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Spectral Properties of the Koopman Operator in the Analysis of Nonstationary Dynamical Systems

Author

Mohr, Ryan M.

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Spectral Properties of the Koopman Operator in the Analysis of Nonstationary Dynamical Systems

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requirements for the degree of

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by

Ryan M. Mohr

Committee in charge:

Professor Igor Mezić, Chair
Professor Bassam Bamieh
Professor Jeffrey Moehlis
Professor Mihai Putinar

June 2014

The dissertation of Ryan M. Mohr is approved:

Bassam Bamieh

Jeffrey Moehlis

Mihai Putinar

Igor Mezić, Committee Chair

May 2014

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To my family and dear friends

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Curriculum Vitae

Education

2006 – PhD Candidate in Mechanical Engineering, University of California, Santa Barbara, USA

2001 – 2006 Bachelor of Science, Mechanical Engineering & Physics (*Majors*), Mathematics (*Minor*), *Summa Cum Laude*, Virginia Commonwealth University, Virginia, USA

Relevant Experience

Research

2007 – Graduate Student Researcher, UC Santa Barbara, Research group of Dr. Igor Mezić

2005 – 2006 Undergraduate Researcher, Virginia Commonwealth University, Control and Mechatronic Systems Laboratory, Research group of Dr. Kam Leang

2004 – 2005 Undergraduate Researcher, Medical College of Virginia, Smooth Muscles Laboratory, Research group of Dr. Paul Ratz (Dept. Biochemistry, Med. Col. of Virginia), Co-PI: Dr. John Speich (Dept. Mech Eng., Virginia Commonwealth Univ.)

Teaching

2006 – 2012 Teaching Assistant, UC Santa Barbara, 6 academic quarters; courses: “Mechanical Vibrations (Lead TA)”, “Nonlinear Phenomena” (twice), “System Identification and Control”, “ME Design II”, “Sensors”

Fall 2005 Laboratory Assistant, Virginia Commonwealth University; course: “Mechatronics”

Professional

Jun 2004 – Aug 2005 Advanced Fiber Systems, BSME Cooperative Education Engineer, *Kevlar SBU, Spruance Plant, DuPont, Richmond, Virginia, USA*; Created and maintained Kevlar Production database and analysis programs for a Six Sigma project; analysis of historical meter pump performance and reliability

Fall 2004 Advanced Fiber Systems, BSME Cooperative Education Engineer, *Nomex SBU, Spruance Plant, DuPont, Richmond, Virginia, USA*; Product portfolio optimization study, created analysis program identifying opportune areas for improvement

Academic Activities

Speaker

May 2013 SIAM Conference on Applications of Dynamical Systems DS13, Snowbird, UT, USA; *Koopman Mode Analysis of Networks*, Minisymposium Talk (MS60)

May 2011 SIAM Conference on Applications of Dynamical Systems DS11, Snowbird, UT, USA; *Low-dimensional models and anomaly detection for TCP-like networks using the Koopman operator*; Contributed Talk (CP15)

Dec 2010 49th IEEE Conference on Decision and Control, Atlanta , GA, USA, *The Use of Ergodic Theory in Designing Dynamics for Search Problems*; Presentation and a Proceedings Paper [see MM10]

May 2009 SIAM Conference on Applications of Dynamical Systems DS09, Snowbird, UT, USA; *Designing Search Dynamics Robust Under Sensor Uncertainty*; Minisymposium Talk (MS33)

Attendee

Jan 2008 Dynamics Days, Knoxville, TN, USA;

Dec 2007 Workshop on *Chaos and Ergodicity of Realistic Hamiltonian Systems*, Centre de recherches mathématiques (CRM), Université de Montréal, Montréal, Québec, Canada;

May 2007 SIAM Conference on Applications of Dynamical Systems DS07, Snowbird, UT, USA;

Member of Professional Societies

2013 – present American Mathematical Society (AMS), Student Member

2006 – present Society for Industrial and Applied Mathematics (SIAM), Student Member

Awards

June 2013 *Outstanding Teaching Assistant for the Academic Year 2012/13*; Department of Mechanical Engineering, UC Santa Barbara

May 2006 *Invited Speaker*; School of Engineering Graduation Ceremony, Virginia Commonwealth University

Fall 2005 *Best Physics Student*; Dept. of Physics, Virginia Commonwealth University

2004 *National Dean's List*; 2003–2004 Edition

2003 *National Dean's List*; 2002–2003 Edition

2001 – 2006 *Henrico County Merit Scholarship* for full tuition and fees for duration of undergraduate engineering education

Fields of Study

Major Field: Dynamical Systems

Studies in Operator and Spectral Theory with Professor Igor Mezić

Studies in Ergodic Theory with Professor Igor Mezić

Abstract

Spectral Properties of the Koopman Operator in the Analysis of Nonstationary Dynamical Systems

by

Ryan M. Mohr

The dominating methodology used in the study of dynamical systems is the geometric picture introduced by Poincaré. The focus is on the structure of the state space and the asymptotic behavior of trajectories. Special solutions such as fixed points and limit cycles, along with their stable and unstable manifolds, are of interest due to their ability to organize the trajectories in the surrounding state space.

Another viewpoint that is becoming more prevalent is the operator-theoretic / functional-analytic one which describes the system in terms of the evolution of functions or measures defined on the state space. Part I of this doctoral dissertation focuses on the Koopman, or composition, operator that determines how a function on the state space evolves as the state trajectories evolve. Most current studies involving the Koopman operator have dealt with its spectral properties that are induced by dynamical systems that are, in some sense, stationary (in the probabilistic sense). The dynamical systems studied are either measure-preserving or initial conditions for trajectories are restricted to an attractor for the system. In these situations, only the

point spectrum on the unit circle is considered; this part of the spectrum is called the unimodular spectrum. This work investigates relaxations of these situations in two different directions. The first is an extension of the spectral analysis of the Koopman operator to dynamical systems possessing either dissipation or expansion in regions of their state space. The second is to consider switched, stochastically-driven dynamical systems and the associated collection of semigroups of Koopman operators.

In the first direction, we develop the Generalized Laplace Analysis (GLA) for both spectral operators of scalar type (in the sense of Dunford) and non spectral operators. The GLA is a method of constructing eigenfunctions of the Koopman operator corresponding to non-unimodular eigenvalues. It represents an extension of the ergodic theorems proven for ergodic, measure-preserving, on-attractor dynamics to the case where we have off-attractor dynamics. We also give a general procedure for constructing an appropriate Banach space of functions on which the Koopman operator is spectral. We explicitly construct these spaces for attracting fixed points and limit cycles. The spaces that we introduce and construct are generalizations of the familiar Hilbert Hardy spaces in the complex unit disc.

In the second direction, we develop the theory of switched semigroups of Koopman operators. Each semigroup is assumed to be spectral of scalar-type with unimodular point spectrum, but possibly non-unimodular continuous spectrum. The functions evolve by applying one semigroup for a period of time, then switching to another semigroup. We develop an approximation of the vector-valued function evolution by a linear approximation in the vector space that the functions map into. A basis for this linear approximation is constructed from the vector-valued modes that are coefficients of the projections of the vector-valued observable onto scalar-valued eigenfunctions of the Koopman operator. The unmodeled modes show up as noisy dynamics in the output space. We apply this methodology to traffic matrices of an Internet Service Provider's (ISP's) network backbone. Traffic matrices measure the traffic volume moving between an ingress and egress router for the network's backbone. It is shown that on each contiguous interval of time in which a single semigroup acts the modal dynamics are deterministic and periodic with Gaussian or nearly-Gaussian noise superimposed.

Part II of the dissertation represents a divergence from the first part in that it does not deal with the Koopman operator explicitly. In the second part, we consider the problem of using exponentially mixing dynamical systems to generate trajectories for an agent to follow in its search for a physical target in a large domain. The domain is a compact subset of the n-dimensional Euclidean space $\mathbb{R}^n$. It is assumed that the size of the target is unknown and can take any value in some continuous range. Furthermore, it is assumed that the target can be located anywhere in the domain with equal probability.

We cast this problem as one in the field of quantitative recurrence theory, a relatively new sub-branch of ergodic theory. We give constructive proofs for upper bounds of hitting times of small metric balls in $\mathbb{R}^n$ for mixing transformations of various speeds. The upper bounds and limit laws we derive say, approximately, that the hitting time is bounded above by some constant multiple of the inverse of the measure of the metric ball. From these results, we derive upper bounds for the expected hitting time, with respect to the range of target sizes $[\delta, V)$, to be of order $\mathcal{O}(-\ln\delta)$. First order, continuous time dynamics are constructed from discrete time mixing transformations and upper bounds for these hitting times are shown to be proportional to the discrete time case.

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Chapter I

Introduction

[1]A first introduction to nonlinear dynamical systems, such as Wiggins [Wig03], focuses on differentiable dynamics with an eye towards the asymptotic nature of trajectories. Special solutions such as fixed points, limit cycles, and periodic orbits take a primary role due their organization of trajectories in the state space. More generally, attractors and invariant manifolds help to decompose a system and develop an understanding of its behavior. Most techniques are local in nature — stability via linearization or, more generally, perturbation theory; the local coordinate changes arising from the theory of normal forms; or bifurcation theory, as examples — and require some regularity of the governing differential equations. Many theorems hold for C^r-differentiable systems with at least C^2 often being required.

The standard techniques allow practitioners to build good intuition about a sys-

[1]Portions of this chapter are similar to and reprinted with permission from M. Budišić, R. Mohr, and I. Mezić "Applied Koopmanism," Chaos. 22 (4), 047510-1 – 047510-33 (2012), doi:10.1063/1.4772195. Copyright 2012.

tem's behavior. However, the geometric approach is not always well-suited to the analysis of systems that arise in practice. Some of this difficulty stems from the low dimensional nature of many of the standard techniques — Poincaré-Bendixson's theorem being a particularly blatant example whose result only holds in the plane. With work, some theorems can be extended to higher dimensions. However, the normal route to analyzing moderate to large dimensional systems is to reduce the complexity in some way. These techniques can include exploiting special symmetries in the system, finding integrals (or other invariants) of motion to reduce the degrees of freedom, or concepts such as center manifold theory. Unfortunately, these techniques are not always applicable or if they are, finding the object of interest is intractable. In a generic large-scale, complex system, many of the standard techniques break down or applying them can be tricky.

Many of the standard, rigorous analysis techniques require explicit access to the governing differential equations and state space. For example, rigorously proving stability via linearization or Lyapunov functions requires the vector field to be known. While simulations can suggest properties such as stability of a fixed point or other solution, they are not rigorous proofs. As the dimension or complexity of the system grows, the potential for exotic or unwanted behavior increases and making conclusions based on simulations becomes a more delicate issue. In short, many of the techniques of the geometric approach cannot be considered as "data-driven" methods.

Even when the governing equations are known explicitly, the geometric picture's focus on single trajectories may not be the best viewpoint to take. In systems that exhibit any of the hierarchies of ergodic behavior — ergodicity, mixing, exactness — there is but a single trajectory that is dense in the state space. More generally, for a measure preserving dynamical system, the state space can be decomposed into ergodic components which are regions invariant under the dynamics for which the dynamics restricted to that region is ergodic. With regards to numerical conclusions, the exponential divergence of trajectories in a chaotic system makes the focus on a single trajectory meaningless since any error in the initial conditions — finite machine precision, for example — quickly grows and the simulated trajectory is nowhere near the true trajectory. In these cases, focus on the statistics of ensembles of trajectories gives a more robust analysis. The book by Lasota and Mackey [LM94] gives a good introduction to the study of such dynamics using densities.

An alternative viewpoint to the geometric one is the functional one, where the focus is on studying the behavior of elements of some function class whose common domain is the state space. In general, we refer to elements of the function class as *observables* on the system and the function class as the *space of observables*. For analysis purposes, we require that the function class have both algebraic and topological structures; the linear structure of a vector space and some type of complete topology at the very least. Most commonly, the topological structure is induced by a norm, in which case the function class is a Banach space, however, locally convex topologies induced by a family of seminorms arise as well.

Focus is now on how observables of the function class evolve over time due to the dynamics of the system and the relation of this function evolution to the behavior of the trajectories in the state space. The dynamical system induces a linear operator, the *Koopman operator*, that drives the evolution of an observable via composition with the flow map. If f is an observable and Φ_t ($t \geq 0$) is the flow map, the evolution of observables is formally given via the family of mappings $f \mapsto f \circ \Phi_t$. This mapping gives the action of the Koopman operator on the space of observables. Obviously, the space of observables must be large enough so that it is closed under the action of the operator, i.e., the mapping is a self-map of the space of observables. This is dictated by the properties of the flow map. For example, if we require that the function class consist of C^r-differentiable functions, then Φ_t should also be a C^r-differentiable family. To be more explicit, if we denote the vector space of observables by $\mathscr{F}$ and it is closed under composition with the flow maps Φ_t, then we have the family of Koopman operators $U_t : \mathscr{F} \to \mathscr{F}$ given by

$$U_t(f) = f \circ \Phi_t \qquad (t \geq 0). \tag{I.0.1}$$

Due to the linear structure of the space of observables and the definition of the Koopman operators, it is immediately clear that the Koopman operator is linear, even when the dynamical system is nonlinear. Given some mild conditions[2] on the space of observables, the linear dynamics induced by the Koopman operator are able to capture the full nonlinear behavior of the underlying dynamical system. This is contrasted

[2]For example, it is necessary that the space of observables be able to separate points.

to typical linearization techniques in the state space, such as Taylor expansion about a fixed point, that discard a lot of information and cannot resolve the nonlinear contributions of the dynamics. In order to do this, however, the Koopman operator acts on an infinite dimensional space even when the dynamical system acts in finite dimensional Euclidean space. Given the linearity of the operator, and the fact that in many cases the operator is bounded, all the tools of functional analysis can be readily applied. Furthermore, as in the case of finite dimensional linear systems, such as when studying linear vibration theory, determining the spectral properties of the Koopman operator can give a lot of insight into the behavior of the system.

However, given that the Koopman operator is defined in terms of the flow map — an object that is often difficult or impossible to construct — one may wonder as to the value of going about things in this manner. First, if we are dealing with a discrete time dynamical system, $T : \mathcal{M} \to \mathcal{M}$, the flow map Φ_n is just the map T^n evolving the state space dynamics, which gives us an explicit functional form for the Koopman operator. In this case, (I.0.1) becomes

$$U^n(f) \equiv U_n(f) = f \circ T^n. \tag{I.0.2}$$

In the continuous time case and a fixed system of ordinary differential equations, the flow map just codifies the solutions of initial value problems of the dynamical system. Therefore, if we generate trajectories in some manner, we can recover the action of the Koopman operator. All that is really required is some mechanism that takes a state space point as an input and outputs a trajectory. This represents a certain black-box point of view for the Koopman operator and makes analysis of a system in this manner suitable when explicit access to the governing dynamical equations are unavailable; such is the case with legacy codes or time series of numerical or experimental data. With this viewpoint, analysis with the Koopman operator has a certain “data-driven” flavor.

Unfortunately, not everything comes up roses. We have already mentioned that in order to capture the full nonlinear behavior of the dynamical system, the function space we have to work in is very large. The choice of the finite dimensional approximation becomes important in numerical computations (for example, see [BM12; Bud12]). Furthermore, the new functional/algebraic viewpoint, and the results obtained therein, are somewhat divorced from our hard-won intuition from the geometric setting since the connection of the functional/algebraic results with a geometric meaning is not always immediate obvious. We do mention, however, that this is not always true; there are standard characterization of ergodic and mixing dynamics in terms of the Koopman operator[3]. In fact, from a dynamical systems perspective, the strongest connection between properties of the Koopman operator and behavior of the underlying dynamical system occurs when the dynamics are restricted to an attractor.

Spectral properties of composition operators have proven useful in analyzing con-

[3]See Lasota and Mackey [LM94] or Reed and Simon [RS72] for these formulations. Lasota and Mackey [LM94] focuses on the properties of averages and does not relate the spectrum of the operator to the various types of ergodic behavior. In addition to the behavior of averages, Reed and Simon [RS72] gives spectral characterizations of ergodic and mixing dynamics.

crete systems. By constructing eigenfunctions, either explicitly or implicitly, the practitioner has gained insight into the behavior of their particular system. Typically, rather than a full spectral decomposition, a so-called Koopman mode analysis is pursued [Mez05]. This amounts to considering the spectral decomposition from a specific initial condition.

The Koopman mode analysis has seen the most application in the fluids community. The connection between the Koopman operator and fluid dynamics was worked out in [Row+09]. This work connected the spectral properties of the Koopman operator with the Dynamic Mode Decomposition (DMD) of P. Schmid — a numerical algorithm, first introduced in [SS08], to analyze fluid dynamics via snapshots of the velocity profile. A number of applications in fluid dynamics have followed — see [DHS12; NME11; Sch+11; Sch10; SS11; TAM+11; Tu+12] or the recent review article [Mez13], and the references therein, for a subset for a small subset of the results.

While the bulk of applications have been in the fluids literature via the DMD algorithm and its variants, applications in other areas have also been pursued. Other applied studies, such as energy efficiency in buildings [Eis+10; GEM12], power-systems analysis [SM11; SM12], and neurodynamics [MM12] have used spectral properties and the Koopman mode decomposition of the associated composition operator.

In most of these studies, it is implicitly assumed that the dynamics considered are on the attractor and, therefore, the eigenfunctions corresponding to unimodular (unit modulus) eigenvalues are important. A general procedure for constructing such eigenfunctions is through the use of infinite Fourier averages

$$\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} e^{-i\omega k} U_\Phi^k, \quad (\omega \in \mathbb{R}). \tag{I.0.3}$$

The existence of these limits can be guaranteed in certain situations by various ergodic theorems, Wiener and Wintner's extension of Birkhoff's pointwise ergodic theorem [Ass03; WW41] and Yosida's extension of the mean ergodic theorems [Yos78] having primacy (see ch. II.3.2).

Often though, it is quite useful to know the spectrum and eigenfunctions for non-measure preserving dynamics. For example, the recent work in neurodynamics on isochrons and isostables [MMM13] leverage the non-unimodular spectrum of the Koopman operator. The works of Cowen, MacCluer, and others on the spectra of composition operators with analytic symbols acting on (weighted) Hardy Hilbert spaces $H^2(\mathbb{D})$ or Bergman spaces $A^p(\mathbb{D})$ in the complex unit disc (or more generally, the polydisc) can be considered as other examples [AL04; CM94; GGL09; Hyv+13; Rid73; SM93]. Unfortunately, it is not immediately clear that such dynamical systems arise naturally when considering applied problems and, additionally, there seems not to be a clear general procedure on constructing non-unimodular eigenfunctions as there is for unimodular eigenfunctions through the application of equation (I.0.3).

I.1 Contributions of this dissertation

Previous work on the theory and application of the Koopman operator has dealt with dynamics that are stationary in some sense, whether this be because the dynamics are measure-preserving or are restricted to an attractor. In each of these cases the spectrum of the associated Koopman operator is fixed and unitary. In the first part of the dissertation, we pursue two different avenues that relax these conditions. The first deals with allowing the spectrum of the dynamics to be non-unimodular. This case arises when we have dissipation or expansion in the dynamics. The second avenue deals with families of Koopman operators, each possessing unimodular point spectrum, that are switched between as the system evolves. Part of the motivation for this approach is in the understanding system wide behavior of internet-type network traffic flows.

In chapters III and IV, we develop the generalized Laplace analysis (GLA) that can be used to construct eigenfunctions of the Koopman operator for eigevalues off the unit circle using Laplace averages. This can be regarded as an extension of the existing Koopman mode analysis, valid for unimodular spectrum, to non-unimodular spectrum. Chapter III develops this theory for spectral operators of scalar-type. This is done for arbitrary linear operators satisfying the spectral assumption and not just for the Koopman operator. A key theorem used in proving a our results is Yosida's mean ergodic theorem. Yosida's mean ergodic theorem can be used to construct eigenfunctions corresponding to eigenvalues on the unit circle. Our use of the theorem to construct eigenfunctions for off-unit-circle eigenvalues can be regarded, in some sense, as an extension of the ergodic theorem to dynamics that are more general than the normal measure-preserving dynamics or contraction operators that are considered in ergodic theory. All of these theorems are proven explicitly for discrete time, but we do give discuss the continuous time case. With a few additional assumptions on the semigroup of operators, the continuous time analog of the discrete case also holds. Chapter IV develops the same type of GLA theorems but for non-spectral operators. In order to prove the results, we cannot consider the entire observable space. We construct the absolutely summable subspace for a collection of eigenvalues possessing certain properties. The subspace consists of absolutely summable series of eigenfunctions. The GLA theorems are proven for observables in this subspace. We also develop of GLA-type theorem for when the eigenvalues possess generalized eigenfunctions. Time averages can still be used to construct the generalized eigenfunction but a different scaling factor has to be used due to an algebraic time dependence in the evolution of observables.

In chapter V, we specialize to the Koopman operator. We first start off with two key properties of eigenfunctions of the Koopman operator that deal with their algebraic and topological properties. In particular, pointwise products of two eigenfunctions of the Koopman operator is again an eigenfunction if the product still belongs to the observable space. In short, if the eigenfunctions are closed in the observable space under pointwise products, they form a semigroup. The second key property relates the eigenvalues and eigenfunctions of topologically conjugate dynamical systems. If we find an eigenfunction of the Koopman operator associated to one dynamical system, we can immediately construct an eigenfunction for the other Koopman operator by composing with the conjugacy map. The new eigenfunction has the same eigenvalue as the original one. This is particular useful for determining the spectrum and eigenfunctions for the Koopman operator associated to a nonlinear dynamical system. We can find the spectrum and eigenfunctions for the Koopman operator of the linearized system fairly easily. This gives part of the spectrum for the Koopman operator of the nonlinear system and we can get the associated eigenfunction as soon as we know the conjugacy map from the nonlinear system to the linear one. The rest of the chapter gives a method of constructing a space of observables for a dynamical system possessing an attractor on which the Koopman operator is spectral and satisfies the assumptions of chapter III. This space of functions takes the form as a power series with coefficients taking values in a Banach space of observables. The Banach space consists of functions supported on the attractor of the dynamical system, whereas the indeterminates of the power series correspond to stable directions. These functions can be thought of as a generalization the normal H^2 Hardy functions in the unit disc which is why we call them $\mathscr{B}$-Hardy functions. The construction is accomplished by first finding a set of formal eigenfunctions of the Koopman operator. This set of (principle) eigenfunctions is used to generate an algebra which is then completed under a certain polynomial norm. Construction of a space of observables for any topologically conjugate system can be done using a pullback construction which is just composing any observable in the original space with the conjugacy map between the two dynamical systems.

In chapter VII, we switch gears. Whereas the previous three chapters dealt with the case of operators with non-unimodular spectrum, the current chapter deals with families of Koopman operators all having purely unimodular point spectrum. The system is evolved by applying an operator from the family for a certain period of time, then switching to another operator in the family which drives the systems for another (not necessarily equal) period of time. The contiguous period of time that an operator acts to drive the system is called a stationary component. We develop a spectral decomposition for this type of switch and hold dynamics and spectral projections for the Koopman operator acting on a stationary component are approximated by finite Fourier averages over the stationary component. We linearly approximate the dynamics the output space by building a basis from the Koopman modes computed from the finite time Fourier averages. Since we are trying to approximate the nonlinear dynamics in the state space by linear dynamics in the output space of the observables, there will be some "spectral leakage" from other modes and the continuous part of the spectrum into the Koopman modes forming the basis. We are able to characterize this contribution to each Koopman mode in the basis by first computing the skew projection onto a Koopman mode in the basis and then subtracting acting the analytic prediction of the dynamics that come from the standard Koopman mode analysis.

This residual time series is the contribution of the other modes and the continuous part of the spectrum to that particular Koopman basis mode. This framework is applied to traffic matrices of the Abilene network. We show that the residual time series for each Koopman mode and stationary component have Gaussian or nearly Gaussian distribution. Therefore, the traffic matrices can be simply approximated by a system consisting of deterministic oscillatory components (corresponding to the point spectrum) superimposed with Gaussian or nearly-Gaussian noise.

Part II of this dissertation represents a divergence from the first part as it does not explicitly deal with the Koopman operator. The work in this part of the dissertation deals with searching for (physical) targets when we possess very little information about them other than they are located in some (large) region. Specifically, we look at using exponentially mixing transformations to design search dynamics to look for targets that can be located with uniform probability in the search domain and whose size is a nearly-uniform random variable in some interval $[\delta, V)$. We are interested in the search performance when using these mixing transformations. This problem is formulated abstractly as a problem in the field of quantitative recurrence, a recently new sub-field of ergodic theory concerned with the distribution of recurrence and hitting times for various type of sets in the state space and various classes of dynamical systems. In the course our work, we introduce the concepts of multivalued maps and $\mathfrak{B}$-regularity, the later guaranteeing the existence of time averages of continuous functions, and develop concepts of ergodicity and mixing for multivalued $\mathfrak{B}$-regular maps. We prove constructive upper bounds for the hitting times for uniformly and non-uniformly exponentially mixing maps for targets of a fixed size. Expected search times for targets taking sizes in $[\delta, V)$ are easily computed from the hitting time results and have order $\mathcal{O}(\delta^{-1})$. Limit laws for the uniformly exponentially mixing maps are proven for δ going to zero. First order continuous time search dynamics are generated from the discrete time maps and it is shown that the hitting time results hold with the same upper bound scaled by some constant the depends on the physical search domain.

Part I

Spectral Properties of the Koopman Operator in the Analysis of Nonstationary Dynamical Systems

Chapter II

Mathematical Prerequisites

This short chapter can be skimmed on a first reading and referred back to as needed. The first two subsections collect some standard definitions and concepts from dynamical systems and functional analysis that will be needed. Most of this material of this section can be found in standard references such as [BN00; DS58; KH97; Lax02; RS72; Yos78], and [Wig03].

Concepts are broken out into their own headings for ease of reference.

II.1 Dynamical systems, flow maps, and observables

Let $\mathcal{M}$ be a metric space. We will call $\mathcal{M}$ the state space. A discrete time dynamical system on $\mathcal{M}$ is given by a family of maps $\Phi_n : \mathcal{M} \to \mathcal{M}$, where $n \in \mathbb{T}$

and $\mathbb{T}$ is either $\mathbb{Z}$ or $\mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We assume the family of maps have the properties

$$\begin{aligned} \Phi_n \circ \Phi_m &= \Phi_{n+m} \quad (\forall m, n \in \mathbb{T}), \\ \Phi_0 &= \mathrm{Id}_{\mathcal{M}}, \end{aligned} \tag{II.1.1}$$

where $\mathrm{Id}_{\mathcal{M}}$ is the identity map on $\mathcal{M}$. The relations in (II.1.1) defines a *semigroup property* of the family. The map $\Phi : \mathbb{T} \times \mathcal{M} \to \mathcal{M}$ defined by $\Phi(n, x) = \Phi_n(x)$ is called the *flow map*. Often, the flow map is induced by a single map $T : \mathcal{M} \to \mathcal{M}$. Then, $\Phi_n := T^n = \underbrace{T \circ \cdots \circ T}_{n \text{ times}}$. The dynamical system will be written $(\mathcal{M}, \Phi)$.

A continuous time dynamical system is given by the family of maps $\Phi_t : \mathcal{M} \to \mathcal{M}$, where the family is now parameterized by $t \in \mathbb{T} \subset \mathbb{R}$. Usually, $\mathbb{T}$ will either be all of $\mathbb{R}$ or the positive semiaxis $\mathbb{R}^+ = [0, \infty)$. When the continuous time dynamical system is induced by a system of ordinary differential equations (ODE's), the flow map $\{\Phi_t\}_{t \in \mathbb{T}}$ codifies solutions of the initial value problems (IVP's) for the ODE's; i.e., $\Phi_t(x)$ is the solution of the IVP

$$\begin{aligned} \frac{du}{dt} &= \dot{u} = F(u) \\ u(0) &= x, \end{aligned} \tag{II.1.2}$$

at time $t \in \mathbb{T}$ and where $u \in \mathcal{M}$ and $F : \mathcal{M} \to \mathcal{M}$ is the system of ODE's.

Let $\mathscr{F}$ be a vector space of functions $f : \mathcal{M} \to \mathbb{C}$. The space $\mathscr{F}$ is called the *space of observables* whereas an element f of $\mathscr{F}$ is called an *observable*. The space $\mathscr{F}$ is a space of of vector-valued observables if its elements are maps from $\mathcal{M}$ into a Banach space $\mathscr{B}$. Usually, we consider the space of vector-valued observables

$\mathscr{F}^m = \underbrace{\mathscr{F} \times \cdots \times \mathscr{F}}_{m \text{ times}}$, so that $\mathscr{B} = \mathbb{C}^m$. If $\mathscr{F}$ is closed under pointwise products defined by the bilinear map $* : \mathscr{F} \times \mathscr{F} \to \mathscr{F}$ given by $(f * g)(x) = f(x) \cdot g(x)$[1], then $(\mathscr{F}, +, *)$ is an associative, abelian ring. The product $f * g$ will usually be written fg. If the constant functions are in $\mathscr{F}$, then it is an associative, abelian ring with a unit.

Example II.1.1. *Let $\mathscr{F}$ be the space of observables consisting of $\mathbb{R}$-valued functions on the real line integrable with respect to the Lebesgue measure. Then $\mathscr{F}$ has no unit since the constant functions are not integrable.* □

II.2 Basic concepts from linear functional analysis

Let $\mathscr{X}$ and $\mathscr{Y}$ be (real or complex) vector spaces.

Definition II.2.1 (Linear operator, Space of linear operators)**.** A map $U : \mathscr{X} \to \mathscr{Y}$ is a linear if for all vectors $x, y \in \mathscr{X}$ and scalars α, β, $U(\alpha x + \beta y) = \alpha U(x) + \beta U(y)$. The set of all linear maps from $\mathscr{X}$ to $\mathscr{Y}$ is denoted by $L(\mathscr{X}, \mathscr{Y})$. When $\mathscr{Y} = \mathscr{X}$, the we write $L(\mathscr{X})$ instead.

Definition II.2.2 (Linear functional)**.** If $\mathscr{Y}$ in the above definition is the scalar field over which $\mathscr{X}$ is defined, then U is called a linear functional. The space of linear functionals on $\mathscr{X}$ is denoted by $\mathscr{X}'$. Some times it is called the algebraic dual.

[1]The product on the right hand side is normal multiplication in $\mathbb{C}$.

Definition II.2.3 (Adjoint)**.** Let $U : \mathscr{X} \to \mathscr{Y}$ be linear and let $\mathscr{X}'$ be the algebraic dual of $\mathscr{X}$. The adjoint of U is the linear map $U' : \mathscr{Y}' \to \mathscr{X}'$ is defined for all $f \in \mathscr{Y}'$ and $x \in \mathscr{X}$ by $(U'f)(x) = f(Ux)$.

II.2.1 Topologies.

Various topologies, on both $\mathscr{X}$ and the space of linear operators from $\mathscr{X}$ to $\mathscr{X}$ will be useful.

Definition II.2.4 (Normed space, Banach space)**.** Let $\mathscr{X}$ be a vector space. A map $\|\cdot\| : \mathscr{X} \to \mathbb{R}$ is called a norm if it has the following properties:

(i) for all $x \in \mathscr{X}$, $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,

(ii) for all scalars α and $x \in \mathscr{X}$, $\|\alpha x\| = |\alpha| \, \|x\|$,

(iii) for all $x, y \in \mathscr{X}$, $\|x + y\| \leq \|x\| + \|y\|$.

We call $(\mathscr{X}, \|\cdot\|)$ a normed space. If the normed space is complete, it is called a Banach space.

Definition II.2.5 (Normed topology)**.** Fix $x \in \mathscr{X}$ and $\varepsilon > 0$. Let $V(x, \varepsilon) = \{y \in \mathscr{X} \mid \|x - y\| < \varepsilon\}$. All sets of this form define a basis for a topology on $\mathscr{X}$. This topology is called the norm topology on $\mathscr{X}$.

Definition II.2.6 (Seminorms)**.** Let $\mathscr{X}$ be a vector space. A function $p : \mathscr{X} \to \mathbb{R}$ is a called a *seminorm* if it has the following properties:

(i) $p(x) \geq 0$, (nonnegative)

(ii) $p(\beta x) = |\beta| \, p(x), \quad \forall \beta \in \mathbb{C}$, (homogeneity)

(iii) $p(x+y) \leq p(x) + p(y)$, (triangle inequality).

The only difference between norms and seminorms is that the seminorm conditions do not imply that $x = 0$ if $p(x) = 0$.

Definition II.2.7 (Separating family)**.** Let $\mathscr{A}$ be an index set and $P = \{p_\alpha : \mathscr{X} \to \mathbb{R} \mid \alpha \in \mathscr{A}\}$ a family of seminorms. The family is said to separate points if for all $x \neq 0$, there is a $\gamma(x) \in \mathscr{A}$ such that $p_{\gamma(x)}(x) \neq 0$.

Equivalently, if P is a separating family of seminorms, then $p_\alpha(x) = 0$ for all $\alpha \in \mathscr{A}$ implies $x = 0$.

Definition II.2.8 (Locally convex topology)**.** Assume $P = \{p_\alpha : \mathscr{X} \to \mathbb{R}\}_{\alpha \in A}$ is a separating family of seminorms. Let $I \subset A$ be a finite subset and $\varepsilon > 0$ an arbitrary number. For $x \in \mathscr{X}$, define

$$V(x; I, \varepsilon) := \{y \mid y \in \mathscr{X}, p_{\alpha_i}(y - x) < \varepsilon, i \in I\}\,. \tag{II.2.1}$$

All sets of the form (II.2.1) form a neighborhood base at x. Call a set *open* if it is formed from arbitrary unions of sets of the form (II.2.1). This defines a topology on $\mathscr{X}$ called the locally convex topology. A vector space $\mathscr{X}$ equipped with this topology is called a locally convex linear topological (LCLT) space.

The locally convex topology and the norm topology will sometimes be called the strong topology.

II.2.2 Continuity and operator norms.

Definition II.2.9 (Operator norm, bounded operator)**.** Let $\mathscr{X}$ and $\mathscr{Y}$ be two normed spaces and let $U \in L(\mathscr{X}, \mathscr{Y})$. Then we define a norm for U by

$$\|U\| = \sup_{x \neq 0} \frac{\|Ux\|}{\|x\|} = \sup_{\|x\|=1} \|Ux\| . \tag{II.2.2}$$

The norm in the numerator of the fraction is the norm on $\mathscr{Y}$ and the norm in the denominator is the one on $\mathscr{X}$. An operator $U \in L(\mathscr{X}, \mathscr{Y})$ is called bounded if $\|U\| < \infty$.

Definition II.2.10 (Continuous operators and continuous dual in LCLT spaces)**.** Let $\mathscr{X}$ and $\mathscr{Y}$ be locally convex linear topological spaces and $U : \mathscr{X} \to \mathscr{Y}$ a linear operator. Then U is continuous if for every set $B \subset \mathscr{Y}$ that is open in the topology of $\mathscr{Y}$, $U^{-1}(B)$ is an open set in the topology of $\mathscr{X}$. The subspace of $L(\mathscr{X}, \mathscr{Y})$ consisting of all continuous linear operators is denoted by $\mathscr{L}(\mathscr{X}, \mathscr{Y})$. If $\mathscr{Y}$ is the scalar field for $\mathscr{X}$, then $\mathscr{L}(\mathscr{X}, \mathscr{Y})$ is the space of all continuous linear functionals. This space is denoted by $\mathscr{X}^*$ and is called the continuous dual of $\mathscr{X}$.

It is an elementary result that a linear operator between two normed spaces is bounded if and only if it is continuous with respect to the norm topologies.

It is easy to see that $x \in \mathscr{X}$ defines a linear functional on $\mathscr{X}^*$. Let ℓ_x be the map from $\mathscr{X}^*$ to the scalar field defined by $\ell_x(f) = f(x)$. The linearity and continuity of this map follows from the continuity of $f \in \mathscr{X}^*$. The map taking $x \mapsto \ell_x$ embeds $\mathscr{X}$ into $\mathscr{X}^{**}$.

Definition II.2.11. $\mathscr{X}$ is reflexive if $\mathscr{X}$ is isometrically isomorphic to $\mathscr{X}^{**}$ under the embedding map.

Definition II.2.12 (Weak topology on $\mathscr{X}$)**.** Let $x \in \mathscr{X}$, $\{f_i\}_{i \in I}$ an arbitrary finite subset of $\mathscr{X}^*$, and $\varepsilon > 0$. Define the set

$$V(x, I, \varepsilon) = \{y \mid y \in \mathscr{X}, |f_i(y - x)| < \varepsilon\} . \tag{II.2.3}$$

All sets of this form define a basis for the weak topology on $\mathscr{X}$. This is the weakest topology such that all $f \in \mathscr{X}^*$ are continuous.

The weak topology on $\mathscr{X}$ is a locally convex topology defined by taking the family of seminorms to be $p_f(x) = |f(x)|$ where $f \in \mathscr{X}^*$. It is weaker than the strong topology, hence the name.

Definition II.2.13 (Strong convergence)**.** Let $\mathscr{X}$ be a LCLT space. A sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathscr{X}$ converges to $x \in \mathscr{X}$ in the locally convex topology if for every neighborhood $V(x; I, \varepsilon)$ of x, there is an $N_{I,\varepsilon} \in \mathbb{N}$ such that $x_n \in V(\psi; I, \varepsilon)$ for all $n \geq N_{I,\varepsilon}$. This is equivalent to the conditions $p_\alpha(\psi_n - \psi) \to 0$ as $n \to \infty$ for all $\alpha \in A$.

Definition II.2.14 (Weak convergence)**.** Let $\mathscr{X}$ be a LCLT space. A sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathscr{X}$ is said to converge weakly if for every $f \in \mathscr{X}^*$, the sequence $\{f(x_n)\}_{n \in \mathbb{N}}$ is convergent. It is said to converge weakly to $x \in \mathscr{X}$, if for all $f \in \mathscr{X}^*$, $\lim_{n \to \infty} |f(x_n - x)| = 0$.

Definition II.2.15 (Weak sequential compactness)**.** A space LCLT space $\mathscr{X}$ is said to be weakly sequentially compact if every sequence in $\mathscr{X}$ has a subsequence weakly convergent to an element of $\mathscr{X}$. The space is said to be locally weakly sequentially compact if every point of $\mathscr{X}$ ha a weakly sequentially compact neighborhood.

If $\mathscr{X}$ is a separable and reflexive Banach space, it follows from the sequential version of the Banach-Alaoglu theorem that it is also locally sequentially weakly compact.

Definition II.2.16 (Strongly continuous semigroup)**.** Let $\mathscr{X}$ be a Banach space. For each $t \geq 0$, let $U(t) : \mathscr{X} \to \mathscr{X}$ be a bounded linear operator. Furthermore, assume this family of operators satisfies

(i) $U(t+s) = U(t)U(s)$, for all $t, s \geq 0$ and

(ii) $U(0) = I$, where I is the identity operator on $\mathscr{X}$.

Such a family is called a semigroup of operators. If, in addition, it is true that for all $x \in \mathscr{X}$, that $\lim_{t\to 0} \|U(t)x - x\| = 0$, then the family is said to be strongly continuous at 0.

Definition II.2.17 (Equicontinuous family)**.** Consider the family of operators formed by taking powers of $U \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$; $\mathscr{T} := \{U^n\}_{n\in\mathbb{N}}$. The family is called *equicontinuous* if for every continuous seminorm p, there is a continuous seminorm p' such that for all $\psi \in \mathscr{F}$

$$\sup_{n\geq 1} p(U^n\psi) \leq p'(\psi).$$

Example II.2.18. *Examples of equicontinuous families.*

(i) The topology of $\mathscr{F}$ is given by a norm $\|\cdot\|$ and $U : \mathscr{F} \to \mathscr{F}$ has $\|U\| \leq 1$.

(ii) Let $\mathscr{F} = \ell^\infty(\mathbb{R}^{\mathbb{N}})$*. Define* $Ue_j = \alpha_j e_{j+1}$*, where* $\{\alpha_j\}_1^\infty$ *is a real sequence converging to 0. Then* $\|U^n\|_\infty \leq \sup_{F \subset \mathbb{N}, F \text{ finite}} \prod_{j \in F} |\alpha_j| < \infty$*, for all* $n \in \mathbb{N}$*. Note that if* $\sup |\alpha_j| > 1$*, then* $\|U\|_\infty > 1$*.*

□

Example II.2.19. *Let* $\{\ell_\alpha : \mathscr{X} \to \mathbb{C}\}_{\alpha \in A}$ *be a set of linear functionals on the normed space* $\mathscr{X}$ *that separates points. Define a family of seminorms by*

$$p_\alpha(f) := |\ell_\alpha(f)| .$$

The locally convex topology induced by the seminorms is exactly the weak topology of $\mathscr{X}$ *induced by the linear functionals.* □

Example II.2.20. *Let* $\mathscr{F}$ *be the set of real analytic functions on the real line and let* $\mathscr{K}$ *be the collection of compact sets of the real line that contain an open interval. Define a family of seminorms by*

$$p_K(\psi) = \sup_{x \in K} |\psi(x)| , \quad (K \in \mathscr{K}). \tag{II.2.4}$$

This gives an uncountable collection of seminorms. However, the topology generated by this family of equivalent to the topology generated by $\{p_{K_i}\}_{i \in \mathbb{N}}$ *where* $\{K_i\}$ *is an increasing family compact intervals such that* $\mathbb{R} = \cup_{i \in \mathbb{N}} K_i$*.* □

Example II.2.21 (Continuation of example II.2.20)**.** *Let* $\{\psi_n\}_{n \geq 1}$ *be a sequence in* $\mathscr{F}$ *converging to* $\psi \in \mathscr{F}$*. This means that for every set* $V(0; I, \varepsilon)$ *of the form* (II.2.1)*,* *there is an* $N_{I,\varepsilon} \in \mathbb{N}$ *such that* $\psi_n - \psi \in V(0; I, \varepsilon)$ *for* $n \geq N_{I,\varepsilon}$*. This implies that for every compact set* $K \in \mathscr{K}$ *and every* $\varepsilon > 0$*, there is an* $N_{K,\varepsilon} \in \mathbb{N}$ *such that*

$$p_K(\psi_n - \psi) = \sup_{x \in K} |\psi_n(x) - \psi(x)| < \varepsilon,$$

for all $n \geq N_{K,\varepsilon}$*. Therefore, a sequence convergent in the locally convex topology converges uniformly on compact sets. The family of seminorms given by* (II.2.4) *specifies the topology of compact convergence.* □

II.2.3 Operator topologies

The norm defined by (II.2.2) can be used to define a topology on the space of linear operators.

Definition II.2.22 (Uniform operator topology)**.** Sets of the form

$$V(U, \varepsilon) = \{A \in \mathscr{L}(\mathscr{X}, \mathscr{Y}) \mid \|A - U\| < \varepsilon\} \qquad (\varepsilon > 0, U \in \mathscr{L}(\mathscr{X}, \mathscr{Y})), \quad \text{(II.2.5)}$$

form a basis for the uniform operator topology, where the norm is given by (II.2.2).

Definition II.2.23 (Strong operator topology)**.** Let $\mathscr{X}$ be a LCLT space. For each $x \in \mathscr{X}$ define $p_x : \mathscr{L}(\mathscr{X}, \mathscr{Y}) \to \mathscr{Y}$ by

$$p_x(U) = Ux. \quad \text{(II.2.6)}$$

The strong operator topology is the weakest topology on $\mathscr{L}(\mathscr{X}, \mathscr{Y})$ such that all maps of the form (II.2.6) are continuous. A neighborhood base at $U \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ is given by sets of the form

$$V(U,\varepsilon) = \{A \mid A \in \mathscr{L}(\mathscr{X},\mathscr{Y}), \forall i \in I, \|p_{x_i}(U - A)\| < \varepsilon\} \tag{II.2.7}$$

where $\varepsilon > 0$, and $\{x_i\}_{i=1}^n$ is an arbitrary finite collection of elements of $\mathscr{X}$.

The weak operator topology can also be defined, but we will not have occasion to use it.

II.2.4 Closed linear operators

Definition II.2.24 (Graph of an operator)**.** Let $\mathscr{X}$ be a Banach space and $\mathscr{D} \subset \mathscr{X}$ be a dense subspace of $\mathscr{X}$. Let $U : \mathscr{D} \to \mathscr{X}$ be a linear operator. The set of ordered pairs

$$\Gamma_U := \{(f, Uf) \mid f \in \mathscr{D}\} \tag{II.2.8}$$

is called the *graph* of U.

The graph is a subset of $\mathscr{X} \times \mathscr{X}$. The cartesian product $\mathscr{X} \times \mathscr{X}$ can naturally be viewed as a vector space. With this viewpoint, it is the external direct sum of the two copies of $\mathscr{X}$ and will be denoted by $\mathscr{X} \oplus \mathscr{X}$. Since U is linear, then Γ_U is a linear subspace of $\mathscr{X} \oplus \mathscr{X}$. A norm on $\mathscr{X} \oplus \mathscr{X}$ can be given by $\|(x, y)\| = \|x\| + \|y\|$.

Definition II.2.25 (Closed operator)**.** The operator U is *closed* if Γ_U is a closed subspace of $\mathscr{X} \oplus \mathscr{X}$. Equivalently, U is closed if for every sequence $\{x_n\} \subset \mathscr{D}$ of points satisfying

(i) $x_n \to x \in \mathscr{X}$ and

(ii) $Ux_n \to y \in \mathscr{X}$,

the following two conditions are are true:

(i) $x \in \mathscr{D}$ and

(ii) $y = Ux$.

II.2.5 The Koopman operator.

Definition II.2.26 (Observables)**.** Let $F = \{f : \mathcal{M} \to \mathbb{C}\}$ be a vector space of functions on $\mathcal{M}$. The set $\mathscr{F}$ is called the space of observables on $\mathcal{M}$, while $f \in \mathscr{F}$ is called an observable on $\mathcal{M}$.

Definition II.2.27 (Discrete time Koopman operator)**.** Let $\Phi : \mathbb{N}_0 \times \mathcal{M} \to \mathcal{M}$ be a flow map and $\mathscr{F}$ a space of observables on $\mathcal{M}$ with the property that for all $f \in \mathscr{F}$, $f \circ \Phi(n, \cdot) \in \mathscr{F}$ for all $n \in \mathbb{N}_0$. For each $n \in \mathbb{N}_0$, define $U_n : \mathscr{F} \to \mathscr{F}$ by $(U_n f)(x) = f \circ \Phi(n, x)$. This family is called the (discrete time) Koopman semigroup. When Φ is generated by composing a fixed $T : \mathcal{M} \to \mathcal{M}$. The we call $U_T f = f \circ T$ the Koopman operator. In this case, T is called the symbol of the Koopman operator.

Definition II.2.28 (Continuous time Koopman operator)**.** Let $\Phi : [0, \infty) \times \mathcal{M} \to \mathcal{M}$, be a (continuous time) flow map and $\mathscr{F}$ a space of observables on $\mathcal{M}$ with the property that for all $f \in \mathscr{F}$, $f \circ \Phi(t, \cdot) \in \mathscr{F}$ for all $t \geq 0$. For each $t \in \mathbb{N}_0$, define $U_t : \mathscr{F} \to \mathscr{F}$ by $(U_t f)(x) = f \circ \Phi(t, x)$. This family is called the (continuous time) Koopman semigroup.

II.3 Resolvent and spectrum.

Let $\mathscr{X}$ be a LCLT space and $\mathscr{D} \subset \mathscr{X}$ a dense subspace. Fix a linear operator $U : \mathscr{D} \to \mathscr{X}$.

Definition II.3.1 (Resolvent operator)**.** Let $\lambda \in \mathbb{C}$. The linear operator $R(U;\lambda)$: $\operatorname{Im}[\lambda I - U] \to \mathscr{D}$ defined by $R(U;\lambda) = (\lambda I - U)^{-1}$ is called the resolvent (at λ) when it exists.

This resolvent operator may not exist for all $\lambda \in \mathbb{C}$. The domain of the resolvent $R(U;\lambda)$ is always taken to be $\operatorname{Im}(\lambda I - U)$.

Definition II.3.2 (Resolvent set)**.** Let $\rho(U) \subset \mathbb{C}$ be the set of complex numbers λ such that $\operatorname{Im}[\lambda I - U]$ is dense in $\mathscr{X}$ and $R(U;\lambda)$ exists and is continuous is called the resolvent set.

Definition II.3.3 (Spectrum)**.** The complement of $\rho(U)$, denoted by $\sigma(U)$, is the spectrum of U.

The spectrum decomposes into three sets depending on how the definition of the resolvent set fails.

Definition II.3.4 (Point spectrum)**.** The point spectrum, denoted $\sigma_p(U)$, is the set of $\lambda \in \mathbb{C}$ for which the resolvent does not exist on $\operatorname{Im}[\lambda I - U]$.

For λ to be in the point spectrum, there must be a $0 \neq x \in \mathscr{D}$ such that $Ux = \lambda x$. Hence the point spectrum just contains the eigenvalues of U.

Definition II.3.5 (Continuous spectrum)**.** The continuous spectrum, denoted $\sigma_c(U)$, is the set of $\lambda \in \mathbb{C}$ for which the resolvent $R(U;\lambda)$ exists, $\text{Im}[\lambda I - U]$ is dense in $\mathscr{X}$, but the $R(U;\lambda)$ is discontinuous.

Definition II.3.6 (Residual spectrum)**.** The residual spectrum, denoted $\sigma_r(U)$, is the set of $\lambda \in \mathbb{C}$ for which the resolvent $R(U;\lambda)$ exists, but $\text{Im}[\lambda I - U]$ is not dense in $\mathscr{X}$.

Definition II.3.7 (Koopman eigenvalue)**.** Let U_t be either a discrete or continuous time Koopman semigroup. A number $\lambda \in \mathbb{C}$ is called an eigenvalue of the semigroup if there is a nonzero $f \in \mathscr{F}$ such that $U_t f = \lambda^t f$ for all times t.

Definition II.3.8 (unimodular spectrum)**.** Let $U_t : \mathscr{F} \to \mathscr{F}$ be a Koopman semigroup. Let $\lambda \in \sigma(U)$. The λ is said to be part of the unimodular spectrum if $|\lambda| = 1$. If λ is in the unimodular point spectrum, the associated eigenfunction is called an unimodular eigenfunction.

Sometimes the unimodular spectrum is called the unitary spectrum; this, however, does not imply that the associated operator is unitary. Therefore, the unimodular terminology leads to less confusion and we endeavor to stick to this terminology in this dissertation.

II.3.1 Koopman mode decomposition

In the analysis of dynamical systems, it is often the case that we only need to know the behavior from a single (or small set of) initial conditions. In this case, it is often advantageous to consider vector-valued observables $\boldsymbol{f} \in \mathscr{F}^m$. Such cases include fluid dynamics or heat flow simulations where the system is simulated from some fixed initial and boundary conditions. If the initial condition is on (or very close) to an attractor, then decomposition of the vector-valued observable into unimodular eigenfunctions can be accomplished by taking the Fourier averages like in the previous subsection. Since we are simulating from a single initial condition, the pointwise ergodic theorems put the analysis on a solid theoretical footing. For a vector-valued observable $\boldsymbol{f} = [f_1, \ldots, f_m]^\mathsf{T}$, $f_i \in \mathscr{F}$, formed entirely from unimodular eigenfunctions, we get the so-called *Koopman mode decomposition*

$$U^n \boldsymbol{f}(x) = \sum_j e^{i\omega_j n} \begin{bmatrix} c_{1,j}(\boldsymbol{f}) \\ \vdots \\ c_{m,j}(\boldsymbol{f}) \end{bmatrix} \varphi_j(x) = \sum_j e^{i\omega_j n} \boldsymbol{c}_j(\boldsymbol{f}) \varphi_j(x) \tag{II.3.1}$$

In this notation, $\varphi_j : \mathcal{M} \to \mathbb{C}$ is an eigenfunction of U at the unimodular eigenvalue $\lambda_j = e^{i\omega_j}$ and $c_{i,j}(\boldsymbol{f})$ is the coefficient of φ_j in the expansion of f_i. The complex m-vector $\boldsymbol{c}_j(\boldsymbol{f}) = [c_{1,j}(\boldsymbol{f}), \ldots, c_{m,j}(\boldsymbol{f})]^\mathsf{T}$ is called the *Koopman mode corresponding to* φ_j. Of course, to arrive at this representation, we have implicitly assumed that $N\left(e^{i\omega_j} I - U\right)$ is one-dimensional and, additionally, that none of the f_i's have components that are either part of the continuous or residual spectrum or are nonspectral.

The space $\mathbb{C}^m$ is called the *output space*. The $\boldsymbol{c}_j(\cdot)$ above can be regarded as a mapping from the observable space $\mathscr{F}^m$ into the output space $\mathbb{C}^m$. Analyzing the dynamics in the output space allows insight into the dynamics of the system. A number of applied studies, mentioned earlier in this chapter, have made extensive use of the Koopman mode decomposition.

II.3.2 Ergodic theorems and the construction of unimodular eigenfunctions

We restrict our attention to the case when the dynamical system is generated by a fixed map $T : \mathcal{M} \to \mathcal{M}$. One would like a procedure to construct eigenfunctions of an operator. Eigenfunctions corresponding to unimodular eigenvalues can be obtained via averaging processes. An *infinite time average* of an observable gives an eigenfuntion at 1. Arguing formally for a moment, let $f \in \mathscr{F}$ be a bounded function and

$$\langle f \rangle := \lim_{n\to\infty} n^{-1} \sum_{0}^{n-1} U^n f. \qquad \text{(II.3.2)}$$

Then

$$U^m \langle f \rangle := U^m \left[\lim_{n\to\infty} n^{-1} \sum_{0}^{n-1} U^n f \right] = \lim_{n\to\infty} n^{-1} \sum_{0}^{n-1} U^{n+m} f = \langle f \rangle$$

for any fixed $m \geq 0$. To obtain an eigenfunctions at $\lambda = e^{i\omega}$ ($\omega \in \mathbb{R}$), we can compute the *Fourier average*

$$\langle f \rangle_\omega = \lim_{n\to\infty} n^{-1} \sum_{0}^{n-1} \lambda^{-n} U^n f = \lim_{n\to\infty} n^{-1} \sum_{0}^{n-1} e^{-i\omega n} U^n f. \qquad \text{(II.3.3)}$$

A similar computation as that for the regular time average shows that indeed $\langle f \rangle_\omega$ is an eigenfunction at $\lambda = e^{i\omega}$. Similar computations hold for continuous time dynamical systems by replacing the sum with the appropriate integrals. When exactly these formal averages exist rigorously in a function class and in what sense (almost everywhere, uniformly, in norm, in a locally convex topolology, etc.) and when they define projection operators form a set of results under the heading of ergodic theorems.

From a purely algebraic viewpoint, there is the ergodic theorem of Birkhoff guaranteeing almost everywhere convergence of infinite time averages and the strengthening of this theorem proved by Wiener and Wintner showing the almost everywhere convergence of Fourier averages with respect to any frequency. Both of these theorems are proved in the case of measure preserving dynamical systems[2] and observables integrable with respect to the preserved measure.

Theorem II.3.9 (Birkhoff's pointwise ergodic theorem [Pet83])**.** *Let $T : \mathcal{M} \to \mathcal{M}$ be a measure preserving dynamical system of the finite measure space $(\mathcal{M}, \Sigma, \nu)$. Then for any $f \in L^1(\mathcal{M}, \nu)$,*

$$\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} f(T^n(x)) = \mathbb{E}\left[f \mid \mathcal{C}\right],$$

exists for ν-almost every $x \in \mathcal{M}$, where $\mathcal{C} \subset \Sigma$ is the sub-σ-algebra of T-invariant sets and $\mathbb{E}\left[f \mid \mathcal{C}\right]$ is the conditional expectation with respect to $\mathcal{C}$. When T is ergodic[3], then

$$\langle f \rangle := \lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} f(T^n(x)) = \int_{\mathcal{M}} f(x) d\nu(x).$$

[2] The quadruple $(\mathcal{M}, \Sigma(\mathcal{M}), \nu, T)$, where $\Sigma(\mathcal{M})$ is a σ-algebra of subsets of $\mathcal{M}$ and ν is a finite measure on the σ-algebra, is a measure preserving dynamical system if $\nu(T^{-1}A) = \nu(A)$ for all $A \in \Sigma(\mathcal{M})$. Here $T^{-1}(A)$ is the pre-image of A.

[3] T is ergodic if all T-invariant sets have trivial measure; $\nu(C) = 0$ or $\nu(\mathcal{M})$.

Theorem II.3.10 (Wiener & Wintner's ergodic theorem [WW41]). *Let $T : \mathcal{M} \to \mathcal{M}$ be an (essentially) one-to-one, measure preserving dynamical system of the finite measure space $(\mathcal{M}, \Sigma, \nu)$. Then for any $f \in L^1(\mathcal{M}, \nu)$*

$$\langle f \rangle_\omega (x) := \lim_{n\to\infty} (2n+1)^{-1} \sum_{k=-n}^{n} e^{-ik\omega} f(T^k x)$$

exists for every $-\infty < \omega < \infty$ and ν-almost every $x \in \mathcal{M}$. The set of points x where the limit does not exist is independent of ω. The limit vanishes if $e^{i\omega}$ is not in the point spectrum.

An easy consequence of these theorems is that the limit functions are unimodular eigenfunctions of the Koopman operator. For measure preserving dynamical systems, the associated family of Koopman operators is an equicontinuous family with respect to the L^1-norm. In particular, $\|Uf\|_{L^1} \leq \|f\|_{L^1}$. The continuity of this family allows one to interchange limits. This allows one to show

$$U \langle f \rangle (x) = \langle f \rangle (x)$$

and

$$U \langle f \rangle_\omega (x) = e^{i\omega} \langle f \rangle_\omega (x),$$

respectively, where equality is almost everywhere. Additionally, since we are in a finite measure space, $L^2(\nu) \subset L^1(\nu)$, so, in particular, these theorems hold almost everywhere for the square integrable observables.

From the topological perspective, two basic ergodic theorems are given by von Neumann and Yosida, respectively, and are usually differentiated from the above pointwise ergodic theorems by calling them mean ergodic theorems. The mean ergodic theorem of von Neumann applies to any unitary operator on a Hilbert space and not just the Koopman operator. It implies that averages of the operator converge strongly[4] to the orthogonal projection onto the eigenspace at 1.

Theorem II.3.11 (von Neumann's mean ergodic theorem [RS72]). *Let $\mathscr{H}$ be a Hilbert space and $U : \mathscr{H} \to \mathscr{H}$ a unitary operator. Let P be the orthogonal projection onto $\{\psi \in \mathscr{H} \mid U\psi = \psi\} =: N(I - U)$. Then for any $f \in \mathscr{H}$*

$$\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} U^n f = Pf. \tag{II.3.4}$$

Yosida's theorem is more general and deals with equicontinuous families of operators on locally convex spaces. The weakest form shows that if for some fixed element of the space the averages have a weakly convergent subsequence, then the averages converge strongly to the limit function and this limit point is invariant under the family.

Theorem II.3.12 (Yosida's mean ergodic theorem [Yos78]). *Let $\mathscr{F}$ be a locally convex linear topological space and $U : \mathscr{F} \to \mathscr{F}$ a continuous linear operator. Assume that the family $\{U^n\}_{n\in\mathbb{N}}$ is equicontinuous. Define $A_n f := n^{-1} \sum_{k=1}^{n} U^k f$. Assume for a fixed $g \in \mathscr{F}$ that there is a subsequence of $\{A_n g\}_{n\in\mathbb{N}}$ that is weakly convergent to $g_0 \in \mathscr{F}$. Then $A_n g \to g_0$ in the locally convex topology and $Ug_0 = g_0$.*

[4] A sequence of operators $A_n : \mathscr{F} \to \mathscr{F}$ is said to converge strongly to the operator $B : \mathscr{F} \to \mathscr{F}$, in the Banach space, if $\|A_n f - Bf\| \to 0$ for all $f \in \mathscr{F}$.

None of these theorems imply that the operator has a spectral decomposition[5]; they do not say anything about the structure of the full space. Rather, these theorems allow one to construct unimodular eigenfunctions from an arbitrary observable. If the observable we are considering is an infinite linear combination of unimodular eigenfunctions, then, in principle, that particular observable can be fully decomposed into its spectral elements and we know everything about its evolution. However, an arbitrary observable may not be an infinite linear combination of spectral elements.

[5]Although, this is implied for the von Neumann mean ergodic theorem since U is a unitary operator on a Hilbert space

Chapter III

Generalized Laplace Analysis (GLA) for Spectral Operators of Scalar-Type

Our main goal is to show that certain averages of an operator construct non-unimodular eigenfunctions. This chapter and the next look at the problem in the abstract by considering operators on both a separable Banach space and more generally a locally convex linear topological (LCLT) space. In this chapter, we deal exclusively with continuous operators on an LCLT space determined by a spectral measure. These are the operators that are spectral (in the sense of Dunford) of scalar type. The results of this chapter follow from applying Yosida's mean ergodic theorem (MET) in a setting where the spectrum is non-unimodular and using the machinery of spectral measures. Since Yosida's MET underpins the proofs of the GLA theorems, a section is devoted to its statement and discussion. In particular, some natural restrictions on the spectrum arise when one wants to use the MET in an iterative manner in the case of a non-unimodular spectrum. In the next section, we look at continuous spectral operators of scalar type on a LCLT space and prove some GLA theorems there. When reasonable, we will give definitions of objects for the Banach space case along side those of the locally convex case, since the Banach space setting is in general more familiar. For the spectral operators, a resolution of the identity and the functional calculus play a large role. Once the technical machinery is out of the way, the proofs of the GLA theorems are easy consequences.

III.1 Yosida's mean ergodic theorem

III.1.1 The ergodic theorem

Let $\mathscr{F}$ be a complex vector space. Let Γ be an index set and $\{p_\gamma : \mathscr{F} \to \mathbb{R}\}_{\gamma\in\Gamma}$ a separating family of seminorms. Recall, that together these mean that for all $\gamma \in \Gamma$, scalars α, and $f, g \in \mathscr{F}$ that (i) $p_\gamma(f) \geq 0$, (ii) $p_\gamma(\alpha f) = |\alpha|\, p_\gamma(f)$, (iii) $p_\gamma(f+g) \leq p_\gamma(f) + p_\gamma(g)$, and (iv) for every $f \neq 0$, there is a $p_{\gamma(f)} \in \Gamma$ such that $p_{\gamma(f)}(f) \neq 0$.

Define on $\mathscr{F}$ the locally convex topology (LCT) generated by the seminorms. Neighborhood bases of the LCT have the form

$$V(f;\varepsilon) := \{g \in \mathscr{F} \mid p_{\gamma_i}(g-f) < \varepsilon, \forall i \in I\},$$

where $I \subset \Gamma$ is an arbitrary finite subset and ε is an arbitrary positive number. Note that a sequence $\{f_n\} \subset \mathscr{F}$ converges to $f \in \mathscr{F}$ if and only if for all $\gamma \in \Gamma$, $p_\gamma(f_n - f) \to 0$ as $n \to \infty$. This topology will be called the strong topology on $\mathscr{F}$ and with this topology $\mathscr{F}$ is a locally convex linear topological space (LCLTS).

Denote by $\mathscr{L}(\mathscr{F})$ the set of all linear operators on $\mathscr{F}$ continuous with respect to the strong topology. The set $\mathscr{F}^*$, the topological dual to $\mathscr{F}$, is the space of all linear functionals continuous with respect to the locally convex topology. The weak topology on $\mathscr{F}$, denote by $\sigma(\mathscr{F}, \mathscr{F}^*)$ is generated by the family of seminorms $\{p_\varphi = |\varphi| \mid \varphi \in \mathscr{F}^*\}$.

Fix $U \in \mathscr{L}(\mathscr{F})$. The resolvent $\rho(U) \subset \mathbb{C}$ of U is the set of $\xi \in \mathbb{C}$ such that $\mathrm{Im}(\xi I - U)$ is dense in $\mathscr{F}$ and $(\xi I - U)^{-1}$ exists and is continuous. The complement of this set is the spectrum, denoted $\sigma(U)$. The point spectrum $\sigma_p(U)$ is the set of $\lambda \in \sigma(U)$ for which $(\lambda I - U)$ does not have an inverse. When the operator U is understood, it will be dropped from the notation. We denote for $\Lambda \subset \sigma$

$$|\Lambda| := \{|\lambda| \mid \lambda \in \Lambda\} \tag{III.1.1}$$

and the spectral radius of Λ as

$$\|\Lambda\|_\infty := \sup_{\lambda \in \Lambda} |\lambda| \,. \tag{III.1.2}$$

Definition III.1.1 (Equicontinuous family of linear operators)**.** Consider a fixed linear operator $U \in \mathscr{L}(\mathscr{F})$. The family $\{U^n\}_{n\in\mathbb{N}}$ is called equicontinuous if for every continuous (with respect to the LCT) seminorm $p : \mathscr{F} \to \mathbb{R}$, there is a continuous seminorm $p' : \mathscr{F} \to \mathbb{R}$ such that

$$\sup_{n \geq 1} p(U^n f) \leq p'(f). \tag{III.1.3}$$

for every $f \in \mathscr{F}$.

Definition III.1.2 (Averaging operators)**.** For each $n \in \mathbb{N}$, define the averaging operator $A_n : \mathscr{L}(\mathscr{F}) \to \mathscr{L}(\mathscr{F})$ as

$$A_n(U) := n^{-1} \sum_{k=0}^{n-1} U^k. \tag{III.1.4}$$

One of the major tools of this paper is Yosida's extension of the mean ergodic theorem.

Theorem III.1.3 (Yosida's mean ergodic theorem, [Yos78])**.** *Let $\{U^n\}_{n \in \mathbb{N}}$ be an equicontinuous family of linear operators defined on a locally convex linear topological space $\mathscr{F}$. Fix $\psi \in \mathscr{F}$. If there exists as subsequence $\{n_i\} \subset \mathbb{N}$ and an $\psi_0 \in \mathscr{F}$ such that $\lim_{i \to \infty} \langle A_{n_i}(U)\psi - \psi_0, \varphi^* \rangle = 0$ for all $\varphi^* \in \mathscr{F}^*$, then $U\psi_0 = \psi_0$ and $\lim_{n \to \infty} A_n(U)\psi = \psi_0$ exists in the strong topology.*

Futhermore, if $\mathscr{F}$ is locally sequentially weakly compact, then $P := \lim_{n \to \infty} A_n(U)$, with the limit taken in the strong operator topology, defines a continuous projection operator onto $N(I - U)$ commuting with U and giving the direct sum decomposition $\mathscr{F} = N(I - U) \oplus \overline{\mathrm{Im}(I - U)}$.

III.1.2 Remarks on Yosida's mean ergodic theorem

(i) The theorem states that if the averages of an element have a weakly convergent subsequence, this implies that the full average converges strongly to the limit.

(ii) The mean ergodic theorem applies, in particular, to when $\mathscr{F}$ is a Banach space as this is just a specialized example of a LCLTS.

(iii) Note that the first result does not define a projection operator on $\mathscr{F}$ since some ψ may not have a weakly convergent subsequence of averages. To define the spectral projection and get a direct sum decomposition of the space, every $\psi \in \mathscr{F}$ must have some subsequence of $\{A_n(U)\psi\}_{n\in\mathbb{N}}$ that is weakly convergent. This is guaranteed by $\mathscr{F}$ being locally sequentially weakly compact. In particular, if $\mathscr{F}$ is a separable reflexive Banach space, it is locally sequentially weakly compact. This follows directly from the sequential version of the Banach-Alaoglu theorem.

(iv) If $|\lambda| = 1$ and $\{U^n\}$ is equicontinuous, then $\{\lambda^{-n}U^n\}_{n\in\mathbb{N}}$ is also equicontinuous. If the sequence $\{A_n(\lambda^{-1}U)\psi\}_{n\in\mathbb{N}}$ has a weakly convergent subsequence, then

$$\lim_{n\to\infty} A_n(\lambda^{-1}U)\psi = \psi_\lambda \in \mathscr{F}$$

exists, $\lambda^{-1}U\psi_\lambda = \psi_\lambda$, and hence ψ_λ is an element of the eigenspace at λ.

(v) Suppose $\mathscr{F}$ is a Banach space and suppose $\{U^n\}_{n\in\mathbb{N}}$ $(U \in \mathscr{L}(\mathscr{F}))$ is not equicontinuous in the norm (e.g. $1 < \|U\| < \infty$). We always have the spectral radius as a lower bound on the norm of U:

$$\|\sigma(U)\|_\infty := \sup_{\lambda\in\sigma(U)} |\lambda| \leq \|U\| \,.$$

To get an element of an eigenspace using the above results, there must be some $\mu \in \sigma(U)$ such that $\left\{(\mu^{-1}U)^n\right\}$ is equicontinuous. This happens only if $|\mu| = \|U\|$.

In particular, suppose $\sigma(U)$ has a sequence of eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ such that $\{|\lambda_j|\}_{j\in\mathbb{N}}$ is a strictly increasing sequence, $|\lambda_j| < \|U\|$, and $\lim_{j\to\infty} |\lambda_j| = \|U\|$. For λ_j, let $\varphi \in N(\lambda_{j+1}I - U)$ and $\|\varphi\| = 1$. Then

$$\left\|\left(\lambda_j^{-1}U\right)^n\right\| \geq \left\|\left(\lambda_j^{-1}U\right)^n \varphi\right\| = \left|\frac{\lambda_{j+1}}{\lambda_j}\right|^n$$

Therefore $\left\{\left(\lambda_j^{-1}U\right)^n\right\}$ cannot be an equicontinuous family.

(vi) Suppose $\|U\| \leq 1$ and $\mathscr{F}$ is a locally sequentially weakly compact Banach space. Suppose $\lambda_{1,2}$ are in the point spectrum $\sigma_p(U)$ and satisfy $|\lambda_2| < |\lambda_1| = \|U\|$. Also assume that U has a continuous portion of the spectrum, $\mu \in \sigma_c(U)$, and $|\lambda_2| < |\mu| < |\lambda_1| = \|U\|$. The above results can be applied to any $f \in \mathscr{F}$ using $\lambda_1^{-1}U$ to get a projection onto $N(\lambda_1 - U)$.

We would also like to use the same procedure to get the projections onto $N(\lambda_2 I - U)$. Recall that we had the direct sum decomposition $\mathscr{F} = \overline{\mathrm{Im}(\lambda_1 I - U)} \oplus N(\lambda_1 I - U)$. Consider elements in $\overline{\mathrm{Im}(\lambda_1 I - U)}$ and restrict U to this subspace (which we will denote as U_2). We cannot use the averaging above to get the projection since the continuous part of the spectrum prevents $\left\{(\lambda_2^{-1}U_2)^n\right\}$ from being equicontinuous. This is because the norm of U_2 satisfies $|\lambda_2| < |\mu| \leq \|U_2\|$. Therefore, if we want to apply the averaging procedure to compute the projections, U must either be restricted to subspaces spanned by eigenspaces or all elements of the point spectrum must have greater modulus than other parts of the spectrum. Additionally, if $|\sigma_p(U)| = \{|\lambda| \mid \lambda \in \sigma(U)\}$ is a discrete set, then we can guarantee that some eigenvalue's modulus attains the spectral radius. This condition means that the point spectrum is contained in separated circles in the complex plane.

Given these remarks, in order to apply the mean ergodic theorem in an iterative manner, some assumptions on the spectrum must be made. Loosely, the point spectrum must be larger than the non-point spectrum and additionally, whenever any set of eigenvalues is removed from the spectrum, there is an eigenvalue that achieves the supremum over the remaining set (if the remaining set contains an eigenvalue). To this end we define the following objects. The first definition is standard.

Definition III.1.4 (Peripheral spectrum). $\lambda \in \sigma(U)$ is in the peripheral spectrum of U if the modulus of λ is equal to the spectral radius; $|\lambda| = \|\sigma(U)\|_\infty$. If $B \subset \mathbb{C}$, then the peripheral spectrum of $\sigma(U) \cap B$ is defined as $\{\lambda \in \sigma(U) \cap B \mid |\lambda| = \|\sigma(U) \cap B\|_\infty\}$.

Definition III.1.5 (Dominating point spectrum). For $r > 0$, let $\mathbb{D}_r$ be the open disc of radius r centered at 0 in the complex plane and let $\sigma(U; \mathbb{D}_r) := \mathbb{D}_r \cap \sigma(U)$. If there exists an $R > 0$ such that $\sigma(U) \setminus \mathbb{D}_R$ is not empty and for every $r > R$ we have:

(i) if $\sigma(U; \mathbb{D}_r) \cap \sigma_p(U) \neq \emptyset$, then the peripheral spectrum of $\sigma(U; \mathbb{D}_r)$ is not empty, and

(ii) the set $\sigma(U) \setminus \mathbb{D}_r$ consists only of eigenvalues,

then U is said to have dominating point spectrum.

We note that the second condition of this definition implies that the peripheral spectrum of $\sigma(U; \mathbb{D}_r)$ consists only of eigenvalues if it is not empty. Basic properties of a dominating point spectrum are given in the following lemma.

Lemma III.1.6. *Assume $U \in \mathscr{L}(\mathscr{F})$ has a dominating point spectrum. Then*

(i) every pair $\lambda \in \sigma_p$ and $\xi \in \sigma \setminus \sigma_p$ with $|\lambda| > R$ satisfies $|\xi| \leq |\lambda|$,

(ii) no sequence of eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}} \subset \sigma_p$ has both strictly increasing moduli and satisfies "$\lim_{n\to\infty} |\lambda_n|$ exists", and

(iii) if $\Lambda \subset \sigma_p$ is any set of eigenvalues, then $\sigma \setminus \Lambda$ has dominating point spectrum.

Proof. (i) Either $|\xi| \leq |\lambda|$ or $|\xi| > |\lambda| > R$. In the first case, this is what we are trying to prove, so assume the second case holds. Then $\xi \in \sigma \setminus \mathbb{D}_{|\lambda|}$ which contradicts that this set contains only eigenvalues. This contradiction gives the result.

(ii) Consider the sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \sigma_p$ satisfying $|\lambda_n| < |\lambda_{n+1}|$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} |\lambda_n| = r$. Then $\|\sigma(U; \mathbb{D}_r)\|_\infty = r$ and no eigenvalue having modulus equal to r is in $\sigma(U; \mathbb{D}_r)$ since $\mathbb{D}_r$ is open. Therefore, the peripheral spectrum of $\sigma(U; \mathbb{D}_r)$ contains no eigenvalues. But, $\sigma(U; \mathbb{D}_r) \cap \sigma_p$ is not empty. This contradicts the first condition in definition III.1.5.

(iii) Assume that $\sigma_p \setminus \Lambda$ is not empty, since otherwise the conditions for a dominating point spectrum are trivially fulfilled. Fix $r \in \mathbb{R}^+$. If $(\sigma \setminus \Lambda) \cap \mathbb{D}_r$, contains no eigenvalues, then we need to do nothing. Suppose, then, that this set does contain an eigenvalue. Put $\alpha = \sup |\lambda|$, where the supremum is taken over all eigenvalues in $(\sigma \setminus \Lambda) \cap \mathbb{D}_r$, and let $\{\lambda_n\}_{n \in \mathbb{N}}$ be an increasing, maximizing sequence of eigenvalues in this set; i.e. (a) $\forall n \in \mathbb{N}$, $|\lambda_n| \leq |\lambda_{n+1}|$ and (b) $\lim_{n \to \infty} |\lambda_n| = \alpha$. Since the sequence of moduli of eigenvalues has a limit, then by result (ii) of this lemma, this sequence cannot be strictly increasing; there must some point $n \in \mathbb{N}$ such that $|\lambda_n| = |\lambda_{n+1}|$. We claim that there is an $M \in \mathbb{N}$ such that $|\lambda_M| = |\lambda_{M+m}|$ for all $m \geq 1$. Assume to the contrary. Put $N_1 = 1$. Then there exists $m_1 \geq 1$ such that $|\lambda_{N_1}| < |\lambda_{N_1+m_1}|$. Put $N_2 = N_1 + m_1$. Suppose, we have chosen $N_1, \ldots, N_j$ in this manner. Then there exists $m_j \geq 1$ such that $\left|\lambda_{N_j}\right| < \left|\lambda_{N_j+m_j}\right|$. Put $N_{j+1} = N_j + m_j$. By induction, we have constructed a subsequence $\left\{\lambda_{N_j}\right\}_{j \in \mathbb{N}}$ that has strictly increasing modulus and $\lim_{j \to \infty} \left|\lambda_{N_j}\right| = \alpha$. This contradicts (ii). Therefore, some tail of $\{\lambda_n\}$ has constant modulus (namely α). Therefore, a peripheral eigenvalue exists. Finally, $(\sigma \setminus \Lambda) \setminus \mathbb{D}_r$ consists only of eigenvalues since $(\sigma \setminus \Lambda) \setminus \mathbb{D}_r \subset \sigma \setminus \mathbb{D}_r$ and $\sigma \setminus \mathbb{D}_r$ only contains eigenvalues.

□

In particular, assuming that U has a dominating point spectrum guarantees that situation in remark (vi) on the mean ergodic theorem does not occur. Result (ii) of lemma III.1.6 rules out the case in remark (v) on the mean ergodic theorem.

III.2 Spectral measures

The section develops the framework for scalar type spectral operators, where "spectral" is understood in the sense of Dunford [Dun54]. Resolutions of the identity and the associated functional calculus will underpin most of the analysis. We develop them now. This material is adapted from [Dun54], [Sch62], and [Wal65] and theorems will be stated without proof. We refer the reader to the references for a more detailed exposition.

III.2.1 Spectral measures in Banach spaces.

Let $\mathcal{B}(\mathbb{C})$ be the σ-algebra of Borel sets in the complex plane. Let $\mathscr{L}(\mathscr{F})$ be the set of bounded linear operators on the complex Banach space $\mathscr{F}$.

Definition III.2.1 (Spectral measure for Banach spaces)**.** A spectral measure $E : \mathcal{B}(\mathbb{C}) \to \mathscr{L}(\mathscr{F})$ is a homomorphism from the Borel sets into the projection operators on $\mathscr{F}$ and has the properties [Dun54]

$$\begin{aligned}
&E(A \cap B) = E(A)E(B), \quad E(A \cup B) = E(A) + E(B) - E(A \cap B), \\
&E(A^c) = I - E(A), \qquad E(\emptyset) = 0, \quad E(\mathbb{C}) = I \qquad \text{(III.2.1)} \\
&\|E(A)\| \le K,
\end{aligned}$$

for all Borel sets A, B.[1] □

The final property says that the family of projections determined by the spectral measure is a uniformly bounded family of operators.

[1] The union of two commuting projections $P_1 \cup P_2$ is understood to mean $P_1 + P_2 - P_1 P_2$.

Definition III.2.2 (Spectral operator)**.** An operator $U : \mathscr{F} \to \mathscr{F}$ is a spectral operator if there is a spectral measure $E : \mathcal{B}(\mathbb{C}) \to \mathscr{L}(\mathscr{F})$ such that

$$UE(A) = E(A)U \qquad \text{and} \qquad \sigma(U, E(A)\mathscr{F}) \subset \overline{A} \tag{III.2.2}$$

for all $A \in \mathcal{B}(\mathbb{C})$ and for every $\psi \in \mathscr{F}$ and $\psi^* \in \mathscr{F}^*$, $\langle E(A)\psi, \psi^* \rangle$ is a countably additive $\mathbb{C}$-valued set function on $\mathcal{B}(\mathbb{C})$. The notation $\sigma(U, E(A)\mathscr{F})$ means the spectrum of U when considered as an operator acting on $E(A)\mathscr{F}$. A spectral measure satisfying these properties is called a *resolution of the identity.* □

To a spectral operator, there is an equibounded family of projection operators commuting with the operator and if we consider the projection determined by a set in the complex plane, the spectrum of the operator when restricted to the subspace determined by the projection must be contained in the original set. These properties follow intuitively from the finite dimensional case; if you restrict an operator to an eigenspace, the spectrum of the restricted operator is just the eigenvalue of the corresponding to the eigenspace.

Example III.2.3. *Let $A : \mathbb{C}^d \to \mathbb{C}^d$ be a normal matrix with simple spectrum $\{\lambda_1, \ldots, \lambda_d\}$ and the orthogonal projections $P_j : \mathbb{C}^d \to \mathbb{C}^d$ defined by $P_j \boldsymbol{x} = \langle \boldsymbol{x}, \boldsymbol{v}_j \rangle$, where $\boldsymbol{v}_j$ is the unit norm eigenvector associated λ_j. Let $D \in \mathcal{B}(\mathbb{C})$ and define $E : \mathcal{B}(\mathbb{C}) \to \mathscr{L}(\mathbb{C}^d)$ by*

$$E(\cdot) := \sum_{j=1}^{d} \delta_{\lambda_j}(\cdot) P_j,$$

where $\delta_{\lambda_j}(\cdot)$ *is the Dirac delta measure supported at* λ_j*. For a set* $D \in \mathcal{B}(\mathbb{C})$*, this measure takes the operator value*

$$E(D) := \sum_{\{j|\lambda_j \in D\}} P_j.$$

This set function defines a spectral measure for A*. The spectral measure commutes with* A *since each* P_j *does and the condition* $\sigma(A, E(D)\mathbb{C}^d) \subset \overline{D}$ *follows from*

$$AE(D)\boldsymbol{v}_i = A\left(\sum_{\{j|\lambda_j \in D\}} P_j \boldsymbol{v}_i\right) = \begin{cases} \lambda_i \boldsymbol{v}_i & \lambda_i \in D \\ 0 & \lambda_i \notin D. \end{cases}$$

The countable additivity follows from the spectrum being finite and the countable additivity of the Dirac measures, which implies that only a finite number of sets from a countable disjoint collection can contain elements of the spectrum. Any set not containing an element of the spectrum gives, by definition, the zero operator. Since the operator is normal, the eigenvectors are orthogonal and $\|E(D)\| \leq 1$ *for all* $D \in \mathcal{B}(\mathbb{C})$[2]. □

An important result regarding resolutions of the identity is the following theorem from Dunford [Dun54]. It gives a continuous functional calculus for the resolution of the identity.

Theorem III.2.4 (Theorem 7, Dunford [Dun54])**.** *Let* E *be a spectral measure whose support is a compact set* σ*. Then, for every scalar function* f *continuous on* σ*, the*

[2] *If* A *was merely diagonalizable and not normal, then the upper bound for the projections would be some constant greater than 1 since skew projections can be norm increasing.*

Riemann integral $\int_\sigma f(z)E(dz)$ exists in the uniform operator topology, and

$$\left\| \int_\sigma f(z)E(dz) \right\| \leq \sup_{z\in\sigma} |f(z)|\, \nu(E), \tag{III.2.3}$$

where $\nu(E)$ is a constant depending only on E. Additionally, for any two continuous functions f and g we have the multiplicative identity

$$\left[\int_\sigma f(z)E(dz) \right] \cdot \left[\int_\sigma g(z)E(dz) \right] = \int_\sigma f(z)g(z)E(dz). \tag{III.2.4}$$

We now have a representation of spectral operators.

Theorem III.2.5 (Canonical decomposition of spectral operators, [Dun54])**.** *An operator U is spectral if and only if it is given by the sum $U = S + J$, where S is an operator of scalar-type, which by definition, has the form*

$$S = \int zE_S(dz), \tag{III.2.5}$$

E_S is the resolution of the identity for S, and J is a generalized nilpotent operator commuting with S that, by definition, satisfies $\|J^n\|^{1/n} \to 0$. This decomposition is unique, the spectrums of U and S are identical, and the resolutions of the identity for U and S are the same.

For $U : \mathscr{F} \to \mathscr{F}$ a bounded spectral operator, $\sigma(U)$ is compact and all the preceding theorems apply. Finally, we have the analytic functional calculus. For an function f, analytic and single-valued on $\sigma(U)$,

$$f(U) = \sum_{j=0}^{\infty} \frac{J^j}{j!} \int_{z\in\sigma} f^{(j)}(z)E(dz)$$

exists in the uniform operator topology. An operator is said to be of type m if

$$f(U) = \sum_{j=0}^{m} \frac{J^j}{j!} \int_{z \in \sigma} f^{(j)}(z) E(dz)$$

for all f analytic on σ. Scalar operators are of type 0, so for such operators, these equations reduce to

$$f(U) = \int_{z \in \sigma} f(z) E(dz). \tag{III.2.6}$$

In fact, by theorem III.2.4, f can be a continuous function rather than just an analytic one if U is of scalar type.

III.2.2 Spectral measures in locally convex linear topological spaces

As this paper is concerned with general procedures in constructing eigenfunctions for spectral operators of scalar type on LCLT spaces, we need the concept of a spectral measure for these spaces. This machinery is contained in this section. This section follows [Wal65] and [Sch62] closely. The reader should consult those references for a more detailed treatment.

Let $(\mathscr{F}, \tau)$ be a locally convex linear topological space with topology τ, $\mathscr{S}$ is a set, and Σ a σ-algebra of subsets of $\mathscr{S}$.

Definition III.2.6 (Spectral measure triple [Wal65])**.** A triple $(\mathscr{S}, \Sigma, \mu)$, where μ is a set function from Σ to $\mathscr{L}(\mathscr{F})$ which is countable additive in the weak operator topology, that satisfies

(i) $\mu(\mathscr{S}) = I \in \mathscr{L}(\mathscr{F})$

(ii) for A_1 and A_2 in Σ, $\mu(A_1 \cap A_2) = \mu(A_1) \cdot \mu(A_2) \in \mathscr{L}(\mathscr{F})$

is called a spectral measure triple.

The spectral measure is said to be equicontinuous if the values in $\mathscr{L}(\mathscr{F})$ that μ takes on Σ are equicontinuous. If $(\mathscr{S}, \Sigma, \mu)$ is an equicontinuous spectral measure triple and $\mathscr{F}$ is sequentially complete in its topology, the integrals of $\mathbb{C}$-valued, bounded, Σ-measurable functions can be defined [Wal65]. The integral is multiplicative since for simple f and g, $\int fgd\mu = \int fd\mu \cdot \int gd\mu$ follows from condition (ii) in the definition of the spectral measure triple. Hence the map $f \mapsto U_f$ from (equivalence classes) of $\mathbb{C}$-valued, bounded, Σ-measurable functions to $\mathscr{L}(\mathscr{F})$ is a homomorphism. The integral gives a representation of the bounded $\mathbb{C}$-valued Σ-measurable functions as an algebra of linear operators on $\mathscr{F}$. This algebra of operators is called the *spectral algebra associated with* $(\mathscr{S}, \Sigma, \mu)$ and will be denoted by $\mathscr{A}$. Elements of this algebra are spectral operators and will be called spectral elements.

By a change of measure $E = \mu \circ f^{-1}$, we get the familiar representation of a scalar-type spectral operator (in the sense of Dunford) as an integral against a spectral measure with domain in $\mathcal{B}(\mathbb{C})$;

$$U = \int_{\mathscr{S}} f(s) d\mu(s) = \int_{\mathbb{C}} zE(dz). \tag{III.2.7}$$

The measure E depends only on U and its support is the spectrum of U. The support of E is contained in a compact set since f is bounded by assumption.

In order to apply the mean ergodic theorem, we will need to guarantee that certain families of spectral operators are equicontinuous. The most useful result in this regard is the following proposition, due to Walsh.

Proposition III.2.7 (Proposition 2.1, [Wal65])**.** *Let $\mathscr{F}$ be a locally convex space and $(\mathscr{S}, \Sigma, \mu)$ an equicontinuous spectral measure triple. Then the set*

$$\left\{ \int_{\mathscr{S}} f d\mu \,\middle|\, f \ \Sigma\text{-measurable}, 0 \leq |f| \leq 1 \right\} \tag{III.2.8}$$

is an equicontinuous family.

Schaefer [Sch62] has given an operational calculus for spectral measures. If $\mathscr{A} \subset \mathscr{L}(\mathscr{F})$ is a weakly semi-complete locally convex algebra, $U = \int f d\mu$ a spectral element and $g : \mathbb{C} \to \mathbb{C}$ a bounded complex Baire function, the integral $\int_{\mathscr{S}} (g \circ f) d\mu$ defines another spectral element

$$g(U) = \int_{\mathscr{S}} (g \circ f) d\mu = \int_{\mathbb{C}} g(z) E(dz) \in \mathscr{A}. \tag{III.2.9}$$

This is the operational calculus for spectral elements in locally convex spaces. If g is continuous on the spectrum of U, we get the spectral mapping theorem $\sigma[g(U)] = g[\sigma(U)]$ [Sch62]. The spectral algebras we will be most concerned about are algebras of the form

$$\mathscr{A}(f) := \left\{ \int_{\mathscr{S}} (g \circ f) d\mu \,\middle|\, g : \mathbb{C} \to \mathbb{C} \text{ is a polynomial} \right\}.$$

Each polynomial is bounded since $\mathrm{Im}(f) \subset \mathbb{C}$ is bounded.

The reader may wonder what is gained from defining the abstract spectral measure triple $(\mathscr{S}, \Sigma, \mu)$ in the case of locally convex linear topological spaces, rather than the form $\int_{\sigma(U)} zE(dz)$ that is familiar from scalar-type spectral operators on Banach spaces. This abstract framework allows, for example, the treatment of products of commuting operators with random spectrum.

Example III.2.8. *Let $\mathscr{S} = \{0, 1\}$. Let $\{f_1, f_2\}$ be a pair of functions from $\mathscr{S}$ into $\mathbb{C}$. The spectral measure μ takes values as a projection operator on $\mathscr{F}$. Assume that both $P_0 = \mu(\{0\})$ and $P_1 = \mu(\{1\})$ are projections onto one-dimensional subspaces and $\mathscr{F} = \mathrm{Im}(P_0) \oplus \mathrm{Im}(P_1)$.*

Let $f_1(0) = \lambda_0$ and $f_1(1) = \lambda_1$ and similarly for f_2 but with ω replacing λ. To this pair of functions is associated a pair of operators U_1 and U_2 acting on $\mathscr{F}$. Both of these operators have eigenspaces corresponding to $\mathrm{Im}(P_0)$ and $\mathrm{Im}(P_1)$. Consider a finite length sequence consisting of f_1 and f_2 with f_1 appearing m_1 times and f_2 appearing m_2 times. By the multiplicative property of the spectral integral and the commutative property of the functions

$$
\begin{aligned}
(U_{i_1} \cdots U_{i_{m_1+m_2}}) &= \int_{\mathscr{S}} (f_{i_1} \cdots f_{i_{m_1+m_2}})(s) d\mu(s) \\
&= \int_{\mathscr{S}} f_1^{m_1}(s) \cdot f_2^{m_2}(s) d\mu(s) \\
&= \sum_{s=0}^{1} (\lambda_s^{m_1} \omega_s^{m_2}) P_s.
\end{aligned}
$$

where $i_j \in \{1, 2\}$ for $j = 1, \ldots, m_1 + m_2$. The operator resulting from the product is a spectral operator with eigenvalues $\gamma_1 = \lambda_1^{m_1} \omega_1^{m_2}$ and $\gamma_2 = \lambda_2^{m_1} \omega_2^{m_2}$. It is straightforward to take a larger collection of functions so that we can get more possible composite eigenvalues. □

Projections onto different parts of the spectrum take the form of integrating against indicator functions.

Lemma III.2.9. *Let* Γ *be a Baire subset of* $\mathbb{C}$. *Let* $\chi_\Gamma : \sigma(U) \to \mathbb{C}$ *be the indicator function for* $\Gamma \cap \sigma(U)$. *Then* $P_\Gamma = \int_{\sigma(U)} \chi_\Gamma(z) E(dz) = \int_{\Gamma \cap \sigma(U)} E(dz)$ *is a projection operator.*

Proof. Since $\chi_\Gamma \circ f$ is an indicator function for $f^{-1}(\Gamma)$, by the multiplicative property of the integral

$$P_\Gamma^2 = \left[\int_{\mathscr{S}} (\chi_\Gamma \circ f) d\mu\right] \cdot \left[\int_{\mathscr{S}} (\chi_\Gamma \circ f) d\mu\right] = \int_{\mathscr{S}} (\chi_\Gamma \circ f)^2 d\mu = \int_{\mathscr{S}} (\chi_\Gamma \circ f) d\mu = P_\Gamma.$$

□

III.3 Generalized Laplace Analysis

Let $(\mathscr{S}, \Sigma, \mu)$ be a spectral measure triple, $\mathscr{F}$ a separable, locally sequentially weakly compact, locally convex linear topological space. Fix a bounded $\mathbb{C}$-valued Σ-measurable function f and denote the associated spectral operator by U

$$U = \int_{\mathscr{S}} f(s) d\mu(s) = \int_{z \in \sigma} z E(dz) \in \mathscr{L}(\mathscr{F}). \tag{III.3.1}$$

Since f is bounded, the spectrum of U is contained in a compact set in the complex plane.

Definition III.3.1. We say that U has a point spectrum concentrated on isolated circles if every point of the set

$$\left\{r \in \mathbb{R} \;\middle|\; \exists s \in \mathscr{S}, |f(s)| = r, (f(s)I - U)^{-1} \text{ does not exist}\right\}$$

is an isolated point.

By definition, $f(s)$ in the above definition is an eigenvalue of U. Then $|f(s)|$ is the modulus of the eigenvalue. The above definition states that the set consisting of the moduli of all the eigenvalues of the operator forms a discrete set. This definition does not exclude the situation when eigenvalues are dense on a circle. Such a situation arises naturally for dynamical systems possessing a limit cycle, as will be seen later.

III.3.1 GLA for dominating point spectrum

Lemma III.3.2. *Assume the peripheral spectrum of U contains an eigenvalue $\lambda \in \sigma_p(U)$. Let $g_n : \sigma(U) \to \mathbb{C}$ be $g_n(z) = \lambda^{-n} z^n$. Then $\{g_n(U)\}_{n\in\mathbb{N}} = \{\lambda^{-n} U^n\}_{n\in\mathbb{N}}$ is an equicontinuous family of operators.*

Proof. Since f is bounded, then σ is contained in a compact set in $\mathbb{C}$. For each n, the support of g_n is a compact set, g_n is a continuous and, therefore, a bounded Baire function on σ. Since λ is a peripheral eigenvalue, then $|g_n(z)| \leq 1$ for all $z \in \sigma$ since by definition $|z| \leq |\lambda|$ for all $z \in \sigma$. If $f : \mathscr{S} \to \mathbb{C}$ is the function associated with U, then $|(g_n \circ f)(s)| \leq 1$ for all $n \in \mathbb{N}$ and $s \in \mathscr{S}$. By proposition III.2.7, $\left\{g_n(U) = \int_{\mathscr{S}} (g_n \circ f) d\mu\right\}_{n\in\mathbb{N}}$ is an equicontinuous family. By the functional calculus, $g_n(U) = \lambda^{-n} U^n$. □

Proposition III.3.3. *Assume the peripheral spectrum of U contains an eigenvalue $\lambda \in \sigma_p(U)$. Then*

$$A_n(\lambda^{-1}U) := n^{-1} \sum_{k=1}^{n} \lambda^{-k} U^k \tag{III.3.2}$$

converges to the projection operator $P_\lambda := E(\{\lambda\})$ onto $N(\lambda I - U)$ in the strong operator topology; that is, for every continuous seminorm p and every element $\psi \in \mathscr{F}$

$$\lim_{n\to\infty} p(A_n(\lambda^{-1}U)\psi - P_\lambda \psi) = 0.$$

Furthermore, the projection P_λ commutes with U and we have the direct sum decomposition $\mathscr{F} = N(\lambda I - U) \oplus \overline{\mathrm{Im}(\lambda I - U)}$.

Proof. By lemma III.3.2, $\{(\lambda^{-1}U)^n\}$ is an equicontinuous family of operators on a locally sequentially weakly compact LCLT space since $(\lambda^{-1}U)^n = g_n(\lambda^{-1}U)$, where g_n is the function in the above lemma. By Yosida's mean ergodic theorem, the averages of this family converge to a projection operator onto the λ-eigenspace. Since projections take the form of indicator functions on subsets of $\sigma \subset \mathbb{C}$ (lemma III.2.9), then this projection takes the form $E(\{\lambda\})$. □

Remark III.3.4. *Proposition III.3.3 is true for any other eigenvalue in the peripheral spectrum, say $\lambda' \in \sigma_p$. For this λ', the proposition can be applied to obtain the projection onto $N(\lambda' I - U)$. Additionally, $N(\lambda' I - U) \subset \overline{\mathrm{Im}(\lambda I - U)}$. Since $\overline{\mathrm{Im}(\lambda I - U)}$ is U-invariant and $\mathscr{F}$ is locally sequentially weakly compact, then $(\lambda')^{-n}(U')^n$ is a equicontinuous family of operators on the locally sequentially weakly compact space $\overline{\mathrm{Im}(\lambda I - U)}$, where U' is a the restriction of U to $\overline{\mathrm{Im}(\lambda I - U)}$. By proposition III.3.3,*

$\overline{\mathrm{Im}(\lambda I - U)} = N(\lambda' I - U') \oplus \overline{\mathrm{Im}(\lambda I - U')}$. *Since* $N(\lambda' I - U') = N(\lambda' I - U)$ *and* $\mathrm{Im}(\lambda' I - U') = \mathrm{Im}((\lambda' I - U)(I - P_\lambda))$, *then*

$$\mathscr{F} = N(\lambda I - U) \oplus N(\lambda' I - U) \oplus \overline{\mathrm{Im}((\lambda' I - U)(I - P_\lambda))}.$$

□

Corollary III.3.5. *Let the peripheral spectrum of* U *consist of only (at most countably many) eigenvalues. Then*

$$\mathscr{F} = \left[\bigoplus_{j\geq 1} N(\lambda_j I - U)\right] \oplus \mathscr{F}' \tag{III.3.3}$$

where $\{\lambda_j\}$ *is some labeling of the peripheral spectrum and* $\mathscr{F}'$ *is the subspace of* $\mathscr{F}$ *corresponding to the part of the spectrum not in the peripheral spectrum. The subspace* $\mathscr{F}'$ *has the form*

$$\mathscr{F}' = \overline{\left(I - \sum_{j\geq 1} E(\{\lambda_j\})\right)}. \tag{III.3.4}$$

The above results can be combined to give a recursive procedure for computing projections onto eigenspaces for any $\psi \in \mathscr{F}$. In order to do this, we must compute the projection onto eigenspaces corresponding to eigenvalues of largest modulus first, subtract these from ψ, then compute the projections for eigenvalues of the next largest modulus.

Theorem III.3.6 (Generalized Laplace Analysis)**.** *Let* $\sigma(U)$ *have dominating point spectrum and assume that the point spectrum is concentrated on isolated circles in the complex plane (def. III.3.1). Let* λ *be an eigenvalue for* U*. Denote* $E(\{\lambda\})$ *by* P_λ*.*

Then

$$P_\lambda = \lim_{n\to\infty} A_n \left(\lambda^{-1} U(I - \sum_{\mu\in\Omega} P_\mu) \right) = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n-1} \lambda^{-n} U^n \left(I - \sum_{\mu\in\Omega} P_\mu \right) \qquad \text{(III.3.5)}$$

exists in the strong topology and where $\Omega := \{\mu \in \sigma_p(U) \mid |\mu| > |\lambda|\}$*. Furthermore, if* Ω *is a finite set, then* $P_\lambda\psi$ *can be obtained via a finite recursion processes by computing* $P_\mu\psi$ *with* (III.3.5) *for each* $\mu \in \Omega$ *and subtracting it from* ψ*.*

Proof. Fix $\lambda \in \sigma_p(U)$ and let $\Omega = \{\mu \in \sigma(U) \mid |\mu| > |\lambda|\}$. Since $\sigma(U)$ has dominating point spectrum, Ω consists only of eigenvalues. By property (iii) of lemma III.1.6, $\sigma(U) \setminus \Omega$ has dominating point spectrum. Since the point spectrum of U is concentrated on isolated circles, $h(z) = 1 - \chi_\Omega(z)$ is a bounded, continuous function on $\sigma(U)$ and $h(U) = \int_{\sigma(U)} h(z)E(dz) = I - E(\Omega)$. By the spectral mapping theorem with the continuous function $z \mapsto zh(z)$, $\sigma(U(I - E(\Omega))) = (\sigma(U) \setminus \Omega) \cup \{0\}$. Since $\sigma(U) \setminus \Omega$ has dominating point spectrum, so does $\sigma(U(I - E(\Omega)))$. Since $\lambda \in \sigma(U(I - E(\Omega)))$ and $\|U(I - E(\Omega))\|_\infty \le |\lambda|$, then λ is a peripheral eigenvalue for $U(I - E(\Omega))$.

Let $g_n(z) = \lambda^{-n} z^n (1 - \chi_\Omega(z))$. Then g_n is a continuous function satisfying $|g_n(z)| \le 1$ for all $n \in \mathbb{N}$ and $z \in \sigma$ since it is 0 on Ω and λ is a peripheral eigenvalue in $\sigma(U) \setminus \Omega$. Therefore, g_n is a Baire function for every n. If $f : \mathscr{S} \to \mathbb{C}$ is the function associated with U, then $|(g_n \circ f)(s)| \le 1$ for all $n \in \mathbb{N}$ and $s \in \mathscr{S}$. Therefore, by proposition III.2.7, $\{g_n(U)\}_{n\in\mathbb{N}}$ is an equicontinuous family.

By the multiplicative property of the spectral integrals

$$\begin{aligned}\left[\lambda^{-1}U(I-E(\Omega))\right]^n &= \int_\sigma \lambda^{-n}z^n(1-\chi_\Omega(z))^n E(dz)\\ &= \left(\int_\sigma \lambda^{-n}z^n E(dz)\right)\cdot\left(\int_\sigma (1-\chi_\Omega(z))E(dz)\right)\\ &= \lambda^{-n}U^n\left(I-E(\Omega)\right).\end{aligned}$$

Since the point spectrum is at most countable and Ω contains only eigenvalues, then Ω is a countable union of singleton sets of eigenvalues, $\Omega = \bigcup_{\mu\in\Omega}\{\mu\}$. Since the spectral measure is countably additive, then $E(\Omega) = \sum_{\mu\in\Omega} E(\{\mu\})$. Therefore, $\lambda^{-n}U^n\left(I-E(\Omega)\right) = \mu^{-n}U^n(I-\sum_{\mu\in\Omega}E(\{\mu\})$.

By proposition III.3.3, the right side of (III.3.5) converge strongly to a $\lambda^{-1}U(I-\sum_{\mu\in\Omega}E(\{\mu\}))$-invariant function, say $\psi_0\in\mathscr{F}$. Then $U(I-\sum_{\mu\in\Omega}E(\{\mu\}))\psi_0=\lambda\psi_0$. We claim that ψ_0 is in the nullspace of $E(\{\mu\})$ for every $\mu\in\Omega$. Suppose this was not the case and $\psi_0\neq 0$. Then there is some $\mu'\in\Omega$ such that $E(\{\mu'\})\psi_0=\varphi\neq 0$. But $E(\{\mu'\})$ is the projection onto $N(\mu' I-U)$. Therefore, we get $U(I-E(\{\mu'\}))\psi_0 = U\psi_0-\mu'\psi_0=\lambda\psi_0$. Then $U\psi_0=(\mu'+\lambda)\psi_0$, which implies that $\psi_0\in N((\mu'+\lambda)I-U)$. But since $\lambda\neq 0$, this implies that $\psi_0\in N((\mu'+\lambda)I-U)\cap N(\mu' I-U)$. Consequently, $\psi_0=0$, contrary to assumption. Therefore, ψ_0 is in the nullspace of $\sum_{\mu\in\Omega}E(\{\mu\})$ and we get that $U\psi_0=\lambda\psi_0$. Consequently, the averages converge strongly to the projection $E(\{\lambda\})$ onto $N(\lambda I-U)$. □

III.3.2 GLA for minimal point spectrum

The above results gave a procedure for constructing eigenfunctions when the point spectrum dominated the rest of the spectrum. Unfortunately, this situation does not hold in a number of cases of interest. Consider a dynamical system with an attractor. In this setting, the spectrum on the unit circle corresponds to the attractor. Since the system is asymptotically stable, eigenvalues corresponding to eigenfunctions supported off-attractor are contained strictly inside the unit circle. In this system, the point spectrum may not dominate the spectrum since there may be parts of the continuous spectrum contained in the unit circle. If we wish to project onto the off-attractor, stable eigenspaces, we need to modify the above GLA procedure which was valid in the presence of a dominating point spectrum. The general idea is to consider the inverse operator U^{-1}. If U has point spectrum inside the unit circle, then U^{-1} has point spectrum outside the unit circle via the spectral mapping theorem. The GLA theorems of the last section can then be applied to U^{-1} to obtain projections onto the stable directions of the attractor. Proposition III.3.8 formalizes this.

Definition III.3.7 (Minimal point spectrum)**.** Let $U \in \mathscr{A}$. We say that $\sigma(U)$ has a minimal point spectrum if $\sigma^{-1}(U) := \{\lambda^{-1} \mid \lambda \in \sigma(U)\} \in \mathbb{C} \cup \{\infty\}$ has a dominating point spectrum.

Proposition III.3.8 (Inverse GLA)**.** *Let $U = \int_\sigma zE(dz) \in \mathscr{A}$ have a minimal point spectrum. Additionally, assume that the point spectrum is concentrated on isolated* *circles and the spectrum satisfies* $0 < C^{-1} \leq \inf_{\xi \in \sigma(U)} |\xi|$. *Then for* $\lambda \in \sigma_p(U)$,

$$E(\{\lambda\}) = \lim_{n\to\infty} A_n \left(\lambda U^{-1}(I - E(\Omega))\right) \tag{III.3.6}$$

where $\Omega = \{\xi \in \sigma(U) \mid |\xi| < |\lambda|\}$.

Proof. Since $0 < C^{-1} \leq \inf_{\xi \in \sigma(U)} |\xi|$, then $i(z) = z^{-1}$ is continuous and bounded on $\sigma(U)$. By the multiplicative property of the integral

$$U \cdot i(U) = \int_{\sigma(U)} z \cdot i(z) E(dz) = I = \int_{\sigma(U)} i(z) \cdot z E(dz) = i(U) \cdot U.$$

Therefore $i(U) = U^{-1}$. By the spectral mapping theorem $\sigma(U^{-1}) = \{\xi^{-1} \mid \xi \in \sigma(U)\}$. Therefore, U^{-1} has a bounded, dominating point spectrum that is concentrated on isolated circles. By the change of measure $E_1 := E \circ i^{-1}$, we have the representation of U^{-1} as the integral

$$U^{-1} = \int_{\sigma(U^{-1})} w E_1(dw).$$

Fix $\lambda \in \sigma_p(U)$ and let $\Omega' = \{\xi^{-1} \in \sigma(U^{-1}) \mid |\xi^{-1}| > |\lambda^{-1}|\}$. By proposition III.3.6,

$$E_1(\{\lambda^{-1}\}) = \lim_{n\to\infty} A_n \left((\lambda^{-1})^{-1} U^{-1} \left(I - \sum_{\xi^{-1} \in \Omega'} E_1(\{\xi^{-1}\}) \right) \right)$$

with convergence in the strong topology. Since $N(\lambda I - U) = N(\lambda^{-1} I - U^{-1})$, then $E(\{\lambda\}) = E_1(\{\lambda^{-1}\})$ for all $\lambda \in \sigma_p(U)$. Consequently,

$$E(\{\lambda\}) = \lim_{n\to\infty} A_n \left(\lambda U^{-1} \left(I - \sum_{\xi \in \Omega} E(\{\xi\}) \right) \right) = \lim_{n\to\infty} A_n \left(\lambda U^{-1} \left(I - E(\Omega)\right)\right)$$

where $\Omega = \{\xi \in \sigma(U) \mid |\xi| < |\lambda|\}$. ☐

Remark III.3.9 (GLA for strongly continuous groups)**.** *The GLA theorems can be extended to hold for strongly continuous groups of spectral operators on locally sequentially weakly compact Banach spaces. In this setting, the Laplace averages of the group take the form*

$$A_\alpha(\lambda^{-1}G(\cdot))f := \alpha^{-1}\int_0^\alpha \lambda^{-t}G(t)f. \tag{III.3.7}$$

The extension of the GLA theorems in this case rest upon a version of Yosida's mean ergodic theorem for strongly continuous semigroups of operators $\{G(t)\}$, $t \in \mathbb{R}^+$, that roughly says that if $n^{-1}G(n)f$ converges to 0 and the averages of the semigroup do not behave too badly between integer times, then the limit of the continuous time averages $\alpha^{-1}\int_0^\alpha G(t)$ converges in the strong operator topology as $\alpha \to \infty$ to a projection operator on the subspace of elements that are fixed points of the semigroup (see ch. VIII.7, thm. 1 of [DS58], and its corollary). More precisely, if $G(t)$ is a strongly continuous one-parameter semigroup that is assumed to be strongly integrable on every finite interval and it additionally satisfies

(i) $\lim_{n\to\infty} n^{-1}G(n)f = 0$, *for all* $f \in \mathscr{F}$,

(ii) $\|A_\alpha(G(\cdot))\| \le K$ *for all* $\alpha \ge 0$,

(iii) for each f in a fundamental set in $\mathscr{F}$, the set $\{A_\alpha(G(\cdot)f)\}_{\alpha>0}$ is weakly sequentially compact,

then the averages $A_\alpha(G(\cdot))$ converge as $\alpha \to \infty$ in the strong operator topology.

A second result used in the extension is due to Lange and Nagy [LN94] on the representation of strongly continuous groups *of scalar-type operators with spectrums* *contained in the unit circle as a spectral integral over a common spectral measure; i.e., the group of operators has a representation as* $G(t) = \int_{\mathbb{R}} e^{it\lambda} E(d\lambda)$. *A sketch of the continuous time GLA proof is as follows.*

If we assume that there is a set of isolated circles in the complex plane and that each $G(t)$ *in the group of operators is spectral with point spectrum restricted to these circles and furthermore that every operator from the group has a dominating point spectrum with the same* R *(see def. III.1.5), then we can scale the family by an eigenvalue* $\lambda = \rho_1 e^{i\omega}$, *where* ρ_1 *is the radius of the largest circle containing the spectrum, and construct a new family of spectral operators* $G_1(t) := (\rho_1 e^{i\omega})^{-t} G(t)$. *This new family will have part of the spectrum contained in the unit circle and the rest strictly inside. This new group can be split into two groups, the unimodular part* $U_1(t)$, *with spectrum contained in the unit circle, and the dissipative part* $D_1(t)$, *with spectrum contained strictly inside and bounded away from the unit circle uniformly in* t. *Then* $(\rho_1 e^{i\omega})^{-t} G(t) = G_1(t) = U_1(t) + D_1(t)$. *Lange and Nagy's representation of the unitary part allows us to write* $U_1(t) = \int_{\mathbb{R}} e^{it\lambda} E_1(d\lambda)$ *and the above extension of Yosida's mean ergodic theorem allows us to prove that averages of* $U_1(t)$ *converge to the projection operators onto the subspace of* $U_1(t)$*-invariant elements. An average of the dissipative part converges to zero with order* $O(\alpha^{-1})$ *since the norm of this average has order* $\alpha^{-1} \int_0^{\alpha} O(e^{-\beta t}) dt$. *Here,* $\beta > 0$ *is related to the gap between the unit circle and the largest circle contained in* $\mathbb{D}$ *on which the spectrum of* $D_1(t)$ *is concentrated. Combined, the above arguments give that the Laplace averages*

$A_\alpha(e^{-i\omega}\rho_1^{-1}G(\cdot)) = \alpha^{-1}\int_0^\alpha(\rho_1 e^{i\omega})^{-t}G(t)dt$ converge in the strong operator topology to the subspace of elements such that $G(t)\psi = (\rho_1 e^{i\omega})^t\psi$. $\square$

Chapter IV

Generalized Laplace Analysis (GLA) for Non-Spectral Operators

In this chapter, we consider non-spectral operators. We introduce a subspace of absolutely summable series of eigenfunctions on which the analysis will proceed. This subspace will be constructed as an internal (algebraic) direct sum of eigenspaces corresponding to a certain subset of the spectrum. Each eigenspace in the construction can be thought of as "coordinates" for the subspace. Projections from the absolutely summable subspace onto the "coordinates" may not be bounded operators. However, we will see that the operator restricted to this subspace admits a spectral decomposition.

Section IV.1 constructs the absolutely summable subspace, defines the projection operators, and shows that a Laplace average converges strongly to the appropriate projection operator on a subspace of the absolutely summable subspace. In section IV.2, the previous section's result on the equivalence of an averaging operator and a projection operator on subspaces of the absolutely summable subspace is extended to prove the Generalized Laplace Analysis (GLA) theorem (theorem IV.2.1) that gives a recursive method to compute the spectral decomposition of Banach space elements that are in the absolutely summable subspace. Section IV.3 extends the GLA theorems to the situation where the operator has nilpotent components. In particular, the case of uniformly finite algebraic multiplicity[1] is considered. In order to do this, the definition of the absolutely summable subspace has to be extended. We also introduce the concepts of algebraic Koopman modes as certain nilpotent operators in addition to nonvanishing generalized eigenfunctions.

IV.1 Preliminaries

IV.1.1 Absolutely summable subspaces

Let $\mathscr{B}$ be a separable Banach space whose norm is denoted by $\|\cdot\|_{\mathscr{B}}$. Let $\mathscr{D}(U)$ be a dense linear subspace of $\mathscr{B}$ and let $U : \mathscr{D}(U) \to \mathscr{B}$ be a closed, linear operator. The nullspace of U will be denoted by $N(U) := \{f \in \mathscr{D}(U) \mid Uf = 0\}$ and the range by $\operatorname{Im}(U) = \{Uf \in \mathscr{B} \mid f \in \mathscr{D}(U)\}$. Let $\lambda \in \mathbb{C}$ be in the point spectrum of U. If $0 < |\lambda| < 1$, $N(\lambda I - U)$ is called a stable eigenspace; if $|\lambda| > 1$, $N(\lambda I - U)$ is called an unstable eigenspace; and if $|\lambda| = 1$, $N(\lambda I - U)$ is called the center eigenspace.

[1] $\exists m \in \mathbb{N}, \forall \lambda \in \Lambda,\ N((\lambda I - U)^m) = N\left((\lambda I - U)^{m+1}\right)$.

The dimension of an eigenspace will be denoted by $d(\lambda) = \dim[N(\lambda I - U)]$, with $d(\lambda) = \infty$ being possible.

Let $\Lambda \subset \sigma_p$, be an at most countable, bounded set of distinct eigenvalues of U not containing 0 and let

$$|\Lambda| = \{|\lambda| \mid \lambda \in \Lambda\} \tag{IV.1.1}$$

and

$$\|\Lambda\|_\infty = \sup\{|\lambda| \mid \lambda \in \Lambda\}. \tag{IV.1.2}$$

Assume $|\Lambda| \setminus \{0\}$ is a discrete set. Let $\{\rho_i \mid i = 1, \ldots, R\}$, where $R \in \mathbb{N} \cup \{+\infty\}$, denote the distinct values of $|\Lambda|$, ordered such that $\rho_i > \rho_{i+1} > 0$ for all i. Then define $\Lambda_i \subset \Lambda$ as

$$\Lambda_i = \{\lambda \in \Lambda \mid |\lambda| = \rho_i\}. \tag{IV.1.3}$$

For simplicity, we will denote $N(\lambda I - U)$ by $E_{i,\lambda}$, whenever $\lambda \in \Lambda_i$.

Let $\mathscr{X}_i$ be the internal direct sum

$$\mathscr{X}_i := \bigoplus_{\lambda \in \Lambda_i} E_{i,\lambda} = \left\{ \{x_\lambda\}_{\lambda \in \Lambda_i} \;\middle|\; x_\lambda \in N(\lambda I - U), \sum_{\lambda \in \Lambda_i} \|x_\lambda\|_{\mathscr{B}} < \infty \right\} \tag{IV.1.4}$$

Define a norm $\|\cdot\|_i : \mathscr{X}_i \to \mathbb{R}$ by

$$\left\|\{x_\lambda\}_{\lambda \in \Lambda_i}\right\|_i := \sum_{\lambda \in \Lambda_i} \|x_\lambda\|_{\mathscr{B}}. \tag{IV.1.5}$$

It is easy to show that $(\mathscr{X}_i, \|\cdot\|_i)$ is a Banach space. Let the linear map $\pi_i : \mathscr{X}_i \to \mathscr{D}(U)$ be defined by

$$\pi_i(\{x_\lambda\}_{\lambda \in \Lambda_i}) = \sum_{\lambda \in \Lambda_i} x_\lambda \tag{IV.1.6}$$

and let $E_{i,abs} = \pi_i(\mathscr{X}_i)$. Since the series is absolutely convergent in the Banach space $\mathscr{B}$, then it is convergent in $\mathscr{B}$ and is contained in the closure $\overline{E_{i,abs}} \subset \mathscr{D}(U)$. The subspace $E_{i,abs}$ is an algebraic direct sum, but not necessarily a topological direct sum since the projections onto each eigenspace forming $E_{i,abs}$ may not be bounded, as the next example shows.

Example IV.1.1. *Let $\{E_j\}_{j=1}^{\infty}$ be sequence of eigenspaces for U with the property that*

$$\inf_{\substack{x\in E_1, y\in E_j\\ \|x\|=\|y\|=1}} \|x-y\| =: \delta_j > 0 \qquad \text{and} \qquad \delta_j \to 0.$$

The second condition implies that the subspace $\sum_{j=2}^{\infty} E_j$ generated from $\{E_j\}_2^{\infty}$ is not closed. Let $\varepsilon_j \to 0$. For each $j \geq 2$ there exists $x_j \in E_1$, $y_j = E_j$ satisfying $\|x_j\| = \|y_j\| = 1$ and $\|x_j - y_j\| < \delta_j + \varepsilon_j$. Consider the the projection P_1 from the absolutely summable subspace formed from $\{E_j\}_1^{\infty}$ into E_1. For all $j \geq 2$, $x_j - y_j$ is in the absolutely summable subspace and

$$\|P_1\| \geq \left\| P_1\left(\frac{x_j - y_j}{\|x_j - y_j\|}\right)\right\| = \frac{\|x_j\|}{\|x_j - y_j\|} > \frac{1}{\delta_j + \varepsilon_j}.$$

Therefore, P_1 is an unbounded projection.

A concrete example is for $U : \mathscr{D} \to \ell^1(\mathbb{R}^{\mathbb{N}})$ where $U\boldsymbol{e}_{2k-1} = \boldsymbol{e}_{2k-1}$ and $U\boldsymbol{x}_k = \lambda_k \boldsymbol{x}_k$, for all $k \in \mathbb{N}$, where $\boldsymbol{x}_k = \boldsymbol{e}_{2k-1} + 2^{-k}\boldsymbol{e}_{2k}$ and $\lambda_k \neq 1$ and $\lim_{k\to\infty} = \lambda_1$. E_1 is the infinite dimensional subspace with basis $\{\boldsymbol{e}_{2k-1}\}_{k=1}^{\infty}$ and $E_k = \{\alpha \boldsymbol{x}_k \mid \alpha \in \mathbb{R}\}$, $k \geq 2$. In this case, the separation is $\delta_k = 2^{-k}$ for $k \geq 2$. The domain $\mathscr{D} \subset \ell^1$ is the space of sequences expanded in terms of $\boldsymbol{e}_{2k-1}$ and $\boldsymbol{x}_k$ such that their image under U has finite 1-norm (with respect to the canonical basis). □

The eigenspaces in example IV.1.1 exhibited a property which characterized the bounded projections. In essence, the closer two subspaces are, the larger the norms of the corresponding projections.

Definition IV.1.2 (Separated subspaces)**.** Let M and N be two subspaces. M and N are said to be *separated* if there exists a $\delta > 0$ such that for all $x \in M$ and $y \in N$ with $\|x\|_{\mathscr{B}} = \|y\|_{\mathscr{B}} = 1$, then $\|x - y\|_{\mathscr{B}} \geq \delta$. M and N are said to be separated with constant δ.

Definition IV.1.3 (Uniformly separated family of subspaces)**.** Let A be some index set and $\mathscr{V} = \{V_\alpha \mid V_\alpha \subset \mathscr{B}, \alpha \in A\}$ be a family of subspaces. The collection $\mathscr{V}$ is said to be a *uniformly separated family of subspaces* if there is a $\delta > 0$ such that for all $\alpha \in A$, V_α and $\operatorname{lin}\{V_\beta \mid \forall \beta \in A, \beta \neq \alpha\}$ are separated subspaces with constant δ. Each V_α is said to be uniformly separated in $\mathscr{V}$.

It is easy to show that $\mathscr{V}$ is a family of subspaces, then P_α is bounded if and only if $\mathscr{V}$ is a uniformly separated family.

Assumption IV.1.4. *For the rest of the chapter it will be assumed that the eigenspaces corresponding to eigenvalues in* Λ *are a uniformly separated family.* $\square$

Define

$$\mathscr{X} := \bigoplus_{i=1}^{R} \mathscr{X}_i = \left\{ \{z_i\}_{i=1}^{R} \;\middle|\; z_i \in \mathscr{X}_i, \sum_{i=1}^{R} \|z_i\|_i < \infty \right\} \tag{IV.1.7}$$

Define a norm $|||\cdot||| : \mathscr{X} \to \mathbb{R}$ by

$$|||z||| = \sum_{i=1}^{R} \|z_i\|_i \,. \tag{IV.1.8}$$

Since each $(\mathscr{X}_i, \|\cdot\|_i)$ is a Banach space, it can be shown that $(\mathscr{X}, |||\cdot|||)$ is a Banach space.

Note that $z_i \in \mathscr{X}_i$, for each $i \in \{1, \ldots, R\}$, and hence z_i is a sequence $z_i = \{z_{i,\lambda}\}_{\lambda \in \Lambda_i}$. Then (IV.1.8) is equivalently

$$|||z||| = \sum_{i=1}^{R} \sum_{\lambda \in \Lambda_i} \|z_{i,\lambda}\|_{\mathscr{B}}. \tag{IV.1.9}$$

Define the linear map $\pi : \mathscr{X} \to \mathscr{D}(U)$ by

$$\pi(z) = \sum_{i=1}^{R} \pi_i(z_i) = \sum_{i=1}^{R} \sum_{\lambda \in \Lambda_i} z_{i,\lambda}, \tag{IV.1.10}$$

for $z \in \mathscr{X}$. Finally, let

$$E_{abs} := \pi(\mathscr{X}) = \left\{ \pi(z) = \sum_{i=1}^{R} \sum_{\lambda \in \Lambda_i} z_{i,\lambda} \;\middle|\; z \in \mathscr{X} \right\}. \tag{IV.1.11}$$

Again, we note that $\pi : \mathscr{X} \to E_{abs}$ is bounded.

Each $f \in E_{abs}$ has a unique expansion of the form $f = \sum_{i=1}^{R} \sum_{\lambda \in \Lambda_i} f_{i,\lambda}$, where $f_{i,\lambda} \in E_{i,\lambda}$ and the double series is absolutely summable. Define for $f = \sum_{i=1}^{R} \sum_{\lambda \in \Lambda_i} f_{i,\lambda} \in E_{abs}$ the family of linear maps

$$P_{i,\lambda}(f) := \begin{cases} f_{i,\lambda} & \text{if } \lambda \in \Lambda_i \\ 0 & \text{if } \lambda \notin \Lambda_i \end{cases} \tag{IV.1.12}$$

and $P_i : E_{abs} \to E_{i,abs}$ by

$$P_i(f) = \sum_{\lambda \in \Lambda_i} P_{i,\lambda}(f). \tag{IV.1.13}$$

Furthermore,

$$f = \sum_{i=1}^{R} P_i(f) = \sum_{i=1}^{R} \sum_{\lambda \in \Lambda_i} P_{i,\lambda}(f), \tag{IV.1.14}$$

for all $f \in E_{abs}$. Since the expansion of f is unique, the projection operators defined in (IV.1.12) and (IV.1.13) are well-defined. However, these projections may not be bounded.

The following proposition shows that, on E_{abs}, U admits a spectral decomposition.

Proposition IV.1.5. *Let $f \in E_{abs}$. Then since U is closed*

$$U(f) = \sum_{i=1}^{R} \sum_{\lambda \in \Lambda_i} \lambda \, P_{i,\lambda}(f). \tag{IV.1.15}$$

In particular, since $\|\Lambda\|_\infty < \infty$, then $U(f) \in E_{abs}$ and hence $U : E_{abs} \to E_{abs}$.

Proof. Let $f = \sum_{i=1}^{R} P_i(f) \in E_{abs}$. Consider $P_i(f) = \sum_{\lambda \in \Lambda_i} P_{i,\lambda}(f)$. Let $\{K_{i,n}\}$ be an increasing sequence of finite sets contained in Λ_i such that

$$f^{(i,n)} := \sum_{\lambda \in K_{i,n}} P_{i,\lambda}(f) \to \sum_{\lambda \in \Lambda_i} P_{i,\lambda}(f) = P_i(f)$$

as $n \to \infty$. Applying U to $f^{(i,n)}$ gives,

$$U(f^{(i,n)}) = \sum_{\lambda \in K_{i,n}} \lambda P_{i,\lambda}(f)$$

and $U(f^{(i,n)}) \to \sum_{\lambda \in \Lambda_i} \lambda P_{i,\lambda}(f)$ as $n \to \infty$ since

$$\left\| U(f^{(i,n)}) - \sum_{\lambda \in \Lambda_i} \lambda P_{i,\lambda}(f) \right\|_{\mathscr{B}} \le \rho_i \sum_{\lambda \in \Lambda_i \setminus K_n} \|P_{i,\lambda}(f)\|_{\mathscr{B}}$$

for all $n \in \mathbb{N}$. Since U is closed,

$$U(P_i(f)) = \sum_{\lambda \in \Lambda_i} \lambda P_{i,\lambda}(f). \tag{IV.1.16}$$

Furthermore,

$$\|U(P_i(f))\|_{\mathscr{B}} \leq \rho_i \sum_{\lambda \in \Lambda_i} \|P_{i,\lambda}(f)\|_{\mathscr{B}}. \tag{IV.1.17}$$

For any $m \in \mathbb{N}$, $m \leq R$, define $f^{(m)} = \sum_{i=1}^{m} P_i(f)$. Then $\lim_{m\to\infty} f^{(m)} = f$ and $U(f^{(m)}) = \sum_{i=1}^{m} U(P_i(f))$. Furthermore, $\lim_{m\to\infty} U(f^{(m)}) = \sum_{i=1}^{R}\sum_{\lambda\in\Lambda_i} \lambda P_{i,\lambda}(f)$, since

$$\begin{aligned}\left\|U(f^{(m)}) - \sum_{i=1}^{R}\sum_{\lambda\in\Lambda_i} \lambda P_{i,\lambda}(f)\right\|_{\mathscr{B}} &= \left\|\sum_{i=m+1}^{R}\sum_{\lambda\in\Lambda_i} \lambda P_{i,\lambda}(f)\right\|_{\mathscr{B}} \\ &\leq \|\Lambda\|_\infty \sum_{i=m+1}^{R}\sum_{\lambda\in\Lambda_i} \|P_{i,\lambda}(f)\|_{\mathscr{B}}.\end{aligned}$$

for all $m \in \mathbb{N}$ and $f \in E_{abs}$. Since U is closed,

$$U(f) = U(\sum_{i=1}^{R}\sum_{\lambda\in\Lambda_i} P_{i,\lambda}(f)) = \sum_{i=1}^{R}\sum_{\lambda\in\Lambda_i} \lambda P_{i,\lambda}(f), \tag{IV.1.18}$$

completing the proof. □

IV.1.2 Averaging operators

For the *closed*, linear operator $U_\Lambda : E_{abs} \to E_{abs}$ and any scalar α, define the family of averaging operators $\{A_n(\alpha U_\Lambda)\}_{n\in\mathbb{N}}$, $A_n(\alpha U_\Lambda) : E_{abs} \to E_{abs}$, by

$$A_n(\alpha U_\Lambda) = \frac{1}{n}\sum_{k=0}^{n-1} \alpha^k U_\Lambda^k. \tag{IV.1.19}$$

Since U_Λ is closed, then $A_n(\alpha U_\Lambda)$ is also a closed linear operator for each $n \in \mathbb{N}$.

The following lemma has the form of a mean ergodic theorem on $E_{i,abs}$.

Lemma IV.1.6. *Fix $\mu \in \Lambda_i$. Then $A_n(\mu^{-1}U_\Lambda)$ converges strongly in $E_{i,abs}$ to $P_{i,\mu}$; i.e., for any $f \in E_{i,abs}$*

$$P_{i,\mu}(f) = \lim_{n\to\infty} A_n(\mu^{-1}U_\Lambda)(f) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mu^{-k}U_\Lambda^k f. \tag{IV.1.20}$$

Proof. Let $f = E_{i,abs}$. Then $f = \sum_{\lambda\in\Lambda_i} P_{i,\lambda}(f)$ and $\sum_{\lambda\in\Lambda_i} \|P_{i,\lambda}(f)\|_{\mathscr{B}} < \infty$. For any finite subset $F \subset \Lambda_i$ containing μ, we have the inequality

$$\begin{aligned}\left\|A_n(\mu^{-1}U_\Lambda)(f) - P_{i,\mu}(f)\right\|_{\mathscr{B}} &\le \left\|\frac{1}{n}\sum_{k=0}^{n-1}\sum_{\lambda\in F\setminus\{\mu\}} \frac{\lambda^k}{\mu^k}P_{i,\lambda}(f)\right\|_{\mathscr{B}} \\ &\quad + \frac{1}{n}\sum_{k=0}^{n-1}\sum_{\Lambda_i\setminus F}\left\|\frac{\lambda^k}{\mu^k}P_{i,\lambda}(f)\right\|_{\mathscr{B}}.\end{aligned}$$

Since $f \in E_{i,abs}$, we can choose, for every $\varepsilon > 0$, a finite F so that $\sum_{\Lambda_i\setminus F} \|P_{i,\lambda}(f)\|_{\mathscr{B}} \le \varepsilon$.

Since λ and μ are in Λ_i, then $|\lambda| = |\mu|$ and $\frac{\lambda}{\mu} = e^{i\theta_\lambda}$ for some $\theta_\lambda \in [0, 2\pi)$. Therefore,

$$\begin{aligned}\left\|A_n(\mu^{-1}U_\Lambda)(f) - P_{i,\mu}(f)\right\|_{\mathscr{B}} &\le \left\|\sum_{\lambda\in F\setminus\{\mu\}} \frac{1}{n}(\sum_{k=0}^{n-1} e^{i\theta_\lambda k})P_{i,\lambda}(f)\right\|_{\mathscr{B}} + \varepsilon \\ &\le \sum_{\lambda\in F\setminus\{\mu\}} \left|\frac{1}{n}(\sum_{k=0}^{n-1} e^{i\theta_\lambda k})\right| \|P_{i,\lambda}(f)\|_{\mathscr{B}} + \varepsilon.\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\left\|A_n(\mu^{-1}U_\Lambda)(f) - P_{i,\mu}(f)\right\|_{\mathscr{B}} &\le \sum_{\lambda\in F\setminus\{\mu\}} \lim_{n\to\infty}\left|\frac{1}{n}(\sum_{k=0}^{n-1} e^{i\theta_\lambda k})\right| \|P_{i,\lambda}(f)\|_{\mathscr{B}} + \varepsilon \\ &\le \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof. □

Remark IV.1.7. *Lemma IV.1.6 does not exclude the case where μ is an accumulation point for Λ_i. However, for the λ's that are arbitrarily close to μ, the contribution of $P_{i,\lambda}(f)$ to the sum defining f is arbitrarily small.* □

IV.2 Generalized Laplace Analysis

The following theorem gives the form of the projection operators onto the nonunitary eigenspaces as a type of Laplace average. In essence, we extract the maximally unstable mode first and subtract it off from the dynamics. A necessary condition for the averages to be well-defined is that Λ must be a bounded set; otherwise, there would not be a maximally unstable mode and the construction would fail.

Theorem IV.2.1 (Generalized Laplace Analysis)**.** *Let $\mathscr{B}$ be a separable Banach space under the norm $\|\cdot\|_{\mathscr{B}}$ and $U : \mathscr{D}(U) \to \mathscr{B}$ a closed linear operator such that $\sigma_p(U) \neq \{0\}$ and is nonempty. Furthermore, let $\Lambda \subset \sigma(U)$, have dominating point spectrum (see def. III.1.5) that is concentrated on isolated circles. Let Λ_i be defined by (IV.1.3).*

For any $i \in \{1, \ldots, R\}$ and $\mu \in \Lambda_i \cap \sigma_p(U)$, define $U_{i,\mu} : E_{abs} \to E_{abs}$ by

$$U_{i,\mu} := \mu^{-1} U_\Lambda \left(I - \sum_{j=1}^{i-1} P_j \right), \tag{IV.2.1}$$

where P_j and U_Λ are given by (IV.1.13) and (IV.1.15), respectively. Then $A_n(U_{i,\mu})$ converges strongly to $P_{i,\mu}$ in $(E_{abs}, \|\cdot\|_{\mathscr{B}})$, where $P_{i,\mu}$ is given by (IV.1.12); for any $f \in E_{abs}$

$$P_{i,\mu}(f) = \lim_{n\to\infty} A_n(U_{i,\mu})(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} U_{i,\mu}^k(f). \tag{IV.2.2}$$

Proof of theorem IV.2.1. Fix $i \in \{1, \dots, R\}$, $\mu \in \Lambda_i$, and $f \in E_{abs}$. f has the unique representation $f = \sum_{j=1}^{R} P_j(f) = \sum_{j=1}^{R} \sum_{\lambda \in \Lambda_j} P_{j,\lambda}(f)$. Note that $I - \sum_{j=1}^{i-1} P_j$ is a projection onto $E_{abs} \cap \operatorname{lin} E_{j,abs}{}_{j=i}^{R}$ since

$$(I - \sum_{j=1}^{i-1} P_j)^2(f) = (I - \sum_{j=1}^{i-1} P_j)(\sum_{j=i}^{R} P_j(f)) = \sum_{j=i}^{R} P_j(f).$$

Since $\operatorname{lin} E_{j,abs}{}_{j=i}^{R}$ is U_Λ-invariant,

$$[U_\Lambda(I - \sum_{j=1}^{i-1} P_j)]^k = U_\Lambda^k(I - \sum_{j=1}^{i-1} P_j).$$

and

$$\mu^{-k} U_\Lambda^k (I - \sum_{j=1}^{i-1} P_j)(f) = \mu^{-k} U_\Lambda^k (\sum_{j=i}^{R} \sum_{\lambda \in \Lambda_j} P_{j,\lambda}(f)) = \sum_{j=i}^{R} \sum_{\lambda \in \Lambda_j} \left(\frac{\lambda}{\mu}\right)^k P_{j,\lambda}(f). \quad \text{(IV.2.3)}$$

Since $A_n(U_{i,\mu})$ is a linear operator

$$\begin{aligned} A_n(U_{i,\mu})(f) &= A_n(\mu^{-1} U_\Lambda)(\sum_{j=i}^{R} \sum_{\lambda \in \Lambda_j} P_{j,\lambda}(f)) \\ &= \left[A_n(\mu^{-1} U_\Lambda)(\sum_{\lambda \in \Lambda_i} P_{i,\lambda}(f)) \right] + \left[A_n(\mu^{-1} U_\Lambda)(\sum_{j=i+1}^{R} \sum_{\lambda \in \Lambda_j} P_{j,\lambda}(f)) \right] \end{aligned} \quad \text{(IV.2.4)}$$

Expanding the second term in (IV.2.4) and using (IV.2.3), we get

$$\begin{aligned} \left\| A_n(\mu^{-1} U_\Lambda)(\sum_{j=i+1}^{R} \sum_{\lambda \in \Lambda_j} P_{j,\lambda}(f)) \right\|_{\mathscr{B}} &\le \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=i+1}^{R} \sum_{\lambda \in \Lambda_j} \left| \frac{\lambda}{\mu} \right|^k \| P_{j,\lambda}(f) \|_{\mathscr{B}} \\ &\le \left[\frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{\rho_{i+1}}{\rho_i} \right)^k \right] \left[\sum_{j=i+1}^{R} \sum_{\lambda \in \Lambda_j} \| P_{j,\lambda}(f) \|_{\mathscr{B}} \right]. \end{aligned}$$

Since $f \in E_{abs}$, then $\sum_{j=i+1}^{R} \sum_{\lambda \in \Lambda_j} \| P_{j,\lambda}(f) \|_{\mathscr{B}} = C_i < \infty$. Since $\rho_{i+1} < \rho_i$, then

$$\lim_{n \to \infty} A_n(\mu^{-1} U_\Lambda)(\sum_{j=i+1}^{R} \sum_{\lambda \in \Lambda_j} P_{j,\lambda}(f)) = 0. \quad \text{(IV.2.5)}$$

Consider the first term in (IV.2.4). Since $\sum_{\lambda\in\Lambda_i} P_{i,\lambda}(f)$ is in $E_{i,abs}$, then lemma IV.1.6 gives that

$$\lim_{n\to\infty} A_n(U_{i,\mu})(\sum_{\lambda\in\Lambda_i} P_{i,\lambda}(f)) = P_{i,\mu}(f) \tag{IV.2.6}$$

Taking the limit of (IV.2.4) and inserting (IV.2.5) and (IV.2.6) into the resulting expression gives that

$$\lim_{n\to\infty} A_n(U_{i,\mu})(f) = P_{i\mu}(f).$$

This completes the proof. □

Remark IV.2.2. *These results for closed operators on Banach spaces apply, in particular, to previously studied cases where U was the Koopman operator associated with a invertible transformation $T : X \to X$ preserving a measure ν. In those settings, the Koopman operator was a bounded operator of norm 1 acting on $L^2 = L^2(X, \mu)$. The closed graph theorem then guaranteed that the Koopman operator was closed.*

In fact, if Λ is a set of eigenvalues such that $|\lambda_j| = 1$ for all $\lambda_j \in \Lambda$, then $U_{i,\mu}$ can be taken to be U_Λ instead of (IV.2.1) and then (IV.2.2) reduces to a normal Fourier average. □

IV.3 Generalized Laplace analysis in the presence of nilpotentcy

This section extends the results of the previous section to include the case when the eigenvalues have finite algebraic multiplicity. Let

$$K_{\lambda,m} := N\left((U-\lambda I)^m\right). \tag{IV.3.1}$$

Definition IV.3.1 (Uniformly finite algebraic multiplicity)**.** For each $\lambda \in \Lambda$, the algebraic multiplicity of λ, denoted by $a(\lambda)$, is the smallest integer $k \geq 1$ such that $K_{\lambda,k} = K_{\lambda,k+1}$; i.e,

$$K_{\lambda,a(\lambda)-1} \subsetneq K_{\lambda,a(\lambda)} = K_{\lambda,a(\lambda)+1}. \tag{IV.3.2}$$

An operator $U : \mathscr{D}(U) \to \mathscr{B}$ is said to have *uniformly finite algebraic multiplicity in* $\Lambda \subset \sigma_p(U)$ if

$$a(\Lambda) := \max\left\{a(\lambda) \mid \lambda \in \Lambda\right\} < \infty. \tag{IV.3.3}$$

If we are thinking in terms of an "infinite matrix" in Jordan block form, then uniformly finite algebraic multiplicity means that there is some finite size that every Jordan block is smaller than.

Assumption IV.3.2. *For the rest of this section, we assume that* $U : \mathscr{D}(U) \to \mathscr{B}$ *has a uniformly finite algebraic multiplicity on* Λ *and that* $\Lambda \subset \sigma(U)$ *has a dominating point spectrum concentrated on isolated circles.* □

Let

$$V_{\lambda,m} := \begin{cases} K_{\lambda,1}, & \text{for } m = 1 \\ \\ K_{\lambda,m} \ominus K_{\lambda,m-1}, & \text{for } m \in \{2,\ldots,a(\lambda)\} \end{cases} \tag{IV.3.4}$$

For each $f \in V_{\lambda,m}$, $(U - \lambda I)^{m-1} f \neq 0$ and $(U - \lambda I)^m f = 0$. Redefine $E_{i,\lambda}$ to be

$$E_{i,\lambda} := \bigoplus_{m=1}^{a(\lambda)} V_{\lambda,m}. \tag{IV.3.5}$$

Each $f \in E_{i,abs}$ has a unique expansion of the form $f = \sum_{m=1}^{a(\lambda)} f_{\lambda,m}$, where $f_{\lambda,m} \in V_{\lambda,m}$. Define the projection operators $Q_{\lambda,m} : E_{i,\lambda} \to V_{\lambda,m}$ by

$$Q_{\lambda,m}(\sum_{n=1}^{a(\lambda)} f_{\lambda,n}) = f_{\lambda,m}. \tag{IV.3.6}$$

Define $N_\lambda : E_{i,\lambda} \to E_{i,\lambda}$ by the restriction

$$N_\lambda := (U - \lambda I)|E_{i,\lambda}, \tag{IV.3.7}$$

for $\lambda \in \Lambda_i$. N_λ is nilpotent on $E_{i,\lambda}$.

Construct $\mathscr{X}_i$, $E_{i,abs}$, $\mathscr{X}$, and E_{abs} as before (see sec. IV.1.1) using this new definition of $E_{i,\lambda}$. An element $f_i \in E_{i,abs}$, has an expansion of the form

$$f_i = \sum_{\lambda \in \Lambda_i} f_{i,\lambda} = \sum_{\lambda \in \Lambda_i} \sum_{m=1}^{a(\lambda)} Q_{\lambda,m}(f_{i,\lambda}) \tag{IV.3.8}$$

where $f_{i,\lambda} \in E_{i,\lambda}$ and $\sum_{\lambda \in \Lambda_i} \|f_{i,\lambda}\|_{\mathscr{B}} < \infty$. An element $f \in E_{abs}$ has an expansion of the form

$$f = \sum_{i=1}^{R} f_i = \sum_{i=1}^{R} \sum_{\lambda \in \Lambda_i} f_{i,\lambda} = \sum_{i=1}^{R} \sum_{\lambda \in \Lambda_i} \sum_{m=1}^{a(\lambda)} Q_{\lambda,m}(f_{i,\lambda}). \tag{IV.3.9}$$

where

$$\sum_{i=1}^{R} \|f_i\|_i = \sum_{i=1}^{R} \sum_{\lambda \in \Lambda_i} \|f_{i,\lambda}\|_{\mathscr{B}} < \infty.$$

Define the family of projection operators $P_{i,\lambda} : E_{abs} \to E_{i,\lambda}$ by

$$P_{j,\mu}(\sum_{i=1}^{R} \sum_{\lambda \in \Lambda_i} f_{i,\lambda}) = f_{j,\mu} = \sum_{m=1}^{a(\mu)} Q_{\mu,m}(f_{j,\mu}) \tag{IV.3.10}$$

and $P_j : E_{abs} \to E_{j,abs}$ by

$$P_j(\sum_{i=1}^{R} f_i) = f_j = \sum_{\lambda \in \Lambda_j} f_{j,\lambda} = \sum_{\lambda \in \Lambda_j} \sum_{m=1}^{a(\lambda)} Q_{\lambda,m}(f_{j,\lambda}) \tag{IV.3.11}$$

Definition IV.3.3 (Algebraic Koopman Mode)**.** For $i \in \{1, \ldots, R\}$, $\lambda \in \Lambda_i$, and $p \in \{0, \ldots, a(\lambda) - 1\}$, the algebraic Koopman mode is the linear operator $\Omega_{i,\lambda,p}$: $E_{abs} \to \bigoplus_{j=1}^{a(\lambda)-p} V_{\lambda,j}$ given by

$$\Omega_{i,\lambda,p}(f) := \sum_{m=p+1}^{a(\lambda)} N_\lambda^p Q_{\lambda,m}(P_{i,\lambda}(f)) \tag{IV.3.12}$$

where N_λ is given by (IV.3.7) and $f \in E_{abs}$.

Lemma IV.3.4. *For fixed $i \in \{1, \ldots, R\}$ and $\lambda \in \Lambda_i$, the algebraic Koopman mode is a projection for $p = 0$ and nilpotent for $p > 0$.*

Proof. Let p and q be elements of $\{0, \ldots, a(\lambda) - 1\}$. Note that $\Omega_{i,\lambda,p}(f) \in E_{i,\lambda} \subset E_{abs}$.

Then

$$\begin{aligned}\Omega_{i,\lambda,q}(\Omega_{i,\lambda,p}(f)) &= \sum_{m=q+1}^{a(\lambda)} N_\lambda^q Q_{\lambda,m}(P_{i,\lambda}(\Omega_{i,\lambda,p}(f))) \\ &= \sum_{m=q+1}^{a(\lambda)} N_\lambda^q Q_{\lambda,m}(\Omega_{i,\lambda,p}(f)) \\ &= \sum_{m=q+1}^{a(\lambda)} N_\lambda^q Q_{\lambda,m}(\sum_{n=p+1}^{a(\lambda)} N_\lambda^p Q_{\lambda,n}(P_{i,\lambda}(f))) \\ &= \sum_{m=q+1}^{a(\lambda)} N_\lambda^q(\sum_{n=p+1}^{a(\lambda)} Q_{\lambda,m} N_\lambda^p Q_{\lambda,n}(P_{i,\lambda}(f))).\end{aligned}$$

Since $Q_{\lambda,m}$ is a projection onto $V_{\lambda,m}$, then $Q_{\lambda,m}N_\lambda^p Q_{\lambda,n} \neq 0$ if and only if $m = n - p$ since $N_\lambda^p Q_{\lambda,n}(P_{i,\lambda}(f)) \in V_{\lambda,n-p}$. It is only possible for the operator $Q_{\lambda,m}N_\lambda^p Q_{\lambda,n}$ to be nonzero when $m \leq a(\lambda) - p$. When $n = m + p$,

$$\begin{aligned}Q_{\lambda,m}N_\lambda^p Q_{\lambda,n}(P_{i,\lambda}(f))) &= Q_{\lambda,m}N_\lambda^p Q_{\lambda,m+p}(P_{i,\lambda}(f))) \\ &= N_\lambda^p Q_{\lambda,m+p}(P_{i,\lambda}(f)))\end{aligned}$$

Plugging this back in gives

$$\sum_{m=q+1}^{a(\lambda)} N_\lambda^q(\sum_{n=p+1}^{a(\lambda)} Q_{\lambda,m}N_\lambda^p Q_{\lambda,n}(P_{i,\lambda}(f))) = \sum_{m=q+1}^{a(\lambda)-p} N_\lambda^q N_\lambda^p Q_{\lambda,m+p}(P_{i,\lambda}(f)).$$

Therefore,

$$\Omega_{i,\lambda,q}(\Omega_{i,\lambda,p}(f)) = \sum_{m=q+1}^{a(\lambda)-p} N_\lambda^{q+p} Q_{\lambda,m+p}(P_{i,\lambda}(f)) \in \bigoplus_{j=1}^{a(\lambda)-p-q} V_{\lambda,j}.$$

In particular, putting $q = p$ and proceeding inductively gives

$$\begin{aligned}\Omega_{i,\lambda,p}^j(f) &= \sum_{m=p+1}^{a(\lambda)-(j-1)p} N_\lambda^{jp} Q_{\lambda,m+(j-1)p}(P_{i,\lambda}(f)) \\ &= \sum_{m=1}^{a(\lambda)-jp} N_\lambda^{jp} Q_{\lambda,m+jp}(P_{i,\lambda}(f)).\end{aligned}$$

When $p > 0$ there is some $j \in \mathbb{N}$ such that $a(\lambda) \leq jp$ and hence $\Omega^j_{i,\lambda,p}(f) = 0$. When $p = 0$, $\Omega^j_{i,\lambda,0}(f) = \sum_{m=1}^{a(\lambda)} Q_{\lambda,m}(P_{i,\lambda}(f)) =: \Omega_{i,\lambda,0}(f)$ for all $j \in \mathbb{N}$. $\square$

Definition IV.3.5. Fix $f \in E_{abs}$ such that $P_{i,\lambda}(f) \neq 0$. Let $i \in \{1, \ldots, R\}$ and $\lambda \in \Lambda_i$. Define $V_{i,\lambda}(f) \subset E_{i,abs}$ to be the finite dimensional vector space

$$V_{i,\lambda}(f) := \operatorname{lin}\left\{N_\lambda^p Q_{\lambda,m}(P_{i,\lambda}(f)) \mid p \in \{0, \ldots, a(\lambda) - 1\}, m \in \{1, \ldots, a(\lambda)\}\right\}.$$

The functional $\|\cdot\|_{V_{i,\lambda}(f)} : V_{i,\lambda}(f) \to \mathbb{R}$ defined by

$$\|g\|_{V_{i,\lambda}(f)} := \sum_{p=0}^{a(\lambda)-1} \sum_{m=p+1}^{a(\lambda)} \|N_\lambda^p Q_{\lambda,m}(g)\|_{\mathscr{B}} \tag{IV.3.13}$$

is a norm on $V_{i,\lambda}(f)$.

Since $V_{i,\lambda}(f)$ is finite-dimensional and $\|\cdot\|_{\mathscr{B}}$ restricted to $V_{i,\lambda}(f)$ is a norm on $V_{i,\lambda}(f)$, then there exists a $C_{i,\lambda}(f) > 0$ such that

$$\|g\|_{V_{i,\lambda}(f)} \leq C_{i,\lambda}(f)\|g\|_{\mathscr{B}} \tag{IV.3.14}$$

for all $g \in V_{i,\lambda}(f)$, where

$$C_{i,\lambda}(f) := \sum_{m=1}^{a(\lambda)} \sum_{p=0}^{m-1} \|N_\lambda^p Q_{\lambda,m}\|_{\mathscr{B}} \tag{IV.3.15}$$

Definition IV.3.6 (Non-vanishing generalized eigenfunctions)**.** Let $f \in E_{abs}$. If

$$\sup\left\{C_{i,\lambda}(f)\right\} < \infty, \tag{IV.3.16}$$

where the supremum is taken over all $i \in \mathbb{N}$ satisfying $1 \leq i \leq R$ and over all $\lambda \in \Lambda_i$ such that $P_{i,\lambda}(f) \neq 0$, we say that f has *non-vanishing generalized eigenfunctions*.

Remark IV.3.7. *Note that the set* $\{C_{i,\lambda}\}$ *is uniformly bounded over* $i \in \{1, \dots, R\}$ *and* $\lambda \in \Lambda_i$ *if and only if the sets* $\{\|N_\lambda\|_{\mathscr{B}}\}$ *and* $\left\{\|Q_{\lambda,m}\|_{\mathscr{B}}\right\}$ *are uniformly bounded over* $\lambda \in \Lambda$ *and* $m \in \{1, \dots, a(\lambda)\}$*. The projections* $\{Q_{\lambda,m}\}$ *are bounded if and only if the all generalized eigenspaces* $\{V_{\lambda,m}\}$ *are uniformly separated for all* $\lambda \in \Lambda$ *and* m*. If* N_λ *is bounded, then*

$$\frac{\left\|N_\lambda^{m-1} Q_{\lambda,m}(f)\right\|_{\mathscr{B}}}{\|Q_{\lambda,m}(f)\|_{\mathscr{B}}} \leq C_{\lambda,m}$$

Since $N_\lambda^{m-1} Q_{\lambda,m}(f)$ *is an eigenfunction of* U *and* $Q_{\lambda,m}(f)$ *is the generalized eigenfunction that gets mapped into it, if the above bound is uniform it says that no generalized eigenfunction has arbitrarily small norm compared to its associated eigenfunction. This gives rise to the name non-vanishing generalized eigenfunction.* □

Lemma IV.3.8. *Let* $f \in E_{abs}$ *have non-vanishing generalized eigenfunctions. Then,*

$$\sum_{i=1}^{R} \sum_{\lambda \in \Lambda_i} \sum_{p=0}^{a(\lambda)-1} \|\Omega_{i,\lambda,p}(f)\|_{\mathscr{B}} \leq C \sum_{i=1}^{R} \sum_{\lambda \in \Lambda_i} \|P_{i,\lambda}(f)\|_{\mathscr{B}} < \infty, \tag{IV.3.17}$$

where C *is the supremum given in* (IV.3.16)*.*

Proof. For convenience, let $n_\lambda = \min(k, a(\lambda)-1)$. Then for $k \geq 0$,

$$\begin{aligned}
\sum_{i=1}^{R}\sum_{\lambda\in\Lambda_i}\sum_{p=0}^{n_\lambda}\|\Omega_{i,\lambda,p}(f)\|_{\mathscr{B}} &= \sum_{i=1}^{R}\sum_{\lambda\in\Lambda_i}\sum_{p=0}^{n_\lambda}\left\|\sum_{m=p+1}^{a(\lambda)} N_\lambda^p Q_{\lambda,m}(P_{i,\lambda}(f))\right\|_{\mathscr{B}} \\
&\leq \sum_{i=1}^{R}\sum_{\lambda\in\Lambda_i}\left(\sum_{p=0}^{n_\lambda}\sum_{m=p+1}^{a(\lambda)}\|N_\lambda^p Q_{\lambda,m}(P_{i,\lambda}(f))\|_{\mathscr{B}}\right) \\
&= \sum_{i=1}^{R}\sum_{\lambda\in\Lambda_i}\|P_{i,\lambda}(f)\|_{V_{i,\lambda}(f)} \\
&\leq \sum_{i=1}^{R}\sum_{\lambda\in\Lambda_i} C_{i,\lambda}(f)\|P_{i,\lambda}(f)\|_{\mathscr{B}} \\
&\leq C\sum_{i=1}^{R}\sum_{\lambda\in\Lambda_i}\|P_{i,\lambda}(f)\|_{\mathscr{B}} < \infty.
\end{aligned}$$

□

Lemma IV.3.9. *Let $f \in E_{abs}$. Then for $k \geq 0$,*

$$U^k(f) = \sum_{i=1}^{R}\sum_{\lambda\in\Lambda_i}\sum_{p=0}^{\min(k,a(\lambda)-1)}\binom{k}{p}\lambda^{k-p}\Omega_{i,\lambda,p}(f). \qquad \text{(IV.3.18)}$$

If f has non-vanishing generalized eigenfunctions, the series converges absolutely for all $k \geq 0$.

Proof. $f \in E_{abs}$ implies that

$$f \in \sum_{i=1}^{R}\sum_{\lambda\in\Lambda_i} P_{i,\lambda}(f) = \sum_{i=1}^{R}\sum_{\lambda\in\Lambda_i}\sum_{m=1}^{a(\lambda)} Q_{\lambda,m}(P_{i,\lambda}(f))$$

and

$$\sum_{i=1}^{R}\sum_{\lambda\in\Lambda_i}\|P_{i,\lambda}(f))\|_{\mathscr{B}} < \infty.$$

Then

$$U^k(f) = \sum_{i=1}^{R}\sum_{\lambda\in\Lambda_i} U^k(P_{i,\lambda}(f)) = \sum_{i=1}^{R}\sum_{\lambda\in\Lambda_i} U^k(\sum_{m=1}^{a(\lambda)} Q_{\lambda,m}(P_{i,\lambda}(f)))). \qquad \text{(IV.3.19)}$$

Consider $U^k(\sum_{m=1}^{a(\lambda)} Q_{\lambda,m}(P_{i,\lambda}(f)))$:

$$U^k(\sum_{m=1}^{a(\lambda)} Q_{\lambda,m}(P_{i,\lambda}(f))) = (\lambda I + N_\lambda)^k(\sum_{m=1}^{a(\lambda)} Q_{\lambda,m}(P_{i,\lambda}(f)))$$
$$= \sum_{p=0}^{k} \binom{k}{p} \lambda^{k-p} \sum_{m=1}^{a(\lambda)} N_\lambda^p Q_{\lambda,m}(P_{i,\lambda}(f)).$$

For fixed p, $N_\lambda^p Q_{\lambda,m} = 0$ if $m < p+1$. Then

$$U^k(\sum_{m=1}^{a(\lambda)} Q_{\lambda,m}(P_{i,\lambda}(f))) = \sum_{p=0}^{k} \binom{k}{p} \lambda^{k-p} \sum_{m=p+1}^{a(\lambda)} N_\lambda^p Q_{\lambda,m}(P_{i,\lambda}(f))$$

Furthermore, the sum $\sum_{m=p+1}^{a(\lambda)} N_\lambda^p Q_{\lambda,m}(P_{i,\lambda}(f)) = 0$ if $p \geq a(\lambda)$. Theforefore,

$$U^k(\sum_{m=1}^{a(\lambda)} Q_{\lambda,m}(P_{i,\lambda}(f))) = \sum_{p=0}^{\min(k,a(\lambda)-1)} \binom{k}{p} \lambda^{k-p} \sum_{m=p+1}^{a(\lambda)} N_\lambda^p Q_{\lambda,m}(P_{i,\lambda}(f))$$
$$= \sum_{p=0}^{\min(k,a(\lambda)-1)} \binom{k}{p} \lambda^{k-p} \Omega_{i,\lambda,p}(f).$$

Inserting this into (IV.3.19) gives

$$U^k(f) = \sum_{i=1}^{R} \sum_{\lambda \in \Lambda_i} \sum_{p=0}^{\min(k,a(\lambda)-1)} \binom{k}{p} \lambda^{k-p} \Omega_{i,\lambda,p}(f).$$

For all $k \geq 0$, $L_k < \infty$, where

$$L_k := \max\left\{ \binom{k}{p} \max(\|\Lambda\|_\infty^k, 1) \;\middle|\; p \in \{0, \ldots, \min(k, a(\lambda)-1)\} \right\} \tag{IV.3.20}$$

Then, if $f \in E_{abs}$ has non-vanishing generalized eigenfunction, the series converges by lemma IV.3.8. □

The condition that $\sup C_{i,\lambda}(f) < \infty$ is crucial. If this was not true, then for some $f \in E_{abs}$, (IV.3.18) would not be convergent. The following example shows a particular case of this. In this example, a sequence of generalized eigenfunctions converging to 0 will allow for an unbounded sequence of coefficients while still maintaining convergence of the series. However, upon iteration by the operator, these large coefficients will eventually multiply an eigenfunction having norm one, leading to a divergent series.

Example IV.3.10. *Consider $U : \ell^1(\mathbb{C}^{\mathbb{N}}) \to \ell^1(\mathbb{C}^{\mathbb{N}})$ such that each eigenvalue $\{\lambda_j \mid j \in \mathbb{N}\}$ satisfies $\lambda_j = e^{i\theta_j}$, $\theta_j \in [0, \pi/2]$, and has only one corresponding eigenfunction $\varphi_{j,1} = \boldsymbol{e}_{2j-1}$ of norm 1 and one generalized eigenfunction $\varphi_{j,2} = 2^{-(2j+1)}\boldsymbol{e}_{2j}$ with $(U - \lambda_j I)\varphi_{j,2} = \varphi_{j,1}$ and where $\{\boldsymbol{e}_i\}_1^\infty$ are the canonical basis vectors for $\ell^1(\mathbb{C}^{\mathbb{N}})$. In this example, $R = 1$ and $a(\lambda) = 2$.*

Let $f = \sum_{j=1}^\infty \sum_{m=1}^2 \Theta_{j,m}(f)\varphi_{j,m}$ where $\Theta_{j,m}(f) = 2^{-(j+1)} \|\varphi_{j,m}\|_1^{-1}$ for $m = 1, 2$. Then

$$\|f\|_1 = \sum_{j=1}^\infty \sum_{m=1}^2 |\Theta_{j,m}(f)| \, \|\varphi_{j,m}\|_1 = \sum_{j=1}^\infty 2^{-j} = 1.$$

We first show that $\left\| \sum_{p=0}^1 \sum_{m=p+1}^2 \binom{1}{p} \lambda_j^{1-p} N_{\lambda_j}^p Q_{\lambda_j,m}(P_{1,\lambda_j}(f)) \right\|_1$ is bounded below by a divergent series.

$$\begin{aligned}
\left\|P_{1,\lambda_j}(f)\right\|_1 &= \left\| \sum_{p=0}^1 \sum_{m=p+1}^2 \binom{1}{p} \lambda_j^{1-p} N_{\lambda_j}^p Q_{\lambda_j,m}(P_{1,\lambda_j}(f)) \right\|_1 \\
&= \left\| \sum_{m=1}^2 \sum_{p=0}^{m-1} \lambda_j^{1-p} N_\lambda^p Q_{\lambda,m}(P_{1,\lambda_j}(f)) \right\|_1 \\
&= \left\| \sum_{m=1}^2 \sum_{p=0}^{m-1} \lambda_j^{1-p} N_\lambda^p (\Theta_{j,m}(f)\varphi_{j,m}) \right\|_1 \\
&= \|\lambda_j \Theta_{j,1}(f)\varphi_{j,1} + \lambda_j \Theta_{j,2}(f)\varphi_{j,2} + \Theta_{j,2}(f)\varphi_{j,1}\|_1 \\
&\geq |\lambda_j \Theta_{j,1}(f) + \Theta_{j,2}(f)| \, \|\varphi_{j,1}\|_1 ,
\end{aligned}$$

where we have used $\binom{1}{p} = 1$ *for* $p = 0, 1$, $\varphi_{j,1} = \boldsymbol{e}_{2j-1}$, *and* $\varphi_{j,2} \in \operatorname{lin} \boldsymbol{e}_{2j}$. *Since* $\Theta_{j,m}(f)$ *is real and positive for all* $j \geq 1$ *and* $m \in \{1, 2\}$ *and* $\lambda_j = e^{i\theta_j}$, $\theta_j \in [0, \pi/2]$, *then* $|\lambda_j \Theta_{j,1}(f) + \Theta_{j,2}(f)| \geq |\Theta_{j,2}(f)|$. *Therefore,*

$$\left\| \sum_{p=0}^{1} \sum_{m=p+1}^{2} \binom{1}{p} \lambda_j^{1-p} N_{\lambda_j}^p Q_{\lambda_j,m}(P_{1,\lambda_j}(f)) \right\|_1 \geq |\Theta_{j,2}(f)| = 2^j.$$

Using this in (IV.3.18), *for* $k = 1$, *gives*

$$\begin{aligned} \|U(f)\|_1 &= \sum_{j=1}^{\infty} \left\| \sum_{p=0}^{1} \binom{1}{p} \lambda^{1-p} \Omega_{1,\lambda,p}(f) \right\|_1 \\ &= \sum_{j=1}^{\infty} \left\| \sum_{p=0}^{1} \sum_{m=p+1}^{2} \binom{1}{p} \lambda_j^{1-p} N_{\lambda_j}^p Q_{\lambda_j,m}(P_{1,\lambda_j}(f)) \right\|_1 \\ &\geq \sum_{j=1}^{\infty} 2^j \end{aligned}$$

Hence, $U(f)$ *is a divergent series.* □

Remark IV.3.11. *Note that the time-varying term multiplying the algebraic Koopman mode* $\Omega_{i,\lambda,p}(f)$ *is of order* $\mathcal{O}(\lambda^k k^p)$. *Hence for fixed* i *and* λ, $\Omega_{i,\lambda,a(\lambda)-1}$ *has the largest algebraic growth of its corresponding time-dependent term and* $\Omega_{i,\lambda,0}$ *has the least. In particular,*

$$\frac{\binom{k}{p}}{\binom{k}{q}} = \mathcal{O}(k^{p-q}). \tag{IV.3.21}$$

Moreover, call $\Omega_{i,\mu,p}(f)$ *more unstable than* $\Omega_{j,\lambda,q}(f)$ *if*

$$\lim_{k \to \infty} \left| \frac{\mu^k k^p}{\lambda^k k^q} \right| = \infty. \tag{IV.3.22}$$

□

Definition IV.3.12. Let Λ_i be defined by (IV.1.3). Let

$$a(\Lambda_i) := \max\{a(\lambda) \mid \lambda \in \Lambda_i\} \tag{IV.3.23}$$

and for $p \in \{0, \ldots, a(\Lambda_i) - 1\}$, define

$$\Lambda_{i,p} := \{\lambda \in \Lambda_i \mid a(\lambda) = p + 1\}. \tag{IV.3.24}$$

Lemma IV.3.13. *Let $f \in E_{abs}$ have non-vanishing generalized eigenfunctions. For all $k \geq a(\Lambda)$*

$$U^k(f) = \sum_{i=1}^{R} \sum_{p=0}^{a(\Lambda_i)-1} \sum_{\lambda \in \Lambda_{i,p}} \binom{k}{p} \lambda^{k-p} \Omega_{i,\lambda,p}(f) \tag{IV.3.25}$$

is a convergent series and where $a(\Lambda)$ is given by (IV.3.3).

Proof. From its defintion, $\Lambda_{i,p} \cap \Lambda_{i,q} = \emptyset$ for $p \neq q$ and $\Lambda_i = \bigcup_{p=0}^{a(\Lambda_i)-1} \Lambda_{i,p}$. Then, formally,

$$\sum_{i=1}^{R} \sum_{\lambda \in \Lambda_i} \sum_{p=0}^{a(\lambda)-1} \binom{k}{p} \lambda^{k-p} \Omega_{i,\lambda,p}(f) = \sum_{i=1}^{R} \sum_{p=0}^{a(\Lambda_i)-1} \sum_{\lambda \in \Lambda_{i,p}} \binom{k}{p} \lambda^{k-p} \Omega_{i,\lambda,p}(f).$$

Convergence of this series still needs to be shown.

Since

$$\sum_{i=1}^{R} \sum_{\lambda \in \Lambda_i} \sum_{p=0}^{a(\lambda)-1} \binom{k}{p} \lambda^{k-p} \Omega_{i,\lambda,p}(f)$$

is absolutely convergent by lemma IV.3.9, then $\sum_{\lambda \in \Lambda_i}$ and $\sum_{p=0}^{a(\lambda)-1}$ commute and the commuted infinite sum converges to the same limit. □

We now come to the main result of this section. The following theorem represents an extension of theorem IV.2.1 to the current setting.

Theorem IV.3.14 (Computation of algebraic Koopman modes.)**.** *Let $f \in E_{abs}$ and assume $U_\Lambda : E_{abs} \to \mathscr{B}$ has uniformly finite algebraic multiplicity in Λ and f has non-vanishing generalized eigenfunctions.*

Let $j \in \{1, \dots, R\}$, $q \in \{0, \dots, a(\Lambda_j) - 1\}$, and $\mu \in \Lambda_{j,q}$. Define, for $k \geq 0$

$$U^k_{j,q}(f) = U^k(I - \sum_{i=1}^{j-1} P_i)(f) - \sum_{p=q+1}^{a(\Lambda_j)-1} \sum_{\lambda \in \Lambda_{j,p}} \binom{k}{p} \lambda^{k-p} \Omega_{j,\lambda,p}(f) \tag{IV.3.26}$$

Then

$$\Omega_{j,\mu,q}(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \binom{k}{q}^{-1} \mu^{-k} U^k_{j,q}(f) \tag{IV.3.27}$$

Proof. Fix $\varepsilon > 0$. First note that for $k \geq a(\Lambda)$ and using (IV.3.25), (IV.3.26) becomes

$$U^k_{j,q}(f) = \sum_{p=0}^{q} \sum_{\lambda \in \Lambda_{j,p}} \binom{k}{p} \lambda^{k-p} \Omega_{j,\lambda,p}(f) + \sum_{i=j+1}^{R} \sum_{p=0}^{a(\Lambda_i)-1} \sum_{\lambda \in \Lambda_{i,p}} \binom{k}{p} \lambda^{k-p} \Omega_{i,\lambda,p}(f)$$

or equivalently

$$\binom{k}{q}^{-1} \mu^{-(k-q)} U^k_{j,q}(f) - \Omega_{j,\mu,q} = \sum_{\lambda \in \Lambda_{j,q} \setminus \{\mu\}} \frac{\binom{k}{q}}{\binom{k}{q}} \frac{\lambda^{k-q}}{\mu^{k-q}} \Omega_{j,\lambda,q}(f) \tag{IV.3.28}$$

$$+ \sum_{p=0}^{q-1} \sum_{\lambda \in \Lambda_{j,p}} \frac{\binom{k}{p}}{\binom{k}{q}} \frac{\lambda^{k-p}}{\mu^{k-q}} \Omega_{j,\lambda,p}(f) \tag{IV.3.29}$$

$$+ \sum_{i=j+1}^{R} \sum_{p=0}^{a(\Lambda_i)-1} \sum_{\lambda \in \Lambda_{i,p}} \frac{\binom{k}{p}}{\binom{k}{q}} \frac{\lambda^{k-p}}{\mu^{k-q}} \Omega_{i,\lambda,p}(f) \tag{IV.3.30}$$

Consider the term on the right side of (IV.3.28). Taking the average of this term divided by $\binom{k}{q}^{-1} \mu^{-(k-q)}$ and starting at $k = a(\Lambda)$, we get

$$\begin{aligned} &\frac{1}{n} \sum_{k=a(\Lambda)}^{n-1} \sum_{\lambda \in \Lambda_{j,q} \setminus \{\mu\}} \frac{\binom{k}{q}}{\binom{k}{q}} \frac{\lambda^{k-q}}{\mu^{k-q}} \Omega_{j,\lambda,q}(f) \\ &\quad = \frac{n - a(\Lambda)}{n} \frac{1}{n - a(\Lambda)} \sum_{k=a(\Lambda)}^{n-1} \sum_{\lambda \in \Lambda_{j,q} \setminus \{\mu\}} \frac{\lambda^{k-q}}{\mu^{k-q}} \Omega_{j,\lambda,q}(f). \end{aligned} \tag{IV.3.31}$$

Since $\sum_{\lambda\in\Lambda_{j,q}\setminus\{\mu\}}\Omega_{j,\lambda,q}(f)$ is absolutely convergent, by lemma IV.3.8, there is a finite subset $F_1 \subset \Lambda_{j,q}\setminus\{\mu\}$ such that

$$\sum_{\lambda\in(\Lambda_{j,q}\setminus\{\mu\})\setminus F_1} \|\Omega_{j,\lambda,q}(f)\|_{\mathscr{B}} < \frac{\varepsilon}{5}.$$

Then, using the fact that $\mu\in\Lambda_{j,q}$ so that $\left(\frac{\lambda}{\mu}\right)^{k-q} = e^{i\theta_\lambda}$, for all $\lambda\in\Lambda_{j,q}$, taking the norm of (IV.3.31) gives

$$\begin{aligned}&\left\|\frac{1}{n}\sum_{k=a(\Lambda)}^{n-1}\sum_{\lambda\in\Lambda_{j,q}\setminus\{\mu\}}\frac{\binom{k}{q}}{\binom{k}{q}}\frac{\lambda^{k-q}}{\mu^{k-q}}\Omega_{j,\lambda,q}(f)\right\|_{\mathscr{B}}\\ &\qquad < \frac{n-a(\Lambda)}{n}\left(\sum_{\lambda\in F_1}\left\|\left(\frac{1}{n-a(\Lambda)}\sum_{k=a(\Lambda)}^{n-1}e^{i\theta_\lambda(k-q)}\right)\Omega_{j,\lambda,q}(f)\right\|_{\mathscr{B}}+\frac{\varepsilon}{5}\right).\end{aligned}\tag{IV.3.32}$$

Since the term in the $\|\cdot\|_{\mathscr{B}}$ on the right side of (IV.3.32) converges to zero, there exists an $N_1\in\mathbb{N}$, such that for all $n\ge N_1$,

$$\left\|\frac{1}{n}\sum_{k=a(\Lambda)}^{n-1}\sum_{\lambda\in\Lambda_{j,q}\setminus\{\mu\}}\frac{\binom{k}{q}}{\binom{k}{q}}\frac{\lambda^{k-q}}{\mu^{k-q}}\Omega_{j,\lambda,q}(f)\right\|_{\mathscr{B}} < \frac{2\varepsilon}{5}\tag{IV.3.33}$$

Consider the average of the second term, (IV.3.29). We estimate it as follows:

$$\begin{aligned}&\left\|\frac{1}{n}\sum_{k=a(\Lambda)}^{n-1}\sum_{p=0}^{q-1}\sum_{\lambda\in\Lambda_{j,p}}\frac{\binom{k}{p}}{\binom{k}{q}}\frac{\lambda^{k-p}}{\mu^{k-q}}\Omega_{j,\lambda,p}(f)\right\|_{\mathscr{B}}\\ &\qquad\le \frac{n-a(\Lambda)}{n}\frac{1}{n-a(\Lambda)}\sum_{k=a(\Lambda)}^{n-1}\sum_{p=0}^{q-1}\sum_{\lambda\in\Lambda_{j,p}}\left\|\frac{\binom{k}{p}}{\binom{k}{q}}\frac{\lambda^{k-p}}{\mu^{k-q}}\Omega_{j,\lambda,p}(f)\right\|_{\mathscr{B}}\\ &\qquad\le \frac{n-a(\Lambda)}{n}\frac{1}{n-a(\Lambda)}\sum_{k=a(\Lambda)}^{n-1}q\rho_j^{q-p}\frac{\binom{k}{q-1}}{\binom{k}{q}}\left(\sum_{\lambda\in\Lambda_{j,p}}\|\Omega_{j,\lambda,p}(f)\|_{\mathscr{B}}\right)\\ &\qquad\le \frac{n-a(\Lambda)}{n}\left(q\rho_j^{q-p}\sum_{\lambda\in\Lambda_{j,p}}\|\Omega_{j,\lambda,p}(f)\|_{\mathscr{B}}\right)\frac{1}{n-a(\Lambda)}\sum_{k=a(\Lambda)}^{n-1}\frac{\binom{k}{q-1}}{\binom{k}{q}}\end{aligned}$$

where we have used that $|\mu| = |\lambda| = \rho_j$ for all $\lambda\in\Lambda_{j,p}$, $p = 0,\ldots,q-1$. Since $\frac{\binom{k}{q-1}}{\binom{k}{q}} = \mathcal{O}(k^{-1})$ and $\sum_{\lambda\in\Lambda_{j,p}}\|\Omega_{j,\lambda,p}(f)\|_{\mathscr{B}}$ is finite, then there is $N_2\in\mathbb{N}$ such that for all $n \geq N_2$,

$$\left\| \frac{1}{n} \sum_{k=a(\Lambda)}^{n-1} \sum_{p=0}^{q-1} \sum_{\lambda \in \Lambda_{j,p}} \frac{\binom{k}{p}}{\binom{k}{q}} \frac{\lambda^{k-p}}{\mu^{k-q}} \Omega_{j,\lambda,p}(f) \right\|_{\mathscr{B}} < \frac{\varepsilon}{5}. \tag{IV.3.34}$$

Consider the average of the third term, (IV.3.30).

$$\begin{aligned}
&\left\| \frac{1}{n} \sum_{k=a(\Lambda)}^{n-1} \sum_{i=j+1}^{R} \sum_{p=0}^{a(\Lambda_i)-1} \sum_{\lambda \in \Lambda_{i,p}} \frac{\binom{k}{p}}{\binom{k}{q}} \frac{\lambda^{k-p}}{\mu^{k-q}} \Omega_{i,\lambda,p}(f) \right\|_{\mathscr{B}} \\
&\quad \leq \frac{1}{n} \sum_{k=a(\Lambda)}^{n-1} \sum_{i=j+1}^{R} \sum_{p=0}^{a(\Lambda_i)-1} \sum_{\lambda \in \Lambda_{i,p}} \frac{\binom{k}{p}}{\binom{k}{q}} \frac{|\lambda|^{k-p}}{|\mu|^{k-q}} \|\Omega_{i,\lambda,p}(f)\|_{\mathscr{B}} \\
&\quad \leq \frac{1}{n} \sum_{k=a(\Lambda)}^{n-1} \sum_{i=j+1}^{R} \sum_{p=0}^{a(\Lambda_i)-1} \frac{\binom{k}{p}}{\binom{k}{q}} \frac{\rho_{j+1}^{k-p}}{\rho_j^{k-q}} \left(\sum_{\lambda \in \Lambda_{i,p}} \|\Omega_{i,\lambda,p}(f)\|_{\mathscr{B}} \right) \\
&\quad \leq \frac{1}{n} \sum_{k=a(\Lambda)}^{n-1} \left(\frac{\rho_{j+1}}{\rho_j} \right)^{k-q} \sum_{i=j+1}^{R} \left(\sum_{p=0}^{a(\Lambda_i)-1} \frac{\binom{k}{p}}{\binom{k}{q}} \rho_{j+1}^{q-p} \left(\sum_{\lambda \in \Lambda_{i,p}} \|\Omega_{i,\lambda,p}(f)\|_{\mathscr{B}} \right) \right).
\end{aligned}$$

Let $t(k) := \max \left\{ \frac{\binom{k}{p}}{\binom{k}{q}} \rho_{j+1}^{q-p} \,\middle|\, p \in \{0, \ldots, a(\Lambda) - 1\} \right\}$. Since $\binom{k}{p}$ is a positive integer-valued polynomial in k for any p, then $t(k)$ is some positive rational function of k. Since $\binom{k}{q} \geq 1$ for all $k \geq 1$, then $t(k) \leq C_t k^m$ for some $m \in \mathbb{N}$ and constant C_t. Additionally, $\sum_{i=j+1}^{R} \left(\sum_{p=0}^{a(\Lambda_i)-1} \left(\sum_{\lambda \in \Lambda_{i,p}} \|\Omega_{i,\lambda,p}(f)\|_{\mathscr{B}} \right) \right) =: C_{j+1} < \infty$. Therefore,

$$\begin{aligned}
&\left\| \frac{1}{n} \sum_{k=a(\Lambda)}^{n-1} \sum_{i=j+1}^{R} \sum_{p=0}^{a(\Lambda_i)-1} \sum_{\lambda \in \Lambda_{i,p}} \frac{\binom{k}{p}}{\binom{k}{q}} \frac{\lambda^{k-p}}{\mu^{k-q}} \Omega_{i,\lambda,p}(f) \right\|_{\mathscr{B}} \\
&\qquad \leq C_t C_{j+1} a(\Lambda) \frac{n - a(\Lambda)}{n} \frac{1}{n - a(\Lambda)} \sum_{k=a(\Lambda)}^{n-1} k^m \left(\frac{\rho_{j+1}}{\rho_j} \right)^{k-q}.
\end{aligned}$$

Since $\rho_{j+1} < \rho_j$, then $k^m \left(\frac{\rho_{j+1}}{\rho_j} \right)^{k-q}$ converges exponentially fast to zero as k goes to ∞. Therefore, there exists N_3 such that

$$\left\| \frac{1}{n} \sum_{k=a(\Lambda)}^{n-1} \sum_{i=j+1}^{R} \sum_{p=0}^{a(\Lambda_i)-1} \sum_{\lambda \in \Lambda_{i,p}} \frac{\binom{k}{p}}{\binom{k}{q}} \frac{\lambda^{k-p}}{\mu^{k-q}} \Omega_{i,\lambda,p}(f) \right\|_{\mathscr{B}} < \frac{\varepsilon}{5} \tag{IV.3.35}$$

for all $n \geq N_3$.

Taking the average of the left side of (IV.3.28), equations (IV.3.33), (IV.3.34), and (IV.3.35) can be used to show that

$$\left\| \frac{1}{n} \sum_{k=a(\Lambda)}^{n-1} \binom{k}{q}^{-1} \mu^{-k} U_{j,q}^{k}(f) - \Omega_{j,\mu,q}(f) \right\|_{\mathscr{B}} < \frac{4\varepsilon}{5} \tag{IV.3.36}$$

for all $n \geq \max(N_1, N_2, N_3)$.

Finally, consider

$$\begin{aligned} &\left\| \frac{1}{n} \sum_{k=0}^{n-1} \binom{k}{q}^{-1} \mu^{-k} U_{j,q}^{k}(f) - \Omega_{j,\mu,q}(f) \right\|_{\mathscr{B}} \\ &\quad \leq \left\| \frac{1}{n} \sum_{k=0}^{a(\Lambda)-1} \binom{k}{q}^{-1} \mu^{-k} U_{j,q}^{k}(f) \right\|_{\mathscr{B}} + \left\| \frac{1}{n} \sum_{k=a(\Lambda)}^{n-1} \binom{k}{q}^{-1} \mu^{-k} U_{j,q}^{k}(f) - \Omega_{j,\mu,q}(f) \right\|_{\mathscr{B}} \end{aligned} \tag{IV.3.37}$$

There exists $N_4 \geq \max(N_1, N_2, N_3)$ such that

$$\left\| \frac{1}{n} \sum_{k=0}^{a(\Lambda)-1} \binom{k}{q}^{-1} \mu^{-k} U_{j,q}^{k}(f) \right\|_{\mathscr{B}} < \frac{\varepsilon}{5}. \tag{IV.3.38}$$

for all $n \geq N_4$. Using (IV.3.36) and (IV.3.38) in (IV.3.37) gives

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} \binom{k}{q}^{-1} \mu^{-k} U_{j,q}^{k}(f) - \Omega_{j,\mu,q}(f) \right\|_{\mathscr{B}} < \varepsilon$$

for all $n \geq N_4$. □

Remark IV.3.15. *The Laplace averages, (IV.3.27), are not projection operators in general since $U_{j,q}$ is defined in terms of the nilpotent operators $\Omega_{i,\lambda,p}$. Furthermore, the construction of the averaging operators fails if the algebraic eigenspace has infinite multiplicity.* □

IV.4 Some comments on Schauder bases

Bases in finite dimensional vector spaces are a very useful concept. In the finite dimensional case, the spectral decomposition of an operator is equivalent to finding a basis of eigenvectors for the space. The concept of basis extends to infinite dimensional vector spaces as well. These (Hamel) bases are usually uncountable and their existence depends on Zorn's lemma. The more useful concept in Banach spaces is that of a Schauder basis.

Let $\mathscr{B}$ be a Banach space. A sequence of elements $\{e_n\}_{n=1}^{\infty}$ is called a Schauder basis if for every $x \in \mathscr{B}$ there is a unique set of scalars $\{x_n\}_1^{\infty}$ such that

$$x = \sum_{n=1}^{\infty} x_n e_n. \tag{IV.4.1}$$

Equivalently, for every $\varepsilon > 0$, there is an $M \in \mathbb{N}$ such that

$$\left\| x - \sum_{n=1}^{N} x_n e_n \right\| < \varepsilon \tag{IV.4.2}$$

for all $N \geq M$. Necessarily, for a Banach space to have a basis, it must be separable. There are, however, examples of separable reflexive Banach spaces that do not have a Schauder basis [Enf73]. The convergence implies that $\|x\|_0 := \sup_{N\geq 1} \left\| \sum_{n=1}^{N} x_n e_n \right\|$ is finite. It is also a norm on $\mathscr{B}$, and it can be shown equivalent to $\|\cdot\|$;

$$\|x\| \leq \|x\|_0 \leq K \|x\|.$$

The smallest K satisfying this relation is called the basis constant and it is a measure of the "orthogonality" of the basis elements. This orthogonality is in the Banach space sense rather than the Hilbert space sense which would depend on an inner product. The equivalence of these norms shows that the coordinate projections $P_n(x) = x_n e_n$ are a uniformly bounded family.

In the original formulation of the GLA theorems in the non spectral case, the constructions relied upon the notion of a Schauder basis. Unfortunately, pursuing this line of thought leads to problems. We assumed that $\mathscr{B}$ had a Schauder basis and implicitly assumed that every infinite dimensional subspace, and in particular sums of eigenspaces, had a basis, which unfortunately cannot be guaranteed even in nice spaces. For example, each ℓ^p sequence space $(1 \leq p < \infty)$ has a basis. However, for every $p \neq 2$, the sequence space contains an infinite dimensional subspace that has no basis [Jam82]. This is in direct contradiction to our intuition in Hilbert spaces.

However, it can be shown that every infinite dimensional Banach space has an infinite dimensional subspace possessing a Schauder basis. This is shown by constructing an "almost orthogonal" sequence of unit norm vectors in $\mathscr{B}$. These vectors then are a basis for their closed linear span and are called a basic sequence. The construction relies on the following result.

Lemma IV.4.1. *Let $\mathscr{B}$ be an infinite dimensional (not necessarily separable) Banach space and F a finite dimensional subspace of $\mathscr{B}$. Let $\varepsilon > 0$. Then there exists an $x \in \mathscr{B}$ such that*

(i) $\|x\| = 1$

(ii) for all $y \in F$ and all scalars α, we have $\|y\| \leq (1 + \varepsilon) \|y + \alpha x\|$.

Such a vector $x \in \mathscr{B}$ is called "almost orthogonal" to F. A basic sequence is constructed by starting with some x_1 of unit norm and using the above procedure lemma to construct an almost orthogonal x_2. The process is then repeated to get $\{x_n\}_{n\in\mathbb{N}}$. Performed with a certain sequence of $\{\varepsilon_n\}_{n\in\mathbb{N}}$, the resulting sequence can be constructed to have a basis constant of 2.

While this is a nice result (even if the entire space cannot be recovered in this way), for the proofs of the GLA theorems we often need to switch the order of the summation. This is equivalent to permuting the basis. Unfortunately, it can very well be the case that a permutation of a basis fails to be a basis. Therefore, what we would really want is what is called an unconditional basis. These are bases that continue to be bases under any permutation. A slightly stronger property is that expansions in the basis are absolutely summable. If a space has an absolutely summable basis, it is isomorphic to an ℓ^1-space (compare this with the construction of the absolutely summable subspace in section IV.1). Thus we are led to the question of whether every infinite dimensional space has an infinite dimensional subspace with an unconditional basis; i.e., there always exists an unconditional basic sequence. This was a long standing conjecture in Banach space theory [Jam82]. It was finally proved in the negative by Gowers and Maurey [GM93]. Those authors constructed a separable reflexive Banach space containing no unconditional basis sequences. The space as exhibited the property that was termed hereditarily indecomposable. What this means is that if $X = Y \oplus Z$ is a topological direct sum (the projections from $X \mapsto Y$ and

$X \mapsto Z$ are bounded), then Y and Z cannot both be infinite dimensional. Stated in another way, every infinite dimensional subspace is arbitrarily close to every other infinite dimensional subspace.

It was also shown in [GM93] that the only bounded operators were of the form $U = \lambda I + S$, where S is a strictly singular operator. S is strictly singular if its restriction to any infinite dimensional subspace fails to be an isomorphism; the restriction fails to injective on its range. The only possible eigenvalue is λ and eigenvectors must be in the kernel of S. For example, let y and z be points such that $Sy = Sz$. Then $x = y - z$ is in the kernel of S and an eigenvector of U;

$$Ux = U(y - z) = \lambda x + S(y - z) = \lambda x + Sy - Sz = \lambda x.$$

Therefore, if U is a bounded operator and its point spectrum has more than 1 point in it, then the sum of its eigenspaces must contain an unconditional basic sequence. On the other hand, if we allow an unbounded operator, then there are examples where the eigenspaces do not have an unconditional basis (see example IV.1.1).

Chapter V

Dynamical Systems and the Koopman Operator

[1]This chapter connects the abstract results of the previous section with dynamical systems. The operator we are interested in is the Koopman operator associated with a dynamical system $(\mathscr{X}, \Phi)$, where $\Phi : \mathscr{X} \to \mathscr{X}$ (we have yet to put any structure on Φ). Recall that the Koopman operator is the composition operator $U_\Phi \psi = \psi \circ \Phi$, for all ψ in some space of functions $\mathscr{F}$. In particular, we will show that the conditions on the set of eigenvalues Λ (namely, it being a bounded set and concentrated on isolated circles) are natural and are satisfied for the examples we consider.

For problems on the attractor, it is natural to consider the space $L^2(\nu)$, where ν a probability measure supported on the attractor that is preserved by Φ. This case

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has been treated extensively in the literature. However, this space is not particularly natural for dissipative dynamics. For example, consider a dynamical system on the real line with the origin as a globally attracting fixed point. The invariant measure for this system is a delta measure supported at zero, in which case, the associated Hilbert space of observables, $L^2(\mathbb{R}, \delta_0)$, is isomorphic to $\mathbb{R}$; every function agreeing at zero is equivalent regardless of their values away from zero. The space $L^2(\mathbb{R}, \delta_0)$ cannot give any information about the dynamics away from the attractor. As seen in example V.1.4 below, natural observables for dissipative dynamics are polynomials and their completions under certain norms.

The first section of this chapter is devoted to some algebraic and topological properties of eigenfunctions of the Koopman operator. The first main result (thm. V.1.1) states basically that eigenfunctions of the Koopman operator form semigroups under pointwise products (as long as the products belong to $\mathscr{F}$) and monoids if $\mathscr{F}$ contains the constant functions. The second main result (thm. V.1.8) is a relation between the eigenfunctions of Koopman operators associated with topologically conjugate systems. If an eigenfunction is found for one of the systems, then an eigenfunction for the other system can be obtained by composing with the conjugacy map.

In section V.2, the results of section V.1 are used to construct what we call the $\mathscr{B}$-Hardy spaces $H^2_{\mathscr{B}}$ that arise naturally for dissipative dynamical systems. These spaces can loosely be considered as generalizations of the familiar Hilbert Hardy spaces in the disc $H^2(\mathbb{D})$. On these $\mathscr{B}$-Hardy spaces, the Koopman operator is a spectral operator of scalar-type. First, polynomials over normed commutative rings are defined since they arise naturally from principle eigenfunctions of the Koopman operator. It is also shown that under a completion with an ℓ^2 polynomial norm that the completed space is locally sequentially weakly compact as is required to apply the GLA theorems. This holds true as long as the Banach space in which the ring is dense is separable and reflexive. This result is contained in proposition V.2.7. Sections V.2.2 and V.2.3 construct eigenfunctions and spaces of observables for attracting hyperbolic fixed points in $\mathbb{C}^d$ and limit cycles in $\mathbb{R}^2$, respectively. Topological conjugacies are leveraged to use eigenfunctions corresponding to the linearized dynamics to construct eigenfunctions and spaces of observables for the nonlinear system. It is shown that these spaces are identifiable with spaces of polynomials over normed unital commutative rings. Spectral measures for the systems are constructed that satisfy the properties assumed in section III.2.1. Additionally, a short remark connects the space of observables of an attracting fixed point for the $d = 1$ case with Hardy Hilbert spaces in the unit disc $H^2(\mathbb{D})$ that are often studied in the context of composition operators with analytic symbols defined in the unit disc. Loosely, the spaces we construct can be viewed as generalizations of these spaces.

V.1 Algebraic and topological properties of Koopman eigenfunctions.

The first main result of this section states basically that eigenfunctions of the Koopman operator form semigroups under pointwise products (as long as the products belong to $\mathscr{F}$) and monoids if $\mathscr{F}$ contains the constant functions. The second main result is a relation between the eigenfunctions of Koopman operators associated with topologically conjugate systems. If an eigenfunction is found for one of the systems, then an eigenfunction for the other system can be obtained by composing with the conjugacy map.

Theorem V.1.1 (Semigroup of eigenfunctions)**.** *Let $\mathscr{F}$ be a space of observables for the dynamical system $(\mathcal{M}, \Phi)$ and let $U_\Phi : \mathscr{F} \to \mathscr{F}$ be the associated Koopman operator.*

(i) Let φ and ψ be two eigenfunctions of U_Φ at eigenvalues λ and μ, respectively. If the pointwise product $\varphi \cdot \psi$ is in $\mathscr{F}$ and nonzero, then $\varphi \cdot \psi$ is an eigenfunction of U_Φ at eigenvalue $\lambda\mu$.

$$U_\Phi \varphi = \lambda \varphi \quad \text{and} \quad U_\Phi \psi = \mu \psi \qquad \Longrightarrow \qquad U_\Phi(\varphi \cdot \psi) = \lambda\mu(\varphi \cdot \psi). \quad \text{(V.1.1)}$$

(ii) Let $\mathcal{G}$ be a set of nonzero eigenpairs (λ, φ) of U_Φ with the property that any finite subset $\{\varphi_1, \dots, \varphi_m\} \subset \mathcal{G}$ satisfies $\prod_{i=1}^m \varphi_i = \varphi_1 \cdots \varphi_m \neq 0$ and all finite pointwise products of elements of $\mathcal{G}$ are in $\mathscr{F}$. Then $\mathcal{G}$ is a generator for a

semigroup of eigenfunctions of U_Φ, which will be denoted by $\mathcal{P}_{\mathcal{G}}$. If the constant functions are in $\mathscr{F}$, then $\mathcal{P}_{\mathcal{G}}$ is a monoid.

Proof. For any $f \in \mathscr{F}$, define $\mathcal{S}_f = \{x \in X \mid |f(x)| > 0\}$. Let $(\lambda, \varphi) \in \mathcal{G}$. Then $\mathcal{S}_{\varphi^n} = \mathcal{S}_\varphi$ for all $n \in \mathbb{N}$. Therefore, $\varphi \neq 0$ if and only if $\varphi^n \neq 0$ for all $n \in \mathbb{N}$.

To show that $(\lambda\mu, \varphi \cdot \psi)$ is an eigenpair of U_Φ when both (λ, φ) and (μ, ψ) are elements of $\mathcal{G}$, compute

$$U_\Phi(\varphi \cdot \psi)(x) = \varphi(\Phi(x)) \cdot \psi(\Phi(x)) = \lambda\varphi(x)\mu\psi(x) = \lambda\mu(\varphi \cdot \psi)(x).$$

Note that $\varphi \cdot \psi \neq 0$ by hypothesis on $\mathcal{G}$. Therefore, $(\lambda\mu, \varphi \cdot \psi)$ is a nonzero eigenpair of U_Φ and, in particular, (λ^n, φ^n) is a nonzero eigenpair of U for all $n \in \mathbb{N}$.

Now let $(\lambda_i, \varphi_i)_1^m \subset \mathcal{G}$, $(n_1, \ldots, n_m) \in \mathbb{N}^m$, and consider $h = \prod_{i=1}^m \varphi_i^{n_i}$. We claim that h is a nonzero eigenfunction of U. That h is nonzero follows from hypothesis on $\mathcal{G}$; i.e., $\mathcal{S}_{\varphi_i^{n_i}} = \mathcal{S}_{\varphi_i}$, any finite product of elements of $\mathcal{G}$ are nonzero, and hence $\varphi_1^{n_1} \cdots \varphi_m^{n_m} \neq 0$. That h is an eigenfunction follows from

$$Uh(x) = U(\prod_{i=1}^m \varphi_i^{n_i})(x) = \prod_{i=1}^m \varphi_i^{n_i}(Tx). = \prod_{i=1}^m \lambda_i^{n_i}\varphi_i^{n_i}(x) = (\prod_{i=1}^m \lambda_i^{n_i})h(x).$$

The semigroup generated by $\mathcal{G}$ is given by

$$\mathcal{P}_{\mathcal{G}} = \left\{ \left(\prod_{i=1}^m \lambda_i^{n_i}, \prod_{i=1}^m \varphi_i^{n_i} \right) \middle| (\lambda_i, \varphi_i) \subset \mathcal{G}, m \in \mathbb{N}, n_i \in \mathbb{N} \right\}.$$

By the previous work, elements of $\mathcal{P}_{\mathcal{G}}$ are nonzero eigenfunctions of U. □

Corollary V.1.2. *Let (λ, φ) be a nonzero eigenpair of U_Φ such that for any $m \in \mathbb{N}_0$, the m-term pointwise product φ^m is in $\mathscr{F}$. Let $\mathcal{G} = \{(\lambda, \varphi)\}$. Then, letting $\mathcal{P}_\varphi$ denote* *the semigroup generated by* $\mathcal{G}$*,*

$$\mathcal{P}_\varphi = \{(\lambda^n, \varphi^n) \mid n \in \mathbb{N}_0\}\,. \tag{V.1.2}$$

We note that if we considered generalized eigenfunctions, then these results do not hold (see example V.1.6).

Definition V.1.3 (Principle eigenfunctions)**.** Let E be a semigroup of eigenfunctions of the Koopman operator. Let $\mathcal{G}$ be a minimal generator for E. The elements of $\mathcal{G}$ are called *principle eigenfunctions of* U_Φ *in* E.

The principle eigenfunctions are not necessarily unique. For example, consider a Koopman operator with eigenvalue $\lambda = e^{i2\pi p/q}$ with $q \geq 3$ and $\varphi \neq 1$ the associated eigenfunction. Then $\lambda^{q+1} = \lambda$ and therefore $\varphi^{q+1} = \varphi$ and $\varphi^q \equiv 1$. Then $\mathcal{P}_\varphi$ is a cyclic group of finite order. It can easily be shown that $\varphi^{q-1} = \varphi^{-1}$ is also a generator for $\mathcal{P}_\varphi$ and, since $q \geq 3$ and $\varphi \neq 1$, $\varphi^{q-1} \neq \varphi$. Therefore, both φ and φ^{q-1} are principle eigenfunctions of U in $E = \{1, \varphi, \varphi^2, \ldots, \varphi^{q-1}\}$.

Example V.1.4 shows that analytic observables are natural for dissipative systems.

Example V.1.4. *Let* $X = \mathbb{R}$ *and* $T(x) = \lambda x$*, where* $0 < \lambda < 1$*. Let* $\mathscr{F}_X = C(X)$ *and* $\varphi(x) := x$*. Then*

$$(U_T\varphi)(x) = \varphi(\lambda x) = \lambda x = \lambda\varphi(x).$$

Hence φ *is an eigenfunction of* U_T *at eigenvalue* λ*. By corollary V.1.2, any finite product* φ^m *is also an eigenfunction of* U_T *at eigenvalue* λ^m*. This can easily be verified* *by the computation*

$$(U_T\varphi^m)(x) = \varphi^m(\lambda x) = (\lambda x)^m = \lambda^m\varphi^m(x).$$

The eigenfunction φ is the principle eigenfunction of the system.

Let f be the analytic function

$$f(x) = \sum_{m=0}^{\infty} c_m(f)\varphi^m(x),$$

where $c_m(f) = \frac{f^{(m)}(0)}{m!}$. The partial sums $f_n(x) = \sum_{m=0}^{n} c_m(f)\varphi^m$ converge to f uniformly on compact sets. Therefore, the natural observables to take for this system are the analytic functions. □

Example V.1.5 (Koopman eigenfunctions corresponding to a finite-dimensional, diagonalizable linear system)**.** *Let $Y = \mathbb{C}^d$ with the Euclidean norm and A a $d \times d$ diagonalizable matrix. Define the dynamics as $\boldsymbol{y}_{k+1} = \mathsf{A}\boldsymbol{y}_k$. Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\}$ be a basis of eigenvectors of A corresponding to nonzero eigenvalues $\{\lambda_j\}_1^d$. Then $\boldsymbol{y} = \sum_1^d c_j\boldsymbol{v}_j$.*

Let $\{\boldsymbol{w}_j\}_{j=1}^d$ be the adjoint basis to $\{\boldsymbol{v}_j\}_{j=1}^d$; then $\langle \boldsymbol{v}_j, \boldsymbol{w}_k\rangle = \delta_{jk}$ and $\boldsymbol{w}_j$ is an eigenvector of A^* at eigenvalue $\bar{\lambda}_j$. Define observables by the linear functionals

$$\varphi_j(\boldsymbol{y}) = \langle \boldsymbol{y}, \boldsymbol{w}_j\rangle \tag{V.1.3}$$

for all $\boldsymbol{y}$ and $j = 1, \ldots, d$.

The action of the Koopman operator on φ_j is

$$(U_{\mathsf{A}}\varphi_j)(\boldsymbol{y}) = \varphi_j(\mathsf{A}\boldsymbol{y}) = \langle \mathsf{A}\boldsymbol{y}, \boldsymbol{w}_j\rangle = \langle \boldsymbol{y}, \mathsf{A}^*\boldsymbol{w}_j\rangle = \left\langle \boldsymbol{y}, \bar{\lambda}_j\boldsymbol{w}_j\right\rangle = \lambda_j\varphi_j(\boldsymbol{y}).$$

Hence φ_j is a nonzero eigenfunction of U_{A}. Additionally, the product $(\varphi_1 \cdots \varphi_d)$ is not equivalently the zero functional. Therefore, theorem V.1.1 implies that for any $(m_1, \ldots, m_d) \subset \mathbb{N}_0^d$,

$$\left(\prod_{j=1}^{d} \lambda_j^{m_j}, \prod_{j=1}^{d} \varphi_j^{m_j} \right) \tag{V.1.4}$$

is an eigenpair for the corresponding Koopman operator. □

Example V.1.6 (Koopman eigenfunctions of a finite-dimensional, linear system having generalized eigenvectors)**.** *Let $\mathsf{A} : \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator and let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be a basis of generalized eigenvectors satisfying*

$$\begin{aligned}
(\mathsf{A} - \lambda I)\boldsymbol{v}_1 &= \mathbf{0}, \\
(\mathsf{A} - \lambda I)\boldsymbol{v}_j &= \boldsymbol{v}_{j-1}, \quad (j \geq 2)
\end{aligned}$$

where $0 \neq \lambda \in \mathbb{C}$. Let $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n\}$ be the adjoint basis corresponding to $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$; i.e, $\langle \boldsymbol{v}_i, \boldsymbol{w}_j \rangle = \delta_{i,j}$. It can easily be shown that

$$\begin{aligned}
(\mathsf{A}^* - \bar{\lambda} I)\boldsymbol{w}_n &= \mathbf{0}, \\
(\mathsf{A}^* - \bar{\lambda} I)\boldsymbol{w}_j &= \boldsymbol{w}_{j+1}, \quad (j < n),
\end{aligned}$$

where A^* is the adjoint to A and $\bar{(\cdot)}$ denotes complex conjugation. Let U_{A} be the Koopman operator corresponding to A and define $\varphi_j : \mathbb{C}^n \to \mathbb{C}$ by $\varphi_j(\boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{w}_j \rangle$. Then

$$U_{\mathsf{A}}\varphi_n(\boldsymbol{x}) = \varphi_n(\mathsf{A}\boldsymbol{x}) = \langle \mathsf{A}\boldsymbol{x}, \boldsymbol{w}_n \rangle = \langle \boldsymbol{x}, \mathsf{A}^*\boldsymbol{w}_n \rangle = \left\langle \boldsymbol{x}, \bar{\lambda}\boldsymbol{w}_n \right\rangle = \lambda \left\langle \boldsymbol{x}, \boldsymbol{w}_n \right\rangle = \lambda\varphi_n(\boldsymbol{x}).$$

Additionally, for $1 \le j < n$,

$$U_{\mathsf{A}}\varphi_j(\boldsymbol{x}) = \langle \boldsymbol{x}, \mathsf{A}^*\boldsymbol{w}_j \rangle = \left\langle \boldsymbol{x}, \bar{\lambda}\boldsymbol{w}_j + \boldsymbol{w}_{j+1} \right\rangle = \lambda \langle \boldsymbol{x}, \boldsymbol{w}_j \rangle + \langle \boldsymbol{x}, \boldsymbol{w}_{j+1} \rangle$$
$$= \lambda\varphi_j(\boldsymbol{x}) + \varphi_{j+1}(\boldsymbol{x}).$$

Therefore $\{\varphi_1, \ldots, \varphi_n\}$ *are generalized eigenfunctions of the Koopman operator satisfying*

$$\begin{aligned} (U - \lambda I)\varphi_n &= 0, \\ (U - \lambda I)\varphi_j &= \varphi_{j+1}, \quad (1 \le j < n). \end{aligned} \tag{V.1.5}$$

We note that contrary to the case of products of eigenfunctions, products of generalized eigenfunctions are not necessarily generalized eigenfunctions. That is, the set of generalized eigenfunctions is not closed under pointwise products. □

Definition V.1.7 (Pullback space of observables)**.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be locally compact metric spaces and $(\mathcal{M}_1, \Phi_1)$ and $(\mathcal{M}_2, \Phi_2)$ be continuous dynamical systems. Assume Φ_1 and Φ_2 are topologically conjugate under the homeomorphism $h : \mathcal{M}_1 \to \mathcal{M}_2$; i.e., $\Phi_1 = h^{-1} \circ \Phi_2 \circ h$. Assume $U_{\Phi_2} : \mathscr{F}_{\mathcal{M}_2} \to \mathscr{F}_{\mathcal{M}_2}$. Then

$$\mathscr{F}_{\mathcal{M}_1} := \mathscr{F}_{\mathcal{M}_2} \circ h = \{f \circ h \mid f \in \mathscr{F}_{\mathcal{M}_2}\} \tag{V.1.6}$$

is called the pullback space of observables.

Theorem V.1.8 (Eigenfunctions of topologically conjugate dynamics)**.** *Let* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be locally compact metric spaces. Let* $\Phi_1 : \mathcal{M}_1 \to \mathcal{M}_1$ *and* $\Phi_2 : \mathcal{M}_2 \to \mathcal{M}_2$ *be continuous maps that are topologically conjugate under the homeomorphism* h :

$$
\begin{array}{ccc}
\mathcal{M}_1 & \xrightarrow{\Phi_1} & \mathcal{M}_1 \\
{\scriptstyle h}\big\downarrow & & \big\uparrow{\scriptstyle h^{-1}} \\
\mathcal{M}_2 & \xrightarrow{\Phi_2} & \mathcal{M}_2
\end{array}
$$

Figure V.1: The commutative diagram corresponding to the spaces in definition V.1.7 and theorem V.1.8.

$\mathcal{M}_1 \to \mathcal{M}_2$; i.e., $\Phi_1 = h^{-1} \circ \Phi_2 \circ h$. Let $U_{\Phi_2} : \mathscr{F}_{\mathcal{M}_2} \to \mathscr{F}_{\mathcal{M}_2}$ be the Koopman operator associated with the dynamical system $(\mathcal{M}_2, \Phi_2)$, where $\mathscr{F}_{\mathcal{M}_2} \subset C(Y, \mathbb{C})$. Let $\mathscr{F}_{\mathcal{M}_2} \subset C(X, \mathbb{C})$ and $U_{\Phi_1} : \mathscr{F}_{\mathcal{M}_1} \to \mathscr{F}_{\mathcal{M}_1}$.

Then if $\lambda \in \mathbb{C}$ is an eigenvalue of U_{Φ_2} having finite algebraic multiplicity and $\varphi \in \mathscr{F}_{\mathcal{M}_2}$ is an associated generalized eigenfunction, then $\varphi \circ h$ is a generalized eigenfunction of U_{Φ_1} if $\varphi \circ h \in \mathscr{F}_{\mathcal{M}_1}$ with eigenvalue λ.

Remark V.1.9. *The theorem continues to hold if we have continuous dynamical systems and associated semigroups of Koopman operators.* □

Proof. By definition $\lambda \in \mathbb{C}$ has finite algebraic multiplicity if there is some $m < \infty$ such that $N\left((U_{\Phi_2} - \lambda I)^m\right) = N\left((U_{\Phi_2} - \lambda I)^{m+1}\right)$. Let $\varphi \in \mathscr{F}_{\mathcal{M}_2}$ be a generalized eigenfunction of U_{Φ_2} corresponding to λ; i.e., $(U_{\Phi_2} - \lambda I)^m \varphi(y) = 0$, for all $y \in \mathcal{M}_2$.

Assume $\varphi \circ h$ is in $\mathscr{F}_{\mathcal{M}_1}$. We claim that $(U_{\Phi_1} - \lambda I)^m (\varphi \circ h)(x) = 0$ for all $x \in \mathcal{M}_1$.

Fix $x \in \mathcal{M}_1$ and let $y = h(x)$. The claim follows from the chain of equalities:

$$\begin{aligned}
(U_{\Phi_1} - \lambda I)^m (\varphi \circ h)(x) &= \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} U_S^k (\varphi \circ h)(x) \\
&= \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} (\varphi \circ h \circ \Phi_1^k)(x) \\
&= \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} (\varphi \circ \Phi_2^k \circ h)(x) \\
&= \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} (\varphi \circ \Phi_2^k)(y) \\
&= \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} U_{\Phi_2}^k (\varphi)(y) \\
&= (U_{\Phi_2} - \lambda I)^m \varphi(y) \\
&= 0.
\end{aligned}$$

Therefore, $\varphi \circ h$ is a generalized eigenfunction of U_S. □

Example V.1.10. *Consider the differential equation*

$$\begin{aligned}
\dot{x} &= -x + x^3 \\
\dot{y} &= -y,
\end{aligned}$$

$(x, y) \in \mathbb{R}^2$. *The system has equilibrium points at* $(x^*, y^*) = (0,0), (\pm 1, 0)$, *where* $(0,0)$ *is asymptotically stable with basin of attraction* $D = (-1,1) \times \mathbb{R}$. *The solution of the nonlinear system is*

$$\begin{aligned}
x(t) &= \frac{x_0}{\sqrt{(1 - x_0^2)e^{2(t-t_0)} + x_0^2}} \\
y(t) &= y_0 e^{-(t-t_0)}
\end{aligned} \tag{V.1.7}$$

and the solution to the linearization about $(0,0)$ *is*

$$u(t) = u_0 e^{-(t-t_0)}$$
$$v(t) = v_0 e^{-(t-t_0)} \tag{V.1.8}$$

Fix a time-step $0 < \Delta t \ll 1$ *and put* $\lambda = e^{-\Delta t} < 1$*. Then defining* $x_k := x(k\Delta t)$ *(and similarly for* y*), we obtain the nonlinear map* $T = (T_1, T_2) : \mathbb{R}^2 \to \mathbb{R}^2$,

$$T_1^k(x_0) = x_k = \frac{\lambda^k x_0}{\sqrt{(1-x_0^2) + \lambda^{2k} x_0^2}}$$
$$T_2^k(y_0) = y_k = y_0 \lambda^k.$$

and the linear map $\mathsf{A} = (\mathsf{A}_1, \mathsf{A}_2) : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathsf{A}_1^k u_0 = u_k = \lambda^k u_0$$
$$\mathsf{A}_2^k v_0 = v_k = \lambda^k v_0.$$

A homeomorphism $h : D \to \mathbb{R}^2$ *that makes the nonlinear and linear maps,* T *and* A*, topologically conjugate can be found using equations* (V.1.7) *and* (V.1.8)*. Let* $h(x,y) = (h_1(x,y), h_2(x,y)) = (u,v)$*. Note that since the dynamics for* y *are already linear we can identity* y *and* v *everywhere so that* $v = h_2(x,y) = y$*. To get* $h_1(x,y)$*, we identify* $x_0 = u_0 = r$ *for* $0 < r$ *very close to the origin, find the homeomorphism, and then let* $r \to 0$*. For any fixed* $r > 0$*, we get the map*

$$u = h_{1,r}(x,y) = x\sqrt{\frac{1-r^2}{1-x^2}}.$$

Then

$$(u,v) = h(x,y) = \left(\frac{x}{\sqrt{1-x^2}}\,,\, y\right) \tag{V.1.9}$$

makes the dynamics topologically conjugate

According to corollary V.1.2, the eigenfunctions of the linearized Koopman operator are $\varphi(u,v) = u^m v^n$, with eigenvalues λ^{m+n}, for $m,n \in \mathbb{N}_0$. Therefore, the eigenfunctions for the nonlinear system are of the form

$$\psi(x,y) = (\varphi \circ h)(x,y) = \varphi\left(\frac{x}{\sqrt{1-x^2}}, y\right) = \frac{x^m y^n}{(1-x^2)^{m/2}} \tag{V.1.10}$$

and have the eigenvalues λ^{m+n}. □

Example V.1.11. *This example shows using theorem V.1.8 in the case when the linearized dynamics posses generalized eigenvectors. Let $\boldsymbol{x} = (x_1, \ldots, x_4) \in \mathbb{R}^+ \times [0,1) \times \mathbb{R}^+ \times [-\pi,\pi)$, and consider the discrete-time dynamical systems given by*

$$S(\boldsymbol{x}) = \begin{bmatrix} f_1(x_1) \\ f_2(x_2) \\ f_3(x_3,x_4) \\ f_4(x_4) \end{bmatrix} = \begin{bmatrix} x_1 + \frac{1}{2}(1-x_1) \\ x_2 + \omega, \quad \text{mod } 1 \\ x_3[\lambda^2 + 2\lambda\cos(x_4)\sin(x_4) + \sin^2(x_4)]^{1/2} \\ \arctan\left(\dfrac{\lambda\sin(x_4)}{\lambda\cos(x_4)+\sin(x_4)}\right) \end{bmatrix}, \tag{V.1.11}$$

where $\omega \in (\mathbb{R}/\mathbb{Z}) \setminus \mathbb{Q}$ and $0 < \lambda < 1$. This system has a globally attracting limit cycle in the complex plane defined by the first two coordinates ($z = x_1 \exp(i2\pi x_2) \in \mathbb{C}$). Consider the function

$$h(\boldsymbol{x}) = h(x_1,x_2,x_3,x_4) = \begin{bmatrix} x_1 \\ x_2 \\ x_3\cos x_4 \\ x_3 \sin x_4 \end{bmatrix} \tag{V.1.12}$$

and the dynamical system

$$T(\boldsymbol{y}) = \begin{bmatrix} y_1 + \frac{1}{2}(1-y_1) \\ y_2 + \omega, \quad \mod 1 \\ \lambda y_3 + y_4 \\ \lambda y_4 \end{bmatrix} = \begin{bmatrix} y_1 + \frac{1}{2}(1-y_1) \\ y_2 + \omega, \quad \mod 1 \\ \boldsymbol{g}(y_3, y_4) \end{bmatrix} \tag{V.1.13}$$

where $\boldsymbol{y} = (y_1, \ldots, y_4) \in \mathbb{R} + \times [0,1) \times \mathbb{R}^2$ *and* $\boldsymbol{g}(y_3, y_4) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} y_3 \\ y_4 \end{bmatrix}$. *Then*

$$(h \circ S)(\boldsymbol{x}) = (T \circ h)(\boldsymbol{x}).$$

Additionally, $\boldsymbol{v}_3 = (0,0,1,0)$ *is an eigenvector of* T *and* $\boldsymbol{v}_4 = (0,0,0,1)$ *is a generalized eigenvector of* T *satisfying* $T\boldsymbol{v}_4 = \lambda \boldsymbol{v}_4 + \boldsymbol{v}_3$. *Following example V.1.6, we form the adjoint basis, which in this case is just the canonical basis (e.g.* $\boldsymbol{e}_3 = (0,0,1,0)$*). Then* $\varphi_3(\boldsymbol{y}) := \langle \boldsymbol{y}, \boldsymbol{e}_3 \rangle$ *and* $\varphi_4(\boldsymbol{y}) := \langle \boldsymbol{y}, \boldsymbol{e}_4 \rangle$ *are generalized eigenfunctions of* U_T *(see example V.1.6). Since* S *and* T *are conjugate through* h*, then theorem V.1.8 guarantees that* $\varphi_i \circ h$ *(*$i = 3,4$*) are generalized eigenfunctions of* U_S. *Using* (V.1.12),

$$(\varphi_3 \circ h)(\boldsymbol{x}) = \langle h(\boldsymbol{x}), \boldsymbol{e}_3 \rangle = \left\langle \begin{bmatrix} x_1 \\ x_2 \\ x_3 \cos x_4 \\ x_3 \sin x_4 \end{bmatrix}, \boldsymbol{e}_3 \right\rangle = x_3 \cos x_4$$

and similarly $(\varphi_4 \circ h)(\boldsymbol{x}) = x_3 \sin x_4$. □

V.2 Function spaces for dissipative dynamical systems

Assume that $\Phi : \mathcal{M} \to \mathcal{M}$ has $\ell < \infty$ attracting fixed points $\{x_j \mid j = 1, \ldots, \ell\}$ and let D_j be the basin of attraction of x_j, where the basin of attraction is defined as the largest open set in X such that $\Phi^k(x) \to x_j$ for all $x \in D_j$. Then $D_i \cap D_j = \emptyset$ for $i \neq j$ and we can consider the new space $D = D_1 \cup \cdots \cup D_\ell$ and the restricted dynamical system $(D, \Phi|D)$. A space of observables on B, can be given by the direct sum

$$\mathscr{F}_B = \mathscr{F}_{B_1} \oplus \cdots \oplus \mathscr{F}_{B_\ell} \subset C(B, \mathbb{C}). \tag{V.2.1}$$

We have yet to specify exactly what the $\mathscr{F}_{B_j}$'s look like.

Remark V.2.1. *More generally, $\mathcal{M}$ can be decomposed into basins of attraction for more general compact attractors A_j. A space of observables on $B = \bigcup_{j=1}^{\ell} B_j$ can be given like in (V.2.1), where B_j is the basin of attraction for the attractors.* □

Example V.2.2. *This example shows why we must construct the function algebras as a direct sum on the basins of attraction, rather than defining the observables on all of X. Let X be the real line and $S : \mathbb{R} \to \mathbb{R}$ be defined by the continuous map*

$$S(x) = \begin{cases} \lambda_1(x - x_1) + x_1, & (x < x_2) \\ x, & (x = x_2) \\ \lambda_2(x - x_3) + x_3, & (x > x_2). \end{cases}$$

where $x_1 < x_2 < x_3$ and $\lambda_1 \neq \lambda_2$ are positive real numbers. A candidate eigenfunction is $\varphi_{1,m}(x) = (x - x_1)^m$; for $x < x_2$,

$$U_S\varphi_{1,m}(x) = \varphi_{1,m}(\lambda_1(x - x_1) + x_1) = [\lambda_1(x - x_1)]^m = \lambda_1^m \varphi_{1,m}(x).$$

However,

$$U_S\varphi_{1,m}(x) = \begin{cases} \varphi_{1,m}(x), & (x = x_2) \\ [\lambda_2(x - x_2) + x_2 - x_1]^m, & (x > x_2). \end{cases}$$

Therefore, if we allow $\varphi_{1,m}$ to be defined on all of $\mathbb{R}$, it cannot be an eigenfunction of the Koopman operator. In fact, if we define the domain for the observables to be all of $\mathbb{R}$, the only continuous eigenfunctions of U_S are the constant functions which are associated with the eigenvalue 1. □

V.2.1 Polynomials over normed commutative rings

Let $(\mathscr{R}, +, \cdot)$ be a normed unital commutative ring with the norm $\|\cdot\|$. We additionally assume that $(\mathscr{R}, +)$ has a vector space structure over $\mathbb{C}$. Thus $\mathscr{R}$ with the two binary operations and multiplication by scalars is an associative algebra.

Definition V.2.3. The space of polynomials over the ring $\mathscr{R}$ in the indeterminates $x_1, \ldots, x_d$ is defined by

$$\mathscr{R}[x] := \left\{ \sum_{|k| \leq K} \psi_k \cdot x^k \;\middle|\; K \in \mathbb{N}_0, \psi_k \in \mathscr{R} \right\}, \tag{V.2.2}$$

where $x = (x_1, \ldots, x_d)$, $k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$, $|k| := \sum_{j=1}^{d} |k_j|$, and $x^k := x_1^{k_1} \cdots x_d^{k_d}$.

These are just regular polynomials with coefficients taking values in $\mathscr{R}$ rather than $\mathbb{R}$ or $\mathbb{C}$. The space $\mathscr{R}[x]$ is a normed unital commutative ring under the normal operations of addition and convolution products for polynomials. The space $\mathscr{R}[x]$ is isomorphic to the sequence space

$$c_{00}(\mathscr{R}^{\mathbb{N}_0^d}) = \left\{ \{\psi_k\}_{k\in\mathbb{N}_0^d} \;\middle|\; \psi_k \in \mathscr{R}, F \subset \mathbb{N}_0^d \text{ finite}, \forall k \in F^c, \psi_k = 0 \right\}$$

under the ring isomorphism $i : \mathscr{R}[x] \to c_{00}(\mathscr{R}^{N_0^d})$ given by $i(\sum_{k\in\mathbb{N}_0^d} \psi_k \cdot x^k) = \{\psi_k\}_{k\in\mathbb{N}_0^d}$. Only finitely many terms of the sequence are nonzero. The norm $\|\cdot\|$ on $\mathscr{R}$ induces a norm on $\mathscr{R}[x]$:

$$\left\| \sum_{k\in\mathbb{N}_0^d} \psi_k x^k \right\|_{\mathscr{R},2} = (\sum_{k\in\mathbb{N}_0^d} \|\psi_k\|^2)^{1/2}. \tag{V.2.3}$$

Expression (V.2.3) gives a norm for the sequence space as well. Under this norm, the isomorphism i is an isometric isomorphism. Define the coordinate projections $\pi_m : \mathscr{R}[x] \to \mathscr{R}$ by

$$\pi_m \left(\sum_{k\in\mathbb{N}_0^d} \psi_k x^k \right) = \psi_m, \qquad (\forall m \in \mathbb{N}_0^d). \tag{V.2.4}$$

Definition V.2.4. Let $\mathscr{B}$ be a separable reflexive Banach space under the norm $\|\cdot\|$. Let

$$\ell^2 \left(\mathscr{B}^{\mathbb{N}_0^d} \right) := \left\{ \{\psi_k\}_{k\in\mathbb{N}_0^d} \;\middle|\; \psi_k \in \mathscr{B}, \sum_{k\in\mathbb{N}_0^d} \|\psi_k\|^2 < \infty \right\}. \tag{V.2.5}$$

Definition V.2.5 ($\mathscr{B}$-Hardy space)**.** If $\mathscr{B}$ a separable reflexive Banach space under the norm $\|\cdot\|$ and $(\mathscr{R}, +)$ is a dense subspace of $\mathscr{B}$, denote the completion of $\mathscr{R}[x]$ under the ℓ^2 polynomial norm (V.2.3) by $H^2_{\mathscr{B}}$. The completion is

$$H^2_{\mathscr{B}} = \left\{ \sum_{k\in\mathbb{N}_0^d} \psi_k x^k \;\middle|\; \psi_k \in \mathscr{B}, \sum_{k\in\mathbb{N}_0^d} \|\psi_k\|^2 < \infty \right\}. \tag{V.2.6}$$

Lemma V.2.6. *Let* $(\mathscr{R}, +)$ *be a dense subspace of a separable reflexive Banach space* $\mathscr{B}$ *under the norm* $\|\cdot\|$*. Then* $H^2_{\mathscr{B}}$ *is isometrically isomorphic to* $\ell^2\left(\mathscr{B}^{\mathbb{N}_0^d}\right)$*.*

Proof. Let $c_{00}(\mathscr{B}^{\mathbb{N}_0^d})$ be the subspace of $\ell^2\left(\mathscr{B}^{\mathbb{N}_0^d}\right)$ consisting of elements having only finitely many nonzero terms. Since $(\mathscr{R}, +)$ is dense in $\mathscr{B}$, then $\mathscr{R}[x]$ is isometrically isomorphic to a subspace dense in $c_{00}(\mathscr{B}^{\mathbb{N}_0^d})$. Since $c_{00}(\mathscr{B}^{\mathbb{N}_0^d})$ is dense in $\ell^2\left(\mathscr{B}^{\mathbb{N}_0^d}\right)$, then $\mathscr{R}[x]$ is also isometrically isomorphic to a subspace dense in $\ell^2\left(\mathscr{B}^{\mathbb{N}_0^d}\right)$. □

Proposition V.2.7. *Let* $(\mathscr{R}, +)$*, as a vector space, be a dense subspace of a separable, reflexive Banach space. Then* $H^2_{\mathscr{B}}$ *is locally sequentially weakly compact.*

Proof. Since $\mathscr{B}$ is separable and reflexive, so is $H^2_{\mathscr{B}}$ ([Woj91], p.44). Fix $\psi \in H^2_{\mathscr{B}}$ and consider the closed neighborhood $U = \left\{\varphi \in H^2_{\mathscr{B}} \,\middle|\, \|\varphi - \psi\|_{\mathscr{R},2} \leq \varepsilon\right\}$. This is a bounded neighborhood of ψ. Since $H^2_{\mathscr{B}}$ is both a separable and reflexive Banach space, the sequential version of the Banach-Alaoglu theorem implies that this neighborhood is sequentially weakly compact. Since this is true for every ψ, then $H^2_{\mathscr{B}}$ is locally sequentially weakly compact. □

In this dissertation, $\mathscr{B}$ will be a separable reflexive Banach space of functions whose domain is the attractor of the dynamical system. The normed ring $\mathscr{R}$ will be a dense subspace of $\mathscr{B}$ where the ring multiplication is given by pointwise products of functions.

We should remark that while $H^2_{\mathscr{B}}$ is a well-defined Banach space, elements of it are not necessarily well-defined observables on $\mathscr{X}$ even though all polynomials in $\mathscr{R}[x]$

are. For example, let $\mathscr{X} = \mathbb{R}^d$. Consider a point of $\mathbb{R}^d$ having at least one coordinate x_i satisfying $|x_i| > 1$. Then there are infinitely many elements of $H^2_{\mathscr{B}}$ such that the infinite series diverges if we substitute the above point in for the indeterminates. Hence, not all elements of $H^2_{\mathscr{B}}$ define finite-valued functions on $\mathbb{R}^d$. On the other hand, if $\mathscr{X}$ is contained strictly in the open unit cube of $\mathbb{R}^d$, then every series in $H^2_{\mathscr{B}}$ is convergent, since $|x_i| < 1$ for every point $x = (x_1, \ldots, x_d)$. If we restrict our attention to bounded subsets B in $\mathbb{R}^d$ containing the origin, by a change of variables g which maps B into a set $g(B)$ that is contained strictly in the unit cube, we get a well-defined space of observables $H^2_{\mathscr{B}} \circ g$, where $H^2_{\mathscr{B}}$ is defined on the open unit cube.

V.2.2 Asymptotically stable fixed points in $\mathbb{C}^d$

Linear Dynamical System.

Example V.2.8 (Stable diagonalizable linear system)**.** *Let $\mathscr{Y} = \mathbb{C}^d$ with the Euclidean norm and A a $d \times d$ diagonalizable matrix. Define the dynamics as $\boldsymbol{y}_{k+1} = \mathsf{A}\boldsymbol{y}_k$. Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\}$ be a basis of eigenvectors of A corresponding to nonzero eigenvalues $\{\lambda_j\}_1^d$. Then $\boldsymbol{y} = \sum_1^d c_j \boldsymbol{v}_j$.*

Let $\{\boldsymbol{w}_j\}_{j=1}^d$ be the adjoint basis to $\{\boldsymbol{v}_j\}_{j=1}^d$; then $\langle \boldsymbol{v}_j, \boldsymbol{w}_k \rangle = \delta_{jk}$ and $\boldsymbol{w}_j$ is an eigenvector of A^* at eigenvalue $\bar{\lambda}_j$. Define observables by the linear functionals

$$\varphi_j(\boldsymbol{y}) = \langle \boldsymbol{y}, \boldsymbol{w}_j \rangle \tag{V.2.7}$$

for all $\boldsymbol{y}$ *and* $j = 1, \ldots, d$*. The observable* φ_j *is a nonzero eigenfunction of* U_{A} *since*

$$(U_{\mathsf{A}}\varphi_j)(\boldsymbol{y}) = \varphi_j(\mathsf{A}\boldsymbol{y}) = \langle \mathsf{A}\boldsymbol{y}, \boldsymbol{w}_j \rangle = \langle \boldsymbol{y}, \mathsf{A}^*\boldsymbol{w}_j \rangle = \left\langle \boldsymbol{y}, \bar{\lambda}_j \boldsymbol{w}_j \right\rangle = \lambda_j \varphi_j(\boldsymbol{y}).$$

Additionally, the product $(\varphi_1 \cdots \varphi_d)$ *is not equivalently the zero functional. Therefore, the semigroup property for eigenfunctions (thm. V.1.1) implies that for any* $(m_1, \ldots, m_d) \subset \mathbb{N}_0^d$*,*

$$\left(\prod_{j=1}^{d} \lambda_j^{m_j}, \prod_{j=1}^{d} \varphi_j^{m_j} \right) \tag{V.2.8}$$

is an eigenpair for the corresponding Koopman operator.

Let $\mathcal{G} = \{\varphi_j : \mathscr{Y} \to \mathbb{C} \mid \forall j = 1, \ldots, d\}$*, be called the principle eigenfunctions of* U_{A} *defined in (V.2.7) above. Define the sets of functions*

$$\mathscr{P}_{\mathcal{G}} := \left\{ c : \mathscr{Y} \to \mathbb{C} \;\middle|\; c(\boldsymbol{y}) = \prod_{j=1}^{d} \varphi_j^{m_j}(\boldsymbol{y}), (m_1, \ldots, m_d) \subset \mathbb{N}_0^d \right\}$$

and

$$\mathscr{R}[\hat{\boldsymbol{y}}] := \left\{ \sum_{i=1}^{n} \alpha_i c_i(\boldsymbol{y}) \;\middle|\; n \in \mathbb{N}, \alpha_i \in \mathbb{C}, c_i \in \mathscr{P}_{\mathcal{G}} \right\}. \tag{V.2.9}$$

Then, $\mathscr{R}[\hat{\boldsymbol{y}}]$*, where* $\hat{\boldsymbol{y}} = (\varphi_1(\boldsymbol{y}), \ldots, \varphi_d(\boldsymbol{y}))$*, is the space of polynomials over the normed unital commutative ring* $L^2(\mathbb{C}^d, \delta_0)$*, where* δ_0 *is the Dirac measure supported at* $\boldsymbol{y} = 0$*. The ring* $\mathscr{R}[\hat{\boldsymbol{y}}]$ *is isomorphic to* $\mathbb{C}[\hat{\boldsymbol{y}}]$*. Furthermore,* $\mathscr{R}[\hat{\boldsymbol{y}}]$ *is the space of finite linear combinations of eigenfunctions of the Koopman operator corresponding to the dynamical system* $(\mathscr{Y}, \mathsf{A})$*.*

To see this, put $\mathsf{V} = [\boldsymbol{v}_1, \cdots, \boldsymbol{v}_d]$ *and* $\mathsf{W} = [\boldsymbol{w}_1, \cdots, \boldsymbol{w}_d]$*. Define the new coordinates* $\hat{\boldsymbol{y}} = [\hat{y}_1, \ldots, \hat{y}_d]^{\mathsf{T}}$ *by the mapping* $\hat{\boldsymbol{y}} = g(\boldsymbol{y}) = \mathsf{W}^*\boldsymbol{y}$*, so that, in particular,* $\hat{y}_j = \varphi_j(\boldsymbol{y})$*. This is invertible since* W *is and the inverse is given by* $\boldsymbol{y} = g^{-1}(\hat{\boldsymbol{y}}) = \mathsf{V}\hat{\boldsymbol{y}}$*.*

By definition of the functions $c_i \in \mathscr{P}_{\mathcal{G}}$*,* $c_i(\boldsymbol{y}) = \prod_{j=1}^d \varphi_j^{m_{j,i}}(\boldsymbol{y}) = \prod_{j=1}^d \hat{y}_j^{m_{j,i}} = \hat{\boldsymbol{y}}^{\boldsymbol{m}_i}$ *for* $\boldsymbol{m}_i \in \mathbb{N}_0^d$*. Since* $L^2(\mathbb{C}^d, \delta_0) \cong \mathbb{C}$*, then every element of* $\mathscr{R}[\hat{\boldsymbol{y}}]$ *has the form* $\sum_{i=1}^n \alpha_i \hat{\boldsymbol{y}}^{\boldsymbol{m}_i}$ *which is a polynomial in the indeterminates* $\hat{y}_1, \dots, \hat{y}_d$ *with coefficient in* $\mathbb{C} \cong L^2(\mathbb{C}^d, \delta)$*. Therefore,* $\mathscr{R}[\hat{\boldsymbol{y}}]$ *is the space of polynomials over the normed unital commutative ring* $L^2(\delta_0)$ *and* $\mathscr{R}[\hat{\boldsymbol{y}}] \cong \mathbb{C}[\hat{\boldsymbol{y}}]$*. The final result follows directly from* $\mathscr{P}_{\mathcal{G}}$ *consisting of eigenfunctions of the Koopman operator and the definition of* $\mathscr{R}[\hat{\boldsymbol{y}}]$*.* □

Proposition V.2.9. *If* $\mathscr{R}[\hat{\boldsymbol{y}}]$ *is the ring of polynomials (V.2.9), then the associated Koopman operator* U_{A} *leaves* $\mathscr{R}_K[\hat{\boldsymbol{y}}]$ *invariant for all* $K \in \mathbb{N}_0$*.*

Proof. Recall $\mathscr{R}_K[\hat{\boldsymbol{y}}]$ is the space of polynomials having multi-degree modulus no greater than K. This space has the form

$$\mathscr{R}_K[\hat{\boldsymbol{y}}] = \left\{ \sum_{|\boldsymbol{k}| \le K} \psi_{\boldsymbol{k}} (\hat{y}_1^{k_1} \cdots \hat{y}^{k_d}) \right\} = \left\{ \sum_{|\boldsymbol{k}| \le K} \psi_{\boldsymbol{k}} \left(\varphi_1^{k_1} \cdots \varphi_d^{k_d} \right) (\boldsymbol{y}) \right\},$$

where φ_j are the eigenfunctions in (V.2.7) of the Koopman operator and $\psi_{\boldsymbol{k}} \in \mathbb{C}$. Then

$$\begin{aligned} U \left[\sum_{|\boldsymbol{k}| \le K} \psi_{\boldsymbol{k}} \left(\varphi_1^{k_1} \cdots \varphi_d^{k_d} \right) (\boldsymbol{y}) \right] &= \sum_{|\boldsymbol{k}| \le K} \psi_{\boldsymbol{k}} \left(\lambda_1^{k_1} \cdots \lambda_d^{k_d} \right) \left(\varphi_1^{k_1} \cdots \varphi_d^{k_d} \right) (\boldsymbol{y}) \\ &= \sum_{|\boldsymbol{k}| \le K} \hat{\psi}_{\boldsymbol{k}} \left(\varphi_1^{k_1} \cdots \varphi_d^{k_d} \right) (\boldsymbol{y}) \\ &= \sum_{|\boldsymbol{k}| \le K} \hat{\psi}_{\boldsymbol{k}} (\hat{y}_1^{k_1} \cdots \hat{y}^{k_d}). \end{aligned}$$

where $\hat{\psi}_{\boldsymbol{k}} := \left(\lambda_1^{k_1} \cdots \lambda_d^{k_d} \right) \cdot \psi_{\boldsymbol{k}}$ is in $\mathbb{C}$, since $\psi_k \in \mathbb{C}$. □

Corollary V.2.10. $U_{\mathsf{A}} : H^2_{\mathscr{B}} \to H^2_{\mathscr{B}}$ *is spectral, where* $H^2_{\mathscr{B}}$ *is the completion of the ring (V.2.9) under the norm (V.2.3).*

Nonlinear Dynamical System.

Construction of observables for the nonlinear dynamical system can be constructed through a conjugacy map and a pullback construction (def. V.1.7). Assume $\Phi(x) = \mathsf{A}x + v(x)$, where $\mathsf{A} := D\Phi(0)$ is the linearization of Φ around the origin, A is diagonalizable and all its eigenvalues are contained strictly inside the unit circle, and $v : \mathbb{R}^d \to \mathbb{R}^d$ is a C^2-function. Then $(\mathbb{R}^d, \Phi)$ is a nonlinear dynamical system having an asymptotically stable fixed point at 0. Let $D \subset \mathbb{R}^d$ be the basin of attraction for the fixed point; it is possibly unbounded. Let $(\mathbb{R}^d, \mathsf{A})$ be the linearization of the dynamics around the fixed point. Due to a theorem of Lan and Mezić (thm. 2.4, [LM13]), there is a diffeomorphism $h : D \to h(D)$ under which the nonlinear and linear dynamics are topologically conjugate; $h \circ \Phi = \mathsf{A} \circ g$. Let $B \subset \mathscr{X}$ be a simply connected, bounded, positively invariant open set in $\mathscr{X}$ such that $h(B) \subset Q_r \subset \mathscr{Y}$, where Q_r is a cube in $\mathscr{Y}$. Scaling Q_r to the unit cube Q_1 via the smooth diffeomorphism $g : Q_r \to Q_1$ gives $(g \circ h)(B) \subset Q_1$. Then if $\psi \in \mathscr{F}$ is an eigenfunction for $U_{\hat{\mathsf{A}}} : \mathscr{F} \to \mathscr{F}$ at λ, then $\psi \circ g \circ h$ is an eigenfunction for U_Φ at eigenvalue λ (see thm. V.1.8). The observable space for U_Φ will be given as $\mathscr{F} \circ g \circ h = \{\varphi \circ g \circ h \mid \varphi \in \mathscr{F}\}$. See figure V.2 to for a schematic.

Let $\mathscr{R}[\hat{\boldsymbol{y}}]$ be the polynomials defined in example V.2.8 with the indeterminates taking values in $D_1 := (g \circ h)(B) \subset Q_1$. Take the completion of this space, as in lemma V.2.6, and denote it as $H^2_{\mathscr{B}}(D_1)$. Since all the coordinates have a modulus strictly smaller than 1, this space can be identified with a well-defined Banach space

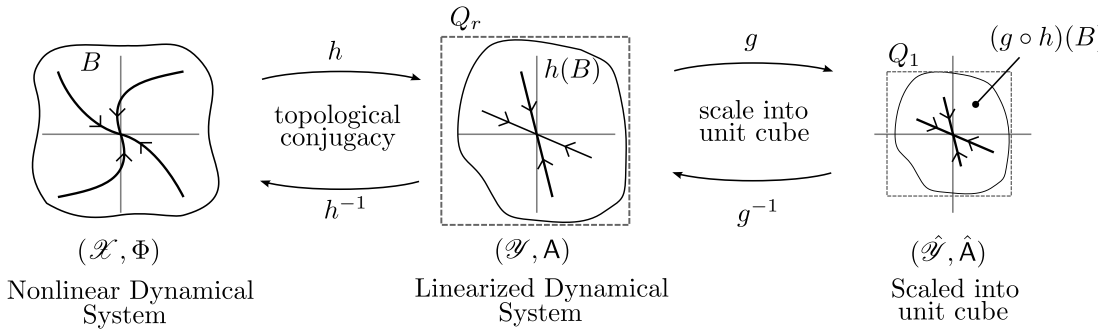


Figure V.2: Chain of topological conjugacies used to construct eigenfunctions in the basin of attraction B of the fixed point for the Koopman operator corresponding to the nonlinear system. The existence if $h : B \to h(B)$ is guaranteed by a theorem in [LM13].

of functions having domain $D_1 = (g \circ h)(B)$. Using the conjugacies, we can define a Banach space of observables for the nonlinear dynamical system having domain B;

$$\mathscr{F}_\Phi := H^2_{\mathscr{B}} \circ g \circ h := \left\{ (\psi \circ g \circ h)(\boldsymbol{x}) \;\middle|\; \psi \in H^2_{\mathscr{B}}(D_1), \boldsymbol{x} \in B \right\}. \tag{V.2.10}$$

By proposition V.2.7, this space is locally sequentially weakly compact Banach space in addition to being separable.

Proposition V.2.11. *Let $U_\Phi : \mathscr{F}_\Phi \to \mathscr{F}_\Phi$ be the Koopman operator associated with Φ. Assume $\Phi(x) = \mathsf{A}x + v(x)$, where A is a diagonalizable matrix with eigenvalues $\{\lambda_1, \ldots, \lambda_d\}$ that satisfy $|\lambda_i| < 1$. Then U_Φ is a spectral operator of scalar type and the spectrum of U_Φ is*

$$\sigma(U_\Phi) = \left\{ \prod_{i=1}^{d} \lambda_i^{k_i} \;\middle|\; \forall i \in \{1, \ldots, d\}, k_i \in \mathbb{N}_0 \right\}. \tag{V.2.11}$$

Proof. Elements of $\mathscr{F}_\Phi$ are infinite linear combinations of eigenfunctions of U_Φ. This follows from the construction of $H^2_{\mathscr{B}}$ as infinite linear combinations of eigenfunctions of U_{A}, that Φ and A are topologically conjugate in the entire basin of attraction by theorem 2.4 of [LM13], and theorem V.1.8. The spectral measure $E : \sigma(U) \to \mathscr{L}(\mathscr{F}_\Phi)$ is given by

$$E(\Lambda)\psi = \sum_{\left\{\boldsymbol{k}\in\mathbb{N}_0^d \,\middle|\, \boldsymbol{\lambda}^{\boldsymbol{k}}\in\Lambda\right\}} \alpha_{\boldsymbol{k}} \cdot \left[(\varphi_1 \cdots \varphi_d)^{\boldsymbol{k}} \circ g \circ h\right], \qquad (\psi \in \mathscr{F}_\Phi), \tag{V.2.12}$$

where $\Lambda \in \mathcal{B}(\mathbb{C})$, $\psi = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \alpha_{\boldsymbol{k}} \cdot \left[(\varphi_1 \cdots \varphi_d)^{\boldsymbol{k}} \circ g \circ h\right]$, and $(\varphi_1 \cdots \varphi_d)^{\boldsymbol{k}} = (\varphi_1^{k_1} \cdots \varphi_d^{k_d})$ for $\boldsymbol{k} = (k_1, \ldots, k_d)$. Since the norm on this space is given by the ℓ^2-norm on coefficients, then its easy to show that $\|E\| \leq 1$, so that, in particular, the projections defined by the spectral measure form an equicontinuous family of operators. It is straight forward to verify all the remaining properties spectral measures. □

Since $U_\Phi : \mathscr{F}_\Phi \to \mathscr{F}_\Phi$ is spectral and has dominating point spectrum concentrated on isolated rings, the GLA theorems can be applied in this space.

Remark V.2.12. *Put $d = 1$ in example V.2.8. It can be shown that $\varphi(y) = y$ is an eigenfunction of the Koopman operator for $\lambda = \mathsf{A}$. This implies that $\varphi_k(y) = y^k$ is also an eigenfunction at eigenvalue λ^k. Then elements of the completion $H^2_{\mathscr{B}}$ of $\mathscr{R}[\hat{\boldsymbol{y}}]$, identified with a space of observables on $\mathbb{D}$, is*

$$H^2_{\mathbb{C}}(\mathbb{D}) = \left\{ \sum_{k=0}^{\infty} \alpha_k y^k \,\middle|\, \alpha_k \in \mathbb{C}, \sum_{k\geq 0} |\alpha_k|^2 < \infty, y \in \mathbb{D} \right\},$$

since $\mathscr{B} = \mathbb{C}$. This is the Hardy Hilbert space of observables $H^2(\mathbb{D})$. The spectrum of *the Koopman operator on this space is*

$$\left\{\lambda^k \mid k = 0, 1, 2, \ldots\right\}, \tag{V.2.13}$$

For a nonlinear dynamical system topologically conjugate to this linear one, we get as the pullback space of observables

$$\mathscr{F} = H^2_{\mathbb{C}}(\mathbb{D}) \circ g \circ h, \tag{V.2.14}$$

where g and h are the same conjugacy maps from above. The composition operator on this space is spectral. Furthermore, by Poincaré's linearization theorem [AL88], if the nonlinear dynamical system is a holomorphic diffeomorphism with a fixed point at zero and the linearization of the map at the fixed point has modulus strictly less than 1, then the conjugacy map $g \circ h$ is biholomorphic in a neighborhood of zero. As long as for all $n \geq 0$, f^n is invertible on its image and the inverse is analytic, the local conjugacy can be extended biholomorphically to the entire disc [LM13]. Then the map $f \mapsto f \circ g \circ h$ defines a composition operator on $H^2(\mathbb{D})$ (thm. 3.2.1(i), [SM93]). Additionally, by part (iii) of the same theorem, this composition operator is a bijection. Therefore, the pullback space of observables $\mathscr{F}$ for the analytic map is equivalent to $H^2(\mathbb{D})$. □

Remark V.2.13. *In Cowen and MacCluer [CM94], the spectra of composition operators on weighted Hardy spaces are investigated. In one particular result, the authors prove that when $\Phi : \mathbb{D} \to \mathbb{D}$ is an univalent, holomorphic map that is not an automorphism and satisfies $\Phi(\mathbb{D}) \subset \mathbb{D}$ and $\Phi(0) = 0$, then the spectrum of the associated*

Koopman operator on (the classical Hardy space) $H^2(\mathbb{D})$ is

$$\sigma(U_\Phi) = \{\lambda \mid |\lambda| \leq \tilde{\rho}\} \cup \left\{\Phi'(0)^k \;\middle|\; k = 1, 2, \ldots\right\} \cup \{1\}, \tag{V.2.15}$$

where $\tilde{\rho}$ is the essential spectral radius of Φ and $\Phi'(0) \neq 0$ is the derivative of Φ at 0. Note that $\Phi'(0)$ is the eigenvalue of the linearized dynamics around the fixed point at 0. The conditions put on Φ merely guarantee that 0 is a globally attracting (in the disc) fixed point.

Clearly, the last two sets of (V.2.15) are equivalent to (V.2.13) and merely come from the linearized dynamics. The linear dynamics $w \mapsto \Phi'(0)w$ and the nonlinear dynamics $z \mapsto \Phi(z)$ are topologically conjugate in the disc under some diffeomorphism $g : \mathbb{D} \to \mathbb{D}$; i.e., $g \circ \Phi = \Phi'(0) \circ g$. This paper defines the pullback space (V.2.14), on which the Koopman operator corresponding to the nonlinear dynamics Φ is spectral, whereas Cowen and MacCluer consider the composition operator acting on $H^2(\mathbb{D})$ and as a result obtain an additional term in the spectrum, namely $\{\lambda \mid |\lambda| \leq \tilde{\rho}\}$. Using the same argument as in the previous remark, if the conjugacy is biholomorphic in the disc, then $\mathscr{F} \equiv H^2(\mathbb{D})$. As we have shown, the composition operator has only point spectra. This is a sharpening of the result by Cowen and MacCluer when applied to this specific class of maps.

Even if the conjugacy is not biholomorphic and we consider the composition operator U_Φ on $H^2(\mathbb{D})$ rather than the pullback space $\mathscr{F} = H^2_{\mathscr{B}} \circ g \circ h$, we can apply the GLA theorems in this paper to construct the eigenfunctions as long as $\tilde{\rho} < \Phi'(0)$. In this case, U_Φ has a dominating point spectrum (take $R = \tilde{\rho}$ in def. III.1.5). □

V.2.3 Asymptotically stable limit cycles in $\mathbb{R}^2$

Consider a stable limit cycle in the plane, topologically conjugate in a neighborhood of the limit cycle to the linearized system

$$\begin{aligned} \dot{x} &= \rho(s)x \\ \dot{s} &= 1 \end{aligned} \tag{V.2.16}$$

where $x \in \mathbb{R}$, $s \in S^1 = \mathbb{Z}/2\pi$, and $\rho(s)$ is 2π periodic. Letting $\Phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ be the flow map of the system, the continuous time Koopman semigroup is formally defined as $U(t)f = f \circ \Phi_t$. Eigenfunctions of the semigroup are functions φ taking the form $U(t)\varphi = \lambda^t \varphi$. We call $\lambda \in \mathbb{C}$ an eigenvalue of the Koopman semigroup.

It was shown in [LM11] that the Koopman semigroup associated with the dynamical system (V.2.16) has eigenfunctions of the form,

$$\begin{aligned} g_m(x,s) &= x^m e^{-m\int_0^s (\rho(\bar{s})-\rho^*)d\bar{s}}, \quad (m \in \mathbb{N}_0) \\ h_n(x,s) &= e^{ins}, \quad (n \in \mathbb{Z}) \end{aligned} \tag{V.2.17}$$

with eigenvalues

$$\lambda_m = e^{m\rho^*}, \text{ and}$$

$$\mu_n = e^{in},$$

respectively, and where $\rho^* = (2\pi)^{-1}\int_0^{2\pi} \rho(s)ds$. We have that $\rho^* < 0$ since the limit cycle is asymptotically stable.

By the semigroup property of eigenfunctions, $b_{m,n}(x,s) = g_m(x,s) \cdot h_n(x,s)$ is an eigenfunction having eigenvalue $e^{(m\rho^*+in)}$. Let $\mathscr{V}$ be the subspace given by the linear span of elements of the form $b_{m,n} = g_m \cdot h_n$;

$$\begin{aligned}\mathscr{V} &:= \left\{ \sum_{k=1}^{K} a_k (g_{m_k} \cdot h_{n_k}) \;\middle|\; K \in \mathbb{N}, a_k \in \mathbb{C}, m_k \in \mathbb{N}_0, n_k \in \mathbb{Z} \right\} \\ &= \left\{ \sum_{k=0}^{K} a_k x^k e^{-k \int_0^s (\rho(\bar{s}) - \rho^*) d\bar{s}} e^{i n_k s} \;\middle|\; K \in \mathbb{N}_0, n_k \in \mathbb{Z}, a_k \in \mathbb{C} \right\}.\end{aligned} \tag{V.2.18}$$

We show that $\mathscr{V}$ is contained in a space of polynomials over a normed unital commutative ring $\mathscr{R}$ that is dense in $L^2(S^1, \mu)$, where μ is the normalized Haar measure $d\mu = (2\pi)^{-1} ds$. Elements of $\mathscr{V}$ can be written in the form

$$\mathscr{V} = \left\{ \sum_{k=0}^{K} x^k e^{-k \int_0^s (\rho(\bar{s}) - \rho^*) d\bar{s}} \sum_{|n| \le N_k} a_{k,n} e^{ins} \;\middle|\; K \in \mathbb{N}_0, a_{k,n} \in \mathbb{C}, n \in \mathbb{Z}, N_k \in \mathbb{N}_0 \right\}. \tag{V.2.19}$$

Define $\mathscr{W}$ as

$$\mathscr{W} := \left\{ e^{-m \int_0^s (\rho(\bar{s}) - \rho^*) d\bar{s}} \sum_{|n| \le N} a_n e^{ins} \;\middle|\; m \in \mathbb{N}_0, N \in \mathbb{N}_0, a_n \in \mathbb{C} \right\} \tag{V.2.20}$$

and $\mathscr{R}$ as

$$\mathscr{R} := \operatorname{lin} \mathscr{W} = \left\{ \sum_{k=1}^{K} b_k w_k(s) \;\middle|\; K \in \mathbb{N}, w_k \in \mathscr{W}, b_k \in \mathbb{C} \right\}. \tag{V.2.21}$$

Lemma V.2.14. *$\mathscr{R}$ is a normed unital commutative ring under pointwise products of functions with norm $\|f\|_2 = (\frac{1}{2\pi} \int_0^{2\pi} |f(s)|^2 \, ds)^{1/2}$.*

Proof. The unit function $\mathbf{1}$, taking values $\mathbf{1}(s) = 1$, is contained in $\mathscr{W}$ and has norm $\|\mathbf{1}\|_2 = 1$; take $m = 0$ and $a_n = 0$ for $|n| \neq 0$ and $a_0 = 1$ in (V.2.20). Similarly, for any $c \in \mathbb{C}$, $c\mathbf{1}$ is in $\mathscr{W}$ and has finite norm. This implies that the constant functions are in $\mathscr{R}$. Clearly, products of elements in $\mathscr{W}$ commute. This implies that products of elements from $\mathscr{R}$ commute. We must show that these products are also in $\mathscr{R}$ and have finite norm. Let w_j and w_k be in $\mathscr{W}$. It is a straightforward calculation to show that $(a_j w_j(s))(b_k w_k(s))$ is in $\mathscr{W}$ for any $a_j, b_k \in \mathbb{C}$. It follows that $\mathscr{R}$ is closed under pointwise products. Now let $g(s)$ be in $\mathscr{R}$. This function is a finite linear combinations of elements of $\mathscr{W}$. But elements of $\mathscr{W}$ are continuous functions on the compact interval $[0, 2\pi]$. Consequently, $g(s)$ is also a continuous function on the compact interval. Therefore, there is some constant C_g such that $\sup_{0 \leq s \leq 2\pi} |g(s)| \leq C_g$. Thus $\|g\|_2 \leq C_g$. Now, let $h(s)$ be another function in $\mathscr{R}$. We have shown that $g(s) \cdot h(s)$ is in $\mathscr{R}$. Furthermore, $g(s)h(s)$ is a continuous function on the compact interval $[0, 2\pi]$. Therefore, $\sup |g(s)h(s)| \leq C_{gh}$ for $s \in [0, 2\pi]$ and the product also has finite norm. □

Lemma V.2.15. *$\mathscr{W}$ is dense in $L^2(S^1, \mu)$. Consequently, $\mathscr{R}$ is dense in $L^2(S^1, \mu)$.*

Proof. $\{e^{ins}\}_{n \in \mathbb{Z}}$ is an orthonormal basis for $L^2(S^1, \mu)$. Put $\xi(s) = \int_0^s (\rho(\bar{s}) - \rho^*) \, d\bar{s}$, for $s \in [0, 2\pi]$. Since $\xi(s)$ is a continuous function on a closed interval, it is bounded; $|\xi(s)| \leq M$ for all s. Therefore, $0 < e^{-kM} \leq e^{-k\xi(s)} \leq e^{kM}$, where c_k is a constant depending on k. It follows that $e^{k\xi(s)}$ is a positive bounded function bounded away from 0.

Fix $f \in L^2(S^1, \mu)$. Since $e^{k\xi(s)}$ is a bounded function, it follows that $g_k(s) := e^{k\xi(s)} f(s)$ is in $L^2(S^1, \mu)$. For each $\varepsilon > 0$, there is a trigonometric polynomial such that $\left\| g_k - \sum_{|n| \leq N} a_n e^{ins} \right\| < e^{kM} \varepsilon$. Therefore, $\left\| f - e^{-k\xi(s)} \sum_{|n| \leq N} a_n e^{ins} \right\| < \varepsilon$. □

Lemma V.2.16. *$\mathscr{V} = \mathscr{R}[x]$.*

Proof. This follows directly from the definitions of $\mathscr{V}$, $\mathscr{W}$, and $\mathscr{R}$. □

Completing $\mathscr{R}[x]$ under the polynomial norm (V.2.3) gives the space consisting of elements of the form $\sum_{k=0}^{\infty} \psi_k x^k$ where $\psi_k \in L^2(S^1, \mu)$, and $\sum_{k=0}^{\infty} \|\psi_k\|_2^2 < \infty$. These are well-defined functions for $|x| < 1$. The completion $\mathscr{F} := H^2_{L^2}$ is not a ring. Furthermore, $\mathscr{V}$ is dense in $H^2_{L^2}$.

Spectrum and spectral measure for the Koopman semigroup on $H^2_{L^2}$.

The linear space $\mathscr{V}$ is generated by products of eigenfunctions of the Koopman semigroup and is dense in $H^2_{L^2}$. Eigenvalues of these eigenfunctions are of the form

$$\gamma_{k,n} = e^{(k\rho^* + in)}, \quad (k \in \mathbb{N}_0, \forall n \in \mathbb{Z}). \tag{V.2.22}$$

The set $\{\gamma_{k,n}\}$ is dense on isolated rings in the complex plane. Since $\rho^* < 1$, then $|\gamma_{k,n}| \leq 1$. It is easy to check that $\|U(t)f\|_{\mathscr{R},2} \leq \|f\|_{\mathscr{R},2}$ for all $f \in \mathscr{V}$ and $t \geq 0$. Since $\mathscr{V}$ is dense in $H^2_{L^2}$, we can extend $U(t)$ to $H^2_{L^2}$ by continuity.

Let $\mathscr{V}_K$ be the subspace of $\mathscr{V}$ consisting of all elements for which the degree of x is at most K. That is, in the definition (V.2.19) of $\mathscr{V}$ fix K rather than allowing it to run over all values of $\mathbb{N}_0$.

Lemma V.2.17. *The Koopman semigroup leaves $\mathscr{V}_K$ invariant.*

Proof. $\mathscr{V}_K$ is generated by eigenfunctions of the Koopman operator. □

Proposition V.2.18. *The Koopman semigroup corresponding to the dynamical system (V.2.16) is a spectral operator of scalar type on $\mathscr{F} = H^2_{L^2}$.*

Proof. We show this by constructing explicitly the spectral measure E so that $U(t)f = \int_{\mathbb{C}} z^t E(dz) f$ for $f \in H^2_{L^2}$. Let $\mathcal{B}(\mathbb{C})$ be the σ-algebra of Borel sets in the complex plane. Let $D \in \mathcal{B}(\mathbb{C})$ and define

$$I(D) := \{(k,n) \in \mathbb{N}_0 \times \mathbb{Z} \mid \gamma_{k,n} \in D\},$$

where $\gamma_{k,n}$ is defined in (V.2.22). This is the set of all indices corresponding to the eigenvalues $\gamma_{k,n}$ that are contained in D. Define the set function $E : \mathcal{B}(\mathbb{C}) \to \mathscr{L}(\mathscr{V})$ as

$$E(D)f = \sum_{\substack{(k,n)\in I(D)\\ k\le K, |n|\le N}} c_{k,n}(f) x^k e^{-k\int_0^s (\rho(\bar{s})-\rho^*)d\bar{s}} e^{ins}, \tag{V.2.23}$$

where

$$f(x,s) = \sum_{k=0}^{K} \sum_{|n|\le N} c_{k,n}(f) x^k e^{-k\int_0^s (\rho(\bar{s})-\rho^*)d\bar{s}} e^{ins}$$

is the unique representation of $f \in \mathscr{V}$. The set function E commutes with $U(t)$ for all t and

$$\|E(D)f\|^2_{\mathscr{R},2} := \sum_{\substack{(k,n)\in I(D)\\ k\le K, |n|\le N}} |c_{k,n}(f)|^2 \left\| e^{-k\int_0^s (\rho(\bar{s})-\rho^*)d\bar{s}} e^{ins} \right\|_2^2 \le \|f\|^2_{\mathscr{R},2}$$

Therefore, E is an equibounded set function commuting with $U(t)$. It is easy to show that E satisfies all the other properties of the spectral measure given in (III.2.6). Since $E(D)$ is a bounded linear operator on $\mathscr{V}$ and $\mathscr{V}$ is dense in $H^2_{L^2}$, it can be extended by continuity to all of $H^2_{L^2}$. □

Remark V.2.19. *The same pullback space of observables corresponding to bounded open sets in the basin of attraction can be constructed as was done for the attracting* *fixed point. Theorem 2.6 of [LM13] guarantees the existence of the required topological conjugacy valid in the entire basin of attraction of the limit cycle for appropriately regular nonlinear maps* Φ *in* $\mathbb{R}^2$ *possessing a stable limit cycle.* $\square$

Chapter VI

Disscussion of the GLA and Future Work

In the analysis of nonlinear dynamical systems, spectral analysis of the Koopman operator offers much insight into the system. However, most of the spectral theory has dealt with the Koopman operator associated with a measure-preserving or non-dissipative system. Many of the simplest examples of dynamical systems, and many practical systems, have dissipation or expansion present in them, and the traditional L^2-spectral theory of the measure-preserving case cannot handle these systems. In order to fill this gap, this paper extended the theory to scalar-type spectral operators on locally convex linear topological spaces having a point spectrum that is not restricted to the unit circle. Projections onto these eigenspaces can be recovered by Laplace averages of the operator. When the point spectrum is contained in the unit circle, the Laplace averages of this paper reduce to the well-known Fourier averages used to compute projections for unitary operators in a Hilbert space.

The results in this paper, however, do not give a full spectral picture. There are restrictions on the placement of both the eigenvalues and the continuous parts of the spectrum in the complex plane. For example interleaving of the continuous and point spectrum cannot be immediately treated with the GLA theorems of this paper. In particular, we must assume the point spectrum is concentrated on isolated circles in the complex plane and additionally the operator either has a dominating or minimal point spectrum. Given some dynamical system, it is not clear a priori whether the corresponding Koopman operator satisfies these conditions and, if it does, what exactly a typical observable might look like.

These questions were answered here for finite-dimensional, dynamical systems possessing hyperbolic attracting fixed points in $\mathbb{C}^d$ and attracting limit cycles in $\mathbb{R}^2$. When the dynamics are restricted to some bounded set in a basin of attraction for a fixed point, a space of observables can be constructed so that the spectrum of the Koopman operator possesses the properties assumed in this paper, namely that $\sigma(\mathrm{U})$ has dominating or minimal point spectrum and eigenvalues are concentrated on isolated circles. Formal eigenfunctions for the linearized system were used to generate a space of observables. As an algebraic object, this space was a subspace of the space of polynomials over a normed unital commutative ring. The indeterminates of these polynomials corresponded to coordinates corresponding to stable directions of the attractor while the coefficients took values in a normed ring consisting of observables supported on the attractor. This polynomial space was completed using an ℓ^2 polynomial norm to a Banach space that was identifiable with a space of observables defined on a bounded subset in the basin of attraction. Observables for the topologically conjugate nonlinear system were merely the result of composing the space of observable for the Koopman operator of the linearized system with the conjugacy maps. We conjecture that for any bounded attractor with a basin of attraction that can be properly "coordinatized", the natural space of observables for the Koopman operator will be a space of polynomials (and its completion) over a normed unital commutative ring, with the ring formed from observables supported on the attractor.

Numerical considerations were not treated in this paper. Efficient numerical algorithms are needed to make the results of this paper useful in applied settings. This will be the subject of future work. To this end, we mention work that has been done on discrete Laplace transforms by Rokhlin [Rok88], Strain [Str92], and Anderson [And13]. It may be that these algorithms can be leveraged into a package to compute the projections. Speed and numerical stability will be primary concerns since these algorithms require good estimates of the eigenvalues. Three other possible directions to pursue are the development of iterative methods using, for example, (i) Krylov subspace-type techniques and the methods of matrix-free linear algebra [Saa11; TD97], (ii) using the conjugacy method to explicitly construct eigenfunctions and project an observable onto them, or (iii) using polynomial approximations of indicator functions on the spectrum. The first technique is conceptually closer to Laplace averages in the sense that it would extract the most unstable modes first, whereas the second technique would allow the projection onto a particular eigenfunction at the cost of additional computational complexity in finding the eigenfunctions. In general, the eigenfunctions are the roots of a nonlinear operator equation that can be solved using an operator version of the Newton method [KH97]. Furthermore, the discrete Laplace transforms and the matrix-free methods could be classified as a "data-driven" method, since, in principle, all that is required is a sequence of observations and not an explicit representation of the dynamical system. The third technique rests on the operational calculus for spectral operators and that spectral projections are determined by integrating the spectral measure against indicator functions (see lemma III.2.9). This final possibility, in principle, allows one to extract both the point spectrum and continuous parts of the spectrum even if the spectrum does not satisfy the conditions in this paper. Furthermore, this is also a data driven technique as we only need iterations of an observable to compute this. Questions revolve around best polynomial approximations to the indicator functions, which polynomial basis allows for stable and efficient computation of the approximating polynomial in addition to the parsimony of the approximation; i.e., for a fixed approximation error, which polynomial basis has an approximating polynomial satisfying the error bound with the polynomial having minimal degree?

In summary, this paper introduced a method for constructing eigenfunctions of a spectral operator that can be considered as an extension of mean ergodic theorems to dissipative or expanding systems; we showed certain basic examples of dynamical systems and spaces of observables on them that gave rise to spectral Koopman operators on which the GLA theorems could be applied; and, additionally, we showed that these spaces of observables were each identifiable with a completion of a space of polynomials over a normed unital commutative ring, with the ring elements being observables defined on the attractor and the ring being dense in separable, reflexive Banach space. The observable spaces for the linearized dynamics took the form of a generalized Hardy space where instead of the coefficients being in $\mathbb{C}$, they took values in the Banach space. The space of observables for the nonlinear dynamical system were constructed by composing the space of observables for the linearized system with the conjugacy map between the nonlinear and linear system. The exact form of this pullback space is determined by the conjugacy map.

Chapter VII

Switched Families of Koopman Operators with Unitary Spectrum

While the previous chapters dealt we spectral analysis of the Koopman operator having non-unimodular spectrum, the present chapter represents a switch in this viewpoint. Broadly, the work in this dissertation can be thought of as Koopman mode analysis for non-ergodic dynamics. One of the methods of getting non-ergodicity is to introduce some dissipation in the dynamics which results in a portion of the spectrum of the associated Koopman operator being non-unimodular. This was discussed in the preceding chapters. Another way to get non-ergodic behavior, and still retain an unimodular spectrum, is to introduce discontinuous switching between Koopman operators with unimodular point spectrum. This is pursued in this chapter.

The motivation for this approach is comes from the study of internet-type networks and the desire to understand their dynamical behavior from a network-level viewpoint (rather than the router or individual link level). A network-wide understanding of the dynamics of traffic in internet service provider (ISP) backbones is important for a number of reasons. Tasks such as debugging performance problems and capacity planning [Fel+01] as well as verifying network performance under specific controls and achieving a target level quality of service [Che07], all require accurate models.

While unproven, most practitioners assume at least wide sense stationarity of the traffic when developing statistical models. The difficulty in modeling internet traffic arises, in part, from other characteristics of its statistical nature. Such traffic exhibits a high degree of variability, or burstiness [Che07]. Additionally, long-range dependence (LRD) and self-similarity have been claimed for a variety of scenarios, with long range dependence being characterized by a non-summable autocorrelation function, $\sum_{k=0}^{\infty} \rho(k) = \infty$, with a power law, $\rho(k) \sim C_\rho k^{-\alpha}$, often assumed ($C_\rho > 0$, $\alpha \in (0,1)$) [DC05]. The related concept of self-similarity of the traffic has also been claimed [CB97; Err+02; Lel+93]. While both LRD and self-similarity are hard to measure empirically because of sparse sampling in the regimes giving rise to these phenomena, empirical evidence for LRD have been found in a number of detailed studies (see [DC05] and the references therein).

In order to capture these types of statistics, a number of models for traffic sources have been proposed. Briefly, we mention simple ON-OFF models with heavy-tailed periods, Markov modulated Possion processes, stochastic fluid models and fractional

Brownian motion, and ARIMA time series models (see [Che07] for a discussion).

It has been shown that self-similarity and LRD can arise from aggregating many ON-OFF source models with heavy-tailed periods [HRS98; TWS97]. Alternatively, its been shown that using chaotic maps to drive the behavior of sources also gives rise to self-similarity [Err+02; ES95].

The use of chaotic dynamical systems gives a connection between network dynamics and the Koopman operator that is used in the analysis of dynamical systems from a spectral viewpoint. The Koopman operator is an infinite-dimensional linear operator that acts on some space of observables defined on the state space of a dynamical system. Formally, the Koopman operator drives the evolution of the observable by composing it with the dynamical system. Instead of tracking state space trajectories, we now track trajectories of observables. From a practitioners standpoint, this is important since the observable can be considered a sensor probe on the system.

Products of level sets of eigenfunctions of the Koopman operator give a structural decomposition of the state space [Bud12; Mez05]. In many cases though, approximations of the dynamics in the measurement space are of more interest. In an ISP backbone, the dynamical system is a stochastically driven networked dynamical system in which each node's (router's) dynamics consists of the TCP/IP and whatever other protocols the router is running. The state space is a complicated subset of the product of buffer lengths, internal routers states, packet round trip times, etc., and would be unreasonable to write down explicitly for moderately-sized networks. In short, a decomposition of the full state space would give little insight into the system. Alternatively, the Koopman mode analysis of vector-valued observables decomposes the dynamics in the measurement space into products of vector-valued coefficients of eigenfunctions of the Koopman operator and their eigenvalues.

The theory behind the Koopman operator is well-developed for the case when the dynamical system is measure-preserving and the operator is unitary on an L^2 space of observables. In these cases, the operator is spectral and we can get a full resolution of the dynamics by computing spectral projections. In this paper, we extend the methodology to the case of a family of operators which the system switches between. In particular, we assume there is some, finite or countable, family of Koopman operators driving the evolution of some fixed vector-valued observable. The operators are assumed to have point spectrum restricted to the unit circle, but the operators are not assumed to be unitary; in particular, the eigenfunctions are not assumed orthogonal. Furthermore, the dynamics are assumed to evolve by a switch-and-hold process; the observable's dynamics are driven for some amount of time under the actions of one operator before switching to another in the family, under which, the dynamics evolve for a possibly different amount of time.

As we are interested in a network-wide view of the dynamics, the vector-valued observable we consider are traffic matrices of a network. Traffic matrices measure the amount of traffic flowing between pairs of input and output routers of an ISP's backbone. A single element of the traffic matrix corresponds to the amount of data flowing between a specific input and output router in a short window of time. Calling the contiguous time-frame during which a single operator acts a stationary component, Koopman mode analysis shows that on stationary components, traffic matrices of the Abilene network decompose simply into oscillating modes superimposed with gaussian or nearly gaussian noise. As one would expect, strong peaks in the dynamics exists at daily and weekly periods. Between stationary components, these periods still exists, but with varying degrees of intensity. Furthermore, while the noise is nearly gaussian, its intensity from one stationary component to another varies wildly.

Section VII.1 develops the mathematical theory of the switched Koopman families. Relevant definitions and the representation of a vector-valued wide-sense stationary as a spectral integral are given. The useful concept of switch and hold dynamics for a family of wide-sense stationary stochastic processes is defined. Section VII.1.2 develops the Koopman basis algorithm used for approximating the dynamics in the output space. In section VII.2, the Koopman mode analysis is applied to a dataset of traffic matrices for the Abilene network. The final section concludes the chapter with a few remarks and suggestions for future work.

VII.1 Families of Koopman operators

Let $(\Omega, \mathcal{A}, p)$ be a probability space. A weaker form of stationarity that does not require the preservation of a measure is stationarity in the wide sense. It is defined for sequences L^2 functions.

Definition VII.1.1. For $x, y \in L^2_{\mathbb{C}}(\Omega, p)$ define the expectation

$$\mathbb{E}[x] = \int_{\Omega} x(w)p(dw), \tag{VII.1.1}$$

and the covariance

$$\mathsf{cov}(x, y) = \langle x, y\rangle = \int_{\Omega} x(w)\overline{y(w)}p(dw). \tag{VII.1.2}$$

Definition VII.1.2 (Wide-sense stationary sequence (WSS))**.** Let $x_n \in L^2_{\mathbb{C}}(\Omega, p)$. The sequence $\{x_n\}_{n\in\mathbb{Z}}$ is stationary in the wide-sense if for all $n \in \mathbb{Z}$,

(i) $\mathbb{E}[x_n] = \mathbb{E}[x_0]$, and

(ii) $\mathsf{cov}(x_{k+n}, x_k) = \mathsf{cov}(x_n, x_0)$, for all $k \in \mathbb{Z}$.

Definition VII.1.3. Let $\mathscr{F} = \bigotimes_{i=1}^{d} L^2_{\mathbb{C}}(\Omega, p)$. $\boldsymbol{x} \in \mathscr{F}$ is given by $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_d$ where $x_i \in L^2_{\mathbb{C}}(\Omega, p)$ for $i = 1, \ldots, d$. If $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{F}$, then

(i) $\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1) \otimes \cdots \otimes (x_d + y_d)$, and

(ii) $\alpha\boldsymbol{x} = \alpha x_1 \otimes \cdots \otimes \alpha x_d$, for all $\alpha \in \mathbb{C}$.

If $U : L^2_{\mathbb{C}}(\Omega, p) \to L^2_{\mathbb{C}}(\Omega, p)$ is a linear operator, then define $U : \mathscr{F} \to \mathscr{F}$ by $U\boldsymbol{x} = Ux_1 \otimes \cdots \otimes Ux_d$.

Let $\mathscr{F}_{\mathbb{R}}$ be the subspace of $\mathscr{F}$ consisting of those $\boldsymbol{x} \in \mathscr{F}$ such that $x(w) \in \mathbb{R}^d$ for every $w \in \Omega$.

Definition VII.1.4 (Vector-valued WSS)**.** If $\boldsymbol{x}_n = x_{1,n} \otimes \cdots \otimes x_{d,n} \in \mathscr{F}$, $n \in \mathbb{Z}$, is a sequence, we say $\{\boldsymbol{x}_n\}_{z\in\mathbb{Z}}$ is wide sense stationary if for each i, $\{x_{i,n}\}_{n\in\mathbb{Z}}$ is WSS in the sense of definition VII.1.2

Let $\{\boldsymbol{x}_n\}_{n\in\mathbb{Z}} \subset \mathscr{F}$ be a vector-valued WSS sequence. We assume this sequence is given by the action of a family of Koopman operators U_n on a fixed $\boldsymbol{x} \in \mathscr{F}$ and that each U_n is spectral:

$$\boldsymbol{x}_n = U_n\boldsymbol{x} = \int_{\mathbb{C}} \lambda E_n(d\lambda)\boldsymbol{x}, \tag{VII.1.3}$$

where $E_n : \mathcal{B}(\mathbb{C}) \to \mathscr{L}(\mathscr{F})$ is a projection-valued measure (see def. III.2.1). Each E_n is assumed to have compact support and $\{E_n\}$ is an equibounded family; there is a $K > 0$ such that for all $\Delta \in \mathcal{B}(\mathbb{C})$ and $n \in \mathbb{Z}$, $\|E_n(\Delta)\| \leq K$. We decompose $\mathbb{C}$ into the point spectrum for U_n and its complement. Then

$$U_n\boldsymbol{x} = \int_{\mathbb{C}} \lambda E_n(d\lambda)\boldsymbol{x} = \sum_{\lambda\in\sigma_p(U_n)} \lambda E_n(\{\lambda\})\boldsymbol{x} + \int_{\mathbb{C}\setminus\sigma_p(U_n)} \lambda E_n(d\lambda)\boldsymbol{x}. \tag{VII.1.4}$$

If we assume that $\sigma_p(U_n) = \sigma_p(U_m)$ for all $m, n \in \mathbb{Z}$ and furthermore that the point spectrum is restricted to the unit circle and the eigenspaces for U_n and U_m are identical, then we get that (VII.1.4) takes the form

$$\boldsymbol{x}_n = U_n\boldsymbol{x} = \sum_{e^{i2\pi\theta}\in\sigma_p} e^{i2\pi\theta n} P_\theta \boldsymbol{x} + \int_{\mathbb{C}\setminus\sigma_p} \lambda E_n(d\lambda)\boldsymbol{x}, \tag{VII.1.5}$$

where σ_p is the common point spectrum and $P_\theta = E_n(\{e^{i2\pi\theta}\})$ is the projection operator from $\mathscr{F}$ onto the eigenspace V_θ corresponding to $e^{i2\pi\theta}$.

The representation (VII.1.5) allows a cyclic process in $\mathbb{C}^d$ superimposed with "noise". The cyclic nature comes from the point spectrum restricted to the unit circle. The "noisy" part comes from the spectral integral over $\mathbb{C} \setminus \sigma_p$. Consider a one-dimensional eigenspace V_θ corresponding to $e^{i2\pi\theta}$ with $\psi \in \mathscr{F}$ a corresponding eigenfunction. For every $w \in \Omega$, both $\mathrm{lin}[\psi(w)]$ and $\mathrm{lin}[E_n(\Delta)\boldsymbol{x}(w)]$ are subspaces of

$\mathbb{C}^d$ for each $n \in \mathbb{Z}$ and $\Delta \in \mathcal{B}(\mathbb{C})$. Assume for some $w \in \Omega$, that for each $n \in \mathbb{Z}$ there is a $\Delta \in \mathcal{B}(\mathbb{C}) \setminus \sigma_p$ such that $\operatorname{lin}[E_n(\Delta)\boldsymbol{x}(w)] \supset \operatorname{lin}[\psi(w)]$. Then there are sets $\Delta_n \in \mathcal{B}(\mathbb{C}) \setminus \sigma_p$ such that $\emptyset \neq \operatorname{lin}[E_n(\Delta_n)x(w)] \subset \operatorname{lin}[\psi(w)]$. In such a case, for all $n \in \mathbb{Z}$,

$$e^{i2\pi\theta n} P_\theta \boldsymbol{x}(w) + \int_{\Delta_n} \lambda E_n(d\lambda)\boldsymbol{x}(w) \in \operatorname{lin}[\psi(w)].$$

The sequence of Δ_n's for which this is true is determined by how E_n changes with n. As Δ_n varies, we get a process on $\operatorname{lin}[\psi(w)]$ that is deterministic plus the contribution given by the integral over Δ_n.

Now if we also assume that $\{\boldsymbol{x}_n\}$ is in the subspace $\mathscr{F}_\mathbb{R}$ of $\mathscr{F}$ taking values in $\mathbb{R}^d$, then the point spectrum occurs in complex conjugate pairs and we can replace the spectral integral $\int_{\mathbb{C}\setminus\sigma_p} \lambda E_n(d\lambda)$ with a different spectral integral over $\mathbb{R}$. The (VII.1.5) becomes

$$\boldsymbol{x}_n = U_n \boldsymbol{x} = \sum_{e^{i2\pi\theta} \in \sigma_p} e^{i2\pi\theta n} P_\theta \boldsymbol{x} + \int_\mathbb{R} \lambda F_n(d\lambda)\boldsymbol{x}, \tag{VII.1.6}$$

where $F_n : \mathcal{B}(\mathbb{R}) \to \mathscr{L}(\mathscr{F}_\mathbb{R})$ and $P_{-\theta}\boldsymbol{x} = \overline{P_\theta \boldsymbol{x}}$ for all θ. For each Borel set $\Delta \in \mathcal{B}(\mathbb{R})$, $F(\Delta)$ is a projection from $\mathscr{F}_\mathbb{R}$ into $\mathscr{F}_\mathbb{R}$ and, furthermore, $\|F_n(\Delta)\| \leq K$ for all $n \in \mathbb{Z}$ and $\Delta \in \mathcal{B}(\mathbb{R})$ since $\{E_n\}$ is equibounded. Assuming that

$$\hat{F}(\varphi)\boldsymbol{x} = \lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} e^{-i2\pi\varphi k} \int_\mathbb{R} \lambda F_k(d\lambda)\boldsymbol{x} \tag{VII.1.7}$$

exists, the infinite Fourier average gives

$$P_\varphi \boldsymbol{x} + \hat{F}(\varphi)\boldsymbol{x} = \lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} e^{-i2\pi\varphi k} U_k \boldsymbol{x} = \lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} e^{-i2\pi\varphi k} \boldsymbol{x}_k. \tag{VII.1.8}$$

Also, $\hat{F}(-\varphi) = \overline{\hat{F}(\varphi)}$. For $\hat{F}(\varphi)$ small, $P_\varphi \boldsymbol{x} \approx \lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} e^{-i2\pi\varphi k} U_k \boldsymbol{x}$.

VII.1.1 Switch and hold dynamics

Let $\mathcal{A}$ be an index set. For each $\alpha \in \mathcal{A}$, let $\{\boldsymbol{x}_{n,\alpha}\}_{n\in\mathbb{Z}}$ be a WSS sequence in $\mathscr{F}$ satisfying all the assumptions above and assume $\boldsymbol{x}_{n,\alpha} = U_{n,\alpha}\boldsymbol{x}$ for some fixed $\boldsymbol{x} \in \mathscr{F}_{\mathbb{R}}$. According to (VII.1.6), for each $\alpha \in \mathcal{A}$, we have

$$U_{n,\alpha}\boldsymbol{x} = \sum_{e^{i2\pi\theta}\in\sigma_{p,\alpha}} e^{i2\pi\theta n} P_{\theta,\alpha}\boldsymbol{x} + \int_{\mathbb{R}} \lambda F_{n,\alpha}(d\lambda)\boldsymbol{x}$$

where $\sigma_{p,\alpha}$ is the common point spectrum of the family of Koopman operators $\{U_{n,\alpha}\}_{n\in\mathbb{Z}}$ associated with $\{\boldsymbol{x}_{n,\alpha}\}_{n\in\mathbb{Z}}$, $P_{\theta,\alpha}$'s are the corresponding eigenprojections, and $F_{n,\alpha}$ is the associated real-valued spectral measure.

Definition VII.1.5 (Hold times)**.** Let $\boldsymbol{h} = \{h_i \in \mathbb{N}\}_{i\in\mathbb{N}}$, be a set of integers called the hold times. Let $t_0 = 0$ and for $j \geq 1$, define $t_j := \sum_{i=1}^{j} h_i$. We call $[t_j, t_{j+1})$ a contiguous block.

Each $t_j \in \mathbb{N}_0$ represents the times at which the dynamics switch. Let $\{\alpha_j\}_{j\geq 1}$ be a sequence from $\mathcal{A}$. We define a switched stochastic process $\{X_n\}_{n\in\mathbb{N}_0} \subset \mathscr{F}_{\mathbb{R}}$ by

$$X_n = U_{n,\alpha_j}\boldsymbol{x} = \sum_{e^{i2\pi\theta}\in\sigma_{p,\alpha_j}} e^{i2\pi\theta n} P_{\theta,\alpha_j}\boldsymbol{x} + \int_{\mathbb{R}} \lambda F_{n,\alpha_j}(d\lambda)\boldsymbol{x}, \qquad n \in [t_{j-1}, t_j). \tag{VII.1.9}$$

Given the finite nature of the contiguous blocks, the infinite average of (VII.1.8) must be replaced with a finite average:

$$\begin{aligned} P_{\varphi,\alpha_j}\boldsymbol{x} + \hat{F}_{\alpha_j}(\varphi)\boldsymbol{x} &\approx \frac{1}{h_j} \sum_{k=t_{j-1}}^{t_j-1} e^{-i2\pi\varphi k} U_{k,\alpha_j}\boldsymbol{x} \\ &= \frac{1}{h_j} \sum_{k=t_{j-1}}^{t_j-1} e^{-i2\pi\varphi k} X_n \end{aligned} \tag{VII.1.10}$$

We denote the approximate Koopman mode of a realization (as determined by the finite Fourier average (VII.1.10)) as $\boldsymbol{p}_{\varphi,\alpha_j}$, then

$$\boldsymbol{p}_{\varphi,\alpha_j} := \frac{1}{h_j} \sum_{k=t_{j-1}}^{t_j-1} e^{-i2\pi\varphi k} X_n(w), \quad (w \in \Omega). \tag{VII.1.11}$$

The error between the projection and the finite average is order $\mathcal{O}(h_j^{-1})$. Additionally, a fast fourier transform can be used to compute the finite average in $\mathcal{O}(h_j \log h_j)$ time; in this case φ is of the form $\frac{s}{h_j}$ for $s = 0, \ldots, h_j - 1$. In this case, denote $\varphi_s = \frac{s}{h_j}$ and P_{φ_s,α_j} as $P_{s,j}$ and $\boldsymbol{p}_{\varphi_s,\alpha_j}$ as $\boldsymbol{p}_{s,j}$.

VII.1.2 Linear approximation of the dynamics in the output space

Note that the X_n given in (VII.1.9) is an $\mathbb{R}^d$-valued process. Our goal is to approximate realizations of the process in its output space $\mathbb{R}^d$. This is done by building a basis for $\mathbb{C}^d$ on each continuous block $[t_{j-1}, t_j)$ from the approximate Koopman modes of (VII.1.11); that is, to each continuous block there corresponds a basis for $\mathbb{C}^d$ built using the approximate Koopman modes computed on the contiguous block.

Fix $j \geq 1$ and consider the contiguous block $[t_{j-1}, t_j)$. Let $\{\boldsymbol{p}_{s,j} \mid s = 0, \ldots, h_j - 1\}$ be the approximate Koopman modes defined in (VII.1.11) and computed via a fast Fourier transform. Let $\|\boldsymbol{p}_{s,j}\|_{\mathbb{C}^n}$ be its norm. Order the $\boldsymbol{p}_{s,j}$'s by decreasing norm: $\boldsymbol{p}_{s_1,j} \succcurlyeq \boldsymbol{p}_{s_2,j} \succcurlyeq \cdots \succcurlyeq \boldsymbol{p}_{s_{h_j},j}$, where $\boldsymbol{p}_{s_i,j} \succcurlyeq \boldsymbol{p}_{s_k,j}$ if $\|\boldsymbol{p}_{s_i,j}\|_{\mathbb{C}^n} \geq \|\boldsymbol{p}_{s_k,j}\|_{\mathbb{C}^n}$. Note that h_j gives the number of Koopman modes. The basis is built by iteratively adding $\boldsymbol{p}_{s,j}$'s to the basis if they are not a linear combination of the preceding $\boldsymbol{p}_{s,j}$'s.

Theorem VII.1.6 (Koopman Basis algorithm)**.** *A basis for the output space is computed as follows:*

(1) ***Input:*** $\left\{\boldsymbol{p}_{s_{\ell},j} \mid \ell = 1, \ldots, h_j\right\} \subset \mathbb{C}^d$*.*

(2) Let $B_0 = \emptyset$*,* $\ell = 1$*.*

(3) While $\ell < h_j$

  (a) Compute

$$B_{\ell+1} = \begin{cases} B_\ell \cup \boldsymbol{p}_{s_{\ell+1},j}, & \text{if } \operatorname{rank}\{B_\ell \cup \boldsymbol{p}_{s_{\ell+1},j}\} > \operatorname{rank}\{B_\ell\}, \\ B_\ell, & \text{otherwise} \end{cases}.$$

  (b) $\ell \leftarrow \ell + 1$*.*

(4) If $\operatorname{rank}\{B_{h_j}\} < d$*, complete* B_{h_j} *to a basis for* $\mathbb{C}^d$ *by choosing an orthonormal basis* Q *for the orthogonal complement of* $\operatorname{lin} B_{h_j}$*.*

(5) $\mathsf{B}_j := B_{h_j} \cup Q$*.*

(6) ***Output:*** B_j*.*

Projections onto Koopman basis modes. Adjoint bases. Modal dynamics.

Let $\boldsymbol{b}_{\ell,j} \in \mathbb{C}^d$ denote the elements of B_j. Approximate (VII.1.9), at $w \in \Omega$, by projecting $\{X_n'(w)\}_{n=t_{j-1}}^{t_j-1}$ onto B_j, where $X_n' = X_n(w) - \langle X_n(w)\rangle$ and $\langle X_n(w)\rangle$ is the time average (over the variable n) of the realization $X_n(w)$ over the interval $[t_{j-1}, t_j)$; i.e., we wish to represent $X_n(w)$ in the output space as

$$X_n(w) = \langle X_n(w)\rangle + \sum_{\ell=1}^{d} c_{\ell,j}(n)\boldsymbol{b}_{\ell,j}.$$

To project the data $\{X'_n(w)\}_n$ onto B_j, compute an adjoint basis $\mathsf{W}_j = \{\boldsymbol{w}_{1,j}, \ldots, \boldsymbol{w}_{d,j}\}$, where each $\boldsymbol{w}_{\ell,j} \in \mathbb{C}^d$ satisfies $\langle \boldsymbol{b}_{k,j}, \boldsymbol{w}_{\ell,j} \rangle_{\mathbb{C}^d} = \delta_{k,\ell}$ and $\delta_{k,\ell}$ is the Dirac delta function.

Project X'_n onto $\boldsymbol{b}_{\ell,j}$ using the adjoint basis. Letting $c_{\ell,j}(n)$ denote the value of this projection, the coefficient can be written as

$$c_{\ell,j}(n) := \langle X'_n(w), \boldsymbol{w}_{\ell,j} \rangle_{\mathbb{C}^d} = e^{i2\pi\theta_\ell n} + \zeta_{\ell,j}(n),$$

where $\zeta_{\ell,j}(n)$ is the "noise" for the ℓ^{th} Koopman basis mode for the j^{th} contiguous block and is a combination of the continuous part of the spectrum and any Koopman modes $P_{k,j}\boldsymbol{x}(w) \in \mathbb{R}^d$ having components along $\boldsymbol{w}_{\ell,j}$. The complex exponential term comes from the Koopman mode associated with the eigenvalue $\lambda = e^{i2\pi\theta_\ell}$. Let

$$\mathscr{R}_{\ell,j} = \left\{ k \in \{0, \ldots, h_j - 1\} \;\middle|\; \langle P_{k,j}\boldsymbol{x}(w), \boldsymbol{w}_{\ell,j} \rangle_{\mathbb{C}^d} \neq 0 \right\}, \tag{VII.1.12}$$

be the set of labels those Koopman modes that have components along $\boldsymbol{p}_{\ell,j}$. Then using (VII.1.9)

$$\zeta_{\ell,j}(n) = \sum_{k \in \mathscr{R}_{\ell,j}} e^{i2\pi\theta_k n} \langle P_{k,\ell}\boldsymbol{x}(w), \boldsymbol{w}_{\ell,j} \rangle_{\mathbb{C}^d} + \int_R \lambda \left\langle F_{n,\alpha_j}(d\lambda)\boldsymbol{x}(w), \boldsymbol{w}_{\ell,j} \right\rangle_{\mathbb{C}^d} \tag{VII.1.13}$$

Note that $\zeta_{\ell,j}(n)$ contains the error between $P_{\ell,j}\boldsymbol{x}(w)$ and $\boldsymbol{p}_{\ell,j}$ arising from the finite time Fourier average.

The modal dynamics for mode $\boldsymbol{b}_{\ell,j}$ are given by

$$P_{\ell,j}[X'_n(w)] = c_{\ell,j}(n)\boldsymbol{b}_{\ell,j} = e^{i2\pi\theta_\ell n}\boldsymbol{b}_{\ell,j} + \zeta_{\ell,j}(n)\boldsymbol{b}_{\ell,j}$$

where

$$P_{\ell,j}[\cdot] := \langle \cdot, \boldsymbol{w}_{\ell,j} \rangle_{\mathbb{C}^n} \boldsymbol{b}_{\ell,j}$$

is the projection operator from $\mathbb{C}^d$ onto the j^{th} Koopman basis mode[1]. The modal "noise" is computed via

$$\zeta_{\ell,j}(n)\boldsymbol{b}_{\ell,j} = c_{\ell,j}(n)\boldsymbol{b}_{\ell,j} - e^{i2\pi\theta_\ell k}\boldsymbol{b}_{\ell,j}.$$

The first term on the right side is the projection of the dynamics onto the mode and the second term is the analytic prediction of the evolution.

To get real modal dynamics take the projection

$$[P_{\ell,j} + P_{-\ell,j}](\cdot) = \langle \cdot, \boldsymbol{w}_{\ell,j}\rangle_{\mathbb{C}^d}\, \boldsymbol{b}_{\ell,j} + \langle \cdot, \boldsymbol{w}_{-\ell,j}\rangle_{\mathbb{C}^d}\, \boldsymbol{b}_{-\ell,j}; \tag{VII.1.14}$$

the projection onto the j^{th} Koopman basis mode and its complex conjugate. Since the Koopman modes must come in complex conjugate pairs, the real modal "noise" is then given by

$$\boldsymbol{\xi}_{\ell,j}(n) := \underbrace{[P_{\ell,j} + P_{-\ell,j}]\,(X'_k)}_{\text{computed from data}} - \underbrace{\left(e^{i2\pi\theta_\ell n}\boldsymbol{b}_{\ell,j} + e^{-i2\pi\theta_\ell n}\overline{\boldsymbol{b}_{\ell,j}}\right)}_{\text{analytic prediction of modal dynamics}} . \tag{VII.1.15}$$

VII.2 Traffic matrices and the Abilene network

We apply the above model to compute the modal dynamics and modal noise corresponding to traffic matrices of an ISP network.

A typical internet-type network consists of a collection of routers and links between them. The routers consist of border routers that connect to other networks and internal router that form the paths within the domain. In the simplest implementation

[1] There is an abuse of notation here for $P_{\ell,j}$

of TCP traffic, the router dynamics are deterministic[2], whereas the whole system is stochastically driven by the user page requests.

Let $\mathscr{N}$ be a network. Let $\mathscr{I}$ be the set of input routers to the network and $\mathscr{O}$ the set of output routers for the network. There can be internal routers that do not connect to routers outside of the network. Let $\mathscr{P} = \mathscr{I} \times \mathscr{O}$ be the set of source/destination (s/d) pairs.

A traffic matrix captures the amount of traffic flowing between s/d pairs. An index p corresponds to a single source/destination pair of routers. Then each entry, $X_t(p)$, of the traffic matrix captures the amount of network traffic entering the input router and leaving through the paired output router at time t.

Unfortunately, measuring traffic matrices directly is prohibitively expensive and thus they must be inferred from readily obtained link flow counts (obtained via SNMP[3] measurements [Zha+05]). If Y_t are the link counts at time t, then the the link counts and the traffic matrix are related via the identity [Zha+05]

$$Y_t = \mathbf{A} X_t.$$

Here Y_t has dimension equal to the number of links in the systems and $\mathbf{A}$ is the routing matrix. $\mathbf{A}_{\ell p} = 1$ if s/d pair p uses link ℓ and 0 otherwise. Since Y_t can be measured and $\mathbf{A}$ is easily obtained, determining the traffic matrix X_t becomes an inference problem. Unfortunately, there are generally many more s/d pairs than link

[2]In some implementations of congestion control, routers can employ randomized strategies such a Random Early Marking where some packets are randomly assigned a tag that drops them from the queue [LL99].

[3]SNMP stands for Simple Network Management Protocol.

measurements resulting in an ill-posed system. Many different methods for estimating the traffic matrices have been developed (e.g. Tomogravity or those methods based on Principal Component Analysis or Kalman filtering; see [Sou+05] for an overview), each introducing different additional data and assumptions.

In practice, traffic matrices are not instantaneous entities, but the aggregate over some time frame. This induces a discrete-time process X_n, where $X_n(p)$ is the amount of traffic flowing between input router i_p and output router j_p during the k^{th} time interval. The network has a finite capacity. Therefore, $|X_n(p)| \leq C$ for some constant $C > 0$ and all p, k.

In this section, we will be considering traffic matrices of the Abilene network. The Abilene network is an Internet2 high-performance backbone network enabling the development of advanced internet applications. It connects regional network aggregation points and provides Internet2 access for 200+ universities, corporate, and affiliated institutions [Int05]. At the time the dataset was collected, there were 12 such aggregation points (see figure VII.1 and table VII.1); the output space for our model is thus $\mathbb{R}^{144}$. Therefore $X_n \in \mathbb{R}^{144}$ for all $n \geq 0$. We apply the above switched Koopman family analysis to a dataset consisting of traffic matrices collected over 24 weeks in 2004. Each sample X_n is the aggregate flow over 5 minutes. The data used set can be found at [Zha04].

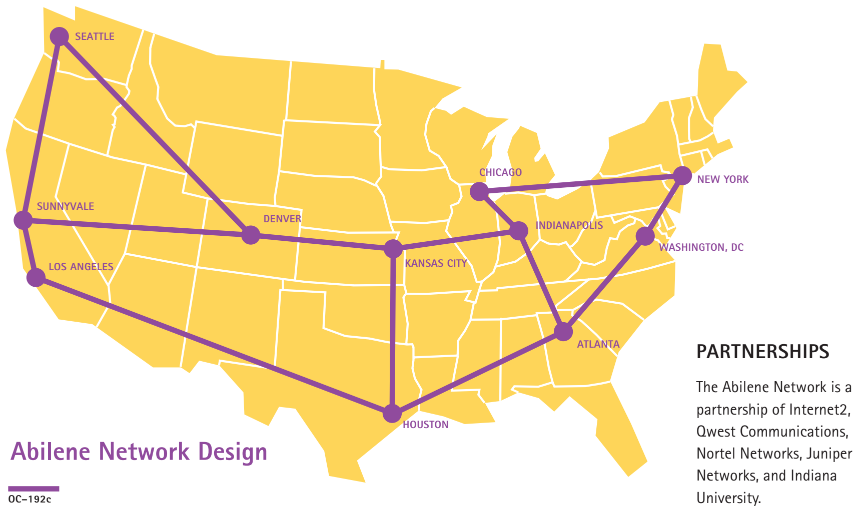


Figure VII.1: Abilene research network. From [Int05]

VII.2.1 Spectrum of Abilene traffic matrices.

Koopman modes for contiguous time frames of 4 weeks (8064 data sample) were computed using (VII.1.10). Figure VII.2 shows the results with the traffic mean (0 cycles/week) removed for the first 4 weeks. The plot shows the intensity of the Koopman modes corresponding to each frequency. On the horizontal axis, the frequency in cycles per week is plotted, whereas points on the vertical axis correspond to source/destination pairs. For a fixed frequency on the horizontal axis, the vertical streak above it is the intensity of the associated Koopman mode. The intensity of the Koopman is given by taking the modulus of each element of the mode. A strong vertical streak in this plot means that the network has a significant periodic component at that frequency; most of the s/d pairs oscillate at that frequency. As can be

Table VII.1: Input and Ouput Routers to the Abilene Network

Input Links	Link Index	Output Links	Link Index
*, ATLA-M5	31	ATLA-M5, *	32
*, ATLAng	33	ATLAng, *	34
*, CHINng	35	CHINng, *	36
*, DNVRng	37	DNVRng, *	38
*, HSTNng	39	HSTNng, *	40
*, IPLSng	41	IPLSng, *	42
*, KSCYng	43	KSCYng, *	44
*, LOSAng	45	LOSAng, *	46
*, NYCMng	47	NYCMng, *	48
*, SNVAng	49	SNVAng, *	50
*, STTLng	51	STTLng, *	52
*, WASHng	53	WASHng, *	54

seen, a strong 24 hour cycle can be seen to be embedded in the traffic; this is the vertical streak plotted at 7 cycles/week in the figure. Figure VII.3, corresponding to weeks 9 through 12, shows that in addition to a daily cycle, there is a weekly cycle as exhibited by the vertical streak at 1 cycle/week. Although not shown here, cycles corresponding to 12 and 8 hour periods (14 and 21 cycles/week, respectively) also appear. One would expect this type of behavior as it correlates with human behavior. Strong horizontal streaks in the plot signify the "bursty" behavior of internet traffic. This "burstiness" phenomena is exhibited in the time trace of the traffic volume between s/d pairs in figure VII.4. Such a time trace is typical. The horizontal streaks should not be surprising since we are basically taking a Fourier transform of something that looks like a delta measure. It can also be seen from figures VII.2 and VII.3, that the first 12 s/d pairs carry comparatively little traffic as exhibited by weak color of the plot. These 12 s/d pairs correspond to the input link ATLA-M5 input link.

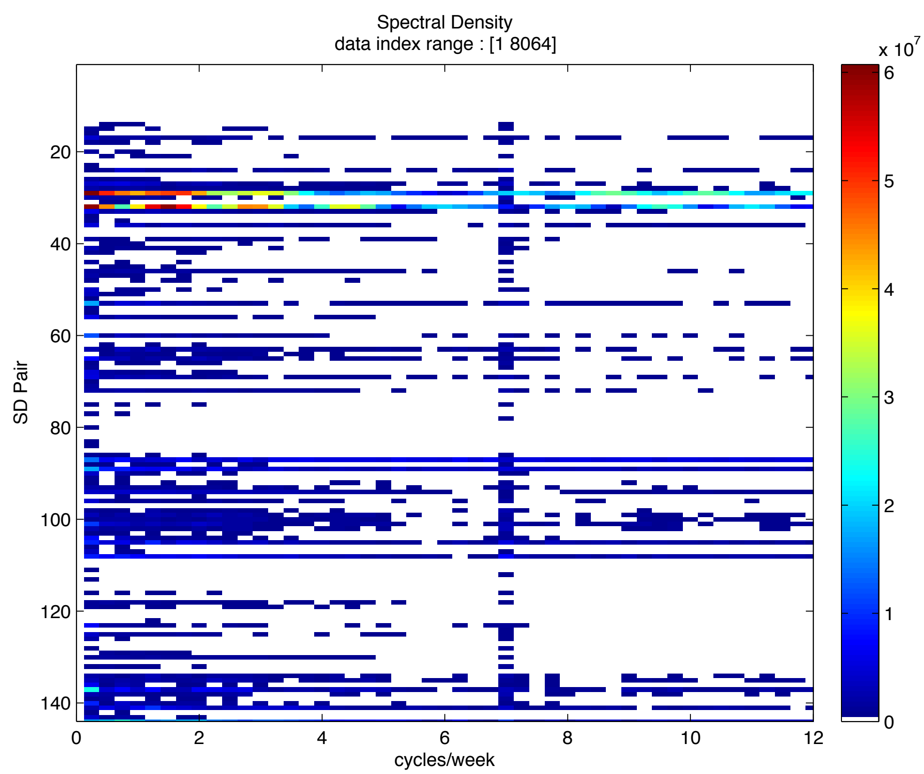


Figure VII.2: Modulus of elements of the Koopman modes for weeks 1 through 4.

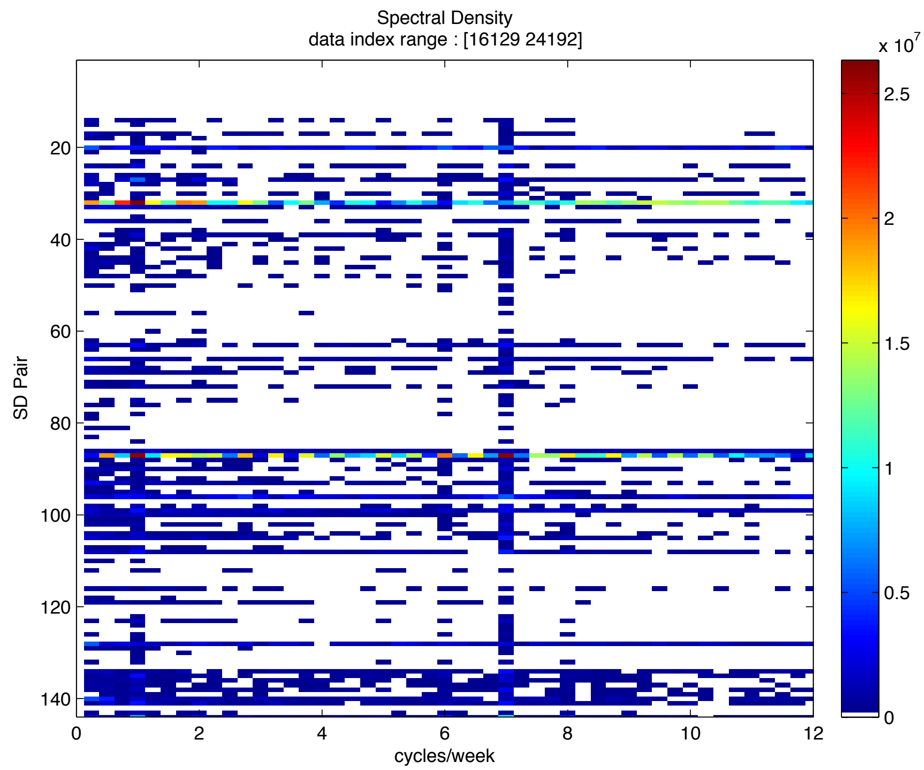


Figure VII.3: Modulus of elements of the Koopman modes for weeks 9 through 12.

VII.2.2 Real modal dynamics for the Abilene network. Distributions of the real modal noise, $\boldsymbol{\xi}_{\ell,j}(n)$.

A time window of 4 weeks was chosen to compute projections over. This corresponded to 8064 samples. The Koopman modes were computed with the Fourier averages (VII.1.10). For each contiguous time frame, Koopman modes and Koopman bases were recomputed via the process in theorem VII.1.6 and the data (in that time window) was projected onto the new basis modes.

Histograms of the real modal noise $\{\boldsymbol{\xi}_{\ell,j}(n)\}$, $n = 0, \ldots, K$ are also shown. The

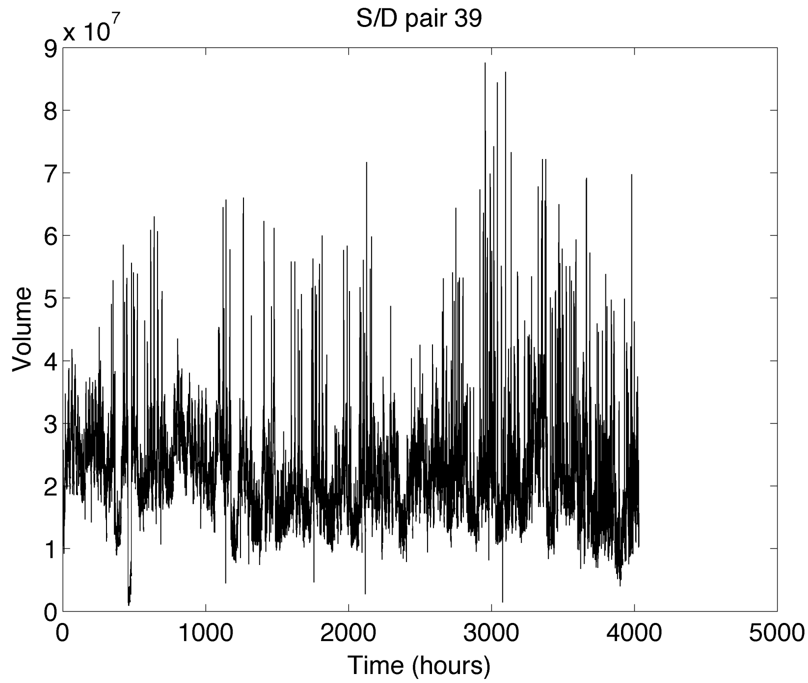


Figure VII.4: Traffic volume on s/d pair 39.

bin widths of the histogram were chosen to minimize the estimated L^2 risk function:

$$h^* = \arg\min_h \frac{2\overline{m} - v}{h^2}$$

where $\overline{m}$ and v are the mean and biased variance of a histogram with bin width h:

$$\overline{m} = \frac{1}{I_h}\sum_{i=1}^{I_h} m_i \qquad \text{and} \qquad v = \frac{1}{I_h}\sum_{i=1}^{I_h}(m_i - \overline{m})^2,$$

where m_i is the count of bin i and I_h is the number of bins when the bin width is h. Then h^* minimizes the discrepancy between the histogram, H, and the true density function, f, where the discrepancy is $\mathbb{E}\left[\int (H(x) - f(x))^2 dx\right]$. This obviates the need to manually select the bin size, at the expense of run-time complexity [FD81].

Let $\mathsf{M}^n_{\pm f}$ denote the the real modal dynamics at time k corresponding to f cycles/week:

$$\mathsf{M}^n_{\pm f} := \left(\mathsf{P}_{f \text{ cycle/week}} + \mathsf{P}_{-f \text{ cycle/week}}\right)[X'_n]\,.$$

For each $n \in \mathbb{N}_0$, $\mathsf{M}^n_{\pm f} \in \mathbb{R}^{144}$;

$$\mathsf{M}^n_{\pm f} = \begin{bmatrix} \mathsf{M}^n_{\pm f}(1) \\ \vdots \\ \mathsf{M}^n_{\pm f}(144) \end{bmatrix}$$

where $\mathsf{M}^k_{\pm 1}(s)$, $s = 1, \ldots, 144$, corresponds to the real modal dynamics at time k for s/d pair s.

Figure VII.5a, shows $\mathsf{M}^n_{\pm 1}(32)$ for the first 4 week window for the real modal dynamics corresponding to ± 1 cycle/week. The blue in the figure, corresponds to the projection of the full dynamics onto the real mode (eq. (VII.1.14)), whereas the red curve is the analytic prediction of the modal dynamics for s/d pair 32 (the second term on the right side of (VII.1.15)). One can see that this choice of window was not the correct one, with a switch in the dynamics occurring just before the two week mark. The left image of figure VII.5b shows the empirical distribution of the real modal noise in the blue histogram and a normal distribution with parameters estimated from the empirical distribution in red. The real modal noise is defined by (VII.1.15). The right image of figure VII.5b shows is a normality plot. Again the data is in blue with the red line corresponding to the normal distribution. The more the blue data points follow the red line, the closer the data is to a normal distribution. As can be seen the data is approximately normal, being linear between the probabilities

0.05 and 0.95. Figures VII.6 - VII.10 show the plots analogous to figure VII.5 for each subsequent four week window. As seen, the real modal noise is approximately Gaussian. This situation is typical for all source destination pairs and frequencies.

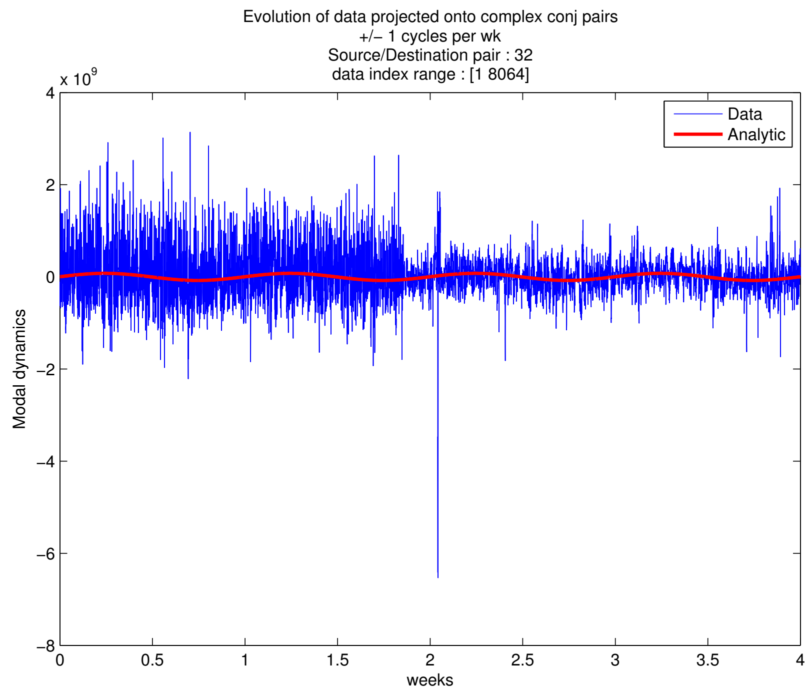
In figure VII.8a, it looks like the analytic prediction (red line) is a flat line with no periodicity, which would be incorrect. However, this is an artifact of the plot scale. In the 13 - 16 week window, the order of the magnitude of the noise is $\sim 10^{10}$ which is much greater than the order of magnitude of the analytic prediction, resulting in the seemingly flat line.

Figure VII.11 show the same modal projections smoothed over a 6 hr window using a triangular weight function; i.e.,

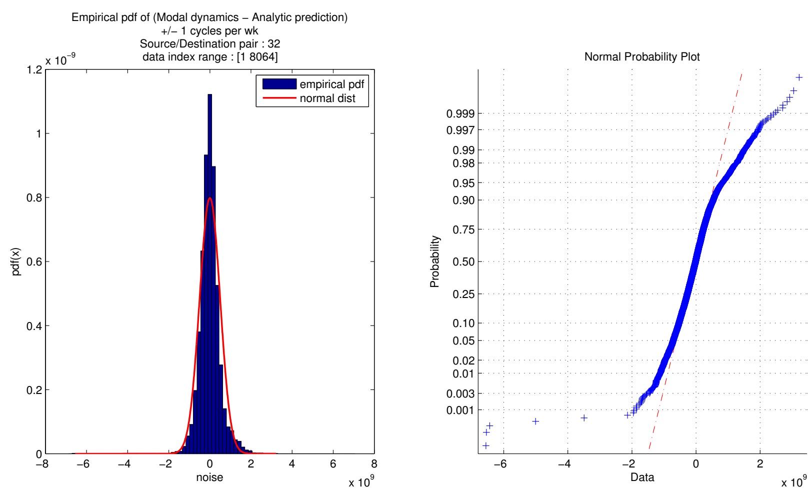
$$\tilde{\mathsf{M}}^n_{\pm f} = (\mathsf{M}_{\pm f} * \Phi)(n) = \begin{bmatrix} \sum_j \mathsf{M}^j_{\pm f}(1)\Phi(n-j) \\ \vdots \\ \sum_j \mathsf{M}^j_{\pm f}(144)\Phi(n-j) \end{bmatrix}$$

where $\tilde{\mathsf{M}}^n_{\pm f}$ denotes the smoothed modal projection, Φ is the weight function which satisfies $\|\Phi\|_1 = 1$, and $*$ denotes convolution. The smoothing kills off part of the contribution due to the noise and it is clearer that the oscillatory behavior of the data follows the analytic prediction. The only exception is weeks 13 to 16, where it looks like the analytic prediction is a flat line. This is due to the difference in scales between the analytic prediction and the noise and follows directly from the discussion of figure VII.8a.

Figures VII.12 - VII.17 show the corresponding results for the 7 cycle/week Koop-

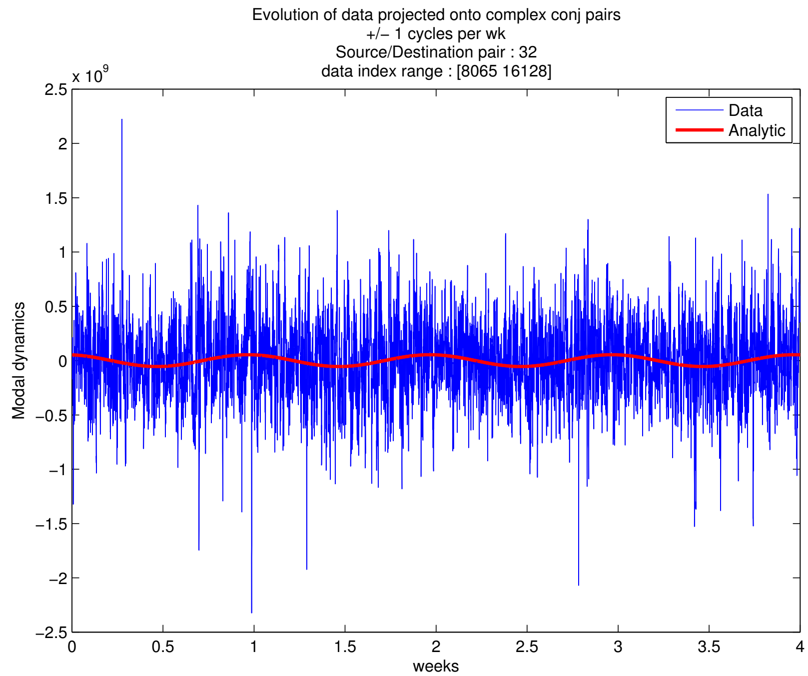


(a) Real modal dynamics. Blue is the projection, $\mathsf{M}^k_{\pm 1}(32)$, of the data onto the Koopman mode. Red is the analytic prediction of the modal dynamics.

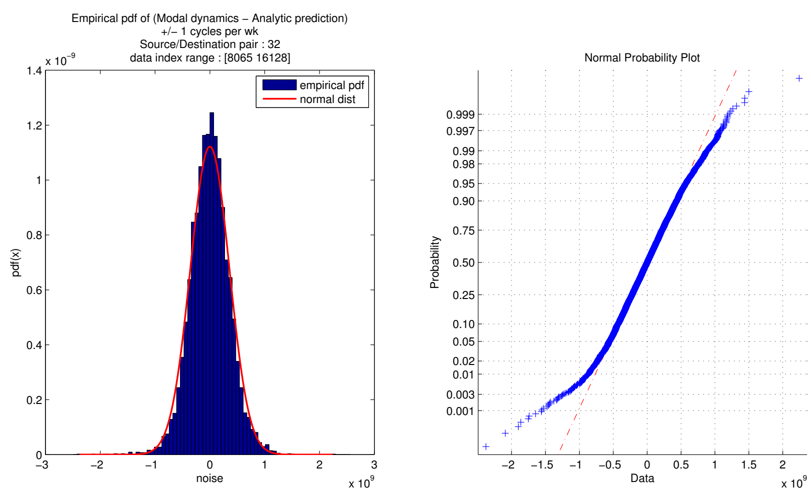


(b) Distribution of the noise (see eq. (VII.1.15)).

Figure VII.5: Weeks 1 through 4. ±1 cycle/ week mode. S/D pair 32.

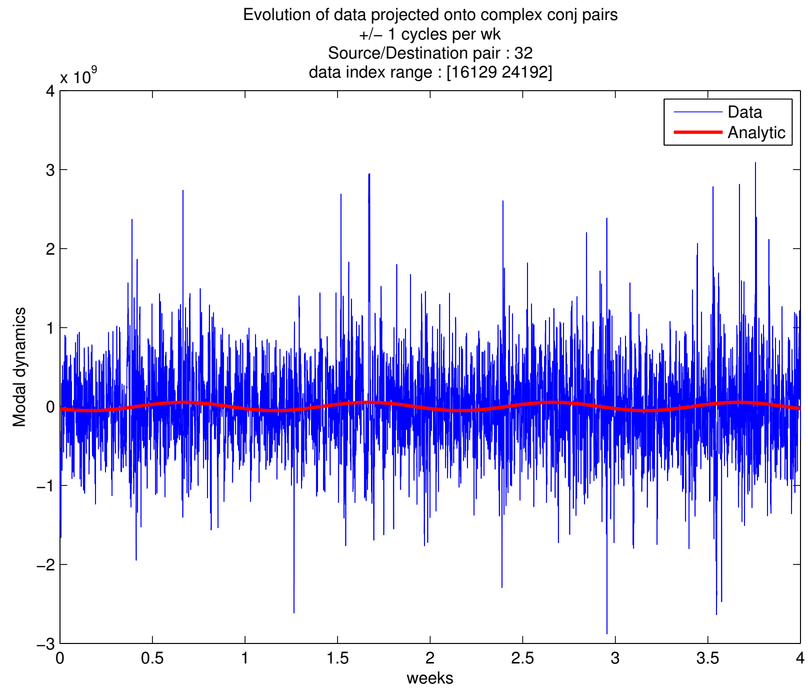


(a) Real modal dynamics. Blue is the projection, $\mathsf{M}^k_{\pm 1}(32)$, of the data onto the Koopman mode. Red is the analytic prediction of the modal dynamics.

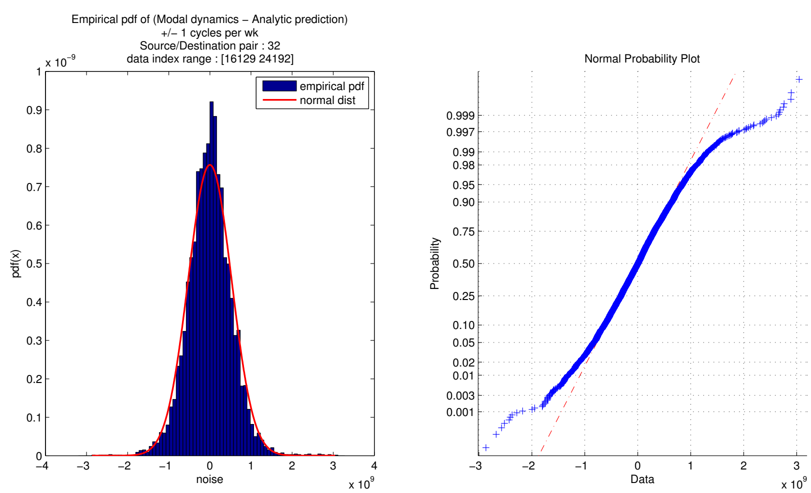


(b) Distribution of the noise (see eq. (VII.1.15)).

Figure VII.6: Weeks 5 through 8. ±1 cycle/ week mode. S/D pair 32.

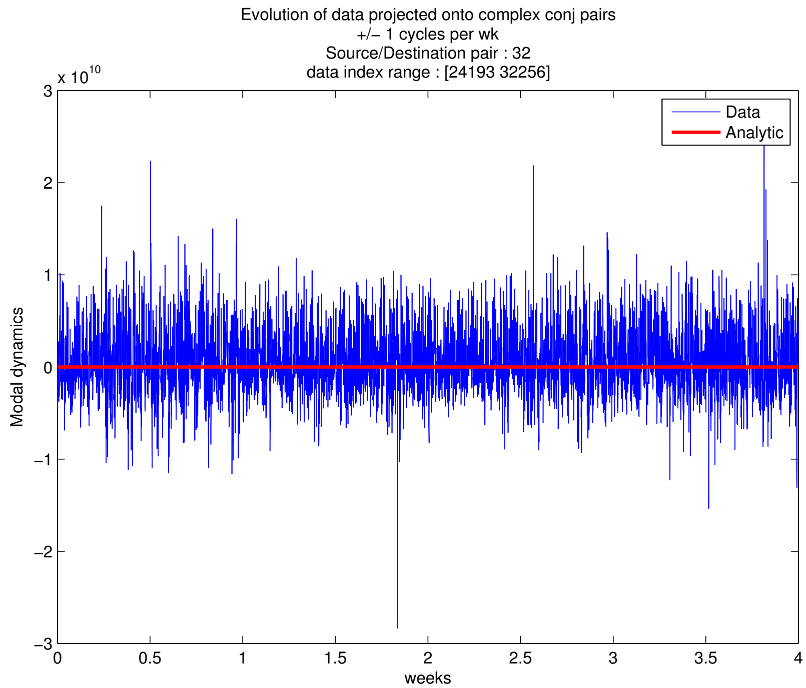


(a) Real modal dynamics. Blue is the projection, $\mathsf{M}^k_{\pm 1}(32)$, of the data onto the Koopman mode. Red is the analytic prediction of the modal dynamics.

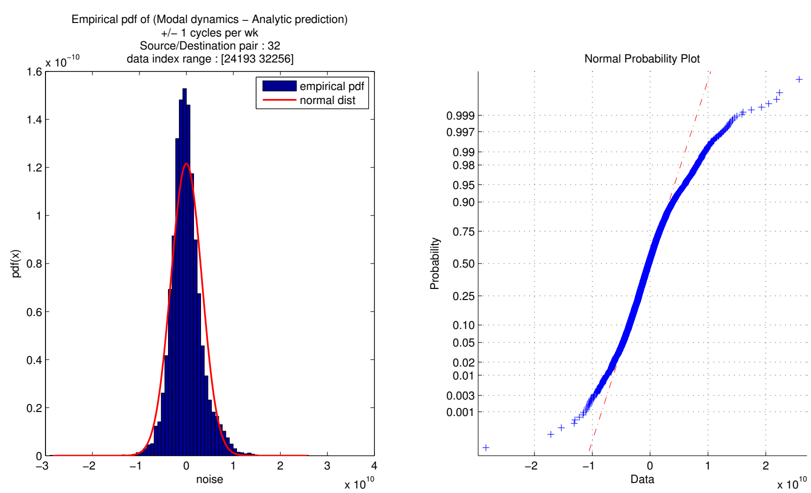


(b) Distribution of the noise (see eq. (VII.1.15)).

Figure VII.7: Weeks 9 through 12. ±1 cycle/ week mode. S/D pair 32.

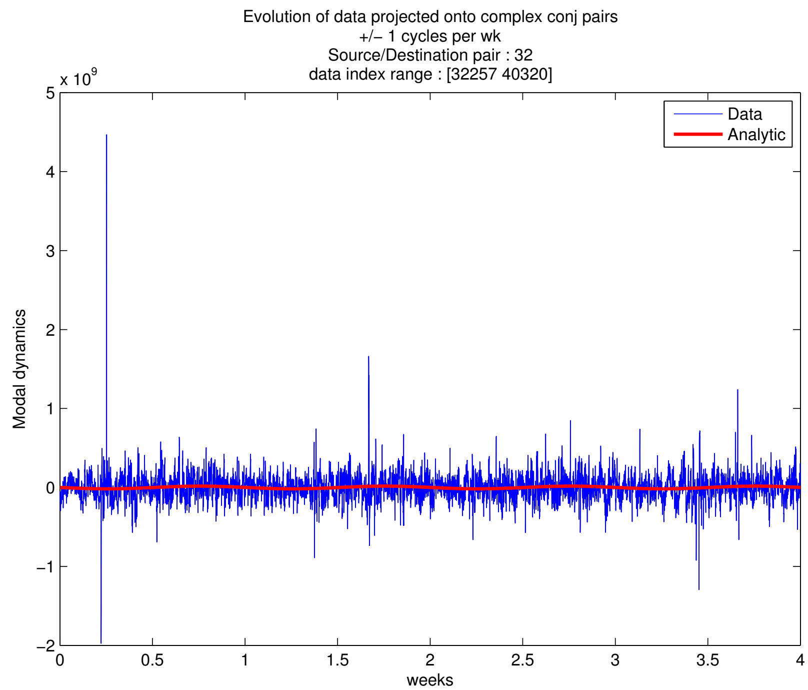


(a) Real modal dynamics. Blue is the projection, $\mathsf{M}^{k}_{\pm 1}(32)$, of the data onto the Koopman mode. Red is the analytic prediction of the modal dynamics.

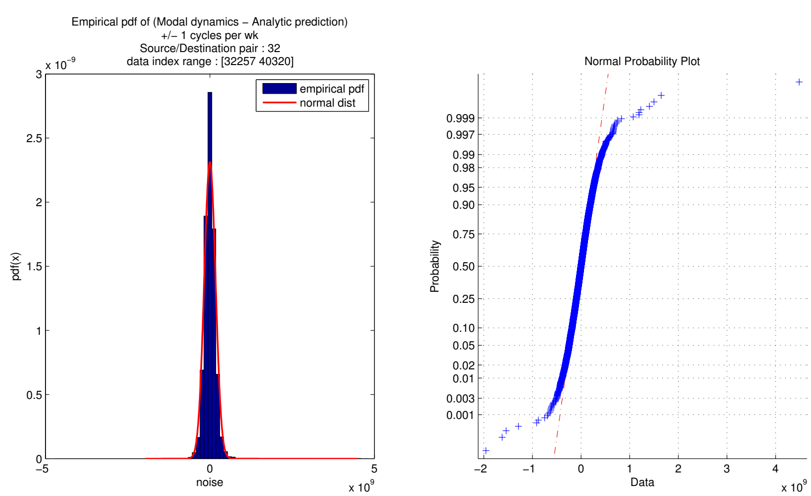


(b) Distribution of the noise (see eq. (VII.1.15)).

Figure VII.8: Weeks 13 through 16. ±1 cycle/ week mode. S/D pair 32.

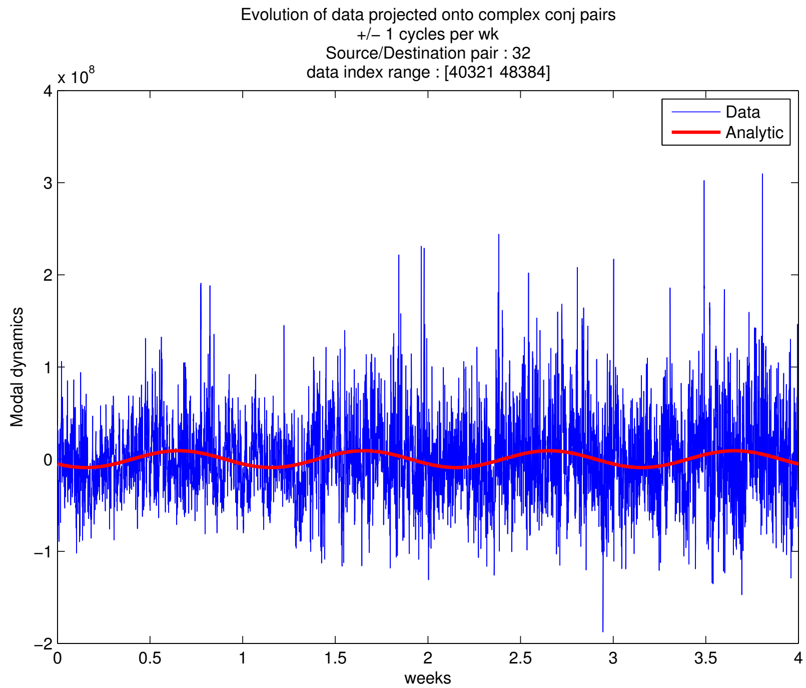


(a) Real modal dynamics. Blue is the projection, $\mathsf{M}^k_{\pm 1}(32)$, of the data onto the Koopman mode. Red is the analytic prediction of the modal dynamics.

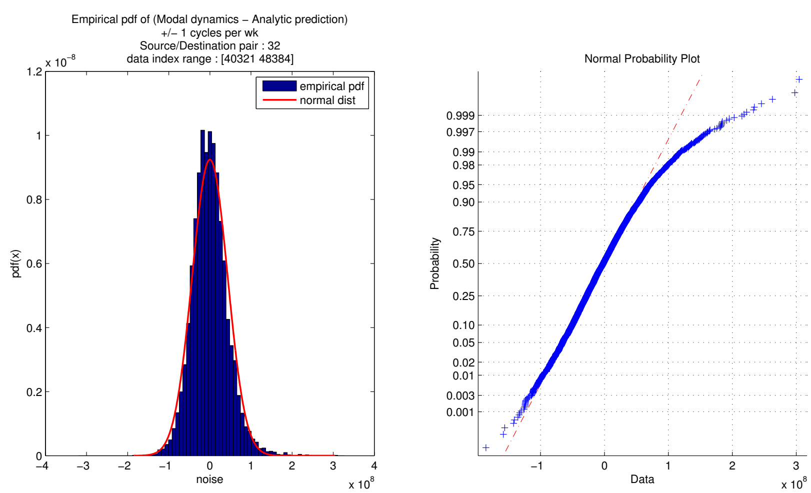


(b) Distribution of the noise (see eq. (VII.1.15)).

Figure VII.9: Weeks 17 through 21. ±1 cycle/ week mode. S/D pair 32.

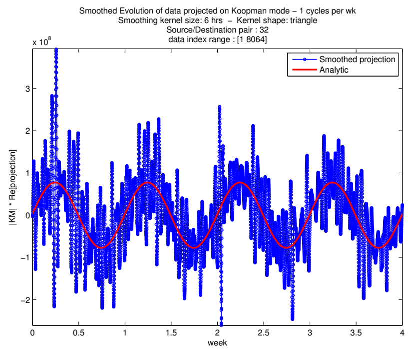


(a) Real modal dynamics. Blue is the projection, $\mathsf{M}^k_{\pm 1}(32)$, of the data onto the Koopman mode. Red is the analytic prediction of the modal dynamics.

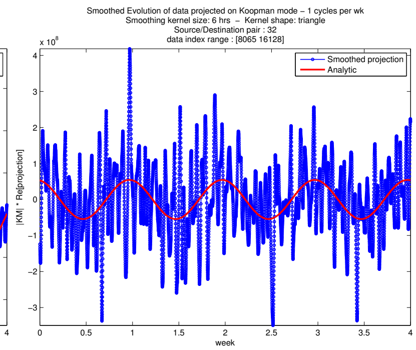


(b) Distribution of the noise (see eq. (VII.1.15)).

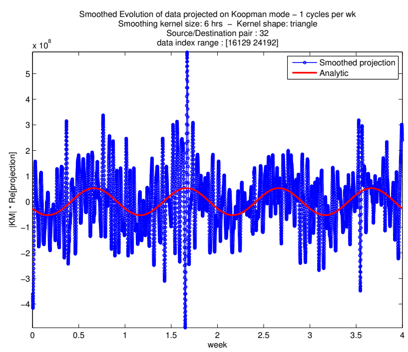
Figure VII.10: Weeks 21 through 24. ±1 cycle/ week mode. S/D pair 32.



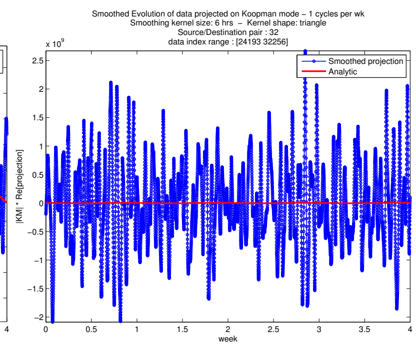
(a) Weeks 1 - 4.



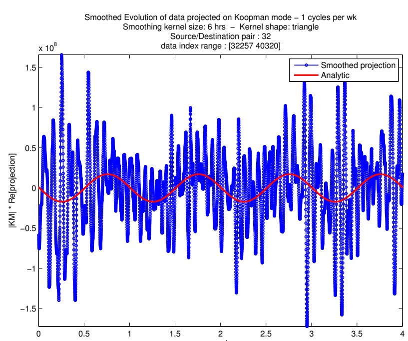
(b) Weeks 5 - 8.



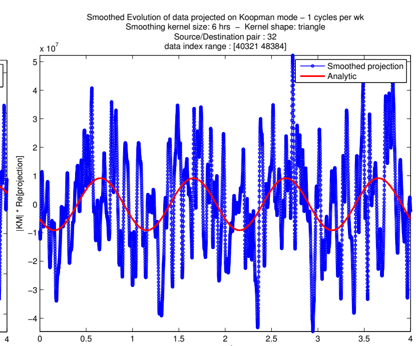
(c) Weeks 9 - 12.



(d) Weeks 13 - 16.



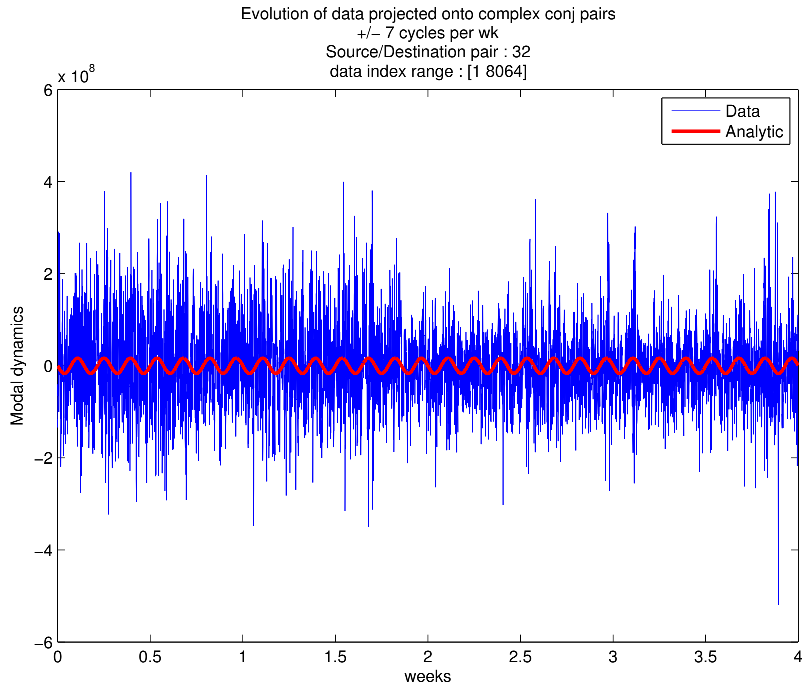
(e) Weeks 17 - 20.



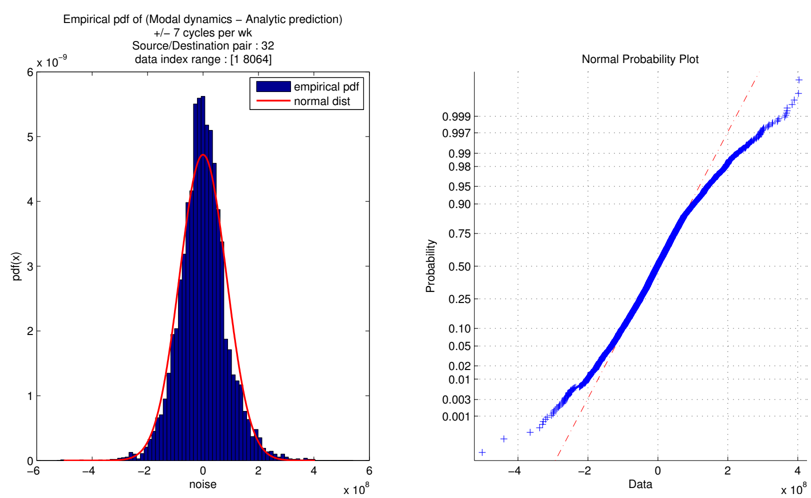
(f) Weeks 21 - 24.

Figure VII.11: Smoothed real modal dynamics (±1 cycle/wk) for S/D pair 32. The smoothed dynamics are in blue, the analytic prediction is in red.

man mode and figure VII.18 shows the smoothed versions. The same qualitative conclusions can be drawn, the difference being that the noise does not seems to be as close to Gaussian as for the previous mode.

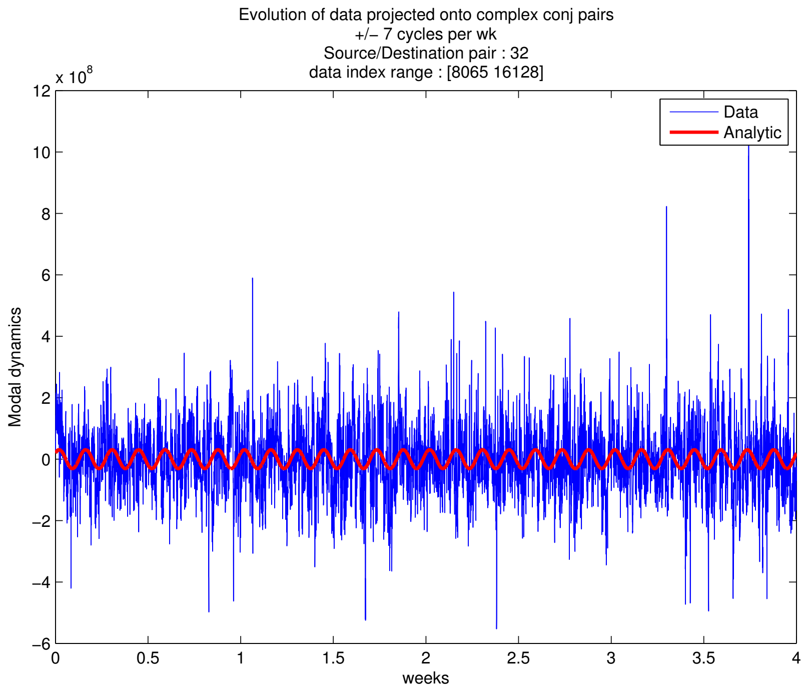


(a) Real modal dynamics. Blue is the projection, $\mathsf{M}^{k}_{\pm 7}(32)$, of the data onto the Koopman mode. Red is the analytic prediction of the modal dynamics.

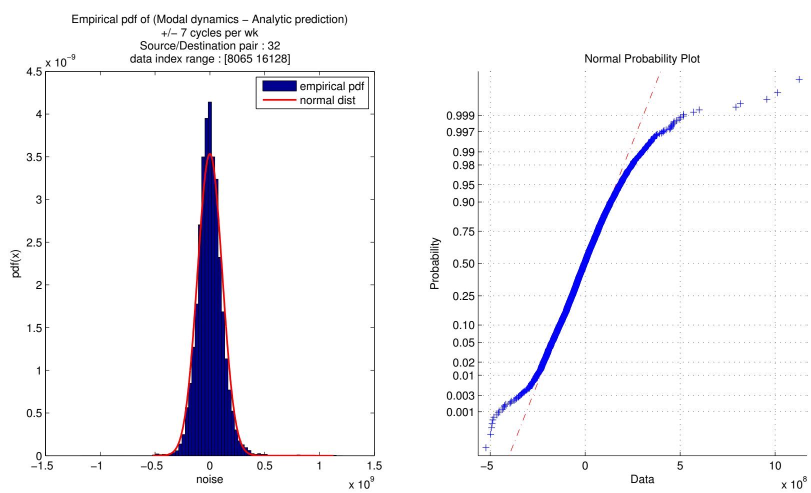


(b) Distribution of the noise (see eq. (VII.1.15)).

Figure VII.12: Weeks 1 through 4. ± 7 cycle/ week mode. S/D pair 32.

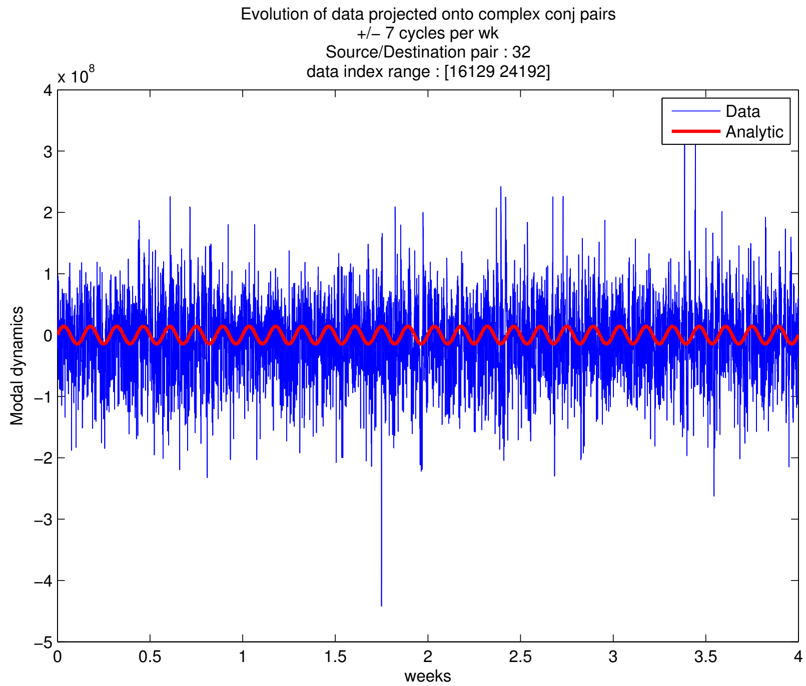


(a) Real modal dynamics. Blue is the projection, $\mathsf{M}^k_{\pm 7}(32)$, of the data onto the Koopman mode. Red is the analytic prediction of the modal dynamics.

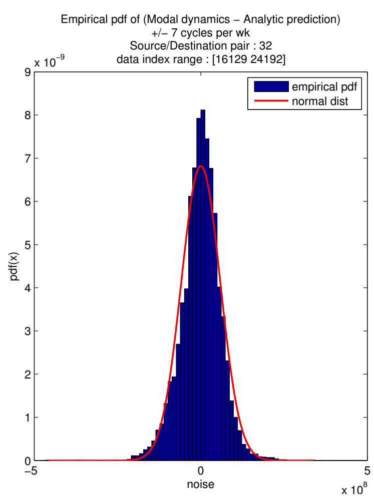


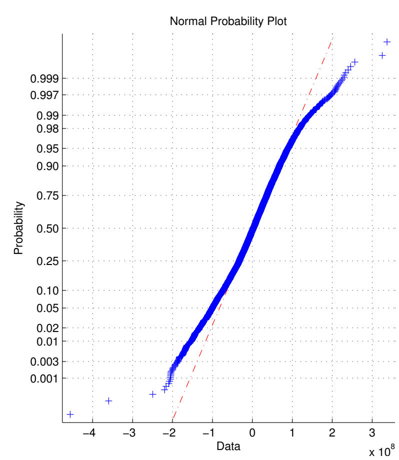
(b) Distribution of the noise (see eq. (VII.1.15)).

Figure VII.13: Weeks 5 through 8. ±7 cycle/ week mode. S/D pair 32.



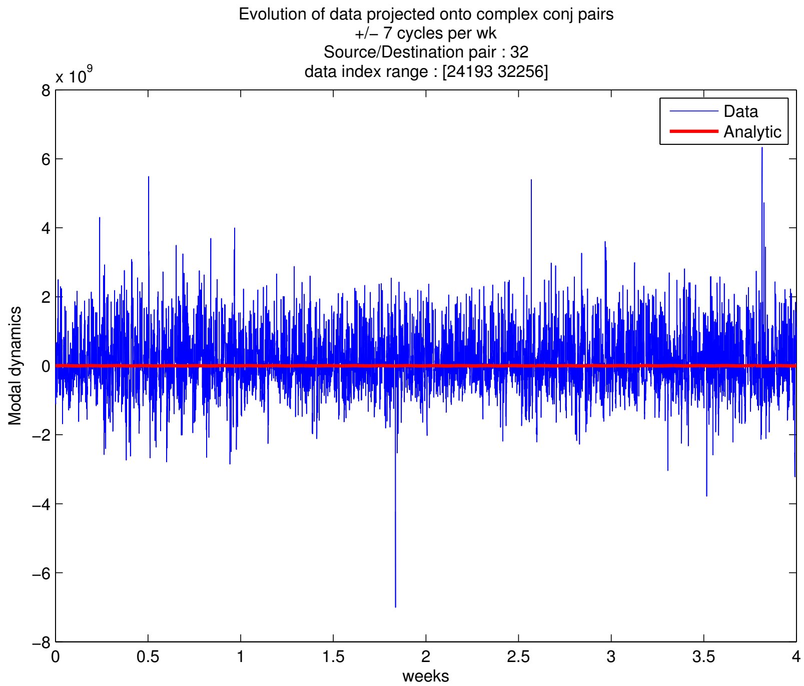
(a) Real modal dynamics. Blue is the projection, $\mathsf{M}^k_{\pm 7}(32)$, of the data onto the Koopman mode. Red is the analytic prediction of the modal dynamics.



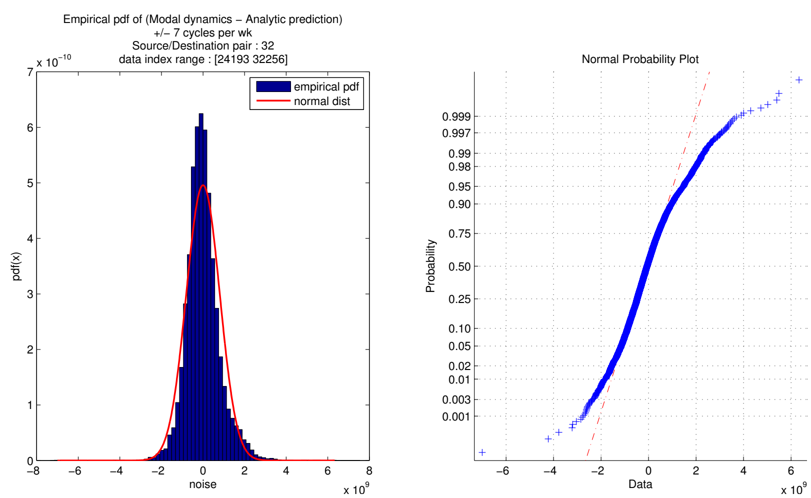


(b) Distribution of the noise (see eq. (VII.1.15)).

Figure VII.14: Weeks 9 through 12. ±7 cycle/ week mode. S/D pair 32.

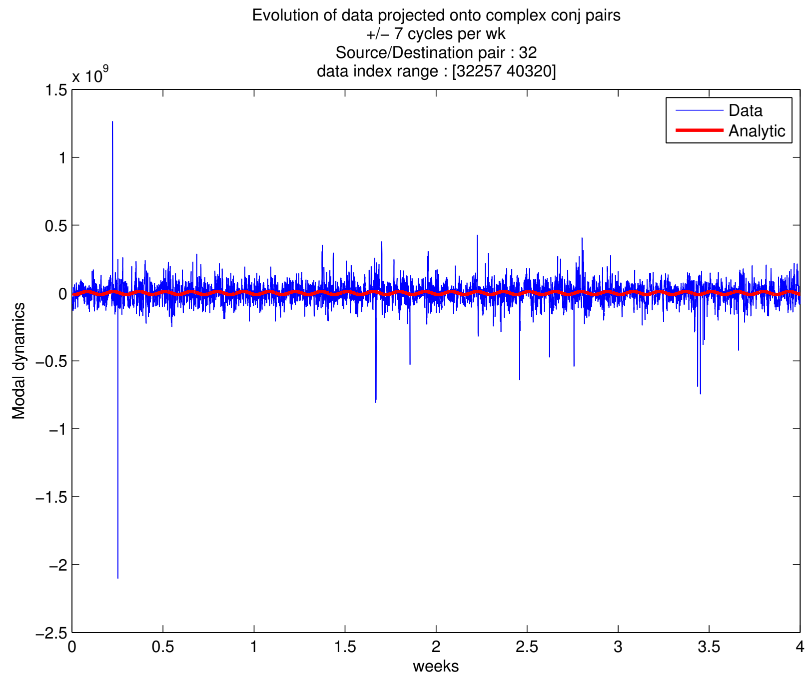


(a) Real modal dynamics. Blue is the projection, $\mathsf{M}^k_{\pm 7}(32)$, of the data onto the Koopman mode. Red is the analytic prediction of the modal dynamics.

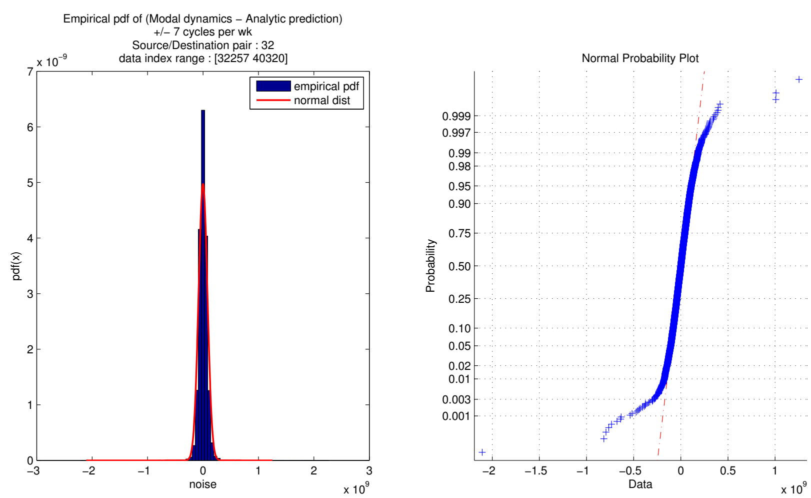


(b) Distribution of the noise (see eq. (VII.1.15)).

Figure VII.15: Weeks 13 through 16. ±7 cycle/ week mode. S/D pair 32.

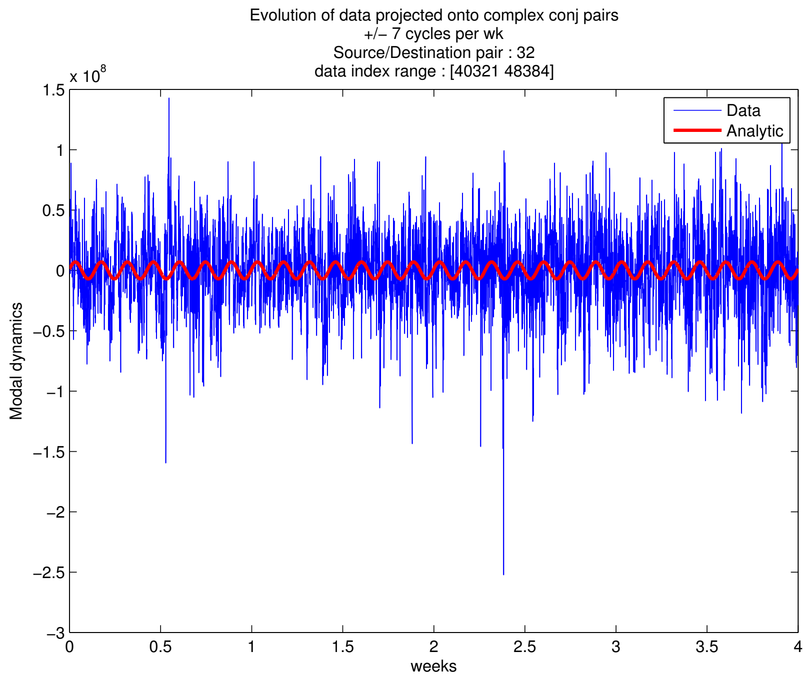


(a) Real modal dynamics. Blue is the projection, $\mathsf{M}^k_{\pm 7}(32)$, of the data onto the Koopman mode. Red is the analytic prediction of the modal dynamics.

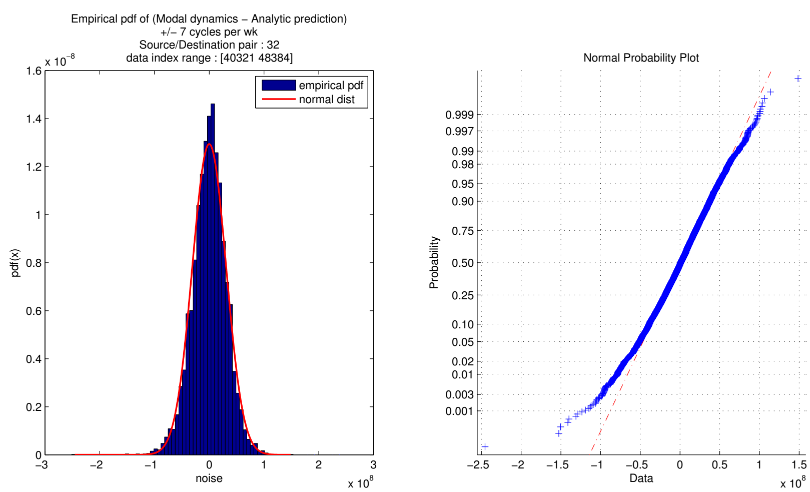


(b) Distribution of the noise (see eq. (VII.1.15)).

Figure VII.16: Weeks 17 through 21. ±7 cycle/ week mode. S/D pair 32.

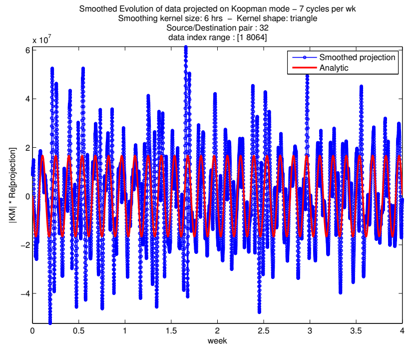


(a) Real modal dynamics. Blue is the projection, $\mathsf{M}^k_{\pm 7}(32)$, of the data onto the Koopman mode. Red is the analytic prediction of the modal dynamics.

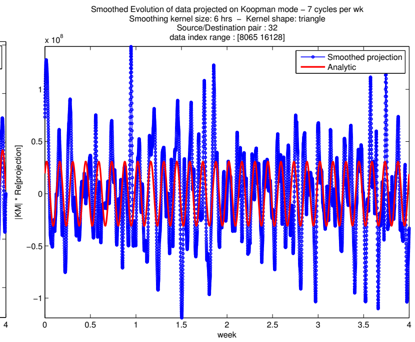


(b) Distribution of the noise (see eq. (VII.1.15)).

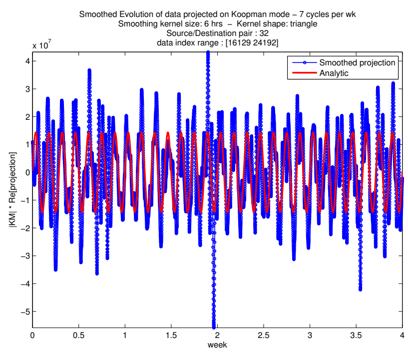
Figure VII.17: Weeks 21 through 24. ±7 cycle/ week mode. S/D pair 32.



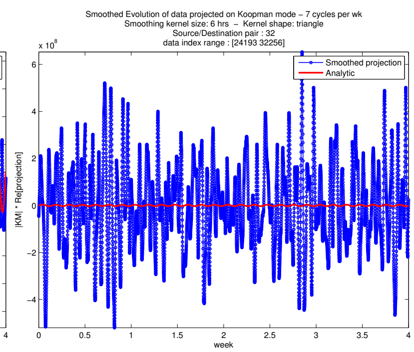
(a) Weeks 1 - 4.



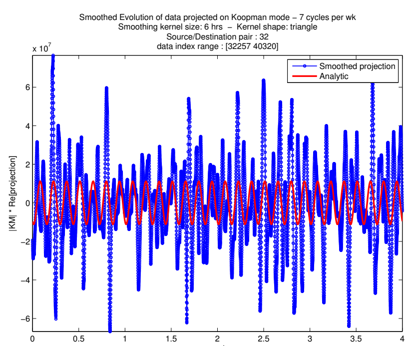
(b) Weeks 5 - 8.



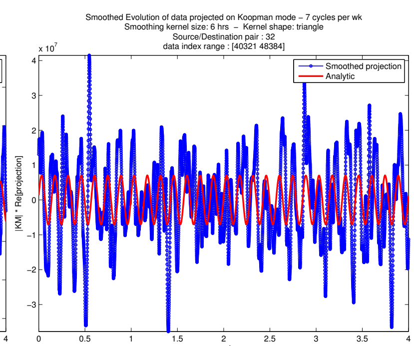
(c) Weeks 9 - 12.



(d) Weeks 13 - 16.



(e) Weeks 17 - 20.



(f) Weeks 21 - 24.

Figure VII.18: Smoothed real modal dynamics (± 7 cycles/wk) for S/D pair 32. The smoothed dynamics are in blue, the analytic prediction is in red.

VII.3 Some remarks on future work

Using the switched Koopman family model, we see that the dynamics over contiguous time blocks can be represented by a superposition of noisy oscillating modes. The noise is Gaussian or nearly Gaussian in nature. The magnitude depends on the behavior of the real spectral measure $F_{n,\alpha}$ (eq. (VII.1.9)), which is not well-characterized at the moment. More work needs to be done in this area. A possible approach is the POD modeling of the continuous part of spectrum discussed in the next paragraph. In this chapter, the contiguous time frame was chosen heuristically. In figure VII.5a, one can see the problem with choosing the contiguous time frames in such a way. It behooves us to have a more robust (and automatic) method of choosing the contiguous time frames. One approach is to implement some type of statistical hypothesis testing in the numerical code to detect a switch. This approach would provide confidence bounds on the certainty a switch has occurred.

While the Koopman mode analysis provides a good way to compute the contribution of the point spectrum, representations of the continuous part of the spectrum have been lacking in the literature. This work has computed the modal noise which can be thought of as a spectral leakage of the continuous spectrum into the point spectrum. This "leakage" occurs because we are looking at a linear approximation in the output space of the nonlinear dynamics. In the future, it may be better to represent the contribution of the continuous spectral integral in the output space as a stochastic process on POD modes. Recall the we had the stochastic process in $\mathbb{R}^d$

$$X_n(w) = U_n \boldsymbol{x}(w) = \sum_{e^{i2\pi\theta} \in \sigma_p} e^{i2\pi\theta n} P_\theta \boldsymbol{x}(w) + \int_{\mathbb{R}} \lambda F_n(d\lambda) \boldsymbol{x}(w). \qquad \text{(VII.3.1)}$$

The first summation can be computed using a Fourier average. We would like to approximate the spectral integral against F_n. First compute the residual, data set

$$R_n(w) = X_n(w) - \sum_{e^{i2\pi\theta} \in \sigma_p} e^{i2\pi\theta n} \boldsymbol{p}_\theta$$

where $\boldsymbol{p}_\theta$ are the approximate Koopman modes of the data computed via a fast Fourier transform. The residual $R_n(w)$ is a sequence in $\mathbb{R}^d$ and represents the contribution of the continuous part of the spectrum. Compute the POD modes of the sequence $\{R_n(w)\} \subset \mathbb{R}^d$. Denote them by $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_s$, where $s \leq d$ is the dimension of the subspace that $\{R_n(w)\}$ generates. We use these POD modes to generate a stochastic process. Let $P_{\boldsymbol{m}_j}$ be the orthogonal projection in $\mathbb{R}^d$ onto $\operatorname{lin}\{\boldsymbol{m}_j\}$. Consider the time-dependent coefficient corresponding to this POD mode.

$$c_{n,j} = P_{\boldsymbol{m}_j}(R_n(w)).$$

This is a real valued sequence. Furthermore, we can compute its cumulative probability distribution (cdf) $\boldsymbol{\xi}_j : \mathbb{R} \to [0,1]$. Then we get a stochastic process, $\xi_j(n)\boldsymbol{m}_j$, where $C_j(n)$ is an IID random variable drawn from the cdf $\boldsymbol{C}_j$. Doing this for every POD mode gives a representation of the dynamics as a stochastic process in $\mathbb{R}^d$ as

$$Y_n(w) = \sum_{e^{i2\pi\theta} \in \sigma_p} e^{i2\pi\theta n} P_\theta \boldsymbol{x}(w) + \sum_{j=1}^{s} \xi_j(n) \boldsymbol{m}_j,$$

where again each $\xi_j(\cdot)$ is an IID random variable drawn from the cdf $\boldsymbol{C}_j$. We have represented the contribution of the spectral integral in $\mathbb{R}^d$ by the stochastic process

$$Z_n = \sum_{j=1}^{s} \xi_j(n) \boldsymbol{m}_j.$$

Part II

Exponentially Mixing Transformations for Search of Targets of Nonuniform Size

Chapter VIII

Introduction

In the context of search for physical objects, path-coverage problems are common, arising whenever the situation requires a guarantee that the target be found. Numerous algorithms have been developed to handle path-coverage problems. Most, however, can be split along two lines: (i) heuristic and random algorithms, and (ii) complete algorithms.

With heuristic algorithms, the search agent is programmed with a simple set of local behaviors, for example, repulsion from objects and other search agents or leader following [PF07]. Random algorithms rely on some type of stochastic mechanism in the generation of search dynamics such as the selection of a random reflection angle [Gag03; LR88; LR89] or the use random walks. In either case, such simple interactions can give rise to quite complicated behavior which is desirable in some situations such as the presence of a moving, adversarial target. Much work has gone into understanding how such simple rules give rise to large scale coordinated motion, particularly in the context of swarms [BVA07a; BVA07b; Olf06] or consensus [BCM09; OFM07]. While in practice these algorithms can perform quite well, many times there is no provable guarantee on full coverage of the space.

Complete algorithms on the other hand do have such guarantees. Most of these algorithms rely on a full or approximate partition of the search space, termed a cellular decomposition, in which each cell of the partition is such that it is fully covered by the sensor footprint or can be covered by simple motion such as a raster scan of the area by the agent. The cells are then identified as vertices in a graph and the problem of path planning reduces finding a spanning tree [Cho01]. The space is covered once all cells have been visited. For shortest path problems, this is equivalent to solving the Traveling Salesman Problem or the Covering Salesman Problem [Cho01].

While existing algorithms can perform quite well for these types of problems, they have only dealt with fixed target sizes (or targets with little variation in size) and there often exists additional assumptions on the distribution for the target location. In short, there is a fair amount of information available prior to designing the search dynamics. This work is concerned with path-coverage problems where the target's volume varies by orders of magnitude and the target can be located anywhere in the search domain with equal probability. This type of path-coverage is, in some sense, at the limits of available information. Our goal is to pursue a detailed investigation of the expected search time for a target in such a situation, obtaining provable upper bounds.

As a simple example of this type, consider the following search problem. Let $\mathcal{M}$ be a closed and bounded set in the plane having large area. Let the target be a circle whose center is a random variable having a uniform distribution on $\mathcal{M}$ and whose area is a random variable having a uniform distribution on $(\delta, V]$; it is not assumed that the exact values of δ and V are known, but $\delta \ll V \ll \text{area}(\mathcal{M})$.

Problems of this type naturally arise in applications such as satellite imagery, searching for oil or mineral deposits, or detecting structural faults or defects [Kol82]. On the less serious side, the children's game Battleship is also of this type. Unfortunately, existing algorithms are ill-suited this class of search problems. For heuristic and random algorithms, upper bounds for the search time are generally difficult or impossible to prove when targets have a fixed size. This issue is exacerbated when target sizes are allowed to vary. Complete algorithms based on cellular decompositions are inherently restricted to a fixed target or sensor size since cell sizes are chosen so that the sensor radius completely covers the cell. When the target size is allowed to vary, the choice of a cell size is far from straightforward. In the case when the minimum target volume approaches zero, cellular decomposition algorithms can perform arbitrarily poorly. Good performance depends on knowing the target size, but by definition of the search problem, this is unknown information.

The following search problem will be treated in this part of the dissertation. Let the search domain be $\mathcal{M} = \mathbb{T}^n$, the n-dimensional torus. Let $\mathcal{B}(\mathcal{M})$ be the Borel

σ-algebra. The target is represented by a closed, metric ball $B(c, r)$, where the center is a random variable $c \sim \mathcal{U}(\mathcal{M})$, where $\mathcal{U}(\mathcal{M})$ is the uniform distribution on $\mathcal{M}$. Fix $V > 0$ and let δ satisfy $0 < \delta < V$. Define the family of functions, $\nu_\delta : (\delta, V] \times (0, \infty) \to \mathbb{R}$, parameterized by $\delta \in (0, V]$, by

$$\nu_\delta(u, \xi) = \frac{u^\xi}{V^{1+\xi} - \delta^{1+\xi}}. \tag{VIII.0.2}$$

For a fixed ξ, each ν_δ is a probability distribution function (pdf) on the interval $(\delta, V]$. Each ν_δ is called a "nearly" uniform distribution, since, as $\xi \to 0$, the pdf converges pointwise to the uniform distribution. Let *vol* denote the Lebesgue measure on $\mathcal{M}$. The radius of the target is a random variable such that $vol(B(c, \cdot)) \sim \nu_\delta$, for any $c \in \mathcal{M}$ and any fixed δ and ξ. Finally, to divorce sensor issues from dynamics, we consider the case of a perfect sensor; the target is assume found the first time the search trajectory enters the target set.

We approach the above search problem as a problem in the ergodic theory of dynamical systems. We investigate the expected search time for different classes of mixing transformations and processes on $\mathcal{M}$. Path-coverage algorithms based around these transformations will automatically be complete since the dynamics are ergodic over the domain. Rigorous definitions of these concepts will be given later. Specifically, results of the paper include the following:

(i) Constructive upper bounds for the hitting time of a target are obtained (the hitting time will be defined precisely later). These bounds are uniform over the target locations and uniform for each initial condition in a "good" set of initial conditions. The bounds depend on the size of the "good" set, the speed of mixing, and the size of the target.

(ii) The hitting time upper bounds are used to show that the hitting time behaves like the inverse of the volume of the target, for small targets. This relation holds for every target location and almost every initial condition for the searcher.

(iii) Limit laws (as $\delta \to 0$) are derived showing that the expected search time behaves like $\mathcal{O}(-\log \delta)$. These results are proven for the case when the target volume is distributed against the "nearly" uniform distribution ν_δ.

In a broader context, the hitting time results of this work belong to a body of results on quantitative recurrence theory, a relatively new subfield of ergodic theory. While it has been known since the work of Poincaré, that almost every point of a positive measure set will return to that set infinitely often under the action of a measure-preserving transformation, results on return rates for specific classes of transformations took much longer to be discovered. Modern quantitative recurrence results include limit laws for specific classes of dynamical systems; which classes of transformation have properties such as strongly Borel-Cantelli sequences and the Shrinking Target Property; and the relation of recurrence and hitting times to the dimension theory of measures (for a subset of results see, e.g., [Aba04; Bos93; BS01; CK01; Coe97; FMP12; Gal06; GK07]).

In addition to saying something about search times for, say, unmanned aerial vehicles or robots, the results contained here relate to sampling and Monte Carlo-type numerical integration. In Monte Carlo integration, one wants to compute the definite integral $\int_B f(x)d\mu(x)$. For some function f and some measure μ over some some domain B. An approximation can be computed by independently drawing a set of points $\{x_1, \ldots, x_N\}$ from the distribution corresponding to μ and approximating the integral with the finite sum $N^{-1}\sum_{n=1}^{N} f(x_n)$ [RR04]. A basic question asks how many samples are needed to give a good approximation to the true integral?

Many times samples are generated by an appropriate Markov chain, but they can also be generated by a discrete-time mixing map. For the mixing map, a loose theoretical bound on the number of samples needed to well-approximate the integral can be computed. If the integration domain $B \subset \mathbb{R}^d$ is compact, it can be covered by a finite number of small balls. The results of this paper give a bound on the hitting time for each of these balls and we can easily get an upper bound on the time needed to hit all of the balls by multiplying the individual upper bound by the number of covering balls. Of course for this to work, the measure the map preserves must match the measure we would like to integrate against. If this is not the case, an appropriate map must be constructed from the original mixing map. There are a number of standard methods to do this such as the Metropolis-Hastings algorithm or the Gibbs sampler (see, e.g., [RR04]).

The rest of the text is organized as follows. We begin, in section IX.1, with the discrete-time setting and investigate two different processes for the 1D case of the search problem. The analysis here is simple and is intended to motivate the concepts defined later. This investigation leads to the relevant notions needed to generalize the analysis to higher dimensions. We go on to define the notions of $\mathfrak{B}$-regularity, ergodicity, and mixing in section IX.2 so that both maps and multi-valued processes can be treated simultaneously. Upper bounds are then proved for a exponentially mixing transformations and these are used to prove limit laws for infinitely small targets (sec. IX.3). Asymptotic laws for the expected search time are proved in section IX.4 for any transformation satisfying the limit laws of section IX.3 for the hitting time. Finally, in section IX.5, the discrete-time results are pushed to continuous first-order dynamics.

Chapter IX

Limit laws and expected search times for mixing dynamical systems

IX.1 Preliminary discussion of the problem

We begin by fixing some concepts and notation that will be used throughout the paper. Let $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ be a Borel measurable space. Let $T : \mathcal{M} \to \mathcal{M}$ be some fixed, discrete-time transformation. As usual $\mathbb{N}$ denotes the natural numbers $\{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition IX.1.1 (Hitting time)**.** Let $A \in \mathcal{B}(\mathcal{M})$. The *hitting time* of A under T

when starting from y is denoted by $\tau_T[A, y]$ and defined by

$$\tau_T[A, y] = \inf\{k \in \mathbb{N}_0 : T^k(y) \cap A \neq \emptyset\}. \tag{IX.1.1}$$

When its clear which transformation is being discussed, T is dropped from the notation and just write $\tau[A, y]$. More properly, τ_T is called the first hitting time, but for brevity will stick to the above terminology.

Definition IX.1.2 (Expected hitting time)**.** Let the target be $B(c, r)$, the closed ball of radius r, centered at c. The *expected hitting time*, $\mathcal{H}(c, s, \delta)$, of a target whose volume is distributed against the "nearly" uniform distribution, ν_δ, when the searcher starts from s is defined as

$$\mathcal{H}(c, s, \delta) = \int_\delta^V \tau[B(c, r(u)), s]\, \nu_\delta(u, \xi)\, du. \tag{IX.1.2}$$

Note that the radius of the target is a function of the volume, the integration variable.

It will be useful in our analysis to define the set of initial conditions that have not found the target by time t.

Definition IX.1.3 (Exception set)**.** Let

$$\mathcal{E}(c, r, t) = \{s \in \mathcal{M} : \tau[B(c, r), s] > t\}\ . \tag{IX.1.3}$$

$\mathcal{E}(c, r, t)$ is called the *exception set at time t*.

Remark IX.1.4. *A partial ordering, "$\preccurlyeq$", can be defined on $\mathcal{E}(x, r, t)$ by set inclusion. Then both $\mathcal{E}(c, \cdot, t)$ and $\mathcal{E}(c, r, \cdot)$ are decreasing functions. So if $r_k \geq r_{k+1}$, then $\mathcal{E}(c, r_k, t) \preccurlyeq \mathcal{E}(c, r_{k+1}, t)$, and in particular*

$$\mathcal{E}(c,r,t) = \bigcup_{u \geqslant r} \mathcal{E}(c,u,t), \tag{IX.1.4}$$

while if $t_k \leq t_{k+1}$*, then* $\mathcal{E}(c,r,t_k) \succcurlyeq \mathcal{E}(c,r,t_{k+1})$.

We also remark that if the transformation is continuous, then, since $B(c,r)$ *is closed,* $\mathcal{E}(c,r,t)$ *is an open set.* □

Remark IX.1.5. *Suppose* T *is Lipschitz continuous with Lipschitz constant* α*. If* $\tau[B(c,r),y] = k$*, and* $T^k(y)$ *is in the interior of* $B(c,r)$*, then there is a positive Lebesgue measure set containing* y *whose points have hitting time equal to* k*:*

Let

$$\rho_k := \min\{\left\|T^j(y) - B(c,r)\right\| : 0 \leq j \leq k-1\}$$

and $d_k := \left\|T^k(y) - \partial B(c,r)\right\|$*, where* $\partial B(c,r)$ *is the boundary of* $B(c,r)$*. Set* $L = \max(1,\alpha)$*. For any* $\|z\| < L^{-k}\min\{\rho_k, d_k\}$*,*

$$\left\|T^k(y+z) - T^k(y)\right\| \leq L^k \|z\| < \left\|T^k(y) - \partial B(c,r)\right\|;$$

hence $T^k(y+z) \in B(c,r)$*. Additionally, for* $0 \leq j \leq k-1$*,*

$$\left\|T^j(y+z) - T^j(y)\right\| \leq L^j \|z\| < L^{j-k}\rho_k \leq \rho_k.$$

Hence $T^j(y+z) \notin B(c,r)$ *for* $0 \leq j \leq k-1$*. Therefore, for all such* z*,* $\tau[B(c,r), y+z] = k$*.* □

In this paper, limit laws of the following types are shown:

$$\limsup_{vol(B(c,r))\to 0} \frac{\ln \tau[B(c,r),s]}{-\ln vol(B(c,r))} \leq 1 \tag{IX.1.5}$$

and

$$\lim_{\xi\to 0}\limsup_{\delta\to 0}\frac{\mathcal{H}(c,s,\delta)}{-\ln\delta}\leq const. \tag{IX.1.6}$$

We first analyze two algorithms for a simple case of the search problem where $\mathcal{M}=\mathbb{T}$, the one-dimensional torus. Without loss of generality, we can identify $\mathbb{T}$ with the unit interval, mod 1. In this setting, a target, $B(c,r)$, will be an interval of length $2r$. We assume the interval length is distributed uniformly in $(\delta, V]$; i.e., $vol(B(c,r)) \sim \nu_\delta(\cdot, 0)$. While the analysis is simple, it nevertheless highlights a few important points.

IX.1.1 Rational Lawnmower Algorithm

Fix some $N \in \mathbb{N}$ and let $\Delta = N^{-1}$. The lawnmower algorithm is given by the rational shift $T^k(s) = s + k\Delta \mod 1$, for $k \in \mathbb{N}_0$. It is so named because its generalization to the square gives trajectories that mimic the path one takes when mowing a lawn.

What is a suitable step size to choose? First let us look at the initial conditions that never find a given target. Fix some $c \in [0,1)$ and some $N \in \mathbb{N}$ such that $\delta < N^{-1} \leq V$. Assume the target's volume, $vol(B(c,r))$, satisfies $0 < vol(B(c,r)) < N^{-1}$. Then in the interval $[c-r, c-r+N^{-1}]$, the measure of the points not in target interval is $N^{-1} - 2r = N^{-1} - vol(B(c,r))$. Since all points have period N, this implies that the total measure of the points that never find the target is

$$vol(\mathcal{E}(c,r,\infty)) = N\cdot(N^{-1} - vol(B(c,r))) = 1 - N\cdot vol(B(c,r)) > 0.$$

Hence, if the searcher's initial condition is distributed uniformly at random in $[0, 1)$, there is a positive probability of never finding the target if its size is in the range (δ, N^{-1}). Furthermore, as $vol(B(c, r))) \to 0$, almost every point never finds the target.

To guarantee finding the target we must have that $N^{-1} \leq \delta$. For simplicity assume $1/\delta$ is an integer and fix $N^{-1} = \delta$. The expected hitting time of a target is computed as follows. Without loss of generality, we can assume that the searcher starts at $s = 0$. Fix $c \in [0, 1)$. Since the volume of the target is $2r$, the expected search time becomes

$$\mathcal{H}(c, 0, \delta) = \frac{2}{V - \delta} \int_{\delta/2}^{V/2} \tau[B(c, r), 0]\, dr.$$

Note that for $c \leq \frac{\delta}{2}$ or $c \geq 1 - \frac{\delta}{2}$, $0 \in B(c, r)$ for all $r > \frac{\delta}{2}$. Therefore, for $r > \frac{\delta}{2}$, $\tau[B(c, r), 0] = 0$.

Assume $c \in (\frac{\delta}{2}, 1 - \frac{\delta}{2})$. Then for all $r \geq \frac{\delta}{2}$, let $k \in \mathbb{N}_0$ be the smallest nonnegative integer such that $c - r \leq kN^{-1}$ or equivalently the smallest nonnegative integer such that $N(c - r) \leq k$. Then $kN^{-1} \in [c - r, c + r] \mod 1$. Therefore, for $\frac{\delta}{2} \leq r \leq c$

$$\tau[B(c, r), 0] = k < N(c - r) + 1,$$

and for $r \geq c$, $\tau[B(c, r), 0] = 0$.

Therefore, for any $c \in [0, 1)$,

$$\tau[B(c, r), 0] \leq \begin{cases} 0, & \text{if } r \geq c \\ N(c - r) + 1, & \text{if } \frac{\delta}{2} < r < c \end{cases}. \tag{IX.1.7}$$

If $A = \min\{c, \frac{V}{2}\}$, then

$$\int_{\delta/2}^{V/2} \tau[B(c,r),0]\,dr = \int_{\delta/2}^{A} \tau[B(c,r),0]\,dr.$$

Then

$$\mathcal{H}(c,0,\delta) \le \frac{2}{V-\delta}\int_{\delta/2}^{A} (N(c-r)+1)\ dr = \frac{2}{\delta}\left[\frac{(A-\frac{\delta}{2})(c+\delta) - \frac{1}{2}(A^2-(\frac{\delta}{2})^2)}{V-\delta}\right],$$

for any $c \in [0,1)$. Since the dynamics are shift-invariant and the above is true for any $c \in [0,1)$, the expected search time, $\mathcal{H}(c,s,\delta)$, is of order $\mathcal{O}(1/\delta)$.

The preceding discussion highlighted two important points. First, a suitable choice of N requires knowledge of the target's volume, information completely lacking in the problem formulation. Second, any choice of N gives dynamics that do not densely fill the domain, leaving gaps that a sufficiently small target can hide in. □

IX.1.2 Subdivision Algorithm in 1D

The q-adic subdivision algorithm on $[0,1)$ is defined in the following way. Let $q \in \{2,3,\ldots\}$ be called the *rank*[1] and k the subdivision level. Let $P_0 = \emptyset$ and $\zeta_0 = \emptyset$. For each $k \in \mathbb{N}$, define

(i) $P_k = \left\{\frac{j}{q^k}\right\}_{j=0}^{q^k-1}$, and

(ii) $\zeta_k = P_k \setminus P_{k-1}$.

Define

$$S_k(x) := x + \zeta_k, \quad \mod 1.$$

[1] $q = 2$ corresponds to the dyadic subdivision algorithm.

The subdivision algorithm is then $S = \{S_1(x), S_2(x), \ldots\}$; i.e. iterate through all points in $S_1(x)$, then all points in $S_2(x)$, and so on.

The subdivision protocol creates a regular grid with edge lengths $\Delta = q^{-k}$. To guarantee finding a target of radius r, we need to find the smallest k that gives a $2r$-net; hence we need to chose k as the smallest integer giving $q^{-k} \leq 2r$. An easy computation shows that k satisfies

$$\log_q \left(\frac{1}{2r}\right) \leq k < \log_q \left(\frac{1}{2r}\right) + 1.$$

At the k^{th} subdivision level, $\#\{P_k\} = q^{kn}$ points have been searched. Hence $\tau[B(c,r),s] \leq q^{kn}$ for all c. Using the upper bound for k gives

$$\tau[B(c,r),s] \leq \frac{q}{2r}. \tag{IX.1.8}$$

Since $vol(B(c,r)) = 2r$, (IX.1.8) becomes

$$\tau[B(c,r),s] \leq \frac{q}{vol(B(c,r))}, \tag{IX.1.9}$$

and

$$\begin{aligned}\mathcal{H}(c,s,\delta) &= \frac{1}{V-\delta}\int_\delta^V \tau[B(c,r(u)),s]\,du \\ &\leq \frac{q}{V-\delta}\log\frac{V}{\delta}.\end{aligned}$$

The expected search time $\mathcal{H}(c,s,\delta)$ is $\mathcal{O}(-\log\delta)$, with the constant only depending on the rank of the subdivision algorithm. Comparing this with the result from the lawnmower algorithm shows that the subdivision algorithm performs much better. □

The subdivision algorithm exhibited two different characteristics relevant to the search problem. The first was that the algorithm came arbitrarily close to any point in the domain. In fact, the density of the points was uniform in the infinite time limit. The second was that the algorithm was inherently multi-scale. On average larger targets are found first, and if targets are not found at a certain size, the partition size was refined. The multi-scale property of the subdivision algorithm allowed to speedup versus the lawnmower algorithm.

IX.2 $\mathfrak{B}$-regular systems

The subdivision algorithm exhibited two different characteristics that are relevant to the general search problem. The first was that the algorithm came arbitrarily close to any point in the domain. The second was that the density of points was approximated the Lebesgue measure in the limit.

In dynamical systems, the first property is ergodicity. For a map to be ergodic, a measurable space is required, as well as a map preserving a finite measure and that any invariant set has trivial measure. Unfortunately, the subdivision algorithm is not a function strictly speaking; it is a multi-valued process. The traditional definition of ergodicity does not cover such cases, even though intuitively speaking, the subdivision map preserves the Lebesgue measure. The definition of ergodicity must be modified somewhat to cover these exceptional cases. It will, however, reduce to the usual case for single-valued maps.

We start by defining multivalued maps and their time averages in a general setting. Let $\mathscr{F}_c$ be the σ-algebra of countable and co-countable sets; every set in $\mathscr{F}_c$ is either countable or its complement is countable. Recall that the counting measure $\mu_c : \mathscr{F}_c \to [0, \infty]$ is defined by

$$\mu_c(A) = \begin{cases} \sum\limits_{x \in A} 1, & A \text{ is a finite set,} \\ \\ \infty, & \text{otherwise.} \end{cases} \tag{IX.2.1}$$

Definition IX.2.1 (Multi-valued map)**.** Let $\mathcal{M}$ be a compact metric space. A *multi-valued map* is a function $T : \mathscr{F}_c \times \mathbb{N}_0 \to \mathscr{F}_c$ with the following properties:

(i) for each $x \in \mathcal{M}$, there is a sequence of nonempty *finite* sets $\{A_{x,k}\}$, such that $T(\{x\}, k) = A_{x,k}$,

(ii) for every $F \in \mathscr{F}_c$ and $k \in \mathbb{N}_0$, $T(F, k) = \bigcup\limits_{x \in F} T(\{x\}, k)$.

(iii) for every $x \in \mathcal{M}$, $T(\{x\}, 0) = A_{x,0} = \{x\}$, and

(iv) $T(\{x\}, k) = T(T(\{x\}, j), k - j)$ for all $0 \leq j \leq k$.

A map $S : \mathcal{M} \to \mathcal{M}$ can be treated in this framework since we can define $T(\{x\}, k)$ to be just the singleton set $\left\{S^k(x)\right\}$. For this reason, we will usually write $T^k(x)$ for $T(\{x\}, k)$ so that notation is consistent for both maps and multi-valued processes.

The last two conditions in the above definition just require that the multi-valued process satisfies the semigroup property. In the new notation, the second condition takes the form $T^k(x) = T^{k-j}(T^j(x))$, for any $0 \leq j \leq k$.

Definition IX.2.2. Let $T : \mathscr{F}_c \times \mathbb{N}_0 \to \mathscr{F}_c$ be a multi-valued map. Let $d_{\mathscr{F}} : \mathscr{F}_c \times \mathscr{F}_c$ be a pseudo-metic for $\mathscr{F}_c$ such that $d_{\mathscr{F}}(T^j(x), T^k(y)) = 0$ implies $T^j(x) = T^k(y)$. The multi-valued process is,

(i) *continuous at* $x \in \mathcal{M}$, if for every $\varepsilon > 0$ and $k \in \mathbb{N}_0$, there is a $\delta_{x,k} > 0$ such that $d_{\mathscr{F}}(T^k(x), T^k(y)) < \varepsilon$ for any $y \in \mathcal{M}$ such that $d(x, y) < \delta_{x,k}$.

(ii) *continuous* if its continuous at all $x \in \mathcal{M}$.

(iii) *uniformly continuous* if it is continuous and for each $k \in \mathbb{N}_0$, $\delta_{x,k} = \delta_{y,k} = \delta_k$ for all $x, y \in \mathcal{M}$.

(iv) *equicontinuous at* $x \in \mathcal{M}$ it is continuous at $x \in \mathcal{M}$ and $\delta_{x,j} = \delta_{x,k} = \delta_x$ for all $j, k \in \mathbb{N}_0$.

(v) *equicontinuous* if it is equicontinuous at every point $x \in \mathcal{M}$.

If we let $C(\mathcal{M})$ be the space of continuous functions on $\mathcal{M}$, we have the following:

Definition IX.2.3 ($\mathfrak{B}$-regularity)**.** Let T be a continuous multi-valued map. Let $\mu_c : \mathscr{F}_c \to [0, \infty]$ be the counting measure. For each $f \in C(\mathcal{M})$, define

$$\mathbb{E}\left[f \circ T^k(x)\right] = \frac{1}{\mu_c(T^k(x))} \sum_{y \in T^k(x)} f(y). \tag{IX.2.2}$$

We call the tuple $(T, \mathcal{M})$ a $\mathfrak{B}$-*regular system* if for each $f \in C(\mathcal{M})$, the limit

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E}\left[f \circ T^k(x)\right] \tag{IX.2.3}$$

exists for almost every $x \in \mathcal{M}$ with respect to the Lebesgue measure. If $(T, \mathcal{M})$ is a $\mathfrak{B}$-regular system, then T is called $\mathfrak{B}$-regular for short.

Since $T^k(x)$ is a finite set for all $x \in \mathcal{M}$ and $k \in \mathbb{N}_0$ and $|f| < \infty$ for $f \in C(\mathcal{M})$, (IX.2.2) is well-defined.

Remark IX.2.4. *If $T : \mathcal{M} \to \mathcal{M}$ is a (single-valued) map and for each $f \in C(\mathcal{M})$, the limit $\lim \frac{1}{N} \sum_{k=0}^{n-1} f \circ T^k(x)$ exists for Lebesgue almost every $x \in \mathcal{M}$, then $(T, \mathcal{M})$ is a $\mathfrak{B}$-regular system and in this case, $\mathbb{E}\left[f \circ T^k(x)\right] = f \circ T^k(x)$. Systems possessing an SRB measure satisfy this condition [You02].* □

Let Σ denote all the points in $\mathcal{M}$ where the time average (IX.2.3) does not exist for at least one $f \in C(\mathcal{M})$. If the system is $\mathfrak{B}$-regular, Σ has Lebesgue measure 0. The $\mathfrak{B}$-regularity of the system allows a family of positive linear functionals, parameterized by points in $\mathcal{M} \setminus \Sigma$, to be defined on $C(\mathcal{M})$ by setting

$$\Lambda_x(f) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}\left[f \circ T^k(x)\right]. \tag{IX.2.4}$$

For a fixed $x \in \mathcal{M} \setminus \Sigma$, the Riesz representation theorem gives a positive, regular measure, μ_x, so that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}\left[f \circ T^k(x)\right] = \Lambda_x(f) = \int f \, d\mu_x \tag{IX.2.5}$$

for all $f \in C(\mathcal{M})$. With this measure, $L^1(\mathcal{M}, \mu_x)$ can be defined and the limit (IX.2.5) exists for $L^1(\mu_x)$-functions.

Definition IX.2.5 ($\mathfrak{B}$-regular, ergodic measure)**.** Let $A \subset \mathcal{M}$ be Borel measurable. A $\mathfrak{B}$-regular system $(T, \mathcal{M})$ is said to be *ergodic on* A if every $x \in A \setminus \Sigma$ generates the same measure μ. T is said to be *ergodic* if A can be taken to be $\mathcal{M}$ and in that case μ is called an *ergodic measure*.

If the system is ergodic on A, then for every $x \in A \backslash \Sigma$, the time average is identical to the μ-spatial average over A for $f \in L^1(\mu)$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}\left[f \circ T^k(x)\right] = \int_A f\, d\mu. \tag{IX.2.6}$$

Ergodic invariant measures are import because they characterize the residency times of trajectories in sets; for $f = \chi_A$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}\left[\chi_A(T^k(x))\right] = \mu(A). \tag{IX.2.7}$$

The term on the left is the asymptotic relative frequency of the time the trajectory spends in the set A and the relation says that this relative frequency is equal to the measure of A.

Given the implication (IX.2.7) of ergodicity and that the target can be located anywhere in the domain, we would like a search transformation with an ergodic measure μ such that every nonempty, open set has positive measure. Motivated by these physical considerations, we make the following assumptions for the rest of the paper:

Assumption IX.2.6. *$T : \mathcal{M} \to \mathcal{M}$ is a $\mathfrak{B}$-regular, multi-valued map that is ergodic on $\mathcal{M}$, and whose ergodic measure μ is absolutely continuous with respect to the Lebesgue measure. Furthermore, assume that the Radon-Nikodym derivative of μ satisfies*

$$0 < \alpha \leq \frac{d\mu}{dx} \leq \beta < \infty, \tag{IX.2.8}$$

everywhere on $\mathcal{M}$. □

Definition IX.2.7 (Mixing multi-valued map)**.** Let $T : \mathcal{M} \to \mathcal{M}$ be a $\mathfrak{B}$-regular process and μ a measure. Define $\mu(T^{-j}A \cap T^{-k}B)$ as

$$\mu(T^{-j}A \cap T^{-k}B) = \int \mathbb{E}\left[\chi_A \circ T^j(x)\right] \cdot \mathbb{E}\left[\chi_B \circ T^k(x)\right]\, d\mu. \tag{IX.2.9}$$

Now let T be an ergodic multi-valued process preserving μ. T is said to be *mixing* if

$$\lim_{n\to\infty} \mu(A \cap T^{-k}B) = \mu(A)\mu(B). \tag{IX.2.10}$$

Mixing dynamical systems asymptotically look like an independent process. Of course, without knowing the speed at which the limit is approached, little can be said quantitatively in terms of the hitting times of sets. Therefore, we need to make an assumption on the rate of approach to the limit. To this end, we introduce the notion of exponential mixing.

Definition IX.2.8 (Uniform exponential mixing)**.** A transformation is *uniformly exponentially mixing* if

$$\left|\mu(A \cap T^{-k}B) - \mu(A)\mu(B)\right| \le C\mu(A)\mu(B)e^{-\gamma k}\,. \tag{IX.2.11}$$

Nonuniform exponential mixing is defined as

$$\left|\mu(A \cap T^{-k}B) - \mu(A)\mu(B)\right| \le C\mu(B)e^{-\gamma k}\,. \tag{IX.2.12}$$

The positive parameter γ is called the *mixing rate*.

Uniform exponential mixing must hold for sets of arbitrary measure. We can relax this by defining a scale parameter, and defining (non-)uniform mixing down to this scale.

Definition IX.2.9 (Scale of a set)**.** Fix a Borel measurable set $A \subset \mathbb{R}^n$ and let $R = R_1 \times \cdots \times R_n$ denote an n-dimensional rectangle. Define $\beta \in \mathbb{R}$ to be a

$$\beta = \sup_{R \subset \overline{A}} \sup \{s \leq 1 : s \leq R_i \text{ for } 1 \leq i \leq n\} \qquad \text{(IX.2.13)}$$

where $\overline{A}$ is the closure of A. A is said to have scale β.

Definition IX.2.10. Let A, B be Borel measurable sets and $\{T^k\}$ a multi-valued map. Fix a scale β. T is said to be *uniformly exponentially mixing down to scale* β if

$$\left|\mu(A \cap T^{-k}B) - \mu(A)\mu(B)\right| \leq C_\beta \mu(A)\mu(B)e^{-\gamma k}\,. \qquad \text{(IX.2.14)}$$

for Borel measurable sets A, B having scales no less than β. C_β is a constant only depending on β.

Nonuniform exponential mixing down to scale β is defined similarly.

In general, we can define mixing with an arbitrary rate.

Definition IX.2.11 (Summable mixing)**.** Let $\Phi : \mathbb{N}_0 \to \mathbb{R}^+$ be a positive, monotonically decreasing sequence such that $\sum_{k=1}^{\infty} \Phi(k) < \infty$. We obtain *uniform summable mixing* and *nonuniform summable mixing* by replacing $Ce^{-\gamma k}$ with $\Phi(k)$ in IX.2.11 and IX.2.12 respectively.

Definition IX.2.12 (n-dimensional, q-adic subdivision algorithm)**.** Let ζ be a partition of $[0, 1)$; i.e, a finite set of distinct points $\zeta = \{p_0, \ldots, p_m\}$ ordered such that $0 \leq p_i < p_{i+1} < 1$ for $i = 0, \ldots, m-1$. Let $\mathscr{P}_m$ be the set of all partitions on $\mathbb{R}$ which have m points and $\mathscr{P}$ the space of all finite partitions on $\mathbb{R}$. Let $|\zeta|$ be the number of points in the partition. For any $q \in \{2, 3, \ldots\}$, define the maps map $I_q : \mathscr{P} \to \mathscr{P}$, $E_q : \mathscr{P} \to \mathscr{P}$, and $S_q : \mathscr{P} \to \mathscr{P}$ by

$$I_q(\zeta) = \bigcup_{i=1}^{|\zeta|-1} \bigcup_{j=0}^{q-1} \left\{ p_{i-1} + j \left(\frac{p_i - p_{i-1}}{q} \right) \right\}, \tag{IX.2.15}$$

$$E_q(\zeta) = \bigcup_{j=0}^{q-1} \left\{ p_m + j \left(\frac{1 + p_0 - p_m}{q} \right) \right\} \mod 1,$$

and,

$$S_q(\zeta) = I_q(\zeta) \cup E_q(\zeta), \tag{IX.2.16}$$

respectively.

Define

$$S_q^0(\zeta) = \zeta. \tag{IX.2.17}$$

and

$$S_q^{k+1}(\zeta) = S_q \circ S_q^k(\zeta) := (I_q \circ I_q^k(\zeta)) \cup (E_q \circ E_q^k(\zeta)) \tag{IX.2.18}$$

for $k \geq 0$.

Fix a dimension $n \in \mathbb{N}$. Let $(\zeta_1, \ldots, \zeta_n)$ be a set of partitions of the $[0, 1)$. Define $\boldsymbol{S}_q : \mathscr{P}^n \to \mathscr{P}^n$, by

$$\boldsymbol{S}_q(\zeta_1 \times \cdots \times \zeta_n) = S_q(\zeta_1) \times \cdots \times S_q(\zeta_n) \tag{IX.2.19}$$

With this definition, we define

$$\boldsymbol{S}_q^0(\zeta_1 \times \cdots \times \zeta_n) = \zeta_1 \times \cdots \times \zeta_n.$$

and

$$\boldsymbol{S}_q^{k+1} = \boldsymbol{S}_q \circ \boldsymbol{S}_q^k = (S_q \circ S_q(\zeta_1)) \times \cdots \times (S_q \circ S_q(\zeta_n))$$

for $k \geq 0$.

For $\boldsymbol{x} = (x_1, \ldots, x_n) \in [0,1)^n$, define the partitions $\zeta_i = \{x_i\}$ and $\boldsymbol{\zeta} = \zeta_1 \times \cdots \times \zeta_n$. The n-dimensional subdivision process of rank q is defined for $k \geq 0$ by

$$T_q^k(\boldsymbol{x}) = \boldsymbol{S}_q^k(\boldsymbol{\zeta}). \tag{IX.2.20}$$

By construction, the subdivision process T_q has the semigroup property. Additionally, let $\zeta \in \mathscr{P}_m$. Then $|\zeta| = m$, and $|S_q(\zeta)| = qm$. Therefore S_q maps $\mathscr{P}_m$ into $\mathscr{P}_{qm}$. In terms of the subdivision process $\left|S_q^k(\zeta_i)\right| = q^k$, since the initial partition is an element of $\mathscr{P}_1$, and $\left|\boldsymbol{S}_q^k(\boldsymbol{\zeta})\right| = q^{kn}$.

Example IX.2.13 (n-dimensional, q-adic subdivision algorithm)**.** *This example shows that the generalization of the subdivision algorithm to $\mathbb{T}^n$ is a uniformly exponentially mixing multi-valued process preserving the Lebesgue measure.*

Recall μ_c is the counting measure. Let $\boldsymbol{x} = (x_1, , \ldots, x_n)$. Note that $\mu_c(T^k(\boldsymbol{x})) = q^{kn}$ since

$$S_q^k(\{x_i\}) = \left\{x_i, x_i + q^{-k}, \ldots, x_i + (q^k - 1)q^{-k}\right\}. \tag{IX.2.21}$$

To ease notation, let $\zeta_i^k = S_q^k(\{x_i\})$; then $T^k(\boldsymbol{x}) = \zeta_1^k \times \cdots \times \zeta_n^k$.

For any $f \in C(\mathbb{T}^n)$,

$$\mathbb{E}\left[f \circ T^k(\boldsymbol{x})\right] = \frac{1}{q^{kn}} \sum_{\boldsymbol{y} \in T^k(\boldsymbol{x})} f(\boldsymbol{y}).$$

Since

$$\lim_{k \to \infty} \mathbb{E}\left[f \circ T^k(\boldsymbol{x})\right] = q^{-kn} \sum_{\boldsymbol{y} \in T^k(\boldsymbol{x})} f(\boldsymbol{y}) = \int f(\boldsymbol{y})\, d\boldsymbol{y}$$

then

$$\frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\left[f\circ T^k(\boldsymbol{x})\right]\to\int f(\boldsymbol{y})\,d\boldsymbol{y}.$$

as $N\to\infty$. *Hence, the subdivision algorithm is a* $\mathfrak{B}$*-regular system. The limit is independent of the initial condition* $\boldsymbol{x}$*, implying that the system is ergodic with respect to the Lebesgue measure.*

Note that $\mathbb{E}\left[f\circ T^k(\boldsymbol{x})\right]\to vol(B)$ *by letting* $f=\chi_B$*, the indicator function of* B*, which implies that* $vol(A\cap T^{-k}B)\to vol(A)vol(B)$*. Therefore, the subdivision algorithm is mixing.*

We now turn to showing that the subdivision algorithm is uniformly exponentially mixing for rectangles $\boldsymbol{R}=R_1\times\cdots\times R_n$ *such that* $|R_d|=b_d-a_d\geq\beta$*, for* $d=1,\ldots,n$*, and where* $R_d=[a_d,b_d)$*,* $a_d<b_d$*, and* $0\leq\beta\leq 1$*. Recall that* $vol(A\cap T^{-k}B)$ *is defined by* (IX.2.9)*. We will show that for such a rectangle* R*,*

$$vol(\boldsymbol{R})(1-C_\beta e^{-\gamma k})<\mathbb{E}\left[\chi_{\boldsymbol{R}}\circ T^k(\boldsymbol{x})\right]<vol(\boldsymbol{R})(1+C_\beta e^{-\gamma k}).$$

for large enough k*.*

Consider, the one-dimensional case first. Let $R_1=[a_1,b_1)$ *where we assume* $0<a_1<b_1<1$*. We need to estimate the number of points from* $\zeta_1^k=\left\{jq^{-k}\right\}_{j=0}^{q^k-1}$ *belonging to* R_1*. Let* $n_k(R_1):=\sum_{p\in\zeta_1^k}\chi_{R_1}(p)$*. We estimate* $n_k(R_1)$ *as follows. Assume* k *is large enough so that for some point* $p'\in\zeta_1^k$*,* $b_1<p'<1$ *and* $b_1-a_1>q^{-k}$*. Define* j' *(see fig. IX.1) as the integer satisfying*

$$(j'-1)q^{-k}<a_1\leq j'q^{-k}.$$

Then $n_k(R_1)$ *satisfies*

$$(j' + n_k(R_1) - 1)q^{-k} < b_1 \leq (j' + n_k(R_1))q^{-k}.$$

Combining these two expressions, it can be shown

$$|R_1|\, q^k - 1 \leq n_k(R_1) < |R_1|\, q^k + 1 \tag{IX.2.22}$$

where $|R_1| = b_1 - a_1$.

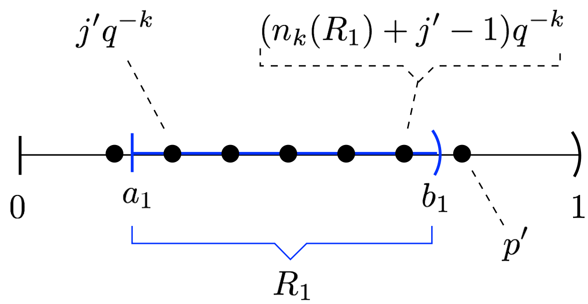


Figure IX.1: The number $n_k(R_1)$ points contained in R_1.

Now consider the n*-dimensional case. Let* $\boldsymbol{R} = R_1 \times \cdots \times R_n$*, where* $R_d = [a_d, b_d)$*. Again it is assumed that* $0 < a_d < b_d < 1$ *and* k *is large enough that there is a point of* ζ_d^k *between* b_d *and 1 and* $b_d - a_d > q^{-k}$*. Let* $N_k(\boldsymbol{R})$ *be the number of points of* $T^k(\boldsymbol{x}) = \bigtimes_1^n \zeta_d^k$ *that are in* $\boldsymbol{R}$*. Note that*

$$N_k(\boldsymbol{R}) = n_k(R_1) \cdots n_k(R_n).$$

Using (IX.2.22)*, we get*

$$N_k(\boldsymbol{R}) \geq vol(\boldsymbol{R}) q^{kn} \prod_{d=1}^{n} \left(1 - \frac{1}{|R_d|\, q^k}\right)$$

and

$$N_k(\boldsymbol{R}) < vol(\boldsymbol{R})q^{kn} \prod_{i=d}^{n} \left(1 + \frac{1}{|R_d|\, q^k}\right)$$

Since $|R_d| \geq \beta$,

$$vol(\boldsymbol{R})q^{kn} \left(1 - \beta^{-1}q^{-k}\right)^n \leq N_k(\boldsymbol{R}) < vol(\boldsymbol{R})q^{kn} \left(1 + \beta^{-1}q^{-k}\right)^n \tag{IX.2.23}$$

and

$$\mathbb{E}\left[\chi_{\boldsymbol{R}} \circ T^k(\boldsymbol{x})\right] = \frac{1}{q^{kn}} \sum_{\boldsymbol{y} \in T^k(\boldsymbol{x})} \chi_{\boldsymbol{R}}(\boldsymbol{y}) = \frac{N_k(\boldsymbol{R})}{q^{kn}} \tag{IX.2.24}$$

then

$$vol(\boldsymbol{R}) \left(1 - \beta^{-1}q^{-k}\right)^n \leq \mathbb{E}\left[\chi_{\boldsymbol{R}} \circ T^k(\boldsymbol{x})\right] < vol(\boldsymbol{R}) \left(1 + \beta^{-1}q^{-k}\right)^n \tag{IX.2.25}$$

There exists $C_\beta > 0$ *(depending on* β*) and* $\gamma > 0$ *such that*

$$1 - C_\beta e^{-\gamma k} \leq \left(1 - \beta^{-1}q^{-k}\right)^n \quad \textit{and} \quad \left(1 + \beta^{-1}q^{-k}\right)^n \leq 1 + C_\beta e^{-\gamma k}$$

Therefore

$$vol(\boldsymbol{R})(1 - C_\beta e^{-\gamma k}) \leq \mathbb{E}\left[\chi_{\boldsymbol{R}} \circ T^k(\boldsymbol{x})\right] < vol(\boldsymbol{R})(1 + C_\beta e^{-\gamma k})$$

and

$$vol(A)vol(\boldsymbol{R})(1 - C_\beta e^{-\gamma k}) \leq vol(A \cap T^{-k}(\boldsymbol{R})) < vol(A)vol(\boldsymbol{R})(1 + C_\beta e^{-\gamma k}).$$

Hence the subdivision map is uniformly exponentially mixing down to scale β *on rectangles and hence Borel sets.* □

Example IX.2.14 (q-adic maps of the interval)**.** *We use a theorem from [Liv95] to show that the q-adic map* $T(x) = qx \mod 1$*, for* $q \in \mathbb{N} \setminus \{1\}$*, is exponentially mixing for targets having scale greater than a fixed number. The theorem is actually a result for the decay of correlations of a class of functions having some regularity properties. Unfortunately, it cannot be strengthened to the uniform exponential condition above because the bound depends on the derivative of one of the observables. It can, however, give "*ε*-independence". More precisely, this means that for any* $\varepsilon > 0$*, we can show that*

$$\left|vol(A \cap T^{-k}B) - vol(A)vol(B)\right| < \varepsilon + C(\varepsilon)vol(A)vol(B)e^{-\gamma k} \qquad \text{(IX.2.26)}$$

where $C(\varepsilon)$ *is a constant depending on* ε *such that* $C(\varepsilon) \to \infty$ *as* $\varepsilon \to 0$*.*

We have

Theorem IX.2.15 ([Liv95])**.** *There exist* $K, r \in \mathbb{R}^+$*,* $\Lambda \in (0,1)$*, such that, for* $f \in L^1([0,1))$*,* $g \in C^1([0,1))$ *satisfying* $\int_0^1 g(x)\,dx = 1$*,*

$$\left|\int_0^1 (f \circ T^n(x))g(x)\,dx - \int_0^1 f(x)\varphi_*(x)\,dx\right| \leq K \left\|f\right\|_1 (\left\|g\right\|_1 + r \left\|g'\right\|_\infty)\Lambda^n \quad \text{(IX.2.27)}$$

where $\varphi_*(x)$ *is the density, with respect to Lebesgue, of the invariant measure.*

To show (IX.2.26), consider the intervals A *and* B *in* $[0,1)$*. Let* $f = \chi_B$ *and* $h = \chi_A/vol(A)$ *and define* $A_{+\varepsilon} = [a - \varepsilon/2, b + \varepsilon/2]$ *where* $A = [a,b]$*. Let* $\varphi \in C^1$ *be defined so that* $0 \leq \varphi \leq 1$*,* φ *is strictly monotonic on* $A_{+\varepsilon} \setminus A$*, and*

$$\varphi(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A_{+\varepsilon} \end{cases}.$$

Define $g = \varphi / \|\varphi\|_1$. Then

$$\left| \int_0^1 (\chi_B \circ T^k(x)) h(x)\, dx - \int_0^1 \chi_B(x) \varphi_*(x)\, dx \right|$$
$$\leq \left| \int_0^1 (\chi_B \circ T^k(x)) \left(h(x) - g(x)\right) dx \right| + \left| \int_0^1 (\chi_B \circ T^k(x))\, g(x)\, dx - \int_0^1 \chi_B \varphi_*(x)\, dx \right|$$
$$\leq \|\chi_B\|_\infty \|g - h\|_1 + \left| \int_0^1 (\chi_B \circ T^k(x))\, g(x)\, dx - \int_0^1 \chi_B \varphi_*(x)\, dx \right|$$

By definition of g and $A_{+\varepsilon}$, we have $\|g - h\|_1 < \varepsilon$, while the second term can be bounded using Theorem 2.5. Therefore

$$\left| \int_0^1 (\chi_B \circ T^k(x)) h(x)\, dx - \int_0^1 \chi_B(x) \varphi_*(x)\, dx \right| < \varepsilon + K \|\chi_B\|_1 (\|g\|_1 + r \|g'\|_\infty) \Lambda^k$$

Since $\|g\|_1 = 1$, $h = \chi_A / vol(A)$, and $\varphi_* = 1$ for q-adic maps,

$$\left| vol(A \cap T^{-k} B) - vol(A) vol(B) \right| < \varepsilon\, vol(A) + K vol(A) vol(B)(1 + r \|g'\|_\infty) \Lambda^K$$
$$< \varepsilon + K vol(A) vol(B)(1 + r \|g'\|_\infty) \Lambda^k.$$

Since $g = 0$ on the complement of $A_{+\varepsilon}$ and increases to 1 on A, we can choose g such that we can bound g' by a constant depending on ε. Therefore,

$$\left| vol(A \cap T^{-k} B) - vol(A) vol(B) \right| < \varepsilon + C(\varepsilon) vol(A) vol(B) e^{-\gamma k}$$

where $\gamma = -\ln \Lambda$ and $C(\varepsilon)$ bounds $K(1 + r \|g'\|_\infty)$. This completes the example. □

Finally, the hitting time for discrete-time transformations (def. IX.1.1), needs to amended for multi-valued processes. To do this, we need a way to order the elements of each set $T^k(y)$. This is straight forward since each set is finite; for each k, put a subset of $T^k(y)$ into one-to-one correspondence with a finite subset of $\mathbb{N}$ and use the normal ordering on $\mathbb{N}$.

Definition IX.2.16 (Hitting time for multivalued maps)**.** Let $(T, \mathcal{M})$ be a $\mathfrak{B}$-regular system. For each $k \in \mathbb{N}_0$, let $D_{y,k} \subset T^k(y)$ and $g_{y,k} : D_{y,k} \to \{1, 2, \ldots, \mu_c(D_{y,k})\}$ be a bijection. Let A be a measurable set. Define

$$m := \inf \{k \in \mathbb{N}_0 : D_{y,k} \cap A \neq \emptyset\} \tag{IX.2.28}$$

and

$$\ell := \inf \left\{ j \in g_{y,m}(D_{y,m}) : g_{y,m}^{-1}(\{\mathbb{N} \leq j\}) \cap A \neq \emptyset \right\}. \tag{IX.2.29}$$

The hitting time of A when starting from y is defined as

$$\tau[A, y] = \sum_{k=0}^{m-1} \mu_c(D_{y,k}) + \ell - 1. \tag{IX.2.30}$$

Recall that μ_c is the counting measure (see eq. (IX.2.1)).

Remark IX.2.17. *In the context of search, the bijections $g_{y,k}$ tells the searcher the order that the points of $T^k(y)$ should be visited. The domains of the $g_{y,k}$'s are defined as (not necessarily strict) subsets of $T^k(y)$ so that some points of $T^k(y)$ can be ignored by the searcher.* □

Remark IX.2.18. *In the case of a single-valued, $\mathfrak{B}$-regular map T, the above definition reduces to the standard hitting time definition* (IX.1.1)*.* □

Remark IX.2.19. *For the n-dimensional subdivision process, define $D_{y,0} = T^0(y) = \{y\}$ and*

$$D_{y,k} = T^k(y) \setminus T^{k-1}(y)$$

for $k \geq 1$. *Since* $T^k(y) \subset T^{k+1}(y)$ *for the subdivision process, then* $D_{y,k} \cap D_{y,j} = \emptyset$ *for* $j \neq k$ *and*

$$T^k(y) = D_{y,0} \cup \cdots \cup D_{y,k}.$$

The integer m *represents the first subdivision level that has points intersecting the target. The term* $\sum_{k=0}^{m-1} \mu_c(D_{y,k})$ *counts all the points searched in the previous subdivision levels, while the term* $\mu_c(g_m^{-1}(\{\mathbb{N} \leq \ell\}))$ *counts the number of points of the* m^{th} *subdivision level before and including the first point that is an element of the target. Hence,* $\tau[A, y] \leq \mu_c(T^m(y)) = q^{mn}$. □

IX.3 Hitting times for mixing transformations

Our end goal is to show relations of the form (IX.1.6) for mixing transformations. The first step in exhibiting this is to show

$$\limsup_{\mu(B(c,r))\to 0} \frac{\ln \tau[B(c,r), y]}{-\ln \mu(B(c,r))} \leq 1 \tag{IX.3.1}$$

for almost all y and where μ an ergodic measure preserved by the transformation. Later, the equivalence of μ and *vol* will be used show that (IX.3.1) is equivalent to the relation (IX.1.5) which is used in proving (IX.1.6).

Before moving on, we remark that the lower bound

$$\liminf_{\mu(B(c,r))\to 0} \frac{\ln \tau[B(c,r), y]}{-\ln \mu(B(c,r))} \geq 1$$

holds for any measure-preserving dynamical system, not just the ergodic ones. The proof can be found in [GK07]. This implies that for any measure-preserving dynamical system satisfying (IX.3.1), that the limit exists and is equal to 1; i.e., the hitting time scales like $\tau[B, y] \sim \mu(B)^{-1}$.

Results showing (IX.3.1) already exist in the quantitative recurrence literature for various types of systems and assumptions on T. However, those results are generally arrived at by nonconstructive means. Since many classes of transformations have the above behavior, it becomes important to differentiate them using their multiplier functions. The function φ in the lemma below differentiates transformations satisfying (IX.3.1) and its form can only be obtained using constructive proofs, which are given in this work.

Lemma IX.3.1. *Assume that μ is a non-atomic measure. For a fixed $c, y \in \mathcal{M}$, (IX.3.1) holds if and only if there is a function $\varphi : [0,1] \to [0, \infty)$ such that*

(i) $\lim_{x \downarrow 0} \dfrac{\ln \varphi(x)}{-\ln x} = 0,$ *and*

(ii) $\tau[B(c,r), y] \leq \varphi(\mu(B(c,r)))\mu(B(c,r))^{-1}$

hold for $\mu(B(c,r))$ sufficiently small. Furthermore, if (IX.3.1) holds, we can show that for any $\varepsilon > 0$, there is a $U = U_\varepsilon > 0$ such that $\tau[B(c,r), y] \leq \mu(B(c,r))^{-(1+\varepsilon)}$ for all $\mu(B(c,r)) \leq U$.

Proof. Fix $c, y \in \mathcal{M}$ and let $B_r = B(c,r)$. If such a φ exists it is easy to see that (IX.3.1) holds. Conversely, if (IX.3.1) holds, we can construct φ with the required properties as follows. Note that since μ contains no atoms, $\mu(B_r) \to 0$ if $r \to 0$. The lim sup inequality implies that for any $n \in \mathbb{N}$, there is an $R_n > 0$ such that for all $r \leq R_n$, $\dfrac{\ln \tau[B_r, y]}{-\ln \mu(B_r)} < 1 + \dfrac{1}{n}$. Note that we can choose $\{R_n\}$ such that $R_n > R_{n+1}$ and

$R_n \downarrow 0$. Define $\varphi(\mu(B_r))$ as follows. If $r > R_1$, put $\varphi(\mu(B_r)) = \mu(B_r)^{-1}$. For $r \leq R_1$, let $n(r)$ be the largest $n \in \mathbb{N}$ such that $R_n \geq r$. Define $\varphi(\mu(B_r)) = \mu(B_r)^{-\frac{1}{n(r)}}$. Note that as $r \downarrow 0$, $n(r) \to \infty$. Obviously, $\lim \dfrac{\ln \varphi(\mu(B_r))}{-\ln \mu(B_r)} = 0$. Additionally, for all $r \leq R_1$, $\dfrac{\ln \tau[B_r, y]}{-\ln \mu(B_r)} < 1 + \dfrac{1}{n(r)}$ which implies that for $r \leq R_1$,

$$\tau[B_r, y] < \mu(B_r)^{-\frac{1}{n(r)}} \mu(B_r)^{-1} = \varphi(\mu(B_r))\mu(B_r)^{-1}$$

as required.

We now show $\tau[B_r, y] \leq \mu(B)^{-(1+\varepsilon)}$. Fix $\varepsilon > 0$ and assume (IX.3.1) holds. Let φ be constructed as above. If we show that $\varphi(\mu(B_r))\mu(B_r)^{\varepsilon}$ goes to 0 as r goes to 0, we will be done since $\tau[B_r, y] \leq \varphi(\mu(B_r))\mu(B)^{-1} = \varphi(\mu(B_r))\mu(B_r)^{\varepsilon}\mu(B_r)^{-(1+\varepsilon)}$. Note that $\mu(B_r)^{\varepsilon} \ln \frac{1}{\mu(B_r)} \to 0$ as $\mu(B_r) \to 0$ by L'Hôpital's rule. Note that,

$$\varphi(\mu(B_r))\mu(B_r)^{\varepsilon} = \frac{\varphi(\mu(B_r))}{\ln \frac{1}{\mu(B_r)}} \mu(B_r)^{\varepsilon} \ln \frac{1}{\mu(B_r)}.$$

By the construction of φ and the L'Hôpital argument above, the above expression limits to 0 as $\mu(B_r) \to 0$. Hence for $\mu(B_r)$ sufficiently small, $\varphi(\mu(B_r))\mu(B_r)^{\varepsilon} < 1$. This completes the proof. □

IX.3.1 Uniformly exponentially mixing transformations

We first show (IX.3.1) holds for uniformly exponentially mixing transformations. The proofs given for this case are templates for the weaker forms of mixing defined above. The first lemma shows that the size of the exception set, $\mathcal{E}(c, r, t)$, decays at an exponential rate, suggesting that the process looks almost independent after a short time period. The rate of decay depends only on the mixing rate γ and the measure of the target.

Lemma IX.3.2. *Assume $T : \mathcal{M} \to \mathcal{M}$ satisfies (IX.2.11) and fix $c \in \mathcal{M}$. Then, there exists an integer $p \geq 1$ and a $\lambda > 0$, both depending on the radius of the target r, the measure μ preserved by T, and the mixing rate γ, such that for $t > p$*

$$\mu(\mathcal{E}(c, r, t)) \leq e^{-\lambda t} . \tag{IX.3.2}$$

Proof. Fix $c \in \mathcal{M}$ and $r > 0$ and denote $B(c, r)$ by B. We can write $E_t := \mathcal{E}(c, r, t)$ as

$$E_t = (\mathcal{M} \setminus B) \cap T^{-1}(\mathcal{M} \setminus B) \cap \cdots \cap T^{-t}(\mathcal{M} \setminus B) .$$

Let p be a positive integer that will be chosen later. For all $t > p$,

$$E_t \subset A \cap T^{-p} A \cap \cdots \cap T^{-mp} A =: C_m \tag{IX.3.3}$$

where $A := \mathcal{M} \setminus B$ and $m = \lfloor t/p \rfloor$, the integer part of t/p. Apply the mixing condition (IX.2.11) to the p-skeleton (IX.3.3) to get the relation

$$\begin{aligned} \mu(C_m) = \mu(A \cap T^{-p} A \cap \cdots \cap T^{-mp} A) &\leq \mu(A \cap T^{-p} C_{m-1}) \\ &\leq (1 + Ce^{-\gamma p}) \mu(A) \mu(C_{m-1}). \end{aligned}$$

Proceeding inductively, we get

$$\begin{aligned} \mu(C_m) \leq \left[(1 + Ce^{-\gamma p}) \mu(A)\right]^m \mu(C_0) &\leq \mu(A)^{m+1} (1 + Ce^{-\gamma p})^m \\ &\leq \mu(\mathcal{M} \setminus B)^{m+1} (1 + Ce^{-\gamma p})^{m+1} , \end{aligned}$$

where we have used that $C_0 = A$. Since $E_t \subset C_m$,

$$\mu(E_t) \leq \left[\mu(\mathcal{M} \setminus B)(1 + Ce^{-\gamma p})\right]^{m+1} . \tag{IX.3.4}$$

Since $\mu(\mathcal{M} \setminus B) = 1 - \mu(B) < 1$ for all r and $e^{-\gamma k} \downarrow 0$ as $k \to \infty$, then for each r there will be a $p = p(r)$ that makes $\mu(\mathcal{M} \setminus B)(1 + Ce^{-\gamma n}) < 1$.

Accordingly, p can be chosen so that $(1 - \mu(B))(1 + Ce^{-\gamma p}) \leq 1 - \frac{1}{2}\mu(B)$. A short calculation shows that it is sufficient to take p as

$$p \geq -\frac{1}{\gamma} \ln \left(\frac{1}{2C} \frac{\mu(B)}{1 - \mu(B)} \right) . \tag{IX.3.5}$$

Since p is defined as a positive integer, take $p = 1$ if the right side of (IX.3.5) is not positive.

Let p be chosen as such and put $\kappa := 1 - \frac{1}{2}\mu(B)$. Using (IX.3.4), E_t can be bounded above by

$$\mu(E_t) \leq \kappa^{m+1} \leq \kappa^{t/p} = e^{(\ln \kappa)t/p} .$$

Putting

$$\lambda = -\frac{1}{p} \ln \kappa \equiv -\frac{1}{p} \ln \left(1 - \frac{1}{2}\mu(B) \right) \tag{IX.3.6}$$

gives the desired result. □ □

Remark IX.3.3. *We call the integer p defined in the lemma above the* gap time *and interpret it as the time the system needs to reach independence. To see this, we note that p is chosen so that $(1 - \mu(B))(1 + Ce^{-\gamma p}) \leq 1 - \frac{1}{2}\mu(B)$. A short calculation* *shows that* $Ce^{-\gamma p} \leq \frac{\mu(B)}{2(1-\mu(B))}$ *which is approximately* $\frac{1}{2}\mu(B)$ *for small targets. Using this with the mixing condition* (IX.2.11), *we get that*

$$\mu(A \cap T^{-p}B) \leq \mu(A)\mu(B)(1 + Ce^{-\gamma p}) \lesssim \mu(A)\mu(B)(1 + \frac{\mu(B)}{2}) \approx \mu(A)\mu(B)$$

for small $\mu(B)$.

Note that the gap time p *from* (IX.3.5) *is a decreasing function of the mixing rate* γ. *A large* γ *corresponds to a fast mixing rate, meaning that the system approaches independence rapidly. Intuitively, we would expect that the corresponding gap time* p *would be small. This relation is made precise by* (IX.3.5).

The gap time is also a decreasing function of the measure of the target set. As the measure decreases to zero, p *approaches infinity. Given that* p *is the time to reach approximate independence and that it increases as the measure of the target decreases, we can interpret this to mean that the transformation becomes independent in a hierarchical fashion. Meaning, if* $\mu(B)$ *is relatively large,* p *is small and if we look at only large target sets, the process is almost independent after a short time. However, if we use the same* p *and look at smaller sets, the process may not look independent yet. Looking at it from the other direction, as* p *increases, the process looks indepedent at smaller and smaller scales. More precisely,* p *is approximately* $\ln \frac{1}{\mu(B)}$ *for small targets. Thus, if we shrink the target so that its new measure* $\mu(B')$ *satisfies* $\mu(B') = \mu(B)^2$, *then* p *approximately doubles.* $\square$

Remark IX.3.4. *From* (IX.3.5) *and* (IX.3.6)*, we get that*

$$\lambda \approx \gamma \left(\frac{\ln \frac{2}{2-\mu(B)}}{\ln \frac{2c(1-\mu(B))}{\mu(B)}} \right).$$

For fixed γ, λ goes to 0 as the measure goes to 0. Since λ controls the rate of decrease of the exception set's measure, this result says that the measure of the points that have not found the target by time t decreases very slowly for small targets. This is agrees with our intuition.

If we let H_t be the set of points that have hitting times less or equal to t, then $H_{t+1} = H_t \cup \{y : \tau[B, y] = t + 1\}$. But $\{y : \tau[B, y] = t + 1\} \subset T^{-(t+1)}B$, so that $\mu(H_t) \leq \mu(H_{t+1}) \leq \mu(H_t) + \mu(B)$. Therefore, at each time step, the measure of the exception set $\mathcal{E}(c, r, t)$ decreases by a quantity smaller than $\mu(B)$. For small $\mu(B)$, this decreases very slowly. □

The next result gives a lower bound on the time needed to wait if we want to guarantee finding a target of a fixed size from a set of initial conditions of a certain size.

Lemma IX.3.5. *Assume $T : \mathcal{M} \to \mathcal{M}$ satisfies* (IX.2.11)*. Fix $c \in \mathcal{M}$, $r > 0$, and let $B = B(c, r)$. If ϵ satisfies $0 < \epsilon \leq 1 - \frac{1}{2}\mu(B)$, then $\mu(\mathcal{E}(c, r, t)) < \epsilon$ for*

$$t > p \, \frac{|\ln \epsilon|}{\left|\ln\left(1 - \frac{1}{2}\mu(B)\right)\right|} \,. \tag{IX.3.7}$$

where p is chosen as in lemma IX.3.2.

Proof. This is a straight forward computation using (IX.3.2) and (IX.3.6). □

A combination of the above two lemmas gives an upper bound for the hitting time of a target for an appropriately large set of initial conditions for the searcher. In particular, if we wish to find the target with probability 0.95, set $\varepsilon = 0.05$ and use (IX.3.8) and (IX.3.9) to find the time needed to wait.

Proposition IX.3.6. *Assume $T : \mathcal{M} \to \mathcal{M}$ is uniformly exponentially mixing preserving a measure μ equivalent to the Lebesgue measure. Fix any $c \in \mathcal{M}$, $r > 0$, and let $B = B(c, r) \subset \mathcal{M}$. Then for any $0 < \epsilon < 1 - \frac{1}{2}\mu(B)$, there exists a set $\mathscr{G}(c, r)$, with $\mu(\mathscr{G}(c, r)) > 1 - \varepsilon$, such that for $y \in \mathscr{G}(c, r)$,*

$$\tau[B, y] \leq \varphi(\mu(B)) \cdot \mu(B)^{-1} \tag{IX.3.8}$$

where

$$\varphi(\mu(B)) = 2\,|\ln \varepsilon| \left[\frac{1}{\gamma} \ln \left(\frac{2C(1 - \mu(B))}{\mu(B)}\right) + 1\right] + \mu(B). \tag{IX.3.9}$$

Furthermore,

$$\mu\left(\left\{y \in \mathcal{M} : \limsup_{\mu(B) \to 0} \frac{\ln \tau[B, y]}{- \ln \mu(B)} \leq 1\right\}\right) = 1\,. \tag{IX.3.10}$$

Before starting the proof, we remark that $\dfrac{\ln \varphi(\mu(B))}{- \ln \mu(B)}$ converges to 0 as $\mu(B) \to 0$.

Proof. Fix $c \in \mathcal{M}$ and denote $B(c, r)$ as B. Fix ϵ such that $0 < \epsilon < 1 - \frac{1}{2}\mu(B)$. Choosing

$$p := \inf \left\{k \in \mathbb{N} : k \geq \frac{1}{\gamma} \ln \frac{2C(1 - \mu(B))}{\mu(B)}\right\}, \tag{IX.3.11}$$

we have that

$$\frac{1}{\gamma} \ln \frac{2C(1 - \mu(B))}{\mu(B)} \leq p < \frac{1}{\gamma} \ln \frac{2C(1 - \mu(B))}{\mu(B)} + 1. \tag{IX.3.12}$$

Furthermore, choose $t \in \mathbb{N}$ such that

$$t := \inf \left\{ k \in \mathbb{N} : k > \frac{2\left|\ln \epsilon\right|}{\mu(B)} \left[\frac{1}{\gamma} \ln \left[\frac{2C(1-\mu(B))}{\mu(B)} \right] + 1 \right] \right\}. \tag{IX.3.13}$$

Since for all $0 < x < 1$, $|\ln(1-x)| > x$, then putting $x = \mu(B)/2$ gives

$$t > \left[\frac{1}{\gamma} \ln \left[\frac{2C(1-\mu(B))}{\mu(B)} \right] + 1 \right] \frac{|\ln \epsilon|}{\frac{1}{2}\mu(B)} > p \frac{|\ln \epsilon|}{\left|\ln\left(1 - \frac{1}{2}\mu(B)\right)\right|}. \tag{IX.3.14}$$

Given (IX.3.11), (IX.3.14), and lemma IX.3.5, $\mu(\mathcal{E}(c,r,t)) < \epsilon$ so that $\mu(\mathcal{M} \setminus \mathcal{E}(c,r,t)) > 1 - \epsilon$. Put $\mathscr{G}(c,r) = \mathcal{M} \setminus \mathcal{E}(c,r,t)$. For $y \in \mathscr{G}(c,r)$, lemma IX.3.5 implies $\tau[B,y] \leq t$ and by (IX.3.13),

$$t \leq \frac{2\left|\ln \epsilon\right|}{\mu(B)} \left[\frac{1}{\gamma} \ln \left[\frac{2C(1-\mu(B))}{\mu(B)} \right] + 1 \right] + 1.$$

This is equivalent to (IX.3.8) and (IX.3.9).

We now show that (IX.3.10) holds. We show that,

$$\mu\left(\left\{ y \in \mathcal{M} : \limsup_{\mu(B) \downarrow 0} \frac{\ln \tau[B,y]}{-\ln \mu(B)} > 1 \right\} \right) = 0. \tag{IX.3.15}$$

We first note that

$$\begin{aligned} \ln \varphi(\mu(B)) &= \ln \left[2\left|\ln \varepsilon\right| \left[\frac{1}{\gamma} \ln \left(\frac{2C(1-\mu(B))}{\mu(B)} \right) + 1 \right] + \mu(B) \right] \\ &\leq \ln(2\left|\ln \varepsilon\right|) + \ln \frac{1}{\gamma} + \ln \ln \left(\frac{2C'}{\mu(B)} \right) + 2 \end{aligned}$$

where $C' \geq C$ is a constant chosen so that $\frac{1}{\gamma} \ln(2C') \geq 1$.

Fix $i \in \mathbb{N}$. Define the sequence $\{B_j\}_{j=2}^{\infty}$ as $B_j := B(c, r_j)$, where r_j is such that $\mu(B_j) = 1/j$. Define $\epsilon_j = \mu(B_j)^2 \equiv 1/j^2$, so that $\epsilon_j < 1 - \frac{1}{2}\mu(B_j)$ for all j. Then

$$\begin{aligned} \ln \tau[B_j, y] &\leq \ln\left(\varphi(\mu(B_j)) \cdot \mu(B_j)^{-1}\right) \\ &\leq -\ln \mu(B_j) + \ln(4\left|\ln j\right|) + \ln \frac{1}{\gamma} + \ln \ln \left(\frac{2C'}{\mu(B_j)} \right) + 2. \end{aligned}$$

By lemma IX.3.5, this holds on a set of measure greater than $1-\epsilon_j = 1-j^{-2}$. Therefore,

$$\frac{\ln\tau[B_j,y]}{-\ln\mu(B_j)} \leq 1 + \frac{\ln(4\,|\ln j|)}{-\ln\mu(B_j)} + \frac{\ln\frac{1}{\gamma}}{-\ln\mu(B_j)} + \frac{\ln\ln\left(\frac{2C}{\mu(B_j)}\right)}{-\ln\mu(B_j)} + \frac{2}{-\ln\mu(B_j)}. \quad \text{(IX.3.16)}$$

on a set of measure greater than $1-j^{-2}$.

Define

$$\varphi_j(y) := \begin{cases} \dfrac{\ln\tau[B_j,y]}{-\ln\mu(B_j)} - 1, & \text{if } \frac{\ln\tau[B_j,y]}{-\ln\mu(B_j)} - 1 \geq 0, \\ 0 & \text{otherwise} \end{cases},$$

so that $\{\varphi_j\}_2^\infty$ is a sequence of non-negative functions. Let

$$\mathcal{F}_j := \left\{ y \in \mathcal{M} : \varphi_j(y) > \frac{1}{i} \right\}$$

and define

$$\mathcal{F} := \limsup_{j\to\infty} \mathcal{F}_j \equiv \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \mathcal{F}_k = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \left\{ y \in \mathcal{M} : \varphi_k(y) > \frac{1}{i} \right\}.$$

Note that it is always true that

$$\left\{ y \in \mathcal{M} : \limsup_{j\to\infty} \varphi_j(y) > \frac{1}{i} \right\} \subset \mathcal{F},$$

so we only need to show that $\mu(\mathcal{F}) = 0$. Note that $\bigcup_{k=j}^{\infty} \left\{ y \in \mathcal{M} : \varphi_k(y) > \frac{1}{i} \right\}$ is a monotonically decreasing sequence of sets in the variable j, and since μ is continuous on monotonic sequences of sets,

$$\mu(\mathcal{F}) = \lim_{j\to\infty} \mu\left(\bigcup_{k=j}^{\infty} \mathcal{F}_k \right) \leq \lim_{j\to\infty} \sum_{k=j}^{\infty} \mu(\mathcal{F}_k).$$

If we can show that $\sum_{j=1}^{\infty} \mu(\mathcal{F}_j) < \infty$, we will be done since then $\lim_{j\to\infty} \sum_{k=j}^{\infty} \mu(\mathcal{F}_k) = 0$.

Consider (IX.3.16). Since we have chosen $\mu(B_j) = 1/j$, then for large enough j, the expression on the right side of (IX.3.16) will be less than or equal to $1 + 1/i$, and this will be true on a set of measure greater than $1 - 1/j^2$. This implies that for such j, $\varphi_j(y) \leq 1/i$ on a set of measure greater than $1 - 1/j^2$. Therefore, for sufficiently large j, $\mu(\mathcal{F}_j) < 1/j^2$, and hence $\{\mu(\mathcal{F}_j)\}$ is summable, as required. Therefore,

$$\mu\left(\left\{y \in \mathcal{M} : \limsup_{j\to\infty} \varphi_j(y) > \frac{1}{i}\right\}\right) = 0.$$

Since $i \in \mathbb{N}$ was arbitrary, the above holds for all $i \in \mathbb{N}$. Since

$$\left\{y \in \mathcal{M} : \limsup_{j\to\infty} \varphi_j(y) > 0\right\} = \bigcup_{i=1}^{\infty} \left\{y \in \mathcal{M} : \limsup_{j\to\infty} \varphi_j(y) > \frac{1}{i}\right\},$$

then $\mu\left(\left\{y \in \mathcal{M} : \limsup_{j\to\infty} \varphi_j(y) > 0\right\}\right) = 0$.

We have shown that (IX.3.15) holds for a specific decreasing sequence $\{B_j\}$. Now we need to strengthen it to hold for arbitrary $\mu(B)$. This is easily done by noting that for any $B(c,r)$, there is some j such that $B_j \subset B(c,r) \subset B_{j-1}$. Then the following holds for every y:

$$\frac{\ln \tau[B_{j-1}, y]}{\ln(j-1)} \frac{\ln(j-1)}{\ln j} < \frac{\ln \tau[B(c,r), y]}{-\ln \mu(B(c,r))} < \frac{\ln \tau[B_j, y]}{\ln j} \frac{\ln j}{\ln(j-1)}.$$

Both the lower bound and upper bound have the same lim sup and therefore

$$\limsup_{\mu(B(c,r))\to 0} \frac{\ln \tau[B(c,r), y]}{-\ln \mu(B(c,r))} = \limsup_{j\to\infty} \frac{\ln \tau[B_j, y]}{-\ln \mu(B_j)}.$$

This completes the proof. □

Remark IX.3.7. *The results can be restate in the following "confidence" form: For a fixed* $B = B(c, r)$,

$$\mu\left(\left\{y \in \mathcal{M} : \tau[B, y] \leq \varphi(\mu(B)) \cdot \mu(B)^{-1}\right\}\right) > 1 - \varepsilon$$

□

Remark IX.3.8. *In the above proposition, we fixed an* $\varepsilon > 0$ *and to this* ε *there corresponded a* t *satisfying* (IX.3.7) *and a set of initial conditions* $\mathcal{E}(c, r, t)$ *such that* $\mu(\mathcal{E}(c, r, t)) < \varepsilon$ *and for* $y \notin \mathcal{E}(c, r, t)$, $\tau[B, y] \leq t$.

Alternatively, for a fixed $y \in \mathcal{M}$, ε *can be made a function of* y. *Do this as follows. Assume the hitting time of* B *from* y *is finite. Let* $n = \tau[B, y]$. *Let* $\mathscr{G}(k) = \bigcup_{j=0}^{k} T^{-j}(B)$ *and define* $\zeta(y) = \mu(\mathscr{G}(n))$. *Note that* $\mathscr{G}(k)$ *is the set of* y *having hitting times less than or equal to* k. *There are two cases: either* $1 - \zeta(y) > 1 - \frac{1}{2}\mu(B)$, *or* $1 - \zeta(y) \leq 1 - \frac{1}{2}\mu(B)$.

In the first case, take $\varepsilon = 1 - \frac{1}{2}\mu(B)$. *From lemma IX.3.5, we can take* $t = p + 1$. *For this* t, $\mu(\mathcal{E}(c, r, t)) < \varepsilon$. *Since* $\varepsilon < 1 - \zeta(y) = \mu(\mathcal{E}(c, r, n))$, *then* $\mu(\mathcal{E}(c, r, t)) < \mu(\mathcal{E}(c, r, n))$. *Since* $\mathcal{E}(c, r, \cdot)$ *is a monotonically decreasing sequence of sets, the previous inequality implies* $t > n$. *Therefore,* $y \notin \mathcal{E}(c, r, t)$.

In the second case, define $\varepsilon = 1 - \zeta(y)$. *The conditions of lemma IX.3.5 again hold. If we let* t *satisfy* (IX.3.7), *then* $\mu(\mathcal{E}(c, r, t)) < \varepsilon = 1 - \zeta(y) = \mu(\mathcal{E}(c, r, n))$. *By the same argument as above,* $t > n$ *and* $y \notin \mathcal{E}(c, r, t)$.

In either case, (IX.3.8) *and* (IX.3.9) *still hold, only now* ε *is a function of* y. *For fixed* y, *this is just a constant and we can regard* $\tau[B(c, r), y]$ *as a function of the* *radius r only*[2]*. Hence, for any fixed y having a finite hitting time for B, $\tau[B,y] = \mathcal{O}(\frac{1}{\mu(B)}\ln\frac{1}{\mu(B)})$.* □

IX.3.2 Nonuniform exponentially mixing transformations

The uniform exponential mixing condition used in the previous sections can be changed to the weaker form of exponential mixing given by (IX.2.12) and similar results will hold. In this case, we would get a decay rate for $\mu(\mathcal{E}(c,r,t))$ that depends on $(\mu(\mathcal{M}-B))^{-1}$ instead of (IX.3.4). Regardless, p can still be chosen so that $\mu(\mathcal{E}(c,r,t)) \leq (1-\frac{1}{2}\mu(B))^{m+1}$. The following two lemma make this precise and are analogues to lemma IX.3.2.

Lemma IX.3.9. *Assume $T:\mathcal{M}\to\mathcal{M}$ satisfies (IX.2.12) and fix $c\in\mathcal{M}$. Then, there exists an integer $p\geq 1$ and a $\lambda>0$, both depending on the radius of the target r, the measure μ preserved by T, and the mixing rate γ, such that for $t>p$*

$$\mu(\mathcal{E}(c,r,t)) \leq e^{-\lambda t}. \tag{IX.3.17}$$

Proof. Define B, E_t, p, m, and C_m as in lemma IX.3.2. Letting $B^c=\mathcal{M}\setminus B$, we have

$$\begin{aligned}\mu(E_t) \leq \mu(C_m) &\leq \mu(B^c)\mu(C_{m-1}) + C\mu(C_{m-1})e^{-\gamma p}\\ &= \left(1+\frac{C}{\mu(B^c)}e^{-\gamma p}\right)\mu(B^c)\mu(C_{m-1})\end{aligned}$$

[2] *We always assume that the center c of the ball is fixed.*

Proceeding inductively,

$$\begin{aligned}\mu(E_t) &\le \left(1+\frac{C}{\mu(B^c)}e^{-\gamma p}\right)^m \mu(B^c)^m\mu(C_0)\\ &\le \left[\left(1+\frac{C}{\mu(B^c)}e^{-\gamma p}\right)\mu(B^c)\right]^{m+1}\\ &= \left[\left(1+\frac{C}{1-\mu(B)}e^{-\gamma p}\right)(1-\mu(B))\right]^{m+1}\end{aligned}$$

By choosing $p \in \mathbb{N}$ so that

$$p \ge \frac{1}{\gamma}\ln\frac{2C}{\mu(B)} \tag{IX.3.18}$$

we have

$$\left[\left(1+\frac{C}{1-\mu(B)}e^{-\gamma p}\right)(1-\mu(B))\right] \le 1-\frac{\mu(B)}{2}.$$

Then

$$\mu(E_t) \le \left[1-\frac{\mu(B)}{2}\right]^{m+1} \le \left(1-\frac{\mu(B)}{2}\right)^{t/p}$$

Putting $\lambda = -\frac{1}{p}\ln(1-\frac{\mu(B)}{2})$, gives $\mu(E_t) \le e^{-\lambda t}$. □

Despite this weaker form of mixing, Lemma IX.3.5 remains unchanged and we still have an analogue of proposition IX.3.6. The difference is that the premultiplier φ will not be given by (IX.3.9), but some other function of $\mu(B)$.

Proposition IX.3.10. *Assume $T : \mathcal{M} \to \mathcal{M}$ satisfies (IX.2.12) and preserves a measure μ equivalent to the Lebesgue measure. Fix any $c \in \mathcal{M}$, $r > 0$, and let $B = B(c,r) \subset \mathcal{M}$. Then for any $0 < \epsilon < 1 - \frac{1}{2}\mu(B)$, there exists a set $\mathscr{G}(c,r)$, with*

$\mu(\mathscr{G}(c,r)) > 1-\varepsilon$, such that for $y \in \mathscr{G}(c,r)$,

$$\tau[B,y] \leq \varphi(\mu(B)) \cdot \mu(B)^{-1} \tag{IX.3.19}$$

where

$$\varphi(\mu(B)) = 2\,|\ln\varepsilon| \left[\frac{1}{\gamma}\ln\frac{2C}{\mu(B)} + 1\right] + \mu(B). \tag{IX.3.20}$$

Furthermore,

$$\mu\left(\left\{y \in \mathcal{M} : \limsup_{\mu(B)\downarrow 0} \frac{\ln\tau[B,y]}{-\ln\mu(B)} \leq 1\right\}\right) = 1\,. \tag{IX.3.21}$$

As the proof proceeds exactly the same as for Proposition IX.3.6, except that p is now given by the infimum of integers satisfying (IX.3.18), rather than (IX.3.11), it will not be repeated.

IX.3.3 Summable mixing systems

While everything above was derived for an exponentially mixing maps, the same procedure can be used for uniform summable mixing transformations of the form

$$\left|\mu(A \cap T^{-k}B) - \mu(A)\mu(B)\right| \leq C\mu(A)\mu(B)\Phi(k)\,, \tag{IX.3.22}$$

where $\Phi(k)$ is a summable, monotonically decreasing sequence of positive numbers. In this case, there will not be a nice expression for the time p we must wait for the process to look nearly independent as we do in (IX.3.5). We will however have the condition

$$\mu(\mathcal{E}(c,r,t)) \leq [(1-\mu(B))(1+C\Phi(p))]^{m+1}.$$

which is analogous to (IX.3.4). As before, p can be chosen so that

$$(1-\mu(B))(1+C\Phi(p)) \leq 1 - \frac{1}{2}\mu(B).$$

IX.3.4 Measures equivalent to the Lebesgue measure

Thus far, we have talked about asymptotic laws in terms of the preserved measure μ or, equivalently, the "μ-size" of the target. However, when one thinks of searching for a target, it is natural to talk about its size in terms of its volume. The equivalence of the measures allows us to transform statements in terms of μ, which is natural for the transformation, into statements in terms of the Lebesgue measure, which is natural for the problem specification. This is the content of the following proposition.

Proposition IX.3.11. *If $T : \mathcal{M} \to \mathcal{M}$ satisfies (IX.3.1) and μ satisfies (IX.2.8), then*

$$\limsup_{vol(B(c,r))\to 0} \frac{\tau[B(c,r),y]}{-\ln vol(B(c,r))} \leq 1\,. \tag{IX.3.23}$$

Proof. Fix $c \in \mathcal{M}$. Let $B = B(c,r)$. By lemma IX.3.1 above, there is a $\varphi(\mu(B))$ having the properties $\frac{\ln \varphi(\mu(B))}{-\ln \mu(B)} \to 0$ as $\mu(B) \to 0$, and, for all $\mu(B)$ sufficiently small, $\tau[B,y] < \varphi(\mu(B))\mu(B)^{-1}$. By the assumption, (IX.2.8), on the ergodic measure μ, $d\mu = f\,dx$ and $0 < \alpha \leq f(x) \leq \beta$ for all $x \in \mathcal{M}$. Using this to bound $\mu(B)^{-1}$, we get for $\mu(B)$ sufficiently small

$$\tau[B,y] < \alpha\,\varphi(\mu(B))\,vol(B)^{-1}$$

This implies that

$$\frac{\ln \tau[B,y]}{-\ln vol(B)} < \frac{\ln \alpha}{-\ln vol(B)} + \frac{\ln \varphi(\mu(B))}{-\ln vol(B)} + 1. \qquad \text{(IX.3.24)}$$

By (IX.2.8), $\frac{-\ln \mu(B)}{-\ln vol(B)}$ is bounded, so that

$$\frac{\ln \varphi(\mu(B))}{-\ln vol(B)} = \frac{\ln \varphi(\mu(B))}{-\ln \mu(B)} \frac{-\ln \mu(B)}{-\ln vol(B)} \to 0$$

and hence

$$\limsup_{vol(B)\to 0} \frac{\ln \tau[B,y]}{-\ln vol(B)} \leq 1, \qquad \text{(IX.3.25)}$$

which proves the result. □

IX.4 Expected search time

In the previous sections, we showed relations of the form

$$\limsup_{vol(B)\to 0} \frac{\ln \tau[B(c,r),y]}{-\ln vol(B(c,r))} \leq 1 \qquad \text{(IX.4.1)}$$

for every c and almost every y. We now compute the expected search time for these classes of transformations. The main result states that the expected search time behaves like $\mathcal{H}(c,y,\delta) = \mathcal{O}(\ln \frac{1}{\delta})$.

Proposition IX.4.1. *If $T : \mathcal{M} \to \mathcal{M}$ is a transformation satisfying (IX.4.1) and $\nu_\delta(u;\xi)$ is the nearly uniform distribution on target sizes given by (VIII.0.2), then for every c and almost every y*

$$\lim_{\xi\to 0} \limsup_{\delta\to 0} \frac{\mathcal{H}(c,y,\delta)}{-\ln \delta} \leq \frac{1}{V}\ . \qquad \text{(IX.4.2)}$$

Proof. Fix $c \in \mathcal{M}$. Let $B = B(c, r)$. $\xi > 0$ is the parameter of the nearly uniform distribution. Per lemma IX.3.1 (where we have replaced μ with m), there is a $u_\xi > 0$ such that for almost every y, $\tau[B, y] \leq m(B)^{-(1+\xi)}$ if $m(B) \leq u_\xi$. If $r(u)$ is the radius needed to make the volume of the ball equal to u, then we write $u = m(B(c, r(u))$. Assume that δ satisfies $0 < \delta < u_\xi$. We can decompose the definition of $\mathcal{H}(c, y, \delta)$ as

$$\mathcal{H}(c, y, \delta) = \int_\delta^{u_\xi} \nu_\delta(u; \xi)\, \tau[B(c, r(u)), y]\, du + \int_{u_\epsilon}^{V} \nu_\delta(u; \xi)\, \tau[B(c, r(u)), y]\, du\,. \tag{IX.4.3}$$

Consider the first integral on the second line.

$$\int_\delta^{u_\xi} \nu_\delta(u; \xi)\, \tau[B(c, r(u)), y]\, du \leq \int_\delta^{u_\xi} K_\delta u^\xi u^{-(1+\xi)}\, du = K_\delta \ln \frac{u_\xi}{\delta}\,. \tag{IX.4.4}$$

where $K_\delta := \dfrac{1+\xi}{V^{1+\xi} - \delta^{1+\xi}}$.

Consider the second integral in (IX.4.3). For all $u \in (u_\epsilon, V)$,

$$\tau[B(c, r(u)), y] \leq \tau[B(c, r(u_\xi)), y] \leq u_\xi^{-(1+\xi)}.$$

Therefore

$$\int_{u_\epsilon}^{V} \nu_\delta(u; \xi)\, \tau[B(c, r(u)), y]\, du \leq \left[\frac{V^{1+\xi} - u_\xi^{1+\xi}}{V^{1+\xi} - \delta^{1+\xi}}\right] u_\xi^{-(1+\xi)}\,. \tag{IX.4.5}$$

From (IX.4.4) and (IX.4.5)

$$\frac{\mathcal{H}(c, y, \delta)}{-\ln \delta} \leq K_\delta - K_\delta \frac{\ln u_\xi}{\ln \delta} + \left[\frac{V^{1+\xi} - u_\xi^{1+\xi}}{V^{1+\xi} - \delta^{1+\xi}}\right] \frac{u_\xi^{-1-\xi}}{-\ln \delta}$$

for all $0 < \delta < u_\xi$, and hence

$$\limsup_{\delta \to 0} \frac{\mathcal{H}(c, y, \delta)}{-\ln \delta} \leq \frac{(1+\xi)}{V^{1+\xi}}.$$

Since this is true for any $\xi > 0$,

$$\lim_{\xi \to 0} \limsup_{\delta \to 0} \frac{\mathcal{H}(c, y, \delta)}{-\ln \delta} \leq \frac{1}{V} ,$$

and the proof is complete. □

IX.4.1 Simulations

The first set of simulations are computations of the hitting times for the 1-dimensional subdivision and q-adic examples above (see IX.1.2 and example IX.2.14, respectively). For each simulation, a fixed set of $M = 100$ initial conditions for the searcher was chosen to be uniformly distributed in $[0, 1)$. For the j^{th} initial condition, a trajectory $\{y_{j,0}, y_{j,1}, \ldots, y_{j,N-1}\}$ ($N = 10^6$) was generated using the appropriate process. A center $c \in [0, 1)$ for the target was fixed. For each $k \in \{0, \ldots, N-1\}$, $r_{j,k}$ was defined as

$$r_{j,k} = \min \{|y_{j,i} - c| : 0 \leq i \leq k\} . \tag{IX.4.6}$$

Each $r_{j,k}$ measures the closest the trajectory came to the target center in the time interval $[0, 1, \ldots, k]$. Note that for any $r \geq r_{j,k}$, the hitting time satisfies $\tau[B(c, r), y_{j,0}] \leq k$. This gave a decreasing sequence $r_{j,0} \geq r_{j,1} \geq \cdots \geq r_{j,N-1}$.

Define R_k as

$$R_k = \max \{r_{j,k} : 1 \leq j \leq M\} . \tag{IX.4.7}$$

The variable R_k gave a lower bound for the smallest target size having hitting time equal to k, uniform over the chosen initial conditions; i.e., for all $r \geq R_k$ and all initial conditions $y_{j,0}$ $(j = 1, \ldots, M)$, $\tau[B(c,r), y_{j,0}] \leq k$. If enough initial conditions are taken, then k is an estimate for an upper bound, uniform over almost all initial conditions, for the hitting time of a target centered at c with volumes greater than $vol(k) = 2R_k$.

Figures IX.2a - IX.2c show the hitting time for the 1-dimensional subdivision process (ranks $2, 3, 4$) defined in IX.1.2. The plots are of k-vs-$vol(k)$ and include simulations results for 100 different target centers, chosen uniformly at random in $[0, 1)$. Results of the simulations are plotted in color (lower curve). The upper bound for the 1D simulations (equation (IX.1.9)) is plotted in black (upper curve). The data is plotted on a log-log scale and shows the $1/vol(B)$ order for the hitting time. Note that the bound is tight since for any subdivision level j, there exists a target $B(c, r)$ whose hitting time is exactly q^j.

The same simulations were performed for the q-adic maps ($q = 2, 3, 4$). The difference were the initial conditions used. Since, for an integer q, the q-adic map will eventually map any rational point to the fixed point at 0, an irrational point very close to the specified initial conditions was taken instead. This can be done by representing a number in $[0, 1)$ as a q-ary expansion; $y = \sum_{j=0}^{\infty} a_k q^{-(k+1)}$, where $a_k \in \{0, \ldots, q-1\}$ are called the q-bits. The q-adic map is the shift operator on the sequence. For a rational y, the first 64 q-bits were computed. The rest of the sequence was filled with integers taken uniformly at random from $\{0, \ldots, q-1\}$. This guaranteed that the irrational point was within q^{-64} of the specified initial condition.

Figures IX.3a - IX.3c show results of the q-adic simulations in color (lower curve). The analytic upper bound given by (IX.3.8) and (IX.3.9) is plotted in black (upper curve). The constants in (IX.3.9) were chosen as $C = q$, $\gamma = \ln q$, and $\varepsilon = 0.01$. Note that the mixing rate, γ, was replaced with the Lyapunov exponent of the map. For topologically mixing piecewise expanding Markov maps with slopes of the same sign (of which, the q-adic maps are examples), the mixing rate is bounded above by the Lyapunov exponent; $\gamma \leq \Lambda$ [SBJ13]. Therefore, by replacing the mixing rate, γ, with the Lyapunov exponent in (IX.3.9) gives a tighter bound. The data is again plotted on a log-log scale to show the $\frac{1}{vol(B)} \ln(\frac{1}{vol(B)})$ order for the hitting time. Note that these bounds are not as tight as for the subdivision map. This is due to having to estimate the measure of the exception set in the course of the proof of (IX.3.8) by choosing an appropriate gap time. Many gap times are feasible and it was never claimed that the one chosen was optimal. A slight increase in the distance of the data and the upper bound can be seen as we progress to smaller target volumes. This suggests that the order of the hitting time is actually smaller than the $\frac{1}{vol(B)} \ln(\frac{1}{vol(B)})$ of the proof.

Figures IX.4a and IX.4b show the expected hitting time $\mathcal{H}(c, y, \delta)$ averaged over target centers and initial conditions for the searcher for the 1-dimensional subdivision algorithms and q-adic maps, respectively. On the y-axis is plotted an approximation of

$$\mathbb{E}\left[\mathcal{H}(c, y, \delta)\right] = \frac{1}{0.01 - \delta} \int_{\delta}^{0.01} \mathbb{E}\left[\tau(B(u))\right] du \qquad \text{(IX.4.8)}$$

where the maximum target volume was chosen as $V = 10^{-2}$ and

$$\mathbb{E}\left[\tau(B(u))\right] = \int_0^1 \int_0^1 \tau[B(c, r(u)), y]\, dc\, dy. \tag{IX.4.9}$$

In simulations, (IX.4.9) was approximated by Monte Carlo integration over pairs (c_p, y_p) in the square $[0,1) \times [0,1)$, so that

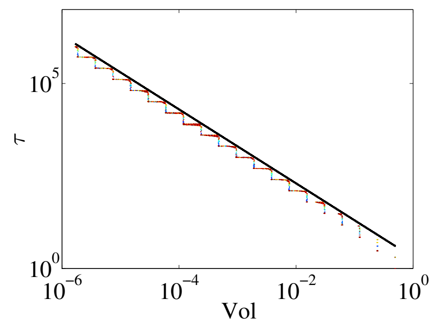
$$\mathbb{E}\left[\tau(B(u))\right] \approx \frac{1}{P} \sum_{p=1}^{P} \tau[B(c_p, r(u)), y_p], \tag{IX.4.10}$$

where $P = 2 \times 10^3$. Since $\tau[B(c, r(u)), y]$ is piecewise constant with respect to u, the integral (IX.4.8) was approximated by a Riemann sum. The partition size was chosen as $\Delta u = \dfrac{10^{-2} - 10^{-6}}{10^4}$. Then $u_j := 10^{-2} - j\Delta u$, for $j = 0, \ldots, 10^4 - 1$. For each u_k,

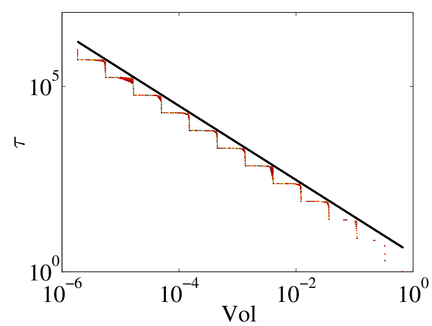
$$\mathbb{E}\left[\mathcal{H}(c, y, u_k)\right] \approx \frac{1}{V - u_k} \sum_{j=1}^{k} \mathbb{E}\left[\tau(B(u_j))\right] \Delta u$$

Note that this is an approximation from above.

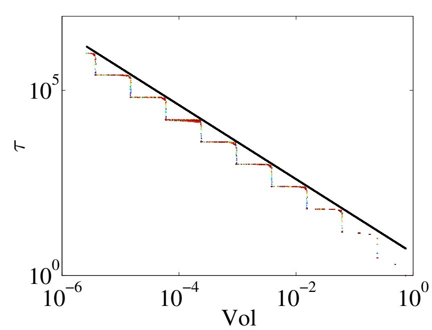
Figure IX.4a clearly shows the separation between the subdivision algorithms as predicted by (IX.1.9). The expected hitting times for the q-adic maps do not share the same clean separation. Both figures IX.4a and IX.4b, however, show the asymptotic behavior of the algorithms as $\mathcal{O}(-\ln \delta)$ as predicted by (IX.4.2).



(a) Rank 2

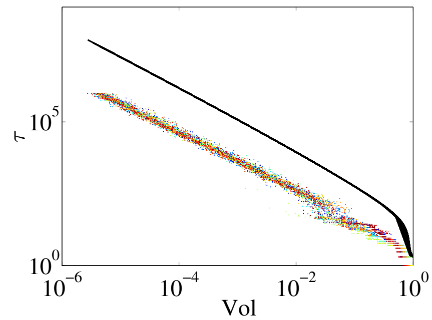


(b) Rank 3

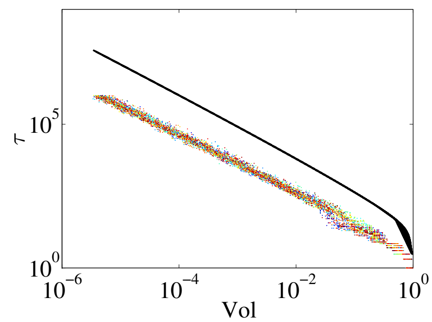


(c) Rank 4

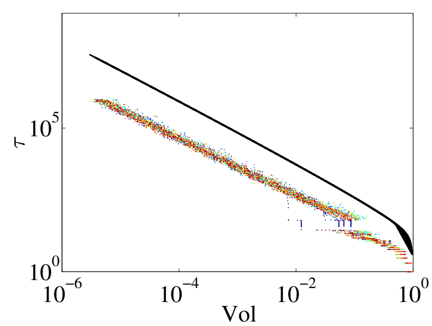
Figure IX.2: Hitting time of the 1D subdivision algorithms. Hitting time from simulations for 100 target centers are plotted in color (stair-step, lower curve). The upper bound given by (IX.1.9) is plotted in black (upper line).



(a) $q = 2$

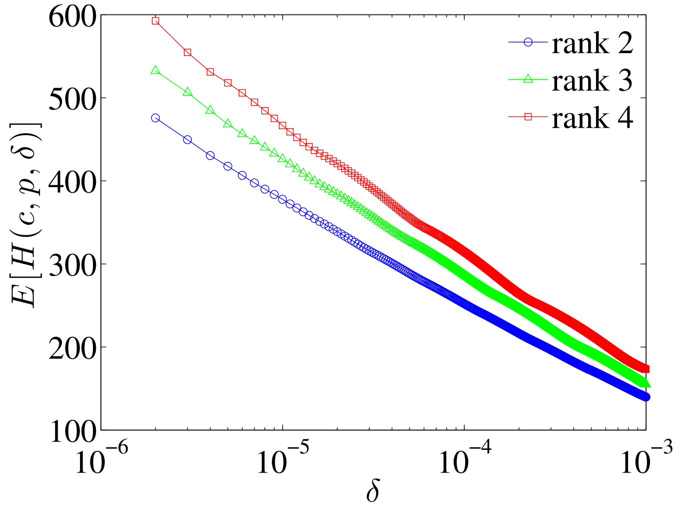


(b) $q = 3$

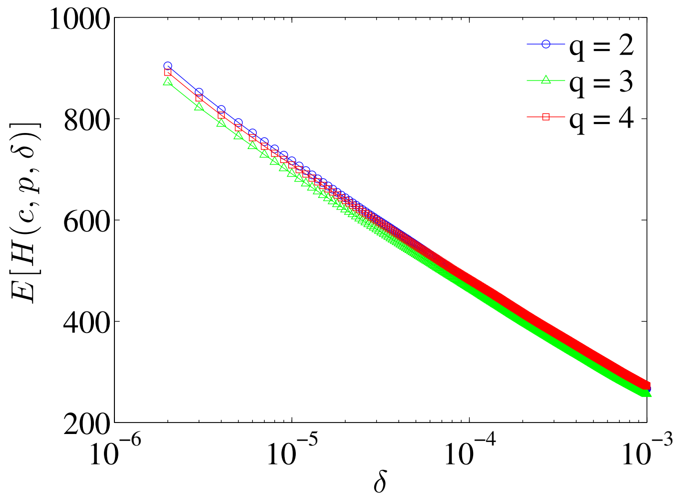


(c) $q = 4$

Figure IX.3: Hitting time of the q-adic maps. Hitting times from simulations for 100 target centers are plotted in color (lower curve). The upper bound given by (IX.3.8) is plotted in black (upper curve).



(a) 1D subdivision algorithms.



(b) 1D q-adic maps.

Figure IX.4: The expected hitting time $\mathcal{H}(c, y, \delta)$ averaged over c and y.

IX.5 Continuous-time dynamics

In the previous sections, it was assumed the searcher evolved according to a discrete-time map. In this section, continuous-time search transformations are constructed using a discrete-time map. The basic idea is to use the discrete-time map to generate *waypoints* that a searcher moves between according to some type of continuous dynamics. In general, it cannot be guaranteed that the continuous-time transformation has the same ergodic measure as the discrete-time map. However, the upper bound on the hitting time in the discrete-time case can be used to derive an upper-bound for the continuous-time case. In passing from discrete-time to continuous-time, some of the tightness on the upper bound is lost. As we are interested in the case when $V \ll \operatorname{diam} \mathcal{M}$, it is natural to approximate the dynamics of the searcher by a first-order differential equation.

IX.5.1 First-order dynamics.

Let $T : \mathcal{M} \to \mathcal{M}$, $\mathcal{M} = \mathbb{T}^n$, be an ergodic discrete time map with a preserved measure μ absolutely continuous with respect to the Lebesgue measure and having a strictly positive, bounded Radon-Nikodym derivative. Let the searcher's position be denoted $\vec{s} = (s_1, \ldots, s_n) \in \mathcal{M}$. The searcher's dynamics are defined

$$\begin{aligned}
\dot{\vec{s}}(t) &= -u_{\min} \frac{\vec{s}(t) - \vec{w}(t)}{\|\vec{s}(t) - \vec{w}(t)\|} \\
\vec{w}(t) &= T^{k(t)}(\vec{y}) \\
k(t) &\leftarrow \begin{cases} k(t), & \vec{s}(t) \neq \vec{w}(t) \\ k(t)+1, & \vec{s}(t) = \vec{w}(t) \end{cases}
\end{aligned} \tag{IX.5.1}$$

where $k(0) = 0$, $\vec{y} \in \mathcal{M}$, and $u_{\min} > 0$. We call $\vec{w}_k = T^k(\vec{y})$ the k^{th} waypoint. Note that $k(t)$ is an $\mathbb{N}_0$-valued jump-process driven by the transformation T that updates when the searcher reaches a waypoint.

Let $\{t_k\}_0^\infty$ be the times at which the searcher arrives at the k^{th} waypoint; i.e., $\vec{s}(t_k) = T^k(\vec{y})$ and $t_k \geq t_{k-1}$. Denoting the hitting time for the discrete-time map T as $\tau_d[B(c,r), \vec{y}]$ and the hitting time for the continuous-time system as $\tau_c[B(c,r), \vec{s}]$ allows us to write

$$\tau_c[B(c,r), \vec{s}] := \inf\{t \geq 0 : \|s(t) - c\| < r, t \in \mathbb{R}\} \leq t_0 + \sum_{k=1}^{\tau_d[B(c,r),\vec{y}]} t_k - t_{k-1}. \tag{IX.5.2}$$

This merely says that the first time the continuous trajectory enters the set $B(c,r)$ must be less than the sum of the inter-arrival times between waypoints, where the sum is taken up to the hitting time for the discrete-time map T. This is easily seen since $T^{\tau_d[B(c,r),y]}(y) \in B(c,r)$ and the right side of (IX.5.2) defines the time the system has been running. Therefore, if the inter-arrival times $\{t_k - t_{k-1}\}$ can be bounded, an upper bound for the continuous-time case can be derived from the discrete-time case.

The bound on the inter-arrival times can be computed by noting that $\|\dot{s}(t)\| = u_{\min}$ for all $t \geq 0$ and

$$t_k - t_{k-1} \leq \frac{\operatorname{diam} \mathcal{M}}{u_{\min}}.$$

When $\mathcal{M} = [0, 1]^n$, (IX.5.2) becomes

$$\tau_c[B(c, r), y] \leq \frac{\sqrt{n}}{u_{\min}} \tau_d[B(c, r), y]. \tag{IX.5.3}$$

IX.6 Discussion and Conclusions

Our results show, when searching for a target while having almost no prior information, that one can do much better than when using a naïve (rational) lawnmower algorithm. In particular, we investigated the hitting times for discrete-time exponentially mixing transformations, obtaining explicit estimates for systems with exponential mixing rates. The finite-time "confidence" results were extend to show that asymptotically, the hitting time behaved as $\tau[B, y] = \mathcal{O}(vol(B)^{-1})$ and the expected hitting time $\mathcal{H}(c, y, \delta) = \mathcal{O}(-\ln \delta)$ for every target center c and almost every initial condition y for the searcher. These results are very similar to what one finds when looking at iid processes. In a certain sense, mixing transformation are no different than iid processes and much of the intuition from the iid case carries over to the mixing case.

By constructing continuous-time dynamics by using a discrete-time transformation, the discrete-time results could be pushed to the continuous-time case by looking at a finite sum of the inter-arrival times between the “waypoints” generated by the discrete-time map. For first-order dynamics and bounded domains, the inter-arrival times could be bounded above so that the hitting time for the continuous dynamics were just a constant multiplying the hitting time for the discrete-time transformation.

While this work investigated exponentially mixing transformations and used the lack of ergodicity of the rational lawnmower algorithm to justify this focus, the reader will notice that a perfectly acceptable modification to the lawnmower algorithm is to use an irrational shift parameter instead of a rational one. It is well known that irrational shifts on the circle are ergodic, but not mixing (see [LM94] or any standard text on ergodic theory). Hitting and recurrence times of almost every irrational shift on the unit interval also satisfy limit laws of the type (IX.1.5) for dyadic intervals [KS03], even though these transformations have fundamentally different behavior than mixing transformations. Although not shown here, numerical simulations have shown that irrational rotations on the circle perform better than their mixing counterparts in terms of the search problem, even though both have the same asymptotic law. However, much of the analysis for irrational shifts depends on the continued fraction representation of the shift parameter [Kim06; Kim07; KS03]. Unfortunately, extensions of continued fractions to higher dimensions are difficult. This makes analysis of irrational shifts in higher dimensions problematic and, at this time, the author is unaware of any analytic recurrence or hitting time results for irrational shifts in higher dimensions. This is contrasted with the results in this paper which hold in arbitrary dimensions as long as the appropriate mixing conditions are satisfied.

We wish to mention one other point. The results in the paper relied on the fact that the transformation chosen to generate the search dynamics was exponentially mixing. A number of simple classes of transformation have been shown to have this mixing property, usually by showing that the associated Perron-Frobenius operator has a spectral gap between 1 and the magnitude of the next largest eigenvalue. Similar results for higher-dimensional expanding maps have been shown as well (see, for example, [Liv12]). Given some dynamics in a higher-dimensional space, if one would like to use the results in this chapter, one would first need to prove that the transformation is exponentially mixing and estimate its mixing rate, tasks that are not simple for a general system. It would be desirable to have a constructive procedure for specifying a transformation in arbitrary dimension with a specified mixing rate. The authors are unaware of any such procedure at this time. This will be the subject of future work.

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Authored by R.M. Mohr

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