Title
Mean curvature flows with forcing and degenerate nonlinear parabolic equations

Author
Kwon, Dohyun

Publication Date
2020

UNIVERSITY OF CALIFORNIA
Los Angeles

Mean curvature flows with forcing and degenerate nonlinear parabolic equations

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Dohyun Kwon

2020

ABSTRACT OF THE DISSERTATION

Mean curvature flows with forcing and degenerate nonlinear parabolic equations

by

Dohyun Kwon

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2020

Professor Christina Kim, Chair

Degenerate diffusion equations and their interface dynamics have received a lot of attention in the past couple of decades. In particular, surfaces moving with curvature dependent velocities and discontinuous diffusion intensities naturally appear in physical and biological models. In this dissertation, we study global well-posedness and geometry of two equations: mean curvature flows with forcing and degenerate nonlinear parabolic equations with discontinuous diffusion coefficients. Both problems have gradient flow structures in the space of sets and the Wasserstein space, respectively, which are useful to study the global-time behavior.

In Chapter 1, we develop a parabolic version of the Aleksandrov and Serrin's moving plane methods for mean curvature flow with forcing. With the class of forcing which bounds the volume of evolving sets away from zero and infinity, we show that a strong version of star-shapedness is preserved over time. Based on this geometric property, we prove that volume preserving mean curvature flow starting from a star-shaped set converges to a ball.

Chapter 2 is devoted to the study of degenerate parabolic equations with discontinuous diffusion intensities. We show the existence and uniqueness of the solutions in the sense of distributions. Our notion of solutions allows us to give a fine characterization of the emerging critical regions, observed previously in numerical experiments.

This dissertation has been resulted in the publications [KK20a], [KK20b] and [KM19].

The dissertation of Dohyun Kwon is approved.

Georg Menz

Wilfrid Dossou Gangbo

Stanley J. Osher

Christina Kim, Committee Chair

University of California, Los Angeles

2020

To my family

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

First of all, I would like to express my deepest appreciation to my advisor Professor Inwon Kim. Her unwavering support, valuable advice and patience in guiding me are greatly appreciated and will not be forgotten. She has inspired me to become an independent researcher and this dissertation would not have been possible without her guidance.

I would like to extend my sincere thanks to my committee Professors Stanley Osher, Wilfrid Gangbo and Georg Menz. They generously gave their time to offer me valuable comments toward improving my work. Moreover, I very much appreciate my collaborators Alpár Mészáros, Sun-Ho Choi, Norbert Požár and Matt Jacobs for constructive comments and insightful discussions.

Throughout my doctoral studies, I have had the great privilege of engaging and working with many people at UCLA. I would like to thank Jiajun Tong, Levon Nurbekyan, Flavien Léger, Chenchen Mou and Paul Zhang for their support and helpful discussions. Furthermore, many thanks to my fellow students Stephen Ge, Adam Azzam, Sangchul Lee, Kyung Ha, Bohyun Kim, Wonjun Lee, Bumsu Kim, Younghak Kwon and Raymond Chu. I am also grateful to Maggie Albert, Maida Bassili, and Martha Contreras for their administrative support.

Most importantly, I am eternally thankful for my family. I have had great support from my family. I could not have done this dissertation without their support and love. I am extremely grateful to my father Man Bok Kwon and my mother Haeja Lee for encouraging me in all of my pursuits and inspiring me to follow my dreams. I also wish to thank my brother Yonghyun Kwon for always answering my various questions and providing his insight.

Lastly, I'm deeply indebted to my lovely wife Ji Eun Hwang. I can't express how lucky and how fortunate I feel to have her as my wife. She has been my best friend and dearest companion and lover. Her constant support and profound belief in me made this dissertation possible. I very much appreciate all the patience and support that she has provided me through this journey.

VITA

2012	B.S. in Mathematical Sciences, B.A. in Economics, Seoul National University
2016–2020	Teaching Assistant, Department of Mathematics, UCLA

PUBLICATIONS

1. Danielle Hilhorst, Yong-Jung Kim, Dohyun Kwon, and Thanh Nam Nguyen. Dispersal towards food: the singular limit of an Allen-Cahn equation. *Journal of Mathematical Biology*, 76(3):531-565, 2018

2. Dohyun Kwon and Alpár Mészáros. Degenerate nonlinear parabolic equations with discontinuous diffusion coefficients. *arXiv preprint arXiv:1907.11319*, 2019

3. Inwon Kim and Dohyun Kwon. On mean curvature flow with forcing. *Communications in Partial Differential Equations*, 45(5):414-455, 2020

4. Inwon Kim and Dohyun Kwon. Volume preserving mean curvature flow for star-shaped sets. *Calculus of Variations and Partial Differential Equations*, 59(2):81, 2020

CHAPTER 1

Mean curvature flows with forcing

1.1 Introduction

Consider an evolving set $(\Omega_t)_{t>0}$ in $\mathbb{R}^n$ moving by the motion law

$$V = -H + \lambda \quad \text{on } \partial\Omega_t. \tag{1.1.1}$$

Here, $V = V(x,t)$ and $H = H(x,t)$ respectively denote the outward normal velocity and the mean curvature of $\partial\Omega_t$ at $x \in \partial\Omega_t$, where H is set to be positive if Ω_t is convex at the point.

We are interested in the global-time description of the flow, including its well-posedness. In general, due to the low-dimensional nature of the interface, finite-time topological singularities are expected even for interfaces starting out with smooth shapes. On the other hand, (1.1.1) is a parabolic flow, and thus parabolic regularity theory applies once we know that the evolving boundary $\partial\Omega_t$ is locally a graph. Thus our first goal is to establish an a priori graph property of $\partial\Omega_t$ by studying the geometry of the evolution.

1.1.1 Volume-dependent forcing

The volume-dependent forcing $\lambda = \lambda(|\Omega_t|)$ we consider keeps the volume of Ω_t bounded away from zero and infinity. With such choices of forcing we will show that a strong version of star-shapedness property holds for Ω_t at all $t > 0$ if initially true. To state the main results, let us begin with discussing the assumptions on the forcing.

Assumption A. $\lambda : \mathbb{R}^+ \to \mathbb{R}$ is locally Lipschitz continuous and satisfies $\limsup_{R\to\infty} \frac{\lambda(|B_R|)}{R} < \infty$. In addition, there exists $\rho > 0$ such that $\lambda(|\Omega|) > \frac{n-1}{\rho}$ for all $\Omega \subset \overline{B}_{5\rho}$.

The first part of the assumption is necessary to show that the evolution is unique and the set does not spread to $\mathbb{R}^n$ in finite time. The second part puts a sufficient penalty on shrinkage of the evolution, and is used in showing that the evolution always contains a small ball $B_\rho(0)$ if initially so (Lemma 1.3.9). With the parameter ρ given from above assumption, we assume that Ω_0 has *ρ-reflection* (see Definition 1.3.3). The ρ-reflection property should be interpreted as a quantitative smallness requirement on the Lipschitz norm distance between Ω_0 and the nearest ball (see Lemma 1.3.4). We adopt Definition 1.2.6 as the notion of solutions for (1.1.1). Our first result states the preservation of the ρ-reflection property in Theorem 1.3.8. The proof is based on the reflection maximum principle as well as various barrier arguments based on Assumption A.

Theorem 1. *Suppose that Ω_0 has ρ-reflection and $(\Omega_t)_{t\geq 0}$ is a solution of* (1.1.1)*. Then, Ω_t has ρ-reflection at all times $t \geq 0$. In particular there exists $r_1 = r_1(\rho) > 0$ such that Ω_t is star-shaped with respect to a ball $B_{r_1}(0)$ for all $t \geq 0$.*

Let us remark that this geometric result does not extend to the classical mean curvature flow where $\lambda = 0$. With zero forcing and with star-shaped initial set, solutions of (1.1.1) have been shown to hold certain semi-convexity estimates by Smoczyk [Smo98] and Lin [Lin15]. While these estimates allow classification of possible singularities for the flow in terms of blow-up limits, it remains open whether the initially star-shaped flow stays star-shaped beyond the initial time even with zero forcing.

With the a priori geometric property of the flow, we next discuss existence and uniqueness of the flow (1.1.1) based on its variational structure. A formal calculation yields the energy inequality

$$\frac{d}{dt}J(t) = -\int_{\partial\Omega_t} V^2 dS, \tag{1.1.2}$$

where $J(t) = \mathrm{Per}(\Omega_t) - \Lambda(|\Omega_t|)$ with Λ the anti-derivative of λ and V as given in (1.1.1). From (1.1.2) one expects Ω_t to flow toward a stationary point of the energy as time grows. We will make this observation rigorous by generating a discrete-time approximation (or "minimizing movement") that satisfies the energy dissipation. The aforementioned a priori geometric property enables the uniform convergence of its discrete time approximations, to guarantee that in the continuum limit we recover a smooth solution.

While the variational approach yields the minimizing movements approximation as well as the asymptotic analysis of the flow, viscosity solutions are more suited for geometric arguments. To take advantage of both approaches we will show that the variational flow is, in a sense, a viscosity solution of (1.1.1). This idea of combining the two approaches were previously used for the mean curvature flow in [Cha04], but in our problem the standard maximum principle does not apply for (1.1.1), and thus the notion of viscosity solutions needs to be modified from the standard one. Indeed our main novelty in the analysis is to combine these two approaches to address geometric motions which do not satisfy a comparison principle but still is of parabolic nature. For free boundary problems this combination has been introduced in [FK14], where the presence of bulk pressure plays a crucial role in the analysis.

From Theorem 1 and the volume bound it follows that Ω_t has locally Lipschitz boundary which is uniform in time. This fact endows sufficient compactness for the evolution that makes it possible for the discrete-time variational scheme to approximate the flow, in particular establishing the global existence results in Theorem 1.4.21.

Theorem 2. *Suppose that Ω_0 has ρ-reflection. Then, there exists a unique solution $(\Omega_t)_{t\geq 0}$ of* (1.1.1) *that is bounded and has smooth boundary for every $t > 0$. $(\Omega_t)_{t\geq 0}$ can be approximated locally uniformly by minimizing movements with constraints.*

1.1.2 Volume preserving flow

Next, we consider

$$V = -H + \lambda(t) \text{ on } \Gamma_t := \partial\Omega_t, \quad |\Omega_t| = |\Omega_0| = 1. \tag{1.1.3}$$

In smooth setting, $\lambda : [0, +\infty) \to \mathbb{R}$ satisfies $\int_{\Gamma_t} V dS = 0$ so that the evolution satisfies $|\Omega_t| = |\Omega_0|$, i.e.

$$\lambda(t) = \frac{1}{\mathrm{Per}(\Omega_t)} \int_{\partial\Omega_t} H d\sigma = \fint_{\partial\Omega_t} H d\sigma. \tag{1.1.4}$$

There are two main difficulties to study the global behavior of the flow (1.1.3) in general settings. First the evolution may go through topological changes, and secondly the formula (1.1.4) does not hold for λ in less than $C^{1,\alpha}$ settings. The first difficulty motivates us to study geometric properties that are preserved by the flow, and the second requires new ideas to obtain sufficient compactness to establish convergence to equilibrium.

In variational setting, (1.1.3) can be formulated based on its energy dissipation structure for the perimeter energy with volume preserving constraint. Using this structure [MSS16] and [Tak17] showed the existence of general distribution solution of (1.1.3). For our interest in geometric properties of solutions, we instead work with a modified version of viscosity solutions, where we consider an implicit choice of λ so that the volume of the evolving set is preserved over time.

Motivated by recent results [MSS16], our strategy is to approximate (1.1.3) by the following flow as $\delta \to 0$:

$$\begin{cases} V &= -H + \lambda_\delta(t), \quad \lambda_\delta(t) := \gamma_\delta(|\Omega_t|) \text{ on } \Gamma_t, \\ \Omega_0^\delta &= \Omega_0 \end{cases} \tag{1.1.5}$$

where $\gamma_\delta : \mathbb{R}^+ \to \mathbb{R}$ for $\delta > 0$ is defined by

$$\gamma_\delta(s) := \frac{1}{\delta}(1 - s). \tag{1.1.6}$$

Let us mention that the comparison principle does not hold for both (1.1.3) and (1.1.5), so the notion of solutions should be understood as viscosity solutions with a priori given $\lambda_\delta(t)$ (see Definition 1.2.6 and Definition 1). Compared to the original flow (1.1.3), (1.1.5) holds an advantage that $\lambda_\delta(t)$ only depends on $|\Omega_t|$, thus it can be handled with little information on the regularity of Γ_t, which makes it easier to handle with viscosity solutions theory. The existence and uniqueness for viscosity solutions of (1.1.5) were proved in Theorem 2.

Below we summarize the main results in Theorems 1.5.1 & 1.5.6, Corollary 1.5.7 and Theorem 1.5.16. We assume the following geometric condition on the initial data:

$$\Omega_0 \text{ has } \rho\textit{-reflection} \text{ for some } \rho \in [0, (5c_n)^{-1}),\ c_n = |B_1|^{1/n} \text{ and } |\Omega_0| = 1. \tag{1.1.7}$$

Theorem 3. *Under the assumption* (1.1.7)*, there exists a viscosity solution* $((\Omega_t^\infty)_{t\geq 0}, \lambda_\infty)$ *of* (1.1.3) *approximated by solutions* $\{((\Omega_t^\delta)_{t\geq 0}, \lambda_\delta)\}_{\delta>0}$ *of* (1.1.5) *with the following properties:*

(a) For any finite time $T > 0$*, we have*

$$\max_{0\leq t\leq T} d_H(\Omega_t^\delta, \Omega_t^\infty) \to 0 \text{ and } \lambda_\delta \rightharpoonup \lambda_\infty \text{ in } L^2([0,T]) \text{ as } \delta \to 0$$

along a subsequence.

(b) There exists $r, \delta_0 > 0$ *such that for all* $t \geq 0$ *and* $\delta \in (0, \delta_0)$ *both* Ω_t^∞ *and* Ω_t^δ *contain the ball* $B_r(0)$ *and stay star-shaped with respect to it.*

(c) $(\Omega_t^\infty)_{t\geq 0}$ *uniformly converges to a ball of volume* 1*, modulo translation, i.e.,*

$$\inf_{x_0\in\mathbb{R}^n} d_H(\Omega_t^\infty, \mathcal{B}(x_0)) \to 0 \text{ as } t \to \infty$$

where $\mathcal{B}(x_0)$ *is the ball of unit volume centered at* x_0*.*

The upper bound of ρ in (1.1.7) follows from the condition $|\Omega_0| > |B_{5\rho}|$ in Proposition 1.5.2 and ρ can be chosen large if the initial volume is large.

Let us briefly discuss our assumption (1.1.7), the main ingredients and challenges in the context of literature. It is well known ([Hui87]) that convexity is preserved in the flow (1.1.3), and the global-time behavior of convex evolution, as well as exponential convergence to the unit ball, has been studied in the smooth case [Hui87] and for anisotropic flow [And01] and [BCC09]. Our goal is understanding the evolution of star-shaped sets. While it is suspected that star-shapedness is preserved in the evolution, it remains open to be proved. In Theorem 1, we show that this property is preserved in the flow with volume-dependent forcing, which includes (1.1.5). In particular this property implies (b) for Ω_t^δ, as well as an equi-continuity over time, yielding the first part of (a). It should be pointed out that our geometric arguments should be incorporated with the variational methods, since the underlying gradient flow structure of (1.1.3) and (1.1.5) provides both existence and asymptotic convergence results for both problems. For this reason our construction of solutions for (1.1.3)-(1.1.5) employs constrained minimizing movements with admissible sets only for star-shaped sets, which differs from the standard constructions.

To yield the second part of (a), we obtain uniform L^2 bound for λ_δ, largely following the variational arguments in [MSS16], adapted to our constrained minimizing movements described above. The main difficulty is the lack of the uniform L^∞ bound on λ_∞. The bounds for λ_δ correlates to that of the *total curvature* $\int_{\partial\Omega_t} HdS$. An L^∞ bound for λ_∞ along with the geometric property of Ω_t would invoke parabolic regularity theory for curvature flows to yield smoothness of the flow, which in turn yields sufficient compactness to discuss the asymptotic behavior of the flow.

For us there is only L^2 estimates are available on λ_∞, which is inherited from λ_δ's (see Section 1.5.2). For this reason, we fall short of obtaining regularity of $\partial\Omega_t^\infty$ that goes beyond Lipschitz. In particular this necessitates a notion of viscosity solutions of $V = -H + \lambda$ for a priori given $\lambda \in L^1_{loc}([0,\infty))$ (Definition 1.2.12). Moreover, to assert that the limit $(\Omega_t^\infty, \lambda_\infty)$ solves (1.1.3), our notion needs to stay stable under weak convergence of λ in L^2. Once such notion is established for prescribed λ, we can introduce a notion of viscosity solutions of (1.1.3).

Definition 1. The pair $((\Omega_t)_{t\geq 0}, \lambda)$ is a viscosity solution of (1.1.3) if $|\Omega_t| = |\Omega_0|$ and $(\Omega_t)_{t\geq 0}$ is a viscosity solution (See Definition 1.2.13) of $V = -H + \lambda(t)$.

The extended notion for prescribed λ, Definition 1.2.12, enables us to analyze geometric properties of $((\Omega_t)_{t\geq 0}, \lambda)$ for $\lambda \in L^1_{loc}([0,\infty))$. Notions of viscosity solutions for time-integrable operator are previously introduced in [Ish85], [Bou08a] and [Bou08b]. These previous notions however do not allow stability under weak convergence of operators, and thus in this aspect our notion is new. Our notions however coincide with the previous ones as a consequence of its stability properties, see Remark 1.2.21.

Higher regularity of volume preserving mean curvature flows remains open. Note that gradient and curvature estimates of volume preserving mean curvature flows were proven in [Ath97] for the rotationally symmetric case and [Hui87] for the convex case. However, in both cases the uniform boundedness of λ was essential. As we obtain the Lipschitz graph property of Ω_t^∞, we may apply interior estimates for classical mean curvature flows from [EH91] and [Eck04], but similar obstacles on the estimates for λ are expected. With higher regularity, uniqueness of the solution for (1.1.3) may be shown by dilation arguments as in [Gig06] and [BCC09].

We are short of proving smoothness of Ω_t^∞ beyond its Lipschitz graph property, though we expect it to be true. Note that in non-smooth or non-convex setting, perimeter difference may not converge into zero as Hausdorff distance converges to zero. This poses a challenge for proving asymptotic convergence of Ω_t^∞. Our proof of perimeter convergence in the asymptotic limit uses both the uniform L^2 estimates of mean curvature and star-shapedness (See Lemma 1.5.18). [ES98] and [AKS10] show global well-posedness and exponential convergence if the initial condition is sufficiently close to a round sphere in Hölder norm and Sobolev norm, respectively. Similar results were proven for sufficiently small traceless second fundamental form of the initial condition in [Li09]. We mention that our result is not a perturbative one but most of existing results on asymptotic convergence require regularity of the interface to be smoother than $C^{1,\alpha}$.

1.1.3 Outline

In section 1.2, we give a definition on the notion of "viscosity solutions" for (1.1.1) in terms of its level-set formulation. To do so we first discuss the mean curvature flows with a priori fixed forcing,

$$V = -H + \eta(t). \tag{1.1.8}$$

Our solution Ω_t of (1.1.1) is then defined as the viscosity solution of (1.1.8) where $\eta(t)$ coincides with $\lambda(|\Omega_t|)$. In section 1.2.2, we extend the notion to $\eta \in L^1_{loc}([0,\infty))$ and establish its well-posedness by comparison principle. Then we are able to define the notion of solutions for our original flow (1.1.3).

In section 1.3 we show that (1.1.1) preserves the ρ-reflection property. As in [FK14] our arguments are based on reflection comparisons. More precisely, for given $\nu, x_0 \in \mathbb{R}^n$ define $\Pi^+_{\nu,x_0} := \{x + x_0 : x \cdot \nu \geq 0\}$ and $\Pi_\nu := \partial \Pi^+_{\nu,x_0}$. Since the normal velocity law (1.1.1) is preserved with respect to spatial reflections, comparison principle applies in the region

$\Pi_\nu^+ \times [0,\infty)$ to Ω_t and Ω_t^{ν,x_0}, the reflected version of Ω_t with respect to Π_ν. It follows that if

$$\Omega_0^{\nu,x_0} \subset \Omega_0 \text{ in } \Pi_\nu^+, \tag{1.1.9}$$

then such property is preserved for later times. We will show that this property and the ρ-reflection property imply that $\partial\Omega_t$ is locally Lipschitz, as long as Ω_t contains a small neighborhood of the origin. Recall that (1.1.1) does not satisfy classical comparison principle. This is why we resort exclusively to this particular type of comparison arguments.

Section 1.4.1 yields uniqueness of solutions for (1.1.1). The proof is based on small-time uniqueness for star-shaped solutions of (1.1.8), and the Lipschitz continuity of λ given by Assumption A. In section 1.4.2, based on the discrete-time minimizing movement, we generate a *flat flow* of (1.1.1) characterized as the continuum gradient flow of the energy functional $J(E)$ given in (1.1.2). Let us mention that, due to the lack of comparison principle, we need strong convergence of the discrete flow to characterize the continuum limit. To this end we impose geometric constraints to the minimizing movement to generate sufficient compactness on the discrete flow: see Definition 1.4.13. Section 1.4.3 discusses coincidence of the two notions of solutions. Based on Proposition 1.4.17, we show in Theorem 1.4.21 that the flat flow is the unique viscosity solution of (1.1.8) with $\eta(t) = \lambda(|\Omega_t|)$.

In section 1.5.1 we introduce the approximation by (1.1.5) constructed by a constrained minimizing movement. Based on their geometric properties, we establish the first part of Theorem 3 (a) for the limiting set Ω_∞. Section 1.5.2 completes the statement of Theorem 3(a) and (b) by establishing a uniform L^2 bound of λ_δ, using the variational construction of solutions for (1.1.5). This leads to the weak convergence of λ_δ to λ_∞, While following the outline given by [MSS16], our construction of local variation is more delicate (Lemma 1.5.12 and Lemma 1.5.13), since the perturbed set needs to stay within our geometric constraints. Finally in section 1.5.3 we prove Theorem 3(c), by establishing the perimeter convergence of Ω_t^∞ as $t \to \infty$, using the L^2 bound on λ_∞ obtained in section 1.5.2.

1.2 Viscosity solutions

Equation (1.1.1) can be formulated in terms of level sets, which allows us to introduce the notion of viscosity solutions for the flow. More precisely, for $Q := \mathbb{R}^n \times (0,\infty)$ and $u : Q \to \mathbb{R}$, let us define

$$\Omega_t = \Omega_t(u) := \{x \in \mathbb{R}^n \mid u(x,t) > 0\} \text{ for } t \geq 0$$

and consider the following corresponding PDE of mean curvature flows with forcing:

$$\frac{u_t}{|Du|}(x,t) = \nabla \cdot \left(\frac{Du}{|Du|}\right)(x,t) + \lambda[|\Omega_t(u)|] \text{ for } (x,t) \in Q. \tag{1.2.1}$$

In this section, we introduce a weak notion of solutions for (1.2.1). To this end we first introduce $\eta(\cdot) : [0,+\infty) \to \mathbb{R}$ as an *a priori* known continuous function of time t, and consider

$$\frac{u_t}{|Du|}(x,t) = \nabla \cdot \left(\frac{Du}{|Du|}\right)(x,t) + \eta(t) \tag{1.2.2}$$

with initial data

$$u(x,0) = u_0(x) := \chi_{\Omega_0} - \chi_{\Omega_0^C} \text{ for } x \in \mathbb{R}^n. \tag{1.2.3}$$

We begin by a list of definitions.

- $Q := \mathbb{R}^n \times [0,\infty)$, $\quad Q_T := \mathbb{R}^n \times (0,T]$. For an open set $U \subset \mathbb{R}^n$, we define the parabolic cylinder $U_T := U \times (0,T]$ and the parabolic boundary of U_T, $\partial_p U_T := \overline{U}_T - U_T$.
- $D_r(x_0,t_0) := B_r(x_0) \times (t_0 - r^2, t_0]$, $\quad \partial_p D_r := (\overline{B}_r(x_0) \times \{t_0\}) \cup (\partial B_r(x_0) \times [t_0 - r^2, t_0])$.
- $C_{r,h}(x) := x + [-h,h] \times B_r^{n-1}(0)$, $C^+_{r,h}(x) := x + [0,h] \times B_r^{n-1}(0)$, $B_r^{n-1}(0) := \{x \in \mathbb{R}^{n-1}, |x| \leq 1\}$.
- We denote $\mathcal{S}^{n\times n}$ as the space of $n \times n$ real symmetric matrices.
- For $u : L \subset \mathbb{R}^d \to \mathbb{R}$ we denote its semicontinuous envelopes $u_*, u^* : \overline{L} \to \mathbb{R}$ by

$$u_*(x) := \lim_{\epsilon \downarrow 0} \inf_{\substack{|x-y|<\epsilon,\\ y\in L}} u(y) \quad \text{ and } u^*(x) := \lim_{\epsilon \downarrow 0} \sup_{\substack{|x-y|<\epsilon,\\ y\in L}} u(y). \tag{1.2.4}$$

- For a sequence of functions $\{u_k\}_{k\in\mathbb{N}}$ on Q,

$$\limsup_{k\to\infty}{}^* u_k(x,t) := \lim_{j\to\infty} \sup\left\{u_k(y,s) : k \geq j, \quad |y-x| \leq \frac{1}{j}, \quad |s-t| \leq \frac{1}{j}\right\}, \tag{1.2.5}$$

$$\liminf_{k\to\infty}{}_* u_k(x,t) := \lim_{j\to\infty} \inf\left\{u_k(y,s) : k \geq j, \quad |y-x| \leq \frac{1}{j}, \quad |s-t| \leq \frac{1}{j}\right\}. \tag{1.2.6}$$

- For a function $h : Q \to \mathbb{R}$ we denote its positive set by $\Omega_t(h) := \{x \in \mathbb{R}^n : h(x,t) > 0\}$ for $t \geq 0$.

- For a set U in $\mathbb{R}^d$ and $d \in \mathbb{N}$, we denote the signed distance function by

$$\mathrm{sd}(x,U) := \delta(x,U) - \delta(x,U^C). \tag{1.2.7}$$

We use the convention that $\mathrm{sd}(x,U) := \infty$ if U is empty and $\mathrm{sd}(x,U) := -\infty$ if U^c is empty.

1.2.1 Viscosity Solutions for continuous forcing

Now we recall the definition viscosity solutions for equations (1.2.2). Let us denote $A := (\mathbb{R}^n \setminus \{0\}) \times \mathcal{S}^{n\times n} \times [0,\infty)$ and define $F : A \to \mathbb{R}$ by

$$F(p,X,t) := \mathrm{trace}\left(\left(I - \frac{p}{|p|} \otimes \frac{p}{|p|}\right) X\right) + \eta(t)|p|.$$

Then, the equation (1.2.2) can be rewritten in the form of

$$u_t = F(Du, D^2u, t).$$

Since the set A is dense in $\mathbb{R}^n \times \mathcal{S}^{n\times n} \times [0,\infty)$, the envelopes F_* and F^* are well-defined in $\mathbb{R}^n \times \mathcal{S}^{n\times n} \times [0,\infty)$ with value in $\mathbb{R} \cup \{\pm\infty\}$.

Recall a test function from [IS13, Definition 3.2]. We say that a function $\phi : Q \to \mathbb{R}$ is *a test function* on Q is if ϕ is C^2 with respect to x and C^1 with respect to t.

Definition 1.2.1. [CGG91, Definition 2.1], [Bar13, Definition 6.1]

- A function $u : Q \to \mathbb{R}$ is *a viscosity subsolution* of (1.2.2) if $u^* < +\infty$ and for any test function ϕ on Q that touches u^* from above at (x_0,t_0) we have

$$\phi_t(x_0,t_0) \leq F^*(D\phi(x_0,t_0), D^2\phi(x_0,t_0), t_0).$$

- A function $u : Q \to \mathbb{R}$ is *a viscosity supersolution* of (1.2.2) if $u_* > -\infty$ and for any test function ϕ on Q that touches u_* from below at (x_0, t_0) we have

$$\phi_t(x_0, t_0) \geq F_*(D\phi(x_0, t_0), D^2\phi(x_0, t_0), t_0).$$

- A function $u : Q \to \mathbb{R}$ is *a viscosity solution* of (1.2.2) with initial data $u_0 : \mathbb{R}^n \to \mathbb{R}$ if u^* is *a viscosity subsolution* and u_* is *a viscosity supersolution*, and if $u^* = (u_0)^*$ and $u_* = (u_0)_*$ at $t = 0$.

- For any $\eta \in C([0, +\infty))$, $(\Omega_t)_{t \geq 0}$ is *a viscosity solution (subsolution or supersolution, respectively)* of

$$V = -H + \eta(t), \tag{1.2.8}$$

if $u := \chi_{\Omega_t} - \chi_{\Omega_t{}^c}$ is *a viscosity solution (subsolution or supersolution, respectively)* of (1.2.2)-(1.2.3).

Theorem 1.2.2.

1. *[GGI91, Theorem 2.1] Let $T > 0$ and U be a bounded domain in $\mathbb{R}^n$. Let u and v be a bounded subsolution and supersolution, respectively, of* (1.2.2)*. If $u^* \leq v_*$ on $\partial_p U_T$, then we have $u^* \leq v_*$ on U_T.*

2. *[BSS93, Theorem 1.1] For a given bounded domain $\Omega_0 \subset \mathbb{R}^n$ and uniformly continuous initial data $u_0 : \mathbb{R}^n \to \mathbb{R}$ such that $\Omega_0 = \{x \in \mathbb{R}^n : u_0(x) = 0\}$, there exists a unique viscosity solution u of* (1.2.2)*, which is uniformly continuous in Q.*

3. *[BSS93, Theorems 1.1] Let u and v be a uniformly continuous subsolution and supersolution, respectively, of* (1.2.2) *in Q. If $u(\cdot, 0) \leq v(\cdot, 0)$ in $\mathbb{R}^n$, then we have $u \leq v$ in Q.*

The following lemma is a consequence of the stability properties of viscosity solutions: see for instance Lemma 6.1 in [CIL92].

Lemma 1.2.3. *For $n \in \mathbb{N}$, let $u_n := \chi_{\Omega_t^n} - \chi_{(\Omega_t^n)^C}$ be a viscosity solution of* (1.2.2) *in Q. If $\partial\Omega_t^n$ converges to $\partial\Omega_t$ as $n \to \infty$ in Hausdorff distance, uniformly for all $t \geq 0$, then $u := \chi_{\Omega_t} - \chi_{\Omega_t^C}$ is a viscosity solution of* (1.2.2)*.*

Note that (1.2.2) is *g*eometric, that is F satisfies the scaling invariance

$$F(ap, aX + bp \otimes p, t) = aF(p, X, t) \tag{1.2.9}$$

for $a > 0$, $b \in \mathbb{R}$, $p \in \mathbb{R}^n$, $X \in \mathcal{S}^{n\times n}$ and $t \geq 0$. Thus, (1.2.2) has the following invariance of geometric equations.

Theorem 1.2.4. *[Gig06, Theorem 4.2.1] Let u and v be a subsolution and supersolution, respectively, of* (1.2.2)*. If $\phi : \mathbb{R} \to \mathbb{R}$ is upper semicontinous and nondecreasing, then the composite function $\phi \circ u$ is also a subsolution. Similarly, if $\phi : \mathbb{R} \to \mathbb{R}$ is lower semicontinous and nondecreasing, then $\phi \circ v$ is also a supersolution.*

Let v be a continuous viscosity solution of (1.2.2) with uniformly continuous initial data $u_0 : \mathbb{R}^n \to \mathbb{R}$ such that $\Omega_0 = \{x \in \mathbb{R}^n : u_0(x) = 0\}$. Based on the invariance in Theorem 1.2.4 and the stability of viscosity solutions in [CIL92, Lemma 6.1], we obtain a discontinuous viscosity solution u of (1.2.2) and (1.2.3) given by

$$u(x, t) = \chi_{\Omega_t(u)} - \chi_{(\Omega_t(u))^C} \text{ and } \Omega_t(u) = \Omega_t(v) \text{ for all } t \geq 0 \tag{1.2.10}$$

(See [BSS93, Theorem 2.1]). Note that $\Omega_t(u)$ satisfies (1.1.8) if $\partial\Omega_t$ is C^2. We will thus consider the set Ω_t obtained from the above viscosity solutions formulation as a weak notion of sets evolving by (1.1.8).

Remark 1.2.5. Note that in Theorem 1.2.2(1), we need $u^* \leq v_*$ at the initial time, so this theorem does not yield the uniqueness for discontinuous solutions. Indeed solutions of the form (1.2.10) may be non-unique due to the "fattening" of the zero level set, see the discussion in [BSS93], [ESS92], [Gig06] and [SS93]. We will show in section 1.4.1 that our solutions are unique under the geometric constraint on the initial data.

Definition 1.2.6.

- A function $u : Q \to \mathbb{R}$ is *a viscosity subsolution (supersolution)* of (1.2.1) and (1.2.3) if u is a viscosity subsolution (supersolution) of (1.2.2) and (1.2.3) with continuous and bounded $\eta(t) = \lambda(|\Omega_t(u)|)$. A function u is *a viscosity solution* of (1.2.1) and (1.2.3) if u is a viscosity solution of (1.2.2) and (1.2.3).

- For $\eta \in C([0, +\infty))$, $((\Omega_t)_{t\geq 0}, \lambda)$ be a viscosity solution of

$$V = -H + \lambda(|\Omega_t|) \tag{1.2.11}$$

 if $(\Omega_t)_{t\geq 0}$ is a viscosity solution of (1.2.8) with $\lambda(|\Omega_t|) = \eta(t)$.

Remark 1.2.7. For (1.2.1) and (1.2.3), the comparison principle fails, and thus viscosity solutions theory cannot be directly applied. Indeed the well-posedness of (1.2.1) and (1.2.3) will be established later in section 1.4.2.

Next we introduce a regularization that is often used in free boundary problems (see e.g. [CS05] and Lemma 3.1 in [Kim03]). This is useful in our geometric analysis in sections 1.4.1 and 1.4.3.

Lemma 1.2.8. *Consider a continuous function $l : [0, \infty) \to \mathbb{R}$ with $L(t) := \int_0^t l(s)ds \leq A$ in $[0, T]$. Let u be a viscosity supersolution of* (1.2.2)*. Then, the function*

$$\tilde{u}(x, t) := \inf_{y \in \overline{B}_{A-L(t)}(x)} u(y, t),$$

is a viscosity supersolution of

$$\tilde{u}_t = F(D\tilde{u}, D^2\tilde{u}, t) + l(t)|D\tilde{u}| \quad \text{in } Q_T. \tag{1.2.12}$$

Similarly, let u be a viscosity subsolution of (1.2.2)*. Then, the function*

$$\hat{u}(x, t) := \sup_{y \in \overline{B}_{A-L(t)}(x)} u(y, t)$$

is a viscosity subsolution of

$$\hat{u}_t = F(D\hat{u}, D^2\hat{u}, t) - l(t)|D\hat{u}| \quad \text{in } Q_T. \tag{1.2.13}$$

Proof. Let us show that the function $\tilde{u}$ is a viscosity supersolution of (1.2.12), the subsolution part can be proved with parallel arguments. For simplicity we will only present the proof for the case $l(t) = c > 0$, in which case $T = A/c$.

Suppose a test function ϕ touches $\tilde{u}_*$ from below at $(x_0, t_0) \in Q_T$. It holds that

$$\tilde{u}_*(x_0,t_0) - \phi(x_0,t_0) = 0 \text{ and } \tilde{u}_*(x,t) - \phi(x,t) \geq 0 \text{ in } \mathcal{N}_\delta(x_0,t_0) := B_\delta(x_0) \times (t_0-\delta, t_0] \tag{1.2.14}$$

for some $\delta > 0$. From the construction of $\tilde{u}_*$, there exists $x_1 \in \mathbb{R}^n$ such that

$$|x_1 - x_0| \leq A - ct_0 \text{ and } \tilde{u}_*(x_0,t_0) = u_*(x_1,t_0). \tag{1.2.15}$$

If $D\phi(x_0,t_0) = 0$, then it suffices to show that

$$\phi_t(x_0,t_0) \geq F_*(D\phi(x_0,t_0), D^2\phi(x_0,t_0), t_0). \tag{1.2.16}$$

We choose the shifted test function $\psi(x,t) := \phi(x - x_1 + x_0, t)$ and claim that ψ touches u_* from below at (x_1, t_0). As $c > 0$, (1.2.15) yields that

$$|x_1 - x_0| \leq A - ct \text{ for all } t \in (t_0 - \delta, t_0]. \tag{1.2.17}$$

From (1.2.14), we have $u_*(x_1,t_0) - \psi(x_1,t_0) = 0$. From we have $u_*(x_1,t_0) - \psi(x_1,t_0) = 0$ again and (1.2.17), it holds that

$$u_*(x,t) - \psi(x,t) \geq \tilde{u}_*(x - x_1 + x_0, t) - \phi(x - x_1 + x_0, t) \text{ for any } (x,t) \in \mathcal{N}_\delta(x_1,t_0) \tag{1.2.18}$$

which yields the claim. Since u_* is a viscosity supersolution of (1.2.2) we have the corresponding PDE inequality for ψ at (x_1, t_0), which translates to (1.2.16).

Next, we suppose that $D\phi(x_0,t_0) \neq 0$. If $|x_1 - x_0| < A - ct_0$, then $u(\cdot, t_0)$ is constant in a small neighborhood of x_0 in $\mathbb{R}^n$ and it holds that $D\phi(x_0,t_0) = 0$. Thus, we have $|x_1 - x_0| = A - ct_0$. We claim that the shifted test function $\psi(x,t) := \phi(x - (A - ct)\vec{n}, t)$ touches u_* from below at (x_1, t_0) where

$$\vec{n} := \frac{x_1 - x_0}{|x_1 - x_0|}.$$

First, note that $x_1 - (A - ct_0)\vec{n} = x_0$ and thus $u_*(x_1, t_0) - \psi(x_1, t_0) = 0$. Furthermore, if we choose $\varepsilon = \frac{1}{2}\min\{\delta, t_0\}$, then

$$(x - (A - ct)\vec{n}, t) = (x - x_1 + x_0 + c(t - t_0)\vec{n}, t) \in \mathcal{N}_\delta(x_0, t_0) \quad \text{for all } (x, t) \in \mathcal{N}_\varepsilon(x_1, t_0). \tag{1.2.19}$$

(1.2.14) and (1.2.19) imply that

$$u_*(x, t) - \psi(x, t) \geq \tilde{u}_*(x - (A - ct)\vec{n}, t) - \phi(x - (A - ct)\vec{n}, t) \geq 0 \quad \text{for all } (x, t) \in \mathcal{N}_\varepsilon(x_1, t_0),$$

which yields the claim.

As described in the first case, since u_* is a viscosity supersolution of (1.2.2) we have the corresponding PDE inequality for ψ at (x_1, t_0), which translates to

$$\phi_t(x_0, t_0) + cD\phi(x_0, t_0) \cdot \vec{n} \geq F_*(D\phi(x_0, t_0), D^2\phi(x_0, t_0), t_0). \tag{1.2.20}$$

Since $D\phi(x_0, t_0) \neq 0$ and the level set $\{x \in \mathbb{R}^n : \phi(x, t_0) = \phi(x_0, t_0)\}$ touches $\partial\Omega_{t_0}(\tilde{u})$ from inside at x_0, $-D\phi(x_0, t_0)$ is parallel to the outward normal $\vec{n}$ of $\partial\Omega_{t_0}(\tilde{u})$ at x_0. Therefore, (1.2.20) yields

$$\phi_t(x_0, t_0) \geq F_*(D\phi(x_0, t_0), D^2\phi(x_0, t_0), t_0) + c|D\phi(x_0, t_0)|.$$

Now we can conclude that the function $\tilde{u}$ is viscosity supersolution of (1.2.12).

In general, we choose the shifted test function $\psi(x, t) := \phi(x - x_1 + x_0, t)$ or $\phi(x - L(t)\vec{n}, t)$ and apply the parallel arguments to conclude. □

The following lemma will be used in section 3 to ensure uniform continuity of $\Omega_t(u)$ over time in Hausdorff distance.

Lemma 1.2.9. *Let u be a bounded viscosity solution of* (1.2.2) *given by the form* (1.2.10). *Then the following holds for $0 < \delta < \frac{1}{\|\eta\|_\infty}$: If $B_{2\delta}(x_0) \subset (\Omega_{t_0}(u))^C$ (or $\Omega_{t_0}(u)$), then $B_\delta(x_0) \subset (\Omega_t(u))^C$ (or $\Omega_t(u)$) for $t_0 \leq t \leq t_0 + \frac{\delta^2}{n}$.*

Proof. We will verify the case when $B_{2\delta}(x_0)$ lies outside of $\Omega_{t_0}(u)$, since the rest follows from a parallel barrier argument. Let us compare u with a radial barrier ϕ defined by

$$\phi := -\chi_{B_{r(t)}(x_0)} + \chi_{B_{r(t)}(x_0)^C},$$

where $r : \left[t_0, t_0 + \frac{\delta^2}{n}\right) \to \mathbb{R}$ solves $r(t_0) := 2\delta, r'(t) := -\frac{n-1}{\delta} - \|\eta\|_\infty$. By assumption $u^*(x, t_0) \leq \phi_*(x, t_0)$.

Let us show that ϕ is a viscosity supersolution for $t_0 \leq t \leq t_0 + \frac{\delta^2}{n}$. Since ϕ is a radial function, the normal velocity on $\partial\Omega_t(\phi)$ is equal to $-r'(t)$, and the mean curvature on $\partial\Omega_t(\phi)$ is $-\frac{n-1}{r(t)}$. Moreover, we have

$$r'(t) = -\frac{n-1}{\delta} - \|\eta\|_\infty \geq -\frac{n}{\delta}. \tag{1.2.21}$$

Since $r(t_0) = 2\delta$, it follows that $r(t) \geq \delta$ if $t_0 \leq t \leq t_0 + \frac{\delta^2}{n}$. Therefore, it holds that for $t_0 \leq t \leq t_0 + \frac{\delta^2}{n}$

$$-r'(t) = \frac{n-1}{\delta} + \|\eta\|_\infty \geq \frac{n-1}{r(t)} + \eta(t) \tag{1.2.22}$$

and we conclude. Now by Theorem 1.2.2(1), $u^* \leq \phi_*$ for $t_0 \leq t \leq t_0 + \frac{\delta^2}{n}$ and thus $B_\delta(x_0 + \delta\nu)$ lies outside of $\Omega_t(u)$ for $t_0 \leq t \leq t_0 + \frac{\delta^2}{n}$. □

1.2.2 Viscosity Solutions for L^1_{loc} forcing

In the level set formulation, Ω_t is given by $\Omega_t(u) := \{x \in \mathbb{R}^n : u(x,t) > 0\}$ where $u : Q \to \mathbb{R}$ solves the following equation:

$$u_t = F(Du, D^2u) + \lambda(t)|Du| \tag{1.2.23}$$

where $F : (\mathbb{R}^n \setminus \{0\}) \times \mathcal{S}^{n \times n} \to \mathbb{R}$ is given by

$$F(p, X) := \text{trace}\left(\left(I - \frac{p}{|p|} \otimes \frac{p}{|p|}\right) X\right) \tag{1.2.24}$$

with initial data

$$u(x, 0) = u_0(x) := \chi_{\Omega_0} - \chi_{\Omega_0^C} \text{ for } x \in \mathbb{R}^n. \tag{1.2.25}$$

We recall definitions of classical solutions and equivalent definitions of viscosity solutions of (1.2.23) with fixed $\lambda \in C([0, +\infty))$.

Definition 1.2.10. Consider a cylinder $D_r \subset Q$ and F given in (1.2.24).

- A function $\phi \in C^{2,1}(D_r)$ is *a classical subsolution* in D_r of (1.2.23) if it holds that

$$\phi_t \le F_*(D\phi, D^2\phi) + \lambda|D\phi| \text{ in } D_r. \tag{1.2.26}$$

- A function $\phi \in C^{2,1}(D_r)$ is *a classical supersolution* in D_r of (1.2.23) if it holds that

$$\phi_t \ge F^*(D\phi, D^2\phi) + \lambda|D\phi| \text{ in } D_r. \tag{1.2.27}$$

- We say that $\phi \in C^{2,1}(D_r)$ is *a classical strict subsolution (supersolution, respectively)* on D_r of (1.2.23) if the strict inequality of (1.2.26) ((1.2.27), respectively) holds in D_r

Definition 1.2.11. [CS05, Definition 7.2]

- A function $u : Q \to \mathbb{R}$ is *a viscosity subsolution* of (1.2.23) if $u^* < \infty$ and for $D_r \subset Q$ and for every classical strict supersolution $\phi \in C^{2,1}(D_r)$, $u^* < \phi$ on $\partial_p D_r$ implies $u^* < \phi$ in $\overline{D}_r$.

- A function $u : Q \to \mathbb{R}$ is *a viscosity supersolution* of (1.2.23) if $u_* > -\infty$ and $D_r \subset Q$ and for every classical strict subsolution $\phi \in C^{2,1}(D_r)$, $u_* > \phi$ on $\partial_p D_r$ implies $u_* > \phi$ in $\overline{D}_r$.

In this section, we develop a notion of viscosity solutions for (1.2.23) for a fixed λ in $L^1_{loc}([0,\infty))$. Some notations are in order. For a continuous function $\gamma : [0, +\infty) \to [0, +\infty)$, the sup convolution $\widehat{u}(\cdot;\gamma)$ and inf convolution $\widetilde{u}(\cdot;\gamma)$ is given by

$$\widehat{u}(x,t;\gamma) := \sup_{y \in \overline{B}_{\gamma(t)}(x)} u(y,t), \tag{1.2.28}$$

$$\widetilde{u}(x,t;\gamma) := \inf_{y \in \overline{B}_{\gamma(t)}(x)} u(y,t). \tag{1.2.29}$$

Note that $\widehat{u^*} = (\widehat{u})^*$ and $\widetilde{u_*} = (\widetilde{u})_*$ (See Lemma A.3.8).

Definition 1.2.12. For $\lambda \in L^1_{loc}([0,\infty))$, $\Lambda(t) := \int_0^t \lambda(s)ds$ and F given in (1.2.24),

- A function $u : Q \to \mathbb{R}$ is *a viscosity subsolution* of (1.2.23) if $u^* < \infty$ and for any $0 \le t_1 < t_2$ and $\Theta \in C^1((t_1,t_2)) \cap C([t_1,t_2])$ such that $\Theta \ge \Lambda$ in $[t_1,t_2]$, a function $\widehat{u} = \widehat{u}(\cdot;\Theta - \Lambda)$ given in (1.2.28) is a viscosity subsolution of

$$u_t = F(Du, D^2u) + \Theta'|Du| \tag{1.2.30}$$

in $\mathbb{R}^n \times (t_1, t_2)$ in the sense of Definition 1.2.1.

- A function $u : Q \to \mathbb{R}$ is *a viscosity supersolution* of (1.2.23) if $u_* > -\infty$ and for any $0 \le t_1 < t_2$ and $\Theta \in C^1((t_1,t_2)) \cap C([t_1,t_2])$ such that $\Theta \le \Lambda$ in $[t_1,t_2]$, a function $\widetilde{u} = \widetilde{u}(\cdot; -\Theta + \Lambda)$ given in (1.2.29) is a viscosity supersolution of (1.2.30) in $\mathbb{R}^n \times (t_1, t_2)$ in the sense of Definition 1.2.1.

- A function $u : Q \to \mathbb{R}$ is *a viscosity solution* of (1.2.23)-(1.2.25) (or (1.2.70)) if u^* is *a viscosity subsolution* of (1.2.23) and u_* is *a viscosity supersolution* of (1.2.23), and if $u^* = (u_0)^*$ and $u_* = (u_0)_*$ at $t = 0$.

We also define the corresponding notion of viscosity solutions for sets.

Definition 1.2.13. For $\lambda \in L^1_{loc}([0,\infty))$, $(\Omega_t)_{t\ge 0}$ is *a viscosity solution (subsolution or supersolution, respectively)* of $V = -H + \lambda(t)$ if $u := \chi_{\Omega_t} - \chi_{{\Omega_t}^c}$ is *a viscosity solution (subsolution or supersolution, respectively)* of (1.2.23)-(1.2.25) in the sense of Definition 1.2.12.

Remark 1.2.14. Note that for $\lambda \in C([0,+\infty))$, Definition 1.2.12 coincides with Definition 1.2.1. First of all, Lemma 1.2.8 implies that a viscosity subsolution (supersolution, respectively) in the sense of Definition 1.2.1 is that in the sense of Definition 1.2.12. On the other hand, if $\lambda \in C([0,+\infty))$, then $\Lambda \in C^1([0,+\infty))$. Thus, we can choose $\Theta = \Lambda$. As $\widehat{u}(\cdot;0) = \widetilde{u}(\cdot;0) = u$ in Q, we conclude that a viscosity subsolution (supersolution, respectively) in the sense of Definition 1.2.12 is that in the sense of Definition 1.2.1.

In the rest of this section, we develop existence and uniqueness results for (1.2.23). We first show the comparison principle in Theorem 1.2.15, which yields uniqueness (Corollary 1.2.16). Moreover, we show the stability of viscosity solutions of $V = -H + \lambda_k(t)$ for $\{\lambda_k\}_{k\in\mathbb{N}} \subset L^1_{loc}([0,\infty))$ when a sequence of time integrals of λ_k converges. This yields existence (Corollary 1.2.20).

Theorem 1.2.15. *For $\lambda \in L^1_{loc}([0,+\infty))$, let $u : Q \to \mathbb{R}$ and $v : Q \to \mathbb{R}$ be a viscosity subsolution and supersolution of* (1.2.23)*, respectively, in the sense of Definition 1.2.12. If for some $r > 0$ and $(x_0,t_0) \in Q$ such that $D_r(x_0,t_0) \subset Q$ we have $u^* \le v_*$ on $\partial_p D_r(x_0,t_0)$,* *then*

$$u^* \le v_* \ \textit{on}\ D_r(x_0, t_0). \tag{1.2.31}$$

Proof. For simplicity, consider $(x_0, t_0) = (0, r^2)$ and denote $D_r := D_r(0, r^2) = B_r(0) \times (0, r^2]$. Note that we may assume the following, by adding a small constant to v:

$$u^* < v_* \text{ on } \partial_p D_r. \tag{1.2.32}$$

1. Let us show that there exists $\varepsilon_1 > 0$ such that

$$\widehat{u}^*(\cdot; \varepsilon_1) < \widetilde{v}_*(\cdot; \varepsilon_1) \text{ on } \partial_p D_r. \tag{1.2.33}$$

Suppose that (1.2.33) does not hold for all $\varepsilon_1 > 0$. Then, there exists a sequence $\{\xi_k^1 = (x_k, t_k)\}_{k\in\mathbb{N}} \subset \partial_p D_r$ such that

$$\widehat{u}^* \left(\xi_k^1; \frac{1}{k} \right) \ge \widetilde{v}_* \left(\xi_k^1; \frac{1}{k} \right). \tag{1.2.34}$$

By the semicontinuity of u^* and v_*, there exists $\{(\xi_k^2, \xi_k^3)\}_{k\in\mathbb{N}} \subset Q \times Q$ such that

$$|\xi_k^1 - \xi_k^2| \le \frac{1}{k}, \quad |\xi_k^1 - \xi_k^3| \le \frac{1}{k} \tag{1.2.35}$$

and

$$u^*(\xi_k^2) \ge v_*(\xi_k^3). \tag{1.2.36}$$

By compactness of $\overline{D}_{r+1}$, there exists a subsequence $\{k_i\}_{i\in\mathbb{N}}$ and $(\xi_\infty^2, \xi_\infty^3) \in Q \times Q$ such that $\{(\xi_{k_i}^2, \xi_{k_i}^3)\}_{i\in\mathbb{N}}$ converges to $(\xi_\infty^2, \xi_\infty^3)$. From (1.2.35) and the closedness of $\partial_p D_r$, we conclude that $\xi_\infty^2 = \xi_\infty^3 \in \partial_p D_r$. From (1.2.36) and the semicontinuity of u^* and v_*, it holds that

$$u^*(\xi_\infty^2) \ge \limsup_{i\to\infty} u^*(\xi_{k_i}^2) \ge \liminf_{i\to\infty} v_*(\xi_{k_i}^3) \ge v_*(\xi_\infty^3) = v_*(\xi_\infty^2). \tag{1.2.37}$$

This contradicts to (1.2.32).

2. Note that $C^1([0, r^2])$ is dense in $C([0, r^2])$. There exists $\Theta \in C^1([0, r^2])$ such that

$$\sup_{t\in[0,r^2]} |\Lambda(t) - \Theta(t)| \le \frac{\varepsilon_1}{2} \tag{1.2.38}$$

where $\varepsilon_1 > 0$ is given in Step 1. Then, $\widehat{u}^*(\cdot; \frac{\varepsilon_1}{2} + \Theta(t) - \Lambda(t))$ and $\widetilde{v}_*(\cdot; \frac{\varepsilon_1}{2} - \Theta(t) + \Lambda(t))$ are well-defined in D_r. Note that $\widehat{u}^*$ and $\widetilde{v}_*$ given above are respectively viscosity subsolution and supersolutions of (1.2.30).

From (1.2.38) and (1.2.33), it holds that

$$\widehat{u}^* \left(\cdot; \frac{\varepsilon_1}{2} + \Theta(t) - \Lambda(t)\right) \leq \widehat{u}^*(\cdot; \varepsilon_1) < \widetilde{v}_*(\cdot; \varepsilon_1) \leq \widetilde{v}_* \left(\cdot; \frac{\varepsilon_1}{2} - \Theta(t) + \Lambda(t)\right) \tag{1.2.39}$$

on $\partial_p D_r$. From comparison principle for (1.2.30) in [CGG91, Theorem 4.1], we conclude that

$$\widehat{u}^* \left(\cdot; \frac{\varepsilon_1}{2} + \Theta(t) - \Lambda(t)\right) \leq \widetilde{v}_* \left(\cdot; \frac{\varepsilon_1}{2} - \Theta(t) + \Lambda(t)\right) \tag{1.2.40}$$

on D_r, which implies (1.2.31). □

Corollary 1.2.16. *For $\lambda \in L^1_{loc}([0, +\infty))$, let $u : Q \to \mathbb{R}$ be a viscosity solution of* (1.2.23) *in the sense of Definition 1.2.12. If $u^* = u_* = g$ on $\partial_p D_r$ for $g \in C(\partial_p D_r)$, then u is uniquely determined in D_r by g.*

Next we develop stability results for $\{\lambda_k\}_{k\in\mathbb{N}}$ such that $\{\Lambda_k\}_{k\in\mathbb{N}}$ uniformly converges to Λ_∞ where

$$\{\lambda_k\}_{k\in\mathbb{N}\cup\{+\infty\}} \subset L^1_{loc}([0, +\infty)) \text{ and } \Lambda_k(t) := \int_0^t \lambda_k(s)ds \text{ for } k \in \mathbb{N} \cup \{+\infty\}. \tag{1.2.41}$$

Note that for $T > 0$, the weak convergence of $\{\lambda_k\}_{k\in\mathbb{N}}$ in $L^p([0, T])$ for any $p \in (1, \infty]$ implies the uniform convergence of $\{\Lambda_k\}_{k\in\mathbb{N}}$ in $C([0, T])$. Thus, we obtain stability results for a weakly converging sequence in $L^p([0, T])$ for any $p \in (1, \infty]$ from Theorem 1.2.17 below. This results will be used Corollary 1.5.7.

Theorem 1.2.17. *For $\{\lambda_k\}_{k\in\mathbb{N}\cup\{+\infty\}}$ and $\{\Lambda_k\}_{k\in\mathbb{N}\cup\{+\infty\}}$ given in* (1.2.41)*, assume that $\{\Lambda_k\}_{k\in\mathbb{N}}$ locally uniformly converges to Λ_∞. Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence of viscosity subsolutions (supersolutions, respectively) of* (1.2.23) *with $\lambda = \lambda_k$ for all $k \in \mathbb{N}$ in the sense of Definition 1.2.12. If $u := \limsup_{k\to\infty}{}^* u_k < \infty$ ($u := \liminf_{k\to\infty}{}_* u_k > -\infty$, respectively), then u is a viscosity subsolution (supersolution, respectively) of* (1.2.23) *with $\lambda = \lambda_\infty$ in the sense of Definition 1.2.12.*

Proof. We only show the subsolution part, since the rest can be shown with parallel arguments. Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence of viscosity subsolutions.

1. Choose any $0 \le t_1 < t_2$ and $\Theta \in C^1((t_1,t_2)) \cap C([t_1,t_2])$ such that $\Theta \ge \Lambda_\infty$ in $[t_1,t_2]$. Let us show that $\widehat{u}(\cdot;\Theta-\Lambda_\infty)$ given in (1.2.28) is a viscosity subsolution of (1.2.30). From the equivalent definition of viscosity solutions in Definition 1.2.11, it is enough to show that for any $D_r \subset \mathbb{R}^n \times (t_1,t_2)$

$$\widehat{u}^*(\cdot;\Theta-\Lambda_\infty) < \phi \text{ in } D_r \tag{1.2.42}$$

where $\phi \in C^{2,1}(D_r)$ is a classical strict supersolution of (1.2.30) given in Definition 1.2.10 such that

$$\widehat{u}^*(\cdot;\Theta-\Lambda_\infty) < \phi \text{ on } \partial_p D_r. \tag{1.2.43}$$

First, as $u < +\infty$ and u is upper semicontinuous, we get $\widehat{u}^* < \infty$. Next, by the upper semicontinuity of u^*, there exists $\varepsilon_2 > 0$ such that

$$\widehat{u}^*(\cdot;\Theta-\Lambda_\infty) < \phi - 3\varepsilon_2 \quad \text{on } \partial_p D_r. \tag{1.2.44}$$

From the upper semicontinuity again, there exists $\varepsilon_1 > 0$ such that

$$\widehat{u}^*(\cdot;\varepsilon_1+\Theta-\Lambda_\infty) < \phi - 2\varepsilon_2 \quad \text{on } \partial_p D_r. \tag{1.2.45}$$

By uniform convergence of Λ_k, there exists $k_1 \in \mathbb{N}$ such that for all $k > k_1$, it holds that

$$\|\Lambda_\infty - \Lambda_k\|_{L^\infty([t_1,t_2])} < \frac{\varepsilon_1}{2}. \tag{1.2.46}$$

By definition, $\widehat{u}_k = \widehat{u}_k(\cdot;\varepsilon_1+\Theta-\Lambda_k)$ is a viscosity subsolutions of (1.2.30) in (t_1,t_2) for all $k > k_1$.

2. Let us show that there exists $k_2 \in \mathbb{N}$ such that $k_2 > k_1$ and

$$\widehat{u}_k^*(\cdot;\varepsilon_1+\Theta-\Lambda_k) < \phi - \varepsilon_2 \text{ on } \partial_p D_r \text{ for all } k \ge k_2 \tag{1.2.47}$$

where k_1 is given in Step 1. Suppose that such k_2 does not exist. Then, there exists a sequence $\{k_i\}_{i\in\mathbb{N}}$ converging to infinity and $\left\{\xi^1_{k_i} = (x_{k_i}, t_{k_i})\right\}_{i\in\mathbb{N}} \subset \partial_p D_r$ such that $k_i \ge k_1$ and

$$\widehat{u}_{k_i}^*(\xi^1_{k_i};\varepsilon_1+\Theta-\Lambda_{k_i}) \ge \phi(\xi^1_{k_i}) - \varepsilon_2 \text{ for all } i \in \mathbb{N}. \tag{1.2.48}$$

By the upper semicontinuity of u^*, there exists $\{\xi^2_{k_i}\}_{i\in\mathbb{N}} \subset Q$ such that

$$|\xi^2_{k_i} - \xi^1_{k_i}| \le \varepsilon_1 + \Theta - \Lambda_{k_i} \quad \text{and} \quad u^*_{k_i}(\xi^2_{k_i}) \ge \phi(\xi^1_{k_i}) - \varepsilon_2. \tag{1.2.49}$$

Furthermore, there exists $\{\xi^3_{k_i}\}_{i\in\mathbb{N}} \subset Q$ such that

$$|\xi^3_{k_i} - \xi^2_{k_i}| \le \frac{1}{k_i} \quad \text{and} \quad u_{k_i}(\xi^3_{k_i}) + \varepsilon_2 \ge u^*_{k_i}(\xi^2_{k_i}). \tag{1.2.50}$$

From (1.2.49) and (1.2.50), we get

$$|\xi^3_{k_i} - \xi^1_{k_i}| \le \varepsilon_1 + \Theta - \Lambda_{k_i} + \frac{1}{k_i} \quad \text{and} \quad u_{k_i}(\xi^3_{k_i}) \ge \phi(\xi^1_{k_i}) - 2\varepsilon_2. \tag{1.2.51}$$

As $\{\xi^1_{k_i}\}_{i\in\mathbb{N}} \subset \partial_p D_r$, (1.2.46) and (1.2.51) imply that

$$\{\xi^3_{k_i}\}_{i\in\mathbb{N}} \subset \overline{D}_{\widehat{r}} \quad \text{where} \quad \widehat{r} = r + 2\varepsilon_1 + \|\Theta - \Lambda_\infty\|_{L^\infty([t_1,t_2])} + 1. \tag{1.2.52}$$

From compactness of $\overline{D}_{\widehat{r}}$, there exists a subsequence $\{k_{i_j}\}_{j\in\mathbb{N}}$ and $(\xi^1_\infty, \xi^3_\infty) \subset Q \times Q$ such that $\{(\xi^1_{k_{i_j}}, \xi^3_{k_{i_j}})\}_{j\in\mathbb{N}}$ converges to $(\xi^1_\infty, \xi^3_\infty)$. (1.2.51) implies that

$$|\xi^3_\infty - \xi^1_\infty| \le \varepsilon_1 + \Theta - \Lambda_\infty \tag{1.2.53}$$

and

$$u(\xi^3_\infty) \ge \limsup_{j\to\infty} u_{k_{i_j}}(\xi^3_{k_{i_j}}) \ge \limsup_{j\to\infty} \phi(\xi^1_{k_{i_j}}) - 2\varepsilon_2 = \phi(\xi^1_\infty) - 2\varepsilon_2. \tag{1.2.54}$$

This contradicts to (1.2.45) and we conclude (1.2.47).

3. From Step 1 and (1.2.47), comparison principle in Theorem 1.2.15 implies that

$$\widehat{u}^*_k(\cdot; \varepsilon_1 + \Theta - \Lambda_k) < \phi - \varepsilon_2 \text{ in } D_r \text{ for all } k \ge k_2 \tag{1.2.55}$$

where ε_1 and ε_2 are given in (1.2.45), and k_2 is given in (1.2.47). The above and (1.2.46) imply that

$$u^*_k(y,t) < \phi(x,t) - \varepsilon_2 \text{ for all } (x,t) \in D_r \text{ and } y \in B_{\frac{\varepsilon_1}{2} + \Theta(t) - \Lambda_\infty(t)}(x) \text{ for all } k \ge k_2 \tag{1.2.56}$$

and we conclude (1.2.42). □

Let us construct radial barriers of (1.2.23).

Lemma 1.2.18. *Let $\lambda \in L^1([0,+\infty))$ and $\Lambda : [0,+\infty) \to \mathbb{R}$ defined by*

$$\Lambda(t) := \int_0^t \lambda(s)ds. \tag{1.2.57}$$

For a constant $c > \|\Lambda\|_{L^\infty([0,+\infty))}$, define $\zeta^- : Q \to \mathbb{R}$ and $\zeta^+ : Q \to \mathbb{R}$ by

$$\zeta^-(x,t;\Lambda,c) := -\chi_{\{x\in\mathbb{R}^n:|x|<c-\Lambda(t)\}}(x) \text{ and } \quad \zeta^+(x,t;\Lambda,c) := \chi_{\{x\in\mathbb{R}^n:|x|<c+\Lambda(t)\}}(x). \tag{1.2.58}$$

Then, ζ^- and ζ^+ are respectively a viscosity subsolution and a viscosity supersolution of (1.2.23) *in the sense of Definition 1.2.12.*

Proof. Let us show that ζ^- is a viscosity subsolution of (1.2.23) only. The respective one can be shown by parallel arguments.

Choose any $0 \le t_1 < t_2$ and $\Theta \in C^1((t_1,t_2)) \cap C([t_1,t_2])$ such that $\Theta \ge \Lambda$ in $[t_1,t_2]$. Let us show that $\widehat{\zeta^-}(\cdot;\Theta-\Lambda)$ given in (1.2.28) is a viscosity subsolution of (1.2.30). Note that we have

$$\widehat{\zeta^-}(x,t;\Theta-\Lambda) = -\chi_{\mathcal{N}_t}(x) \text{ where } \mathcal{N}_t := \{x \in \mathbb{R}^n : |x| < c - \Theta(t)\} \tag{1.2.59}$$

in Q.

Suppose that $\phi \in C^{2,1}(Q)$ touches $\widehat{\zeta^-}$ from above at (x_0,t_0). First, consider the case $|x_0| \ne c - \Theta(t_0)$. In this case, as $\mathcal{N}_t$ given in (1.2.59) moves continuously in time, $\widehat{\zeta^-}$ is constant near (x_0,t_0). Thus, it holds that

$$\phi_t(x_0,t_0) \le 0, \quad D\phi(x_0,t_0) = 0, \text{ and } D^2\phi(x_0,t_0) \ge 0. \tag{1.2.60}$$

The ellipticity of F given in (1.2.24) and (1.2.60) implies

$$\phi_t(x_0,t_0) \le F^*(D\phi(x_0,t_0), D^2\phi(x_0,t_0)) + \Theta'(t_0)|D\phi(x_0,t_0)|. \tag{1.2.61}$$

Let us consider the case $|x_0| = c - \Theta(t_0)$. If x_0 is a local minimum point of $\phi(\cdot,t_0)$ in $\mathbb{R}^n$, then by the parallel arguments above, we get (1.2.60) and (1.2.61). More precisely, in this case, we have $\phi(x_0,t_0) = 0$, $D\phi(x_0,t_0) = 0$ and $D^2\phi(x_0,t_0) \ge 0$. We claim that $\phi_t(x_0,t_0) \le 0$. As $\Theta \in C^1((t_1,t_2))$ and $\phi \in C^{2,1}(Q)$, we have for $t \in [0,+\infty)$

$$c - \Theta(t) = |x_0| + \Theta'(t)(t_0 - t) + o(|t - t_0|) \text{ as } t \to t_0 \tag{1.2.62}$$

and

$$\phi(x,t) = \phi_t(x_0,t_0)(t-t_0) + (x-x_0)^T D^2\phi(x_0,t_0)(x-x_0) + o(|t-t_0|) + o(|x-x_0|^2) \tag{1.2.63}$$

as $(x,t) \to (x_0,t_0)$ for $(x,t) \in Q$. As ϕ touches $\widehat{\zeta^-}$ from above at (x_0,t_0), there exists $\varepsilon > 0$ such that

$$\phi(x,t) \geq \widehat{\zeta^-}(x,t) \text{ for } (x,t) \text{ such that } |x-x_0| < 2|\Theta'(t_0)|\varepsilon \text{ and } t \in (t_0-\varepsilon, t_0). \tag{1.2.64}$$

For $t \in (t_0-\varepsilon, t_0)$ we define

$$y = y(t) := \begin{cases} |c-\Theta(t)|\frac{x_0}{|x_0|} & \text{if } x_0 \neq 0, \\ |c-\Theta(t)|e_1 & \text{if } x_0 = 0. \end{cases} \tag{1.2.65}$$

Note that $\widehat{\zeta^-}(y(t),t) = 0$ for $t_0 - \varepsilon < t < t_0$, and from (1.2.62) we have $|y(t)-x_0| = |\Theta'(t_0)(t_0-t)| + o(|t-t_0|)$. Thus (1.2.63) and (1.2.64) yield that for a sufficiently small $\varepsilon_1 > 0$ we have

$$\phi(y(t),t) = \phi_t(x_0,t_0)(t-t_0) + o(|t-t_0|) \geq \widehat{\zeta^-}(y(t),t) = 0 \text{ as } t \to t_0 \text{ for } t_0 - \varepsilon_1 < t < t_0. \tag{1.2.66}$$

We conclude that $\phi_t(x_0,t_0) \leq 0$ and therefore (1.2.61) has been verified.

Now, suppose that x_0 is not a local minimum point of $\phi(\cdot,t_0)$. Then, both $\mathcal{N}_t$ given in (1.2.59) and a sublevel set O_t of ϕ defined by

$$O_t := \{x \in \mathbb{R}^n : \phi(x,t) < \phi(x_0,t_0)\} \tag{1.2.67}$$

are nonempty near (x_0,t_0). By comparing the normal velocity and mean curvature of the level sets $\mathcal{N}_t$ and O_t, we conclude that

$$\frac{\phi_t}{|D\phi|}(x_0,t_0) \leq \Theta'(t_0) \text{ and } \nabla \cdot \left(\frac{D\phi}{|D\phi|}\right)(x_0,t_0) \geq \frac{n-1}{|x_0|} > 0 \tag{1.2.68}$$

which implies (1.2.61). □

Let us recall C_a from [CGG91] for $\mathcal{N} \subset \mathbb{R}^k$, $k \in \mathbb{N}$ and $a \in \mathbb{R}$,

$$C_a(\mathcal{N}) := \{g \in C(\mathcal{N}) : g - a \text{ has compact support in } \mathcal{N}\} \tag{1.2.69}$$

and consider continuous initial data $g \in C_a(\mathbb{R}^n)$,

$$u(x,0) = u_0(x) := g(x) \text{ for } x \in \mathbb{R}^n. \tag{1.2.70}$$

such that $\{x \in \mathbb{R}^n : g(x) > 0\} = \Omega_0$ and $\{x \in \mathbb{R}^n : g(x) < 0\} = (\overline{\Omega_0})^C$.

From Theorem 1.2.15 and Theorem 1.2.17 combining with radial barriers in Lemma 1.2.18, we get existence and uniqueness of (1.2.23) with continuous initial data.

Theorem 1.2.19. *For $\lambda \in L^1([0,T])$ and $T > 0$, there is a unique viscosity solution u in $C_a(Q_T)$ of* (1.2.23)-(1.2.70) *in the sense of Definition 1.2.12.*

Proof. As $C^1([0,T])$ is dense in $C([0,T])$, there exists $\{\Theta_k\}_{k\in\mathbb{N}} \subset C^1([0,T])$ such that $\{\Theta_k\}_{k\in\mathbb{N}}$ uniformly converges to Λ in $C([0,T])$. From the existence of viscosity solutions in [CGG91, Theorem 6.8] of

$$u_t = F(Du, D^2u) + (\Theta_k)'|Du| \text{ in } Q \tag{1.2.71}$$

with initial data (1.2.70), there exists a sequence of viscosity solutions $\{u_k\}_{k\in\mathbb{N}} \subset C_a(Q_T)$ of (1.2.71)-(1.2.70). Here, F and C_a are given in (1.2.24) and (1.2.69), respectively.

Define $u^+ := \limsup_{k\to\infty}{}^* u_k$ and $u^- := \liminf_{k\to\infty}{}_* u_k$. As $g \in C_a(\mathbb{R}^n)$, from Theorem 1.2.15 we have $\|u_k\|_{L^\infty} \leq \|g\|_{L^\infty}$ and thus $\|u^\pm\|_{L^\infty} < +\infty$. We claim furthermore that the supports of $\{u_k - a\}_{k\in\mathbb{N}}$ are uniformly bounded in Q_T for all $k \in \mathbb{N}$ and thus that $u^\pm - a$ is compactly supported in Q_T. To check this claim, one can compare $\{u_k - a\}_{k\in\mathbb{N}}$ with radial barriers

$$(\|g\|_{L^\infty} + |a|)\, \zeta^\pm(\cdot; \Theta_k, c_0)$$

with $\zeta^\pm$ given in Lemma 1.2.18 and with sufficiently large $c_0 > 0$.

Let us show that

$$u^+ = u^- \quad \text{in } Q_T. \tag{1.2.72}$$

First, by definition of lim sup* and lim inf$_*$ in (1.2.5), it holds that

$$u^+ \geq u^- \quad \text{in } Q_T. \tag{1.2.73}$$

On the other hand, from the uniform convergence of $\{\Theta_k\}_{k\in\mathbb{N}}$ to Λ in $C([0,T])$ and Theorem 1.2.17, u^+ and u^- are a viscosity subsolution and supersolution of (1.2.23)-(1.2.70), respectively. Recall that g is continuous in $\mathbb{R}^n$ and $u_k = g$ at $t = 0$ for all $k \in \mathbb{N}$. Indeed, by comparing solutions of (1.2.23) with radial barriers, we can check that $(u^+)^* = (u^-)_* = g$ at $t = 0$. In the following argument we assume that $g \geq 0$ since otherwise we can always add a constant. To check this, suppose that $g(x_0) > c$ for some constant c. Due to the continuity of g there is a small ball $B_r(x_0)$ which lies in the set $\{g > c\}$. Now consider the radial barrier

$$\phi_k(x,t) := c\chi_{B_{r(t)}(x_0)}, \text{ where } r_k(t) := r - \frac{2(n-1)t}{r} + \Theta_k(t) - \Theta_k(0).$$

Using the fact that Θ_k uniformly converges to Λ in $C([0,T])$ as $k \to \infty$, one can choose a small $\delta > 0$ independent of k and $k_0 \in \mathbb{N}$ such that $r_k(t) > \frac{r}{2}$ for all $0 \leq t \leq \delta$ and $k \geq k_0$. Thus ϕ_k is a well-defined subsolution of (1.2.71) for $0 \leq t \leq \delta$ and $k \geq k_0$. Since $\phi_k(\cdot,0) \leq g = u_k(\cdot,0)$, it follows from the comparison principle of (1.2.71) that $\phi_k(\cdot,t) \leq u_k(\cdot,t)$ for all $k \geq k_0$ and for $0 \leq t \leq \delta$. Thus we can conclude that

$$c \leq u_k(y,t) \quad \text{for all } k \geq k_0, \quad |y - x_0| < \frac{r}{2} \text{ and } 0 \leq t \leq \delta.$$

This yields that $g(x_0) \leq (u^\pm)_*(x_0,0)$. Since x_0 was arbitrary it follows from that $g \leq (u^\pm)_*(\cdot,0)$. A parallel argument replacing ϕ by another radial barrier of the form $\max g - (\max g - b)\chi_{B_{r(t)}(x_0)}$ in the set $\{g < b\}$ for some b, we can conclude that $(u^\pm)^*(\cdot,0) \leq g$.

Thus, $(u^+)^* = (u^-)_* = g$ at $t = 0$ and Theorem 1.2.15 implies

$$u^+ \leq u^- \quad \text{in } Q_T. \tag{1.2.74}$$

Therefore, we get (1.2.72) from (1.2.73) and (1.2.74). From Corollary 1.2.16, we conclude that $u^+(= u^-)$ is a unique viscosity solution in $C_a(\mathbb{R}^n)$ of (1.2.23)-(1.2.70). □

From parallel arguments in the proof of [BSS93, Theorem 2.1], we conclude existence of (1.2.23)-(1.2.25).

Corollary 1.2.20. *For $\lambda \in L^1_{loc}([0,+\infty))$, there exists a viscosity solution of* (1.2.23)-(1.2.25) *in the sense of Definition 1.2.12.*

Remark 1.2.21. Note that for continuous λ in (1.2.23), our notion coincides in Definition 1.2.12 with that of [Ish85], [Bou08b] and [Bou08b]. While the settings in these papers are different than ours, both our and their notions are shown to be stable under strong L^1-convergence of operators. Thus we expect that our notion coincides with an appropriate extension of [Ish85] for (1.2.23) with $\lambda \in L^1_{loc}([0,+\infty))$.

1.3 Geometry of the flow

In this section we study geometric properties of evolution of (1.2.2), following a strong notion of star-shapedness, *ρ-reflection*. This property, introduced in [FK14], is useful for problems which satisfy the reflection comparison principle (See Theorem 1.3.5 below).

1.3.1 Geometric properties

Definition 1.3.1. A bounded set Ω in $\mathbb{R}^n$ is *star-shaped with respect to a ball* $B_r(0)$ if for any point $y \in B_r(0)$, Ω is star-shaped with respect to y. Let

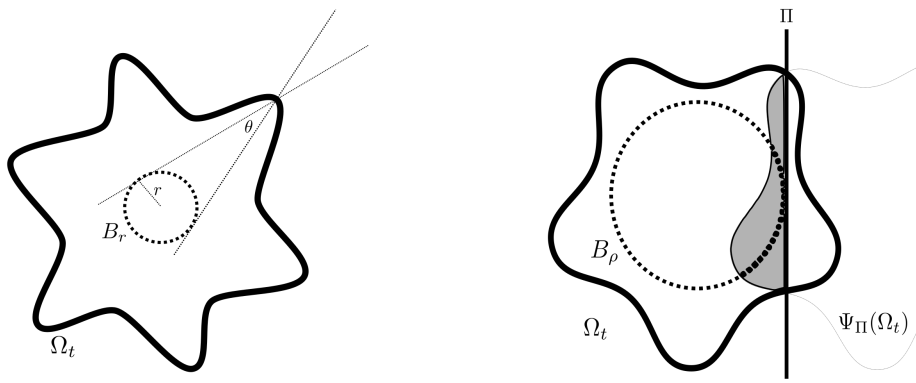
$$S_r := \{\Omega : \textit{star-shaped with respect to } B_r(0)\} \text{ and } S_{r,R} := S_r \cap \{\Omega : \Omega \subset B_R(0)\}.$$

The following lemma is immediate from the interior and exterior cone properties of sets in S_r.

Lemma 1.3.2. *For a continuously differentiable and bounded function $\phi : \mathbb{R}^n \to \mathbb{R}$, let us denote the positive set of ϕ by $\Omega(\phi)$. Let us assume that $\Omega(\phi)$ contains $B_r(0)$ and $D\phi \neq 0$ on $\partial\Omega(\phi)$. Then the set $\Omega(\phi)$ is in S_r if and only if*

$$x \cdot \vec{n}_x = x \cdot \left(-\frac{D\phi}{|D\phi|}(x)\right) \geq r \textit{ for all } x \in \partial\Omega(\phi),$$

where $\vec{n}_x$ denotes the outward normal of $\partial\Omega(\phi)$ at x.



(a) Star-shapedness (b) ρ-reflection

Figure 1.1: Geometric properties

Next we proceed to define the reflection property. For a hyperplane $\Pi = \Pi_\nu(s) := \{x : x \cdot \nu = s\}$, let Ψ_Π denote the corresponding reflection, i.e.,

$$\Psi_{\Pi(s)}(x) := x - 2\langle x - s\nu, \nu\rangle\nu. \tag{1.3.1}$$

For Π that doesn't contain the origin, we denote the half-spaces divided by Π by Π^+ and Π^-, where Π^- contains the origin.

Definition 1.3.3. [FK14, Definition 10] bounded, open set Ω has *ρ-reflection* if

(i) Ω contains $\overline{B_\rho(0)}$ and

(ii) Ω satisfies, for all $\nu \in S^{n-1}$ and all $s > \rho$,

$$\Psi_{\Pi_\nu(s)}(\Omega \cap \Pi_\nu^+(s)) \subset \Omega \cap \Pi_\nu^-(s).$$

The *ρ-reflection* property can be viewed as a smallness condition on the Lipschitz norm distance between $\partial\Omega$ and the nearest ball (see the Appendix in [FK14]). The following lemma states several properties and the relationship between the two concepts introduced above, *ρ-reflection* and S_r (See Figures 1.1, 1.2 and [FK14, Figure 2]).

Lemma 1.3.4. *[FK14, Lemma 3, 9, 10, 24]*

1. *For a bounded domain Ω containing $B_r(0)$, the followings are equivalent:*

(i) $\Omega \in S_r$.

(ii) There exists $\varepsilon_0 = \varepsilon_0(r) > 0$ *such that*

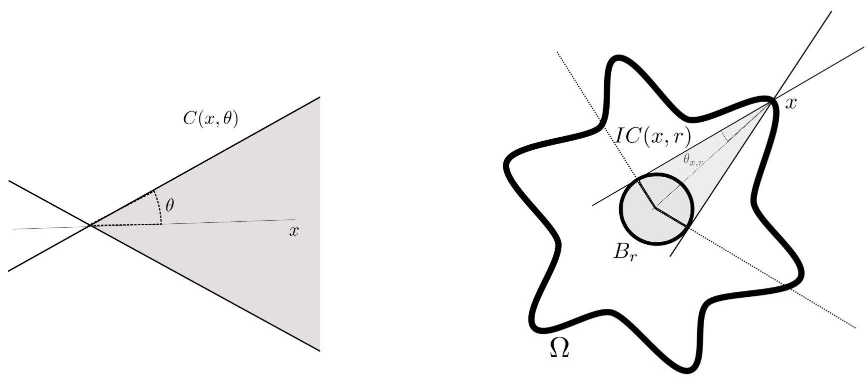
$$\Omega \subset\subset \bigcap_{|z|\le\delta\varepsilon} [(1+\varepsilon)\Omega + z] \text{ for all } 0 < \varepsilon < \varepsilon_0 \text{ and } 0 < \delta < r, \tag{1.3.2}$$

(iii) For all $x \in \Omega$, *there is an interior cone to* Ω*:*

$$IC(x,r) := \Big((x + C(-x,\theta_{x,r})) \cap C(x, \frac{\pi}{2} - \theta_{x,r})\Big) \cup B_r(0) \subset \Omega \text{ for } |x| \ge r \tag{1.3.3}$$

where $C(x,\theta)$ *is a cone in the direction* x *with opening angle* θ *for* $x \in \mathbb{R}^n$ *and* $\theta \in [0,\pi]$,

$$C(x,\theta) := \{y \mid \langle x,y\rangle > \cos\theta |x||y|\} \text{ and } \theta_{x,r} := \arcsin\frac{r}{|x|} \in \Big[0, \frac{\pi}{2}\Big]. \tag{1.3.4}$$



(a) A cone in the direction x with opening angle θ

(b) An interior cone to Ω

Figure 1.2: The cone property

(iv) For all $x \in \Omega^C$, *there is an exterior cone to* Ω*:*

$$EC(x,r) := x + C(x,\theta_{x,r}) \subset \Omega^C \text{ where } \theta_{x,r} = \arcsin\frac{r}{|x|} \in \Big[0, \frac{\pi}{2}\Big]. \tag{1.3.5}$$

2. *Suppose that* Ω *has* ρ*-reflection. Then* $\Omega \in S_r$ *with*

$$r = (\inf_{x\in\partial\Omega} |x|^2 - \rho^2)^{1/2}. \tag{1.3.6}$$

Moreover

$$\sup_{x\in\partial\Omega} |x| - \inf_{x\in\partial\Omega} |x| \le 4\rho. \tag{1.3.7}$$

3. *Suppose that* Ω *is in* $S_{r,R}$. *If there exists* $\rho > 0$ *such that* $\overline{B_\rho(0)} \subset \Omega$ *and* $\rho^2 \geq 5(R^2 - r^2)$, *then* Ω *has* ρ*-reflection.*

Theorem 1.3.5. *(Reflection Comparison) Suppose that* Ω_0 *has* ρ*-reflection. Let* u *be a bounded viscosity solution of* (1.2.2) *given by the form* (1.2.10). *Let* Π *be a hyperplane in* $\mathbb{R}^n$ *such that* $\Pi \cap B_\rho(0) = \emptyset$. *Then the reflected function* $u(\Psi_\Pi(x), t)$ *is also a bounded viscosity solution in* $\Pi^- \times (0, \infty)$. *Moreover*

$$\Psi_\Pi(\Omega_t \cap \Pi^+) \subset \Omega_t \cap \Pi^- \text{ for all } t \geq 0 \text{ if true at } t = 0. \tag{1.3.8}$$

Proof. It is easy to see that $u(\Psi_\Pi(x), t)$ is also a viscosity solution of (1.2.2) since F is independent of x.

To show (1.3.8), we will use the comparison principle in $\Pi^- \times [0, \infty)$. To do so it is easier for us to consider a continuous version of u, i.e. let $\tilde{u}$ be the unique viscosity solution of (1.2.2) with uniformly continuous initial data $\tilde{u}(x, 0)$ defined by $\tilde{u}(x, 0) := -\min\{\mathrm{sd}(x, \Omega_0), 2R\}$, where R is chosen large enough that $\Omega_0 \subset\subset B_R$. As u is given by the form (1.2.10), Theorem 1.2.4 combined with the uniqueness implies that $\Omega_t(\tilde{u})$ is equal to $\Omega_t(u)$ for all $t \geq 0$.

Note that Theorem 1.2.2(2) implies that $\tilde{u}$ is uniformly continuous. As $\tilde{u}(\cdot, 0)$ is bounded in $\mathbb{R}^n$, we apply Theorem 1.2.2(3) to conclude that $\tilde{u}$ is bounded in Q. Since $\tilde{u}(\Psi_\Pi(x), 0) \leq \tilde{u}(x, 0)$ in Π^- and $\tilde{u}(\Psi_\Pi(x), 0) = \tilde{u}(x, 0)$ on Π, Theorem 1.2.2(1) applies to $\tilde{u}(x, t)$ and $\tilde{u}(\Psi_\Pi(x), t)$ to yield

$$\tilde{u}(\Psi_\Pi(x), t) \leq \tilde{u}(x, t)$$

for all $x \in \Pi^-$ and $t \geq 0$. Therefore (1.3.8) follows. □

Theorem 1.3.6. *Suppose that* Ω_0 *has* ρ*-reflection. Let* u *be a bounded viscosity solution of* (1.2.2) *given by the form* (1.2.10). *Let* $I = [0, T)$ *be the maximal interval satisfying* $\overline{B_\rho} \subset \Omega_t(u)$. *Then,* $\Omega_t(u)$ *has* ρ*-reflection for* $t \in I$.

Proof. From the definition of *ρ-reflection*, it is enough show that, for any unit vector ν in $\mathbb{R}^n$,

$$\Psi_{\Pi_\nu(\rho)}(\Omega_t(u) \cap \Pi_\nu^+(\rho)) \subset \Omega_t(u) \cap \Pi_\nu^-(\rho) \text{ for } t \in I. \tag{1.3.9}$$

Since $\Omega_0(u)$ has *ρ-reflection*, (1.3.9) holds at $t = 0$, and we can conclude by Theorem 1.3.5. $\square$

In the next section, we will show that $\Omega_t(u) \in S_{r,R}$ in $[0, T]$ if it starts with some geometric restriction for the initial data. This leads to the following regularity of $\Omega_t(u)$ over time.

Corollary 1.3.7. *Let u be a bounded viscosity solution of (1.2.2) given by the form (1.2.10). Assume that $\Omega_0 \in S_{r,R}$ and $|\eta(t)| \leq K$ in $[0, T]$. Then, there exists $\mathcal{M}_1 = \mathcal{M}_1(r, R, K, T)$ such that*

$$\sup_{x \in \partial\Omega_t(u)} d(x, \partial\Omega_0) \leq \mathcal{M}_1 t^{\frac{1}{2}} \text{ for } t \in [0, T]. \tag{1.3.10}$$

In particular, if $\Omega_t(u) \in S_{r,R}$, then there exists $C = C(r, R, K, T)$ such that we have

$$d_H(\partial\Omega_t(u), \partial\Omega_s(u)) \leq C(s-t)^{\frac{1}{2}} \text{ for all } 0 \leq t \leq s \leq T. \tag{1.3.11}$$

Proof. Choose $\delta \in \left(0, \min\left\{\frac{1}{K}, \frac{r}{2}\right\}\right)$ and $t \in [0, T]$. We claim that

$$\sup_{x \in \partial\Omega_t(u)} d(x, \partial\Omega_s(u)) \leq \frac{2R\delta}{r} \text{ for all } s \in I := \left[t, \min\left\{t + \frac{\delta^2}{n}, T\right\}\right]. \tag{1.3.12}$$

As $\Omega_t(u) \in S_{r,R}$, there exists $x_1 = x_1(s) \in \partial\Omega_t(u)$ such that

$$\sup_{x \in \partial\Omega_t(u)} d(x, \partial\Omega_s(u)) = d(x_1, \partial\Omega_s(u)) \text{ for } s \in I. \tag{1.3.13}$$

Let $y = \left(1 - \frac{2\delta}{r}\right) x$ and $z = \left(1 + \frac{2\delta}{r}\right) x$. From the interior and exterior cone properties in Lemma 1.3.4, it holds that

$$B_{2\delta}(y) \subset \Omega_t(u) \text{ and } B_{2\delta}(z) \subset \Omega_t(u)^C.$$

As the assumption in Lemma 1.2.9 is satisfied, we conclude that $y \in \Omega_s(u)$ and $z \in \Omega_s(u)^C$ for all $s \in I$. As $\Omega_s(u) \in S_{r,R}$ in I, there exists $x_2 \in \partial\Omega_s(u)$ such that

$$|x_1 - x_2| \leq \max\{|x_1 - y|, |x_1 - z|\} \leq \frac{2R\delta}{r}. \tag{1.3.14}$$

(1.3.13) and (1.3.14) imply (1.3.12). Thus, we get (1.3.10). As $\Omega_s(u), \Omega_t(u) \in S_{r,R}$, we apply Lemma A.3.6 to conclude (1.3.11). $\square$

1.3.2 Preservation of the ρ-*reflection* property

In this subsection, we suppose that there exists a viscosity solution u of our original equation (1.2.1) in the sense of Definition 1.2.6, and show the preservation of the ρ-*reflection* property. As a consequence, star-shapedness of $\Omega_t(u)$ is preserved for all time. Existence of this solution will be given later in section 1.4.2 and 1.4.3.

Theorem 1.3.8. *Suppose that Ω_0 has ρ-reflection. Assume that there exists a bounded viscosity solution u given by the form* (1.2.10) *of* (1.2.1) *and* (1.2.3)*. Then $\Omega_t(u)$ has ρ-reflection for all $t \geq 0$. In particular there exists $r_1 > 0$ such that Ω_t is star-shaped with respect to a ball $B_{r_1}(0)$ for all $t \geq 0$.*

The proof of above theorem consists of Theorem 1.3.6 and Lemma 1.3.9. In Lemma 1.3.9, we show that the maximal interval I in Theorem 1.3.6 is $[0, \infty)$.

Lemma 1.3.9. *Let u and Ω_0 be as given in above theorem. Then, there exists $a > 0$ depending on Ω_0 such that $B_{(1+a)\rho} \subset \Omega_t(u)$ for all $t \geq 0$.*

Proof. Since Ω_0 has ρ-*reflection*, $B_{(1+a)\rho} \subset \Omega_0$ for some $a > 0$. Due to Assumption A and the continuity of λ, one can choose a small $a > 0$ such that

$$\lambda[|\Omega|] > \frac{n-1}{\rho} \text{ for sets contained in } B_{(5+a)\rho}. \tag{1.3.15}$$

Suppose that $B_{(1+a)\rho}$ is not contained in $\Omega_{t_*}(u)$ at some $t_* > 0$. Then, there exists $t_0 \in (0, t_*)$ such that $\partial\Omega_t(u)$ touches from outside $\partial B_{(1+a)\rho}$ at (x_0, t_0) for the first time. Then, by (1.3.7) in Lemma 1.3.4, we have

$$\sup_{x \in \partial\Omega_{t_0}(u)} |x| \leq 4\rho + \inf_{x \in \partial\Omega_{t_0}(u)} |x| = (5+a)\rho,$$

and thus $\Omega_{t_0}(u)$ is contained in $B_{(5+a)\rho}$. Hence it follows from (1.3.15) that

$$\lambda[|\Omega_{t_0}(u)|] > \frac{n-1}{\rho} > H[B_{(1+a)\rho}]. \tag{1.3.16}$$

where $H[B_{(1+a)\rho}]$ is the mean curvature of $\partial B_{(1+a)\rho}$.

Consider $\phi(x) := -\left(\frac{|x|}{(1+a)\rho}\right)^2$. Note that (1.3.16) and $|x_0| = (1+a)\rho$ yield

$$\nabla \cdot \left(\frac{D\phi}{|D\phi|}\right)(x_0) + \lambda[|\Omega_{t_0}(u)|] = -H[B_{(1+a)\rho}] + \lambda[|\Omega_{t_0}(u)|] > 0 \tag{1.3.17}$$

Hence $\psi(x,t) := \phi(x)$ is a strict subsolution of (1.2.2) with $\eta(t) = \lambda[|\Omega_t(u)|]$ in a small neighborhood of (x_0, t_0).

On the other hand, we have $\psi \leq 0$ in Q and $\psi \leq -1$ outside of $B_{(1+a)\rho}$. Recall that u is given by the form (1.2.10). As $\Omega_{t_0}(u)$ touches $B_{(1+a)\rho}$ at (x_0, t_0) for the first time, ψ touches u_* from below at (x_0, t_0) and we have

$$\nabla \cdot \left(\frac{D\psi}{|D\psi|}\right)(x_0, t_0) + \lambda[|\Omega_{t_0}(u)|] \leq \psi_t(x_0, t_0) = 0 \tag{1.3.18}$$

and this contradicts to (1.3.17). □

Proof of Theorem 1.3.8. First note that $\Omega_t(u)$ has *ρ-reflection* thanks to Lemma 1.3.9 and Theorem 1.3.6 applied to $u(x,t)$ and $\eta(t) = \lambda[|\Omega_t(u)|]$. Moreover from (1.3.6) in Lemma 1.3.4, $\Omega_t(u) \in S_r$ for

$$r = \left(\inf_{x\in\partial\Omega} |x|^2 - \rho^2\right)^{1/2} \geq r_1 := \rho(a^2 + 2a)^{1/2}. \tag{1.3.19}$$

Hence $\Omega_t(u)$ is *star-shaped with respect to a ball* B_{r_1} for all $t \geq 0$. □

A particular consequence of Theorem 1.3.8 is that $\partial\Omega_t(u)$ is a locally Lipschitz graph. This, in combination with Lemma A.2.1, yields that the evolution is indeed $C^{1,1}$:

Corollary 1.3.10. *Let u and Ω_0 be as in Theorem 1.3.8. Then $\Omega_t(u)$ has $C^{1,1}$-boundary for all $t > 0$. In particular its principal curvatures are bounded by $O(1 + 1/\sqrt{t})$.*

Next we note that, with the sublinear growth condition imposed on λ, $\Omega_t(u)$ is uniformly bounded in finite time.

Lemma 1.3.11. *Let u and Ω_0 be as given in Theorem 1.3.8. Then, there exists $R_1 = R_1(T) > 0$ such that $\Omega_t(u) \subset B_{R_1}$ in $[0,T]$.*

Proof. By Assumption A, there exists a constant $C_1 > 0$ such that $\lambda[|B_R|] \leq C_1 R$ for all $R \geq \rho$. Since Ω_0 is bounded, there exists $\hat{R} > \rho$ such that $\Omega_0 \subset\subset B_{\hat{R}}$. Let us compare u with a radial barrier $\phi : Q \to \mathbb{R}$ defined by

$$\phi(x,t) := \chi_{B_{r(t)}}(x) - \chi_{B^C_{r(t)}}(x) \text{ for } (x,t) \in Q,$$

where $r : [0,T] \to \mathbb{R}$ is defined by $r(t) := \hat{R}e^{(C_1+1)t}$. Note that $\Omega_0(u) \subset\subset \Omega_0(\phi)$, and $r'(t) = (C_1+1)r(t)$.

Choose $\varepsilon \in (0, \hat{R}C_1^{-1})$ and let us show that $\Omega_t(u) \subset B_{r(t)+\varepsilon}$ for all time. Suppose it is false, then we have

$$t_0 := \sup\{t : \Omega_s(u) \subset B_{r(s)+\varepsilon} \text{ for } 0 \leq s \leq t\} < +\infty. \tag{1.3.20}$$

By Corollary 1.3.7, $\partial\Omega_t(u)$ evolves continuously in time and thus

$$\partial\Omega_s(u) \cap \partial B_{r(s)+\varepsilon} \neq \emptyset. \tag{1.3.21}$$

Combining (1.3.20) with Lemma 1.3.9, we have $|B_\rho| \leq |\Omega_t(u)| \leq |B_{r(t)+\varepsilon}|$ in $[0,t_0]$. Furthermore, as $r(t) \geq \hat{R} > \rho$, it holds that

$$\lambda[|\Omega_t(u)|] \leq C_1(r(t)+\varepsilon) \leq r'(t) + \frac{n-1}{r(t)}. \tag{1.3.22}$$

Therefore, ϕ is a viscosity supersolution of (1.2.2) with $\eta(t) = \lambda[|\Omega_t(u)|]$ in $[0,t_0]$. Note that $u^* \leq \phi_*$ at $t = 0$. From Theorem 1.2.2(1) we have $u^* \leq \phi_*$ in $[0,t_0)$ and thus

$$\overline{\Omega}_t(u) \subset B_{r(t)} \text{ in } [0,t_0). \tag{1.3.23}$$

By Corollary 1.3.7 again, $\partial\Omega_t(u)$ evolves continuously in time and thus we have $\Omega_t(u) \subset B_{r(t_0)}$ in $[0,t_0]$, which contradicts (1.3.21).

As a consequence, we conclude that

$$\Omega_t(u) \subset B_{R_1} \text{ where } R_1(T) := \hat{R}e^{(C_1+1)T} + \varepsilon \tag{1.3.24}$$

in $[0,T]$. ☐

We finish this section with some properties of our solutions that will be used later. The following corollary holds due to the fact that Ω_0 has *ρ-reflection* and therefore for small $\varepsilon > 0$ the sets $\Omega_0^{\varepsilon,+} := (1+\varepsilon)\Omega_0$ and $\Omega_0^{\varepsilon,-} := (1+\varepsilon)^{-1}\Omega_0$ satisfy $\rho(1+O(\varepsilon))$-reflection.

Corollary 1.3.12. *Let u, Ω_0 and r_1 be as given in Theorem 1.3.8 and R_1 as given in Lemma 1.3.11. Then for sufficiently small $\varepsilon > 0$ viscosity solutions $u^\pm$ of (1.2.1) starting from $\Omega_0^{\varepsilon,\pm}$ have their positive sets $\Omega_t(u^\pm)$ in $S_{r_1-O(\varepsilon),R_1+O(\varepsilon)}$ in $[0,T]$.*

Lemma 1.3.13. *Let u, Ω_0 and r_1 be as given in Theorem 1.3.8 and R_1 as given in Lemma 1.3.11. Then, there exists positive constants $\tilde{K}_\infty = \tilde{K}_\infty(r_1,R_1,T)$ and $\tilde{K}_{1/2} = \tilde{K}_{1/2}(r_1,R_1,T)$ such that the following holds for all t,s in $[0,T]$:*

$$\Big|\lambda[|\Omega_t(u)|] - \lambda[|\Omega_s(u)|]\Big| \leq \tilde{K}_{1/2}|t-s|^{\frac{1}{2}} \tag{1.3.25}$$

and

$$\Big|\lambda[|\Omega_t(u)|]\Big| \leq \tilde{K}_\infty.$$

Proof. From Lemma 1.3.9 and 1.3.11, $|\Omega_t|$ is bounded away from zero and infinity, and thus λ is bounded. Next, by the Lipschitz continuity of λ and the last inequality of (A.3.1) in Lemma A.3.1, there exists $C_1(r_1,R_1,T)$ such that

$$\Big|\lambda[|\Omega_t(u)|] - \lambda[|\Omega_s(u)|]\Big| \leq C_1 d_H(\Omega_t(u),\Omega_s(u)) \text{ for } t,s \in [0,T].$$

From the above inequality and Hölder continuity in Corollary 1.3.7, we conclude (1.3.25). □

Finally, let us show Lipschitz continuity of $|\Omega_t|$ in time for the later purpose in Lemma 1.4.12.

Lemma 1.3.14. *Let u, Ω_0 and r_1 be as given in Theorem 1.3.8, R_1 as given in Lemma 1.3.11, and $\tilde{K}_\infty$ as given in Lemma 1.3.13. Then there exists $C = C(r_1,R_1,\tilde{K}_\infty)$ such that we have*

$$||\Omega_t(u)| - |\Omega_s(u)|| \leq C\left(1+\frac{1}{\sqrt{t}}\right)|s-t| \text{ for } 0 \leq t \leq s \leq T. \tag{1.3.26}$$

Proof. First, by Corollary 1.3.10, all principal curvatures are bounded by $M(t) := C_1(1+1/\sqrt{t})$ for some constant $C_1 = C_1(r_1,R_1,\tilde{K}_\infty)$. Thus, there exist interior and exterior balls of radius $M(t)^{-1}$ on each point of $\partial\Omega_t(u)$ for all $t > 0$. As described in Corollary 1.3.7, we apply Lemma 1.2.9 in these balls to conclude that

$$d_H(\partial\Omega_t(u), \partial\Omega_s(u)) \leq C_2 \left(1 + \frac{1}{\sqrt{t}}\right) |s - t| \text{ for } 0 \leq t \leq s \leq T.$$

for some $C_2 = C_2(r_1, R_1, \tilde{K}_\infty)$. Recall from the first and last inequalities of (A.3.1) in Lemma A.3.1 that the volume difference is bounded by the Hausdorff distance. Thus, we conclude that there exists $C = C(r_1, R_1, \tilde{K}_\infty)$ satisfying (1.3.26). □

1.4 Mean curvature flow with volume dependent forcing

1.4.1 Uniqueness of the flow

In this section, we show the uniqueness for solutions of (1.2.1) and (1.2.2) with given initial data (1.2.3). As pointed out in Remark 1.2.5, the comparison principle (Theorem 2.2) does not deliver the uniqueness for a discontinuous viscosity solution, due to the possible fattening phenomena of level sets. We show that our flow (1.1.1) can be uniquely determined when the initial data has *ρ-reflection*. We follow the argument of [BCC09], where the uniqueness result is shown for convex evolution of volume-preserving flow.

In section 1.4.1.1, we show the short-time uniqueness result for (1.2.2) in Theorem 1.4.3 for a star-shaped initial data Ω_0. We define appropriate convolutions to perturb solutions (see Definition 1.4.4) and show that our perturbation preserves sub- and supersolution properties for (1.2.2). These perturbations are more delicate than those used in [Gig06] due to the presence of the time-dependent forcing η. We use these perturbations to obtain the uniqueness results. At this point, it is crucial to find a uniform interval $[0, t_1]$ where these convolutions are well defined in this interval (see Lemma 1.4.6). It remains open whether the flow (1.2.2) stays unique beyond the interval.

In section 1.4.1.2, we show the global-time uniqueness for (1.2.1) when its initial data has *ρ-reflection* (see Theorem 1.4.9). Here we know that any evolution, if exists, preserves the *ρ-reflection* property, which we use to iterate the short-time uniqueness result from the previous subsection. The key step is to estimate the difference between $\lambda[|\Omega_t(u)|]$ and

$\lambda[|\Omega_t(v)|]$ for two possible solutions (see Lemma 1.4.12).

1.4.1.1 Short-time uniqueness

Definition 1.4.1. [BSS93, Definition 2.1] For a function $u : Q \to \mathbb{R}$ and $t \geq 0$, we say that $\Omega_t(u) = \{u(\cdot, t) > 0\}$ is *regular* if the closure of $\Omega_t(u)$ is $\{x \in \mathbb{R}^n : u(x,t) \geq 0\}$, and the interior of $\{x \in \mathbb{R}^n : u(x,t) \geq 0\}$ is $\Omega_t(u)$.

Note that for $t \geq 0$, if $\Omega_t(u)$ is regular, then the interface $\{x \in \mathbb{R}^n : u(x,t) = 0\}$ has an empty interior.

Lemma 1.4.2. *[BSS93, Theorem 2.1] Let $u : Q \to \mathbb{R}$ be a viscosity solution of* (1.2.2) *and* (1.2.3). *Then, $\Omega_t(u)$ is regular for all $t \geq 0$ if and only if there exists a unique solution in Q of* (1.2.2) *with initial data $u(x,0) = u_0(x) := \chi_{\Omega_0} - \chi_{\Omega_0^C}$.*

Recall from section 1.2 that

$$K_\infty := \|\eta\|_{L^\infty([0,\infty))}. \tag{1.4.1}$$

We define $t_1 = t_1(r, K_\infty)$ by

$$t_1 := \frac{r}{10K_\infty} \tag{1.4.2}$$

and we will show the following theorem in this section.

Theorem 1.4.3. *Suppose that the initial set Ω_0 is in S_r. Then, there is exactly one bounded viscosity solution u of* (1.2.2) *and* (1.2.3) *in $[0, t_1]$ where t_1 is given in* (1.4.2). *Moreover, $\Omega_t(u)$ is regular in $[0, t_1]$.*

We begin the proof with some definitions.

Definition 1.4.4. For $\varepsilon, r > 0$ and $L : [0, +\infty) \to \mathbb{R}$, let us define a maximal time $T_1 = T_1(\varepsilon, r, L)$ by

$$T_1 := \sup\{s > 0 : L(t) < r\varepsilon/2 \text{ for all } t \in [0, s]\}; \tag{1.4.3}$$

$$\underline{u}(x,t;\varepsilon,r,L) := \inf\left\{u\left(\frac{y}{1+\varepsilon},\frac{t}{(1+\varepsilon)^2}\right)\ \middle|\ y\in\overline{B_{r\varepsilon/2-L(t)}(x)}\right\};$$

and

$$\overline{u}(x,t;\varepsilon,r,L) := \sup\left\{u\left(\frac{y}{1-\varepsilon},\frac{t}{(1-\varepsilon)^2}\right)\ \middle|\ y\in\overline{B_{r\varepsilon/2-L(t)}(x)}\right\}$$

Lemma 1.4.5. *Let u be a bounded viscosity solution of* (1.2.2) *and* (1.2.3) *with forcing η and $\Omega_0 \in S_r$, and let $\eta_\varepsilon(t) := (1+\varepsilon)^{-1}\eta(t/(1+\varepsilon)^2)$. Let $\underline{u}$ and $\overline{u}$ be as given above with $L \in C^1([0,\infty))$. Then the following holds in $(0,T_1)$ in the sense of viscosity solutions:*

$$\frac{\underline{u}_t}{|D\underline{u}|}(x,t) \geq \nabla\cdot\left(\frac{D\underline{u}}{|D\underline{u}|}\right)(x,t)+\eta_\varepsilon(t)+L'(t) \tag{1.4.4}$$

and

$$\frac{\overline{u}_t}{|D\overline{u}|}(x,t) \leq \nabla\cdot\left(\frac{D\overline{u}}{|D\overline{u}|}\right)(x,t)+\eta_{-\varepsilon}(t)-L'(t). \tag{1.4.5}$$

Moreover, if $\varepsilon \leq \varepsilon_0(r)$ for $\varepsilon_0(r)$ given in (1.3.2)*, we have*

$$\Omega_0(\overline{u}) \subset\subset \Omega_0(u) \subset\subset \Omega_0(\underline{u}). \tag{1.4.6}$$

Proof. First, let us denote $v(x,t) := u\left(\frac{x}{1+\varepsilon},\frac{t}{(1+\varepsilon)^2}\right)$. Then, v is a viscosity solution of

$$\frac{v_t}{|Dv|}(x,t) = \nabla\cdot\left(\frac{Dv}{|Dv|}\right)(x,t)+\eta_\varepsilon(t).$$

and thus Lemma 1.2.8 implies (1.4.4). Parallel arguments holds for $\overline{u}$.

On the other hand, if $\Omega_0(u)$ is in S_r then Lemma 1.3.4 yields, for all $\varepsilon \leq \varepsilon_0(r)$,

$$\Omega_0(u) \subset\subset \bigcap_{|z|\leq r\varepsilon/2}[(1+\varepsilon)\Omega_0(u)+z] = \Omega_0(\underline{u}), \tag{1.4.7}$$

and

$$\Omega_0(\overline{u}) = \bigcup_{|z|\leq r\varepsilon/2}[(1-\varepsilon)\Omega_0(u)+z] \subset\subset \Omega_0(u). \tag{1.4.8}$$

□

Lemma 1.4.6. *Let η and η_ε be as given in Lemma 1.4.5, and let $t_1 = r/10K_\infty$ be as given in (1.4.2). Then for the choice of $L(t) = \int_0^t -\eta_\varepsilon(s) + \eta(s)ds$ or $L(t) = \int_0^t \eta_{-\varepsilon}(s) - \eta(s)ds$ and for $0 < \varepsilon \le 1/4$, we have*

$$T_1 = T_1(\varepsilon, r, L) \ge t_1 \text{ for } 0 < \varepsilon < 1/4.$$

Proof. 1. First, let us choose $L(t) = \int_0^t -\eta_\varepsilon(s) + \eta(s)ds$ and estimate the function L by the change of variables.

$$\begin{aligned} L(t) &= \int_0^t \eta(s) - \frac{1}{1+\varepsilon}\eta\left(\frac{s}{(1+\varepsilon)^2}\right) ds, \\ &= \int_0^t \eta(s)ds - (1+\varepsilon)\int_0^{\frac{t}{(1+\varepsilon)^2}} \eta(s)ds, \\ &= \int_{\frac{t}{(1+\varepsilon)^2}}^t \eta(s)ds - \varepsilon\int_0^{\frac{t}{(1+\varepsilon)^2}} \eta(s)ds. \end{aligned}$$

Therefore, we conclude that for $\varepsilon \in (0, 1/4)$

$$|L(t)| \le K_\infty t\left(\frac{\varepsilon^2 + 2\varepsilon}{(1+\varepsilon)^2}\right) + K_\infty \varepsilon t < 5K_\infty \varepsilon t. \tag{1.4.9}$$

2. Similarly, let us choose $L(t) = \int_0^t \eta_{-\varepsilon}(s) - \eta(s)ds$, then for $\varepsilon \in (0, 1/4)$

$$\begin{aligned} |L(t)| &= \Big| \int_t^{\frac{t}{(1-\varepsilon)^2}} \eta(s)ds - \varepsilon\int_0^{\frac{t}{(1-\varepsilon)^2}} \eta(s)ds \Big|, \\ &\le K_\infty t\left(\frac{2\varepsilon - \varepsilon^2}{(1-\varepsilon)^2}\right) + K_\infty \varepsilon t\frac{1}{(1-\varepsilon)^2} < 5K_\infty \varepsilon t. \end{aligned}$$

3. By definition of T_1 we have $L(T_1) = r\varepsilon/2$. Thus $5K_\infty \varepsilon t_1 = r\varepsilon/2 = L(T_1) < 5K_\infty \varepsilon T_1$. □

Lemma 1.4.5 and Lemma 1.4.6 imply the following.

Lemma 1.4.7. *Let u and Ω_0 be as given in Lemma 1.4.5 and let $0 < \varepsilon \le \varepsilon_0(r)$. For t_1 given in (1.4.2), $\underline{u}$ with the choice of $L(t) = \int_0^t -\eta_\varepsilon + \eta$ is a viscosity supersolution of (1.2.2) in $(0, t_1]$. Similarly, $\overline{u}$ with $L(t) = \int_0^t \eta_{-\varepsilon} - \eta$ is a subsolution of (1.2.2) in $(0, t_1]$. Moreover it holds that $\overline{u} \le u \le \underline{u}$ in $[0, t_1]$.*

Proof. By Lemma 1.4.6, $\underline{u}$ and $\overline{u}$ are well-defined in $[0, t_1]$. So, we could apply Lemma 1.4.5 and comparison principle in Theorem 1.2.2(1) for (1.2.2) in $[0, t_1]$ to conclude. □

Proof of Theorem 1.4.3: Suppose that u and v are two bounded solutions of (1.2.2) and $u(\cdot, 0) = v(\cdot, 0)$ in $\mathbb{R}^n$. Let us construct $\underline{u}$ and $\overline{u}$ as in Lemma 1.4.7. As $\Omega_0(\overline{u}) \subset\subset \Omega_0(v) = \Omega_0(u) \subset\subset \Omega_0(\underline{u})$ from (1.4.6), we have $\overline{u}^*(\cdot, 0) \leq v_*(\cdot, 0)$ and $v^*(\cdot, 0) \leq \underline{u}_*(\cdot, 0)$ in $\mathbb{R}^n$. By Lemma 1.4.7 and the comparison principle in Theorem 1.2.2(1), it holds that $\overline{u} \leq v \leq \underline{u}$ in $[0, t_1]$. Sending ε to zero, we conclude that $u = v$ in $[0, t_1]$. □

Lastly, for the next subsection let us state the following lemma.

Lemma 1.4.8. *Let u and Ω_0 be as given in Lemma 1.4.5. Then for $0 < \varepsilon \leq \varepsilon_0(r)$ and $0 \leq t \leq t_1$ we have*

$$(1-\varepsilon)\Omega_{t/(1-\varepsilon)^2}(u) \subset \Omega_t(u) \subset (1+\varepsilon)\Omega_{t/(1+\varepsilon)^2}(u).$$

where t_1 is given (1.4.2).

Proof. Lemma 1.4.7 implies that $\Omega_t(\overline{u}) \subset \Omega_t(u) \subset \Omega_t(\underline{u})$ in $[0, t_1]$. Moreover we have, by definition,

$$(1-\varepsilon)\Omega_{t/(1-\varepsilon)^2}(u) \subset \Omega_t(\overline{u}), \text{ and } \Omega_t(\underline{u}) \subset (1+\varepsilon)\Omega_{t/(1+\varepsilon)^2}(u).$$

□

1.4.1.2 Uniqueness of mean curvature flows with forcing

In this subsection, we show the uniqueness of our original equation (1.2.1). Here is the main theorem of this subsection.

Theorem 1.4.9. *Suppose that Ω_0 has ρ-reflection. Then, there exists at most one bounded viscosity solution of* (1.2.1) *and* (1.2.3).

Let u and v be two bounded viscosity solutions of (1.2.1) and (1.2.3), and let $\eta(t; u) := \lambda[|\Omega_t(u)|]$ and $\eta(t; v) := \lambda[|\Omega_t(v)|]$. Fix $T > 0$. Recall from Theorem 1.3.8 and Lemma 1.3.11

that both $\Omega_t(u)$ and $\Omega_t(v)$ are in S_{r_1,R_1} in $[0,T]$ where r_1 and R_1 are given in (1.3.19) and (1.3.24), respectively. From Lemma 1.3.13 that there exists a uniform bound of $\eta(t;u)$ and $\eta(t;v)$ in $[0,T]$,

$$\tilde{K}_\infty := \||\eta(t;u)| + |\eta(t;v)|\|_{L^\infty([0,T])} < \infty. \tag{1.4.10}$$

Recall $\eta_\varepsilon(t) := (1+\varepsilon)^{-1}\eta(t/(1+\varepsilon)^2)$ and define

$$L_1(t) := \int_0^t -\eta_\varepsilon(s;u) + \eta(s;v)ds \text{ and } L_2(t) := \int_0^t \eta_{-\varepsilon}(s;u) - \eta(s;v)ds \tag{1.4.11}$$

Definition 1.4.10. For $\varepsilon \in (0, \frac{1}{4})$, let us define

$$\tilde{T}_1 = \tilde{T}_1(\varepsilon, r_1, L_1, L_2) := \sup\left\{s \in (0,T] : L_1(t), L_2(t) < \frac{r_1\varepsilon}{2} \text{ for all } t \in [0,s]\right\} \tag{1.4.12}$$

where r_1 is given in (1.3.19). Remind that r_1 is chosen so that $\Omega_t(u)$ and $\Omega_t(v)$ are in S_{r_1,R_1} for all $t \in [0,T]$.

Let $\underline{u} = \underline{u}(\cdot;\varepsilon, r_1, L_1)$ and $\overline{u} = \overline{u}(\cdot;\varepsilon, r_1, L_2)$ be as given in Definition 1.4.4. The construction of L_1 and L_2 and Lemma 1.4.5 readily yields the following lemma.

Lemma 1.4.11. *$\underline{u}$ and $\overline{u}$ are a viscosity supersolution, and subsolution, respectively, of* (1.2.2) *with $\eta = \eta(;v)$ in $(0,\tilde{T}_1)$. Moreover, it holds that $\overline{u} \le v \le \underline{u}$ in $[0,\tilde{T}_1]$. Here, $\tilde{T}_1$ is given in* (1.4.12).

Lemma 1.4.12. *There exists $t_2 > 0$ such that for any $\varepsilon \in (0, \frac{1}{4})$,*

$$\tilde{T}_1 = \tilde{T}_1(\varepsilon, r_1, L_1, L_2) > t_2 \tag{1.4.13}$$

where $\tilde{T}_1$ is given in (1.4.12).

Proof. Let $t_1(r_1, \tilde{K}_\infty) = \frac{r_1}{5\tilde{K}_\infty}$ be as given in (1.4.2). If $\tilde{T}_1 \ge t_1$ for all $\varepsilon \in (0, \frac{1}{4})$, we take $t_2 = t_1$. If $\tilde{T}_1 < t_1$ for some $\varepsilon \in (0, \frac{1}{4})$, Lemma 1.4.8 implies that in $[0, \tilde{T}_1)$

$$(1-\varepsilon)\Omega_{t/(1-\varepsilon)^2}(u) \subset \Omega_t(u) \subset (1+\varepsilon)\Omega_{t/(1+\varepsilon)^2}(u). \tag{1.4.14}$$

Lemma 1.4.11 implies that $\Omega_t(\overline{u}) \subset \Omega_t(v) \subset \Omega_t(\underline{u})$ in $[0, \tilde{T}_1)$. Thus as shown in Lemma 1.4.8, the following holds for $0 \leq t < \tilde{T}_1$:

$$(1-\varepsilon)\Omega_{t/(1-\varepsilon)^2}(u) \subset \Omega_t(\overline{u}) \subset \Omega_t(v) \subset \Omega_t(\underline{u}) \subset (1+\varepsilon)\Omega_{t/(1+\varepsilon)^2}(u). \tag{1.4.15}$$

By subtracting $\eta(s;u)$ and adding the same term,

$$L_1(t) = \int_0^t \eta(s;v) - \eta_\varepsilon(s;u)ds = \int_0^t \eta(s;v) - \eta(s;u)ds + \int_0^t \eta(s;u) - \eta_\varepsilon(s;u)ds. \tag{1.4.16}$$

As Lemma 1.4.6, the second term is bounded by $5\tilde{K}_\infty \varepsilon t$. As for the first term, from Lipschitz continuity of λ for some $C_1 > 0$,

$$\mathcal{I}_1 := \left|\int_0^t \eta(s;v) - \eta(s;u)ds\right| \leq \int_0^t \Big|\lambda[|\Omega_s(v)|] - \lambda[|\Omega_s(u)|]\Big|ds \leq C_1 \int_0^t \Big||\Omega_s(v)| - |\Omega_s(u)|\Big|ds$$

By (1.4.14)-(1.4.15) and Lemma 1.3.11,

$$\begin{aligned}\mathcal{I}_1 &\leq C_1 \int_0^t \Big|\big|(1-\varepsilon)\Omega_{s/(1-\varepsilon)^2}(u)\big| - \big|(1+\varepsilon)\Omega_{s/(1+\varepsilon)^2}(u)\big|\Big|ds \\ &\leq C_1 \int_0^t \Big|\big|\Omega_{s/(1-\varepsilon)^2}(u)\big| - \big|\Omega_{s/(1+\varepsilon)^2}(u)\big|\Big|ds + C_2\varepsilon t\end{aligned}$$

for some constant $C_2 = C_2(R_1, T)$. By Lemma 1.3.14, we conclude that $\mathcal{I}_1$ is bounded by $C_3\varepsilon t$ for some constant $C_3 = C_3(r_1, R_1, T, \tilde{K}_\infty)$. Therefore, we have $L_1(t) < (C_3 + 5\tilde{K}_\infty)\varepsilon t$ in $[0, \tilde{T}_1]$. By similar arguments, the bound holds for L_2 as well in $[0, \tilde{T}_1]$.

Finally, by continuity of L_1 and L_2, we have $L_1(\tilde{T}_1) = r_1\varepsilon/2$ or $L_2(\tilde{T}_1) = r_1\varepsilon/2$. In both cases, it holds that

$$r_1\varepsilon/2 = L_1(\tilde{T}_1)(\text{ or } L_2(\tilde{T}_1)) < (C_3 + 5\tilde{K}_\infty)\tilde{T}_1\varepsilon,$$

so we conclude with

$$\tilde{T}_1 \geq t_2 = t_2(r_1, R_1, T, \tilde{K}_\infty) := \frac{r_1}{2C_3 + 10\tilde{K}_\infty}. \tag{1.4.17}$$

□

Proof of Theorem 1.4.9: The first part is parallel to the proof of Theorem 1.4.3. Let u and v be two viscosity solutions of (1.2.1) and (1.2.3). By Lemma 1.4.11 and Lemma 1.4.12, it holds that $\overline{u} \leq v \leq \underline{u}$ in $[0, t_2]$ where t_2 is given in (1.4.13). We can now send ε to zero to conclude that $u = v$ in $[0, t_2]$.

Next let us consider the corresponding convolutions of $\underline{u}$ and $\overline{u}$ in the time interval $t_0 + [0, t_2] \subset [0, T]$ for $t_0 > 0$ and t_2 given in (1.4.13). Note that t_2 given in (1.4.13) does not depend on t_0 because both $\Omega_t(u)$ and $\Omega_t(v)$ are in S_{r_1,R_1} for all $t \in [0, T]$. Thus, we can iterate the step 1 for $t_0 = kt_2$ on $kt_2 + [0, t_2]$, $k \in \mathbb{N}$ and, conclude that $u = v$ in $[0, T]$. □

1.4.2 Construction of flat flows

In this section, we construct a flat flow for (1.2.1), which coincides our notion of viscosity solutions. Our approach is based on minimizing movements first introduced by Almgren-Taylor-Wang [ATW93] (see also [LS95], [Cha04], [BCC09]).

As in [FK14], we introduce a gradient flow with geometric constraint, corresponding to the preservation of star-shapedness obtained in Theorem 1.3.8. Our constraint is crucial to ensure the strong (in Hausdorff distance) convergence of the minimizing movements, which enables geometric analysis of the limiting flow. On the other hand the constraint also poses technical challenges when we show the coincidence of flat flows with viscosity solutions (See Proposition 1.4.17 and Corollary 1.4.18).

1.4.2.1 Constrained Minimizing Movements

Recall the following energy functional associated with (1.2.1),

$$J(E) = \mathrm{Per}(E) - \Lambda[|E|]. \tag{1.4.18}$$

where the function $\Lambda(s)$ is an anti-derivative of $\lambda(s)$, and $\mathrm{Per}(E)$ denotes the perimeter of E. For the sets E and F in $\mathbb{R}^n$, we use the pseudo-distance defined by

$$\tilde{d}(F, E) := \left(\int_{E \triangle F} d(x, \partial E) dx \right)^{\frac{1}{2}}, \quad E \triangle F := (E \setminus F) \cup (F \setminus E).$$

We consider minimizing movements for (1.2.2) in a finite time interval $[0, T]$ with initial data (1.2.3) with the admissible sets S_{r_0,R_0} with

$$r_0 < r_1 = r_1(\rho, a) = \rho(a^2 + 2a)^{1/2} \text{ and } R_0 > R_1 \tag{1.4.19}$$

where r_1 is given in (1.3.19) and $R_1 = R_1(T)$ in (1.3.24). Recall that ρ is given in Definition 1.3.3 and a is given in Lemma 1.3.9. The dependence of R_1 on T is the reason why we restrict the discussion in this and next section to the finite time interval. For simplicity we will omit the time dependence in R_1 and thus in R_0.

Definition 1.4.13. For $h > 0$, T_h is defined by

$$T_h(E) \in \arg\min_{F \in S_{r_0,R_0}} I_h(F; E), \quad I_h(F; E) := J(F) + \frac{1}{h}\tilde{d}^2(F, E),$$

The existence of a minimizer, $T_h(E)$ follows from Lemma A.3.1, A.3.2 and A.3.3.

The constrained minimizing movement E_t^h of J for $t \in [0, T]$ with initial set E_0 can be defined by

$$E_t^h := T_h^{[t/h]}(E_0).$$

Here, T^m for $m \in \mathbb{N}$ is the m-th functional power.

Definition 1.4.14. A function $w := \chi_{E_t} - \chi_{E_t^C}$ is *a flat flow* of (1.2.2) and (1.2.3) if $E_0 = \Omega_0$ and there exists a sequence $h_k \to 0$ such that

$$d_H(E_t, E_t^{h_k}) \to 0$$

locally uniformly in time as k goes to infinity.

To show the existence of a flat flow, let us show compactness property of the constrained minimizing movements.

Lemma 1.4.15. *The constrained minimizing movement E_t^h in Definition 1.4.13 satisfies the following inequality for $0 < t < s \leq T$ and for some $\mathcal{K}_2 = \mathcal{K}_2(r_0, R_0)$:*

$$\tilde{d}^2(E_s^h, E_t^h) \leq \mathcal{K}_2(s - t)(J(E_t^h) - J(E_s^h)) \tag{1.4.20}$$

and, as a consequence,

$$d_H(E_s^h, E_t^h)^{n+1} \leq \mathcal{K}_1\mathcal{K}_2(s - t). \tag{1.4.21}$$

Proof. We will use the triangle-like inequality (see e.g. Lemma 17, [FK14]):

$$\frac{\tilde{d}^2(F_{k+1}, F_1)}{k} \lesssim_{r,R} \sum_{j=1}^{k} \tilde{d}^2(F_{j+1}, F_j) \text{ for } F_1, ..., F_{k+1} \in S_{r,R}. \tag{1.4.22}$$

Suppose that $t \in [Kh, (K+1)h)$ and $s \in [(K+L)h, (K+L+1)h)$ for some K and $L > 0$. By the construction of E_t^h in Definition 1.4.13 for $k \in N$,

$$J(E^h_{(k-1)h}) - J(E^h_{kh}) \geq \frac{1}{h}\tilde{d}^2(E^h_{kh}, E^h_{(k-1)h}).$$

By summing both sides from $k = K+1$ to $k = K+L$,

$$\begin{aligned} J(E^h_{Kh}) - J(E^h_{(K+L)h}) &\geq \sum_{k=K+1}^{K+L} \frac{1}{h}\tilde{d}^2(E^h_{kh}, E^h_{(k-1)h}), \\ &\gtrsim_{r,R} \frac{1}{Lh}\tilde{d}^2(E^h_{(K+L)h}, E^h_{Kh}), \end{aligned}$$

where the last inequality follows from (1.4.22). (1.4.21) follows from Lemma A.3.2. □

One can apply Lemma 1.4.15 and compactness of star-shaped sets (Lemma A.3.1, A.3.2 and A.3.3) to obtain the following:

Theorem 1.4.16. *There exists at least one flat flow w of* (1.2.2) *and* (1.2.3) *in the sense of Definition 1.4.14.*

1.4.2.2 Barrier property under star-shapedness

Next we establish a "restricted barrier property" for a flat flow with respect to a classical subsolution and supersolution of (1.2.2) with $\eta(t) = \lambda[|\Omega_t(w)|]$. The proof of this proposition is rather technical and follows that of [GK11]: see Appendix A.1. In a different setting, similar results are shown in [CMN19] and [CMP15].

Proposition 1.4.17. *(Restricted barrier property) Let w be a flat flow of* (1.2.2) *with the admissible set constraint parameters r_0 and R_0 satisfying* (1.4.19). *For any $r > r_0$ and $R < R_0$, suppose that there exists a test function ϕ on Q_T such that ϕ is a classical subsolution in Q_T of* (1.2.2) *with $\eta(t) = \lambda[|\Omega_t(w)|]$, $|D\phi| \neq 0$ on $\partial\Omega_t(\phi)$ and $\Omega_t(\phi) \in S_{r,R}$ in $[0,T]$. If $\Omega_0(\phi) \subset\subset \Omega_0(w)$, then*

$$\Omega_t(\phi) \subset\subset \Omega_t(w) \quad \textit{for all } t \in [0,T].$$

Similarly, suppose that there exists a test function ψ on Q_T such that ψ is a classical supersolution in Q_T of (1.2.2) *with $\eta(t) = \lambda[|\Omega_t(w)|]$, $|D\psi| \neq 0$ on $\partial\Omega_t(\psi)$ and $\Omega_t(\psi) \in S_{r,R}$ in $[0,T]$. If $\Omega_0(w) \subset\subset \Omega_0(\psi)$, then*

$$\Omega_t(w) \subset\subset \Omega_t(\psi) \quad \textit{for all } t \in [0,T].$$

In the proof of Proposition 1.4.17, we only use the properties of the classical solution ϕ in small neighborhood of (x_0, t_0), thus we can deduce the following localized barrier property of the flat flow.

Corollary 1.4.18. *Let w be a flat flow of* (1.2.2) *with the admissible set constraint parameter r_0 and R_0 satisfying* (1.4.19)*. If there exists a test function ϕ on Q_T such that ϕ touches w from below at (x_0, t_0), $|x_0| < R_0$, $|D\phi|(x_0, t_0) \neq 0$ and $-x_0 \cdot \frac{D\phi}{|D\phi|}(x_0, t_0) > r_0$. then*

$$\frac{\phi_t}{|D\phi|}(x_0, t_0) \geq \nabla \cdot \left(\frac{D\phi}{|D\phi|} \right)(x_0, t_0) + \eta(t_0).$$

Similarly, if there exists a test function ψ on Q_T such that ψ touches w from above at (x_0, t_0), $|x_0| < R_0$, $|D\psi|(x_0, t_0) \neq 0$ and $-x_0 \cdot \frac{D\psi}{|D\psi|}(x_0, t_0) > r_0$ then

$$\frac{\psi_t}{|D\psi|}(x_0, t_0) \leq \nabla \cdot \left(\frac{D\psi}{|D\psi|} \right)(x_0, t_0) + \eta(t_0).$$

1.4.3 Existence of the flow

Our goal in this section is to show the existence of a viscosity solution for (1.2.1). Let us give a brief summary of this section. We will show that a flat flow coincides with the corresponding viscosity solution as long as the viscosity solution is star-shaped (Proposition 1.4.19). Ensuring this star-shaped property for the viscosity solution (Proposition 1.4.20) is the last step leading to the coincidence result (Theorem 1.4.21).

We first show a comparison result between a flat flow and the corresponding viscosity solution of (1.2.2). We use the doubling argument in [CIL92] and [Kim05] which preserves the star-shaped geometry of the level sets of the solutions.

Proposition 1.4.19. *Let w be a flat flow of* (1.2.2) *with the admissible set constraint parameter r_0 and R_0 satisfying* (1.4.19)*. Suppose that there exists a viscosity subsolution*

$u : Q_T \to \mathbb{R}$ of (1.2.2) with $\eta(t) = \lambda[|\Omega_t(w)|]$ such that $\Omega_t(u)$ is in $S_{r,R}$ for all $t \in [0,T]$ for some $r > r_0$ and $R < R_0$. If $\Omega_0(u) \subset\subset \Omega_0(w)$, then

$$\Omega_t(u) \subset\subset \Omega_t(w) \qquad \textit{for all } t \in [0,T].$$

Similarly, suppose that there exists a viscosity supersolution $u : Q_T \to \mathbb{R}$ of (1.2.2) with $\eta(t) = \lambda[|\Omega_t(w)|]$ such that $\Omega_t(u)$ is in $S_{r,R}$ for all $t \in [0,T]$ for some $r > r_0$ and $R < R_0$. If $\Omega_0(w) \subset\subset \Omega_0(u)$, then

$$\Omega_t(w) \subset\subset \Omega_t(u) \quad \textit{for all } t \in [0,T].$$

Proof. The proof follows the outline of [Kim05], where the comparison principle is shown for a nonlocal mean-curvature flow.

For $c, \delta > 0$, let us consider

$$Z(x,t) := \sup_{|z| \le c - \delta t} u(x+z,t) \text{ and } 0 \le t \le \frac{c}{\delta},$$

where c is chosen sufficiently small so that $\Omega_0(Z) \subset\subset \Omega_0(w)$. Due to Lemma 1.2.8, the function Z is a viscosity subsolution of

$$u_t = F(Du, D^2u, t) - \delta|Du|.$$

We will show Proposition 6.2 by showing that for any $\delta > 0$ and $0 \le t \le c/\delta$ we have

$$\Omega_t(Z) \subset\subset \Omega_t(w). \tag{1.4.23}$$

Note that for any $z \in \mathbb{R}^n$ such that $|z| \le c$, the interior cone $IC(x,r)$ given in (1.3.3) satisfies $IC(x+z, r-c) \subset IC(x,r) + z$ (See Lemma A.3.5). Thus, by the equivalence relation in Lemma 1.3.4, $\Omega_t(u) \in S_{r,R}$ implies that $\Omega_t(u) + z \in S_{r-c,R+c}$ for all $|z| \le c$ and thus

$$\Omega_t(Z) = \bigcup_{|z| \le c - \delta t} [\Omega_t(u) + z] \in S_{r-c,R+c}.$$

Thus, $\Omega_t(Z) \in S_{r_0+c,R_0-c}$ for $0 < c \le \min\left\{\frac{r-r_0}{2}, \frac{R_0-R}{2}\right\}$.

Suppose (1.4.23) is false, then we have

$$t_0 := \sup\{t : \Omega_s(Z) \subset\subset \Omega_s(u) \text{ for } 0 \le s \le t\} \in (0, c/\delta).$$

Due to Lemma 1.3.13 and Lemma 1.4.15, both sets $\partial\Omega_t(Z)$ and $\partial\Omega_t(w)$ evolve continuously in time. Hence, $\partial\Omega(Z)$ touches $\partial\Omega(w)$ from inside for the first time at $t = t_0 \in (0, \frac{\varepsilon}{\delta})$.

For $\varepsilon \in (0, \frac{\delta}{2n})$, let us define $\tilde{Z} := \chi_{\text{bar}\Omega(Z)}$ and $\tilde{W} := \chi_{\Omega(w)}$ and

$$\Phi_\varepsilon(x, y, t) := \tilde{Z}(x, t) - \tilde{W}(y, t) - \frac{|x - y|^4}{4\varepsilon} - \frac{\varepsilon}{2(t_0 - t)}.$$

Let d_0 be distance between $\partial\Omega_0(Z)$ and $\partial\Omega_0(w)$. Since $\tilde{Z} - \tilde{W}$ is bounded, we can choose a sufficiently small $\varepsilon << d_0^4$ such that $\Phi(x, y, 0) < 0$ for all x and y.

Since the function $\tilde{Z} - \tilde{W}$ is upper semicontinuous and bounded above by zero for all $t < t_0$, the function $\Phi_\varepsilon(x, y, t)$ has a local maximum at $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ in $\mathbb{R}^n \times [0, t_0)$ for any ε. By Hölder continuity of $\partial\Omega(Z)$ and $\partial\Omega(w)$ from Lemma 1.3.13 and Lemma 1.4.15, there exists $x_1 \in \partial\Omega_{t_0-\varepsilon}(\tilde{Z})$ and $y_1 \in \partial\Omega_{t_0-\varepsilon}(\tilde{W})$ such that $|x_1 - y_1| \leq K\varepsilon^{\frac{1}{2}}$ where K depends on Hölder constants of $\partial\Omega(Z)$ and $\partial\Omega(w)$. For $\varepsilon << K^{-4}$, it holds that $\Phi(x_\varepsilon, y_\varepsilon, t_\varepsilon) > \Phi(x_1, y_1, t_0-\varepsilon) > \frac{1}{3}$, and thus $t_\varepsilon \in (0, t_0)$. Also, $\Phi(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ is uniformly bounded from below in ε, and thus it holds that $|x_\varepsilon - y_\varepsilon| = O(\varepsilon^{\frac{1}{4}})$.

Moreover, since $\tilde{Z} - \tilde{W} > \Phi > \frac{1}{3}$ at $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$, we conclude that $x_\varepsilon \in \Omega_{t_\varepsilon}(\tilde{Z})$, $y_\varepsilon \in \Omega_{t_\varepsilon}(\tilde{W})^C$. As t_0 is the first touching point and $t_\varepsilon < t_0$, it holds that $|x_\varepsilon - y_\varepsilon| > 0$. On the other hand, $\tilde{Z}(x, t_\varepsilon) - \tilde{W}(y, t_\varepsilon) = 1$ for all $(x, y) \in \Omega_{t_\varepsilon}(\tilde{Z}) \times \Omega_{t_\varepsilon}(\tilde{W})^C$, and thus $(x_\varepsilon, y_\varepsilon)$ is a maximizer of the third term $-\frac{|x-y|^4}{4\varepsilon}$ in $\Omega_{t_\varepsilon}(\tilde{Z}) \times \Omega_{t_\varepsilon}(\tilde{W})^C$. We conclude that x_ε and y_ε are on $\partial\Omega_{t_\varepsilon}(\tilde{Z})$ and $\partial\Omega_{t_\varepsilon}(\tilde{W})$, respectively.

Then, as equation (2.9) in [Kim05], there exist quadratic test functions $\phi^\varepsilon(x, t)$ and $\psi^\varepsilon(x, t)$ such that

$$\begin{cases} \phi^\varepsilon(x, t) := [a_\varepsilon(t - t_\varepsilon) + p_\varepsilon \cdot (x - x_\varepsilon) + \frac{1}{2}(x - x_\varepsilon)^T X_\varepsilon (x - x_\varepsilon)]_+ \geq \tilde{Z}(x, t) & \text{in } N_1^\varepsilon, \\ \psi^\varepsilon(y, t) := [b_\varepsilon(t - t_\varepsilon) + q_\varepsilon \cdot (y - y_\varepsilon) + \frac{1}{2}(y - y_\varepsilon)^T Y_\varepsilon (y - y_\varepsilon)]_+ \leq \tilde{W}(y, t) & \text{in } N_2^\varepsilon, \end{cases} \tag{1.4.24}$$

where constants $a_\varepsilon, b_\varepsilon \in \mathbb{R}$, $p_\varepsilon, q_\varepsilon = \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + O(\varepsilon^2) \in \mathbb{R}^n \setminus \{0\}$, $X_\varepsilon, Y_\varepsilon \in S^{n\times n}$, neighborhoods

N_1^ε of $(x_\varepsilon, t_\varepsilon)$ and N_2^ε of $(y_\varepsilon, t_\varepsilon)$ satisfying the inequalities:

$$\begin{cases} a_\varepsilon - b_\varepsilon & \geq 0, \\ X_\varepsilon - Y_\varepsilon & \leq \varepsilon |p_\varepsilon| I, \\ ||p_\varepsilon| - |q_\varepsilon|| & \leq \varepsilon^2 \min\{1, |p_\varepsilon|^2\}, \\ |p_\varepsilon - q_\varepsilon| & \leq \varepsilon^2 \min\{1, |p_\varepsilon|^2\}. \end{cases} \tag{1.4.25}$$

Since $\tilde{Z}$ is a viscosity solution and ϕ^ε touches $\tilde{Z}$ from above at $(x_\varepsilon, t_\varepsilon)$, it holds that

$$\frac{a_\varepsilon}{|p_\varepsilon|} = \frac{\phi_t^\varepsilon}{|D\phi^\varepsilon|}(x_\varepsilon, t_\varepsilon) \leq \nabla \cdot \left(\frac{D\phi^\varepsilon}{|D\phi^\varepsilon|} \right)(x_\varepsilon, t_\varepsilon) + \eta(t_\varepsilon) - \delta = \frac{1}{|p_\varepsilon|} \left(\text{trace}(X_\varepsilon) - \frac{p_\varepsilon^T X_\varepsilon p_\varepsilon}{|p_\varepsilon|^2} \right) + \eta(t_\varepsilon) - \delta.$$

By inequalities (1.4.25) and the ellipticity of the operator, $\text{trace}(X) - \frac{p^T X p}{|p|^2}$, it can be seen that

$$\begin{aligned} \frac{b_\varepsilon}{|p_\varepsilon|} \leq \frac{a_\varepsilon}{|p_\varepsilon|} &\leq \frac{1}{|p_\varepsilon|} \left(\text{trace}(X_\varepsilon) - \frac{p_\varepsilon^T X_\varepsilon p_\varepsilon}{|p_\varepsilon|^2} \right) + \eta(t_\varepsilon) - \delta, \\ &\leq \frac{1}{|p_\varepsilon|} \left(\text{trace}(Y_\varepsilon) - \frac{p_\varepsilon^T Y_\varepsilon p_\varepsilon}{|p_\varepsilon|^2} \right) + \eta(t_\varepsilon) - \frac{\delta}{2}. \end{aligned}$$

Thus, by (1.4.25), for sufficiently small $\varepsilon > 0$, it holds that

$$\frac{b_\varepsilon}{|q_\varepsilon|} \leq \frac{1}{|q_\varepsilon|} \left(\text{trace}(Y_\varepsilon) - \frac{q_\varepsilon^T Y_\varepsilon q_\varepsilon}{|q_\varepsilon|^2} \right) + \eta(t_\varepsilon) - \frac{\delta}{4}. \tag{1.4.26}$$

Moreover, as $\Omega_t(\tilde{Z}) \in S_{r_0+c, R_0-c}$, $|x_\varepsilon| < R_0 - c$ and Lemma 1.3.2 implies that

$$x_\varepsilon \cdot \left(-\frac{p_\varepsilon}{|p_\varepsilon|} \right) \geq r_0 + c.$$

There exists sufficiently small ε_0 such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$|y_\varepsilon| < R_0, \text{ and } y_\varepsilon \cdot \left(-\frac{q_\varepsilon}{|q_\varepsilon|} \right) > r_0. \tag{1.4.27}$$

This contradicts Corollary 1.4.18. since ψ^ε touches $\tilde{W}$ from below at $(y_\varepsilon, t_\varepsilon)$, but satisfies (1.4.26) and (1.4.27). □

Next we will show that viscosity solutions u of (1.2.2) has a short time star-shapedness property.

Proposition 1.4.20. *(Short-time star-shapedness) Suppose that Ω_0 has ρ-reflection. Let r_0 and R_0 satisfy* (1.4.19)*, and $\tilde{K}_\infty = \tilde{K}_\infty(r_0, R_0, T)$ be as in Lemma 1.3.13. For $r > r_0 > 0$ and $0 < R < R_0$, suppose that $\overline{B}_{(1+\beta)\rho} \subset \Omega_0$ and $\Omega_0 \in S_{r,R}$ for $r = \rho(\beta^2 + 2\beta)$. Then, for all $t \in [0, t_1]$, it holds that for some $\hat{r} > r_0$ and $\hat{R} < R_0$*

$$\Omega_t \in S_{\hat{r},\hat{R}}. \tag{1.4.28}$$

where

$$t_1 = t_1(r, R, \tilde{K}_\infty, T) := \frac{1}{2}\left(\min\left\{\frac{\sqrt{r^2+\rho^2} - \sqrt{r_0^2+\rho^2}}{\mathcal{M}_1}, \frac{R_0 - R}{\mathcal{M}_1}\right\}\right)^2 \tag{1.4.29}$$

Here, $\mathcal{M}_1$ is given in Corollary 1.3.7.

Now we are ready to prove our main theorem.

Theorem 1.4.21. *Suppose that Ω_0 has ρ-reflection. Let w be a flat flow of* (1.2.1) *and* (1.2.3) *and let u be the unique viscosity solution of* (1.2.2) *and* (1.2.3) *with $\eta(t) = \lambda[|\Omega_t(w)|]$. Then $w = u$ in Q. In other words, w is the unique viscosity solution of* (1.2.1) *and* (1.2.3)*.*

Proof. The existence and short time uniqueness of u for the above choice of $\eta(t)$ follows by Theorem 1.2.2 and Theorem 1.4.3. Let r_0 and R_0 satisfy (1.4.19), and $\tilde{K}_\infty = \tilde{K}_\infty(r_0, R_0, T)$ be as in Lemma 1.3.13.

Recall that $\Omega_0 \in S_{r_1,R_1}$ where r_1 and R_1 are given in (1.3.19) and (1.3.24). Let us first show that $u = w$ in the small time interval $I = [0, t_1]$ for $t_1 = t_1(r, R, \tilde{K}_\infty, T)$ given in (1.4.29). As Corollary 1.3.12, we can make Ω_0 strictly smaller $\Omega_0^{\varepsilon,-}$ or bigger $\Omega_0^{\varepsilon,+}$ by dilation and can still make it stay in $S_{r_\varepsilon,R_\varepsilon}$ with $r_\varepsilon = r_1 - O(\varepsilon) > r_0$ and $R_\varepsilon = R_1 + O(\varepsilon) > R_0$, where ε can be chosen arbitrarily small such that $r_\varepsilon - r_0 > \frac{r_1 - r_0}{2}$ and $R_0 - R_\varepsilon > \frac{R_0 - R_1}{2}$. Let us choose to make the domain strictly bigger, $\Omega_0^{\varepsilon,+}$, we can apply Proposition 1.4.20 to ensure that the corresponding viscosity solution u^ε of (1.2.2) satisfies, for some $r > r_0$ and $R < R_0$,

$$\Omega_t(u^\varepsilon) \in S_{r,R} \text{ for } t \in I.$$

We can then apply Proposition 1.4.19 to u^ε and w to yield that

$$\Omega_t(w) \subset \Omega_t(u^\varepsilon) \text{ for } t \in I. \tag{1.4.30}$$

Now to send $\varepsilon \to 0$, note that $\Omega_t(u^\varepsilon)$ satisfies Hölder continuity, Corollary 1.3.7. Thus along a sequence $\varepsilon = \varepsilon_n \to 0$, $\Omega_t(u^\varepsilon)$ converges to a domain $\Omega_t \in S_{r,R}$ uniformly with respect to d_H in the time interval I. Lemma 1.2.3 then yields that the corresponding level set function u for Ω_t is the unique viscosity solution of (1.2.2) with the initial data u_0. From (1.4.30) we have

$$\Omega_t(w) \subset \Omega_t = \Omega_t(u) \text{ for } t \in I.$$

Similarly, using $\Omega_0^{\varepsilon,-}$ instead of $\Omega_0^{\varepsilon,+}$ we can conclude that $\Omega_t(u) \subset \Omega_t(w)$ and thus it follows that they are equal sets for the time interval I.

3. Once we know that $u = w$ in I, we know that $\eta(t)$ equals $\lambda[|\Omega_t(u)|]$ in I, and thus Theorem 1.3.8 and Lemma 1.3.11 applies and now we know that $\Omega_t(u) \in S_{r_1,R_1}$ for $t \in I$. Now we can repeat the argument at $t = t_1$ over the time interval $t_1 + I$, using the fact that $\Omega_{t_1}(u) \in S_{r_1,R_1}$. Now we can repeat above arguments to obtain that $w = u$ for all times. □

1.5 Volume preserving mean curvature flow

1.5.1 Approximation

In this section a solution of (1.1.3) will be constructed. We first show that a family of viscosity solutions $(\Omega_t^\delta)_{t\geq 0}$ of (1.1.5) for small $\delta > 0$ is equicontinuous in the Hausdorff distance, based on the geometric properties of $(\Omega_t^\delta)_{t\geq 0}$. This yields the uniform convergence of $(\Omega_t^\delta)_{t\geq 0}$ along a subsequence. We will conclude in Section 1.5.2 that their limit is a viscosity solution of (1.1.3). Our focus here is to obtain the uniform estimates that stay independent of $0 < \delta << 1$.

Below is the main theorem of this section. As before, we assume the geometric condition on the initial data (1.1.7).

Theorem 1.5.1. *Let $(\Omega_t^\delta)_{t\geq 0}$ be a viscosity solution of* (1.1.5)*. Then, there exists $(\Omega_t^\infty)_{t\geq 0} \subset S_{r_1,R_1}$ for some $0 < r_1 < R_1$ such that*

$$d_H(\Omega_t^\delta, \Omega_t^\infty) \to 0 \textit{ as } \delta \to 0 \tag{1.5.1}$$

locally uniformly in time along a subsequence. As a consequence, $|\Omega_t^\infty| = 1$ *for all* $t > 0$.

Let us briefly explain the outline of proof. We first show that for a small δ (1.1.5) is well-posed and Ω_t^δ is star-shaped with respect to a ball (See Definition 1.3.1) in Proposition 1.5.2. In Proposition 1.5.5, based on geometric properties in Lemma A.3.2, we show that Ω_t^δ is Hölder continuous with respect to time. Then, we apply the compactness of $(\partial S_{r,R}, d_H)$ in Lemma A.3.3 to find a converging subsequence.

Proposition 1.5.2. *Let*

$$\delta_0 := \frac{\rho(1 - |B_{5\rho}|)}{n-1}. \tag{1.5.2}$$

Then the following statements hold for $0 < \delta < \delta_0$.

1. *There is a unique viscosity solution* $((\Omega_t^\delta)_{t\geq 0}, \lambda_\delta)$ *of* (1.1.5) *such that* Ω_t^δ *is bounded with* $C^{1,1}$ *boundary for all* $t > 0$.

2. Ω_t^δ *has* ρ*-reflection at all times* $t \geq 0$*. Moreover* $\Omega_t^\delta \in S_{r_1,R_1}$ *where* r_1, R_1 *only depends on* Ω_0.

Proof. Note that $1 - |B_{5\rho}| > 0$ from (1.1.7), and thus we get $\delta_0 > 0$. We check that γ_δ in (1.1.6) satisfies Assumption A for all $\delta \in (0, \delta_0)$. Since γ_δ is decreasing and δ_0 is given in (1.5.2), we have

$$\gamma_\delta(|\Omega|) \geq \gamma_\delta(|B_{5\rho}|) > \frac{n-1}{\rho} \text{ for all } \Omega \subset \overline{B}_{5\rho} \text{ and all } \delta \in (0, \delta_0)\,. \tag{1.5.3}$$

On the other hand, γ_δ satisfies

$$\limsup_{R\to\infty} \frac{\gamma_\delta(|B_R|)}{R} = -\infty < \infty \tag{1.5.4}$$

and we conclude.

From Theorems 1 and 2, the problem (1.1.5) is well-posed and $(\Omega_t^\delta)_{t\geq 0}$ has *ρ-reflection* for all $\delta \in (0, \delta_0)$. Furthermore, (1.3.19) implies that $(\Omega_t^\delta)_{t\geq 0} \subset S_{r_1}$ where $r_1 = r_1(\Omega_0)$ is given by

$$r_1 := \rho(\beta_1^2 + 2\beta_1)^{\frac{1}{2}} \tag{1.5.5}$$

for some $\beta_1 > 0$ such that $\overline{B}_{(1+\beta_1)\rho} \subset \Omega_0$. On the other hand due to Lemma 1.3.4, $\Omega_0 \subset\subset B_{R_1}$ and $\lambda_\delta(t) < 0$ if $\sup_{x\in\Omega_t^\delta} |x| \geq R_1$, where

$$R_1 := 5\rho + w_n^{-\frac{1}{n}} \text{ and } \ w_n := |B_1(0)|. \tag{1.5.6}$$

A barrier argument with B_{R_1} yields that $\Omega_t^\delta \subset B_{R_1}$ for all $t \geq 0$ and all $\delta \in (0, \delta_0)$. □

Definition 1.5.3.

- *The one-step discrete gradient flow with a time step* $h > 0$, $T = T(\cdot; h, \delta) \subset \mathbb{R}^n$, is defined by

$$T(E; h, \delta) \in \underset{F\in S_{r_0,R_0}}{\arg\min}\, \mathcal{J}_\delta(F) + \frac{1}{h}\tilde{d}^2(F, E), \quad \mathcal{J}_\delta(\Omega) := \mathrm{Per}(\Omega) + \frac{1}{2\delta}(1 - |\Omega|)^2, \tag{1.5.7}$$

  where *the pseudo-distance* $\tilde{d}$ is given by

$$\tilde{d}(F, E) := \left(\int_{E\triangle F} d(x, \partial E)dx\right)^{\frac{1}{2}}, \tag{1.5.8}$$

  Here, r_0 and R_0 are constants such that

$$r_0 \in (0, r_1) \text{ and } R_0 > R_1 \tag{1.5.9}$$

  for r_1 and R_1 given in Proposition 1.5.2

- *The discrete gradient flow with a time step* $h > 0$ *and the initial set* E_0, $E_t = E_t(h, \delta) \subset \mathbb{R}^n$, can be defined by for $t \in [0, +\infty)$

$$E_t = E_t(h, \delta) := T^{[t/h]}(E_0; h, \delta). \tag{1.5.10}$$

  Here, T^m for $m \in \mathbb{N}$ is the mth functional power.

Recall from Theorem 2 that $(\Omega_t^\delta)_{t\geq 0}$ can be approximated locally uniformly by the above discrete flow.

Proposition 1.5.4. *Let* δ_0 *be given in* (1.5.2). *Then for* $0 < \delta < \delta_0$ *and for* $0 \leq t_1 < t_2$ *we have*

$$\lim_{h\to 0} \sup_{t\in[t_1,t_2]} d_H(E_t(h, \delta), \Omega_t^\delta) = 0. \tag{1.5.11}$$

Next, we show the Hölder continuity in time in Proposition 1.5.5. Lemma A.3.2 and Lemma 1.4.15 imply uniform Hölder continuity in time with respect to δ.

Proposition 1.5.5. *There exists $\mathcal{K}_3 = \mathcal{K}_3(r_0, R_0)$, which is independent on $\delta > 0$ such that for all $0 < t_1 < t_2$, it holds that*

$$d_H(\Omega_{t_1}^\delta, \Omega_{t_2}^\delta) \leq \mathcal{K}_3(t_2 - t_1)^{\frac{1}{n+1}} \operatorname{Per}(\Omega_0)^{\frac{1}{n+1}}. \tag{1.5.12}$$

Proof. Note that $\mathcal{J}_\delta(E_t)$ is nonnegative and decreases in time from the construction of E_t in Definition 1.5.3. Thus, Lemma 1.4.15 implies that

$$\widetilde{d}^2(E_{t_2}, E_{t_1}) \leq \mathcal{K}_2(t_2 - t_1)(\mathcal{J}_\delta(E_{t_1}) - \mathcal{J}_\delta(E_{t_2})) \leq \mathcal{K}_2(t_2 - t_1)\mathcal{J}_\delta(\Omega_0) \tag{1.5.13}$$

for all $0 < t_1 < t_2$ and $\mathcal{K}_2 = \mathcal{K}_2(r_0, R_0)$ given in Lemma 1.4.15. Note that $|\Omega_0| = 1$ implies

$$\mathcal{J}_\delta(\Omega_0) = \operatorname{Per}(\Omega_0) + \frac{1}{2\delta}(1 - |\Omega_0|)^2 = \operatorname{Per}(\Omega_0) \tag{1.5.14}$$

for all $\delta > 0$. From Lemma A.3.2 and (1.5.14), there exists $\mathcal{K}_3 = \mathcal{K}_3(r_0, R_0)$ such that for all $0 < t_1 < t_2$

$$d_H(E_{t_2}, E_{t_1}) \leq \mathcal{K}_3(t_2 - t_1)^{\frac{1}{n+1}} \operatorname{Per}(\Omega_0)^{\frac{1}{n+1}}. \tag{1.5.15}$$

From Proposition 1.5.4 it follows that $E_t = E_t(h, \delta)$ converges to Ω_t^δ locally uniformly as $h \to 0$. Therefore

$$d_H(E_{t_2}, E_{t_1}) \to d_H(\Omega_{t_2}^\delta, \Omega_{t_1}^\delta) \text{ as } h \to 0. \tag{1.5.16}$$

Thus, from (1.5.15) and (1.5.16), we conclude (1.5.12). □

Proof of Theorem 1.5.1.

Recall from Proposition 1.5.2 that $(\Omega_t^\delta)_{t\geq 0} \subset S_{r_1,R_1}$ for all $t \geq 0$ and $\delta \in (0, \delta_0)$. Here, r_1, R_1, δ_0 are given in (1.5.5), (1.5.6) and (1.5.2), respectively. Moreover, Lemma A.3.1 and the Hölder continuity from Proposition 1.5.5 yield that a family of evolving sets, $t \mapsto \partial\Omega_t^\delta$, for $\delta \in (0, \delta_0)$ is equicontinuous in $(\partial S_{r_1,R_1}, d_H)$. Lemma A.3.3 applies to obtain a subsequential convergence of $(\Omega_t^\delta)_{t\geq 0}$: there exists a sequence $\{\delta_i\}_{i\in\mathbb{N}}$ such that

$$d_H(\Omega_t^{\delta_i}, \Omega_t^\infty) \to 0 \tag{1.5.17}$$

locally uniformly in time as i goes to infinity for some $(\Omega_t^\infty)_{t\geq 0} \subset S_{r_1,R_1}$. By Lemma A.3.1, we conclude that $|\Omega_t^\infty| = 1$ for all $t \geq 0$. ☐

1.5.2 Uniform L^2 estimates of the Lagrange multiplier and existence

In this section we establish uniform L^2 estimates of λ_δ which yields the weak L^2 limit of λ_δ in Theorem 1.5.6. Combining with the stability of viscosity solutions in Theorem 1.2.17, we show the existence of solution of (1.1.3) in Corollary 1.5.7. Following the outline given in [MSS16], we obtain the estimates for our constrained discrete gradient flow defined in (1.5.10). Our new challenge lies in constructing local variations given in Definition 1.5.11 which stays in our admissible set S_{r_0,R_0} (See Lemma 1.5.12 and Lemma 1.5.13).

Theorem 1.5.6. *Let $\delta \in (0, \delta_0)$ for δ_0 given in* (1.5.2) *and λ_δ be given in* (1.1.5). *There exists $\sigma_1 = \sigma_1(r_0, R_0) > 0$ such that*

$$\|\lambda_\delta\|^2_{L^2([0,T])} \leq \sigma_1(\mathrm{Per}(\Omega_0) + T). \tag{1.5.18}$$

Here, r_0 and R_0 are given in (1.5.9). *As a consequence, $\{\lambda_\delta\}_{\delta\in(0,\delta_0)}$ weakly converges to some λ_∞ in $L^2([0,T])$ as $\delta \to 0$ along a subsequence satisfying* (1.5.1).

Before proving the above theorem, let us show the existence of a viscosity solution of (1.1.3).

Corollary 1.5.7. $((\Omega_t^\infty)_{t\geq 0}, \lambda_\infty)$ *is a viscosity solution (See Definition 1) of* (1.1.3). *Here, $(\Omega_t^\infty)_{t\geq 0}$ and λ_∞ are given in Theorem 1.5.1 and Theorem 1.5.6, respectively.*

Proof. Recall from Theorem 1.5.1 and Theorem 1.5.6 that for any $T > 0$ we have, along a subsequence,

$$\max_{0\leq t\leq T} d_H(\Omega_t^\delta, \Omega_t^\infty) \to 0 \text{ and } \lambda_\delta \rightharpoonup \lambda_\infty \text{ in } L^2([0,T]) \text{ as } \delta \to 0.$$

Thus, $\{\Lambda_\delta(t)\}_{\delta\in(0,\delta_0)}$ locally uniformly converges to $\Lambda_\infty(t) := \int_0^t \lambda_\infty(s)ds$ along a subsequence as $\delta \to 0$ where $\Lambda_\delta(t) := \int_0^t \lambda_\delta(s)ds$ for $t \in [0, +\infty)$.

Note that $(\Omega_t^\delta)_{t\geq 0}$ given in Proposition 1.5.2 is a viscosity solution of $V = -H + \lambda^\delta(t)$ in the sense of Definition 1.2.13 (See Remark 1.2.14). From Theorem 1.2.17 and Lemma A.3.7, we conclude that $(\Omega_t^\infty)_{t\geq 0}$ is a viscosity solution of $V = -H + \lambda_\infty(t)$. On the other hand, from Theorem 1.5.1, $|\Omega_t^\infty| = |\Omega_0|$ for all $t \geq 0$. Thus, we conclude that $((\Omega_t^\infty)_{t\geq 0}, \lambda_\infty)$ is a viscosity solution of (1.1.3). □

Let us briefly explain the outline of proof of Theorem 1.5.6. First, in Proposition 1.5.10, we show that $\|d(\cdot, \partial E)\|_{L^2(\partial F)}$ is bounded by $\widetilde{d}(F, E)$ given in (1.5.8) up to a constant for any sets $E, F \in S_{r,R}$. The proof of Proposition 1.5.10 is based on the density estimates and the Besicovitch's Covering Theorem.

On the other hand, we recall the discrete scheme $E_t = E_t(h, \delta)$ in (1.5.10) and define the corresponding Lagrange multiplier $\lambda_\delta^h(t)$ in (1.5.35). In Proposition 1.5.14, we show that the $\lambda_\delta^h(t)$ is bounded by $\frac{1}{h}\|d(\cdot, \partial E_{t-h})\|_{L^2(\partial E_t)}$ up to a constant. By combining Propositions 1.5.10 and 1.5.14 with the inequality from Lemma 1.4.15, we conclude that L^2 norm of λ_δ^h is uniformly bounded with respect to $h > 0$. Here, we construct a local variation (See Lemma 1.5.12 and Lemma 1.5.13) in order to find the Euler-Lagrange equation.

Here is density estimates for sets in $S_{r,R}$. We postpone the proof into the Appendix A.3 as the proof is classical.

Lemma 1.5.8. *For $E \in S_{r,R}$ and $0 < r < R$, the following holds: there exists $\varepsilon_0 = \varepsilon_0(r, R)$, $\eta_i = \eta_i(r, R)$ for $i = 1, 2$ and 3 such that for all $\varepsilon \in [0, \varepsilon_0]$ and $x \in \partial E$*

$$\eta_1 \varepsilon^n \leq \min\{|B_\varepsilon(x) \setminus E|, |E \cap B_\varepsilon(x)|\} \tag{1.5.19}$$

and

$$\eta_3 \varepsilon^{n-1} \leq \mathrm{Per}(E; B_\varepsilon(x)) \leq \eta_2 \varepsilon^{n-1} \tag{1.5.20}$$

where

$$\mathrm{Per}(E; F) := \sup\left\{\int_E \mathrm{div} T(x) dx : T \in C_c^1(F; \mathbb{R}^n), \sup_F |T| \leq 1\right\}.$$

Note that for any $F \subset \mathbb{R}^n$ and $E \subset \mathbb{R}^n$, which has a Lipschitz boundary, it holds that

$$P(E; F) = \mathcal{H}^{n-1}(F \cap \partial E) \tag{1.5.21}$$

(See Remark 9.5 and Example 12.6 in [Mag12]).

Lemma 1.5.9. *[EG92, Theorem 1.27][Mag12, Theorem 5.1] (Besicovitch's Covering Theorem) There exists a positive constant $\xi = \xi(n)$ with the following property: if $\mathcal{F}$ is a family of closed balls of $\mathbb{R}^n$ with positive radii, and the set $\mathcal{N}$ of the centers of the balls in $\mathcal{F}$ is bounded, then there exists at most countable $\mathcal{F}_1, \ldots, \mathcal{F}_\xi$ subfamilies of disjoint balls in $\mathcal{F}$ such that*

$$\mathcal{N} \subset \bigcup_{j=1}^{\xi} \bigcup_{B \in \mathcal{F}_j} B.$$

The density estimates in Lemma 1.5.8 and the Besicovitch's Covering Theorem in Lemma 1.5.9 imply the following proposition. A similar inequality was proven for the discrete gradient flow in the proof of [MSS16, Lemma 3.6]. We extend this results for sets in $S_{r,R}$.

Proposition 1.5.10. *For $E, F \in S_{r,R}$ and $0 < r < R$, the following holds: for some $\sigma_2 = \sigma_2(r, R)$*

$$\int_{\partial F} d^2(x, \partial E) d\sigma \leq \sigma_2 \widetilde{d}^2(F, E). \tag{1.5.22}$$

Here, $\widetilde{d}$ is given in (1.5.8).

Proof. 1. For all $i \in \mathbb{Z}$, define

$$\mathcal{D}_i := \{x \in \mathbb{R}^n : 2^i < d(x, \partial E) \leq 2^{i+1}\} \text{ and } \delta_0 := \min\left\{\frac{\varepsilon_0}{R}, 1\right\} \tag{1.5.23}$$

where ε_0 is given in Lemma 1.5.8. Let us show that there exists $c_1 = c_1(r, R)$ such that for all $x_i \in \mathcal{D}_i \cap \partial F$

$$\mathcal{I}_1 \leq c_1 \mathcal{I}_2, \quad \mathcal{I}_1 := \int_{\partial F \cap B_{2^{i-1}\delta_0}(x_i)} d^2(x, \partial E) d\sigma \text{ and } \mathcal{I}_2 := \int_{(E \triangle F) \cap B_{2^{i-1}\delta_0}(x_i)} d(x, \partial E) dx. \tag{1.5.24}$$

As $E, F \subset B_R$, it holds that for $2^i > 2R$,

$$\mathcal{D}_i \cap \partial F = \emptyset. \tag{1.5.25}$$

Thus, it is enough to consider $i \leq \log_2 R + 1$. Then, it holds that

$$2^{i-1}\delta_0 \leq R\delta_0 \leq \varepsilon_0. \tag{1.5.26}$$

For any $x \in B_{2^{i-1}\delta_0}(x_i)$ and $x_i \in \mathcal{D}_i$, it hold that

$$2^{i-1} \le d(x, \partial E) \le 2^{i+2}. \tag{1.5.27}$$

Therefore, $\mathcal{I}_1$ and $\mathcal{I}_2$ are bounded as follows;

$$\mathcal{I}_1 \le \mathrm{Per}(F; B_{2^{i-1}\delta_0}(x_i))2^{2i+4} \text{ and } \mathcal{I}_2 \ge |(E\triangle F) \cap B_{2^{i-1}\delta_0}(x_i)|2^{i-1}\delta_0. \tag{1.5.28}$$

By (1.5.20) in Lemma 1.5.8, (1.5.26) and (1.5.28), it holds that

$$\mathcal{I}_1 \le \eta_2(2^{(i-1)}\delta_0)^{n-1}2^{2i+4} = \eta_2 2^{i(n+1)-n+5}\delta_0^{n-1}. \tag{1.5.29}$$

On the other hand, as $B_{2^{i-1}\delta_0}(x_i) \subset E$ or $B_{2^{i-1}\delta_0}(x_i) \subset E^c$, it holds that

$$|(E\triangle F) \cap B_{2^{i-1}\delta_0}(x_i)| = \begin{cases} |B_{2^{i-1}\delta_0}(x_i) \setminus F| & \text{if } B_{2^{i-1}\delta_0}(x_i) \subset E, \\ |B_{2^{i-1}\delta_0}(x_i) \cap F| & \text{if } B_{2^{i-1}\delta_0}(x_i) \subset E^c. \end{cases} \tag{1.5.30}$$

From (1.5.19) in Lemma 1.5.8 and (1.5.26), in both cases, we have

$$\mathcal{I}_2 \ge \eta_1 2^{(i-1)(n+1)}\delta_0^n = \eta_1 2^{i(n+1)-n-1}\delta_0^n. \tag{1.5.31}$$

From (1.5.29) and (1.5.31), (1.5.24) holds for $c_1 := \frac{2^6\eta_2}{\eta_1\delta_0}$.

2. Let $\mathcal{F} := \{B_{2^{i-1}\delta_0}(x_i) : x_i \in D_i\}$. Then, by Lemma 1.5.9, there exists $\mathcal{F}_1, \dots, \mathcal{F}_\xi$ subfamilies of disjoint balls in $\mathcal{F}$ such that each family $\mathcal{F}_j$ is at most countable and

$$\partial F \cap D_i \subset \bigcup_{j=1}^{\xi} \bigcup_{B\in\mathcal{F}_j} B. \tag{1.5.32}$$

From (1.5.32) and (1.5.24) in Step 1, it holds that

$$\mathcal{I}_3 := \int_{\partial F\cap D_i} d^2(x,\partial E)d\sigma \le \sum_{j=1}^{\xi}\sum_{B\in\mathcal{F}_j}\int_{\partial F\cap B} d^2(x,\partial E)d\sigma \le c_1\sum_{j=1}^{\xi}\sum_{B\in\mathcal{F}_j}\int_{(E\triangle F)\cap B} d(x,\partial E)dx. \tag{1.5.33}$$

As (1.5.27) implies $B \subset \mathcal{D}_{i-1} \cup \mathcal{D}_i \cup \mathcal{D}_{i+1}$ for all $B \in \mathcal{F}$ and $\mathcal{F}_j$ is a family of disjoint balls, we conclude that

$$\mathcal{I}_3 \le c_1\sum_{j=1}^{\xi}\int_{(E\triangle F)\cap(\mathcal{D}_{i-1}\cup\mathcal{D}_i\cup\mathcal{D}_{i+1})} d(x,\partial E)dx = c_1\xi\int_{(E\triangle F)\cap(\mathcal{D}_{i-1}\cup\mathcal{D}_i\cup\mathcal{D}_{i+1})} d(x,\partial E)dx. \tag{1.5.34}$$

3. From $\bigcup_{i\in\mathbb{Z}}\mathcal{D}_i=\mathbb{R}^n$, (1.5.25) and (1.5.34), it holds that

$$\int_{\partial F} d^2(x,\partial E)d\sigma = \sum_{i\in\mathbb{Z}}\int_{\partial F\cap\mathcal{D}_i} d^2(x,\partial E)d\sigma,$$
$$\leq c_1\xi\sum_{i\in\mathbb{Z}}\int_{(E\triangle F)\cap(\mathcal{D}_{i-1}\cup\mathcal{D}_i\cup\mathcal{D}_{i+1})} d(x,\partial E)dx = 3c_1\xi\widetilde{d}^2(F,E).$$

Thus, (1.5.22) holds for $\sigma_2 := 3c_1\xi$. □

Now, let us find the Euler-Lagrange equation as [MSS16, Lemma 3.7] and [Mag12, Theorem 17.20]. Consider the discrete flow $E_t=E_t(h,\delta)$ given in (1.5.10) and define the Lagrange multiplier at each time step,

$$\lambda_\delta^h(t) := \gamma_\delta(|E_t(h,\delta)|). \tag{1.5.35}$$

Definition 1.5.11. [Mag12, Chapter 17.3] We say that $\{f_s\}_{-\varepsilon_1<s<\varepsilon_2}$ is *a local variation in A* for an open set A if for a fixed $-\varepsilon_1<s<\varepsilon_2$ and $\varepsilon_1,\varepsilon_2>0$, $f_s:\mathbb{R}^n\to\mathbb{R}^n$ is a diffeomorphism of $\mathbb{R}^n$ such that

$$f_0(x)=x \text{ for all } x\in\mathbb{R}^n, \tag{1.5.36}$$
$$\{x\in\mathbb{R}^n : f_s(x)\neq x\}\subset\subset A \text{ for all } -\varepsilon_1<s<\varepsilon_2. \tag{1.5.37}$$

Let us denote *the initial velocity of* $\{f_s\}_{-\varepsilon_1<s<\varepsilon_2}$ by

$$\Psi(x) := \frac{\partial f_s}{\partial s}\Big|_{s=0}(x). \tag{1.5.38}$$

Recall the first variation of perimeter and volume from Theorem 17.5 and Proposition 17.8 in [Mag12]. For $E\in S_{r,R}$, it holds that

$$\mathrm{Per}(f_s(E)) = \mathrm{Per}(E) + s\int_{\partial E}\mathrm{div}_{\partial E}\Psi d\mathcal{H}^{n-1} + O(s^2) \text{ and} \tag{1.5.39}$$
$$|f_s(E)| = |E| + s\int_{\partial E}\Psi\cdot\vec{n}d\mathcal{H}^{n-1} + O(s^2) \tag{1.5.40}$$

where $\mathrm{div}_{\partial E}$ is the boundary divergence on ∂E defined by

$$\mathrm{div}_{\partial E}\Psi(x) := \mathrm{div}\Psi(x) - (\vec{n}\cdot\nabla\Psi\vec{n})(x) \tag{1.5.41}$$

for $x \in \partial E$. On the other hand, the first variation of $\widetilde{d}$ is as follows,

$$\widetilde{d}^2(f_s(E), F) = \widetilde{d}^2(E, F) + s \int_{\partial E} \mathrm{sd}(x, \partial F)\Psi \cdot \vec{n} d\mathcal{H}^{n-1} + O(s^2) \tag{1.5.42}$$

from (3.1) in [MSS16] and Proposition 17.8 in [Mag12].

In our case, the constraints S_{r_0,R_0} gives some difficulties when we choose the local variation. The following two lemmas construct the local variations within the constraint. Here we use interior and exterior cone properties of $S_{r,R}$, (1.3.3) and (1.3.5), from Lemma 1.3.4. The first lemma discusses creating a larger perturbed set by dilation. For $a > 0$, let us denote $aE := \{x : a^{-1}x \in E\}$.

Lemma 1.5.12. *Let $E_t(h, \delta)$ be given in* (1.5.10)*, δ_0 in* (1.5.2)*, and r_0, R_0 in* (1.5.9)*. Then for $0 < \delta < \delta_0$ there exist $h^* = h^*(\delta)$ and a constant $s_1 > 0$ such that for all $h \in (0, h^*)$ and $s \in [0, s_1)$ we have*

$$(1 + s)E_t(h, \delta) \in S_{r_0,R_0} \text{ for } t \in [0, T]. \tag{1.5.43}$$

Proof. From Proposition 1.5.2, $B_{r_1}(0) \subset \Omega^\delta_t \subset B_{R_1}(0)$ for all $t \geq 0$. Let us first show that there exists $h^* = h^*(\delta) > 0$ such that for all $h \in (0, h^*)$, $t \in [0, T]$ we have

$$B_{r_2}(0) \subset E_t(h, \delta) \subset B_{R_2}(0) \quad \text{where} \quad R_2 := \frac{R_0 + R_1}{2} \text{ and } r_2 := \frac{r_0 + r_1}{2}. \tag{1.5.44}$$

By the uniform convergence of $E_t(h, \delta)$ in $[0, T]$ from Proposition 1.5.4, there exists $h^* = h^*(\delta)$ such that

$$d_H(E_t(h, \delta), \Omega^\delta_t) \leq \min \left\{ \frac{R_0 - R_1}{4}, \frac{r_0 - r_1}{4} \right\} \tag{1.5.45}$$

for all $h \in (0, h^*)$ and $t \in [0, T]$. This implies (1.5.44). From (1.5.44), we conclude that for all $s \in [0, s_1)$

$$(1 + s)E_t(h, \delta) \in B_{R_0} \text{ where } s_1 := \frac{R_0}{R_2} - 1. \tag{1.5.46}$$

As $E_t(h, \delta) \in S_{r_0}$, (1.3.3) imply that for all $x \in \partial E_t(h, \delta)$

$$IC(r, x) \subset E_t(h, \delta). \tag{1.5.47}$$

Since $(1+s)IC(r,x) = IC((1+s)r,(1+s)x))$, we conclude that for all $x \in \partial((1+s)E_t(h,\delta))$

$$IC((1+s)r,x) \subset (1+s)E_t(h,\delta). \tag{1.5.48}$$

As $IC(r,x) \subset IC((1+s)r,x)$, (1.5.43) holds for $s \in [0,s_1)$. □

Generating a smaller set that stays in S_{r_0,R_0} turns out to be more delicate. For this we need perturbations that preserve $\partial B_{r_0}(0)$ and shrinks outside of $B_{r_0}(0)$. To stay within S_{r_0,R_0} we must ensure that the angles of interior cone and exterior cone given in (1.3.3) and (1.3.5) do not decrease for the perturbed set. This is what we prove with a specific choice of the perturbation $\mathcal{G}_s$ below.

Lemma 1.5.13. *Let $E_t(h,\delta)$ with $h \in (0,h^*)$ and $0 < \delta < \delta_0$, where δ_0, r_0, R_0 and h^* are given in the previous lemma. Define the following functions in Q:*

$$\psi(x,t) := \chi_{E_t(h,\delta)}(x) - \chi_{E_t(h,\delta)^C}(x) \text{ and } \mathcal{G}_s[\psi](x,t) := \psi((1+s(|x|^2-r_0^2))x,t).$$

Then there exists $s_2 > 0$ such that

$$\{x \in \mathbb{R}^n : \mathcal{G}_s[\psi](x,t) > 0\} \in S_{r_0,R_0} \text{ for } s \in [0,s_2) \text{ and } t \in [0,T]. \tag{1.5.49}$$

Proof. Let us fix $t \in [0,T]$. We may assume that E_t has a C^1 boundary. Then, there is a C^1 function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that

$$\{\phi > 0\} = E_t, \quad \{\phi = 0\} = \partial E_t, \quad \{\phi < 0\} = \overline{E}_t^C, \text{ and } D\phi \neq 0 \text{ on } \partial E_t. \tag{1.5.50}$$

First note that as $E_t \in S_{r_0,R_0}$ we have $B_{r_0} \subset \{\mathcal{G}_s[\phi] > 0\} \subset B_{R_0}$. To show that $\{\mathcal{G}_s[\phi] > 0\}$ is in S_r, from Lemma 1.3.2 it is enough to show that

$$D\mathcal{G}_s[\phi](x) \neq 0 \text{ and } -\frac{D\mathcal{G}_s[\phi]}{|D\mathcal{G}_s[\phi]|}(x) \cdot x \geq r_0 \text{ for all } x \in \{\mathcal{G}_s[\phi] = 0\}. \tag{1.5.51}$$

For the rest of the proof we assume that $x \in \{\mathcal{G}_s[\phi] = 0\}$.

Denote $P_s(x) := 1 + s(|x|^2 - r_0^2)$ so that we can write $\mathcal{G}_s[\phi](x) = \phi(P_s(x)x)$, and thus $P_s(x)x \in \{\phi = 0\}$ with $D\phi(P_s(x)x) \neq 0$. Observe that

$$1 \leq P_s(x) \leq \frac{3}{2} \text{ for } 0 \leq s < s_2 := \frac{1}{2(R_0^2 - r_0^2)}. \tag{1.5.52}$$

Since

$$D\mathcal{G}_s[\phi](x) = 2s(D\phi(P_s(x)x) \cdot x)x + P_s(x)D\phi(P_s(x)x), \tag{1.5.53}$$

we have

$$|D\mathcal{G}_s[\phi](x)|^2 = P_s(x)^2|D\phi(P_s(x)x)|^2 + 4s(|x|^2s + P_s(x))(D\phi(P_s(x)x) \cdot x)^2. \tag{1.5.54}$$

(1.5.52) yields

$$|D\mathcal{G}_s[\phi](x)|^2 \geq |D\phi(P_s(x)x)|^2 > 0 \text{ for all } s \in [0, s_2),$$

and thus the first condition of (1.5.51) is satisfied.

Let us now show the second condition of (1.5.51). As $\{\phi > 0\} \in S_{r_0,R_0}$ and $P_s(x)x \in \{\phi = 0\}$, Lemma 1.3.2 implies

$$-\frac{D\phi}{|D\phi|}(P_s(x)x) \cdot (P_s(x)x) \geq r_0. \tag{1.5.55}$$

From (1.5.53), we have

$$-D\mathcal{G}_s[\phi](x) \cdot x = -(2|x|^2s + P_s(x))D\phi(P_s(x)x) \cdot x. \tag{1.5.56}$$

Then, (1.5.52), (1.5.55) and (1.5.56) imply that $-D\mathcal{G}_s[\phi](x) \cdot x$ is positive. Thus, it is enough to show that

$$\mathcal{I}_1 := (-D\mathcal{G}_s[\phi](x) \cdot x)^2 - r_0^2|D\mathcal{G}_s[\phi](x)|^2 \geq 0 \text{ for all } s \in [0, s_2) \text{ and } x \in \{\mathcal{G}_s[\phi] = 0\}. \tag{1.5.57}$$

From (1.5.54) and (1.5.55), it holds that

$$r_0^2|D\mathcal{G}_s[\phi](x)|^2 \leq \left(P_s(x)^4 + 4r_0^2s(|x|^2s + P_s(x))\right)(D\phi(P_s(x)x) \cdot x)^2. \tag{1.5.58}$$

From (1.5.56) and (1.5.58) it follows that

$$\frac{\mathcal{I}_1}{(D\phi(P_s(x)x) \cdot x)^2} \geq \left(P_s(x)^2 + 4|x|^2s(|x|^2s + P_s(x))\right) - \left(P_s(x)^4 + 4r_0^2s(|x|^2s + P_s(x))\right). \tag{1.5.59}$$

Using $s(|x|^2 - r_0^2) = P_s(x) - 1$ and factorizing the above, we conclude

$$\frac{\mathcal{I}_1}{(D\phi(P_s(x)x) \cdot x)^2} \geq (P_s(x) - 1)(-P_s(x)^3 - P_s(x)^2 + 4P_s(x) + 4|x|^2s). \tag{1.5.60}$$

From (1.5.52), we conclude that $\mathcal{I}_1 \geq 0$ for all $s \in [0, s_2)$ and $x \in \{\mathcal{G}_s[\phi] = 0\}$. □

From Lemma 1.5.12 and 1.5.13, we get the following estimates. In the proof of Proposition 1.5.14, we use two initial velocities of local variations introduced in Lemma 1.5.12 and Lemma 1.5.13.

Proposition 1.5.14. *There exists* $\sigma_3 = \sigma_3(r_0, R_0)$ *and* $\sigma_4 = \sigma_4(r_0, R_0)$ *such that for all* $t \in [0, T]$ *and* $h \in (0, h^*)$,

$$|\lambda_\delta^h(t)|^2 \leq \sigma_3 + \frac{\sigma_4}{h^2} \int_{\partial E_t(h,\delta)} d^2(x, \partial E_{t-h}) d\sigma. \tag{1.5.61}$$

Here, h^* *is given in Lemma 1.5.12 and* $E_t(h, \delta)$ *is given in Proposition 1.5.4. Also,* r_0 *and* R_0 *are given in* (1.5.9).

Proof. 1. First, show that if $f_s(E_t) \in S_{r_0,R_0}$ for all $s \in [0, s_0)$, then it holds that

$$\lambda_\delta^h(t) \int_{\partial E_t} \vec{n} \cdot \Psi d\sigma \leq \int_{\partial E_t} \operatorname{div}_{\partial E_t} \Psi + \frac{1}{h} \operatorname{sd}(x, \partial E_{t-h}) \vec{n} \cdot \Psi d\sigma. \tag{1.5.62}$$

As E_t is a minimizer of $\mathcal{J}_\delta(\cdot) + \frac{1}{h}\tilde{d}^2(\cdot, E_{t-h})$ on S_{r_0,R_0}, (1.5.39), (1.5.40), and (1.5.41) imply that

$$s\lambda_\delta^h(t) \int_{\partial E_t} \vec{n} \cdot \Psi d\sigma \leq s \int_{\partial E_t} \operatorname{div}_{\partial E_t} \Psi d\sigma + \frac{s}{h} \int_{\partial E_t} \operatorname{sd}(x, \partial E_{t-h}) \vec{n} \cdot \Psi d\sigma + O(s^2) \tag{1.5.63}$$

for all $s \in [0, s_0)$. Dividing both sides by $s > 0$ and sending s to zero, we conclude (1.5.62).

2. Let us find the upper bound of $\lambda_\delta^h(t)$. Recall $f_s(x) := x + sx$ in Lemma 1.5.12. Then, $f_s(E_t) \in S_{r_0,R_0}$ for $s \in [0, s_1)$ and $\Psi(x) = x$. From (1.5.62) in Step 1 and $r_0 \leq \vec{n} \cdot x \leq R_0$ on ∂E_t, it holds that

$$\lambda_\delta^h(t) \leq \frac{\int_{\partial E_t} \operatorname{div}_{\partial E_t} \Psi + \frac{1}{h} \operatorname{sd}(x, \partial E_{t-h}) \vec{n} \cdot \Psi d\sigma}{\int_{\partial E_t} \vec{n} \cdot \Psi d\sigma} \leq \frac{n-1}{r_0} + \frac{R_0}{r_0 \operatorname{Per}(E_t)} \frac{1}{h} \int_{\partial E_t} \operatorname{sd}(x, \partial E_{t-h}) d\sigma. \tag{1.5.64}$$

3. Let us construct the lower bound. Define $g, f : \mathbb{R}^n \times [0, s_2) \to \mathbb{R}^n$ by

$$g_s(x) = g(x, s) := (1 + s(|x|^2 - r_0^2))x \text{ and } f_s(x) = f(x, s) := (g_s)^{-1}(x) \tag{1.5.65}$$

where s_2 is given in (1.5.52) in Lemma 1.5.13. As $g(f(x, s), s) = x$ and $Dg_s|_{s=0} = I$, it holds that

$$\frac{\partial f_s}{\partial s}\Big|_{s=0}(x) = -\frac{\partial g_s}{\partial s}\Big|_{s=0}(x) = -(|x|^2 - r_0^2)x. \tag{1.5.66}$$

From the above and (1.5.38), the initial velocity is

$$\Psi(x) = -(|x|^2 - r_0^2)x. \tag{1.5.67}$$

From Lemma 1.5.13, $f_s(E_t) \in S_{r_0,R_0}$ for $s \in [0, s_2)$. By (1.5.62) and $\Psi \cdot \vec{n} \le 0$ on ∂E_t, it holds that

$$\lambda_\delta^h(t) \ge \frac{\int_{\partial E_t} \mathrm{div}_{\partial E_t}\Psi + \frac{1}{h}\mathrm{sd}(x, \partial E_{t-h})\vec{n} \cdot \Psi d\sigma}{\int_{\partial E_t} \vec{n} \cdot \Psi d\sigma}. \tag{1.5.68}$$

Note that from (1.5.44)

$$-R_0(R_0^2 - r_0^2) \le \Psi \cdot \vec{n} \le -r_0(r_2^2 - r_0^2) \text{ and } -(n+1)(R_0^2 - r_0^2) \le \mathrm{div}_{\partial E_t}\Psi \le -(n-1)(r_2^2 - r_0^2). \tag{1.5.69}$$

From (1.5.68) and (1.5.69), we conclude that

$$\lambda_\delta^h(t) \ge \frac{(n-1)(r_2^2 - r_0^2)}{R_0(R_0^2 - r_0^2)} - \frac{1}{h\mathrm{Per}(E_t)} \int_{\partial E_t} d(x, \partial E_{t-h}) d\sigma. \tag{1.5.70}$$

4. From (1.5.64) and (1.5.70), there exists $c_1 = c_1(r_0, R_0)$ and $c_2 = c_2(r_0, R_0)$ such that

$$|\lambda_\delta^h(t)| \le c_1 + \frac{c_2}{h\mathrm{Per}(E_t)} \int_{\partial E_t} d(x, \partial E_{t-h}) d\sigma. \tag{1.5.71}$$

From (1.5.71), $(a+b)^2 \le 2(a^2 + b^2)$ for $a, b \in \mathbb{R}$ and the Hölder's inequality, it holds that

$$|\lambda_\delta^h(t)|^2 \le 2c_1^2 + \frac{2c_2^2}{h^2\mathrm{Per}(E_t)^2}\left(\int_{\partial E_t} d(x, \partial E_{t-h}) d\sigma\right)^2 \le 2c_1^2 + \frac{2c_2^2}{h^2\mathrm{Per}(E_t)} \int_{\partial E_t} d(x, \partial E_{t-h})^2 d\sigma. \tag{1.5.72}$$

By the isoperimetric inequality and $B_{r_0} \subset E_t$, we have $\mathrm{Per}(E_t) > c_3$ for some $c_3 = c_3(r_0)$, we conclude that (1.5.61) holds for

$$\sigma_3 := 2c_1^2 \text{ and } \sigma_4 := \frac{2c_2^2}{c_3}. \tag{1.5.73}$$

□

Proof of Theorem 1.5.6.

Let us show that $\|\lambda_\delta^h\|^2_{L^2([0,T])}$ is uniformly bounded for all $h \in (0, h^*)$ and all $\delta \in (0, \delta_0)$. Here, $h^* = h^*(\delta)$ is given in Lemma 1.5.12.

By Proposition 1.5.14 and Proposition 1.5.10, it holds that

$$\|\lambda_\delta^h\|_{L^2([0,T])}^2 \le \sigma_3 T + \frac{\sigma_2\sigma_4}{h^2}\int_0^T \widetilde{d}^2(E_t, E_{t-h})dt \le \sigma_3 T + \frac{\sigma_2\sigma_4}{h}\sum_{k=1}^{[\frac{T}{h}]}\widetilde{d}^2(E_{kh}, E_{(k-1)h}). \quad (1.5.74)$$

Note that Lemma 1.4.15 implies

$$\frac{1}{h}\sum_{k=1}^{[\frac{T}{h}]}\widetilde{d}^2(E_{kh}, E_{(k-1)h}) \le \mathcal{K}_2\sum_{k=1}^{[\frac{T}{h}]}(\mathcal{J}_\delta(E_{(k-1)h}) - \mathcal{J}_\delta(E_{kh})) = \mathcal{K}_2(\mathcal{J}(\Omega_0) - \mathcal{J}(E_{[\frac{T}{h}]h})) \le \mathcal{K}_2\mathrm{Per}(\Omega_0). \quad (1.5.75)$$

Thus, (1.5.74) and (1.5.75) imply that

$$\|\lambda_\delta^h\|_{L^2([0,T])}^2 \le \sigma_3 T + \sigma_2\sigma_4\mathcal{K}_2\mathrm{Per}(\Omega_0) \quad (1.5.76)$$

for all $h \in (0, h^*)$.

By the uniform continuity of $E_t(h,\delta)$ in $[0,T]$ from Proposition 1.5.4, λ_δ^h given in (1.5.35) uniformly converges to λ_δ given in (1.1.5) in $[0,T]$ along a subsequence. Thus, we conclude that (1.5.18) holds for

$$\sigma_1 := \max\{\sigma_3, \sigma_2\sigma_4\mathcal{K}_2\}. \quad (1.5.77)$$

Here, σ_2 is given in Proposition 1.5.14, σ_3 and σ_4 are given in Proposition 1.5.10 and $\mathcal{K}_2$ is given in Lemma 1.4.15. For $\delta_i \in (0, \delta_0)$ given in Theorem 1.5.1, λ_{δ_i} is uniformly bounded in $L^2([0,T])$ for all $i \in \mathbb{N}$. Thus, by the Banach-Alaoglu Theorem, there exists a subsequence δ_{i_j} of δ_i in Theorem 1.5.1 such that $\lambda_{\delta_{i_j}}$ weakly converges to λ_∞ in $L^2[0,T]$. □

For the later purpose in Section 1.5.3, let us also construct L^2 estimates in $[t_0, t_0+T]$ for all $t_0 \ge 0$.

Corollary 1.5.15. *Let $\delta \in (0, \delta_0)$ for δ_0 given in* (1.5.2) *and λ_δ be given in* (1.1.5).

$$\|\lambda_\delta\|_{L^2([t_0,t_0+T])}^2 \le \sigma_1(\mathrm{Per}(\Omega_0) + T) \quad (1.5.78)$$

where σ_1 is given in (1.5.77).

Proof. As $\mathcal{J}_\delta(\Omega_t^\delta)$ given in (1.5.7) decreases in time, $\mathcal{J}_\delta(\Omega_t^\delta)$ is bounded by $\mathcal{J}_\delta(\Omega_0) = \mathrm{Per}(\Omega_0)$ for all $\delta > 0$ and $t \geq 0$. From (1.5.74) and (1.5.75) in the proof of Theorem 1.5.6, we have

$$\|\lambda_\delta^h\|_{L^2([t_0,t_0+T])}^2 \leq \sigma_1(\mathrm{Per}(\Omega_0) + T) \tag{1.5.79}$$

where σ_1 is given in (1.5.77). As the proof of Theorem 1.5.6, we conclude (1.5.78). □

1.5.3 Convergence

In this section, we discuss the large-time behavior of $(\Omega_t^\infty)_{t\geq 0}$ given in Theorem 1.5.1. Here is the main theorem in this section.

Theorem 1.5.16. $(\Omega_t^\infty)_{t\geq 0}$ *given in Theorem 1.5.1 uniformly converges to a ball of volume* 1*, modulo translation. More precisely*

$$\inf\left\{d_H(\Omega_t^\infty, B_{r_\infty}(x)) : x \in \overline{B}_{r_1}(0)\right\} \to 0 \text{ as } t \to \infty, \tag{1.5.80}$$

where r_1 *is given in Proposition 1.5.2,* $r_\infty := (w_n)^{-\frac{1}{n}}$ *and* w_n *is a volume of an unit ball in* $\mathbb{R}^n$*.*

Intuitively this convergence is due to the flow's formal gradient flow structure with respect to the perimeter energy. Unfortunately, due to the lack of uniform regularity for Ω_t^δ with respect to $\delta > 0$, we are not able to directly show that Ω_t^∞ is the gradient flow of the perimeter energy in the space of sets with unit volume. Hence we instead utilize the gradient flow structure for the δ-flow, as given in section 4, to show this convergence.

The main estimate in the analysis is Lemma 1.5.18, where we bound the difference of total perimeter with respect to their differences in Hausdorff distance, in the class of star-shaped sets with their total curvature in L^2. Based on this estimate, we can proceed to show in (1.5.110) that the time integral of δ-energy converges to the time integral of the perimeter energy. This now establishes the link between the gradient flow structure of δ-flow and the limit flow, and the asymptotic convergence follows.

For $k \in \mathbb{N}$ we consider $((U_t^k)_{t\geq 0}, \eta^k)$ defined by

$$U_t^k := \Omega_{t+k}^\infty \text{ and } \eta^k(t) := \lambda_\infty(t+k). \tag{1.5.81}$$

Here, $(\Omega_t^\infty)_{t\geq 0}$ and λ_∞ are given in Theorem 1.5.1 and Theorem 1.5.6, respectively.

Proposition 1.5.17. *There exists a subsequence $\{k_i\}_{i\in\mathbb{N}}$ such that $\{(U_t^{k_i})_{t\geq 0}\}_{i\in\mathbb{N}}$ locally uniformly converges to $(U_t^\infty)_{t\geq 0} \subset S_{r_1,R_1}$ and $\{\eta^{k_i}\}_{i\in\mathbb{N}}$ weakly converges to η^∞ in $L^2([0,T])$ for all $T > 0$. As a consequence, $(U_t^\infty)_{t\geq 0}$ is a viscosity solution (See Definition 1.2.13) of $V = -H + \eta^\infty(t)$. Here, r_1 and R_1 are given in Proposition 1.5.2.*

Proof. Theorem 1.5.1 and Proposition 1.5.5 imply that for all $0 < k_1 < k_2$

$$d_H(U_t^{k_1}, U_t^{k_2}) \leq \mathcal{K}_3(k_2 - k_1)^{\frac{1}{n+1}}\mathrm{Per}(\Omega_0)^{\frac{1}{n+1}} \tag{1.5.82}$$

where $\mathcal{K}_3$ is given in Proposition 1.5.5. Since $U_t^k \subset S_{r_1,R_1}$ by Theorem 1.5.1, we can apply Lemma A.3.3 to obtain a subsequential limit of $\{(U_t^k)_{t\geq 0}\}_{k\in\mathbb{N}}$.

On the other hand the Banach-Anaoglu theorem can be applied to $\{\eta^k\}_{k\in\mathbb{N}}$ due to the uniform L^2 estimates obtained in Corollary 1.5.15. Thus, for any $T > 0$ there exists a subsequence $\{k_i\}_{i\in\mathbb{N}}$ such that $\{(U_t^{k_i})_{t\geq 0}\}_{i\in\mathbb{N}}$ locally uniformly converges to $(U_t^\infty)_{t\geq 0} \subset S_{r_1,R_1}$ and $\{\eta^{k_i}\}_{i\in\mathbb{N}}$ weakly converges to η^∞ in $L^2([0,T])$. Note that Υ^{k_i} converges locally uniformly to Υ^∞, where $\Upsilon^k(t) := \int_0^t \eta^k(s)ds$ for $k \in \mathbb{N}\cup\{+\infty\}$.

Recall that $(U_t^k)_{t\geq 0}$ is a viscosity solution of $V = -H + \eta^k(t)$. From Theorem 1.2.17 and Lemma A.3.7, $(U_t^\infty)_{t\geq 0}$ is a viscosity solution of $V = -H + \eta^\infty(t)$. □

Now, in Lemma 1.5.18, we estimates the time integral of the perimeter difference for two evolving sets $(\Omega_t^j)_{t\geq 0} \subset S_{r,R}$ and $j \in \{1,2\}$.

Lemma 1.5.18. *For $j \in \{1,2\}$, consider $(\Omega_t^j)_{t\geq 0} \subset S_{r,R}$ for $R > r > 0$ such that $(\partial\Omega_t^j)_{t>0}$ are smooth. Suppose that there exists a constant $\mathcal{W} < +\infty$ such that for $T > 0$ and $j \in \{1,2\}$*

$$\int_0^T \int_{\partial\Omega_t^j} H(x,t)^2 d\sigma dt < \mathcal{W} \tag{1.5.83}$$

where $H(x,t)$ is the mean curvature at $x \in \partial\Omega_t^j$. Then, there exists a constant $\overline{m} = \overline{m}(r,R,T,\mathcal{W}) > 0$ such that

$$\left(\int_0^T \mathrm{Per}(\Omega_t^1) - \mathrm{Per}(\Omega_t^2)dt\right)^2 \leq \overline{m} \sup_{t\in[0,T]} d_H(\Omega_t^1, \Omega_t^2). \tag{1.5.84}$$

Proof. As $\{\Omega_t^j\}_{j\in\{1,2\}} \subset S_{r,R}$ are smooth for $t>0$, there exist two smooth functions $u_1, u_2 : B_r^{n-1}(0)\times[0,T]\to\mathbb{R}$ such that for $j=1,2$

$$\partial\Omega_t^j \cap C_{r,R}^+(0) = \{(u_j(y',t),y') : y'\in B_r^{n-1}(0)\} \tag{1.5.85}$$

where $C_{r,R}^+(0) := [0,R]\times B_r^{n-1}(0)$. Furthermore, from $\Omega_1,\Omega_2\in S_{r,R}$ again, there exists a constant $c_1 = c_1(r,R)$ such that

$$\|u_1-u_2\|_{L^\infty(B_r^{n-1}(0)\times[0,T])} \le c_1 \sup_{t\in[0,T]} d_H(\Omega_t^1,\Omega_t^2) \text{ and } \|\nabla u_j\|_{L^\infty(B_r^{n-1}(0)\times[0,T])}\le c_1 \text{ for } j=1,2. \tag{1.5.86}$$

1. Let us first show that there exists $\overline{m}_1=\overline{m}_1(r,R,T,\mathcal{W})$ for $\mathcal{W}$ given in (1.5.83)

$$\|\vec{n}_1-\vec{n}_2\|^2_{L^2(B_r^{n-1}(0)\times[0,T])} \le \overline{m}_1 \sup_{t\in[0,T]} d_H(\Omega_t^1,\Omega_t^2) \text{ where } \vec{n}_j := \frac{(1,\nabla u_j)}{\sqrt{1+|\nabla u_j|^2}} \text{ for } j=1,2. \tag{1.5.87}$$

As $\vec{n}_1$ and $\vec{n}_2$ are unit vectors, we get the following by the direct computation,

$$|\vec{n}_1-\vec{n}_2|^2 = 2(1-\vec{n}_1\cdot\vec{n}_2), \tag{1.5.88}$$

$$\le (\sqrt{1+|\nabla u_1|^2}+\sqrt{1+|\nabla u_2|^2})(1-\vec{n}_1\cdot\vec{n}_2), \tag{1.5.89}$$

$$= ((\sqrt{1+|\nabla u_1|^2})\vec{n}_1 - (\sqrt{1+|\nabla u_2|^2})\vec{n}_2)\cdot(\vec{n}_1-\vec{n}_2) = \nabla(u_1-u_2)\cdot(\vec{n}_1'-\vec{n}_2') \tag{1.5.90}$$

where $\vec{n}_j'$ is the last $n-1$ components of $\vec{n}_j$ given by

$$\vec{n}_j' := \frac{\nabla u_j}{\sqrt{1+|\nabla u_j(x)|^2}} \text{ for } j\in\{1,2\}. \tag{1.5.91}$$

Note that the mean curvature at $(u_j(x,t),x)\in\partial\Omega_t^j$ for $x\in B_r^{n-1}(0)$ is given by

$$H((u_j(x,t),x),t) = \nabla\cdot\vec{n}_j'(x,t). \tag{1.5.92}$$

From (1.5.83), there exists $c_2=c_2(r,R,\mathcal{W})$ such that for $j\in\{1,2\}$

$$\|\nabla\cdot\vec{n}_j'\|_{L^2(B_r^{n-1}(0)\times[0,T])}\le c_2. \tag{1.5.93}$$

From integration by parts, we have

$$\mathcal{I}_1 := \int_{B_r^{n-1}(0)\times[0,T]} \nabla(u_1-u_2)\cdot(\vec{n}_1'-\vec{n}_2')dxdt, \tag{1.5.94}$$

$$= \int_{\partial B_r^{n-1}(0)\times[0,T]} (u_1-u_2)(\vec{n}_1'-\vec{n}_2')\cdot\nu d\sigma dt - \int_{B_r^{n-1}(0)\times[0,T]} (u_1-u_2)\cdot(\nabla\cdot(\vec{n}_1'-\vec{n}_2'))dxdt \tag{1.5.95}$$

where ν is the outward normal vector on $\partial B_r^{n-1}(0)$. By applying the Hölder inequality at each terms and using (1.5.86) and (1.5.93), we have

$$|\mathcal{I}_1| \le (2\mathrm{Per}(B_r^{n-1})T + \|\nabla\cdot(\vec{n}_1'-\vec{n}_2')\|_{L^2(B_r^{n-1}(0)\times[0,T])}|B_r^{n-1}|^{\frac{1}{2}}T^{\frac{1}{2}})\|u_1-u_2\|_{L^\infty(B_r^{n-1}(0)\times[0,T])}, \tag{1.5.96}$$

$$\le (2\mathrm{Per}(B_r^{n-1})T + 2c_2|B_r^{n-1}|^{\frac{1}{2}}T^{\frac{1}{2}})c_1 \sup_{t\in[0,T]} d_H(\Omega_t^1,\Omega_t^2). \tag{1.5.97}$$

From (1.5.90) and (1.5.97), we conclude (1.5.87) with

$$\overline{m}_1 := 2c_1\left(\mathrm{Per}(B_r^{n-1})T + c_2|B_r^{n-1}|^{\frac{1}{2}}T^{\frac{1}{2}}\right). \tag{1.5.98}$$

2. Let us show that there exists $\overline{m}_2 = \overline{m}_2(r,R,T,\mathcal{W})$ for $\mathcal{W}$ given in (1.5.83)

$$(\mathcal{I}_2)^2 \le \overline{m}_2 \sup_{t\in[0,T]} d_H(\Omega_t^1,\Omega_t^2) \text{ where } \mathcal{I}_2 := \int_0^T \mathrm{Per}(\Omega_1; C_{r,R}^+(0)) - \mathrm{Per}(\Omega_2; C_{r,R}^+(0))dt. \tag{1.5.99}$$

Recall from (1.5.21) and Theorem 9.1 in [Mag12], we have

$$\int_0^T \mathrm{Per}(\Omega_t^j; C_{r,R}^+(0))dt = \int_{B_r^{n-1}(0)\times[0,T]} \sqrt{1+|\nabla u_j|^2}dxdt = \int_{B_r^{n-1}(0)\times[0,T]} (1,\nabla u_j)\cdot\vec{n}_j dxdt \tag{1.5.100}$$

where $\{u_j\}_{j\in\{1,2\}}$ and $\{n_j\}_{j\in\{1,2\}}$ are given in (1.5.85) and (1.5.87), respectively. By adding and subtracting the same term in $\mathcal{I}_2$, we have the identity

$$\mathcal{I}_2 = \mathcal{I}_3 + \mathcal{I}_4 \tag{1.5.101}$$

where

$$\mathcal{I}_3 := \int_{B_r^{n-1}(0)\times[0,T]} (1,\nabla u_1)\cdot(\vec{n}_1-\vec{n}_2)dxdt \tag{1.5.102}$$

and

$$\mathcal{I}_4 := \int_{B_r^{n-1}(0)\times[0,T]} ((1,\nabla u_1)-(1,\nabla u_2))\cdot \vec{n}_2 dxdt = \int_{B_r^{n-1}(0)\times[0,T]} \nabla(u_1-u_2)\cdot \vec{n}_2' dxdt. \tag{1.5.103}$$

Here, $\{n_j\}_{j\in\{1,2\}}$ and $\{n_j'\}_{j\in\{1,2\}}$ are given in (1.5.87) and (1.5.91), respectively.

By applying (1.5.86) and (1.5.87) and the Hölder inequality, we get

$$\mathcal{I}_3^2 \le (1+c_1^2)|B_r^{n-1}|T\overline{m}_1 \sup_{t\in[0,T]} d_H(\Omega_t^1,\Omega_t^2). \tag{1.5.104}$$

where c_1 and $\overline{m}_1$ are given in (1.5.86) and (1.5.98). On the other hand, by the similar arguments in (1.5.95)

$$\mathcal{I}_4 \le \overline{m}_1 \sup_{t\in[0,T]} d_H(\Omega_t^1,\Omega_t^2) \tag{1.5.105}$$

where $\overline{m}_1$ is given in (1.5.98). As $(\Omega_t^j)_{t\ge 0}\subset S_{r,R}$ for $j\in\{1,2\}$, we have

$$\sup_{t\in[0,T]} d_H(\Omega_t^1,\Omega_t^2) \le 2R. \tag{1.5.106}$$

Thus, (1.5.105) and (1.5.106) imply that

$$\mathcal{I}_4^2 \le 2\overline{m}_1^2 R \sup_{t\in[0,T]} d_H(\Omega_t^1,\Omega_t^2). \tag{1.5.107}$$

From (1.5.101) combining with (1.5.104) and (1.5.107), we have

$$\mathcal{I}_2^2 \le 2(\mathcal{I}_3^2+\mathcal{I}_4^2) \le 2((1+c_1^2)|B_r^{n-1}|T\overline{m}_1 + 2\overline{m}_1^2 R) \sup_{t\in[0,T]} d_H(\Omega_t^1,\Omega_t^2). \tag{1.5.108}$$

Thus, we conclude (1.5.99) for

$$\overline{m}_2 := 2\overline{m}_1\left((1+c_1^2)|B_r^{n-1}|T + 2\overline{m}_1 R\right). \tag{1.5.109}$$

Here, c_1 and $\overline{m}_1$ are given in (1.5.86) and (1.5.98).

3. As every sets in $S_{r,R}$ can be covered by a finite number of cylinders $C_{r,R}^+(0)$ after some rotations, (1.5.99) implies (1.5.84). □

From the estimates in Lemma 1.5.18 and our approximation from $(\Omega_t^\delta)_{t\ge 0}$ in Theorem 1.5.1, we conclude that the limit flow $(U_t^\infty)_{t\ge 0}$ is stationary.

Proposition 1.5.19. $(U_t^\infty)_{t\geq 0}$ *given in Proposition 1.5.17 is stationary.*

Proof. 1. Let us show that there exists $\mathcal{E}^\infty : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\mathcal{E}^\infty(k) := \lim_{\delta\to 0} \mathcal{E}^\delta(k) \text{ where } \mathcal{E}^\delta(k) := \int_0^T \mathcal{J}_\delta(\Omega^\delta_{t+k})dt. \tag{1.5.110}$$

It is enough to show that $\{\mathcal{E}^\delta(k)\}_{\delta>0}$ is a Cauchy sequence as $\delta \to 0$ for all $k \in [0, +\infty)$. As Ω_t^δ is smooth for $t > 0$ from Proposition 1.5.2 and Ω_t^δ is a gradient flow of $\mathcal{J}_\delta$, we have

$$\int_{t_0}^{t_0+T} \int_{\partial\Omega_t^\delta} V^2 d\sigma dt = \mathcal{J}_\delta(\Omega^\delta_{t_0}) - \mathcal{J}_\delta(\Omega^\delta_{t_0+T}) \leq \mathrm{Per}(\Omega_0) \tag{1.5.111}$$

where V is the normal velocity at $x \in \partial\Omega_t^\delta$. As $H = \lambda - V$, Corollary 1.5.15 and (1.5.111) implies the uniform bound on $\|H\|_{L^2(t_0,t_0+T;L^2(\partial\Omega_t^\delta))}$.

As $\mathcal{J}_\delta(\Omega_t^\delta) = \mathrm{Per}(\Omega_t^\delta) + 2\delta\lambda_\delta(t)^2$, Lemma 1.5.18 and Corollary 1.5.15 imply that for $\delta_1 > \delta_2 > 0$

$$\left|\mathcal{E}^{\delta_1}(k) - \mathcal{E}^{\delta_2}(k)\right| \leq \left|\int_0^T \mathrm{Per}(\Omega^{\delta_1}_{t+k}) - \mathrm{Per}(\Omega^{\delta_2}_{t+k})dt\right| + 2\delta_1\|\lambda_{\delta_1}\|^2_{L^2([k,k+T])} + 2\delta_2\|\lambda_{\delta_2}\|^2_{L^2([k,k+T])}, \tag{1.5.112}$$

$$\leq c\left(\sup_{t\in[0,T]} d_H(\Omega_t^{\delta_1}, \Omega_t^{\delta_2})^{\frac{1}{2}} + \delta_1 + \delta_2\right) \tag{1.5.113}$$

where a constant c is given by

$$c := \max\left\{\overline{m}^{\frac{1}{2}}, 2\sigma_1(\mathrm{Per}(\Omega_0) + T)\right\}. \tag{1.5.114}$$

Here, σ_1 and $\overline{m}$ are given in (1.5.77) and (1.5.84), respectively. From Theorem 1.5.1, we conclude (1.5.110).

2. Lemma 1.4.15 and the smoothness of Ω_t^δ for $t > 0$ from Proposition 1.5.2 imply that for $s, k \in [0, +\infty)$

$$\int_0^T \widetilde{d}^2(\Omega^\delta_{t+k+s}, \Omega^\delta_{t+k})dt \leq s\mathcal{K}_2(\mathcal{E}^\delta(k) - \mathcal{E}^\delta(k+s)) \tag{1.5.115}$$

where $\mathcal{K}_2$ is given in Lemma 1.4.15. Taking δ into zero, (1.5.110) and Theorem 1.5.1 imply that for $s, k \in [0, +\infty)$

$$\int_0^T \widetilde{d}^2(U^k_{t+s}, U^k_t)dt \leq s\mathcal{K}_2(\mathcal{E}^\infty(k) - \mathcal{E}^\infty(k+s)) \tag{1.5.116}$$

where U_t^k is given in (1.5.81).

Note that as $\mathcal{E}^\delta(k)$ is monotone decreasing for all $\delta > 0$, $\mathcal{E}^\infty(k)$ is also monotone decreasing in k. Taking k into ∞, we get for $s \in [0, +\infty)$

$$\int_0^T \widetilde{d}^2(U_{t+s}^\infty, U_t^\infty)dt \le s\mathcal{K}_2(\inf_{k>0} \mathcal{E}^\infty(k) - \inf_{k>s} \mathcal{E}^\infty(k)) = 0 \tag{1.5.117}$$

and we conclude. □

Proof of Theorem 1.5.16.

1. Let η^∞ and U_t^∞ be as given in Proposition 1.5.17. We denote U_t^∞ by U^∞ since we know that it is stationary from the last proposition. We will show that η^∞ is independent of time as well. Let us argue by contradiction, and suppose $\eta^\infty(t_1) \neq \eta^\infty(t_2)$ for two Lebesgue points $t_1 < t_2$ in $[0, +\infty)$. We may assume that $\eta^\infty(t_1) < \eta^\infty(t_2)$. As t_1 and t_2 are Lebesgue points of η^∞, there exists $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for any $\delta \in (0, \delta_1]$, we have

$$\frac{\Upsilon(t_1+\delta) - \Upsilon(t_1)}{\delta} < \frac{\eta^\infty(t_1) + \eta^\infty(t_2)}{2} < \frac{\Upsilon(t_2+\delta) - \Upsilon(t_2)}{\delta} - \varepsilon_1 \text{ where } \Upsilon(t) := \int_0^t \eta^\infty(s)ds.$$

Therefore, for $\delta \in [0, \delta_1]$, we have

$$\Upsilon(t_1+\delta) \le \Theta_1(t_1+\delta) \text{ and } \Upsilon(t_2+\delta) \ge \Theta_2(t_2+\delta) + \varepsilon_1\delta \tag{1.5.118}$$

where

$$\Theta_i(t) := \frac{1}{2}(\eta^\infty(t_1) + \eta^\infty(t_2))(t - t_i) + \Upsilon(t_i) \text{ for } i \in \{1, 2\}. \tag{1.5.119}$$

From Proposition 1.5.17, $u(x) := \chi_{U^\infty}(x) - \chi_{(U^\infty)^C}(x)$ is a viscosity solution of $V = -H + \eta^\infty(t)$. Let us define $v_i : \mathbb{R}^n \times [0, \delta_1] \to \mathbb{R}$ for $i \in \{1, 2\}$ by

$$v_1(x,t) := \widetilde{u}(x; (-\Upsilon + \Theta_1)(t + t_1)) \text{ and } v_2(x,t) := \widehat{u}\left(\frac{x}{1-\varepsilon_2}; (\Upsilon - \Theta_2)(t + t_2)\right), \tag{1.5.120}$$

where $\widetilde{u}$ and $\widehat{u}$ are as defined in (1.2.28) and (1.2.29) and ε_2 is a constant satisfying

$$\varepsilon_2 \in \left(0, \min\left\{\frac{\delta_1\varepsilon_1}{2R_1}, 1\right\}\right). \tag{1.5.121}$$

Note that by (1.5.118) and (1.5.119) v_1 and v_2 are each a viscosity supersolution and subsolution of $V = -H + \frac{1}{2}(\eta^\infty(t_1) + \eta^\infty(t_2))$ in $[0, \delta_1]$.

As v_1+1 and v_2+1 have compact support in $\mathbb{R}^n \times [0,\delta_1]$, there exists $R_2>0$ such that

$$v_1(x,t) = v_2(x,t) = -1 \text{ for all } x \in B_{R_2/2}(0)^C \text{ and } t \in [0,\delta_1]. \tag{1.5.122}$$

We claim that

$$(v_1)_* \geq (v_2)^* \text{ in } B_{R_2}(0) \times [0,\delta_1]. \tag{1.5.123}$$

As $\Upsilon - \Theta_1(t_1) = (\Upsilon - \Theta_2)(t_2) = 0$, we have $v_1(x,0) := \chi_{U^\infty}(x) - \chi_{(U^\infty)^C}(x)$ and $v_2(x,0) := \chi_{(1-\varepsilon_2)U^\infty}(x) - \chi_{((1-\varepsilon_2)U^\infty)^C}(x)$. From the fact that $U^\infty \in S_{r_1,R_1}$ and $\varepsilon_2 > 0$, we observe that $(v_1)_* \geq (v_2)^*$ at $t=0$. Combining this with (1.5.122), we have

$$(v_1)_* \geq (v_2)^* \text{ in } \partial_p(B_{R_2}(0) \times [0,\delta_1]) \tag{1.5.124}$$

and thus Theorem 1.2.15 yields (1.5.123).

Next, we claim that

$$v_1\left(x, \frac{\delta_1}{2}\right) \leq v_2\left(x, \frac{\delta_1}{2}\right) \text{ for all } x \in \mathbb{R}^n. \tag{1.5.125}$$

As $U^\infty \in S_{r_1,R_1}$, we have $U^\infty \subset \bigcup_{|z|\leq \varepsilon_2 R_1} ((1-\varepsilon_2)U^\infty + z)$. From this and (1.5.121), it holds that

$$u(x) \leq \widehat{u}\left(\frac{x}{1-\varepsilon_2}; \frac{\varepsilon_1\delta_1}{2}\right) \text{ for all } x \in \mathbb{R}^n. \tag{1.5.126}$$

As v_1 and v_2 are inf and sup convolutions of u, respectively, and $\Upsilon(t_2+t) - \Theta_2(t_2+t) \geq \varepsilon_1 t$ for all $t \in [0,\delta_1]$ from (1.5.118), we have

$$v_1(x,t) \leq u(x) \text{ and } \widehat{u}\left(\frac{x}{1-\varepsilon_2}; \varepsilon_1 t\right) \leq v_2(x,t) \text{ for all } x \in \mathbb{R}^n \text{ and } t \in [0,\delta_1]. \tag{1.5.127}$$

From (1.5.126) and (1.5.127), we get (1.5.125).

Then, (1.5.123) and (1.5.125) yield that

$$v_1\left(x, \frac{\delta_1}{2}\right) \leq v_2\left(x, \frac{\delta_1}{2}\right) \leq (v_2)^*\left(x, \frac{\delta_1}{2}\right) \leq (v_1)_*\left(x, \frac{\delta_1}{2}\right) \text{ for any } x \in B_{R_2}(0).$$

Combining the above with (1.5.122), we have

$$(v_1)^*\left(\cdot, \frac{\delta_1}{2}\right) = (v_1)_*\left(\cdot, \frac{\delta_1}{2}\right) \text{ and } (v_2)^*\left(\cdot, \frac{\delta_1}{2}\right) = (v_2)_*\left(\cdot, \frac{\delta_1}{2}\right) \text{ in } \mathbb{R}^n. \tag{1.5.128}$$

Since $U^\infty \in S_{r_1,R_1}$, it follows that $v_1\left(\cdot, \frac{\delta_1}{2}\right)$ and $v_2\left(\cdot, \frac{\delta_1}{2}\right)$ are characteristic functions and they are discontinuous. This contradicts to (1.5.128).

2. As $(U_t^\infty)_{t\geq 0}$ and η^∞ are stationary from Proposition 1.5.19 and Step 1, we conclude that U_t^∞ is a viscosity solution of the elliptic problem,

$$H = \eta^\infty. \tag{1.5.129}$$

As $(U_t^\infty)_{t\geq 0} \subset S_{r_1,R_1}$ from Proposition 1.5.17, $(U_t^\infty)_{t\geq 0}$ can be locally represented by graphs. Then, the regularity of (1.5.129) in [GT15, Corollary 16.7] implies that $U_{t_1}^\infty$ is smooth. As $(U_t^\infty)_{t\geq 0} \subset S_{r_1,R_1}$, we conclude that $U_t^\infty = B_{r_\infty}(x)$ in $[0, +\infty)$ for some $x \in \overline{B}_{r_1}(0)$ where r_∞ given in Theorem 1.5.16. Therefore, every sequence of $(\Omega_t^\infty)_{t\geq 0}$ has a subsequence converging to $B_{r_\infty}(x)$ for some $x \in \overline{B}_{r_1}(0)$, we conclude (1.5.80). □

CHAPTER 2

Degenerate parabolic equations with discontinuous diffusion intensities

2.1 Introduction

Consider a degenerate parabolic equation with a discontinuous diffusion coefficient: for a density function $\rho : [0,T] \times \Omega \to [0,+\infty)$

$$\begin{cases} \partial_t \rho - \Delta\varphi(\rho) - \nabla \cdot (\nabla\Phi\rho) = 0 & \text{in } (0,T) \times \Omega, \\ (\nabla\varphi(\rho) + \nabla\Phi\rho) \cdot \vec{n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \rho(0,\cdot) = \rho_0 & \text{in } \Omega. \end{cases} \tag{2.1.1}$$

Here, $T > 0$ is a given time horizon, $\Omega \subset \mathbb{R}^d$ is the closure of a bounded convex open set with smooth boundary, $\Phi : \Omega \to \mathbb{R}$ is a given Lipschitz continuous potential function and $\rho_0 \in \mathscr{P}(\Omega)$ is a nonnegative Borel probability measure. The diffusion intensity function $\varphi : [0,+\infty) \to [0,+\infty)$ is supposed to be monotone and to have a discontinuity at $\rho = 1$. There are many studies devoted to theories and applications of the diffusion intensity including *self-organized criticality* in physics (See [BJ92]). However, its continuous-time model as in (2.1.1) has not been well understood.

We rely on the gradient flow structure of (2.1.1) in the space of probability measures, when equipped with the distance W_2 arising in the Monge-Kantorovich optimal transport problem. To (2.1.1), we associate an entropy functional $\mathcal{E} : \mathscr{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}$, defined as

$$\mathcal{E}(\rho) := \begin{cases} \displaystyle\int_\Omega \mathcal{S}(\rho(x))\mathrm{d}x + \int_\Omega \Phi(x)\mathrm{d}\rho(x), & \text{if } S(\rho) \in L^1(\Omega), \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.1.2}$$

where $S : [0, +\infty) \to \mathbb{R}$ is a given function. At the formal level, the relationship between φ and S can be written as

$$\varphi(\rho) = \rho \mathcal{S}'(\rho) - \mathcal{S}(\rho) + \mathcal{S}(1) \text{ and } \varphi'(\rho) = \rho S''(\rho), \quad \text{if } \rho \neq 1.$$

We observe that the discontinuity of φ at $\rho = 1$ corresponds to the non-differentiability of S. Furthermore, as φ is monotone, we impose that S is convex and the multiple values of φ can be represented by the subdifferential of S. In this sense, we consider S to be given which satisfies the following assumption.

Assumption A. $\mathcal{S} : [0, +\infty) \to \mathbb{R}$ is superlinear, continuous and strictly convex. Furthermore, $\mathcal{S}$ is twice continuously differentiable in $\mathbb{R}^+ \setminus \{1\}$.

In our analysis, we rely on the classical *minimizing movements* scheme of De Giorgi (see also [JKO98] and [San17]). This, for a given $\rho_0 \in \mathscr{P}(\Omega)$ (and for a small parameter $\tau > 0$ and $N \in \mathbb{N}$ such that $N\tau = T$) iteratively constructs $(\rho_k)_{k=0}^N$ as

$$\rho_{k+1} = \text{argmin} \left\{ \mathcal{E}(\rho) + \frac{1}{2\tau} W_2^2(\rho_k, \rho) : \ \rho \in \mathscr{P}(\Omega) \right\}, \ k \in \{0, \dots, N-1\}. \tag{2.1.3}$$

In order to give a precise description of the optimality conditions associated to (2.1.3), we introduce a function p_k which encodes the 'transition' between the phases $\{\rho_k < 1\}$ and $\{\rho_k > 1\}$ through the *critical region* $\{\rho_k = 1\}$. This is very much inspired by the derivation of the *pressure variable* in recent models studying crowd movements under density constraints (see in [MRS10], [DMS16], [MS16]). Because of this similarity, we sometimes use the abused terminology of *pressure* to refer to the variable p.

After obtaining the necessary compactness results, we pass to the limit with the time discretization parameter $\tau \downarrow 0$ and we recover a PDE (which precisely describes the weak distributional solutions of (2.1.1)) satisfied by the limit quantities (ρ, p). This formally reads as

$$\begin{cases} \partial_t \rho - \Delta(L_S(\rho, p)) - \nabla \cdot (\nabla \Phi \rho) = 0, & \text{in } (0, T) \times \Omega, \\ \rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\ (\nabla(L_S(\rho, p)) + \nabla \Phi \rho) \cdot \vec{n} = 0, & \text{in } [0, T] \times \partial\Omega. \end{cases} \tag{2.1.4}$$

Here, the operator L_S is defined pointwisely for functions $(\rho, p) : [0,T] \times \Omega \to \mathbb{R}$ by

$$L_{\mathcal{S}}(\rho,p)(t,x) := [\rho(t,x)\mathcal{S}'(\rho(t,x)) - \mathcal{S}(\rho(t,x)) + \mathcal{S}(1)]\, \mathbb{1}_{\{\rho\neq 1\}}(t,x) + p(t,x)\mathbb{1}_{\{\rho=1\}}(t,x) \tag{2.1.5}$$

and the pressure variable $p : [0,T] \times \Omega \to \mathbb{R}$ satisfies

$$\begin{cases} p(t,x) = \mathcal{S}'(1-) & \text{if } 0 \le \rho(t,x) < 1, \\ p(t,x) \in [\mathcal{S}'(1-), \mathcal{S}'(1+)] & \text{if } \rho(t,x) = 1, \\ p(t,x) = \mathcal{S}'(1+) & \text{if } \rho(t,x) > 1. \end{cases} \tag{2.1.6}$$

Starting with Section 2.4, we consider general entropies. Assumptions are made on the growth of S in the two different phases $\{\rho < 1\}$ and $\{\rho > 1\}$. First, we impose

Assumption B.

$$\mathcal{S} : [0,+\infty) \to \mathbb{R} \text{ satisfies } \frac{\rho^{m-2}}{\sigma_2} < \mathcal{S}''(\rho) \text{ if } \rho \in (0,1) \text{ for some } m \ge 1 \text{ and } \sigma_2 > 0. \tag{2.1.7}$$

The imposed summability assumption on the initial data $\rho_0 \in \mathscr{P}(\Omega)$ plays also a crucial role in our analysis. If $\rho_0 \in L^\infty(\Omega)$, it turns out that the entire iterated sequence $(\rho_k)_{k=1}^N$ obtained in the scheme (2.1.3) remains essentially uniformly bounded, provided the potential Φ is regular enough. This fact does not depend on the differentiability of S and it is well-known in the literature (see [San15]). In this case, imposing only the assumption (2.1.7) on S is enough to obtain the well-posedness of (2.1.4)-(2.1.6).

Theorem 1 (Theorems 2.3.1, 2.4.2, 2.5.1, 2.5.5 and Theorem 2.6.1)**.** *Suppose that* (2.1.7) *holds and* Φ *satisfies* (2.2.4)*. For* $\rho_0 \in L^\infty(\Omega)$*, there exists* $\rho \in L^\infty([0,T] \times \Omega)$*,* $\rho^m \in L^2([0,T]; H^1(\Omega))$ *and* $p \in L^2([0,T]; H^1(\Omega)) \cap L^\infty([0,T] \times \Omega)$ *such that* (ρ, p) *is a unique solution of* (2.1.4)-(2.1.6) *in the sense of distributions.*

The other case is when we only impose that ρ_0 has finite energy, i.e. $\mathcal{E}(\rho_0) < +\infty$. We show that the iterated sequence will have improved summability estimates for $k \in \{1, \ldots, N\}$ (see in Lemma 2.2.11), provided S satisfies the additional growth condition (2.1.8b)-(2.1.8a) below. These summability estimates on the iterated sequence will be enough to obtain the necessary a priori estimates and pass to the limit as $\tau \downarrow 0$ to obtain a weak solution to (2.1.4)-(2.1.6).

For general initial data such that $\mathcal{E}(\rho_0) < +\infty$ we shall impose the following additional growth condition on S.

Assumption C.

$$\mathcal{S} : [0, +\infty) \to \mathbb{R} \text{ satisfies } \frac{\rho^{r-2}}{\sigma_1} \le \mathcal{S}''(\rho) \text{ if } \rho \in (1, +\infty) \text{ and} \tag{2.1.8a}$$

$$\mathcal{S}''(\rho) \le \sigma_1 \rho^{r-2} \text{ if } \rho \in (1, +\infty) \text{ for some } r, \sigma_1 \ge 1. \tag{2.1.8b}$$

Notice that under (2.1.8) and $r > 1$, $\mathcal{E}(\rho_0) < +\infty$ is equivalent to $\rho_0 \in L^r(\Omega)$. Similarly to Theorem 1, we can formulate the corresponding well-posedness result.

Theorem 2 (Theorems 2.3.1, 2.4.2, 2.5.1, 2.5.5 and Theorem 2.6.1)**.** *Suppose that* (2.1.7) *and* (2.1.8) *are fulfilled and*

$$m < r + \frac{\beta}{2} \tag{2.1.9}$$

hold true for $\beta > 1$ (its precise value is given in (2.2.29)*). For $\rho_0 \in \mathscr{P}(\Omega)$ such that $\mathcal{E}(\rho_0) < +\infty$, there exists $\rho \in L^\beta([0,T] \times \Omega)$ and $p \in L^2([0,T]; H^1(\Omega)) \cap L^\infty([0,T] \times \Omega)$ such that (ρ, p) is a solution of* (2.1.4)-(2.1.6) *in the sense of distributions. Furthermore, we have*

$$\rho^{m-\frac{1}{2}} \in L^2([0,T]; H^1(\Omega)), \ \text{if } m \le r \ \text{and } \rho^{m-\frac{1}{2}} \in L^q([0,T]; W^{1,q}(\Omega)) \ \text{if } r < m < r + \frac{\beta}{2}$$

for some $q \in (1,2)$. If in addition $\beta \ge 2r$, then the pair (ρ, p) is unique.

2.2 The minimizing movement scheme and optimality conditions

$\Omega \subset \mathbb{R}^d$ is given, as the closure of a bounded, convex open set with smooth boundary. $\mathscr{P}(\Omega)$ denotes the space of Borel probability measures on Ω and $\mathscr{L}^d$ stands the Lebesgue measure on $\mathbb{R}^d$. We also use the notation $\mathscr{P}^{\rm ac}(\Omega) := \left\{\mu \in \mathscr{P}(\Omega) : \ \mu \ll \mathscr{L}^d \llcorner \Omega\right\}$. $T > 0$ is a fixed time horizon and we often use the notation $Q := [0,T] \times \Omega$.

As $\mathcal{S}'$ is strictly increasing in $\mathbb{R}^+ \setminus \{1\}$ from Assumption A, $\mathcal{S}'(0+)$ and $\mathcal{S}'(1\pm)$ are well-defined in $\mathbb{R} \cup \{-\infty\}$ and $\mathbb{R}$, respectively, as follows.

$$\mathcal{S}'(0+) := \lim_{\varepsilon \to 0+} \mathcal{S}'(\varepsilon), \quad \mathcal{S}'(1-) := \lim_{\varepsilon \to 1-} \mathcal{S}'(\varepsilon) \text{ and } \mathcal{S}'(1+) := \lim_{\varepsilon \to 1+} \mathcal{S}'(\varepsilon). \tag{2.2.1}$$

In particular, we have that $S'(1-) < S'(1+)$.

We define the corresponding internal energy $\mathcal{J} : \mathscr{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$\mathcal{J}(\rho) := \begin{cases} \displaystyle\int_\Omega \mathcal{S}(\rho(x))\mathrm{d}x & \text{if } \rho \in \mathscr{P}^{\mathrm{ac}}(\Omega), \\ +\infty & \text{otherwise} \end{cases} \tag{2.2.2}$$

Furthermore, we suppose that there is given $\Phi : \Omega \to \mathbb{R}$ a potential function in $W^{1,\infty}(\Omega)$ and the associated potential energy $\mathcal{F} : \mathscr{P}(\Omega) \to \mathbb{R}$ given by

$$\mathcal{F}(\rho) := \int_\Omega \Phi(x)\mathrm{d}\rho(x).$$

Let $\rho_0 \in \mathscr{P}(\Omega)$ be given and consider a time discretization parameter $\tau > 0$ and $N \in \mathbb{N}$ such that $N\tau = T$. We define the *minimizing movements* $(\rho_k)_{k=1}^N$ of $\mathcal{J} + \mathcal{F}$ as follows: for $k \in \{1, \ldots, N\}$ set,

$$\rho_k := \underset{\rho \in \mathscr{P}(\Omega)}{\arg\min} \left\{ \mathcal{J}(\rho) + \mathcal{F}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_{k-1}) \right\}, \tag{2.2.3}$$

Note that the existence and uniqueness of the solutions in the minimization problems (2.2.3) follow from standard compactness, lower semicontinuity and convexity arguments.

In what follows, in our analysis we differentiate two cases with respect to the summability assumption on ρ_0. Since these need slightly different arguments, we separate them in two different subsections. In particular, if one assumes L^∞ summability on ρ_0, the presented results will hold true under no additional assumptions on S (other than in Assumption A). However, in (2.2.3) we can allow general measure initial data, in which case an additional growth condition (see (2.1.8)) has to be imposed on S in order to obtain the same optimality conditions.

2.2.1 Optimality conditions for $\rho_0 \in L^\infty(\Omega)$

Lemma 2.2.1. *If Φ is non-constant, let us assume that $\Phi \in C^1(\overline{\Omega})$ and*

$$\nabla\Phi(x_0)\cdot\vec{n}(x_0) > 0, \quad \forall x_0 \in \partial\Omega \quad \textit{and} \quad \nabla\Phi \in BV(\Omega;\mathbb{R}^d) \ \textit{with} \ [\Delta\Phi]_+ \in L^\infty(\Omega) \qquad (2.2.4)$$

where $\vec{n}$ stands for the outward normal vector to $\partial\Omega$ and $[\Delta\Phi]_+$ denotes the positive part of the measure $\Delta\Phi$. Let $(\rho_k)_{k=1}^N$ be constructed via the scheme (2.2.3)*. Then we have*

$$\|\rho_k\|_{L^\infty} \le \|\rho_{k-1}\|_{L^\infty}\left(1+\tau\|[\Delta\Phi]_+\|_{L^\infty}\right)^d \le \|\rho_0\|_{L^\infty}\left(1+\tau\|[\Delta\Phi]_+\|_{L^\infty}\right)^{kd} \le \|\rho_0\|_{L^\infty} e^{dT\|[\Delta\Phi]_+\|_{L^\infty}},$$

$\forall k \in \{1,\dots,N\}$.

Remark 2.2.2. Let us notice that the second part of assumption (2.2.4) is sharp and it is very much related to the ones imposed in the work of Ambrosio (see [Amb04]), as an improvement of the classical DiPerna-Lions theory ([DL89]), on transport equations with BV vector fields.

Proof of Lemma 2.2.1. The proof of this result in the case when $\Phi \equiv 0$ is essentially the same as the proof of [San15, Proposition 7.32] (since that proof is not assuming any differentiability on S).

For general Φ, we use some ideas from the proof of [CS17, Theorem 1]. Let us approximate S with a sequence $(S_\varepsilon)_{\varepsilon>0}$ of smooth convex functions such that $S''_\varepsilon \ge c_\varepsilon > 0$ for any $\varepsilon > 0$ with $S'_\varepsilon(0+) = -\infty$. Let Φ_ε be a smooth approximation of Φ which satisfies (2.2.4) and such that $\Phi_\varepsilon \to \Phi$, $\nabla\Phi_\varepsilon \to \nabla\Phi$, uniformly as $\varepsilon \downarrow 0$ and $\|[\Delta\Phi_\varepsilon]_+\|_{L^\infty} \le \|[\Delta\Phi]_+\|_{L^\infty}$, for $\varepsilon > 0$. Let ρ_k^ε be the unique solution of (2.2.3), when we replace S with S_ε and Φ by Φ_ε. Writing down the optimality conditions we obtain

$$S'_\varepsilon(\rho_k^\varepsilon) + \Phi_\varepsilon + \frac{\phi_k^\varepsilon}{\tau} = C \text{ a.e.},$$

where $\phi_k^\varepsilon \in \mathcal{K}(\rho_k^\varepsilon, \rho_{k-1})$. Let us suppose that $\phi_k^\varepsilon \in C^{2,\alpha}(\Omega)$, otherwise we approximate ρ_{k-1} by strictly positive $C^{0,\alpha}$ measures (and ρ_k^ε is Lipschitz continuous and strictly positive), and we use Caffareli's regularity theory to deduce the desired regularity for the potential.

Now, let x_0 a maximum point of ρ_k^ε. From the previous equality, since S'_ε is strictly increasing, we certainly have that x_0 is a minimum point of $\Phi_\varepsilon + \frac{\phi_k^\varepsilon}{\tau}$.

We claim that $x_0 \notin \partial\Omega$. Indeed, if x_0 would belong to $\partial\Omega$, we would have that

$$(\nabla\phi_k^\varepsilon(x_0) + \tau\nabla\Phi_\varepsilon(x_0)) \cdot \vec{n}(x_0) \leq 0.$$

However, by the convexity of Ω, we have that $(x_0 - \nabla\phi_k^\varepsilon(x_0)) \cdot \vec{n}(x_0) \leq 0$, from where $\nabla\phi_k^\varepsilon(x_0) \cdot \vec{n}(x_0) \geq 0$. This fact together with the assumption (2.2.4) yields a contradiction. Indeed, from the uniform convergence of $\nabla\Phi_\varepsilon \to \nabla\Phi$, we have that

$$\nabla\Phi_\varepsilon(x_0) \cdot \vec{n} \geq \nabla\Phi(x_0) \cdot \vec{n} - \varepsilon > 0,$$

for sufficiently small $\varepsilon > 0$.

Therefore, the maximum point x_0 of ρ_k^ε belongs to the interior of Ω. This implies that $\Delta\phi_k^\varepsilon(x_0) + \tau\Delta\Phi_\varepsilon(x_0) \geq 0$. Using the Monge-Ampère equation we find

$$\begin{aligned}
\|\rho_k^\varepsilon\|_{L^\infty} &= \rho_k^\varepsilon(x_0) = \rho_{k-1}(x_0 - \nabla\phi_k^\varepsilon(x_0))\det\left(I_d - D^2\phi_k^\varepsilon(x_0)\right) \leq \|\rho_{k-1}\|_{L^\infty}(1 - \Delta\phi_k^\varepsilon(x_0))^d \\
&\leq \|\rho_{k-1}\|_{L^\infty}(1 + \tau\Delta\Phi_\varepsilon(x_0))^d \leq \|\rho_{k-1}\|_{L^\infty}(1 + \tau\|[\Delta\Phi_\varepsilon]_+\|_{L^\infty})^d \\
&\leq \|\rho_{k-1}\|_{L^\infty}(1 + \tau\|[\Delta\Phi]_+\|_{L^\infty})^d \leq \|\rho_0\|_{L^\infty}(1 + \tau\|[\Delta\Phi]_+\|_{L^\infty})^{kd} \leq \|\rho_0\|_{L^\infty} e^{dT\|[\Delta\Phi]_+\|_{L^\infty}},
\end{aligned}$$

where in the first inequality we have used the inequality between the arithmetic and geometric means. Since the last three bounds depend only on the data, these will also remain valid also in the limit $\varepsilon \downarrow 0$ (since the minimizers of both the approximated and the original problems are unique). Therefore the thesis of the lemma follows. $\square$

Now, we state the main result of this subsection on the first order necessary optimality conditions for the problems in (2.2.3).

Theorem 2.2.3. *Suppose that $\rho_0 \in L^\infty(\Omega)$. For all $k \in \{1, \ldots, N\}$, there exists $\mathcal{C} = \mathcal{C}(k) \in \mathbb{R}$ and $\overline{\phi}_k \in \mathcal{K}(\rho_k, \rho_{k-1})$ such that*

$$\begin{cases} \mathcal{C} - \frac{\overline{\phi}_k}{\tau} - \Phi \leq \mathcal{S}'(0+) & \text{in } \{\rho_k = 0\}, \\ \mathcal{C} - \frac{\overline{\phi}_k}{\tau} - \Phi \in [\mathcal{S}'(1-), \mathcal{S}'(1+)] & \text{in } \{\rho_k = 1\}, \\ \mathcal{C} - \frac{\overline{\phi}_k}{\tau} - \Phi = \mathcal{S}' \circ \rho_k & \text{otherwise.} \end{cases} \tag{2.2.5}$$

Here, $\mathcal{K}(\rho_k, \rho_{k-1})$ is given in Definition A.4.1. Also, $\mathcal{S}'(0+)$ and $\mathcal{S}'(1\pm)$ are given in (2.2.1). Note that $\rho_k > 0$ a.e. (See Lemma A.4.5) if $S'(0+) = -\infty$, and in this case the first inequality in (2.2.5) is not present.

The proof of the previous results relies on the precise derivation of the subdifferential of the corresponding objective functional in (2.2.3). Let us point out that the subdifferential of sum is not always the sum of subdifferentials (see for instance [San15, Example 7.22]). Therefore, we need to carefully choose the domain of definition of $\mathcal{J}$. In the spirit of Lemma 2.2.1, we consider it as a functional on $L^\infty(\Omega)$ instead of $\mathscr{P}(\Omega)$. The additive property of subdifferentials on $L^\infty(\Omega)$ holds under suitable conditions (cf. [ET76]).

Proposition 2.2.4. *For all $k \in \{1, \dots, N\}$ we have*

$$\partial\left(\mathcal{J}(\rho) + \mathcal{F}(\rho) + \frac{1}{2\tau}W_2^2(\rho, \rho_{k-1})\right)\bigg|_{\rho=\rho_k} = \partial\mathcal{J}(\rho_k) + \Phi + \frac{1}{2\tau}\partial(W_2^2(\rho, \rho_{k-1}))|_{\rho=\rho_k} \quad (2.2.6)$$

Proof. To simplify the writing, we consider only the case $k = 1$. Let us check that $\mathcal{J}$ and $W_2^2(\cdot, \rho_0)$ satisfy the assumptions in Lemma A.5.4. The convexity of $\mathcal{S}$ implies that of $\mathcal{J}$. Also, the continuity of $\mathcal{J}$ in $L^\infty(\Omega)$ follows from the continuity of S. From Lemma A.5.3, we conclude $\mathcal{J} \in \Gamma(L^\infty(\Omega))$. We have the same conclusion for the functional $\mathcal{F}$ (which is actually linear in ρ).

Let us show that $W_2^2(\cdot, \rho_0) \in \Gamma(L^\infty(\Omega))$. Define $H : L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$H(\phi) := -\int_\Omega \phi^c \mathrm{d}\rho_0. \quad (2.2.7)$$

Proposition A.4.4 implies that $H^* : L^\infty(\Omega) \to \mathbb{R} \cup \{+\infty\}$ is given (in the sense of (A.5.1)) by

$$H^* = \frac{1}{2}W_2^2(\cdot, \rho_0) \text{ on } L^\infty(\Omega). \quad (2.2.8)$$

We conclude $W_2^2(\cdot, \rho_0) \in \Gamma(L^\infty(\Omega))$.

Lastly, choose $A \subseteq \Omega$ a Borel set such that $\mathscr{L}^d(A) \neq 1$ and define

$$\hat{\mu} := \frac{1}{\mathscr{L}^d(A)}\mathbb{1}_A. \quad (2.2.9)$$

$\mathcal{J}(\hat{\mu})$, $\mathcal{F}(\hat{\mu})$ and $W_2^2(\hat{\mu}, \rho_0)$ are finite. Furthermore, by the continuity of $\mathcal{S}$ in $\mathbb{R}^+$, $\mathcal{J}$ is continuous at $\hat{\mu}$. In the same way $\mathcal{F}$ is also continuous at $\hat{\mu}$. Thus, we conclude (2.2.6) from Lemma A.5.4. $\square$

Next, let us find the subdifferential of $W_2^2(\cdot, \rho_{k-1})$. While this subdifferential is expected to be the set of Kantarovich potentials $\mathcal{K}(\rho_k, \rho_{k-1})$, it is not straight forward to conclude about this as we consider the subdifferential for the functional on $L^\infty(\Omega)$. We rely on the ideas from [San15, Proposition 7.17], tailored to our setting.

Lemma 2.2.5. *[San15, Lemma 7.15] Let $\mathfrak{X}$ be a Banach space and $H : \mathfrak{X} \to \mathbb{R} \cup \{+\infty\}$ be convex and lower semicontinuous. Set $H^*(y) = \sup_{x \in \mathfrak{X}} \{\langle x, y\rangle_{\mathfrak{X},\mathfrak{X}^*} - H(x)\}$. Then, we have*

$$\partial H^*(y) = \arg\max_{x \in \mathfrak{X}} \{\langle x, y\rangle_{\mathfrak{X},\mathfrak{X}^*} - H(x)\}. \tag{2.2.10}$$

Lemma 2.2.6. *$H : L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ given in* (2.2.7) *is convex and l.s.c.*

Proof. The proof of convexity of H is the same as in [San15, Proposition 7.17], where one needs to change only the definition of φ^c using essential infima.

Let us show now that H is l.s.c. For this, let $\varphi \in L^1(\Omega)$ and $(\varphi_n)_{n\in\mathbb{N}}$ a sequence in $L^1(\Omega)$ such that $\varphi_n \to \varphi$ strongly in $L^1(\Omega)$ as $n \to +\infty$.

Notice first that by definition,

$$-\varphi^c(y) \geq \varphi(y), \text{ a.e. in } \Omega,$$

from where we have that $H(\varphi) > -\infty$. Because of the strong L^1 convergence, we know that there exists a subsequence of $(\varphi_n)_{n\in\mathbb{N}}$ (that we do not relabel), which is converging pointwise a.e. in Ω to φ. We shall work with this sequence from now on.

Writing the previous inequality for φ_n^c and φ_n, we have that

$$\liminf_{n\to+\infty} -\varphi_n^c(y) \geq \liminf_{n\to+\infty} \varphi_n(y) = \varphi(y), \text{ a.e. in } \Omega,$$

where we used the fact that $\varphi_n(y) \to \varphi(y)$ a.e. in Ω, as $n \to +\infty$.

Let us define $g : \Omega \to \mathbb{R}\cup\{+\infty\}$ as $g(y) := \liminf_{n\to+\infty} -\varphi_n^c(y)$. Notice that this is measurable function. Indeed, $(-\varphi_n^c)_{n\in\mathbb{N}}$ is a sequence of measurable functions (infima of measurable functions), and using Fatou's lemma for the non-negative sequence of measurable functions $(-\varphi_n^c - \varphi_n)_{n\in\mathbb{N}}$, one concludes that g is measurable and

$$\int_\Omega \varphi(y)\rho_0(y)\mathrm{d}y \leq \int_\Omega g(y)\rho_0(y)\mathrm{d}y \leq \liminf_{n\to+\infty} H(\varphi_n).$$

Claim. $\varphi(y) \leq -\varphi^c(y) \leq g(y)$ for a.e. $y \in \Omega$.

Proof of the claim. Actually the first inequality was shown before, thus we show only the second one. Thus, by Egorov's theorem, we have that for any $\delta > 0$ there exists a measurable set $B_\delta \subseteq \Omega$ such that $\mathscr{L}^d(B_\delta) < \delta$ and $(\varphi_n)_{n\in\mathbb{N}}$ converges uniformly to φ as $n \to +\infty$ on $\Omega \setminus B_\delta$. Let us fix a small $\delta > 0$. We have furthermore that for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\varphi(x) - \varepsilon \leq \varphi_n(x) \leq \varphi(x) + \varepsilon$$

for a.e. $x \in \Omega \setminus B_\delta$ and $n \geq N_\varepsilon$. Because of this, we have the following chain of inequalities for all $n \geq N_\varepsilon$

$$-\varphi_n^c(y) = \sup_{x\in\Omega} \left\{\varphi_n(x) - |x-y|^2\right\} \geq \sup_{x\in\Omega\setminus B_\delta} \left\{\varphi_n(x) - |x-y|^2\right\} \geq \sup_{x\in\Omega\setminus B_\delta} \left\{\varphi(x) - \varepsilon - |x-y|^2\right\}.$$

Taking $\liminf\limits_{n\to+\infty}$ of both sides, one obtains

$$g(y) \geq \sup_{x\in\Omega\setminus B_\delta} \left\{\varphi(x) - \varepsilon - |x-y|^2\right\}$$

for a.e. $y \in \Omega$. By the arbitrariness of ε and δ (in this order), one gets that

$$g(y) \geq \sup_{x\in\Omega} \left\{\varphi(x) - |x-y|^2\right\} = -\varphi^c(y),$$

as we claimed.

Notice that we have proved the following: if $(\varphi_n)_{n\in\mathbb{N}}$ is converging to φ in $L^1(\Omega)$, then there exists a subsequence $(\varphi_{n_j})_{j\in\mathbb{N}}$ of the original sequence such that

$$H(\varphi) \leq \liminf_{j\to+\infty} H(\varphi_{n_j}).$$

This statement actually implies the l.s.c. of H on the full sequence. Indeed, observe that by the definition of lim inf, there exists a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ of the original sequence such that

$$\liminf_{n\to+\infty} H(\varphi_n) = \lim_{k\to+\infty} H(\varphi_{n_k}).$$

We have shown previously that there exists a subsequence $(\varphi_{n_{k_j}})_{j\in\mathbb{N}}$ of $(\varphi_{n_k})_{k\in\mathbb{N}}$ such that

$$H(\varphi) \leq \liminf_{j\to+\infty} H(\varphi_{n_{k_j}}).$$

On the other hand

$$\liminf_{j\to+\infty} H(\varphi_{n_{k_j}}) = \lim_{j\to+\infty} H(\varphi_{n_{k_j}}) = \lim_{k\to+\infty} H(\varphi_{n_k}) = \liminf_{n\to+\infty} H(\varphi_n),$$

thus the l.s.c. of H follows. □

Proposition 2.2.7. *For all $k \in \{1, \ldots, N\}$ we have*

$$\frac{1}{2}\partial(W_2^2(\rho, \rho_{k-1}))\big|_{\rho=\rho_k} = \mathcal{K}(\rho_k, \rho_{k-1}) \tag{2.2.11}$$

Proof. To simplify the notation, we set $k = 1$. Recall from Proposition A.4.4 that

$$\frac{1}{2}\partial(W_2^2(\rho, \rho_0))|_{\rho=\rho_1} = \partial H^*(\rho_1) \tag{2.2.12}$$

for H given in (2.2.7). From Lemma 2.2.5 and Lemma 2.2.6, it holds that

$$\partial H^*(\rho_1) = \text{argmax}_{\phi\in L^1(\Omega)} \left\{ \int_\Omega \phi \mathrm{d}\rho_1 + \int_\Omega \phi^c \mathrm{d}\rho_0 \right\}. \tag{2.2.13}$$

From Definition A.4.1, we conclude. □

Lastly, let us compute the subdifferential of $\mathcal{J}$ based on [Roc71]. Before, we need the following preparatory result.

Lemma 2.2.8. *[Roc71, Corollary 1B] Let ψ and Ψ be given as in* (A.5.2)*. Assume that $\psi(\mu(x))$ is majorized by a summable function of x for at least one $\mu \in L^\infty(\Omega)$ and that $\psi^*(\zeta(x))$ is majorized by a summable function of x for at least one $\zeta \in L^1(\Omega)$. Then, an element $\xi \in L^\infty(\Omega)^*$ belongs to $\partial\Psi(\mu)$ given in* (A.5.3) *if and only if $\xi^{ac}(x) \in \partial\psi(\mu(x))$ for a.e. $x \in \Omega$ where ξ^{ac} is the absolutely continuous component of ξ, and the singular component ξ^s of ξ attains its maximum at μ over*

$$\{\nu \in L^\infty(\Omega) : \Psi(\nu) < +\infty\}.$$

Proposition 2.2.9. *For ρ_k is given in* (2.2.3)*, if $\xi \in \partial\mathcal{J}(\rho_k) \cap L^1(\Omega)$, then it holds that*

$$\xi \in \begin{cases} [-\infty, \mathcal{S}'(0+)] & \textit{a.e. in } \{\rho_k = 0\}, \\ [\mathcal{S}'(1-), \mathcal{S}'(1+)] & \textit{a.e. in } \{\rho_k = 1\}, \\ \mathcal{S}' \circ \rho_k & \textit{a.e. in } \{\rho_k \neq 1\}. \end{cases} \tag{2.2.14}$$

Proof. Let us show that $\mathcal{S}$ and $\mathcal{S}^*$ satisfies assumptions on Lemma 2.2.8. Let $\mu = \zeta = \frac{1}{\mathscr{L}^d(\Omega)}\mathbb{1}_\Omega$, then $\mathcal{S}(\mu)$ is finite, and thus in $L^1(\Omega)$. On the other hand, as $\mathcal{S}$ is superlinear, $\mathcal{S}^* < +\infty$ in $[0, +\infty)$. Therefore, for any constant $c \in \mathbb{R}$, $\mathcal{S}^*(c) \in L^1(\Omega)$.

By Lemma 2.2.8, $\xi^{\mathrm{ac}}(x) \in \partial\mathcal{S}(\rho_k(x))$ a.e., where ξ^{ac} is the absolutely continuous part of ξ. From the direct computation of $\partial\mathcal{S}(\rho_k(x))$, we conclude that ξ^{ac} satisfies the right hand side of (2.2.14). As $\xi \in L^1(\Omega)$, the singular part of ξ is zero, $\xi^{\mathrm{ac}} = \xi$ and we conclude (2.2.14). □

Proof of Theorem 2.2.3. We only consider the case that $k = 1$. By the optimality of ρ_1 in (2.2.3), it holds that

$$0 \in \partial\left(\mathcal{J}(\rho_1) + \mathcal{F}(\rho) + \frac{1}{2\tau}W_2^2(\rho_1, \rho_0)\right) \tag{2.2.15}$$

From Proposition 2.2.4 and Proposition 2.2.7, there exists $\xi \in \partial\mathcal{J}(\rho_1)$, $\overline{\phi}_1 \in \mathcal{K}(\rho_1, \rho_0)$ and $\mathcal{C} \in \mathbb{R}$ such that

$$\xi + \frac{\overline{\phi}_1}{\tau} + \Phi - \mathcal{C} = 0 \text{ a.e. on } \Omega. \tag{2.2.16}$$

As $\overline{\phi}_1, \Phi \in L^1(\Omega)$, $\xi \in \partial\mathcal{J}(\rho_1) \cap L^1(\Omega)$, Proposition 2.2.9 implies (2.2.5). □

2.2.2 Optimality conditions for $\rho_0 \in \mathscr{P}(\Omega)$ having finite energy

In this subsection we are imposing (2.1.8). Let us show first that $\mathcal{J}$ satisfying the additional assumption in (2.1.8) defines a continuous functional on $L^r(\Omega)$. In the previous subsection, the continuity of $\mathcal{J}$ in $L^\infty(\Omega)$ directly follows from the continuity of S.

Lemma 2.2.10. *Let $\mathcal{J}$ be given in* (2.2.2) *satisfying* (2.1.8b)*. Then $\mathcal{J}$ is continuous in $L^s(\Omega)$ for all*

$$s > r \text{ if } r = 1, \text{ and } s \geq r \text{ if } r > 1. \tag{2.2.17}$$

Proof. From (2.1.8b), there exists $c > 0$ such that for all $\rho \in [0, +\infty)$ (since S is also continuous, hence uniformly bounded on $[0, 1]$)

$$|S(\rho)| \leq c(\rho^s + 1). \tag{2.2.18}$$

for all s satisfying (2.2.17).

Consider a sequence $\{\mu_i\}_{i\in\mathbb{N}} \subset L^s(\Omega)$ such that

$$\mu_i \to \mu \text{ in } L^s(\Omega) \text{ as } i \to \infty \tag{2.2.19}$$

These exists a subsequence $\{\mu_{i_j}\}_{j\in\mathbb{N}} \subset \{\mu_i\}_{i\in\mathbb{N}}$ such that

$$\mu_{i_j} \to \mu \text{ a.e. as } j \to \infty. \tag{2.2.20}$$

From (2.2.18), it holds that for all $j \in \mathbb{N}$

$$0 \le c(|\mu_{i_j}|^s + 1) - |S(\mu_{i_j})| \le c(|\mu_{i_j}|^s + 1) \pm S(\mu_{i_j}). \tag{2.2.21}$$

Let us apply Fatou's lemma into $c(|\mu_{i_j}|^s + 1) + S(\mu_{i_j})$. From (2.2.19), (2.2.20) and the continuity of S, it holds that

$$\int_\Omega c(|\mu(x)|^s + 1) + S(\mu(x))\mathrm{d}x \le \liminf_{j\to\infty} \int_\Omega c(|\mu_{i_j}(x)|^s + 1) + S(\mu_{i_j}(x))\mathrm{d}x, \tag{2.2.22}$$

$$\le \int_\Omega c(|\mu(x)|^s + 1)\mathrm{d}x + \liminf_{j\to\infty} \int_\Omega S(\mu_{i_j})\mathrm{d}x. \tag{2.2.23}$$

and we have

$$\mathcal{J}(\mu) \le \liminf_{j\to\infty} \mathcal{J}(\mu_{i_j}).$$

Similarly to the argument at the end of the proof of Lemma 2.2.6, we conclude the lower semicontinuity along the full sequence, therefore

$$\mathcal{J}(\mu) \le \liminf_{i\to\infty} \mathcal{J}(\mu_i). \tag{2.2.24}$$

Applying Fatou's lemma again into $c(|\mu_{i_j}|^s + 1) - S(\mu_{i_j})$, we get

$$\mathcal{J}(\mu) \ge \limsup_{j\to\infty} \mathcal{J}(\mu_{i_j}), \tag{2.2.25}$$

and as before, we deduce the upper semicontinuity along the full sequence. Therefore (2.2.24) and (2.2.25) imply that $\mathcal{J}(\mu) = \lim_{j\to\infty} \mathcal{J}(\mu_{i_j})$ and $\mathcal{J}$ is continuous in $L^s(\Omega)$. □

In what follows, we show that the minimizers of the of the minimizing movements scheme (2.2.3) enjoy higher order summability estimates (which are independent of ρ_0, but depend on τ). These will play a crucial role later when deriving the optimality conditions.

Lemma 2.2.11. *Suppose that S satisfies Assumption A and* (2.1.8a)*. Let $\rho_k \in \mathscr{P}(\Omega)$ be the minimizer in* (2.2.3)*. Then $\rho_k \in L^\beta(\Omega)$, where $\beta := (2r-1)d/(d-2)$, if $d \geq 3$. If $d = 2$ then the statement is true for any $\beta < +\infty$ and $\beta = +\infty$ if $d = 1$.*

Remark 2.2.12. Let us notice that the previous lemma gives an improvement on the summability of ρ_k. Indeed, in case when the internal energy is of logarithmic entropy type, we know a priori that $\rho_k \in L^1(\Omega)$, while in the case of power like entropies, we have a priori $\rho_k \in L^r(\Omega)$. In contrast to these, we clearly improve the summability exponents in both cases.

Proof of Lemma 2.2.11. For $\varepsilon > 0$ let $S_\varepsilon : [0, +\infty) \to \mathbb{R}$ smooth, strictly convex such that $S''_\varepsilon \geq c_\varepsilon > 0$ (for some $c_\varepsilon > 0$), $S'_\varepsilon(0+) = -\infty$ and $S_\varepsilon \to S$ uniformly as $\varepsilon \to 0$. Let ρ^ε_k be the unique minimizer of the problem

$$\inf_{\rho \in \mathscr{P}(\Omega)} \left\{ \mathcal{E}_\varepsilon(\rho) := \int_\Omega S_\varepsilon(\rho) \mathrm{d}x + \mathcal{F}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_{k-1}) \right\}. \tag{2.2.26}$$

By the assumptions on S_ε, classical results imply that ρ^ε_k is Lipschitz continuous.

Without loss of generality, we can assume that S_ε satisfies the growth (2.1.8a) if $\rho > 2$. We can write the optimality condition

$$S''_\varepsilon(\rho^\varepsilon_k) \nabla \rho^\varepsilon_k + \nabla \Phi + \frac{\nabla \varphi^\varepsilon_k}{\tau} = 0 \ \text{ a.e.}, \tag{2.2.27}$$

where φ^ε_k is a Kantorovich potential in the transport of ρ^ε_k onto ρ_{k-1}. From here, there exists a constant $C > 0$ (depending only on r and σ_1) such that

$$\int_\Omega |S''_\varepsilon(\rho^\varepsilon_k) \nabla \rho^\varepsilon_k|^2 \rho^\varepsilon_k \mathrm{d}x \leq C \left(\|\nabla \Phi\|^2_{L^\infty} + \frac{1}{\tau^2} W_2^2(\rho^\varepsilon_k, \rho_{k-1}) \right).$$

And in particular, for any $\ell > 2$, we have by setting $\Omega_\ell := \{\rho^\varepsilon_k > \ell\}$,

$$\int_{\Omega_\ell} |\nabla (\rho^\varepsilon_k)^{r-1/2}|^2 \mathrm{d}x \leq C \left(\|\nabla \Phi\|^2_{L^\infty} + \frac{1}{\tau^2} W_2^2(\rho^\varepsilon_k, \rho_{k-1}) \right). \tag{2.2.28}$$

We know that the optimizers ρ_k^ε are Lipschitz continuous on their supports, therefore the super-level sets Ω_ℓ are open.

Moreover, once again using the fact that ρ_k^ε is Lipschitz, we have that there exists $\delta > 0$ such that

$$\mathrm{dist}(\partial\Omega_\ell, \overline{\Omega}_{2\ell}) \geq 2\delta.$$

Indeed, otherwise if one supposes the contrary, then for any $n \in \mathbb{N}$, there exist $x_n \in \partial\Omega_\ell$ and $y_n \in \Omega_{2\ell}$ such that $\mathrm{dist}(x_n, y_n) < \frac{1}{n}$, then one would have that $|\rho_k^\varepsilon(x_n) - \rho_k^\varepsilon(y_n)| \leq \frac{1}{n}\|\nabla\rho_k^\varepsilon\|_{L^\infty(\Omega_\ell)} \to 0$, as $n \to +\infty$. However, this would be a contradiction since $\rho_k^\varepsilon(x_n) = \ell$ and $\rho_k^\varepsilon(y_n) \geq 2\ell$.

Now, by defining $\Omega_{\ell,\delta} := \{\chi_{\Omega_{2\ell}} \star \eta_\delta > s\}$ for some $s \in (0, 1/2)$ to be set later (where $\eta_\delta : \mathbb{R}^d \to \mathbb{R}$ is a mollifier obtained from a smooth even kernel $\eta : \mathbb{R}^d \to \mathbb{R}$ – such that $\int_{\mathbb{R}^d} \eta \mathrm{d}x = 1$, $\eta \geq 0$ and $\mathrm{spt}(\eta) \subset B_1(0)$ – by $\eta_\delta := \eta(\cdot/\delta)$), we have that $\Omega_{2\ell} \subset \overline{\Omega}_{\ell,\delta} \subset \Omega_\ell$, $\Omega_{\ell,\delta}$ is an open set, and by Sard's theorem it has smooth boundary for $\mathscr{L}^1$-a.e. $s \in (0, 1/2)$. We choose such an s.

We have in particular from (2.2.28) that

$$\int_{\Omega_{\ell,\delta}} |\nabla(\rho_k^\varepsilon)^{r-1/2}|^2 \mathrm{d}x \leq C\left(\|\nabla\Phi\|_{L^\infty}^2 + \frac{1}{\tau^2}W_2^2(\rho_k^\varepsilon, \rho_{k-1})\right).,$$

and so the Sobolev embedding theorem implies (since ρ_k^ε is only uniformly bounded in $L^r(\Omega)$) that $(\rho_k^\varepsilon)^{r-1/2} \in L^{2^*}(\Omega_{\ell,\delta})$ from where $\rho_k^\varepsilon \in L^\beta(\Omega_{\ell,\delta})$, where $\beta := 2^*(r - 1/2)$, if $d \geq 3$ and $\beta < +\infty$ arbitrary if $d = 2$ and β can be taken $+\infty$ if $d = 1$. He we use the notation $2^* = 2d/(d - 2)$.

From the above construction we can claim that $\rho_k^\varepsilon \in L^\beta(\Omega)$. Indeed, we have

$$\begin{aligned}\int_\Omega (\rho_k^\varepsilon)^\beta \mathrm{d}x &= \int_{\{\rho_k^\varepsilon \leq \ell\}} (\rho_k^\varepsilon)^\beta \mathrm{d}x + \int_{\Omega_{\ell,\delta}} (\rho_k^\varepsilon)^\beta \mathrm{d}x + \int_{\Omega_\ell \setminus \Omega_{\ell,\delta}} (\rho_k^\varepsilon)^\beta \mathrm{d}x \\ &\leq (2^\beta + 1)\ell^\beta \mathscr{L}^d(\Omega) + C\left(\|\nabla\Phi\|_{L^\infty}^2 + \frac{1}{\tau^2}W_2^2(\rho_k^\varepsilon, \rho_{k-1})\right).\end{aligned}$$

Let us underline that this bound only depends on $W_2^2(\rho_k^\varepsilon, \rho_{k-1})$.

Now, it is easy to see that because $S_\varepsilon \to S$ uniformly, we have that the objective functional in (2.2.26) Γ-convergences to the objective functional in the original problem as $\varepsilon \downarrow 0$, w.r.t.

the weak-$*$ convergence of probability measures. Indeed, take a sequence $(\rho^\varepsilon)_{\varepsilon>0}$ and ρ in $\mathscr{P}(\Omega)$ such that $\rho^\varepsilon \overset{\star}{\rightharpoonup} \rho$ as $\varepsilon \downarrow 0$. Notice that by the construction of the approximation S_ε, if $\mathcal{E}_\varepsilon(\rho^\varepsilon) \le C$ (for a constant independent of ε), then we have that $(\rho^\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^r(\Omega)$. By the uniform convergence $S_\varepsilon \to S$, we have that for any $\delta > 0$ there exists ε_0 such that

$$S(\rho^\varepsilon) \le S_\varepsilon(\rho^\varepsilon) + \delta,\ \forall \varepsilon < \varepsilon_0.$$

Therefore

$$\mathcal{E}(\rho) \le \liminf_{\varepsilon\downarrow 0} \mathcal{E}(\rho^\varepsilon) \le \liminf_{\varepsilon\downarrow 0} \mathcal{E}_\varepsilon(\rho^\varepsilon) + \delta \mathscr{L}^d(\Omega),$$

so the Γ-liminf inequality follows by the lower semicontinuity of the energy $\mathcal{E}$ and the arbitrariness of $\delta > 0$. For the Γ-limsup inequality, we use a constant sequence $\rho^\varepsilon = \rho$ as a recovery sequence such that $\mathcal{E}_\varepsilon(\rho)$ is finite for all $\varepsilon > 0$. Clearly $\lim_{\varepsilon\downarrow 0} \mathcal{E}_\varepsilon(\rho) = \mathcal{E}(\rho)$.

Finally, since both ρ_k and ρ_k^ε, the solutions of the original and the approximated problems, respectively are unique, when $\varepsilon \downarrow 0$ we find that ρ_k also has the $L^\beta(\Omega)$ bound. The thesis of the lemma follows. □

Let us notice that in Lemma 2.2.11 the L^β bounds on ρ_k depends only on $\frac{1}{\tau^2} W_2^2(\rho_k, \rho_{k-1})$ and the data. Therefore, when considering the piecewise constant interpolated curves $(\rho^\tau)_{\tau>0}$ (see their precise definition in (2.3.9) below), and integrating them in time and space, we find the following very important lemma.

Lemma 2.2.13. *Suppose that $\rho_0 \in \mathscr{P}(\Omega)$ with $\mathcal{J}(\rho_0) < +\infty$ and (2.1.8) hold. The curves $(\rho^\tau)_{\tau>0}$ are uniformly bounded in $L^\beta(Q)$ for β given in*

$$\beta := \begin{cases} (2r-1)\frac{d}{d-2} & \text{if } d \ge 3, \\ (0,\infty) & \text{if } d = 2 \\ +\infty & \text{if } d = 1. \end{cases} \tag{2.2.29}$$

Proof. Let β as in the statement of the lemma and let $(\rho^\tau)_{\tau>0}$ stand for the piecewise constant interpolations as defined in (2.3.9). Then, Lemma 2.2.11 implies that

$$\int_0^T \int_\Omega (\rho^\tau)^\beta \mathrm{d}x\mathrm{d}t = \tau \sum_{k=1}^N \int_\Omega (\rho^\tau)^\beta \mathrm{d}x \le \tau N C + C \sum_{k=1}^N \frac{1}{\tau} W_2^2(\rho_k, \rho_{k-1}),$$

where $C > 0$ depends only on the data and Ω. Since $\tau N = T$ and $\sum_{k=1}^{N} \frac{1}{\tau} W_2^2(\rho_k, \rho_{k-1})$ is uniformly bounded (see Lemma 2.3.6), we conclude. □

Under the above assumption, we show a result parallel to Theorem 2.2.3.

Theorem 2.2.14. *Suppose that $\rho_0 \in \mathscr{P}(\Omega)$ such that $\mathcal{E}(\rho_0) < +\infty$ and (2.1.8) hold. Then, for all $k \in \{1, \dots, N\}$ there exists $\mathcal{C} = \mathcal{C}(k) \in \mathbb{R}$ and $\overline{\phi}_k \in \mathcal{K}(\rho_k, \rho_{k-1})$ satisfying (2.2.5). Here, $\mathcal{K}(\rho_k, \rho_{k-1})$ and ρ_k are given in Definition A.4.1 and (2.2.3), respectively.*

We recall the following lemma from [Roc68] and [Roc71] and compute the subdifferential of $\mathcal{J}$ explicitly. In comparison to the previous subsection, it holds that $(L^r(\Omega))^* = L^{r'}(\Omega)$ for $r \in (1, +\infty)$ where $r' := \frac{r}{r-1}$ and thus the argument below is simpler than Lemma 2.2.8.

Lemma 2.2.15. *[Roc68, Theorem 2], [Roc71, Equations (1.11) & (1.12)] Let ψ and Ψ be given as in (A.5.2). Assume that $\psi(\mu(x))$ is majorized by a summable function of x for at least one $\mu \in L^\infty(\Omega)$ and that $\psi^*(\zeta(x))$ is majorized by a summable function of x for at least one $\zeta \in L^1(\Omega)$. Then, an element $\xi \in L^r(\Omega)^*$ belongs to $\partial\Psi(\mu)$ given in (A.5.3) if and only if $\xi(x) \in \partial\psi(\mu(x))$ for a.e. $x \in \Omega$.*

Proof of Theorem 2.2.14. Let us set $k = 1$. The first part of the proof is parallel to Proposition 2.2.4 and Proposition 2.2.7. Let us show

$$\partial\left(\mathcal{J}(\rho) + \mathcal{F}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_0)\right)\bigg|_{\rho=\rho_1} = \partial\mathcal{J}(\rho_1) + \Phi + \frac{1}{\tau}\mathcal{K}(\rho_1, \rho_0) \tag{2.2.30}$$

where $\mathcal{K}$ is given in Definition A.4.1 and the subdifferential is defined in Definition A.5.1. Recall $\Gamma(\cdot)$ from Definition A.5.2 and its equivalent property in Lemma A.5.3. Note that $\mathcal{J} \in \Gamma(L^r(\Omega))$ follows from the convexity of S and Lemma 2.2.10. The same is true for $\mathcal{F}$.

Let us underline that it is crucial that we have a priori bounds on the optimizers of (2.2.3) in $L^\beta(\Omega)$ for some $\beta > 1$. Indeed, Lemma 2.2.11 yields that even if $r = 1$ (which corresponds to the logarithmic entropy type interaction energy), we have that the optimizers satisfy $\rho_k \in L^\beta(\Omega)$. In this case, without loss of generality, one considers the continuity of $\mathcal{J}$ and $\mathcal{F}$ in $L^\beta(\Omega)$. Otherwise, we gain $L^r(\Omega)$ bounds simply from the growth condition on S at $+\infty$, hence we can also refer to the continuity of $\mathcal{J}$ in this space.

Furthermore, from Proposition A.4.4, we have

$$H^* = \frac{1}{2}W_2^2(\cdot, \rho_0) \text{ on } L^\beta(\Omega) \tag{2.2.31}$$

for $H : L^{\beta'}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ given in (2.2.7) and $\beta' := \frac{\beta}{\beta-1}$. Thus we get $W_2^2(\cdot, \rho_0) \in \Gamma(L^\beta(\Omega))$. Lastly, by the parallel argument in Lemma 2.2.6, H is also in $\Gamma(L^{\beta'}(\Omega))$. From Lemma A.5.4 and Lemma 2.2.5, we conclude (2.2.30).

The rest of the proof is parallel to that of Theorem 2.2.3. From (2.2.30) and Lemma 2.2.15, there exists $\xi \in \partial\mathcal{J}(\rho_1)$ satisfying (2.2.14), $\overline{\phi}_1 \in \mathcal{K}(\rho_1, \rho_0)$ and $\mathcal{C} \in \mathbb{R}$ such that

$$\xi + \frac{\overline{\phi}_1}{\tau} + \Phi - \mathcal{C} = 0 \text{ a.e. on } \Omega. \tag{2.2.32}$$

and we conclude (2.2.5). □

2.3 Linear diffusion with discontinuities

In this section we show the well-posedness of (2.1.4) in the most simple case considered, i.e. when the associated internal energy is an entropy of logarithmic type. We give a fine characterization of the 'critical phase' $\{\rho = 1\}$ via a scalar *pressure field*, inspired from recent works on crowd motion models with hard congestion effects (see for instance [MRS10,MS16]). In the next sections we shall see how the results and ideas from this sections will be important to build solutions for problems with more general nonlinearities.

In this section, we assume that $\mathcal{S} : [0, +\infty) \to \mathbb{R}$ is defined by

$$\mathcal{S}(\rho) := \begin{cases} \rho \log \rho, & \text{for } \rho \in [0, 1], \\ 2\rho \log \rho, & \text{for } \rho \in (1, +\infty). \end{cases} \tag{2.3.1}$$

Let us notice that S defines a continuous superlinear function on $\mathbb{R}^+$ with $S'(1-) = 1$ and $S'(1+) = 2$.

Our main theorem from this section can be formulated as follows.

Theorem 2.3.1. *For $\rho_0 \in \mathscr{P}(\Omega)$ such that $\mathcal{J}(\rho_0) < +\infty$ and S given in (2.3.1), there exists $\rho \in L^1(Q) \cap AC^2([0, T]; \mathscr{P}(\Omega))$ and $p \in L^2([0, T]; H^1(\Omega)) \cap L^\infty(Q)$ with $\sqrt{\rho} \in L^2([0, T]; H^1(\Omega))$*

such that (ρ, p) *is a weak solution of*

$$\begin{cases} \partial_t \rho - \Delta(\rho p) - \nabla \cdot (\nabla \Phi \rho) = 0, & \text{in } (0,T) \times \Omega, \\ \rho(0, \cdot) = \rho_0, & \text{in } \Omega, \\ (\nabla(\rho p) + \nabla \Phi) \cdot \vec{n} = 0, & \text{in } [0,T] \times \partial\Omega, \end{cases} \tag{2.3.2}$$

in the sense of distribution. Furthermore, (ρ, p) *satisfies*

$$\begin{cases} p(t,x) = 1 & \text{a.e. in } \{0 < \rho(t,x) < 1\}, \\ p(t,x) \in [1,2] & \text{a.e. in } \{\rho(t,x) = 1\}, \\ p(t,x) = 2 & \text{a.e. in } \{\rho(t,x) > 1\}. \end{cases} \tag{2.3.3}$$

If in addition $\rho_0 \in L^\infty(\Omega)$ *and* Φ *satisfies* (2.2.4)*, then* $\rho \in L^2([0,T]; H^1(\Omega)) \cap L^\infty(Q)$.

In the proof of the previous theorem we rely on the minimizing movements scheme associated to the gradient flow of $\mathcal{J}$, defined in (2.2.3). As technical tools, we define different interpolations between the discrete in time densities $(\rho_k)_{k=0}^N$ and obtain a weak solution of (2.3.2) by sending $\tau \downarrow 0$. The new pressure term p arrises from the Wasserstein subdifferential of $\mathcal{J}$ and its 'nontrivial' value on the set $\{\rho = 1\}$ is due to the non-differentiability of S at $s_0 = 1$.

Definition 2.3.2. Let $(\rho_k)_{k=1}^N$ be given by the minimizing movement scheme (2.2.3) and let $\overline{\phi}_k \in \mathcal{K}(\rho_k, \rho_{k-1})$. For $k \in \{1, \ldots, N\}$, let us define $p_k : \Omega \to \mathbb{R}$ and $p^\tau : Q \to \mathbb{R}$ by

$$p_k = p_k(\cdot;\tau) := \begin{cases} \max\{\mathcal{C} - \frac{\overline{\phi}_k}{\tau} - \Phi, \mathcal{S}'(1-)\} & \text{in } \rho_k^{-1}([0,1)), \\ \mathcal{C} - \frac{\overline{\phi}_k}{\tau} - \Phi & \text{in } \rho_k^{-1}(\{1\}), \\ \min\{\mathcal{C} - \frac{\overline{\phi}_k}{\tau} - \Phi, \mathcal{S}'(1+)\} & \text{in } \rho_k^{-1}((1,+\infty)). \end{cases} \quad \text{and} \quad p^\tau(t,x) := p_k(x;\tau) \tag{2.3.4}$$

for $(t,x) \in ((k-1)\tau, k\tau] \times \Omega$, where the constant $C \in \mathbb{R}$ might be different at each step.

Remark 2.3.3. Let us notice that Lemma A.4.5 yields $\rho_k > 0$ a.e., therefore $(p_k)_{k=1}^N$ is well-defined also on the sets $\rho_k^{-1}([0,1))$. From the above definition, the optimality condition in Theorem 2.2.3 can be simplified as follows.

Lemma 2.3.4. *For all $k \in \{1, \ldots, N\}$, there exists $C \in \mathbb{R}$ such that*

$$p_k(1 + \log \rho_k) + \frac{\overline{\phi}_k}{\tau} + \Phi = \mathcal{C} \quad a.e. \tag{2.3.5}$$

Proof. Note that a subdifferential $\partial\mathcal{S}(\rho)$ of $\mathcal{S} : [0, +\infty) \to \mathbb{R}$ is given by

$$\partial\mathcal{S}(\rho) = \begin{cases} 1 + \log \rho & \text{for } 0 < \rho < 1, \\ [1, 2] & \text{for } \rho = 1, \\ 2(1 + \log \rho) & \text{for } \rho > 1. \end{cases} \tag{2.3.6}$$

Thus, Theorem 2.2.3 and (2.3.4) imply

$$p_k = \begin{cases} 1 & \text{in } \rho_k^{-1}((0, 1)), \\ \mathcal{C} - \frac{\overline{\phi}_k}{\tau} - \Phi \in [1, 2] & \text{in } \rho_k^{-1}(\{1\}), \\ 2 & \text{in } \rho_k^{-1}((1, +\infty)). \end{cases} \qquad \text{a.e.} \tag{2.3.7}$$

Thus, we simplify (2.2.5) into (2.3.5). □

An easy consequence of the above constructions is the following result.

Lemma 2.3.5. *For $k \in \{1, \ldots, N\}$, ρ_k, p_k and $\overline{\phi}_k \in \mathcal{K}(\rho_k, \rho_{k-1})$ are Lipschitz continuous in Ω. Here, ρ_k and p_k are given in* (2.2.3) *and* (2.3.4), *respectively.*

Proof. 1. Let us show that $\overline{\phi}_k$ is Lipschitz continuous in Ω. From [San15, Theorem 1.17] we have that $\overline{\phi}_k$ shares the modulus of continuity of the cost $(x, y) \mapsto |x - y|^2$. On the one hand, as $\overline{\Omega}$ is compact, we conclude that $\overline{\phi}_k$ is Lipschitz continuous. On the other hand, (2.3.7) together with (2.2.5) imply that p_k is Lipschitz continuous.

2. From (2.3.5) in Lemma 2.3.4, we have that

$$\rho_k(x) = \exp\left\{\frac{1}{p_k(x)}\left(\mathcal{C} - \frac{\overline{\phi}_k(x)}{\tau} - \Phi\right) - 1\right\} \text{ a.e.} \tag{2.3.8}$$

As p_k, $\overline{\phi}_k$ and Φ are Lipschitz continuous and p_k has a lower bound $+1$ from (2.3.7), (2.3.8) implies that ρ_k is Lipschitz continuous. □

As technical tools, similarly as it is done in the framework of models developed for instance in [MRS10,MS16,San15], we introduce two different kinds of interpolations between the objects in the title of the subsection.

Piecewise constant interpolations. Let us define $\rho^\tau, p^\tau : Q \to \mathbb{R}$ and $\mathbf{v}^\tau, \mathcal{E}^\tau : Q \to \mathbb{R}^d$ as follows

$$\begin{cases} \rho^\tau(t,x) := \rho_k(x;\tau), \\ \mathbf{v}^\tau(t,x) := \dfrac{1}{\tau}\nabla\overline{\phi}_k(x), & \text{for } (t,x) \in ((k-1)\tau, k\tau] \times \Omega \text{ and } k \in \{1,\dots,N\}, \\ \mathcal{E}^\tau(t,x) := \rho^\tau(t,x)\mathbf{v}^\tau(t,x) \end{cases} \tag{2.3.9}$$

for $(\rho_k)_{k=1}^N$ obtained in (2.2.3) and $\overline{\phi}_k \in \mathcal{K}(\rho_k, \rho_{k-1})$ given in Theorem 2.2.3 and let p^τ defined as in (2.3.4).

By standard arguments on gradient flows (see for instance [San15, Proposition 8.8], [MS16, Lemma 3.5]), we have the following.

Lemma 2.3.6. *It holds that*

$$\frac{1}{2\tau}\sum_{k=1}^N W_2^2(\rho_k, \rho_{k-1}) = \frac{1}{2\tau}\sum_{k=1}^N \int_\Omega |\nabla\overline{\phi}_k|^2 \, d\rho_k(x) \leq \mathcal{J}(\rho_0) - \inf \mathcal{J}. \tag{2.3.10}$$

Furthermore, there exists a constant $C > 0$ such that for any $0 \leq s < t \leq T$

$$W_2(\rho^\tau(t), \rho^\tau(s)) \leq C(t-s+\tau)^{\frac{1}{2}}. \tag{2.3.11}$$

Proposition 2.3.7. *Let $(\rho^\tau)_{\tau>0}$ and $(p^\tau)_{\tau>0}$ given* (2.3.9) *and* (2.3.4)*, respectively. We have the followings.*

(1) $(p^\tau)_{\tau>0}$ is uniformly bounded in $L^2([0,T];H^1(\Omega)) \cap L^\infty(Q)$;

(2) $(\sqrt{\rho^\tau})_{\tau>0}$ is uniformly bounded in $L^2([0,T];H^1(\Omega))$;

(3) if in addition $\rho_0 \in L^\infty(\Omega)$ and Φ satisfies (2.2.4)*, then $(\rho^\tau)_{\tau>0}$ is uniformly bounded in $L^2([0,T];H^1(\Omega)) \cap L^\infty(Q)$.*

Proof. 1. Clearly, by construction, $(p^\tau)_{\tau>0}$ is uniformly bounded. Furthermore, if $\rho_0 \in L^\infty(\Omega)$, then Lemma 2.2.1 implies that $(\rho^\tau)_{\tau>0}$ is uniformly bounded by a constant depending only on the data for all $t \in [0,T]$.

2. Now, let us show that $(\nabla\sqrt{\rho^\tau})_{\tau>0}$ and $(\nabla p^\tau)_{\tau>0}$ are uniformly bounded in $L^2(Q)$. Let $\overline{\phi}_k \in \mathcal{K}(\rho_k, \rho_{k-1})$. Lemma 2.3.5 implies that $\overline{\phi}_k$, ρ_k and p_k are Lipschitz continuous functions, and therefore by Rademacher's theorem one can differentiate these function a.e. in Ω. Note that $\{\rho_k \neq 1\}$ is an open by by the continuity of ρ_k in Lemma 2.3.5 and thus (2.3.7) implies

$$\nabla p_k = 0 \text{ a.e. in } \{\rho_k \neq 1\}. \tag{2.3.12}$$

Therefore, we get

$$\log \rho_k \nabla p_k = 0 \text{ and } (\rho_k - 1)\nabla p_k = 0 \quad \text{ a.e.} \tag{2.3.13}$$

Next, we claim that

$$\nabla p_k \cdot \nabla \rho_k = 0 \quad \text{ a.e. in } \Omega. \tag{2.3.14}$$

From (2.3.7), the above holds in the open set $\{\rho_k \neq 1\}$ and in the interior of $\{\rho_k = 1\}$, but we point out that $\partial\{\rho_k = 1\}$ may have positive measure even though ρ_k is Lipschitz continuous. In order to show (2.3.14) in Ω, we apply the coarea formula and (2.3.12). As ρ_k is Lipschitz and ∇p_k is in $L^1(\Omega)$, we could use the coarea formula in [KP08, Corollary 5.2.6] and conclude that

$$\int_\Omega |\nabla p_k||\nabla \rho_k| \mathrm{d}x = \int_{\mathbb{R}} \int_{(\rho_k)^{-1}(s)} |\nabla p_k| \mathrm{d}\mathscr{H}^{d-1} \mathrm{d}s. \tag{2.3.15}$$

where $\mathscr{H}^{d-1}$ stands for the $(d-1)$-dimensional Hausdorff measure. From (2.3.12), we conclude (2.3.14).

Differentiating (2.3.5) and applying (2.3.13) and (2.3.14), we have

$$-\frac{\nabla\overline{\phi}_k}{\tau} - \nabla\Phi = \nabla(p_k(1+\log\rho_k)) + \nabla\Phi = \nabla p_k + \frac{p_k}{\rho_k}\nabla\rho_k \quad \text{ a.e.} \tag{2.3.16}$$

From (2.3.16) and (2.3.14) again, we have

$$2\rho_k \left(\frac{|\nabla\overline{\phi}_k|^2}{\tau^2} + |\nabla\Phi|^2 \right) \geq |\nabla p_k|^2 + \frac{p_k^2}{\rho_k}|\nabla\rho_k|^2 \quad \text{ a.e.,} \tag{2.3.17}$$

from where we can write

$$2\rho_k \left(\frac{|\nabla\overline{\phi}_k|^2}{\tau^2} + |\nabla\Phi|^2 \right) \geq |\nabla p_k|^2 + p_k^2|\nabla\sqrt{\rho_k}|^2 \quad \text{ a.e.}$$

As $p_k \in [1,2]$ (from (2.3.7)), we have

$$\int_\Omega \left(|\nabla p_k|^2 + |\nabla \sqrt{\rho_k}|^2\right) \le 2\int_\Omega \frac{|\nabla \overline{\phi}_k|^2}{\tau^2}\rho_k \mathrm{d}x + 2\mathscr{L}^d(\Omega)\|\nabla\Phi\|^2_{L^\infty}.$$

From Lemma 2.3.6, we conclude that $(\sqrt{\rho^\tau})_{\tau>0}$ and $(p^\tau)_{\tau>0}$ are uniformly bounded in $L^2([0,T];H^1(\Omega))$ for all $\tau > 0$.

Moreover, if $\rho_0 \in L^\infty(\Omega)$, we have $\|\rho_k\| \le \|\rho_0\|_{L^\infty(\Omega)} e^{dT\|[\Delta\Phi]_+\|_{L^\infty}}$ (from Lemma 2.2.1), and therefore from (2.3.17) we get

$$\int_\Omega |\nabla p_k|^2 \mathrm{d}x + \int_\Omega \frac{1}{\|\rho_0\|_{L^\infty(\Omega)} e^{dT\|[\Delta\Phi]_+\|_{L^\infty}}}|\nabla \rho_k|^2 \mathrm{d}x \le C, \tag{2.3.18}$$

from where we have $(\rho^\tau)_{\tau>0}$ is uniformly bounded in $L^2([0,T];H^1(\Omega))$. □

Corollary 2.3.8. *Let $(\rho^\tau)_{\tau>0}$ and $(p^\tau)_{\tau>0}$ be as in the previous proposition. There exists $p \in L^2([0,T];H^1(\Omega))$ and $\rho \in L^1(Q)$ such that*

$$\rho^\tau \to \rho \text{ in } L^1(Q), \text{ as } \tau \downarrow 0, \tag{2.3.19}$$

and

$$p^\tau \rightharpoonup p \text{ in } L^2([0,T];H^1(\Omega)), \text{ as } \tau \downarrow 0. \tag{2.3.20}$$

along a subsequence. If in addition $\rho_0 \in L^\infty(\Omega)$ and Φ satisfies (2.2.4)*, then we also have $\rho \in L^2([0,T];H^1(\Omega))$ and $\rho^\tau \to \rho$ in $L^2(Q)$, as $\tau \downarrow 0$.*

Proof. The weak sequential compactness of $(p^\tau)_{\tau>0}$ follows from the uniform boundedness in $L^2([0,T];H^1(\Omega))$ in the previous proposition. Also, as $(\rho^\tau)_{\tau>0}$ has the 'quasi-Hölder' type estimates in Lemma 2.3.6 and $(\sqrt{\rho^\tau})_{\tau>0}$ is uniformly bounded in $L^2([0,T];H^1(\Omega))$, we conclude the strong compactness of $(\rho^\tau)_{\tau>0}$ in $L^1(Q)$ by a consequence of a modified version of the classical Aubin-Lions lemma in Lemma A.6.2, ofter used in similar context (see for instance [DM14, Proposition 4.8] and [Lab17, Proposition 5.2]). If $\rho_0 \in L^\infty(\Omega)$, the last statement simply follows from the similar arguments. □

As a consequence of the above results, we have the following.

Lemma 2.3.9. *(ρ, p) given in Proposition 2.3.7 satisfies* (2.3.3)*.*

Proof. 1. Let (ρ^τ, p^τ) be defined in (2.3.9) and (2.3.4). First, from (2.3.7), we have

$$(p^\tau - 2)(\rho^\tau - 1)_+ = (p^\tau - 1)(\rho^\tau - 1)_- = 0 \text{ in } Q. \tag{2.3.21}$$

As it holds that

$$|(\rho^\tau - 1)_+ - (\rho - 1)_+| \le |\rho^\tau - \rho| \text{ and } |(\rho^\tau - 1)_- - (\rho - 1)_-| \le |\rho^\tau - \rho|, \tag{2.3.22}$$

Proposition 2.3.7 implies that both $(\rho^\tau - 1)_+ \to (\rho - 1)_+$ and $(\rho^\tau - 1)_- \to (\rho - 1)_-$ in $L^1(Q)$ as $\tau \downarrow 0$ (up to passing to a subsequence).

2. Let us show that for a.e. $t \in [0, T]$

$$\int_\Omega (p(t,x) - 2)(\rho(t,x) - 1)_+ \mathrm{d}x = 0 \text{ and } \int_\Omega (p(t,x) - 1)(\rho(t,x) - 1)_- \mathrm{d}x = 0. \tag{2.3.23}$$

We only show the first one as the parallel arguments work for the second one. From (2.3.21), we have

$$0 = \int_Q (p^\tau(t,x) - 2)(\rho^\tau(t,x) - 1)_+ \mathrm{d}x\mathrm{d}t. \tag{2.3.24}$$

Recall that up to passing to a subsequence, $(p^\tau)_{\tau>0}$ convergences weakly$-\star$ in $L^\infty(Q)$ (see Proposition 2.3.7) and $((\rho^\tau(t,x) - 1)_+)_{\tau>0}$ converges strongly (from Step 1) in $L^1([0, T \times \Omega])$ as $\tau \downarrow 0$. Combining these with (2.3.24), we conclude the first equation of (2.3.23).

As $p^\tau \in [1, 2]$ for p^τ given in (2.3.4), we have $p \in [1, 2]$ a.e. in Q. Thus, (2.3.23) implies that

$$(p - 2)(\rho - 1)_+ = (p - 1)(\rho - 1)_- = 0 \quad \text{a.e.} \tag{2.3.25}$$

and we conclude (2.3.3). □

Proposition 2.3.10. *Let $\mathcal{E}^\tau$ be given in* (2.3.9). *Then up to passing to a subsequence, $(\mathcal{E}^\tau)_{\tau>0}$ weakly-$\star$ converges to*

$$\mathcal{E} := -\nabla(p\rho) - \nabla\Phi\rho, \quad in \quad \mathscr{D}'(Q; \mathbb{R}^d),$$

as $\tau \downarrow 0$ where and (ρ, p) is given in Proposition 2.3.7.

Proof. For any test function $\zeta \in C_c^\infty(Q;\mathbb{R}^d)$, we claim that up to passing to a subsequence,

$$\mathcal{I} := \int_Q \zeta \cdot \mathrm{d}(\mathcal{E}^\tau - \mathcal{E}) \to 0, \text{ as } \tau \downarrow 0 \tag{2.3.26}$$

From (2.3.13), we have $\log \rho^\tau \nabla p^\tau = 0$ in a.e. in Q and thus it holds that

$$-\mathcal{E}^\tau = p\nabla\rho^\tau + \rho^\tau(1+\log\rho^\tau)\nabla p^\tau + \nabla\Phi\rho^\tau = \nabla(\rho^\tau p^\tau) + \nabla\Phi\rho^\tau. \tag{2.3.27}$$

By the weak convergence of $(\rho^\tau)_{\tau>0}$ to ρ, we already have that

$$\int_Q \zeta\cdot\nabla\Phi \mathrm{d}\rho^\tau \mathrm{d}t \to \int_Q \zeta\cdot\nabla\Phi\mathrm{d}\rho\mathrm{d}t,\ \tau\downarrow 0,$$

we only focus on the other term. By integration by parts and and from the fact that $\zeta \in C_c^\infty(Q;\mathbb{R}^d)$, we study thus

$$\mathcal{I}_1 = \int_Q (\rho^\tau p^\tau - \rho p)\nabla\cdot\zeta \mathrm{d}x\mathrm{d}t \tag{2.3.28}$$

By subtracting and adding the same term in the above equation, we get

$$\mathcal{I}_1 = \mathcal{I}_2 + \mathcal{I}_3 \text{ where } \mathcal{I}_2 = \int_Q (\rho^\tau - \rho)p^\tau\nabla\cdot\zeta\mathrm{d}x\mathrm{d}t \text{ and } \mathcal{I}_3 = \int_Q \rho(p^\tau - p)\nabla\cdot\zeta\mathrm{d}x\mathrm{d}t \tag{2.3.29}$$

From the Hölder inequality, we have

$$|\mathcal{I}_2| \le \|\rho^\tau - \rho\|_{L^1(Q)}\|p^\tau\|_{L^\infty(Q)}\|\nabla\cdot\zeta\|_{L^\infty(Q)}. \tag{2.3.30}$$

As $\rho^\tau \to \rho$ in $L^1(Q)$ as $\tau\downarrow 0$ and $\|p^\tau\|_{L^\infty(Q)}$ is uniformly bounded (Proposition 2.3.7), we conclude $\mathcal{I}_2 \to 0$ as $\tau \downarrow 0$. On the other hand, as $p^\tau \overset{\star}{\rightharpoonup} p$ in $L^\infty(Q)$ as $\tau\downarrow 0$ (Proposition 2.3.7), and $\rho \in L^1(Q)$ we have $\mathcal{I}_3 \to 0$ as $\tau \downarrow 0$ as well, and thus we conclude (2.3.26). □

To arrive to the continuous in time in the time continuous PDE satisfied by (ρ, p) from Proposition 2.3.7, as technical tools (inspired from [MRS10, MS16, San15]), we introduce a geodesic interpolation between $(\rho_k)_{k=1}^N$ and we consider the corresponding velocities and momenta as well.

More precisely, we define $\widetilde{\rho}^\tau : [0,T] \to \mathscr{P}(\Omega)$, $\tilde{\mathbf{v}}^\tau, \widetilde{\mathbf{E}}^\tau \in \mathscr{M}(Q;\mathbb{R}^d)$ as follows: for $t \in ((k-1)\tau, k\tau]$ and $k \in \{1,\dots,N\}$

$$\begin{cases} \widetilde{\rho}^\tau(t,x) := \left(\frac{k\tau - t}{\tau}v^\tau(t,x) + \mathrm{id}\right)_\# \rho^\tau(t,x)), \\ \tilde{\mathbf{v}}^\tau(t,x) := \mathbf{v}^\tau(t,x)\circ\left(\frac{k\tau-t}{\tau}\mathbf{v}^\tau(t,x)+\mathrm{id}\right)^{-1}, \\ \widetilde{\mathbf{E}}^\tau(t,x) := \widetilde{\rho}^\tau(t,x)\tilde{\mathbf{v}}^\tau(t,x), \end{cases} \tag{2.3.31}$$

where ρ^τ and $\mathbf{v}^\tau$ are given in (2.3.9).

Following the very same steps as in From [San15, Lemma 8.9] and [MS16, Step 2 in Theorem 3.1], we have the following.

Lemma 2.3.11. *We have that*

(i) $(\widetilde{\rho}^\tau)_{\tau>0}$ *is uniformly bounded in* $AC([0,T];\mathscr{P}(\Omega))$*;*

(ii) there exists $C>0$ *such that* $\displaystyle\int_0^T\int_\Omega |\tilde{v}^\tau|^2\,d\tilde{\rho}^\tau_t\,dt \le C$*;*

(iii) $(\widetilde{\boldsymbol{E}}^\tau)_{\tau>0}$ *is uniformly bounded in* $\mathscr{M}(Q;\mathbb{R}^d)$*.*

As a consequence, we have that along a subsequence

(iv) $\displaystyle\sup_{t\in[0,T]} W_2(\widetilde{\rho}^\tau_t,\rho_t)\to 0,$ *as* $\tau\downarrow 0$*,*

(v) $\widetilde{\boldsymbol{E}}^\tau \overset{\star}{\rightharpoonup} \mathcal{E}$*, in* $\mathscr{M}(Q;\mathbb{R}^d)$*, as* $\tau\downarrow 0$*,*

where ρ *is given in Proposition 2.3.7 and* $\mathcal{E}$ *is given in Proposition 2.3.10.*

Now, we are ready to prove Theorem 2.3.1.

Proof of Theorem 2.3.1. Let us underline that the main reason for introducing the interpolations $(\widetilde{\rho}^\tau,\widetilde{\mathbf{E}}^\tau)$ is that by construction, they satisfy the PDE

$$\begin{cases} \partial_t\tilde{\rho}^\tau + \nabla\cdot\widetilde{\mathbf{E}}^\tau = 0, & \text{in } (0,T)\times\Omega,\\ \tilde{\rho}^\tau(0,\cdot)=\rho_0, & \text{in } \Omega,\\ \widetilde{\mathbf{E}}^\tau\cdot\vec{n}=0, & \text{on } [0,T]\times\partial\Omega, \end{cases} \tag{2.3.32}$$

in the distributional sense. Then, Lemma 2.3.11 and Proposition 2.3.10 allow us to conclude that (ρ,p) satisfies (2.3.2) in the distributional sense. Last, from Lemma 2.3.9, we conclude that (ρ,p) satisfies (2.3.3). The thesis of the theorem follows. □

2.4 Linear diffusion on $\{\rho < 1\}$ and porous medium type diffusion on $\{\rho > 1\}$

As we will see below, in this section the diffusion coefficients and the diffusion rates are not necessarily supposed to be the same in the regions $\{\rho < 1\}$ and $\{\rho > 1\}$. Therefore, a technical difficulty arrises, because of the lack of a simple way (as in (2.3.5)) to derive the first order necessary optimality conditions for the minimizing movement scheme. To overcome this issue, instead, we use a particular decomposition for $\mathcal{S}$, which allows us to use the construction from Section 2.3.

In this section too, we impose Assumption A. If $\rho_0 \notin L^\infty(\Omega)$, we impose additionally (2.1.8). Furthermore, throughout this section we suppose also the following: $\mathcal{S} : [0, +\infty) \to \mathbb{R}$ satisfies

$$\frac{\rho^{-1}}{\sigma_2} \leq \mathcal{S}''(\rho) \text{ in } (0,1) \tag{2.4.1}$$

for some constant $\sigma_2 > \sigma_1$ for σ_1 given in (2.1.8a). This corresponds to (2.1.7) with $m = 1$.

A direct consequence of the above assumption is the following result.

Lemma 2.4.1. $\mathcal{S} : [0, +\infty) \to \mathbb{R}$ *satisfies*

$$\mathcal{S}'(0+) = -\infty \tag{2.4.2}$$

Proof. Integrating (2.4.1) from $\frac{1}{2}$ to ρ, it holds that

$$\mathcal{S}'\left(\frac{1}{2}\right) - \mathcal{S}'(\rho) \geq \frac{1}{\sigma_2}\left(\log\frac{1}{2} - \log\rho\right) \tag{2.4.3}$$

As $\sigma_2 > 0$, we conclude that

$$\mathcal{S}'(\rho) \leq \mathcal{S}'\left(\frac{1}{2}\right) - \frac{1}{\sigma_2}\log\frac{1}{2} + \frac{1}{\sigma_2}\log\rho \quad \to \quad -\infty \qquad \text{as } \rho \to 0^+. \tag{2.4.4}$$

□

Our main theorem from this section reads as:

Theorem 2.4.2. *Suppose that* (2.1.8) *and* (2.4.1) *hold true. For* $\rho_0 \in \mathscr{P}(\Omega)$ *such that* $\mathcal{J}(\rho_0) < +\infty$*, there exists* $\rho \in L^\beta(Q) \cap AC^2([0,T];\mathscr{P}(\Omega))$ *for* β *given in* (2.2.29) *and* $p \in L^2([0,T];H^1(\Omega)) \cap L^\infty(Q)$ *with* $\sqrt{\rho} \in L^2([0,T];H^1(\Omega))$ *such that* (ρ,p) *is a weak solution of*

$$\begin{cases} \partial_t \rho - \Delta(L_S(\rho,p)) - \nabla\cdot(\nabla\Phi\rho) = 0, & \text{in } (0,T)\times\Omega, \\ \rho(0,\cdot) = \rho_0, & \text{in } \Omega, \\ (\nabla(L_S(\rho,p)) + \nabla\Phi\rho)\cdot\vec{n} = 0, & \text{in } [0,T]\times\partial\Omega, \end{cases} \tag{2.4.5}$$

in the sense of distribution. Furthermore, (ρ,p) *satisfies for a.e.* $(t,x) \in Q$

$$\begin{cases} p(t,x) = \mathcal{S}'(1-) & \text{if } 0 < \rho(t,x) < 1, \\ p(t,x) \in [\mathcal{S}'(1-), \mathcal{S}'(1+)] & \text{if } \rho(t,x) = 1, \\ p(t,x) = \mathcal{S}'(1+) & \text{if } \rho(t,x) > 1. \end{cases} \tag{2.4.6}$$

If in addition $\rho_0 \in L^\infty(\Omega)$ *and* Φ *satisfies* (2.2.4)*, we can drop* (2.1.8) *from the statement and we obtain that* $\rho \in L^2([0,T];H^1(\Omega)) \cap L^\infty(Q)$.

Let us briefly explain the outline of the proof. First, we define S_a and $S_b : [0,+\infty) \to \mathbb{R}$ by

$$S_a(\rho) := \begin{cases} \mathcal{S}'(1-)\rho\log\rho, & \text{for } \rho \in [0,1], \\ \mathcal{S}'(1+)\rho\log\rho, & \text{for } \rho \in (1,+\infty), \end{cases} \tag{2.4.7}$$

and

$$S_b(\rho) := \mathcal{S}(\rho) - S_a(\rho). \tag{2.4.8}$$

We show the convexity of S_a and twice differentiability of S_b in Lemma 2.4.4. This particular decomposition will be useful when deriving optimality conditions in our minimizing movement scheme. Under (2.4.1), we are able to apply similar arguments as the ones in Section 2.3.

We point out that Lemma 2.4.1 implies the positivity of ρ_k a.e. (See Lemma A.4.5). From Theorem 2.2.3 and (2.4.1), ρ_k satisfies the following lemma.

Lemma 2.4.3. *Let $(\rho_k)_{k=1}^N$ be obtained via the minimizing movement scheme* (2.2.3)*. For $k \in \{1, \dots, N\}$ and $\overline{\phi}_k \in \mathcal{K}(\rho_k, \rho_{k-1})$ given in Theorem 2.2.3, we have that*

$$\rho_k = \begin{cases} 1, & \text{in } f_k^{-1}([\mathcal{S}'(1-), \mathcal{S}'(1+)]), \\ (\mathcal{S}')^{-1} \circ f_k, & \text{otherwise,} \end{cases} \tag{2.4.9}$$

where $f_k := \mathcal{C} - \frac{\overline{\phi}_k}{\tau} - \Phi$, and $\mathcal{S}'(0+)$ and $\mathcal{S}'(1\pm)$ are given in (2.2.1)*. In particular, ρ_k is Lipschitz continuous in Ω and its Lipschitz constant might degenerate when $\tau \downarrow 0$.*

Proof. 1. As $\mathcal{S}'$ is strictly increasing function in $\mathbb{R}^+ \setminus \{1\}$ from Assumption A, (2.2.5) implies that $\rho_k(x) = 1$ for $x \in f_k^{-1}([\mathcal{S}'(1-), \mathcal{S}'(1+)])$. Also, as $\mathcal{S}'$ is invertible in $\mathbb{R}^+ \setminus \{1\}$, therefore (2.2.5) implies

$$\rho_k(x) = (\mathcal{S}')^{-1} \circ f_k(x) \text{ for } x \in f_k^{-1}((-\infty, \mathcal{S}'(1-)) \cup (\mathcal{S}'(1+), +\infty)) \tag{2.4.10}$$

and we conclude (2.4.9).

2. Let us show that ρ_k is continuous in Ω. Define $\widehat{(\mathcal{S}')^{-1}} : \mathbb{R} \to \mathbb{R}$ by

$$\widehat{(\mathcal{S}')^{-1}} = \begin{cases} 1, & \text{in } [\mathcal{S}'(1-), \mathcal{S}'(1+)], \\ (\mathcal{S}')^{-1}, & \text{otherwise.} \end{cases} \tag{2.4.11}$$

Note that from (2.4.9), we have

$$\rho_k = \widehat{(\mathcal{S}')^{-1}} \circ f_k. \tag{2.4.12}$$

From the continuity and invertibility of $\mathcal{S}'$ in $\mathbb{R}^+ \setminus \{1\}$, we conclude that $\widehat{(\mathcal{S}')^{-1}}$ is continuous in $\mathbb{R}$. Furthermore, from Lemma 2.3.5 we know that $\overline{\phi}_k$ is Lipschitz continuous (and Φ is Lipschitz contonuous by assumption), therefore f_k is Lipschitz continuous. From (2.4.12), we conclude that ρ_k is continuous.

Lastly, as $\mathcal{S}$ is strictly convex and twice differentiable in $\mathbb{R}^+ \setminus \{1\}$ from Assumption A, $(\mathcal{S}')^{-1}$ is differentiable in $(-\infty, \mathcal{S}'(1-)) \cup (\mathcal{S}'(1+), +\infty)$ and on this set we have

$$((\mathcal{S}')^{-1})' = \frac{1}{\mathcal{S}'' \circ (\mathcal{S}')^{-1}}. \tag{2.4.13}$$

Thus, from (2.4.1) and (2.4.13), we conclude that a.e. in $\rho_k^{-1}(\mathbb{R}^+ \setminus \{1\})$ we can compute

$$|\nabla \rho_k| = \frac{|\nabla f_k|}{\mathcal{S}''(\rho_k)} \leq \sigma_2 \|\rho_k\|_{L^\infty(\Omega)} |\nabla f_k| \tag{2.4.14}$$

As f_k is Lipschitz continuous and ρ_k is continuous, we conclude that ρ_k is Lipschitz continuous in Ω. □

The following properties hold for S_a and S_b.

Lemma 2.4.4. *S_a is convex and continuous in $\mathbb{R}^+$. Also, S_b is continuously differentiable and S_b' is locally Lipschitz continuous in $\mathbb{R}^+$. In particular, we have*

$$S_b(1) = \mathcal{S}(1) \text{ and } S_b'(1) = 0. \tag{2.4.15}$$

Proof. From convexity of $\mathcal{S}$, it holds that $\mathcal{S}'(1-) < \mathcal{S}'(1+)$ and thus S_a is convex. It is obviously also continuous by construction.

On the other hand, by the construction in (2.4.7), $S_b(\rho)$ is differentiable on $\mathbb{R}^+ \setminus \{1\}$. Let us show that $S_b(\rho)$ is differentiable at $\rho = 1$. By differentiating (2.4.7) on $\mathbb{R}^+ \setminus \{1\}$, we have that

$$S_a'(\rho) = \begin{cases} \mathcal{S}'(1-)(1+\log\rho), & \text{for } \rho \in (0,1), \\ \mathcal{S}'(1+)(1+\log\rho), & \text{for } \rho \in (1,+\infty), \end{cases} \tag{2.4.16}$$

Therefore, we conclude that

$$S_b'(1-) = \mathcal{S}'(1-) - S_a'(1-) = 0 \text{ and } S_b'(1+) = \mathcal{S}'(1+) - S_a'(1+) = 0 \tag{2.4.17}$$

and S_b is continuously differentiable in $\mathbb{R}^+$. As both $\mathcal{S}'$ and S_a' are locally Lipschitz in $\mathbb{R}^+ \setminus \{1\}$, S_b' is also locally Lipschitz continuous in $\mathbb{R}^+ \setminus \{1\}$. As S_b' is continuous, we conclude that S_b' is locally Lipschitz continuous in $\mathbb{R}^+$. Lastly, $S_b(1) = \mathcal{S}(1)$ follows from $S_a(1) = 0$. □

Lemma 2.4.5. *Let $(\rho_k)_{k=1}^N$ be obtained via the minimizing movement scheme* (2.2.3) *and let $(p_k)_{k=1}^N$ be constructed in* (2.3.4). *For $k \in \{1,\dots,N\}$, we have that*

$$p_k(1+\log\rho_k) + S_b'(\rho_k) + \frac{\overline{\phi}_k}{\tau} + \Phi = \mathcal{C}, \text{ a.e. in } \Omega. \tag{2.4.18}$$

Proof. We first note that Lemma 2.4.1 implies that $\rho_k > 0$ a.e. in Ω (see also Lemma A.4.5). From Theorem 2.2.3, we have

$$p_k = \begin{cases} \mathcal{S}'(1-), & \text{in } \rho_k^{-1}((0,1)), \\ \mathcal{C} - \frac{\overline{\phi}_k}{\tau} - \Phi, & \text{in } \rho_k^{-1}(\{1\}), \\ \mathcal{S}'(1+), & \text{in } \rho_k^{-1}((1,+\infty)). \end{cases} \tag{2.4.19}$$

As $S_b'(1) = 0$, (2.4.18) holds in $\rho_k^{-1}(\{1\})$ by (2.4.19).

Lastly, from (2.4.19), in $\rho_k^{-1}(\mathbb{R}^+ \setminus \{1\})$ we have that

$$S_a'(\rho_k) = p_k(1 + \log \rho_k). \tag{2.4.20}$$

As $\mathcal{S}' = S_a' + S_b'$ in $\rho_k^{-1}(\mathbb{R}^+ \setminus \{1\})$, we conclude (2.4.18) from Proposition 2.2.9. □

Remark 2.4.6. As S_b is differentiable, in the previous proof we also used the fact

$$\partial \mathcal{S} = \partial S_a + S_b', \tag{2.4.21}$$

the proof of which can be found for instance in [Kru03, Corollary 1.12.2].

Similarly as in Section 2.3, we construct piecewise constant and continuous in time interpolations $(\rho^\tau, \mathbf{v}^\tau, \mathcal{E}^\tau)$ and $(\tilde{\rho}^\tau, \tilde{\mathbf{v}}^\tau, \widetilde{\mathbf{E}}^\tau)$. Similarly to Proposition 2.3.7, we can formulate the following result.

Proposition 2.4.7. *$(\rho^\tau)_{\tau>0}$ and $(p^\tau)_{\tau>0}$ satisfy the exact same bounds as in Proposition 2.3.7.*

Proof. Let us notice first that the uniform boundedness of $(p^\tau)_{\tau>0}$ in $L^\infty(Q)$ follows from the construction in (2.4.19).

Let us show the other estimates from Proposition 2.3.7. Note that both S_b' and ρ_k are locally Lipschitz continuous (as we have shown in Lemma 2.4.4 and Lemma 2.4.3). Thus, Lemma 2.4.5 implies that

$$-\frac{\nabla \overline{\phi}_k}{\tau} - \nabla \Phi = \nabla p_k + \left(\frac{p_k}{\rho_k} + S_b''(\rho_k) \right) \nabla \rho_k, \quad \text{a.e. in } \Omega. \tag{2.4.22}$$

By the parallel computation as in (2.3.17), we conclude that

$$2\rho_k \frac{|\nabla \overline{\phi}_k|^2}{\tau^2} + 2\rho_k |\nabla \Phi|^2 \geq |\nabla p_k|^2 + \rho_k \left(\frac{p_k}{\rho_k} + S_b''(\rho_k) \right)^2 |\nabla \rho_k|^2 \tag{2.4.23}$$

From Lemma 2.4.10 below, we have

$$\rho_k \left(\frac{p_k}{\rho_k} + S_b''(\rho_k) \right)^2 |\nabla \rho_k|^2 \geq \frac{1}{\sigma_2^2 \rho_k} |\nabla \rho_k|^2 \text{ a.e. in } \Omega. \tag{2.4.24}$$

The rest of arguments is parallel to Step 3 in Proposition 2.3.7, thus we conclude the thesis of the proposition. □

Corollary 2.4.8. *Up to passing to subsequences, the sequences $(\rho^\tau)_{\tau>0}$ and $(p^\tau)_{\tau>0}$ converge in the same sense as in Corollary 2.3.8.*

Remark 2.4.9. From (2.4.22), we have

$$2\rho_k \frac{|\nabla \overline{\phi}_k|^2}{\tau^2} + 2\rho_k |\nabla \Phi|^2 \geq \frac{|\nabla (F(\rho_k, p_k))|^2}{\rho_k}, \text{ where } F(\rho, p) := p\rho + \rho S_b'(\rho) - S_b(\rho). \tag{2.4.25}$$

Then, if $\rho_0 \in L^\infty(\Omega)$, this observation together with the uniform L^∞ bounds on ρ^τ imply uniform $L^2([0,T]; H^1(\Omega))$ bounds on $F(\rho^\tau, p^\tau)$.

As the proof of Proposition 2.3.7, we rely on the coarea formula when proving the following result.

Lemma 2.4.10. *For $(\rho_k)_{k=1}^N$ and $(p_k)_{k=1}^N$ given in* (2.2.3) *and* (2.3.4)*, it holds that*

$$|p_k + \rho_k S_b''(\rho_k)||\nabla \rho_k| \geq \frac{1}{\sigma_2} |\nabla \rho_k| \textit{ a.e. in } \Omega. \tag{2.4.26}$$

Proof. If $x \in \{\rho_k \neq 1\}$, then (2.4.20) implies that

$$\frac{p_k(x)}{\rho_k(x)} + S_b''(\rho_k(x)) = S_a''(\rho_k(x)) + S_b''(\rho_k(x)) = \mathcal{S}''(\rho_k(x)). \tag{2.4.27}$$

From (2.4.1), we conclude

$$|p_k + \rho_k S_b''(\rho_k)| \geq \frac{1}{\sigma_2} \text{ a.e. in } \{\rho_k \neq 1\}. \tag{2.4.28}$$

Recall that as ρ_k is Lipschitz continuous from Lemma 2.5.6 and thus

$$\nabla \rho_k = 0 \text{ a.e. in } \{\rho_k = 1\}$$

(see for instance [EG92, Theorem 4.(iv), Section 4.2.2]). Therefore, we conclude (2.4.26). □

Proof of Theorem 2.4.2. As and initial observation, let us remark that by similar arguments as in Lemma 2.3.11, one obtains the same estimates for the continuous in time interpolations $(\tilde{\rho}^\tau, \tilde{\mathbf{v}}^\tau, \widetilde{\mathbf{E}}^\tau)$, and by passing to the limit as $\tau \downarrow 0$, we obtain a continuity equation of the form

$$\partial_t \rho + \nabla \cdot \mathcal{E} = 0.$$

Since the limits of $(\tilde{\rho}^\tau, \widetilde{\mathbf{E}}^\tau)$ and $(\rho^\tau, \mathcal{E}^\tau)$ are the same, it remains to identify the limit of the latter one to get the precise form of our limit equation.

1. From direct computation as in (2.3.27), we obtain that

$$-\mathcal{E}^\tau = \rho^\tau \nabla(S_b'(\rho^\tau) + p^\tau(1 + \log \rho^\tau)) + \rho^\tau \nabla \Phi = \nabla(\rho^\tau S_b'(\rho^\tau) - S_b(\rho^\tau) + S_b(1) + p^\tau \rho^\tau) + \rho^\tau \nabla \Phi \tag{2.4.29}$$

From Proposition 2.4.7 and Corollary 2.4.8 we can claim that

$$\nabla(\rho^\tau S_b'(\rho^\tau) - S_b(\rho^\tau) + S_b(1) + p^\tau \rho^\tau) \to \nabla(\rho S_b'(\rho) - S_b(\rho) + S_b(1) + p\rho), \tag{2.4.30}$$

as $\tau \downarrow 0$ in the sense of distribution. Indeed, using the strong $L^1(Q)$ compactness of $(\rho^\tau)_{\tau>0}$ and the weak-$\star$ compactness of $(p^\tau)_{\tau>0}$ in $L^\infty(Q)$, we can pass to the limit $\rho^\tau p^\tau$. Recall that $(\rho^\tau)_{\tau>0}$ in uniformly bounded in $L^\beta(Q)$ for β given in (2.2.29). As $r < \beta$, Corollary 2.4.8 yields the convergence of $(\rho^\tau)_{\tau>0}$ in $L^r(Q)$. As the growth rate of $\rho S_b'(\rho)$ and $S_b(\rho)$ is r, we conclude that $\rho^\tau S_b'(\rho^\tau) - S_b(\rho^\tau) \to \rho S_b'(\rho) - S_b(\rho)$ in $L^1(Q)$ as $\tau \downarrow 0$.

2. Let us show that

$$\rho S_b'(\rho) - S_b(\rho) + S_b(1) + p\rho = L_S(\rho, p), \tag{2.4.31}$$

By parallel arguments as in Lemma 2.3.9, we conclude that (ρ, p) satisfies (2.4.6). Thus, it holds that

$$\rho S_a'(\rho) - S_a(\rho) = p\rho, \text{ a.e. in } \rho^{-1}(\mathbb{R}^+ \setminus \{1\}) \tag{2.4.32}$$

and we conclude (2.4.31) a.e. in $\rho^{-1}(\mathbb{R}^+ \setminus \{1\})$. From (2.4.30) and (2.4.31), we conclude (2.4.5).

Furthermore, from Lemma 2.4.4, we obtain that

$$\rho S_b'(\rho) - S_b(\rho) + S_b(1) + p\rho = p \text{ in } \rho^{-1}(\{1\}). \tag{2.4.33}$$

and we conclude (2.4.31) a.e. in $\rho^{-1}(\{1\})$. □

2.5 Porous medium type diffusion on $\{\rho < 1\}$ and general diffusion on $\{\rho > 1\}$

Similarly to the classical porous medium equation, in this section we do not expect solutions to be fully supported. As in Section 2.3, let us first study an example with a particular nonlinearity.

2.5.1 Same diffusion exponent

In this subsection, we suppose that $\mathcal{S} : [0, +\infty) \to \mathbb{R}$ is defined by

$$\mathcal{S}(\rho) := \begin{cases} \dfrac{\rho^m}{m-1}, & \text{for } \rho \in [0,1], \\ \dfrac{2\rho^m}{m-1} - \dfrac{1}{m-1}, & \text{for } \rho \in (1, +\infty). \end{cases} \tag{2.5.1}$$

where $m > 1$.

Our main theorem in this section can be formulated as follows.

Theorem 2.5.1. *For $\rho_0 \in \mathscr{P}(\Omega)$ such that $\mathcal{J}(\rho_0) < +\infty$ and S given in (2.5.1), there exists $\rho \in L^\beta(Q) \cap AC^2([0,T]; (\mathscr{P}(\Omega), W_2))$ and $p \in L^2([0,T]; H^1(\Omega)) \cap L^\infty(Q)$ with $\rho^{m-\frac{1}{2}} \in L^2([0,T]; H^1(\Omega))$ such that (ρ, p) is a weak solution of*

$$\begin{cases} \partial_t \rho - \Delta([(m-1)\rho^m + 1]\frac{p}{m}) - \nabla \cdot (\nabla\Phi\rho) = 0, & \text{in } (0,T) \times \Omega, \\ \rho(0,\cdot) = \rho_0, & \text{in } \Omega, \\ (\nabla([(m-1)\rho^m + 1]\frac{p}{m}) + \nabla\Phi\rho) \cdot \vec{n} = 0, & \text{in } [0,T] \times \partial\Omega, \end{cases} \tag{2.5.2}$$

in the sense of distribution. Furthermore, (ρ, p) satisfies

$$\begin{cases} p(t,x) = \frac{m}{m-1} & a.e. \text{ in } \{0 < \rho(t,x) < 1\}, \\ p(t,x) \in \left[\frac{m}{m-1}, \frac{2m}{m-1}\right] & a.e. \text{ in } \{\rho(t,x) = 1\}, \\ p(t,x) = \frac{2m}{m-1} & a.e. \text{ in } \{\rho(t,x) > 1\}. \end{cases} \tag{2.5.3}$$

In addition, if $\rho_0 \in L^\infty(\Omega)$ and Φ satisfies (2.2.4), then $\rho \in L^\infty(Q)$ and $\rho^m \in L^2([0,T]; H^1(\Omega))$.

Let us recall the definition of $(\rho_k)_{k=1}^N$ and $(p_k)_{k=1}^N$ from (2.2.3) and (2.3.4), respectively. Let us underline that in the setting of this section due to the structure of the nonlinearity we typically expect $\mathrm{spt}(\rho_k)$ to be a proper subset of Ω, unlike in the case of Lemma A.4.5 which was used in Section 2.3 and Section 2.4. For this reason, we expect the Lipschitz continuity of ρ_k^{m-1} instead of ρ_k.

Lemma 2.5.2. *For all $k \in \{1, \dots, N\}$, there exists $C \in \mathbb{R}$ such that*

$$\rho_k^{m-1} p_k = \left(\mathcal{C} - \frac{\overline{\phi}_k}{\tau} - \Phi \right)_+ \quad a.e. \tag{2.5.4}$$

In particular, p_k and ρ_k^{m-1} are Lipschitz continuous. Here, $\overline{\phi}_k$ is given in Theorem 2.2.3.

Proof. Note that

$$\partial \mathcal{S}(\rho) = \begin{cases} \frac{m}{m-1}\rho^{m-1} & \text{for } 0 < \rho < 1, \\ \left[\frac{m}{m-1}, \frac{2m}{m-1}\right] & \text{for } \rho = 1, \\ \frac{2m}{m-1}\rho^{m-1} & \text{for } \rho > 1. \end{cases} \quad \text{and} \quad p_k = \begin{cases} \frac{m}{m-1} & \text{in } \rho_k^{-1}([0,1)), \\ \mathcal{C} - \frac{\overline{\phi}_k}{\tau} - \Phi & \text{in } \rho_k^{-1}(\{1\}), \\ \frac{2m}{m-1} & \text{in } \rho_k^{-1}((1,+\infty)). \end{cases} \quad \text{a.e.} \tag{2.5.5}$$

for p_k given in (2.3.4). Then, Theorem 2.2.14 implies that

$$\rho_k^{m-1} p_k + \frac{\overline{\phi}_k}{\tau} + \Phi = \mathcal{C} \text{ a.e. on } \mathrm{spt}(\rho_k) \tag{2.5.6}$$

for some constant $C \in \mathbb{R}$.

Moreover, if $\rho_k = 0$ a.e. on some set $A \subset \Omega$, then Theorem 2.2.14 and $\mathcal{S}'(0+) = 0$ from (2.5.5) imply that

$$\mathcal{C} - \frac{\overline{\phi}_k}{\tau} - \Phi \leq 0 \text{ a.e. in } A, \tag{2.5.7}$$

and we conclude (2.5.4).

Next, recall that $\overline{\phi}_k$ is Lipschitz continuous (as shown in Lemma 2.3.5). From this and (2.5.5), we conclude that p_k and $\left(\mathcal{C} - \frac{\overline{\phi}_k}{\tau} - \Phi \right)_+$ are Lipschitz continuous as well. As p_k is Lipschitz continuous and has a positive lower bound $\frac{m}{m-1}$ (from (2.5.5) and (2.2.5)), we conclude that ρ_k^{m-1} is also Lipschitz continuous. □

Lemma 2.5.3. *Let $(\rho^\tau)_{\tau>0}, (p^\tau)_{\tau>0}$ stand for the piecewise constant interpolations given in* (2.3.9) *and* (2.3.4), *respectively. Then $((\rho^\tau)^{m-\frac{1}{2}})_{\tau>0}$ and $(p^\tau)_{\tau>0}$ are uniformly bounded in $L^2([0,T];H^1(\Omega))$.*

Proof. From Lemma 2.5.2, it holds that

$$\mathcal{I}_1 := -\rho_k^{\frac{1}{2}}\nabla\Phi - \rho_k^{\frac{1}{2}}\frac{\nabla\overline{\phi}_k}{\tau} = \rho_k^{\frac{1}{2}}\nabla(\rho_k^{m-1}p_k) \ \text{ a.e.} \tag{2.5.8}$$

As p_k and ρ_k^{m-1} are Lipschitz continuous from Lemma 2.5.2, we have

$$\mathcal{I}_1 = \rho_k^{\frac{1}{2}}p_k\nabla(\rho_k^{m-1}) + \rho_k^{m-\frac{1}{2}}\nabla p_k \ \text{ a.e. on } \operatorname{spt}(\rho_k). \tag{2.5.9}$$

Furthermore, since we have the Lipschitz continuity of ρ_k^{m-1} and (2.5.5), we apply the parallel argument in the proof of Proposition 2.3.7 and conclude that

$$(\rho_k^{m-\frac{1}{2}} - 1)\nabla p_k = 0 \text{ and } \nabla(\rho_k^{m-1})\cdot\nabla p_k = 0 \ \text{ a.e. on } \Omega \tag{2.5.10}$$

From (2.5.9) and (2.5.10), we have that

$$\mathcal{I}_1^2 = p_k^2|\rho_k^{\frac{1}{2}}\nabla(\rho_k^{m-1})|^2 + |\nabla p_k|^2 \ \text{ a.e. on } \operatorname{spt}(\rho_k). \tag{2.5.11}$$

As $p_k \geq \frac{m}{m-1}$ a.e. in Ω as in (2.5.5), we conclude that

$$\mathcal{I}_1^2 \geq \left(\frac{m}{m-1}\right)^2 |\rho_k^{\frac{1}{2}}\nabla(\rho_k^{m-1})|^2 + |\nabla p_k|^2 \ \text{ a.e. on } \operatorname{spt}(\rho_k). \tag{2.5.12}$$

From (2.5.10) it holds also that $\nabla p_k = 0$ a.e. on $\operatorname{spt}(\rho_k)^c = \{\rho_k = 0\}$. Furthermore, as ρ_k^{m-1} is Lipschitz continuous (see Lemma 2.5.2), we have

$$\rho_k^{\frac{1}{2}}\nabla(\rho_k^{m-1}) = 0 \ \text{ a.e. on } \operatorname{spt}(\rho_k)^c. \tag{2.5.13}$$

Therefore, (2.5.12) holds a.e. on Ω.

On the other hand, applying Lemma 2.3.6, it holds that

$$\int_0^T\int_\Omega \mathcal{I}_1^2 \mathrm{d}x\mathrm{d}t \leq 2\left(\mathcal{J}(\rho_0) - \inf\mathcal{J}\right) + T\mathscr{L}^d(\Omega)\|\nabla\Phi\|_{L^\infty} \tag{2.5.14}$$

As $\rho_k^{\frac{1}{2}}\nabla(\rho_k^{m-1}) = \frac{m-1}{m-\frac{1}{2}}\nabla(\rho_k^{m-\frac{1}{2}})$ and $(\rho^\tau)_{\tau>0}$ is uniformly bounded in $L^\beta(Q)$ (with $\beta > m - 1/2$, see Lemma 2.2.11) we conclude that $((\rho^\tau)^{m-\frac{1}{2}})_{\tau>0}$ and $(p^\tau)_{\tau>0}$ are uniformly bounded in $L^2([0,T];H^1(\Omega))$ (since $(p^\tau)_{\tau>0}$ is also uniformly bounded) and therefore we conclude. □

As a consequence of Lemma 2.5.3 and Lemma A.6.2, we have the following convergence.

Corollary 2.5.4. *Let $(\rho^\tau)_{\tau>0}$ and $(p^\tau)_{\tau>0}$ be as in the previous lemma. Then, there exists $\rho \in L^m(Q)$ and $p \in L^2([0,T];H^1(\Omega))$ with $\rho^{m-\frac{1}{2}} \in L^2([0,T];H^1(\Omega))$, such that*

$$\rho^\tau \to \rho \text{ in } L^m(Q), \text{ as } \tau \downarrow 0, \tag{2.5.15}$$

and

$$p^\tau \rightharpoonup p \text{ in } L^2([0,T];H^1(\Omega)), \text{ as } \tau \downarrow 0. \tag{2.5.16}$$

along a subsequence.

Proof of Theorem 2.5.1. Note that (2.5.5) implies (2.5.3) for (ρ^τ, p^τ). Then, a similar argument as the one in Lemma 2.3.9 together with the convergence results from Corollary 2.5.4 reveals that (ρ, p) satisfies (2.5.3).

Furthermore, from Lemma 2.5.2, we can write that

$$\mathcal{E}^\tau = \rho^\tau \mathbf{v}^\tau = -\rho^\tau \nabla((\rho^\tau)^{m-1}p^\tau) - \nabla\Phi\rho^\tau = -\left\{(m-1)p^\tau(\rho^\tau)^{m-1}\nabla\rho^\tau + (\rho^\tau)^m \nabla p^\tau\right\} - \nabla\Phi\rho^\tau \tag{2.5.17}$$

Note that (2.5.5) implies

$$((\rho^\tau)^m - 1)\nabla p^\tau = 0 \text{ a.e.} \tag{2.5.18}$$

From (2.5.18), we conclude that

$$(m-1)p^\tau(\rho^\tau)^{m-1}\nabla\rho^\tau + (\rho^\tau)^m\nabla p^\tau = (m-1)p^\tau(\rho^\tau)^{m-1}\nabla\rho^\tau + \frac{1}{m}\left\{(m-1)(\rho^\tau)^m + 1\right\}\nabla p^\tau, \tag{2.5.19}$$

$$= \frac{1}{m}\nabla\left([(m-1)(\rho^\tau)^m + 1]p^\tau\right) \tag{2.5.20}$$

As described in Proposition 2.3.10, up to passing to a subsequence and using the weak-$\star$ convergence of $(p^\tau)_{\tau>0}$ in $L^\infty(Q)$ and strong convergence of $((\rho^\tau)^m)_{\tau>0}$ in $L^1(Q)$ from Corollary 2.5.4, we conclude that $(\mathcal{E}^\tau)_{\tau>0}$ converges to

$$\mathcal{E} := -\frac{1}{m}\nabla\left([(m-1)\rho^m + 1]p\right) - \nabla\Phi\rho \tag{2.5.21}$$

in $\mathscr{D}'(Q;\mathbb{R}^d)$, as $\tau \downarrow 0$ where (ρ, p) is given in Corollary 2.5.4. The rest of argument is parallel to the proof of Theorem 2.3.1.

A last remark is that if $\rho_0 \in L^\infty(\Omega)$, then clearly $\rho \in L^\infty(Q)$ and thus $\rho^m \in L^2([0,T];H^1(\Omega))$.

□

2.5.2 General cases

In this subsection, we suppose that Assumption A and (2.1.7) hold for some $r \geq 1$ and $\mathcal{S} : [0,+\infty) \to \mathbb{R}$ satisfies (2.1.8) for some $m > 1$ and a constant $\sigma_2 > 0$,

$$\frac{\rho^{m-2}}{\sigma_2} < \mathcal{S}''(\rho) \text{ for all } \rho \in (0,1). \tag{2.5.22}$$

Note that $\mathcal{S}$ can be any function satisfying the assumptions, and in particular in the case of $r = 1$, S behaves as the logarithmic entropy when $\rho > 1$.

Our main theorem from this section reads as:

Theorem 2.5.5. *Suppose that* (2.1.8), (2.5.22) *and*

$$m < r + \frac{\beta}{2} \tag{2.5.23}$$

hold true for β given in (2.2.29). *For $\rho_0 \in \mathscr{P}(\Omega)$ such that $\mathcal{J}(\rho_0) < +\infty$, there exists $\rho \in L^\beta(Q)$ and $p \in L^2([0,T];H^1(\Omega)) \cap L^\infty(Q)$ such that (ρ, p) is a weak solution of*

$$\begin{cases} \partial_t \rho - \Delta(L_S(\rho,p)) - \nabla \cdot (\nabla \Phi \rho) = 0, & \text{in } (0,T) \times \Omega, \\ \rho(0,\cdot) = \rho_0, & \text{in } \Omega, \\ (\nabla(L_S(\rho,p)) + \nabla\Phi\rho) \cdot \vec{n} = 0, & \text{in } [0,T] \times \partial\Omega, \end{cases} \tag{2.5.24}$$

in the sense of distribution. Furthermore, (ρ, p) satisfies

$$\begin{cases} p(t,x) = \mathcal{S}'(1-) & \text{if } 0 \leq \rho(t,x) < 1, \\ p(t,x) \in [\mathcal{S}'(1-), \mathcal{S}'(1+)] & \text{if } \rho(t,x) = 1, \\ p(t,x) = \mathcal{S}'(1+) & \text{if } \rho(t,x) > 1. \end{cases} \tag{2.5.25}$$

Here, L_S is given in (2.1.5)*. In particular,*

$$\rho^{m-\frac{1}{2}} \in L^2([0,T];H^1(\Omega)) \text{ if } m \le r \text{ and } \rho^{m-\frac{1}{2}} \in L^q([0,T];W^{1,q}(\Omega)) \text{ if } r < m < r + \frac{\beta}{2} \tag{2.5.26}$$

for $q \in (1,2)$ given in (2.5.55)*. If in addition $\rho_0 \in L^\infty(\Omega)$ and Φ satisfies* (2.2.4)*, we can drop* (2.1.8) *and* (2.5.23) *from the statement and we obtain $\rho \in L^\infty(Q)$ and $\rho^m \in L^2([0,T];H^1(\Omega))$.*

First, using similar ideas as in Section 2.4, we choose a constant l such that

$$1 < l < \beta \tag{2.5.27}$$

for β given in (2.2.29) and split the function S into S_a and $S_b : [0,+\infty) \to \mathbb{R}$ defined by

$$S_a(\rho) := \begin{cases} \frac{\mathcal{S}'(1-)(\rho^l-1)}{l}, & \text{for } \rho \le 1, \\ \frac{\mathcal{S}'(1+)(\rho^l-1)}{l}, & \text{for } \rho > 1, \end{cases} \tag{2.5.28}$$

and

$$S_b(\rho) := \mathcal{S}(\rho) - S_a(\rho). \tag{2.5.29}$$

Note that $\mathcal{S}'(1+) > \mathcal{S}'(1-)$. Then, as shown in Lemma 2.4.4, we conclude that S_a is convex and continuous in $[0,+\infty)$. Also, S_b is continuously differentiable and S_b' is locally Lipschitz continuous in $[0,+\infty)$.

Let us recall the definition of $(\rho_k)_{k=1}^N$ and $(p_k)_{k=1}^N$ from (2.2.3) and (2.3.4). Also, recall the definition of $\overline{\phi}_k$ given in Theorem 2.2.3.

Lemma 2.5.6. *For all $k \in \{1,\dots,N\}$, there exists $C \in \mathbb{R}$ such that*

$$\rho_k^{l-1}p_k + S_b'(\rho_k) = \left(\mathcal{C} - \frac{\overline{\phi}_k}{\tau} - \Phi\right)_+ \quad a.e. \tag{2.5.30}$$

In particular, p_k and ρ_k^{m-1} are Lipschitz continuous in Ω. Also, ρ_k is locally Lipschitz continuous in spt(ρ_k).

Proof. First we notice that $\overline{\phi}_k$ is Lipschitz continuous (cf. Lemma 2.3.5), p_k and $f_k := \mathcal{C} - \frac{\overline{\phi}_k}{\tau} - \Phi$ are Lipschitz continuous. Furthermore, (2.5.30) follows from the parallel argument in the proof of Lemma 2.5.2.

Next, let us show that ρ_k^{m-1} is Lipschitz continuous. From (2.5.30), it holds that for

$$\rho_k = \begin{cases} 1 & \text{in } f_k^{-1}[S'(1-), S'(1+)], \\ (S')^{-1}(f_k) & \text{otherwise.} \end{cases} \quad \text{a.e.} \tag{2.5.31}$$

As f_k are continuous, ρ_k is continuous on each regions,

$$f_k^{-1}[S'(1-), S'(1+)], f_k^{-1}(-\infty, S'(1-)) \text{ and } f_k^{-1}(S'(1+), +\infty). \tag{2.5.32}$$

Let us show that ρ_k is continuous on the boundary between two regions. By the continuity of f_k, it holds that

$$f_k = \begin{cases} S'(1-) & \text{in } \partial f_k^{-1}(-\infty, S'(1-)) \cap \partial f_k^{-1}[S'(1-), S'(1+)], \\ S'(1+) & \text{in } \partial f_k^{-1}(S'(1+), +\infty) \cap \partial f_k^{-1}[S'(1-), S'(1+)]. \end{cases} \tag{2.5.33}$$

As $(S')^{-1}(S'(1-)) = (S')^{-1}(S'(1+)) = 1$, (2.5.31) and (2.5.33) show the continuity of ρ_k on the boundary between two regions in (2.5.32). Thus, we conclude that ρ_k is continuous in Ω

Furthermore, as S is strictly convex, S' is invertible in $(1, +\infty)$. From (2.4.14), it holds that

$$|\nabla(\rho_k)^{m-1}| = (m-1)\rho_k^{m-2}|\nabla\rho_k| = (m-1)\rho_k^{m-2}\frac{|\nabla f_k|}{S''(\rho_k)} \text{ a.e. in spt}(\rho_k). \tag{2.5.34}$$

From (2.5.22), we have

$$|\nabla(\rho_k)^{m-1}| \leq \sigma_2(m-1)|\nabla f_k| \text{ a.e. in } \{x \in \Omega : 0 < \rho_k < 1\} \tag{2.5.35}$$

and from the assumption (2.1.8a)

$$|\nabla(\rho_k)^{m-1}| \leq \sigma_1(m-1)|\nabla f_k|\rho_k^{m-r} \leq \sigma_1(m-1)|\nabla f_k| \max\{\|\rho_k\|_{L^\infty(\Omega)}^{m-r}, 1\} \tag{2.5.36}$$

a.e. in $\{x \in \Omega : \rho_k > 1\}$. Therefore, we conclude that ρ_k^{m-1} is Lipschitz continuous in Ω.

Lastly, the following identity

$$|\nabla\rho_k| = \frac{1}{(m-1)\rho_k^{m-2}}|\nabla(\rho_k)^{m-1}| \text{ a.e. in spt}(\rho_k) \tag{2.5.37}$$

shows that ρ_k is locally Lipschitz continuous in $\text{spt}(\rho_k)$. □

Proposition 2.5.7. *Let $(\rho^\tau)_{\tau>0}, (p^\tau)_{\tau>0}$ stand for the piecewise constant interpolations given in (2.3.9) and (2.3.4), respectively. Then, $(p^\tau)_{\tau>0}$ is uniformly bounded in $L^2([0,T];H^1(\Omega))$.*

(1) If $r \geq m$, then $((\rho^\tau)^{m-\frac{1}{2}})_{\tau>0}$ is uniformly bounded in $L^2([0,T];H^1(\Omega))$.

(2) If $r < m < r + \frac{\beta}{2}$, then $((\rho^\tau)^{m-\frac{1}{2}})_{\tau>0}$ is uniformly bounded in $L^q([0,T];W^{1,q}(\Omega))$ for some $q \in (1,2)$.

(3) If in addition $\rho_0 \in L^\infty(\Omega)$ and Φ satisfies (2.2.4), then $((\rho^\tau)^m)_{\tau>0}$ is also uniformly bounded in $L^2([0,T];H^1(\Omega))$ for any $m > 1$ and $r \geq 1$.

Proof. From Lemma 2.5.6, it holds that

$$\mathcal{I}_1 := -\rho_k^{\frac{1}{2}}\frac{\nabla\overline{\phi}_k}{\tau} - \rho_k^{\frac{1}{2}}\nabla\Phi = \rho_k^{\frac{1}{2}}\nabla(\rho_k^{l-1}p_k + S_b'(\rho_k)) \text{ a.e.} \tag{2.5.38}$$

We follow the very same steps and in the proof of Lemma 2.5.2 (where we also use (2.5.9) and (2.5.10)). Therefore, we have

$$\mathcal{I}_1 = \frac{l-1}{m-1}\rho_k^{l-m+\frac{1}{2}}p_k\nabla(\rho_k^{m-1}) + \rho_k^{l-\frac{1}{2}}\nabla p_k + \rho_k^{\frac{1}{2}}\nabla(S_b'(\rho_k)) \text{ a.e. on spt}(\rho_k). \tag{2.5.39}$$

Note that

$$\rho_k^{\frac{1}{2}}\nabla(S_b'(\rho_k)) = \frac{1}{m-1}\rho_k^{\frac{5}{2}-m}S_b''(\rho_k)\nabla(\rho_k^{m-1}) \text{ a.e. on spt}(\rho_k) \tag{2.5.40}$$

From (2.5.39) and (2.5.40), it holds that

$$\mathcal{I}_1 = \frac{1}{(m-1)\rho_k^{m-2}}\left((l-1)\rho_k^{l-2}p_k + S_b''(\rho_k)\right)\rho_k^{\frac{1}{2}}\nabla(\rho_k^{m-1}) + \rho_k^{l-\frac{1}{2}}\nabla p_k \text{ a.e. on spt}(\rho_k). \tag{2.5.41}$$

We can apply (2.5.10) and conclude (since $\nabla p_k = 0$ a.e. in $\{\rho_k \neq 1\}$) that

$$\mathcal{I}_1^2 = \frac{1}{(m-1)^2\rho_k^{2m-4}}\left((l-1)\rho_k^{l-2}p_k + S_b''(\rho_k)\right)^2\rho_k|\nabla(\rho_k^{m-1})|^2 + |\nabla p_k|^2 \text{ a.e. on spt}(\rho_k) \tag{2.5.42}$$

(1) If $r \geq m$, then Lemma 2.5.8 below implies

$$\mathcal{I}_1^2 \geq \frac{\sigma_3^2}{(m-1)^2}|\nabla(\rho_k^{m-\frac{1}{2}})|^2 + |\nabla p_k|^2 \text{ a.e. on spt}(\rho_k). \tag{2.5.43}$$

for σ_3 given in (2.5.47). By the parallel argument in Lemma 2.5.3, we conclude the uniform bound in $L^2([0,T];H^1(\Omega))$.

(2) If $r < m < r+\frac{\beta}{2}$, then Lemma 2.5.9 below yields the uniform bound of $(\nabla(\rho^\tau)^{m-\frac{1}{2}})_{\tau>0}$ in $L^q(Q)$ for q given in (2.5.55). On the other hand, as $2r-1 \leq \beta$, it holds that

$$\left(m-\frac{1}{2}\right)q = \frac{m-\frac{1}{2}}{\frac{m-r}{\beta}+\frac{1}{2}} = \beta\frac{2m-1}{2m-2r+\beta} \leq \beta, \tag{2.5.44}$$

As ρ^τ is uniformly bounded in $L^\beta(Q)$ from Lemma 2.2.13, $(\rho^\tau)^{m-\frac{1}{2}}$ is uniformly bounded in $L^q(Q)$.

(3) From Lemma 2.5.10, we conclude that

$$\mathcal{I}_1^2 \geq \frac{\sigma_4^2}{(m-1)^2}|\nabla(\rho_k^m)|^2 + |\nabla p_k|^2 \quad \text{a.e. on } \operatorname{spt}(\rho_k). \tag{2.5.45}$$

The same argument as before yields that $((\rho^\tau)^m)_{\tau>0}$ is uniformly bounded in $L^2([0,T];H^1(\Omega))$.

□

Lemma 2.5.8. *Let us suppose that we are in the setting of Proposition 2.5.7. If $r \geq m$, it holds that*

$$\left|\frac{1}{\rho_k^{m-2}}\left((l-1)\rho_k^{l-2}p_k + S_b''(\rho_k)\right)\right|\rho_k^{\frac{1}{2}}|\nabla(\rho_k^{m-1})| \geq \sigma_3|\nabla(\rho_k^{m-1/2})|, \tag{2.5.46}$$

where

$$\sigma_3 := \frac{m-1}{m-\frac{1}{2}}\min\left\{\frac{1}{\sigma_1},\frac{1}{\sigma_2}\right\}. \tag{2.5.47}$$

Proof. We claim that

$$\left|\frac{1}{\rho_k^{m-2}}\left((l-1)\rho_k^{l-2}p_k + S_b''(\rho_k)\right)\right| \geq \min\left\{\frac{1}{\sigma_1},\frac{1}{\sigma_2}\right\} \text{ in } \{\rho_k \neq 1\} \tag{2.5.48}$$

Recall that

$$S_a''(\rho_k) = \begin{cases}(l-1)S'(1-)\rho_k^{l-2} \text{ if } \rho_k < 1,\\ (l-1)S'(1+)\rho_k^{l-2} \text{ if } \rho_k > 1,\end{cases} \tag{2.5.49}$$

and thus by the definition of p_k (see (2.3.4)) we have

$$(l-1)\rho_k^{l-2}p_k + S_b''(\rho_k) = S_a''(\rho_k) + S_b''(\rho_k) = S''(\rho_k) \quad \text{a.e. in } \{\rho_k \neq 1\} \tag{2.5.50}$$

Thus, (2.5.22) implies that

$$\frac{S''(\rho_k)}{\rho_k^{m-2}} \geq \frac{1}{\sigma_2} \;\text{ a.e. in } \{0 < \rho_k < 1\}. \tag{2.5.51}$$

Furthermore, as $r \geq m$, (2.1.8a) implies

$$\frac{S''(\rho_k)}{\rho_k^{m-2}} \geq \frac{\rho_k^{r-m}}{\sigma_1} \geq \frac{1}{\sigma_1} \;\text{ a.e. in } \{\rho_k > 1\}. \tag{2.5.52}$$

and we conclude (2.5.48).

Recall that ρ_k^{m-1} is Lipschitz continuous from Lemma 2.5.6. Thus, we have

$$\nabla(\rho_k^{m-1}) = 0 \;\text{ a.e. in } \{\rho_k = 1\} \tag{2.5.53}$$

(see for instance [EG92, Theorem 4(iv), Section 4.2.2]). As $\rho_k^{\frac{1}{2}}\nabla(\rho_k^{m-1}) = \frac{m-1}{m-\frac{1}{2}}\nabla(\rho_k^{m-\frac{1}{2}})$, (2.5.46) follows from (2.5.48) and (2.5.53). □

Lemma 2.5.9. *Let us suppose that we are in the setting of Proposition 2.5.7. If* $r < m < r + \frac{\beta}{2}$, *then*

$$\left\| \frac{1}{\rho_k^{m-2}} \left((l-1)\rho_k^{l-2}p_k + S_b''(\rho_k)\right) |\nabla(\rho_k^{m-\frac{1}{2}})| \right\|_{L^2(\Omega)} \geq C\|\nabla(\rho_k^{m-\frac{1}{2}})\|_{L^q(\Omega)} \tag{2.5.54}$$

for some $q \in (1,2)$ *and a constant* $C > 0$.

Proof. From the relation between r and m, the constant q defined by

$$q := \frac{1}{\frac{m-r}{\beta} + \frac{1}{2}} \tag{2.5.55}$$

is in the interval $(1,2)$. As shown in (2.5.50), it holds that

$$\mathcal{I}_2 := \frac{1}{\rho_k^{m-2}} \left((l-1)\rho_k^{l-2}p_k + S_b''(\rho_k)\right) |\nabla(\rho_k^{m-\frac{1}{2}})| = \frac{S''(\rho_k)}{\rho_k^{m-2}}|\nabla(\rho_k^{m-\frac{1}{2}})| \;\text{ a.e. in } \{\rho_k \neq 1\} \tag{2.5.56}$$

In $\{0 < \rho_k < 1\}$, (2.5.51) implies that

$$\|\mathcal{I}_2\|_{L^2(\{0<\rho_k<1\})} \geq \frac{1}{\sigma_2} \left\|\nabla(\rho_k^{m-\frac{1}{2}})\right\|_{L^2(\{0<\rho_k<1\})} \tag{2.5.57}$$

for σ_2 given in (2.5.22). As $q \in (1, 2)$ and the domain is compact, the Hölder inequality yields that

$$\|\mathcal{I}_2\|_{L^2(\{0<\rho_k<1\})} \geq \frac{|\Omega|^{\frac{1}{2}-\frac{1}{q}}}{\sigma_2} \|\nabla(\rho_k^{m-\frac{1}{2}})\|_{L^q(\{0<\rho_k<1\})}. \tag{2.5.58}$$

Next, we claim that

$$\|\mathcal{I}_2\|_{L^2(\{\rho_k>1\})} \geq C\|\nabla(\rho_k^{m-\frac{1}{2}})\|_{L^q(\{\rho_k>1\})} \tag{2.5.59}$$

for some constant $C > 0$.

From (2.1.8a) and (2.5.56), it holds that

$$\|\mathcal{I}_2\|_{L^2(\{\rho_k>1\})} = \left\|\rho_k^{2-m} S''(\rho_k)\nabla(\rho_k^{m-\frac{1}{2}})\right\|_{L^2(\{\rho_k>1\})} \geq \frac{1}{\sigma_1}\left\|\rho_k^{r-m}\nabla(\rho_k^{m-\frac{1}{2}})\right\|_{L^2(\{\rho_k>1\})} \tag{2.5.60}$$

On the other hand, as

$$\frac{1}{2} + \frac{m-r}{\beta} = \frac{1}{q},$$

the Hölder inequality yields that

$$\left\|\rho_k^{r-m}\nabla(\rho_k^{m-\frac{1}{2}})\right\|_{L^2(\{\rho_k>1\})} \|\rho_k^{m-r}\|_{L^{\frac{\beta}{m-r}}(\{\rho_k>1\})} \geq \|\nabla(\rho_k^{m-\frac{1}{2}})\|_{L^q(\{\rho_k>1\})} \tag{2.5.61}$$

As ρ_k is uniformly bounded in $L^\beta(\Omega)$ from Lemma 2.2.11, ρ_k^{m-r} is uniformly bounded in $L^{\frac{\beta}{m-r}}(\Omega)$. From (2.5.60) and (2.5.61), we conclude (2.5.59).

Lastly, as (2.5.53) holds true, (2.5.54) follows from (2.5.58) and (2.5.59).

□

Lemma 2.5.10. *Let us suppose that we are in the setting of Proposition 2.5.7. If $\rho_0 \in L^\infty(\Omega)$ and Φ satisfies* (2.2.4), *then it holds that*

$$\left|\frac{1}{\rho_k^{m-2}}\left((l-1)\rho_k^{l-2}p_k + S_b''(\rho_k)\right)\right|\rho_k^{\frac{1}{2}}|\nabla(\rho_k^{m-1})| \geq \sigma_4|\nabla(\rho_k^m)|, \tag{2.5.62}$$

where

$$\sigma_4 := \frac{m-1}{m}\min\left\{\frac{1}{\sigma_1}, \frac{1}{\sigma_2}\right\}\min\left\{\left(\|\rho_0\|_{L^\infty}e^{dT\|\Delta\Phi\|_{L^\infty}}\right)^{-\frac{1}{2}}, \left(\|\rho_0\|_{L^\infty}e^{dT\|\Delta\Phi\|_{L^\infty}}\right)^{r-m-\frac{1}{2}}\right\}. \tag{2.5.63}$$

Proof. Recall from Lemma 2.2.1 that if $\rho_0 \in L^\infty(\Omega)$, then we have

$$\|\rho_k\|_{L^\infty} \le \|\rho_0\|_{L^\infty} e^{dT\|\Delta\Phi\|_{L^\infty}} =: C. \tag{2.5.64}$$

On the other hand, from (2.5.50) and $\nabla(\rho_k^{m-1}) = \frac{m-1}{m}\nabla(\rho_k^m)$, it holds that

$$\mathcal{I}_3 := \frac{1}{\rho_k^{m-2}}\left((l-1)\rho_k^{l-2}p_k + S_b''(\rho_k)\right)\rho_k^{\frac{1}{2}}\nabla(\rho_k^{m-1}) = \frac{m-1}{m}\frac{S''(\rho_k)}{\rho_k^{m-\frac{3}{2}}}\nabla(\rho_k^m) \text{ a.e. in } \{\rho_k \neq 1\}. \tag{2.5.65}$$

Then, (2.5.51) and (2.5.64) yield that

$$|\mathcal{I}_3| \ge \frac{m-1}{m\sigma_2}\rho_k^{-\frac{1}{2}}|\nabla(\rho_k^m)| \ge \frac{m-1}{m\sigma_2}C^{-\frac{1}{2}}|\nabla(\rho_k^m)| \text{ a.e. in } \{0 < \rho_k < 1\}. \tag{2.5.66}$$

Furthermore, (2.1.8b) and (2.5.64) imply that

$$|\mathcal{I}_3| \ge \frac{m-1}{m\sigma_1}\rho_k^{r-m-\frac{1}{2}}|\nabla(\rho_k^m)| \ge \frac{m-1}{m\sigma_1}C^{-\frac{1}{2}}\min\{C^{r-m},1\}|\nabla(\rho_k^m)| \text{ a.e. in } \{\rho_k > 1\}. \tag{2.5.67}$$

Lastly, as (2.5.53) holds, (2.5.62) follows from (2.5.66) and (2.5.67). □

Corollary 2.5.11. *Let $(\rho^\tau)_{\tau>0}$ and $(p^\tau)_{\tau>0}$ be as in the previous proposition and* (2.5.23) *hold. There exists $\rho \in L^\beta(Q)$ and $p \in L^2([0,T];H^1(\Omega))$ such that*

$$\rho^\tau \to \rho \text{ in } L^s(Q), \text{ as } \tau \downarrow 0, \tag{2.5.68}$$

and

$$p^\tau \rightharpoonup p \text{ in } L^2([0,T];H^1(\Omega)), \text{ as } \tau \downarrow 0. \tag{2.5.69}$$

along a subsequence for any $s \in (0,\beta)$ and β given in (2.2.29).

Proof. Recall that Lemma 2.2.13 yields that $(\rho^\tau)_{\tau>0}$ is uniformly bounded in $L^\beta(Q)$. In both cases $r \ge m$ and $r < m < r + \frac{\beta}{2}$, Lemma A.6.2 and Proposition 2.5.7 yield $(\rho^\tau)_{\tau>0}$ is precompact in $L^s(Q)$ for any $s \in (0,\beta)$.

Indeed, first, we consider the case $r < m < r + \frac{\beta}{2}$. We apply Proposition 2.5.7(2) and Lemma A.6.2(1) to conclude that $(\rho^\tau)_{\tau>0}$ converges to ρ in $L^{(m-\frac{1}{2})q^*}(Q)$ along a subsequence, where $q^* := \frac{qd}{d-q}$ and $q \in (1,2)$ is given in Proposition 2.5.7(2). Note that a direct computation shows that

$$q^* = \frac{2r-1}{2m-1}\frac{2d}{d-2} = \frac{\beta}{m-1/2}. \tag{2.5.70}$$

By a similar argument, we conclude the strong convergence of $(\rho^\tau)_{\tau>0}$ in $L^s(Q)$ along a subsequence, also in the case when $r \geq m$. $\square$

Proof of Theorem 2.5.5. Note that by the direct computation as in (2.5.20) and (2.4.29), we have

$$-\mathcal{E}^\tau = -\rho^\tau \mathbf{v}^\tau = \rho^\tau \nabla((\rho^\tau)^{l-1} p^\tau + S_b'(\rho^\tau)) + \rho^\tau \nabla\Phi, \tag{2.5.71}$$

$$= \nabla \left(\frac{1}{l}((l-1)(\rho^\tau)^l + 1)p^\tau + \rho^\tau S_b'(\rho^\tau) - S_b(\rho^\tau) + S_b(1) \right) + \rho^\tau \nabla\Phi \tag{2.5.72}$$

Then, we have $-\mathcal{E}^\tau = \nabla L_S(\rho^\tau, p^\tau) + \rho^\tau \nabla\Phi$ for L_S given in (2.1.5). Since $l, r < \beta$ from (2.5.23), Corollary 2.5.11 yields that $(\rho^\tau)^l$, $\rho^\tau S_b'(\rho^\tau)$ and $S_b(\rho^\tau)$ converge in $L^1(Q)$ as $\tau \downarrow 0$. As p^τ is uniformly bounded, we conclude that

$$-\mathcal{E}^\tau \to \nabla \left(\frac{1}{l}((l-1)\rho^l + 1)p^\tau + \rho S_b'(\rho) - S_b(\rho) + S_b(1) \right) + \rho\nabla\Phi, \text{ as } \tau \to 0 \tag{2.5.73}$$

along a subsequence in $\mathscr{D}'(Q; \mathbb{R}^d)$. Note that we have $\rho \in L^\beta$ from the uniform boundedness in Lemma 2.2.13 and $p \in L^2([0,T]; H^1(\Omega)) \cap L^\infty(Q)$ from Proposition 2.5.7. As

$$L_S(\rho, p) = \frac{1}{l}((l-1)\rho^l + 1)p^\tau + \rho S_b'(\rho) - S_b(\rho) + S_b(1) \tag{2.5.74}$$

for L_S given in (2.1.5), we conclude that (ρ, p) satisfies (2.5.24). The rest of argument is parallel to Theorem 2.4.2. $\square$

2.6 Uniqueness via an L^1-contraction

We construct an L^1 contraction result, inspired by [DM16, Section 3] and [Vaz07, Theorem 6.5]. In particular, this will imply the uniqueness of the solution of (2.4.5)-(2.4.6) and (2.5.24)-(2.5.25). Let us underline the fact that because of the generality of the previous two problems, on the one hand, the techniques from [DM16, Section 3] do not apply directly. On the other hand, because of the presence of the critical regimes $\{\rho^i = 1\}$, $i = 1, 2$, the construction from [Vaz07, Theorem 6.5] does not apply directly either. Therefore, we develop a careful combination of these two approaches to be able to provide an L^1-contraction for all the systems considered previously, with general initial data.

Theorem 2.6.1. *Let* $(\rho^1, p^1), (\rho^2, p^2)$ *be solutions to* (2.1.4)-(2.1.6) *with initial conditions* $\rho^1_0, \rho^2_0 \in \mathscr{P}(\Omega)$ *such that* $\mathcal{J}(\rho^i_0) < +\infty$, $i = 1, 2$. *Suppose that* $L_S(\rho^i, p^i) \in L^2(Q)$, $i = 1, 2$ *(or equivalently* $\rho^i \in L^{2r}(Q)$, $i = 1, 2$*). Then we have*

$$\|\rho^1_t - \rho^2_t\|_{L^1(\Omega)} \le \|\rho^1_0 - \rho^2_0\|_{L^1(\Omega)}, \ \mathscr{L}^1 - a.e.\ t \in [0, T].$$

Remark 2.6.2. It worth noticing that the assumption $L_S(\rho^i, p^i) \in L^2(Q)$, $i = 1, 2$ in the statement of the previous theorem seems quite natural in the setting of L^1-type contractions for porous medium equations (see [Vaz07]). In our setting, because of the $L^\beta(Q)$ estimates on ρ^i (where β is defined in (2.2.29)) and because of the L^r-type growth condition on L_S at $+\infty$, this assumption is fulfilled already if $\beta \ge 2r$. In the same time, no assumption is needed if the initial data is in $L^\infty(\Omega)$, since in that case L^∞ estimates hold true for ρ^i_t for a.e. $t \in [0, T]$ (see Lemma 2.2.1).

Proof. Let (ρ^1, p^1) and (ρ^2, p^2) be two solutions to (2.1.4)-(2.1.6) with initial data ρ^1_0 and ρ^2_0 respectively. Let $\varphi \in C^2_c((0, T] \times \Omega)$ and using the notation

$$\mathcal{I}(\varphi, t) := \int_\Omega \varphi_t \left(\rho^1_t - \rho^2_t\right) \mathrm{d}x$$

we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(\varphi, t) = \int_\Omega \partial_t\varphi(\rho^1 - \rho^2) + \varphi\partial_t(\rho^1 - \rho^2)\mathrm{d}x$$

Now, using the equation (2.4.5) and by integrating the above expression on $(0, t)$, we get

$$\mathcal{I}(\varphi, t) = \mathcal{I}(\varphi, 0) + \int_0^t \int_\Omega \partial_s\varphi(\rho^1 - \rho^2) + \Delta\varphi(L_{\mathcal{S}}(\rho^1, p^1) - L_{\mathcal{S}}(\rho^2, p^2)) - \nabla\varphi \cdot \nabla\Phi(\rho^1 - \rho^2)\mathrm{d}x\mathrm{d}s \tag{2.6.1}$$

$$= \mathcal{I}(\varphi, 0) + \int_0^t \int_\Omega (L_{\mathcal{S}}(\rho^1, p^1) - L_{\mathcal{S}}(\rho^2, p^2)) \left[A\partial_s\varphi + \Delta\varphi - A\nabla\Phi \cdot \nabla\varphi\right] \mathrm{d}x\mathrm{d}s,$$

where we use the notation

$$A := \frac{\rho^1 - \rho^2}{L_{\mathcal{S}}(\rho^1, p^1) - L_{\mathcal{S}}(\rho^2, p^2)}, \tag{2.6.2}$$

with the convention $A = 0$, when $L_{\mathcal{S}}(\rho^1, p^1) = L_{\mathcal{S}}(\rho^2, p^2)$. Note that Lemma 2.6.3 below implies that if $L_{\mathcal{S}}(\rho^1, p^1) = L_{\mathcal{S}}(\rho^2, p^2)$ a.e., then $\rho^1 = \rho^2$ and $p^1 = p^2$ a.e. Furthermore, on this very particular set actually there is no contribution in the integral on the right hand side of (2.6.1), so it is meaningful to set $A = 0$ there. Also, because of the monotonicity property of the operator $L_{\mathcal{S}}$ (see Lemma 2.6.3), we have that $A \geq 0$ a.e. in Q.

Similarly to the arguments from [DM16, Section 3], for $\zeta : \Omega \to \mathbb{R}$ smooth with $|\zeta| \leq 1$, we consider the dual backward equation as

$$\begin{cases} A\partial_t \varphi + \Delta\varphi - A\nabla\Phi \cdot \nabla\varphi = 0, & \text{in } (0,T) \times \Omega, \\ \nabla\varphi \cdot \vec{n} = 0, & \text{on } (0,T) \times \partial\Omega, \\ \varphi(T, \cdot) = \zeta, & \text{in } \Omega. \end{cases} \tag{2.6.3}$$

Let us notice that if we are able to construct a suitable (weak) solution φ to (2.6.3), for which the computations in (2.6.1) remain valid, we can deduce the L^1-contraction result, after optimizing w.r.t. ζ. In general one cannot hope for smoothness of A, and so (2.6.3) is degenerate. Therefore, we introduce suitable approximations which will allow to construct smooth test function.

Let us define two Borel sets

$$E_1 := \{\rho^1 \geq 1/2\} \cup \{\rho^2 \geq 1/2\}$$

and $E_2 := Q \setminus E_1$. We suppose that both sets E_1 and E_2 have positive measures w.r.t. $\mathscr{L}^{d+1}$, otherwise we simply do not consider the negligible one in the consideration below. First, by Lemma 2.6.4, we have that $A \llcorner E_1$ is bounded. Second we have the following

Claim. $A^{-1} \llcorner E_2 \in L^2(E_2)$.

Proof of the claim. Let us notice that we can write

$$\begin{aligned} E_2 &= \big(\{\rho^1 < 1/2\} \cap \{\rho^2 \geq 1/2\}\big) \cup \big(\{\rho^1 \geq 1/2\} \cap \{\rho^2 < 1/2\}\big) \cup \big(\{\rho^1 < 1/2\} \cap \{\rho^2 < 1/2\}\big) \\ &:= E_2^1 \cup E_2^2 \cup E_2^3. \end{aligned}$$

We further decompose $E_2^1 := (\{\rho^1 < 1/2\} \cap \{1/2 \leq \rho^2 < 1\}) \cup (\{\rho^1 < 1/2\} \cap \{\rho^2 \geq 1\}) =: E_1^{11} \cup E_1^{12}$. For a.e. $q \in E_1^{11}$ we have

$$A^{-1}(q) = \frac{L_S(\rho^1(q), p^1(q)) - L_S(\rho^2(q), p^2(q))}{\rho^1(q) - \rho^2(q)} = \tilde{\rho}(q) S''(\tilde{\rho}(q))$$

where $\tilde{\rho}(q)$ is between $\rho^1(q)$ and $\rho^2(q)$. Since restricted to E_1^{11} both ρ^1 and ρ^2 are bounded by 1, we have that $A^{-1} \llcorner E_1^{11} \in L^\infty(E_1^{11})$.

For a.e. $q \in E_1^{12}$ we have

$$A^{-1}(q) = \frac{L_S(\rho^1(q), p^1(q)) - L_S(\rho^2(q), p^2(q))}{\rho^1(q) - \rho^2(q)} \leq 2|L_S(\rho^1(q), p^1(q)) - L_S(\rho^2(q), p^2(q))|,$$

since restricted to this set $|\rho^1(q) - \rho^2(q)| \geq 1/2$ a.e. Therefore, by our assumption on $L_S(\rho^i, p^i)$ we have that $A^{-1} \llcorner E_2^{12} \in L^2(E_2^{12})$. Therefore, $A^{-1} \llcorner E_1^1 \in L^2(E_2^1)$

Similarly, we can draw the same conclusion in the case of E_2^2, and so $A^{-1} \llcorner E_2^2 \in L^2(E_2^2)$.

For a.e. $q \in E_2^3$, we conclude similarly as in the case of E_2^{11}, i.e. we have that

$$A^{-1}(q) = \frac{L_S(\rho^1(q), p^1(q)) - L_S(\rho^2(q), p^2(q))}{\rho^1(q) - \rho^2(q)} = \tilde{\rho}(q) S''(\tilde{\rho}(q)),$$

where $\tilde{\rho}(q)$ is between $\rho^1(q)$ and $\rho^2(q)$. Since restricted to E_2^3 both ρ^1 and ρ^2 are bounded by $1/2$, we have that $A^{-1} \llcorner E_2^3 \in L^\infty(E_2^3)$.

Therefore, combining all the previous arguments, one obtains that $A^{-1} \llcorner E_2 \in L^2(E_2)$, and the claim follows.

Let $\varepsilon > 0$ and let $K_1 := \|A\mathbb{1}_{E_1}\|_{L^\infty(Q)}$. Let $A_1^\varepsilon := \max\{\varepsilon, A\mathbb{1}_{E_1}\}$. Then, we have $\varepsilon \leq A_1^\varepsilon \leq K_1$ and $\|A_1^\varepsilon - A\mathbb{1}_{E_1}\|_{L^\infty(Q)} \leq \varepsilon$. In the same time, for $0 < \delta \leq K$ given, let $A_2^\varepsilon = A_2^\varepsilon(\delta, K)$ be smooth such that $\delta \leq (A_2^\varepsilon)^{-1} \leq K$ and

$$(A_2^\varepsilon)^{-1} \to [(A\mathbb{1}_{E_2})^{-1}]_{\delta,K} \text{ strongly in } L^q(E_2), \quad \text{as } \varepsilon \downarrow 0, \tag{2.6.4}$$

for any $q \in [1, +\infty)$ and in particular, $A_\varepsilon^{-1} \overset{\star}{\rightharpoonup} [(A\mathbb{1}_{E_2})^{-1}]_{\delta,K}$ weakly-$\star$ in $L^\infty(E_2)$ as $\varepsilon \downarrow 0$. Here, for a nonnegative function $f : Q \to [0, +\infty)$ we use the notation $f_{\delta,K} := \min\{\max\{f, \delta\}, K\}$.

Now, let us define $A_\varepsilon : Q \to [0, +\infty)$ as

$$A_\varepsilon := \begin{cases} A_1^\varepsilon, & \text{a.e in } E_1, \\ A_2^\varepsilon, & \text{a.e. in } E_2. \end{cases}$$

By construction $\min\{\varepsilon; 1/K\} \leq A_\varepsilon \leq \max\{K_1, 1/\delta\}$. For $\theta > 0$ let A_θ (which depends also on ε, δ and K) be a smooth approximation of A_ε such that

$$\begin{aligned} &\min\{\varepsilon; 1/K\} \leq A_\theta \leq \max\{K_1, 1/\delta\}, \text{ in } Q; \\ &\varepsilon \leq A_\theta \leq K_1, \text{ a.e. in } E_1; \\ &1/K \leq A_\theta \leq 1/\delta, \text{ a.e. in } E_2; \end{aligned} \tag{2.6.5}$$

and $A_\theta \to A_\varepsilon$ strongly in $L^q(Q)$ for any $q \in [1, +\infty)$ and in particular

$$A_\theta \overset{\star}{\rightharpoonup} A_\varepsilon \text{ weakly} - \star \text{ in } L^\infty(Q), \text{ as } \theta \downarrow 0. \tag{2.6.6}$$

Moreover, we have

$$A_\theta^{-1} \to [(A\mathbb{1}_{E_2})^{-1}]_{\delta,K} \text{ in } L^q(E_2),\ \forall\, q \in [1, +\infty) \text{ and } A_\theta^{-1} \overset{\star}{\rightharpoonup} [(A\mathbb{1}_{E_2})^{-1}]_{\delta,K} \text{ in } L^\infty(E_2), \text{ as } \max\{\theta, \varepsilon\} \downarrow 0. \tag{2.6.7}$$

To check this last claim, we argue as follows:

$$\begin{aligned} \|A_\theta^{-1} - [(A\mathbb{1}_{E_2})^{-1}]_{\delta,K}\|_{L^q(E_2)} &\leq \|A_\theta^{-1} - (A_2^\varepsilon)^{-1}\|_{L^q(E_2)} + \|(A_2^\varepsilon)^{-1} - [(A\mathbb{1}_{E_2})^{-1}]_{\delta,K}\|_{L^q(E_2)} \\ &= \|(A_\theta - A_2^\varepsilon)/(A_\theta A_2^\varepsilon)\|_{L^q(E_2)} + \|(A_2^\varepsilon)^{-1} - [(A\mathbb{1}_{E_2})^{-1}]_{\delta,K}\|_{L^q(E_2)} \\ &\leq K^2\|A_\theta - A_2^\varepsilon\|_{L^q(E_2)} + \|(A_2^\varepsilon)^{-1} - [(A\mathbb{1}_{E_2})^{-1}]_{\delta,K}\|_{L^q(E_2)} \to 0, \end{aligned}$$

as $\max\{\theta, \varepsilon\} \downarrow 0$, by the construction of A_θ and A_2^ε. We conclude similarly about the weak-$\star$ convergence as well.

Since $\Phi \in W^{1,\infty}(\Omega)$, we consider a smooth approximation of it, $(\Phi_\theta)_{\theta>0}$ such that $\nabla\Phi_\theta \to \nabla\Phi$, as $\theta \downarrow 0$, strongly in $L^{2r'}(\Omega)$.

Let us consider the regularized dual equation which reads as

$$\begin{cases} \partial_t \varphi_\theta + (1/A_\theta)\Delta\varphi_\theta - \nabla\Phi_\theta \cdot \nabla\varphi_\theta = 0, & \text{in } (0,T) \times \Omega, \\ \nabla\varphi_\theta \cdot \vec{n} = 0, & \text{on } (0,T) \times \partial\Omega, \\ \varphi_\theta(T, \cdot) = \zeta, & \text{in } \Omega. \end{cases} \tag{2.6.8}$$

Let φ_θ be the smooth solution of (2.6.8), when the coefficient function is A_θ and we use this in (2.6.1) as

$$\begin{aligned}
\mathcal{I}(\varphi_\theta, T) - \mathcal{I}(\varphi_\theta, 0) &= \int_0^T \int_\Omega \partial_s \varphi_\theta(\rho^1 - \rho^2) + \Delta\varphi_\theta(L_{\mathcal{S}}(\rho^1, p^1) - L_{\mathcal{S}}(\rho^2, p^2)) - \nabla\varphi_\theta \cdot \nabla\Phi(\rho^1 - \rho^2) \mathrm{d}x \mathrm{d}s \\
&= \int_{E_1} \partial_s \varphi_\theta(\rho^1 - \rho^2) + \Delta\varphi_\theta(L_{\mathcal{S}}(\rho^1, p^1) - L_{\mathcal{S}}(\rho^2, p^2)) - \nabla\varphi_\theta \cdot \nabla\Phi(\rho^1 - \rho^2) \mathrm{d}\mathscr{L}^{d+1} \\
&+ \int_{E_2} \partial_s \varphi_\theta(\rho^1 - \rho^2) + \Delta\varphi_\theta(L_{\mathcal{S}}(\rho^1, p^1) - L_{\mathcal{S}}(\rho^2, p^2)) - \nabla\varphi_\theta \cdot \nabla\Phi(\rho^1 - \rho^2) \mathrm{d}\mathscr{L}^{d+1} \\
&= \int_{E_1} (L_{\mathcal{S}}(\rho^1, p^1) - L_{\mathcal{S}}(\rho^2, p^2)) \left[A \partial_s \varphi_\theta + \Delta\varphi_\theta - A\nabla\Phi \cdot \nabla\varphi_\theta\right] \mathrm{d}\mathscr{L}^{d+1} \\
&+ \int_{E_2} (\rho^1 - \rho^2) \left[\partial_s \varphi_\theta + A^{-1}\Delta\varphi_\theta - \nabla\Phi \cdot \nabla\varphi_\theta\right] \mathrm{d}\mathscr{L}^{d+1} =: \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}$$

It remains to show that both $|\mathcal{I}_1|$ and $|\mathcal{I}_2|$ can be made arbitrary small. Because ϕ_θ solves (2.6.8) with the coefficient function A_θ, we have

$$\begin{aligned}
\mathcal{I}_1 &= \int_{E_1} (L_{\mathcal{S}}(\rho^1, p^1) - L_{\mathcal{S}}(\rho^2, p^2)) \left[A \partial_s \varphi_\theta + \Delta\varphi_\theta - A\nabla\Phi \cdot \nabla\varphi_\theta\right] \mathrm{d}\mathscr{L}^{d+1} \\
&- \int_{E_1} (L_{\mathcal{S}}(\rho^1, p^1) - L_{\mathcal{S}}(\rho^2, p^2)) A \left[\partial_s \varphi_\theta + A_\theta^{-1}\Delta\varphi_\theta - \nabla\Phi_\theta \cdot \nabla\varphi_\theta\right] \mathrm{d}\mathscr{L}^{d+1} \\
&= \int_{E_1} (L_{\mathcal{S}}(\rho^1, p^1) - L_{\mathcal{S}}(\rho^2, p^2))(A_\theta - A) A_\theta^{-\frac{1}{2}} A_\theta^{-\frac{1}{2}} \Delta\varphi_\theta \mathrm{d}\mathscr{L}^{d+1} \\
&+ \int_{E_1} (L_{\mathcal{S}}(\rho^1, p^1) - L_{\mathcal{S}}(\rho^2, p^2)) A \nabla\varphi_\theta \cdot (\nabla\Phi_\theta - \nabla\Phi) \mathrm{d}\mathscr{L}^{d+1},
\end{aligned}$$

From here, by (2.6.5) we have

$$\begin{aligned}
|\mathcal{I}_1| &\leq \varepsilon^{-\frac{1}{2}} \|A_\theta^{-\frac{1}{2}} \Delta\varphi_\theta\|_{L^2(Q)} \left(\int_{E_1} |L_{\mathcal{S}}(\rho^1, p^1) - L_{\mathcal{S}}(\rho^2, p^2)|^2 |A_\theta - A|^2 \mathrm{d}\mathscr{L}^{d+1}\right)^{\frac{1}{2}} \\
&+ \int_0^T \int_\Omega |\rho^1 - \rho^2| |\nabla\varphi_\theta| |\nabla\Phi_\theta - \nabla\Phi| \mathrm{d}x \mathrm{d}t.
\end{aligned}$$

By Lemma 2.6.5(1), the summability assumption on $\rho^i \in L^{2r}(Q)$ and the approximation $\nabla\Phi_\theta \to \nabla\Phi$, in $L^{2r'}(\Omega)$ as $\theta \downarrow 0$, we conclude that the second term in the previous inequality tends to 0 as $\theta \downarrow 0$. By Lemma 2.6.5(2), we have that $\|A_\theta^{-\frac{1}{2}} \Delta\varphi_\theta\|_{L^2(Q)} \leq C$ for some constant independent of θ and ε. Furthermore, by (2.6.6), by the summability assumption on $L_S(\rho^i, p^i)$ and by the construction of A_1^ε, for θ small enough we have

$$\begin{aligned}
&\int_{E_1} |L_{\mathcal{S}}(\rho^1, p^1) - L_{\mathcal{S}}(\rho^2, p^2)|^2 |A_\theta - A|^2 \mathrm{d}\mathscr{L}^{d+1} \\
&\leq 2\int_{E_1} |L_{\mathcal{S}}(\rho^1, p^1) - L_{\mathcal{S}}(\rho^2, p^2)|^2 |A_\theta - A_1^\varepsilon|^2 \mathrm{d}\mathscr{L}^{d+1} + 2\int_{E_1} |L_{\mathcal{S}}(\rho^1, p^1) - L_{\mathcal{S}}(\rho^2, p^2)|^2 |A_1^\varepsilon - A|^2 \mathrm{d}\mathscr{L}^{d+1} \\
&\leq \varepsilon^2 + C\varepsilon^2,
\end{aligned}$$

for some constant independent of ε, θ, K and therefore by the arbitrariness of ε, we conclude that $\mathcal{I}_1 = 0$.

In the case of $\mathcal{I}_2$ we argue as follows.

$$\begin{aligned}
\mathcal{I}_2 = &\int_{E_2} (\rho^1 - \rho^2) \left[\partial_s \varphi_\theta + A^{-1}\Delta\varphi_\theta - \nabla\Phi \cdot \nabla\varphi_\theta\right] \mathrm{d}\mathscr{L}^{d+1} \\
&- \int_{E_2} (\rho^1 - \rho^2) \left[\partial_s \varphi_\theta + A_\theta^{-1}\Delta\varphi_\theta - \nabla\Phi_\theta \cdot \nabla\varphi_\theta\right] \mathrm{d}\mathscr{L}^{d+1} \\
&= \int_{E_2} (\rho^1 - \rho^2)(A^{-1} - A_\theta^{-1}) A_\theta^{\frac12} A_\theta^{-\frac12} \Delta\phi_\theta \mathrm{d}\mathscr{L}^{d+1} \\
&+ \int_{E_2} (\rho^1 - \rho^2)\nabla\varphi_\theta \cdot (\nabla\Phi_\theta - \nabla\Phi) \mathrm{d}\mathscr{L}^{d+1} \\
&= \int_{E_2} (\rho^1 - \rho^2)(A^{-1} - A_{\delta,K}^{-1}) A_\theta^{\frac12} A_\theta^{-\frac12} \Delta\phi_\theta \mathrm{d}\mathscr{L}^{d+1} + \int_{E_2} (\rho^1 - \rho^2)(A_{\delta,K}^{-1} - A_\theta^{-1}) A_\theta^{\frac12} A_\theta^{-\frac12} \Delta\phi_\theta \mathrm{d}\mathscr{L}^{d+1} \\
&+ \int_{E_2} (\rho^1 - \rho^2)\nabla\varphi_\theta \cdot (\nabla\Phi_\theta - \nabla\Phi) \mathrm{d}\mathscr{L}^{d+1} \\
&=: \mathcal{I}_{21} + \mathcal{I}_{22} + \mathcal{I}_{23}
\end{aligned}$$

In the case of $\mathcal{I}_{23}$, we argue exactly as in the case of the second term of $\mathcal{I}_1$ to conclude that this term tends to 0 as $\theta \downarrow 0$. As for the other terms, let us notice that by the definition of $A_{\delta,K}^{-1}$ (on E_2), we have that

$$\left|A^{-1} - A_{\delta,K}^{-1}\right| = \begin{cases} \delta & \text{a.e. in } \{0 \le A^{-1} < \delta\} \cap E_2, \\ 0 & \text{a.e. in } \{\delta \le A^{-1} \le K\} \cap E_2, \\ A^{-1} - K & \text{a.e. in } \{K \le A^{-1}\} \cap E_2, \end{cases} \tag{2.6.9}$$

and thus

$$\left|A^{-1} - A_{\delta,K}^{-1}\right| \le \delta + (A^{-1} - K)_+, \text{ a.e. in } E_2. \tag{2.6.10}$$

Therefore, since $A_\theta^{\frac12} \le \delta^{-\frac12}$, we obtain

$$|\mathcal{I}_{21}| \le \|A_\theta^{-\frac12}\Delta\phi_\theta\|_{L^2(Q)} \delta^{-\frac12} \left(\delta\|\rho^1 - \rho^2\|_{L^2(E_2)} + \|(\rho^1 - \rho^2)(A^{-1} - K)\|_{L^2(\{K \le A^{-1}\} \cap E_2)}\right) \to 0,$$

as $K \to +\infty$ and $\delta \downarrow 0$ (in this order). This is true indeed, by Lemma 2.6.5(2) and by the fact that

$$\int_{\{K\leq A^{-1}\}\cap E_2}(\rho^1-\rho^2)^2(A^{-1}-K)^2\mathrm{d}\mathscr{L}^{d+1}\leq\int_{\{K\leq A^{-1}\}\cap E_2}(\rho^1-\rho^2)^2(A^{-1})^2\mathrm{d}\mathscr{L}^{d+1}$$
$$\leq\int_{\{K\leq A^{-1}\}\cap E_2}(L_S(\rho^1,p^1)-L_S(\rho^2,p^2))^2\mathrm{d}\mathscr{L}^{d+1}$$

Since $A^{-1}\in L^2(E_2)$, by Chebyshev's inequality $\mathscr{L}^{d+1}(\{K\leq A^{-1}\}\cap E_2)\to 0$, as $K\to+\infty$, so by the summability of $L^2_S(\rho^i,p^i)$ we deduce that for K large enough last term in the last inequality is smaller than δ^2. Therefore, by the arbitrariness of δ, we conclude that $\mathcal{I}_{21}$ has to be zero.

To show that $|\mathcal{I}_{22}|$ can be made arbitrary small, using again $A_\theta^{\frac{1}{2}}\leq\delta^{-\frac{1}{2}}$ a.e. on E_2 and Lemma 2.6.5(2), we have

$$|\mathcal{I}_{22}|^2\leq\delta^{-1}C\int_{E_2}(\rho^1-\rho^2)^2(A^{-1}_{\delta,K}-A^{-1}_\theta)^2\mathrm{d}\mathscr{L}^{d+1}.$$

By the fact that $A^{-1}_{\delta,K},A^{-1}_\theta\in L^\infty(E_2)$, $\rho^1,\rho^2\in L^2(E_2)$ and by the weak-$\star$ convergence of A^{-1}_θ to $A^{-1}_{\delta,K}$ in $L^\infty(E_2)$, we conclude that for θ small enough, the r.h.s. of the previous inequality is smaller than δ, therefore by the arbitrariness of δ we conclude that $\mathcal{I}_{22}=0$. □

Lemma 2.6.3. *Let $(\rho^1,p^1),(\rho^2,p^2)$ satisfy* (2.4.6). *Then L_S (defined in* (2.1.5)*) defines a monotone operator in the sense that*

$$\text{if } \rho^1(x)<\rho^2(x),\ \text{ then } L_{\mathcal{S}}(\rho^1,p^1)(x)<L_{\mathcal{S}}(\rho^2,p^2)(x). \tag{2.6.11}$$

In particular, for $x\in\Omega$, if

$$L_{\mathcal{S}}(\rho^1,p^1)(x)=L_{\mathcal{S}}(\rho^2,p^2)(x), \tag{2.6.12}$$

then $\rho^1(x)=\rho^2(x)$ and $p^1(x)=p^2(x)$.

Proof. First of all, if we have (2.6.12) and $\rho^1(x)=\rho^2(x)$, then (2.1.5) and (2.4.6) imply $p^1(x)=p^2(x)$. Thus, it is enough to show that $\rho^1(x)=\rho^2(x)$. We claim that L_S is a monotone operator in the sense of (2.6.11). Note that $\rho\mapsto\rho\mathcal{S}'(\rho)-\mathcal{S}(\rho)$ is strictly increasing in $\mathbb{R}^+\setminus\{1\}$ because it holds that

$$\partial_\rho(\rho\mathcal{S}'(\rho)-\mathcal{S}(\rho))=\rho\mathcal{S}''(\rho)>0 \text{ in } \mathbb{R}^+\setminus\{1\} \tag{2.6.13}$$

from the strict convexity of $\mathcal{S}$ in Assumption A. Therefore, (2.6.11) holds if $\rho^1(x), \rho^2(x) \in (0,1)$ or $\rho^1(x), \rho^2(x) \in (1,+\infty)$.

Consider the case that $\rho^1(x) = 1 < \rho^2(x)$. Recall from Assumption A that $\mathcal{S}$ and $\mathcal{S}'$ are continuous in $\mathbb{R}^+$ and $\mathbb{R}^+ \setminus \{1\}$, respectively. As $\rho \mapsto \rho\mathcal{S}'(\rho) - \mathcal{S}(\rho)$ is strictly increasing in $(1,+\infty)$, it holds that

$$L_{\mathcal{S}}(\rho^2, p^2) = \rho^2(x)\mathcal{S}'(\rho^2(x)) - \mathcal{S}(\rho^2(x)) + \mathcal{S}(1) > \lim_{\rho\to 1+} \rho\mathcal{S}'(\rho) - \mathcal{S}(\rho) + \mathcal{S}(1) = \mathcal{S}'(1+) \tag{2.6.14}$$

$$\geq p^1(x) = L_S(\rho^1, p^1)(x).$$

From (2.6.14) and (2.4.6), we conclude (2.6.11) if $\rho^1(x) = 1 < \rho^2(x)$. Similar arguments hold for $\rho^1(x) < \rho^2(x) = 1$. Lastly, by combining the inequalities in (2.6.11) for two cases, $\rho^1(x) = 1 < \rho^2(x)$ or $\rho^1(x) < 1 = \rho^2(x)$, we conclude (2.6.11) for $\rho^1(x) < 1 < \rho^2(x)$. □

Lemma 2.6.4. *We differentiate two cases.*

(1) Assume $m = 1$ for m given in (2.1.7)*. Let (ρ^1, p^1) and (ρ^2, p^2) satisfy* (2.4.6)*, then we have*

$$0 \leq A \leq \max\{\sigma_1, \sigma_2\}, \ \text{a.e. in } Q \tag{2.6.15}$$

for $A = A(\rho^1, p^1, \rho^2, p^2)$ given in (2.6.2) *and σ_1, σ_2 are from Assumption* (2.1.7)*-*(2.1.8)*.*

(2) Let $m > 1$. If there exist $c_0 > 0$ and a Borel set $E \subseteq Q$ such that $\rho^1, \rho^2 \geq c_0$ a.e. on E, then $A \llcorner E \in L^\infty(E)$ and $A \leq \max\left\{\sigma_1, \frac{\sigma_2}{c_0^{m-1}}\right\}$ a.e. in E.

Proof. Let us recall that $L_{\mathcal{S}}(\rho, p)(t, x) := [\rho(t,x)\mathcal{S}'(\rho(t,x)) - \mathcal{S}(\rho(t,x)) + \mathcal{S}(1)]\, \mathbb{1}_{\{\rho\neq 1\}}(t,x) + p(t,x)\mathbb{1}_{\{\rho=1\}}(t,x)$ from (2.1.5). The non-negativity of A follows from the monotonicity of $L_{\mathcal{S}}$ shown in Lemma 2.6.3. We fix $q = (t,x) \in Q$ a Lebesgue for ρ^1, ρ^2, p^1, p^2 and assume that $\rho^1(t,x) \geq \rho^2(t,x)$. If $q \in \{\rho^1 = 1\} \cap \{\rho^2 = 1\}$ there is nothing to check, since $A(q) = 0$ in both cases.

Let us show (1).

Case 1. If $q \in (\{\rho^1 > 1\} \cap \{\rho^2 > 1\}) \cup (\{\rho^1 < 1\} \cap \{\rho^2 < 1\})$ we have that

$$\rho^1(q)\mathcal{S}'(\rho^1(q)) - \mathcal{S}(\rho^1(q)) - \rho^2(q)\mathcal{S}'(\rho^2(q)) + \mathcal{S}(\rho^2(q)) = \tilde{\rho}S''(\tilde{\rho})(\rho^1(q) - \rho^2(q)),$$
$$\geq \min\left\{\frac{1}{\sigma_1}, \frac{1}{\sigma_2}\right\}(\rho^1(q) - \rho^2(q)),$$

where $\tilde{\rho}$ is a constant between $\rho^1(q)$ and $\rho^2(q)$. Therefore, we get that $A(q) \leq \max\{\sigma_1, \sigma_2\}$.

Case 2. If $q \in \{\rho^1 > 1\} \cap \{\rho^2 = 1\}$ we have from (2.4.6) that

$$\rho^1(q)\mathcal{S}'(\rho^1(q)) - \mathcal{S}(\rho^1(q)) + S(1) - p^2(q) \geq \rho^1(q)\mathcal{S}'(\rho^1(q)) - \mathcal{S}(\rho^1(q)) - (\mathcal{S}'(1+) - \mathcal{S}(1)). \tag{2.6.16}$$

As $\rho \mapsto \rho\mathcal{S}'(\rho) - \mathcal{S}(\rho)$ is continuous in $[1, \rho^1(q)]$ and differentiable in $(1, \rho^1(q))$, the mean value theorem yields that

$$\rho^1(q)\mathcal{S}'(\rho^1(q)) - \mathcal{S}(\rho^1(q)) - p^2(q) \geq \tilde{\rho}S''(\tilde{\rho})(\rho^1(q) - 1) \geq \frac{1}{\sigma_1}(\rho^1(q) - 1), \tag{2.6.17}$$

where $\tilde{\rho}$ is between 1 and $\rho^1(q)$. Parallel arguments show (2.6.15) on the region $\{\rho^1 = 1\} \cap \{\rho^2 < 1\}$.

Case 3. If $q \in \{\rho^1 > 1\} \cap \{\rho^2 < 1\}$ from similar arguments as in Case 2, we have that

$$\rho^1(q)\mathcal{S}'(\rho^1(q)) - \mathcal{S}(\rho^1(q)) - (\mathcal{S}'(1+) - \mathcal{S}(1)) \geq \frac{1}{\sigma_1}(\rho^1(q) - 1) \tag{2.6.18}$$

and

$$(\mathcal{S}'(1-) - \mathcal{S}(1)) - [\rho^2(q)\mathcal{S}'(\rho^2(q)) - \mathcal{S}(\rho^1(q))] \geq \frac{1}{\sigma_2}(1 - \rho^2(q)) \tag{2.6.19}$$

As $\mathcal{S}'(1+) \geq \mathcal{S}'(1-)$, we conclude that

$$L_{\mathcal{S}}(\rho^1, p^1)(q) - L_{\mathcal{S}}(\rho^2, p^2)(q) \geq \sigma_1(\rho^1(q) - 1) + \sigma_2(1 - \rho^2(q)) = \min\left\{\frac{1}{\sigma_1}, \frac{1}{\sigma_2}\right\}(\rho^1(q) - \rho^2(q)). \tag{2.6.20}$$

The proof of (2) follows the very same steps as the one of (1). By the lower bound $c_0 > 0$ on the densities in E, we conclude that $A \leq \max\left\{\sigma_1, \frac{\sigma_2}{c_0^{m-1}}\right\}$. □

Lemma 2.6.5. *Let $\varepsilon > 0$ and let φ_ε be a smooth solution to* (2.6.8). *Then there exists a constant $C = C(T, \|\nabla\zeta\|_{L^2}) > 0$ such that*

(1) $\sup_{t\in[0,T]}\|\nabla\varphi_\varepsilon\|_{L^2(\Omega)} \le C$*;*

(2) $\|A_\varepsilon^{-\frac{1}{2}}\Delta\varphi_\varepsilon\|_{L^2(Q)} \le C$*.*

Proof. The proof of this results follows the same lines as the one of [DM16, Lemma 3.1], therefore we omit it. □

Corollary 2.6.6. *Let* $\rho_0 \in \mathscr{P}(\Omega)$ *satisfy* $\mathcal{J}(\rho_0) < +\infty$*. A solution pair to* (2.1.4)-(2.1.6) *such that* $L_S(\rho,p) \in L^2(Q)$ *is uniquely determined by* ρ_0*.*

Proof. From the contraction result in Theorem 2.6.1 we deduce the uniqueness of ρ. Now suppose that there exists to pressure fields p^1, p^2 solving (2.4.5) with the same ρ. Taking the difference of these two equations we get

$$\Delta(L_{\mathcal{S}}(\rho,p^1) - L_{\mathcal{S}}(\rho,p^2)) = 0, \text{ in } \mathscr{D}'((0,T)\times\Omega).$$

For a.e. $t \in [0,T]$ and for any $\varphi \in C_c^2(\Omega)$ we have that

$$0 = \int_\Omega (L_{\mathcal{S}}(\rho_t,p_t^1) - L_{\mathcal{S}}(\rho_t,p_t^2))\Delta\varphi \mathrm{d}x = \int_{\{\rho_t=1\}} (p_t^1 - p_t^2)\Delta\varphi \mathrm{d}x,$$

where in the last equality we used the fact that $p_t^1 = p_t^2$ a.e. in $\{\rho_t < 1\} \cup \{\rho_t > 1\}$. By the arbitrariness of φ we conclude that $p_t^1 = p_t^2$ a.e. on $\{\rho_t = 1\}$ and therefore the uniqueness of p follows. □

APPENDIX A

Appendix

A.1 Proof of Proposition 1.4.17

Proof of Proposition 1.4.17. 1. We will prove the case $w < \psi$ at $t = 0$, parallel proof holds for the other case.

2. First, let us assume that $\Omega_t(\phi)$ touches $\Omega_t(w)$ from inside for the first time at $t = t_0$ at $x_0 \in \Omega_{t_0}(w)$. Our goal is to make a perturbation of $\Omega_t(w)$ using $\Omega_t(\phi)$, which leads to a contradiction with the gradient flow property of w. To this end, let $\tilde{\phi}$ be a parallel translation of ϕ in the direction of normal vector at x_0, $\vec{n}_{x_0}$, so that $\Omega_{t_0}(\tilde{\phi})$ has nonempty intersection with the complement of $\Omega_t(w)$:

$$\tilde{\phi}(x,t) := \phi\left(x - \delta\left(e + (t - t_0)\right)\vec{n}_{x_0}, t\right). \tag{A.1.1}$$

Here, $e > 0$ will be chosen in next step. Then, $U_t := \Omega_t(\tilde{\phi}) \setminus \Omega_t(w)$ is nonempty at t_0 and we have

$$\frac{\tilde{\phi}_t}{|D\tilde{\phi}|}(x_0, t_0) \leq \nabla \cdot \left(\frac{D\tilde{\phi}}{|D\tilde{\phi}|}\right)(x_0, t_0) + \eta(t_0) - \delta. \tag{A.1.2}$$

For any $\varepsilon \in (0, \frac{\delta}{8+4C})$ where C is defined in (A.1.8), there exists sufficiently small $e \in (0, \frac{r_1 - r_0}{2})$ such that (a) $e \leq d_H(\Omega_t(\phi), \Omega_t(w))$ in $[t_0 - 4e, t_0 - 2e]$, (b) $|U_t| < \varepsilon$ in $[t_0 - 4e, t_0]$, and (c)

$$\frac{\tilde{\phi}_t}{|D\tilde{\phi}|}(x,t) \leq \nabla \cdot \left(\frac{D\tilde{\phi}}{|D\tilde{\phi}|}\right)(x,t) + \eta(t) - \frac{\delta}{4} \text{ and } \left|\frac{\tilde{\phi}_t}{|D\tilde{\phi}|}(x,t) - \frac{\tilde{\phi}_t}{|D\tilde{\phi}|}(x,t_0)\right| < \frac{\varepsilon}{2} \tag{A.1.3}$$

in $\mathcal{N}_\varepsilon \times [t_0 - 4e, t_0]$ where $\mathcal{N}_\varepsilon := \{x : d(x, U_s) < \varepsilon \text{ for all } t_0 - 4e \leq s \leq t_0\}$.

Note that (a) implies $\Omega_t(\tilde{\phi}) \subset\subset \Omega_t(w)$ in $[t_0 - 4e, t_0 - 2e]$. By definition of w and Lemma A.3.1, there exists sufficiently small $h \in (0, e)$ such that *the constrained minimizing* *movements* E_t^h starting from $\Omega_0(w)$ satisfies the following relations: $\Omega_t(\phi) \subset E_t^h$ in $[t_0 - 4e, t_0 - 2e]$ and

$$|U_t^h| < \varepsilon, \left|\lambda[|E_t^h|] - \lambda[|\Omega_t(w)|]\right| < \varepsilon, \text{ and } d_H(U_t^h, U_t) < \varepsilon \text{ in } [t_0 - 4e, t_0] \qquad \text{(A.1.4)}$$

$U_t^h := \Omega_t(\tilde{\phi}) - E_t^h$.

Then, there exists $k \in \mathbb{N}$ such that $\Omega_{t_0-hk}(\phi) \subset E_{t_0-hk}^h$ and $U_{t_0-h(k-1)}^h$ is nonempty. By $\Omega_t(\phi) \subset E_t^h$ in $[t_0 - 4e, t_0 - 2e]$, we have $t_1 := t_0 - h(k-1) \geq t_0 - 2e$. Also, by (A.1.4), $U_{t_1}^h \subset \mathcal{N}_\varepsilon$ and thus (A.1.3) holds in $U_{t_1}^h$.

3. For simplicity let us denote sets

$$F_0 := E_{t_1-h}^h, F_h := E_{t_1}^h, \tilde{U} := U_{t_1}^h \text{ and } \tilde{F}_h := E_{t_1}^h \cup \tilde{U}. \qquad \text{(A.1.5)}$$

Let us show that $\tilde{F}_h \in S_{r_0,R_0}$. First, as $e \leq \frac{r_1 - r_0}{2}$, $\Omega_{t_1}(\tilde{\phi}) \in S_{r_0}$. Moreover, $E_{t_1}^h \in S_{r_0}$, and thus $\tilde{F}_h \in S_{r_0}$. On the other hand, since $\tilde{F}_h \subset F_h$,

$$d_H(\partial(\tilde{F}_h \cap F_0), \partial F_0) \leq d_H(\partial(F_h \cap F_0), \partial F_0) \leq Mh, \qquad \text{(A.1.6)}$$

Next, let us show that $I_h(F_h; F_0) > I_h(\tilde{F}_h, F_0)$. Let us write out the difference of the energies:

$$\begin{aligned} &I_h(F_h; F_0) - I_h(\tilde{F}_h; F_0) \\ &= \left(\text{Per}(E_h) - \text{Per}(\tilde{E}_h)\right) + \left(-\Lambda[|F_h|] + \Lambda[|\tilde{F}_h|]\right) + \frac{1}{h}\left(\tilde{d}^2(F_h, F_0) - \tilde{d}^2(\tilde{F}_h, F_0)\right). \end{aligned}$$

Let us estimate the first term

$$\mathcal{I}_1 := \text{Per}(F_h) - \text{Per}(\tilde{F}_h) \geq \int_{\partial F_h/\partial \tilde{F}_h} d\sigma - \int_{\partial \tilde{F}_h/\partial F_h} d\sigma$$

Let $\vec{n}$ be the outward normal vector at each point of $\partial F_h/\partial \tilde{F}_h$ and $\partial \tilde{F}_h/\partial F_h$. Note that, $-\frac{D\tilde{\phi}}{|D\tilde{\phi}|}(\cdot, t_1) \cdot \vec{n} \leq 1$ on $\partial F_h/\partial \tilde{F}_h$ and $-\frac{D\tilde{\phi}}{|D\tilde{\phi}|}(\cdot, t_1) \cdot \vec{n} = 1$ on $\partial \tilde{F}_h/\partial F_h$, and thus

$$\mathcal{I}_1 \geq \int_{\partial F_h/\partial \tilde{F}_h} -\frac{D\tilde{\phi}}{|D\tilde{\phi}|}(x, t_1) \cdot \vec{n} d\sigma - \int_{\partial \tilde{F}_h/\partial F_h} -\frac{D\tilde{\phi}}{|D\tilde{\phi}|}(x, t_1) \cdot \vec{n} d\sigma = \int_{\partial \tilde{U}} \frac{D\tilde{\phi}}{|D\tilde{\phi}|}(x, t_1) \cdot \vec{n} d\sigma.$$

Note that outward normal of $\tilde{U}$ is opposite to that of $\partial F_h/\partial \tilde{F}_h$. Finally, by divergence theorem, we conclude that

$$\mathcal{I}_1 \geq \int_{\tilde{U}} \nabla \cdot \frac{D\tilde{\phi}}{|D\tilde{\phi}|}(x,t_1)dx \tag{A.1.7}$$

Next, since $\Lambda(\cdot)$ is $C^{1,1}$, we have

$$\mathcal{I}_2 := -\Lambda[|F_h|] + \Lambda[|\tilde{F}_h|] \geq \lambda[|F_h|]|\tilde{U}| - C|\tilde{U}|^2 \text{ where } C := \sup_{|B_{r_0}|\leq z \leq |B_R|} |\lambda'(z)| \tag{A.1.8}$$

Lastly we have

$$\mathcal{I}_3 := \frac{1}{h}\tilde{d}^2(F_h, F_0) - \frac{1}{h}\tilde{d}^2(\tilde{F}_h, F_0) = -\frac{1}{h}\int_{U^h} \mathrm{sd}(x, F_0)dx \tag{A.1.9}$$

where $\mathrm{sd}(x,\Omega)$ is the signed distance function given in (1.2.7). Since $\Omega_{t_1-h}(\tilde{\phi}) \subset F_0$, it holds that $\mathrm{sd}(x, F_0) \leq \mathrm{sd}(x, \Omega_{t_1-h}(\tilde{\phi}))$ for all $x \in \mathbb{R}^n$. Moreover, since (A.1.3) holds in $\tilde{U}$, we have

$$\mathcal{I}_3 \geq -\frac{1}{h}\int_{\tilde{U}} \mathrm{sd}(x, \Omega_{t_1-h}(\tilde{\phi}))dx \quad \geq -\int_{\tilde{U}} \frac{\tilde{\phi}_t}{|D\tilde{\phi}|}(x,t_1) + \varepsilon dx, \tag{A.1.10}$$

Putting all terms together, we have

$$\mathcal{I}_4 := I_h(F_h; F_0) - I_h(\tilde{F}_h; F_0) \geq \int_{\tilde{U}} \left(\nabla \cdot \frac{D\tilde{\phi}}{|D\tilde{\phi}|}(x,t_1) - \frac{\tilde{\phi}_t}{|D\tilde{\phi}|}(x,t_1) + \lambda[|F_h|] \right) dx - \varepsilon|\tilde{U}| - C|\tilde{U}|^2,$$

Applying (A.1.3) and (A.1.4), it holds that

$$\mathcal{I}_4 \geq \int_{\tilde{U}} \left(\frac{\delta}{4} - \lambda[|\Omega_{t_1}(w)|] + \lambda[|F_h|] \right) dx - \varepsilon|\tilde{U}| - C|\tilde{U}|^2 \geq |\tilde{U}| \left(\frac{\delta}{4} - 2\varepsilon - C|\tilde{U}| \right) > 0$$

where the last inequality follows from the fact that $\varepsilon < \frac{\delta}{8+4C}$ and $|\tilde{U}| \leq \varepsilon$. □

A.2 Regularity

In this section, we use notation from [Hui84] and [Hui87]. Let $\partial\Omega_0$ be represented locally by some diffeomorphism, $F_0 : U \subset \mathbb{R}^{n-1} \to F_0(U) \subset \partial\Omega_0$. Then, (1.1.8) can be formulated into

$$\begin{cases} \frac{\partial}{\partial t} F(x,t) &= (\eta(t) - H(x,t)) \cdot \vec{n}(x,t), \quad \text{for } x \in U, t \geq 0 \\ F(\cdot, 0) &= F_0 \end{cases} \tag{A.2.1}$$

The induced metric, its inverse matrix, and the second fundamental form are denoted by $\{g_{ij}\}$, $\{g^{ij}\}$ and $A = \{h_{ij}\}$. Note that g_{ij} and h_{ij} can be computed as follows:

$$g_{ij} = \left(\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_i} \right), \quad h_{ij} = -\left(\vec{n}, \frac{\partial^2 F}{\partial x_i \partial x_j} \right), \tag{A.2.2}$$

We use the following notion for the trace of the second fundamental from,

$$H = g^{ij} h_{ij}, \quad |A|^2 = g^{ij} g^{kl} h_{ik} h_{jl}, \text{ and } C = g^{ij} g^{kl} g^{mn} h_{ik} h_{lm} h_{nj}.$$

The following lemma is parallel to Theorem 3.1 in [EH91] and Lemma 3.2 in [SW10].

Lemma A.2.1. *Let $u(x,t)$ be a solution of*

$$\frac{\partial u}{\partial t} = \sqrt{1+|Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}} \right) + \eta(t)\sqrt{1+|Du|^2} \tag{A.2.3}$$

in $Q_R = B_R(0) \times [0, R^2]$. Then for $0 < t \leq R^2$, we have the interior gradient estimate

$$|D^2 u|^2(0,t) \leq K(1 + \sup_{Q_R} |Du|^6)(\frac{1}{R^2} + \frac{1}{t}) \tag{A.2.4}$$

where the constant $K = K(\|u\|_{L^\infty(Q_R)}, \|\eta\|_{L^\infty([0,R^2])})$.

Proof. First, by Corollary 1.2 in [Hui87], it holds that $(\frac{\partial}{\partial t} - \triangle)(|A|^2) = -2|\nabla A|^2 + 2|A|^4 - 2\eta C$.

Let us denote $v = \sqrt{1+|Du|^2}$. As Lemma 1.1 in [EH91] and Lemma 3.2 in [SW10], the function v satisfies the equation

$$v_t = \triangle v - |A|^2 v - \frac{2}{v}|\nabla v|^2. \tag{A.2.5}$$

Let us define $\phi(r) := \frac{r}{1-\delta r}$ and $g := |A|^2 \phi(v^2)$. Then, by the direct computation motivated from of Lemma 3.2 in [SW10] and Theorem 3.1 in [EH91], we have

$$\mathcal{I}_1 := \left(\frac{\partial}{\partial t} - \triangle \right) g = (-2|\nabla A|^2 + 2|A|^4 - 2\eta C)\phi(v^2) + \left(-|A|^2 v - \frac{2}{v}|\nabla v|^2 \right) \times \frac{2v|A|^2}{(1-\delta v^2)^2}$$

Note that $\delta^2 \phi(v^2) = \frac{1}{1-\delta v^2} - 1$, it holds that

$$\begin{aligned}
\mathcal{I}_1 &= -2\delta g^2 - 2|\nabla A|^2 \phi(v^2) + \frac{-4|A|^2|\nabla v|^2}{(1-\delta v^2)^2} - 2\eta C\phi(v^2), \\
&= -2\delta g^2 - 2|\nabla A|^2 \phi(v^2) + \left(\frac{-2\delta|\nabla v|^2 g}{(1-\delta v^2)} + \frac{-2|A|^2|\nabla v|^2}{(1-\delta v^2)} \right) + \frac{-2|A|^2|\nabla v|^2}{(1-\delta v^2)^2} - 2\eta C\phi(v^2).
\end{aligned}$$

Now, choose $\delta := \frac{1}{2}\inf_{Q_R} v^{-2}$. Applying Young's inequality and $\nabla g = 2A\nabla A\phi(v^2) + 2v|A|^2\phi'(v^2)\nabla v$,

$$\phi v^{-3}\langle \nabla g, \nabla v\rangle \leq |\nabla A|^2\phi(v^2) + \frac{|A|^2|\nabla v|^2}{1-\delta v^2} + \frac{|A|^2|\nabla v|^2}{(1-\delta v^2)^2}.$$

Finally, from Young's inequality and $\phi(v^2) \geq v^2$, the last term of $\mathcal{I}_1$ is bounded by

$$|-2\eta C\phi(v^2)| \leq 2K_1 g^{3/2}|v| \leq \delta g^2 + \frac{K_1^2 g v^2}{\delta} \tag{A.2.6}$$

for some constant $K_1 := K_1(\|\eta\|_{L^\infty([0,R^2])}) > 0$.

Putting all together, it holds that

$$\left(\frac{\partial}{\partial t} - \triangle\right) g \leq -2\delta g^2 + \frac{-2\delta|\nabla v|^2 g}{(1-\delta v^2)} - 2\phi v^{-3}\langle\nabla g, \nabla v\rangle + \delta g^2 + \frac{K_1^2 g v^2}{\delta}.$$

The rest of proof is parallel to Theorem 3.1 in [EH91] and Lemma 3.2 in [SW10]. Taking a cutoff function as in [EH91], $\psi = \psi(r) = (R^2 - r)^2$ where $r = r(X,t)$ satisfies $r(X,0) \leq \frac{R^2}{2}$,

$$\left|\left(\frac{\partial}{\partial t} - \triangle\right) r\right| \leq K_2 \text{ and } |\nabla r|^2 \leq K_2 r$$

on $X = F(x,t)$ for some constant $K_2 = K_2(\|u\|_{L^\infty(Q_R)}, \|\eta\|_{L^\infty([0,R^2])}) > 0$. It holds that

$$\left(\frac{\partial}{\partial t} - \triangle\right)[tg\psi] \leq -\delta g^2\psi t - \vec{b}\cdot\nabla(tg\psi) + c\left(\left(1 + \frac{1}{\delta v^2}\right) r + R^2\right) tg + g\psi + \frac{K_1^2 g v^2}{\delta}\psi t$$

where $\vec{b} = \vec{b}(v,\psi,\phi)$ and $c = c(K_2)$ is a constant (See equations (21) and (23) in [EH91] for details).

Let t_0 be a maximizer of $m(T) := \sup\limits_{0\leq t\leq T}\ \sup\limits_{r(x,t)\leq R^2} tg\psi$. Then, by parallel computation in Theorem 3.1 in [EH91], we conclude that

$$\delta g^2\psi t_0 \leq c\left(\left(1 + \frac{1}{\delta v^2}\right) r + R^2\right) t_0 g + g\psi + \frac{K_1^2 g v^2}{\delta}\psi t_0.$$

Note that $\frac{R^4}{2} \leq \psi \leq R^4$ at $t = 0$, $\phi(v^2) \geq v^2 \geq 1$, and $v^2 \leq \frac{1}{\delta}$. Thus, it holds that

$$|A|^2 \leq \frac{2}{\delta R^4}\left(cR^2\left(2 + \frac{1}{\delta}\right) + \frac{R^4}{T} + \frac{K_1^2 R^4}{\delta^2}\right) \leq K\left(1 + \frac{1}{\delta^3}\right)\left(\frac{1}{T} + \frac{1}{R^2}\right) \tag{A.2.7}$$

where $K = K(K_1, c)$, thus we conclude. □

A.3 Geometric properties

Lemma A.3.1. *[FK14, Lemma 23 and 24] Let us consider two sets* $\Omega_1, \Omega_2 \in S_{r,R}$ *for* $R > r > 0$*. Then the following holds:*

$$d_H(\Omega_1, \Omega_2) \leq d_H(\partial\Omega_1, \partial\Omega_2), \quad d_H(\partial\Omega_1, \partial\Omega_2) \lesssim_{r,R} d_H(\Omega_1, \Omega_2), \quad |\Omega_1 \Delta \Omega_2| \lesssim_{r,R} d_H(\Omega_1, \Omega_2), \tag{A.3.1}$$

$$\left|\tilde{d}(\Omega_1, E) - \tilde{d}(\Omega_2, E)\right|, \left|\tilde{d}(E, \Omega_1) - \tilde{d}(E, \Omega_2)\right| \lesssim_{r,R} d_H(\Omega_1, \Omega_2) \text{ for any } E \in S_{r,R},$$

Lemma A.3.2. *Let us consider two sets* $\Omega_1, \Omega_2 \in S_{r,R}$ *for* $R > r > 0$*. Then the following holds for* $\mathcal{K}_1 = \mathcal{K}_1(r, R) = w_n^{-1}\left(\frac{4R}{r}\right)^{n+1}$*:*

$$d_H(\Omega_1, \Omega_2)^{n+1} \leq \mathcal{K}_1 \tilde{d}^2(\Omega_1, \Omega_2) \text{ and } d_H(\Omega_1, \Omega_2)^{n+1} \leq \mathcal{K}_1 \tilde{d}^2(\Omega_2, \Omega_1) \tag{A.3.2}$$

Proof. Due to the first inequality of (A.3.1) in Lemma A.3.1, it is enough to show that

$$d_H(\partial\Omega_1, \partial\Omega_2)^{n+1} \leq w_n^{-1}\left(\frac{4R}{r}\right)^{n+1} \tilde{d}^2(\Omega_1, \Omega_2) \text{ and } d_H(\partial\Omega_1, \partial\Omega_2)^{n+1} \leq w_n^{-1}\left(\frac{4R}{r}\right)^{n+1} \tilde{d}^2(\Omega_2, \Omega_1).$$

Without loss of generality, let us assume that $d_H(\partial\Omega_1, \partial\Omega_2) = \sup_{x\in\partial\Omega_1} d(x, \partial\Omega_2)$. Since $\partial\Omega_1$ and $\partial\Omega_2$ are compact, there exists $x_1 \in \partial\Omega_1$ and $x_2 \in \partial\Omega_2$ such that $\sup_{x\in\partial\Omega_1} d(x, \partial\Omega_2) = d(x_1, \partial\Omega_2) = |x_1 - x_2|$. Since $\Omega_2 \in S_r$, there exists $y \in \partial\Omega_2$ such that x_1 and y are parallel. Note that we have $d(x_1, \partial\Omega_2) \leq |x_1 - y|$.

We argue for the case $|x_1| < |y|$. Since $x_1 \in \partial\Omega_1$ and $y \in \partial\Omega_2$, there exists an exterior cone $EC(x_1, r)$ and an interior cone $IC(y, r)$ given in (1.3.3) and (1.3.5) such that $EC(x_1, r) \cap IC(y, r) \subset \Omega_2 \setminus \Omega_1$. Note that, for $\theta \in (0, \frac{\pi}{2})$ such that $\sin(\theta) = \frac{r}{R}$, we have

$$(x_1 + C(x_1, \theta)) \cap (y + C(-y, \theta)) \subset EC(x_1, r) \cap IC(y, r).$$

Note also that there is $\delta = \delta(r, R)$ such that

$$B_{2\delta|x_1-y|}\left((x_1 + y)/2\right) \subset (x_1 + C(x_1, \theta)) \cap (y + C(-y, \theta)).$$

Specifically, as x_1 and y are parallel, the above inequality holds for

$$\delta(r, R) = \frac{\sin(\theta)}{4} = \frac{r}{4R}. \tag{A.3.3}$$

Then, it holds that

$$\tilde{d}^2(\Omega_1, \Omega_2) \geq \int_{\Omega_1 \triangle \Omega_2} d(x, \partial\Omega_2) dx \geq \int_{B_{\delta|x_1-y|}((x_1+y)/2)} \delta|x_1 - y| dx = w_n \delta^{n+1} |x_1 - y|^{n+1}.$$

The same inequality holds for $\tilde{d}^2(\Omega_2, \Omega_1)$ and thus we can conclude. Lastly, if $|x_1| < |y|$, then we can apply the parallel arguments in $(x_1 + C(-x_1, \theta)) \cap (y + C(y, \theta)) \subset \Omega_1 \setminus \Omega_2$. □

Lemma A.3.3. *[FK14, Lemma 24] The metric space $(\partial S_{r,R}, d_H)$ is compact:*

1. *Suppose that $\Gamma_j \in (\partial S_{r,R}, d_H)$ for some $r, R > 0$ and all $j \in \mathbb{N}$. Then $\{\Gamma_j\}_{j \in \mathbb{N}}$ has a subsequence that converges and any subsequential limit is also in $\partial S_{r,R}$.*

2. *Let I be a compact interval in $\mathbb{R}$ and $\Gamma_j : I \to \partial S_{r,R}$ for $j \in \mathbb{N}$ is an equicontinuous sequence of paths in $(\partial S_{r,R}, d_H)$. Then, there is a subsequence of the $\Gamma_j(\cdot)$ that converges uniformly on I on a path $\Gamma : I \to (\partial S_{r,R}, d_H)$.*

Lemma A.3.4. *For $r > 0$ and $x \in \mathbb{R}^n$ such that $|x| \geq r$, it holds that*

$$IC(x, r) = \{\alpha x + (1 - \alpha) y : \alpha \in (0, 1), y \in B_r(0)\}. \tag{A.3.4}$$

Here, $IC(\cdot, \cdot)$ is given in (1.3.3).

Proof. The proof is based on the geometry of interior cones describe in Figure 1.2. Let us show that

$$\mathcal{N} := \{\alpha x + (1 - \alpha) y : \alpha \in (0, 1), \quad y \in B_r(0)\} \subset IC(x, r). \tag{A.3.5}$$

For $z \in \mathcal{N}$, we fix $\alpha \in (0, 1)$ and $y \in B_r(0)$ satisfying $z := \alpha x + (1 - \alpha) y$. If $z \in B_r(0)$, then it can be checked that $z \in IC(x, r)$. Let us assume that $z \in B_r(0)^C$ and show that

$$z \in (x + C(-x, \theta_{x,r})) \cap C\left(x, \frac{\pi}{2} - \theta_{x,r}\right). \tag{A.3.6}$$

Note that $x + C(-x, \theta_{x,r})$ is a convex set and $y \in B_r(0) \subset x + C(-x, \theta_{x,r})$ (See Figure 1.2) and thus $z \in x + C(-x, \theta_{x,r})$. It remains to show that

$$z \in C\left(x, \frac{\pi}{2} - \theta_{x,r}\right). \tag{A.3.7}$$

As $y \in B_r(0)$ and $z \in B_r(0)^C$, there two intersection points z_1 and z_2 between $\partial B_r(0)$ and the line passing through y and z such that

$$z_i := \alpha_i x + (1-\alpha_i)y \in \partial B_r(0) \text{ for } i = 1, 2 \text{ and } |x - z_1| < |x - z_2|$$

for some $\alpha_1 \in (0, \alpha]$ and $\alpha_2 < 0$. As z_1 and z_2 are intersection points between a circle and a line, it holds that

$$|x|^2 - r^2 = |x - z_1||x - z_2| \text{ and thus } |x - z_1| < \sqrt{|x|^2 - r^2}.$$

As $x \in C\left(x, \frac{\pi}{2} - \theta_{x,r}\right)$ and $d(x, \partial C\left(x, \frac{\pi}{2} - \theta_{x,r}\right)) = \sqrt{|x|^2 - r^2}$ (See Figure 1.2), we conclude that $z_1 \in C\left(x, \frac{\pi}{2} - \theta_{x,r}\right)$. As $C\left(x, \frac{\pi}{2} - \theta_{x,r}\right)$ is a convex set, we conclude (A.3.7) and thus (A.3.5) holds.

The opposite relation can be shown by similar geometric arguments. As $B_r(0) \subset \mathcal{N}$, it suffices to show that

$$z \subset \mathcal{N} \text{ for all } z \in \left\{(x + C(-x, \theta_{x,r})) \cap C\left(x, \frac{\pi}{2} - \theta_{x,r}\right)\right\} \setminus B_r(0).$$

Consider a line passing through x and z, we can find a point $y \in B_r(0)$ such that $z = \alpha x + (1-\alpha)y$ for some $\alpha \in (0, 1)$. □

Lemma A.3.5. *For $x, z \in \mathbb{R}^n$ and $r > c > 0$, assume that $|x| \geq r$ and $|z| < c$. Then, it holds that*

$$IC(x + z, r - c) \subset IC(x, r) + z. \tag{A.3.8}$$

Here, $IC(\cdot, \cdot)$ is given in (1.3.3).

Proof. We claim that for $\alpha \in (0, 1)$ and $y \in B_{r-c}(0)$, it holds that

$$\alpha(x + z) + (1-\alpha)y \in IC(x, r) + z. \tag{A.3.9}$$

Note that

$$\alpha(x + z) + (1-\alpha)y - z = \alpha x + (1-\alpha)(y - z).$$

As $y \in B_{r-c}(0)$ and $z \in B_c(0)$, we have $y - z \in B_r(0)$. From Lemma A.3.4, we have (A.3.9). From Lemma A.3.4 again, we conclude (A.3.8). □

Lemma A.3.6. *Let us consider two sets* $\Omega_1, \Omega_2 \in S_{r,R}$ *for* $R > r > 0$. *Then the following holds:*

$$\sup_{x \in \partial\Omega_2} d(x, \partial\Omega_1) \leq \frac{R}{r} \sup_{x \in \partial\Omega_1} d(x, \partial\Omega_2). \tag{A.3.10}$$

Proof. If $\sup_{x \in \partial\Omega_2} d(x, \partial\Omega_1) = 0$, then (A.3.10) holds. We suppose that $\sup_{x \in \partial\Omega_2} d(x, \partial\Omega_1) > 0$. As $\Omega_2 \in S_{r,R}$, there exists $x_2 \in \partial\Omega_2$ such that

$$\sup_{x \in \partial\Omega_2} d(x, \partial\Omega_1) = d(x_2, \partial\Omega_1) =: l > 0. \tag{A.3.11}$$

As a consequence, we have

$$B_l(x_2) \subset \Omega_1^C \text{ and } B_l(x_2) \subset \Omega_1. \tag{A.3.12}$$

Let us assume the former one. As $\Omega_1 \in S_{r,R}$, there exists $x_1 \in \partial\Omega_1$ such that x_1 is in the line segment between the origin and x_2. From (A.3.12), $|x_1 - x_2| \geq l$. From the interior cone property of $S_{r,R}$ in Lemma 1.3.4, it holds that

$$d(x_1, \partial\Omega_2) \geq d(x_1, \partial IC(x_2, r)) \geq \frac{lr}{R} \tag{A.3.13}$$

and we conclude (A.3.10). The latter case in (A.3.12) can be shown by the parallel arguments.

□

Proof of Lemma 1.5.8.

From Lemma 1.3.4, it holds that for all $x \in \partial E$,

$$IC(x, r) \subset E \text{ and } EC(x, r) \subset E^c \tag{A.3.14}$$

where IC is an interior cone given in (1.3.3), and EC is an exterior cone given in (1.3.5). Note that as $|x| \leq R$, the angle of both the interior cone and exterior cone, θ_x, is bounded from below as follows,

$$\theta_x := \arcsin \frac{r}{|x|} \geq \arcsin \frac{r}{R}. \tag{A.3.15}$$

Thus, for $\eta_1(r, R) := |IC(Re_1, r) \cap B_\varepsilon(Re_1)|$, it holds that for $\varepsilon \in (0, r)$

$$\eta_1 \varepsilon^n \leq |IC(x, r) \cap B_\varepsilon(x)| \leq |E \cap B_\varepsilon(x)|. \tag{A.3.16}$$

Here, e_1 is a unit vector in the positive x_1 direction. Similarly, it holds that

$$|B_\varepsilon(x) \setminus E| \geq |B_\varepsilon(x) \cap EC(x,r)| \geq \eta_1 \varepsilon^n. \tag{A.3.17}$$

As $E \in S_{r,R}$, there exists $\varepsilon_0 = \varepsilon(r,R) < r$ such that for all $\varepsilon \in (0, \varepsilon_0)$

$$B_\varepsilon(x) \cap \partial E = (U, f(U)) \tag{A.3.18}$$

up to rotation for some Lipschitz function $f = f_{x,\varepsilon} : U \subset B_\varepsilon^{n-1}(x) \to \mathbb{R}$. Note that as $E \in S_{r,R}$, the Lipschitz constant of f is uniformly bounded by some constant $\Lambda = \Lambda(r,R)$.

From Theorem 9.1 in [Mag12],

$$\mathcal{H}^{n-1}(B_\varepsilon(x) \cap \partial E) = \int_U \sqrt{1 + |\nabla f|^2} dx \leq |U|\sqrt{1+\Lambda^2} \leq n w_n \varepsilon^{n-1}\sqrt{1+\Lambda^2}. \tag{A.3.19}$$

Thus, (1.5.20) holds with $\eta_2(r,R) := n w_n \sqrt{1+\Lambda^2}$. Here, w_n is a volume of a unit ball in $\mathbb{R}^n$. On the other hand, from the isoperimetric inequality in [Mag12, Proposition 12.37] and (1.5.19), we get the lower bound of (1.5.20). □

For $E, F \subset \mathbb{R}^n$, define the Hausdorff distance by

$$d_H(E,F) := \max\left\{\sup_{x\in E} d(x,F), \sup_{x \in F} d(x,E)\right\}. \tag{A.3.20}$$

We say that $A \lesssim_{r,R} B$ if there exists a constant $C = C(r,R) > 0$ depending on r, R such that $A \leq CB$.

Lastly, let us show the following property of characteristic functions.

Lemma A.3.7. *Let $\{(\Omega_t^k)_{t\geq 0}\}_{k\in\mathbb{N}}$ be a sequence of sets in $S_{r,R}$ for $0 < r < R$. Suppose that Ω_t^k converges locally uniformly to Ω_t^∞ on $[0,+\infty)$. For a sequence of functions $\{u_k\}_{k\in\mathbb{N}\cup\{+\infty\}}$ defined by*

$$u_k := \chi_{\Omega_t^k} - \chi_{(\Omega_t^k)^C} \text{ for } k \in \mathbb{N}\cup\{+\infty\}, \tag{A.3.21}$$

it holds that

$$u_\infty^* = \limsup_{k\to\infty}{}^* u_k \text{ and } (u_\infty)_* = \liminf_{k\to\infty}{}_* u_k. \tag{A.3.22}$$

Here, $\limsup^*$ *and* $\liminf_*$ *are given in* (1.2.5).

Proof. Let us show the first equation in (A.3.22) only. The second one can be shown by the parallel arguments.

By uniform convergence of Ω_t^k in a finite interval, for any $j \in \mathbb{N}$, there exists $k_1 > 0$ such that for all $k > k_1$

$$d_H(\Omega_t^k, \Omega_t^\infty) < \frac{1}{j}. \tag{A.3.23}$$

Thus, for any $x \in \Omega_t^\infty$ and $k > k_1$, there exists $y \in \Omega_t^k$ such that $|x - y| < \frac{1}{j}$. Thus, we conclude that

$$\limsup_{k\to\infty}{}^* u_k(x,t) = \lim_{j\to\infty} \sup \left\{ u_k(y,s) : k \geq j, \quad |y - x| \leq \frac{1}{j}, \quad |s - t| \leq \frac{1}{j} \right\} = 1 \tag{A.3.24}$$

and $u_\infty^*(x) = \limsup_{k\to\infty}{}^* u_k(x)$ for $x \in \Omega_t^\infty$.

Note that we have for any sets $\Omega_1, \Omega_2 \in S_{r,R}$

$$d_H(\Omega_1^C, \Omega_2^C) \leq d_H(\partial\Omega_1, \partial\Omega_2). \tag{A.3.25}$$

Combining this with Lemma A.3.1, we conclude that $(\Omega_t^k)^C$ converges locally uniformly to $(\Omega_t^\infty)^C$. By parallel arguments, for any $x \in (\Omega_t^\infty)^C$, we conclude that $\limsup_{k\to\infty}{}^* u_k(x,t) = -1$. As $\limsup_{k\to\infty}{}^* u_k$ is upper semicontinuous, we conclude (A.3.22). □

Lemma A.3.8. *For any function $u : Q \to \mathbb{R}$ and $\Theta \in C([0, +\infty))$, it holds that*

$$\widehat{u^*}(\cdot;\Theta) = \widehat{u}(\cdot;\Theta)^* \tag{A.3.26}$$

and

$$\widetilde{u_*}(\cdot;\Theta) = \widetilde{u}(\cdot;\Theta)_*. \tag{A.3.27}$$

Proof. Let us only show (A.3.26). The parallel arguments imply (A.3.27).

Let us assume that both sides are finite at $(x_0, t_0) \in Q$. We claim that

$$\widehat{u^*}(x_0, t_0; \Theta) \leq \widehat{u}(x_0, t_0; \Theta)^*. \tag{A.3.28}$$

By the upper semicontinuity of $(\widehat{u})^*$, for $\varepsilon > 0$ there exists $\delta \in (0, t_0)$ such that

$$\widehat{u}(\cdot, \cdot; \Theta) < \widehat{u}(x_0, t_0; \Theta)^* + \varepsilon \text{ in } B_\delta(x_0) \times (t_0 - \delta, t_0 + \delta). \tag{A.3.29}$$

From the definition of $\widehat{u}$ in (1.2.28), it holds that

$$u(\cdot,\cdot) < \widehat{u}(x_0,t_0;\Theta)^* + \varepsilon \text{ in } \mathcal{N}_1 := \{(x,t) \in Q : |t-t_0| < \delta \text{ and } |x-x_0| < \delta + \Theta(t)\}. \tag{A.3.30}$$

Furthermore, by the continuity of Θ, for any $y \in \overline{B}_{\Theta(t_0)}(x_0)$ there exists a small neighborhood $\mathcal{N}_2$ of (y,t_0) in Q such that $\mathcal{N}_2 \subset \mathcal{N}_1$. From (A.3.30), we have

$$u(y,t_0)^* \leq \widehat{u}(x_0,t_0;\Theta)^* + \varepsilon \text{ for all } y \in \overline{B}_{\Theta(t_0)}(x_0). \tag{A.3.31}$$

As ε is arbitrary, we conclude (A.3.28).

Next, let us show the opposite inequality,

$$\widehat{u^*}(x_0,t_0;\Theta) \geq \widehat{u}(x_0,t_0;\Theta)^*. \tag{A.3.32}$$

For any $\varepsilon > 0$, let us show that there exists $\delta \in (0,t_0)$ such that

$$\widehat{u^*}(x_0,t_0;\Theta) > u(\cdot,\cdot) - \varepsilon \text{ in } B_{\delta+\Theta(t_0)}(x_0) \times (t_0-\delta, t_0+\delta). \tag{A.3.33}$$

If not, then there exists $\{(y_k,s_k)\}_{k\in\mathbb{N}} \subset Q$ such that

$$|s_k - t_0| \leq \frac{1}{k}, \quad |y_k - x_0| < \frac{1}{k} + \Theta(t_0) \text{ and } \widehat{u^*}(x_0,t_0) \leq u(y_k,s_k) - \varepsilon. \tag{A.3.34}$$

Then, $\{s_k\}_{k\in\mathbb{N}}$ converges to t_0. Also, by compactness of $\overline{B}_{\Theta(t_0)+1}(x_0)$, there exists a subsequence $\{k_i\}_{i\in\mathbb{N}}$ and $y^* \in \overline{B}_{\Theta(t_0)+1}(x_0)$ such that $\{y_{k_i}\}_{i\in\mathbb{N}}$ converges to y^*. Thus, it holds that

$$\widehat{u^*}(x_0,t_0) \leq \limsup_{k\to\infty} u(y_k,s_k) - \varepsilon \leq u^*(y^*,t_0) - \varepsilon. \tag{A.3.35}$$

On the other hand, (A.3.34) implies $y^* \in \overline{B}_{\Theta(t_0)}(x_0)$, which contradicts to (A.3.35). As a consequence, we get (A.3.33).

By (A.3.33) and the continuity of Θ, there exists $\delta_1 \in (0,\delta)$ such that

$$\widehat{u^*}(x_0,t_0;\Theta) > u(\cdot,\cdot) - \varepsilon \text{ in } \{(x,t) \in Q : |t-t_0| < \delta_1 \text{ and } |x-x_0| < \delta_1 + \Theta(t)\}. \tag{A.3.36}$$

As ε is arbitrary, we get (A.3.32) and conclude (A.3.26).

Similar arguments can be applied if either the left hand side or right hand side in (A.3.26) is infinity at $(x_0, t_0) \in Q$. In particular, for any $\varepsilon \in (0, t_0)$, there exists a sequence $\{x_k, t_k\}_{k\in\mathbb{N}}$ in $(t_0 - \varepsilon, t_0 + \varepsilon) \times B_{\Theta(t_0)+\varepsilon}(x_0)$ such that $u(x_k, t_k)$ converging to infinity. This implies that the other side is infinity. □

A.4 Optimal transport toolbox

Let us recall now some basic definitions and results from the theory of optimal transport. Let $\Pi(\mu, \nu)$ be the set of all Borel probability measure π on $\Omega \times \Omega$ such that

$$\pi(A \times \Omega) = \mu(A), \quad \pi(\Omega \times B) = \nu(B) \text{ for all measurable subsets } A, B \subset \Omega. \tag{A.4.1}$$

For $\mu, \nu \in \mathscr{P}_2(\Omega)$ we define the 2-Wasserstein or Monge-Kantorovich distance as

$$W_2(\mu, \nu) := \min \left\{ \int_{\Omega\times\Omega} |x - y|^2 \mathrm{d}\gamma : \gamma \in \Pi(\mu, \nu) \right\}^{\frac{1}{2}} \tag{A.4.2}$$

For $\phi : \Omega \to \mathbb{R}$ measurable, we use the notations

$$\phi^+(x) := \max\{\phi(x), 0\}, \phi^-(x) := \max\{-\phi(x), 0\} \text{ and } \phi^c(x) := \operatorname*{ess\,inf}_{y\in\Omega} \left\{ \frac{1}{2}|x - y|^2 - \phi(y) \right\} \tag{A.4.3}$$

where $x \in \Omega$.

A.4.1 Basic facts from optimal transport

Let us recall the definition and properties of Kantorovich potentials and optimal transport maps. There results are well-known in the literature, we refer for instance to [San15] for the proofs of the statements.

Definition A.4.1. Let $\mu, \nu \in \mathscr{P}(\Omega)$ be given.

1. We say that $\overline{\phi} : \Omega \to \mathbb{R}$ is a Kantorovich potential from μ to ν if $(\overline{\phi}, \overline{\phi}^c)$ is a maximizer of the *Kantorovich* problem:

   $$\sup \left\{ \int_\Omega \phi \mathrm{d}\mu + \int_\Omega \psi \mathrm{d}\nu : (\phi, \psi) \in L^1_\mu(\Omega) \times L^1_\nu(\Omega), \phi(x) + \psi(y) \le \frac{1}{2}|x - y|^2,\ \mu \otimes \nu - \text{a.e. } (x, y) \in \Omega \times \Omega \right\}.$$

   We denote the set of Kantorovich potential from μ to ν by $\mathcal{K}(\mu, \nu)$.

2. We say that a Borel map $T : \Omega \to \Omega$ is a optimal transport map from μ to ν if T is a minimizer of the following problem:

$$\inf\left\{\int_\Omega |x - T(x)|^2 \mathrm{d}\mu : T_\# \mu = \nu\right\}. \tag{A.4.4}$$

Here, $(T_\#\mu)(A) := \mu(T^{-1}(A))$ for any Borel set $A \subseteq \Omega$.

Lemma A.4.2 ([San15])**.** *For $\mu \in \mathscr{P}^{ac}(\Omega)$ and $\nu \in \mathscr{P}(\Omega)$, there exists a Lipschitz continuous Kantorovich potential $\overline{\phi}$ and an optimal transport map T from μ to ν. Also, it holds that*

$$x - T(x) = \nabla\overline{\phi}(x) \text{ for a.e. } x \in spt(\mu) \text{ and } W_2(\mu, \nu) = \|\nabla\overline{\phi}\|_{L^2_\mu}. \tag{A.4.5}$$

Lemma A.4.3. *[Vil03, Theorem 1.3],[San15, Proposition 1.11] Let $\mu, \nu \in \mathscr{P}(\Omega)$. Define $\mathcal{L} : L^1_\mu(\Omega) \times L^1_\nu(\Omega) \to \mathbb{R}$ as*

$$\mathcal{L}(\phi, \psi) := \int_\Omega \phi \, d\mu + \int_\Omega \psi \, d\nu \tag{A.4.6}$$

Then, it holds that

$$\frac{1}{2}W_2^2(\mu, \nu) = \max\left\{\mathcal{L}(\phi, \psi) : (\phi, \psi) \in C_b(\Omega) \times C_b(\Omega),\ \phi(x) + \psi(y) \le \frac{|x - y|^2}{2} \text{ for all } x, y \in \Omega\right\},$$
$$= \sup\left\{\mathcal{L}(\phi, \psi) : (\phi, \psi) \in L^1_\mu(\Omega) \times L^1_\nu(\Omega),\ \phi(x) + \psi(y) \le \frac{|x - y|^2}{2} \text{ for } \mu \otimes \nu \text{ a.e. } (x, y) \in \Omega \times \Omega\right\}.$$

Proposition A.4.4. *For $r \in [1, +\infty]$, let $\mu \in L^r(\Omega) \cap \mathscr{P}(\Omega)$ and $\nu \in \mathscr{P}(\Omega)$. Then, it holds that*

$$\sup_{\phi \in L^{r'}(\Omega)} \mathcal{L}(\phi, \phi^c) = \frac{1}{2}W_2^2(\mu, \nu) \tag{A.4.7}$$

where $r' := \frac{r}{r-1}$ ($r' = 1$ if $r = +\infty$ and $r' = +\infty$ if $r = 1$) and $\mathcal{L}$ is given in (A.4.6).

Proof. 1. Let us show that

$$\frac{1}{2}W_2^2(\mu, \nu) = \mathcal{I}_1 \tag{A.4.8}$$

where $\mathcal{I}_1 :=$

$$\sup\left\{\mathcal{L}(\phi, \psi) : (\phi, \psi) \in L^{r'}(\Omega) \times L^1_\nu(\Omega),\ \phi(x) + \psi(y) \le \frac{|x - y|^2}{2} \text{ for } \mu \otimes \nu \text{ a.e. } (x, y) \in \Omega \times \Omega\right\}. \tag{A.4.9}$$

By Hölder's inequality, it holds that

$$\|\phi\|_{L^1_\mu(\Omega)} = \int_\Omega |\phi(x)|\mu(x)\mathrm{d}x \leq \|\phi\|_{L^{r'}(\Omega)}\|\mu\|_{L^r(\Omega)}. \tag{A.4.10}$$

As $\mu \in L^r(\Omega) \cap \mathscr{P}(\Omega)$, we conclude that

$$L^{r'}(\Omega) \subset L^1_\mu(\Omega) \text{ and thus } C_b(\Omega) \times C_b(\Omega) \subset L^{r'}(\Omega) \times L^1_\nu(\Omega) \subset L^1_\mu(\Omega) \times L^1_\nu(\Omega). \tag{A.4.11}$$

From Lemma A.4.3, we conclude (A.4.8).

2. It remains to show that

$$\sup_{\phi \in L^{r'}(\Omega)} \mathcal{L}(\phi, \phi^c) = \mathcal{I}_1 \tag{A.4.12}$$

for $\mathcal{I}_1$ given in (A.4.9). Indeed, let us notice that by density we have

$$\sup_{\phi \in L^{r'}(\Omega)} \mathcal{L}(\phi, \phi^c) = \sup_{\phi \in C_b(\Omega)} \mathcal{L}(\phi, \phi^c) = \max_{\phi \in C_b(\Omega)} \mathcal{L}(\phi, \phi^c),$$

and the latter two quantities are finite by [San15, Proposition 1.11]. Therefore the thesis of the proposition follows. □

A.4.2 Some properties of minimizers in the minimizing movements scheme and optimality conditions

Lemma A.4.5. *For ρ_k given in* (2.2.3) *and $\mathcal{S}$ satisfying* (2.4.2), *it holds that $\rho_k > 0$ a.e.*

Proof. The proof is inspired by [San15, Lemma 8.6]. The difference is that we consider the sub-differential of $\mathcal{S}$ instead of its derivative.

1. For simplicity, let us use the notation $\mu := \rho_k$ and consider a competitor

$$\mu_1 := \frac{1}{\mathscr{L}^d(\Omega)}. \tag{A.4.13}$$

Define $\mu_\varepsilon := (1-\varepsilon)\mu + \varepsilon\mu_1$ for $\varepsilon \in (0,1)$. From convexity of Wasserstein distance, we have

$$\begin{aligned}\mathcal{I}_1 := \mathcal{J}(\mu) - \mathcal{J}(\mu_\varepsilon) &\leq \frac{1}{2\tau}W_2^2(\mu_\varepsilon, \rho_{k-1}) - \frac{1}{2\tau}W_2^2(\mu, \rho_{k-1}),\\ &\leq \varepsilon\left\{\frac{1}{2\tau}W_2^2(\mu_1, \rho_{k-1}) - \frac{1}{2\tau}W_2^2(\mu, \rho_{k-1})\right\}\end{aligned}$$

The compactness of Ω implies

$$\mathcal{I}_1 \leq C_1\varepsilon \text{ for some } C_1 > 0. \tag{A.4.14}$$

2. Set $A := \{x \in \Omega : \mu > 0\}$ and $B := \{x \in \Omega : \mu = 0\}$. Let us show that $\mathscr{L}^d(B) = 0$. For sufficiently small $\varepsilon > 0$, it holds that $\varepsilon\mu_1 < 1$ and thus

$$\mathcal{I}_1 = \int_A \mathcal{S}(\mu(x)) - \mathcal{S}(\mu_\varepsilon(x)) + \Phi[\mu(x) - \mu_\varepsilon(x)]\mathrm{d}x + (\mathcal{S}(0) - \mathcal{S}(\varepsilon\mu_1))\mathscr{L}^d(\mathcal{B}) - \varepsilon\frac{1}{\mathscr{L}^d(\Omega)}\int_B \Phi\mathrm{d}x \tag{A.4.15}$$

By convexity of $\mathcal{S}$, it holds that

$$\mathcal{I}_1 \geq \varepsilon\int_A [\xi_\varepsilon(x) + \Phi](\mu(x) - \mu_1)\mathrm{d}x + (\mathcal{S}(0) - \mathcal{S}(\varepsilon\mu_1))\mathscr{L}^d(B) - \varepsilon\frac{1}{\mathscr{L}^d(\Omega)}\int_B \Phi\mathrm{d}x, \tag{A.4.16}$$

where $\xi_\varepsilon(x) \in \partial\mathcal{S}(\mu_\varepsilon(x))$.

From (A.4.14), we conclude that for all $\xi_\varepsilon(x) \in \partial\mathcal{S}(\mu_\varepsilon(x))$

$$\mathcal{I}_2 := \int_A [\xi_\varepsilon(x) + \Phi](\mu(x) - \mu_1)\mathrm{d}x + \frac{1}{\varepsilon}(\mathcal{S}(0) - \mathcal{S}(\varepsilon\mu_1))\mathscr{L}^d(\mathcal{B}) \leq C_1 + C. \tag{A.4.17}$$

Note that by the convexity of S, its subdifferential is monotone, therefore for all $\varepsilon \in [0, 1]$,

$$(\xi_\varepsilon(x) - \xi_1)(\mu_\varepsilon(x) - \mu_1) \geq 0,$$

and thus

$$\xi_\varepsilon(x)(\mu(x) - \mu_1) \geq \xi_1(\mu(x) - \mu_1). \tag{A.4.18}$$

for a.e. $x \in \Omega$ where $\xi_1 \in \partial\mathcal{S}(\mu_1)$. Therefore,

$$\mathcal{I}_2 \geq \int_A [\xi_1 + \Phi](\mu(x) - \mu_1)\mathrm{d}x + \frac{1}{\varepsilon}(\mathcal{S}(0) - \mathcal{S}(\varepsilon\mu_1))\mathscr{L}^d(B)$$

Since $\mathcal{S}'(0+) = -\infty$ from (2.4.2), the right hand side blows up as ε goes to zero unless $\mathscr{L}^d(B) = 0$. As $\mathcal{I}_2$ is bounded by $C_1 + C$ from (A.4.17), we conclude that $\mathscr{L}^d(B) = 0$, and thus $\mu > 0$ a.e. □

A.5 Some results from convex analysis

For a Banach space $\mathfrak{X}$ and $F : \mathfrak{X} \to \mathbb{R} \cup \{\pm\infty\}$, we say that $F^* : \mathfrak{X}^* \to \mathbb{R} \cup \{\pm\infty\}$ is a Legendre transform of F if

$$F^*(y) := \sup_{x \in \mathfrak{X}} \{\langle x, y \rangle_{\mathfrak{X},\mathfrak{X}^*} - F(x)\} \text{ for } y \in \mathfrak{X}^*. \tag{A.5.1}$$

Here, $\mathfrak{X}^*$ stands for the topological dual space of $\mathfrak{X}$. We will denote by $C_b(\Omega)$ the space of bounded continuous functions in Ω. In the derivation of optimality conditions associated to the minimizing movement schemes, in Section A.5, we use subdifferential calculus in $L^r(\Omega)$ ($r \in [1, +\infty]$) spaces. Let us recall some basic results on this.

Let us recall the definition of subdifferentials on $L^r(\Omega)^*$ for $r \in [1, +\infty]$.

Definition A.5.1. [Roc71, (1.9), (1.10) & (1.13)] For $\psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, $r \in [1, +\infty]$ and $\Psi : L^r(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\Psi(\mu) := \int_\Omega \psi(\mu(x))\mathrm{d}x, \tag{A.5.2}$$

we say that $\xi \in L^r(\Omega)^*$ belongs to the subdifferential of Ψ at $\mu \in L^r(\Omega)$ if

$$\Psi(\nu) \geq \Psi(\mu) + \langle \xi, \nu - \mu \rangle_{L^r(\Omega)^*, L^r(\Omega)} \tag{A.5.3}$$

for every $\nu \in L^r(\Omega)$. We denote by $\partial\Psi(\mu)$ the set of subdifferentials of Ψ at the point $\mu \in L^r(\Omega)$.

Definition A.5.2. [ET76, Definition 1.3.1] Let $\mathfrak{X}$ be a Banach space. The set of functions $F : \mathfrak{X} \to \mathbb{R} \cup \{\pm\infty\}$ which are pointwise supremum of a family of continuous affine function is denoted by $\Gamma(\mathfrak{X})$.

Lemma A.5.3. *[ET76, Proposition 1.3.1] The following properties are equivalent to each other:*

1. *$F \in \Gamma(\mathfrak{X})$*

2. *F is a convex lower semicontinuous function from $\mathfrak{X}$ into $\mathbb{R} \cup \{\pm\infty\}$ and if F takes the value $-\infty$, then F is identically equal to $-\infty$.*

Lemma A.5.4. *[ET76, Proposition 1.5.6] If $F_1, F_2 \in \Gamma(\mathfrak{X})$ and if there exists $\hat{\mu} \in \mathfrak{X}$ such that $F_1(\hat{\mu}), F_2(\hat{\mu}) < +\infty$ and either F_1 or F_2 is continuous at $\hat{\mu}$, then it holds that*

$$\partial F_1(\mu) + \partial F_2(\mu) = \partial(F_1 + F_2)(\mu) \text{ for all } \mu \in \mathfrak{X}. \tag{A.5.4}$$

A.6 An Aubin-Lions lemma and some of its consequences

In [RS03] the authors presented the following version of the classical Aubin-Lions lemma (see [Aub63]):

Theorem A.6.1. *[RS03, Theorem 2] Let B be a Banach space and $\mathcal{U}$ be a family of measurable B-valued function. Let us suppose that there exist a normal coercive integrand $\mathfrak{F} : (0,T) \times B \to [0, +\infty]$, meaning that*

(1) $\mathfrak{F}$ is $\mathscr{B}(0,T) \otimes \mathscr{B}(B)$-measurable, where $\mathscr{B}(0,T)$ and $\mathscr{B}(B)$ denote the σ-algebgras of the Lebesgue measurable subsets of $(0,T)$ and of the Borel subsets of B respectively;

(2) the maps $v \mapsto \mathfrak{F}_t(v) := \mathfrak{F}(t,v)$ are l.s.c. for a.e. $t \in (0,T)$;

(3) $\{v \in B : \mathfrak{F}_t(v) \leq c\}$ are compact for any $c \geq 0$ and for a.e. $t \in (0,T)$,

and a l.s.c. map $g : B \times B \to [0, +\infty]$ with the property

$$[u, v \in D(\mathfrak{F}_t),\ g(u,v) = 0] \Rightarrow u = w,\ \text{for a.e. } t \in (0,T).$$

If

$$\sup_{u\in\mathcal{U}} \int_0^T \mathfrak{F}(t, u(t))\, dt < +\infty \quad \text{and} \quad \limsup_{h\downarrow 0} \sup_{u\in\mathcal{U}} \int_0^{T-h} g(u(t+h), u(t))\, dt = 0,$$

then $\mathcal{U}$ is relatively compact in $\mathscr{M}(0,T;B)$.

Many recent papers (including [KM18,Lab17]) on gradient flows in the Wasserstein space used the previous theorem to gain pre-compactness of interpolated curves. In our setting we use the following result.

Lemma A.6.2. *Let $T > 0$ and let $q \in [1, +\infty)$ and $n > 0$ be such that $nq^* > 1$, where $q^* := \frac{qd}{d-q}$ (with the convention $q^* \in (0, +\infty)$ is arbitrary if $q \geq d$, and therefore, $n > 0$ and*

$nq^* > 1$ can also be arbitrary). Suppose that $(\rho^\tau)_{\tau>0}$ is a sequence of curves on $[0,T]$ with values in $\mathscr{P}(\Omega)$ and suppose that there exists $C > 0$ such that

$$W_2^2(\rho_t^\tau, \rho_s^\tau) \le C|t - s + \tau|, \ \forall\ 0 \le s < t \le T \tag{A.6.1}$$

and $((\rho^\tau)^n)_{\tau>0}$ is uniformly bounded in $L^q([0,T]; W^{1,q}(\Omega))$ by C. We suppose moreover that there exists $\beta \ge 1$ such that $\|\rho_t^\tau\|_{L^\beta(\Omega)} \le C$ for a.e. $t \in [0,T]$.

(1) Then, $(\rho^\tau)_{\tau>0}$ is pre-compact in $L^\gamma(Q)$, with $1 \le \gamma \le \beta$ if $\beta < nq^*$ and $1 \le \gamma < nq^*$, if $\beta \ge nq^*$.

(2) If in addition, $(\rho^\tau)_{\tau>0}$ is uniformly bounded in $L^{\beta_2}(Q)$ for some $\beta_2 > \gamma$ (where γ is given in (1)), then $(\rho^\tau)_{\tau>0}$ is pre-compact in $L^{\gamma_2}(Q)$, for any $1 \le \gamma_2 < \beta_2$.

Proof. Let us use the previously stated Aubin-Lions lemma, i.e. Theorem A.6.1. Let $1 \le \alpha < q^*$ be fixed (that we set up later) and let us set $B := L^{n\alpha}(\Omega)$, $\mathfrak{F} : L^{n\alpha}(\Omega) \to [0, +\infty]$ defined as

$$\mathfrak{F}(\rho) := \begin{cases} \|\rho^n\|_{W^{1,q}(\Omega)}, & \text{if } \rho^n \in W^{1,q}(\Omega),\ \rho \in \mathscr{P}(\Omega), \\ +\infty, & \text{otherwise} \end{cases}$$

and $g : L^{n\alpha}(\Omega) \times L^{n\alpha}(\Omega) \to [0, +\infty]$ defined as

$$g(\mu, \nu) := \begin{cases} W_2(\mu, \nu), & \text{if } \mu, \nu \in \mathscr{P}(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

In this setting, $(\rho^\tau)_{\tau>0}$ and $\mathfrak{F}$ satisfy the assumptions of Theorem A.6.1. Indeed, from the assumption, one has in particular that $\displaystyle\int_0^T \|(\rho_t^\tau)^n\|_{W^{1,q}(\Omega)}^q \mathrm{d}t \le C$. The injection $W^{1,q}(\Omega) \hookrightarrow L^\alpha(\Omega)$ is compact for any $1 \le \alpha < q^*$, the injection $i : s \mapsto s^{\frac{1}{n}}$ is continuous from $L^\alpha(\Omega)$ to $L^{n\alpha}(\Omega)$ and the sub-level sets of $\rho \mapsto \|\rho^n\|_{W^{1,q}(\Omega)}$ are compact in $L^{n\alpha}(\Omega)$.

Moreover, by the fact that g defines a distance on $D(\mathfrak{F})$ and from (A.6.1), one has that g also satisfies the assumptions from Theorem A.6.1, hence the implication of the theorem holds and one has that $(\rho^\tau)_{\tau\ge 0}$ is pre-compact in $\mathscr{M}(0, T; L^{n\alpha}(\Omega))$. Let us notice that (A.6.1) implies that there exists $\rho \in C([0,T]; \mathscr{P}(\Omega))$ such that up to passing to a subsequence $(\rho_\tau)_{\tau>0}$

converges uniformly (w.r.t. W_2) to ρ as $\tau > 0$. Up to passing to another subsequence, ρ is the limit also in $\mathscr{M}(0,T;L^{n\alpha}(\Omega))$.

From our assumption, we know that $\|\rho_t^\tau\|_{L^\beta(\Omega)} \leq C$ for a.e. $t \in [0,T]$. Now, if $\beta < nq^*$, then setting α such that $n\alpha = \beta$, Lebesgue's dominated convergence theorem implies the strong pre-compactness of $(\rho^\tau)_{\tau>0}$ in $L^\beta(Q)$. Otherwise, Lebesque's dominated convergence implies the strong pre-compactness in $L^\gamma(Q)$ for any $1 \leq \gamma < nq^*$. This concludes the proof of (1).

To show (2), we notice that (1) already implies that $\rho^\tau \to \rho$, strongly in $L^\gamma(Q)$ as $\tau \downarrow 0$ and in particular a.e. in Q. Furthermore, by the by the uniform bounds in $L^{\beta_2}(\Omega)$, with $\beta_2 > \gamma$, for any $1 \leq \gamma_2 < \beta_2$ we have that

$$\int_Q (\rho^\tau)^{\gamma_2}\,\mathrm{d}x\mathrm{d}t \leq \left(T\mathscr{L}^d(\Omega)\right)^{1-\frac{\gamma_2}{\beta_2}} \|\rho^\tau\|_{L^{\beta_2}}^{\gamma_2},$$

which implies that $(\rho^\tau)^{\gamma_2}$ is uniformly integrable on Q. Therefore, Vitali's convergence theorem yields the claim. □

REFERENCES

[AKS10] Dimitra C Antonopoulou, Georgia Karali, and Israel Michael Sigal. "Stability of spheres under volume-preserving mean curvature flow." *Dynamics of Partial Differential Equations*, **7**(4):327–344, 2010.

[Amb04] Luigi Ambrosio. "Transport equation and Cauchy problem for BV vector fields and applications." *Journées équations aux dérivées partielles*, pp. 1–11, 2004.

[And01] Ben Andrews. "Volume-preserving anisotropic mean curvature flow." *Indiana University Mathematics Journal*, **50**(2):783–827, 2001.

[Ath97] Maria Athanassenas. "Volume-preserving mean curvature flow of rotationally symmetric surfaces." *Commentarii Mathematici Helvetici*, **72**(1):52–66, 1997.

[ATW93] Fred Almgren, Jean E Taylor, and Lihe Wang. "Curvature-driven flows: a variational approach." *SIAM Journal on Control and Optimization*, **31**(2):387–438, 1993.

[Aub63] Jean-Pierre Aubin. "Un théorème de compacité." *C. R. Acad. Sci. Paris*, **256**:5042–5044, 1963.

[Bar13] Guy Barles. "An introduction to the theory of viscosity solutions for first-order Hamilton-Jacobi equations and applications." In *Hamilton-Jacobi equations: approximations, numerical analysis and applications*, pp. 49–109. Springer, Heidelberg, Berlin, Heidelberg, 2013.

[BCC09] Giovanni Bellettini, Vicent Caselles, Antonin Chambolle, and Matteo Novaga. "The volume preserving crystalline mean curvature flow of convex sets in." *Journal de Mathématiques Pures et Appliquées*, **92**(5):499–527, 2009.

[BJ92] Péter Bántay and Imre M Jánosi. "Avalanche dynamics from anomalous diffusion." *Physical Review Letters*, **68**(13):2058, 1992.

[Bou08a] Mariane Bourgoing. "Viscosity solutions of fully nonlinear second order parabolic equations with L^1 dependence in time and Neumann boundary conditions." *Discrete & Continuous Dynamical Systems-A*, **21**(3):763–800, 2008.

[Bou08b] Mariane Bourgoing. "Viscosity solutions of fully nonlinear second order parabolic equations with L^1 dependence in time and Neumann boundary conditions. Existence and applications to the level-set approach." *Discrete & Continuous Dynamical Systems-A*, **21**(4):1047–1069, 2008.

[BSS93] Guy Barles, H Mete Soner, and Panagiotis E Souganidis. "Front propagation and phase field theory." *SIAM Journal on Control and Optimization*, **31**(2):439–469, 1993.

[CGG91] Yun Gang Chen, Yoshikazu Giga, and Shun'ichi Goto. "Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations." *Journal of Differential Geometrys*, **33**(3):749–786, 1991.

[Cha04] Antonin Chambolle. "An algorithm for mean curvature motion." *Interfaces and Free Boundaries*, **6**(2):195–218, 2004.

[CIL92] Michael G Crandall, Hitoshi Ishii, and Pierre-Louis Lions. "User's guide to viscosity solutions of second order partial differential equations." *Bulletin of the American Mathematical Society*, **27**(1):1–67, 1992.

[CMN19] Antonin Chambolle, Massimiliano Morini, Matteo Novaga, and Marcello Ponsiglione. "Existence and uniqueness for anisotropic and crystalline mean curvature flows." *Journal of the American Mathematical Society*, 2019.

[CMP15] Antonin Chambolle, Massimiliano Morini, and Marcello Ponsiglione. "Nonlocal curvature flows." *Archive for Rational Mechanics and Analysis*, **218**(3):1263–1329, 2015.

[CS05] Luis A Caffarelli and Sandro Salsa. *A geometric approach to free boundary problems*, volume 68. American Mathematical Soc., 2005.

[CS17] José-Antonio Carrillo and Filippo Santambrogio. "L^∞ estimates for the JKO scheme in parabolic-elliptic Keller-Segel systems." *arXiv preprint arXiv:1709.04201*, 2017.

[DL89] Ronald J DiPerna and Pierre-Louis Lions. "Ordinary differential equations, transport theory and Sobolev spaces." *Inventiones Mathematicae*, **98**(3):511–547, 1989.

[DM14] Marco Di Francesco and Daniel Matthes. "Curves of steepest descent are entropy solutions for a class of degenerate convection–diffusion equations." *Calculus of Variations and Partial Differential Equations*, **50**(1-2):199–230, 2014.

[DM16] Simone Di Marino and Alpár Richárd Mészáros. "Uniqueness issues for evolution equations with density constraints." *Mathematical Models and Methods in Applied Sciences*, **26**(9):1761–1783, 2016.

[DMS16] Simone Di Marino, Bertrand Maury, and Filippo Santambrogio. "Measure sweeping processes." *Journal of Convex Analysis*, **23**(2):567–601, 2016.

[Eck04] Klaus Ecker. *Regularity theory for mean curvature flow.* Springer Science & Business Media, Boston, MA, 2004.

[EG92] Lawrence Craig Evans and Ronald F Gariepy. *Measure theory and fine properties of functions.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[EH91] Klaus Ecker and Gerhard Huisken. "Interior estimates for hypersurfaces moving by mean curvature." *Inventiones Mathematicae*, **105**(3):547–569, 1991.

[ES98] Joachim Escher and Gieri Simonett. "The volume preserving mean curvature flow near spheres." *Proceedings of the American Mathematical Society*, **126**(9):2789–2796, 1998.

[ESS92] Lawrence C Evans, H Mete Soner, and Panagiotis E Souganidis. "Phase transitions and generalized motion by mean curvature." *Communications on Pure and Applied Mathematics*, **45**(9):1097–1123, October 1992.

[ET76] Ivar Ekeland and Roger Temam. *Convex analysis and variational problems.* Translated from the French. Studies in Mathematics and its Applications, Vol. 1. North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1976.

[FK14] William M Feldman and Inwon Kim. "Dynamic stability of equilibrium capillary drops." *Archive for Rational Mechanics and Analysis*, **211**(3):819–878, 2014.

[GGI91] Yoshikazu Giga, Shun'ichi Goto, Hitoshi Ishii, and M-H Sato. "Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains." *Indiana University Mathematics Journal*, pp. 443–470, 1991.

[Gig06] Yoshikazu Giga. *Surface evolution equations: a level set approach*, volume 99 of *Monographs in Mathematics*. Birkhäuser Basel, Basel, 2006.

[GK11] Natalie Grunewald and Inwon Kim. "A variational approach to a quasi-static droplet model." *Calculus of Variations and Partial Differential Equations*, **41**(1-2):1–19, 2011.

[GT15] David Gilbarg and Neil S Trudinger. *Elliptic partial differential equations of second order*. Springer, 2015.

[Hui84] Gerhard Huisken. "Flow by mean curvature of convex surfaces into spheres." *Journal of Differential Geometry*, **20**(1):237–266, 1984.

[Hui87] Gerhard Huisken. "The volume preserving mean curvature flow." *Journal für die reine und angewandte Mathematik (Crelles Journal)*, **1987**(382):35–48, 1987.

[IS13] Cyril Imbert and Luis Silvestre. "An introduction to fully nonlinear parabolic equations." In *An introduction to the Kähler-Ricci flow*, pp. 7–88. Springer, 2013.

[Ish85] Hitoshi Ishii. "Hamilton-Jacobi equations with discontinuous Hamiltonians on arbitrary open sets." *Bull. Fac. Sci. Eng. Chuo Univ*, **28**(28):1985, 1985.

[JKO98] Richard Jordan, David Kinderlehrer, and Felix Otto. "The variational formulation of the Fokker-Planck equation." *SIAM Journal on Mathematical Analysis*, **29**(1):1–17, 1998.

[Kim03] Inwon Kim. "Uniqueness and existence results on the Hele-Shaw and the Stefan problems." *Archive for Rational Mechanics and Analysis*, **168**(4):299–328, 2003.

[Kim05] Inwon Kim. "A free boundary problem with curvature." *Communications in Partial Differential Equations*, **30**(1-2):121–138, 2005.

[KK20a] Inwon Kim and Dohyun Kwon. "On mean curvature flow with forcing." *Communications in Partial Differential Equations*, **45**(5):414–455, 2020.

[KK20b] Inwon Kim and Dohyun Kwon. "Volume preserving mean curvature flow for star-shaped sets." *Calculus of Variations and Partial Differential Equations*, **59**(2):81, 2020.

[KM18] Inwon Kim and Alpár Richárd Mészáros. "On nonlinear cross-diffusion systems: an optimal transport approach." *Calculus of Variations and Partial Differential Equations*, **57**:1–40, 2018.

[KM19] Dohyun Kwon and Alpár Richárd Mészáros. "Degenerate nonlinear parabolic equations with discontinuous diffusion coefficients." *arXiv preprint arXiv:1907.11319*, 2019.

[KP08] Steven G Krantz and Harold R Parks. *Geometric integration theory.* Springer Science & Business Media, 2008.

[Kru03] A Ya Kruger. "On Fréchet subdifferentials." *Journal of Mathematical Sciences*, **116**(3):3325–3358, 2003.

[Lab17] M. Laborde. "On some non linear evolution systems which are perturbations of Wasserstein gradient flows." In *Topological Optimization and Optimal Transport*, Radon Ser. Comput. Appl. Math. 2017.

[Li09] Haozhao Li. "The volume-preserving mean curvature flow in Euclidean space." *Pacific Journal of Mathematics*, **243**(2):331–355, December 2009.

[Lin15] Longzhi Lin. "Mean curvature flow of star-shaped hypersurfaces." *arXiv:1508.01225v1 [math.DG]*, 2015.

[LS95] Stephan Luckhaus and Thomas Sturzenhecker. "Implicit time discretization for the mean curvature flow equation." *Calculus of Variations and Partial Differential Equations*, **3**(2):253–271, 1995.

[Mag12] Francesco Maggi. *Sets of finite perimeter and geometric variational problems.* An Introduction to Geometric Measure Theory. Cambridge University Press, 2012.

[MRS10] Bertrand Maury, Aude Roudneff-Chupin, and Filippo Santambrogio. "A macroscopic crowd motion model of gradient flow type." *Mathematical Models and Methods in Applied Sciences*, **20**(10):1787–1821, 2010.

[MS16] Alpár Richárd Mészáros and Filippo Santambrogio. "Advection-diffusion equations with density constraints." *Analysis & PDE*, **9**(3):615–644, 2016.

[MSS16] Luca Mugnai, Christian Seis, and Emanuele Spadaro. "Global solutions to the volume-preserving mean-curvature flow." *Calculus of Variations and Partial Differential Equations*, **55**(1):Art. 18–23, 2016.

[Roc68] Ralph Rockafellar. "Integrals which are convex functionals." *Pacific Journal of Mathematics*, **24**:525–539, 1968.

[Roc71] Ralph Rockafellar. "Integrals which are convex functionals. II." *Pacific Journal of Mathematics*, **39**(2):439–469, 1971.

[RS03] Riccarda Rossi and Giuseppe Savaré. "Tightness, integral equicontinuity and compactness for evolution problems in Banach spaces." *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, **2**(2):395–431, 2003.

[San15] Filippo Santambrogio. *Optimal transport for applied mathematicians*, volume 87 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, Cham, 2015. Calculus of variations, PDEs, and modeling.

[San17] Filippo Santambrogio. "{Euclidean, metric, and Wasserstein} gradient flows: an overview." *Bulletin of Mathematical Sciences*, **7**(1):87–154, 2017.

[Smo98] Knut Smoczyk. "Starshaped hypersurfaces and the mean curvature flow." *Manuscripta Mathematica*, **95**(2):225–236, 1998.

[SS93] H Mete Soner and Panagiotis E Souganidis. "Singularities and uniqueness of cylindrically symmetric surfaces moving by mean curvature." *Communications in Partial Differential Equations*, **18**(5-6):859–894, 1993.

[SW10] Weimin Sheng and Xu-Jia Wang. "Regularity and singularity in the mean curvature flow." In *Trends in partial differential equations*, volume 10 of *Advanced Lectures in Mathematics*, pp. 399–436. Higher Education Press and International Press, 2010.

[Tak17] Keisuke Takasao. "Existence of weak solution for volume preserving mean curvature flow via phase field method." *Indiana University Mathematics Journal*, **66**:2015–2035, 2017.

[Vaz07] J. L. Vázquez. *The porous medium equation: mathematical theory*. Oxford University Press, 2007.

[Vil03] Cédric Villani. *Topics in optimal transportation*. American Mathematical Soc., 2003.