UNIVERSITY OF CALIFORNIA AT LOS ANGELES

INCREASE IN THE PERIOD OF WAVES TRAVELING OVER LARGE DISTANCES: WITH APPLICATIONS TO TSUNAMIS, SWELL, AND SEISMIC SURFACE WAVES

A dissertation submitted in partial satisfaction of
the requirements for the degree
Doctor of Philosophy
in
Physical Oceanography
by

Walter H. Munk

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ABSTRACT OF THE DISSERTATION

Increase in the Period of Waves Traveling over Large Distances: With Applications to Tsunamis, Swell, and Seismic Surface Waves

An expression for the increase in wave period during long distance propagation is derived from very general assumptions. To illustrate the general nature of the solution, application is made to three different geophysical phenomena: first to the tsunami from the Aleutian earthquake of 1st April, 1946; then to the long forerunners of the swell recorded at the wave station in Pendeen, England; finally to the seismic surface waves from the Montana earthquake of 28 June, 1925 and a smaller Mexican shock in 1943. In the case of the tsunami and the swell, observations and theory are in good agreement. For the seismic surface waves the theory gives at least the right order of magnitude. Application of the theory to the period increase of swell seems furthermore to provide a simple, rational basis for locating and tracking storms at sea by means of swell observations, and may therefore be of interest in weather forecasting.

(This work represents results of research carried out for the Hydrographic Office, the Office of Naval Research, and the Bureau of Ships of the Navy Department under contract with the University of California.)

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PUBLICATIONS

Internal Waves in the Gulf of California. *Journal of Marine Research,* Vol. IV, No. 1, 81-91 (1941)

Empirical and Theoretical Relations between Wind, Sea and Swell, (with H. U. Sverdrup). *Trans. Am. Geoph. Union* (in press)

Theoretical and Empirical Relations in Forecasting Breakers and Surf (with H. U. Sverdrup). *Trans. Am. Geoph. Union* (in press).

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Increase in the Period of Waves Traveling over Large Distances: with Application to Tsunamis, Swell, and Seismic Surface Waves. *Trans. Am. Geoph. Union* (in press).

Refraction of Ocean Waves: A Process Linking Underwater Topography to Beach Erosion (with M. A. Traylor). *Journal of Geology* (in press).

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UNIVERSITY OF CALIFORNIA AT LOS ANGELES

Increase in the Period of Waves Traveling over Large Distances: With Applications to Tsunamis, Swell, and Seismic Surface Waves

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Events that are merely described and have no apparent relation to others may as well be forgotten, and in fact usually are.

Sir Harold Jeffreys
Theory of Probability

Introduction

Changes in wave period have been noticed for many types of wave propagation. Gutenberg (1929) has summarized some of the observations and gives as examples the propagation of swell, microseisms, ordinary seismic waves, and sound waves from explosions. Changes in wave period appear to be truly geophysical phenomena, noticeable only when dealing with waves traveling for long times over long distances.

The present paper attempts to explain these changes as the consequence of wave dispersion, that is, as the result of the dependence of wave velocity upon wave period. The only assumptions that must be made are that (1) individual waves maintain their identity, and (2) at some given time the wave period changes with distance. Under these conditions it is shown that dispersion must lead to an increase in period of an individual wave. Although the concepts of "dispersion" and "period increase" have been previously connected by several authors, they seem not to have noted this effect upon individual waves, and for that reason have sought explanation in internal friction rather than dispersion (Richter, 1943).

The effect of dispersion on wave period is first considered from a qualitative point of view. The rate of period increase is then derived quantitatively in very general terms from the assumption that the wave crests maintain their identity. The solutions are applied to three specific problems for which they can be checked by observations.

The first application deals with tsunamis, a type of wave motion not ordinarily considered dispersive. However, a slight dependence of the velocity upon period suffices to bring about measurable changes in period over long distances, and the observed changes can be brought into accord with theory. A method is suggested whereby the nature and size of the original disturbance may be estimated from the tidal record at any given station.

The second application deals with the long, low waves which appear to precede the visible swell from a storm. These forerunners of the swell, which have been recorded at newly established wave stations at Pendeen, England, and Woods Hole, Massachusetts, obey the general dispersion law. The theory can also be applied to trace storms across the ocean by means of these wave records. The computed path of two actual storms is shown to be in good agreement with the observations on weather maps. The application towards a practical system of storm warning is briefly discussed.

The third application deals with seismic surface waves. Good records from a large number of seismographic stations makes it possible to measure the period changes with somewhat greater precision than for tsunamis and swell, but this advantage is more than offset by the lack of a dependable analytic expression for dispersion. The relationship between velocity and period has to be found from observations, and these give only a very approximate answer. Within the limitations thus imposed, theory and observations of the period increase of seismic surface waves are consistent.

The principles can also be applied to other types of wave motion, indeed to any wave motion for which individual waves maintain their identity, and for which the velocity depends to any degree, no matter how slight, upon wave period. Observed changes in period may offer a powerful tool for computing the dependence of wave velocity upon period and thereby describing some of the characteristics of the transmitting medium. The wide applicability derives from the general nature of the basic equation, to which Rossby has applied the descriptive term "wave equation of continuity".

Only the kinematics of wave motion will be considered, that is, no attention will be paid to problems involving wave height. The relationships between group velocity, phase velocity and energy propagation have been investigated in a fundamental paper by Rossby (1945) to which we shall make frequent reference.

The Nature of the Period Increase

A number of attempts have been made to explain the increase in wave period in terms of a shift in spectrum brought about by viscosity. This point of view has been summarized by Richter (1943) [italics mine]:

"Some extension of theory is required to account for the frequent observation that the prevailing periods of all types of seismic waves increase with distance. This cannot wholly be due to the normal spreading of an impulsive disturbance with filtering out of the shorter periods. .....Surface waves show dispersion, the theory showing that velocity should vary with period; but this cannot account for the increase with distance of the period of a clearly indentifiable wave group. ..... The required effect will be given by some type of viscosity, or internal friction."

The reason why this approach has not lead to tangible results lies perhaps in a misunderstanding regarding the true significance of the period spectrum for an impulsively generated disturbance. The shift of this spectrum has, in general, nothing to do with viscosity, and furthermore does not relate to the changes of the recorded wave period except through a series of very complex mathematical manipulations. The misunderstanding can be avoided by referring directly to the classical investigations by Cauchy and Poisson (Lamb, 1932) on waves produced by a local surface disturbance in deep water.

For an initial displacement of the free surface, which at the time t = 0 is assumed to be confined to the immediate neighborhood of the origin (x = 0), the disturbance of the sea surface at any time t and distance x is given by the Fourier integral

$$\eta(x,t) = \frac{8\pi}{g}\int_{\infty}^{0} \frac{1}{T_c^3} \cos\left(\frac{4\pi^2 x}{g T_c^2}\right) \cos\left(\frac{2\pi t}{T_c}\right) dT_c$$

The symbol T_c denotes the period of the underlying spectrum. By means of several ingenious approximations the foregoing integral has been evaluated, and

$$\eta(x,t) = (t/2)\sqrt{g/\pi x^3}\ \cos\ (gt^2/4x - \pi/4)$$

provided gt^2 is large compared to x. In regions where this formula holds, changes in height and period from wave to wave are very gradual. For a fixed value of x a wave crest passes at a time t_1, if

$$gt_1^2/4x - \pi/4 = 2\pi n$$

and the following crest at a time t_2, if

$$gt_2^2/4x - \pi/4 = 2\pi(n+1)$$

where n is any integer. Let T_r, the recorded wave period, be defined as the time interval between successive maxima (or minima) at a fixed point. It follows that

$$T_r \equiv t_2 - t_1 = 4\pi x/g\bar{t}$$

where $\bar{t}$ is the mean of t_1 and t_2.

This very brief review of the classical method should serve to point out the difference between the period T_c of the underlying spectrum, and the period T_r of the recorded waves. Accordingly we shall speak of component periods and recorded periods. It is important to note that the underlying wave system is not in any simple manner discernible in the recorded waves, and may indeed be regarded as a mathematical abstraction.

It is not proper to speak of an increase in the component periods, since all periods between zero and infinity are assumed to be present at all times in the Fourier integral expression. However, different component periods are associated with variable amounts of energy. At first much of the energy is concentrated in the shorter component periods, but a gradual shift takes place toward the longer component periods. This shift follows directly from the dynamics of the problem, and does not involve, as is frequently supposed, the higher attenuation of short component periods. Furthermore, a knowledge of this energy shift, and of the selective attenuation if the latter is important enough to be included, does not offer any direct evidence concerning an increase in the recorded periods. But by dealing directly with the recorded wave periods an increase in period can be explained by the following elementary reasoning.

A group of n waves arrives at station A. The first crest passes at a time $t = 0$, succeeding crests at diminishing intervals $T_1, T_2, \ldots T_n$. This is typical of wave systems generated impulsively. Assume that the recorded period T_r at A decreases linearly with time, according to the equation

$$T_r = T_1 - mt .$$

The upper portion of figure 1 shows such a wave system for a constant amplitude. The record is assumed to move from left to right, so that t increases from right to left.

Assume furthermore that the propagation is dispersive; for this simple illustration let the wave velocity C be proportional to the recorded period:

$$C = pT_r$$

The first wave has the longest recorded period and the greatest velocity. It will travel over a small distance X between stations A and B in a short time interval.

$$\Delta t_1 = X/C_1 = X/pT_1$$

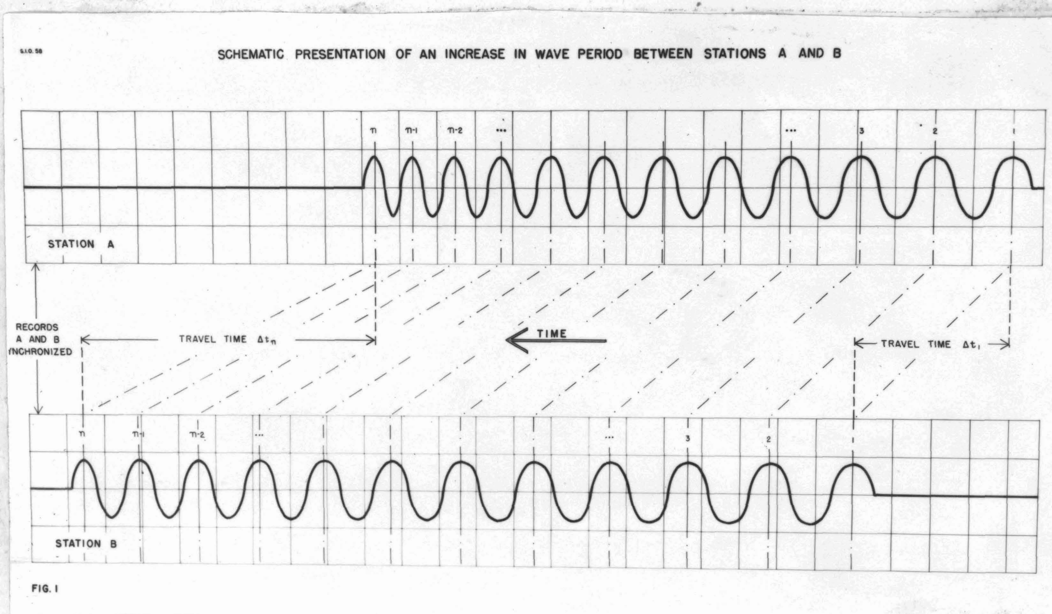

Fig. 1--Excess in velocity of longer waves over velocity of shorter waves causes stretching of entire wave group.

The last wave, which arrives at A at the time t_n, travels much slower, and will take a time interval

$$\Delta t_n = X/C_n = X/p(T_1 - mt_n)$$

to travel from A to B. In figure 1, $T_n = (1/2)T_1$, and therefore $\Delta t_n = 2\Delta t_1$. The duration of the record at B will exceed the duration at A by $\Delta t_n - \Delta t_1$. Since the number of waves in the record are the same, the average increase in the recorded period equals $(\Delta t_n - \Delta t_1)/n$.

The foregoing discussion would have been more nearly accurate if C_i in the expression for Δt_i had not been computed from the period T_i at station A, but the mean period of the i^{th} wave between stations A and B. But even without the precise differential formulation which follows in the next section, the above considerations show that the wave group "stretches". The result is not merely a shift in spectrum from long to short periods; it is an actual increase of the average recorded period of the waves in the group, and of all individual waves comprising the group.

The formulation of the problem depends therefore upon whether one deals directly with the physical deformation of the transmitting medium, such as we have done, or whether one considers a spectrum of underlying, constant period, wave systems, as in the Cauchy and Poisson analyses. In the first case one deals with the recorded period, representing the time interval between two successive maxima (or minima) at a fixed point; in the second case one deals with component periods which are found by generalizing a given portion of a record into a Fourier integral. In the first case wave periods increase, in the second case they remain constant.

For a complete solution, involving wave height as well as wave period, one must follow the method of Cauchy and Poisson. The problem in its simplest two-dimensional form is not an easy one, and attempts to extend the investigation to the case of a finite initial disturbance (Burnside, 1888) or finite depth (Pidduck, 1912) have met with great difficulties. By dealing directly with the recorded deformations, without recourse to any underlying spectrum, certain problems can be treated which are too complicated for a complete analysis of the Cauchy-Poisson type.[1] The method permits the computation of the period increase for individual waves. Good agreement between computed and observed values offers strong evidence that the phenomenon is inherent in the kinematics of wave motion, and is not related to the viscous properties of the transmitting medium.

The following section is devoted to a mathematical formulation of these principles. The recorded wave period will be referred to simply as period and denoted by T.

Derivation of Basic Equations

Changes in wave length

The wave length, L, of a wave traveling with (phase) velocity C along the positive x-axis changes at the rate

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial x} C \qquad (1)$$

---

[1] "In any other problem of wave-motion in a dispersive medium we can do little more than argue by analogy to the special problem of waves in deep water .....". A. E. H. Love, 1911

This change can also be found from the rate at which the wave "stretches" due to the difference in velocity of two adjoining crests:

$$\frac{dL}{dt} = \frac{\partial C}{\partial x} L \tag{2}$$

Eliminating the total derivative between (1) and (2) gives the following important relationship:

$$\frac{\partial L}{\partial t} + \frac{\partial L}{\partial x} C = \frac{\partial C}{\partial x} L \tag{3}$$

We shall be concerned in general with cases where the wave velocity depends on wave length L and depth h only, or

$$C = C(L, h) \tag{4}$$

where $L = L(x, t)$, and the bottom profile along the path of travel is given by $h = h(x)$.

Substituting the identity

$$\frac{\partial C}{\partial x} = \frac{\partial C}{\partial L} \frac{\partial L}{\partial x} + \frac{\partial C}{\partial h} \frac{dh}{dx} \tag{5}$$

into equation (3) gives

$$\frac{\partial L}{\partial t} + \left(C - L \frac{\partial C}{\partial L}\right) \frac{\partial L}{\partial x} = \frac{\partial C}{\partial h} \frac{dh}{dx} L \tag{6}$$

If V_L is defined by the equation

$$\frac{\partial L}{\partial t} + \frac{\partial L}{\partial x} V_L = 0 \tag{7}$$

then to an observer traveling at velocity V_L the wave length remains constant. According to (6) and (7)

$$V_L = C - L\frac{\partial C}{\partial L} - L\frac{dh}{dx}\frac{\frac{\partial C}{\partial h}}{\frac{\partial L}{\partial x}} \tag{8}$$

For the special case of constant depth, dh/dx = 0, and the partial derivative can be replaced by a total derivative

$$V_L' = C - L\frac{dC}{dL} \tag{9}$$

The V has been primed to denote the case of constant depth.

Equation (9) represents the well-known equation for group velocity at constant depth, and for this special case has been derived by Lamb (1932) in a somewhat similar manner. Rossby (1945) derives equation (9) by equating the rate of increase of wave crests within a certain distance Δx to the excess of waves entering at x over those leaving at $x + \Delta x$. In analogy with the problem in hydrodynamics he uses the very descriptive term "wave continuity equation". Rossby also generalizes equation (9) to include some of our results.

These derivations identify the physical meaning of the term group velocity with the velocity of the locus of a point in whose immediate vicinity the wave length remains constant. The same equation is usually derived from the interference pattern between two wave systems of equal amplitude and slightly different wave length, but that approach is unnecessarily restricted. The only assumption which has been made here is that the waves are "conservative", i.e., that individual crests maintain their identity.

Changes in wave period

Wave length, velocity and period are related by the identity

$$L = C\,T \tag{10}$$

which permits restating the fundamental law (3) in terms of T and C rather than L and C:

$$C \frac{\partial T}{\partial t} = T \frac{\partial C}{\partial t} + \left(C \frac{\partial T}{\partial x} + T \frac{\partial C}{\partial x} \right) C = \frac{\partial C}{\partial x} CT \tag{11}$$

or

$$\frac{\partial T}{\partial t} + \frac{T}{C} \frac{\partial C}{\partial t} + C \frac{\partial T}{\partial x} = 0$$

For the general case assume, as before

$$C = C\,(T,\ h)\ ,\quad h = h\,(x) \tag{12}$$

Therefore

$$\frac{\partial C}{\partial t} = \frac{\partial C}{\partial T} \frac{\partial T}{\partial t}$$

and equation (11) becomes

$$\frac{\partial T}{\partial t} + \frac{C}{1 + \frac{T}{C} \frac{\partial C}{\partial T}} \frac{\partial T}{\partial x} = 0 \tag{13}$$

To an observer traveling at a velocity

$$V_T = \frac{C}{1 + \frac{T}{C} \frac{\partial C}{\partial T}} \tag{14}$$

the wave period remains constant. It is remarkable that the bottom slope does not enter explicitly into the expression for V_T as it did for V_L, so that, in general, $V_T \neq V_L$. It is simpler to deal with wave period than wave length, because then the equations are the same for the cases of constant and variable depth, or, by our previous notation

$$V_T' = V_T = V_L' \tag{15}$$

In all subsequent equations $\underline{V_T}$ will be denoted simply by $\underline{V}$ and referred to as the group velocity.

<u>The wave period of any system of waves in which the crests retain their identity must satisfy the "equation of continuity" (13)</u> Writing this equation in the abbreviated form

$$\frac{\partial T}{\partial t} + V \frac{\partial T}{\partial x} = 0 \tag{16}$$

its solution can be put down at once:

$$T = T\left(t - \int (1/V)\,dx\right) \tag{17}$$

It is not possible to solve for the period explicitly, at least not in the general case, since $\underline{T}$ is contained in $\underline{V}$ and thus appears on both sides of the equation. As far as the integration is concerned, on the other hand, $\underline{V}$ can be considered a function of $\underline{h}$, or of $\underline{x}$ only, and

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial[t - \int (1/V)dx]}, \quad \frac{\partial T}{\partial x} = \frac{\partial T}{\partial[t - \int (L/V)dx]}\left(-\frac{T}{V}\right) \tag{18}$$

It can be seen that (17) satisfies the differential equation (16). The limits of integration in equation (17) are, as yet, undetermined because no provisions have been made to fix the coordinate system. In most cases it is convenient to let $t = 0$, $x = 0$, designate the time and position of the initial wave generation. If it were possible to measure a period $\underline{T_o}$ at or very near the origin, then according to the foregoing convention $T_o = T(0)$.

Equations (16) and (17) define the "field" of wave period in the xt plane, and will be called <u>field equations.</u> By applying the field solution (17) to problems dealing with tsunamis and swell severe restrictions are imposed upon the dependence of period on distance x and time t. It will be shown that these restrictions lead to results in agreement with observations.

In many problems it is more convenient to express the relationships in terms of individual waves. For each individual wave the "travel-time" curve, $x = x(t)$, is fixed, and the transformation from the field equations to the individual wave equation requires a line integral in the xt plane.

Equations for individual waves

To derive changes in period for individual waves, equation (16) can be combined with either of the equations

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} C , \quad \frac{dT}{dx} = \frac{\partial T}{\partial x} + \frac{\partial T}{\partial t} \frac{1}{C}$$

to give the following sets of relationships:

$$\frac{dT}{dt} = -\left(\frac{C}{V} - 1\right) \frac{\partial T}{\partial t} = (C - V) \frac{\partial T}{\partial x} \qquad (20\ a,b)$$

$$\frac{dT}{dx} = -\left(\frac{1}{V} - \frac{1}{C}\right) \frac{\partial T}{\partial t} = \left(1 - \frac{V}{C}\right) \frac{\partial T}{\partial x} \qquad (21\ a,b)$$

The significance of the total and partial derivatives must be kept in mind: the partial derivatives refer either to changes with distance at a given instant, $\partial T/\partial x$; or changes with time at a given location, $\partial T/\partial t$. The variables x and t are independent.

The total derivatives refer to changes of an individual wave which has traveled a distance x in a time t since it has been generated. For individual waves x and t are not independent. Indeed they are related by the equations

$$C = C\left[T, x\right]_{\text{(equ. 12)}} = C\left[x, t\right]_{\text{(equ. 17)}} \tag{22}$$

which together with the definition of wave velocity

$$C = \frac{dx}{dt} \tag{23}$$

make solutions of the type t = t (x) possible.

This distance-travel relationship for <u>individual</u> waves can in turn be substituted into (17) to give solutions of the form

$$T = T(x) \quad \text{or} \quad T = T(t) \tag{24}$$

Equations (20) and (21) give the differential expression, equation (24) the integral expression, for the period of individual waves. The meaning of these very general remarks can best be illustrated with specific examples.

Application To Tsunamis

<u>Theory</u>

For low waves in water of depth <u>h</u> the velocity is given by the equations

$$C^2 = \frac{gL}{2\pi} \tanh \frac{2\pi h}{L} \tag{25a}$$

or

$$C = \frac{gT}{2\pi} \tanh \frac{2\pi h}{CT} \tag{25b}$$

Equations (25) were originally derived for an infinitely long wave train of constant period, but it has been shown by Cauchy and Poisson (Lamb, 1932) that they apply also to a wave system of gradually varying period. It has been customary to state that in a very shallow water, that is water where,

$$h < 0.05 L \tag{26}$$

equations (25) reduce to the well-known form

$$C^2 = gh \equiv C_o^2 \tag{27}$$

according to which the velocity depends upon depth only, and the propagation is therefore non-dispersive.

Condition (26) is certainly fulfilled by all known tsunamis and accordingly equation (27) has been applied for computing travel times. These attempts have been remarkably successful.

There are, however, certain secondary phenomena associated with the propagation over long distances, such as a very gradual increase in wave period, for which equation (27) is no longer a satisfactory approximation. The first two terms in the expansion of a hyperbolic tangent are

$$\tanh \alpha = \alpha - \frac{1}{3}\alpha^3 \tag{28}$$

and to the same degree of approximation equation (27) becomes

$$C^2 = C_o^2 \left(1 - \frac{1}{3}\alpha^2 \right) \tag{29}$$

where

$$\alpha = \frac{2\pi h}{L} = \frac{2\pi h}{CT} \tag{30}$$

For small values of α^2

$$\sqrt{1 - \frac{1}{3}\alpha^2} = 1 - \frac{1}{6}\alpha^2, \qquad \frac{1}{1 - \frac{1}{6}\alpha^2} = 1 + \frac{1}{6}\alpha^2 \tag{31}$$

and

$$\frac{C_o - C}{C_o} = \frac{1}{6}\alpha^2 \tag{32}$$

Percentage differences in the velocity of very long waves are therefore of the order of α^2.

The rotation of the earth also introduces dispersion, and for tsunamis of half hour period or longer this effect cannot be neglected. But in the case of the short period tsunami of 1 April 1946, the earth's rotation increases the total dispersion by less than one percent.

Differentiating (25b) gives

$$\frac{\partial C}{\partial T} = \frac{g}{2\pi} \frac{\tanh\alpha = \alpha \operatorname{sech}^2\alpha}{1 + \frac{C_o^2}{C^2} \operatorname{sech}^2\alpha} \tag{33}$$

which can, in view of (29) and the expansion

$$\operatorname{sech}\alpha = 1 - \frac{1}{2}\alpha^2 \qquad \operatorname{sech}^2\alpha = 1 - \alpha^2 \tag{34}$$

be reduced to the following form:

$$\frac{\partial C}{\partial T} = \frac{\alpha^3}{6\pi} \tag{35}$$

Substituting into equation (14) gives

$$V = C_o(1 - \frac{1}{2}\alpha^2) \tag{36}$$

or, in terms of percentage differences

$$\frac{C - V}{C_o} = \frac{1}{3}\alpha^2 \tag{37}$$

A comparison with (32) leads to the interesting result

$$C - V = 2\ (C_o - C) \tag{38}$$

so that for any given wave period the group velocity is smaller than the wave velocity by twice the amount that the wave velocity is smaller than the velocity of infinitely long waves.

Substituting (36) into (20) and (21) gives the following relationships for the changes in period of individual waves.

$$\frac{dT}{dt} = -\frac{1}{3}\alpha^2 \frac{\partial T}{\partial t} = \frac{1}{3}\alpha^2 c_o \frac{\partial T}{\partial x} \qquad (39a,b)$$

$$\frac{dT}{dx} = -\frac{1}{3}\alpha^2 \frac{1}{c_o}\frac{\partial T}{\partial t} = \frac{1}{3}\alpha^2 \frac{\partial T}{\partial x} \qquad (40a,b)$$

The general integral (17) can now be written in the form

$$T = T\left(t - t_o - \frac{1}{T^2} I\right) \qquad (41)$$

where

$$t_o = \frac{1}{\sqrt{g}}\int h^{-1/2}\, dx, \qquad I = \frac{2\pi^2}{g^{3/2}}\int h^{1/2}\, dx \qquad (42)$$

If the bottom profile along the path of travel is approximated by n short straight line segments, from x_o, h_o to x_1, h_1; from x_1, h_1 to x_2, h_2; ....., from x_{n-1}, h_{n-1} to x_n, h_n, then in the interval (i-1) to i

$$h = h = \frac{h_i - h_{i-1}}{x_i - x_{i-1}} (x - x_i - 1) \qquad (43)$$

and equations (42) can be integrated along each line segment:

$$t_o = \frac{2}{\sqrt{g}} \sum_{i=1}^{n} \frac{1}{\sqrt{h_i} + \sqrt{h_{i-1}}} (x_i - x_{i-1}),$$

$$L = \frac{4\pi^2}{3\, g^{3/2}} \sum_{i=1}^{n} \frac{h_i + \sqrt{h_i\, h_{i-1}} + h_{i-1}}{\sqrt{h_i} + \sqrt{h_{i-1}}} (x_i - x_{i-1}) \qquad (44)$$

By tabulating corresponding values of h_i, x_i the integrals can be evaluated fairly rapidly.

The travel time t_o of an infinitely long wave can be computed according to

$$t_o = \int (1/C_o)\ dx \tag{45}$$

By setting $\Delta t = t - t_o$ equation (41) can be written in the more convenient form

$$T = T\ (\Delta t - I/T^2) \tag{46}$$

If the first arrival of the tsunami is clearly indicated on the record, and if the initial wave period exceeds 10 minutes, $\underline{\Delta t}$ may, with sufficient approximation, be taken as the time interval after the first recorded arrival of the tsunami at a given station.

One particular form of the solution (46), and one that satisfies dimensional considerations, is

$$T = \frac{1}{a}\ (\Delta t - I/T^2) \tag{47}$$

where $\underline{a}$ is an arbitrary constant. Equation (47) represents a cubic equation in $\underline{T}$

$$T^3 - \frac{(\Delta t)\ \ T^2}{a} + \frac{I}{a} = 0 \tag{48}$$

with one or three distinct real roots, depending upon whether the discriminant

$$D = \frac{4\ I}{a}\ \frac{\Delta t^2}{a} - 27\ \frac{I}{a}^{2} \tag{49}$$

is positive or negative.

The simplest possible solution is found by setting a = 0 in (47):

$$T = \sqrt{\frac{I}{\Delta t}} \tag{50*}$$

It is worth noting that even this simple solution exhibits some of the desired physical characteristics. For the earliest arrival of the waves $\underline{\Delta t}$ approaches zero, and $\underline{T}$ approaches infinity. For negative values of $\underline{\Delta t}$, that is, for times preceding the arrival of the first wave, the solution for $\underline{T}$ is properly imaginary.

Whether equations (47) or (50) are valid can be decided by plotting the relationship (46) from observational data. If the relationship is linear, equation (47) may be used, and if the line furthermore coincides with the $\underline{T}$ axis, equation (50) may be used.

Observations

Figure 2A shows the general refraction pattern of the tsunamis formed by the submarine earthquake of April 1, 1946. The dashed curves give the positions of the crests at stated time intervals, and the solid lines, which are normal to the crests, give the paths of travel towards various stations. There are two alternate routes into Honolulu and La Jolla. The graph, which was prepared by the Department of Engineering, University of California, (1946), probably represents the first attempt of applying the principles of refraction diagrams (Hydrographic Office, 1945) to a study of tsunamis. Fig. 2A also gives the approximate paths of two other tsunamis recorded at Honolulu about twenty-five years ago.

Figure 2B shows the depth profiles along various paths of travels; from these, $\underline{t_0}$ and $\underline{I}$ have been evaluated according to (43) and (44), and summarized in table 1 together with other pertinent data.

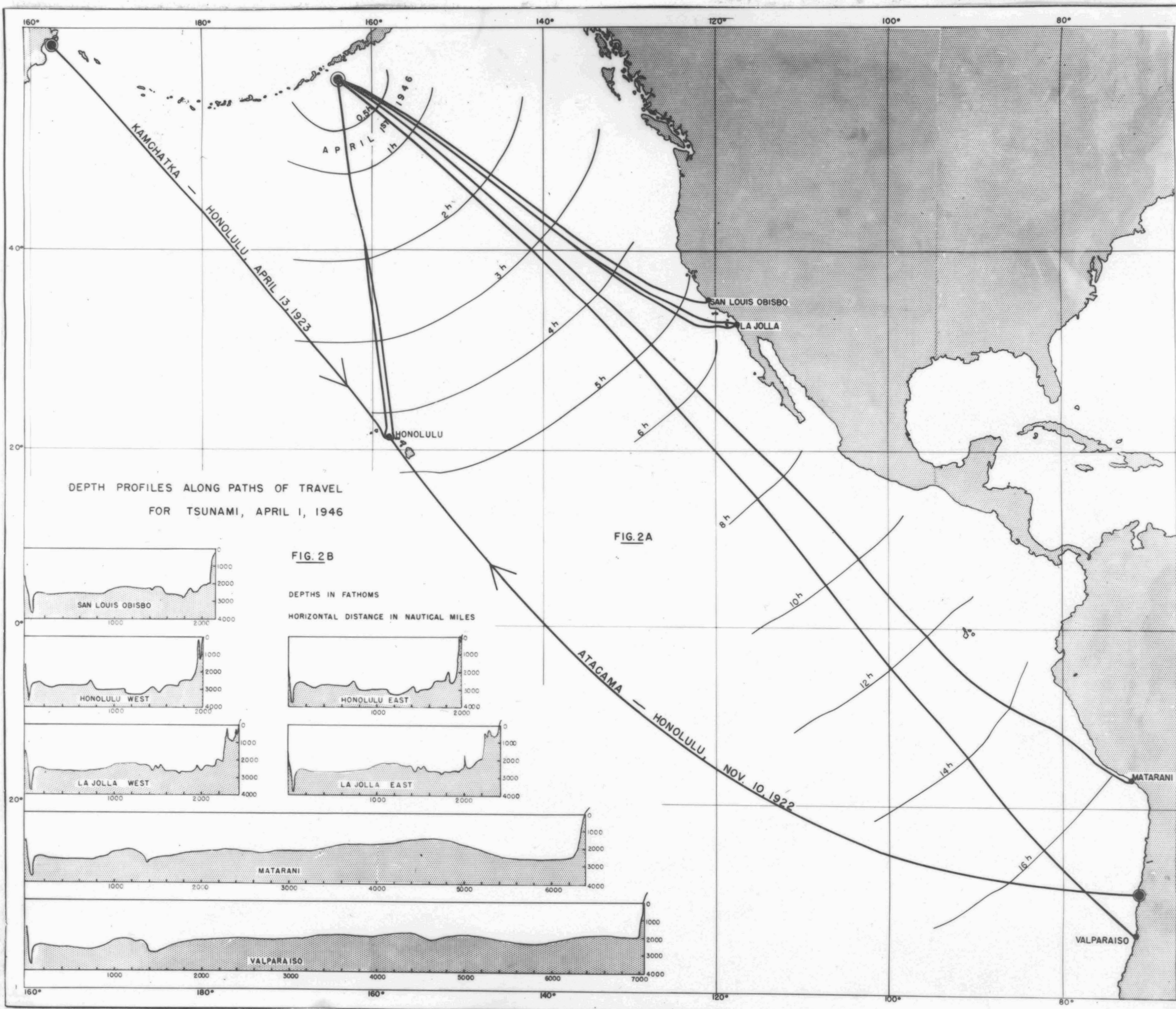

Fig. 2--Paths of tsunamis for three major submarine earthquakes. Figure 2B shows vertical profiles along the paths of travel of the 1 April 1946 tsunami.

Table 1

Data for Tsunamis

Date	Time	Epicenter	Station	x naut. m.	t_o h. m	I min.[3]
1 Apr. 46	$12^h28.9^m$	Unimac Is. 54°N 164°W	Honolulu, East	1976	4 34	725
			Hololulu, West	2025	4 34	789
			Honolulu, Mean	2000		757
			San Luis Obispo	2168	5 36	789
			La Jolla, East	2410	6 11	849
			La Jolla, West	2441	6 11	868
			La Jolla, Mean	2425		859
			Matarani	6389	16 26	2220
			Valparaiso	7050	18 07	2471
13 Apr.23	05^h17^m Honolulu time	Kamchatka 56°N 162°E	Honolulu	2756	7 23	1101
10 Nov.22	00^h03^m Honolulu time	Chile	Honolulu	5992	14 57	2079

Commander Green, USCGS, has kindly put at our disposal the tidal records which formed the basis of his recent discussion (Green, 1946). The five best records have been selected, and the changes of period with time at the five corresponding stations have been found in the following manner.

Consecutive crests and troughs were numbered (fig. 3A), and plotted against $\underline{\Delta t}$, the time interval after the initial arrival of the tsunami (fig. 3B). The period was then determined from the slope of the curves in figure 3B, and plotted in 3C. In addition to this method, groups of particularly well defined crests and troughs were selected from the tidal record, the period measured directly, and then plotted in figure 3C.

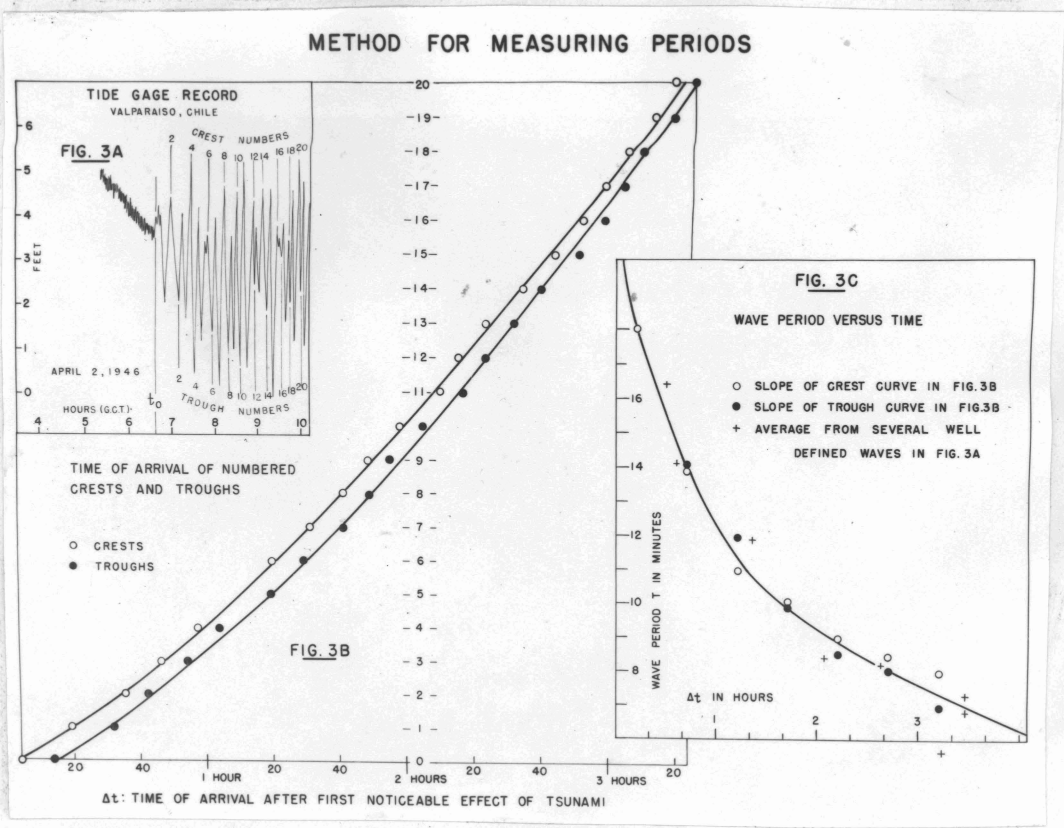

Fig. 3--See Text.

Figure 4A gives the relationship of period with time at the five selected stations. The relatively short periods of the April 1st tsunami, interference between waves traveling over different paths and the superposition of seiches greatly reduce the accuracy with which periods can be measured. Certain portions of the records at San Luis Obispo and La Jolla had to be omitted because of severe seiche formation. At Honolulu a three-minute seiche is predominant after the first hour, and the measured periods from the nearby Waima River water stage recorder have been taken into account in drawing the curve. It would be desirable to apply more refined methods of period analysis, but these are made difficult by the slow recording speed (1 hour=1" inch) and the resulting extreme vertical exaggeration of the record.

Comparison between theory and observations

The observed field of wave period is in qualitative agreement with the theoretical considerations. The period increases with distance of travel, but decreases with time at every given station.

For a quantitative check, corresponding values of $\underline{T}$, $\underline{\Delta t}$ (fig. 4A) and I (table 1) were substituted in the general field solution (46) and plotted in figure 4B.

In accordance with the theory all observations pertaining to the April 1st tsunami, from Hawaii to Chile, obey the same general relationship, while observations of the two other tsunamis follow a different relationship. Thus the indeterminateness of the function in the general solution (46) depends upon the nature of the original disturbance. Preliminary considerations indicate that long periods

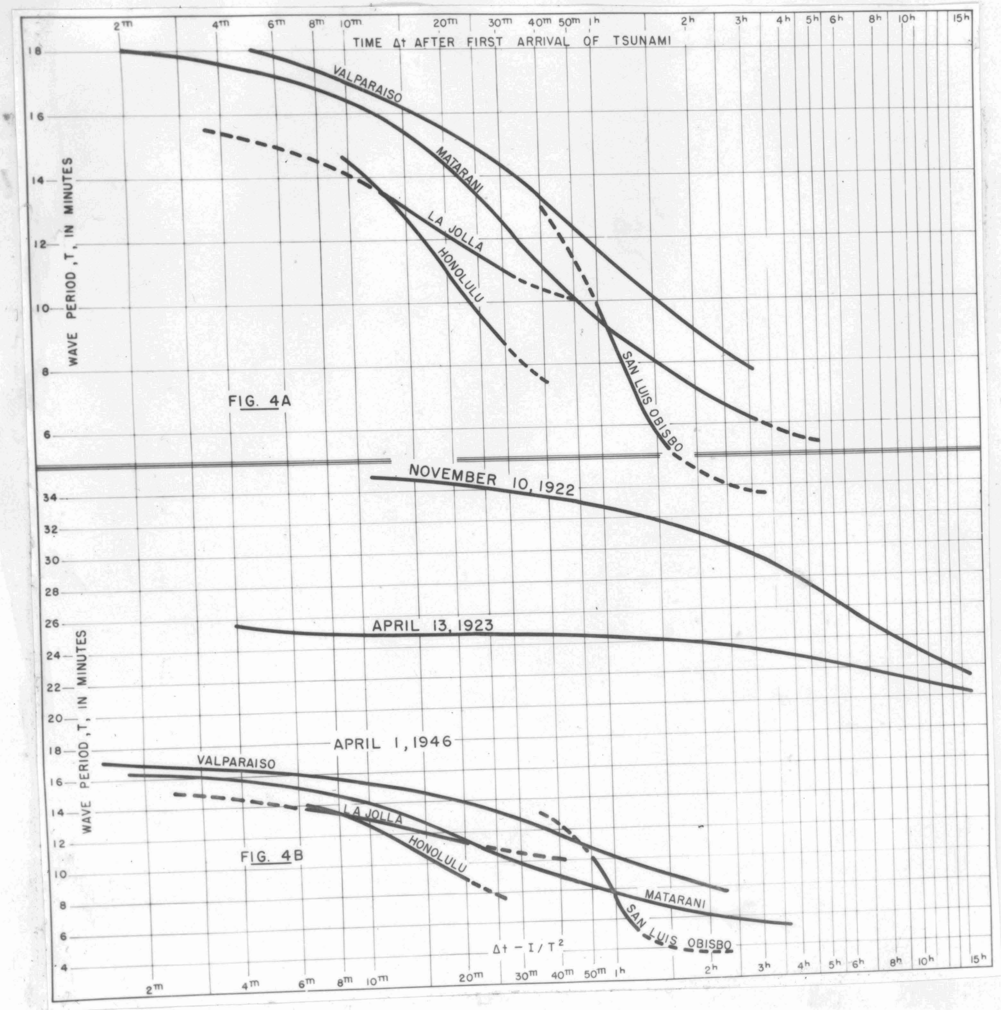

Fig. 4A--Period against time of April 1 tsunami at various stations.

Fig. 4B--Plot of general solution (equ. 17) for three different earthquakes.

in fig. 4B indicate a large horizontal extent of the initial disturbance. Perhaps the size and shape of the initial disturbance could be estimated by assuming certain forms of the initial deformation, or initial impulse, for which an analysis of the Cauchy-Poisson type can be carried through, and then constructing a network of lines on fig. 4B corresponding to various types and sizes of the initial boundary conditions. A comparison of this network with observed relationships, such as the ones in fig. 4B, may yield information regarding the nature and size of the original disturbance.

A feature which is obscured by the logarithmic scale of the abcissa is that the leading wave, especially the portion of the wave preceeding the first crest, is associated with small values of $\Delta t - I/T^2$, and large values of T. Therefore a (47) must be small and the approximate solution (50) is applicable, so that the periods of the very first waves, the "forerunners", are approximately proportional to $\sqrt{I}$ and inversely proportional to $\sqrt{t}$.

From any one complete record it is possible to compute the periods at any of the other stations. Assuming the Matarani record to define the relationship for the April 1st tsunami, the periods for all other stations have been computed for various values of Δt, and plotted against the observed periods (fig. 5). The observed wave periods in the North Pacific tend to be somewhat shorter, those at Valparaiso longer than the computed periods. The same conclusion follows directly from equation (40a), which gives an average increase in the period of the first wave of 1.3 minutes in 5000 miles. Green (1946) has measured the time interval between the first crest and

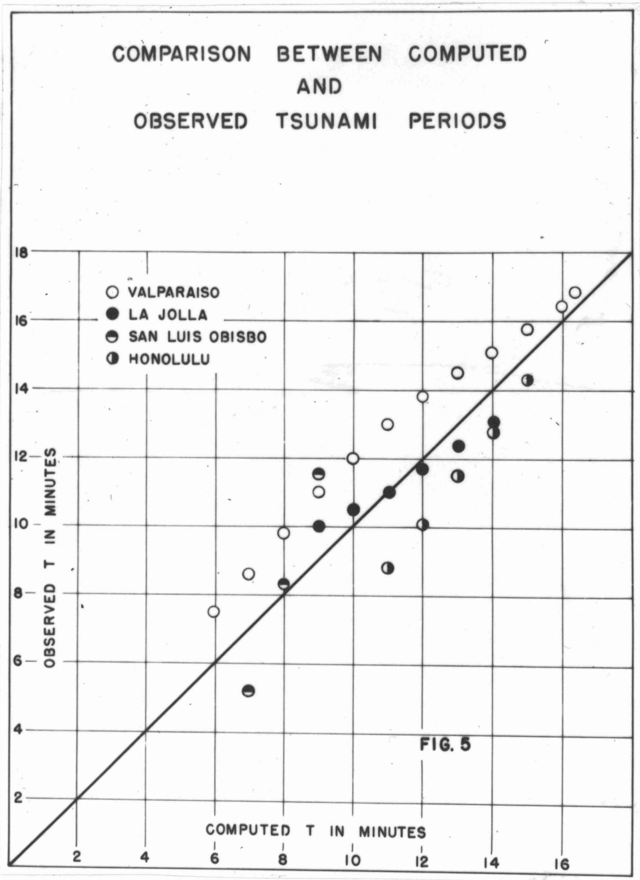

Fig. 5--Computed wave periods are based upon equation 17 and the observed periods at Matarani, Peru.

first trough and on that basis estimates the period increase of the first wave to amount to 2.0 minutes in 5000 miles. The observational data are too inaccurate to determine whether this discrepancy is significant, and whether it therefore indicates a more rapid increase in period than the increase due to dispersion.[2]

Application to Swell

Theory

For low waves in deep water, the velocity depends upon wave length alone, or period alone,

$$C = \sqrt{\frac{gL}{2\pi}} = \frac{gT}{2\pi} \qquad (51)$$

and the following formulae are therefore much simpler than in the case of tsunamis. Equations (8) and (14) lead at once to the well-known relationships

$$V_T = V_L = V = \frac{C}{2} \qquad (52)$$

Equations (20) and (21) give

$$\frac{dT}{dt} = -\frac{\partial T}{\partial t} = \frac{C}{2}\frac{\partial T}{\partial x}, \quad \frac{dT}{dx} = -\frac{1}{C}\frac{\partial T}{\partial t} = \frac{1}{2}\frac{\partial T}{\partial x} \qquad (53)$$

Equation (17) reduces to

$$T = T\left(t - \sigma \frac{x}{T}\right) \qquad (54)$$

where $\sigma = 4\pi/g$ (dimensions t^2/L). The integral (54) has been given by Rossby (1945).

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[2] "... the amount of increase in period is difficult to determine from the present records. For instance the table shown on page 7 of my AGU paper indicates an increase of around 2 minutes. But by dropping the shortest period station from the California group, and the longest period station from the Peru-Chile group, the mean value of increase in period would be 1 1/3 minutes from California to South America." (C. K. Green, July 18, 1946, personal communication to the author.)

The special case corresponding to (47) gives

$$T = \frac{t}{2a} (1 \pm u) \tag{55}$$

where

$$u = \sqrt{1 - 4\pi ax/t^2} \; ;$$

if furthermore a = 0 (equ. 50)

$$T = \sigma \frac{x}{t} \tag{56}$$

which is a solution of the Cauchy-Poisson wave problem (Lamb, 1932, pp. 384-398).

It will be shown that the special solution (56) is in good agreement with observations of the "forerunners" of swell from a distant storm. This is assumed to justify the use of the simple solution for all further derivations, though the agreement might also have been expected from the analogy with the forerunners of the tsunami.

The following discussion deals therefore exclusively to the forerunners, and is not applicable to the visible swell. Some of the following derivations could have been given for the general case, but the not inconsiderable difficulties involved in dealing with a moving source are greatly simplified by working with the special solution.

To derive the equations for <u>individual</u> waves, <u>T</u> can be eliminated between (51) and (56)

$$\frac{dx}{dt} \equiv C = \frac{gT}{2} = \frac{2x}{t}$$

Integrating gives

$$\frac{x}{x_D} = \left(\frac{t}{t_D} \right)^2 \tag{57}$$

$$\frac{T}{T_D} = \frac{t}{t_D} = \sqrt{\frac{x}{x_D}} \tag{58}$$

$$\frac{\partial T}{\partial t} = -\frac{\sigma x}{t^2} = -\frac{T}{t} \tag{59}$$

The subscript "D" may refer to any fixed point on the wave's journey, but will here be taken to denote conditions at the wave station. If T_D and $(\partial T/\partial t)_D$ are determined from the wave record, then the distance and time of travel can be computed from (59):

$$x_D = \frac{{T_D}^2}{\sigma(-\partial T/\partial t)_D}, \quad t_D = \frac{T_D}{-(\partial T/\partial t)_D} \tag{60a,b}$$

<u>Waves from a moving storm</u>

Application of these equations to the study of swell is difficult, because one must deal with a moving source of finite size which <u>continuously</u> emits new waves. It will be assumed that the problem can be treated as one involving an infinite number of instantaneous sources, each independently emitting a wave field according to the assumptions made earlier. Let <u>x</u> and <u>t</u> retain their former meanings, designating the total distance and time, respectively, that any wave has traveled, The fixed coordinated system will be denoted by χ and τ, χ representing distance from the wave origin to the wave station, and <u>t</u> the time on some standard scale, for example GCT.

The movement of the storm center is represented by the equation

$$\chi_c = \phi(\tau) \tag{61}$$

Let $\delta\chi$ be the distance of a point of wave origin from the storm center, so that

$$\chi = \chi_c + \delta\chi \tag{62}$$

Waves arriving at the wave station at any time $\underline{\tau}$ were generated at a time τ - t when the storm center was located at

$$\mathcal{X}_c = \phi \ (\tau - t) \tag{63}$$

To determine the period <u>at the wave station</u> at a time $\underline{\tau}$, set

$$x = \mathcal{X} = \phi \ (\tau - t) = \delta \mathcal{X}$$

and substitute into (56)

$$T = \sigma \ \frac{\phi \ (\tau - t) \ = \delta \mathcal{X}}{t} \tag{64}$$

At any time $\underline{\tau}$ a <u>spectrum</u> of wave periods must be present at the wave station, due to two factors:

(a) Waves arrive simultaneously from different sectors of the storm (different values of $\delta\mathcal{X}$).

(b) Waves arrive simultaneously, although they were generated at different times (different values of $\underline{t}$).

The total width of the period "band" is given by

$$\Delta T = \frac{\partial T}{\partial(\delta\mathcal{X})} \ F = \frac{\partial T}{\partial t} \ Q \tag{65}$$

where $\underline{F}$ is the effective length of the storm fetch at a time (τ - t), and $\underline{Q}$ is the time interval during which waves perceptible at the station were generated. Partial differentiation of (64) leads to the expressions

$$\frac{\partial T}{\partial(\delta\mathcal{X})} \ F = \frac{\sigma F}{t}, \quad \frac{\partial T}{\partial t} Q = - \frac{\sigma}{t} \ (V - U) \ Q \tag{66a,b}$$

where

$$U = - \frac{\partial \phi}{\partial(\tau - t)} \tag{67}$$

is the rate of storm movement towards the wave station at the time (τ - t), and V $= \mathcal{X}/t$ the group velocity.

Equations (66) may also be interpreted to give the width in the period band due to the instantaneous actions of a finite storm (66a), and due to a moving point-source (66b), respectively. The longest periods in the band are those of the earliest perceptible waves from the furthest sector of the storm; the shortest periods are due to the waves generated latest near the leading edge of the storm.

The shift in periods of the band as a whole will be interpreted with the aid of an actual example. It will be shown that the shift is related to the position and speed of the storm and can therefore provide clues regarding changes in the meteorological situation. The width and appearance of the band should provide some information about the size and character of the storm itself.

Observations

Unfortunately no simultaneous wave records from two or more well separated stations are available to test the theory. The proposed wave-recording stations at La Jolla, California, and San Nicolas Island, 120 miles WNW from La Jolla can be expected to give valuable information. Observation shows that the increase in swell period for an individual wave may be of the order of 2 seconds a day (Sverdrup and Munk, in press). According to (53), for an average period of 10 seconds,

$$\frac{\partial T}{\partial t} = -2 \text{ seconds/day} = -2.3 \times 10^{-5} \text{ sec/sec}$$

$$\frac{\partial T}{\partial x} = 0.94 \times 10^{-6} \text{ sec/foot} = 6 \times 10^{-3} \text{ sec/n.mile}$$

The corresponding period difference over 120 miles is about 0.7 seconds. Both $\partial T/\partial t$ and $\partial T/\partial x$ can therefore be measured with existing instruments. Comparison of the observation with the foregoing considerations would show whether the assumptions made in this report are applicable to swell, and especially whether the significant waves, that is, the highest waves present and the ones that determine the character of the surf, maintain their identity.

Some confirmation of the applicability of the theory to a long swell can be found in records from a single wave station at Pendeen, England.[3] The waves were measured by means of a differential pressure recorder on the ocean bottom at an average depth of 114 feet. The records were then analyzed for period by a frequency analyzer, specially designed for that purpose at the Admiralty Research Laboratory.

Figure 6 shows a set of period analyses of twenty-minute records taken at intervals of two hours. The abscissa simply give wave periods in seconds, but the ordinates are more difficult to interpret, depending both upon the frequency of occurrence of a certain period and upon the corresponding wave height.

The band of gradually diminishing periods in fig. 6 is the result of an intense cyclone which formed off the coast of Cuba, 3000 miles from Pendeen, on March 10, 1945, and traveled in a general NE direction, passing west of the British Isles. The positions of the forward and rear edges of the storm are given in figure 7A. These were determined from 6-hourly weather maps, following the rules for measuring fetches for wave forecasting (Hydrographic Office, 1944). Four representative weather maps are included in fig. 7.[4]

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[3]The Oceanographic Research Group, Admiralty Research Laboratory, Teddington, England, have kindly placed at our disposal unpublished records of swell at Pendeen, near Lands End.

[4]The weather maps were drawn and analyzed at the U.S. Navy Weather Central, Washington, D.C.

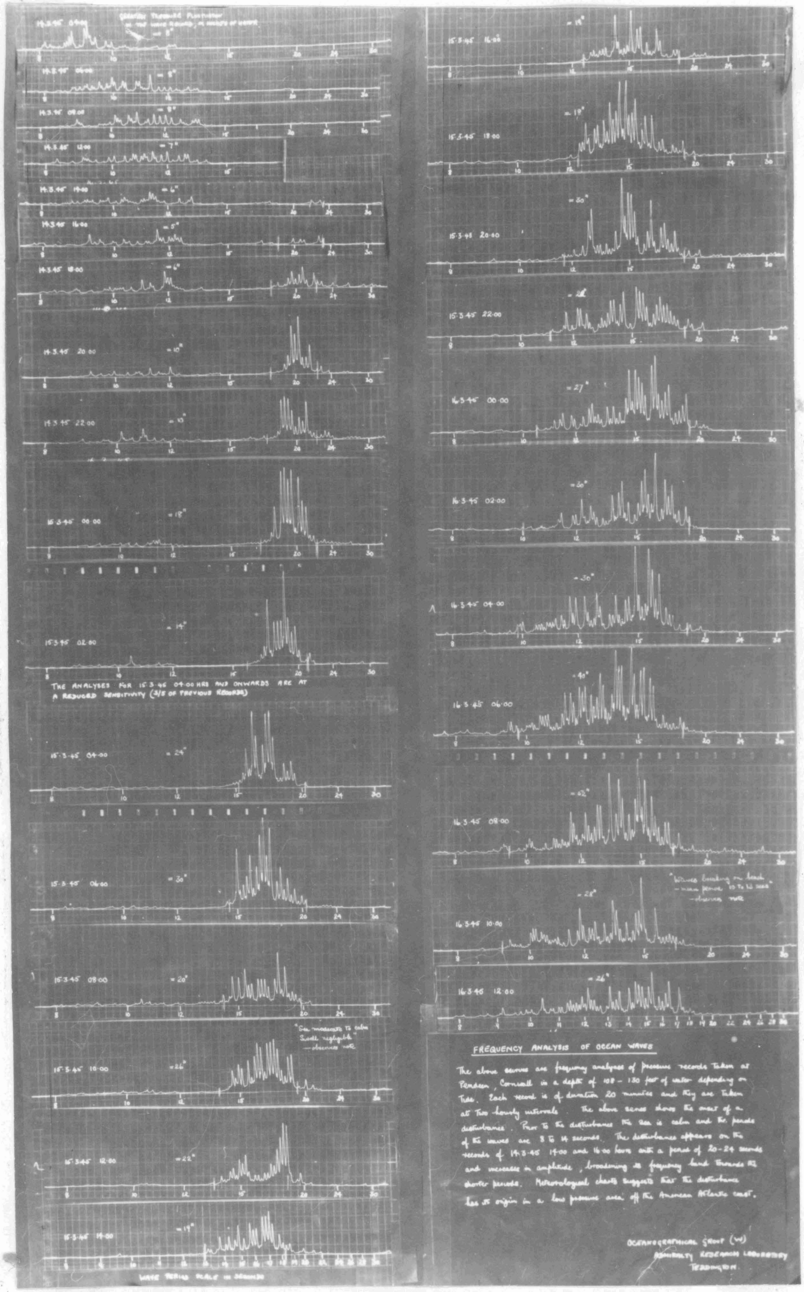

Fig. 6--Spectrogram of waves at Pendeen, England, at two-hourly intervals from March 14, 0400 to March 16, 1200.

The first indications of the storm on the wave record are some "low-frequency noises" at 1400 GCT, March 14th. At that time visual observations at Pendeen noted only the presence of 10 to 12 second waves from local winds. The long "forerunners" of the swell were too low to be visible. The upper and lower limits of the period band have been read off the record and drawn on fig. 7B to emphasize the similarity between the recorded wave periods and the observed storm paths. A relationship between the width of the period band and the fetch is also apparent. The implications of equations (64)-(66) are therefore borne out by the observations. It will be shown, however, that the similarity of the curves in fig. 7 may be misleading, and that great care must be exercised in the interpretation of the wave records.

Comparison between theory and observations

We shall first deal with the lower boundary of the period band. The initial waves from the storm of March 10-13 reached Pendeen around 1600, March 14th, with a minimum period of 19 seconds. Let this time and period be designated by τ_0 and $T\tau_0$, and let x_0 and $\tau_0 - t_0$ refer to the distance from Pendeen and the time of wave origin. Then the period recorded at the time τ_0 equals:

$$T_{\tau_0} = \frac{\delta x_0}{t_0} \tag{68}$$

Assume that the "instantaneous" disturbance which gave rise to the waves described in (68) was the only existing source of the waves present; then slower waves with shorter periods would follow the faster longer waves, so that the period at the wave station would

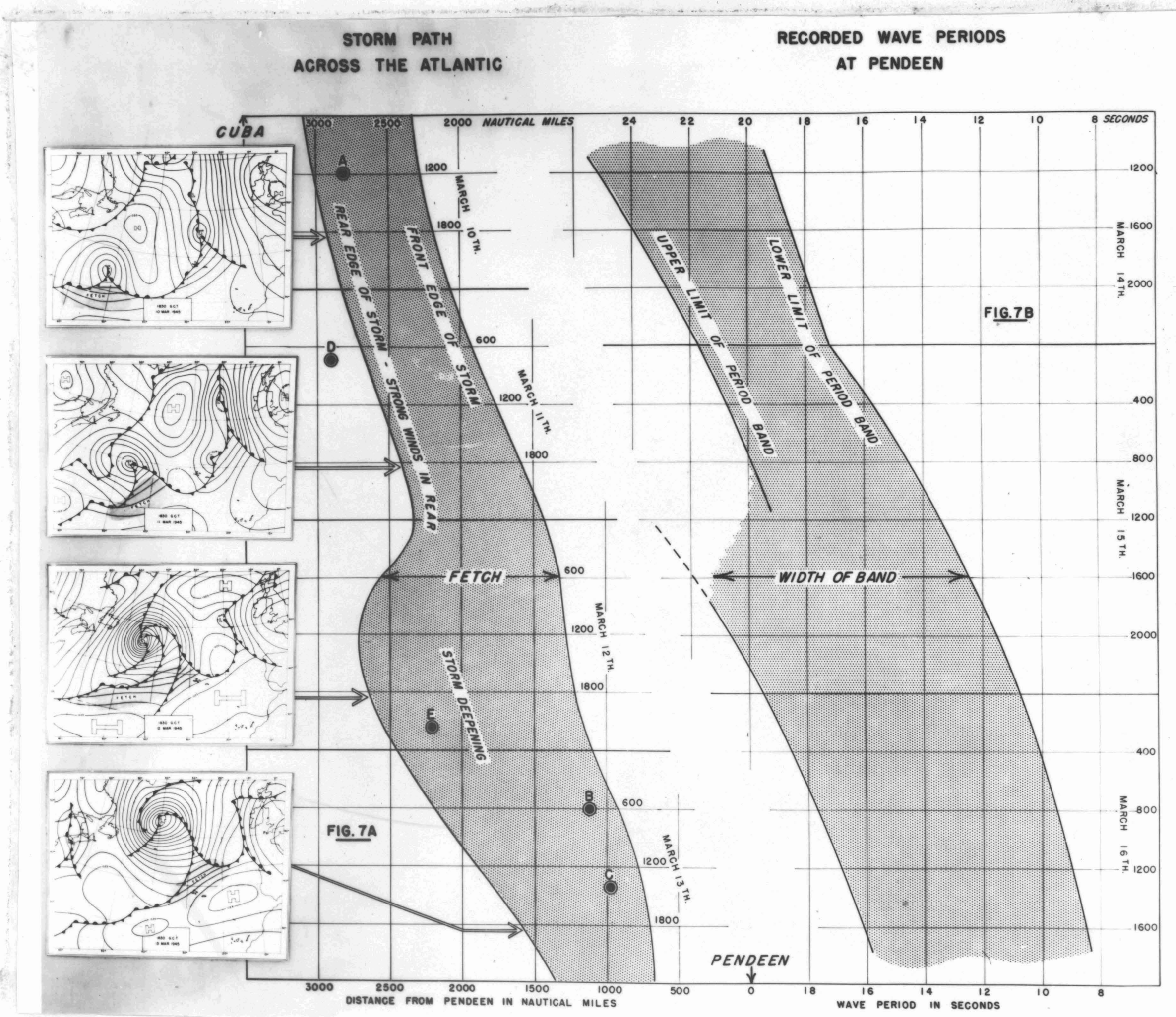

Fig. 7--Storm path across Atlantic is similar in appearance to recorded wave periods at Pendeen. Points A--E in figure 7A are locations of wave origin computed from wave records, suggesting possibility of using wave observations for tracking storms.

decrease with time according to the equation

$$T_\tau = \frac{\sigma x_0}{t} \tag{69}$$

where $(\tau - t) = (\tau_0 - t_0)$ is the time of the disturbance. Therefore as long as the lower limit of the band is defined by the waves from a single instantaneous disturbance,

$$\frac{\partial T}{\partial \tau} = \frac{\partial T}{\partial t} \tag{70}$$

Let another instantaneous disturbance at a time $\tau - t + \Delta\tau$ and distance $x' = x - \Delta x$ generate a system of waves which arrives at Pendeen simultaneously with the first system, thus having traveled a time interval $t' = t - \Delta\tau$. The period of the waves from the second disturbance at the time τ equals

$$T'_\tau = \frac{\sigma(x - \Delta x)}{t - \Delta\tau}$$

and

$$T'_\tau - T_\tau = \frac{\sigma}{t - \Delta\tau}\left(\frac{x}{t} - \frac{\Delta\tau}{\Delta t}\right)\Delta\tau \tag{71}$$

As the time and distance between the two disturbances is decreased without limits, equation (71) approaches the differential equation (66b) for a moving point source. According to (51), (52) and (56)

$$\frac{x}{t} = V = \frac{T}{\sigma}, \tag{72}$$

and

$$\frac{\Delta x}{\Delta t} = \overline{U} \tag{73}$$

can be considered the average velocity of the storm movement between any two points. Therefore <u>T'</u> is higher than, equal to, or lower than <u>T</u> according to whether <u>V</u> is larger than, equal to, or smaller than $\overline{U}$.

For T = 19 seconds, V = 29 knots, while $\overline{U}$ during the first day of storm was about 17 knots. Since the forerunners of the swell travel at considerable speed, the initial decrease of the lower limit of the band results from the characteristics of a single wave system from the earliest perceptible disturbance and is therefore similar to the decrease of the tsunami period at a fixed station. The time and position of the origin of this wave system can be plotted as a point in fig. 8A, where it should fall within the observed storm track. The point will be called the "focus" of a certain portion of the wave-band boundary.

The circles in fig. 8 give the lower limits of the period band, as they were measured on the Pendeen record. The position of the focus is computed in the following manner:

1. At the time τ = March 14, 1800, fig. 8 shows T = 18.2 seconds, $\partial T/\partial \tau$ = - 0.18 sec/hour.
2. Equations (70) and (60) give $\mathcal{X}$ = 2800 n.miles, t = 102 hours.
3. March 14, 1800 minus 102 hours is March 10, 1200. The position of the focus is shown by the large circle marked "a" in fig. 7A, and falls well within the observed storm track.
4. Setting $\mathcal{X}_0$ = 2800, and τ = March 10, 1200 + t in equ. 69 plot T_τ for various values of t in fig. 8.

On March 15, at 0000, the lower band changes its slope abruptly. The wave system from the original disturbance no longer constitutes the lower boundary of the period band. The new focus can be located by the same methods as before, and one obtains $\underline{\mathcal{X}}$ = 1100 n.miles, on March 13, at 0600. It is remarkable that the storm during the interval from March 10, 1200 to March 13, 0600 does not contribute anything to the lower boundary of the period band. The final portion of

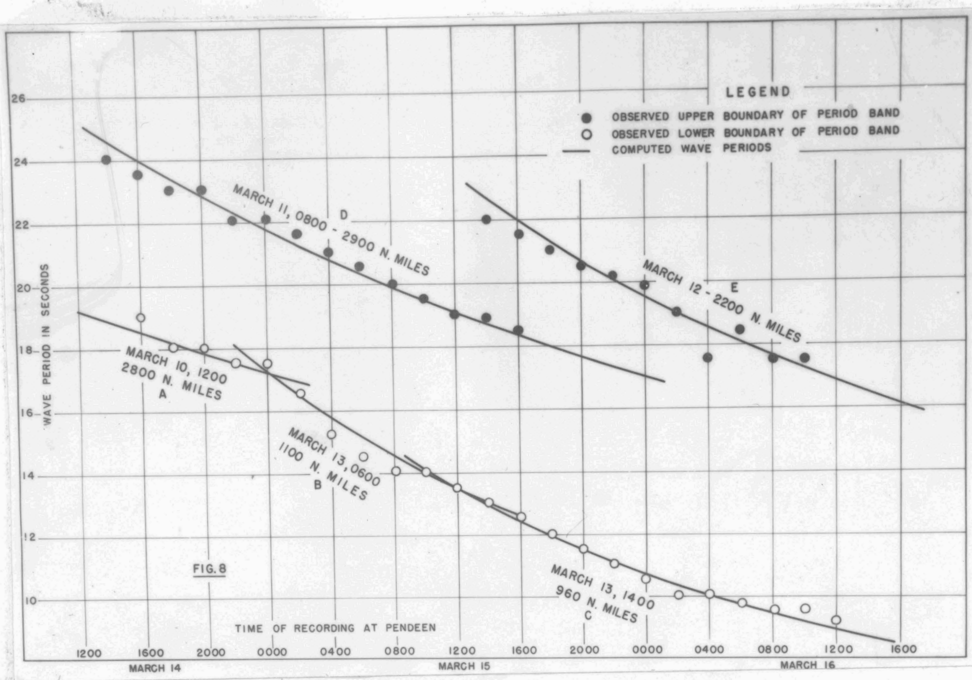

Fig. 8--Method for locating points A--E in figure 7. (see text)

the record can be fitted by a single curve from focus C at x = 960 nautical miles, March 13, 1400.

The distance between the first two foci is 1700 n.miles, the time interval 66 hours, and $\overline{U}$ equals therefore 25.8 knots or 43 feet per second. According to (72) and (73) the two associated curves in fig. 8 should intersect at $T = \sigma \overline{U}$ = 16.8 seconds. Actually the intersection takes place at T = 17.2 seconds.

The foregoing considerations lead to the following rules for the interpretation of the lower boundary of the period band:

1. During any time interval of continuously decreasing slope the boundary is determined by the wave system from a single focus. The focus can be located by measuring T and $\partial T/\partial t$ at any point within the interval, and applying equations (60) and (70).

2. A discontinuous increase in slope indicates an abrupt change from one focus to another. The intersection $T = \sigma \overline{U}$ gives the mean velocity $\overline{U}$ between the two foci.

3. A gradual increase in slope can be formed only by an envelope to a family of curves from adjoining foci. This situation can arise only in the case of a decelerating storm movement, according to the equation $U = T/\sigma$, where T is the recorded period along the envelope.

These rules constitute a possible means of tracking storms. It will be noticed, however, that focus A in fig. 7A is located 500 miles behind the storm front, the other two foci about 200 miles behind the front. This discrepancy must be expected since waves generated at the front itself could not gather sufficient energy to be perceptible at the wave station; furthermore the distance between the point of wave origin and the storm front must be larger for the first disturbance, which is almost three thousand miles from Pendeen.

The upper boundary of the period band depends upon the distribution of energy among waves of various period and upon the sensitivity of the recording instrument. If a storm emits a complete spectrum of waves, as may indeed be the case, then waves whose periods are 5 minutes or longer would travel as shallow water waves similar to tsunamis from an earthquake, and would constitute the upper limit of the period band. But even if these very long period waves should be of height similar to recorded waves they would not record because of the unfavorable response characteristics of the instrument.

Since the upper limit of the period band is well defined, we have applied the same methods as to the lower boundary. Between 1400 and 1800 on March 15 two upper boundaries can be recognized on the record, both of which are marked. The time and position of foci D and E have been computed and plotted on fig, 7A. Focus D lies somewhat behind the far boundary of the storm track, focus E somewhat short of it.

Fig. 9 illustrates the application of the method to a more complicated meteorological situation, where it appears possible to recognize and place two simultaneous storms from the wave record. The good agreement between theory and observations bears out the validity of the special solution (56) and of the relation (58) according to which the period of an individual forerunner is proportional to the travel time and to the square-root of the travel distance.

Although it is too early for any definite predictions, the theory for the increase in wave period seems to provide a rational

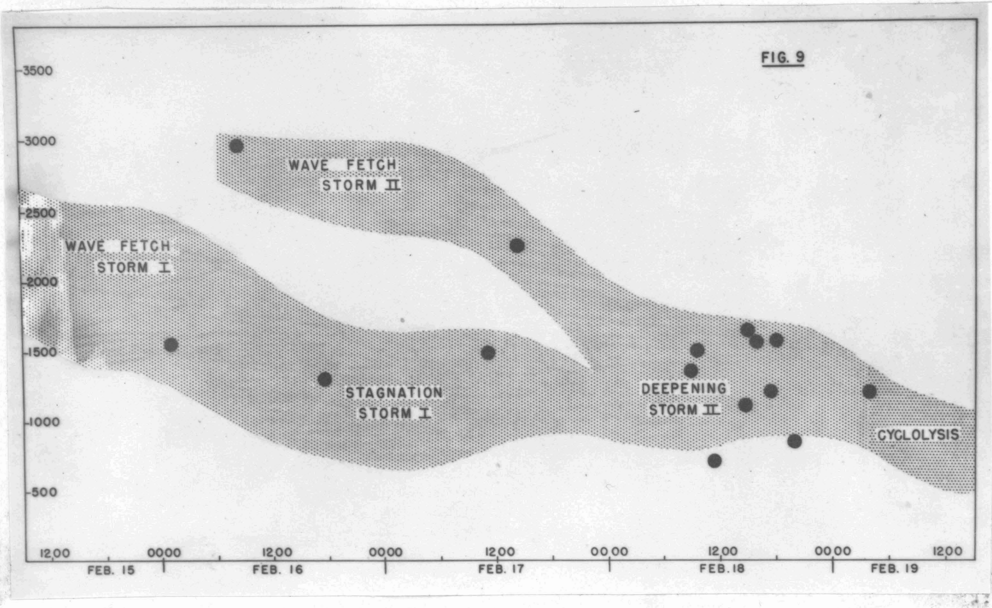

Fig. 9--Shaded bands show paths of two storms across the Atlantic. Points give position of wave origin computed from wave records at Pendeen.

basis for the interpretation of wave records as an aid to weather forecasting. The forerunners of the swell are especially suitable for a "seaborne storm warning" because they arrive much sooner than the ordinary swell, because they give a simple record since their group velocity exceeds the velocity of storm travel, and because the special solution (56) is most likely to be applicable. For these reasons it would be advisable to work with periods even higher than those recorded at Pendeen. If the recording unit should be placed into deeper water where it is out of reach from the ordinary swell, then it could be designed to record the low forerunner with great sensitivity. The proposed wave recording station at the Scripps Institution is designed to meet some of these requirements.

Application to Seismic Waves

Theory

The velocity of seismic surface waves, that is, of Love and Rayleigh waves, depends greatly upon the structural unit over which the waves are transmitted. Thus the average velocity of Love waves through the basin of the Pacific Ocean is roughly 15% higher than across continental America.

Within each structural unit the velocity depends also on the wave period for the following reason: the transmitting medium consists usually of horizontal layers with different elastic constants. Short waves are transmitted through a thin upper layer and travel with its characteristic velocity; long waves spread their energy into lower layers and travel at a velocity approximately equal to a weighted mean velocity of the disturbed layers.

Unfortunately it is not possible, as it was for the tsunami and swell, to write a simple analytical expression for the relationship between period and velocity. Even for the simplest case of two uniform layers the expression is complicated, and for assumptions conforming more closely to actual conditions the theory becomes too cumbersome to be useful here (Meissner, 1926).

It is easier to work directly with empirical relationships such as those compiled by Gutenberg and Richter (1936) from a large number of observations. These empirical relationships are between group velocity and period, but a knowledge of the wave velocity is also required for the application of our theory. If the elastic constants are functions of depth only, and do not vary along the path, then the group velocity is defined by

$$V = \frac{C}{1 + \frac{T}{C}\frac{dC}{dT}} \qquad (14)$$

Multiplying numerator and denominator by C dT, and setting

$$L = C\,T \qquad (10)$$

one obtains $dL/L^2 = dT/VT^2$. Integrating between the limits T and ∞ gives

$$\frac{1}{L} = \int_T^{\infty} \frac{dT}{VT^2}$$

and finally, going back to (10):

$$\frac{1}{C} = T\int_T^{\infty} \frac{dT}{VT^2} \qquad (74)$$

The rate of period increase of an individual wave

$$\frac{dT}{dx} = \left(\frac{1}{V} - \frac{1}{C}\right)\left(-\frac{\partial T}{\partial t}\right) \tag{21a}$$

can now be computed from equation (74), the empirical relationship $V = V(T)$, and the measured decrease of period with time at a given station.

General observations

Macelwane (1923), who also made the problem of period increase the subject of his doctor's dissertation, obtained an average value of $dT/dx = 1.4 \times 10^{-8}$ sec/cm for the California earthquake of 31 January, 1922. This figure represents the increase in period of waves occurring two and four minutes after the first arrival of surface waves, and this increase must be somewhat larger than the corresponding increase of individual waves. Gutenberg (1939) p. 367, suggests an equation of the form

$$T^2 = T_o^2 + \frac{ax}{V^3} \tag{75}$$

for the period increase of an individual wave, and gives a value of $a \approx 10^{10}$ cm^2/sec for seismic surface waves. For various values of T, and corresponding values of V (fig. 1, Gutenberg and Richter, 1936) equation 75 gives

Table 2

T sec	V km/sec	dT/dx sec/cm
10	3.45	1.2×10^{-8}
15	3.45	0.8×10^{-8}
20	3.5	0.6×10^{-8}
25	3.6	0.4×10^{-8}

The cauda of the surface waves increases in period at the rate of $dT/dx = 1.4 \times 10^{-8}$ sec/cm (fig. 5, Gutenberg and Richter, 1936).

Byerly (1926) has made a careful study of the Montana earthquake of 28 June 1925, and his data indicates dT/dx to lie between 1.4×10^{-8} and 1.8×10^{-8} sec/cm.

All these estimates deal with long distance propagation from major shocks, and they consistently lead to an increase of the order of 1 second per 1000 km. To compare this value with theory the clearest records from American stations[5] have been selected from Byerly's report on the Montana earthquake (Byerly 1926), and the average relationship between period and time plotted in fig. 10D. The relationship V = V(T) is again taken from fig. 1, Gutenberg and Richter (1936), and the wave velocity is found by numerical integration of (74):

Table 3

T seconds	V km/sec	C km/sec (equ.74)	$-\partial T/\partial t$ sec/sec (fig.10B)	dT/dx sec/cm
10	3.56	3.76	.058	0.9×10^{-8}
12	3.52	3.80	.110	2.3×10^{-8}
14	3.50	3.85	.18	4.8×10^{-8}

The computed and observed period increases are therefore of the same order of magnitude.

<u>The Mexican earthquake, 22 Feb. 1943</u>

Figure 10A shows records of this shock at the various stations maintained by the Pasadena Seismological Laboratory[6]. Tracing individual waves from station to station by making use of the known

[5] Pasadena, Lick, Berkeley, and St. Louis.

[6] Dr. Richter of the Seismological Laboratory, Pasadena, has gone to considerable trouble in selecting a set of records suitable for this discussion.

distances and wave velocities has been attempted but the identification, which is difficult, especially for the furthest station, is not essential to the following analysis.

Figure 10B shows the relationship between period and time at the four stations. Figure 10C is a plot of the basic solution (17) provided that the transmission characteristics are uniform in the x-direction, so that

$$T = T(t - \frac{x}{V}) \qquad (76)$$

where $\underline{t}$ is time (GCT), $\underline{V}$ is the group velocity, and $\underline{x}$ is the excess of the travel distance over the distance to La Jolla. The theory is confirmed by the fact that observations for the four stations obey a single relationship. The procedure is analogous to the one followed earlier in the analysis of the April 1st tsunami. The similarity between the curves in figures 4B and 10C is striking and suggests the possibility that the nature of the general solution (76) may serve as a means of estimating the horizontal extent of the initial seismic disturbance.

Conclusions

The theory for the increase in wave period leads to two expressions which are especially suitable for comparison with observations. The <u>differential</u> form

$$\frac{dT}{dx} = \left(\frac{1}{V} - \frac{1}{C}\right) \left(-\frac{\partial T}{\partial t}\right) \qquad (21a)$$

gives the period increase with distance of an individual wave of group velocity $\underline{V}$, wave velocity $\underline{C}$, and rate of period decrease at a fixed point $\underline{\partial T/\partial t}$. Between two stations separated by a finite

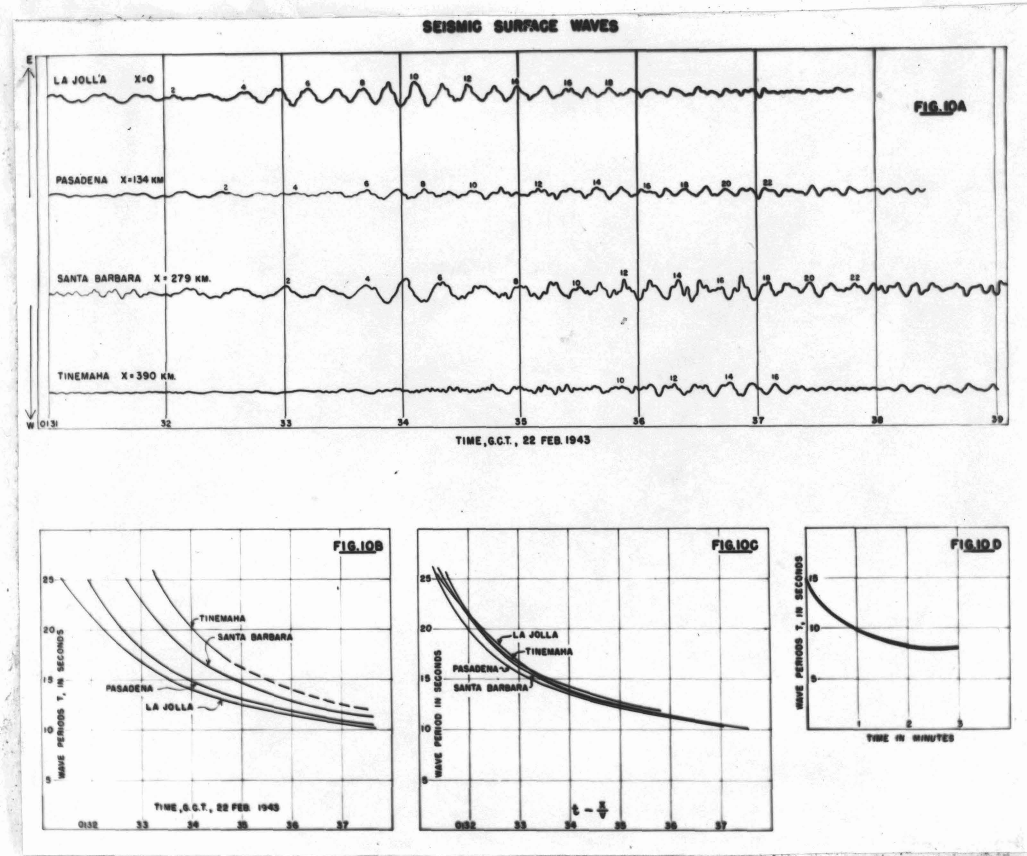

Fig. 10--10A shows records of Mexican shock at four southern California stations. 10B and 10D show period against time for the Montana and Mexican earthquakes, respectively. 10C gives the general solution (76) for the Mexican earthquake.

distance average values have to be used. The integrated form for the wave field requires that from a single disturbance all waves at any time t and distance x must obey the same functional relationship

$$T = T\left(t - \int \frac{dx}{V}\right) \tag{17}$$

Both forms have been applied successfully to the examples in this report. However, each application poses its own particular problem. For the tsunami the variation in depth along the path of travel must be taken into account. In the case of the forerunners of the swell, complexities associated with a moving continuous source must be considered. For the seismic surface waves the dispersion relationship must be taken from empirical evidence.

Yet in spite of the fact that we have dealt with different geophysical phenomena the increase in period seems to follow the same physical law. The nature of the functional relationship in (17) is strikingly similar for the tsunami and the seismic surface waves. In all instances the period increase is slow except for the leading waves, whose periods are approximately proportional to their travel time and to the square-root of their travel distance. From the experience gained it seems likely that other phenomena dealing with long-distance wave propagation can be brought into the same theoretical framework.

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