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### Author

Finkelstein, Jerome.

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**University of California**  
**Ernest O. Lawrence**  
**Radiation Laboratory**

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**UNPHYSICAL UNITARITY OF PARTIAL-WAVE AMPLITUDES**

**Berkeley, California**

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Jerome Finkelstein

April 27, 1965

## UNPHYSICAL UNITARITY OF PARTIAL-WAVE AMPLITUDES\*

Jerome Finkelstein

Lawrence Radiation Laboratory  
University of California  
Berkeley, California

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The unitarity relation for a partial-wave scattering amplitude  $(A_l)_{ab}$  is

$$\text{Im}[A_l(s)]_{ab} = \sum_n \frac{q_n}{s^{1/2}} [A_l^*(s)]_{an} [A_l(s)]_{nb} \theta(s-s_n), \quad (1)$$

where  $a, b,$  and  $n$  are channel indices, and

$$A(s, t) = \sum_l (2l+1) A_l(s) P_l(z).$$

It is often convenient to define

$$(A_l)_{ab} = q_a^l q_b^l (B_l)_{ab}$$

and write the unitarity relation

$$\text{Im}[B_l(s)]_{ab} = \sum_n \frac{q_n^{2l+1}}{s^{1/2}} [B_l^*(s)]_{an} [B_l(s)]_{nb} \theta(s-s_n). \quad (1')$$

Unless  $a$  and  $b$  both denote the lowest channel, it may be necessary to use the unitarity relation below the physical threshold. In this note I wish to point out that, below threshold Eqs. (1) and (1') are incompatible for odd  $l$ , and that (1') is implied by unphysical unitarity of the full amplitude. In practice the amplitude  $B_l$  is used;<sup>1</sup> however, so far as I am aware, nobody has justified its use from general S-matrix principles.

Equations (1) and (1') imply, respectively,

$$\text{Im}[A_l^{-1}(s)] = -\rho(s) \quad (2)$$

and

$$\text{Im}[B_l^{-1}(s)] = -\beta(s), \quad (2')$$

where  $\rho$  and  $\beta$  are both diagonal matrices. Assume they are both correct, and take  $a \neq b$ , with  $s$  between the thresholds for  $a$  and  $b$ . Then from (2), we have

$$0 = \text{Im}(A_l^{-1})_{ab} = \text{Im} q_a^l q_b^l (B_l^{-1})_{ab} = q_a^l q_b^l \text{Re}(B_l^{-1})_{ab}$$

for odd  $l$ . But from (2'),  $\text{Im}(B_l^{-1})_{ab} = 0$ , and so  $(B_l^{-1})_{ab} = 0$ . Hence, for odd  $l$ , (1) and (1') are compatible only in the trivial case in which there is no inelastic scattering.

I wish to derive (1') when  $s$  is between the two thresholds, say  $s_a < s < s_b$ ; the other cases can be done similarly. Neglecting spin, the unphysical unitarity relation is<sup>2</sup>

$$\text{Im} A_{ab}(s, t) = \frac{1}{4\pi} \sum_n \frac{q_n}{s^{1/2}} \int d\Omega_n A_{an}^*(s, z_{an}) A_{nb}(s, z_{nb}) \theta(s - s_n), \quad (3)$$

where  $s$  is above the lowest threshold, and  $t$  is real. The left side of Eq. (3) is

$$\frac{i}{2t} \sum_l (2l+1) [q_a^l q_b^l (B_l(s))_{ab} P_l(z) - (q_a^l q_b^l)^* (B_l^*(s))_{ab} P_l^*(z)]. \quad (4)$$

Since  $q_a$  is real and  $q_b$  and  $z$  are imaginary,  $(q_a^l q_b^l)^* = (-1)^l q_a^l q_b^l$  and  $P_l^*(z) = (-1)^l P_l(z)$ , and so (4) is equal to

$$\sum_l (2l + 1) q_a^l q_b^l P_l(z) \text{Im} [B_l(s)]_{ab}. \quad (5)$$

The right side of Eq. (3) is

$$\frac{1}{4\pi} \sum_n \frac{q_n}{s^{1/2}} \theta(s - s_n) \sum_{l', l''} (2l' + 1) (2l'' + 1) q_a^{l'} q_n^{l'+l''} q_b^{l''} (B_{l'}^*)_{an} (B_{l''})_{nb} \\ \times \int d\Omega_n P_{l'}^*(z_{an}) P_{l''}(z_{nb}). \quad (6)$$

Since channels a and n are physical,  $z_{an}$  is real, and the integral in (6) is

$$\int_0^{2\pi} d\phi_{an} \int_{-1}^1 dz_{an} P_{l'}(z_{an}) \\ \times P_{l''} \left[ z_{ab} z_{an} - (1 - z_{ab}^2)^{\frac{1}{2}} (1 - z_{an}^2)^{\frac{1}{2}} \cos \phi_{an} \right]. \quad (7)$$

Now (7) is clearly an analytic function of  $z_{ab}$ ; when  $z_{ab}$  is real (when channel b is physical), it equals  $4\pi (2l'+1)^{-1} \delta_{l', l''} P_{l'}(z_{ab})$ , and therefore it equals that for all values of  $z_{ab}$ . Then (6) is equal to

$$\sum_n \frac{q_n}{s^{1/2}} \theta(s - s_n) \sum_{l'} (2l' + 1) q_n^{2l'} q_a^{l'} q_b^{l'} P_{l'}(z_{ab}) (B_{l'}^*)_{an} (B_{l'})_{nb} \quad (8)$$

Equating (8) and (5), and assuming sufficient analyticity in  $t$  to separate the partial waves, we arrive at (1').

I would like to thank Professor Geoffrey F. Chew for suggesting this problem.

FOOTNOTES AND REFERENCES

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1. For example, W. R. Frazer and J. R. Fulco, Phys. Rev. Letters 2, 365 (1959); F. Zachariasen and C. Zemach, Phys. Rev. 128, 849 (1962); J. R. Fulco, G. L. Shaw, and D. Y. Wong, Phys. Rev. 137, B1242 (1965).
2. D. I. Olive, Phys. Rev. 135, B745 (1964).



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