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Author
Chung, Victor.

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Victor Chung
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Victor Chung

Lawrence Radiation Laboratory
University of California
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ABSTRACT

The infrared divergences of quantum electrodynamics are eliminated to all orders of perturbation theory in the matrix elements by an appropriate choice of initial and final soft photon states. The coherent state formalism of Glauber is used to parameterize these states. It is shown that the condition for the cancellation of the divergences requires that these states belong to representations of the canonical commutation rules which are unitarily inequivalent to the usual Fock representation.

A comparison to lowest order is made between this treatment and the conventional treatment whereby the divergences are cancelled in the total cross section.

I. INTRODUCTION

The matrix element in quantum electrodynamics for the scattering from an initial state containing a finite number of electrons and photons into a similar final state contains an integral which diverges logarithmically for small momentum k . The conventional treatment of this "infrared divergence" has been to sum the cross sections over all possible final states consistent with experimental measurements. In particular, when all states with any number of soft photons with momenta below the threshold of observability are considered, the divergences cancel, and the calculated cross sections are consistent with experiment. It is therefore possible to attribute the original divergence in the matrix element to the inappropriate choice of initial and final states to represent the experimental situation. In an actual scattering experiment, an indefinite number of soft photons are emitted, so that in some sense, states which are eigenstates of the number operator are unphysical.

In this paper, we shall show that there exists a representation of the photon states for quantum electrodynamics which appears more appropriate for describing scattering than the usual Fock representation in that the matrix elements do not have infrared divergences. These states are not eigenstates of the number operator, and are parameterized in a manner similar to that used by Glauber,[1] Bargmann,[2] and others. When certain conditions of convergence are imposed, the states can be shown to form irreducible representations of the canonical commutation rules for the "in" and "out" fields which are unitarily inequivalent to the usual Fock representation. Similar results have been obtained by Shroer[3] in certain model field theories.

In the absence of known solutions to the renormalized field equations, we make no pretence to mathematical rigor. In particular the Feynman-Dyson perturbation techniques are used throughout, and most questions of order in limiting procedures, etc., are treated heuristically.

The parameterization of the states and its relationship to the usual occupation number parameterization are introduced in Section II. We shall make use of the algebra of states developed in Glauber's paper.[1] In Section III the cancellation of the divergences to second order is demonstrated in order to illustrate the methods to be used in the succeeding sections. Section IV will summarize the parts of the conventional treatment of infrared divergences which we shall need. This section is based on a more complete discussion made in the article by Yennie, Frautschi, and Suura.[4] A calculation of the matrix elements for potential scattering in Section V shows that the divergences indeed cancel to all orders. In Section VI, the structure and the physical meaning of the representations are examined. Then we show that by squaring the matrix elements and summing over the final states, results are obtained in low order which agree with those obtained by Yennie et al.[4] by the conventional treatment. Some extensions and generalizations of our treatment are carried out in the appendices.

II. PARAMETERIZATION OF THE PHOTON STATES

The properties of the states which we will find convenient to use have been discussed by several other authors[1,2] in different contexts from the one in which we intend to use them.

Let $\{f_i(k)\}$ be a complete and orthonormal set of functions defined on some region Ω of momentum space including $\underline{k} = 0$ (perhaps all of momentum space). A typical state "belonging to the $\underline{i}$ th mode" is defined by

$$|\alpha_i\rangle = \frac{\exp(\alpha_i a_i^\dagger)}{\exp(\frac{1}{2}|\alpha_i|^2)} |0\rangle = \exp(-\tfrac{1}{2}|\alpha_i|^2) \sum_n \frac{(\alpha_i a_i^\dagger)^n}{n!} |0\rangle, \quad (1)$$

where

$$a_i^\dagger = \int d^3k \; f_i(k) \; a^+(k) \quad (2)$$

is an "in" or "out" creation operator.

In this expression α_i is a complex number which can take on any value in the complex plane, $a^\dagger(k)$ is the photon creation operator which obeys the commutation rules

$$\left[a(k),\; a^\dagger(k')\right] = \delta(k-k'),\; \left[a(k),\; a(k')\right] = \left[a^\dagger(k),\; a^\dagger(k')\right] = 0, \quad (3)$$

and $|0\rangle$ is the state with no photons; $a_i^\dagger$ obeys the commutation rules

$$\left[a_i, a_j^\dagger\right] = \delta_{ij},\; \left[a_i, a_j\right] = \left[a_i^\dagger,\; a_j^\dagger\right] = 0. \quad (4)$$

From the commutation rules, it is a trivial matter to show that these states are eigenfunctions of the destruction operator

$$a(k) \mid \alpha_i \rangle = \alpha_i f_i(k) \mid \alpha_i \rangle \qquad (5)$$

or

$$a_i \mid \alpha_i \rangle = \alpha_i \mid \alpha_i \rangle,$$

and that the mean number of "photons" is

$$\langle \alpha_i | \hat{N} | \alpha_i \rangle = \int d^3k \, \langle \alpha_i | \, a^\dagger(k) \, a(k) \, | \alpha_i \rangle = |\alpha_i|^2. \qquad (6)$$

It is sometimes useful to note that the state in Equation (1) can be "created" by a unitary operator

$$D(\alpha_i) = \exp \left[\alpha_i a_i^\dagger - \alpha_i^* a_i \right], \qquad (7)$$

which has the following "translation" property:

$$D(\alpha_i) \, D(\beta_i) = \exp \left[\tfrac{1}{2} (\alpha_i \beta_i^* - \alpha_i^* \beta_i) \right] D(\alpha_i + \beta_i). \qquad (8)$$

The states defined in this manner are nonorthogonal; the overlap between two states $|\alpha_i\rangle$ and $|\beta_i\rangle$ is given by

$$|\langle \alpha_i | \beta_i \rangle|^2 = \exp \{ -|\alpha_i - \beta_i|^2 \} . \qquad (9)$$

However, it follows from (9) that the states are normalized, i.e.,

$$\langle \alpha_i | \alpha_i \rangle = 1 . \tag{10}$$

Another property which these states possess is completeness. In fact it is easy to show that

$$\frac{1}{\pi} \int d^2\alpha_i \, |\alpha_i\rangle \langle \alpha_i| = \sum_{n_i} |n_i\rangle \langle n_i| = I , \tag{11}$$

where the state denoted by n_i is an eigenstate of the number of photons which have the momentum distribution described by the function $f_i(k)$, and $d^2\alpha = d(\mathrm{Re}\,\alpha_i)\, d(\mathrm{Im}\,\alpha_i)$ is real.

An arbitrary state of the ith mode has an expansion in terms of the n-photon states of the form

$$|\rangle = \sum_n c_n |n\rangle = \sum_n c_n \frac{(a_i^\dagger)^n}{(n!)^{\frac{1}{2}}} |0\rangle , \tag{12}$$

where $\sum_n |c_n|^2 = 1$. We associate with each such state an analytic function,

$$h(z) = \sum_n c_n \frac{z^n}{(n!)^{\frac{1}{2}}} . \tag{13}$$

Equation (12) may then be rewritten as

$$|h\rangle = h(a_i^\dagger) \, |0\rangle . \tag{14}$$

Using (11), we can expand $|h\rangle$ in terms of the new states:

$$|h\rangle = \frac{1}{\pi}\int d^2\alpha_i \; |\alpha_i\rangle \langle\alpha_i| \; h(a_i^\dagger) \; |0\rangle$$
$$= \frac{1}{\pi}\int d^2\alpha_i \; |\alpha_i\rangle \; h(\alpha_i^*) \exp(-\tfrac{1}{2}|\alpha_i|^2) \; . \tag{15}$$

In (15) we have used the fact that the states $|\alpha_i\rangle$ are eigenstates of the destruction operator a_i :

$$a_i|\alpha_i\rangle = \alpha_i|\alpha_i\rangle \quad . \tag{16}$$

In a similar fashion, the adjoint state vectors $\langle g|$ can be shown to possess an analogous expansion,

$$\langle g| = \frac{1}{\pi}\int \left[g(\beta_i^*)\right]^* \langle\beta_i|\exp(-\tfrac{1}{2}|\beta_i|^2) \; d^2\beta_i \; . \tag{17}$$

A basis for the whole electromagnetic field is the set of vectors formed from all direct products of the states $|\alpha_i\rangle$ of the individual modes where the α_i are allowed to range over the whole complex plane

$$|\rangle = \prod_i |\alpha_i\rangle \equiv |\{\alpha_i\}\rangle \; , \tag{18}$$

and the mean number of photons in such a state is

$$\sum_i |\alpha_i|^2 \quad . \tag{19}$$

Equations (13), (15), and (18) ensure that states containing a finite number of photons (the usual Fock representation) can be expanded in terms of the states $|\{\alpha_i\}\rangle$ which satisfy $\sum_i |\alpha_i|^2 < \infty$. This will be shown in Section VI. However, this restriction will not be imposed in the discussion that follows, i.e., we shall allow for the possibility that there exist states in which the average number of photons is not bounded.

III. CANCELLATION OF THE INFRARED DIVERGENCES TO SECOND ORDER

In this section we shall demonstrate the feasibility of our scheme for cancelling the infrared divergences in the scattering amplitude, and compare it with the conventional method for handling the divergences in the cross section. In both cases, a simple calculation will be made in second order of the electronic charge e_0.

As a concrete example, let us consider an electron scattering from a potential. The zeroth order matrix element, given by the diagram in Fig. III-1 (a), we shall call $M_0(p,p')$. The second order corrections to this process are given by the diagrams in Fig. III-1 (b), (c) and (d). Straightforward application of the Feynman rules with appropriate renormalization will show that the latter three diagrams contribute[4]

$$M_1 = M_0 \, \alpha \, \beta + m_1 \tag{20}$$

where α is the fine structure constant, m_1 is finite, and

$$B = \frac{i}{(2\pi)^3} \int \frac{d^4k}{k^2 - \Lambda^2} \left(\frac{2p'_\mu - k_\mu}{2p'\cdot k - k^2} - \frac{2p_\mu - k_\mu}{2p\cdot k - k^2} \right)^2 \tag{21}$$

p' and p are the electron momenta in the final and initial state respectively. Letting the fictitious photon mass Λ go to zero, we see that B is a logarithmically divergent integral as $k \to 0$.

The conventional solution to this dilemma is to say that the very low energy soft photons in the final state have been neglected, i.e. we should also have computed the diagrams illustrated in Fig.III-2 (b) and (c) . If the momentum and polarization of the soft photon are k and $e^{(\lambda)}$ respectively, then the contribution of these two diagrams is

$$\frac{e_0}{[(2\pi)^3\ 2k_0]^{\frac{1}{2}}}\left[\frac{p'\cdot e^{(\lambda)}}{k\cdot p'}-\frac{p\cdot e^{(\lambda)}}{k\cdot p}\right]M_0+K(k) \tag{22}$$

where K will always denote some function or constant which does not contribute to the infrared divergence.

Let Ω denote that region of the photon's momentum which is below the threshold of observation, i.e. the region which defines the photon as being "soft". Then to second order, the cross section σ is given by

$$\sigma = (M_0+\alpha\, B\, M_0+m_1)^2+\sum_{\lambda}\int_{\Omega} d^3k\ \frac{e_0^{\,2}}{(2\pi)^3\ 2k_0}\left[\frac{p'\cdot e^{(\lambda)}}{k\cdot p'}-\frac{p\cdot e^{(\lambda)}}{k\cdot p}\right]^2 \times M_0^{\,2}+K+\cdots$$

$$= M_0^{\,2}+2\alpha\, B\, M_0^{\,2}+2\alpha\,\tilde{B}\, M_0^{\,2}+K+(\text{higher order terms in } e_0^{\,2})$$

$$= M_0^{\,2}\,(1+2\alpha\, B+2\alpha\,\tilde{B})+\cdots \tag{23}$$

where

$$\tilde{B} = \frac{-1}{8\pi^2}\int_{\Omega}\frac{d^3k}{k_0}\left(\frac{p'_\mu}{k\cdot p'}-\frac{p_\mu}{k\cdot p}\right)^2 . \tag{24}$$

Note that $\tilde{B}$ is logarithmically divergent and, in fact, exactly cancels the divergence in B. The cross section is finite.

It is possible to carry out this treatment to all orders of e_0, and the result one obtains by summing over the contributions from the emission of 1, 2, $\cdots, \infty$ photons is[4]

$$\sigma = \exp\{2\alpha B + 2\alpha \tilde{B}\} \hat{\sigma} \qquad (25)$$

where $\hat{\sigma}$ is a series in e_0 which is term by term infrared divergence free.

We shall now repeat the calculation for the scattering process in which the photon states are parameterized as in Section II. Thus the initial and final states of the photon are specified by the sequences of complex numbers $\{\alpha_a^\lambda\}$ and $\{\gamma_a^\lambda\}$ respectively. The additional superscript λ distinguishes the two polarizations of the photon.

The initial state may be expanded in the following manner:

$$|\{\alpha_a^\lambda\}\rangle = \exp\{-\tfrac{1}{2}\sum_{a,\lambda} |\alpha_a^\lambda|^2\} \exp\{\sum_{a,\lambda} \alpha_a^\lambda a_a^{\lambda +}\}|0\rangle \qquad (26)$$

$$= (1 - \tfrac{1}{2}\sum_{a,\lambda} |\alpha_a^\lambda|^2 + \cdots)(I + \sum_{a,\lambda} \alpha_a^\lambda a_a^{\lambda \dagger} + \cdots)|0\rangle$$

where

$$a_a^{\lambda \dagger} = \int_\Omega d^3k \, f_a(k) \, e^{(\lambda)}(k) \, a^\dagger(k) \qquad (27)$$

and $e^{(\lambda)}(k)$ is the polarization vector. For the final state, substitute γ_a^{λ} for α_a^{λ}.

We must now consider all diagrams in Fig. III-1 and Fig. III-2. With some foresight we will find out that α_a^{λ} and γ_a^{λ} are proportional to e_0. Therefore, to second order, the contribution of Fig. III-1 to the scattering matrix element is

$$M_0\left(1 - \tfrac{1}{2} \sum_{a,\lambda} |\alpha_a^{\lambda}|^2\right)\left(1 - \tfrac{1}{2} \sum_{a,\lambda} |\gamma_a^{\lambda}|^2\right) + \alpha B M_0 + m_1 .$$

The diagram of Fig. III-2 (a) gives us

$$M_0 \left(\sum_{\lambda,a} \alpha_a^{\lambda} \gamma_a^{\lambda *} \right).$$

Fig. III-2 (b) and (c) contribute

$$M_0 \sum_{\lambda,a} \int_{\Omega} \frac{d^3k}{[(2\pi)^3 2k_0]^{\frac{1}{2}}} e_0 \left(\frac{p' \cdot e^{(\lambda)}}{k \cdot p'} - \frac{p \cdot e^{(\lambda)}}{k \cdot p} \right) f_a^*(k) \gamma_a^{\lambda *} + K$$

while in a similar manner, Fig. III-2 (c) and (d) contribute

$$- M_0 \sum_{\lambda,a} \int_{\Omega} \frac{d^3k}{[(2\pi)^3 2k_0]^{\frac{1}{2}}} e_0 \left(\frac{p' \cdot e^{(\lambda)}}{k \cdot p'} - \frac{p \cdot e^{(\lambda)}}{k \cdot p} \right) f_a(k) \alpha_a^{\lambda} + K .$$

Thus the total contribution to the matrix element to second order is

$$M = M_0\left(1 - \tfrac{1}{2} \sum_{a,\lambda} |\alpha_a^{\lambda}|^2 - \tfrac{1}{2} \sum_{a,\lambda} |\gamma_a^{\lambda}|^2\right) + \alpha \beta M_0 + M_0\left(\sum_{a,\lambda} \alpha_a^{\lambda} \gamma_a^{\lambda *}\right)$$

$$+ M_0 \left\{ \sum_{\lambda,a} \int \frac{d^3k}{[(2\pi)^3 2k_0]^{\frac{1}{2}}} e_0 \left(\frac{p'}{k \cdot p} - \frac{p}{k \cdot p} \right) \cdot e^{(\lambda)} \left(f_a^*(k) \gamma_a^{*\lambda} - f_a(k) \alpha_a^{\lambda} \right) \right\} + K + \text{higher order terms in } e .$$

Now suppose we define the following sequences of constants:

$$\beta_{ia}^{\lambda} \equiv \int_{\Omega} d^3k\, f_a^*(k) \left(\frac{p \cdot e^{(\lambda)}}{k \cdot p}\right) \frac{1}{[(2\pi)^3\, 2k_0]^{\frac{1}{2}}} \quad ,$$
$$\beta_{fa}^{\lambda} \equiv \int_{\Omega} d^3k\, f_a^*(k) \left(\frac{p' \cdot e^{(\lambda)}}{k \cdot p'}\right) \frac{1}{[(2\pi)^3\, 2k_0]^{\frac{1}{2}}} \qquad (28)$$

and we let

$$\alpha_a^{\lambda} = \beta_{ia}^{\lambda} \, , \; \gamma_a^{\lambda} = \beta_{fa}^{\lambda} \quad . \qquad (29)$$

Then

$$M = M_0 \left(1 + \tfrac{1}{2} \sum_{a,\lambda} |\beta_{ia}^{\lambda} - \beta_{fa}^{\lambda}|^2 + \alpha\, B\right) + K + \cdots$$
$$= M_0\, (1 + \alpha\, B + \alpha\, \tilde{B}) + K + \cdots \qquad (30)$$

The last step follows from the orthonormality of the f_a's .

Note that for this particular choice of photon states, the infrared divergence has been removed from the scattering amplitude to lowest order. It should however be emphasized that in order to make this cancellation work, it was not necessary to define the sequences of coefficients $\{\beta_{ia}^{\lambda}\}$ and $\{\beta_{ia}^{\lambda}\}$ exactly as in Equation (28). What mattered was that the differences $\{\beta_{fa}^{\lambda} - \beta_{ia}^{\lambda}\}$ be as defined in Equation (28) for infinitesimally small k , i.e.,

$$\beta_{fa}^{\lambda} - \beta_{ia}^{\lambda} = \int_{\Omega} d^3k\, f_a^*(k)\, I(k) \quad , \qquad (28')$$

where

$$\lim_{|k| \to 0} I(k) = \frac{1}{[(2\pi)^3\, 2k_0]^{\frac{1}{2}}} \left(\frac{p' \cdot e^{(\lambda)}}{k \cdot p'} - \frac{p \cdot e^{(\lambda)}}{k \cdot p} \right). \qquad (28'')$$

This is essentially equivalent to saying that if we had let

$$\alpha_a{}^{\lambda} = \beta_{ia}{}^{\lambda} + \epsilon_{ia}{}^{\lambda} , \ \gamma_a{}^{\lambda} = \beta_{fa}{}^{\lambda} + \epsilon_{fa}{}^{\lambda} \text{ where } \beta_{fa}{}^{\lambda} \text{ and } \beta_{ia}{}^{\lambda}$$

are defined by Equation (28), then convergence would be assured if

$$\sum_{\lambda, a} |\epsilon_{fa}{}^{\lambda} - \epsilon_{ia}{}^{\lambda}|^2 < \infty .$$

In fact, the generalization of the scheme to all orders of e_0 will show that this last convergence condition continues to be true.

IV. SEPARATION OF THE INFRARED FACTORS

The following exposition of the conventional separation of the infrared parts from the matrix element can be found in the review article by Yennie et al.[4] We will summarize here those parts of the conventional treatment which we shall use in our own scheme. For simplicity, we again study the example of an electron scattering from a potential, although similar results can be obtained for more general situations.

Consider a process in which there are a fixed number of photons and an electron of momentum p in the initial state, and a fixed number of photons with the scattered electron of momentum p' in the final state. The photons may or may not have interacted with the electron line. The complete matrix element for this process is given by

$$M(\underset{\sim}{p},\underset{\sim}{p}') = \sum_{n=0}^{\infty} M_n(\underset{\sim}{p},\underset{\sim}{p}') , \qquad (31)$$

where $M_n(\underset{\sim}{p},\underset{\sim}{p}')$ corresponds to the sum of all diagrams in which there are n virtual photons which can be distinguished from the potential interactions in the "basic process" M_0 . The real photon variables have been suppressed.

The quantity $\rho_n(k_1,\cdots k_n)$ is defined by the relation:

$$M_n = \frac{1}{n!} \int \cdots \int \prod_{i=1}^{n} \frac{d^4k}{k_i^2 - \Lambda^2 + i\epsilon} \, \rho_n(k_1,\cdots k_n) , \qquad (32)$$

where Λ is the photon mass which we allow to approach zero later. It has been shown that ρ_n is of the form[4]

$$\rho_n(k_1,\cdots k_n) = S(k_n)\,\rho_{n-1}(k_1,\cdots,k_{n-1}) + \xi^{(1)}(k_1,\cdots,k_{n-1}\,;\,k_n), \quad (33)$$

where $S(k_n)$ contains the k_n infrared divergence, and can have the form

$$S(k_n) = \frac{1}{2}\left[\frac{ie^2}{(2\pi)^4}\right]\left(\frac{2p'_\mu - k_\mu}{2p'\cdot k-k^2} - \frac{2p_\mu - k_\mu}{2p\cdot k-k^2}\right)^2 . \quad (34)$$

The remainder $\xi^{(1)}$ has no infrared divergence in k_n, and its infrared divergence in the other k's has not been made worse by the separation.

By iteration of Equation (33), $\rho_n(k_1,\cdots,k_n)$ can be expressed as a sum over all permutations of the k's:

$$\rho_n(k_1,\cdots,k_n) = \sum_{\text{perm}}\sum_{r=0}^{n}\frac{1}{r!(n-r)!}\prod_{i=1}^{r}S(k_i)\,\xi_{n-r}(k_{r+1},\cdots,k_n). \quad (35)$$

The functions ξ_r are noninfrared and symmetrical in the k's. If we adopt the definitions

$$\alpha B(\underset{\sim}{p},\underset{\sim}{p}') \equiv \int \frac{d^4k\,S(k)}{k^2-\Lambda^2}, \quad (36a)$$

$$m_r(\underset{\sim}{p},\underset{\sim}{p}') \equiv \frac{1}{r!}\int\prod_{i=1}^{r}\frac{d^4k_i}{k_i^2}\,\xi_r(k_1,\cdots,k_n), \quad (36b)$$

then substitution of (32), (35), (36a), and (36b) into (31) results in the simple expression:

$$M = \exp(\alpha B) \sum_{n=0}^{\infty} m_n . \qquad (37)$$

In this expression $m_o = \rho_o = \xi_o = M_o$. The m_n's in (37) are divergence-free, so that the whole infrared divergence has been isolated in the argument αB of the exponential. For future reference, we can write down the form of $\mathrm{Re}(\alpha B)$ which follows from (34) and (36):

$$\mathrm{Re}(\alpha B) = \frac{e^2}{4(2\pi)^3} \int \frac{d^3k}{(k^2 + \Lambda^2)^{\frac{1}{2}}} \left(\frac{2p'_\mu - k_\mu}{2p'\cdot k - \Lambda^2} - \frac{2p_\mu - k_\mu}{2p\cdot k - \Lambda^2} \right)^2 . \qquad (38)$$

The extraction of the infrared contribution to the matrix element for the emission of real photons has a form similar to that in Equation (33). In this case we let $\tilde{\rho}_n(k_1, \cdots, k_n)$ be the matrix element corresponding to the emission or absorption of n undetectable photons with momenta $k_1, \cdots k_n$, and for some arbitrary order in the virtual photon corrections. It has been shown that[4]

$$\tilde{\rho}_n(k_1, \cdots, k_n) = \pm \tilde{S}(k_n)\, \tilde{\rho}_{n-1}(k_1, \cdots k_n) + \tilde{\xi}^{(1)}(k_1, \cdots, k_{n-1} ; k_n), \qquad (39)$$

where $\tilde{S}(k_n)$ is the factor containing the infrared divergence, and has the form

$$\tilde{S}^{\lambda}(k) = \frac{e}{\left[2(2\pi)^3 k_o\right]^{\frac{1}{2}}} \left[\frac{p'\cdot e^{(\lambda)}}{k\cdot p'} - \frac{p\cdot e^{(\lambda)}}{p\cdot k}\right], \tag{40}$$

and the (+) and (-) signs correspond to emission and absorption respectively. Again the remainder $\tilde{\xi}^{(1)}$ is divergence-free in k_n, and the divergences in the other k's is no worse for the separation.

It can be shown that the iteration of (39) leads to the form

$$\tilde{\rho}_n(k_1,\cdots,k_n) = \sum_{\text{perm}} \sum_{r=0}^{n} (-1)^m \frac{1}{r!(n-r)!} \prod_{i=1}^{r} \tilde{S}(k_i)\, \tilde{\xi}_{n-r}(k_{r+1},\cdots,k_n), \tag{41}$$

where the functions $\tilde{\xi}$ are noninfrared and symmetrical in the k's, and m corresponds to the number of absorbed photons.

V. CANCELLATION OF INFRARED DIVERGENCES TO ALL ORDERS

In this section we return to our own scheme to eliminate the infrared divergences. Some combinatorial algebra will be needed to calculate the matrix element for the transition from a state of electron momentum p and photon "quantum numbers" $\{\alpha_a^\lambda\}$ to a state of electron momentum p' and photon "quantum numbers" $\{\gamma_c^\lambda\}$, where again

$$|\{\alpha_a^\lambda\}\rangle = \prod_a \frac{\exp \sum_\lambda \left[\alpha_a^\lambda \int_\Omega d^3k \, f_a(k) \, e_\mu^{(\lambda)}(k) \, a^{(\lambda)+}(k)\right]}{\exp\left[\sum_\lambda \frac{1}{2} |\alpha_a^\lambda|^2\right]} |0\rangle \, . \qquad (42)$$

Consider all the diagrams represented by Figure V.1, in which there are m real photons absorbed by the electron line, m' real photons emitted by the electron line, and ℓ photons which do not interact with the electron at all.

The matrix element for the process $|\{\alpha_a^{\lambda}\}, p_i\rangle \rightarrow |\{\gamma_c^{\lambda'}\}, p_f\rangle$ is then a sum over all diagrams of the kind shown in Figure V.1 for all values of m, m', and ℓ, and with the proper factors determined by Equations (42).

Considerations which enter the calculation of this matrix element are explained below:

(a) There is an infrared divergent factor $e^{\alpha B}$ due to the virtual photon corrections. (See Equation (37)).

(b) The overlap of the ℓ initial-state noninteracting photons with the ℓ final-state noninteracting photons contributes a factor

$$\ell!\left[\sum_{\substack{\mu,a,c,\\ \lambda,\lambda'}} \alpha_a^{\lambda}\gamma_c^{*\lambda'}\int d^3k\, f_c^*(k)f_a(k)e_\mu^{(\lambda)}(k)e_\mu^{(\lambda')}(k)\right]^{\ell} = \ell!\left[\sum_{\substack{\lambda,\lambda',\\ a,c}} \alpha_a^{\lambda}\gamma_c^{*\lambda'}\delta_{\lambda\lambda'}\delta_{ac}\right]^{\ell} . \qquad (43)$$

(c) Equation (41) gives the contribution due to the interaction of m initial-state photons and m' final-state photons with the electron line:

$$\tilde{\rho}^{\{\lambda\}}_{m+m'}(k_1,\cdots,k_{m+m'}) = \sum_{\substack{\text{perm}\\ k's}} \sum_{t=0}^{m+m'} (-1)^m \left(\prod_{i=1}^{t} \tilde{S}^{(\lambda)}(k_i)\right)$$

$$\times\ \xi^{\{\lambda\}}_{m+m'-t}(k_{t+1},\cdots,k_{m+m'})\ \frac{1}{t!(m+m'-t)!} . \qquad (44)$$

(d) Contribution (c) must be integrated over the momentum distribution that is obtained from the formal expansion of the initial and final states (See Equation 42):

$$\left[\left(\prod_{r=1}^{m} \sum_{\lambda,a} \alpha_a^{\lambda} \int d^3k_r f_a(k_r)\right)\left(\prod_{r'=m+1}^{m+m'} \sum_{\lambda',c} \gamma^{*\lambda'}_c \int d^3k_{r'} f^*_c(k_{r'})\right) \tilde{\rho}_{m+m'}^{\{\lambda,\lambda'\}} (k_1, \cdots, k_{m+m'})\right]. \tag{45}$$

(e) The formal expansion of Equation (42) also leads to the factors

$$\frac{1}{(m+\ell)!} \; \frac{1}{(m'+\ell)!} \; \exp\left(-\tfrac{1}{2} \sum_{a,\lambda} |\alpha_a^{\lambda}|^2\right) \exp\left(-\tfrac{1}{2} \sum_{c,\lambda'} |\gamma_c^{\lambda'}|^2\right). \tag{46}$$

(f) In addition to the above, there is a combinatorial factor which accounts for the number of ways that $(m+\ell)$ initial-state photons and $(m'+\ell)$ final-state photons can be distributed among m initial-state interacting photons, m' final-state interacting photons, and ℓ noninteracting photons:

$$\frac{(m+\ell)!}{m! \; \ell!} \; \frac{(m'+\ell)!}{m'! \; \ell!} \; .$$

After summing over all numbers m, m', and ℓ, we arrive at the following expression for the matrix element $\tilde{M}$:

$$\tilde{M} = e^{\alpha B} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{m'=0}^{\infty} \left[\frac{1}{(m+\ell)!} \quad \frac{1}{(m'+\ell)!}\right]\left[\frac{(m+\ell)!\,(m'+\ell)!}{m!\,m'!\,\ell!\,\ell!}\right]$$

$$\times\left[\exp\left(-\tfrac{1}{2}\sum_{\lambda,a} |\alpha_a{}^{\lambda}|^2\right)\exp\left(-\tfrac{1}{2}\sum_{\lambda,a}|\gamma_a^{\lambda}|^2\right)\right]\left[\ell!\right]\left[\sum_{\lambda,a} \alpha_a{}^{\lambda}\, \gamma^*{}_a{}^{\lambda}\right]^{\ell}$$

$$\times\left[\left(\prod_{r=1}^{m} \sum_{\lambda,a} \alpha_a{}^{\lambda}\int d^3k_r f_a(k_r)\right)\left(\prod_{r'=m+1}^{m+m'} \sum_{\lambda',c} \gamma^*{}_c{}^{\lambda'}\int d^3k_{r'} f_c^*(k_{r'})\right)\right.$$

$$\left.\tilde{\rho}_{m+m'}^{\{\lambda,\lambda'\}}(k_1,\cdots,k_{m+m'})\right] \quad . \tag{47}$$

Another factor corresponding to the contribution from the scattering of photons by photons could have been included explicitly, but since this term does not contribute to the cancellation of infrared divergences, nor does add to the divergences, it has not been considered in this analysis.

Making the appropriate cancellations, and combining the terms with a little bit of careful counting, we arrive at the expression

$$M = e^{\alpha B} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{m'=0}^{\infty} \frac{1}{m'!}\left[\sum_{a,\lambda} \alpha_a^{\lambda}\, \gamma_a^{*\lambda}\right]^{\ell}\left\{\sum_{j=0}^{m} \sum_{j'=0}^{m'} \frac{m!}{j!(m-j)!} \quad \frac{m'!}{j'!(m'-j')!}\right.$$

$$\left.\times\left[-\sum_{a,\lambda} \alpha_a{}^{\lambda}(\tilde{S}^{\lambda}, f_a)_{\Omega}\right]^j \left[\sum_{c,\lambda'} \gamma_c^{*\lambda'}(f_c^*, \tilde{S}^{\lambda'})_{\Omega}\right]^{j'} P_{m-j,m'-j'}(p_i, p_f)\right\}$$

$$\times\left[\exp\left(-\tfrac{1}{2}\sum_{\lambda,a} |\alpha_a{}^{\lambda}|^2\right) \exp\left(-\tfrac{1}{2}\sum_{\lambda,a} |\gamma_a{}^{\lambda}|^2\right)\right] , \tag{48}$$

where

$$(\tilde{S}^{\lambda}, f_a)_{\Omega} \equiv \int_{\Omega} d^3k \, \tilde{S}^{(\lambda)}(k) \, f_a(k) \, ,$$

$$(f_c^*, \tilde{S}^{\lambda'})_{\Omega} \equiv \int_{\Omega} d^3k \, \tilde{S}^{(\lambda')}(k) \, f_c^*(k) \, ,$$

and

$$P_{j,j'} \equiv (-1)^j \left(\prod_{r=1}^{j} \int d^3k \sum_{\lambda,a} \alpha_a{}^{\lambda} f_a(k_r) \right) \tag{49}$$

$$\times \left(\prod_{r'=1}^{j'} \int d^3k_{r'} \sum_{\lambda',c} \gamma_c^{*\lambda'} f_c^*(k_{r'}) \right) \tilde{\xi}^{\{\lambda\}}_{j+j'} (k_1, \cdots k_{j+j'}) \, .$$

Defining the "residuals" $m_{j,j'}$ by

$$m_{j,j'}(p_i, p_f) = \frac{P_{j,j'}}{j!j'!} \, , \tag{50}$$

and reordering the sums in (52), we can write

$$\tilde{M} = e^{\alpha B} \exp\left[-\tfrac{1}{2} \sum_{\lambda,a} |\alpha_a{}^{\lambda}|^2 \right] \exp\left[-\tfrac{1}{2} \sum_{\lambda,a} |\gamma_a{}^{\lambda}|^2 \right] \exp\left[\sum_{\lambda,a} \alpha_a{}^{\lambda} \gamma_a^{*\lambda} \right]$$

$$\times \exp\left[-\sum_{\lambda,a} \alpha_a{}^{\lambda} (\tilde{S}^{\lambda}, f_a)_{\Omega} \right] \exp\left[\sum_{\lambda,c} \gamma_c^{*\lambda} (f_c^*, \tilde{S}^{\lambda})_{\Omega} \right] \left\{ \sum_{m,m'=0}^{\infty} m_{m,m'} \right\} \tag{51}$$

To simplify the notation further, we define the coefficients $\beta_a{}^{\lambda}$ by

$$\beta_a{}^{\lambda} \equiv (f_a^*, \tilde{S}^{\lambda})_{\Omega} \, . \tag{52}$$

Since the function $\tilde{S}$ is real, Equation (55) becomes

$$\tilde{M} = e^{\alpha B} \exp\left[-\tfrac{1}{2} \sum_{\lambda, a} |\alpha_a^\lambda|^2\right] \exp\left[-\tfrac{1}{2} \sum_{\lambda, a} |\gamma_a^\lambda|^2\right] \exp\left[\sum_{\lambda, a} \alpha_a^\lambda \gamma_a^{*\lambda}\right] \exp\left[- \sum_{\lambda, a} \alpha_a^\lambda \beta_a^{*\lambda}\right]$$

$$\times \exp\left[\sum_{\lambda, a} \beta_a^\lambda \gamma_a^{*\lambda}\right] \left\{ \sum_{m, m'=0}^{\infty} m_{m, m'} (p_i, p_f) \right\} . \tag{53}$$

By Equations (28) and (52),

$$\beta_a^\lambda = \beta_{fa}^\lambda - \beta_{ia}^\lambda . \tag{54}$$

It is convenient to transform from the variables α_a^λ , γ_a^λ to the variables ϵ_{ia}^λ , ϵ_{fa}^λ defined by

$$\gamma_a^\lambda = \beta_{fa}^\lambda + \epsilon_{fa}^\lambda , \quad \alpha_a^\lambda = \beta_{ia}^\lambda + \epsilon_{ia}^\lambda . \tag{55}$$

Thus

$$\tilde{M} = e^{\alpha B} \left\{ \exp \sum_{a, \lambda} \left[-\tfrac{1}{2} |\beta_{ia}^\lambda + \epsilon_{ia}^\lambda|^2 - \tfrac{1}{2} |\beta_{fa}^\lambda + \epsilon_{fa}^\lambda|^2 + (\beta_{ia}^\lambda + \epsilon_{ia}^\lambda)(\beta_{fa}^\lambda + \epsilon_{fa}^\lambda)^* \right.\right.$$

$$\left.\left. - (\beta_{ia}^\lambda + \epsilon_{ia}^\lambda) \beta_a^{*\lambda} + \beta_a^\lambda (\beta_{fa}^\lambda + \epsilon_{fa}^\lambda)^* \right] \right\} \left\{ \sum_{m, m'=0}^{\infty} m_{m, m'} \right\}$$

$$= e^{\alpha B} \left\{ \exp \sum_{a, \lambda} \left[+ \tfrac{1}{2} |\beta_f - \beta_i|^2 - \tfrac{1}{2} |\epsilon_f - \epsilon_i|^2 + i \operatorname{Im}(\beta_i^* \epsilon_i + \beta_f \epsilon_f - \beta_i \beta_f^* + \epsilon_i \epsilon_f^*) \right] \right\} \tag{56}$$

$$\times \left\{ \sum_{m, m'=0}^{\infty} m_{m, m'} \right\} ,$$

where the mode and polarization indices have been suppressed for convenience. Using the definitions of β_{ia}^{λ} and β_{fa}^{λ}, and orthonormality as we did in Section III, we arrive at the important result

$$\tilde{M} = e^{(\alpha B + \alpha\tilde{B})} \exp\left\{-\frac{1}{2}\sum_{\lambda,a} |\epsilon_{fa}^{\lambda} - \epsilon_{ia}^{\lambda}|^2\right\} e^{i\phi} \left\{ \sum_{m,m'=0}^{\infty} m_{m,m'} \right\} \qquad (57)$$

where ϕ is real.

The argument of the first exponential was shown in Section IV to be infrared divergenceless in the limit of zero photon mass. The third exponential has modulus unity, and the last sum is term by term divergence-free. If the possible states of the system are restricted by the condition

$$\sum_{\lambda,a} |\epsilon_{fa}^{\lambda} - \epsilon_{ia}^{\lambda}|^2 < \infty \, , \qquad (58)$$

the second exponential is nonzero, but less than or equal to unity. With this condition satisfied the infrared divergences have been eliminated. The interpretation of this restriction is discussed in the next section.

VI. INTERPRETATION OF THE PHOTON STATES

In the beginning of this section, we will show that Equation (58) defines a separable Hilbert space. To do this, we study a related space $\mathfrak{H}$ which will turn out to be identical to the ordinary Fock space $\mathfrak{H}_{\infty}$. Translations like Equation (55) will not change the intrinsic properties of this space. Finally a calculation of the total cross section will relate this whole discussion to experiment.

Much of the mathematical material here will be treated heuristically, but a more rigorous formulation of some of the statements can be found in the papers by V. Bargmann.[2,5]

We will define a separable Hilbert space $\mathfrak{H}$ in the following manner: Let $\{\theta_i\}$ be an infinite sequence of complex numbers. A set of "principal vectors" $|\{\theta_i\}\rangle$ is then defined by the equation

$$
\begin{aligned}
|\{\theta_i\}\rangle &= \prod_i |\theta_i\rangle = \prod_i \exp\left[-\tfrac{1}{2}|\theta_i|^2\right] \exp\left[\theta_i a_i^{\dagger}\right] |0\rangle \\
&= \prod_i \exp\left(-\tfrac{1}{2}|\theta_i|^2\right) \exp\left[\theta_i \int f_i(k)\, a^{\dagger}(k)\right] |0\rangle
\end{aligned}
\tag{59}
$$

and the condition

$$
\sum_i |\theta_i|^2 < \infty . \tag{60}
$$

The elements of $\mathcal{H}$ are taken to be the closure of all finite linear combinations of the principal vectors.

From (59) and the commutation rules for $a^+(k)$, the inner product of two elements, $|f\rangle = \sum_{j=1}^{p} \lambda_j \, |\{\theta_i^{(j)}\}\rangle$ and $|f'\rangle = \sum_{k=1}^{q} \mu_k \, |\{\theta_i^{(k)}\}\rangle$, is given by

$$\langle f|f'\rangle = \sum_{j,k} \lambda_j^* \mu_k \left\{ \exp\left[\sum_i \theta_i^{*(j)} \theta_i^{(k)}\right] \exp\left[-\tfrac{1}{2}\sum_i |\theta_i^{(j)}|^2\right] \exp\left[-\tfrac{1}{2}\sum_i |\theta_i^{(k)}|^2\right]\right\}. \tag{60}$$

In particular, the inner product of two principal vectors $|\{\theta_i^{(j)}\}\rangle$ and $|\{\theta_i^{(k)}\}\rangle$ has the property

$$|\langle\{\theta_i^{(j)}\}|\,\{\theta_i^{(k)}\}\rangle|^2 = \left|\exp\left[\sum_i \theta_i^{(j)}\theta_i^{(k)}\right] \exp\left[-\tfrac{1}{2}\sum_i |\theta_i^{(j)}|^2\right] \exp\left[-\tfrac{1}{2}\sum_i |\theta_i^{(k)}|^2\right]\right|^2$$

$$= \exp\left\{-\sum_i |\theta_i^{(j)} - \theta_i^{(k)}|^2\right\}, \tag{61}$$

so that the principal vectors are all normalized to unit length. Moreover, by Equation (60), no two principal vectors are normal to each other.

The properties of the space could in fact have been derived by using Equation (61) instead of Equation (59), but we wish to retain the connection with the previous sections of this paper.

The separability of $\mathcal{H}$ follows from the existence of a countable sequence of vectors which is dense in $\mathcal{H}$. Let $|\{\theta_i\}\rangle$ be any principal vector. From Equation (59), it is known that for any $\delta_N > 0$ there exists an integer N such that $\sum_{i>N} |\theta_i|^2 < \delta_N$.

Moreover, for any $i < N$ and $\delta > 0$ it is always possible to find rational numbers $\{R_i\}$ such that $|\theta_i - R_i|^2 < \delta$.

Consider a principal vector $|\{\theta_i'\}\rangle$ such that $\theta_i' = 0$ for $i > N$, and $\theta_i' = R_i$ for $i < N$. Let $\sum_i |\theta_i|^2 = A^2$. Then

$$\sum_i |\theta_i' - \theta_i|^2 \leqslant N\delta + \delta_N$$

and

$$\begin{aligned}\left|\operatorname{Im} \sum_i \theta_i^* \theta_i'\right| &= \left|\operatorname{Im} \sum_i \theta_i^* \theta_i + \operatorname{Im} \sum_i \theta_i^* (\theta_i' - \theta_i)\right| \\ &\leqslant \left|\sum_i \theta_i^* (\theta_i' - \theta_i)\right| \\ &\leqslant \sqrt{\sum_i |\theta_i|^2 \sum_i |\theta_i' - \theta_i|^2} \\ &\leqslant A\sqrt{N\delta + \delta_N} \quad .\end{aligned}$$

Thus,

$$\begin{aligned}\Big|\, |\{\theta_i\}\rangle - |\{\theta_i'\}\rangle \Big|^2 &= 2 - \langle\{\theta_i\}|\{\theta_i'\}\rangle - \langle\{\theta_i'\}\,|\{\theta_i\}\rangle \\ &= 2 - \exp\left[-\tfrac{1}{2}\sum_i |\theta_i|^2\right]\exp\left[-\tfrac{1}{2}\sum_i |\theta_i'|^2\right] \\ &\quad \times \left\{\exp\left[\sum_i \theta_i^* \theta_i'\right] + \exp\left[\sum_i \theta_i^{*\prime}\, \theta_i\right]\right\} \\ &= 2 - \exp\left[-\tfrac{1}{2}\sum_i |\theta_i' - \theta_i|^2\right]\left\{\exp\left[i \operatorname{Im} \sum_i \theta_i^* \theta_i'\right]\right. \\ &\qquad \left. + \exp\left[i \operatorname{Im} \sum_i \theta_i^{*\prime}\, \theta_i\right]\right\} \\ &= 2\left\{1 - \left(\exp\left[-\tfrac{1}{2}\sum_i |\theta_i' - \theta_i|^2\right]\right)\left(\cos\left[\operatorname{Im} \sum_i \theta_i^* \theta_i'\right]\right)\right\} \\ &\leqslant (A^2 + 1)(N\delta + \delta_N) \qquad (62)\end{aligned}$$

Since $|\{\theta_i\}\rangle$, δ_N, and δ were arbitrary, we have shown that any principal vector can be approximated by another principal vector belonging to a denumerable set. The denumerable set which consists of all finite sums of principal vectors like $|\{\theta_i'\}\rangle$ is dense in $\mathcal{H}$.

In the case of massless soft photons, there is no reason to restrict the photon states by Equation (59). Let $\{\theta_i^{(o)}\}$ be a sequence of complex numbers which are not square-summable, i.e.,

$$\sum_i |\theta_i^{(o)}|^2 \not< \infty . \tag{63}$$

Then the states defined by the complex numbers $\{\theta_i\}$, and which satisfy the condition

$$\sum_i |\theta_i - \theta_i^{(o)}|^2 < \infty , \tag{64}$$

form a separable Hilbert space $\mathcal{H}^{(o)}$ with all the properties of $\mathcal{H}$, except Equation (60). $\mathcal{H}^{(o)}$ is unitarily inequivalent to $\mathcal{H}$, i.e., it forms a unitarily inequivalent representation of the canonical commutation rules.

In Section II we discussed the connection between the Fock states and the principal vectors for a single mode. We will now briefly study the relationship between the occupation number parameterization and the principal vector parameterization.

The states in the Fock space $\mathfrak{F}_\infty$ are specified by a set M of infinite sequences of non negative integers $\{m_i\}$, or "occupation numbers" of which a finite number are different from zero, and the limit sequences of such vectors. An orthonormal basis of Fock space is given by

$$|u_{\{m\}}\rangle = \prod_i \frac{\left(a_i^\dagger\right)^{m_i}}{\sqrt{m_i\,!}} \quad |0\rangle \quad . \tag{65}$$

An arbitrary state $|f\rangle$ of $\mathfrak{F}_\infty$ is given by

$$|f\rangle = \sum_{\{m\}\epsilon M} \gamma_{\{m\}} \, |u_{\{m\}}\rangle \quad , \tag{66}$$

where the complex coefficients $\gamma_{\{m\}}$ satisfy

$$\sum_{\{m\}\epsilon M} \gamma_{\{m\}}^{\;2} < \infty \quad . \tag{67}$$

At this point, it should be apparent that $\mathfrak{F} \subset \mathfrak{F}_\infty$, since

$$|\{\theta_i\}\rangle = \sum_{\{m\}\epsilon M} \prod_i \frac{(\theta_i a_i^\dagger)^{m_i}}{m_i\,!} \, |0\rangle = \sum_{\{m\}\epsilon M} \prod_i \frac{(\theta_i)^{m_i}}{\sqrt{m_i\,!}} \, |u_{\{m\}}\rangle \tag{68}$$

and

$$\sum_{\{m\}\epsilon M} \left| \prod_i \frac{(\theta_i)^{m_i}}{\sqrt{m_i\,!}} \right|^2 = \prod_i \left(\sum_{m=0}^{\infty} \frac{|\theta_i|^{2m_i}}{m_i\,!} \right) = \left(\exp \sum_i |\theta_i|^2\right) < \infty \, . \tag{69}$$

The scalar product of two vectors $|f\rangle$ and $|f'\rangle$ in $\mathcal{H}_\infty$ can be obtained from (65) and (66):

$$\langle f|f'\rangle = \sum_{\{m\}\in M} \gamma^*_{\{m\}} \gamma'_{\{m\}} . \tag{70}$$

Let $Q_n\{m_i\}$ be the truncated sequence

$$Q_n\{m_i\} = (m_1, m_2, \cdots, m_n, 0, 0, \cdots), \tag{71}$$

and define a projection on $\mathcal{H}_\infty$ by

$$E_n |f\rangle = E_n \sum_{\{m\}\in M} \gamma_{\{m\}} |u_{\{m\}}\rangle = \sum_{\{m\}\in M_n} \gamma_{\{m\}} |u_{\{m\}}\rangle, \tag{72}$$

where M_n is the set of all sequences of the form given by Equation (71). Then it follows from Equation (67) that $E_n|f\rangle$ converges strongly to $|f\rangle$ as $n \to \infty$. We will show that $E_n|f\rangle$ is contained in $\mathcal{H}$, which implies that $\mathcal{H}_\infty \equiv \mathcal{H}$.

The expansion of $E_n|f\rangle$ in terms of the principal vectors follows directly from Equation (15):

$$E_n|f\rangle = \sum_{\{m\}\in M_n} \gamma_{\{m\}} \prod_{i=1}^{n} \int d\mu_i \; |(\theta_1, \theta_2, \cdots, \theta_n, 0, 0, \cdots)\rangle \frac{(\theta_i^*)^{m_i}}{\sqrt{m_i!}} , \tag{73}$$

where

$$d\mu_i = \exp\left[-\tfrac{1}{2} |\theta_i|^2\right] d(\mathrm{Re}\,\theta_i)\, d(\mathrm{Im}\,\theta_i) . \tag{74}$$

From Equation (62), it is clear that principal vectors $|\{\theta_i\}\rangle$ which do not satisfy Equation (60) are orthogonal to $E_n|f\rangle$. Therefore $E_n|f\rangle \in \mathfrak{F}$ and the result $\mathfrak{F}_\infty = \mathfrak{F}$ follows, i.e., the Fock space built from states with a finite number of photons, and the space of principal vectors satisfying Equation (60), are the same space.

In order to satisfy the requirement Equation (58) for finite matrix elements, it will be necessary not to restrict the scattering states to $\mathfrak{F}$. For if the initial state were in $\mathfrak{F}$, i.e., the $\{\alpha_a^{\lambda}\}$ of Equation (42) satisfied the condition

$$\sum_{\lambda,a} |\alpha_a^{\lambda}|^2 < \infty, \tag{75}$$

then the final state parameterized by the sequence of complex numbers $\{\gamma_a^{\lambda}\}$ would be given by

$$\gamma_a^{\lambda} = \alpha_a^{\lambda} + (\beta_{fa}^{\lambda} - \beta_{ia}^{\lambda}) + (\epsilon_{fa}^{\lambda} - \epsilon_{ia}^{\lambda}), \tag{76}$$

where we have used Equation (55), and the ϵ's would satisfy

$$\sum_{\lambda,a} |\epsilon_{fa}^{\lambda} - \epsilon_{ia}^{\lambda}|^2 < \infty . \tag{77}$$

But we know from Section III that

$$\sum_{\lambda,a} |\beta_{fa}^{\lambda} - \beta_{ia}^{\lambda}|^2 = \frac{1}{2} \sum_{\ell=1}^{2} \int |\tilde{S}^{(\ell)}(k)|^2 \, d^3k = \alpha \tilde{B}, \tag{78}$$

and $\tilde{B} \to \infty$ as the photon mass approaches zero. Therefore

$$\sum_{\lambda, a} |\gamma_a{}^\lambda|^2 \to \infty \qquad (79)$$

as the photon mass approaches zero. Thus the final state cannot belong to $\mathfrak{F}$.

Nevertheless, the coefficients $\{\gamma_a{}^\lambda\}$ define a final state. It must be that the final state belongs to a representation of the canonical commutation rules inequivalent to the Fock space $\mathfrak{F}$ whose most outstanding feature is that the average number of photons is finite. Not any final state will do, however, for the boundaries of this new space $\mathfrak{F}'$ are restricted by the condition Equation (58).

One of many ways to satisfy Equation (58) which preserves symmetry between the initial and final states is to write these state as

$$\begin{aligned} \alpha_a{}^\lambda &= \epsilon_{ia}{}^\lambda + \beta_{ia}{}^\lambda + \epsilon_{oa}{}^\lambda \,, \\ \gamma_a{}^\lambda &= \epsilon_{fa}{}^\lambda + \beta_{fa}{}^\lambda + \epsilon_{oa}{}^\lambda \,, \end{aligned} \qquad (80)$$

and to restrict the states by

$$\sum_{\lambda, a} |\epsilon_{ia}{}^\lambda|^2 < \infty \,, \quad \sum_{\lambda, a} |\epsilon_{fa}{}^\lambda|^2 < \infty \,. \qquad (81)$$

Then we would get different theories by different choices of the sequence $\{\epsilon_{oa}^{\lambda}\}$. With such a choice, the photon states would have a dependence upon the momenta of the participating electrons.

In a practical calculation where one wants to treat, for example, the scattering of an electron with the emission of hard photons, the hard photons can be dealt with by the conventional occupation-number parameterization, while the soft photons are described in terms of the translated principal vectors. More specifically, consider the calculation of the cross section for an electron of momentum p_i scattering into a state with an electron of momentum p_f plus several hard photons. The incoming electron is associated with a photon field described by a sequence $\{\alpha_a^{\lambda}\}$ and the outgoing electron has a photon field $\{\gamma_a^{\lambda}\}$. In Equation (57) the "basic matrix element" $m_{o,o}$ corresponds to diagrams with only the detectable real photons and those virtual photons necessary for the process to occur. The terms m_{ij} for $i,j = 0$ contain the effects of the noninfrared parts of the real and virtual soft photons to higher order in the coupling constant.

To lowest order in the noninfrared photons, the squared matrix element for a particular diagram m_{oo} is from Equation (57):

$$|\tilde{M}|^2 = e^{2(\mathrm{Re}\,\alpha B + \alpha\tilde{B}_{\Omega})} \exp\left[-\sum_{\lambda,a} |\epsilon_{fa}^{\lambda} - \epsilon_{ia}^{\lambda}|^2\right] |m_{o,o}|^2 . \qquad (82)$$

We can then sum over final states. The result (to lowest order) is independent of the initial state:

$$\sum_{\substack{\text{final}\\ \text{states}}} |\tilde{M}|^2 = e^{2(\mathrm{Re}\ \alpha B + \alpha\tilde{B}_\Omega)} |m_{o,o}|^2 \lim_{n\to\infty} (\pi^{-1})^n \left[\int d^2\epsilon_f \quad e^{-|\epsilon_f|^2} \right]^n$$

$$= e^{2(\mathrm{Re}\ \alpha B + \alpha\tilde{B}_\Omega)} |m_{o,o}|^2 \quad . \tag{83}$$

The remaining exponential contains part of the effect of the choice of $\tilde{S}_i(k)$ and the "region of resolution" Ω, and we obtain a similar result to what Yennie et al. obtained (for a non-energy-conserving potential). In fact, the "reason" why the results are the same is that in the summation over all final states in the conventional treatment of the infrared divergence, the main contributions came from states which were not in the usual Fock space, but were in a reducible representation of the canonical commutation rules without any restrictions on the sequence $\{m\}$ of occupation numbers. In particular, the separable space of final states $\mathfrak{F}'$ is contained in this larger, nonseparable space.

In the above computation, and in Section V, the resolution regions for the initial and final states were assumed to be the same. One can argue that the resolution region of the initial state can be made arbitrarily small, but finite, by waiting a sufficiently long time before the scattering experiment. The situation where the initial resolution region is smaller than the final state resolution region is discussed in the Appendix.

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APPENDIX A

GENERALIZATION OF THE CANCELLATION TO SEVERAL ELECTRON LINES

For simplicity, only the case of a single electron line interacting with a potential was treated in Section V. The generalization to several interacting electron lines will be outlined here for completeness. The extension to other processes such as positron scattering, pair production, etc., is obvious and will not be done here.

Since the cancellation of the infrared divergence in this more complicated situation requires a proliferation of subscripts and superscripts, all notation pertaining to the photon polarization will be suppressed until the final steps of the proof.

The initial state will consist of N incoming electrons with momenta $p_1, p_2, \cdots, p_\nu, \cdots, p_N$ and a photon state $\{\alpha_a\}$. These will scatter into final state of N outgoing electrons with momenta $p'_1, p'_2, \cdots, p'_\nu, \cdots, p'_N$ and a photon state $\{\gamma_a\}$. The "resolution regions" Ω of the initial and final state are assumed to be identical. The interaction is illustrated in Fig. A-1.

The matrix element is given by the following considerations:

(a) Due to virtual photon corrections there is an infrared divergent factor $F = \exp(\alpha B)$, where the generalized definition of B is[4]

$$B = \frac{-i}{8\pi^3} \int \frac{d^4k}{k^2 - \Lambda^2} \sum_{i<j}^{2N} \theta_i \theta_j \left(\frac{2p_i\theta_i - k}{k^2 - 2k \cdot p_i\theta_i} + \frac{2p_j\theta_j + k}{k^2 + 2k \cdot p_j\theta_j} \right)^2$$

and $\theta_i = +(-)$ if electron line i is outgoing (incoming). The sum is taken over all $2N$ outgoing and incoming electron lines.

(b) The overlap of ℓ initial state noninteracting photons with ℓ final state noninteracting photons contributes a factor

$$\ell!\ \left(\sum_a \alpha_a \gamma^*_a\right)^\ell$$

(c) The interaction of m initial state photons and m' final state photons with the N electron lines gives

$$\tilde{\rho}(\{m^\nu\}\{m'^\nu\}) = \prod_{\nu=1}^{N} \sum_{t=0}^{m^\nu + m'^\nu} (-1)^{m^\nu} \sum_{\substack{\text{perm}\\ \text{k's}}} \left(\prod_{i=1}^{t} \tilde{S}(k_i^\nu) \right)$$

$$\times\, \tilde{\xi}_{m^\nu + m'^\nu - t} \left(k^\nu_{t+1}, \cdots, k^\nu_{m^\nu + m'^\nu}\right) \frac{1}{t!\ (m^\nu + m'^\nu - t)!} \quad .$$

The superscript ν indicates which of the N electron lines participates in the interaction. Thus

$$m = \sum_{\nu=1}^{N} m^\nu \ , \quad m' = \sum_{\nu=1}^{N} m'^\nu \quad .$$

Note that the above expression is exactly the same as the expression (c) in Section V, except that a product is taken over the contributions from N electron lines.

(d) The photon momenta k in (c) must be integrated over the momentum distribution given by the set of functions f_a

$$\left\{ \prod_{\nu=1}^{N} \left[\left(\prod_{r=1}^{m^\nu} \sum_a \alpha_a \int d^3k_r^\nu \, f_a(k_r^\nu) \right) \left(\prod_{r'=m^\nu+1}^{m^\nu+m'^\nu} \sum_{a'} \gamma^*_{a'} \int d^3k^\nu_{r'} \, f^*_{a'}(k^\nu_{r'}) \right) \right] \times \tilde{\rho}(\{m^\nu\}\{m'^\nu\} \; ; \; \{k^\nu\}) \right\}$$

(e) Formal expansion of Equation (42) again leads to the factors

$$\frac{1}{(m + \ell)!} \quad \frac{1}{(m + \ell)!} \quad \exp[-\tfrac{1}{2} \sum_a |\alpha_a|^2] \; \exp[-\tfrac{1}{2} \sum_{a'} |\gamma_{a'}|^2]$$

(f) In addition to the above, there is a combinatorial factor which accounts for the number of ways that the $(m + \ell)$ initial state photons and the $(m + \ell)$ final state photons can be distributed among the N electron lines and the noninteracting photons.

$$\frac{(m + \ell)!}{(\prod_\nu m^\nu!)\ell!} \quad \frac{(m' + \ell)!}{(\prod_\nu m'^\nu!)\ell!} \quad .$$

To obtain the matrix element M, the contributions are summed over all values of m^ν, m'^ν and ℓ.

$$M = F \sum_{\ell=0}^{\infty} \prod_{\nu=1}^{N} \sum_{m^\nu=0}^{\infty} \sum_{m'^\nu=0}^{\infty} \left[\frac{1}{(m + \ell)!} \quad \frac{1}{(m' + \ell)!} \right]$$

$$\times \left[\frac{(m + \ell)!}{(\prod_\nu m^\nu !)\ell!} \quad \frac{(m' + \ell)!}{(\prod_\nu m'^\nu !)\ell!} \right]$$

$$\times \left[\exp\left[-\tfrac{1}{2} \sum_a |\alpha_a|^2\right] \exp\left[-\tfrac{1}{2} \sum_a |\gamma_a|^2\right] \left[\ell!(\sum_a \alpha_a \gamma_a^*)^\ell\right] \right.$$

$$\times \left\{ \prod_{\nu=1}^{N} \left[\left(\prod_{r=1}^{m^\nu} \sum_a \alpha_a \int d^3k_r^\nu \, f_a(k_r^\nu) \right) \left(\prod_{r'=m^\nu+1}^{m^\nu+m'^\nu} \sum_{a'} \gamma_{a'}^* \int d^3k_{r'}^* \, f_{a'}^*(k_{r'}^\nu) \right) \right] \right.$$

$$\left. \times \, \tilde{\rho}(\{m^\nu\} \{m'^\nu\} ; \{k^\nu\}) \right\} .$$

The last factor in the braces can be reduced by separating out the divergent terms:

$$\left\{ \prod_{\nu=1}^{N} \left[\sum_{j^\nu=0}^{m^\nu} \sum_{j'^\nu=0}^{m'^\nu} \frac{m^\nu !}{j^\nu!(m^\nu - j^\nu)!} \frac{m'^\nu !}{j'^\nu!(m'^\nu - j'^\nu)!} \left(-\sum_a \alpha_a(\tilde{S}, f_a)\right)^{j^\nu} \right. \right.$$

$$\left. \left. \left(\sum_a \gamma_a^*(f_a^*, \tilde{S})^{j'^\nu} \right) P(m^\nu - j^\nu, m'^\nu - j'^\nu ; p_\nu, p'_\nu) \right] \right\}$$

where

$$P(j^\nu, j'^\nu ; p^\nu, p'^\nu)$$

$$= \left[(-1)^{j_\nu} \left(\prod_{r=1}^{j^\nu} \int d^3k_r \sum_a \alpha_a f_a(k) \right) \left(\prod_{r'=j^\nu+1}^{j^\nu+j'^\nu} \int d^3k_{r'} \sum_{a'} \gamma_{a'}^* f_{a'}^*(k) \right) \right.$$

$$\left. \times \, \hat{\xi}_{j^\nu + j'^\nu} (k_1, \ldots, k_{j^\nu + j'^\nu} : p_\nu, p'_\nu) \right] .$$

Again one defines a divergence-free "residual" by

$$m(j^{\nu}, j'^{\nu}) = \frac{P(j^{\nu}, j'^{\nu})}{j^{\nu}!\ j'^{\nu}!}$$

so that

$$M = F\left\{\exp\left[-\tfrac{1}{2}\sum_a |\alpha_a|^2\right]\exp\left[-\tfrac{1}{2}\sum_a |\gamma_a|^2\right]\exp\left[\sum_a \alpha_a\,\gamma_a{}^*\right]\right\}$$

$$\times \prod_{\nu}\left\{\left[\exp\left[-\sum_a \alpha_a\,\beta_a^{\nu *}\right]\exp\left[\sum_a \gamma_a^*\,\beta_a^{\nu}\right]\right]\left[\sum_{j^{\nu}=0}^{\infty}\ \sum_{j'^{\nu}=0}^{\infty} m(j^{\nu}, j'^{\nu})\right]\right\}$$

where as usual,

$$\beta_a{}^{\nu} \equiv (f_a^*, \tilde{S}) = \int_{\Omega} d^3k\ f_a^*(k)\ \frac{e_0}{[(2\pi)^3\ 2k_0]^{\frac{1}{2}}}\left[\frac{p'_{\nu}\cdot e}{k\cdot p'_{\nu}} - \frac{p_{\nu}\cdot e}{k\cdot p_{\nu}}\right].$$

Now suppose we again define the sequence of numbers

$$\beta_{ia}{}^{\lambda} \equiv \int_{\Omega} d^3k\ f_a^*(k)\ \frac{e_0}{[(2\pi)^3\ 2k_0]^{\frac{1}{2}}}\left[\frac{p_{\nu}\cdot e}{k\cdot p_{\nu}}\right]$$

$$\beta_{fa}{}^{\nu} \equiv \int_{\Omega} d^3k\ f_a^*(k)\ \frac{e_0}{[(2\pi)^3\ 2k_0]^{\frac{1}{2}}}\left[\frac{p'_{\nu}\cdot e}{k\cdot p'_{\nu}}\right]$$

and transform from the variables α_a, γ_a to ϵ_{ia}, ϵ_{fa} :

$$\alpha_a = \sum_{\nu} \beta_{ia}{}^{\nu} + \epsilon_{ia}, \quad \gamma_a = \sum_{\nu} \beta_{fa}{}^{\nu} + \epsilon_{fa}$$

Then a little additional algebra will exhibit the result

$$M = F \exp\left[\frac{1}{2} \sum_a \left| \sum_{\nu=1}^{N} \beta_{fa}^{\nu} - \beta_{ia}^{\nu}\right|^2\right] \exp\left[- \frac{1}{2} \sum_a \left|\epsilon_{fa} - \epsilon_{ia}\right|^2\right]$$

$$\exp[i\phi] \prod_{\nu} \left[\sum_{j^{\nu}=0}^{\infty} \sum_{j'^{\nu}=0}^{\infty} m(j^{\nu}, j'^{\nu})\right] .$$

The orthonormality of the f_a's allows us to write

$$\left[\frac{1}{2} \sum_a \left| \sum_{\nu=1}^{N} \beta_{fa}^{\nu} - \beta_{ia}^{\nu} \right|^2\right] = \alpha \tilde{B}$$

where the generalized definition of $\tilde{B}$ is

$$\tilde{B} = - \frac{1}{8\pi^2} \int_{\Omega} \frac{d^3k}{(k^2 + \Lambda^2)^{\frac{1}{2}}} \left(\sum_{\nu=1}^{N} \frac{p'_{\nu}}{k \cdot p'_{\nu}} - \frac{p_{\nu}}{k \cdot p_{\nu}} \right)^2 .$$

Thus

$$M = \exp[\alpha \beta + \alpha \tilde{B}] \exp[- \frac{1}{2} \sum_a |\epsilon_{fa} - \epsilon_{ia}|^2] \exp[i\phi]$$

$$\times \prod_{\nu} \left[\sum_{j^{\nu}=0}^{\infty} \sum_{j'^{\nu}=0}^{\infty} m(j^{\nu} , j'^{\nu})\right] .$$

Since the infrared divergences in B and $\tilde{B}$ cancel,[4] the matrix element is finite for

$$\sum_a |\epsilon_{fa} - \epsilon_{ia}|^2 < \infty$$

APPENDIX B

THE PROJECTION OF ENERGY-MOMENTUM EIGENSTATES

For completeness, the procedure for projecting out energy-momentum eigenstates will be given here. Consider a typical soft photon state described by a principal vector $|\{\alpha_a{}^\lambda\}\rangle$:

$$|\{\alpha_a^\lambda\}\rangle = \exp\left[-\tfrac{1}{2}\sum_{\lambda,a}|\alpha_a^\lambda|^2\right]\exp\left[\sum_{\lambda,a}\alpha_a^\lambda\int_\Omega d^3k\ f_a(k)\ e^{(\lambda)}(k)\ a^{(\lambda)+}(k)\right]|0\rangle \qquad (B.1)$$

$$= \exp\left[-\tfrac{1}{2}\sum_{\lambda,a}|\alpha_a^\lambda|^2\right]\sum_{n=0}^{\infty}\frac{1}{n!}\left[\sum_{a,\lambda}\alpha_a^\lambda\int_\Omega d^3k_1\ f_a(k_1)e^{(\lambda)}{}_a{}^{(\lambda)+}(k)\right]\cdots$$

$$\times\left[\sum_{a,\lambda}\alpha_a{}^\lambda\int_\Omega d^3k_n\ f_a(k_n)\ e^{(\lambda)}{}_a{}^{(\lambda)+}(k_n)\right]\ |0\rangle.$$

Using the formula

$$\frac{1}{2\pi}\int e^{ixy}\,dy = \delta(x), \qquad (B.2)$$

it is then possible to project out from $|\{\alpha_a{}^\lambda\}\rangle$ the eigenstates of energy momentum:

$$P(E,\underset{\sim}{K})\ |\{\alpha_a{}^\lambda\}\rangle = \exp\left[-\tfrac{1}{2}\sum_{\lambda,a}|\alpha_a{}^\lambda|^2\right]\sum_{n=0}^{\infty}\frac{1}{n!}$$

$$\times\left\{\int d^3k_1\cdots d^3k_n \prod_{i=1}^{n}\left[\sum_{a,\lambda}\alpha_a^\lambda f_a(k_i)e^{(\lambda)}(k_i)a^{(\lambda)+}(k_i)\right]\delta(E-\sum_i\omega_i)\delta^3(\underset{\sim}{K}-\sum_i \underset{\sim}{k}_i)\right\}|0\rangle$$

$$= \exp\left[-\tfrac{1}{2}\sum_{\lambda,a}|\alpha_a{}^\lambda|^2\right]\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{(2\pi)^4}\int dy\, d^3\underset{\sim}{x}\, e^{iEy}\, e^{i\underset{\sim}{K}\cdot\underset{\sim}{x}}$$

$$\times\left\{\int d^3k_1\cdots d^3k_n \prod_i\left[\sum_{\lambda,a}\alpha_a^\lambda e^{-i\omega y}\, e^{+i\underset{\sim}{k}_i\cdot x} f_a(k_i)\, e^{(\lambda)}(k_i)\, a^{(\lambda)+}(k_i)\right]\right\}|0\rangle$$

$$= \frac{1}{(2\pi)^4}\int dy\, d^3\underset{\sim}{x}\,\exp\left[-\tfrac{1}{2}\sum_{\lambda,a}|\alpha_a^\lambda|^2\right]e^{iEy-i\underset{\sim}{K}\cdot\underset{\sim}{x}}\exp\left[\sum_{\lambda,a}\alpha_a^\lambda\int d^3k\, e^{-i\omega y}\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}\right.$$

$$\left.\times\ f_a(k)\, e^{(\lambda)}(k)\, a^{(\lambda)+}(k)\right] .$$

$$= \frac{1}{(2\pi)^4}\int d^4x\ \exp\left[-\tfrac{1}{2}\sum_{\lambda,a}|\alpha_a{}^\lambda|^2\right]e^{iK\cdot x}\exp\left[\sum_{\lambda,a}\alpha_a{}^\lambda\int d^3k\, e^{-ik\cdot x}\right.$$

$$\left.\times\ f_a(k)\, e^{(\lambda)}(k)\, a^{(\lambda)\dagger}(k)\right] . \qquad \text{(B.3)}$$

This equation may now be used to calculate the matrix element between states in which there is a constraint on the total energy-momentum of the soft photons. For example, consider a process where no soft photons are present in the initial state, but because of energy-momentum conservation the total energy-momentum K of the soft photons in the final state is fixed.

From Equation (B.3) one sees that the only effect on our calculations in Section V is making the replacement

$$\gamma_a^{\lambda} f_a(k) \longrightarrow \gamma_a^{\lambda} f_a(k) \; e^{-ik\cdot x} \qquad (B.4)$$

and requiring an additional integration over the variable x:

$$\frac{1}{(2\pi)^4} \int d^4x \; e^{iK\cdot x} \; \cdots$$

We make the following new definitions:

$$P_{j,j'}(x) \equiv (-1)^j \left(\prod_{r=1}^{j} \int d^3k \sum_{\lambda,a} \alpha_a^{\lambda} f_a(k_r) \right)$$

$$\times \left(\prod_{r'=1}^{j'} \int d^3k_{r'} \sum_{\lambda'_1 c} \gamma_c^{*\lambda'} f_c^*(k_{r'}) \, e^{+ik_{r'}\cdot x} \, \tilde{\xi}_{j+j'}^{\{\lambda\}}(k_1, \cdots, k_{j+j'}) \right) \qquad (49')$$

$$\beta_a'^{\lambda}(x) \equiv \left(f_a^* \, e^{+ik\cdot x}, \, S^{\lambda} \right)_{\Omega} = \int_{\Omega} d^3k \, \breve{S}^{(\lambda)} \, f_c^*(k) \, e^{+ik\cdot x} \qquad (52')$$

so that Equation (53) becomes

$$\tilde{M} = \int d^4x \, e^{iK\cdot x} \, e^{\alpha B} \exp\left[-\tfrac{1}{2} \sum_{\lambda,a} |\gamma_a^{\lambda}|^2 \right] \exp\left[\sum_{\lambda,a} \beta_a'^{\lambda} \gamma_a^{*\lambda} \right]$$

$$\times \left\{ \sum_{m,m'}^{\infty} m_{m,m'} \right\} . \qquad (B.5)$$

By rearranging the terms slightly, we find

$$\widetilde{M} = \int d^4x\ e^{iK\cdot x}\ e^{\alpha B} \exp \sum_{\lambda, a} \Big[\tfrac{1}{2} |\beta_a{}^\lambda|^2 + (\beta_a'{}^\lambda(x) - \beta_a^\lambda)\ \beta_a^{*\lambda} - \tfrac{1}{2} |\gamma_a^\lambda - \beta_a^\lambda|^2$$

$$+ (\beta_a'{}^\lambda(x) - \beta_a{}^\lambda)\ (\gamma_a{}^\lambda - \beta_a{}^\lambda)^* + i\ \mathrm{Im}\ (\gamma_a{}^\lambda - \beta_a{}^\lambda)^*\ \beta_a{}^\lambda \Big] \quad .$$

The second sum in the argument of the last exponential is finite:

$$\sum_{\lambda, a} (\beta_a'{}^\lambda(s) - \beta_a{}^\lambda)\ \beta_a^{*\lambda} = \sum_{\lambda} \int d^3k \quad |\widetilde{S}^{(\lambda)}(k)|^2\ (e^{-ik\cdot x} - 1) < \infty \quad .$$

If the condition $\sum_{\lambda, a} |\gamma_a{}^\lambda - \beta_a{}^\lambda|^2 < \infty$ is imposed, then the third and fourth terms in the argument are also finite. Thus the cancellation of the infrared divergences is again established.

Squaring the matrix element and summing over final states for the lowest order term as we did in Section VI, one arrives at the form

$$\sum_{\substack{\text{final} \\ \text{states}}} |\widetilde{M}|^2 = \delta(0)\ e^{2\alpha\ B + 2\alpha\ \widetilde{B}}\ |m_{oo}|^2\ \frac{1}{(2\pi)^4} \int d^4x \quad e^{iK\cdot x + \alpha\ D} \qquad (B.6)$$

where

$$D = -\frac{1}{4\pi^2} \int_\Omega \frac{d^3k}{k_0} \left(\frac{p'_\mu}{k\cdot p'} - \frac{p_\mu}{k\cdot p} \right)^2 (e^{-ik\cdot x} - 1) \ . \qquad (B.7)$$

This is the same general result obtained by Yennie et al.[4]

APPENDIX C

INITIAL AND FINAL STATES WITH DIFFERENT RESOLUTION REGIONS

Suppose that we are dealing with the situation where the resolution region Ω for the initial state is smaller than the resolution region Ω' for the final state $(\Omega \subset \Omega')$. In other words, the threshold for detecting low-energy photons is lower before the scattering experiment than afterwards. Then an infrared divergenceless matrix element in a form analogous to Equation (64) may be obtained with very little additional complication.

Let us define a domain D of momentum space such that

$$\Omega' = \Omega \cup D \quad \text{where} \quad \Omega \cup D = 0 .$$

Then we suppose that there exists a complete set of orthnormal functions $\{g_j(k)\}$ defined on D. A typical final state is now given by

$$|f\rangle = |\{\gamma_i, \gamma_j'\}\rangle = \frac{\exp\left[\prod_i \gamma_i \int_\Omega d^3k\ f_i(k) a^+(k)\right]}{\exp\left[\frac{1}{2}\sum_i |\gamma_i|^2\right]} \ \frac{\exp\left[\prod_j \gamma_j' \int_D d^3k\ g_j(k) a^+(k)\right]}{\exp\left[\frac{1}{2}\sum_j |\gamma_j|^2\right]} |0\rangle . \tag{C:1}$$

The indices having to do with polarization have been suppressed. The result of the modification is that in the derivation of Equation (55), one must make the substitutions

$$\gamma_a^{\lambda} \int_{\Omega} f_a \, e^{(\lambda)} \longrightarrow \gamma_a^{\lambda} \int_{\Omega} f_a \, e^{(\lambda)} + \gamma_a'^{\lambda} \int_{D} g_a \, e^{(\lambda)} ,$$

so that Equation (55) is correct only if we have on the right-hand side the additional factor

$$\exp\left[-\tfrac{1}{2} \sum_{\lambda, c} |\gamma_c'^{\lambda}|^2 \right] \exp\left[\sum_{\lambda, c} \gamma_c'^{*\lambda} (g_c^{*}, \tilde{S}^{(\lambda)})_D \right],$$

and the divergenceless sum

$$\sum_{m,m'=0}^{\infty} m_{m,m'}$$

contains integrals of the functions $\{g_c\}$ over the addition region D.

We now define new coefficients $\beta_a'^{\lambda}$ and variables $\epsilon_{fa}'^{\lambda}$ in a similar way to Equations (59) and (61),

$$\beta_a'^{\lambda} = (g_a^{*}, \tilde{S}^{(\lambda)})_D \quad , \quad \gamma_a'^{\lambda} = \beta_a'^{\lambda} + \epsilon_{fa}'^{\lambda} . \tag{C.2}$$

Then the additional (non-infrared-divergent) factor becomes

$$\exp\left[\sum_{\lambda, a} \left(\tfrac{1}{2} |\beta_a'^{\lambda}|^2 - |\epsilon_{fa}'^{\lambda}|^2 + i \, \mathrm{Im} \, \epsilon_{fa}^{*'\lambda} \beta_a'^{\lambda} \right) \right] .$$

It is natural to define

$$\alpha \, \tilde{B}_D = \exp\left[\sum_{\lambda, a} \tfrac{1}{2} |\beta_a'^{\lambda}|^2 \right] , \tag{C.3}$$

so that Equation (64) becomes

$$\tilde{M} = e^{(\alpha B + \alpha\tilde{B}_{\Omega} + \alpha\tilde{B}_{D})} \exp\left[- \tfrac{1}{2} \sum_{\lambda, a} \left|\epsilon_{fa}{}^{\lambda} - \epsilon_{ia}{}^{\lambda}\right|^{2} \right] . \tag{C.4}$$

$$\times \exp\left[-\tfrac{1}{2} \sum_{\lambda, a} \left|\epsilon'_{fa}{}^{\lambda}\right|^{2}\right] e^{i\phi} \sum_{m,m'=0}^{\infty} m_{m,m'}$$

In this expression the infrared divergences cancel in the sum $\alpha B + \alpha B_{\Omega}$ of the argument of the first exponential. The term $e^{\alpha\tilde{B}_D}$ accounts for the difference in resolution regions. The condition for finite matrix elements is now

$$\sum_{\lambda, a} \left|\epsilon_{fa}{}^{\lambda} - \epsilon_{ia}{}^{\lambda}\right|^{2} < \infty ,$$

$$\sum_{\lambda, a} \left|\epsilon'_{fa}{}^{\lambda}\right|^{2} < \infty . \tag{C.5}$$

FOOTNOTES AND REFERENCES

* This work was performed under the auspices of the United States Atomic Energy Commission.

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FIGURE CAPTIONS

Fig. III-1. Contributions to the second-order virtual photon radiative corrections. Diagram (a) corresponds to the original uncorrected matrix element M_0 .

Fig. III-2. Contributions to the second-order corrections due to emission or absorption of real soft photons. Diagram (a) accounts for the possibility that the photon does not interact with the electrons at all.

Fig. V-1. Representation of ℓ noninteracting real soft photons, m real soft photons absorbed by the electron line, and m' real soft photons emitted by the electron line.

Fig. A-1. Representation of ℓ noninteracting real soft photons, m^{ν} real soft photons absorbed by the νth electron, and m'^{ν} real soft photons emitted by the νth electron.

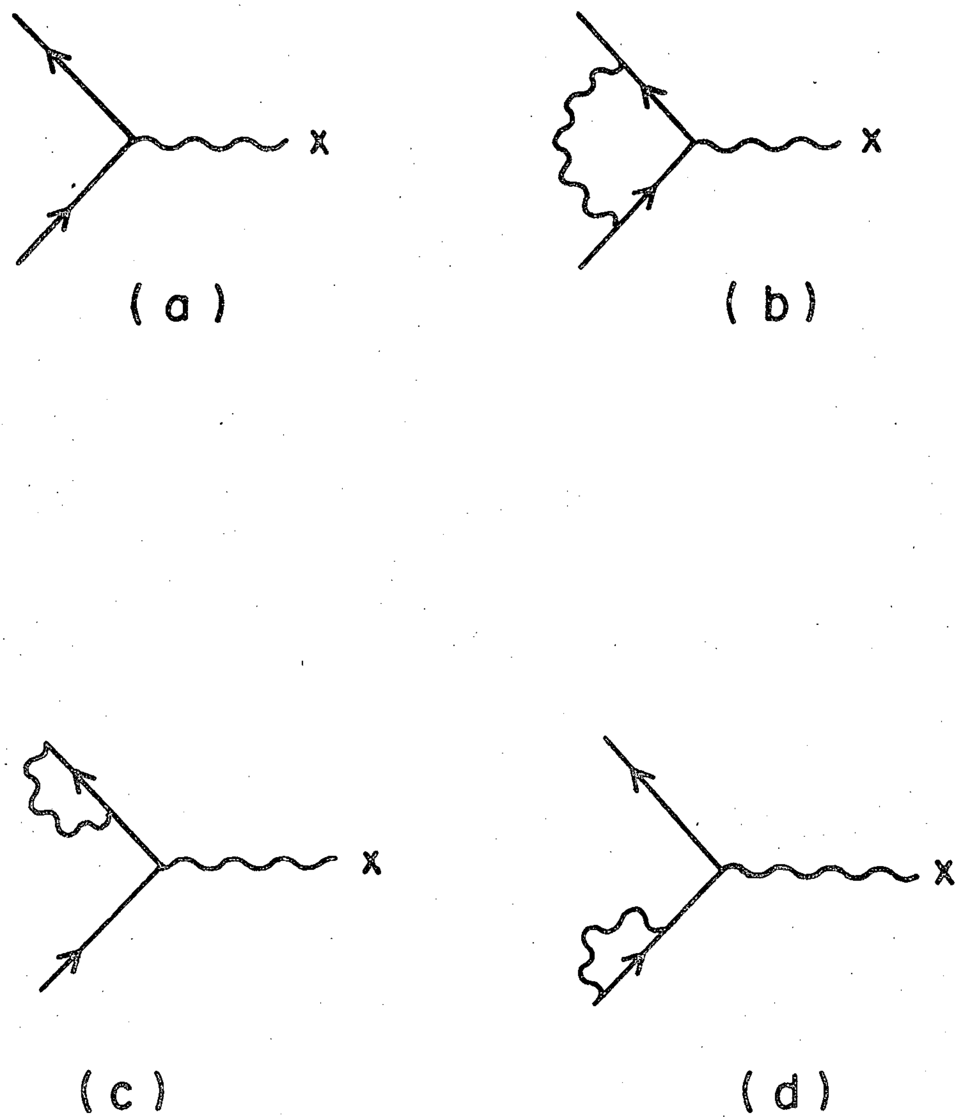

Fig. III-1. Contributions to the second-order virtual photon radiative corrections. Diagram (a) corresponds to the original uncorrected matrix element M_0 .

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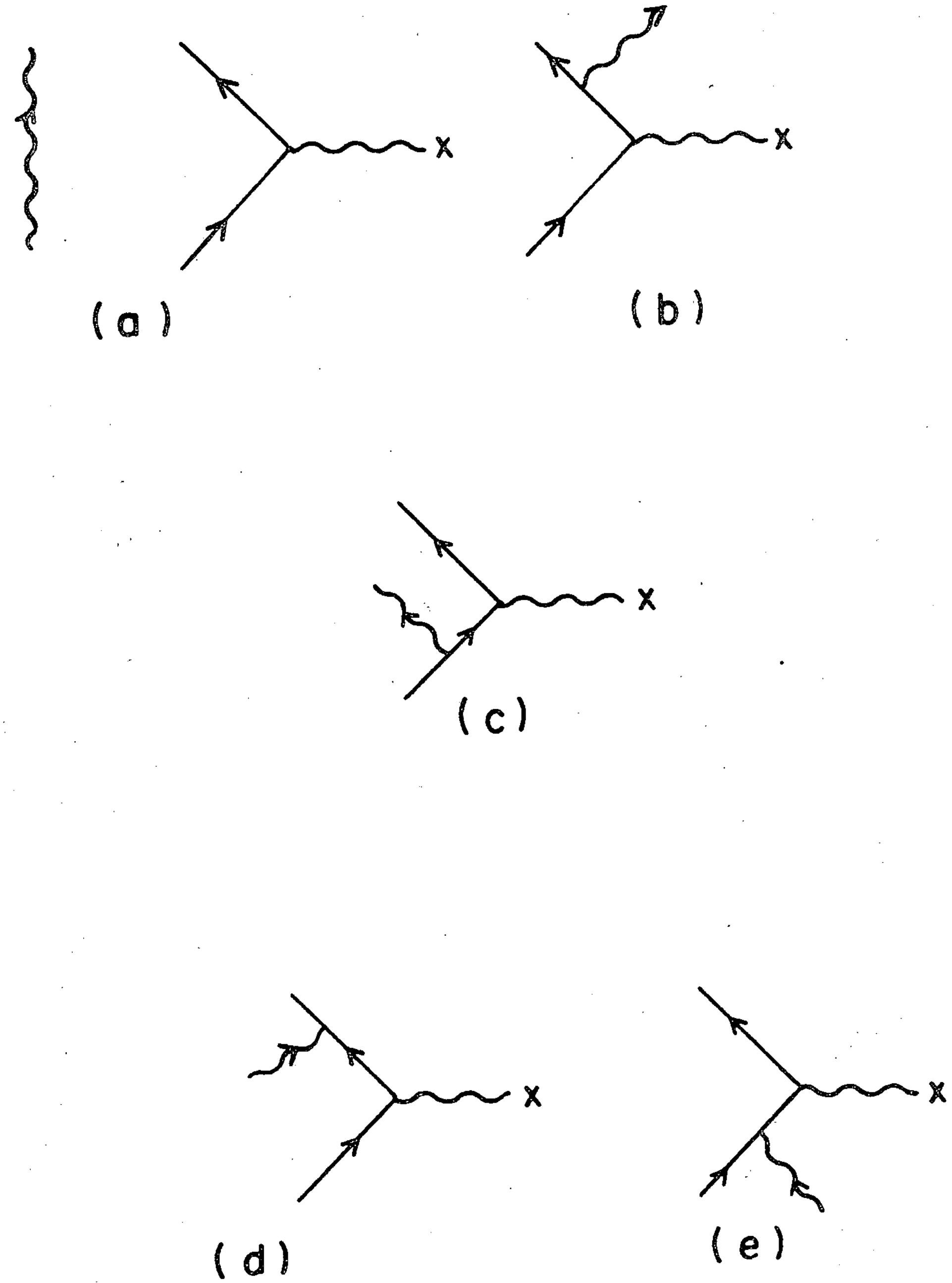

Fig. III-2. Contributions to the second-order corrections due to emission or absorption of real soft photons. Diagram (a) accounts for the possibility that the photon does not interact with the electrons at all.

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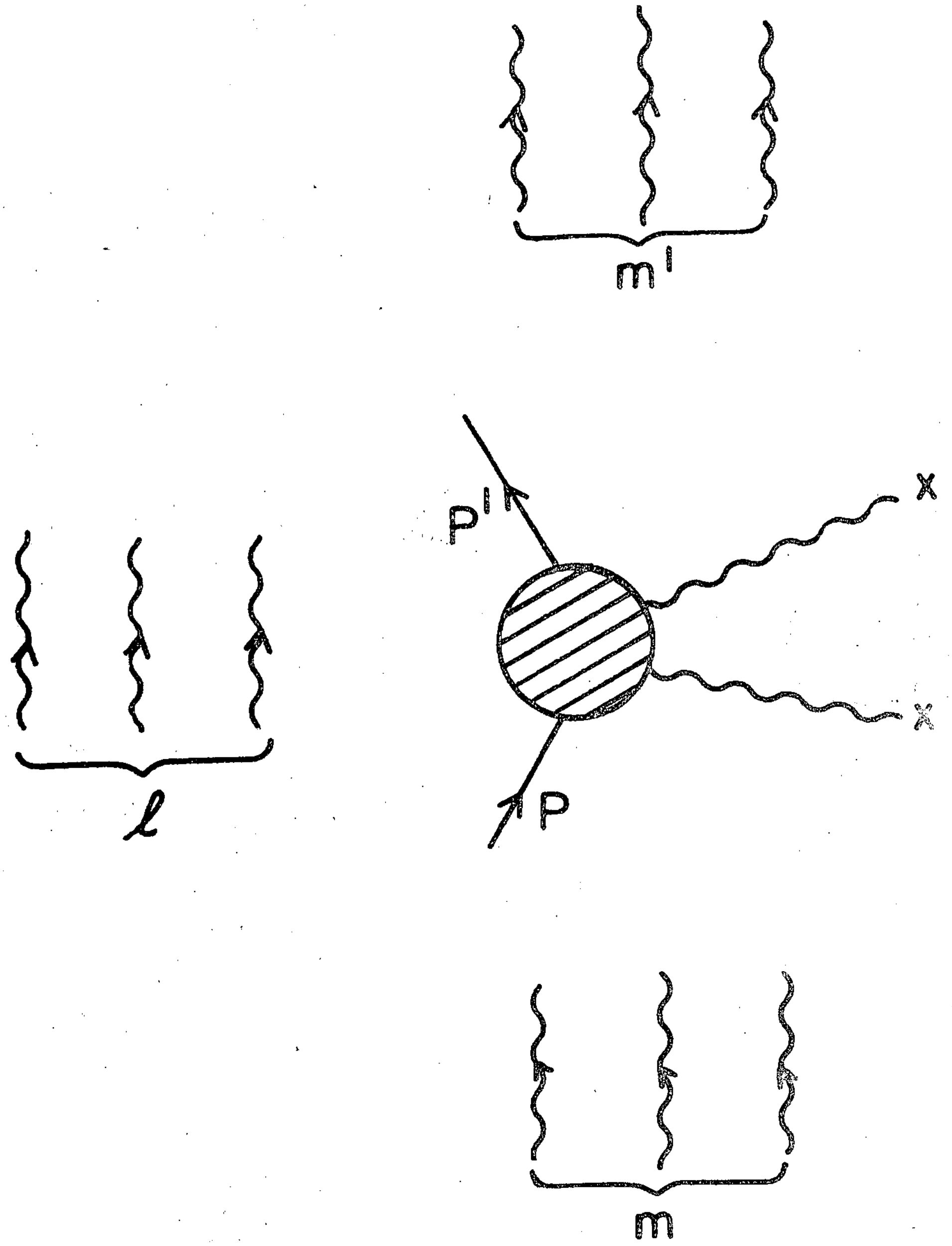

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Fig. V-1. Representation of ℓ noninteracting real soft photons, m real soft photons absorbed by the electron line, and m' real soft photons emitted by the electron line.

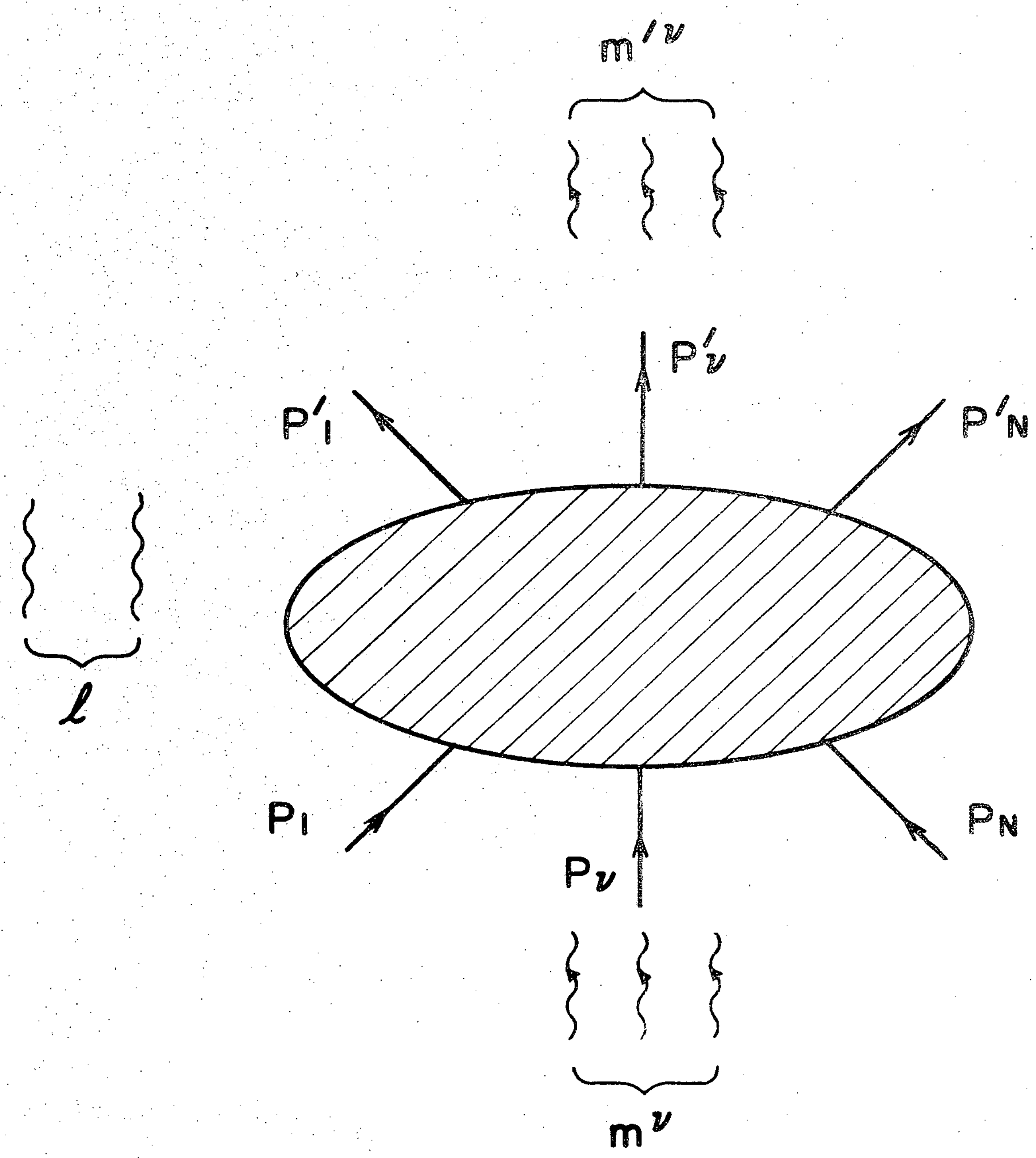

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Fig. A-1. Representation of ℓ noninteracting real soft photons, m^{ν} real soft photons absorbed by the νth electron, and m'^{ν} real soft photons emitted by the νth electron.