI}(L)$, and in particular $a \in I(x)$. By Lemma 3.1.8, $I(x) \setminus a \subseteq I(y)$. Since $I(y)$ is the set of elements in $E \setminus T(y)$ which are not the root of a circuit disjoint from $T(y)$, any new elements in $I(y) \setminus I(x)$ are elements b which are the root of a circuit (C, b) with $a \in C$. Since the ordering on E is confluent, the root b is maximal in C, so $b > a$.

This shows that $I(y)$ consists of the elements in $I(x) \setminus a$ plus a (potentially empty) set of elements S all of which are larger than a. The ordering $I(x) < I(y)$ follows, and the general fact for y not covering x follows by induction on the length of a maximal chain between x and y. □

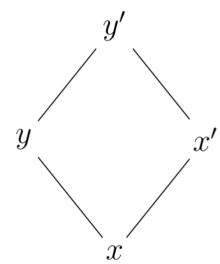
The main structural result of this section is Proposition 3.3.15, which is similar to the work of Armstrong in [5] characterizing *supersolvable matroids*. In fact, our result can be derived from Armstrong's Theorem 2.13, which lists several conditions which are equivalent to S_n EL-shellability of a join-distributive lattice. Our result in particular shows that the condition "$(E, \mathcal{F})$ is a confluent antimatroid" is also equivalent to the conditions listed in Armstrong's theorem.

We provide an independent proof of Proposition 3.3.15 for the reader's convenience. The proof has the particular advantage of more directly relating S_n EL-labelings with the natural labelings of antimatroids without needing to pass through the theory of supersolvable lattices.

We begin by proving the following lemma.

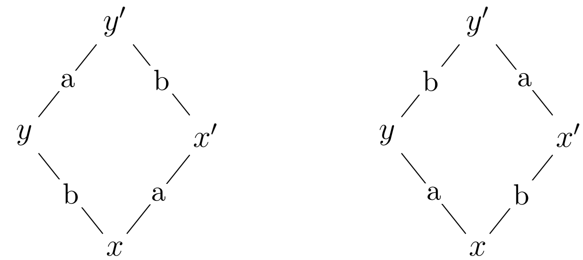
Lemma 3.3.14. *Let L be a join-distributive lattice. Then any S_n EL-labeling of L is equivalent to the natural edge labeling of L for some ordering of its labels.*

Proof. Let $\epsilon : \mathrm{Cov}(L) \to [n]$ be an S_n EL-labeling of L, and let $e : \mathrm{Cov}(L) \to \mathrm{MI}(L)$ denote the natural edge labeling of L. First we prove that for any diamond of elements $x, y, x', y' \in L$ as below, we have that $\epsilon(x, x') = \epsilon(y, y')$.



To see this, suppose that $\mathfrak{m}$ is a maximal chain of L which includes the covering relations $x \lessdot x' \lessdot y'$, and let $\mathfrak{m}'$ be the maximal chain of L which is identical to $\mathfrak{m}$ except that it replaces the covering relations $x \lessdot x' \lessdot y'$ with the relations $x \lessdot y \lessdot y'$. Then the edge labels of $\mathfrak{m}$ and $\mathfrak{m}'$ form permutations of $[n]$, and the edge labels below x and above y' in each chain are identical.

In particular, since both are permutations, the sets of labels $\{\epsilon(x, x'), \epsilon(x', y')\}$ and $\{\epsilon(x, y), \epsilon(y, y')\}$ are the same, say $\{a, b\}$ with $a < b$. Since ϵ is an S_n EL-labeling, exactly one chain in the interval $[x, y']$ is in increasing order, which means that ϵ gives one of the two labelings:



In either case, $\epsilon(x, x') = \epsilon(y, y')$, as we wished to show.

Now let $x, x' \in L$ be a covering pair, $x \lessdot x'$, let $y \in \mathrm{MI}(L)$ be the edge label $e(x, x')$, and let y' be the unique element covering y in L. We will show that in this case, $\epsilon(x, x') = \epsilon(y, y')$.

To see this, note that $x \leq y$, and let $\mathfrak{m}$ be a maximal chain between x and y, given by $x = z_0 \lessdot z_1 \lessdot \cdots \lessdot z_k = y$. If $k = 0$, then $x = y$ and the desired relation holds trivially. Otherwise, by the interval property without upper bounds, there exist elements $z'_i \gtrdot z_i$ with $e(z_i, z'_i) = y$, and we observe a parallel chain $\mathfrak{m}'$ given by $x' = z'_0 \lessdot z'_1 \lessdot \cdots \lessdot z'_k = y'$. Then each pair of coverings $z_i \lessdot z'_i$ and $z_{i+1} \lessdot z'_{i+1}$ form a diamond of elements as in the previous argument, and so $\epsilon(z_i, z'_i) = \epsilon(z_{i+1}, z'_{i+1})$ for each i. This shows that $\epsilon(x, x') = \epsilon(y, y')$.

Finally, let $\mathfrak{m}$ now denote the unique increasing maximal chain of L in the labeling ϵ, given by $0 = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = 1$. In particular, since the labels of $\mathfrak{m}$ are an increasing permutation of $[n]$, we have that $\epsilon(x_{i-1}, x_i) = i$ for each i. Then each covering in this chain corresponds with the meet irreducible $y_i = e(x_{i-1}, x_i)$, which is covered by a unique element y'_i. By the above argument, $\epsilon(y_i, y'_i) = \epsilon(x_i, x'_i) = i$ as well.

In particular, this implies that for any covering relation $x \lessdot x'$ in L, the label $\epsilon(x, x')$ is given by the label $\epsilon(y_i, y'_i) = i$, where $e(x, x') = y_i$. Thus $\epsilon(x, x') = \varphi(e(x, x'))$ for the bijection $\varphi : \mathrm{MI}(L) \to [n]$ given by $y_i \mapsto i$, and we see that ϵ is equivalent to e under the ordering induced by φ. $\square$

Applying this lemma, we can demonstrate the equivalence of confluence and S_n EL-shellability for join-distributive lattices. We will prove in two parts the following:

Proposition 3.3.15. *A join-distributive lattice is S_n EL-shellable if and only if it is confluent.*

Lemma 3.3.16. *If L is a confluent antimatroid, then the natural edge labeling of L is an S_n EL-labeling for any confluent ordering.*

Proof. Fix a confluent ordering of $E = \mathrm{MI}(L)$, and as usual, let $e : \mathrm{Cov}(L) \to E$ denote the natural edge labeling of L. The fact that the sequence of labels of any maximal chain gives a permutation of E is clear from the fact that the union of the edge labels of a maximal chain is equal to $E = T(1)$.

Thus it is sufficient to show that every interval $[x, y]$ has a unique increasing maximal chain. Further, since the edge labels of any maximal chain in $[x, y]$ are a permutation of $T(y) \setminus T(x)$ and determine the chain uniquely, it is enough to prove that there is a chain whose edge labels are the increasing sequence of the elements of $T(y) \setminus T(x)$.

For this, we proceed by induction on the size of $T(y) \setminus T(x)$. If $x = y$, then the empty chain is sufficient, so suppose that $x < y$, and let $a = \min(T(y) \setminus T(x))$.

For any $z \in [x, y]$, we have that $I(z)$ is lex greater than or equal to $I(x)$ in the sense of Lemma 3.3.13. Further, if we denote $J = I(x) \setminus T(y)$, then we have $J \subseteq I(z)$ by the antimatroid interval property without upper bounds.

Thus the smallest element of lexicographic divergence between $I(x)$ and $I(z)$ must be an element b of $I(x) \cap T(y)$ which is contained in $I(x)$ but not in $I(z)$. In particular we have $b \in T(y) \setminus T(x)$. Since a is smallest in $T(y) \setminus T(x)$, if $a \notin I(x)$, then the smallest element of divergence between $I(x)$ and $I(z)$ is larger than a, so $a \notin I(z)$.

However, this holds for any $z \in [x, y]$, so if it were the case that $a \notin I(x)$, then we would conclude that there are no edges in $[x, y]$ labeled by a, which would imply that $a \notin T(y)$, a contradiction. Thus we must have $a \in I(x)$.

In particular, this means that there is an element x' covering x such that $T(x') = T(x) \cup a$, and by induction, there is a unique increasing chain in the interval $[x', y]$, whose labels are the increasing permutation of the elements in $T(y) \setminus (T(x) \cup a)$. Appending this chain to the covering relation $x \lessdot x'$ gives an increasing chain in $[x, y]$, and completes the proof. $\square$

Lemma 3.3.17. *If L is a non-confluent join-distributive lattice, then L is not S_n EL-shellable.*

Proof. Let $(E, \mathcal{F})$ be the associated antimatroid of L, and suppose that L is non-confluent. Then for any ordering of E, there is a rooted circuit C whose root is not maximal in C.

Suppose that nevertheless, L is S_n EL-shellable. By Lemma 3.3.14, an S_n EL-labeling corresponds with the natural labeling $e : \mathrm{Cov} \to E$ for some ordering of E. With respect to that ordering, there is a rooted circuit (C, a) of $\mathcal{F}$ such that $a \neq \max(C)$.

Let $b = \max(C)$. By Proposition 2.1.26, the stem $C \setminus a$ of C is in the blocker for the clutter of stems

$$\mathcal{C}_a^* = \{D \setminus a \,:\, (D, a) \text{ an antimatroid cocircuit of } \mathcal{F}\}\,.$$

In particular, since a blocker consists of the *minimal* sets intersecting each set in a clutter, we have that $(C \setminus a) \setminus b$ is not in the blocker of $\mathcal{C}_a^*$, and so some antimatroid cocircuit (D, a) must include b in its stem $D \setminus a$.

In particular, D is feasible and corresponds with a join-irreducible element of L where the single feasible set covered by D is $D \setminus a$. If $x \in L$ satisfies $T(x) = D$, then any chain $\mathfrak{m}$ given by $0 = z_0 \lessdot z_1 \lessdot \cdots \lessdot z_k = x$ has edge labels which are a permutation of the elements of D.

Further, since the only feasible set covered by D is $D \setminus a$, we have that $e(z_{k-1}, z_k) = a$. This implies that a comes after b in the sequence of edge labels of $\mathfrak{m}$, and so $\mathfrak{m}$ is not an increasing chain. This contradicts the fact that in an S_n EL-labeling, any interval must have a unique increasing maximal chain. We conclude that no S_n EL-labeling exists, and so a join-distributive lattice which is non-confluent is not S_n EL-shellable. $\square$

Finally, Proposition 3.3.15 allows us to classify the matroidal join-distributive lattices which are the external order for a matroid. Specifically, it is immediate that a matroidal join-distributive lattice L is the external order of an ordered matroid iff it is confluent, in which case the underlying matroid may be ordered by the reverse of any confluent ordering of L. Thus we immediately conclude

Corollary 3.3.18. *A matroidal join-distributive lattice L with corresponding matroid M is the external order for some ordering of M if and only if L is S_n EL-shellable.*

Aggregating our results to this point, we can now state a complete characterization of lattices corresponding with the external order of an ordered matroid.

Theorem 3. *A finite lattice L is isomorphic to the external order $\leq^*_{\text{ext}}$ of an ordered matroid if and only if it is join-distributive, matroidal, and S_n EL-shellable.*

3.4 Deletion and Contraction

We continue by exploring a correspondence between the deletion and contraction operations of matroids and antimatroids which is introduced by the external order construction. In the following, let $(E, \mathcal{F})$ denote an antimatroid, and unless otherwise noted, for $A \subseteq E$ let $\mathcal{F} \setminus A$ and $\mathcal{F} \,/\, A$ denote *antimatroid* deletion and contraction, as defined in Section 2.1.

Definition 3.4.1. We call an element $a \in E$ an **extending element** of $\mathcal{F}$ if a is the root of any circuit of $\mathcal{F}$ which contains it. We say that $A \subseteq E$ is an **extending set** of $\mathcal{F}$ if there is an ordering $A = \{a_1, \ldots, a_k\}$ such that a_i is an extending element of $\mathcal{F} \setminus \{a_1, \ldots, a_{i-1}\}$ for each i.

It is not hard to show that an antimatroid $(E, \mathcal{F})$ is *confluent* (cf. Section 3.3) if and only if E is an extending set. The following lemma relates antimatroid deletion with the standard greedoid deletion and contraction operations.

Lemma 3.4.2. *If $A \in \mathcal{F}$ is a feasible set, then the antimatroid deletion $\mathcal{F} \setminus A$ is equal to the greedoid contraction $\mathcal{F} \,/\, A$. If A is an extending set of $\mathcal{F}$, then the antimatroid deletion $\mathcal{F} \setminus A$ is equal to the greedoid deletion $\mathcal{F} \setminus A$.*

The first part of this lemma is discussed in [18], Section 4, but we will prove both parts here for completeness.

Proof. Because antimatroid and greedoid minors satisfy the usual commutativity properties of minors, in each case it is sufficient to prove the lemma when $A = \{a\}$ is a singleton set.

If $A = \{a\}$ is a feasible set, then $\mathcal{F} \setminus A = \{F \setminus a : F \in \mathcal{F}\}$. On the other hand, the greedoid contraction by $\{a\}$ consists of all sets $G \subseteq E$ such that $G \cup a \in \mathcal{F}$. In particular, the feasible sets F containing a correspond with the feasible sets $G = F \setminus a$ in the greedoid contraction, so any feasible set in the greedoid contraction is also feasible in the antimatroid deletion.

The remaining feasible sets in the antimatroid deletion are sets $F \in \mathcal{F}$ with $a \notin F$. For these sets, note that because $\emptyset \subseteq F$ and $\emptyset$ may be extended to $\{a\}$, we see by the antimatroid interval property without upper bounds that $F \cup a \in \mathcal{F}$ as well. Thus $F = F \setminus a = (F \cup a) \setminus a$ is also feasible in the greedoid contraction.

Now suppose $A = \{a\}$ where a is an extending element of $\mathcal{F}$. One consequence of being an extending element is that for any feasible set F, if $a \in F$, then $F \setminus a$ is feasible.

To see this, let $F \in \mathcal{F}$ be a feasible set containing a, and suppose that $F \setminus a \notin \mathcal{F}$. Then there exists a rooted circuit (C, x) such that $(F \setminus a) \cap C = \{x\}$, and in particular, we have that the root x is not equal to a. Because a is an extending element, we conclude that $a \notin C$. However, this means that $F \cap C = \{x\}$ as well, so we conclude that F is not feasible, a contradiction.

From this we see that the antimatroid deletion $\mathcal{F} \setminus A = \{F \setminus a : F \in \mathcal{F}\}$ is given by the feasible sets of $\mathcal{F}$ which don't contain a. This is exactly the greedoid deletion by $\{a\}$. $\square$

Note that for A feasible, it follows directly that $\mathcal{F} \setminus A$ corresponds with the the interval $[A, E]$ in $\mathcal{F}$ via the map $F \mapsto F \sqcup A$. For A extending, it follows that $E \setminus A$ is feasible, and $\mathcal{F} \setminus A$ is equal to the interval $[\emptyset, E \setminus A]$ in $\mathcal{F}$.

We now show that matroid and antimatroid deletion are in exact correspondence for matroidal antimatroids.

Proposition 3.4.3. *Suppose that $\mathcal{F}$ is matroidal with associated matroid M. Then for $A \subseteq E$, the antimatroid deletion $\mathcal{F} \setminus A$ is matroidal with associated matroid $M \setminus A$. If $\mathcal{F} = \mathcal{F}_{\mathrm{ext}}(M)$ for an ordered matroid M, then $\mathcal{F} \setminus A = \mathcal{F}_{\mathrm{ext}}(M \setminus A)$, where the order on $M \setminus A$ is induced by the order on M.*

Proof. Recall that the circuits of an antimatroid are the minimal non-independent sets, so an antimatroid is matroidal with associated matroid M iff its circuits are the circuits of M.

Now let $\mathcal{C}$ denote the collection of rooted circuits of $\mathcal{F}$. Then the circuits of $\mathcal{F} \setminus a$ are given by

$$\mathcal{C} \setminus a = \{C \in \mathcal{C} : C \cap \{a\} = \emptyset\}.$$

Forgetting the roots, these are exactly the circuits of $M \setminus a$, so we conclude that $\mathcal{F} \setminus a$ is matroidal with associated matroid $M \setminus a$.

Remembering the roots, if M is ordered then $\mathcal{F} = \mathcal{F}_{\mathrm{ext}}(M)$ iff every circuit C of $\mathcal{F}$ has root $x = \min(C)$. This property is preserved by restricting to a subset of the circuits, so we see that if $\mathcal{F} = \mathcal{F}_{\mathrm{ext}}(M)$, then $\mathcal{F} \setminus a = \mathcal{F}_{\mathrm{ext}}(M \setminus a)$. $\square$

Antimatroid contractions do not behave as nicely as deletions with respect to matroid structure — in many cases, contraction does not even preserve the property of being matroidal! However, for certain contraction sets the situation is still favorable.

Proposition 3.4.4. *Suppose that $\mathcal{F}$ is matroidal with associated matroid M.*

- *For A feasible, the antimatroid contraction $\mathcal{F} \,/\, A$ is matroidal with associated matroid $M' = M \,/\, A$.*
- *For A extending, the antimatroid contraction $\mathcal{F} \,/\, A$ is matroidal with associated matroid $M' = M \setminus A$.*

For either case, if $\mathcal{F} = \mathcal{F}_{\mathrm{ext}}(M)$ for an ordered matroid M, then $\mathcal{F} \,/\, A = \mathcal{F}_{\mathrm{ext}}(M')$, where the order on M' is induced by the order on M.

Proof. As in Lemma 3.4.2, it is sufficient to prove these cases when $A = \{a\}$ is a singleton set because of commutativity properties of minors.

If $A = \{a\}$ is a feasible set, then $A \cap C \neq \{a\}$ for any rooted circuit C, and so a is never the root of a circuit of $\mathcal{F}$. In particular, this means that

$$\mathcal{C}(\mathcal{F} \,/\, a) = \min \{(C \setminus a, x) \,:\, (C, x) \in \mathcal{C}(\mathcal{F})\}.$$

The circuits of $M \,/\, a$ are exactly the underlying sets of the rooted circuits of $\mathcal{F} \,/\, a$, so we conclude that $\mathcal{F} \,/\, a$ is matroidal with associated matroid $M \,/\, a$. If M is ordered and $\mathcal{F} = \mathcal{F}_{\mathrm{ext}}(M)$, then any rooted circuit (C', x) of $\mathcal{F} \,/\, a$ corresponds with a rooted circuit (C, x) of $\mathcal{F}$, where $C' = C \setminus a$. Since $\mathcal{F} = \mathcal{F}_{\mathrm{ext}}(M)$, we have $x = \min(C)$, and since $x \neq a$, we have also that $x = \min(C')$, so the root of each circuit of $\mathcal{F} \,/\, a$ is the minimal element of the circuit. This implies that $\mathcal{F} \,/\, a = \mathcal{F}_{\mathrm{ext}}(M \,/\, a)$ for the induced order on $M \,/\, a$.

If $A = \{a\}$ for a an extending element of $\mathcal{F}$, then a is the root of any circuit containing it. In particular this means that

$$\begin{aligned}\mathcal{C}(\mathcal{F} \,/\, a) &= \min \{(C \setminus a, x) \,:\, (C, x) \in \mathcal{C}(\mathcal{F}), x \neq a\} \\ &= \{(C, x) \,:\, (C, x) \in \mathcal{C}(\mathcal{F}), a \notin C\} = \mathcal{C}(\mathcal{F} \setminus a).\end{aligned}$$

Thus in this case, $\mathcal{F} \,/\, a = \mathcal{F} \setminus a$, and the result follows from Proposition 3.4.3. □

Although antimatroid contraction doesn't preserve matroid structure for arbitrary contraction sets, if $\mathcal{F}$ is the external order for an ordered matroid, the resulting set system is related nicely to the external orders for the corresponding matroid deletion and contraction. We start with two lemmas, one due to Dietrich, and the other a short technical lemma on matroid deletions.

Lemma 3.4.5 ([18], Lemma 13)**.** *If $(C, x) \in \mathcal{C}(\mathcal{F})$ and $A \subseteq E$ with $x \notin A$, then there exists a rooted circuit $(C', x) \in \mathcal{C}(\mathcal{F} \,/\, A)$ with $C' \subseteq C \setminus A$.*

Lemma 3.4.6. *Let M be a matroid on ground set E, and let $A \subseteq E$. If $C \in \mathcal{C}(M)$, then for each $x \in C \setminus A$, there exists $C' \in \mathcal{C}(M \,/\, A)$ with $C' \subseteq C$ and $x \in C'$.*

Proof. We induct on the size of A. If $A = \emptyset$, then the lemma holds trivially. Now suppose that $|A| \geq 1$, and let $a \in A$. We will apply a result from [35] Exercise 3.1.3, which states that

- If $a \in C$, then either a is a loop or $C \setminus a$ is a circuit of $M \,/\, a$
- If $a \notin C$, then C is a union of circuits of $M \,/\, a$

Let $C \in \mathcal{C}$, assume without loss of generality that $C \setminus A$ is nonempty, and let $x \in C \setminus A$.

Suppose first that $a \in C$. If a were a loop, this would imply $C = \{a\}$, which contradicts our assumption that $C \setminus A$ is nonempty. By the above, we now have that $C \setminus a$ is a circuit of $M \,/\, a$. In particular, $x \in (C \setminus a) \setminus (A \setminus a)$, so by induction there exists a circuit C' of $M \,/\, A = (M \,/\, a) \,/ (A \setminus a)$ such that $C' \subseteq C \setminus a \subseteq C$ and $x \in C'$. Thus the lemma holds.

Now suppose that $a \notin C$. Then C is a union of circuits of $M \,/\, a$, so in particular there is a circuit $C' \in \mathcal{C}(M \,/\, a)$ with $C' \subseteq C$ and $x \in C'$. Inductively there exists a circuit C'' of $M \,/\, A = (M \,/\, a) \,/ (A \setminus a)$ such that $x \in C''$ and $C'' \subseteq C' \subseteq C$. This completes the proof. $\square$

Using these lemmas, we prove the following.

Proposition 3.4.7. *Let M be an ordered matroid with ground set E, and suppose $\mathcal{F} = \mathcal{F}_{\text{ext}}(M)$ is the external order for M. Then for $A \subseteq E$, we have*

$$\mathcal{F}_{\text{ext}}(M \,/\, A) \subseteq \mathcal{F} \,/\, A \subseteq \mathcal{F}_{\text{ext}}(M \setminus A).$$

Proof. We begin with the left inclusion. Suppose that $F \subseteq E \setminus A$ is not feasible in $\mathcal{F} \,/\, A$, so that there exists a rooted circuit (C, x) of $\mathcal{F} \,/\, A$ such that $F \cap C = x$. Then in particular, $C = C_0 \setminus A$ for a rooted circuit $(C_0, x) \in \mathcal{C}(\mathcal{F})$ with $x \notin A$.

Since $\mathcal{F} = \mathcal{F}_{\text{ext}}(M)$, the set C_0 is a circuit of M, and $x = \min(C_0)$. By Lemma 3.4.6, there exists a circuit $C' \in \mathcal{C}(M \,/\, A)$ with $C' \subseteq C_0 \setminus A = C$ and $x \in C'$. Since $x = \min(C_0)$, we also have $x = \min(C')$, so (C', x) is a rooted circuit of $\mathcal{F}_{\text{ext}}(M \,/\, A)$. In particular we see that $C' \cap F = \{x\}$, so we conclude that F is also not feasible in $\mathcal{F}_{\text{ext}}(M \,/\, A)$.

For the right inclusion, suppose that $F \subseteq E \setminus A$ is not feasible in $\mathcal{F}_{\text{ext}}(M \setminus A)$, so that there exists a rooted circuit (C, x) of $\mathcal{F}_{\text{ext}}(M \setminus A)$ with C disjoint from A and $F \cap C = x$.

Then $(C, x) \in \mathcal{C}(\mathcal{F})$, and by Lemma 3.4.5, there is a circuit $(C', x) \in \mathcal{C}(\mathcal{F} \,/\, A)$ with $C' \subseteq C \setminus A = C$. In particular, $F \cap C' = x$, so we conclude that F is also not feasible in $\mathcal{F} \,/\, A$. $\square$

3.5 Passive Exchanges and Downward Covering Relations

We conclude this chapter by discussing a more subtle structure which emerges in the external order relating to the downward covering relations and basis exchanges with externally passive elements. We begin with a few elementary matroid-theoretic observations which will be useful.

Lemma 3.5.1. *Let $B, B' \subseteq E$ be bases with $B' = B \setminus b \cup x$ for some $b \in B$ and $x \in E \setminus B$. Then*

- $\mathrm{ci}(B, x) = \mathrm{ci}(B', b)$
- $\mathrm{bo}(B, b) = \mathrm{bo}(B', x)$

Proof. For the first statement, note that $B \cup x = B' \cup b$, so since the two fundamental circuits are the unique circuits contained in these unions, they must be identical. The second statement is the dual of the first. □

Lemma 3.5.2. *Let $B, B' \subseteq E$ be bases with $B' = B \setminus b \cup x$ for some $b \in B$ and $x \in E \setminus B$. Further, let $b_0 \in B$ and let $x_0 \in E \setminus B$.*

- *If $b_0 \notin \mathrm{ci}(B, x)$ (equiv. $x \notin \mathrm{bo}(B, b_0)$), then $\mathrm{bo}(B', b_0) = \mathrm{bo}(B, b_0)$.*
- *If $x_0 \notin \mathrm{bo}(B, b)$ (equiv. $b \notin \mathrm{ci}(B, x_0)$), then $\mathrm{ci}(B', x_0) = \mathrm{ci}(B, x_0)$.*

Proof. For the first statement, note that since $b_0 \notin \mathrm{ci}(B, x)$, we can decompose B into disjoint parts

$$B = b_0 \cup (\mathrm{ci}(B, x) \setminus x) \cup U.$$

Then since $B' = B \setminus b \cup x$ is a basis, we have $b \in \mathrm{ci}(B, x)$, so we further have

$$B' = B \setminus b \cup x = b_0 \cup (\mathrm{ci}(B, x) \setminus b) \cup U.$$

Then since $\mathrm{ci}(B, x) \setminus b$ and $\mathrm{ci}(B, x) \setminus x$ have the same span, we see that $\mathrm{span}(B' \setminus b_0) = \mathrm{span}(B \setminus b_0)$, whence equality of the basic cocircuits.

The second statement is the dual of the first. □

Additionally recall the following classical lemma concerning the intersections of circuits with the complement of a flat. Sometimes this result is formulated in a slightly weaker form in terms of circuits and cocircuits.

Lemma 3.5.3. *If $C \subseteq E$ is a circuit and $F \subseteq E$ is a flat, then $|C \setminus F| \neq 1$.*

Proof. Suppose $|C \setminus F| = 1$, and let x be the singleton element in this difference. We have $x \in \mathrm{span}(C \setminus x) = \mathrm{span}(C)$ since C is a minimal dependent set, but the fact that $C \setminus x \subseteq F$ means that $\mathrm{span}(C \setminus x) \subseteq F$, so we also conclude that the span does not contain x, a contradiction. □

We now introduce an operator on the elements of a matroid which allows us to refine the notion of matroid activity and describe in detail the way that arbitrary exchange operations in matroids relate to the external order. The following notation will be used broadly.

Notation. If $A \subset E$ and $x \in E$, we define $A_{>x}$ to be the set $\{a \in A : a > x\}$. Similar notation will be used for other conditions specifying a subset of A filtered by a condition. Such usage should be clear from its context.

In the following, for technical convenience, we will make use of an additional formal symbol "$\mathfrak{m}$" which will play the formal role of "one or more minimal matroid ground set elements in general position for E". In particular, we will often consider the disjoint union $E \cup \mathfrak{m}$, which we consider as an ordered set with $\mathfrak{m} < x$ for each $x \in E$. Thus in particular, we have the notation $E_{>\mathfrak{m}} = E$.

We now define the spread operator.

Definition 3.5.4. For each $A \subseteq E$, define the **spread operator** $\mathrm{Spr}_A \colon E \cup \mathfrak{m} \to 2^E$ by

$$\mathrm{Spr}_A(x) \coloneqq \mathrm{span}(A_{>x}, x) \setminus \mathrm{span}(A_{>x})$$

for $x \in E$, and

$$\mathrm{Spr}_A(\mathfrak{m}) = E \setminus \mathrm{span}(A).$$

In general terms, the spread operator describes the "ordered contribution" from greatest to least of an element x to the span of A. A first observation about the operator is the following.

Lemma 3.5.5. *If $x \in E \cup \mathfrak{m}$ and $A, B \subseteq E$ with $\mathrm{span}(A_{>x}) = \mathrm{span}(B_{>x})$, then $\mathrm{Spr}_A(x) = \mathrm{Spr}_B(x)$.*

We will be particularly interested in the case when A is an independent set.

Lemma 3.5.6. *Let $I \subseteq E$ be independent, and let $x \in E$. Then*

1. *The collection $\{\mathrm{Spr}_I(a) : a \in I\}$ forms a partition of $\mathrm{span}(I)$ with a the unique element of I contained in each set $\mathrm{Spr}_I(a)$.*

2. *If $x \in \mathrm{Spr}_I(a)$ for $a \in I$, then x is externally active if $x < a$, and externally passive if $x > a$.*

Proof. If $I = \{a_1, \ldots, a_k\}$ where the elements are ordered by index, then if A_i denotes the flat

$$A_i = \begin{cases} \mathrm{span}(I_{>a_i}), & 1 \le i \le k \\ \mathrm{span}(I), & i = 0 \end{cases},$$

then

$$A_k \subsetneq A_{k-1} \subsetneq \cdots \subsetneq A_0.$$

In particular, $\mathrm{Spr}_I(a_i) = A_{i-1} \setminus A_i$ for each i, so these sets partition $A_k = \mathrm{span}(I)$, and $a_i \in \mathrm{Spr}_I(a_i)$.

Next, note that for $x \in \mathrm{span}(I) \setminus I$, the element $a \in I$ for which $x \in \mathrm{Spr}_I(a)$ is the smallest element of I which is needed to span x, which in particular is the smallest element of I in the (relative) fundamental circuit $\mathrm{ci}(I, x)$. The element x is externally active iff it is the minimal element of $\mathrm{ci}(I, x)$, hence x is externally active if $x < a$, and externally passive if $x > a$. $\square$

Since $\mathrm{Spr}_I(\mathfrak{m}) = E \setminus \mathrm{span}(I)$, we can extend the partition above to a partition of the whole ground set E by

$$E = \bigcup_{a \in I \cup \mathfrak{m}} \mathrm{Spr}_I(a).$$

We will denote the unique element $a \in I \cup \mathfrak{m}$ for which $x \in \mathrm{Spr}_I(a)$ by $\rho_I(x)$. The above condition for matroid activity can then be restated as $x \in \mathrm{EP}(I)$ iff $x > \rho_I(x)$, and $x \in \mathrm{EA}(I)$ iff $x < \rho_I(x)$. We adopt the convention $\rho_I(\mathfrak{m}) = \mathfrak{m}$, even though we do not include $\mathfrak{m}$ as an element of $\mathrm{Spr}_I(\mathfrak{m})$.

We additionally introduce the following notation for the restriction of this decomposition to the externally passive elements of an independent set.

Definition 3.5.7. Let $I \subseteq E$ be independent, and suppose $x \in E \cup \mathfrak{m}$. Then let

$$\mathrm{EP}_x(I) \coloneqq \mathrm{Spr}_I(x) \cap E_{>x}.$$

In particular,

$$\mathrm{EP}_{\mathfrak{m}}(I) = \mathrm{Spr}_I(\mathfrak{m}) \cap E = E \setminus \mathrm{span}(I).$$

By Lemma 3.5.6, note that $\mathrm{EP}_a(I) = \mathrm{EP}(I) \cap \mathrm{Spr}_I(a)$ for $a \in I$, which consists of the externally passive elements in $\mathrm{span}(I)$ whose fundamental circuit $\mathrm{ci}(I, x)$ has minimal element a. In particular, we obtain the decomposition

$$\mathrm{EP}(I) = \bigcup_{a \in I \cup \mathfrak{m}} \mathrm{EP}_a(I).$$

Remark. Similar definitions and a similar decomposition could be made with respect to the externally *active* elements of an independent set. However, this decomposition does not yield as much insight due to the underlying antimatroid structure of matroid activity. Since antimatroids may be formulated in terms of *rooted circuits*, the *root* (minimal element) of a given circuit holds particular combinatorial significance. For elements in $\mathrm{EP}_a(I)$, a gives the root of their corresponding fundamental circuit (or $\mathfrak{m}$ if they are not spanned by I), and thus the sets $\mathrm{EP}_a(I)$ decompose the externally passive elements of I according to these fundamental circuit roots.

However, an externally active element is by definition the root of its own fundamental circuit, so the element $a \in I$ for which $x \in \mathrm{Spr}_I(a)$ represents the smallest element of I in the fundamental circuit $\mathrm{ci}(I, x)$, which in particular is not the root of the circuit. In terms of antimatroid combinatorics, this means that a is not distinguishable in particular among the other non-root elements in $I \cap \mathrm{ci}(I, x)$, so partitioning externally active elements according to this parameter is not interesting on the level of rooted circuits.

We now define a class of single-element exchanges between elements of an independent set and externally passive elements.

Definition 3.5.8. For $I \subseteq E$ an independent set, define a map $\varepsilon_I \colon \mathrm{EP}(I) \to \mathbb{I}$ by

$$\varepsilon_I(x) = \begin{cases} I \cup x \setminus a, & x \in \mathrm{EP}_a(I) \text{ for } a \in I \\ I \cup x, & x \in \mathrm{EP}_{\mathfrak{m}}(I) \end{cases}.$$

We call $\varepsilon_I(x)$ the **canonical passive exchange** of x with I. Sometimes we will extend the domain of ε_I to include I, in which case $\varepsilon_I(a) = I$ for any $a \in I$.

The fact that $\varepsilon_I(x)$ is an independent set is immediate from the definition. We also see that the spread operator is preserved under this class of exchange in some cases.

Lemma 3.5.9. *Let $I \subseteq E$ be independent, $x \in E \setminus \mathrm{EA}(I)$, and $a = \rho_I(x)$. If y is an element of $E \cup \mathfrak{m}$ with $y \geq x$ or $y < a$, then $\mathrm{Spr}_{\varepsilon_I(x)}(y) = \mathrm{Spr}_I(y)$.*

Proof. Let J denote $\varepsilon_I(x)$. If $x \in I$ the identity is trivial since $J = I$, so suppose that $x \in \mathrm{EP}(I)$. In particular, this implies that $x > a$ and $J_{>x} = I_{>x}$.

If $y \geq x$, then because $J_{>x} = I_{>x}$, we also have $J_{>y} = I_{>y}$. By Lemma 3.5.5, this implies that $\mathrm{Spr}_J(y) = \mathrm{Spr}_I(y)$.

On the other hand, if $y < a$, then a must be an element of I, so $x \in \mathrm{span}(I)$ and $a = \min(\mathrm{ci}(I, x))$. In particular, letting $U = I_{>y} \setminus \mathrm{ci}(I, x)$, we can write $I_{>y} = U \cup \mathrm{ci}(I, x) \setminus x$ and $J_{>y} = U \cup \mathrm{ci}(I, x) \setminus a$. Since these sets have the same span as $U \cup \mathrm{ci}(I, x)$, we see again by Lemma 3.5.5 that $\mathrm{Spr}_J(y) = \mathrm{Spr}_I(y)$. □

We next explore a useful structure underlying an ordered independent set which is revealed by inclusion relations between spread sets. As expected, the situation is not interesting for externally active elements.

Lemma 3.5.10. *If $I \subseteq E$ is independent and $x \in E$, then $x \in \mathrm{EA}(I)$ if and only if $\mathrm{Spr}_I(x) = \emptyset$.*

Proof. Note that an element $x \in E$ is externally active iff $x \in I$ and x is minimal in $\mathrm{ci}(I, x)$, which is the case exactly when x is spanned by the elements of I greater than it. In this case, $\mathrm{span}(I_{>x} \cup x) = \mathrm{span}(I_{>x})$, so $\mathrm{Spr}_I(x) = \emptyset$. □

On the other hand, a significantly richer structure is observed for the externally passive elements.

Lemma 3.5.11. *Let $I \subseteq E$ be independent, and let $x, y \in E \setminus \mathrm{EA}(I)$. We have*

1. *If $a \in I \cup \mathfrak{m}$, then $x \in \mathrm{Spr}_I(a)$ if and only if $\mathrm{Spr}_I(x) \subseteq \mathrm{Spr}_I(a)$.*

2. *If $y \in \mathrm{Spr}_I(x)$, then $\rho_I(y) = \rho_I(x)$.*

3. *If* $y \geq x$*, then* $y \in \mathrm{Spr}_I(x)$ *if and only if* $\mathrm{Spr}_I(y) \subseteq \mathrm{Spr}_I(x)$.

4. *If* $y < x$*, then* $y \in \mathrm{Spr}_I(x)$ *implies* $\mathrm{Spr}_I(y) \supseteq \mathrm{Spr}_I(x)$.

Proof. To begin, note that if $x \in E \setminus \mathrm{EA}(I)$, then $\mathrm{Spr}_I(x)$ is nonempty, and in particular, $x \in \mathrm{Spr}_I(x)$. From this, the reverse direction of part 1 is immediate since we have $x \in \mathrm{Spr}_I(x) \subseteq \mathrm{Spr}_I(a)$.

For the forward direction of part 1, we will divide the argument into two cases, $a = \mathfrak{m}$ and $a \in I$. First suppose that $a = \mathfrak{m}$, so that $x \in \mathrm{Spr}_I(\mathfrak{m}) = E \setminus \mathrm{span}(I)$. Let $J = \varepsilon_I(x) = I \cup x$, so that in particular $\mathrm{Spr}_J(x) = \mathrm{Spr}_I(x)$ by Lemma 3.5.9. If $y \in \mathrm{Spr}_J(x)$ with $y \neq x$, then $y \in \mathrm{span}(J)$, and $x \in \mathrm{ci}(J, y)$. In particular, $y \notin \mathrm{span}(J \setminus x) = \mathrm{span}(I)$, so we also have $y \in \mathrm{Spr}_I(\mathfrak{m})$, which proves the inclusion.

Now suppose $a \in I$, so that $x \in \mathrm{Spr}_I(a) \subseteq \mathrm{span}(I)$. If $x \in I$, then we must have $x = a$ since a is the only element of I in $\mathrm{Spr}_I(a)$, which of course gives the desired inclusion.

If $x \notin I$, then we have $a = \min(\mathrm{ci}(I, x))$. To show that $\mathrm{Spr}_I(x) \subseteq \mathrm{Spr}_I(a)$, we will show that (i) $\mathrm{Spr}_I(x) \subseteq \mathrm{span}(I_{>a} \cup a)$, and (ii) $\mathrm{Spr}_I(x)$ is disjoint from $\mathrm{span}(I_{>a})$, from which the inclusion follows directly.

For (i), note that

$$\mathrm{Spr}_I(x) \subseteq \mathrm{span}(I_{>x} \cup x) \subseteq \mathrm{span}(I_{>a} \cup x) = \mathrm{span}(I_{>a} \cup a),$$

where the last equality follows because both $I_{>a} \cup x$ and $I_{>a} \cup a$ contain all but one element of $\mathrm{ci}(I, x)$, and thus have the same span as $I_{>a} \cup ax$.

For (ii), suppose that $\mathrm{Spr}_I(x)$ contains an element $y \in \mathrm{span}(I_{>a})$. In particular, $y \neq x$ since $x \notin \mathrm{span}(I_{>a})$. Letting $J = \varepsilon_I(x)$, we have $x \in J$, and by Lemma 3.5.9, $\mathrm{Spr}_J(x) = \mathrm{Spr}_I(x)$. Since $y \in \mathrm{Spr}_J(x)$ with $y \neq x$, we have $y \in \mathrm{span}(J) \setminus J$ and $x = \min(\mathrm{ci}(J, y) \setminus y)$. Then

$$\mathrm{ci}(J, y) \subseteq J_{>x} \cup xy = I_{>x} \cup xy \subseteq I_{>a} \cup xy.$$

However, this implies that $\mathrm{ci}(J, y) \setminus \mathrm{span}(I_{>a}) = x$, so the intersection of a circuit with the complement of a flat in particular has exactly one element. This contradicts Lemma 3.5.3, so we conclude that $\mathrm{Spr}_I(x)$ must be disjoint from $\mathrm{span}(I_{>a})$.

For part 2, let $a = \rho_I(x)$. Since $x \in \mathrm{Spr}_I(a)$, we have $\mathrm{Spr}_I(x) \subseteq \mathrm{Spr}_I(a)$ by part 1. Thus if $y \in \mathrm{Spr}_I(x)$, then $y \in \mathrm{Spr}_I(a)$, so $\rho_I(y) = a$.

For part 3, apply part 1 to the independent set $J = \varepsilon_I(x)$, noting in particular that $x \in J$, and that $\mathrm{Spr}_J(y) = \mathrm{Spr}_I(y)$ and $\mathrm{Spr}_J(x) = \mathrm{Spr}_I(x)$ by Lemma 3.5.9.

For part 4, suppose that $y < x$ and $y \in \mathrm{Spr}_I(x)$. Letting $J = \varepsilon_I(x)$, we have $\mathrm{Spr}_J(x) = \mathrm{Spr}_I(x)$ by Lemma 3.5.9, so $y \in \mathrm{Spr}_J(x)$. In particular, $y \in \mathrm{span}(J)$, and $x > y$ is the minimal element of J in $\mathrm{ci}(J, y)$. This means that $\mathrm{ci}(J, y)$ is contained in $J_{>x} \cup xy = I_{>x} \cup xy$, so in particular, $x \in \mathrm{span}(I_{>x} \cup y) \subseteq \mathrm{span}(I_{>y} \cup y)$.

Suppose by way of contradiction that $x \in \mathrm{span}(I_{>y})$ as well, and let $a = \rho_I(x) \in I$. Then since $y \in \mathrm{Spr}_I(x)$ we have $\rho_I(y) = \rho_I(x) = a$ by part 2. However, notice that we have $a > y$ because $x \notin \mathrm{span}(I_{>a})$ but $x \in \mathrm{span}(I_{>y})$, so we see by Lemma 3.5.6 that y is externally active. This contradicts $y \in E \setminus \mathrm{EA}(I)$, so we conclude $x \notin \mathrm{span}(I_{>y})$. Since we've shown

$x \in \operatorname{span}(I_{>y} \cup y)$, we thus have $x \in \operatorname{Spr}_I(y)$, and by part 3, we obtain $\operatorname{Spr}_I(x) \subseteq \operatorname{Spr}_I(y)$ as desired. □

Lemma 3.5.12. *Let $I \subseteq E$ be independent, $x \in E \setminus \operatorname{EA}(I)$, and $a = \rho_I(x)$. If $y \in E \cup \mathfrak{m}$ with $\rho_I(y) \neq a$, then $\operatorname{Spr}_{\varepsilon_I(x)}(y) \supseteq \operatorname{Spr}_I(y)$.*

Proof. Let J denote $\varepsilon_I(x)$, and let $a' = \rho_I(y)$. If $x \in I$ or if $a' = \mathfrak{m}$, then the desired inclusion is immediate, so assume that $x \in \operatorname{EP}(I)$ and $a' \neq \mathfrak{m}$. From Lemma 3.5.9 we know that $\operatorname{Spr}_J(y) = \operatorname{Spr}_I(y)$ when $y \geq x$ or $y < a$, so assume also that $x > y \geq a$. In particular, this implies $J_{>y} = I_{>y} \cup x$.

Now suppose $z \in \operatorname{Spr}_I(y)$. By Lemma 3.5.11 part 2, we have $\rho_I(z) = a'$. Further, since

$$\operatorname{span}(J_{>y} \cup y) = \operatorname{span}(I_{>y} \cup xy) \supseteq \operatorname{span}(I_{>y} \cup y),$$

we have $z \in \operatorname{span}(J_{>y} \cup y)$. To show that $z \in \operatorname{Spr}_J(y)$, we thus only need to show that $z \notin \operatorname{span}(J_{>y}) = \operatorname{span}(I_{>y} \cup x)$.

Suppose first that $a = \mathfrak{m}$, so that $J = I \cup x$. Since $\rho_I(z) = a' \neq \mathfrak{m}$, we have $z \in \operatorname{span}(I)$, and in particular, $\operatorname{ci}(J, z) = \operatorname{ci}(I, z)$ does not contain x. If we had $z \in \operatorname{span}(I_{>y} \cup x)$, then this would imply $\operatorname{ci}(J, z) \subseteq I_{>y} \cup x$, and thus that $\operatorname{ci}(J, z) \subseteq I_{>y}$ since $x \notin \operatorname{ci}(J, z)$. But this means $z \in \operatorname{span}(I_{>y})$, which contradicts $z \in \operatorname{Spr}_I(y)$. Thus $z \notin \operatorname{span}(I_{>y} \cup x) = \operatorname{span}(J_{>y})$, so we conclude that $z \in \operatorname{Spr}_J(y)$.

Now suppose that $a \in I$, so that $J = I \cup x \setminus a$, and suppose again that $z \in \operatorname{span}(I_{>y} \cup x)$. Then we have

$$\operatorname{span}(I_{>y} \cup x) \subseteq \operatorname{span}(I_{>a} \cup x) = \operatorname{span}(I_{>a} \cup a),$$

where the equality is because both spans contain all but one element of $\operatorname{ci}(I, x)$, and thus have the same span as $I_{>a} \cup ax$. Since $z \in \operatorname{Spr}_I(a')$, we know that $z \notin \operatorname{Spr}_I(a)$, so in particular, we conclude that $z \in \operatorname{span}(I_{>a})$. However, since z is in both $\operatorname{Spr}_I(y)$ and $\operatorname{Spr}_I(a')$, we have that $z \notin \operatorname{span}(I_{>y})$, but $z \in \operatorname{span}(I_{>a'} \cup a') = \operatorname{span}(I_{\geq a'})$ and $z \notin \operatorname{span}(I_{>a'})$. Thus we must have $a < a' \leq y$.

In this case, we conclude that $a \notin \operatorname{ci}(I, z)$ since $a' > a$ is the minimal element of this circuit. However, this implies that $\operatorname{ci}(I, z) = \operatorname{ci}(J, z)$, and in particular, $z \in \operatorname{Spr}_J(a')$. But this is a contradiction with our assumption that $z \in \operatorname{span}(J_{>y}) \subseteq \operatorname{span}(J_{>a'})$. We conclude that $z \notin \operatorname{span}(I_{>y} \cup x) = \operatorname{span}(J_{>y})$, and thus that $z \in \operatorname{Spr}_J(y)$. □

We now have the machinery to describe the local structure of the external order beneath a particular independent set in terms of the spread operator.

Definition 3.5.13. If $I \subseteq E$ is independent, let $<_I$ denote the partial order on $E \setminus \operatorname{EA}(I)$ given by letting $x <_I y$ iff $x > y$ and $\operatorname{Spr}_I(x) \subseteq \operatorname{Spr}_I(y)$.

Lemma 3.5.14. *For $x, y \in E \setminus \operatorname{EA}(I)$, we have $x <_I y$ if and only if $x \in \operatorname{EP}_y(I)$.*

Proof. This follows by applying Lemma 3.5.11 part 3, and noting that both the conditions $x <_I y$ and $x \in \operatorname{EP}_y(I)$ imply that $x > y$. □

Lemma 3.5.15. *Let $I \subseteq E$ be independent, and suppose $x \in \mathrm{EP}_a(I)$ for $a \in I \cup \mathfrak{m}$. Denoting $J = \varepsilon_I(x)$, we have*

1. $\mathrm{EP}_{a'}(J) \supseteq \mathrm{EP}_{a'}(I)$ *for* $a' \in I \cup \mathfrak{m}$, $a' \neq a$
2. $\mathrm{EP}_x(J) = \mathrm{EP}_x(I) \subseteq \mathrm{EP}_a(I)$.
3. $\mathrm{EP}(J) = \mathrm{EP}(I) \setminus \{y \in \mathrm{EP}(I) : y \geq_I x\}$.

Proof. For part 1, let $a' \in I \cup \mathfrak{m}$ with $a' \neq a$. By Lemma 3.5.12, we have $\mathrm{Spr}_J(a') \supseteq \mathrm{Spr}_I(a')$, and thus $\mathrm{EP}_{a'}(J) \supseteq \mathrm{EP}_{a'}(I)$.

For part 2, the equality $\mathrm{EP}_x(J) = \mathrm{EP}_x(I)$ follows from Lemma 3.5.9, and the inclusion $\mathrm{EP}_x(I) \subseteq \mathrm{EP}_a(I)$ is given by Lemma 3.5.11 part 1.

For part 3, we consider the activity of the elements of E according to the decomposition

$$E = I \cup \mathrm{EA}(I) \cup \left(\bigcup_{\alpha \in I \cup \mathfrak{m} \setminus a} \mathrm{EP}_\alpha(I) \right) \cup \mathrm{EP}_a(I).$$

To prove the desired identity for $\mathrm{EP}(J)$, we will need to show that no elements outside of $\mathrm{EP}(I)$ are externally passive with respect to J, and that among the externally passive elements of I, only the elements in $\{y \in \mathrm{EP}(I) : y \geq_I x\}$ are not passive with respect to J.

To this end, first note that if $a \neq \mathfrak{m}$, then $\mathrm{ci}(I, x) = \mathrm{ci}(J, a)$, so the fact that $a = \min(\mathrm{ci}(I, x))$ implies a is minimal in its fundamental circuit in J, hence is externally active. Since all elements of I not equal to a are in J, this proves that no element of I is externally passive with respect to J.

Next suppose that $y \in \mathrm{EA}(I)$. Then we know that $\rho_I(y) \in I$, that $y < \rho_I(y)$ by Lemma 3.5.6, and that $\mathrm{Spr}_I(y) = \emptyset$ by Lemma 3.5.10. If $\rho(y) = a$, then $y < a$, so by Lemma 3.5.9 we have $\mathrm{Spr}_J(y) = \mathrm{Spr}_I(y) = \emptyset$, and we conclude $y \in \mathrm{EA}(J)$ by Lemma 3.5.10. If $\rho_I(y) = a' \in I$ with $a' \neq a$, then by Lemma 3.5.12, $\mathrm{Spr}_J(a') \supseteq \mathrm{Spr}_I(a')$. In particular, $\rho_J(y) = a' > y$, so $y \in \mathrm{EA}(J)$ by Lemma 3.5.6. We conclude that $\mathrm{EA}(I) \subseteq \mathrm{EA}(J)$, so in particular no externally active element of I is externally passive with respect to J.

For the externally passive elements of I, notice that the set $\{y \in \mathrm{EP}(I) : y \geq_I x\}$ consists of elements y with $x \in \mathrm{EP}_y(I) \subseteq \mathrm{Spr}_I(y)$. This implies by Lemma 3.5.11 part 2 that $\rho_I(y) = \rho_I(x) = a$ for each such y, so $y \in \mathrm{EP}_a(I)$. Notice that by part 1, $\mathrm{EP}_{a'}(I) \subseteq \mathrm{EP}_{a'}(J)$ for each $a' \in I \cup \mathfrak{m}$, $a' \neq a$, so the elements of $\mathrm{EP}_{a'}(I)$ remain externally passive with respect to J. Thus to conclude, it is enough to show for $y \in \mathrm{EP}_a(I)$ that y is not externally passive with respect to J if and only if $y \geq_I x$.

For the simplest case, if $y = x$, then $y \in J$ is not externally passive with respect to J as desired. Any other choice of y lies outside of J, so we will proceed to show that if $y \in \mathrm{EP}_a(I)$ with $y \neq x$, then $y \in \mathrm{EA}(J)$ if and only if $y >_I x$.

For the forward direction, suppose $y \in \mathrm{EP}_a(I)$ with $y \in \mathrm{EA}(J)$. Since $y \in \mathrm{EP}_a(I)$, we have $y > a$. We claim also that $y < x$, and that $x \in \mathrm{ci}(J, y)$.

If $a = \mathfrak{m}$, then $y \notin \operatorname{span}(I)$, but since y is externally active with respect to J, we have $y \in \operatorname{span}(J) = \operatorname{span}(I \cup x)$, so $x \in \operatorname{ci}(J, y)$. If $a \neq \mathfrak{m}$, then $a \in \operatorname{ci}(I, y)$, so $y \in \operatorname{bo}(I, a) = \operatorname{bo}(J, x)$, and so $x \in \operatorname{ci}(J, y)$. Thus $x \in \operatorname{ci}(J, y)$ in either of these cases, so if we had $y > x$, this would imply that y is externally passive with respect to J, a contradiction. We conclude that $y < x$ as well.

Notice now that since $y \in \operatorname{EA}(J)$, we have $y \in \operatorname{span}(J_{>y}) = \operatorname{span}(I_{>y} \cup x)$. In particular, $\operatorname{ci}(J, y) \subseteq J_{>y} \cup y$, so we have

$$\operatorname{span}(J_{>y}) = \operatorname{span}(J_{>y} \cup y) = \operatorname{span}(J_{>y} \cup y \setminus x) = \operatorname{span}(I_{>y} \cup y).$$

Thus this span contains x. However, since $x \in \operatorname{EP}_a(I)$, we have that $x \notin \operatorname{span}(I_{>a}) \supseteq \operatorname{span}(I_{>y})$. Thus we conclude that $x \in \operatorname{span}(I_{>y} \cup y) \setminus \operatorname{span}(I_{>y}) = \operatorname{Spr}_I(y)$. By Lemma 3.5.11 part 3, we have $\operatorname{Spr}_I(x) \subseteq \operatorname{Spr}_I(y)$, and since $x > y$ this shows $x <_I y$.

For the reverse direction, suppose that $y \in \operatorname{EP}_a(I)$ and $y >_I x$. Then we have $y < x$, $\operatorname{Spr}_I(y) \supseteq \operatorname{Spr}_I(x)$, and $\rho_I(y) = a$. Letting $J' = \varepsilon_I(y)$, note that by Lemma 3.5.9, we have $\operatorname{Spr}_{J'}(y) = \operatorname{Spr}_I(y)$, so $x \in \operatorname{Spr}_{J'}(y)$. In particular, $x \in \operatorname{span}(J')$, and $y = \min(\operatorname{ci}(J', x))$. Noting that $x \in \operatorname{EP}_y(J')$, we see that $\varepsilon_{J'}(x) = J' \cup x \setminus y$ which in particular is equal to $\varepsilon_I(x)$. This implies that $\operatorname{ci}(J, y) = \operatorname{ci}(J', x)$, so y is minimal in this circuit, and thus is externally active with respect to J. $\square$

From the above proposition, we obtain the following correspondence with the external order.

Proposition 3.5.16. *Let $I \subseteq E$ be independent. Then*

1. *If $J = I \cup x \setminus a$ is independent for $x \in \operatorname{EP}(I)$ and $a \in I \cup \mathfrak{m}$, then $J <^*_{\text{ext}} I$ if and only if $J = \varepsilon_I(x)$.*

2. *The canonical passive exchange map ε_I is an order embedding of $(\operatorname{EP}(I), \leq_I)$ into the external order which preserves covering relations.*

3. *An independent set J is covered by I in the external order if and only if $J = \varepsilon_I(x)$ for $x \in \operatorname{EP}(I)$ maximal with respect to $\leq_I$.*

Proof. For part 1, let $x \in \operatorname{EP}(I)$ and $a \in I \cup \mathfrak{m}$, and denote $J = I \cup x \setminus a$. Then $J = \varepsilon_I(x)$ if and only if $x \in \operatorname{EP}_a(I)$, and in this case $\operatorname{EP}(J) \subsetneq \operatorname{EP}(I)$ by Lemma 3.5.15 part 3, so $J <^*_{\text{ext}} I$.

Now suppose that $J \neq \varepsilon_I(x)$, so that $x \in \operatorname{EP}_{a'}(I)$ for some $a' \neq a$. If $a = \mathfrak{m}$, then $x \in \operatorname{EP}_{a'}(I)$ implies that $x \in \operatorname{span}(I)$, so $J = I \cup x \setminus a$ is not independent. If $a' = \mathfrak{m}$, then $x \notin \operatorname{span}(I)$, so $I \cup x$ is independent, and $I \cup x \setminus a$ does not span a. In this case, we have $a \in \operatorname{EP}(J)$, so in particular, $\operatorname{EP}(J) \nsubseteq \operatorname{EP}(I)$.

Thus suppose that $a, a' \neq \mathfrak{m}$. The fact that $x \in \operatorname{EP}_{a'}(I)$ implies that $a' = \min(\operatorname{ci}(I, x))$. If $a' > a$, this implies $a \notin \operatorname{ci}(I, x)$, from which we conclude that J is not independent. If $a' < a$, then we have $\operatorname{ci}(J, a) = \operatorname{ci}(I, x)$, so since $a' < a$ is minimal in this circuit, this implies that $a \in \operatorname{EP}(J)$, hence $\operatorname{EP}(J) \nsubseteq \operatorname{EP}(I)$ and $J \not<^*_{\text{ext}} I$.

For part 2, first suppose that $x, y \in \mathrm{EP}(I)$ with $x \not\leq_I y$. Then by Lemma 3.5.15 part 3, $y \in \mathrm{EP}(\varepsilon_I(x))$, but $y \notin \mathrm{EP}(\varepsilon_I(y))$. In particular $\mathrm{EP}(\varepsilon_I(x)) \nsubseteq \mathrm{EP}(\varepsilon_I(y))$, so we have $\varepsilon_I(x) \not\leq^*_{\mathrm{ext}} \varepsilon_I(y)$.

Next, let $x, y \in \mathrm{EP}(I)$ with $x \leq_I y$. In particular, any element α of $\mathrm{EP}(I)$ with $\alpha \geq_I y$ also satisfies $\alpha \geq_I x$, so we see that $\mathrm{EP}(\varepsilon_I(x)) \subseteq \mathrm{EP}(\varepsilon_I(y))$ by Lemma 3.5.15 part 3. Thus in this case, $\varepsilon_I(x) \leq^*_{\mathrm{ext}} \varepsilon_I(y)$.

Finally, suppose further that $x <_I y$ is a covering relation, and suppose that J is an independent set with $\varepsilon_I(x) \leq^*_{\mathrm{ext}} J \leq^*_{\mathrm{ext}} \varepsilon_I(y)$. Since $\mathrm{Spr}_I(x) \subseteq \mathrm{Spr}_I(y)$, we have $\rho_I(x) = \rho_I(y) = a$ for some $a \in I \cup \mathfrak{m}$. Let $J_x = \varepsilon_I(x)$ and $J_y = \varepsilon_I(y)$, and denote $J_0 = I \setminus a$, so that $J_x = J_0 \cup x$ and $J_y = J_0 \cup y$. By the antimatroid interval property without upper bounds, we have that $J_0 \subseteq J$, and since matroid rank is monotone in the external order, we must have $J = J_0 \cup z$ for some $z \in \mathrm{EP}(I)$. In particular, $J = I \setminus a \cup z$ with $J \leq^*_{\mathrm{ext}} I$, so by part 1 we must have $J = \varepsilon_I(z)$.

But in this case, $\mathrm{EP}(J) = \mathrm{EP}(I) \setminus \{\alpha \in \mathrm{EP}(I) : \alpha \geq_I z\}$ which implies $y \geq_I z$, and $\mathrm{EP}(J_x) = \mathrm{EP}(I) \setminus \{\alpha \in \mathrm{EP}(I) : \alpha \geq_I x\}$ which implies $z \geq_I x$. Thus $x \leq_I z \leq_I y$, so since y covers x, we must have that either $J = J_x$ or $J = J_y$. We conclude that J_y covers J_x in the external order.

For part 3, suppose first that $J \leq^*_{\mathrm{ext}} I$ is a covering relation. Then I is obtained from J by exchanging some element $x \in J$ with a (maximal) externally active element $a \in \mathrm{bo}(J, x)$, or by removing x if no such externally active element exists. In the latter case set $a = \mathfrak{m}$, and we then have $J = I \cup x \setminus a$. By part 1, we see then that $J = \varepsilon_I(x)$ for some $x \in \mathrm{EP}(I)$. If x were not maximal with respect to $\leq_I$, then there would be some $y >_I x$, and in particular we would have $\mathrm{EP}(J) \subseteq \mathrm{EP}(I) \setminus xy$ by Lemma 3.5.15 part 3. This implies that $|\mathrm{EP}(J)| \leq |\mathrm{EP}(I)| - 2$, which contradicts that $J <^*_{\mathrm{ext}} I$ is a covering relation. We conclude $J = \varepsilon_I(x)$ for some $\leq_I$-maximal element $x \in \mathrm{EP}(I)$.

On the other hand, suppose that $J = \varepsilon_I(x)$ for some $\leq_I$-maximal element $x \in \mathrm{EP}(I)$. Then by Lemma 3.5.15 part 3, $\mathrm{EP}(J) = \mathrm{EP}(I) \setminus \{y \in \mathrm{EP}(I) : y \geq_I x\} = \mathrm{EP}(I) \setminus x$, so we see that $J <^*_{\mathrm{ext}} I$ is a covering relation. □

The end result of this discussion is that we have identified a local neighborhood of independent sets below an independent set I in the external order with the externally passive elements of I via inclusion ordering of the spread sets $\mathrm{Spr}_I(x)$. An example of this structure is illustrative.

Example. Let X denote the matrix

$$X = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix},$$

and label the columns left to right by the numbers 1 to 5. As usual we will also use X to denote the ordered collection of columns of this matrix. Under this ordering, the independent set $I = 2$ has passive set $\mathrm{EP}_X(I) = 1345$. We compute the spread operator of each element in $X \cup \mathfrak{m}$ as follows.

$$\begin{aligned}
\mathrm{Spr}_I(\mathfrak{m}) &= 134\\
\mathrm{Spr}_I(1) &= 134\\
\mathrm{Spr}_I(2) &= 25\\
\mathrm{Spr}_I(3) &= 3\\
\mathrm{Spr}_I(4) &= 14\\
\mathrm{Spr}_I(5) &= 25.
\end{aligned}$$

In particular, $I \cup \mathfrak{m}$ gives the expected partition of X, by $\mathrm{Spr}_I(2) = 25$ and $\mathrm{Spr}_I(\mathfrak{m}) = 134$. For the externally passive elements, we have $5 \in \mathrm{EP}_2(I)$ and $1, 3, 4 \in \mathrm{EP}_{\mathfrak{m}}(I)$, so we get the canonical passive exchanges

$$\begin{aligned}
\varepsilon_I(5) &= 5\\
\varepsilon_I(1) &= 12\\
\varepsilon_I(3) &= 23\\
\varepsilon_I(4) &= 24.
\end{aligned}$$

In particular, these canonical passive exchange independent sets are ordered in the external order according to inclusion ordering of their spread sets, as illustrated in Figure 3.3.

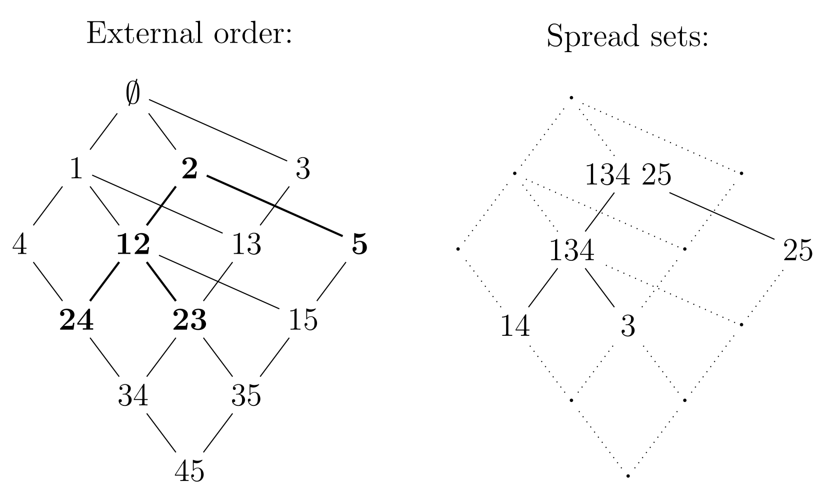


Figure 3.3: The external order of X with the passive exchange neighborhood of I in bold, and the corresponding spread sets associated with the elements of $I \cup \mathfrak{m}$ and $\mathrm{EP}(I)$.

Chapter 4

Applications to Zonotopal Algebra

Having defined the generalized external order and developed some of its fundamental combinatorial properties, we turn now to the theory of zonotopal algebra. In this chapter, we describe the fundamental connection between the external order and the combinatorial structure of zonotopal spaces, and we leverage this connection to improve our understanding of the canonical bases of the zonotopal $\mathcal{P}$- and $\mathcal{D}$-spaces in several settings.

Relevant background in zonotopal algebra is reviewed in Section 2.2. Recall that we will be adopting a convention for matroid activity which is common to the zonotopal algebra literature but uncommon elsewhere: the groundset ordering used for activity in this chapter is *reversed* from the usual convention. For instance, we will say that an element x in an ordered matroid is externally active with respect to a basis B if it is *maximal* in the fundamental circuit $\mathrm{ci}(B, x)$. This does not substantially change the way that matroid activity behaves, but when referencing material from Chapter 3 or Section 2.1, the ordering on the ground set typically needs to be reversed.

In Section 4.1, we begin by providing a direct correspondence between Lenz's forward exchange matroids and the structure of the external order, in particular characterizing the forward exchange property and generalized cocircuits. This lays the groundwork for later applications.

In Section 4.2, we review the two classes of polynomials, the P-polynomials and the D-polynomials, which are used to produce canonical bases for the zonotopal $\mathcal{P}$- and $\mathcal{D}$-spaces, and we summarize the form of these bases in the classical and semi-classical zonotopal settings. The section serves to frame various known notions and structural results for later reference, but we also present a characterization of the canonical $\mathcal{D}$-bases for the internal and semi-internal settings which does not seem to have been addressed explicitly in the literature before.

In Section 4.3 we give a simple description of the behavior of D-polynomials under differential operators, and in Section 4.4 we present a new recursive algebraic construction for these polynomials. The latter provides the first practical algorithm to compute the D-polynomials which is computationally tractible for matrices of moderate size.

Finally, in Section 4.5 we give a new elementary characterization for the adjusted P-

polynomials which form the canonical bases of the internal and semi-internal $\mathcal{P}$-spaces. The characterization is given by a simple projection operation on some of the linear forms in the defining products of the P-polynomials, and in particular this gives an explicit construction for these canonical basis polynomials.

4.1 Forward Exchange Matroids and the External Order

Recall from Section 2.2 that in [30], Lenz defines a class of combinatorial objects called *forward exchange matroids* which capture certain structural properties of ordered matroids which are relevant to the construction of zonotopal spaces. The properties defining these objects can be conveniently restated in terms of the theory of canonical passive exchanges developed in Section 3.5, as follows.

Definition 4.1.1. Let M be an ordered matroid with ground set E and bases $\mathbb{B}$. If $\mathbb{B}' \subseteq \mathbb{B}$, then we say that $\mathbb{B}'$ satisfies the **forward exchange property** if for every $B \in \mathbb{B}'$, and for every $x \in \mathrm{EP}(B)$, the canonical passive exchange $\varepsilon_B(x)$ is also in $\mathbb{B}'$. In this case, the triple $(E, \mathbb{B}, \mathbb{B}')$ is called a **forward exchange matroid**.

In light of the fact that $\varepsilon_B(x) <^*_{\mathrm{ext}} B$ and the canonical passive exchanges include all bases covered by B in the external order, the forward exchange property can be reformulated simply in terms of the external order.

Theorem 4. *A collection of bases $\mathbb{B}' \subseteq \mathbb{B}$ satisfies the forward exchange property if and only if it is downward closed in the external order. That is, if $B \in \mathbb{B}'$ and $B' \leq^*_{\mathrm{ext}} B$, then $B' \in \mathbb{B}'$.*

We additionally can relate the notion of generalized cocircuits of a forward exchange matroid to the external order. For notational convenience, we extend Definition 2.2.20 to allow for arbitrary collections of independent sets.

Definition 4.1.2. If $\mathbb{I}'$ is a collection of independent sets of a matroid, then a set $D \subseteq E$ is called a **generalized $\mathbb{I}'$-cocircuit**, or just an **$\mathbb{I}'$-cocircuit**, if D is inclusion-minimal with $D \cap I \neq \emptyset$ for every $I \in \mathbb{I}'$.

When a collection $\mathbb{B}'$ of bases satisfies the forward exchange property, or more generally when a collection $\mathbb{I}'$ of independent sets is downward closed in the external order, we can give a succinct description for the corresponding generalized cocircuits.

Proposition 4.1.3. *Let $\mathbb{I}'$ be a collection of independent sets in the ordered matroid $M = (E, \mathbb{I})$ which are downward closed in the external order. Then $D \subseteq E$ is a generalized cocircuit with respect to $\mathbb{I}'$ if and only if it is equal to $\mathrm{EP}(J)$ for a $\leq^*_{\mathrm{ext}}$-minimal set J in $\mathbb{I} \setminus \mathbb{I}'$.*

Proof. Suppose first that J is an independent set with $J \notin \mathbb{I}'$. Then because the external order is an antimatroid with feasible sets given by externally passive sets, we know by Lemma 3.1.4 that $\mathrm{EP}(J)$ has nonempty intersection with any $I \in \mathbb{I}'$ since $J \not\leq^*_{\mathrm{ext}} I$.

Now if additionally J is $\leq^*_{\mathrm{ext}}$-minimal outside of $\mathbb{I}'$, we will argue that $\mathrm{EP}(J)$ is a generalized cocircuit with respect to $\mathbb{I}'$ by showing that $\mathrm{EP}(J)$ is minimal among sets intersecting every set in $\mathbb{I}'$.

To see this, let $x \in \mathrm{EP}(J)$. Then the canonical passive exchange independent set $J' = \varepsilon_J(x)$ gives an independent set containing x which is less than J in the external order, and such that $J' \setminus x \subseteq J$. In particular, since J is $\leq^*_{\mathrm{ext}}$-minimal outside of $\mathbb{I}'$, we have $J' \in \mathbb{I}'$, and $J' \cap \mathrm{EP}(J) = \{x\}$. Thus we see that $\mathrm{EP}(J) \setminus x$ does not intersect every basis in $\mathbb{I}'$, and since $x \in \mathrm{EP}(J)$ was arbitrary, we see that $\mathrm{EP}(J)$ is minimal intersecting all bases in $\mathbb{I}'$, hence is a generalized cocircuit for $\mathbb{I}'$.

This proves one direction. Now suppose that $D \subseteq E$ is a generalized cocircuit for $\mathbb{I}'$. We will identify the particular independent set whose externally passive elements are given by D. If $\mathbb{I}' = \emptyset$, then we must have $D = \emptyset = \mathrm{EP}(B_0)$ where B_0 is the lex minimal basis of E. So suppose that $\mathbb{I}'$ is nonempty, so D is nonempty.

Now let I_0 denote the lex minimal basis of E, so that $\mathrm{EP}(I_0) = \emptyset$. In particular since $\mathbb{I}'$ is nonempty, it must contain I_0. Now if $I_i \in \mathbb{I}'$, define I_{i+1} as follows. Let $x \in I_i \cap D$, which exists because D has nonempty intersection with every set in $\mathbb{I}'$, and let I_{i+1} be the independent set covering I_i in the external order with edge label x, so that $\mathrm{EP}(I_{i+1}) = \mathrm{EP}(I_i) \cup x$.

As long as I_{i+1} lies in $\mathbb{I}'$, this construction can be repeated. In particular, for every i, we have $|\mathrm{EP}(I_i)| = i$, and $\mathrm{EP}(I_i) \subseteq D$. Now let j be the first index such that $I_j \notin \mathbb{I}'$. The fact that $I_j \notin \mathbb{I}'$ implies by the preceding discussion that $\mathrm{EP}(I_j)$ has nonempty intersection with every set in $\mathbb{I}'$. Thus if $\mathrm{EP}(I_j)$ is a strict subset of D, then this contradicts the assumption that D is minimal among sets intersecting every set in $\mathbb{I}'$. Thus we must have that $D = \mathrm{EP}(I_j)$.

Finally, if I_j were not $\leq^*_{\mathrm{ext}}$-minimal among independent sets outside of $\mathbb{I}'$, then there would be another independent set $J <^*_{\mathrm{ext}} I_j$ with $J \notin \mathbb{I}'$, and in particular $\mathrm{EP}(J) \subsetneq \mathrm{EP}(I_j) = D$. However, again by the above, $\mathrm{EP}(J)$ has nonempty intersection with each basis in $\mathbb{I}'$, so again this contradicts the fact that D is minimal with respect to this property. We conclude that D is of the required form. $\square$

As a consequence of this characterization, we can give an alternate formulation of the generalized $\mathcal{J}$-ideal in terms of externally passive sets.

Corollary 4.1.4. *If $(X, \mathbb{B}, \mathbb{B}')$ is a forward exchange matroid, then*

$$\mathcal{J}(X, \mathbb{B}') = \mathrm{Ideal}\left\{\ell_{\mathrm{EP}(I)} \,:\, I \subseteq X \text{ independent, } I \notin \mathbb{B}'\right\}.$$

Proof. Recall that $\mathcal{J}(X, \mathbb{B}')$ is defined as the ideal generated by the polynomials ℓ_D where D is a generalized $\mathbb{B}'$-cocircuit. By Proposition 4.1.3, a set D is a generalized $\mathbb{B}'$-cocircuit if and only if $D = \mathrm{EP}(I)$ for an independent set I which is $\leq^*_{\mathrm{ext}}$-minimal outside of $\mathbb{B}'$. Thus any generator ℓ_D of $\mathcal{J}(X, \mathbb{B}')$ is of the form $\ell_{\mathrm{EP}(I)}$ for some $I \notin \mathbb{B}'$. On the other hand, if I is any independent set outside of $\mathbb{B}'$, then $I \geq^*_{\mathrm{ext}} I_0$ for a minimal independent set outside of $\mathbb{B}'$, and in particular $\mathrm{EP}(I) \supseteq \mathrm{EP}(I_0)$. This implies that $\ell_{\mathrm{EP}(I_0)}$ divides $\ell_{\mathrm{EP}(I)}$, and since $\ell_{\mathrm{EP}(I_0)}$ is a generator of $\mathcal{J}(X, \mathbb{B}')$, we see that $\ell_{\mathrm{EP}(I)}$ lies in this ideal. □

In the next section we will discuss how the polynomials $\ell_{\mathrm{EP}(I)}$ play an important role in various parts of zonotopal algebra, in particular forming the canonical bases of the zonotopal $\mathcal{P}$-spaces in many settings.

4.2 Canonical Basis Polynomials

In the following, we will recall the definitions of the canonical polynomial bases of the classical zonotopal $\mathcal{P}$ and $\mathcal{D}$ spaces, and give simpler characterizations of the bases for several zonotopal settings. We begin by introducing notation for a class of polynomials that is used extensively.

Definition 4.2.1. If $I \subseteq X$ is independent, let P_I^X denote the polynomial $\ell_{\mathrm{EP}_X(I)}$ given by the product of linear forms from the set of externally passive elements of I with respect to X. We call the polynomial P_I^X a ***P*-polynomial**. When the matrix X is clear from context, the superscript will sometimes be omitted.

In [23], the notation Q_I is used for the polynomial P_I^X, and the notation $X(I)$ is used for the set $\mathrm{EP}_X(I)$ of externally passive elements.

Recall that the central, external, internal, semi-external, and semi-internal $\mathcal{P}$ and $\mathcal{D}$ spaces are dual under the action of the polynomial differential bilinear form. In each of these classical settings, a canonical polynomial basis exists for the $\mathcal{P}$ space, and we will denote these bases by $\mathcal{B}_{\mathcal{P}}^X$, $\mathcal{B}_{\mathcal{P}_+}^X$, $\mathcal{B}_{\mathcal{P}_-}^X$, $\mathcal{B}_{\mathcal{P}_-}^{X,\mathbb{I}'}$, and $\mathcal{B}_{\mathcal{P}_-}^{X,I_{\mathrm{int}}}$ respectively.

Proposition 4.2.2 (Classical $\mathcal{P}$-space Bases, [23])**.** *The following sets of polynomials are vector space bases of their corresponding zonotopal $\mathcal{P}$-spaces.*

- *Central case:* $\mathcal{B}_{\mathcal{P}}^X := \left\{ P_B^X \;:\; B \in \mathbb{B}(X) \right\}$
- *External case:* $\mathcal{B}_{\mathcal{P}_+}^X := \left\{ P_I^X \;:\; I \in \mathbb{I}(X) \right\} = \left\{ P_B^{X'} \;:\; B \in \mathbb{B}_+(X) \right\}$
- *Internal case:* $\mathcal{B}_{\mathcal{P}_-}^X := \left\{ \widetilde{P}_B^X \;:\; B \in \mathbb{B}_-(X) \right\}$
- *Semi-external case:* $\mathcal{B}_{\mathcal{P}_-}^{X,\mathbb{I}'} := \left\{ P_I^X \;:\; I \in \mathbb{I}' \right\} = \left\{ P_B^{X'} \;:\; B \in \mathbb{B}_+(X, \mathbb{I}') \right\}$
- *Semi-internal case:* $\mathcal{B}_{\mathcal{P}_-}^{X,I_{\mathrm{int}}} := \left\{ \widetilde{P}_B^{X,I_{\mathrm{int}}} \;:\; B \in \mathbb{B}_-(X, I_{\mathrm{int}}) \right\}$

We call these bases the ***canonical zonotopal $\mathcal{P}$-space bases*** *of X.*

In the internal case, the polynomial $\widetilde{P}_B^X$ is defined by

$$\widetilde{P}_B^X := P_B^X - f_B,$$

where $f_B \in \mathcal{J}_-(X)$ is a unique polynomial such that $\widetilde{P}_B^X \in \mathcal{P}_-(X)$. The structure of these polynomials, which were defined abstractly in [23], Section 5, has been to this point poorly understood. In Section 3 of [24], the semi-internal polynomials $\widetilde{P}_B^{X,I_{\mathrm{int}}}$ are shown to exhibit a corresponding structure for an appropriate polynomial f_B in the semi-internal $\mathcal{J}$-ideal. In Section 4.5 we will give an explicit construction for these polynomials in matroid-theoretic terms.

The duality of the $\mathcal{P}$ and $\mathcal{D}$ spaces in particular induces a canonical polynomial basis for each $\mathcal{D}$ space, whose elements are dual to polynomials of the corresponding $\mathcal{P}$ basis. These $\mathcal{D}$ space bases are defined and developed in [30], and in particular are constructed explicitly in terms of a class of functions called multivariate splines.

Definition 4.2.3. If $B \subseteq X$ is a basis, let D_B^X denote the polynomial of $\mathcal{D}(X)$ which is dual to P_B^X under the differential bilinear form. We call the polynomial D_B^X a **central D-polynomial**. The matrix X may sometimes be omitted from this notation when the meaning is clear from context.

In addition, it will be convenient to extend the above definition to include independent sets which span a hyperplane in X.

Notation. If $H \subseteq X$ is a hyperplane, as usual let $\eta(H)$ be a unit normal vector of H in the column space of X, and let n_H^X denote the linear form $\ell_{\eta(H)}$. Additionally, let $\mathbb{I}_{\mathcal{F}}(X)$ denote the collection of independent sets spanning a (facet) hyperplane in X,

$$\mathbb{I}_{\mathcal{F}}(X) := \{I \in \mathbb{I}(X) \,:\, r(I) = r(X) - 1\}.$$

Definition 4.2.4. Let $I \in \mathbb{I}_{\mathcal{F}}(X)$ with $H = \operatorname{span}(I)$, let $D = X \setminus H$ denote the complementary cocircuit of H, and let

$$c_H := |D|! \prod_{d \in D} \langle \ell_d, n_H^X \rangle = \left(n_H^X\right)^{|D|}(\partial)\, \ell_D.$$

Define

$$D_I^X := \frac{1}{c_H} \left(n_H^X\right)^{|D|} D_I^H.$$

We call the polynomial D_I^X a **boundary D-polynomial**.

It is an interesting structural property of the zonotopal spaces that the P polynomials may be naturally defined over arbitrary independent sets, but there is an inherent problem in attempting to define the D polynomials similarly. This is reflected in the fact noted by [23] that $\mathcal{D}_+(X)$ depends not only on the matrix X, but also on the extending basis B_{ext}.

In particular, without referencing a choice of extending basis, there is no way to canonically define D_I^X for all independent sets I so that the polynomials $\left\{D_I^X : I \in \mathbb{I}(X)\right\}$ give a dual basis of $\mathcal{B}_{\mathcal{P}_+}^X$.

We distinguish between the central and boundary D-polynomials because they play distinct roles with respect to the zonotopal spaces. However, a key unifying feature of the two types is the following duality with the P-polynomials.

Lemma 4.2.5. *If $I, I' \subseteq X$ are independent with $I' \in \mathbb{B}(X) \cup \mathbb{I}_{\mathcal{F}}(X)$, then*

$$\left\langle P_I^X, D_{I'}^X \right\rangle = \begin{cases} 1, & I = I' \\ 0, & I \neq I' \end{cases}.$$

Proof. First suppose that I' is a basis. If I is a basis, then the identity follows by the definition of $D_{I'}^X$ as the dual polynomial in $\mathcal{D}(X)$ of $P_{I'}^X$. If I is not a basis, then $P_I^X \in \mathcal{J}(X)$ since it is divisible by $\ell_{X \setminus H}$ for any hyperplane H containing I. In particular, since $D_{I'}^X \in \mathcal{D}(X) = \ker \mathcal{J}(X)$, we have $P_I^X(\partial) D_{I'}^X = 0$, whence the bilinear form is zero.

Now suppose that $I' \in \mathbb{I}_{\mathcal{F}}(X)$ with $H = \operatorname{span}(I')$. If $I \nsubseteq H$, then I contains an element in $X \setminus H$, so $|\mathrm{EP}(I)| \leq |X \setminus H| - 1$. In particular,

$$D_{I'}^X(\partial) P_I^X = \frac{1}{c_H} D_I^H(\partial) \left(n_H^X\right)^{|X \setminus H|}(\partial)\ \ell_{\mathrm{EP}(I)}.$$

However, $n_H^X(\partial)$ kills any linear form corresponding to a vector in H, so since at most $|X \setminus H| - 1$ linear forms in $\ell_{\mathrm{EP}(I)}$ lie outside of H, we see by direct computation of the derivative with the product rule that this derivative is zero, and likewise for the bilinear form.

Finally, suppose that $I \subseteq H$. In this case, we have $\mathrm{EP}_X(I) = (X \setminus H) \cup \mathrm{EP}_H(I)$, where in particular, $\mathrm{EP}_H(I) \subseteq H$. We can then compute the derivative $D_{I'}^X(\partial) P_I^X$ as

$$\begin{aligned} D_{I'}^X(\partial) P_I^X &= \frac{1}{c_H} D_{I'}^H(\partial) \left(n_H^X\right)^{|X \setminus H|}(\partial)\ \ell_{X \setminus H} \ell_{\mathrm{EP}_H(I)} \\ &= \frac{1}{c_H} \left(\left(n_H^X\right)^{|X \setminus H|}(\partial) \ell_{X \setminus H}\right) \left(D_{I'}^H(\partial) \ell_{\mathrm{EP}_H(I)}\right) = D_{I'}^H(\partial) P_I^H. \end{aligned}$$

Here, we can split the differential operator because the action of $n_H^X(\partial)$ kills the linear forms from $\mathrm{EP}_H(I)$, as they correspond to vectors in H. From the above, we see that the derivative is zero when $I \neq I'$ by the preceding argument, considering I' as a basis of H. When $I = I'$, we have by definition of $D_{I'}^H$ that $D_{I'}^H(\partial) P_I^H = \left\langle D_{I'}^H, P_I^H \right\rangle = 1$. □

Using the central D-polynomials, we can describe the canonical bases for the classical zonotopal $\mathcal{D}$-spaces. For notation, we will denote the canonical dual bases of the central, external, and internal $\mathcal{D}$-spaces by $\mathcal{B}_{\mathcal{D}}^X$, $\mathcal{B}_{\mathcal{D}_+}^X$, and $\mathcal{B}_{\mathcal{D}_-}^X$ respectively, and of the semi-external and semi-internal $\mathcal{D}$-spaces by $\mathcal{B}_{\mathcal{D}_+}^{X,\mathbb{I}'}$ and $\mathcal{B}_{\mathcal{D}_-}^{X,I_{\mathrm{int}}}$. These bases are given explicitly by the following.

Proposition 4.2.6. *The* ***canonical zonotopal $\mathcal{D}$-space bases*** *are given by the following sets of polynomials.*

- *Central case:* $\mathcal{B}_{\mathcal{D}}^{X} = \left\{D_B^X : B \in \mathbb{B}(X)\right\}$
- *External case:* $\mathcal{B}_{\mathcal{D}_+}^{X} = \left\{D_B^{X'} : B \in \mathbb{B}_+(X)\right\}$
- *Internal case:* $\mathcal{B}_{\mathcal{D}_-}^{X} = \left\{D_B^X : B \in \mathbb{B}_-(X)\right\}$
- *Semi-external case:* $\mathcal{B}_{\mathcal{D}_+}^{X,\mathbb{I}'} = \left\{D_B^{X'} : B \in \mathbb{B}_+(X,\mathbb{I}')\right\}$
- *Semi-internal case:* $\mathcal{B}_{\mathcal{D}_-}^{X,I_{\text{int}}} = \left\{D_B^X : B \in \mathbb{B}_-(X,I_{\text{int}})\right\}$

The fact that the above canonical $\mathcal{D}$-bases are all given in terms of the unmodified central D-polynomials is somewhat surprising, as it contrasts with the situation for the $\mathcal{P}$-bases, which differ from the P-polynomials in the internal and semi-internal cases. The characterization is given directly for the central, external and semi-external cases by the following proposition, which summarizes arguments of Lenz in [30], Sections 7 and 8.

Proposition 4.2.7 ([30])**.** *Let $\mathbb{B}'$ denote the collection either of central, external, internal, semi-external, or semi-internal bases of X, and let Y denote X for the central, internal, and semi-internal cases, and $X \cup B_{\text{ext}}$ for the external and semi-external cases, where B_{ext} is the extending basis. Then:*

- *$\mathbb{B}'$ satisfies the forward exchange property as a collection of bases of Y.*
- *The zonotopal $\mathcal{J}$-ideal and $\mathcal{D}$-space of the corresponding case are given by the corresponding forward exchange constructions $\mathcal{J}(Y,\mathbb{B}')$ and $\mathcal{D}(Y,\mathbb{B}')$.*
- *The collection of polynomials $\left\{D_B^Y : B \in \mathbb{B}'\right\}$ forms a basis of $\mathcal{D}(Y,\mathbb{B}')$ which is dual to the collection $\left\{P_B^Y : B \in \mathbb{B}'\right\}$ under the differential bilinear form.*

The internal and semi-internal cases of Proposition 4.2.6 are not directly implied by this result due to the unusual nature of the canonical $\mathcal{P}$-bases in the internal settings. However, a few additional observations on the structure of the internal and semi-internal $\mathcal{P}$-basis polynomials allows us to complete the argument.

Proof of Proposition 4.2.6. The considerations of Proposition 4.2.7 prove Proposition 4.2.6 for the central, external, and semi-external cases due to the fact that in these cases, the zonotopal $\mathcal{P}$-space is given by the span of polynomials $\left\{P_B^Y : B \in \mathbb{B}'\right\}$, and thus the polynomials $\left\{D_B^Y : B \in \mathbb{B}'\right\}$ lie in the corresponding $\mathcal{D}$-space and are dual to the canonical $\mathcal{P}$-space basis.

For the internal and semi-internal cases, some extra work is necessary. In these settings, we still know from Proposition 4.2.7 that the polynomials $\left\{D_B^Y : B \in \mathbb{B}'\right\}$ lie in the appropriate zonotopal $\mathcal{D}$-space. However, the canonical basis polynomials for the corresponding zonotopal $\mathcal{P}$-spaces are in general different from P_B^Y, so we need to argue that the polynomials D_B^Y are still dual to the canonical bases.

For notation, let $\widetilde{P}_B^Y$ denote the internal or semi-internal $\mathcal{P}$-basis polynomial associated with $B \in \mathbb{B}'$, given by $\widetilde{P}_B^X$ for the internal case, and $\widetilde{P}_B^{X,I_{\mathrm{int}}}$ for the semi-internal case. In [23] and [24], it is shown that $\widetilde{P}_B^Y$ can be represented as a sum of the form $P_B^Y + J_B$, where J_B is an appropriate polynomial in the corresponding zonotopal $\mathcal{J}$-ideal $\mathcal{J}(Y, \mathbb{B}')$. In particular, for bases $B, B' \in \mathbb{B}'$, we have

$$\left\langle D_B^Y, \widetilde{P}_{B'}^Y \right\rangle = \left\langle D_B^Y, P_{B'}^Y + J_{B'} \right\rangle = \left\langle D_B^Y, P_{B'}^Y \right\rangle + \left\langle D_B^Y, J_{B'} \right\rangle = \left\langle D_B^Y, P_{B'}^Y \right\rangle,$$

where the last equality follows because $D_B^Y \in \mathcal{D}(Y, \mathbb{B}') = \ker \mathcal{J}(Y, \mathbb{B}')$, so any polynomial in $\mathcal{J}(Y, \mathbb{B}')$ kills D_B^Y under the differential bilinear form. Thus the polynomials $\left\{ D_B^Y \,:\, B \in \mathbb{B}' \right\}$ indeed are dual to the canonical $\mathcal{P}$-space basis $\left\{ \widetilde{P}_B^Y \,:\, B \in \mathbb{B}' \right\}$. □

Next we show that the boundary D-polynomials and corresponding P-polynomials give a convenient generating set for the central ideals $\mathcal{I}(X)$ and $\mathcal{J}(X)$.

Lemma 4.2.8. *The central zonotopal $\mathcal{I}$- and $\mathcal{J}$-ideals are given by*

$$\begin{aligned} \mathcal{I}(X) &= \mathrm{Ideal}\left\{ D_I^X \,:\, I \in \mathbb{I}_{\mathcal{F}}(X) \right\}, \\ \mathcal{J}(X) &= \mathrm{Ideal}\left\{ P_I^X \,:\, I \in \mathbb{I}_{\mathcal{F}}(X) \right\}. \end{aligned}$$

In particular, the standard generators of these ideals are given by the above polynomials for the independent sets $\{I_H \,:\, H \in \mathcal{F}(X)\}$ where I_H is the lex minimal basis of H.

Proof. For $\mathcal{I}(X)$, note that the polynomials $\left(n_H^X\right)^{|X \setminus H|}$ for $H \in \mathcal{F}(X)$ form the standard generating set for $\mathcal{I}(X)$, so since all of the polynomials D_I^X are divisible by one of these generators, they are members of the ideal. Further, if I_0 is the lex minimal basis of a hyperplane H, then $D_{I_0}^H = 1$ by duality with the corresponding polynomial $P_{I_0}^H$, so $D_{I_0}^X$ gives a scalar multiple of the standard generator $\left(n_H^X\right)^{|X \setminus H|}$ of $\mathcal{I}(X)$. This shows that all of $\mathcal{I}(X)$ is generated by the polynomials D_I^X.

For $\mathcal{J}(X)$, if $I \in \mathbb{I}_{\mathcal{F}}(X)$ with $\mathrm{span}(I) = H$, then $X \setminus H \subseteq \mathrm{EP}(I)$, so the $\mathcal{J}$-ideal generator $\ell_{X \setminus H}$ divides P_I^X, which thus lies in $\mathcal{J}(X)$. If I_0 is the lex minimal basis of H, then $\mathrm{EP}(I_0) = X \setminus H$, so $P_{I_0}^X$ is the standard generator $\ell_{X \setminus H}$ of $\mathcal{P}(X)$. This in particular shows that all of $\mathcal{J}(X)$ is generated by the polynomials P_I^X. □

To refer to the standard generators of the central zonotopal ideals, we will sometimes use the notation D_H^X and P_H^X to refer to the corresponding generators $D_{I_H}^X$ and $P_{I_H}^X$ as above. Specifically,

$$D_H^X = \left(n_H^X\right)^{|X \setminus H|},$$

and

$$P_H^X = \ell_{X \setminus H}.$$

A useful property of the P- and D-polynomials is that much of their structure is preserved by extending X with a new vector.

Lemma 4.2.9. *If $x \in \mathbb{R}^d$ lies in the column space of X*[1] *and $X \cup x$ denotes the collection obtained by appending x to X as the final element, then for any basis $B \subseteq X$, we have*

$$P_B^{X \cup x} = P_B^X,$$

and

$$D_B^{X \cup x} = D_B^X.$$

In particular, $\mathcal{B}_\mathcal{P}^X \subseteq \mathcal{B}_\mathcal{P}^{X \cup x}$ and $\mathcal{B}_\mathcal{D}^X \subseteq \mathcal{B}_\mathcal{D}^{X \cup x}$. The remaining polynomials of $\mathcal{B}_\mathcal{P}^{X \cup x}$ not contained in $\mathcal{B}_\mathcal{P}^X$ are of the form $P_{I \cup x}^{X \cup x}$ where $I \in \mathbb{I}_\mathcal{F}(X)$ spans a hyperplane avoiding x, and

$$P_{I \cup x}^{X \cup x} = P_I^X.$$

Proof. For the inclusion of $\mathcal{P}$-basis polynomials, let B be a basis of X. In particular, we have $\mathrm{EP}_X(B) = \mathrm{EP}_{X \cup x}(B)$ since x is larger than every element of B hence is externally active, and the activity of elements in X with respect to B is not changed by appending a largest vector to X. This implies that $P_B^X = P_B^{X \cup x}$, which is thus a member of $\mathcal{B}_\mathcal{P}^{X \cup x}$.

Additionally, note that if $P_B^{X \cup x} \in \mathcal{B}_\mathcal{P}^{X \cup x} \setminus \mathcal{B}_\mathcal{P}^X$, then $x \in B$ since otherwise $P_B^{X \cup x}$ is covered by the previous case. In this case, the elements passive with respect to B in $X \cup x$ are the same as those passive with respect to $B \setminus x$ in X, since any element of X outside of $\mathrm{span}(B \setminus x)$ is passive in both cases, and any element in $\mathrm{span}(B \setminus x)$ has activity unchanged by adding x. Since $\mathrm{EP}_{X \cup x}(B) = \mathrm{EP}_x(B \setminus x)$, we thus have $P_B^{X \cup x} = P_{B \setminus x}^X$. The bases of this type correspond with the independent sets $I \subseteq X$ which span a hyperplane avoiding x.

For the inclusion of the $\mathcal{D}$-basis polynomials, first note that $\mathcal{J}(X) \supseteq \mathcal{J}(X \cup x)$ because any cocircuit of $X \cup x$ contains a cocircuit of X. In particular, this implies $\mathcal{D}(X) = \ker \mathcal{J}(X) \subseteq \mathcal{J}(X \cup x) = \mathcal{D}(X \cup x)$, so it is sufficient to show that for any basis B of X, the polynomial D_B^X is dual under the differential bilinear form to $P_B^{X \cup x}$ in $\mathcal{B}_\mathcal{P}^{X \cup x}$.

Thus let $B' \subseteq X \cup x$ be a basis. If $x \notin B'$, then B' is a basis of X, and thus $P_{B'}^{X \cup x} = P_{B'}^X$, and $\left\langle D_B^X, P_{B'}^{X \cup x} \right\rangle = \delta_{B,B'}$. If $x \in B'$, then $\mathrm{EP}_{X \cup x}(B') = \mathrm{EP}_X(B' \setminus x)$, so $P_{B'}^{X \cup x} = P_{B' \setminus x}^X$. In particular, since $B' \setminus x \in \mathbb{I}_\mathcal{F}(X)$, we have $P_{B' \setminus x}^X \in \mathcal{J}(X)$, and $\left\langle D_B^X, P_{B'}^{X \cup x} \right\rangle = 0$. Thus we conclude that D_B^X is dual to $P_B^{X \cup x}$ in $\mathcal{P}(X)$, and so $D_B^X = D_B^{X \cup x}$. □

In the above we argue that the new polynomials of $\mathcal{B}_\mathcal{P}^{X \cup x}$ not already contained in $\mathcal{B}_\mathcal{P}^X$ consist of the P-polynomials corresponding to independent sets $I \in \mathbb{I}_\mathcal{F}(X)$ which span a hyperplane avoiding x. Absent from the lemma is a corresponding statement about the polynomials in $\mathcal{B}_\mathcal{D}^{X \cup x} \setminus \mathcal{B}_\mathcal{D}^X$, and this is because the boundary D-polynomials of X are *not* in general contained in $\mathcal{D}(X \cup x)$. However, we know from Lemma 4.2.5 that these polynomials do at least act dually to the corresponding P-polynomials. This property will be fundamental for Section 4.4, where the boundary D-polynomials of X will be used as the starting point for constructing the new central D-polynomials of $X \cup x$.

[1] An analogous statement is straightforward to state and prove when x increases the rank of X.

4.3 Differential Structure of Central D-Polynomials

We now use the structure of the external order to give a simple description of the behavior of central D-polynomials under differential operators. Recall that $[x]_B^b$ denotes the b-coordinate of a vector x with respect to a basis B. The main result is the following.

Proposition 4.3.1. *If $B \subseteq X$ is a basis, and $b \in B$, then*

$$\ell_b(\partial) D_B = \sum_{B' = B \setminus b \cup x} [b]_{B'}^x D_{B'}, \tag{4.1}$$

where the sum is over bases B' covered by B in the external order.

Proof. Since $\mathcal{D}(X)$ is closed under derivatives in $\mathbb{R}^d$, $\ell_b(\partial) D_B$ can be represented as a linear combination of the polynomials $D_{B'}$ for B' a basis of X. Further, since the polynomials in $\mathcal{B}_\mathcal{D}^X$ are homogeneous of degree given by their rank in the external order $\leq_{\text{ext}}^*$, we note that $\ell_b(\partial) D_B$ is spanned by the polynomials $D_{B'}$ for B' of $\leq_{\text{ext}}^*$-rank one less than that of B.

Suppose $B' = \{b_1', \ldots, b_d'\}$ is such a basis, and let

$$c^{(B')} = \langle P_{B'},\, \ell_b(\partial) D_B \rangle$$

denote the $D_{B'}$ coordinate of $\ell_b(\partial) D_B$. Representing b in the basis B' as $b = \sum_i [b]_{B'}^{b_i'} b_i'$, in particular we have

$$c^{(B')} = \langle P_{B'},\, \ell_b(\partial) D_B \rangle = \langle \ell_b P_{B'},\, D_B \rangle = \sum_i [b]_{B'}^{b_i'} \big\langle \ell_{b_i'} P_{B'},\, D_B \big\rangle.$$

In the last expression, we can write $\ell_{b_i'} P_{B'} = \ell_{\mathrm{EP}(B') \cup b_i'}$. In particular, the set $\mathrm{EP}(B') \cup b_i'$ is the passive set corresponding to the independent set I_i which covers B' in the external order with label b_i', and so this polynomial can be written as P_{I_i}. By Lemma 4.2.5,

$$\langle P_{I_i},\, D_B \rangle = \begin{cases} 1, & I_i = B \\ 0, & I_i \neq B \end{cases},$$

so in particular, $\big\langle \ell_{b_i'} P_{B'},\, D_B \big\rangle$ is nonzero only if $I_i = B$.

Suppose now that $c^{(B')}$ is nonzero. The above implies that B' is covered by B in the external order, is of the form $B' = B \setminus b_0 \cup b_j'$ for some $b_0 \in B$ and some j, and that

$$c^{(B')} = [b]_{B'}^{b_j'}.$$

To complete the proof, we need to show that $b_0 = b$. If this were not the case, then this would imply that $b \in B'$, so in particular

$$[b]_{B'}^{b_i'} = \begin{cases} 1, & b_i' = b \\ 0, & b_i' \neq b \end{cases}.$$

However, under our assumptions, $b_j' \neq b$, so this implies that $c^{(B')} = [b]_{B'}^{b_j'} = 0$, giving a contradiction. $\square$

As a result of this derivative formula, we additionally obtain the following corollaries.

Corollary 4.3.2. *If $B \subseteq X$ is a basis and v a vector, then $\ell_v(\partial)D_B$ is a linear combination of the polynomials $D_{B'}$ where $B' \lessdot^*_{\mathrm{ext}} B$.*

Proof. Represent v in terms of the basis B, and apply Proposition 4.3.1. □

Corollary 4.3.3. *Let $B \subseteq X$ be a basis, and denote $B^{(p)} = \{b \in B : \mathrm{EP}_b(B)$ is nonempty$\}$. Then*

1. *If $b \in B$, then $\ell_b(\partial)D_B$ is nonzero if and only if $b \in B^{(p)}$.*
2. *The polynomials $(\ell_b(\partial)D_B)_{b\in B^{(p)}}$ are linearly independent.*
3. *The linear support of D_B is given by $\mathrm{Supp}(D_B) = \mathrm{span}(B \setminus B^{(p)})^\perp$.*

Proof. By Proposition 3.5.16, B covers a basis B' with edge label b if and only if $B' = \varepsilon_B(x)$ for an element $x \in \mathrm{EP}_b(B)$ which is maximal with respect to the ordering $<_B$ of Definition 3.5.13. Such a basis therefore exists iff $\mathrm{EP}_b(B)$ is nonempty, or when $b \in B^{(p)}$. Proposition 4.3.1 gives the summation formula

$$\ell_b(\partial)D_B = \sum_{B'=B\setminus b\cup x} [b]^x_{B'} D_{B'},$$

so in particular the summation is nonempty if and only if $b \in B^{(p)}$. Since the summation is over linearly independent polynomials $D_{B'}$, the sum is nonzero iff any of the coefficients $[b]^x_{B'}$ are nonzero. However, if $B' = B \setminus b \cup x$ for some x and $B' \lessdot^*_{\mathrm{ext}} B$, then $b \in \mathrm{bo}(B', x)$, so the coefficient $[b]^x_{B'}$ is nonzero. Since this is true for any such basis B', we conclude that $\ell_b(\partial)D_B$ is nonzero exactly when $b \in B^{(p)}$.

For the second part, notice that each basis polynomial $D_{B'}$ for $B' \lessdot^*_{\mathrm{ext}} B$ appears with nonzero coefficient in the summation formula for $\ell_b(\partial)D_B$ for exactly one $b \in B^{(p)}$, so the sums of polynomials $\ell_b(\partial)D_B$ are linearly independent.

For the third part, note that since the vectors v for which $\ell_v(\partial)D_B = 0$ are exactly those which can be represented as a linear combination of vectors in $B \setminus B^{(p)}$, we have from Lemma 2.2.9 that $\mathrm{Supp}(D_B) = \mathrm{span}(B \setminus B^{(p)})^\perp$. □

We also obtain a more refined orthogonality result between P- and D-polynomials on the level of derivatives.

Lemma 4.3.4. *Let $I, B \subseteq X$ with I independent and B a basis. Then $P_I(\partial)D_B$ is nonzero if and only if $I \leq^*_{\mathrm{ext}} B$.*

Proof. First suppose that $I \leq^*_{\mathrm{ext}} B$. In particular, this means that P_I divides P_B, so we have $P_B(\partial) = P_I(\partial)f(\partial)$ for some polynomial f. If $P_I(\partial)D_B = 0$, this would imply

$$1 = \langle P_B,\, D_B\rangle = P_B(\partial)D_B = f(\partial)P_I(\partial)D_B = 0.$$

For the reverse direction, suppose $I \not\leq^*_{\text{ext}} B$. We proceed by induction on the rank of B in the external order. Because $I \not\leq^*_{\text{ext}} B$, the meet $B_0 = I \wedge B$ in the external order is strictly less than I.

Let I' be an independent set with $B_0 \lessdot^*_{\text{ext}} I' \leq^*_{\text{ext}} I$. In particular, we also have $I' \not\leq^*_{\text{ext}} B$, so there exists an element $b \in B \cap \mathrm{EP}(I')$. Since $I' \gtrdot^*_{\text{ext}} B_0$, b is the unique element in $\mathrm{EP}(I')$ which is not in $\mathrm{EP}(B_0) \subseteq \mathrm{EP}(B)$, and in particular, this implies that $b \in B_0$.

By Proposition 4.3.1, $\ell_b(\partial)D_B$ is a linear combination of the polynomials $D_{B'}$ where $B' \lessdot^*_{\text{ext}} B$ and $b \notin B'$. Then we have

$$\begin{aligned} P_I(\partial)D_B &= g(\partial)P_{I'}(\partial)D_B \\ &= g(\partial)P_{B_0}(\partial)\ell_b(\partial)D_B \\ &= g(\partial)P_{B_0}(\partial) \sum_{B'=B\setminus b\cup x} [b]^x_{B'} D_{B'} \\ &= g(\partial) \sum_{B'=B\setminus b\cup x} [b]^x_{B'} P_{B_0}(\partial)D_{B'}. \end{aligned}$$

where $g = P_I/P_{I'} = \ell_{\mathrm{EP}(I)\setminus\mathrm{EP}(I')}$. For B' as in the summation, suppose $B_0 \leq^*_{\text{ext}} B'$. In this case, since $b \in B_0$ and $b \notin \mathrm{EP}(B') \subseteq \mathrm{EP}(B)$, by the antimatroid interval property without upper bounds, this would imply that $b \in B'$, a contradiction. Thus we find that $B_0 \not\leq^*_{\text{ext}} B'$ for each of the B'. By induction, since the rank of B' in the external order is less than that of B, we have that $P_{B_0}(\partial)D_{B'} = 0$ for each B', and thus that the derivative $P_I(\partial)D_B$ must be zero. □

As a particular consequence of the preceding result, we obtain an alternate proof for the result argued by Lenz in [30] that the central D-polynomials of a forward exchange matroid lie in the corresponding generalized $\mathcal{D}$-space.

Corollary 4.3.5. *If* $(X, \mathbb{B}, \mathbb{B}')$ *is a forward exchange matroid, then*

$$\left\{ D^X_B \,:\, B \in \mathbb{B}' \right\} \subseteq \mathcal{D}(X, \mathbb{B}').$$

Proof. By Proposition 4.1.3, the generalized cocircuits of $\mathbb{B}'$ are given by the sets $\mathrm{EP}(I)$, where I is a $\leq^*_{\text{ext}}$-minimal independent set not contained in $\mathbb{B}'$. In particular, the generators of $\mathcal{J}(X, \mathbb{B}')$ are given by the polynomials P_I for these independent sets.

If $B \in \mathbb{B}'$, then for $I \notin \mathbb{B}'$, we have $I \not\leq^*_{\text{ext}} B$ because $\mathbb{B}'$ is downward closed in the external order. Then by Lemma 4.3.4, $P_I(\partial)D_B = 0$. Since this is true in particular for all of the generators P_I of $\mathcal{J}(X, \mathbb{B}')$, we see that $D_B \in \ker \mathcal{J}(X, \mathbb{B}') = \mathcal{D}(X, \mathbb{B}')$. □

4.4 Recursive Construction for the Central $\mathcal{D}$-Basis

A practical difficulty which currently exists with regards to zonotopal $\mathcal{D}$-spaces is that there is no direct algebraic construction for the central D-polynomials. In the following, we present such a construction, which inductively builds the central D-polynomials of a matrix $X \cup x$ by modifying the boundary D-polynomials of X to produce corresponding central D-polynomials of $X \cup x$. In particular, the elementary and inductive nature of the construction provides a practical recursive method for computing D-polynomials of explicitly defined matrices. In Appendix A, we will demonstrate the usage of a concrete implementation of this algorithm written in Python for use with the SageMath open-source mathematics software system.

We begin by introducing some notation describing certain decompositions of polynomial spaces into subpolynomial spaces, and we discuss some related technical properties.

Definition 4.4.1. If Q is a vector subspace of Π with linear support $U = \mathrm{Supp}(Q)$, let $v_1, \ldots, v_k \in U^\perp$ be an orthonormal basis, and let $y_i = \ell_{v_i}$ for each i. Then define

$$Q[y_1, \ldots, y_k] \coloneqq Q \otimes_{\mathbb{R}} \mathbb{R}[y_1, \ldots, y_k].$$

The space $Q[y_1, \ldots, y_k]$ is naturally identified with the polynomials of the form $\sum_\alpha q_\alpha y^\alpha$ where α denotes a multi-index and $q_\alpha \in Q$. In general, this space can be endowed with the differential action of $\mathbb{R}[y_1, \ldots, y_k]$ on the corresponding polynomials, given by

$$p_0(q \otimes p) \coloneqq q \otimes (p_0(\partial)p).$$

In addition, if Q is closed under taking partial derivatives, then this action can be extended to a differential action of Π via

$$p_0(q \otimes p) \coloneqq (p_0(\partial)q \otimes p) + (q \otimes p_0(\partial)p).$$

If we think of $p_0 \in \mathbb{R}[y_1, \ldots, y_k]$ and $q \otimes p \in Q[y_1, \ldots, y_k]$ as polynomials in Π, then it follows from orthogonality conditions and Lemma 2.2.10 that the differential operator $p_0(\partial)$ applied to $q \otimes p$ acts exactly as the first differential action defined above. Specifically, if we decompose $q \otimes p$ as a polynomial in variables y_i with coefficients in Q, the operator $p_0(\partial)$ acts as if the Q-coefficients were constants.

For the following lemma, let Q and R both denote vector subspaces of Π with common support U, and again let $y_1, \ldots, y_k$ denote linear forms corresponding to an orthonormal basis of $U^\perp$. Suppose further that R is identified with the dual space Q' via the mapping $r \mapsto \langle r, \cdot \rangle$, and let $(q_i, r_i)_i$ denote dual vector space bases of Q and R, so that $\langle q_i, r_j \rangle = \delta_{ij}$. The spaces we have in mind for this result are of course the zonotopal $\mathcal{D}$- and $\mathcal{P}$-spaces with their corresponding canonical bases, but we will state lemma more generally.

Lemma 4.4.2. *If $p \in Q[y_1, \ldots, y_k]$, then the y-constant term of p is zero iff*

$$\langle p, r_i \rangle = 0$$

for each i. More generally, if β is a multi-index, then the y^β term of p is zero iff

$$\langle p, y^\beta r_i \rangle = 0.$$

Proof. For the first part, note that since $r_i \in \Pi_U$, any polynomial divisible by y_i for some i is killed by the operation $\langle \cdot, r_i \rangle$, as

$$\langle y_i p_0, r_i \rangle = \langle p_0, y_i(\partial) r_i \rangle = 0.$$

In particular, in terms of its monomial representation $p = \sum_\alpha q_\alpha y^\alpha$, we have that

$$\langle p, r_i \rangle = \langle q_0, r_i \rangle.$$

Since $q_0 \in Q$, we have that $q_0 = 0$ iff $\langle q_0, r_i \rangle = 0$ for each i, as we wanted to show.

For a general multi-index β, notice that

$$\left\langle p, y^\beta r_i \right\rangle = \left\langle y^\beta(\partial) p, r_i \right\rangle.$$

In particular, the y-constant term of $y^\beta(\partial)p$ is a scalar multiple of q_β, the y^β coefficient of p. The result then follows from the previous case. □

For the remainder of the section, we will adopt the following notation. Let X be a $d \times n$ real matrix, let $x \in \mathbb{R}^d$, and let $X \cup x$ denote the $d \times (n+1)$ matrix obtained by appending x as an additional column on the right of X. Additionally, let $H \subseteq X$ denote a fixed hyperplane of X, and let η denote the linear form n_H^X corresponding to a unit normal vector of H.

We continue with a technical lemma which describes a locality property that will be useful for working with boundary D-polynomials.

Lemma 4.4.3. *If H' is a hyperplane of X with $H' \neq H$, then for any polynomial $q \in \mathcal{D}(H)[\eta]$,*

$$\left(P_{H'}^X(\partial)\right)(q) = 0.$$

Proof. It is sufficient to prove $\left(P_{H'}^X(\partial)\right)(\eta^k d) = 0$ for a monomial $d\eta^k$, with $d \in \mathcal{D}(H)$ and $k \geq 0$.

The polynomial $P_{H'}^X$ is the product of linear forms corresponding to the vectors in the cocircuit $C = X \setminus H'$. Expand the above derivative using the product rule, we have

$$\ell_C(\partial)(d\eta^k) = \sum_{C' \subseteq C} \ell_{C'}(\partial) d \cdot \ell_{C \setminus C'}(\partial) \eta^k.$$

In any nonzero term in this expansion, every linear form from a vector in $H \setminus H'$ must be included in C', since the derivative of η by ℓ_h for any vector $h \in H$ is zero. In particular, C' must contain some cocircuit of H: since the the rank of elements of H' in H is strictly less than the rank of H, the complement $H \setminus H'$ is the complement of a proper flat in H and thus contains an H-cocircuit. This implies $\ell_{C'} \in \mathcal{J}(H)$, so since d is in $\mathcal{D}(H)$, the derivative $\ell_{C'}(\partial) d$ is 0. □

Next we introduce the fundamental operation which we will use for the recursive construction of the central D-polynomials.

Lemma 4.4.4. *Let f be a homogeneous polynomial of degree m, and let $q_k \in \mathcal{D}(H)[\eta]$ with $\eta^k \mid f(\partial)q_k$. Define the projection q_{k+1} of q_k by*

$$q_{k+1} \coloneqq q_k - \sum_{I \in \mathbb{B}(H)} \frac{\langle q_k,\, \eta^k f P_I^H \rangle}{\langle \eta^{m+k} D_I^H,\, \eta^k f P_I^H \rangle} \eta^{m+k} D_I^H.$$

Then $q_{k+1} \in \mathcal{D}(H)[\eta]$, and $\eta^{k+1} \mid f(\partial)q_{k+1}$.

Proof. The fact that q_{k+1} is in $\mathcal{D}(H)[\eta]$ is clear from its defining equation. To see that $\eta^{k+1} \mid f(\partial)q_{k+1}$, we apply Lemma 4.4.2 for $\mathcal{D}(H)[\eta]$. Letting $J \in \mathbb{B}(H)$, we have

$$\begin{aligned}
\langle f(\partial)q_{k+1},\, \eta^k P_J^H \rangle &= \langle q_{k+1},\, \eta^k f P_J^H \rangle \\
&= \langle q_k,\, \eta^k f P_J^H \rangle - \sum_{I \in \mathbb{B}(H)} \frac{\langle q_k,\, \eta^k f P_I^H \rangle}{\langle \eta^{m+k} D_I^H,\, \eta^k f P_I^H \rangle} \langle \eta^{m+k} D_I^H,\, \eta^k f P_J^H \rangle \\
&= \langle q_k,\, \eta^k f P_J^H \rangle - \sum_{I \in \mathbb{B}(H)} \frac{\langle q_k,\, \eta^k f P_I^H \rangle}{\langle \eta^{m+k},\, \eta^k f \rangle} \langle \eta^{m+k},\, \eta^k f \rangle \delta_{I,J} \\
&= \langle q_k,\, \eta^k f P_J^H \rangle - \langle q_k,\, \eta^k f P_J^H \rangle = 0.
\end{aligned}$$

From this we conclude that the η^k coefficient of $f(\partial)q_{k+1}$ in $\mathcal{D}(H)[\eta]$ is zero. We have that η^k additionally divides $f(\partial)\eta^{m+k}D_I^H$ for each I by elementary considerations, so we can see that η^k divides $f(\partial)q_{k+1}$ by distributing the differential operation over the defining expression of q_{k+1}. Thus the coefficient of η^i in $f(\partial)q_{k+1}$ is likewise zero for $i < k$, so we can conclude that $\eta^{k+1} \mid f(\partial)q_{k+1}$. □

We can now present the recursive construction for the central D-polynomials of $X \cup x$. Suppose that for each $I_0 \subseteq X$ independent, we have computed the polynomial $D_{I_0}^{\operatorname{span}_X(I_0)}$. By Lemma 4.2.9, we have that

$$D_{I_0}^{\operatorname{span}_X(I_0)} = D_{I_0}^{\operatorname{span}_{X\cup x}(I_0)}.$$

We thus need to compute the polynomials $D_I^{\operatorname{span}_{X\cup x}(I)}$ where $I \subseteq X \cup x$ is an independent set containing x.

To simplify notation, assume without loss of generality that I is a basis of $X \cup x$, and denote this basis by B. For lower rank independent sets, the construction is the same, restricting attention to the relative zonotopal spaces of the submatrix of X whose columns lie in $\operatorname{span}_X(I)$. Since we assume $x \in B$, let $B = I_0 \cup x$ where $I_0 \in \mathbb{I}_{\mathcal{F}}(X)$. Let H now denote the hyperplane spanned by I_0, and let η denote the normal linear form n_H^X. Additionally, let f denote the generator $P_H^{X\cup x}$ of $\mathcal{J}(X \cup x)$ corresponding with the hyperplane H, which is given by $\ell_{(X\cup x)\setminus H}$. Let $m = |(X \cup x) \setminus H|$ denote the degree of f.

Now inductively define polynomials q_i, $i \geq 0$ as follows. Let

$$q_0 \coloneqq D_{I_0}^X = \frac{1}{c_H} \eta^{m-1} D_{I_0}^H$$

be the boundary D-polynomial of I_0, which is an element of $\mathcal{I}(X)$, and define

$$q_{i+1} \coloneqq q_i - \sum_{I \in \mathbb{B}(H)} \frac{\langle q_i, \eta^i f P_I^H \rangle}{\langle \eta^{m+i} D_I^H, \eta^i f P_I^H \rangle} \eta^{m+i} D_I^H. \tag{4.2}$$

We then have the following.

Theorem 5. *The sequence (q_i) stabilizes at entry q_k, where $k = \deg\left(D_{I_0}^H\right) = |\mathrm{EP}_H(I_0)|$, and the stabilizing polynomial q_k is equal to the central D-polynomial $D_B^{X \cup x}$.*

Proof. As noted in Lemma 4.2.5, $q_0 = D_{I_0}^X$ acts dually to $P_{I_0}^X$ among the D- and P-polynomial generators of $\mathcal{I}(X \cup x)$ and $\mathcal{J}(X \cup x)$. Since each polynomial $\eta^{m+i} D_I^H$ is divisible by $\eta^m = D_H^{X \cup x}$, these polynomials are elements of $\mathcal{I}(X \cup x)$, and thus act trivially on polynomials in $\mathcal{P}(X \cup x)$. In particular, this implies that the action of each q_i on $\mathcal{P}(X \cup x)$ is the same as that of q_0, so each q_i acts as the dual of $P_{I_0}^X = P_B^{X \cup x}$ in $\mathcal{B}_{\mathcal{P}}(X \cup x)$.

By induction, each polynomial q_i lies in $\mathcal{D}(H)[\eta]$, so by Lemma 4.4.3, $P_{H'}^{X \cup x}(\partial) q_i = 0$ for any hyperplane $H' \neq H$. We will show additionally that $f(\partial) q_k = P_H^{X \cup x}(\partial) q_k = 0$, which thus implies that $q_k \in \mathcal{D}(X \cup x)$.

To this end, we prove by induction that q_i is homogeneous of degree $k + m - 1$ and satisfies $\eta^i \mid f(\partial) q_i$. This is clearly the case for q_0. If q_i satisfies $\eta^i \mid f(\partial) q_i$, then by Lemma 4.4.4, $\eta^{i+1} \mid f(\partial) q_{i+1}$ as desired. To see that q_{i+1} is homogeneous of degree $k + m - 1$, note that the expression

$$\langle q_i, \eta^i f P_I^H \rangle$$

is a differential bilinear form of two homogeneous polynomials, and thus is nonzero only when their degrees are the same. In particular, this requires that

$$\deg(P_I^H) = \deg(q_i) - \deg(\eta^i f) = k - 1 - i.$$

For each $I \in \mathbb{B}(H)$ contributing nontrivially to the sum in the definition of q_{i+1}, we conclude that $\eta^{m+i} D_I^H$ has degree $(m + i) + (k - 1 - i) = k + m - 1$, and so the summation for q_{i+1} is again homogeneous of degree $k + m - 1$.

A further consequence of this argument is that all of the terms in the summation for q_{i+1} are zero when $i \geq k$, so $q_{i+1} = q_i$ for $i \geq k$, and we see that the sequence stabilizes.

Finally, to see that $q_k \in \mathcal{D}(X \cup x)$, note now that q_k is homogeneous of degree $k + m - 1$, and satisfies $\eta^k \mid f(\partial) q_k$. In particular, $f(\partial) q_k$ is homogeneous of degree $k - 1$, so in order for η^k to divide this expression, the derivative must be zero. We conclude that $f(\partial) q_k = P_H^{X \cup x}(\partial) q_k = 0$, so $q_k \in \mathcal{D}(X \cup x)$.

Finally, note that since $q_k \in \mathcal{D}(X \cup x)$ and acts as the dual of $P_{I_0}^X = P_B^{X \cup x}$ in $\mathcal{B}_{\mathcal{P}}(X \cup x)$, we conclude that q_k is in fact the basis polynomial $D_B^{X \cup x}$ in $\mathcal{B}_{\mathcal{D}}(X \cup x)$. □

Thus for any new independent set $I = I_0 \cup x$ introduced by appending x to X, we can apply this construction to extend the polynomial $D_{I_0}^{F_0}$ to the polynomial D_I^F, where $F_0 = \mathrm{span}_X(I_0)$ and $F = \mathrm{span}_{X \cup x}(I)$. This allows us to compute the central D-polynomials associated to each independent set of $X \cup x$. To construct the central D-polynomials from scratch for a matrix X and all of its flats, it is then sufficient to apply the construction successively for each prefix of the columns of X.

Next, we will use the differential properties of central D-polynomials explored in Section 4.3 to simplify the computation of the polynomials q_i, and eventual to provide a streamlined linear-algebraic method to compute the new central D-polynomials of $X \cup x$.

For the following, let $\mathcal{D}_{I_0}(H)$ denote the linear span of $\{D_I^H : I \in \mathbb{B}(H),\, I \leq^*_{\text{ext}} I_0\}$.

Lemma 4.4.5. *For each i, the polynomial q_i is an element of $\mathcal{D}_{I_0}(H)[\eta]$. In particular, the defining summation of q_i from Equation 4.2 can be taken over independent sets $I \in \mathbb{B}(H)$ with $I \leq^*_{\text{ext}} I_0$.*

Proof. We argue by induction on i. For $i = 0$ the statement is obvious by the definition of q_0, so suppose the statement holds for some $i \geq 0$. We need to argue that if $I \in \mathbb{B}(H)$ with $I \not\leq^*_{\text{ext}} I_0$, then the inner product $\langle q_i,\, \eta^i f P_I^H \rangle$ is zero. To this end, decompose the polynomial $\eta^i f$ as a polynomial in $\Pi_H[\eta]$,

$$\eta^i f = \sum_j p_j \eta^j,$$

where $p_j \in \Pi_H$ for each $j = 0, \ldots, \deg(\eta^i f)$. In particular, we can write

$$\langle q_i,\, \eta^i f P_I^H \rangle = \langle (\eta^i f)(\partial) q_i,\, P_I^H \rangle.$$

Then the expression $(\eta^i f)(\partial) q_i$ can be expanded as

$$(\eta^i f)(\partial) q_i = \Big(\sum_j p_j \eta^j\Big)(\partial) q_i = \sum_j \frac{\partial^j}{\partial \eta^j}\big(p_j(\partial) q_i\big).$$

Since $p_j \in \Pi_H$ for each j, it acts directly on the $\mathcal{D}_{I_0}(H)$-components of q_i (as a polynomial in $\mathcal{D}_{I_0}(H)[\eta]$), which by Corollary 4.3.2 yields polynomials which are again in $\mathcal{D}_{I_0}(H)$. The $\eta^j(\partial)$ operator on the other hand acts as the partial derivative $\frac{\partial^j}{\partial \eta^j}$ for polynomials in $\Pi_H[\eta]$, treating the coefficients in Π_H as constants. In particular, it also preserves membership in $\mathcal{D}_{I_0}(H)[\eta]$.

Thus we conclude that $(\eta^i f)(\partial) q_i$ remains in $\mathcal{D}_{I_0}(H)[\eta]$. In particular, for each monomial $\eta^l D_J^H$, we have that $\langle \eta^l D_J^H,\, P_I^H \rangle$ is nonzero only when $J = I$ and $l = 0$. Since $(\eta^i f)(\partial) q_i$ can be represented as a linear combination of monomials $\eta^l D_J^H$ with $J \leq^*_{\text{ext}} I_0$, the inner product $\langle (\eta^i f)(\partial) q_i,\, P_I^H \rangle = \langle q_i,\, \eta^i f P_I^H \rangle$ is zero when $I \not\leq^*_{\text{ext}} I_0$.

We conclude that the expression $\langle q_i,\, \eta^i f P_I^H \rangle$ can be nonzero only when $I \leq^*_{\text{ext}} I_0$, and thus that q_{i+1} lies in $\mathcal{D}_{I_0}(H)[\eta]$. This completes the inductive argument, and the fact that the summation in Equation 4.2 can be taken over just the independent sets $I \leq^*_{\text{ext}} I_0$ then follows from the form of the polynomials q_i. □

Finally, using this result, we can give an alternate construction for the polynomial $D_B^{X \cup x}$ in terms of an elementary linear-algebraic computation. Specifically, we know from Lemma

4.4.5 that $D_B^{X\cup x} = q_k$ lies in $\mathcal{D}_{I_0}(H)[\eta]$, and since this polynomial is homogeneous, it is given by a unique linear combination of the linearly independent polynomials

$$\left\{\eta^{m-1+j(I)} D_I^H \;:\; I \leq^*_{\text{ext}} I_0,\, j(I) \coloneqq \deg(D_{I_0}^H) - \deg(D_I^H)\right\}.$$

In the above, $\eta^{m-1+j(I)} D_I^H = c_H \eta^{j(I)} D_I^X$, and this is in $\mathcal{I}(X\cup x)$ for every $I <^*_{\text{ext}} I_0$ (for which $j(I) \geq 1$). We thus see that the only component of the linear combination contributing to the inner product $\left\langle D_B^{X\cup x},\, P_B^{X\cup x}\right\rangle = 1$ is the scalar multiple of $\eta^{m-1} D_{I_0}^H$, which must thus be given by $D_{I_0}^B$, since this polynomial has inner product 1 with $P_B^X = P_B^{X\cup x}$ by Lemma 4.2.5.

The remaining coefficients of the linear combination can be determined by the property that $f(\partial) D_B^{X\cup x} = 0$, noting that the linear combination satisfying this property is unique because the sum $\mathcal{I}(X\cup x) + \mathcal{D}(X\cup x)$ is direct. To compute these coefficients explicitly, it is sufficient to determine the linear combination of the polynomials $P_H^{X\cup x}(\partial)\big(\eta^{j(I)} D_I^X\big)$ equal to $P_H^{X\cup x}(\partial) D_{I_0}^X$. Summarizing, we have the following.

Theorem 6. *There exists a unique solution in the parameters (α_I), $I \in \mathbb{B}(H)$ with $I <^*_{\text{ext}} I_0$, for the equation*

$$\sum_{I <^*_{\text{ext}} I_0} \alpha_I \Big[P_H^{X\cup x}(\partial)\big(\eta^{j(I)} D_I^X\big) \Big] = P_H^{X\cup x}(\partial) D_{I_0}^X.$$

For these values of α_I, we have

$$D_B^{X\cup x} = D_{I_0}^X - \sum_{I <^*_{\text{ext}} I_0} \alpha_I \eta^{j(I)} D_I^X.$$

4.5 Explicit Construction for the Internal and Semi-internal $\mathcal{P}$-Bases

We now provide an explicit construction for the semi-internal zonotopal $\mathcal{P}$-space basis polynomials, which in particular includes the internal basis polynomials of $\mathcal{B}^X_{\mathcal{P}_-}$ as a special case. The construction revolves around certain projection operations on the externally passive elements of a basis.

Definition 4.5.1. If $B \subseteq X$ is a basis and $I \subseteq B$, then let $\pi_I^B : \mathbb{R}^d \to \mathbb{R}^d$ denote the orthogonal projection map in the basis B onto $\operatorname{span}(I)$, that is, so that

$$[\pi_I^B(x)]_B^b = \begin{cases} [x]_B^b, & b \in I \\ 0, & b \notin I \end{cases}.$$

We now define the internal projection operator, which will provide us with a concrete description of the relation between the polynomials P_B and the canonical basis of $\mathcal{P}_-(X, I_{\text{int}})$.

Definition 4.5.2. Let $B \in \mathbb{B}_-(X, I_{\text{int}})$ be a semi-internal basis, and let $A \subseteq X \setminus B$. For $x \in A$, let $\iota_A^B(x)$ denote the collection of elements $b \in B$ such that

- $x = \max(\mathrm{bo}(B, b))$
- $\mathrm{bo}(B, b) \setminus b \subseteq A$
- $\mathrm{span}(B \setminus b) \in \mathcal{F}(X, I_{\mathrm{int}})$

Define the **semi-internal projection** of A with respect to B to be the collection

$$\pi_{-}^{B,I_{\mathrm{int}}}(A) := \left\{ \pi^{B}_{B \setminus \iota_A^B(x)}(x) \,:\, x \in A \right\}.$$

Note that in particular since the independent set I_{int} is chosen greedily from the end of X, the properties defining $\iota_A^B(x)$ imply that if this set is nonempty, then $x \in I_{\mathrm{int}}$. It is clear from the definition that a basis element $b \in B$ is contained in at most one set $\iota_A^B(x)$. Additionally, we note that b is contained in some such set iff $\mathrm{bo}(B, b) \setminus b \subseteq A$, and the maximal element $x = \max(\mathrm{bo}(B, b))$ satisfies $x \in I_{\mathrm{int}}$ and $x \neq b$.

We now argue that for a fixed set A, the choice of basis B disjoint from A does not change the semi-internal projection of A.

Lemma 4.5.3. *If $A \subseteq X$ and $B, B' \subseteq X$ are bases disjoint from A, then $\pi_{-}^{B,I_{\mathrm{int}}}(A) = \pi_{-}^{B',I_{\mathrm{int}}}(A)$.*

Proof. Let $x \in A$. We begin by arguing that $\iota_A^B(x) = \iota_A^{B'}(x)$, and that $\mathrm{span}(B \setminus \iota_A^B(x)) = \mathrm{span}(B' \setminus \iota_A^{B'}(x))$.

If $b \in \iota_A^B(x)$ for some $b \in B$, then we have that $A \supseteq \mathrm{bo}(B, b) \setminus b$. Because B' is a basis, it doesn't lie in the hyperplane $X \setminus \mathrm{bo}(B, b)$, so since it is disjoint from A, it must contain b. Since no other element of B' lies in $\mathrm{bo}(B, b)$, the remaining elements must lie in $\mathrm{span}(B \setminus b)$. This means that $\mathrm{bo}(B', b) = \mathrm{bo}(B, b)$, so it is straightforward that $b \in \iota_A^B(x)$.

By repeating this argument for each $b \in \iota_A^B(x)$, we see that $\iota_A^B(x) \subseteq \iota_A^{B'}(x)$, and that $B' \setminus \iota_A^B(x) \subseteq \mathrm{span}(B \setminus \iota_A^B(x))$. By a symmetric argument, we have that $\iota_A^B(x) = \iota_A^{B'}(x)$ and $B \setminus \iota_A^{B'}(x) \subseteq \mathrm{span}(B' \setminus \iota_A^{B'}(x))$, and hence we have the desired equality of linear spans.

Finally, note that the elements $\pi^{B}_{B \setminus \iota_A^B(x)}(x)$ in the definition of $\pi_{-}^{B,I_{\mathrm{int}}}(A)$ are determined by the flat $B \setminus \iota_A^B(x)$ and the vectors in B which lie outside of the flat. Since these parameters are identical for both B and B', we see that the projection maps $\pi^{B}_{B \setminus \iota_A^B(x)}(X) = \pi^{B}_{B' \setminus \iota_A^{B'}(x)}(x)$ are identical, and thus that $\pi_{-}^{B,I_{\mathrm{int}}}(A) = \pi_{-}^{B',I_{\mathrm{int}}}(A)$. $\square$

From this fact, we are able to define a semi-internal projection operator independent of the choice of basis.

Definition 4.5.4. If $A \subseteq X$ is a coindependent set, that is, a set which avoids some basis, let $\pi_{-}^{I_{\mathrm{int}}}(A) := \pi_{-}^{B,I_{\mathrm{int}}}(A)$, where B is any basis disjoint from A.

The algorithm for this projection operation for a set A can be informally described by the following. "When A contains all elements of a semi-internal cocircuit except an element

b which is not maximal, project the maximal element x onto the complementary hyperplane using b."

We now define the polynomials which we will show form the canonical basis of the semi-internal zonotopal spaces.

Definition 4.5.5. If $B \subseteq X$ is a basis, note that $\mathrm{EP}(B)$ avoids B and thus is coindependent, and define

$$\widetilde{\mathrm{EP}}(B, I_{\mathrm{int}}) := \pi_-^{I_{\mathrm{int}}}(\mathrm{EP}(B)),$$

and let

$$\widehat{P}_B^{X,I_{\mathrm{int}}} := \ell_{\widetilde{\mathrm{EP}}(B,I_{\mathrm{int}})}.$$

We will argue that $\widehat{P}_B^{X,I_{\mathrm{int}}}$ is in fact equal to the canonical semi-internal basis polynomial $\widetilde{P}_B^{X,I_{\mathrm{int}}}$, using the following characterization, which is a consequence of the discussion in [24] following the proof of their Lemma 3.6.

Lemma 4.5.6. $\widetilde{P}_B^{X,I_{\mathrm{int}}}$ *is the unique polynomial such that*

- $\widetilde{P}_B^{X,I_{\mathrm{int}}} \in \mathcal{P}_-(X, I_{\mathrm{int}})$*, and*
- $\widetilde{P}_B^{X,I_{\mathrm{int}}} = P_B^{X,I_{\mathrm{int}}} - f_B$ *for a polynomial* $f_B \in \mathcal{J}_-(X, I_{\mathrm{int}})$.

Thus it is enough to prove that that $\widehat{P}_B^{X,I_{\mathrm{int}}}$ satisfies these two characteristics. We begin with the first, in slightly more general form.

Proposition 4.5.7. *If* $A \subseteq X$ *is coindependent and avoids a semi-internal basis* B*, then the polynomial* $\ell_{\pi_-^{I_{\mathrm{int}}}(A)}$ *lies in* $\mathcal{P}_-(X, I_{\mathrm{int}})$.

Proof. Let B be a semi-internal basis disjoint from A, so that $\pi_-^{I_{\mathrm{int}}}(A) = \pi_-^{B,I_{\mathrm{int}}}(A)$, and denote $\widetilde{A} = \pi_-^{B,I_{\mathrm{int}}}(A)$ and $\widetilde{p} = \ell_{\widetilde{A}}$. To prove that $\widetilde{p}$ lies in $\mathcal{P}_-(X, I_{\mathrm{int}}) = \ker \mathcal{I}_-(X, I_{\mathrm{int}})$, we will show that $g(\partial)$ kills $\widetilde{p}$ for each generator g of $\mathcal{I}_-(X, I_{\mathrm{int}})$. By the product rule for derivatives, this is equivalent to showing for each hyperplane H of X that the set $\widetilde{A} \setminus H$ contains at most $|X \setminus H| - 1$ elements if H is not semi-internal, and at most $|X \setminus H| - 2$ elements if H is semi-internal.

To start, fix a hyperplane $H \subseteq X$. In general, the elements of B which lie outside of H reduce the number of elements that can lie in $\widetilde{A} \setminus H$ since B is disjoint from A. However, the semi-internal projection may also map elements of $A \cap H$ outside of H. Let $x_1 < x_2 < \cdots < x_k$ denote the elements of $A \cap H$ which are mapped outside of H in $\widetilde{A}$. In particular, we have

$$\left|\widetilde{A} \setminus H\right| \leq |X \setminus H| - |B \setminus H| + k \tag{4.3}$$

Notice that in order for the projection $\pi_{B \setminus \iota_A^B(x)}^B(x_i)$ to lie outside of H, at least one element of $\iota_A^B(x_i)$ must lie outside of H. Since a basis element can lie in at most one set $\iota_A^B(x)$, this gives the upper bound $k \leq |B \setminus H|$.

Suppose now that $k = |B \setminus H|$. Then exactly one element of $B \setminus H$ must lie in each set $\iota_A^B(x_i)$, so denote this element by b_i. In particular, we have $x_i = \max(\mathrm{bo}(B, b_i))$ for each i. Note however that $x_k \notin \mathrm{bo}(B, b_i)$ for $i < k$, so in particular, $b_i \notin \mathrm{ci}(B, x_k)$ for $i < k$. This implies that $\mathrm{ci}(B, x_k) \setminus H = b_k$, but this is the intersection of the circuit $\mathrm{ci}(B, x_k)$ with the cocircuit $X \setminus H$, and such an intersection cannot be a singleton by Lemma 3.5.3. This contradiction implies that $k < |B \setminus H|$, so for any H we have $\left|\widetilde{A} \setminus H\right| \leq |X \setminus H| - 1$. This bound is sufficient for hyperplanes H which are not semi-internal.

Now suppose further that H is semi-internal, and suppose that $k = |B \setminus H| - 1$, so that $k + 1$ elements of B lie outside of H. As a first case, suppose that $k = 0$, so $B \setminus H$ has a single element b, and $\mathrm{span}(B \setminus b) = H$. Since B and H are semi-internal, we have that $b \neq x := \max(\mathrm{bo}(B, b))$. If $\mathrm{bo}(B, b) \setminus b \subseteq A$, then $b \in \iota_A^B(x)$, so $\pi^B_{B \setminus \iota_A^B(x)}(x) \in H$. Since $k = 0$, the elements of $A \cap H$ all also lie in $\widetilde{A} \cap H$, so neither b nor x contribute to the collection of elements in $\widetilde{A} \setminus H$. We see then that $\left|\widetilde{A} \setminus H\right| \leq |X \setminus H| - 2$ as desired.

We now show that the above case is actually the only possibility. Suppose by way of contradiction that $k \geq 1$. In particular, in this case at least one element of $B \setminus H$ lies in $\iota_A^B(x_i)$ for each i. Suppose however that $\iota_A^B(x_i)$ contains two elements of $B \setminus H$ for some $i \leq k - 1$. Then k of the $k + 1$ elements of $B \setminus H$ lie in the sets $\iota_A^B(x_i)$, $i \leq k - 1$, and so in particular for any such basis element b, we have $x_k \notin \mathrm{bo}(B, b)$, and thus $b \notin \mathrm{ci}(B, x_k)$. As argued above, this implies that $\mathrm{ci}(B, x_k) \cap (X \setminus H)$ consists of only the single remaining element of $B \setminus H$, which gives a contradiction.

Thus for $i \leq k - 1$, the set $\iota_A^B(x_i)$ has exactly one element of $B \setminus H$, which we denote by b_i. Additionally let $b_k, b \in B \setminus H$ denote the remaining two elements. We again have $x_k \notin \mathrm{bo}(B, b_i)$ and thus $b_i \notin \mathrm{ci}(B, x_k)$ for $i \leq k - 1$. Since at least one of b_k, b lies in $\mathrm{ci}(B, x_k)$, and $\mathrm{ci}(B, x_k) \setminus H$ can't be a single element, we see that $\mathrm{ci}(B, x_k) \setminus H = \{b_k, b\}$. Since $\iota_A^B(x_k)$ contains at least one of these two elements, suppose without loss of generality that $b_k \in \iota_A^B(x_k)$.

First suppose that $b \in \iota_A^B(x_k)$. In this case, note that the projection operator $\pi^B_{B \setminus \iota_A^B(x_k)}$ maps x_k to a vector $\tilde{x}_k$ whose basic circuit with respect to B is given by $\mathrm{ci}(B, x_k) \setminus \iota_A^B(x_k)$. We see that the basic circuit $\mathrm{ci}(B, \tilde{x}_k)$ then consists of only elements in H, so $\tilde{x}_k$ lies in H. This contradicts the definition of x_k.

Now suppose that $b \notin \iota_A^B(x_k)$. The elements x_i, as maximal elements of a cocircuit complementary to a semi-internal hyperplane, all lie in I_{int}. Since $b \in \mathrm{ci}(B, x_k)$, we have $x_k \in \mathrm{bo}(B, b)$, so $X \setminus \mathrm{bo}(B, b)$ is a semi-internal hyperplane, and thus $\mathrm{span}(B \setminus b) \in \mathcal{F}(X, I_{\mathrm{int}})$. Since $b \notin \iota_A^B(x_k)$, from the definition of this set we must have either $x_k \neq \max(\mathrm{bo}(B, b))$ or $\mathrm{bo}(B, b) \setminus b \nsubseteq A$.

If $x_k \neq \max(\mathrm{bo}(B, b))$, let $x > x_k$ denote this maximal element. Then $x \notin \mathrm{bo}(B, b_i)$ for $i \leq k$ since x is larger than the maximal element of each of these cocircuits, so $b_i \notin \mathrm{ci}(B, x)$ for each i. This implies that $\mathrm{ci}(B, x) \cap (X \setminus H) = b$ is a single element, a contradiction.

Thus $x_k = \max(\mathrm{bo}(B, b))$, and we must have $\mathrm{bo}(B, b) \setminus b \nsubseteq A$. Now let y be an element in $\mathrm{bo}(B, b) \setminus b$ not contained in A. Since $b_i \in \iota_A^B(x_i)$ for $i \leq k$, we have $\mathrm{bo}(B, b_i) \setminus b_i \subseteq A$.

Since $y \notin B$, this implies $y \notin \mathrm{bo}(B, b_i)$ and thus $b_i \notin \mathrm{ci}(B, y)$ for each i. Since $b \in \mathrm{ci}(B, y)$, we thus have $\mathrm{ci}(B, y) \cap (X \setminus H) = b$ is a single element, a contradiction.

This is the final case to show that if H is semi-internal and $k = |B \setminus H| - 1$, then $k = 0$. If $k < |B \setminus H| - 1$, then by Equation 4.3 above, the desired bound on the size of $\widetilde{A} \setminus H$ holds. This concludes the proof. $\square$

For the second characteristic of Lemma 4.5.6, we give an explicit representation of the difference $P_B^X - \widehat{P}_B^{X,I_{\mathrm{int}}}$ as a linear combination of polynomials in $\mathcal{J}_{-}(X, I_{\mathrm{int}})$. Specifically, we will prove that this difference can be represented as a linear combination of polynomials $P_{B'}^X$ for bases B' which are not semi-internal. We begin with the following lemma about certain exchange operations of matroid bases.

Lemma 4.5.8. *Let $B \subseteq X$ be a basis, and let $b \in B$ such that $\mathrm{bo}(B, b) \setminus b \subseteq \mathrm{EP}(B)$ and b is not maximal in $\mathrm{bo}(B, b)$. Let x denote the maximal element, and let $B' = B \setminus b \cup x$. Then*

1. *The set $\mathrm{EP}(B')$ is given by $\mathrm{EP}(B) \setminus x \cup b$.*
2. *In the dual matroid, we have $X \setminus B' = \varepsilon_{X \setminus B}(b)$ is the dual canonical passive exchange of b with respect to the dual basis $X \setminus B$.*
3. *If $b' \in B \setminus b$ such that $\mathrm{bo}(B, b') \setminus b' \subseteq \mathrm{EP}(B)$ and b' is not maximal in $\mathrm{bo}(B, b')$, and if $x' = \max(\mathrm{bo}(B, b')) \neq x$, then $\mathrm{bo}(B', b') \setminus b' \subseteq \mathrm{EP}(B')$, and $x' = \max(\mathrm{bo}(B', b'))$.*

Proof. For part 1, note that since $x \in \mathrm{bo}(B, b)$, the set B' is a basis. Further, since x is the maximal element of $\mathrm{bo}(B, b) = \mathrm{bo}(B', x)$, we have that the elements of $\mathrm{bo}(B', x) \setminus x$ (which includes b but excludes x) are all externally passive with respect to B'. Finally, for elements y outside of B spanned by $B \setminus b$, the fundament circuit $\mathrm{ci}(B, y)$ is unchanged by exchanging b with x, and so their activity is likewise unchanged.

For part 2, let X^* denote the dual matroid of X. Note first that since b is not maximal in the fundamental cocircuit $\mathrm{bo}(B, b)$, we have that b is internally passive, hence externally passive in X^*. In particular, the canonical passive exchange is defined as the exchange of an externally passive element with the maximal element in its fundamental circuit. In this case, $\mathrm{ci}_{X^*}(X \setminus B, b) = \mathrm{bo}_X(B, b)$, so the maximal element of this (dual) circuit is x by assumption. Thus

$$\varepsilon_{X \setminus B}(b) = (X \setminus B) \setminus x \cup b = X \setminus (B \cup x \setminus b) = X \setminus B'.$$

Finally, for part 3, suppose first that $x \notin \mathrm{bo}(B, b')$. By Lemma 3.5.2, we have $\mathrm{bo}(B, b') = \mathrm{bo}(B', b')$, and since $b, x \notin \mathrm{bo}(B, b')$, this implies by part 1 that $\mathrm{bo}(B', b') \setminus b' \subseteq \mathrm{EP}(B')$.

Now suppose $x \in \mathrm{bo}(B, b')$. Since x' is maximal in $\mathrm{bo}(B, b')$, we know that $x' > x$, and in particular since x is maximal in $\mathrm{bo}(B, b)$, we know that $x' \notin \mathrm{bo}(B, b)$. In particular $b \notin \mathrm{ci}(B, x')$, so again by Lemma 3.5.2, we have $\mathrm{ci}(B, x') = \mathrm{ci}(B', x')$. Since $b' \in \mathrm{ci}(B, x')$, this implies that $x' \in \mathrm{bo}(B', b')$.

Now note that $B \setminus bb' = B' \setminus xb'$, so if an element $y \in X$ is spanned by $B \setminus bb'$, then it in particular can't lie in $\mathrm{bo}(B', b')$. This means that $\mathrm{bo}(B', b') \subseteq \mathrm{bo}(B, b) \cup \mathrm{bo}(B, b')$. Since

x and x' are the respective maximal elements of these fundamental cocircuits and $x' > x$, this implies that $x' = \max(\mathrm{bo}(B', b'))$. Further, all of the elements of $\mathrm{bo}(B, b) \cup \mathrm{bo}(B, b')$ except for x and b' are externally passive in B' by part 1, and since $x \notin \mathrm{bo}(B', b')$, we see that $\mathrm{bo}(B', b') \setminus b'$ consists of entirely externally passive elements of B'. □

Using this, we can prove the following.

Proposition 4.5.9. *If $B \subseteq X$ is a semi-internal basis with respect to I_{int}, then the polynomial $P_B^X - \widehat{P}_B^{X,I_{\mathrm{int}}}$ can be represented as a linear combination of polynomials P_B^X for bases $B \notin \mathbb{B}_-(X, I_{\mathrm{int}})$.*

Proof. For notation, let $\{x_1, \ldots, x_k\}$ denote the elements x of $\mathrm{EP}(B)$ for which the projecting set $\iota^B_{\mathrm{EP}(B)}(x)$ is nonempty, and for each i, let $B^{(i)}$ denote the elements of $\iota^B_{\mathrm{EP}(B)}(x_i)$. Denote $Q = \mathrm{EP}(B) \setminus \{x_1, \ldots, x_k\}$. Then the set $\widetilde{\mathrm{EP}}(B, I_{\mathrm{int}}) = \pi_-^{I_{\mathrm{int}}}(\mathrm{EP}(B))$ is obtained by fixing the elements of Q, and by mapping each element x_i to its projection $\tilde{x}_i$ away from the basis elements in $B^{(i)}$. Namely,

$$x_i \mapsto \tilde{x}_i = x_i - \sum_{b \in B^{(i)}} [x_i]_B^b b.$$

Passing to polynomials, this means that

$$\begin{aligned}
\widehat{P}_B^{X,I_{\mathrm{int}}} &= \ell_{\widehat{\mathrm{EP}}(B,I_{\mathrm{int}})} \\
&= \prod_{x \in Q} \ell_x \cdot \prod_i \ell_{\tilde{x}_i} \\
&= \prod_{x \in Q} \ell_x \cdot \prod_i \left(\ell_{x_i} - \sum_{b \in B^{(i)}} [x_i]_B^b \, \ell_b \right) \\
&= \prod_{x \in Q} \ell_x \cdot \sum_{S \subseteq [k]} \sum_{\substack{b_i \in B^{(i)} \\ i \in S}} (-1)^{|S|} \prod_{i \in S} [x_i]_B^{b_i} \ell_{b_i} \prod_{i \notin S} \ell_{x_i}.
\end{aligned}$$

From this form, we see that $P_B^X - \widehat{P}_B^{X,I_{\mathrm{int}}}$ can be expressed in terms of polynomials ℓ_A where $A \subseteq X$ is a set formed by replacing one or more of the elements x_i in $\mathrm{EP}(B)$ with a corresponding projecting basis element $b_i \in \iota^B_{\mathrm{EP}(B)}(x_i)$. We argue that such a set A can be represented as $\mathrm{EP}(B')$ for some non-semi-internal basis B'.

For concreteness, suppose without loss of generality that A is obtained from $\mathrm{EP}(B)$ by replacing x_i with b_i for $i = 1, \ldots, l$, where $l \geq 1$, so that $A = \mathrm{EP}(B) \setminus \{x_1, \ldots, x_l\} \cup \{b_1, \ldots, b_l\}$. Further, denote

$$A_i = \mathrm{EP}(B) \setminus \{x_1, \ldots, x_i\} \cup \{b_1, \ldots, b_i\},$$

and

$$B_i = B \setminus \{b_1, \ldots, b_i\} \cup \{x_1, \ldots, x_i\}.$$

We will prove by induction on i that for $0 \leq i \leq l$,

- $\mathrm{EP}(B_i) = A_i$
- $x_1, \ldots, x_i$ are internally active in B_i
- $b_{i'} \in \iota_{A_i}^{B_i}(x_{i'})$ for $i < i' \leq l$

The base case $i = 0$ follows by definition, as $B_0 = B$ and $A_0 = \mathrm{EP}(B)$. Suppose then that the properties hold for $i = j$, $j < l$. In particular, since $b_{j+1} \in \iota_{A_j}^{B_j}(x_{j+1})$, we see that b_{j+1} and x_{j+1} satisfy the premises of Lemma 4.5.8. Since $B_{j+1} = B_j \setminus b_{j+1} \cup x_{j+1}$, by part 1 of this lemma, we have that

$$\mathrm{EP}(B_{j+1}) = \mathrm{EP}(B_j) \setminus x_{j+1} \cup b_{j+1} = A_j \setminus x_{j+1} \cup b_{j+1} = A_{j+1}.$$

By Lemma 4.5.8 part 2, we have that $X \setminus B_{j+1}$ is the dual canonical passive exchange $\varepsilon_{X \setminus B_j}(b_{j+1})$ in the dual matroid of X. In particular, since externally active elements in the dual matroid correspond with internally active elements in the original matroid, we have that $\mathrm{IA}(B_j) \subseteq \mathrm{IA}(B_{j+1})$, and that x_{j+1} is internally active in B_{j+1}. Since $x_1, \ldots, x_j \in \mathrm{IA}(B_j)$, this implies that $x_1, \ldots, x_{j+1} \in \mathrm{IA}(B_{j+1})$.

Finally, if $i' > j+1$, then $b_{i'} \in \iota_{A_j}^{B_j}(x_{i'})$ by assumption. In particular, this implies that $b_{i'}$ and $x_{i'}$ satisfy the premises of Lemma 4.5.8 part 3. Thus we have that $\mathrm{bo}(B_{j+1}, b_{i'}) \setminus b_{i'} \subseteq \mathrm{EP}(B_{j+1}) = A_{j+1}$ and $x_{i'}$ is maximal in $\mathrm{bo}(B_{j+1}, b_{i'}) \setminus b_{i'}$. Since $\iota_{A_j}^{B_j}(x_{i'})$ is nonempty, we know that $x_{i'} \in I_{\mathrm{int}}$, and this implies that $\mathrm{bo}(B_{j+1}, b_{i'})$ has nonempty intersection with I_{int}, and thus is complementary to a semi-internal hyperplane. From the definition, we have that $b_{i'} \in \iota_{A_{j+1}}^{B_{j+1}}(x_{i'})$, as desired. This completes the induction.

From this we conclude that A can be represented as the set of externally passive elements of the basis B_l as desired. The fact that this basis is not semi-internal follows because $B_l = B \setminus \{b_1, \ldots, b_l\} \cup \{x_1, \ldots, x_l\}$, and the elements x_i are members of I_{int} which are internally active in B_l. □

We finally can conclude the desired equality with the semi-internal $\mathcal{P}$-space basis polynomials.

Theorem 7. *If $B \subseteq X$ is a semi-internal basis with respect to I_{int}, then*

$$\widehat{P}_B^{X,I_{\mathrm{int}}} = \widetilde{P}_B^{X,I_{\mathrm{int}}}.$$

Proof. By Proposition 4.5.7 using $A = \mathrm{EP}(B)$, we see that $\widehat{P}_B^{X,I_{\mathrm{int}}}$ lies in $\mathcal{P}_-(X, I_{\mathrm{int}})$. By Proposition 4.5.9, $P_B^X - \widehat{P}_B^{X,I_{\mathrm{int}}}$ can be represented as a linear combination of polynomials $P_{B'}^X$ where $B' \notin \mathbb{B}_-(X, I_{\mathrm{int}})$. By Corollary 4.1.4 and the fact that $\mathcal{J}_-(X, I_{\mathrm{int}}) = \mathcal{J}(X, \mathbb{B}_-(X, I_{\mathrm{int}}))$, this implies that the difference lies in $\mathcal{J}_-(X, I_{\mathrm{int}})$. By Lemma 4.5.6, $\widetilde{P}_B^{X,I_{\mathrm{int}}}$ is the unique polynomial satisfying these two properties, so the conclusion follows. □

As a corollary, we use this characterization to describe the semi-internal zonotopal spaces in terms closer resembling the definitions of the central and external cases in [23].

Definition 4.5.10. Denote the collection **semi-internally short** sets by

$$S_{-}(X, I_{\mathrm{int}}) := \{Y \subset X \,:\, Y \text{ is disjoint from some semi-internal basis}\}.$$

Corollary 4.5.11. *The semi-internal zonotopal $\mathcal{P}$-space is given by*

$$\mathcal{P}_{-}(X, I_{\mathrm{int}}) = \operatorname{span}\left\{\ell_{\pi_{-}^{I_{\mathrm{int}}}(Y)} \,:\, Y \in S_{-}(X)\right\}.$$

Further, a homogeneous basis for $\mathcal{P}_{-}(X, I_{\mathrm{int}})$ is given by the collection

$$\left\{\widehat{P}_B^{X,I_{\mathrm{int}}} = \ell_{\pi_{-}(\mathrm{EP}(B))} \,:\, B \in \mathbb{B}_{-}(X, I_{\mathrm{int}})\right\}.$$

Proof. The characterization of the homogeneous basis for $\mathcal{P}_{-}(X, I_{\mathrm{int}})$ is the content of Theorem 7.

If $Y \in S_{-}(X, I_{\mathrm{int}})$, then Y is coindependent and avoids a semi-internal basis. By Proposition 4.5.7, this implies that $\ell_{\pi_{-}^{I_{\mathrm{int}}}(Y)}$ lies in $\mathcal{P}_{-}(X, I_{\mathrm{int}})$. Since the sets $\mathrm{EP}(B)$ for B semi-internal are in $S_{-}(X, I_{\mathrm{int}})$, the given collection contains the canonical semi-internal basis of $\mathcal{P}_{-}(X, I_{\mathrm{int}})$, and thus spans. □

The elementary nature of the semi-internal projection operator which produces the canonical basis polynomials gives us hope that a better construction for zonotopal $\mathcal{P}$-spaces could be formulated for a broader class of forward exchange matroids. However, we suspect that such a construction will rely on a deeper understanding of the interaction between internal activity and the lattice structure of the external order. Regardless of potential difficulties, we propose the following imprecise conjecture.

Conjecture. *Using a generalization of the semi-internal projection operator, an alternative forward exchange $\mathcal{P}$-space can be formulated which preserves duality with the forward exchange $\mathcal{D}$-space, but is generally the kernel of an appropriate forward exchange $\mathcal{I}$-ideal.*

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Appendix A

Software Implementations

A parallel goal of much of the work that went into this dissertation was to produce concrete software to compute the classical central, external and internal zonotopal spaces. The two theoretical challenges to this goal were the unusual structure of the internal $\mathcal{P}$-space basis polynomials, and the lack of combinatorial knowledge about Lenz's dual $\mathcal{D}$-space basis polynomials. Both of these issues were fortunately overcome in the latter sections of Chapter 4, and a full library of tools is now available for working with the external order and the classical zonotopal spaces, which can be found at `https://github.com/bgillesp/sage-zonotopal-algebra`. The library is written in the Python programming language for the SageMath open-source mathematics software system. Currently the software is independent of the SageMath project, but in the future we plan to integrate the code into the SageMath libraries.

In this appendix, we will give a high-level overview of the contents of this software library, with the goal of providing enough detail to make it easy for the casual user of the SageMath project to get started working with these tools. In addition, we will give usage examples and select implementation details for the most important modules. The software can logically be divided into three blocks, which will be discussed respectively in the following sections.

A.1 Ordered Matroids and the External Order

The first block of software involves computations related to ordered matroids and the external order. This functionality is implemented in the file `ordered_matroid.py`, which in particular defines a new `OrderedMatroid` class extending the standard SageMath `Matroid` class with functionality related to a total ordering of the ground set.

After opening an interactive Sage prompt in the directory containing the library Python files, start by constructing an `OrderedMatroid` object from a standard `Matroid` object as follows.

```
sage: from ordered_matroid import OrderedMatroid
sage: X = Matrix(QQ, [[1, 0], [0, 1], [1, 1], [1, 0], [0, 1]]).transpose()
```

```
sage: X
  [1 0 1 1 0]
  [0 1 1 0 1]
sage: M = Matroid(matrix=X)
sage: OM = OrderedMatroid(M)
sage: OM
  Ordered matroid of rank 2 on 5 elements
```

The groundset of a SageMath `Matroid` object is already labeled by natural numbers, so unless otherwise specified using the `ordered_groundset` keyword, the `OrderedMatroid` class uses the ordering induced by this labeling. In this case, the ordering is just the column order of the matrix `X`.

The set of elements active with respect to a given input set is computed as follows.

```
sage: OM.active_elements([2, 4])
  frozenset({0, 1})
sage: OM.active_elements([1, 3, 4])
  frozenset({0, 1, 2})
```

The `active_elements` method computes the generalized activity for an arbitrary subset of the groundset as given in Definition 3.2.3. A priori, the generalized notion of activity requires us to inspect a potentially very large collection of circuits in the matroid to determine when certain circuits satisfy the condition needed to make a groundset element active. The following equivalent formulation of generalized activity simplifies the computation extensively.

Definition A.1.1. If M is an ordered matroid with groundset X, and $A \subseteq X$, let I denote the independent subset of A which is lex maximal among those spanning A. We call the independent set I the **dominant basis** of A.

Proposition A.1.2. *Suppose M is an ordered matroid with groundset X, and that $A \subseteq X$. If I is the dominant basis of A, then $\mathrm{Act}_M(A) = \mathrm{Act}_M(I)$.*

Proof. First suppose $x \in \mathrm{Act}_M(I)$. Then x is the minimal element of a circuit $C \subseteq I \cup x$, and since $I \subseteq A$, we have $C \subseteq A \cup x$. Thus C also demonstrates that $x \in \mathrm{Act}_M(A)$, and thus we have $\mathrm{Act}_M(I) \subseteq \mathrm{Act}_M(A)$.

Now suppose that $x \in \mathrm{Act}_M(A)$, so that x is the minimal element of a circuit $C \subseteq A \cup x$. If $C \setminus x$ contains no elements of A outside of I, then we have $C \subseteq I \cup x$ which implies $x \in \mathrm{Act}_M(I)$ as desired. So suppose that $C \setminus x$ contains elements of $A \setminus I$.

Let a be an element of C with $a \in A \setminus I$, and let $C' = \mathrm{ci}(I, a)$. Because I is lex maximal spanning A, the independent set $I \setminus y \cup a$ for any $y \in C' \setminus a$ is lex smaller than I, which implies that a is minimal in C'. (This argument in fact shows that $A \setminus I \subseteq \mathrm{Act}_M(I)$.)

Now notice that $a \in C \cap C'$ and $x \in C \setminus C'$. Thus by the strong circuit elimination axiom, there exists a circuit $C'' \subseteq C \cup C' \setminus a$ with $x \in C''$. Since x is minimal in C, we have

$x < a$, and since a is minimal in C', we see that x is minimal in C'' as well. Further, since $C' \setminus a \subseteq I$, we have that $C \cup C' \setminus a$ contains strictly fewer elements of $A \setminus I$ than C, and thus this is also true of C''.

By repeating this process, we will eventually obtain a circuit $C^* \subseteq A \cup x$ with minimal element x which contains no elements of $A \setminus I$. Then $C^* \subseteq I \cup x$, which implies that $x \in \mathrm{Act}_M(I)$. We conclude that $\mathrm{Act}_M(A) \subseteq \mathrm{Act}_M(I)$, and thus we obtain equality. □

Remark. An additional consequence of this argument is that for any set $A \subseteq X$, the interval $[I, I \cup \mathrm{EA}(I)]$ of Proposition 3.2.23 which contains A is exactly the interval corresponding to the independent set I which is the dominant basis of A.

Overall what this means is that active elements of an arbitrary subset A of X can be computed by first determining the lex maximal spanning independent set of A, and then by computing the active elements of that independent set.

Returning to the library features, convenience methods for dual ordered matroids (the dual matroid with identical ordering on the ground set) and ordered matroid minors (standard matroid minors with induced ordering on the ground set) are implemented.

```
sage: OM.dual()
  Ordered matroid of rank 3 on 5 elements
sage: OM.minor(deletions=[2,3], contractions=[1])
  Ordered matroid of rank 1 on 2 elements
```

Finally, the library provides a method to construct the generalized external order of an ordered matroid as a SageMath `Poset` object.

```
sage: OM.external_order()
  Finite poset containing 14 elements
```

Given the various choices of conventions and representations of the external order, several parameters are used to specify which variant of the order is needed:

- The `variant` parameter describes the ordering convention of the poset, and allows either of the string values:
  - `"convex geometry"` sets the empty set as the minimal element, giving the ordering convention consistent with Las Vergnas's original formulation of the ordering.
  - `"antimatroid"` sets the empty set as the maximal element, giving the ordering convention consistent with the writing of this dissertation.
- The `representation` parameter describes the set system to use as the underlying representation of the poset, and allows the string values:
  - `"independent"` represents poset elements in terms of their matroid independent set.

  - `"passive"` represents poset elements in terms of their set of externally passive elements.
  - `"convex"` represents poset elements in terms of their convex closure, given by the union of their matroid independent set with their set of externally active elements.

The following gives an example of modified usage of this method using the above parameters.

```
sage: OM.external_order(variant="antimatroid", representation="passive")
  Finite poset containing 14 elements
```

Sage additionally has the ability to render and display Hasse diagrams for finite posets, often yielding a decent graphical representation for small inputs. This can be accomplished with the `show` command of class `FinitePoset`. For best results with the external order, the flag `string_labels=True` may be included in the constructor for easy-to-read vertex labels.

A.2 Polynomial Vector Spaces

The second block of software fills a small hole in the functionality of SageMath polynomials by providing tools to easily work with the vector space structure of polynomial rings. This functionality is implemented in two files:

- `poly_free_module.py` provides the class `PolynomialFreeModule` which implements a SageMath `CombinatorialFreeModule` with a given basis of polynomials, along with additional conversion utilities.

- `monomials.py` provides the class `Monomials` to abstractly represent various classes of monomials from a polynomial ring.

The features in these files are meant to be lightweight extensions of usual SageMath linear algebra functionality. The following is a typical example of usage of these classes.

```
sage: from poly_free_module import PolynomialFreeModule
sage: from monomials import Monomials
sage: P.<x,y> = PolynomialRing(QQ)
sage: basis1 = Monomials(P, degree=(0,2))
sage: basis1
  Monomials with degree at most 2 in Multivariate Polynomial Ring in x, y
  over Rational Field
sage: M1 = PolynomialFreeModule(P, basis1)
sage: list(M1.basis())
  [(1), (x), (y), (x^2), (x*y), (y^2)]
```

```
sage: v = M1((x+y)^2); v
  (y^2) + 2*(x*y) + (x^2)
sage: v.to_vector()
  (0, 0, 0, 1, 2, 1)
sage: v.to_polynomial()
  x^2 + 2*x*y + y^2
```

Here we produce a PolynomialFreeModule object `M1` whose basis consists of the monomials of $\mathbb{Q}[x, y]$ of degree at most 2. This demonstrates the process of creating an element of `M1`, converting it to a vector in terms of the underlying ordered basis, and converting it back to a standard polynomial object.

The next example demonstrates the use of an arbitrary finite basis of polynomials.

```
sage: basis2 = (1, x, x+y, x^2-x, x*y + y^2, y^2)
sage: M2 = PolynomialFreeModule(P, basis2)
sage: list(M2.basis())
  [(1), (x), (x + y), (x^2 - x), (x*y + y^2), (y^2)]
sage: w = M2(1 + x*y); w
  (1) - (y^2) + (x*y + y^2)
sage: w.to_vector()
  (1, 0, 0, 0, 1, -1)
sage: M(1 + x*y^2)
  Traceback (most recent call last)
  ...
  ValueError: Value x*y^2 + 1 is not spanned by the basis polynomials
```

The primary work of this software is to handle basis exchange operations between a specified polynomial basis and the standard monomial basis. In practice, this can save a lot of boilerplate code in settings where it is necessary to work with the vector space structure of polynomial rings.

A.3 Zonotopal Spaces

The final block of software provides methods for computing polynomial bases and generating sets for the zonotopal spaces and ideals in the central, external, and internal zonotopal settings. An object class is defined for each of the three classical zonotopal settings, and these three classes inherit from a common abstract base class which implements functionality that is the same for each setting. This is implemented in six Python files:

- `central_zonotopal_algebra.py` defines the class `CentralZonotopalAlgebra` which implements methods to compute the central zonotopal spaces.

- `external_zonotopal_algebra.py` defines the class `ExternalZonotopalAlgebra` which implements methods to compute the external zonotopal spaces.
- `internal_zonotopal_algebra.py` defines the class `InternalZonotopalAlgebra` which implements methods to compute the internal zonotopal spaces.
- `abstract_zonotopal_algebra.py` defines the class `AbstractZonotopalAlgebra` which acts as an abstract base class for the classes of each of the concrete zonotopal settings.
- `zonotopal_algebra.py` provides a convenience factory method which gives a uniform interface for constructing any of the currently implemented zonotopal algebra objects.
- `poly_utils.py` implements utility methods for constructing polynomials formed by a product of linear forms, and for working with polynomials as differential operators.

The following demonstrates the usage of this package to generate the central zonotopal spaces corresponding with a given matrix.

```
sage: from zonotopal_algebra import ZonotopalAlgebra
sage: X = Matrix(QQ, [[1, 0], [0, 1], [1, 1], [1, 1]]).transpose()
sage: X
  [1 0 1 1]
  [0 1 1 1]
sage: Zc = ZonotopalAlgebra(X, variant="central", varNames="xy")
sage: Zc.I_ideal_gens()
  [y^3, x^3, x^2 - 2*x*y + y^2]
sage: Zc.J_ideal_gens()
  [x^3 + 2*x^2*y + x*y^2, x^2*y + 2*x*y^2 + y^3, x*y]
sage: list(Zc.P_space_basis().values())
  [x^2 + x*y, x, y, x*y + y^2, 1]
sage: list(Zc.D_space_basis().values())
  [1/2*x^2, x, y, 1/2*y^2, 1]
```

The method `D_space_basis` gives a complete implementation of the algorithm discussed in detail in Section 4.4, specifically applying the streamlined formulation of Theorem 6, and makes use of the previously discussed library code for both the external order and for polynomial vector space computations.

Note that since the $\mathcal{P}$- and $\mathcal{D}$-space bases are indexed by matroid bases of X, the return type of the methods `P_space_basis` and `D_space_basis` is a python `dict` object whose keys represent the index matroid bases of X, and whose values are the corresponding polynomials.

Computation for the external and internal cases is accomplished similarly.

```
sage: Ze = ZonotopalAlgebra(X, variant="external", varNames="xy")
```

```
sage: Ze.external_matrix()
  [1 0 1 1|1 0]
  [0 1 1 1|0 1]
sage: Ze.I_ideal_gens()
  [y^4, x^4, x^3 - 3*x^2*y + 3*x*y^2 - y^3]
sage: Ze.J_ideal_gens()
  [x^4 + 2*x^3*y + x^2*y^2, x^2*y, x^2*y^2 + 2*x*y^3 + y^4]
sage: list(Ze.P_space_basis().values())
  [x^2*y + x*y^2,
   x^2*y + 2*x*y^2 + y^3,
   x,
   x^3*y + 2*x^2*y^2 + x*y^3,
   y,
   x^3 + 2*x^2*y + x*y^2,
   x^2 + x*y,
   x*y,
   x*y + y^2,
   1]
sage: list(Ze.D_space_basis().values())
  [-1/6*x^3 + 1/2*x*y^2 - 1/3*y^3,
   1/6*y^3,
   x,
   1/6*x*y^3 - 1/12*y^4,
   y,
   1/6*x^3,
   1/2*x^2,
   -1/2*x^2 + x*y - 1/2*y^2,
   1/2*y^2,
   1]
```

The external zonotopal algebra setting automatically selects an extending basis for the column span of the underlying matrix (the final identity block of `Ze.external_matrix()` in the above example), but a particular extending basis matrix can be specified using the `externalBasisMatrix` parameter.

```
sage: Zi = ZonotopalAlgebra(X, variant="internal", varNames="xy")
sage: Zi.I_ideal_gens()
  [y^2, x^2, x - y]
sage: Zi.J_ideal_gens()
  [x*y + y^2, x]
sage: list(Zi.P_space_basis().values())
  [x + y, 1]
```

```
sage: list(Zi.D_space_basis().values())
  [y, 1]
```

Concluding with two final remarks on implementation, first note that the internal $\mathcal{P}$-space basis is computed using the semi-internal projection operator of Section 4.5, specialized to the internal setting. Second, note that the external and internal $\mathcal{D}$-space bases are computed by first computing an appropriate central $\mathcal{D}$-space basis using the `CentralZonotopalAlgebra` class, and then restricting to the elements indexed by the external and internal bases respectively.