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Authors
Duistermaat, J. J.
Pelayo, A.

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Topology of symplectic torus actions with symplectic orbits

J.J. Duistermaat · A. Pelayo

Abstract We give a concise overview of the classification theory of symplectic manifolds equipped with torus actions for which the orbits are symplectic (this is equivalent to the existence of a symplectic principal orbit), and apply this theory to study the structure of the leaf space induced by the action. In particular we show that if M is a symplectic manifold on which a torus T acts effectively with symplectic orbits, then the leaf space M/T is a very good orbifold with first Betti number $b_1(M/T) = b_1(M) - \dim T$.

1 Introduction

Let M be a compact and connected smooth manifold, provided with a symplectic form σ, a smooth closed nowhere degenerate two-form on M. Let T be a torus which acts smoothly and effectively on M, preserving the symplectic structure. Such

J.J. Duistermaat
Mathematisch Instituut, Universiteit Utrecht, P.O. Box 80 010, 3508 TA Utrecht, The Netherlands
e-mail: J.J.Duistermaat@uu.nl

A. Pelayo (✉)
Mathematics Department, University of California–Berkeley, 970 Evans Hall 3840, Berkeley,
CA 94720-3840, USA
e-mail: apelayo@math.berkeley.edu

T-actions are called *symplectic*, and (M, σ, T) will be called a *symplectic T-manifold*. Two symplectic T-manifolds (M, σ, T) and (M', σ', T) are called *isomorphic* if there exists a T-equivariant diffeomorphism Φ from M onto M' such that σ is equal to the pull-back $\Phi^*(\sigma')$ of σ' by the mapping Φ.

A well studied type of symplectic torus actions are the so called *Hamiltonian* torus actions. A vector field v on M is called Hamiltonian if the contraction $\mathrm{i}_v \sigma$ of σ with v is an exact one-form, that is, there exists a smooth real-valued function f on M such that Hamilton's equation $\mathrm{i}_v \sigma = -\mathrm{d}f$ holds. For every element X of the Lie algebra $\mathfrak{t}$ of T, the infinitesimal action X_M of X on M is a smooth vector field on M. The T-action preserves the symplectic form if and only if for every $X \in \mathfrak{t}$ the one-form $\mathrm{i}_{X_M} \sigma$ is closed. The T-action is called Hamiltonian if its infinitesimal action is Hamiltonian, where the Hamiltonian function $f = \mu_X$ of X_M can be chosen to depend linearly on $X \in \mathfrak{t}$. Then the equation $\langle X, \mu \rangle = \mu_X$, $X \in \mathfrak{t}$, defines a smooth mapping μ from M to the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$, called the *momentum mapping* of the Hamiltonian T-action. The theorem of Atiyah [2, Theorem 1] and Guillemin-Sternberg theorem [13] says that the image $\mu(M)$ of the momentum mapping is equal to the convex hull in $\mathfrak{t}^*$ of the image under μ of the set M^T of fixed points in M for the action of T, where the set $\mu(M^T)$ is finite and therefore $\mu(M)$ is a convex polytope. Note that this implies that $M^T \neq \emptyset$. Delzant [8] proved that if $\dim(T) = n$, then $\mu(M)$ is a so called Delzant polytope, and $\mu(M)$ completely determines the *Delzant space* (M, σ, T). Delzant [8] moreover proved that M is isomorphic to a smooth toric variety with σ equal to a Kähler form on it, and the action of T extends to a holomorphic action of the complexification $T_{\mathbb{C}}$ of T. For this reason a Delzant space is also called a *symplectic toric manifold*. See also Guillemin [14] for a beautiful exposition of this subject.

If the first de Rham cohomology group of M is equal to zero, then every symplectic action on M is Hamiltonian. Nevertheless, in general the assumption that the symplectic torus action is Hamiltonian is very restrictive, as it implies that the action has fixed points and that all its orbits are isotropic submanifolds of M, that is, $\sigma(X_M, Y_M) = 0$ for all $X, Y \in \mathfrak{t}$. Research on Hamiltonian and smooth torus actions has been extensive. Orlik-Raymond's [28, 29] and Pao's [31, 32] studied smooth actions of 2-tori on compact connected smooth 4-manifolds; Karshon and Tolman classified centered complexity one Hamiltonian torus actions in [18] and also studied Hamiltonian torus actions with 2-dimensional symplectic quotients in [17]; Kogan [20] worked on completely integrable systems with local torus actions; most recently, Pelayo and Vũ Ngọc [34, 35] have studied integrable systems on symplectic 4-manifolds in which one component of the integrable system comes from a Hamiltonian circle action. There are many other papers which relate integrable systems and Hamiltonian torus actions, for instance Duistermaat's paper on global action-angle coordinates [9] and Zung's work on the topology of integrable Hamiltonian systems [42, 43].

Although Hamiltonian actions of n-dimensional tori on $2n$-dimensional manifolds are present in many integrable systems in classical mechanics, non-Hamiltonian actions occur also in physics, cf. Novikov's article [27]. At the other extreme of a symplectic Hamiltonian T-action is the case of a symplectic T-action whose principal orbits are symplectic submanifolds of (M, σ), in which case the action does

not have any fixed points and the restriction of the symplectic form to the T-orbits is non-degenerate, which in particular implies that the action is never Hamiltonian. The classification of Pelayo [33], reviewed in the present paper, shows that there are lots of cases where this happens.

If one principal orbit is symplectic, then every orbit is symplectic, and the action is locally free in the sense that all the stabilizer groups are finite subgroups of T. We first describe in Sect. 3.2 the particular case when the action is free, and hence the orbit space M/T is a manifold; in this case the classification is more straightforward, see Proposition 3.2. If the action is not free, then the orbit space M/T is a *good* orbifold (proven in [33]), and the classification of Sect. 3.2 is generalized to this rather more delicate situation in Sect. 3.3. If $\dim M - \dim T = 2$, when the orbifold M/T is an orbisurface, the classification can be given in a stronger, more concrete fashion, see Sect. 5.

This paper contains the following new results: Theorem 1.1, Theorem 3.3, Proposition 4.1, Theorem 4.2, Proposition 4.3, Lemma 4.5 items (iii) and (iv) and Corollary 4.6. In particular, the following topological result is a consequence of Theorem 4.2 and Proposition 4.3.

Theorem 1.1 *If M is compact, connected symplectic manifold on which a torus T acts effectively with symplectic orbits, then the leaf space M/T is a very good orbifold with Betti number* $b_1(M/T) = b_1(M) - \dim T$.

The above result may be considered a symplectic version of the classical work by Kirwan [19] on the computation of the Betti numbers of symplectic quotients in the Hamiltonian case.

There are a few results on non-Hamiltonian symplectic torus actions: McDuff [24] and McDuff and Salamon [25] studied non-Hamiltonian circle actions, and Ginzburg [15] non-Hamiltonian symplectic actions of compact groups under the assumption of a "Lefschetz condition". Benoist [3] proved a symplectic tube theorem for symplectic actions with coisotropic orbits and convexity result in the spirit of the of the Atiyah-Guillemin-Sternberg theorem [3]; Ortega-Ratiu [30] proved a local normal form theorem for symplectic torus actions with coisotropic orbits. These appear to be the most general results prior to the classification of symplectic torus actions with coisotropic principal orbits in Duistermaat-Pelayo [11] and Pelayo [33]. For a concise overview of the classification in [11] and an application to complex and Kähler geometry see [12].

2 Preliminaries

Let (M, σ, T) be a symplectic T-manifold. For every $x \in M$ the orbit $T \cdot x$ of the T-action containing x is a smooth manifold, and the mapping $T \to M : t \mapsto t \cdot x$ induces a diffeomorphism from T/T_x onto $T \cdot x$, where $T_x := \{t \in T \mid t \cdot x = x\}$ denotes the stabilizer subgroup of x in T. The tangent mapping at $1T_x$ of this diffeomorphism is a linear isomorphism from $\mathfrak{t}/\mathfrak{t}_x$ onto the tangent space $\mathrm{T}_x(T \cdot x)$ at x of $T \cdot x$.

Here $\mathfrak{t}$ and $\mathfrak{t}_x$ denote the respective Lie algebras of T and T_x. That is, if $X_M(x)$ denotes the infinitesimal action at x of an element X of the Lie algebra $\mathfrak{t}$ of T, then $T_x(T \cdot x) = \{X_M(x) \mid X \in \mathfrak{t}\}$, $\mathfrak{t}_x = \{X \in \mathfrak{t} \mid X_M(x) = 0\}$, and the aforementioned linear isomorphism $\mathfrak{t}/\mathfrak{t}_x \to T_x(T \cdot x)$ is induced by the linear mapping $\mathfrak{t} \to T_x M : X \mapsto X_M(x)$. For an effective torus action the minimal stabilizer subgroups are the trivial ones $T_x = \{1\}$, in which case the action of T is free at the point x, and the corresponding orbits are called the *principal orbits*. The set M^{reg} of all $x \in M$ such that $T_x = \{1\}$ is an open, dense, and T-invariant subset of M.

The orbit $T \cdot x$ is symplectic if, for every $y \in T \cdot x$, restriction of σ_y to the tangent space $T_y(T \cdot x)$ of the orbit is a symplectic form. That is, if $(T_y(T \cdot x))^{\sigma_y}$ denotes the orthogonal complement of $T_y(T \cdot x)$ in $T_y M$ with respect to the symplectic form σ_y, then $T_y M$ is equal to the direct sum of $T_y(T \cdot x)$ and its symplectic orthogonal complement.

Benoist [3, Lemme 2.1] observed that if u and v are smooth vector fields on M which preserve σ, then their Lie bracket $[u, v]$ is Hamiltonian with Hamiltonian function equal to $\sigma(u, v)$, that is, $i_{[u,v]}\sigma = -d(\sigma(u, v))$. It therefore follows from the commutativity of T that if $X, Y \in \mathfrak{t}$, then $d(\sigma(X_M, Y_M)) = 0$, which means that there is a unique antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ on $\mathfrak{t}$ such that

$$\sigma^{\mathfrak{t}}(X, Y) = \sigma_x(X_M(x), Y_M(x)) \tag{1}$$

for every $x \in M$ and $X, Y \in \mathfrak{t}$. If $X \in \mathfrak{t}_x$, that is $X_M(x) = 0$, then $\sigma^{\mathfrak{t}}(X, Y) = 0$ for every $Y \in \mathfrak{t}$. It follows that $\mathfrak{t}_h \subset \mathfrak{l} \subset \mathfrak{t}$, if $\mathfrak{t}_h$ and $\mathfrak{l}$ denote the sum of all $\mathfrak{t}_x$'s and the kernel of $\sigma^{\mathfrak{t}}$ in $\mathfrak{t}$, respectively.

Assume that for some $x \in M$ the orbit $T \cdot x$ is $\dim T$-dimensional and symplectic. Then $\sigma^{\mathfrak{t}}$ is nondegenerate, which in turn implies that every T-orbit is a symplectic submanifold of (M, σ). Because $\mathfrak{t}_x \subset \ker \sigma^{\mathfrak{t}} = \{0\}$, the closed subgroup T_x of the compact group T is discrete, hence finite. Therefore the action is locally free and every T-orbit is $\dim T$-dimensional.

3 Models for symplectic torus actions with symplectic principal orbits

We give a concise review of Pelayo [33, Chaps. 2–7] with some modifications in the exposition and present a new fact: Theorem 3.3.

We study symplectic actions of the torus T on the symplectic manifold (M, σ) such that *at least one T-orbit is a* $\dim T$-*dimensional symplectic submanifold of* (M, σ). This condition means that there exists $x \in M$ such that $\mathfrak{t}_x = \{0\}$ and the restriction of σ_x to $T_x(T \cdot x)$ is nondegenerate. It follows that the antisymmetric bilinear form $\sigma^{\mathfrak{t}}$ in (1) is nondegenerate, which in turn implies that for every $x \in M$ we have $\mathfrak{t}_x = \{0\}$ and the restriction of σ_x to $T_x(T \cdot x)$ is nondegenerate. That is, *the action of T on M is locally free, and all T-orbits are* $\dim T$-*dimensional symplectic submanifolds of* (M, σ). We denote by σ^T the unique invariant symplectic form σ on the Lie group T such that $\sigma_1 = \sigma^{\mathfrak{t}}$ on $T_1 T = \mathfrak{t}$.

It follows that for each $x \in M$ the symplectic orthogonal complement $\Omega_x := (T_x(T \cdot x))^{\sigma_x}$ of the tangent space of the T-orbit is a complementary linear subspace to $T_x(T \cdot x)$ in $T_x M$, and that the restriction to Ω_x of σ_x is a symplectic form.

Furthermore the Ω_x depend smoothly on $x \in M$, and therefore define a distribution Ω in M, a smooth vector subbundle of the tangent bundle TM of M.

Lemma 3.1 *The distribution Ω is T-invariant and integrable.*

Proof The T-invariance of Ω follows from the T-invariance of σ.

There is a unique $\mathfrak{t}$-valued one-form θ on M, called the *connection form*, such that $\Omega = \ker \theta$ and $\theta(X_M) = X$ for every $X \in \mathfrak{t}$. Ω is integrable if and only if θ is closed. Let X^i, $1 \leq i \leq m := \dim T$ be a basis of $\mathfrak{t}$, and let Y^j be the $\sigma^{\mathfrak{t}}$-dual basis of $\mathfrak{t}$, determined by the equations $\sigma^{\mathfrak{t}}(X^i, Y^j) = \delta_{ij}$ for all $1 \leq i, j \leq m$. Then $\theta = \sum_{i=1}^m i_{X_M^i} \sigma \otimes Y^i$. For every $X \in \mathfrak{t}$ we have $d(i_{X_M}\sigma) = L_{X_M}\sigma - i_{X_M}(d\sigma) = 0$, because σ is T-invariant and closed. Hence θ is closed. $\qquad\square$

Lemma 3.1 leads to the local models of the symplectic T-space described in the paragraph after Theorem 3.5. These local models can also be obtained by applying results of Benoist [3, Proposition 1.9] or Ortega and Ratiu [30, Sects. 7.2–7.4] to the case of a symplectic torus action with symplectic orbits. The proof of Lemma 3.1 in [33] uses these local models.

3.1 The model $T \times_S \mathcal{I}$

Let $\mathcal{I}$ be a maximal connected integral manifold of the distribution Ω, where $\sigma^{\mathcal{I}} := \iota_{\mathcal{I}}^* \sigma$ is a symplectic form on $\mathcal{I}$ and $\iota_{\mathcal{I}}$ denotes the inclusion mapping from $\mathcal{I}$ into M. In other words, $\mathcal{I}$ is a leaf of the symplectic foliation in M of which the tangent bundle is equal to Ω.

Let $S := \{s \in T \mid s \cdot \mathcal{I} = \mathcal{I}\}$, which is a subgroup of T. If we provide $\mathcal{I}$ and S with the leaf topology and the discrete topology, respectively, then the action of S on $\mathcal{I}$ is proper. Because each other leaf is of the form $t \cdot \mathcal{I}$ for some $t \in T$ and T is commutative, the group S does not depend on the choice of the leaf $\mathcal{I}$. Furthermore, because the leaves form a partition of M, we have $t \in S$ if and only if $t \cdot \mathcal{I} \cap \mathcal{I} \neq \emptyset$, the mapping $A : T \times \mathcal{I} \to M : (t, x) \mapsto t \cdot x$ is surjective, and $A(t, x) = A(t', x')$ if and only if there exists $s \in S$ such that $x' = s \cdot x$ and $t' = ts^{-1}$. We let $s \in S$ act on $T \times \mathcal{I}$ by sending (t, x) to $(ts^{-1}, s \cdot x)$. Because S acts freely on T and properly on $\mathcal{I}$, it acts freely and properly on $T \times \mathcal{I}$. Therefore the orbit space $T \times_S \mathcal{I}$ has a unique structure of a smooth manifold such that the canonical projection $\psi : T \times \mathcal{I} \to T \times_S \mathcal{I}$ is a smooth covering map with S as its covering group. Moreover, the unique mapping $\alpha : T \times_S \mathcal{I} \to M$ such that $\alpha \circ \psi = A$ is a diffeomorphism. The symplectic form $\sigma^T \oplus \sigma^{\mathcal{I}}$ on $T \times \mathcal{I}$ is S-invariant, hence there is a unique symplectic form $\sigma^{T \times_S \mathcal{I}}$ on $T \times_S \mathcal{I}$ such that $\psi^*(\sigma^{T \times_S \mathcal{I}}) = \sigma^T \oplus \sigma^{\mathcal{I}}$, when the σ_x-orthogonality of Ω_x and $T_x(T \cdot x)$ implies that $\sigma^{T \times_S \mathcal{I}} = \alpha^*(\sigma)$. Finally, α intertwines the T-action on $T \times_S \mathcal{I}$ induced by the T-action $(t', (t, x)) \mapsto (t't, x)$ on $T \times \mathcal{I}$ with the T-action on M. We conclude that α is an isomorphism of symplectic T-spaces from $(T \times_S \mathcal{I}, \sigma^{T \times_S \mathcal{I}}, T)$ onto (M, σ, T).

The orbit space $\mathcal{I}/S$ for the action of S on $\mathcal{I}$ is provided with the finest topology on the orbit space $\mathcal{I}/S$ such that the canonical projection $\pi_{\mathcal{I}/S} : \mathcal{I} \to \mathcal{I}/S$ is continuous. Because the action of S on $\mathcal{I}$ is proper, the topology on $\mathcal{I}/S$ is Hausdorff.

The mapping $\iota : \mathcal{I}/S \to M/T$ induced by the inclusion maps $\mathcal{I} \to M$ and $S \to T$ is bijective and continuous, and because M/T is compact, it follows that ι is a homeomorphism. This in turn implies that $\mathcal{I}/S$ is compact, that is, *the action of S on $\mathcal{I}$ is cocompact.*

3.2 When T acts freely

We assume in this subsection that the action of T on M is free. We will present a model of the symplectic T-space in which $\mathcal{I}$ and S are replaced by the universal covering of M/T and the monodromy homomorphism from the fundamental group of M/T to T, respectively.

The freeness of the T-action implies that M/T has a unique structure of a smooth manifold of dimension $\dim(M/T) = \dim M - \dim T$ and the canonical projection $\pi : M \to M/T : x \mapsto T \cdot x$ exhibits M as a principal T-bundle, with Ω as a flat infinitesimal connection. Furthermore the action of S on $\mathcal{I}$ is free, $\mathcal{I}/S$ has a unique structure of a smooth manifold such that $\pi_{\mathcal{I}/S} : \mathcal{I} \to \mathcal{I}/S$ is a smooth covering map, and the homeomorphism $\iota : \mathcal{I}/S \to M/T$ is a diffeomorphism. The composition $\iota \circ \pi_{\mathcal{I}/S} : \mathcal{I} \to M/T$, which is a smooth covering map, is equal to the restriction $\pi|_{\mathcal{I}}$ of π to $\mathcal{I}$.

There is a unique symplectic form $\sigma^{\mathcal{I}/S}$ and $\sigma^{M/T}$ on $\mathcal{I}/S$ and M/T, respectively, such that $\sigma^{\mathcal{I}} = \pi_{\mathcal{I}/S}^*(\sigma^{\mathcal{I}/S}) = (\pi|_{\mathcal{I}})^*(\sigma^{M/T})$. The symplectic form $\sigma^{M/T}$ on M/T does not depend on the choice of the leaf $\mathcal{I}$, because T acts transitively on the set of leaves and each $t \in T$ acts as a symplectomorphism from $(\mathcal{I}, \sigma^{\mathcal{I}})$ onto $(t \cdot \mathcal{I}, \sigma^{t \cdot \mathcal{I}})$. Because $\pi|_{\mathcal{I}} = \iota \circ \pi_{\mathcal{I}/S}$, it follows that ι is a symplectomorphism from $(\mathcal{I}/S, \sigma^{\mathcal{I}/S})$ onto $(M/T, \sigma^{M/T})$. In the sequel we simplify the notation by writing $\mathcal{O} = M/T$, $\sigma^{\mathcal{O}} = \sigma^{M/T}$, $\psi = \pi|_{\mathcal{I}}$, or equivalently $\mathcal{O} = \mathcal{I}/S$, $\sigma^{\mathcal{O}} = \sigma^{\mathcal{I}/S}$, $\psi = \pi_{\mathcal{I}/S}$.

Let $x_0 \in \mathcal{I}$ and write $p_0 = \psi(x_0)$. For each loop γ in $\mathcal{O}$ starting and ending at p_0 there is a unique curve λ in $\mathcal{I}$, called the *lift* of γ, such that $\gamma = \psi \circ \lambda$ and λ starts at x_0. The endpoint x_0' of λ belongs to $\psi^{-1}(\{p_0\})$, and therefore there is an $s \in S$ such that $x_0' = s \cdot x_0$, where s is unique because T acts freely on M. Furthermore, because S provided with the discrete topology acts properly, the element $s \in S$ only depends on the homotopy class $[\gamma]$ of γ, and the mapping $\mu : [\gamma] \mapsto s$ is a homomorphism from the fundamental group $\pi_1(\mathcal{O}, p_0)$ to S, called the *monodromy homomorphism*. The action of $\mu([\gamma])$ on $\mathcal{I}$ is the unique deck transformation $\Delta_{[\gamma]}$ of the covering $\psi : \mathcal{I} \to \mathcal{O}$, a diffeomorphism Δ of $\mathcal{I}$ such that $\psi \circ \Delta = \Delta$, such that $\Delta_{[\gamma]}(x_0)$ is equal to the endpoint of the lift of γ. Conversely, because $\mathcal{I}$ is connected, there exists for each $s \in S$ a curve λ in $\mathcal{I}$ running from x_0 to x_0', and because $\psi(x_0') = \psi(x_0) = p_0$ it follows that $\gamma = \psi \circ \lambda$ is a loop in $\mathcal{O}$ starting and ending at p_0. In other words, $\mu(\pi_1(\mathcal{O}, p_0)) = S$. For this reason the subgroup S of T is called the *monodromy group*.

The mapping $\psi_* : \pi_1(\mathcal{I}, x_0) \to \pi_1(\mathcal{O}, p_0) : [\lambda] \mapsto [\psi \circ \lambda]$ is an isomorphism of groups from $\pi_1(\mathcal{I}, x_0)$ onto the kernel $\ker \mu$ of the monodromy homomorphism $\mu : \pi_1(\mathcal{O}, p_0) \to S$. It follows that $\psi_*(\pi_1(\mathcal{I}, x_0))$ is a normal subgroup of $\pi_1(\mathcal{O}, p_0)$, that is, $\psi : \mathcal{I} \to \mathcal{O}$ is a *Galois covering*. The homomorphism $[\gamma] \mapsto \Delta_{[\gamma]}$ from $\pi_1(\mathcal{O}, p_0)$ to the group of deck transformations of $\psi : \mathcal{I} \to \mathcal{O}$ has kernel equal to $\ker \mu = \psi_*(\pi_1(\mathcal{I}, x_0))$, and the image group, isomorphic to $S \simeq \pi_1(\mathcal{O}, p_0)/\psi_*(\pi_1(\mathcal{I}, x_0))$, acts freely and transitively on the fibers of $\psi : \mathcal{I} \to \mathcal{O}$.

The *universal covering* $\widetilde{\mathcal{O}}$ is defined as the space of homotopy classes of curves in $\mathcal{O}$ starting at p_0, where in the homotopies the endpoints of the curves are kept fixed. Let $\pi_{\mathcal{O}} : \widetilde{\mathcal{O}} \to \mathcal{O}$ denote the mapping which assigns to $[\gamma] \in \widetilde{\mathcal{O}}$ the endpoint of γ. Then there is a unique structure of a smooth manifold on $\widetilde{\mathcal{O}}$ such that $\pi_{\mathcal{O}} : \widetilde{\mathcal{O}} \to \mathcal{O}$ is a smooth covering map. The manifold $\widetilde{\mathcal{O}}$ is simply connected, and each covering of $\mathcal{O}$ by a simply connected manifold is isomorphic to $\pi_{\mathcal{O}} : \widetilde{\mathcal{O}} \to \mathcal{O}$. If $c \in \widetilde{\mathcal{O}}$, and choose $\gamma \in c$. Then the endpoint x of the curve λ in $\mathcal{I}$ starting at x_0 such that $\psi \circ \lambda = \gamma$ does not depend on the choice of γ, and the mapping $\pi_{\mathcal{I}} : \widetilde{\mathcal{O}} \to \mathcal{I}$ is a smooth covering map, isomorphic to the universal covering of $\mathcal{I}$. The group structure on $\pi_1(\mathcal{O}, p_0)$ is induced by the concatenation of loops. The concatenating of a loop starting and ending at p_0 with a curve starting at p_0 leads to an action of $\pi_1(\mathcal{O}, p_0)$ on $\widetilde{\mathcal{O}}$. This action is free, and transitive each fiber of $\pi_{\mathcal{O}} : \widetilde{\mathcal{O}} \to \mathcal{O}$. For any $[\gamma] \in \pi_1(\mathcal{O}, p_0)$ and $[\delta] \in \widetilde{\mathcal{O}}$, the definition of the homomorphism $\mu : \pi_1(\mathcal{O}, p_0) \to S$ implies that $\Delta_\gamma \circ \pi_{\mathcal{I}}([\delta]) = \mu([\gamma]) \cdot \pi_{\mathcal{I}}([\delta]) = \pi_{\mathcal{I}}([\gamma] \cdot [\delta])$.

We are now ready to present the following model of our symplectic T-space (M, σ, T).

Proposition 3.2 *The composition $A \circ (1 \times \pi_{\mathcal{I}})$ of the projection $1 \times \pi_{\mathcal{I}} : T \times \widetilde{\mathcal{O}} \to T \times \mathcal{I}$ with the action mapping $A : T \times \mathcal{I} \to M$ is a Galois covering map from $T \times \widetilde{\mathcal{O}}$ onto M, with $\pi_1(\mathcal{O}, p_0)$ as the group of deck transformations, where $[\gamma] \in \pi_1(\mathcal{O}, p_0)$ acts on $T \times \widetilde{\mathcal{O}}$ by sending $(t, [\delta])$ to $(t\mu([\gamma])^{-1}, [\gamma] \cdot [\delta])$. Let $M_{\mathrm{model}} = T \times_{\pi_1(\mathcal{O}, p_0)} \widetilde{\mathcal{O}}$ and $\iota_{\mathrm{model}} : M_{\mathrm{model}} \to M$ denote the $\pi_1(\mathcal{O}, p_0)$-orbit space and the induced map, respectively. (Recall that μ is the monodromy homorphism.)*

There is a unique symplectic form σ_{model} of which the pullback by the canonical projection $T \times \widetilde{\mathcal{O}} \to T \times_{\pi_1(\mathcal{O}, p_0)} \widetilde{\mathcal{O}} = M_{\mathrm{model}}$ is equal to $\sigma^T \oplus \pi_{\mathcal{O}}^*(\sigma^{\mathcal{O}})$. The projection $T \times \widetilde{\mathcal{O}} \to M_{\mathrm{model}}$ intertwines the T-action $(t', (t, [\delta])) \mapsto (t't, [\delta])$ on $T \times \widetilde{\mathcal{O}}$ with the unique action of T on M_{model}. The map ι_{model} is a T-equivariant symplecto-morphism from the symplectic T-space $(M_{\mathrm{model}}, \sigma_{\mathrm{model}}, T)$ onto (M, σ, T).

We conclude this subsection with a construction inspired by Kahn [16, Proof of Corollary 1.4].

Recall the T-invariant connection form θ on M introduced in the proof of Lemma 3.1. θ' is the connection form of any other integrable T-invariant infinitesi-mal connection for the principal T-bundle $\pi : M \to M/T$ if and only if $\theta' = \theta + \eta$, where η is a closed T-invariant $\mathfrak{t}$-valued one-form on M which is *horizontal*, that is, $\eta(X_M) = 0$ for every $X \in \mathfrak{t}$. In other words, $\eta = \pi^*(\beta)$ for a unique closed $\mathfrak{t}$-valued one-form β on M/T, and $\theta' = \eta + \pi^*(\beta)$. If $\mu' : \pi_1(M/T, p_0) \to T$ denotes the monodromy homomorphism defined by $\Omega' = \ker$, then

$$\mu'([\gamma]) = \mu([\gamma]) \exp\left(-\int_\gamma \beta\right)$$

for every $[\gamma] \in \pi_1(M/T, p_0)$.

Because T is commutative, the homomorphism $\mu : \pi_1(M/T, p_0) \to T$ is triv-ial on the *commutator subgroup* C of $\pi_1(M/T, p_0)$, the smallest normal subgroup of $\pi_1(\mathcal{O}, p_0)$ which contains all commutators. It is a theorem of Hurewicz that

C is equal to the kernel of the surjective homomorphism $h_1 : \pi_1(M/T, p_0) \to$ $H_1(M/T, \mathbb{Z})$, called the *Hurewicz homomorphism*, which assigns to each $[\gamma] \in$ $\pi_1(M/T, p_0)$ its homology class. See Hu [22, Theorem 12.8]. It follows that there is a unique homomorphism $\mu_h : H_1(M/T, \mathbb{Z}) \to T$ such that $\mu = \mu_h \circ h_1$.

On the other hand $\int_\gamma \beta = [\beta](h_1([\gamma]))$, where $[\beta] \in H^1(M/T, \mathfrak{t}) \simeq$ $\mathrm{Hom}(H_1(M/T, \mathbb{Z}), \mathfrak{t})$ denotes the image of the de Rham cohomology class of β under the canonical isomorphism from $H^1_{\mathrm{deRham}}(M/T) \otimes \mathfrak{t}$ onto $H^1(M/T, \mathfrak{t})$. For any group H the *torsion subgroup H^{tor} of H* is defined as the group of all elements of H of finite order. Because $\mathfrak{t}^{\mathrm{tor}} = \{0\}$, every homomorphism $H_1(M/T, \mathbb{Z}) \to \mathfrak{t}$ vanishes on $H_1(M/T, \mathbb{Z})^{\mathrm{tor}}$, and therefore $\mu'_h = \mu_h$ on $H_1(M/T, \mathbb{Z})^{\mathrm{tor}}$.

The compactness of M/T implies that $H_1(M/T, \mathbb{Z})$ is a finitely generated commutative group, and therefore the group $H_1(M/T, \mathbb{Z})/H_1(M/T, \mathbb{Z})^{\mathrm{tor}}$ is isomorphic to $\mathbb{Z}^b$ for some $b \in \mathbb{Z}_{\geq 0}$, called the *first Betti number* $b_1(M/T)$ of M/T. In contrast, the group $H_1(M/T, \mathbb{Z})^{\mathrm{tor}}$ is finite, isomorphic to the Cartesian product of finitely many cyclic finite groups. Let e_i, $1 \leq i \leq b$ be a $\mathbb{Z}$-basis of $H_1(M/T, \mathbb{Z})/H_1(M/T, \mathbb{Z})^{\mathrm{tor}}$. For every $1 \leq i \leq b$, choose $[\gamma_i] \in \pi_1(M/T, p_0)$ and $X_i \in \mathfrak{t}$ such that $h_1([\gamma_i]) = e_i \bmod H_1(M/T, \mathbb{Z})^{\mathrm{tor}}$ and $T \ni \mu([\gamma_i]) = \exp(X_i)$. Then there exists a unique homomorphism β from $H_1(M/T, \mathbb{Z})/H_1(M/T, \mathbb{Z})^{\mathrm{tor}}$ to $\mathfrak{t}$ such that $\beta(e_i) = X_i$ for every $1 \leq i \leq b$, and it follows that for this choice of β the corresponding μ' satisfies $\mu'([\gamma_i]) = \mu'_h(h_1([\gamma_i])) = 1$ for every $1 \leq i \leq b$. For any $[\gamma] \in \pi_1(M/T, p_0)$, there exist $c_i \in \mathbb{Z}$ and $\delta \in \pi_1(M/T, p_0)$ such that $h_1([\delta]) \in H_1(M/T, \mathbb{Z})^{\mathrm{tor}}$ and $h_1([\gamma]) = \sum_{i=1}^m c_i h_1([\gamma_i]) + h_1([\delta])$, hence $\mu'([\gamma]) = \mu'([\delta]) = \mu([\delta])$. We therefore have proved the following theorem, in which the last statement is Pelayo [33, Theorem 4.2.1].

Theorem 3.3 *Let Ω' be any flat infinitesimal connection for the principal T-bundle $\pi : M \to M/T$, with the corresponding monodromy homomorphism $\mu' : \pi_1(M/T, p_0) \to T$. Then $\mu'_h = \mu_h$ on the finite group $F := H_1(M/T, \mathbb{Z})^{\mathrm{tor}}$, which implies that $\mu_h(F) = \mu'_h(T) \subset \mu'(\pi_1(M/T, p_0))$. On the other hand, Ω' can be chosen such that $\mu'(\pi_1(M/T, p_0))$ is equal to the finite subgroup $\mu_h(F)$ of T.*

It follows that the principal T-bundle $\pi : M \to M/T$ is T-equivariantly diffeomorphic to $T \times (M/T)$, if and only if $\mu([\gamma]) = 1$ for every $[\gamma] \in h_1^{-1}(F)$. This happens in particular if $H_1(M/T, \mathbb{Z})$ has no torsion. Here $t' \in T$ acts on $T \times (M/T)$ by sending (t, p) to $(t't, p)$.

If Ω' is as in Theorem 3.3, then there is a unique two-form σ' on M such that $\sigma' = \sigma$ on the T-orbits and on the linear complements Ω'_x of the tangent spaces to the T-orbits, but this time Ω'_x is the σ'_x-orthogonal complement of $T_x(T \cdot x)$ in $T_x M$. Then σ' is a T-invariant symplectic form on M, the T-orbits are symplectic submanifolds of M, and Ω' is the distribution of the symplectic orthogonal complements to the tangent spaces of the orbits. The integral manifolds $\mathcal{I}'$ of Ω' are compact if and only if the monodromy group $\mu'(\pi_1(M/T, p_0))$ of Ω' is finite, which according to Theorem 3.3 can always be arranged by means of a suitable choice of the T-invariant symplectic form σ' on M, equal to σ on the T-orbits. We like to think of Theorem 3.3 as telling how the integral manifolds and the monodromy of Ω can be changed when changing the symplectic form in the above manner, without changing the T-action.

Let $\dim T = \dim M - 2$, when M/T is a compact connected oriented surface. Then $\mathrm{H}_1(M/T, \mathbb{Z})$ has no torsion, see for instance Sect. 4.1, and the conclusion is that *every principal T-bundle $\pi : M \to M/T$ which admits a T-invariant symplectic form with symplectic T-orbits is trivial*. This is Pelayo [33, Corollary 4.1.2], proved before by Kahn [16, Corollary 1.4] in the case that $\dim M = 4$.

3.3 Orbifolds

We return to the general case, when there may exist $x \in M \setminus M^{\mathrm{reg}}$, meaning that the stabilizer subgroup T_x, which is finite, is nontrivial.

Let $\mathcal{I}$ denote the maximal integral manifold of Ω such that $x \in \mathcal{I}$. If $t \in T_x$ then $t \cdot x = x$, hence $t \cdot \mathcal{I} \cap \mathcal{I} \neq \emptyset$, and therefore $t \in S = \{s \in T \mid s \cdot \mathcal{I} = \mathcal{I}\}$, see Sect. 3.1. Therefore $\mathcal{I}$ is T_x-invariant. Because the derivative at $(1, x)$ of the covering map $A : T \times \mathcal{I} \to M$ is bijective, there exist neighborhoods U and V of 1 and x in T and $\mathcal{I}$, respectively, such that the restriction to $U \times V$ of A is injective. Because T_x is compact, we can choose V to be T_x-invariant. Suppose that t_j and x_j are sequences in T and V, respectively, such that $x_j \to x$ and $V \ni t_j \cdot x_j \to x$ as $j \to \infty$. Because T is compact, we can arrange that there exists $t_\infty \in T$ such that $t_j \to t_\infty$ as $j \to \infty$. The continuity of A implies that $t_\infty \cdot x = \lim_{j \to \infty} t_j \cdot x_j = x$, hence $t_\infty \in T_x$. if we write $s_j = t_\infty^{-1} t_j$, then $s_j \to 1$ and $s_j \cdot x_j \in V$. If $j >> 1$, then $s_j^{-1} \in U$, $s_j x_j \in V$, and $A(s_j^{-1}, s_j x_j) = A(1, x_j)$, in combination with the injectivity of $A|_{U \times V}$, implies that $s_j = 1$, hence $t_j = t_\infty \in T_x$. The conclusion is that there exists an open T_x-invariant neighborhood $\mathcal{I}_0$ of x in $\mathcal{I}$ such that if $t \in T$ and $t \cdot \mathcal{I}_0 \cap \mathcal{I}_0 \neq \emptyset$, then $t \in T_x$.

It follows that if $M_0 = A(T \times \mathcal{I}_0)$ and $\alpha_0 : T \times_{T_x} V \to M$ is the mapping induced by $A|_{T \times \mathcal{I}_o}$, then α_0 is a T-equivariant symplectomorphism from $(T \times_{T_x} \mathcal{I}_0, \sigma^{T \times_{T_x} \mathcal{I}_0}, T)$ onto $(M_0, \sigma|_{M_0}, T)$. This model of $(M_0, \sigma|_{M_0}, T)$ is equal to the model of Sect. 3.1, with M, $\mathcal{I}$, and S replaced by M_0, $\mathcal{I}_0$, and T_x, respectively.

The set $(\mathcal{I}/S)_0 = \pi_{\mathcal{I}/S}(\mathcal{I}_0)$ is an open neighborhood of $S \cdot x$ in $\mathcal{I}/S$, and the mapping $i_0 : \mathcal{I}_0/T_x \to \mathcal{I}/S$, defined by the inclusion mappings $\mathcal{I}_0 \to \mathcal{I}$ and $T_x \to S$, is a homeomorphism from $\mathcal{I}_0/T_x$ onto $(\mathcal{I}/S)_0$. The homeomorphism $\iota : \mathcal{I}/S \to M/T$ of Sect. 3.1 maps $(\mathcal{I}/S)_0$ onto an open neighborhood $(M/T)_0$ of $T \cdot x$ in M/T. By shrinking the above $\mathcal{I}_0$'s if necessary, we can arrange that these are diffeomorphic to open subsets of $\mathbb{R}^n$, $n = \dim \mathcal{I} = \dim M - \dim T$, and it follows that the $\mathcal{I}_0$'s and corresponding finite groups T_x form an *orbifold atlas* for $\mathcal{I}/S$. We define the orbifold structure on M/T by declaring the homeomorphism $\iota : \mathcal{I}/S \to M/T$ of Sect. 3.1 to be an orbifold isomorphism; this is a different, canonically equivalent way of defining the orbifold structure to [33, Definition 2.3.5], where it was defined using the symplectic tube theorem. Then the map $\psi = \pi_{\mathcal{I}/S} : \mathcal{I} \to \mathcal{I}/S$ is an *orbifold covering* from the smooth manifold $\mathcal{I}$ onto the orbifold $\mathcal{I}/S$, which exhibits $\mathcal{I}/S \simeq M/T$ as a *good orbifold*, with covering group equal to S. Here we have used the terminology and basic properties concerning orbifolds as in Boileau, Maillot and Porti [5, Sects. 2.1.1, 2.1.2].

In the sequel we write $\mathcal{O}$ for the orbifold $\mathcal{I}/S \simeq M/T$. Let $\mathcal{O}^{\mathrm{reg}}$ denote the set of all *regular points of $\mathcal{O}$*, points p in a local orbifold chart with a trivial local group. In terms of the orbifold covering $\psi : \mathcal{I} \to \mathcal{O}$ by means of the smooth manifold $\mathcal{I}$,

we have $p \in \mathcal{O}^{\mathrm{reg}}$ if and only if $T_x = \{1\}$ for every $x \in \psi^{-1}(\{p\})$, that is, $\mathcal{I}^{\mathrm{reg}} := \psi^{-1}(\mathcal{O}^{\mathrm{reg}})$ is equal to the set of all points in $\mathcal{I}$ on which the action of S is free.

There is an orbifold covering $\pi_{\mathcal{O}} : \widetilde{\mathcal{O}}^{\mathrm{orb}} \to \mathcal{O}$, called the *universal orbifold covering* such that for every orbifold covering $\pi' : \mathcal{O}' \to \mathcal{O}$ and points $\widetilde{p}_0 \in \mathcal{O}^{\mathrm{orb}}$, $p_0' \in \mathcal{O}'$ such that $\pi_{\mathcal{O}}(\widetilde{p}_0) = \pi'(p_0')$ is a regular point in $\mathcal{O}$, there exists a unique orbifold covering $\pi_{\mathcal{O}'} : \widetilde{\mathcal{O}}^{\mathrm{orb}} \to \mathcal{O}'$ such that $\pi_{\mathcal{O}} = \pi' \circ \pi_{\mathcal{O}'}$ and $\pi_{\mathcal{O}'}(\widetilde{p}_0) = p_0'$. The universal covering is unique up to orbifold isomorphisms. The group of all orbifold deck transformations, the orbifold automorphisms c of $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ such that $\pi_{\mathcal{O}} \circ c = \pi_{\mathcal{O}}$, is called the *orbifold fundamental group* Γ of $\mathcal{O}$. The action of Γ on $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ is proper. See Thurston [40, Proposition 5.3.3 and Definition 5.3.5].

In our good orbifold case the universality property of $\pi_{\mathcal{O}} : \widetilde{\mathcal{O}}^{\mathrm{orb}} \to \mathcal{O}$ implies that there exists a unique orbifold covering $\pi_{\mathcal{I}} : \widetilde{\mathcal{O}}^{\mathrm{orb}} \to \mathcal{I}$ such that $\pi_{\mathcal{O}} = \psi \circ \pi_{\mathcal{I}}$ and $\pi(\widetilde{p}_0) = x_0$ if x_0 is a base point in $\mathcal{I}$ such that $\psi(x_0) = p_0 = \pi_{\mathcal{O}}(\widetilde{p}_0)$. Because $\mathcal{I}$ is a smooth manifold, $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ is a smooth manifold, diffeomorphic to the universal covering $\widetilde{\mathcal{I}}$ of $\mathcal{I}$, and Γ acts on $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ by means of diffeomorphisms.

The mapping $\pi_{\mathcal{I}}$ intertwines the action of Γ on $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ with a unique action of Γ on $\mathcal{I}$, and there is a unique homomorphism $\mu : \Gamma \to S$, called the *orbifold monodromy homomorphism*, such that $c \cdot x = \mu(c) \cdot x$ for every $c \in \Gamma$ and $x \in \mathcal{I}$. We have $\mu(\Gamma) = S$, and therefore the subgroup S of T is called the *monodromy group*. The homomorphism from Γ to the group of deck transformations of $\pi_{\mathcal{I}} : \widetilde{\mathcal{O}}^{\mathrm{orb}} \to \mathcal{I}$ has kernel equal to $\ker \mu = \psi_*(\Gamma)$ and the image group is isomorphic to $S \simeq \Gamma/\psi_*(\pi_1(\mathcal{I}, x_0))$. It also follows that $\widetilde{\mathcal{O}}^{\mathrm{orb,reg}} := \pi_{\mathcal{O}}^{-1}(\mathcal{O}^{\mathrm{reg}})$ is equal to the set of all points of $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ on which Γ acts freely.

Let x_0 and $\widetilde{p}_0$ be base points in $\mathcal{I}$ and $\widetilde{\mathcal{O}}$, respectively, such that $\psi(x_0) = p_0 = \pi_{\mathcal{O}}(\widetilde{p}_0)$. We have $x_0 \in \mathcal{I}^{\mathrm{reg}}$ and $\widetilde{p}_0 \in \widetilde{\mathcal{O}}^{\mathrm{orb,reg}}$ because $p_0 \in \mathcal{O}^{\mathrm{reg}}$. An *orbifold loop in $\mathcal{O}$ based at p_0* is defined as a loop γ in $\mathcal{O}$ based at p_0 such that $\gamma = \pi_{\mathcal{O}} \circ \widetilde{\gamma}$, where $\widetilde{\gamma}$ is a path in $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ starting at $\widetilde{p}_0$, which is unique when it exists. By definition, *orbifold homotopies* of loops γ in $\mathcal{O}$ based at p_0 correspond to homotopies of the curves $\widetilde{\gamma}$ in $\widetilde{\mathcal{O}}^{\mathrm{orb}}$. Because Γ acts transitively on $\pi_{\mathcal{O}}^{-1}(\{p_0\})$, there exists a $c \in \Gamma$ which maps $\widetilde{p}_0$ to the endpoint of $\widetilde{\gamma}$, where c is unique because Γ acts freely on $\widetilde{p}_0$. Furthermore, any path in $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ from $\widetilde{p}_0$ to $c \cdot \widetilde{p}_0$ is homotopic to $\widetilde{\gamma}$, because $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ is simply connected. It follows that the mapping $[\gamma] \mapsto c$ is an isomorphism, from the group $\pi_1^{\mathrm{orb}}(\mathcal{O}, p_0)$ of all orbifold homotopy classes of homotopy loops in $\mathcal{O}$ based at p_0, onto Γ. We use this isomorphism to identify the two groups, and write $\Gamma = \pi_1^{\mathrm{orb}}(\mathcal{O}, p_0)$ in the sequel.

With these notations and basic facts, we have the following model of (M, σ, T), extending Proposition 3.2 to the case that the T-action is not free.

Theorem 3.4 *The composition $A \circ (1 \times \pi_{\mathcal{I}})$ of the projection $1 \times \pi_{\mathcal{I}} : T \times \widetilde{\mathcal{O}}^{\mathrm{orb}} \to T \times \mathcal{I}$ with the action mapping $A : T \times \mathcal{I} \to M$ is a Galois covering map from $T \times \widetilde{\mathcal{O}}^{\mathrm{orb}}$ onto M, with $\Gamma = \pi_1^{\mathrm{orb}}(\mathcal{O}, p_0)$ as the group of deck transformations, where $c \in \Gamma$ acts on $T \times \widetilde{\mathcal{O}}^{\mathrm{orb}}$ by sending $(t, \widetilde{p})$ to $(t \mu(c)^{-1}, c \cdot \widetilde{p})$. This action is proper and free, where the freeness of the action implies that the action of $\ker \mu$ on $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ is free. Let $M_{\mathrm{model}} = T \times_\Gamma \widetilde{\mathcal{O}}^{\mathrm{orb}}$ and $\iota_{\mathrm{model}} : M_{\mathrm{model}} \to M$ denote the Γ-orbit space and the induced map, respectively. (Recall that μ is the monodromy homomorphism.)*

There is a unique symplectic form σ_{model} *of which the pullback by the canonical projection* $T \times \widetilde{\mathcal{O}}^{\mathrm{orb}} \to T \times_\Gamma \widetilde{\mathcal{O}}^{\mathrm{orb}} = M_{\mathrm{model}}$ *is equal to* $\sigma^T \oplus \pi_{\mathcal{I}}^*(\iota_{\mathcal{I}}^*\sigma)$*, where* $\iota_{\mathcal{I}}$ *denotes the inclusion mapping* $\mathcal{I} \to M$*. The projection* $T \times \widetilde{\mathcal{O}}^{\mathrm{orb}} \to M_{\mathrm{model}}$ *intertwines the* T*-action* $(t', (t, \widetilde{p})) \mapsto (t't, \widetilde{p})$ *on* $T \times \widetilde{\mathcal{O}}^{\mathrm{orb}}$ *with a unique action of* T *on* M_{model}*. The map* ι_{model} *is a* T*-equivariant symplectomorphism from the symplectic* T*-space* $(M_{\mathrm{model}}, \sigma_{\mathrm{model}}, T)$ *onto* (M, σ, T)*.*

In Theorem 3.4 we could have written $\iota_{\mathcal{I}}^*(\sigma) = \psi^*(\sigma^{\mathcal{O}})$ for a unique orbifold symplectic form $\sigma^{\mathcal{O}}$ on $\mathcal{O} \simeq \mathcal{I}/S \simeq M/T$, when $\pi_{\mathcal{I}}^*(\iota_{\mathcal{I}}^*\sigma) = \pi_{\mathcal{O}}^*(\sigma^{\mathcal{O}})$.

The following converse to Theorem 3.4 is our *existence theorem* in the classification of all compact connected symplectic T-spaces (M, σ) with symplectic principal orbits.

Theorem 3.5 *Let* T *be an even-dimensional torus provided with an invariant symplectic form* σ^T*. Let* $\mathcal{O}$ *be a compact and connected good even-dimensional orbifold provided with an orbifold symplectic form* $\sigma^{\mathcal{O}}$*. Finally, let* μ *be a homomorphism from* $\Gamma = \pi_1^{\mathrm{orb}}(\mathcal{O}, p_0)$ *to* T *such that* $\ker \mu$ *acts freely on the orbifold universal covering* $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ *of* $\mathcal{O}$*.*

Then $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ *is a smooth manifold and* $\pi_{\mathcal{O}}^*(\sigma^{\mathcal{O}})$ *is a symplectic form on* $\widetilde{\mathcal{O}}^{\mathrm{orb}}$*. The action* $(c, (t, \widetilde{p})) \mapsto (t\mu(c)^{-1}, c \cdot \widetilde{p})$ *of* Γ *on* $T \times \widetilde{\mathcal{O}}^{\mathrm{orb}}$ *is proper, free, and preserves the symplectic form* $\sigma^T \oplus \sigma^{\widetilde{\mathcal{O}}^{\mathrm{orb}}}$*. Let* π_M *be the canonical projection from* $T \times \widetilde{\mathcal{O}}^{\mathrm{orb}}$ *onto the orbit space* $M := T \times_\Gamma \widetilde{\mathcal{O}}^{\mathrm{orb}}$*. Then there is a unique symplectic form* σ *on the smooth manifold* M *such that* $\pi_M^*(\sigma) = \sigma^T \oplus \pi_{\mathcal{O}}^*(\sigma^{\mathcal{O}})$*.* π_M *intertwines the* T*-action* $(t', (t, \widetilde{p})) \mapsto (t't, \widetilde{p})$ *on* $T \times \widetilde{\mathcal{O}}^{\mathrm{orb}}$ *with a unique* T*-action on* M *which preserves* σ *and has symplectic orbits.*

4 Topology of the orbit space

In the next paragraphs we present a more detailed local model of the symplectic T-action, and draw some conclusions about the singularities of the orbifold $\mathcal{O} \simeq \mathcal{I}/S \simeq M/T$ and the orbifold fundamental group of $\mathcal{O}$. The statements Proposition 4.1, Theorem 4.2, Proposition 4.3, Lemma 4.5 items (iii) and (iv) and Corollary 4.6 are new. Theorem 1.1 follows from Theorem 4.2 and Proposition 4.3.

Note that the local groups of the orbifold $\mathcal{O} \simeq M/T \simeq \mathcal{I}/S$ are the finite stabilizer groups T_x which act on a suitable T_x-invariant open neighborhood $\mathcal{I}_0$ of x in $\mathcal{I}$ by means of symplectomorphisms. It follows from the linearization theorem of Bochner [4] that there is a smooth local coordinate system which maps x to the origin in a vector space $\Omega_x := \mathrm{T}_x\mathcal{I}$, in which the elements of T_x act by linear transformations. It follows from Chaperon [6, Corollary 1] that there exists a local system of coordinates around x, equal to zero at x, in which the symplectic form is constant and the elements of T_x act by means of symplectic linear transformations. This actually holds for an arbitrary continuous action of a compact group G by means of symplectomorphisms, with a fixed point x. For the proof, one first applies the *linearization theorem* of Bochner [4, Theorem 1] in order to arrange that G acts by linear transformations, and then the *equivariant Darboux lemma* of Weinstein [41, Corollary 4.3] in order

arrive at the G-invariant constant symplectic form. For any proper symplectic action of a Lie group G on a symplectic manifold (M, σ) there a exist a G-invariant almost complex structure and Hermitian structure J and h on M, such that σ is equal to the imaginary part of h, see for instance [10, Sect. 15.5]. The same proof yields a T_x-invariant complex structure and Hermitian structure h on Ω_x such that $\sigma = \operatorname{Im} h$. In other words, T_x acts on Ω_x by means of unitary complex linear transformations. Because T_x is commutative, there is an h-orthonormal basis of simultaneous eigenvectors for the T_x-action in Ω_x. If $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ denote the coordinates in Ω_x with respect to this basis, we have $(t \cdot z)_j = \lambda_j(t)z_j$ for every $1 \leq j \leq m$ and $t \in T_x$, where λ_j is a homomorphism from T_x to the multiplicative group $\mathbb{C}^\times$ of all nonzero complex numbers. Because T_x is finite, there is a unique $d_j \in \mathbb{Z}_{>0}$ and homomorphism $l_j : T_x \to \mathbb{Z}/d_j\mathbb{Z}$ such that $\lambda_j(t) = e^{2\pi i l_j(t)/d_j}$ for every $1 \leq j \leq m$ and every $t \in T_x$. Because M^{reg} is dense in M, T_x acts effectively on Ω_x, which means that the homomorphism $T_x \ni t \mapsto (l_1(t), \ldots, l_m(t)) \in (\mathbb{Z}/d_1\mathbb{Z}) \times \ldots \times (\mathbb{Z}/d_m\mathbb{Z})$ is injective.

The model $(T \times_{T_x} \mathcal{I}_0, \sigma^{T \times T_x \mathcal{I}_0}, T)$ with these additional properties is the model of $(M_0, \sigma|_{M_0}, T)$ in the *symplectic tube theorem* of Benoist [3, Proposition 1.9] and Ortega and Ratiu [30, Sects. 7.2–7.4] in the case of a symplectic torus action with symplectic orbits, with $\mathcal{I}_0$ as the slice.

The singular points for the T_x-action in $\mathbb{C}^m$ are the $z \in \mathbb{C}^m$ such that $z_j = 0$ if $l_j(t) \neq 0 \bmod d_j$ for some $t \in T_x$. Therefore the singular set is a union of coordinate subspaces in $\mathbb{C}^m$. The *singular sets* $\mathcal{O}^{\mathrm{sing}}$, $\mathcal{I}^{\mathrm{sing}}$, *and* $\widetilde{\mathcal{O}}^{\mathrm{orb,sing}}$ are defined as the complements in $\mathcal{O}$, $\mathcal{I}$, and $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ of $\mathcal{O}^{\mathrm{reg}}$, $\mathcal{I}^{\mathrm{reg}}$, and $\widetilde{\mathcal{O}}^{\mathrm{orb,reg}}$, respectively. Because $\mathcal{I}^{\mathrm{sing}}$ and $\widetilde{\mathcal{O}}^{\mathrm{orb,sing}}$ are the singular sets in the smooth manifolds $\mathcal{I}$ and $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ for the respective actions of S and Γ, these sets locally are unions of symplectic coordinate subspaces of strictly positive even codimension. It follows that the singular set of the orbifold $\mathcal{O} \simeq \mathcal{I}/S \simeq M/T$ locally is equal to a corresponding union of "coordinate orbifolds", each of which is a symplectic suborbifold of strictly positive even codimension.

Because $\mathcal{I}^{\mathrm{sing}}$ and $\widetilde{\mathcal{O}}^{\mathrm{sing}}$ have codimension ≥ 2 in the smooth manifolds $\mathcal{I}$ and $\widetilde{\mathcal{O}}$, the curves $\gamma^{\mathcal{I}}$ and $\widetilde{\gamma}$ are homotopic to curves in $\mathcal{I}^{\mathrm{reg}}$ and $\widetilde{\mathcal{O}}^{\mathrm{reg}}$, respectively. It follows that the inclusion mapping $\iota^{\mathrm{reg}} : \mathcal{O}^{\mathrm{reg}} \to \mathcal{O}$ induces a surjective homomorphism $\iota^{\mathrm{reg}}_* : \pi_1(\mathcal{O}^{\mathrm{reg}}, p_0) \to \pi_1^{\mathrm{orb}}(\mathcal{O}, p_0)$.

Let N be a small tubular open neighborhood of $\mathcal{O}^{\mathrm{sing}}$ in $\mathcal{O}$ such that p_0 is an interior point of $\mathcal{O} \setminus N$, and the injection from $\mathcal{O} \setminus N$ into $\mathcal{O}^{\mathrm{reg}}$ induces an isomorphism $\pi_1(\mathcal{O} \setminus N, p_0) \xrightarrow{\sim} \pi_1(\mathcal{O}, p_0)$. The boundary ∂N of N is a simplicial complex in the smooth manifold $\mathcal{O}^{\mathrm{reg}}$. Because $\mathcal{O}^{\mathrm{reg}} \setminus N$ is a compact subset of $\mathcal{O}^{\mathrm{reg}}$, it is a simplicial complex and therefore $\pi_1(\mathcal{O}^{\mathrm{reg}} \setminus N, p_0)$ is finitely generated, hence $\pi_1(\mathcal{O}^{\mathrm{reg}}, p_0)$ is finitely generated. Because ι^{reg}_* is surjective, we conclude:

Proposition 4.1 *The orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathcal{O}, p_0)$ of $\mathcal{O}$ is finitely generated. The monodromy group $S = \mu(\pi_1^{\mathrm{orb}}(\mathcal{O}, p_0))$ is a finitely generated subgroup of the torus T.*

Let X be a path connected, simply connected metrizable locally compact topological space and Γ a discrete group acting properly on X. Choose $x_0 \in X$ and write $p_0 = \Gamma \cdot x_0 \in X/\Gamma$. For $c \in \Gamma$, let δ be a path in X from x_0 to $c \cdot x_0$. If

$\pi : X \to X/\Gamma$ denotes the canonical projection, then $\pi \circ \delta$ is a loop in X/Γ based at p_0, and because X is simply connected, its homotopy class does not depend on the choice of γ and we can write $[\gamma] = \varphi(c)$ for a uniquely defined $\varphi(c) \in \pi_1(X/\Gamma, p_0)$. The theorem of Armstrong [1] says that $\varphi : \Gamma \to \pi_1(X/\Gamma, p_0)$ is a surjective homomorphism, with kernel equal to the smallest normal subgroup A of Γ which contains all elements $c \in \Gamma$ which have a fixed point in X. If we apply this to $X = \widetilde{\mathcal{O}}^{\mathrm{orb}}$ and $\Gamma = \pi_1^{\mathrm{orb}}(\mathcal{O}, p_0)$, then X/Γ is canonically identified with $\mathcal{O}$ and $\varphi : \pi_1^{\mathrm{orb}}(\mathcal{O}, p_0) \to \pi_1(\mathcal{O}, p_0)$ is the map obtained by forgetting the orbifold structure. It follows that this map φ is surjective from Γ onto $\pi_1(\mathcal{O}, p_0)$, and that its kernel is equal to the smallest normal subgroup A of Γ which contains all $c \in \Gamma$ such that $c \cdot \widetilde{p} = \widetilde{p}$ for some $\widetilde{p} \in \widetilde{\mathcal{O}}^{\mathrm{orb}}$. If $c \cdot \widetilde{p} = \widetilde{p}$, then $\mu(c) \cdot \pi_{\mathcal{I}}(\widetilde{p}) = \pi_{\mathcal{I}}(c \cdot \widetilde{p}) = \pi_{\mathcal{I}}(\widetilde{p})$. That is, $\mu(c) \in T_x$ if we write $x = \pi_{\mathcal{I}}(\widetilde{p})$. It follows that $\mu(\ker \varphi)$ is contained in the product $T_\bullet$ of all T_x's. Because the local normal form of the T-action, in combination with the compactness of M, implies that there are finitely many stabilizer subgroups T_x, each of which is finite, $T_\bullet$ is a finite subgroup of T.

We recall the surjective Hurewicz homomorphism $\mathrm{h}_1 : \pi_1(\mathcal{O}, p_0) \to \mathrm{H}_1(\mathcal{O}, \mathbb{Z})$ with kernel equal to the commutator subgroup of $\pi_1(\mathcal{O}, p_0)$. Furthermore, an orbifold $\mathcal{O}$ is called *very good* if it is isomorphic, as an orbifold, to the orbit space of a *finite* group action on a smooth manifold. We now have the following orbifold version of Theorem 3.3.

Theorem 4.2 *Let σ' be any T-invariant symplectic form on M such that $(\sigma')^{\mathrm{t}} = \sigma^{\mathrm{t}}$. Let $\Omega', \mathcal{I}', \mu'$, and S' respectively denote the integrable T-invariant distribution of the σ'-orthogonal complements of the tangent spaces of the T-orbits, a maximal integral manifold of Ω', the monodromy homomorphism from $\Gamma = \pi_1^{\mathrm{orb}}(\mathcal{O}, p_0)$ to T, and the monodromy group $S' = \mu'(\Gamma)$, where $\mathcal{O} \simeq \mathcal{I}'/S' \simeq M/T \simeq \mathcal{I}/S$.*

Then the torsion group $F = \mathrm{H}_1(M/T, \mathbb{Z})^{\mathrm{tor}}$ is finite. We have $\mu' = \mu$ on $U := (\mathrm{h}_1 \circ \varphi)^{-1}(F)$ and on Armstrong's subgroup A of Γ, $E := \mu(U)\mu(A) \subset S'$, $\mu(A) = T_\bullet :=$ the product of all T_x's, and

$$\#(T_\bullet) \le \#(E) \le \#(F)\#(T_\bullet).$$

The symplectic form σ' can be chosen such that $S' = E$. With such a choice, $M/T \simeq \mathcal{I}'/S'$ exhibits M/T as a very good orbifold, and $\mathcal{I}'$ is a compact symplectic submanifold of M.

Proof The distributions Ω' are characterized by the property that there exists a closed T-invariant and horizontal $\mathfrak{t}$-valued one-form η on M such that $\Omega'_x = \{v - \eta_x(v) \mid v \in \Omega_x\}$ for every $x \in M$. It follows from the theorem of Koszul [21] that the singular cohomology group $\mathrm{H}^1(M/T, \mathfrak{t}) = \mathrm{Hom}(\mathrm{H}_1(M/T, \mathbb{Z}), \mathfrak{t})$ is isomorphic to the space of closed $\mathfrak{t}$-valued T-invariant horizontal smooth one-forms on M modulo the space of all derivatives of T-invariant smooth $\mathfrak{t}$-valued functions on M. If $[\eta]$ denotes the element of $\mathrm{H}^1(M/T, \mathfrak{t})$ defined by η, then the monodromy homomorphism μ' of Ω' is given by

$$\mu'(c) = \mu(c) \exp(-[\eta]((\mathrm{h}_1 \circ \varphi)(c))), \quad c \in \Gamma.$$

Because $[\eta]$ is a homomorphism from $H_1(\mathcal{O}, \mathbb{Z})$ to the torsion-free additive group $\mathfrak{t}$, it vanishes on the torsion subgroup F of $H^1(\mathcal{O}, \mathbb{Z})$, and therefore $\mu' = \mu$ on U. Since $\mu'(A) = T_\bullet$ is a finite hence discrete subgroup of T, and $\mu'|_A$ depends in a continuous fashion on the element $[\eta]$ of the connected vector space $H_1(\mathcal{O}, \mathfrak{t})$, $\mu'|_A$ is constant as a function of $[\eta] \in H^1(\mathcal{O}, \mathfrak{t})$, and therefore $\mu'|_A = \mu|_A$.

Because Γ is finitely generated, $H_1(\mathcal{O}, \mathbb{Z}) = (h_1 \circ \varphi)(\Gamma)$ is a finitely generated commutative group. This can also be proved directly by observing that the orbit space stratification of M/T implies that the compact space M/T is homeomorphic to a simplicial complex. It follows that $H^1(\mathcal{O}, \mathbb{Z})/F \simeq \mathbb{Z}^b$ where $b = b_1(\mathcal{O}) \in \mathbb{Z}_{\geq 0}$, and that the torsion subgroup F of $H^1(\mathcal{O}, \mathbb{Z})$ is finite. Let $e_j \in H_1(\mathcal{O}, \mathbb{Z})$, $1 \leq j \leq b$, be such that the $e_j + F$ form a $\mathbb{Z}$-basis of $H^1(\mathcal{O}, \mathbb{Z})/F$. Because $h_1 \circ \varphi$ is surjective, we have $h_1 \circ \varphi(c_j) = e_j$ and $h_1 \circ \varphi(d_f) = f$ for suitable $c_j, d_f \in \Gamma$. Since the exponential mapping from $\mathfrak{t}$ to T is surjective, we have $\mu(c_j) = \exp(X_j)$ for suitable $X_j \in \mathfrak{t}$. Finally there exists an η as above such that $[\eta]((h_1 \circ \varphi)(c_j)) = X_j$ hence $\mu'(c_j) = 1$ for every $1 \leq j \leq b$.

For any $c \in C$ there exist $m_j \in \mathbb{Z}$ and $f \in F$ such that

$$h_1 \circ \varphi(c) = \sum_{j=1}^{b} m_j e_j + f = \sum_{j=1}^{b} m_j h_1 \circ \varphi(c_j) + h_1 \circ \varphi(d_f)$$

$$= h_1 \circ \varphi(c_1{}^{m_1} \dots c_b{}^{m_b} d_f).$$

In view of the Hurewicz theorem this is equivalent to $\varphi(c) = \varphi(c_1{}^{m_1} \dots c_b{}^{m_b} d_f)u$ for an element u in the commutator subgroup of $\pi_1(\mathcal{O}, p_0)$. Because of the surjectivity of φ, there exists an element v in the commutator subgroup of $\Gamma = \pi_1^{\text{orb}}(\mathcal{O}, p_0)$ such that $u = \varphi(v)$, and we obtain that $c = c_1{}^{m_1} \dots c_b{}^{m_b} d_f v a$ for some $a \in A = \ker\varphi$. If we apply the homomorphism μ' to the left and the right hand side of this equation, and use that $\mu'(c_j) = 1$, where $\mu'(v) = 1$ because μ' is a homomorphism to the commutative group T, we conclude that $\mu'(c) = \mu'(d_f)\mu'(a) = \mu(d_f)\mu(a)$. $\qquad\square$

Note that M can only be diffeomorphic to $T \times (M/T)$ when M/T is a smooth manifold, that is, T acts freely. This case has been dealt with in Theorem 3.3.

Further investigation of the surjective homomorphisms $\iota^{\text{reg}}{}_* : \pi_1(\mathcal{O}^{\text{reg}}, p_0) \to \Gamma$ and $\varphi : \Gamma \to \pi_1(\mathcal{O}, p_0)$ should lead to more detailed information about the orbifold fundamental group $\Gamma = \pi_1^{\text{orb}}(\mathcal{O}, p_0)$ of $\mathcal{O}$. For orbisurfaces $\mathcal{O}$, this leads to a complete understanding of the group structure of Γ, see Sect. 4.1.

We conclude this subsection with the computation in Proposition 4.3 of the first Betti number of M.

Proposition 4.3 *If all orbits of (M, σ, T) are symplectic, then* $b_1(M) = \dim T + b_1(M/T)$.

Proof Consider the action $((Y, \gamma), (X, \tilde{p})) \mapsto (X - Y, \gamma \cdot \tilde{p}))$ on $\mathfrak{t} \times \widetilde{\mathcal{O}}^{\text{orb}}$ of the discrete subgroup $\Delta := \{(Y, \gamma) \in \mathfrak{t} \times \Gamma \mid \exp Y = \mu(\gamma)\}$ of $\mathfrak{t} \times \Gamma$. If $(X - Y, \gamma \cdot \tilde{p}) = (X, \tilde{p})$, then $Y = 0$ hence $\mu(\gamma) = \exp Y = 1$, and $\gamma \cdot \tilde{p} = \tilde{p}$, and therefore $\gamma = 1$ because $\ker\gamma$ acts freely on $\widetilde{\mathcal{O}}^{\text{orb}}$. It follows that the proper action of Δ is also free, and

the covering $\mathfrak{t} \times \widetilde{\mathcal{O}}^{\mathrm{orb}} \to T \times \widetilde{\mathcal{O}}^{\mathrm{orb}} : (X, \widetilde{p}) \mapsto (\exp X, \widetilde{p})$ induces a diffeomorphism from $(\mathfrak{t} \times \widetilde{\mathcal{O}}^{\mathrm{orb}})/\Delta$ onto $T \times_\Gamma \widetilde{\mathcal{O}}^{\mathrm{orb}} \simeq M$. Because $\mathfrak{t} \times \widetilde{\mathcal{O}}^{\mathrm{orb}}$ is simply connected, we conclude that the fundamental group of M is isomorphic to Δ.

For any group G we denote by $G/\mathrm{C}(G)$ the abelianization of G, where $\mathrm{C}(G)$ is the smallest normal subgroup of G which contains all commutators of elements of G. The first homology group $\mathrm{H}_1(M, \mathbb{Z})$ of M is isomorphic to the abelianization of the fundamental group of M, and therefore isomorphic to $\Delta/\mathrm{C}(\Delta)$. Because μ is a homomorphism from Γ to the commutative group T, we have $\mu = 1$ on $\mathrm{C}(\Gamma)$. Therefore $\Delta' := \{(Y, \gamma \mathrm{C}(\Gamma)) \in \mathfrak{t} \times (\Gamma/\mathrm{C}(\Gamma)) \mid \exp Y = \mu(\gamma)\}$ is a well-defined subgroup of the commutative group $\mathfrak{t} \times (\Gamma/\mathrm{C}(\Gamma)$, equal to the image of Δ under the projection $p : (Y, \gamma) \mapsto (Y, \gamma \mathrm{C}(\gamma))$ from $\mathfrak{t} \times \Gamma$ onto $\mathfrak{t} \times (\Gamma/\mathrm{C}(\Gamma))$. Because $\mathrm{C}(\Delta) = \{0\} \times (\Gamma/\mathrm{C}(\Gamma)) =$ the kernel of $p|_\Delta$, it follows that $p|_\Delta$ induces an isomorphism from $\Delta/\mathrm{C}(\Delta)$ onto Δ'. The restriction to Δ' of the projection $(Y, \gamma \mathrm{C}(\Gamma)) \mapsto \gamma \mathrm{C}(\Gamma)$ is a surjective homomorphism from Δ' to $\Gamma/\mathrm{C}(\Gamma)$ with kernel equal to $T_\mathbb{Z} \times \{0\}$, where $T_\mathbb{Z} = \ker(\exp)$ denotes the integral lattice of T in $\mathfrak{t}$. We have $\mathrm{rank}(T_\mathbb{Z}) = \dim T$. For any subgroup B of a finitely generated commutative group A we have $\mathrm{rank}(A) = \mathrm{rank}(B) + \mathrm{rank}(A/B)$, see for instance Spanier [39, bottom of p. 8]. It follows that $\mathrm{b}_1(M) := \mathrm{rank}\, \mathrm{H}_1(M, \mathbb{Z}) = \mathrm{rank}(\Delta/\mathrm{C}(\Delta)) = \dim T + \mathrm{rank}(\Gamma/\mathrm{C}(\Gamma))$.

Recall the surjective homomorphism $\varphi : \Gamma = \pi_1^{\mathrm{orb}}(\mathcal{O}, p_0) \to \pi_1(\mathcal{O}, p_0)$, of which the kernel is equal to the smallest normal subgroup A of Γ which contains all $\gamma \in \Gamma$ such that $\gamma \cdot \widetilde{p} = \widetilde{p}$ for some $\widetilde{p} \in \widetilde{\mathcal{O}}^{\mathrm{orb}}$. And the surjective homomorphism $\mathrm{h}_1 : \pi_1(\mathcal{O}, p_0) \to \mathrm{H}_1(\mathcal{O}, \mathbb{Z})$ with kernel equal to $\mathrm{C}(\pi_1(\mathcal{O}, p_0))$. It follows that the homomorphism $\mathrm{h}_1 \circ \varphi : \Gamma \to \mathrm{H}_1(\mathcal{O}, \mathbb{Z})$ is surjective and has kernel equal to the smallest normal subgroup of Γ which contains both A and $\mathrm{C}(\Gamma)$. If $\varpi : \Gamma \to \Gamma/\mathrm{C}(\Gamma)$ denotes the canonical projection, then $\mathrm{h}_1 \circ \varphi = \psi \circ \varpi$ for a unique surjective homomorphism $\psi : \Gamma/\mathrm{C}(\Gamma) \to \mathrm{H}_1(\mathcal{O}, \mathbb{Z})$ with kernel equal to $\varpi(A)$. Let B be the smallest subgroup of $\Gamma/\mathrm{C}(\Gamma)$ which contains all $\gamma \mathrm{C}(\Gamma)$ such that $\gamma \cdot \widetilde{p} = \widetilde{p}$ for some $\widetilde{p} \in \widetilde{\mathcal{O}}^{\mathrm{orb}}$. Because each such element γ of Γ has finite order, we have $B \subset (\Gamma/\mathrm{C}(\Gamma))^{\mathrm{tor}}$. On the other hand $A \subset \varpi^{-1}(B)$, hence $\varpi(A) \subset B$, and therefore $\mathrm{rank}(\varpi(A)) = 0$. Therefore $\mathrm{rank}(\Gamma/\mathrm{C}(\Gamma)) = \mathrm{rank}(\varpi(A)) + \mathrm{rank}\, \mathrm{H}_1(\mathcal{O}, \mathbb{Z}) = 0 + \mathrm{b}_1(\mathcal{O}) = \mathrm{b}_1(\mathcal{O})$, where $\mathcal{O}$ is homeomorphic to M/T. $\qquad \square$

4.1 When the orbit space is an orbisurface

In this subsection we assume that $\dim M - \dim T = 2$, so $\mathcal{O} \simeq M/T \simeq \mathcal{I}/S$ is a compact connected symplectic orbisurface. The local normal form discussed after Theorem 3.5 leads to the following conclusions about the orbisurface $\mathcal{O}$.

The singular locus $\mathcal{O}^{\mathrm{sing}}$ is a discrete, hence finite, subset of $\mathcal{O}$. We write $n :=$ #$(\mathcal{O}^{\mathrm{sing}})$, where $n = 0$ if and only if $\mathcal{O}$ is a smooth surface. Let s_i, $1 \leq i \leq n$ be an enumeration of $\mathcal{O}^{\mathrm{sing}}$. For each $1 \leq i \leq n$ there is a unique $o = o_i \in \mathbb{Z}_{>1}$ such that for each $x \in \mathcal{I}$ with $\psi(x) = s_i$ the stabilizer group T_x of the point x in T is cyclic of order o_i, and acts on a small open disk D centered x in $\mathcal{I}$ as multiplication in the complex plane by o_i-th roots of unity. The integer o_i is called the *order of the singular point* s_i. Let c_i denote the unique element of T_x which acts near x as a rotation about the angle $2\pi/o_i$ in the positive direction. A fundamental domain of

the T_x-action is given by a wedge in D with vertex at x and opening angle $2\pi/o_i$, where the corresponding neighborhood of s_i in $\mathcal{O}$ is obtained by identifying the two sides of the wedge by means of the rotation about the angle $2\pi/o_i$, which produces a half-cone with vertex at s_i. It follows that the open neighborhood D/T_x of s_i in $\mathcal{O}$ is homeomorphic to a disc. As this holds for every singular point s_i of $\mathcal{O}$, it follows that $\mathcal{O}$ is homeomorphic to a compact, connected and oriented smooth surface, where the orientation is the one defined by the orbifold symplectic form, the orbifold area form on $\mathcal{O}$. The facts in the following paragraph are classically known (see for instance Seifert and Threlfall [38, Sects. 38, 47], where the notes Nos. 21 and 25 refer to the origins, starting with Poincaré [36]).

The surface $\mathcal{O}$ can be provided with the structure of a simplicial complex. Unless $\mathcal{O}$ is homeomorphic to a sphere, there exists a $g \in \mathbb{Z}_{>0}$ such that the surface can be obtained from the convex hull P in the plane of a regular $4g$-gon in the following way. The boundary is viewed as a cycle of g quadruples of subsequent edges $\alpha_j, \beta_j, \alpha'_j, \beta'_j$, $j \in \mathbb{Z}/g\mathbb{Z}$, where α_j and β_j are positively oriented and α'_j and β'_j are negatively oriented. The surface $\mathcal{O}$ is obtained by identifying, for each $j \in \mathbb{Z}/g\mathbb{Z}$, α_j with α'_j, where the identification respects the orientations. In the surface $\mathcal{O}$ all the vertices of the $4g$-gon correspond to a single point $p_0 \in \mathcal{O}$ which is taken as the base point. The edges α_j and β_j define loops in $\mathcal{O}$ based at p_0. If $[\alpha_j, \beta_j] = \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1}$ denotes the commutator of α_j and β_j, then their concatenation for $1 \le j \le g$ corresponds to the loop which runs along the boundary of the polytope P at its interior side, and therefore is contractible. If we denote the homotopy classes of the loops α_j and β_j by the same letters, these considerations lead to the conclusion that the homomorphism, from the free group generated by the α_j and β_j to $\pi_1(\mathcal{O}, p_0)$, is surjective, with kernel equal to the smallest normal subgroup which contains the cyclic concatenation of the commutators $[\alpha_j, \beta_j]$, $j \in \mathbb{Z}/g\mathbb{Z}$. This is usually expressed by saying that $\pi_1(\mathcal{O}, p_0)$ *is generated by the α_j, β_j, subject to the single relation*

$$[\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g] = 1,$$

where $\pi_1(\mathcal{O}, p_0) = \{1\}$ if $g = 0$. Because $h_1([\alpha_j, \beta_j]) = 0$, the commutative group $H_1(\mathcal{O}, \mathbb{Z})$ is freely generated by the homology classes $h_1(\alpha_j)$, $h_1(\beta_j)$, $j \in \mathbb{Z}/g\mathbb{Z}$. Therefore $H_1(\mathcal{O}, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$, which implies that $H_1(\mathcal{O}, \mathbb{Z})$ has no torsion. Therefore the positive integer g, called the *genus* of the surface $\mathcal{O}$, has the topological interpretation that the first Betti number $b = b_1(\mathcal{O})$ is equal to $2g$. With this interpretation, the two-dimensional sphere has genus $g = 0$. Any oriented compact connected surface of genus $g > 0$ is homeomorphic to a sphere with g handles.

It can be arranged that the singular points s_i, $1 \le i \le n$, lie in the interior of the polytope P. For each i, let γ_i be a loop in $\mathcal{O}$ consisting of a path δ_i from a vertex p_0 of P into the interior of P to a point close to s_i, followed by a circle around s_i in the positive direction and completed by the inverse of δ_i. It can be arranged that the curves γ_i don't intersect each other except at the base point and that the concatenation $\gamma_1 \ldots \gamma_n$ is homotopic in $\mathcal{O}^{\mathrm{reg}} = \mathcal{O} \setminus \mathcal{O}^{\mathrm{sing}}$ to the cyclic concatenation of the commutators $[\alpha_j, \beta_j]$, $j \in \mathbb{Z}/g\mathbb{Z}$. If we denote the homotopy classes of the γ_i by the same letters, then this leads to an isomorphism from Q/R onto $\pi_1(\mathcal{O}, p_0)$, where Q is the free group generated by the α_j, β_j, and γ_i, and R is the smallest normal subgroup of

Q which contains the product of the concatenation of the commutators $[\alpha_j, \beta_j]$ with the inverse of the concatenation of the γ_i. That is, $\pi_1(\mathcal{O}^{\text{reg}}, p_0)$ *is generated by the* α_j, β_j, *and* γ_i, *subject to the single relation*

$$[\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g] = \gamma_1 \ldots \gamma_n,$$

where the left and/or the right hand side is equal to 1 if $g = 0$ and/or $n = 0$.

The surjective homomorphism ι^{reg}_* from $\pi_1(\mathcal{O}^{\text{reg}}, p_0)$ onto $\pi_1^{\text{orb}}(\mathcal{O}, p_0)$, discussed in the paragraphs preceding Proposition 4.1, has kernel equal to the $[\gamma] \in \pi_1(\mathcal{O}^{\text{reg}}, p_0)$ such that there is an orbifold homotopy of γ to the trivial loop. The homotopy can be arranged to be transversal to the singular set, and it follows that γ is homotopic in $\mathcal{O}^{\text{reg}}$ to a concatenation of conjugates of powers of the curves γ_i introduced above. Let c_i denote a small circle around s_i in the positive direction and let $\widetilde{c_i}$ denote its orbifold lift to the orbifold chart near s_i. Then $\widetilde{c_i}$ is equal to a rotation on a small circle around the origin about the angle $2\pi/o_i$, and $c_i{}^k$ has an orbifold contraction in the chart around s_i if and only if $k \in \mathbb{Z}o_i$. It follows that the kernel of ι^{reg}_* is equal to the smallest normal subgroup of $\pi_1(\mathcal{O}^{\text{reg}}, p_0)$ which contains the elements $\gamma_i{}^{o_i}$, $1 \le i \le n$. In other words, $\pi_1^{\text{orb}}(\mathcal{O}, p_0)$ *is generated by the* α_j, β_j, *and* γ_i, *subject to the relations*

$$[\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g] = \gamma_1 \ldots \gamma_n \quad \text{and} \quad \gamma_i{}^{o_i} = 1 \quad \text{for every } 1 \le i \le n, \qquad (2)$$

cf. Scott [37, p. 424]. Note that the kernel of the surjective homomorphism $\varphi : \pi_1^{\text{orb}}(\mathcal{O}, p_0) \to \pi_1(\mathcal{O}, p_0)$ is equal to the smallest normal subgroup containing the γ_i's. This is compatible with the aforementioned presentation of $\pi_1(\mathcal{O}, p_0)$ without γ_i's.

The only bad compact connected oriented orbisurfaces $\mathcal{O}$ are the ones with $g = 0$, where $\mathcal{O}$ is homeomorphic to the two-sphere, and either $n = 1$ or $n = 2$ and $o_1 \neq o_2$. See Scott [37, Theorem 2.3]. Every good compact connected oriented orbisurface is isomorphic to $\widetilde{\mathcal{O}}^{\text{orb}}/\Gamma$, where $\widetilde{\mathcal{O}}^{\text{orb}}$ is the two-sphere with the standard Riemannian structure, the Euclidean plane, or the hyperbolic plane, and Γ is a discrete group of orientation preserving isometries acting on $\widetilde{\mathcal{O}}^{\text{orb}}$. See Thurston [40, Sect. 5.5]. This description has been used to prove that *every good compact connected oriented orbisurface is very good*, see Scott [37, Theorem 2.5]. In the case of the hyperbolic plane = the complex upper half plane, Γ is a cocompact discrete subgroup of $\text{PGL}(2, \mathbb{R})$. That is, Γ is a *Fuchsian group* of which the *signature* $(g; o_1, \ldots, o_n)$ satisfies $o_i < \infty$ for every $1 \le i \le n$.

In each tangent space of $\widetilde{\mathcal{O}}^{\text{orb}}$ there is a unique rotation J about the angle $\pi/2$, with respect to the aforementioned Riemannian structure β and orientation. This defines an almost complex structure J, which is integrable because $\dim \widetilde{\mathcal{O}}^{\text{orb}} = 2$. The imaginary part of the Hermitian structure h defined by β and J is a two-form of which the exterior derivative is equal to zero, again because $\dim \widetilde{\mathcal{O}}^{\text{orb}} = 2$. It follows that h and $\text{Im}\,h$ is a Γ-invariant Kähler structure and symplectic form on $\widetilde{\mathcal{O}}^{\text{orb}}$, and therefore defines an orbifold Kähler structure and orbifold symplectic form on $\mathcal{O}$, respectively.

In the existence result Theorem 3.5 we need a compact and connected good orbisurface $\mathcal{O}$ provided with an orbifold smooth area form without zeros. The area form

determines an orientation of $\mathcal{O}$, and in the previous paragraphs we have described the compact and connected oriented good orbisurfaces.

Lemma 4.4 *Every paracompact oriented orbisurface $\mathcal{O}$ carries an orbifold smooth area form without zeros which is compatible with the orientation.*

Proof For every $p \in \mathcal{O}$ there exists an open neighborhood U_p of p in $\mathcal{O}$ and an orbifold smooth area form σ_p without zeros on U_p which is compatible with the orientation of $\mathcal{O}$. The paracompactness of $\mathcal{O}$ implies that there exists a locally finite smooth partition of unity χ_i, $i \in I$, subordinate to the open covering $\mathcal{U} :=$ $\{U_p \mid p \in \mathcal{O}\}$ of $\mathcal{O}$. That is, for each $i \in I$, χ_i is a non-negative orbifold smooth function on $\mathcal{O}$ with support in a U_{p_i}, the U_{p_i} form a locally finite covering and $\sum_{i \in I} \chi_i = 1$, where the left hand side is viewed as a locally finite sum. Write $\sigma := \sum_{i \in I} \chi_i \sigma_{p_i}$, a locally finite sum. Then σ is an orbifold smooth area form on $\mathcal{O}$. If $p \in \mathcal{O}$, $i \in I$, and $p \in U_i$ then, because σ_{p_i} and σ_p both are orbifold smooth area forms without zeros on $U_{p_i} \cap U_p$ compatible with the orientation of $\mathcal{O}$, there exists an orbifold smooth function φ_i on $U_{p_i} \cap U_p$ such that $\sigma_{p_i} = \varphi_i \sigma_p$ and $\varphi_i > 0$ on $U_{p_i} \cap U_p$. It follows that $\sigma = (\sum_{i \in I} \chi_i \varphi_i)\sigma_p$, where $\sum_{i \in I} \chi_i \varphi_i > 0$ on a neighborhood of p, because $\chi_i \geq 0$ for every $i \in I$ and $\sum_{i \in I} \chi_i = 1$. This proves that σ has no zeros. □

In Theorem 3.5 we also need a homomorphism $\mu : \pi_1^{\mathrm{orb}}(\mathcal{O}, p_0) \to T$ such that the kernel of μ acts freely on the orbifold universal covering $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ of $\mathcal{O}$.

Lemma 4.5 *Let a_j, b_j, $1 \leq j \leq g$, and c_i, $1 \leq i \leq n$, be elements of T. Then*

(i) *There exists a homomorphism $\mu : \Gamma := \pi_1^{\mathrm{orb}}(\mathcal{O}, p_0) \to T$ such that $\mu(\alpha_j) = a_j$, $\mu(b_j) = b_j$, and $\mu(\gamma_i) = c_i$ for all $1 \leq j \leq g$ and $1 \leq i \leq n$, if and only if $c_1 \ldots c_n = 1$ and $c_i{}^{o_i} = 1$ for every $1 \leq i \leq n$. If such μ exists, then it is unique.*

(ii) *The kernel of μ acts freely on $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ if and only if, for each $1 \leq i \leq n$, the order $\mathrm{ord}(c_i)$ of the element c_i in T is exactly equal to o_i.*

(iii) *If elements $c_i \in T$ exist as in (i) and (ii) then, for each $1 \leq i \leq n$, o_i is a factor of the least common multiple m_i of the o_h such that $h \neq i$.*

(iv) *If the n orders o_i satisfy the condition in (iii), then there exists an $(n - 1)$-dimensional torus T with elements $c_i \in T$ as in (i) and (ii).*

Proof (i) There is a unique homomorphism $\widehat{\mu}$ from the free group Q generated by the α_j, β_j, and γ_i, such that $\widehat{\mu}(\alpha_j) = a_j$, $\widehat{\mu}(\beta_j) = b_j$, and $\widehat{\mu}(\gamma_j) = c_j$ for every $1 \leq j \leq g$ and $1 \leq i \leq n$. The exists a unique homomorphism $\mu : \Gamma \to T$ such that $\widehat{\mu}$ is equal to the composition of the canonical homomorphism $Q \to \Gamma$ with μ, if and only if $\widehat{\mu}$ is equal to 1 on the relation subgroup R^{orb}, that is, if and only if $[a_1, b_1] \ldots [a_g, b_g] = c_1 \ldots c_n$ and $c_i{}^{o_i} = 1$ for every $1 \leq i \leq n$. Because T is commutative, $[a_j, b_j] = 1$ for every $1 \leq j \leq g$, and therefore the first condition is equivalent to the equation $c_1 \ldots c_n = 1$.

(ii) An element $\gamma \in \Gamma$ does not act freely on $\widetilde{\mathcal{O}}^{\mathrm{orb}}$ if and only if $\gamma \neq 1$ and $\gamma \in \Gamma_{\widetilde{p}}$, the stabilizer subgroup of some $\widetilde{p} \in \widetilde{\mathcal{O}}^{\mathrm{orb}}$. Because $\widetilde{p}$ is a singular point for the action of Γ on $\widetilde{\mathcal{O}}^{\mathrm{orb}}$, its projection $\pi_{\mathcal{O}}(\widetilde{p})$ to $\mathcal{O}$ is one of the singular points s_i of $\mathcal{O}$. The

description of the orbifold chart near s_i implies that there exists an $\widetilde{s_i} \in \widetilde{\mathcal{O}}^{\mathrm{orb}}$ such that $\pi_{\mathcal{O}}(\widetilde{s_i}) = s_i$ and γ_i is the unique generator of $\Gamma_{\widetilde{s_i}} \simeq \mathbb{Z}/\mathbb{Z}o_i$ which acts near $\widetilde{s_i}$ as a rotation about the angle $2\pi/o_i$. Because the fibers of $\pi_{\mathcal{O}}$ are the Γ-orbits, there exists $c \in \Gamma$ such that $\widetilde{p} = c \cdot \widetilde{s_i}$, hence $\Gamma_{\widetilde{p}} = c\Gamma_{\widetilde{s_i}}c^{-1}$, and therefore $\gamma = c\gamma_i{}^k c^{-1}$ for some $k \in \mathbb{Z}$, where $k \notin \mathbb{Z}o_i$ because $\gamma \neq 1$. Because T is commutative, we have $\mu(\gamma) = \mu(\gamma_i)^k = c_i{}^k = 1$ if and only if $k \in \mathbb{Z}\,\mathrm{ord}(c_i)$, where $o_i \in \mathbb{Z}\,\mathrm{ord}(c_i)$. Therefore $\gamma \notin \ker\mu$ for every $\gamma \in \Gamma$ which does not act freely on $\widetilde{\mathcal{O}}^{\mathrm{orb}}$, if and only if $\mathrm{ord}(c_i) = o_i$ for every $1 \leq i \leq n$.

(iii) Because T is commutative and $c_i{}^{-1} = \prod_{h \neq i} c_h$, we have $c_i{}^{-m_i} = \prod_{h \neq i} c_h{}^{m_i} = 1$, which implies that $m_i \in \mathbb{Z}o_i$.

(iv) $D := \{r/o_1 + \mathbb{Z}, \ldots, r/o_n + \mathbb{Z} \in (\mathbb{R}/\mathbb{Z})^n \mid r \in \mathbb{R}\}$ is a one-dimensional subtorus of $(\mathbb{R}/\mathbb{Z})^n$, and therefore $T := (\mathbb{R}^n/\mathbb{Z}^n)/D$ is an $(n-1)$-dimensional torus. Let $c_i \in T$ be the image under the canonical projection $(\mathbb{R}/Z)^n \to (\mathbb{R}/\mathbb{Z})^n/D$ of the element of which the i-th coordinate is equal to $1/o_i + \mathbb{Z}$, and all the other coordinates are equal to zero. Then $c_1 \ldots c_n = 1$ and $\mathrm{ord}(c_i) = o_i$ for every i, if the orders o_i satisfy the condition in (iii). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.1 Lemma 4.4 and (i), (ii) in Lemma 4.5 imply that any list of ingredients as in [33, Definition 7.3.1] is the list of ingredients of a symplectic orbisurface $\mathcal{O}$ and a homomorphism μ as in the assumptions of Theorem 3.5. In this way Lemma 4.4 and (i), (ii) in Lemma 4.5 lead to a proof of [33, Proposition 7.3.6].

The necessary condition (iii) puts quite severe restrictions on the n-tuples of the orders o_i of the singular points s_i. The condition (iii) excludes the bad orbisurfaces, the cases that $g = 0$ and $n = 1$, or $g = 0$, $n = 2$ and $o_1 \neq o_2$. However, it also excludes many good orbisurfaces.[1]

Corollary 4.6 *Assume that the orbisurface $\mathcal{O}$ is isomorphic to the orbit space M/T of a symplectic action of a torus T on a compact and connected symplectic manifold, with symplectic principal orbits. Let o_i, $1 \leq i \leq n$, denote the orders of the singular points of $\mathcal{O}$. Let $c_i \in T$ be such that, for each $1 \leq i \leq n$, the order of c_i in T is equal to o_i, and the product of the c_i's is equal to 1. Let $T_\bullet$ be the subgroup of T generated by the c_i's. Then the orbisurface $\mathcal{O}$ is isomorphic to $R/T_\bullet$, where R is a compact Riemann surface and the finite group $T_\bullet$ acts on R by means of automorphisms.*

Proof Because $\mathrm{H}_1(\mathcal{O}, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$ is torsionfree, $U = \ker(h_1 \circ \varphi)$ is equal to the normal subgroup of Γ generated by the commutator subgroup C and Armstrong's subgroup A. Because T is commutative, $\mu = 1$ on C, hence $\mu(U)\mu(A) = \mu(A) = T_\bullet$, the subgroup of T generated by the T_x's $=$ the subgroup of T generated by the c_i's. The conclusion therefore follows from Theorem 4.2 with $R = \mathcal{I}'$. $\qquad\square$

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[1] Yael Karshon pointed out that in terms of the prime factor decompositions $o_i = \prod_k p_k{}^{\mu_{i,k}}$, the condition (iii) is equivalent to the condition that for every k, if M_k denotes the maximum of the $\mu_{i,k}$ over all i, then there are at least two distinct i and i' such that $\mu_{i,k} = M_k = \mu_{i',k}$.

5 Classification in the orbisurface case

Theorems 3.4 and 3.5 do not give information on when two symplectic T-manifolds are isomorphic. We begin by explaining that the isomorphisms are induced by orbifold symplectomorphisms of the orbit spaces.

Let $\Phi : (M, \sigma, T) \to (M', \sigma', T)$ be an isomorphism of symplectic T-spaces, where $\dim M - \dim T = 2$ and the T-orbits are symplectic. Let $\mathcal{O} = M/T$ and $\mathcal{O}' = M'/T$ denote the corresponding orbit spaces, which are orbifolds. The T-equivariant mapping $\Phi : M \to M'$ induces the mapping $\varphi : \mathcal{O} \to \mathcal{O}' : T \cdot x \mapsto \Phi(T \cdot x) = T \cdot \Phi(x)$, which is an orbifold symplectomorphism from $(\mathcal{O}, \sigma^{\mathcal{O}})$ onto $(\mathcal{O}', \sigma^{\mathcal{O}'})$.

Conversely, let φ be any an orbifold symplectomorphism from $(\mathcal{O}, \sigma^{\mathcal{O}})$ onto $(\mathcal{O}', \sigma^{\mathcal{O}'})$. For any regular point p_0 of $\mathcal{O}$, let $\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}^{\mathrm{orb}}_{p_0}$ denote the orbifold universal covering of $\mathcal{O}$ defined as the space of orbifold homotopy classes of orbifold curves starting at p_0, where the endpoints of the curves remain fixed. Then the mapping which assigns to an orbifold curve δ in $\mathcal{O}$ its image $\varphi \circ \delta$ in $\mathcal{O}'$ induces a diffeomorphism $\widetilde{\varphi}$ from $\widetilde{\mathcal{O}}$ onto $\widetilde{\mathcal{O}}' = \widetilde{\mathcal{O}}'^{\mathrm{orb}}_{p_0'}$, where $p_0' = \varphi(p_0)$. The mapping $\widetilde{\varphi}$ intertwines the action of $\Gamma := \pi_1^{\mathrm{orb}}(\mathcal{O}, p_0)$ on $\widetilde{\mathcal{O}}$ with the action of $\Gamma' := \pi_1^{\mathrm{orb}}(\mathcal{O}', p_0')$ on $\widetilde{\mathcal{O}}'$ via the isomorphism $\varphi_* : \Gamma \to \Gamma'$, in the sense that $\widetilde{\varphi}(c \cdot \widetilde{p}) = \varphi_*(c) \cdot \widetilde{\varphi}(\widetilde{p})$ for every $c \in \Gamma$ and $\widetilde{p} \in \widetilde{\mathcal{O}}$. If $\mu : \Gamma \to T$ and $\mu' : \Gamma' \to T$ are the respective monodromy homomorphisms, then $\mu = \mu' \circ \varphi_*$. Finally $\pi_{\mathcal{O}}{}^*(\sigma^{\mathcal{O}}) = \widetilde{\varphi}^*(\pi_{\mathcal{O}'}{}^*(\sigma^{\mathcal{O}'}))$.

Therefore, if $(M_{\mathrm{model}}, \sigma_{\mathrm{model}}, T)$ and $(M'_{\mathrm{model}}, \sigma'_{\mathrm{model}}, T)$ are the respective models of (M, σ, T) and (M', σ', T) in Theorem 3.4, where $M_{\mathrm{model}} = T \times_\Gamma \widetilde{\mathcal{O}}$ and $M'_{\mathrm{model}} = T \times_{\Gamma'} \widetilde{\mathcal{O}}'$, then the mapping $\mathrm{Id} \times \widetilde{\varphi} : T \times \widetilde{\mathcal{O}} \to T \times \widetilde{\mathcal{O}}'$ induces an isomorphism Φ_{model} from $(M_{\mathrm{model}}, \sigma_{\mathrm{model}}, T)$ onto $(M'_{\mathrm{model}}, \sigma'_{\mathrm{model}}, T)$. Furthermore, Φ_{model} is equal to Φ after identification of the symplectic T-spaces with their models, in the sense that $\Phi_{\mathrm{model}} \circ \iota_{\mathrm{model}} = \iota'_{\mathrm{model}} \circ \Phi$. It follows that we may assume that $\mathcal{O}' = \mathcal{O}$, when we have to investigate the effect of the orbifold automorphisms φ of $\mathcal{O}$ on the data of the model.

From now on we assume that $\mathcal{O}$ is an orbisurface as discussed in Sect. 4.1. The orbisurface diffeomorphism φ of $\mathcal{O}$ permutes the singular points, while preserving the orders. That is, there is a unique permutation α of $\{1, \ldots, n\}$ such that $\varphi(s_i) = s_{\alpha(i)}$ and $o_{\alpha(i)} = o_i$ for every $1 \leq i \leq n$. Because every such permutation α is realized by an orbisurface φ which is equal to the identity on a neighborhood of the boundary of the polytope P in Sect. 4.1, we restrict the discussion in the sequel to orbifold automorphisms φ of $\mathcal{O}$ which leave each of the singular points fixed.

Let $H_1^{\mathrm{orb}}(\mathcal{O}, \mathbb{Z})$ denote the abelianization of $\Gamma = \pi_1^{\mathrm{orb}}(\mathcal{O}, p_0)$, with the canonical surjective homomorphism $h_1 : \Gamma \to H_1^{\mathrm{orb}}(\mathcal{O}, \mathbb{Z})$. The isomorphism φ_* from $\pi_1^{\mathrm{orb}}(\mathcal{O}, p_0)$ onto $\pi_1^{\mathrm{orb}}(\mathcal{O}, \varphi(p_0))$ induces an automorphism of $H_1^{\mathrm{orb}}(\mathcal{O}, \mathbb{Z})$, which we also denote by φ_*, and which does not depend on the choice of the base point p_0. The torsion subgroup $F = H_1^{\mathrm{orb}}(\mathcal{O}, \mathbb{Z})^{\mathrm{tor}}$ is the finite subgroup generated by the $[\gamma_i] = h_1(\gamma_i)$, subject to the relations $o_i[\gamma_i] = 0$ for every i and $\sum_i [\gamma_i] = 0$. Because $\varphi(s_i) = s_i$, we have $\varphi_*([\gamma_i]) = [\gamma_i]$ for every i, and therefore φ_* is the identity on F.

The topological intersection number of one-dimensional cycles in the oriented surface $\mathcal{O}$ defines a nondegenerate antisymmetric $\mathbb{Z}$-valued bilinear form on $H_1(\mathcal{O}, \mathbb{Z}) \simeq H_1^{\mathrm{orb}}(\mathcal{O}, \mathbb{Z})/F \simeq \mathbb{Z}^{2g}$. Actually, the $\mathbb{Z}$-basis $(\alpha_j) = [\alpha_j] + F$,

$(\beta_j) = [\beta_j] + F$ of $H_1(\mathcal{O}, \mathbb{Z})$ is a symplectic basis with respect to the intersection form, in the sense that $(\alpha_j) \cdot (\alpha_k) = 0$, $(\beta_j) \cdot (\beta_k) = 0$, and $(\alpha_j) \cdot (\beta_k) = \delta_{jk}$. The automorphism φ_*/F of $H_1(\mathcal{O}, \mathbb{Z})$ preserves the intersection form, and therefore is given on any symplectic $\mathbb{Z}$-basis by a symplectic matrix

$$\varphi_*/F \simeq \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z}), \tag{3}$$

in the sense that the $g \times g$-matrices P, Q, R, S have integral entries and satisfy the equations $PQ^t - QP^t = 0$, $RS^t - SR^t = 0$, and $PS^t - QR^t = \mathrm{I}$. It follows that

$$\varphi_*([\alpha_j]) = \sum_{k=1}^{g} (p_j^k[\alpha_k] + q_j^k[\beta_k]) + \sum_{i=1}^{n} u_j^i[\gamma_i],$$

$$\varphi_*([\beta_j]) = \sum_{k=1}^{g} (r_j^k[\alpha_k] + s_j^k[\beta_k]) + \sum_{i=1}^{n} v_j^i[\gamma_i] \tag{4}$$

for suitable $u_j^i, v_j^i \in \mathbb{Z}/\mathbb{Z}o_i$.

Dehn [7] proved that the mapping class group of the topological surface $\mathcal{O}$, the group of isotopy classes of homeomorphisms of $\mathcal{O}$, is generated by transformations which nowadays are called *Dehn twists*. These are diffeomorphisms equal to the identity outside a small annulus around a loop α without self-intersections, and act on $H_1(\mathcal{O}, \mathbb{Z})$ by sending (β) to $(\beta) + ((\alpha) \cdot (\beta))(\alpha)$. As the latter transformations generate $\mathrm{Sp}(2g, \mathbb{Z})$ (see Magnus, Karass and Solitar [23, pp. 178, 355, 356]), every automorphism of $H_1(\mathcal{O}, \mathbb{Z})$ which preserves the intersection form is equal to φ_*/F, for an orbifold automorphism φ of $\mathcal{O}$ which leaves each singular point fixed.

It is shown in Pelayo [33, Sect. 6.4] that for each $1 \leq i \leq n$ there is an orbifold automorphism φ of $\mathcal{O}$ which leaves each singular point fixed, preserves all $[\alpha_j]$'s and $[\beta_j]$'s except one of these, to which it adds $[\gamma_i]$. It follows that every automorphism of $H_1^{\mathrm{orb}}(\mathcal{O}, \mathbb{Z})$ which is equal to the identity on the torsion subgroup F and preserves the intersection form is of the form φ_* for an orbifold automorphism φ of $\mathcal{O}$ which leaves each singular point fixed.

Finally the proof of Moser [26] can be used to show that if σ and σ' are two orbifold area forms on $\mathcal{O}$, then there exists an orbifold automorphism φ of $\mathcal{O}$ such that $\sigma' = \varphi^*(\sigma)$ if and only if $\int_{\mathcal{O}} \sigma = \int_{\mathcal{O}} \sigma'$. Moreover, if this is the case, then φ can be chosen to be orbifold isotopic to the identity, which implies that φ leaves each singular point of $\mathcal{O}$ fixed and acts as the identity on $H_1^{\mathrm{orb}}(\mathcal{O}, \mathbb{Z})$.

The unique homomorphisms $\mu_\mathrm{h}, \mu_\mathrm{h}' : H_1^{\mathrm{orb}}(\mathcal{O}, \mathbb{Z}) \to T$ such that $\mu = \mu_\mathrm{h} \circ h_1$ and $\mu' = \mu_\mathrm{h}' \circ h_1$ do not depend on the choice of p_0, and we have $\mu_\mathrm{h} = \mu_\mathrm{h}' \circ \varphi_*$. If we write $a_j = \mu(\alpha_j)$, $b_j = \mu(\beta_j)$, $c_i = \mu(\gamma_i)$, $a_j' = \mu'(\alpha_j)$, $b_j' = \mu'(\beta_j)$, and $c_i' = \mu'(\gamma_i)$, then it follows from $\mu = \mu' \circ \varphi_*$ and (4) that

$$a_j = \prod_{k=1}^{g} (a_k')^{p_j^k} (b_k')^{q_j^k} \prod_{i=1}^{n} (c_i')^{u_j^i},$$

$$b_j = \prod_{k=1}^{g} (a_k')^{r_j^k} (b_k')^{s_j^k} \prod_{i=1}^{n} (c_i')^{v_j^i}. \tag{5}$$

This leads to the following uniqueness theorem, which corresponds to [33, Proposition 7.2.4].

Theorem 5.1 *Let* $(M_{\text{model}}, \sigma_{\text{model}}, T)$ *and* $(M'_{\text{model}}, \sigma'_{\text{model}}, T)$ *be models constructed from the respective ingredients* g, n, o_i, λ, σ^{t}, a_j, b_j, c_i, *and* g', n', o'_i, λ', $(\sigma')^{\mathrm{t}}$, a'_j, b'_j, c'_i. *Then these two models are isomorphic if and only if* $g = g'$, $n = n'$, $\lambda = \lambda'$, $\sigma^{\mathrm{t}} = (\sigma')^{\mathrm{t}}$, *there exists a permutation* α *of* $\{1, \ldots, n\}$ *such that* $c_i = c'_{\alpha(i)}$ *and* $o_{\alpha(i)} = o_i$ *for every* $1 \leq i \leq n$, *and finally there exist an element of* $\mathrm{Sp}(2g, \mathbb{Z})$ *as in* (3) *and* $u^i_j, v^i_j \in \mathbb{Z}/\mathbb{Z}o_i$, *such that* (5) *holds for every* $1 \leq j \leq g$.

This completes the classification of compact connected symplectic T-spaces with symplectic principal orbits and for which the orbit space is 2-dimensional.

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