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Abstract

Power transfer distribution factors depend on the operating point and topology of an electric power system. However, it is known empirically that, for a fixed topology, the power transfer distribution factors are relatively insensitive to the operating point. We demonstrate this result theoretically for lossless systems in two ways: as an exact result for a system having "series-parallel" topology and a single point of injection and a single point of withdrawal and also as an approximate result for systems of arbitrary topology but having reactive compensation sufficient to keep voltages constant at all busses. To the extent that losses are small, the results apply also for more realistic systems with losses. We also analyze two other distribution factors that more closely relate to thermal and steady-state stability constraints.

1 Introduction

An (incremental) power transfer distribution factor (PTDF) is the relative change in power flow on a particular line due to a change in injection and corresponding withdrawal at a pair of busses. PTDFs depend on the topology of the electric power system, the behavior of controllable transmission system elements as their limits are approached, and on the operating point [1]. That is, PTDFs change when an outage of a line occurs, if a controllable element reaches its control limits, and also as the pattern of injections and withdrawals change the loadings on the lines in the system.

Nevertheless, it is known empirically that, given a fixed topology and ignoring controllable device limits, the PTDFs are relatively insensitive to the levels of injections and withdrawals. See, for example, [2][3, §3.9] for empirical studies of the variation of PTDFs for certain systems.

In this paper we develop theoretical insight into this empirical observation in two forms:

1. showing that the PTDFs are exactly constant in a lossless system having "series-parallel" topology and a single point of injection and single point of withdrawal (Theorem 1) and

2. showing that the PTDFs are approximately constant in a lossless system of arbitrary topology but having reactive compensation sufficient to keep voltage magnitudes constant at all busses (Corollary 4.)

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These two results also apply approximately to slightly lossy systems. When the hypotheses of these results do not hold, the PTDFs can be expected to vary significantly as loadings change.

The second result is due fundamentally to the fact that the sin function has a Taylor expansion with zero quadratic term, so that linearization of the sin function about zero angle results in an error that is cubic and higher order in the angle, not quadratic. Consequently, the variation of the PTDFs with net power injection is quadratic and higher order rather than linear. However, the first result shows that in some circumstances the power flow equations yield PTDFs that are exactly constant independent of injections.

The significance of these results is that, in the context of flowgate right schemes for transmission rights [4, 5], capacity to flow power on a line or a group of lines is sold or leased to users of the transmission system. A transmission system user wishing to inject power at one point and withdraw it at another may want to purchase enough capacity on the line so as to hedge its congestion costs. If the PTDFs for a particular line vary significantly with the flows on the other lines then it is more difficult to predict the amount of capacity needed on each binding "flowgate." Either the risk due to variation of the PTDFs must be borne amongst the sellers and buyers of the transmission capability or conservative capacity limits must be used to compensate for variation of the PTDFs. If the PTDFs are relatively constant, however, then presumably the appropriate power flow capacity on each flowgate could be reserved to hedge the transmission congestion costs.

However, a further issue is that relative constancy of the PTDFs may not be the best measure of the lack of risk of unhedged transmission requirements. This is because, for example, in a thermally limited line the fundamental limiting factor is not literally the power flow down the line but rather the resistive losses in the line, which are proportional to the square of the magnitude of the current. Similarly, in a steady-state stability limited line, the angle across the line (or between a generation center and a demand center) is the limiting factor. For this reason, we investigate two other distribution factors:

- power to current magnitude distribution factors (PIDFs) that measure the (incremental) effect of a change in power injection on the magnitude of the current in a line, and
- power to angle distribution factors (PADFs) that measure the (incremental) effect of a change in power injection on the angle across a line.

These distribution factors relate more closely to thermal and steady-state stability constraints, respectively, than do PTDFs, and we will see that they have similar properties to PTDFs. Under the DC power flow approximation, PTDFs, PIDFs, and PADFs are all proportional to each other. However, in a nonlinear setting, the conditions for the PTDFs to be relatively constant as loadings vary are more stringent than those for PTDFs and PADFs, pointing to technical requirements that must be satisfied for workable flowgate rights on thermally limited transmission lines.

The results are proven for lossless systems with fixed topology. The results for systems of general topology are also dependent on the assumption of there being voltage support at all busses sufficient to maintain constant voltage. In fact, the context of thermal constraints implies that there are losses on the lines and, moreover, in typical systems only relatively few of the busses may have controlled voltages. The applicability of these results to thermally limited lines is therefore limited to lines with relatively small resistance to reactance ratios but which are, nevertheless, thermally limited. The results will not hold where voltage constraints are binding, since by definition there is inadequate reactive support to maintain constant voltage. In particular, the results are unlikely

to hold on lower voltage parts of the transmission system but may be applicable to higher voltage lines having sufficient reactive support and low ratios of resistance to reactance.

The structure of the paper is as follows. Section 2 presents a brief literature survey. In section 3, we discuss the assumptions, the power flow formulation, and formal definition of PTDF. We consider PTDFs in series-parallel networks in section 4. We then consider PTDFs in general network topologies in section 5. In sections 6 and 7, we define and characterize PIDFs and PADFs, respectively. We conclude in section 8.

2 Literature survey

Several authors discuss PTDFs in the context of approximating power flows. For example, Baughman and Schweppe use distribution factors to approximate flows as a function of injections and after a change in the topology of the network [6]. Sauer formulates PTDFs for linear load flows in [7]. Wood and Wollenberg describe the calculation of PTDFs using the DC power flow approxation in [8, Appendix 11A] and also discuss the calculation of PTDFs for outage conditions [8, §11.3.2]. The evaluation of PTDFs at an operating point from the Jacobian of the power flow equations is described in [8, §13.3].

Grijalva analyzes in detail the variation of PTDFs in a three bus, three line example system with voltages maintained constant and also discusses how the PTDFs vary with loading. Grijalva shows that if voltages are maintained constant at all busses then, as loading increases from zero injection conditions, the PTDFs that were largest at zero injection tend to decrease while the PTDFs that were smallest at zero injection tend to increase [3, §3.9]. Generalizing the three bus, three line system, Grijalva discusses PTDFs from a given point of injection and a given point of withdrawal to each of the lines in a cutset of the power system, observing that the sum of the PTDFs across all lines in a cutset must be equal to one. Therefore, increases in PTDFs to some lines must be accompanied by decreases in the PTDFs to other lines. Grijalva observes that PTDFs begin to change significantly as "static transfer capability limits" are approached [3, §3.3 and Figure 3.3].

Grijalva also considers higher order terms in a Taylor expansion of the PTDFs using a rectangular representation for the voltage phasor and evaluates a quadratic approximation to the solution of the power flow [3, §3.9]. We take an analogous approach in section 5; however, we use a polar representation of the voltage phasor and consider a Taylor expansion about the zero injection operating point, which allows for convenient evaluation of the linear terms in the Taylor expansion.

Liu and Gross conduct an empirical study of the variation of PTDFs with injections and with other changes [2]. They show that for the system considered the PTDFs typically change by a relatively small amount as the levels of injections and withdrawals change.

Sauer *et al.* introduce and analyze various distribution factors in [9], including two that are closely related to the PIDFs and PADFs that we consider. In particular, they define distribution factors of current injections to current flows (current transfer distribution factors or CTDFs), noting that the CTDFs are customarily converted to PTDFs. The PIDFs that we define are similar in flavor to the CTDFs except that our interest is in the effect of power injections on current flows.

Sauer *et al.* also consider angle distribution factors under outage conditions, generalizing the PADFs that we consider to the line outage case [9, §5]. The analysis that we present concerning the relative constancy of distribution factors could be applied to the outage distribution factors in [9] and also to the various other distribution factors defined there.

Finally, Fradi *et al.* consider non-linear allocation of quantities to transactions [10]. They emphasize the variation of PTDFs. In contrast, we consider the conditions under which the PTDFs are relatively constant.

3 Assumptions and formulation

The material in this section is based on [8, 11, 12] and is mostly standard. We develop it in detail so that we can precisely state the results to follow. We consider the single phase equivalent of a power system having $n+1$ busses. Bus number 0 is the reference bus and will be assumed to have reference angle of zero, while the other busses are labeled 1 through n. We use the symbols j,k,ℓ,m,s,t,u to index the busses. (We will use the symbol $\sqrt{-1}$ for the square root of minus one.)

Unless otherwise specified, we assume that there is at most a single line between any pair of busses, in which case we can refer to *the* line between, say, busses ℓ and m. When considering "series-parallel" networks in section 4, we will occasionaly have to consider the case of more than one line between a pair of busses. We will develop notation in section 4 to distinguish such lines.

For the analysis in section 4 we will not need to make any assumptions about voltage magnitude, whereas for the analysis in sections 5–7 we will have to assume that voltage magnitudes are constant (so that each bus, besides the reference bus, is a *PV* bus [11, §10.2].) For both analyses, we will have to consider the net power injections at each bus and the voltage angles at each bus explicitly. Consequently, we will explicitly represent net power injections and angles as arguments in the functions that we define to formulate the power flow equations. The voltage magnitudes will not be represented explicitly as arguments, but will be considered parameters.

Let the (ℓ,m) entry of the bus admittance matrix [11] be:

$$G_{\ell m}+\sqrt{-1}B_{\ell m},$$

where we note that the conductances $G_{\ell m}$ and the susceptances $B_{\ell m}$ satisfy:

- $G_{\ell m}\leq 0$ and $B_{\ell m}>0$ for $\ell\neq m$ and
- $G_{\ell\ell}\geq 0$ and the sign of $B_{\ell\ell}$ is indeterminate but typically less than zero unless there is significant shunt capacitance on the line.

Let the net power injected by generation and demand at node ℓ be P_ℓ, so that for generator busses, $P_\ell>0$. Let the voltage magnitude at bus ℓ be $|v_\ell|$ and its angle be Θ_ℓ. Collect the vector of power injections at all the busses, except the reference bus, together into a vector $P\in\mathbb{R}^n$ and collect the vector of angles at all busses, except the reference bus, together into a vector $\Theta\in[-\pi,\pi]^n$.

For every bus ℓ (including the reference bus) define functions $p_\ell:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ by:

$$\forall P\in\mathbb{R}^n,\forall\Theta\in\mathbb{R}^n,p_\ell(P,\Theta)=\sum_{m\in\mathbb{K}(\ell)\cup\{\ell\}}|v_\ell||v_m|[G_{\ell m}\cos(\Theta_\ell-\Theta_m)+B_{\ell m}\sin(\Theta_\ell-\Theta_m)]-P_\ell,$$

where $\mathbb{K}(\ell)$ is the set of busses directly connected to bus ℓ by a line.

Collect the functions p_ℓ for each bus ℓ, except the reference bus, into a vector function $p:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$. Then, given a vector of net injections P, solving the power flow is equivalent to solving for Θ in:

$$p(P,\Theta)=\mathbf{0}, \tag{1}$$

where:

- **0** is the vector of all zeros and
- the injection at the reference bus can be calculated once the vector of angles Θ is known.

Consider a vector of net injections $P^\star$ and a corresponding solution $\Theta^\star$ of the power flow equations (1). We consider the properties of the solution as the vector of net injections is varied about $P^\star$.

We first note that p is infinitely partially differentiable with respect to Θ. Suppose that $\frac{\partial p}{\partial \Theta}(P^\star, \Theta^\star)$ is non-singular. Then by the implicit function theorem [13, §4.4] there exists an infinitely partially differentiable function $\theta : \mathbb{R}^n \to \mathbb{R}^n$ such that in some neighborhood $\mathcal{N}$ of $P = P^\star$, the power flow equations (1) has a solution satisfying:

$$\forall P \in \mathcal{N}, p(P, \theta(P)) = \mathbf{0}.$$

That is, as is well-known, the power flow equations have a well-behaved solution in this neighborhood.

Consider the flow along a line joining busses ℓ and m. Neglecting shunt conductance in a line, we can evaluate the power flowing from bus ℓ into the line joining bus ℓ and m by the function $p_{\ell m} : \mathbb{R}^n \to \mathbb{R}$ defined by:

$$\forall \Theta \in \mathbb{R}^n, p_{\ell m}(\Theta) = |v_\ell||v_m|[G_{\ell m}\cos(\Theta_\ell - \Theta_m) + B_{\ell m}\sin(\Theta_\ell - \Theta_m)] - |v_\ell|^2 G_{\ell m}.$$

If there are losses in the system, so that $G_{\ell m} < 0$, then the flow will be different at different points along the line. As a representative flow for the line joining ℓ to m, we take the average of the flows at the two ends of the line. That is, define $\tilde{p}_{\ell m} : \mathbb{R}^n \to \mathbb{R}$ by:

$$\begin{aligned} \forall \Theta \in \mathbb{R}^n, \tilde{p}_{\ell m}(\Theta) &= \tfrac{1}{2}(p_{\ell m}(\Theta) - p_{m\ell}(\Theta)), \\ &= |v_\ell||v_m| B_{\ell m}\sin(\Theta_\ell - \Theta_m) - \tfrac{1}{2}(|v_\ell|^2 - |v_m|^2) G_{\ell m}. \end{aligned}$$

(We would obtain essentially the same results in the theorems below if we considered the sending end flow or the receiving end flow.) To relate the representative flow to the net injections, we define the function $\hat{p}_{\ell m} : \mathbb{R}^n \to \mathbb{R}$ by:

$$\forall P \in \mathcal{N}, \hat{p}_{\ell m}(P) = \tilde{p}_{\ell m}(\theta(P)).$$

Consider a bus k and a line joining busses ℓ and m. We consider the effect on the representative flow along the line joining ℓ and m of a change in the net injection at bus k from the level $P_k^\star$ (and assuming a corresponding change in the net withdrawal at the reference bus to maintain a solution of the power flow equations.) Following Wood and Wollenberg [8], the (incremental) power transfer distribution factor (PTDF) from injection at bus k to flow on the line joining ℓ to m is the sensitivity:

$$\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(P^\star) = \frac{\partial \tilde{p}_{\ell m}}{\partial \Theta}(\Theta^\star)\frac{\partial \theta}{\partial P_k}(P^\star).$$

For brevity, we call this sensitivity "the PTDF from k to line ℓm."

In general, transactions may involve a change in injection at a bus k and a corresponding change at another bus j (that may not be the reference bus.) In this case, and if the system is lossless, then

the PTDF from injection at bus k and withdrawal at bus j to flow on the line joining ℓ to m is the difference of sensitivities:

$$\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(P^\star) - \frac{\partial \hat{p}_{\ell m}}{\partial P_j}(P^\star) = \frac{\partial \tilde{p}_{\ell m}}{\partial \Theta}(\Theta^\star)\left(\frac{\partial \theta}{\partial P_k}(P^\star) - \frac{\partial \theta}{\partial P_j}(P^\star)\right).$$

For brevity, we call this sensitivity "the PTDF from kj to line ℓm."

In the following sections we calculate the PTDFs from k to line ℓm and from kj to line ℓm. We will specialize to the case of lossless networks, but mention the qualitative effect of losses on the results.

4 Series-parallel networks

In this section, we consider a lossless electric power network with one-line diagram having a specific topological property related to combining branches in a "series-parallel" reduction [14], which is specified in:

Definition 1 A series-parallel reduction of an electrical network is either:

A parallel reduction involving the combining of two parallel lines that both join a given pair of busses into a single line having admittance equal to the sum of the admittances of each line, or,

A series reduction involving the combining of two lines that are incident to a common bus (and where the common bus is not incident to any other line) into a single line having admittance equal to the inverse of the sum of the inverse admittances of the lines. The bus that was common to the two lines before the reduction is deleted from the system in forming the reduced system.

□

We are interested in networks that can be series-parallel reduced to a single line joining busses k and j, as specified in:

Definition 2 Consider a network with two distinguished nodes k and j. Suppose that through a sequence of series-parallel reductions, we can reduce the network to a single line joining busses k and j. We call such a network "kj series-parallel." □

An example of a kj series-parallel network is a "triangular" network consisting of three busses with three lines joining them. Figure 1 illustrates the one-line diagram of such a system. The lines $k1$ and $1j$, having susceptances B_{k1} and B_{1j}, can be series reduced to a line kj, having susceptance $\left(\frac{1}{B_{k1}} + \frac{1}{B_{1j}}\right)^{-1}$ that can be parallel reduced with the other line joining k to j to produce a single line joining k and j. Considerably more complicated topologies than shown in figure 1 can be series-parallel reduced to a single line [14]. However, if the graph of the network has as a sub-graph the complete graph on four vertices then it is not kj series-parallel.

We can now prove:

Theorem 1 *Consider a lossless kj series-parallel network and any particular line ℓm. In such a network, the PTDFs from kj to line ℓm are constant independent of injection at k if the only point of real and reactive power injection in the network is k and the only point of real and reactive power withdrawal in the network is j.*

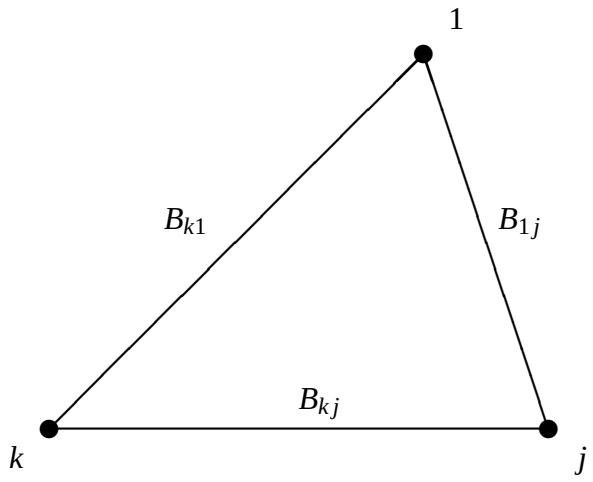

Figure 1: A kj series-parallel system.

Proof We prove by induction on the number of lines L in the (connected) one-line diagram of the network.

If there is only $L = 1$ line in the network (necessarily the line joining bus k to bus j) then the PTDF from kj to line kj is exactly 1 independent of the injection at k since all power injected at k must flow on the line joining k to j.

Suppose that the result is true for all kj series-parallel networks with at most L lines. Consider any kj series-parallel network with $L+1$ lines. Then by definition it can be reduced by a series-parallel reduction to a kj series-parallel network with L lines.

Consider the reduction. Either it was a series reduction or a parallel reduction. We consider each case in turn:

Series reduction involving two lines joining, say, busses ℓ to s and s to m. The reduced network has L lines and is kj series-parallel. By the induction hypothesis, the PTDF from kj to line ℓm in the reduced network is constant independent of injection at k.

Now note that because the system is lossless and there is no real power injected or withdrawn at bus s then, in the system before reduction, the real power flow on the line ℓ to s is the same as the real power flow on the line s to m. Furthermore, these real power flows are the same as the real power flow on the line ℓm after the reduction. Moreover, the combination of the lines does not affect the flow on any other lines since there was no real nor reactive power injection at bus s. That is, the PTDF from kj to line ℓs and the PTDF from kj to line sm in the system before reduction are constant independent of flows and both equal to the PTDF from kj to line ℓm in the system after reduction.

Parallel reduction involving two lines joining, say, busses ℓ to m. To distinguish the lines, we will label the lines in the system before reduction as $\ell m\alpha$ and $\ell m\beta$ having susceptances $B_{\ell m\alpha}$ and $B_{\ell m\beta}$, respectively. We will refer to the line in the reduced system as ℓm and it has susceptance $B_{\ell m} = B_{\ell m\alpha} + B_{\ell m\beta}$.

By the induction hypothesis, in the reduced system the PTDF from kj to line ℓm is constant independent of injection at k. Now note that the lines $\ell m\alpha$ and $\ell m\beta$ share flow between them in proportion to their susceptances and that this sharing is independent of flow along them. In particular, the flows along these lines, which we will denote by $\tilde{p}_{\ell m\alpha}$ and $\tilde{p}_{\ell m\beta}$, respectively, are

given by:

$$\begin{aligned}\forall\Theta\in\mathbb{R}^n, \tilde{p}_{\ell m\alpha}(\Theta) &= |v_\ell||v_m|B_{\ell m\alpha}\sin(\Theta_\ell-\Theta_m),\\ \forall\Theta\in\mathbb{R}^n, \tilde{p}_{\ell m\beta}(\Theta) &= |v_\ell||v_m|B_{\ell m\beta}\sin(\Theta_\ell-\Theta_m),\end{aligned}$$

and the ratio of these is fixed. Moreover, the parallel combination of these lines does not affect the flow on any other lines and the flow $\tilde{p}_{\ell m}$ on the parallel combination in the reduced system satisfies:

$$\begin{aligned}\forall\Theta\in\mathbb{R}^n, \tilde{p}_{\ell m\alpha}(\Theta) &= \frac{B_{\ell m\alpha}}{B_{\ell m}}\tilde{p}_{\ell m},\\ \forall\Theta\in\mathbb{R}^n, \tilde{p}_{\ell m\beta}(\Theta) &= \frac{B_{\ell m\beta}}{B_{\ell m}}\tilde{p}_{\ell m}.\end{aligned}$$

That is, the PTDFs from kj to line $\ell m\alpha$ and from kj to line $\ell m\beta$ are equal to the PTDF from kj to line ℓm in the reduced system multiplied by constant factors $\frac{B_{\ell m\alpha}}{B_{\ell m}}$ and $\frac{B_{\ell m\beta}}{B_{\ell m}}$, respectively.

□

Theorem 1 is restrictive in that it only applies to lossless systems with series-parallel topologies and, furthermore, because it only applies to a system with a single point of real and reactive power injection and a single point of real and reactive power withdrawal, whereas practical applications of PTDFs involve multiple simultaneous points of injection and withdrawal. The assumption of a single point of injection and a single point of withdrawal is utilized in the induction step when a series reduction is made. To the extent that intermediate points of withdrawal and injection involve relatively small injections, the result will hold approximately true in more practical applications.

To emphasize the importance of the assumption of a single point of real and reactive power injection and a single point of real and reactive power withdrawal, consider the three bus, three line network shown in figure 1. If the voltages at all nodes are held constant (potentially involving a large reactive power injection at bus 1) then the PTDFs are no longer independent of the real power injection at k and the real power withdrawal at j. The variation of PTDFs for a three bus, three line network is discussed in detail in [3, §3.9].

The lossless assumption is used in relating the flows on the lines before and after reduction. The result will hold approximately true in lossy systems to the extent that losses are relatively small.

In general, power systems are not necessarily series-parallel in topology. Moreover, reactive power sources are used to maintain bus voltages approximately constant, violating the assumption of a single point of injection and a single point of withdrawal in the system. In the next section, we consider general network topologies and explicitly assume that the voltages are held constant.

5 General network topologies

In the more general case of a lossless network that may not be series-parallel, we calculate an estimate of the PTDF directly, under the assumption that all the voltage magnitudes are constant. That is, we assume that all busses are *PV* busses [11, §10.2] with adequate reactive support to maintain constant voltage. Consider the PTDF from k to line ℓm at some operating point $P^\star, \Theta^\star$

such that the Jacobian $\frac{\partial p}{\partial \Theta}(\Theta^\star)$ is non-singular. Again, using the implicit function theorem, we can solve the power flow equations in a neighborhood $\mathcal{N}$ of $P^\star$ for a solution θ as a function of P.

Suppose that this neighborhood $\mathcal{N}$ of $P^\star$ includes a line segment joining $\mathbf{0}$ and $P^\star$. Then, by Taylor's theorem applied to the derivative of $\hat{p}_{\ell m}$, and assuming that the voltage magnitudes are constant, the PTDF satisfies:

$$\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(P^\star) = \frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0}) + \frac{\partial^2 \hat{p}_{\ell m}}{\partial P_k \partial P}(\mathbf{0})P^\star + o(P^\star), \tag{2}$$

where $o(P^\star)$ is a function such that $\frac{\|o(P^\star)\|}{\|P^\star\|} \to 0$ as $\|P^\star\| \to 0$, and where we note that $\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0})$ is the PTDF from k to line ℓm for zero injections. For a lossless system, $\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0})$ is the PTDF from k to line ℓm calculated according to the DC power flow approximation. In the development that follows, we will show that the vector of coefficients $\frac{\partial^2 \hat{p}_{\ell m}}{\partial P_k \partial P}(\mathbf{0})$ of the linear term in the expression (2) for the PTDF is zero. That is, the PTDF evaluated at $P^\star$ is equal to a constant plus terms that are higher order than linear in $P^\star$. This accounts for the relative constancy of the PTDFs at low to medium load in power systems when voltage constraints are not binding.

We evaluate the terms in the PTDF in the following:

Lemma 2 *Consider a lossless system and a line ℓm. We have the following expressions for the derivatives:*

$$\begin{aligned}
\forall \Theta, \frac{\partial(\Theta_\ell - \Theta_m)}{\partial \Theta}(\Theta) &= [\mathbf{I}_\ell - \mathbf{I}_m]^\dagger, \\
\forall \Theta, \frac{\partial \tilde{p}_{\ell m}}{\partial \Theta}(\Theta) &= |v_\ell||v_m|B_{\ell m}\cos(\Theta_\ell - \Theta_m)[\mathbf{I}_\ell - \mathbf{I}_m]^\dagger, \\
\frac{\partial \tilde{p}_{\ell m}}{\partial \Theta}(\mathbf{0}) &= |v_\ell||v_m|B_{\ell m}[\mathbf{I}_\ell - \mathbf{I}_m]^\dagger, \\
\forall \Theta, \frac{\partial^2 \tilde{p}_{\ell m}}{\partial \Theta^2}(\Theta) &= -|v_\ell||v_m|B_{\ell m}\sin(\Theta_\ell - \Theta_m)[\mathbf{I}_\ell - \mathbf{I}_m][\mathbf{I}_\ell - \mathbf{I}_m]^\dagger, \\
\frac{\partial^2 \tilde{p}_{\ell m}}{\partial \Theta^2}(\mathbf{0}) &= \mathbf{0}, \\
\forall P, \Theta, \frac{\partial p}{\partial P_k}(P, \Theta) &= -\mathbf{I}_k, \\
\forall P, \Theta, \forall t, \frac{\partial^2 p}{\partial P_k \partial P_t}(P, \Theta) &= \mathbf{0}, \\
\forall P, \Theta, \forall s, t, \frac{\partial p_s}{\partial \Theta_t}(P, \Theta) &= \begin{cases} \sum_{u \in \mathbb{K}(s)} |v_s||v_u|B_{su}\cos(\Theta_s - \Theta_u), & \text{if } t = s, \\ -|v_s||v_t|B_{st}\cos(\Theta_s - \Theta_t), & \text{if } t \in \mathbb{K}(s), \end{cases} \\
\forall P, \forall s, t, u, \frac{\partial^2 p_s}{\partial \Theta_t \partial \Theta_u}(P, \mathbf{0}) &= 0,
\end{aligned}$$

where we note that:

- *superscript † denotes transpose,*
- $\mathbf{0}$ *denotes a vector or matrix of all zeros, and*
- $\mathbf{I}_\ell$ *is the vector with all zeros except for a one in the ℓ-th place.*

Proof All of the terms follow from definition of the functions, direct calculation, and substitution. □

Corollary 3 *Consider a lossless system. If* $\dfrac{\partial p}{\partial \Theta}(\mathbf{0},\mathbf{0})$ *is non-singular then* $\dfrac{\partial^2 \hat{p}_{\ell m}}{\partial P_k \partial P}(\mathbf{0}) = \mathbf{0}$.

Proof We note that for a lossless system, $P = \mathbf{0}$ and $\Theta = \mathbf{0}$ is a solution of the power flow equations (1). Again using the implicit function theorem, there is an infinitely partially differentiable function $\theta : \mathbb{R}^n \to \mathbb{R}^n$ such that in some neighborhood $\mathcal{N}_0$ of $P = \mathbf{0}$, the power flow equations (1) has a solution satisfying: $\forall P \in \mathcal{N}, p(P,\theta(P)) = \mathbf{0}$. Differentiating this expression with respect to P_k, we obtain:

$$\begin{aligned} \mathbf{0} &= \frac{\partial p}{\partial P_k}(\mathbf{0},\mathbf{0}) + \frac{\partial p}{\partial \Theta}(\mathbf{0},\mathbf{0})\frac{\partial \theta}{\partial P_k}(\mathbf{0}), \\ &= -\mathbf{I}_k + \frac{\partial p}{\partial \Theta}(\mathbf{0},\mathbf{0})\frac{\partial \theta}{\partial P_k}(\mathbf{0}), \end{aligned}$$

so that since $\dfrac{\partial p}{\partial \Theta}(\mathbf{0},\mathbf{0})$ is non-singular by hypothesis, $\dfrac{\partial \theta}{\partial P_k}(\mathbf{0})$ is well-defined. Differentiating again with respect to P_t:

$$\begin{aligned} \mathbf{0} &= \frac{\partial^2 p}{\partial P_k \partial P_t}(\mathbf{0},\mathbf{0}) + \left[\left[\frac{\partial \theta}{\partial P_k}(\mathbf{0})\right]^\dagger \frac{\partial^2 p_s}{\partial \Theta^2}(\mathbf{0},\mathbf{0})\frac{\partial \theta}{\partial P_t}(\mathbf{0})\right]_{s=1,\ldots,n} + \frac{\partial p}{\partial \Theta}(\mathbf{0},\mathbf{0})\frac{\partial^2 \theta}{\partial P_k \partial P_t}(\mathbf{0}), \\ &= \mathbf{0} + \left[\left[\frac{\partial \theta}{\partial P_k}(\mathbf{0})\right]^\dagger [\mathbf{0}]\frac{\partial \theta}{\partial P_t}(\mathbf{0})\right]_{s=1,\ldots,n} + \frac{\partial p}{\partial \Theta}(\mathbf{0},\mathbf{0})\frac{\partial^2 \theta}{\partial P_k \partial P_t}(\mathbf{0}), \\ &= \frac{\partial p}{\partial \Theta}(\mathbf{0},\mathbf{0})\frac{\partial^2 \theta}{\partial P_k \partial P_t}(\mathbf{0}), \end{aligned}$$

by lemma 2, where the terms of the form $[\cdots]_{s=1,\ldots,n}$ mean a vector having s-th entry given by the term inside the square brackets. Again, since $\dfrac{\partial p}{\partial \Theta}(\mathbf{0},\mathbf{0})$ is non-singular, we have $\dfrac{\partial^2 \theta}{\partial P_k \partial P_t}(\mathbf{0}) = \mathbf{0}$. Also, by lemma 2, $\dfrac{\partial^2 \tilde{p}_{\ell m}}{\partial \Theta^2}(\mathbf{0}) = \mathbf{0}$.

By direct calculation:

$$\begin{aligned}
\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0}) &= \frac{\partial \tilde{p}_{\ell m}}{\partial \Theta}(\mathbf{0})\frac{\partial \theta}{\partial P_k}(\mathbf{0}), \\
\forall t, \frac{\partial^2 \hat{p}_{\ell m}}{\partial P_k \partial P_t}(\mathbf{0}) &= \frac{\partial \tilde{p}_{\ell m}}{\partial \Theta}(\mathbf{0})\frac{\partial^2 \theta}{\partial P_k \partial P_t}(\mathbf{0}) + \left[\frac{\partial \theta}{\partial P_k}(\mathbf{0})\right]^{\dagger} \frac{\partial^2 \tilde{p}_{\ell m}}{\partial \Theta^2}(\mathbf{0})\frac{\partial \theta}{\partial P_t}(\mathbf{0}).
\end{aligned}$$

Substituting in from the terms previously calculated, we obtain: $\forall t, \frac{\partial^2 \hat{p}_{\ell m}}{\partial P_k \partial P_t}(\mathbf{0}) = \mathbf{0}$. □

Corollary 4 *Consider a lossless system with reactive compensation such that all bus voltage magnitudes are constant. Also consider an operating point $P^\star$ sufficiently close to the condition of zero net injection such that for all operating points on the line segment joining $\mathbf{0}$ and $P^\star$ we have that:*

- *the solution of the power flow equations are well-defined and unique and*
- *the Jacobian $\frac{\partial p}{\partial \Theta}$ is non-singular.*

Then the incremental PTDFs at the operating point $P^\star$ differ from the PTDFs calculated from the DC load flow by an error that is small compared to $P^\star$. □

Corollary 4 shows that in a lossless system with constant bus voltage magnitudes, the PTDFs are approximately constant for operating points near to the condition of zero injection; that is, for what we might define as "light" to "medium" flows on the system from the perspective of angle differences across lines.

In contrast, at "heavy" loadings where angles across lines are large, or if the assumption of constant voltage is not maintained, then the error may become large. For example, if there are binding voltage constraints because of a lack of reactive power support, the PTDFs may vary significantly. While this condition is of great interest in the context of flowgate rights for transmission, it is not appropriate to rely on the constancy of PTDFs under these circumstances. We note that some errors in the PTDFs are positive while others are negative since, as discussed in [3, §3.3], the PTDFs from a point of injection and withdrawal to each of the lines in a cutset of a lossless power system must always sum to one.

In practical systems having small but non-zero losses, we observe that in lemma 2 we would have that:

$$\begin{aligned}
\frac{\partial^2 \tilde{p}_{\ell m}}{\partial \Theta^2}(\mathbf{0}) &\approx \mathbf{0}, \\
\forall s,t,u, \frac{\partial^2 p_s}{\partial \Theta_t \partial \Theta_u}(P, \mathbf{0}) &\approx \mathbf{0},
\end{aligned}$$

so that we also have $\frac{\partial^2 \hat{p}_{\ell m}}{\partial P_k \partial P}(\mathbf{0}) \approx \mathbf{0}$, again implying that the PTDFs are relatively insensitive to variations in injections and withdrawals.

6 Power to current distribution factors

Although PTDFs are often discussed in relation to thermally limited lines, in fact it is the heating due to current flowing on the line that determines the thermal limit. Instead of considering the effect of the change of power injection on the power flowing down a line, a more direct measure of the effect on a thermal constraint is the effect of a change in injection on the magnitude of the current flowing down the line. (In practice, the magnitude of the complex flow is often used as a proxy to the magnitude of the current.)

We again maintain the assumption that the voltage magnitude at each bus is constant and assume that losses are small. However, these assumptions are insufficient to yield a result similar to corollary 4. In order for the power to current magnitude distribution factor from k to line ℓm to be approximately constant, we will see that we must additionally require that $|v_\ell| = |v_m|$. The reason for this is that if these voltages are different then there will be reactive power flowing along the line and the PTDFs will change more rapidly with flow.

Moreover, since we are interested in current magnitudes but the current magnitude is not differentiable at the condition of zero current, we will define a "directed" current magnitude that is differentiable and captures the relevant behavior of the current magnitude. We will assume that the power to current distribution factor of interest is relevant to a thermal limit that corresponds to power flowing from bus ℓ to bus m, so that $\Theta_\ell > \Theta_m$ at the operating point.

Ignoring the current flowing in the shunt capacitance of the line, the square of the current magnitude is given by:

$$\begin{aligned}
&\forall \Theta \in \mathbb{R}^n, (G_{\ell m}^2 + B_{lm}^2)(|v_\ell|^2 + |v_m|^2 - 2|v_\ell||v_m|\cos(\Theta_\ell - \Theta_m)) \\
&= (G_{\ell m}^2 + B_{lm}^2)([|v_\ell| - |v_m|]^2 + 2|v_\ell||v_m|[1 - \cos(\Theta_\ell - \Theta_m)]), \\
&= 2|v_\ell||v_m|[1 - \cos(\Theta_\ell - \Theta_m)],
\end{aligned}$$

if $|v_\ell| = |v_m|$.

Paralleling the development in section 3, we define the function $\tilde{i}_{\ell m} : \mathbb{R}^n \to \mathbb{R}$ by:

$$\forall \Theta \in \mathbb{R}^n, \tilde{i}_{\ell m}(\Theta) = \begin{cases} \sqrt{G_{\ell m}^2 + B_{lm}^2}\sqrt{2|v_\ell||v_m|[1 - \cos(\Theta_\ell - \Theta_m)]}, & \text{if } \Theta_\ell > \Theta_m, \\ -\sqrt{G_{\ell m}^2 + B_{lm}^2}\sqrt{2|v_\ell||v_m|[1 - \cos(\Theta_\ell - \Theta_m)]}, & \text{if } \Theta_\ell < \Theta_m, \\ 0, & \text{if } \Theta_\ell = \Theta_m. \end{cases}$$

The function $\tilde{i}_{\ell m}$ is twice partially differentiable and its absolute value is the magnitude of the current on the line ℓm. (Strictly speaking, in the presence of shunt capacitance, this function is equal to the current only at the mid-point of the line between the busses ℓ and m.) To relate the current to the net injections, we define the function $\hat{i}_{\ell m} : \mathbb{R}^n \to \mathbb{R}$ by:

$$\forall P \in \mathcal{N}, \hat{i}_{\ell m}(P) = \tilde{i}_{\ell m}(\theta(P)).$$

The (incremental) power to current magnitude distribution factor (PIDF) from injection at bus k to current magnitude on the line ℓm is the sensitivity:

$$\frac{\partial \hat{i}_{\ell m}}{\partial P_k}(P^\star) = \frac{\partial \tilde{i}_{\ell m}}{\partial \Theta}(\Theta^\star)\frac{\partial \theta}{\partial P_k}(P^\star).$$

For brevity, we call this sensitivity "the PIDF from k to line ℓm."

As in section 5, we have:

$$\frac{\partial \hat{i}_{\ell m}}{\partial P_k}(P^\star) = \frac{\partial \hat{i}_{\ell m}}{\partial P_k}(\mathbf{0}) + \frac{\partial^2 \hat{i}_{\ell m}}{\partial P_k \partial P}(\mathbf{0})P^\star + o(P^\star).$$

Theorem 5 *Consider a lossless system with reactive compensation such that all bus voltage magnitudes are constant. Consider a bus k and a line ℓm and suppose that $|v_\ell| = |v_m|$. Also consider an operating point $P^\star$ sufficiently close to the condition of zero net injection such that for all operating points on the line segment joining $\mathbf{0}$ and $P^\star$ we have that:*

- *the solution of the power flow equations are well-defined and unique and*
- *the Jacobian $\dfrac{\partial p}{\partial \Theta}$ is non-singular.*

Then the incremental PIDFs at the operating point $P^\star$ differ from the PIDFs calculated from the DC load flow by an error that is small compared to $P^\star$. □

Note that under the assumption of constant voltages, the PTDFs and the PIDFs calculated from the DC power flow are proportional to each other. At other operating points, however, the PIDFs can be expected to change more rapidly with flows than the PTDFs unless the condition $|v_\ell| = |v_m|$ is maintained.

Theorem 5 assumes a lossless system; however, the assumption of thermal limits implies losses. Again, we observe that theorem 5 remains approximately true in the presence of losses. The implication of theorem 5 is that in a thermally limited system, to enable flowgate rights to be effective, voltage support must be provided on the constrained lines to make the voltages constant and equal at both ends of each flowgate. If these conditions are not satisfied, then the PIDFs (and indeed the thermal capacity) will vary with loading. In particular,

- the PIDFs will vary as $|v_\ell| - |v_m|$ varies and
- the capacity to transmit real power will fall as the voltage at the receiving end falls (as more of the capacity is used for transmitting reactive power.)

In the extreme, if voltage constraints are binding then flowgate approaches are unlikely to be effective [15, 1]. Fortunately, voltage support is often relatively cheap to provide [16].

7 Power to angle distribution factors

In steady-state stability limited systems, the angle across lines (or between generation and demand) is the limiting factor. Instead of considering the effect of the change of injection on the power flowing down a line, a more direct measure of the effect on a stability limit is the effect of a change in injection on the angle difference across a line. Under the same assumptions as in corollary 4,

$$\begin{aligned}\frac{\partial \theta}{\partial P_k}(P^\star) &= \frac{\partial \theta}{\partial P_k}(\mathbf{0}) + \frac{\partial^2 \theta}{\partial P_k \partial P}(\mathbf{0})P^\star + o(P^\star), \\ &= \frac{\partial \theta}{\partial P_k}(\mathbf{0}) + o(P^\star),\end{aligned}$$

and the PADFs are again relatively constant independent of injections for $P^\star$ close enough to the condition of zero net injections.

8 Conclusion

In this paper we presented conditions for PTDFs to be completely or approximately independent of the injections and withdrawals in a lossless electric power system. For particular topologies, and assuming a single point of injection and a single point of withdrawal, the PTDFs are constant. In more practical systems with arbitrary topology, multiple points of injection and withdrawal, and losses, the PTDFs are relatively independent of injections and withdrawals if topology is fixed, voltages are held constant, the losses are relatively small and the angles across lines not too large.

We also analyzed power to current magnitude distribution factors PIDFs and power to angle distribution factors PADFs. For relative constancy of the PIDFs from k to a line ℓm, we found that we must assume that $|v_\ell| = |v_m|$ in addition to the assumptions for relative constancy of PTDFs and PADFs. That is, we must assume that there is adequate voltage support as a condition for the effectiveness of flowgate rights schemes. In the context of a contingency limited system, this requires that controllable voltage support must also be available under contingency conditions. The conditions for the relative constancy of the distribution factors are stringent and may not hold in typical transmission systems.

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