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The unitarity relation for a partial-wave scattering amplitude $(A_\ell)_{ab}$ is

$$\mathrm{Im}[A_\ell(s)]_{ab} = \sum_n \frac{q_n}{s^{1/2}} [A_\ell^*(s)]_{an} [A_\ell(s)]_{nb} \theta(s-s_n) , \qquad (1)$$

where a, b, and n are channel indices, and

$$A(s,t) = \sum_\ell (2\ell+1) A_\ell(s) P_\ell(z) .$$

It is often convenient to define

$$(A_\ell)_{ab} = q_a^{\ \ell} q_b^{\ \ell} (B_\ell)_{ab}$$

and write the unitarity relation

$$\mathrm{Im}[B_\ell(s)]_{ab} = \sum_n \frac{q_n^{2\ell+1}}{s^{1/2}} [B_\ell^*(s)]_{an} [B_\ell(s)]_{nb} \theta(s-s_n) . \qquad (1')$$

Unless a and b both denote the lowest channel, it may be necessary to use the unitarity relation below the physical threshold. In this note I wish to point out that, below threshold Eqs. (1) and (1') are incompatible for odd ℓ, and that (1') is implied by unphysical unitarity of the full amplitude. In practice the amplitude B_ℓ is used;[1] however, so far as I am aware, nobody has justified its use from general S-matrix principles.

Equations (1) and (1') imply, respectively,

$$\mathrm{Im}[A_\ell^{-1}(s)] = -\rho(s) \tag{2}$$

and

$$\mathrm{Im}[B_\ell^{-1}(s)] = -\not\rho(s)\,, \tag{2'}$$

where ρ and $\not\rho$ are both diagonal matrices. Assume they are both correct, and take $a \neq b$, with s between the thresholds for a and b. Then from (2), we have

$$0 = \mathrm{Im}(A_\ell^{-1})_{ab} = \mathrm{Im}\, q_a^{\ell}\, q_b^{\ell}\, (B_\ell^{-1})_{ab} = q_a^{\ell}\, q_b^{\ell}\, \mathrm{Re}(B_\ell^{-1})_{ab}$$

for odd ℓ. But from (2'), $\mathrm{Im}(B_\ell^{-1})_{ab} = 0$, and so $(B_\ell^{-1})_{ab} \equiv 0$. Hence, for odd ℓ, (1) and (1') are compatible only in the trivial case in which there is no inelastic scattering.

I wish to derive (1') when s is between the two thresholds, say $s_a < s < s_b$; the other cases can be done similarly. Neglecting spin, the unphysical unitarity relation is[2]

$$\mathrm{Im}\, A_{ab}(s,t) = \frac{1}{4\pi} \sum_n \frac{q_n}{s^{1/2}} \int d\Omega_n\, A^*_{an}(s, z_{an})\, A_{nb}(s, z_{nb})\, \theta(s - s_n)\,, \tag{3}$$

where s is above the lowest threshold, and t is real. The left side of Eq. (3) is

$$\frac{1}{2i} \sum_\ell (2\ell+1)\, \left[\, q_a^{\ell}\, q_b^{\ell}\, (B_\ell(s))_{ab}\, P_\ell(z) - (q_a^{\ell}\, q_b^{\ell})^*\, (B_\ell^*(s))_{ab}\, P_\ell^*(z) \right]. \tag{4}$$

Since q_a is real and q_b and z are imaginary, $(q_a^{\ell}\, q_b^{\ell})^* = (-1)^{\ell} q_a^{\ell}\, q_b^{\ell}$ and $P_\ell^*(z) = (-1)^{\ell}\, P_\ell(z)$, and so (4) is equal to

$$\sum_{\ell}(2\ell+1)\, q_a^{\ell}\, q_b^{\ell}\, P_{\ell}(z)\, \mathrm{Im}\,[B_{\ell}(s)]_{ab} \,. \tag{5}$$

The right side of Eq. (3) is

$$\frac{1}{4\pi}\sum_{n}\frac{q_n}{s^{1/2}}\,\theta(s-s_n)\sum_{\ell'\ell''}(2\ell'+1)(2\ell''+1)q_a^{\ell'}\, q_n^{\ell'+\ell''}\, q_b^{\ell''}(B^*_{\ell'})_{an}(B_{\ell''})_{nb}$$

$$\times\int d\Omega_n\, P^*_{\ell'}(z_{an})\, P_{\ell''}(z_{nb}) \,. \tag{6}$$

Since channels a and n are physical, z_{an} is real, and the integral in (6) is

$$\int_0^{2\pi} d\phi_{an}\int_{-1}^{1} dz_{an}\, P_{\ell'}(z_{an})$$

$$\times P_{\ell''}\left[z_{ab}z_{an} - (1-z_{ab}^2)^{\frac{1}{2}}(1-z_{an}^2)^{\frac{1}{2}}\cos\phi_{an}\right]. \tag{7}$$

Now (7) is clearly an analytic function of z_{ab}; when z_{ab} is real (when channel b is physical), it equals $4\pi(2\ell'+1)^{-1}\,\delta_{\ell'\ell''}\,P_{\ell'}(z_{ab})$, and therefore it equals that for all values of z_{ab}. Then (6) is equal to

$$\sum_{n}\frac{q_n}{(s)^{1/2}}\,\theta(s-s_n)\sum_{\ell'}(2\ell'+1)\,q_n^{2\ell'}\,q_a^{\ell'}\,q_b^{\ell'}\,P_{\ell'}(z_{ab})\cdot(B^*_{\ell'})_{an}\quad(8)\quad(B_{\ell'})_{nb}$$

Equating (8) and (5), and assuming sufficient analyticity in t to separate the partial waves, we arrive at (1').

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FOOTNOTES AND REFERENCES

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1. For example, W. R. Frazer and J. R. Fulco, Phys. Rev. Letters 2, 365 (1959); F. Zachariasen and C. Zemach, Phys. Rev. 128, 849 (1962); J. R. Fulco, G. L. Shaw, and D. Y. Wong, Phys. Rev. 137, B1242 (1965).
2. D. I. Olive, Phys. Rev. 135, B745 (1964).