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Essays in Macroeconomics and Finance

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by

Vadim Khramov

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ABSTRACT OF THE DISSERTATION

Essays in Macroeconomics and Finance

by

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Doctor of Philosophy in Economics

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Professor Roger E. Farmer, Chair

The first chapter proposes a method for solving and estimating linear rational expectations models that exhibit indeterminacy. The method implements an idea of moving expectational errors to the set of fundamental shocks, reducing the number of solutions from infinity to one. This transformation allows one to treat indeterminate models as determinate and, therefore, apply standard solution and estimation methods to them. While not all expectational errors have to be moved to the set of fundamental shocks, it is shown that the choice of which expectational errors to move is irrelevant for theoretical solutions of indeterminate models, but is important when an indeterminate model is taken to data. As it is hard to identify empirically which expectational errors lead to indeterminacy, model estimation results might vary, depending on which expectational errors are moved into the set of fundamental shocks. To solve this problem, this chapter provides a simple "rule of thumb," based on a Bayesian model

comparison, for identifying expectational errors that generate indeterminacy. Simulation results support the robustness of this idea. Step-by-step guidelines for implementing this method in the Matlab-based packages Dynare and Gensys are provided.

The second chapter reexamines the source of the Great Moderation by estimating New-Keynesian DSGE models with capital accumulation. In this framework, an increase of the nominal interest rate by the monetary authority influences the cost of renting capital, leading to cost-push inflation. To understand the role of this channel, a model with capital was estimated on U.S. data from 1960 to 2008. By not restricting the monetary policy to be active and allowing indeterminacy to occur, it was found that, in contrast to canonical papers, the Federal Reserve's monetary policy rule remained passive in response to inflation before (1960-1979) and after (1982-2008) the Great Moderation. Bayesian model comparisons enable a declaration that, when capital is added, passive monetary policy with indeterminacy provides a better fit to the data in both subperiods. The results of this chapter suggest that during the Great Moderation structural changes were primarily on the demand side of the economy, supporting the idea of financial innovations.

The third chapter sheds light on a narrow but crucial question in finance: What should be the parameters of a model of the short-term real interest rate? Although models for the nominal interest rate are well studied and estimated, dynamics of the real interest rate are rarely explored. Simple *ad hoc* processes for the short-term real interest rate are usually assumed as building blocks for more sophisticated models. In this chapter, parameters of the real interest rate model are estimated in the broad class of single-factor interest rate diffusion processes on U.S. monthly data. It is shown that the elasticity of interest rate volatility—the relationship between the volatility of changes in the interest rate and its level—plays a crucial role in explaining real interest rate dynamics. The empirical estimates of the elasticity of the real interest rate volatility are found to be about 0.5, much lower than that of the nominal interest rate. These estimates show that the square root process, as in the Cox-Ingersoll-Ross model, provides a good characterization of the short-term real interest rate process.

The dissertation of Vadim Khramov is approved.

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2013

To my parents Valentina and Nikolay,

who provided me all their love and guidance every moment of my life.

To my grandmother Sofia and uncle John, who have always been there for me.

And to my love Ally, who always has hidden strength and radiant beauty.

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Chapter 1 is a version of the forthcoming paper, co-authored with Roger Farmer, who deserves full credit for the main idea, which I encountered during his Monetary Economics course at UCLA. He first presented the method of solving indeterminate models by redefining one or more of the endogenous errors as an exogenous error during his lectures at the DYNARE workshop in Paris on July 2nd, 2010.

All errors are my own.

VITA

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PUBLICATIONS

1. Vadim Khramov. Estimation of DSGE Models with Indeterminacy, Capital Accumulation, and Different Taylor Rules. IMF Working Paper No. 12/83, 2012.

2. Vadim Khramov. Asymmetric Effects of the Financial Crisis: Collateral-Based Investment-Cash Flow Sensitivity Analysis. IMF Working Paper No. 12/97, 2012.

Chapter 1. Note on Solving and Estimating Indeterminate DSGE Models.

1. Introduction

It is well known that linear rational expectations (LRE) models can have an indeterminate set of equilibria under realistic sets of parameters. In this paper, we propose a simple and robust method for solving and estimating linear rational expectations models that exhibit indeterminacy. Our method is built upon the Sims (2000) approach and uses the notion of rational expectation errors as a solution device. By moving expectational errors to the set of fundamental shocks, the number of solutions is reduced from infinity to one, allowing the standard solution algorithms to treat these models as determinate. We provide step-by-step guidelines for implementing this method in the Matlab-based packages Dynare and Gensys.

Using LRE models became very popular in the economic profession in the past decade, making problems of solving and estimating these models important. The baseline solution approach was proposed some time ago by Blanchard and Kahn (1980), who showed that a LRE model can be written as a linear combination of backward-looking and forward-looking solutions. Since then a number of alternative approaches for solving linear rational expectations models have emerged (see, for example, King and Watson (1998), Klein (1999), Uhlig (1999), and Sims (2001)). All of these methods provide a solution only if the dynamics equilibrium path is unique, or, in other words, if the model is determinate. Usually, solution algorithms are based on the idea that under determinacy dynamics of forecast errors are uniquely defined by fundamental shocks, such that the explosive dynamics of real variables are eliminated. A commonly used Sims (2001) solution method for determinate models basically defines expectational errors as functions of exogenous shocks to eliminate explosive components from the dynamics of real variables.

A recent body of literature has emerged that exploits the existence of an indeterminate set of equlibria as a means of better understanding full the dynamics of standard models (see a review in Benhabib and Farmer (1999)). Existence of indeterminacy moved from been a theoretical artifact to a new approach for modeling the self-fulfilling dynamics of business cycles and monetary transmission mechanisms. Benhabib and Farmer (1994) showed that a standard one-sector growth model with increasing returns displays an indeterminate steady state and can be exploited to generate business fluctuations driven by self-fulfilling beliefs. More recent New-Keynesian models exhibit indeterminacy if the monetary authority does not increase the nominal interest rate enough in response to higher inflation (see, for example, Kerr and King (1996), Rotemberg and Woodford (1998), and Christiano and Gust (1999)).

Taking a model with indeterminacy to data has always been a complex task. Farmer and Guo (1994) showed that impulse response functions of major economic variables from an indeterminate model with increasing returns match moments of U.S data better than traditional models. Farmer and Guo (1995) were the first to demonstrate that a general equilibrium model with an indeterminate steady state does a good job of accounting for the propagation mechanism in U.S. data. Lubik and Schorfheide (2004) use a Bayesian approach to estimate the DSGE model on U.S. data and found that U.S. monetary policy post-1982 is consistent with determinacy, whereas the pre-Volcker policy is not.

Solving and estimating indeterminate models is challenging, as non-fundamental shocks may contribute to the variance of economic fluctuations. In this framework, a direct application of Sims' (2001) algorithm, as well as other algorithms, will not deliver a solution, as the Blanchard-Kahn stability conditions would be violated. Therefore, solution methods of indeterminate models are usually based on the idea of adding non-fundamental shocks in the solution of a model. If the equilibrium is not unique, it is possible to construct sunspot equilibria, in which stochastic disturbances that are unrelated to fundamental shocks influence model dynamics. Lubik and Schorfheide (2003) provide simple methods for analyzing the effects of fundamental and sunspot shocks in indeterminate LRE models. They show that imposing certain restrictions on the structure of sunspot shocks is important to characterize the full set of equilibria and indentify model parameters.

Estimation of indeterminate models is sometimes connected with identification issues, as additional non-fundamental shocks might influence the dynamics of real variables differently. Beyer and Farmer (2007) show that sometimes it is not possible to decide whether data is generated by a determinate or an indeterminate model, without imposing additional assumptions regarding the structure of the model. Lubik and Schorfheide (2003) show that not all parameters are unidentifiable when the model is indeterminate and, therefore, additional restrictions are necessary.

This paper provides a coherent transformation method for solving indeterminate models. Our method implements the idea of moving expectational errors to the set of fundamental shocks, reducing the number of solutions from infinity to one. This transformation allows treating indeterminate models as determinate and, therefore, applying standard solution and estimation methods to them. Also, we establish a method for empirically identifying which expectational errors lead to indeterminacy and provide a simple rule of thumb based on a Bayesian model comparison for it. We provide recommendations for choosing parameters of sunspot shock priors. We show that for identification purposes it should not matter which expectational error is moved to the set of fundamental shocks, as parameters of their covariances with fundamentals shocks and variances are functions of each other.

This paper is organized as follows. In Section 2, we provide a general solution method for indeterminate models. In Section 3, we discuss the choice of expectational errors that should be moved to the set of fundamental shocks. In Section 4, we document the fact that there is a one-to-one correspondence between variances and covariances of shocks under different selections of expectational errors that are moved to the set of fundamental shocks. In Section 5, we provide a step-by-step application of the proposed solution method to a simple New-Keynesian model. We provide step-by-step guidelines for implementing this method in the Matlab-based packages Dynare and Gensys in Sections 6 and 7, respectively. Then we conclude.

2. The Solution Method of Indeterminate Models

In this section, we provide a general solution method for linear rational expectations (LRE) models that exhibit indeterminacy. Under indeterminacy, the equilibrium is not unique and non-fundamental (co-called "sunspot") shocks influence model dynamics. One way to deal with indeterminate models is based on using expectational errors as a solution vehicle. The proposed method implements this idea, by moving expectational errors to the set of fundamental shocks. This transformation reduces the number of solutions from infinity to one and allows us to treat indeterminate models as determinate with additional shocks. In this section, a description of this idea is provided.

Consider a k-equation LRE model with p expectational variables and l exogenous shocks. It is typical to add an expectational error equation for each expectational variable, such that the model can be presented as a system of $(k+p)$ equations:[1]

(2.1) $$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi z_t + \Pi \eta_t,$$

where X_t is a $(k+p)\times 1$ vector of variables, which include k endogenous and p expectational variables, z_t is an $l \times 1$ vector of exogenous shocks, and η_t is a $p \times 1$ vector of expectational errors. $\Gamma_0, \Gamma_1, \Psi,$ and Π are matrices of coefficients of the model. $E_{t-1}\left(z_t z_t^T\right) = \Omega$ is an $l \times l$ covariance matrix of exogenous shocks. Each expectational variable, $E_{t-1}X_{i,t}$, has a corresponding expectational (forecast) error equation:

(2.2) $$\eta_{i,t} = X_{i,t} - E_{t-1}X_{i,t}.$$

The model is determinate if the number of expectational variables equals the number of unstable (that are more than one in absolute value) roots of system (2.1).[2] Under determinacy, dynamics of forecast errors are uniquely defined by fundamental shocks, such that explosive dynamics of real variables are eliminated. Sims' (2001) solution algorithm for determinate models basically chooses expectational errors, η_t , as functions of exogenous shocks, z_t , to eliminate explosive components in the dynamics of X_t.

In many cases, the number of expectational variables is bigger than the number of unstable roots of system (2.1). In other words, there are many (a continuum of) equilibrium paths, making the solution of the model indeterminate. In this case, a direct application of Sims' (2001) algorithm will not deliver a solution, as the Blanchard and Kahn (1980) conditions would be violated.

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[1]Although there is no constant term in this model, the conclusions do not change for the model with a non-zero vector of constants.

[2]This condition is the usual case. Sims (2001) provides more general conditions.

The proposed method solves this problem. Under indeterminacy, sunspot shocks affect model dynamics through endogenous expectational errors. Lubik and Schorfheide (2003) show explicitly that in the solution of indeterminate LRE models, dynamics of real economic variables become a function of exogenous shocks and expectational errors. As expectational errors influence real economic variables under indeterminacy, they can be treated as fundamental shocks. The proposed method suggests moving expectational errors to the set of fundamental shocks, reducing the number of solutions from infinity to one. If a model has m degrees of indeterminacy (the difference between the number of stable roots and the number of expectational variables), then m expectational errors should be moved to the set of fundamental shocks. This transformation allows one to treat indeterminate models as determinate and, therefore, to apply standard solution and estimation methods.

Keeping this in mind, consider model (2.1) with m degrees of indeterminacy. The vector η_t can be split in two sub-vectors $\eta_{1,t}$ and $\eta_{2,t}$:

(2.3) $$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi z_t + \begin{bmatrix} \Pi_1 & \Pi_2 \end{bmatrix} \begin{bmatrix} \eta_{1,t} \\ \eta_{2,t} \end{bmatrix},$$

where $\eta_t = \begin{bmatrix} \eta_{1,t} \\ \eta_{2,t} \end{bmatrix}$ and $\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \end{bmatrix}$, such that $\eta_{1,t}$ is an $m \times 1$ vector, consistent with m degrees of indeterminacy of the model.[3]

Our method of solving LRE models with indeterminacy proposes treating m expectational errors as fundamental shocks. One can re-write the system by moving $\eta_{1,t}$ from the vector of expectational shocks to the vector of fundamental shocks:

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[3]By the definition of m, it is always the case that $m \leq p$.

(2.4) $$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \begin{bmatrix}\Psi & \Pi_1\end{bmatrix}\tilde{z}_t + \Pi_2 \eta_{2,t},$$

where $\tilde{z}_t = \begin{bmatrix} z_t \\ \eta_{1,t} \end{bmatrix}$ should be treated as a new vector of fundamental shocks and $\eta_{2,t}$ as a new vector of expectational errors. A new covariance matrix of fundamental shocks is

$E_{t-1}\left(\begin{bmatrix} z_t \\ \eta_{1,t} \end{bmatrix}\begin{bmatrix} z_t \\ \eta_{1,t} \end{bmatrix}^T\right) = \Omega_1$. Note that, by specifying how the new fundamental error $\eta_{1,t}$ covaries with z_t, a particular dynamics path is picked.

As the number of expectational errors was decreased by m, the number of degrees of indeterminacy of the model (2.4) was decreased from m to zero. Therefore, the model (2.4) can be treated as determinate and standard solution methods can be applied to it. As m expectational errors were moved to the set of fundamental shocks, the solution of the model would be a function of these expectational errors, as well as of the function of fundamental shocks. Below, we show how the model can be solved in this case.

Solving the model (2.4) for X_t:[4]

(2.5) $$X_t = \tilde{\Gamma}_1 X_{t-1} + \begin{bmatrix}\tilde{\Psi} & \tilde{\Pi}_1\end{bmatrix}\tilde{z}_t + \tilde{\Pi}_2 \eta_{2,t},$$

where $\tilde{\Gamma}_1 = \Gamma_0^{-1}\Gamma_1$, $\tilde{\Psi} = \Gamma_0^{-1}\Psi$, $\tilde{\Pi}_1 = \Gamma_0^{-1}\Pi_1$, and $\tilde{\Pi}_2 = \Gamma_0^{-1}\Pi_2$

Using the Jordan decomposition $\tilde{\Gamma}_1 = J\Lambda J^{-1}$, the system can be re-written as:

(2.6) $$X_t = J\Lambda J^{-1} X_{t-1} + \begin{bmatrix}\tilde{\Psi} & \tilde{\Pi}_1\end{bmatrix}\tilde{z}_t + \tilde{\Pi}_2 \eta_{2,t},$$

Multiplying both sides by J^{-1}, the system takes the form:

(2.7) $$w_t = \Lambda w_{t-1} + \begin{bmatrix}\hat{\Psi} & \hat{\Pi}_1\end{bmatrix}\tilde{z}_t + \hat{\Pi}_2 \eta_{2,t},$$

---

[4]It is assumes that Γ_0 is invertible. Although it is a special case, it is often observed in practice.

where $w_t = J^{-1}X_{t-1}$, $\hat{\Psi} = J^{-1}\tilde{\Psi}$, $\hat{\Pi}_1 = J^{-1}\tilde{\Pi}_1$, and $\hat{\Pi}_2 = J^{-1}\tilde{\Pi}_2$.

The matrix Λ is diagonal with eigenvalues of $\tilde{\Gamma}_1$ on its diagonal and, therefore, the equations of the system (2.7) are independent of each other. As this model has m degrees of indeterminacy, the matrix $\tilde{\Gamma}_1$ must have $(p-m)$ unstable (larger than one in absolute value) and $(k+p)$ stable (smaller than one in absolute value) eigenvalues.[5] The system (2.7) can be re-arranged in the order such that $(p-m)$ equations related to unstable eigenvalues (elements of Λ that are greater than one in absolute value) are at the bottom. As unstable solutions must be eliminated by adjustment of expectational errors, the solution of the model requires satisfying the following condition:

(2.8) $$\begin{bmatrix}\hat{\Psi}_U & \hat{\Pi}_{1,U}\end{bmatrix}\tilde{z}_t + \hat{\Pi}_{2,U}\eta_{2,t},$$

where $[A]_U$ denotes rows of the matrix A related to unstable roots of matrix Λ.

Equation (2.8) determines the dynamics of expectational errors as functions of the new vector of fundamental shocks $\tilde{z}_{1,t}$. In the solution, the expectational error vector $\eta_{2,t}$ is a function of $\tilde{z}_{1,t}$:[6,7]

(2.9) $$\eta_{2,t} = \hat{\Pi}_{2,U}^{-1}\begin{bmatrix}\hat{\Psi}_U & \hat{\Pi}_{1,U}\end{bmatrix}\tilde{z}_t.$$

Substituting (2.9) in (2.5), the solution of the model takes the form:

(2.10) $$X_t = \tilde{\Gamma}_1 X_{t-1} + \begin{bmatrix}\tilde{\Psi} - \tilde{\Pi}_2\hat{\Pi}_{2,U}^{-1}\hat{\Psi}_U\end{bmatrix}\tilde{z}_t + \begin{bmatrix}\hat{\Pi}_1 - \tilde{\Pi}_2\hat{\Pi}_{2,U}^{-1}\hat{\Pi}_{1,U}\end{bmatrix}\eta_{1,t}.$$

---

[5]A determinate model would have p unstable and k stable eigenvalues.

[6] $\hat{\Pi}_{2,U}$ is a $(p-m)\times(p-m)$ square matrix, as its dimensions are determined by the number of expectational errors, p, minus the number of degrees of indeterminacy, m.

[7] $\hat{\Pi}_{2,U}$ is an invertible matrix to by its construction.

The solution of the LRE model (2.1) under indeterminacy is a function of fundamental shocks $z_{1,t}$, as well as of the vector of expectational shocks $\eta_{1,t}$. An important element that determines the dynamics of the model (in the continuum of solutions under indeterminacy), is the covariance matrix Ω_1 , which specifies how expectational errors $\eta_{1,t}$ covary with the fundamental shocks z_t .

To summarize, the proposed transformation method allows treating indeterminate model as determinate with an extended vector of fundamental shocks. Standard solution techniques can be applied in this case. As some expectational errors become fundamental shocks, they influence the dynamics of real variables under indeterminacy.

3. Choice of Expectational Errors

In the previous section we split the vector η_t into the vectors $\eta_{1,t}$ and $\eta_{2,t}$ and then chose $\eta_{1,t}$ to move to the set of fundamental shocks. This choice was somewhat arbitrary, as it was necessary to move only m out of p expectational errors. In this section, it is shown that the selection of expectational errors, which are chosen to be moved to the vector of fundamental shocks, is irrelevant for the solution. The only difference is that the solution is expressed as a function of different expectational errors under different selections. In other words, there is a linear combination of expectational errors that delivers the same dynamics under different selections of expectational errors.

On order to get an intuitive confirmation of this result, let's first consider a simple case and assume that $\eta_{1,t}$ and $\eta_{2,t}$ have the same dimensions (or that p is even and $m = p/2$). And let's choose $\eta_{2,t}$ instead of $\eta_{1,t}$ to be moved to the set of fundamental shocks. In this case, the model can be solved for $\eta_{1,t}$ as a function of z_t and $\eta_{2,t}$. Parallel to (2.10), the solution of the model takes the form:

(3.1) $$X_t = \tilde{\Gamma}_1 X_{t-1} + \left[\tilde{\Psi} - \tilde{\Pi}_1 \hat{\Pi}_{1,U}^{-1} \hat{\Psi}_U\right] \tilde{z}_t + \left[\hat{\Pi}_2 - \tilde{\Pi}_1 \hat{\Pi}_{1,U}^{-1} \hat{\Pi}_{2,U}\right] \eta_{2,t}.$$

with the covariance matrix Ω_2 , which specifies how the expectational error $\eta_{2,t}$ covariates with fundamental shocks, z_t.

The solution in this case is a function of the fundamental shock $z_{1,t}$ and the expectational error $\eta_{2,t}$, not $\eta_{1,t}$ as in (2.10). Note that dynamics of the expectational errors $\eta_{2,t}$ and $\eta_{1,t}$ are not observed, making the choice of an equilibrium path under indeterminacy somewhat arbitrary.

More generally, the stability conditions of the LRE model (2.4) are determined by the equation (2.8), which is a linear combination of fundamental shocks, z_t , and the expectational errors $\eta_{1,t}$ and $\eta_{2,t}$. A general version of this condition can be written in the form:

(3.2) $$\hat{\Psi}_U z_t + \hat{\Pi}_U \eta_{2,t} = 0.$$

This equation shows that any sub-set of expectational errors can be fundamental shocks in the indeterminate model, as all errors are linearly dependent. If the model has m degrees of indeterminacy, m out of p expectational errors have to be moved from the vector of expectational errors, η_t, to the set of fundamental shocks, z_t. The dynamics of the remaining $(p-m)$ expectational errors become endogenous. As expectational errors in the vector η_t are linearly dependent in (3.2), it does not matter for the solution which of them are moved to the set of fundamental shocks. More precisely, there is a linear combination of expectational errors that would deliver the same solution.

Comparing solutions (2.10) and (3.1), one might expect that there exist such variance of expectational errors and their covariances with fundamental shocks that dynamics of X_t are the same in both cases. In other words, both solutions can produce the same dynamics of real variables under certain restrictions on covariance matrices Ω_1 and Ω_2. We provide a formal characterization of this idea below and give examples in the next sections.

4. Specifying Covariances of Expectational Errors

As discussed in the previous section, the choice of which expectational error to move to the set of fundamental shocks is somewhat arbitrary, as expectational errors are linearly dependent. While the question of which set of expectational errors should be fundamental shocks is irrelevant from the theoretical standpoint, it becomes crucial when one decided to take a model to data. As it is typical to use the Bayesian approach to estimate DSGE models, specification of priors of estimated parameters, including parameters of expectational shocks (if the model is indeterminate) becomes necessary. Estimation results would vary, depending on which expectational error is moved into the set of fundamental shock.

In this section we document the fact that there is a way of specifying priors that reintroduces equivalence, or there is a one-to-one correspondence between variances and covariances of shocks under different selections of expectational errors that are moved to the set of fundamental shocks. In other words, under two choices of two expectational variables, $\eta_{1,t}$ or $\eta_{2,t}$, that are moved to the set of fundamental shocks, their covariances with fundamental shocks and variances (both are elements of matrices Ω_1 and Ω_2) can be expressed as function of each other.

Following the previous section, the vector η_t can be split into two sub-vectors $\eta_{1,t}$ and $\eta_{2,t}$ with dimensions $m\times 1$ and $(p-m)\times 1$, respectively. The condition (3.2) can be written as:

$$\hat{\Psi}_U z_t + \hat{\Pi}_{1,U}\eta_{1,t} + \hat{\Pi}_{2,U}\eta_{2,t} = 0. \tag{4.1}$$

Assuming that the covariance between expectational errors is zero, one can express the variance of one expectational error and its covariance with the fundamental shock as a function of the variance of another expectational shock and its covariance with the fundamental shock. If $\eta_{1,t}$ is decided to be an exogenous shock, the covariance matrix of exogenous shocks includes the variance (matrix) of $\eta_{1,t}$, as well as the covariance matrix of $\eta_{1,t}$ and z_t. In this case, the expectational shock $\eta_{2,t}$ becomes endogenous and its dynamics are a function of $\eta_{1,t}$ and z_t :

$$\eta_{2,t} = -\hat{\Pi}_{2,U}^{-1}\left(\hat{\Psi}_U z_t + \hat{\Pi}_{1,U}\eta_{1,t}\right), \tag{4.2}$$

which can be re-written as:

$$\eta_{2,t} = V_1 z_{1,t}, \tag{4.3}$$

where $V_1 = \left[-\hat{\Pi}_{2,U}^{-1}\hat{\Psi}_U \quad -\hat{\Pi}_{2,U}^{-1}\hat{\Pi}_{1,U}\right]$ and the extended vector $z_{1,t} = \begin{bmatrix} z_t \\ \eta_{1,t} \end{bmatrix}$ with its covariance matrix Ω_1.

As discussed in the previous sections, the solution of the model in this case is a function of the fundamental shock, z_t , and the expectational errors $\eta_{1,t}$. The covariance matrix Ω_1 includes covariances between the elements of the new vector of fundamental shocks $z_{1,t}$. This matrix plays an important role for model estimation, when it is important to estimate the structure of shocks.

Now, consider the extended vector $z_{2,t} = \begin{bmatrix} z_t \\ \eta_{2,t} \end{bmatrix}$. The covariance matrix of its elements can be decomposed into the variance matrices of its components:

$$(4.4) \qquad \Omega_2 = Var_{t-1}(z_{2,t}) = E_{t-1}\left[\begin{bmatrix} z_t \\ \eta_{2,t} \end{bmatrix}\begin{bmatrix} z_t \\ \eta_{2,t} \end{bmatrix}^T\right] = \begin{bmatrix} E_{t-1}[z_t z_t^T] & E_{t-1}[\eta_{2,t} z_t^T] \\ E_{t-1}[\eta_{2,t} z_t^T] & E_{t-1}[\eta_{2,t}\eta_{2,t}^T] \end{bmatrix}.$$

Note that $E_{t-1}[z_t z_t^T]$ is a variance matrix of fundamental shocks, which does not depend on the selection of expectational errors. The covariance matrix $E_{t-1}[\eta_{2,t} z_t^T]$ can be derived, using (4.3):

$$(4.5) \qquad E_{t-1}[\eta_{2,t} z_t^T] = E_{t-1}[V_1 z_{1,t} z_t^T] = V_1 E_{t-1}\begin{bmatrix} z_t z_t^T \\ \eta_{1,t} z_t^T \end{bmatrix}.$$

Also, note that the variance matrix $E_{t-1}[\eta_{2,t}\eta_{2,t}^T]$ is:

$$(4.6) \qquad E_{t-1}[\eta_{2,t}\eta_{2,t}^T] = V_1^T \Omega_1 V_1,$$

where Ω_1 is a covariance matrix of z_t and $\eta_{1,t}$.

These findings are particularly important when one decides to estimate an indeterminate model, using the Bayesian approach, when specification of priors for expectational shock parameters becomes necessary. Equations (4.5) and (4.6) show that elements of the matrix Ω_2 can be expressed as functions of elements of the matrix Ω_1. Therefore, for identification purposes it should not matter which expectational error is moved to the set of fundamental shocks, as parameters of their covariances with fundamentals shocks and variances are functions of each other. In the following sections, we discuss the implications of these results for two models with indeterminacy.

5. Example: a Simple New-Keynesian Model

In this section, we provide a step-by-step application of the proposed solution method to a simple New-Keynesian model, discussed in Lubik and Schorfheide (2003). A canonical version of this model is a system of three equations:

(5.1) $$E_t[x_{t+1}] + \sigma E_t[\pi_{t+1}] = x_t + \sigma R_t,$$

(5.2) $$R_t = \psi \pi_t + \varepsilon_t,$$

(5.3) $$\pi_t = \beta E_t[\pi_{t+1}] + \kappa x_t,$$

where x_t is output, π_t is inflation, R_t is the interest rate, and ε_t is an interest rate shock. The first equation is a consumption-Euler equation, the second one is a monetary policy rule of the monetary authority, and the third one is a New-Keynesian Phillips curve. This model has two forward-looking variables and one fundamental shock. As in the monetary policy rule the nominal interest rate is a function of the current inflation only, this model can be solved analytically.

Substituting R_t with (5.1) and using the expectational error equations ($x_t = \eta_{1,t} + E_{t-1}[x_t]$ and $\pi_t = \eta_{2,t} + E_{t-1}[\pi_t]$), this model can be reduced to a system of two equations:

(5.4) $$E_t[x_{t+1}] + \sigma E_t[\pi_{t+1}] - \eta_{1,t} - E_{t-1}[x_t] - \sigma\psi(\eta_{2,t} + E_{t-1}[\pi_t]) = \sigma\varepsilon_t,$$

(5.5) $$\beta E_t[\pi_{t+1}] = \eta_{2,t} + E_{t-1}[\pi_t] - \kappa(\eta_{1,t} + E_{t-1}[x_t]).$$

In matrix notations, this system can be presented in the standard form (2.1):

(5.6) $$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi z_t + \Pi \eta_t,$$

where $\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \sigma \\ 0 & 0 & 0 & \beta \end{bmatrix}, \Gamma_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \sigma\psi \\ 0 & 0 & -\kappa & 1 \end{bmatrix}, \ \Psi = \begin{bmatrix} 0 \\ 0 \\ \sigma \\ 0 \end{bmatrix}$, and $\Pi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \sigma\psi \\ -\kappa & 1 \end{bmatrix}$.

$X_t = [x_t, \pi_t, E_t[x_{t+1}], E_t[\pi_{t+1}]]'$ is a vector of variables, $z_t = [\varepsilon_t]$ is a vector of fundamental shocks, and $\eta_t = [\eta_{1,t}, \eta_{1,t}]'$ is a vector of expectational shocks.

As a first step towards solving the model, both sides of the system (5.6) are multiplied by Γ_0^{-1}:

(5.7) $$X_t = \Gamma_0^{-1}\Gamma_1 X_{t-1} + \Gamma_0^{-1}\Psi z_t + \Gamma_0^{-1}\Pi\eta_t,$$

where

$$\Gamma_0^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \sigma \\ 0 & 0 & 0 & \beta \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{\sigma}{\beta} \\ 0 & 0 & 0 & \frac{1}{\beta} \end{bmatrix},$$

$$\Gamma_0^{-1}\Gamma_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{\sigma}{\beta} \\ 0 & 0 & 0 & \frac{1}{\beta} \end{bmatrix} * \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \sigma\psi \\ 0 & 0 & -\kappa & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ 0 & 0 & -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix},$$

$$\Gamma_0^{-1}\Psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{\sigma}{\beta} \\ 0 & 0 & 0 & \frac{1}{\beta} \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \\ \sigma \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sigma \\ 0 \end{bmatrix}, \text{ and}$$

$$\Gamma_0^{-1}\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{\sigma}{\beta} \\ 0 & 0 & 0 & \frac{1}{\beta} \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \sigma\psi \\ -\kappa & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}.$$

The system takes the form:

(5.8)
$$\begin{bmatrix} x_t \\ \pi_t \\ E_t[x_{t+1}] \\ E_t[\pi_{t+1}] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta} \\ 0 & 0 & -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \\ E_{t-1}[x_t] \\ E_{t-1}[\pi_t] \end{bmatrix} + \\ + \begin{bmatrix} 0 \\ 0 \\ \sigma \\ 0 \end{bmatrix} z_t + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \eta_t .$$

5.1. Stability

Stability properties of this model are determined by the matrix $\Gamma_0^{-1}\Gamma_1$ with its eigenvalues

$$\lambda_1 = \tfrac{1}{2\beta}\left(\beta + \kappa\sigma + \sqrt{\beta^2 - 2\beta + \kappa^2\sigma^2 + 2\kappa\sigma + 2\kappa\sigma\beta - 4\kappa\sigma\beta\psi + 1} + 1\right),$$

$$\lambda_2 = \tfrac{1}{2\beta}\left(\beta + \kappa\sigma - \sqrt{\beta^2 - 2\beta + \kappa^2\sigma^2 + 2\kappa\sigma + 2\kappa\sigma\beta - 4\kappa\sigma\beta\psi + 1} + 1\right), \text{ and } \lambda_3 = \lambda_4 = 0.$$

As the system has two forward-looking variables, the model is determinate if two eigenvalues are greater than one in absolute value. Note that the eigenvalues λ_3 and λ_4 are always less than one. It can be shown that $\lambda_{1,2} = \frac{1}{2}(1 + \frac{\kappa\sigma+1}{\beta}) \pm \frac{1}{2}\sqrt{(\frac{\kappa\sigma+1}{\beta} - 1)^2 + \frac{4\kappa\sigma}{\beta}(1-\psi)}$. If $\psi > 1$, then eigenvalues $\lambda_1 > 1$ and $\lambda_2 > 1$ and the model is determinate. If $\psi < 1$, one of these eigenvalues is less than one and the model is indeterminate.

5.2. Case of Determinacy

First, consider a standard case when the model is determinate ($\psi > 1$). As the system's matrix is block-triangular, one can focus on the last two equations of system (5.8):

(5.9) $$Y_t = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} Y_{t-1} + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} z_t + \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \eta_t,$$

where $Y_t = \begin{bmatrix} E_t[x_{t+1}] \\ E_t[\pi_{t+1}] \end{bmatrix}$.

As both variables, $E_t[x_{t+1}]$ and $E_t[\pi_{t+1}]$, are forward-looking, the stable solution must satisfy the following condition:

(5.10) $$\begin{bmatrix} \sigma \\ 0 \end{bmatrix} z_t + \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \eta_t = 0.$$

In this case, the vector of expectational errors is endogenous and is a function of the fundamental shock:

(5.11) $$\eta_t = -\begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}^{-1} \begin{bmatrix} \sigma \\ 0 \end{bmatrix} z_t = \tfrac{1}{\kappa\sigma\psi+1} \begin{bmatrix} 1 & \sigma - \sigma\beta\psi \\ \kappa & \beta + \kappa\sigma \end{bmatrix} \begin{bmatrix} \sigma \\ 0 \end{bmatrix} z_t = \\ = -\tfrac{\sigma}{\kappa\sigma\psi+1} \begin{bmatrix} 1 \\ \kappa \end{bmatrix} z_t,$$

implying that:

(5.12) $$\begin{bmatrix} E_t[x_{t+1}] \\ E_t[\pi_{t+1}] \end{bmatrix} = \begin{bmatrix} E_{t-1}[x_t] \\ E_{t-1}[\pi_t] \end{bmatrix} = 0.$$

Re-substituting the expectational errors in the first two equations of system (5.8), the solution for x_t and π_t takes the form:

(5.13) $$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sigma}{\kappa\sigma\psi+1} \\ \kappa\frac{\sigma}{\kappa\sigma\psi+1} \end{bmatrix} z_t = -\frac{\sigma}{\kappa\sigma\psi+1} \begin{bmatrix} 1 \\ \kappa \end{bmatrix} z_t.$$

This example shows that under determinacy dynamics of real variables are determined only by the dynamics of fundamental shocks. Note that the variances of x_t and π_t are linear functions of the variance of z_t : $Var_{t-1}[x_t] = \left(\frac{\sigma}{\kappa\sigma\psi+1}\right)^2 Var_{t-1}(z)$ and $Var_{t-1}[\pi_t] = \left(\frac{\sigma}{\kappa\sigma\psi+1}\right)^2 \kappa^2 Var_{t-1}(z)$.

5.3. Case of Indeterminacy

Next, consider the case of the indeterminate model with one degree of indeterminacy $(\psi < 1)$. Recall the last two equations of the system:

(5.14) $$Y_t = \Gamma^* Y_{t-1} + \Psi^* z_t + \Pi^* \eta_t,$$

where $Y_t = \begin{bmatrix} E_t[x_{t+1}] \\ E_t[\pi_{t+1}] \end{bmatrix}$, $\Gamma^* = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}$, $\Psi^* = \begin{bmatrix} \sigma \\ 0 \end{bmatrix}$, and

$\Pi^* = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}$.

Using the Jordan decomposition $\Gamma^* = J\Lambda J^{-1}$, the system takes the form:

(5.15) $$Y_t = J\Lambda J^{-1} Y_{t-1} + \Psi^* z_t + \Pi^* \eta_t$$

or

(5.16) $$w_t = \Lambda w_{t-1} + J^{-1}\Psi^* z_t + J^{-1}\Pi^* \eta_t,$$

where $w_t = J^{-1} Y_t$.

Let $[A]_{1.}$ denote the first row of the 2x2 matrix A. As the first eigenvalue of the system is more than one, the stability condition is:

(5.17) $$\left[J^{-1}\Psi^*\right]_{1.} z_t + \left[J^{-1}\Pi^*\right]_{1.} \eta_t = 0,$$

which is equivalent to a linear combination of the fundamental shock, z_t, and the expectational errors $\eta_{1,t}$ and $\eta_{2,t}$:

(5.18) $$a_0 z_t + a_1 \eta_{1,t} + a_2 \eta_{2,t} = 0,$$

where $a_0 = \left(\frac{1}{2d}\sigma(d + \beta + \kappa\sigma - 1)\right)$,

$a_1 = \left(\frac{1}{2d}\left(\kappa\frac{\sigma}{\beta}+1\right)(d + \beta + \kappa\sigma - 1) + \frac{1}{d}\frac{\kappa}{\beta}(\sigma - \sigma\beta\psi)\right)$,

$a_2 = \left(\frac{1}{2d}\left(\sigma\psi - \frac{\sigma}{\beta}\right)(d + \beta + \kappa\sigma - 1) - \frac{1}{d\beta}(\sigma - \sigma\beta\psi)\right)$, and

$$d=\sqrt{\beta^2-2\beta+\kappa^2\sigma^2+2\kappa\sigma+2\kappa\sigma\beta-4\kappa\sigma\beta\psi+1}\,.$$

To solve this model under indeterminacy, we have to specify a "sunspot" shock. Following the method discussed in Section 1, one of the expectational errors (either $\eta_{1,t}$ or $\eta_{2,t}$) should be moved to the set of fundamental shocks.

Case 1. Consider $\eta_{1,t}$ to be a fundamental shock. In this case, the expectational error $\eta_{2,t}$ can be expressed as:

$$\eta_{2,t}=-a_2^{-1}\left[a_0 z_t+a_1\eta_{1,t}\right] \tag{5.19}$$

The solution of the model takes the form (see Appendix 1 for details):

$$\begin{aligned}\begin{bmatrix}x_t\\ \pi_t\end{bmatrix}=&\begin{bmatrix}\kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta}\\ -\frac{\kappa}{\beta} & \frac{1}{\beta}\end{bmatrix}\begin{bmatrix}x_{t-1}\\ \pi_{t-1}\end{bmatrix}+\begin{bmatrix}0\\ -a_2^{-1}a_0\end{bmatrix}z_t+\\ &+\begin{bmatrix}1\\ -\frac{\beta}{\beta\psi-1}\frac{d+\beta+\kappa\sigma-1}{d+\beta+\kappa\sigma+1}\end{bmatrix}\eta_{1,t}+\begin{bmatrix}\sigma\\ 0\end{bmatrix}z_{t-1}.\end{aligned} \tag{5.20}$$

Case 2. Consider $\eta_{2,t}$ to be a fundamental shock. In this case, the expectational shock $\eta_{1,t}$ can, likewise, be expressed as:

$$\eta_{1,t}=-a_1^{-1}\left[a_0 z_t+a_2\eta_{2,t}\right] \tag{5.21}$$

The solution of the model takes the form (see Appendix 2 for details):

$$\begin{aligned}\begin{bmatrix}x_t\\ \pi_t\end{bmatrix}=&\begin{bmatrix}\kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta}\\ -\frac{\kappa}{\beta} & \frac{1}{\beta}\end{bmatrix}\begin{bmatrix}x_{t-1}\\ \pi_{t-1}\end{bmatrix}+\begin{bmatrix}-a_1^{-1}a_0\\ 0\end{bmatrix}z_t+\\ &+\begin{bmatrix}-a_1^{-1}a_2\\ 1\end{bmatrix}\eta_{2,t}+\begin{bmatrix}\sigma\\ 0\end{bmatrix}z_{t-1}.\end{aligned} \tag{5.22}$$

5.4. Selection of Expectational Errors

Moving an expectational error (either $\eta_{1,t}$ or $\eta_{2,t}$) to the set of fundamental shocks allows us to reduce the number of solutions to one under indeterminacy, making the model determinate. By specifying how the expectational error covariates with fundamental shocks, a particular sunspot equilibria (dynamic path) is chosen. If the expectational error for $E_t[x_{t+1}]$ becomes a fundamental shock, $\eta_{1,t}$ becomes the only fundamental shock that influences x_t (see (5.20)). From the estimation perspective, if the variance of $\eta_{1,t}$ is known, the variance of z_t is identifiable from the second equation simultaneously with the covariance between $\eta_{1,t}$ and z_t. Likewise, if the expectational error for $E_t[\pi_{t+1}]$ is used as a fundamental shock, $\eta_{2,t}$ becomes the only fundamental shock that influences π_t (see (5.22)). In this case, if the variance of $\eta_{2,t}$ is known, the variance of z_t is identifiable from the second equation simultaneously with the covariance between $\eta_{2,t}$ and z_t.

As mentioned above, the stability of this model is determined by the equation (5.17), which is equivalent to a linear combination between the fundamental shock z_t and the expectational errors $\eta_{1,t}$ and $\eta_{2,t}$:

(5.23) $$a_0 z_t + a_1 \eta_{1,t} + a_2 \eta_{2,t} = 0.$$

This equations shows that any of the two expectational errors can be a fundamental shock in the indeterminate model, as both errors are linearly dependent. Assuming that the covariance between expectational errors is zero, one can express the variance of one expectational shock and its covariance with the fundamental shock as a function of the variance of the other expectational shock and its covariance with the fundamental shock. For example, consider $\eta_{1,t}$ to be a fundamental shock. In this case, $\mathrm{cov}_{t-1}(\eta_{1,t}, z_t)$, $Var_{t-1}(\eta_{1,t})$, and $Var_{t-1}(z_t)$ are know and the expectational shocks $\eta_{2,t}$ can be expressed as a function of fundamental shocks:

(5.24) $$\eta_{2,t} = -a_2^{-1}\left[a_0 z_t + a_1 \eta_{1,t}\right],$$

with the variance:

$$Var_{t-1}(\eta_{2,t}) = Var_{t-1}\left[-a_2^{-1}a_0 z_t - a_2^{-1}a_1\eta_{1,t}\right] = \left(a_2^{-1}a_0\right)^2 Var_{t-1}(z_t) + \left(a_2^{-1}a_1\right)^2 Var_{t-1}(\eta_{1,t}),$$ and its covariance with z_t :

$$\mathrm{cov}_{t-1}(\eta_{2,t}, z_t) = \mathrm{cov}_{t-1}(-a_2^{-1}a_0 z_t - a_2^{-1}a_1\eta_{1,t}, z_t) = -a_2^{-1}a_0 Var_{t-1}(z_t) - a_2^{-1}a_1 \mathrm{cov}_{t-1}(\eta_{2,t}, z_t).$$ If one considers $\eta_{2,t}$ to be a fundamental shock, $\eta_{1,t}$ becomes an endogenous error and $Var_{t-1}(\eta_{1,t})$ and $\mathrm{cov}_{t-1}(\eta_{1,t}, z_t)$ can be expressed as functions of $\mathrm{cov}_{t-1}(\eta_{2,t}, z_t)$, $Var_{t-1}(\eta_{2,t})$, and $Var_{t-1}(z_t)$.

In both cases, the variance of the two expectational shocks and their covariance with the fundamental shock are linearly connected. This simple example shows that the choice of which expectational error to move to the set of fundamental shocks is irrelevant for identification purposes.

6. Solving and Estimating Indeterminate Models in Dynare.

This section describes the application of the proposed method for solving and estimating indeterminate LRE models in Dynare.[8] We start with an example of a code for the canonical New-Keynesian model with two expectational variables and three shocks. Then we discuss a

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[8]Dynare is a Matlab-based software platform for handling a wide class of economic models, in particular dynamic stochastic general equilibrium (DSGE). Visit www.dynare.org for details.

method for choosing which expectational errors that should be moved to the set of fundamental shocks. We propose a simple “rule of thumb” for identifying the expectational errors that lead to indeterminacy. We also provide recommendations for choosing parameters of sunspot shock priors. Finally, we provide simulation tests in Dynare, applying the “rule of thumb.”

6.1. Example: a Dynare code for the Canonical New-Keynesian Model

In this section, we show how to estimate indeterminate models in Dynare and suggest a rule of thumb for choosing which expectational errors should be fundamental shocks when solving indeterminate models. Consider the canonical New-Keynesian model, which can be presented as a system of three equations (details can be found, for instance, in King (2000) and Woodford (2003)):

(6.1) $$x_t = E_t[x_{t+1}] - \tau(R_t - E_t[\pi_{t+1}]) + g_t,$$

(6.2) $$\pi_t = \beta E_t[\pi_{t+1}] + \kappa x_t + z_t,$$

(6.3) $$R_t = \rho_R R_{t-1} + (1-\rho_R)(\psi_1 \pi_t + \psi_2 x_t) + \varepsilon_{R,t},$$

where x_t is output, π_t is inflation, and R_t is the interest rate. The first equation is a consumption-Euler equation, the second one is the New-Keynesian Phillips curve, and the third one is a monetary policy rule. This model has two expectational variables, $E_t[x_{t+1}]$ and $E_t[\pi_{t+1}]$, and three exogenous shocks. Note that, as the monetary policy rule is a function of output and the previous period's interest rate, there is no simple analytical solution for this model, as there was in the previous section. Therefore, numerical methods have to be applied. A standard result is that this model is indeterminate of degree one if the monetary policy is passive ($\psi_1 < 1$) and determinate if the monetary policy is active ($\psi_1 > 1$).

In the case of the determinate model ($\psi_1 > 1$), Dynare solves the model forward for both forward-looking variables and finds the series of expectational errors that does not allow real variables to explode. This path is unique in the determinate model. The Dynare code for this model is relatively standard. In Appendix 3, we provide a simple Dynare code that solves, simulates, saves simulated data series, and then estimates this model based on the simulated data.

In the case of the indeterminate model ($\psi_1 < 1$), running the Dynare code that was applied to the determinate model would show an error with a message "Blanchard-Kahn conditions are not satisfied: indeterminacy." As the model is indeterminate, following the proposed method, one of the two expectational errors (either for $E_t[x_{t+1}]$ or for $E_t[\pi_{t+1}]$) has to be moved to the set of fundamental shocks. There are only two ways to do that and, therefore, two indeterminate models are considered - Model 1, where the expectational error for $E_t[x_{t+1}]$ is a fundamental shock, and Model 2, where the expectational error for $E_t[\pi_{t+1}]$ is a fundamental shock.

Consider, for example, that we want to solve and estimate Model 1. Basically, in comparison with the Dynare code for the determinate model, it is necessary to modify three elements of the code to adapt it for the indeterminate model: variables, parameters, and the model structure. In Table 1, we provide a comparison of the parts of Dynare codes for determinate and two indeterminate models. Note that the variable $E_t[x_{t+1}]$ is normally written as x(+1) in Dynare, such that Dynare treats it as a conditional expectation of this variable at time *t*. Instead of supplying Dynare with x(+1), one can use a new variable *xs* for $E_t[x_{t+1}]$ and make it an endogenous variable in the list of variables. As we want to eliminate one forward-looking variable, an expectational error equation, $E_{t-1}x_t = x_t - \eta_t$, should be added to the model part of the code.

The following changes have to be applied to the determinate model's Dynare code:

1. Add a variable *xs* (for $E_t[x_{t+1}]$) to the set of endogenous variables in the code:

var x, R, pi, ***xs****;*

2. Add an expectational shock (*sunspot*) into the set of exogenous shocks:

varexo er_R, er_g, er_z, ***sunspot****;*

3. Add the variance parameter (*sigmasun*) of the expectational shock into the set of parameters:

parameters tau, kappa, rho_R, psi1, psi2, sigmag, sigmaz, sigmaR, ***sigmasun****;*

4. Instead of x(+1) for $E_t[x_{t+1}]$, use *xs* in the consumption-Euler equation:

*x=****xs****-1/invtau*(R-pi(+1))+sigmag*er_g;*

5. Add the expectational error equation, $E_{t-1}x_t = x_t - \eta_t$, to the model:

xs(-1)=x+sigmasun*sunspot;

Similar steps should be taken to solve and estimate Model 2, where the expectational error for $E_t[\pi_{t+1}]$ is a fundamental shock.

Note that, by substituting expectations of forward-looking variables x(+1) in Model 1 and pi(+1) in Model 2 with xs and pis, respectively, the number of forward-looking variables was decreased by one in each case. This transformation makes Dynare treat these models as determinate. Once this substitution is made, we have to add an equation that describes the dynamics of an expectational error, which we decided to move to the set of exogenous shocks. We add an expectational error equation ($E_{t-1}x_t = x_t - \eta_t$ in Model 1 and $E_{t-1}\pi_t = \pi_t - \eta_t$ in Model 2) in the Dynare code and add a sunspot shock in the set of exogenous shocks. Dynare treats these models as determinate with one additional exogenous (sunspot) shock.

In Appendix 3, we provide complete Dynare codes for the determinate model and two indeterminate models. The codes solve the models, simulate data series, save simulated data series, and then estimate these models based on the simulated data.

Determinate model	Indeterminate model	
	Model 1. Expectational error for $E_t[x_{t+1}]$ is a fundamental shock.	Model 2. Expectational error for $E_t[\pi_{t+1}]$ is a fundamental shock.
Variables		
var x, R, pi; varexo er_R, er_g, er_z;	var x, R, pi, **xs**; varexo er_R, er_g, er_z, **sunspot**;	var x, R, pi, **pis**; varexo er_R, er_g, er_z, **sunspot**;
Parameters		
parameters tau, kappa, rho_R, psi1, psi2, sigmag, sigmaz, sigmaR;	parameters tau, kappa, rho_R, psi1, psi2, sigmag, sigmaz, sigmaR, **sigmasun**;	parameters tau, kappa, rho_R, psi1, psi2, sigmag, sigmaz, sigmaR, **sigmasun**;
Model		

model(linear);
x=x(+1)-tau*(R-pi(+1))+sigmag*er_g;
pi=0.95*pi(+1)+kappa*x+sigmaz*er_z
;
R=rho_R*R(-1)+(1-rho_R)*
(psi1*pi+psi2*x)+sigmaR*er_R;
end;
	model(linear);
	x=**xs**-tau*(R-pi(+1))+sigmag*er_g;
	pi=0.95*pi(+1)+kappa*x+sigmaz*er_z
	;
	R=rho_R*R(-1)+(1-rho_R)*
	(psi1*pi+psi2*x)+sigmaR*er_R;
	xs(-1)=x+sigmasun*sunspot;
	end;
		model(linear);
		x=x(+1)-tau*(R-**pis**)+sigmag*er_g;
		pi=0.95***pis**+kappa*x+sigmaz*er_z
		;
		R=rho_R*R(-1)+(1-rho_R)*
		(psi1*pi+psi2*x)+sigmaR*er_R;
		pis(-1)=pi+sigmasun*sunspot;
		end;

Table 1. Comparison of Dynare codes.

6.2. Which Expectational Errors Should be Fundamental Shocks? The "Rule of Thumb."

In this section, we propose a "rule of thumb" on how to decide which expectational error(s) should be moved to the set of fundamental shocks in the model with p expectational variables and m degrees of indeterminacy. The rule is based on the idea of estimating all possible indeterminate models and then choosing the model that has the highest Bayesian posterior probability. The proposed "rule of thumb" has the following steps:

Step 1. Write a standard Dynare code that would solve a determinate model. (If you run this code, Matlab should stop with a message, "Blanchard-Kahn conditions are not satisfied: indeterminacy.")

Step 2. Determine the number of degrees of indeterminacy of the model, the difference between the number of stable roots and the number of expectational variables (a command *check* can be used for it).

Step 3. In the model with *m* degrees of indeterminacy, *m* expectational errors should become fundamental shocks and *m* expectational error equations should be added in the code. For that, write down all possible combinations of *m* (out of p total)[9] expectational errors that can be moved to the set of fundamental shocks.

Step 4. Write Dynare codes for models with all the different combinations of expectational variables from Step 3. When specifying priors, it is important to impose minimum restrictions on correlation between expectational shocks and other shocks. It is reasonable to use zero average correlations as priors and assume that correlation coefficients have a uniform distribution along the (-1,1) interval.

Step 5. Estimate all models.

Step 6. Provide Bayesian comparisons of these models (command *model_comparison*). Choose the model that has the highest posterior probability as the final model.

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[9]The number of combinations is $C(m,p)=\frac{p!}{(p-m)!m!}$. (Usually this number is relatively small. For example, if the number of degrees of indeterminacy is one and the model has two expectational errors, then there are two ways to move an expectational error to the set of fundamental shocks.)

In the next section, we apply this rule when estimating a canonical New-Keynesian model with two expectational errors. We show that this rule is robust in identifying expectational errors that lead to indeterminacy.

6.3. Applying the "Rule of Thumb."

In this section, the model (6.1)-(6.3) is simulated and estimated in Dynare under indeterminacy (see Dynare codes in Appendix 3). Prior means, standard deviations, and distributions of parameters of the model are specified (see Appendix 4). The monetary policy is assumed to be passive ($\psi_1 < 1$), making the model exhibit one degree of indeterminacy. As there are two expectational errors (for $E_t[x_{t+1}]$ and $E_t[\pi_{t+1}]$), two indeterminate models are considered - Model 1, where the expectational error for $E_t[x_{t+1}]$ is a fundamental shock, and Model 2, where the expectational error for $E_t[\pi_{t+1}]$ is a fundamental shock. First, both models were simulated and data series were generated from both models. Second, both models were estimated on both data series. Third, Bayesian comparisons were provided to evaluate which model better fits the data for each simulated data set.

The simulated and estimated results are presented in Appendix 4. Results of Bayesian comparisons of these models are presented in Table 2 and Table 3. The conclusions are not surprising. If data are generated from Model 1, the Bayesian comparison shows that Model 1 dominates Model 2 with probability of almost one (Table 3). If data are generated from Model 2, Model 2 dominates Model 1 (Table 3). Therefore, Bayesian comparisons of these models allow us to empirically identify expectational errors that lead to indeterminacy. This method gives a clear answer as to the type of model data were generated from and supports the proposed "rule of thumb" in choosing which expectational errors should be fundamental shocks.

	Model 1. Expectational error for $E_t[x_{t+1}]$ is a fundamental shock.	Model 2. Expectational error for $E_t[\pi_{t+1}]$ is a fundamental shock.
Priors	0.5	0.5
Log Marginal Density	-977.2	-982.494
Bayes Ratio	199.8	1
Posterior Model Probability	0.995	0.005

Table 2. Bayesian comparison of the indeterminate models, based on data generated from Model 1.

	Model 1. Expectational error for $E_t[x_{t+1}]$ is a fundamental shock.	Model 2. Expectational error for $E_t[\pi_{t+1}]$ is a fundamental shock.
Priors	0.5	0.5
Log Marginal Density	-848.680435	-828.059766
Bayes Ratio	0	1
Posterior Model Probability	0	1

Table 3. Bayesian comparison of the indeterminate models, based on data generated from Model 2.

7. Solving Indeterminate Models in "Gensys"

The proposed method can be applied when solving LRE models in Gensys, a Matlab code developed by Chris Sims (see Sims (2001)). This code provides a computationally robust solution method, based on the QZ matrix decomposition. The code automatically determines whether the model satisfies Blanchard-Kahn stability/uniqueness conditions and provides solutions only for determinate models. This code requires specifying how expectational errors enter the system. Basically, Gensys requires the specification of matrices $\Gamma_0, \Gamma_1, \Psi,$ and Π.

As an example, consider the standard New-Keynesian Model (5.1)-(5.3), discussed in

Section 5. Recall that in matrix notations this model can be presented as (2.1):

$$\text{(7.1)} \qquad \Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi z_t + \Pi \eta_t,$$

where $X_t = \left[x_t, \pi_t, E_t[x_{t+1}], E_t[\pi_{t+1}]\right]'$, $z_t = [\varepsilon_t], \eta_t = [\eta_{1,t}, \eta_{1,t}]'$,

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \sigma \\ 0 & 0 & 0 & \beta \end{bmatrix}, \Gamma_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \sigma\psi \\ 0 & 0 & -\kappa & 1 \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 \\ 0 \\ \sigma \\ 0 \end{bmatrix}, \text{ and } \Pi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \sigma\psi \\ -\kappa & 1 \end{bmatrix}.$$

Under determinacy, Gensys will provide a solution of this model if matrices $\Gamma_0, \Gamma_1, \Psi,$ and Π are supplied as inputs.

If the model is indeterminate, Gensys will not provide a solution, referring to its non-uniqueness. Using the proposed method, the model can be re-specified in a way, such that one of the expectational errors (either $\eta_{1,t}$ or $\eta_{2,t}$) becomes a fundamental shock. Assuming that $\eta_{1,t}$ is moved to the set of fundamental shocks, then the model takes the form:

$$\text{(7.2)} \qquad \Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi^* z_t^* + \Pi^* \eta_t^*,$$

where $z_t^* = \begin{bmatrix} \varepsilon_t \\ \eta_{1,t} \end{bmatrix}$, $\eta_t^* = \eta_{2,t}$, $\Psi^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ \sigma & 1 \\ 0 & -\kappa \end{bmatrix}$, and $\Pi^* = \begin{bmatrix} 0 \\ 1 \\ \sigma\psi \\ 1 \end{bmatrix}$.

In this case, Gensys should be supplied with the same matrices Γ_0 and Γ_1 and new matrices Ψ^* and Π^*. Under this specification, Gensys treats $\eta_{1,t}$ as the second fundamental shock and the number of expectational errors is decreased by one. Note that, as the model was indeterminate with one degree of indeterminacy, by moving $\eta_{1,t}$ into the set of fundamentals shocks, we “eliminated” one stable root, such that the system in its new specification (7.2) is determinate. In this case, Gensys will provide a solution of this model as a function of ε_t and

$\eta_{1,t}$.

Likewise, if $\eta_{2,t}$ is moved to the set of fundamental shocks, it would be necessary to re-specify matrices Ψ^* and Π^* and Gensys would provide the solution as a function of ε_t and $\eta_{2,t}$.

Conclusion

Solving and estimating indeterminate LRE models has always been an issue, as standard solution algorithms are unable deliver solution for these models. To overcome this problem, this paper provides a coherent and robust method for solving and estimating LRE models with indeterminacy. Our method implements the idea of transforming indeterminate models in such a way that they could be treated as determinate and, therefore, standard solution and estimation methods can be applied. We provide examples of the application of our method to solutions of standard New-Keynesian models, with step-by-step implementation guidelines in Dynare and Gensys.

In our method, some expectational errors have to be treated as fundamental shocks, which raises the question of which set of expectational errors should be moved to the set of fundamental shocks. We provide an evaluation of different choices and show that it does not matter for the solution which expectation errors are moved to the set of fundamental shocks. As it is hard to identify which expectational errors lead to indeterminacy, we establish an empirical method for discovering expectational errors that lead to indeterminacy in data and provide a simple rule of thumb based on a Bayesian model comparison. Simulation results support the robustness of this technique.

This paper may be of particular interest to economists, who deal with estimating and solving LRE models and do not want to restrict their models to be determinate.

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Appendix 1. Case of Indeterminacy 1

Consider the case when $\eta_{1,t}$ is moved to the set of fundamental shocks. As the system (5.8) matrix is block-triangular, we can focus on its last two equations of the system:

(A.1) $$Y_t = \Gamma^* Y_{t-1} + \Psi^* z_t + \Pi^* \eta_t,$$

where $Y_t = \begin{bmatrix} E_t[x_{t+1}] \\ E_t[\pi_{t+1}] \end{bmatrix}$, $\Gamma^* = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}$, $\Psi^* = \begin{bmatrix} \sigma \\ 0 \end{bmatrix}$, and

$$\Pi^* = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}.$$

Step 1. Using Jordan decomposition $\Gamma^* = J^{-1}\Lambda J$, we have:

(A.2) $$w_t = \Lambda w_{t-1} + J^{-1}\Psi^* z_t + J^{-1}\Pi^* \eta_t,$$

where $w_t = J^{-1}\begin{bmatrix} E_t[x_{t+1}] \\ E_t[\pi_{t+1}] \end{bmatrix}$, $J = \begin{bmatrix} 1 & 1 \\ \frac{1}{2\sigma-2\sigma\beta\psi}(\beta+\kappa\sigma-d-1) & \frac{1}{2\sigma-2\sigma\beta\psi}(\beta+\kappa\sigma+d-1) \end{bmatrix}$,

$$\Lambda = \begin{bmatrix} \frac{1}{2\beta}(\beta+\kappa\sigma+d+1) & 0 \\ 0 & \frac{1}{2\beta}(\beta+\kappa\sigma-d+1) \end{bmatrix}, \text{ and}$$

$$J^{-1} = \begin{bmatrix} \frac{1}{2d}(d+\beta+\kappa\sigma-1) & -\frac{1}{d}(\sigma-\sigma\beta\psi) \\ \frac{1}{2d}(d-\beta-\kappa\sigma+1) & \frac{1}{d}(\sigma-\sigma\beta\psi) \end{bmatrix}.$$

Note that with $\lambda_1 = \frac{1}{2\beta}(\beta+\kappa\sigma+d+1) > 1$ and $\lambda_2 = \frac{1}{2\beta}(\beta+\kappa\sigma-d+1) < 1$, where

$$d = \sqrt{\beta^2 - 2\beta + \kappa^2\sigma^2 + 2\kappa\sigma + 2\kappa\sigma\beta - 4\kappa\sigma\beta\psi + 1}.$$

Step 2. Let $[A]_{1.}$ denote the first row of the 2x2 matrix A. As $\lambda_1 > 1$, we would have to eliminate the first row in the system of equations.

(A.3) As $\left[J^{-1}\Psi^*\right]=\begin{bmatrix}\frac{1}{2d}(d+\beta+\kappa\sigma-1) & -\frac{1}{d}(\sigma-\sigma\beta\psi)\\ \frac{1}{2d}(d-\beta-\kappa\sigma+1) & \frac{1}{d}(\sigma-\sigma\beta\psi)\end{bmatrix}\begin{bmatrix}\sigma\\ 0\end{bmatrix}=\begin{bmatrix}\frac{1}{2d}\sigma(d+\beta+\kappa\sigma-1)\\ \frac{1}{2d}\sigma(d-\beta-\kappa\sigma+1)\end{bmatrix}$,

(A.4) $\left[J^{-1}\Psi^*\right]_{1.}=\frac{1}{2d}\sigma(d+\beta+\kappa\sigma-1)$.

And as $J^{-1}\Pi^*=\begin{bmatrix}\frac{1}{2d}(d+\beta+\kappa\sigma-1) & -\frac{1}{d}(\sigma-\sigma\beta\psi)\\ \frac{1}{2d}(d-\beta-\kappa\sigma+1) & \frac{1}{d}(\sigma-\sigma\beta\psi)\end{bmatrix}\begin{bmatrix}\kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta}\\ -\frac{\kappa}{\beta} & \frac{1}{\beta}\end{bmatrix}=$

$$=\begin{bmatrix}\frac{1}{2d}\left(\kappa\frac{\sigma}{\beta}+1\right)(d+\beta+\kappa\sigma-1)+\frac{1}{d}\frac{\kappa}{\beta}(\sigma-\sigma\beta\psi) & \frac{1}{2d}\left(\sigma\psi-\frac{\sigma}{\beta}\right)(d+\beta+\kappa\sigma-1)-\frac{1}{d\beta}(\sigma-\sigma\beta\psi)\\ \frac{1}{2d}\left(\kappa\frac{\sigma}{\beta}+1\right)(d-\beta-\kappa\sigma+1)-\frac{1}{d}\frac{\kappa}{\beta}(\sigma-\sigma\beta\psi) & \frac{1}{d\beta}(\sigma-\sigma\beta\psi)+\frac{1}{2d}\left(\sigma\psi-\frac{\sigma}{\beta}\right)(d-\beta-\kappa\sigma+1)\end{bmatrix}$$, then

$$\left[J^{-1}\Pi^*\right]_{1.}=\begin{bmatrix}\frac{1}{2d}\left(\kappa\frac{\sigma}{\beta}+1\right)(d+\beta+\kappa\sigma-1)+\frac{1}{d}\frac{\kappa}{\beta}(\sigma-\sigma\beta\psi)\\ \frac{1}{2d}\left(\sigma\psi-\frac{\sigma}{\beta}\right)(d+\beta+\kappa\sigma-1)-\frac{1}{d\beta}(\sigma-\sigma\beta\psi)\end{bmatrix}^T.$$

The system takes the form:

(A.5) $$w_t=\Lambda w_{t-1}+J^{-1}\Psi^*z_t+J^{-1}\Pi^*\eta_t=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}w_{t-1}+\begin{bmatrix}\frac{1}{2d}\sigma(d+\beta+\kappa\sigma-1)\\ \frac{1}{2d}\sigma(d-\beta-\kappa\sigma+1)\end{bmatrix}z_t+$$

$$+\begin{bmatrix}\frac{1}{2d}\left(\kappa\frac{\sigma}{\beta}+1\right)(d+\beta+\kappa\sigma-1)+\frac{1}{d}\frac{\kappa}{\beta}(\sigma-\sigma\beta\psi) & \frac{1}{2d}\left(\sigma\psi-\frac{\sigma}{\beta}\right)(d+\beta+\kappa\sigma-1)-\frac{1}{d\beta}(\sigma-\sigma\beta\psi)\\ \frac{1}{2d}\left(\kappa\frac{\sigma}{\beta}+1\right)(d-\beta-\kappa\sigma+1)-\frac{1}{d}\frac{\kappa}{\beta}(\sigma-\sigma\beta\psi) & \frac{1}{d\beta}(\sigma-\sigma\beta\psi)+\frac{1}{2d}\left(\sigma\psi-\frac{\sigma}{\beta}\right)(d-\beta-\kappa\sigma+1)\end{bmatrix}\eta_t$$

Step 3. As the first eigenvalue of the system is more than one, the stability condition is determined by the equation:

(A.6) $\left[J^{-1}\Psi^*\right]_{1.}z_t+\left[J^{-1}\Pi^*\right]_{1.}\eta_t=0,$

or

(A.7) $$\left(\tfrac{1}{2d}\sigma(d+\beta+\kappa\sigma-1)\right)z_t+$$
$$+\left[\tfrac{1}{2d}\left(\kappa\tfrac{\sigma}{\beta}+1\right)(d+\beta+\kappa\sigma-1)+\tfrac{1}{d}\tfrac{\kappa}{\beta}(\sigma-\sigma\beta\psi)\quad \tfrac{1}{2d}\left(\sigma\psi-\tfrac{\sigma}{\beta}\right)(d+\beta+\kappa\sigma-1)-\tfrac{1}{d\beta}(\sigma-\sigma\beta\psi)\right]$$
$$[\eta_{1,t},\eta_{2,t}]^T=0.$$

The equation that determines the dynamics of the expectational errors is:

(A.8)
$$\begin{aligned}&\left(\tfrac{1}{2d}\sigma(d+\beta+\kappa\sigma-1)\right)z_t+\\&+\left(\tfrac{1}{2d}\left(\kappa\tfrac{\sigma}{\beta}+1\right)(d+\beta+\kappa\sigma-1)+\tfrac{1}{d}\tfrac{\kappa}{\beta}(\sigma-\sigma\beta\psi)\right)\eta_{1,t}+\\&+\left(\tfrac{1}{2d}\left(\sigma\psi-\tfrac{\sigma}{\beta}\right)(d+\beta+\kappa\sigma-1)-\tfrac{1}{d\beta}(\sigma-\sigma\beta\psi)\right)\eta_{2,t}=0,\end{aligned}$$

or

(A.9)
$$a_0 z_t + a_1\eta_{1,t} + a_2\eta_{2,t} = 0,$$

where $a_0 = \left(\tfrac{1}{2d}\sigma(d+\beta+\kappa\sigma-1)\right)$,

$a_1 = \left(\tfrac{1}{2d}\left(\kappa\tfrac{\sigma}{\beta}+1\right)(d+\beta+\kappa\sigma-1)+\tfrac{1}{d}\tfrac{\kappa}{\beta}(\sigma-\sigma\beta\psi)\right)$, and

$a_2 = \left(\tfrac{1}{2d}\left(\sigma\psi-\tfrac{\sigma}{\beta}\right)(d+\beta+\kappa\sigma-1)-\tfrac{1}{d\beta}(\sigma-\sigma\beta\psi)\right)$.

Step 4. Consider $\eta_{1,t}$ to be a fundamental shock. In this case, the expectational shocks $\eta_{2,t}$ can be expressed as a function of fundamental shocks:

(A.10)
$$\eta_{2,t} = -a_2^{-1}\left[a_0 z_t + a_1\eta_{1,t}\right].$$

The system takes the form:

(A.11)
$$Y_t = \begin{bmatrix}\kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta}\\ -\frac{\kappa}{\beta} & \frac{1}{\beta}\end{bmatrix}Y_{t-1} + \begin{bmatrix}\sigma\\0\end{bmatrix}z_t + \begin{bmatrix}\kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta}\\ -\frac{\kappa}{\beta} & \frac{1}{\beta}\end{bmatrix}\begin{bmatrix}\eta_{1,t}\\ \eta_{2,t}\end{bmatrix} =$$

$$= \begin{bmatrix}\kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta}\\ -\frac{\kappa}{\beta} & \frac{1}{\beta}\end{bmatrix}Y_{t-1} + \begin{bmatrix}\sigma\\0\end{bmatrix}z_t + \begin{bmatrix}\kappa\frac{\sigma}{\beta}+1\\ -\frac{\kappa}{\beta}\end{bmatrix}\eta_{1,t} + \begin{bmatrix}\sigma\psi-\frac{\sigma}{\beta}\\ \frac{1}{\beta}\end{bmatrix}\left(-a_2^{-1}\left[a_0 z_t + a_1\eta_{1,t}\right]\right) =$$

$$= \begin{bmatrix}\kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta}\\ -\frac{\kappa}{\beta} & \frac{1}{\beta}\end{bmatrix}Y_{t-1} + \begin{bmatrix}\sigma-\left(\sigma\psi-\frac{\sigma}{\beta}\right)a_2^{-1}a_0\\ -\frac{1}{\beta}a_2^{-1}a_0\end{bmatrix}z_t + \begin{bmatrix}\kappa\frac{\sigma}{\beta}+1-\left(\sigma\psi-\frac{\sigma}{\beta}\right)a_2^{-1}a_1\\ -\frac{\kappa}{\beta}-\frac{1}{\beta}a_2^{-1}a_1\end{bmatrix}\eta_{1,t}\ ,$$

where $a_0 = \left(\tfrac{1}{2d}\sigma(d+\beta+\kappa\sigma-1)\right), a_1 = \left(\tfrac{1}{2d}\left(\kappa\tfrac{\sigma}{\beta}+1\right)(d+\beta+\kappa\sigma-1)+\tfrac{1}{d}\tfrac{\kappa}{\beta}(\sigma-\sigma\beta\psi)\right)$,
$a_2 = \left(\tfrac{1}{2d}\left(\sigma\psi-\tfrac{\sigma}{\beta}\right)(d+\beta+\kappa\sigma-1)-\tfrac{1}{d\beta}(\sigma-\sigma\beta\psi)\right)$.

Step 5. The solution for $E_t[x_{t+1}]$ and $E_t[\pi_{t+1}]$ is of the form:

$$\text{(A.12)} \qquad \begin{bmatrix} E_t[x_{t+1}] \\ E_t[\pi_{t+1}] \end{bmatrix} = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} E_{t-1}[x_t] \\ E_{t-1}[\pi_t] \end{bmatrix} + \\ + \begin{bmatrix} \sigma-\left(\sigma\psi-\frac{\sigma}{\beta}\right)\left(a_2^{-1}a_0\right) \\ -\frac{1}{\beta}a_2^{-1}a_0 \end{bmatrix} z_t + \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1-\left(\sigma\psi-\frac{\sigma}{\beta}\right)a_2^{-1}a_1 \\ -\frac{\kappa}{\beta}-\frac{1}{\beta}a_2^{-1}a_1 \end{bmatrix} \eta_{1,t}.$$

Using the definition of expectational errors:

$$\text{(A.13)} \qquad E_{t-1}[x_t] = x_t - \eta_{1,t},$$

$$\text{(A.14)} \qquad E_{t-1}[\pi_t] = \pi_t - \eta_{2,t},$$

$$\text{(A.15)} \qquad E_t[x_{t+1}] = x_{t+1} - \eta_{1,t+1},$$

$$\text{(A.16)} \qquad E_t[\pi_{t+1}] = \pi_{t+1} - \eta_{2,t+1},$$

we have:

$$\text{(A.17)} \qquad \begin{bmatrix} x_{t+1}-\eta_{1,t+1} \\ \pi_{t+1}-\eta_{2,t+1} \end{bmatrix} = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t-\eta_{1,t} \\ \pi_t-\eta_{2,t} \end{bmatrix} + \\ + \begin{bmatrix} \sigma-\left(\sigma\psi-\frac{\sigma}{\beta}\right)\left(a_2^{-1}a_0\right) \\ -\frac{1}{\beta}a_2^{-1}a_0 \end{bmatrix} z_t + \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1-\left(\sigma\psi-\frac{\sigma}{\beta}\right)a_2^{-1}a_1 \\ -\frac{\kappa}{\beta}-a_2^{-1}a_1\frac{1}{\beta} \end{bmatrix} \eta_{1,t}.$$

Moving η_{t+1} to the right hand side and separating η_t :

$$\text{(A.18)} \qquad \begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} - \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \eta_{1,t} \\ \eta_{2,t} \end{bmatrix} + \\ + \begin{bmatrix} \sigma-\left(\sigma\psi-\frac{\sigma}{\beta}\right)\left(a_2^{-1}a_0\right) \\ -\frac{1}{\beta}a_2^{-1}a_0 \end{bmatrix} z_t + \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1-\left(\sigma\psi-\frac{\sigma}{\beta}\right)a_2^{-1}a_1 \\ -\frac{\kappa}{\beta}-a_2^{-1}a_1\frac{1}{\beta} \end{bmatrix} \eta_{1,t} + \begin{bmatrix} \eta_{1,t+1} \\ \eta_{2,t+1} \end{bmatrix},$$

and re-grouping:

(A.19) $$\begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} - \begin{bmatrix} \sigma\psi-\frac{\sigma}{\beta} \\ \frac{1}{\beta} \end{bmatrix} \eta_{2,t} +$$

$$+ \begin{bmatrix} \sigma-\left(\sigma\psi-\frac{\sigma}{\beta}\right)(a_2^{-1}a_0) \\ -\frac{1}{\beta}a_2^{-1}a_0 \end{bmatrix} z_t + \begin{bmatrix} -\left(\sigma\psi-\frac{\sigma}{\beta}\right)a_2^{-1}a_1 \\ -a_2^{-1}a_1\frac{1}{\beta} \end{bmatrix} \eta_{1,t} + \begin{bmatrix} \eta_{1,t+1} \\ \eta_{2,t+1} \end{bmatrix}.$$

As $\eta_{2,t} = -a_2^{-1}\left[a_0 z_t + a_1\eta_{1,t}\right]$, coefficients for $\eta_{1,t}$ and z_t cancel out:

(A.20) $$\begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} z_t + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \eta_{1,t} + \begin{bmatrix} \eta_{1,t+1} \\ \eta_{2,t+1} \end{bmatrix}.$$

Re-writing the system one period backwards:

(A.21) $$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} z_{t-1} + \begin{bmatrix} \eta_{1,t} \\ \eta_{2,t} \end{bmatrix} =$$

$$= \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} z_{t-1} + \begin{bmatrix} \eta_{1,t} \\ -a_2^{-1}\left[a_0 z_t + a_1\eta_{1,t}\right] \end{bmatrix},$$

or:

(A.22) $$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ -a_2^{-1}a_0 \end{bmatrix} z_t + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} z_{t-1} + \begin{bmatrix} 1 \\ -a_2^{-1}a_1 \end{bmatrix} \eta_{1,t}$$

As $-a_2^{-1}a_0 = -\frac{\left(\frac{1}{2d}\sigma(d+\beta+\kappa\sigma-1)\right)}{\left(\frac{1}{2d}\left(\sigma\psi-\frac{\sigma}{\beta}\right)(d+\beta+\kappa\sigma-1)-\frac{1}{d\beta}(\sigma-\sigma\beta\psi)\right)} = -\frac{\beta}{\beta\psi-1}\frac{d+\beta+\kappa\sigma-1}{d+\beta+\kappa\sigma+1}$, the solution takes the form:

(A.23) $$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi-\frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix} +$$
$$+ \begin{bmatrix} 0 \\ -a_2^{-1}a_0 \end{bmatrix} z_t + \begin{bmatrix} 1 \\ -\frac{\beta}{\beta\psi-1}\frac{d+\beta+\kappa\sigma-1}{d+\beta+\kappa\sigma+1} \end{bmatrix} \eta_{1,t} + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} z_{t-1}.$$

Appendix 2. Case of Indeterminacy 2

Consider the case when $\eta_{2,t}$ is moved to the set of fundamental shocks. The step of the solution are similar to those in Appendix 1. From the stability condition (5.18)

(A.24) $$a_0 z_t + a_1 \eta_{1,t} + a_2 \eta_{2,t} = 0,$$

The expectational error $\eta_{1,t}$:

(A.25) $$\eta_{1,t} = -a_1^{-1}\left[a_0 z_t + a_2 \eta_{2,t}\right].$$

The model takes the form:

(A.26) $$Y_t = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} Y_{t-1} + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} z_t + \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \eta_{1,t} \\ \eta_{2,t} \end{bmatrix} =$$

$$= \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} Y_{t-1} + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} z_t + \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 \\ -\frac{\kappa}{\beta} \end{bmatrix} (-a_1^{-1}a_0 z_t - a_1^{-1}a_2\eta_{2,t}) + \begin{bmatrix} \sigma\psi - \frac{\sigma}{\beta} \\ \frac{1}{\beta} \end{bmatrix} \eta_{2,t} =$$

$$= \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} Y_{t-1} + \begin{bmatrix} \sigma - \left(\kappa\frac{\sigma}{\beta}+1\right)a_1^{-1}a_0 \\ \frac{\kappa}{\beta}a_1^{-1}a_0 \end{bmatrix} z_t + \begin{bmatrix} \sigma\psi - \frac{\sigma}{\beta} - \left(\kappa\frac{\sigma}{\beta}+1\right)a_1^{-1}a_0 \\ \frac{\kappa}{\beta}a_1^{-1}a_0 + \frac{1}{\beta} \end{bmatrix} \eta_{2,t}.$$

We found the solution for $E_t[x_{t+1}]$ and $E_t[\pi_{t+1}]$ in the form:

(A.27) $$\begin{bmatrix} E_t[x_{t+1}] \\ E_t[\pi_{t+1}] \end{bmatrix} = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} E_{t-1}[x_t] \\ E_{t-1}[\pi_t] \end{bmatrix} + \\ + \begin{bmatrix} \sigma - \left(\kappa\frac{\sigma}{\beta}+1\right)a_1^{-1}a_0 \\ \frac{\kappa}{\beta}a_1^{-1}a_0 \end{bmatrix} z_t + \begin{bmatrix} \sigma\psi - \frac{\sigma}{\beta} - \left(\kappa\frac{\sigma}{\beta}+1\right)a_1^{-1}a_0 \\ \frac{\kappa}{\beta}a_1^{-1}a_0 + \frac{1}{\beta} \end{bmatrix} \eta_{2,t}.$$

Using expressions for expectational errors:

(A.28) $$E_{t-1}[x_t] = x_t - \eta_{1,t} \ ,$$

(A.29) $$E_{t-1}[\pi_t] = \pi_t - \eta_{2,t} \ ,$$

(A.30) $$E_t[x_{t+1}] = x_{t+1} - \eta_{1,t+1} \ ,$$

(A.31) $$E_t[\pi_{t+1}] = \pi_{t+1} - \eta_{2,t+1} \ ,$$

we obtain:

(A.32) $$\begin{bmatrix} x_{t+1} - \eta_{1,t+1} \\ \pi_{t+1} - \eta_{2,t+1} \end{bmatrix} = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t - \eta_{1,t} \\ \pi_t - \eta_{2,t} \end{bmatrix} +$$
$$+\begin{bmatrix} \sigma - \left(\kappa\frac{\sigma}{\beta}+1\right)a_1^{-1}a_0 \\ \frac{\kappa}{\beta}a_1^{-1}a_0 \end{bmatrix} z_t + \begin{bmatrix} \sigma\psi - \frac{\sigma}{\beta} - \left(\kappa\frac{\sigma}{\beta}+1\right)a_1^{-1}a_0 \\ \frac{\kappa}{\beta}a_1^{-1}a_0 + \frac{1}{\beta} \end{bmatrix} \eta_{2,t} \, .$$

Moving η_{t+1} to the right hand side and separate η_t :

(A.33) $$\begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} - \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \eta_{1,t} \\ \eta_{2,t} \end{bmatrix} +$$
$$+\begin{bmatrix} \sigma - \left(\kappa\frac{\sigma}{\beta}+1\right)a_1^{-1}a_0 \\ \frac{\kappa}{\beta}a_1^{-1}a_0 \end{bmatrix} z_t + \begin{bmatrix} \sigma\psi - \frac{\sigma}{\beta} - \left(\kappa\frac{\sigma}{\beta}+1\right)a_1^{-1}a_0 \\ \frac{\kappa}{\beta}a_1^{-1}a_0 + \frac{1}{\beta} \end{bmatrix} \eta_{2,t} + \begin{bmatrix} \eta_{1,t+1} \\ \eta_{2,t+1} \end{bmatrix} ,$$

and re-grouping:

(A.34) $$\begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} - \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 \\ -\frac{\kappa}{\beta} \end{bmatrix} \eta_{1,t} +$$
$$+\begin{bmatrix} \sigma - \left(\kappa\frac{\sigma}{\beta}+1\right)a_1^{-1}a_0 \\ \frac{\kappa}{\beta}a_1^{-1}a_0 \end{bmatrix} z_t + \begin{bmatrix} -\left(\kappa\frac{\sigma}{\beta}+1\right)a_1^{-1}a_0 \\ \frac{\kappa}{\beta}a_1^{-1}a_0 \end{bmatrix} \eta_{2,t} + \begin{bmatrix} \eta_{1,t+1} \\ \eta_{2,t+1} \end{bmatrix} .$$

As $\eta_{1,t} = -a_1^{-1}a_0 z_t - a_1^{-1}a_2\eta_{2,t}$, coefficients for $\eta_{2,t}$ and z_t cancel out:

(A.35) $$\begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} z_t + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \eta_{2,t} + \begin{bmatrix} \eta_{1,t+1} \\ \eta_{2,t+1} \end{bmatrix} .$$

Re-writing for one period backwards:

$$\text{(A.36)} \qquad \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} z_{t-1} + \begin{bmatrix} \eta_{1,t} \\ \eta_{2,t} \end{bmatrix} =$$

$$= \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} z_{t-1} + \begin{bmatrix} -a_1^{-1}a_0 z_t - a_1^{-1}a_2\eta_{2,t} \\ \eta_{2,t} \end{bmatrix}.$$

The final solution is:

$$\text{(A.37)} \qquad \begin{gathered} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} \kappa\frac{\sigma}{\beta}+1 & \sigma\psi - \frac{\sigma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix} + \\ + \begin{bmatrix} -a_1^{-1}a_0 \\ 0 \end{bmatrix} z_t + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} z_{t-1} + \begin{bmatrix} -a_1^{-1}a_2 \\ 1 \end{bmatrix} \eta_{2,t}, \end{gathered}$$

where $-a_1^{-1}a_0 = -\frac{\left(\frac{1}{2d}\sigma\left(d+\beta+\kappa\sigma-1\right)\right)}{\left(\frac{1}{2d}\left(\kappa\frac{\sigma}{\beta}+1\right)\left(d+\beta+\kappa\sigma-1\right)+\frac{1}{d}\frac{\kappa}{\beta}\left(\sigma-\sigma\beta\psi\right)\right)}$.

Appendix 3. Dynare Codes

Dynare Code 1. Determinate model.

```
// The Standard New-Keynesian Model with Two Forward-looking Variables and One Fundamental Shock
// Determinate model

 // Variables
var x, R, pi;
varexo er_R,er_g,er_z;

 //Parameters
parameters tau, kappa, rho_R, psi1, psi2, sigmag, sigmaz, sigmaR;
tau=0.5;
kappa=0.5;
rho_R=0.5;
psi1=1.5;
psi2=0.25;
sigmaR=1;
sigmag=1;
sigmaz=1;

 //Model
model(linear);
x=x(+1)-tau*(R-pi(+1))+sigmag*er_g;
pi=0.95*pi(+1)+kappa*x+sigmaz*er_z;
R=rho_R*R(-1)+(1-rho_R)*(psi1*pi+psi2*x)+sigmaR*er_R;
end;

 initval;
x=0 ;
pi=0 ;
R=0 ;
er_R=0 ;
er_g=0 ;
er_z=0;
end;

 steady;

 shocks;
var er_R=1;
var er_g=1;
var er_z=1;
corr er_g,er_z =0;
```

```
end ;

 // Simulations
stoch_simul(irf=50,order=1,periods=200);
datatomfile('simudataX',[]);

 // Estimation of parameters
estimated_params;
tau,gamma_pdf,0.5,0.5 ;
kappa,gamma_pdf,0.5,0.2 ;
rho_R, beta_pdf,0.5,0.2 ;
psi1,gamma_pdf,1.5,0.1 ;
psi2,gamma_pdf,0.25,0.15 ;
sigmaR,inv_gamma_pdf,1,1 ;
sigmag,inv_gamma_pdf,1,1 ;
sigmaz,inv_gamma_pdf,1,1 ;
corr er_g, er_z, normal_pdf, 0, 0.4 ;
end;

 varobs x pi R;
estimation(datafile=simudataPI, mode_compute=6,nograph, mh_replic=100000) ;
```

Dynare Code 2. Indeterminate Model 1.

```
// The Standard New-Keynesian Model with Two Forward-looking Variables and One Fundamental Shock
// Indeterminate Model 1. X is replaced by sunspot shock


var x, R, pi, xs;
varexo er_R,er_g,er_z, sunspot;

parameters invtau, kappa,rho_R, psi1, psi2, sigmag, sigmaz, sigmaR, sigmasun;
invtau=2;
kappa=0.5;
rho_R=0.5;
psi1=0.5;
psi2=0.25;
sigmaR=1;
sigmag=1;
sigmaz=2;
sigmasun=0.5;

model(linear);
x=xs-1/invtau*(R-pi(+1))+sigmag*er_g ;
pi=0.95*pi(+1)+kappa*x+sigmaz*er_z ;
```

```
R=rho_R*R(-1)+(1-rho_R)*(psi1*pi+psi2*x)+sigmaR*er_R ;
xs(-1)=x+sigmasun*sunspot;
end;

initval;
x=0 ;
pi=0 ;
R=0 ;
er_R=0 ;
er_g=0 ;
er_z=0;
end;

steady;

shocks;
var er_R=1;
var er_g=1;
var er_z=1;
var sunspot=1;
corr er_g,er_z =0.5 ;
corr sunspot,er_g= 0;
corr sunspot,er_z= 0;
end ;

stoch_simul(irf=50,order=1,periods=200);
datatomfile('simudataX',[]);


estimated_params;
invtau,gamma_pdf,2,0.5 ;
kappa,gamma_pdf,0.5,0.2 ;
rho_R, beta_pdf,0.5,0.2 ;
psi1,gamma_pdf,0.5,0.1 ;
psi2,gamma_pdf,0.25,0.15 ;
sigmaR,inv_gamma_pdf,1,1 ;
sigmag,inv_gamma_pdf,1,1 ;
sigmaz,inv_gamma_pdf,1,1 ;
sigmasun,inv_gamma_pdf,1,1 ;
corr er_g, er_z, normal_pdf, 0, 0.4 ;
corr sunspot, er_g, uniform_pdf,,,-1,1;
corr sunspot, er_z, uniform_pdf,,,-1,1;
end;

estimated_params_bounds ;
psi1, 0, 2 ;
```

```
end ;

varobs x pi R;
estimation(datafile=simudataX, mode_compute=6,nograph, mh_replic=20000,
mh_nblocks=5) ;z= 0;
end ;

stoch_simul(irf=50,order=1,periods=200);
datatomfile('simudataX',[]);


estimated_params;
invtau,gamma_pdf,2,0.5 ;
kappa,gamma_pdf,0.5,0.2 ;
rho_R, beta_pdf,0.5,0.2 ;
psi1,gamma_pdf,0.5,0.1 ;
psi2,gamma_pdf,0.25,0.15 ;
sigmaR,inv_gamma_pdf,1,1 ;
sigmag,inv_gamma_pdf,1,1 ;
sigmaz,inv_gamma_pdf,1,1 ;
sigmasun,inv_gamma_pdf,1,1 ;
corr er_g, er_z, normal_pdf, 0, 0.4 ;
corr sunspot, er_g, uniform_pdf,,,-1,1;
corr sunspot, er_z, uniform_pdf,,,-1,1;
end;

estimated_params_bounds ;
psi1, 0, 2 ;
end ;

varobs x pi R;
estimation(datafile=simudataX, mode_compute=6,nograph, mh_replic=20000,
mh_nblocks=5) ;
corr sunspot, er_z, uniform_pdf,,,-1,1;
end;

estimated_params_bounds ;
psi1, 0, 2 ;
end ;

varobs x pi R;
estimation(datafile=simudataX, mode_compute=6,nograph, mh_replic=20000,
mh_nblocks=5) ;
```

Dynare Code 3. Indeterminate Model 2.

```
// The Standard New-Keynesian Model with Two Forward-looking Variables and One Fundamental Shock
// Indeterminate Model 1. PI is replaced by sunspot shock
var x, R, pi, pis;
varexo er_R,er_g,er_z, sunspot;

parameters invtau, kappa,rho_R, psi1, psi2, sigmag, sigmaz, sigmaR, sigmasun;
invtau=2;
kappa=0.5;
rho_R=0.5;
psi1=0.5;
psi2=0.25;
sigmaR=1;
sigmag=1;
sigmaz=2;
sigmasun=0.5;


model(linear);
x=x(+1)-1/invtau*(R-pis)+sigmag*er_g ;
pi=0.95*pis+kappa*x+sigmaz*er_z ;
R=rho_R*R(-1)+(1-rho_R)*(psi1*pi+psi2*x)+sigmaR*er_R ;
pis(-1)=pi+sigmasun*sunspot;
end;


initval;
x=0 ;
pi=0 ;
R=0 ;
er_R=0 ;
er_g=0 ;
er_z=0;
end;

 steady;

 shocks;
var er_R=1;
var er_g=1;
var er_z=1;
var sunspot=1;
corr er_g,er_z =0.5 ;
corr sunspot,er_g= 0;
corr sunspot,er_z= 0;
```

```
end ;


stoch_simul(irf=50,order=1,periods=200);
datatomfile('simudataPI',[]);


estimated_params;
invtau,gamma_pdf,2,0.5 ;
kappa,gamma_pdf,0.5,0.2 ;
rho_R, beta_pdf,0.5,0.2 ;
psi1,gamma_pdf,0.5,0.1 ;
psi2,gamma_pdf,0.25,0.15 ;
sigmaR,inv_gamma_pdf,1,1 ;
sigmag,inv_gamma_pdf,1,1 ;
sigmaz,inv_gamma_pdf,1,1 ;
sigmasun,inv_gamma_pdf,1,1 ;
corr er_g, er_z, normal_pdf, 0, 0.4 ;
corr sunspot, er_g, uniform_pdf,,,-1,1;
corr sunspot, er_z, uniform_pdf,,,-1,1;
end;


estimated_params_bounds ;
psi1, 0, 2 ;
end ;


varobs x pi R;
estimation(datafile=simudataPI, mode_compute=6,nograph, mh_replic=20000,
mh_nblocks=5) ;
```

Appendix 4. Estimation Results of Simulated Indeterminate Models

				Indeterminate Model 1. Expectational error for $E_t[x_{t+1}]$ is a fundamental shock.		Indeterminate Model 2. Expectational error for $E_t[\pi_{t+1}]$ is a fundamental shock.	
Parameters	Distri-bution	Prior mean	StD	Posterior mean	Confidence interval	Posterior mean	Confidence interval
τ	gamma	0.5	0.5	0.48	[0.28;0.67]	0.34	[0.22;0.44]
κ	gamma	0.5	0.2	0.50	[0.41;0.58]	0.45	[0.38;0.52]
ρ	beta	0.5	0.2	0.52	[0.44;0.60]	0.37	[0.31;0.44]
ψ_1	gamma	0.5	0.1	0.47	[0.40;0.55]	0.52	[0.45;0.58]
ψ_2	gamma	0.25	0.15	0.15	[0.03;0.27]	0.17	[0.03;0.30]
σ_R	invg	1	1	1.10	[0.99;1.19]	1.06	[0.97;1.15]
σ_g	invg	1	1	0.81	[0.49;1.13]	0.73	[0.39;1.02]
σ_z	invg	1	1	0.96	[0.72;1.18]	1.06	[0.69;1.35]
$\sigma_{sunspot}$	invg	1	1	1.00	[0.89;1.12]	2.34	[2.11;2.64]
$\rho_{g,z}$	norm	0	0.4	-0.23	[-0.70;0.24]	-0.15	[-0.79;0.39]
$\rho_{sunspot,g}$	unif	0	0.5774	0.29	[-0.02;0.61]	0.58	[0.29;0.86]
$\rho_{sunspot,z}$	unif	0	0.5774	0.21	[-0.03;0.44]	-0.36	[-0.58;-0.14]
Log data density				-977.020573		-981.963073	

Table 4. Priors and posterior estimation results, based on data generated from Model 1.

				Indeterminate Model 1. Expectational error for $E_t[x_{t+1}]$ is a fundamental shock.		Indeterminate Model 2. Expectational error for $E_t[\pi_{t+1}]$ is a fundamental shock.	
Parameters	Distri-bution	Prior mean	StD	Posterior mean	Confidence interval	Posterior mean	Confidence interval
τ	gamma	0.5	0.5	0.23	[0.16;0.30]	0.40	[0.29;0.51]
κ	gamma	0.5	0.2	0.41	[0.33;0.49]	0.49	[0.40;0.58]
ρ	beta	0.5	0.2	0.63	[0.54;0.71]	0.57	[0.48;0.66]
ψ_1	gamma	0.5	0.1	0.34	[0.25;0.42]	0.45	[0.35;0.55]
ψ_2	gamma	0.25	0.15	0.09	[0.01;0.16]	0.16	[0.02;0.29]
σ_R	invg	1	1	1.04	[0.96;1.13]	1.05	[0.96;1.14]
σ_g	invg	1	1	0.65	[0.49;0.82]	0.67	[0.39;0.90]
σ_z	invg	1	1	0.63	[0.44;0.83]	0.73	[0.47;0.94]
$\sigma_{sunspot}$	invg	1	1	1.10	[0.99;1.20]	1.13	[0.95;1.35]
$\rho_{g,z}$	norm	0	0.4	-0.17	[-0.64;0.32]	-0.05	[-0.64;0.49]
$\rho_{sunspot,g}$	unif	0	0.5774	-0.68	[-0.89;-0.48]	0.13	[-0.19;0.44]
$\rho_{sunspot,z}$	unif	0	0.5774	0.42	[0.19;0.67]	0.25	[0.01;0.48]
Log data density				-849.27		-828.08	

Table 5. Priors and posterior estimation results, based on data generated from Model 2.

Chapter 2. Assessing DSGE Models with Indeterminacy and Capital Accumulation: What Really Happened During the Great Moderation.

Introduction

This paper assesses New-Keynesian Dynamic Stochastic General Equilibrium (DSGE) models with capital accumulation. In this framework, when the monetary authority increases the interest rate, the cost of renting capital likewise increases, leading to cost-push inflation. Bayesian empirical estimates of these models on U.S. data from 1960 to 2008 enable a reconsideration of the sources of the Great Moderation, a reduction in the volatility of business cycle fluctuations starting in the mid-1980s.

Canonical papers consider that there was a change in U.S. the monetary policy rule from passive (when the nominal interest rate increases less than one-to-one with inflation) during the pre-Volcker period to active (when the nominal interest rate increases more than one-to-one with inflation) during the post-1982 period (see, namely, Clarida, Gali, and Gertler (2000) and Lubik and Schorfheide (2004)). This is then pointed to as the main source of the Great Moderation. In contrast to these papers, it was found that, when capital is taken into account, a model with passive monetary policy and indeterminacy provides a better fit to the data before (the pre-Volcker period, 1960-1979) and after (the post-1982 period, 1982-2008) the Great Moderation. In fact, according to the empirical estimates of this paper, little evidence was found in favor of a change in the monetary policy rule between these subperiods—monetary policy remained passive in response to inflation and its response to output changed only slightly.

Structural changes between the two subperiods were found to be primarily on the demand side of the economy. Driven by private market developments and by adjustments in government policy, changes in financial markets enhanced the ability of households to borrow funds and thereby to smooth their spending in the face of swings in income and interest rates. According to the empirical estimates of this paper, the main change was in households' inverse elasticity of intertemporal substitution of consumption, which almost tripled between these subperiods, such that the dynamics of consumption became much less sensitive to interest rates. These findings support the idea of financial innovations, proposed by Dynan, Elmendorf, and Sichel (2006a), as an alternate source of the Great Moderation.

Nevertheless, canonical results indicating a change in the U.S. monetary policy rule from passive during the pre-Volcker period to active during the post-1982 remain popular. However, modeling the transition from passive to active monetary policy is very challenging, as standard New-Keynesian DSGE models exhibit indeterminacy under passive monetary policy. The standard result is that determinacy arises mainly under active monetary policy rules, while passive monetary policy leads to indeterminacy (see, for example, Kerr and King (1996), Rotemberg and Woodford (1998), and Christiano and Gust (1999)). There is a limited set of econometric estimation methods that can be applied if indeterminacy exists and, therefore, the majority of papers *ex-ante* limited themselves to active monetary policy and determinate equilibria models (for example, Smets and Wouters (2003) and Smets and Wouters (2007)). The possibility of indeterminacy was usually ruled out and the Bayesian estimation approach to DSGE models was rarely applied.

Farmer and Guo (1995) were the first to demonstrate that a general equilibrium model with an indeterminate steady state does a good job of accounting for the propagation mechanism in U.S. data. Lubik and Schorfheide (2004) then estimated a three-equation New-Keynesian model with for indeterminacy. Using the Bayesian approach, they estimated the model on U.S. data and found that U.S. monetary policy post-1982 is consistent with determinacy, whereas the pre-Volcker policy is not.

Later research attempts to determine whether there were switches in monetary policy or structural changes in the fundamental parameters of the economy during the great Moderation. Sims and Zha (2006) use a vector autoregression (VAR) approach to estimate a multivariate regime-switching model for U.S. monetary policy. They find that the main changes were in the monetary policy rules. The main drawback of the VAR approach is that, due to rational expectations, agents can anticipate changes in parameters of the economy, leading to inconsistent estimates. Likewise, there are identification issues with the estimation of forward-looking Markov-switching rational expectations models. Beyer and Farmer (2007) argue that it is not always possible to decide whether the data are generated from determinate or indeterminate models. Farmer, Waggoner, and Zha (2008) provide a set of necessary and sufficient conditions for determinacy in a class of forward looking Markov-switching rational expectations models. One method to overcome this problem is to estimate a fully-specified DSGE model that can be re-solved for alternative policy rules. Smets and Wouters (2003) and Smets and Wouters (2007) find most of the structural parameters remained the same before and after the Great Moderation. The biggest difference concerns the variances of the structural shocks. Lubik and Schorfheide (2004) exogenously split data into two sets and show that U.S. monetary policy during the post-1982 period was consistent with determinacy, whereas during the pre-Volcker policy it was consistent with indeterminacy. Schorfheide (2005) estimates a basic New-Keynesian monetary

DSGE model, in which monetary policy follows a regime switching process, and confirms the switch in monetary policy between the pre-Volcker and post-Volker periods. Mavroeidis's (2010) model shows that policy before Volcker led to indeterminacy, however, the model is not accurately identifiable using data after 1979.

To understand the sources of the Great Moderation, this paper extends the canonical New-Keynesian model by adding a no-arbitrage condition between the real return on bonds and the real return on capital and a capital accumulation equation in the standard model. While most of the studies focus on the implications of how the rate of interest affects consumption-savings decisions, the channel by which the rate of interest affects investment decisions has also begun to draw attention recently. The no-arbitrage condition between the real return on bonds and the real return on capital implies that the capital rental rate increases when monetary policy responds to higher inflation by increasing the interest rate. This response increases the cost of renting capital, leading to cost-push inflation. This paper is an attempt to explore the role of this channel in estimating models. In order to clearly understand how capital accumulation influences the results, we expand the canonical model with indeterminacy used in Lubik and Schorfheide (2004) by only one dimension – by adding capital in it.[10] Dupor (2001), Carlstrom and Fuerst (2005), Kurozumi and Zandweghe (2008), Kurozumi (2006), Huang and Meng (2007), and Xiao (2008) study properties of models with capital accumulation and show that this crucial feature changes the stability structure and dynamics of the models, making indeterminacy likely. By estimating the model with capital accumulation and potential for indeterminacy, this paper shows that capital accumulation activity has a strong influence on the model dynamics. Investment

[10] Models with many real and nominal frictions, such as in Christiano, Eichenbaum and Evans (2005), Smets and Wouters (2003), and Smets and Wouters (2007), can be considered to be extensions of the baseline model and, therefore, are not used in this paper to better highlight the role of investment, rather than other frictions, in this framework.

activity changes the monetary transmission mechanisms and allows for the reconsideration and re-estimation of monetary policy.

First, different versions of New-Keynesian models with capital accumulation are simulated, and their dynamic properties are discussed. Though baseline New-Keynesian models have become very popular in the analysis of monetary policy, many authors show that these models are unable to generate enough persistence in inflation and output. Fuhrer and Moore (1995) show that a sticky wage model can generate persistence in the price level but not in the inflation rate. Chari, Kehoe and McGrattan (2000) point out that models with nominal rigidities do not generate enough persistence in output following a monetary shock. The simulated results of this paper show that models with capital accumulation can generate substantial persistence among the major economic variables, as the stock nature of capital adds persistence to the dynamics of all other variables in the models.

Second, models with capital accumulation and indeterminacy are estimated on U.S. data from 1960:I to 2008:I. Using state-space decomposition and the Kalman filter, the overall likelihood of the model is maximized taking into account prior distributions of the parameters, and inferences are made with a likelihood-based approach by adopting the Metropolis-Hastings techniques. The estimated models are compared using the Bayesian approach. While some explanations of the results are consistent with the recent findings of Mavroeidis (2010), the empirical estimates of this paper differ from the results of Lubik and Schorfheide (2004) and Clarida, Gali and Gertler (2000). It is shown that during the Great Moderation there was almost no change in monetary policy rules and it remained passive. Bayesian comparison of the models declared that models with indeterminacy and passive monetary policy dominate determinate models for various periods of U.S. history. Major structural changes were mainly related to consumer behavior. Consumption became more smoothed and a response of consumption to interest rates decreased, supporting the idea of "financial innovations." Although, as in Lubik and Schorfheide (2004), it was found that steady state real interest rates increased and steady state inflation declined, this paper finds very different explanation for this change. Instead of a change in monetary policy this paper shows that higher real returns were related to changes in consumers risk aversion and higher share of capital income in output, which is consistent with empirical evidence.[11]

The paper is structured as follows. In Section 1, a New-Keynesian DSGE model with capital accumulation and different monetary policy rules is derived. In Section 2, different versions of this model are simulated and their dynamic properties are analyzed. In Section 3, the model is fitted to quarterly U.S. data on output, inflation, nominal interest rates, consumption, and capital from 1960:I to 2008:I and the estimation methodology and prior distributions of the parameters are discussed. The empirical results and model comparisons are presented in Section 4. The last section contains concluding remarks.

1. Model

Following Yun (1998), Carlstrom and Fuerst (2005), and Kurozumi and Zandweghe (2008), a New-Keynesian DSGE model with sticky prices and capital accumulation in discrete time is constructed. The economy consists of a large number of households, monopolistically competitive firms, and a monetary authority that changes the nominal interest rate in response to inflation and output.

[11] See, for example, Bental and Demougin (2010) and Gomme and Rupert (2004).

1.1. Households

Households seek to maximize their expected life-time utility function:

$$(1) \qquad E_t \sum_{t=0}^{\infty} \beta^t U(C_t, \frac{M_{t+1}}{P_t},\ 1-L_t),$$

where E_t is the conditional expectations operator on the information set available at date *t*, β is the discount factor, C_t is consumption, M_{t+1} is nominal money holdings and the beginning of the period *(t+1)*, $\frac{M_{t+1}}{P_t}$ is real money balances,[12] and $(1-L_t)$ is leisure.

The utility function is separable in leisure and takes the following functional form:

$$(2) \qquad U(C_t, \frac{M_{t+1}}{P_t}, 1-L_t) = \frac{C_t^{1-\sigma}}{1-\sigma} + \theta \ln \frac{M_{t+1}}{P_t} - \phi L_t .$$

At the beginning of each period *t*, a household has M_t cash balances and B_{t-1} nominal bonds. A household starts period *t* by trading bonds and receiving a lump-sum monetary transfer T_t from the government. A household receives interest payments on bonds B_{t-1} with gross interest rate R_{t-1} and spends money on new bonds B_t. A household also receives real factor payments from the labor market $w_t L_t$ and capital market $[r_t + (1-\delta)]K_t$, receives firm's profits Π_t, and spends money on next period capital K_{t+1} and current consumption C_t at current prices P_t. Each household chooses C_t, M_{t+1}, and L_t to maximize (1) subject to the sequence of intertemporal budget constraints:

$$(3) \qquad M_{t+1} + B_t + P_t C_t + P_t K_{t+1} = M_t + T_t + B_{t-1} R_{t-1} + P_t\{w_t L_t + [r_t + (1-\delta)]K_t\} + \Pi_t .$$

The first order conditions for the household's maximization problem are the following:

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[12] As in recent papers end-of-period money holdings are introduced to be consistent with the Dupor continuous-time analysis (for discussion, see Carlstrom and Fuerst, 2005).

(4) $$\frac{U_C}{U_L} = -\frac{1}{w},$$

(5) $$U_C(t) = \beta E_t \{U_C(t+1)[r_{t+1} + (1-\delta)]\},$$

(6) $$\frac{U_C(t)}{P_t} = \beta R_t E_t\left(\frac{U_C(t+1)}{P_{t+1}}\right), \text{ and}$$

(7) $$\frac{U_m(t)}{U_C(t)} = \frac{R_t - 1}{R_t}.$$

Equation (4) is a standard consumption-labor condition. Equation (5) is the Euler equation of consumption dynamics. Equation (6) is the Fisher equation that connects inflation and the interest rate. Equation (7) is a money demand equation.

1.2. Firms

Firms are monopolistic competitors in the intermediate good market. The final output Y_t is produced from intermediate goods $y_t(i)$ with Dixit-Stiglitz (1977) technology:

(8) $$Y_t = \{\int_0^1 [y_t(i)^{\frac{\eta-1}{\eta}}]di\}^{\frac{\eta}{\eta-1}}.$$

The corresponding demand for an intermediate good possesses constant price elasticity η:

(9) $$y_t(i) = Y_t\left(\frac{P_t(i)}{P_t}\right)^{-\eta},$$

where $P_t(i)$ is the price of the intermediate good and P_t is the price of the final good.

The production function of each firm exhibits constant returns to scale:

(10) $$f(K, L) = K^{\alpha}L^{1-\alpha}.$$

The first order conditions for the cost minimization problem are the following:

(11) $$r_t = z_t f_K(K_t, L_t),$$

(12) $$w_t = z_t f_L(K_t, L_t).$$

where z_t is the marginal cost of production (see Appendix 1 for details).

With the Cobb-Douglas production function (10) the first order conditions take the form:

(13) $$r_t = \alpha z_t Y_t / K_t,$$

(14) $$w_t = (1-\alpha) z_t (Y_t / K_t)^{\frac{-\alpha}{(1-\alpha)}}.$$

The Calvo (1983) staggered pricing model is used, assuming that each period a fraction $(1-\nu)$ of firms gets a signal to set a new price. Therefore, each firm maximizes the sum of discounted profits taking into account the probability of changing its price. Firms choose $P_t(i)$ to maximize expected profits:

(15) $$E_t \sum_{j=t}^{\infty} \left(\frac{\nu}{\prod_{i=0}^{j} R_i} \right)^j \left[\left(\frac{P_t\ (i)}{P_t} \right)^{-\eta} Y_t \left(\frac{P_t\ (i)}{P_t} - z_t \right) \right].$$

The profit maximization conditions give a log-linearized New-Keynesian Phillips Curve[13] of the form:

(16) $$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \lambda \hat{z}_t,$$

where $\hat{\pi}_t$ is inflation and $\lambda = \frac{(1-\upsilon)(1-\beta\upsilon)}{\upsilon}$ is the real marginal cost elasticity of inflation.

[13] See Gali and Gertler (1998) and Clarida, Gali and Gertler (1999) for details.

1.3. Monetary policy rule

Monetary policy reacts to inflation and output with interest rate smoothing:

(17) $$R_t = (R_{t-1})^{\rho_R} \left[R \left(\frac{\pi_t}{\pi} \right)^{\psi_\pi} \left(\frac{Y_t}{Y} \right)^{\psi_Y} \right]^{(1-\rho_R)},$$

where R, π, Y are the steady-state values of the interest rate, inflation, and output, respectively. Parameters ψ_π and ψ_Y are the elasticities of the interest rate by inflation and output, respectively. Interest rate smoothing is introduced with the autocorrelation coefficient ρ_R. In this framework, the monetary policy is active if the nominal interest rate increases more than one-to-one with inflation ($\psi_\pi > 1$), otherwise, it is passive ($\psi_\pi < 1$). Also, a model with $\psi_Y = 0$ would give a standard model with capital in discrete time as in Carlstrom and Fuerst (2005) — an analog of the Dupor (2001) continuous time model.

1.4. Dynamics of the model

The dynamics of the model are represented by a system of first order conditions log-linearized around the steady state for households and firms (18-23), the monetary policy rule (24), shocks of preferences and marginal cost (25-26) (see Appendix 2 for details):

(18) $$\hat{R}_t - E_t \hat{\pi}_{t+1} = \sigma(E_t \hat{C}_{t+1} - \hat{C}_t - \varepsilon_{g,t}),$$

(19) $$\hat{R}_t - E_t \hat{\pi}_{t+1} = [1 - \beta(1-\delta)](E_t \hat{z}_{t+1} + E_t \hat{Y}_{t+1} - \hat{K}_{t+1}),$$

(20) $$\sigma \hat{C}_t = \hat{z}_t + \frac{\alpha}{1-\alpha}(\hat{K}_t - \hat{Y}_t),$$

(21) $$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta \hat{I}_t,$$

(22) $$\hat{Y}_t = s_C \hat{C}_t + s_I \hat{I}_t,$$

(23) $$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \lambda \hat{z}_t + \varepsilon_{z,t} ,$$

(24) $$\hat{R}_t = \rho_R \hat{R}_{t-1} + (1-\rho_R)(\psi_\pi \hat{\pi}_t + \psi_Y \hat{Y}_t) + \varepsilon_{R,t} ,$$

(25) $$\varepsilon_{g,t} = \rho_g \varepsilon_{g,t-1} + \upsilon_{g,t} , \quad \upsilon_{g,t} \text{ is } iid\ (0,\ \sigma_g^2),$$

(26) $$\varepsilon_{z,t} = \rho_z \varepsilon_{z,t-1} + \upsilon_{z,t} , \quad \upsilon_{z,t} \text{ is } iid\ (0,\ \sigma_z^2),$$

Equation (18) is the Euler equation for the household's dynamic optimization problem with a preference shock $\varepsilon_{g,t}$, which follows an AR(1) process with an autocorrelation coefficient of ρ_g (Equation (25)). Equation (19) is the Fisher relation between the nominal interest rate, expected future inflation, and real interest rate, where the latter is determined in the production sector. Equation (20) is the wage-equilibrium relation of the log-linearized equations (4) and (14). Equation (21) is the capital accumulation relation with a depreciation rate δ. Equation (22) is the division of the steady-state output between consumption and investment with shares s_C and s_I, respectively. Equation (23) is a New-Keynesian Phillips curve derived from the Calvo staggered-pricing model with a marginal cost shock $\varepsilon_{z,t}$, which follows an AR(1) process with an autocorrelation coefficient of ρ_z (Equation (26)). Equation (24) is the log-linearized monetary policy rule (17) with an interest rate shock $\varepsilon_{R,t}$. As the money supply is endogenous and the Ricardian equivalence holds in this model, the hidden government budget constraint and the equation for the evolution of government debt are implicitly satisfied.

Straightforward re-arrangements of the variables $\hat{z}_t$ and $\hat{I}_t$ in the model give a system of variables $\hat{C}_t, \hat{R}_t, \hat{\pi}_t, \hat{Y}_t, \hat{K}_t, \varepsilon_{g,t}$, and $\varepsilon_{z,t}$:

(27) $$\hat{C}_t = E_t \hat{C}_{t+1} - \frac{1}{\sigma}\left[\hat{R}_t - E_t \hat{\pi}_{t+1}\right] + \varepsilon_{g,t} ,$$

(28) $$\hat{R}_t - E_t \hat{\pi}_{t+1} = [1-\beta(1-\delta)](\sigma E_t(\hat{C}_{t+1}) + \frac{1}{1-\alpha}\left[E_t \hat{Y}_{t+1} - E_t \hat{K}_{t+1}\right]) ,$$

(29) $$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \frac{\delta}{s_I}(\hat{Y}_t - (1-s_I)\hat{C}_t)\ ,$$

(30) $$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \lambda(\sigma\hat{C}_t - \frac{\alpha}{1-\alpha}(\hat{K}_t - \hat{Y}_t)) + \varepsilon_{z,t}\ ,$$

(31) $$\hat{R}_t = \rho_R \hat{R}_{t-1} + (1-\rho_R)(\psi_\pi \hat{\pi}_{t+k} + \psi_Y \hat{Y}_t) + \varepsilon_{R,t}, \quad \varepsilon_{R,t} \text{ is } iid\ (0,\ \sigma_R^2),$$

(32) $$\varepsilon_{g,t} = \rho_g \varepsilon_{g,t-1} + \upsilon_{g,t}, \quad \upsilon_{g,t} \text{ is } iid\ (0,\ \sigma_g^2),$$

(33) $$\varepsilon_{z,t} = \rho_z \varepsilon_{z,t-1} + \upsilon_{z,t}, \quad \upsilon_{z,t} \text{ is } iid\ (0,\ \sigma_z^2).$$

To compare, the canonical New-Keynesian model can be presented by a system of three equations: the IS equation, the Phillips curve, and a monetary policy rule, similar to equations (27), (30), and (31), respectively. In that way, the interest rate affects output only through the consumption-savings decision of the household and not through the production sector.

Adding capital accumulation to the model real makes interest rate connected to the marginal product of capital, this is contained in the Fisher equation (28). Also, output does not equal consumption in the absence of capital, as in Lubik and Schorfheide (2004), but is split between consumption and investment in this model. This is incorporated in the New-Keynesian Phillips curve equation (30) through the output equation (22). By including investment, this model has the capital accumulation equation (29), which influences interest rates through the equations (26) and (28).

2. Model simulations

The model (27)-(33) exhibits different types of dynamics depending on its parameter values (see Carlstrom and Fuerst (2005), Sosunov and Khramov (2008), Kurozumi and Zandweghe (2008), Kurozumi (2006), Huang and Meng (2007), and Xiao (2008)). Under a wide set of parameters, the model is determinate if the monetary authority implements an active monetary policy ($\psi_{\pi} > 1$), and the model is indeterminate if the monetary policy is passive ($\psi_{\pi} < 1$). Therefore, two major versions of the model (27)-(33), with active and passive monetary policies, are simulated.[14] In the baseline calibration most of the parameter values are the same as prior means used by Lubik and Schorfheide (2004); the rest of the parameters are calibrated according to stylized facts (Table 6). The Matlab-based computer package *Dynare* was used to calculate theoretical moments for the endogenous variables of the model. The simulation results of the two versions of the model with active and passive monetary policy rules and corresponding moments of consumption, interest rate, inflation, output, and capital are presented in Tables 1-4.

The version of the model with passive monetary policy demonstrates substantially higher volatility of interest rate and inflation compared to the version with active monetary policy (Table 1). This can be explained by the existence of indeterminate equilibria in this model. As long as the monetary policy authority is unable to respond sufficiently to changes in inflation by raising interest rates substantially, the volatility of inflation and, therefore, nominal interest rates, is higher. Both models reproduce a similar volatility of capital to that of U.S. data with lower volatilities of consumption, interest rate, and inflation (Table 5).

The crucial differences between the two versions of the model arise from the variance decomposition of shocks, correlation matrices of endogenous variables, and impulse response functions (IRFs). First, the preferences (demand) shock is the main drivers of volatility in the version of the model with passive monetary policy, explaining more than 90 percent of the volatility of endogenous variables (Table 2). In contrast, the marginal cost (supply) shock

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[14] Dynamics of the versions of the model with forward-looking monetary policy rules are similar to versions with current-looking monetary policy rules and, therefore, is not discussed in this paper.

explains more than 50 percent of variance in the model with active monetary policy. These results are similar to the findings of Smets and Wouters (2007), who show that "demand" shocks can explain a substantial share of the variance in output in the general version of a New-Keynesian model.

The two versions of the model demonstrate different correlations among the major variables (Table 3). In the version of the model with active monetary policy, the nominal interest rate plays the role of the active monetary policy instrument and is negatively correlated with output and capital. These theoretical results are consistent with the stabilizing role of a nominal interest rate in the economy. In contrast, in the version of the model with passive monetary policy, the nominal interest rate is positively correlated with output, capital, and consumption. As passive monetary policy is unable to respond sufficiently to shocks, indeterminacy and additional shock propagation. A response of the monetary authority to supply and demand shocks leads to co-movements in the dynamics of the interest rate and real variables as changes in the nominal interest rate are not enough to diminish the effect of shocks and reverse the dynamics of the economy. Therefore, the version of the model with passive monetary policy demonstrates substantially higher volatility among the economic variables, which is consistent with U.S. data for the pre-Volcker period (1960:I to 1979:II). Theoretical IRFs support this intuition (Appendixes 3-4).

As stated in the Introduction, while the baseline "New-Keynesian" models became very popular in the analysis of monetary policy, many papers show that these models are unable to generate enough persistence in inflation and output (see Chari, Kehoe, and McGrattan (2000), Fuhrer and Moore (1995), and Fernandez-Villaverde and Rubio-Ramirez (2004)). First-order autocorrelation coefficients for consumption, interest rate, output, and capital are more than 0.85

in the U.S. data (Appendix 6). Most of the New-Keynesian models fail to replicate even half of these correlation levels (see Rubio-Ramirez and Rabanal (2005)). In contrast, the simulated results of this paper show that models with capital accumulation can, in fact, generate substantial persistence (Table 4). The autocorrelation coefficients for consumption and capital are more than 0.9 in both active and passive monetary policy rule versions of the model. The autocorrelations of nominal variables, such as inflation and the nominal interest rate, are also very high due to the stock nature of capital, which adds persistence to the dynamics of all other variables. The autocorrelation coefficients for consumption and capital are very high, representing substantial consumption smoothing and slow adjustment of capital stock. In the version with passive monetary policy, the interest rate and output autocorrelation coefficients are higher than in the version with active monetary policy, again, due to the fact that passive monetary policy is unable to adjust the interest rate sufficiently to control shock propagation.

	Version with passive monetary policy.		Version with active monetary policy.	
Variable	Mean	St. dev.	Mean	St. dev.
Consumption	0	0.36	0	0.30
Interest rate	0	0.40	0	0.16
Inflation	0	0.50	0	0.30
Output	0	1	0	1
Capital	0	0.46	0	0.53

Table 1. Simulation results of the model. Theoretical moments.

Note: all variables are in log deviations from their steady-state values.

	Version with passive monetary policy.				Version with active monetary policy.		
	Interest rate shock	Preference (demand) shock	Marginal cost (supply) shock	Sunspot shock	Interest rate shock	Preference (demand) shock	Marginal cost (supply) shock
Consumption	0.33	92.75	6.35	0.57	0.21	44.95	54.84
Interest rate	0.72	95.31	3.3	0.67	0	34.74	65.25
Inflation	0.62	96.61	2.03	0.74	0.46	32.89	66.65
Output	0.18	95.5	3.65	0.67	1.49	31.25	67.26
Capital	0.31	93.14	6.01	0.54	0.23	40.38	59.4

Table 2. Simulation results of the model. Variance decomposition (in percent).

Note: all variables are in log deviations from their steady-state values.

Version with passive monetary policy					
Variables	Consumption	Interest rate	Inflation	Output	Capital
Consumption	1	0.6	0.54	0.30	0.98
Interest rate	0.6	1	0.98	0.81	0.73
Inflation	0.54	0.98	1	0.89	0.69
Output	0.30	0.81	0.89	1	0.45
Capital	0.98	0.73	0.69	0.45	1
Version with active monetary policy					
Variables	Consumption	Interest rate	Inflation	Output	Capital
Consumption	1	-0.09	0.01	0.04	0.81
Interest rate	-0.09	1	0.98	-0.96	-0.46
Inflation	0.01	0.98	1	-0.92	-0.35
Output	0.04	-0.96	-0.92	1	0.39
Capital	0.81	-0.46	-0.35	0.39	1

Table 3. Simulation results of the model. Matrix of correlations.

Note: all variables are in log deviations from their steady states.

Version with passive monetary policy					
Order	1	2	3	4	5
Consumption	0.995	0.983	0.965	0.943	0.918
Interest rate	0.885	0.782	0.690	0.609	0.537
Inflation	0.779	0.692	0.614	0.544	0.482
Output	0.384	0.330	0.286	0.248	0.216
Capital	0.991	0.975	0.956	0.933	0.908
Version with active monetary policy					
Order	1	2	3	4	5
Consumption	0.896	0.812	0.745	0.689	0.641
Interest rate	0.723	0.527	0.389	0.292	0.223
Inflation	0.603	0.423	0.297	0.208	0.147
Output	0.799	0.549	0.375	0.253	0.168
Capital	0.992	0.972	0.943	0.910	0.874

Table 4. Simulation results of the model. Coefficients of autocorrelation.

3. Empirical Approach

3.1. Data

The system of equations (27)-(33) is fitted to quarterly postwar U.S. data on output, inflation, nominal interest rates, consumption and capital from 1960:I to 2008:I. Output is a log of real per capita GDP (GDPQ), inflation is the annualized percentage change of CPI-U (PUNEW), and the Federal Funds Rate (FYFF) in percent is used as the nominal interest rate. Real Personal Consumption Expenditures (PCECC96) is used for consumption from the St. Louis Fed database. The time series for capital is constructed using Real Gross Private Domestic Investment (GPDIC96) starting from 1947, taking the initial amount of capital consistent with the steady state level of capital and iterating it forward with a depreciation rate of 2 percent.

To make our empirical analysis comparable to canonical studies, the Hodrick-Prescott (HP) filter is used to remove trends from the consumption, output, and capital series (see the sample moments in Table 5, Appendices 5-6, and Figure 1). Unfortunately, the Hodrick-Prescott filter has substantial limitations, as it eliminates low-frequency movements that can be important and might not properly identify the cyclical versus trend components. Furthermore, while it would appear that the HP filtered series can capture essential features of business cycle fluctuations, filtering can impart some substantively different properties to series. Therefore, the main reason the HP filter is used in this paper is to make the analysis comparable to the canonical papers in this field, in particular, to Clarida, Gali and Gertler (2000) and Lubik and Schorfheide (2004).

Consistent with earlier papers, the data sample 1960:I to 2008:I can be analyzed according to the following sub-samples:[15]

1. the pre-Volcker period (1960:I to 1979:II), usually considered to be a period of passive monetary policy;
2. the post-1982 period (1982:IV to 2008:I), usually considered to be a period of active monetary policy.

		Consumption *	Interest rate	Inflation	Output*	Capital*
Full data sample (1960:I to 2008:I)						
	Mean	-0.004	6.056	4.142	-0.001	-0.014
	Std	1.238	3.259	3.171	1.526	0.742
	StD/Output	0.811	2.136	2.078	1.000	0.486
Pre-Volcker period (1960:I to 1979:II)						
	Mean	0.030	5.473	4.646	-0.012	-0.106
	Std	1.435	2.425	3.359	1.759	0.753
	StD/Output	0.816	1.379	1.910	1.000	0.428
Post-1982 period (1982:IV to 2008:I)						
	Mean	0.083	5.478	3.081	0.024	-0.051
	Std	0.911	2.458	1.849	1.219	0.704
	StD/Output	0.747	2.016	1.517	1.000	0.578

Table 5. Sample moments for quarterly postwar U.S. data on output, inflation, nominal interest rates, consumption, and capital.

* In log deviations from the Hodrick-Prescott filtered trend.

[15] The Volcker disinflation period (1978:III to 1997:IV) is commonly excluded from estimates.

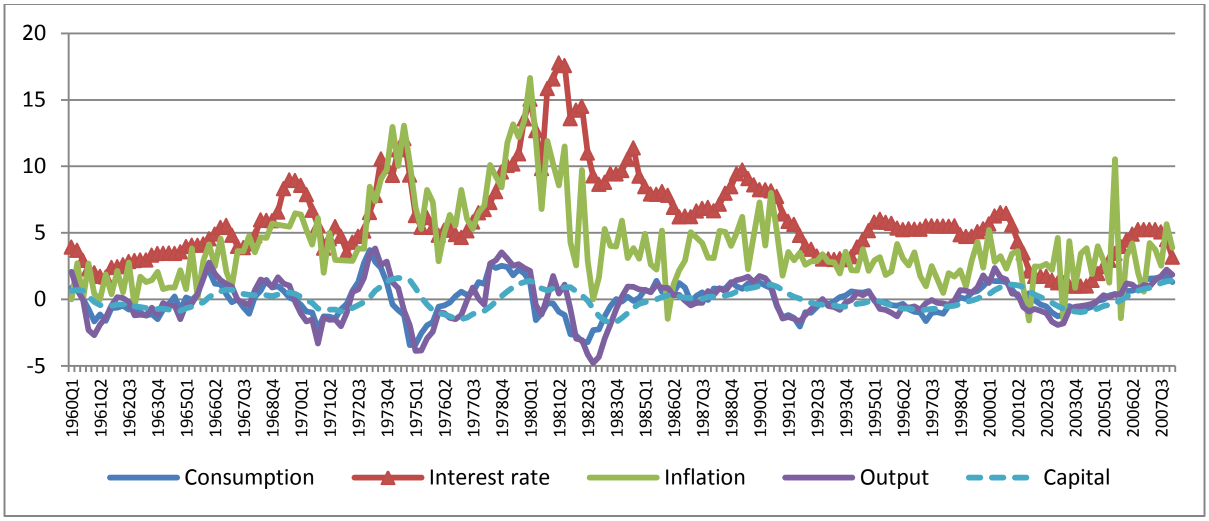

Figure 1. Dynamics of U.S. output, inflation, nominal interest rate, consumption, and capital (in log deviations from the Hodrick-Prescott filtered trend), 1960:I -2008:I.

3.2. Estimation approach

The Bayesian approach is used to estimate the model by constructing prior distributions of the parameters and maximizing the likelihood of the model. The Kalman filter with state and measurement equations is used to fit the data to the model. The Bayesian approach takes advantage of the general equilibrium approach and outperforms GMM and ML in small samples. Furthermore, it does not rely on the identification scheme of the VAR, though does follow the likelihood principle.

The model (27)-(33) is a system of the variables $\hat{C}_t, \hat{R}_t, \hat{\pi}_t, \hat{Y}_t, \hat{K}_t, \varepsilon_{g,t}$, and $\varepsilon_{R,t}$, with a vector of parameters presented in Table 6. The observed capital, consumption, and output deviations from the trends, along with inflation and interest rate are stacked in the vector $y_t = \left[\hat{C}_t, \hat{R}_t, \hat{\pi}_t, \hat{Y}_t, \hat{K}_t\right]^T$, such that the measurement equation is of the form:

$$y_t = \begin{bmatrix} 0 \\ r^* + \pi^* \\ \pi^* \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} s_t ,$$

$$y_t = y^* + Ss_t .$$

The state equation is:

(34) $$s_t = Fs_{t-1} + Q\varepsilon_t ,$$

where r^* and π^* are the steady-state inflation and real interest rate, respectively, $s_t = \left[\hat{C}_t, \hat{R}_t, \hat{\pi}_t, \hat{Y}_t, \hat{K}_t, \varepsilon_{g,t}, \varepsilon_{z,t}\right]^T$ is a vector of system variables, r^* is determined from $\beta = (1+r^*)^{1/4}$, F and Q are the system matrixes, and ε_t is a vector of shocks.

As the posterior distribution of the estimated model is proportional to the product of the likelihood function and the prior, the overall likelihood of the model is maximized taking into account the prior distributions of the parameters and using the state-space decomposition with the Kalman filter. The inference is made with a likelihood-based approach by adopting the random walk Metropolis-Hastings algorithm to obtain 500,000 draws and estimate the moments of the parameter distributions.

3.3. Prior Distributions

The specification of the prior distributions is summarized in Table 6. Most of the priors are the same as in Lubik and Schorfheide (2004). The model is estimated separately for the pre-Volcker period from 1960:I to 1979:II and for the post-1982 period. The Beta distribution is used as a prior for the response of the monetary policy rule to the inflation parameter (ψ_π) centered around 0.5 for indeterminate model and 1.5 for determinate model.

The response of the monetary policy rule to the output parameter (ψ_Y) is centered around 0.25, which is consistent with empirical findings in the range of 0.06 to 0.43. Persistency of the interest rate parameter in the monetary policy rule (ρ_R) is centered around 0.5 and bounded by the beta distribution to be in the interval (0,1). The steady state inflation ($\pi *$) and the interest rate (r*) are centered around 4 and 2 percent per annum, respectively. The real marginal cost elasticity of inflation (λ) is centered around 0.3, assuming that firms reset optimal prices once every three or five quarters, on average. The prior for household risk aversion parameter (σ) is centered around 2, which makes households more risk averse than in the case of logarithmic utility. Shocks of preferences and technology are assumed to follow an AR(1) process with autocorrelation parameters centered around 0.7 and to have a zero prior correlation. Variances of shocks are considered to have inverse gamma distributions. In model exhibits indeterminacy, a sunspot shock is introduced, which is possible correlated with fundamental shocks.

Adding capital and investment activity into the model makes it necessary to specify parameters related to capital accumulation activity and production sector. The priors for capital share in output (α) and investment share in output (s_I) are centered around 0.3 with the standard deviation of 0.1 and are bounded by the beta distribution to be in the interval (0,1).

Parameter		Mean	Std	Distri bution
Monetary policy rule				
Response of monetary policy rule to inflation (determinate model)	ψ_π	1.5	0.20	Gamma
Response of monetary policy rule to inflation (indeterminate model)	ψ_π	0.5	0.20	Gamma
Response of monetary policy rule to output	ψ_Y	0.25	0.15	Gamma
Persistency of interest rate in monetary policy rule	ρ_R	0.50	0.20	Beta
Steady state inflation and real interest rate				
Steady-state inflation	π^*	4.00	2.00	Gamma
Steady-state interest rate	r^*	2.00	1.00	Gamma
Standard model parameters				
Inverse elasticity of intertemporal substitution of consumption	σ	2.00	0.50	Gamma
Real marginal most elasticity of inflation in Calvo model	λ	0.30	0.10	Beta
Persistence of preference shock	ρ_g	0.70	0.10	Beta
Persistence of technology (marginal cost) shock	ρ_z	0.70	0.10	Beta
Capital-related parameters				
Share of capital in output	α	0.33	0.05	Beta
Share of investment in output	s_I	0.30	0.10	Beta
Variance of shocks				
Standard deviation of the interest rate shock	σ_R	0.31	0.16	Inv Gamma
Standard deviation of the preference shock	σ_g	0.38	0.20	Inv Gamma
Standard deviation of the marginal cost shock	σ_z	1.00	0.52	Inv Gamma
Standard deviation of the sunspot shock	σ_s	0.10	0.01	Inv Gamma
Correlation of shocks				
Correlation between technology (marginal cost) and preference shocks	ρ_{gz}	0.00	0.40	Normal
Correlation between sunspot and preference shocks	ρ_{sg}	0.00	0.5774	Uniform
Correlation between sunspot and technology shocks	ρ_{sz}	0.00	0.5774	Uniform

Table 6. Baseline calibration and prior distributions of the parameters of the model.

4. Empirical results

4.1. Model comparison

Models with passive monetary policy ($\psi_\pi < 1$) and active monetary policy ($\psi_\pi > 1$) were estimated separately for the pre-Volcker and post-1982 periods. The two versions of the model are estimated and compared on the two data samples in order to evaluate the odds of each model for a certain period of time. The Bayesian approach is used to evaluate the probability of each model. In a simple two-model case, the ratio of the posterior probabilities of the two models is calculated as:

$$\frac{P(A_1 \mid Y_T)}{P(A_2 \mid Y_T)} = \frac{P(A_1)}{P(A_2)} \frac{P(Y_T \mid A_1)}{P(Y_T \mid A_2)},$$

where $\frac{P(A_1 \mid Y_T)}{P(A_2 \mid Y_T)}$ is the posterior odds ratio, $\frac{P(A_1)}{P(A_2)}$ is the prior odds ratio, and $\frac{P(Y_T \mid A_1)}{P(Y_T \mid A_2)}$ is the Bayes factor that uses the models' estimates.

Equal prior probabilities are assumed for each model and Bayes factor probabilities are calculated using empirical distributions of the estimated parameters:

$$P(Y_T, A_i) = \int_{\theta_A} p(\theta_A \mid Y_T, A_i) p(\theta_A \mid A_i) d\theta_{A_i},$$

$$\hat{p}(Y_T, A_i) = 2\pi^{k/2} \mid \Sigma_{\theta^M} \mid^{-1/2} \; p(\theta_A^M \mid Y_T, A_i) p(\theta_A^M \mid A_i).$$

The probability $p(\theta_A \mid Y_T, A_i)$ is integrated over the set θ_{A_i} of k estimated parameters, assuming a normal distribution for the estimation of $\hat{p}(Y_T, A_i)$.

A comparison of the versions of the model with indeterminacy and determinacy under is presented in Table 7. In contrast to canonical papers, the model with indeterminacy and passive monetary policy dominates the determinate model with a posterior probability of 1.000 for both sample periods. In the next section, the estimation results for the indeterminate model are presented.

	Indeterminacy and passive monetary policy rule	Determinacy and active monetary policy rule
Pre-Volcker period (1960:I to 1979:II)		
Priors	0.50	0.50
Log Marginal Density	-677.52	-2484.83
Posterior probability	1.000	0.000
Post-1982 period (1982:IV to 2008:I)		
Priors	0.50	0.50
Log Marginal Density	-501.76	-4327.34
Posterior probability	1.000	0.000

Table 7. Bayesian comparison of the models.

4.2. Model estimates

As it was shown in the previous section, indeterminate model dominates the determinate model for the pre-Volcker and post-1982 periods. The estimation results of this model on U.S. data from 1960 to 2008 are presented in Table 7. Bayesian empirical estimates show that during the Great Moderation, in contract to canonical papers, there was almost no change in the Fed's monetary policy rule and it remained passive. The priors, the posterior parameter estimates, and the confidence intervals for the model with indeterminacy and passive monetary policy are presented in Table 8. The inference is made with a likelihood-based approach by adopting the Metropolis-Hastings techniques[16].

Regarding monetary policy rule, the estimates of the response of monetary policy to inflation (ψ_π) almost did not change during the Great Moderation, being around 0.581 for the pre-Volcker period and 0.570 for the post-1982 period. These estimates differ substantially from those in Lubik and Schorfheide (2004) or in Clarida, Gali and Gertler (2000), who found that monetary policy was passive during the pre-Volcker period and became active (the estimates of the response of monetary policy to inflation is about 2) during the post-1982 period. This explained by the fact that in the model with capital accumulation there is an additional channel of monetary policy influence through the real interest rate in the production sector. Therefore, the monetary policy can respond less aggressively to changes in inflation to obtain the same goals and remained passive during all times. Furthermore, it was found that the response of the monetary policy rule to output (ψ_Y) did not change either, meaning that there was no change in monetary policy rules.

[16] The Markov Chain Monte Carlo (MCMC) Metropolis-Hastings algorithm with five blocks was used with 500,000 simulations to obtain the inference.

The steady-state inflation and real interest rate estimates of this paper are very similar to Lubik and Schorfheide (2004), showing that real interest rates increased and inflation decreased after the Great Moderation, while the level of nominal interest rates stayed at about the same level. In contrast to canonical models, the model with capital is able to trace higher real return in a higher share of capital income in output, as the share of capital income in output (α) increased from 0.565 to 0.809.[17] Although in canonical models this change would attribute the increase in real interest rates and lower inflation to changes in monetary policy rules, in the model with capital this change is proven to be not due with the changes in monetary policy rules, according to the estimates.

Instead, it was found that during the Great Moderation major structural changes were primarily of the demand side of the economy. Household's inverse elasticity of intertemporal substitution (σ) increased substantially from about 1 during the pre-Volcker period to about 2.7 during the post-1982 period. This result arises from the fact that in the model with capital investment activity permits a breaking of the direct connection between the interest rate and consumption dynamics in the Euler equation, due to the additional no-arbitrage condition between bonds and real sector returns. This allows for the explanation of consumption dynamics not only in terms of shifts in the interest rate but shifts in preferences as well. The empirical findings of this paper show that the dynamics of consumption became much less sensitive to interest rates, as the inverse elasticity of intertemporal substitution increased substantially. Namely, a one percentage point increase in the real interest rate would lead to a three times smaller response of consumption.

[17] This measure of capital income in output is a relative measure, as the model does not take into account human capital and other factor that might bias the estimates.

One of the explanations of this decrease is that different market-driven changes have increased the fraction of households that have access to credit, supporting the idea of financial innovations proposed by Dynan, Elmendorf, and Sichel (2006b). Households that previously had some access to credit have likely gained improved access in terms of both the amount of credit and the consistency of its availability under different macroeconomic conditions. Driven by private market developments and by adjustments in government policy, changes in financial markets enhanced the ability of households to borrow funds and thereby to smooth their spending in the face of swings in income and interest rates. Dynan, Elmendorf, and Sichel (2006a) showed that aggregate consumer spending has become less responsive over time to contemporaneous shifts in aggregate income. Similar results were demonstrated by Dynan, Elmendorf, and Sichel (2006b) on individual household data.

As neither government expenditures nor net exports are included in the model directly, some fluctuations in output are not explained by changes in investment and consumption. This influences the estimates of capital share in output (α), which are about 0.56 and 0.08 for the pre-Volcker and post-1982 periods, respectively. Also, the estimate of the share of investment in output (s_I) is about 0.07 in the baseline specification of the model. The standard deviations estimates (σ_g, σ_z, and σ_R) as well as those for the degree of persistence of shocks(ρ_g and ρ_z) are consistent with other empirical findings.

				Pre-Volcker period (1960-1979)		Post-1982 period (1982-2008)	
	Mean	Std	Distribution	Posterior mean	90 percent CI	Posterior mean	90 percent CI
Monetary policy rule							
ψ_π	0.5	0.20	Gamma	0.581	[0.432,0.735]	0.570	[0.306,0.836]
ψ_Y	0.25	0.15	Gamma	0.398	[0.216,0.558]	0.331	[0.060,0.584]
ρ_R	0.50	0.20	Beta	0.781	[0.700,0.868]	0.959	[0.929,0.986]
Steady state inflation and real interest rate							
$\pi*$	4.00	2.00	Gamma	4.728	[3.300,6.097]	3.786	[0.789,7.103]
$r*$	2.00	1.00	Gamma	0.728	[0.569,0.892]	2.012	[1.617,2.432]
Standard model parameters							
σ	2.00	0.50	Gamma	1.131	[0.699,1.521]	2.647	[2.029,3.293]
λ	0.30	0.10	Beta	0.578	[0.440,0.706]	0.584	[0.468,0.693]
ρ_g	0.70	0.10	Beta	0.628	[0.537,0.730]	0.439	[0.341,0.540]
ρ_z	0.70	0.10	Beta	0.694	[0.598,0.785]	0.916	[0.888,0.943]
Capital-related parameters							
α	0.33	0.05	Beta	0.565	[0.489,0.636]	0.809	[0.760,0.861]
s_I	0.30	0.10	Beta	0.069	[0.065,0.073]	0.065	[0.060,0.071]
Variance of shocks							
σ_R	0.31	0.16	Inv Gamma	0.172	[0.148,0.198]	0.138	[0.121,0.154]
σ_g	0.38	0.20	Inv Gamma	0.273	[0.202,0.342]	0.136	[0.112,0.162]
σ_z	1.00	0.52	Inv Gamma	1.164	[0.820,1.468]	1.000	[0.789,1.199]
σ_s	0.10	0.01	Inv Gamma	0.217	[0.216,0.218]	0.213	[0.208,0.218]
Correlation of shocks							
ρ_{gz}	0.00	0.40	Normal	0.211	[0.201,0.218]	0.195	[0.166,0.218]
ρ_{sg}	0.00	0.5774	Uniform	0.105	[0.084,0.124]	0.095	[0.082,0.108]
ρ_{sz}	0.00	0.5774	Uniform	0.107	[0.086,0.127]	0.100	[0.085,0.116]

Table 8. Priors and posterior estimation results of the model with indeterminacy for the pre-Volcker period (1960:I to 1979:II) and the post-1982 period.

Conclusions

The New-Keynesian model with capital accumulation and the possibility of indeterminacy are simulated and estimated in this paper. Capital accumulation activity introduces new channels of influence for monetary policy on the economy through the no-arbitrage condition between bonds and real sector returns. In canonical models, interest rates affect output solely through the consumption-savings decision of the household in the absence of investment. It is shown in this paper that investment activity changes the monetary transmission mechanisms and allows monetary policy to be passive to achieve the same goals. In this environment, multiple equilibria or indeterminacy are very likely and, therefore, only a limited set of methods can be applied to estimate these models. Many previous papers, did not take this monetary policy channel into account and, therefore, *ex-post* observed active monetary policy.

Introducing capital accumulation and indeterminacy allows for the reconsideration and re-estimation of monetary policy. Different versions of a New-Keynesian model with capital accumulation are simulated and their dynamic properties are discussed. While most of the canonical Keynesian models cannot replicate high autocorrelation levels among the main economic variables, the simulation results of this paper show a model with capital accumulation can generate substantial persistencies in major economic variables. The stock nature of capital adds persistency to the dynamics of all other variables in a model.

The model with capital accumulation was fitted to the quarterly postwar U.S. data on output, inflation, nominal interest rates, consumption, and capital from 1960:I to 2008:I. The versions were estimated separately for the pre-Volcker and post-1982 periods. Bayesian comparisons of the models declared that models with indeterminacy and passive monetary policy dominate determinate models for various periods of U.S. history. In contrast to Lubik and

Schorfheide (2004) and Clarida, Gali and Gertler (2000), the estimates of the response of monetary policy to inflation almost did not change during the Great Moderation, being around 0.581 for the pre-Volcker period and 0.570 for the post-1982 period. Furthermore, it was found that the response of the monetary policy rule to output did not change either, meaning that there was no change in monetary policy rules.

This explained by the fact that in the model with capital accumulation there is an additional channel of monetary policy influence through the real interest rate in the production sector. Monetary policy could respond less aggressively to changes in inflation to obtain the same goals and, therefore, remained passive. Instead, it was found that during the Great Moderation major structural changes were mainly related to consumer behavior. A striking finding of the paper is that the inverse elasticity of intertemporal substitution increased substantially over time in the U.S. from about 1.1 for the pre-Volcker period to about 2.67 for the post-1982 period, meaning that dynamics of consumption became smoother and its response to interest rates decreased, supporting the idea of financial innovations as a source of the Great Moderation.

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Appendix 1. Firm's problem.

The first order conditions for the cost minimization problem are:

$$r_t = z_t f_K(K_t, L_t),$$

$$w_t = z_t f_L(K_t, L_t),$$

where z_t is the marginal cost of production.

Proof:

Minimize $w_t L + r_t K_t$, subject to the restriction: $f(K, L) = \overline{f}$.

The Lagrangian function takes the form:

$$£ = w_t L + r_t K_t - \lambda(f(K, L) - \overline{f})$$

The first order conditions are*:*

$$w_t = \lambda_t f_L(K_t, L_t),$$

$$r_t = \lambda_t f_K(K_t, L_t) \ .$$

Dual problem:

$$£ = c(f(K, L)) - \lambda(f(K_t, L_t) - \overline{f}) \ .$$

The first order conditions are:

$$c'(f(K_t, L_t)) = \lambda_t = z_t \ ,$$

Combining the equation, we have:

$$r_t = z_t f_K(K_t, L_t),$$

$$w_t = z_t f_L(K_t, L_t).$$

Appendix 2. Log-linearization of the model.

Consumption Euler equation

Substituting $U_C(t) = C^{-\sigma}$ in the FOC:

$$\frac{U_C(t)}{P_t} = \beta R_t \left(\frac{U_C(t+1)}{P_{t+1}} \right)$$

we have:

$$\frac{C_t^{-\sigma}}{E_t C_{t+1}^{-\sigma}} = \beta R_t \left(\frac{P_t}{P_{t+1}} \right).$$

In the log-linearized form:

$$(E_t \hat{C}_{t+1} - \hat{C}_t) = \tfrac{1}{\sigma}\left[\hat{R}_t - E_t \hat{\pi}_{t+1}\right],$$

or

$$\hat{C}_t = E_t \hat{C}_{t+1} - \frac{1}{\sigma}\left[\hat{R}_t - E_t \hat{\pi}_{t+1}\right].$$

Fisher equation

From

$$U_C(t) = \beta\{U_C(t+1)[r_{t+1} + (1-\delta)]\}.$$

and from

$$\frac{U_C(t)}{P_t} = \beta R_t \left(\frac{U_C(t+1)}{P_{t+1}} \right)$$

we have

$$\frac{U_C(t)}{U_C(t+1)} = \beta R_t \left(\frac{P_t}{P_{t+1}} \right) = \beta[r_{t+1} + (1-\delta)]$$

Using $r_{t+1} = z_{t+1}\alpha \frac{Y_{t+1}}{K_{t+1}}$, we have:

$$R_t\left(\frac{P_t}{P_{t+1}}\right) = r_{t+1} + (1-\delta).$$

As in the steady state: $r_{ss} = \frac{1}{\beta} - 1 + \delta$, the log-linearized form is:

$$\hat{R}_t - E_t\hat{\pi}_{t+1} = \frac{\frac{1}{\beta}-1+\delta}{\frac{1}{\beta}-1+\delta+(1-\delta)}[E_t\hat{z}_{t+1} + E_t\hat{Y}_{t+1} - \hat{K}_{t+1}] = [1+\beta(1-\delta)]E_t\hat{z}_{t+1} + E_t\hat{Y}_{t+1} - \hat{K}_{t+1}$$

or

$$\hat{R}_t - E_t\hat{\pi}_{t+1} = [1-\beta(1-\delta)](E_t\hat{z}_{t+1} + E_t\hat{Y}_{t+1} - \hat{K}_{t+1})$$

Consumption-labor condition

Using $\frac{U_C}{U_L} = -\frac{1}{w}$ and taking into account that U linear in L,. we have:

$$U_C = \frac{1}{w}$$

and

$$C_t^{\sigma} = z_t(1-\alpha)\frac{Y_t}{L_1}$$

In the log-linearized form:

$$\sigma\hat{C}_t = \hat{z}_t + (\hat{Y}_t - \hat{L}_t)$$

or

$$\hat{C}_t = \frac{1}{\sigma}\hat{z}_t + \frac{1}{\sigma}\frac{\alpha}{1-\alpha}(\hat{K}_t - \hat{Y}_t)$$

Capital accumulation equation

From the capital accumulation equation:

$$K_{t+1} = (1-\delta)K_t + I_t.$$

Using steady-state values:

$$\hat{K}_{t+1} = \tfrac{(1-\delta)K_{ss}}{K_{ss}}\hat{K}_t + \tfrac{I_{ss}}{K_{ss}}\hat{I}_t ,$$

and

$$\tfrac{I_{ss}}{K_{ss}} = \delta ,$$

we have:

$$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta\hat{I}_t .$$

Output

Using the definition of output:

$$Y = C + I$$

we have:

$$\hat{Y}_t = s_C\hat{C}_t + (1-s_C)\hat{I}_t ,$$

where $s_C = \tfrac{C_{ss}}{Y_{ss}}$ is the share of consumption in output

New-Keynesian Phillips curve

From the Calvo model:

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \lambda\hat{z}_t .$$

Monetary policy rule

The log-linearized form:

$$\hat{R}_t = \rho_R\hat{R}_{t-1} + (1-\rho_R)(\psi_\pi E_t\hat{\pi}_{t+k} + \psi_Y\hat{Y}_t) + \varepsilon_{R,t} .$$

Model

$$\hat{C}_t = E_t\hat{C}_{t+1} - \frac{1}{\sigma}\left[\hat{R}_t - E_t\hat{\pi}_{t+1}\right],$$

$$\hat{R}_t - E_t\,\hat{\pi}_{t+1} = [1-\beta(1-\delta)](E_t\,\hat{z}_{t+1} + E_t\,\hat{Y}_{t+1} - \hat{K}_{t+1}),$$

$$\hat{C}_t = \frac{1}{\sigma}\hat{z}_t + \frac{1}{\sigma}\frac{\alpha}{1-\alpha}(\hat{K}_t - \hat{Y}_t),$$

$$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \delta\hat{I}_t,$$

$$\hat{Y}_t = s_C\hat{C}_t + s_I\hat{I}_t,$$

$$\hat{\pi}_t = \beta E_t\,\hat{\pi}_{t+1} + \lambda\hat{z}_t,$$

$$\hat{R}_t = \varphi_R\hat{R}_{t-1} + (1-\varphi_R)(\varphi_\pi E_t\hat{\pi}_{t+j} + \varphi_Y E_t\hat{Y}_{t+j}),$$

substituting $\hat{z}_t = \sigma\hat{C}_t - \frac{\alpha}{1-\alpha}(\hat{K}_t - \hat{Y}_t)$ from (3) and $(\hat{Y}_t - s_C\hat{C}_t)/s_I = \hat{I}_t$ from (5), we have:

$$\hat{C}_t = E_t\hat{C}_{t+1} - \frac{1}{\sigma}\left[\hat{R}_t - E_t\hat{\pi}_{t+1}\right],$$

$$\hat{R}_t - E_t\hat{\pi}_{t+1} = [1-\beta(1-\delta)](\sigma E_t(\hat{C}_{t+1}) + \frac{1}{1-\alpha}\left[E_t\hat{Y}_{t+1} - E_t\hat{K}_{t+1}\right]),$$

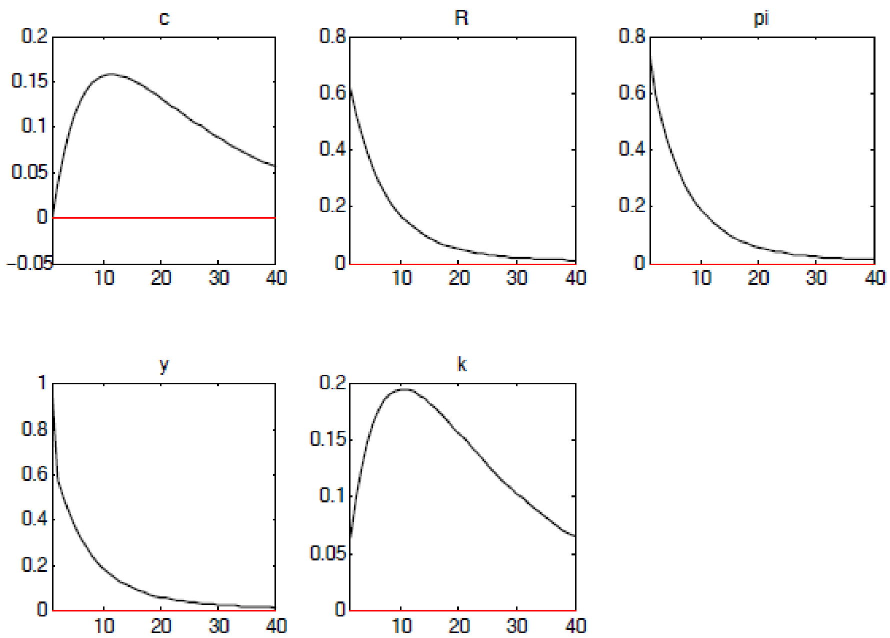
$$\hat{K}_{t+1} = (1-\delta)\hat{K}_t + \frac{\delta}{s_I}(\hat{Y}_t - (1-s_I))\hat{C}_t),$$

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \lambda(\sigma\hat{C}_t - \frac{\alpha}{1-\alpha}(\hat{K}_t - \hat{Y}_t)),$$

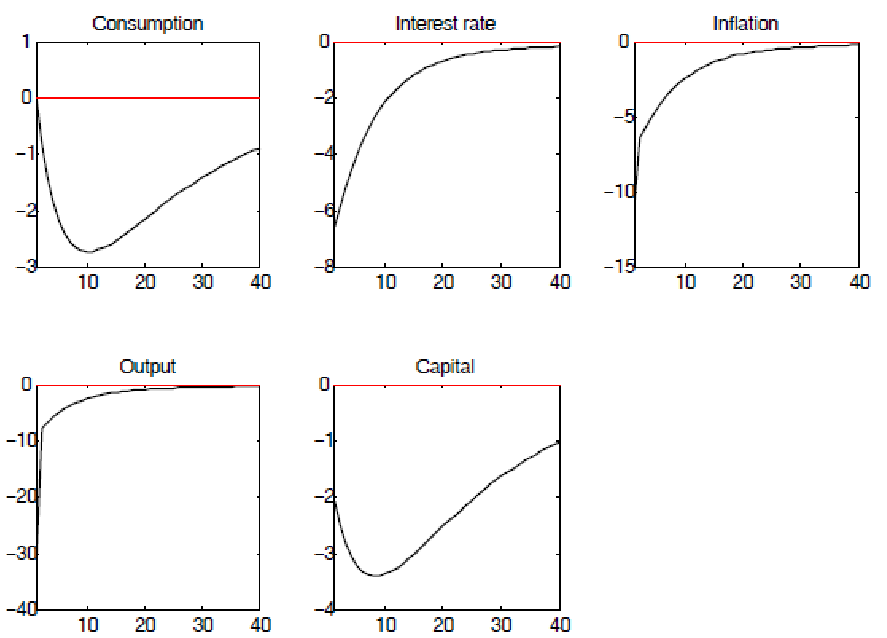
$$\hat{R}_t = \rho_R\hat{R}_{t-1} + (1-\rho_R)(\varphi_\pi E_t\hat{\pi}_{t+k} + \varphi_Y E_t\hat{Y}_{t+j}) + \varepsilon_{R,t}.$$

Appendix 3. Theoretical IRFs, model with indeterminacy.

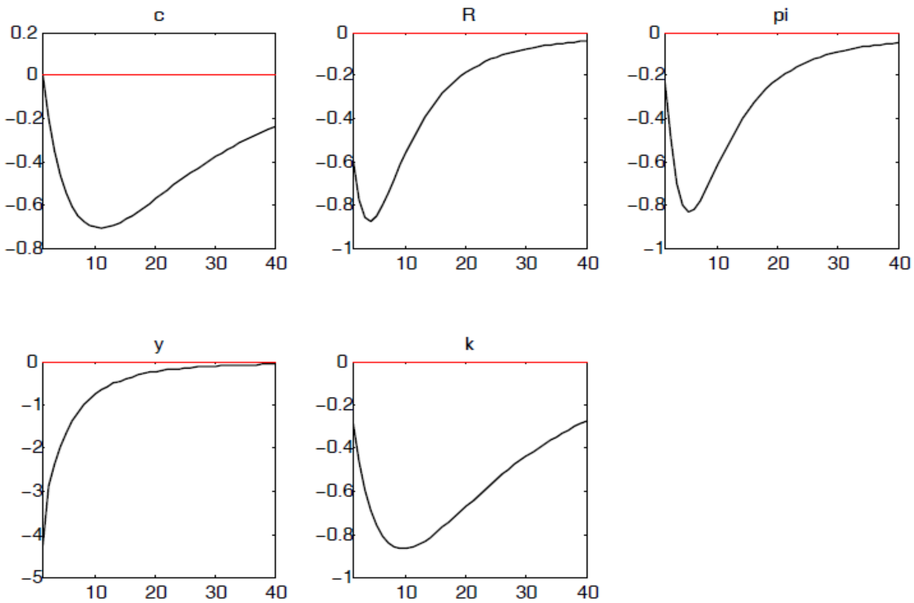
Theoretical IRFs to the interest rate shock.



Theoretical IRFs to the preference shock.

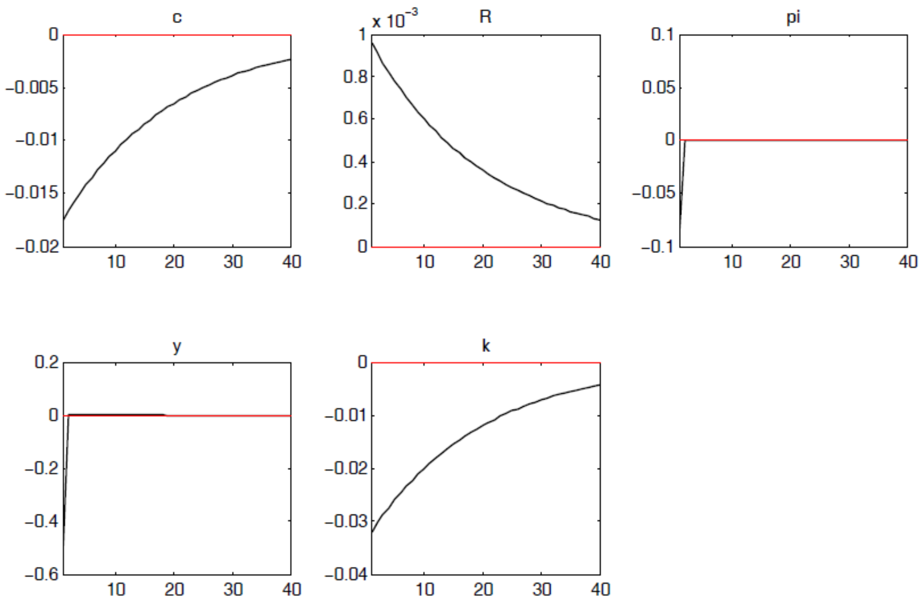


Theoretical IRFs to the marginal cost shock.

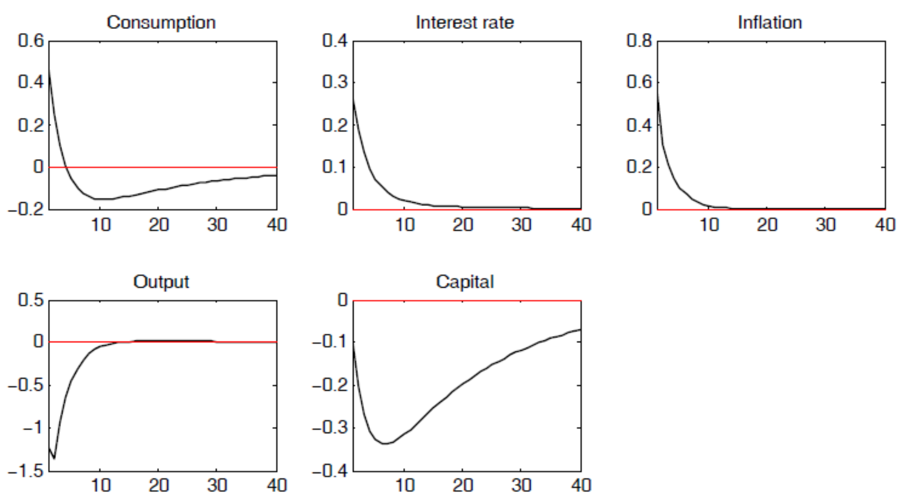


Appendix 4. Theoretical IRFs, model with determinacy.

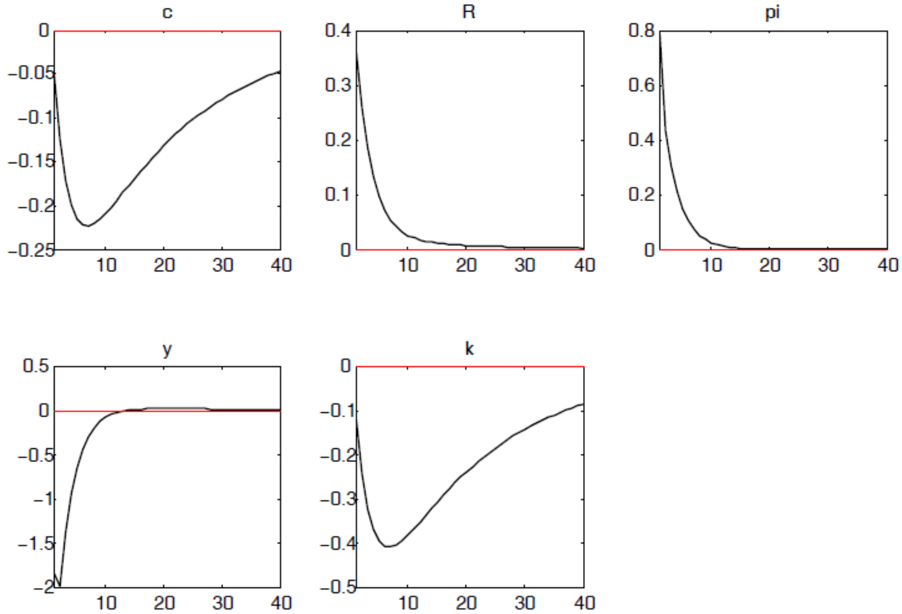
Theoretical IRFs to the interest rate shock.



Theoretical IRFs to the preference shock.



Theoretical IRFs to the marginal cost shock.



Appendix 5. Empirical correlation matrices.

Full sample (1960:I to 2008:IV)					
	Consumption	Interest rate	Inflation	Output	Capital
Consumption	1.000	0.031	0.185	0.869	0.272
Interest rate	0.031	1.000	0.652	0.202	0.413
Inflation	0.185	0.652	1.000	0.288	0.380
Output	0.869	0.202	0.288	1.000	0.461
Capital	0.272	0.413	0.380	0.461	1.000

Pre-Volcker period (1960:I to 1979:II)					
	Consumption	Interest rate	Inflation	Output	Capital
Consumption	1.000	0.284	0.243	0.888	0.127
Interest rate	0.284	1.000	0.867	0.397	0.569
Inflation	0.243	0.867	1.000	0.288	0.410
Output	0.888	0.397	0.288	1.000	0.403
Capital	0.127	0.569	0.410	0.403	1.000

Post-1978 period (1978:III to 1997:IV)					
	Consumption	Interest rate	Inflation	Output	Capital
Consumption	1.000	-0.075	0.312	0.860	0.271
Interest rate	-0.075	1.000	0.667	0.168	0.466
Inflation	0.312	0.667	1.000	0.481	0.559
Output	0.860	0.168	0.481	1.000	0.464
Capital	0.271	0.466	0.559	0.464	1.000

Post-1982 period (1982:IV to 2008:I)					
	Consumption	Interest rate	Inflation	Output	Capital
Consumption	1.000	0.205	0.273	0.864	0.631
Interest rate	0.205	1.000	0.265	0.193	0.091
Inflation	0.273	0.265	1.000	0.268	0.145
Output	0.864	0.193	0.268	1.000	0.594
Capital	0.631	0.091	0.145	0.594	1.000

Appendix 6. Empirical autocorrelations.

Full sample (1960:I to 2008:IV)

Lag	Consumption	Interest rate	Inflation	Output	Capital
1	0.871	0.952	0.669	0.858	0.949
2	0.711	0.885	0.671	0.670	0.836
3	0.527	0.830	0.691	0.449	0.677
4	0.317	0.769	0.574	0.253	0.492
5	0.125	0.701	0.520	0.066	0.297

Pre-Volcker period (1960:I to 1979:II)

Lag	Consumption	Interest rate	Inflation	Output	Capital
1	0.859	0.906	0.765	0.831	0.931
2	0.667	0.757	0.728	0.617	0.791
3	0.431	0.624	0.650	0.360	0.600
4	0.168	0.500	0.579	0.148	0.390
5	-0.067	0.353	0.483	-0.061	0.175

Post-1982 period (1982:IV to 2008:I)

Lag	Consumption	Interest rate	Inflation	Output	Capital
1	0.841	0.960	-0.003	0.825	0.946
2	0.677	0.904	0.168	0.618	0.841
3	0.532	0.837	0.280	0.403	0.703
4	0.358	0.762	-0.026	0.229	0.545
5	0.224	0.683	0.032	0.084	0.385

Chapter 3. Estimating Parameters of Short-Term Real Interest Rate Models.

1. Introduction

Modeling and estimating the volatility of interest rates has significant implications in finance, particularly in pricing bonds, options, and other derivatives. While there is some degree of theoretical and empirical consensus about models for the nominal interest rate, only recently has research tended toward the simultaneous analysis of these main components of the nominal interest rate–real interest rate, expected inflation, and inflation risk premia– though, primarily focusing on the latter two (see, for example, Haubrich, Pennacchi, and Ritchken (2012), Ang, Bekaert, and Wei (2008), and Grishchenko and Huang (2012)). Some papers focus on the term-structure of real interest rates, while dynamics of the real interest rate at the short end of the yield curve are barely studied. An *ad hoc* process for the short-term real interest rate is usually assumed as a building block for more sophisticated models.

To my knowledge, this is the first paper that attempts to shed light on a narrow but crucial question in finance: *What should be the parameters of a model of the short-term real interest rate?* By estimating single-factor models for the short-term real interest rate, it is shown that the relationship between the volatility of changes in the interest rate and its level–called the elasticity of interest rate volatility–plays a crucial role in explaining real interest rate dynamics. Model comparison shows that a square root interest rate process (as in Cox, Ingersoll, and Ross (1985)) is enough to capture the dependence of volatility on the level of interest rates. Many models fail to incorporate this feature, though it should an important assumption according to the empirical results of this paper.

A number of interest rate models that are commonly used to price and hedge interest-rate-dependent securities begin with an assumed process for the instantaneous short-term interest rate. These models differ most notably in the volatility structure assumed to govern interest rate movements. Many empirical papers focus on nominal interest rates and do not consider the fact that two major components of the nominal interest rate are the real interest rate and expected inflation.[18] Researchers have developed many models for the short-term nominal interest rate (see the discussion of nominal interest rate models in Dai and Singleton (2000) and Dai and Singleton (2003)),[19] but fewer models were developed for the real interest rate (see, for example, the discussion in Ang, Bekaert, and Wei (2008)).

There is some understanding of the sources of inflation and factors that can influence it, as well as the way policymakers can forecast and control it, though only a small number of papers devote attention to real interest rates. Although theoretical research often assumes that the real interest rate is constant, empirical estimates for the real interest rate process vary between constancy (Fama (1975)), mean-reverting behavior (Hamilton (1985)), and a unit root process (Rose (1988)). There seems to be greater consensus on the fact that the real interest rate variation mainly affects the short end of the term structure and expected inflation and inflation risk premia influence long-term interest rates (see, among others, Fama (1990) and Mishkin (1990)). Ang, Bekaert, and Wei (2008) show that real interest rates are quite variable at short maturities but smooth and persistent at long maturities. Haubrich, Pennacchi, and Ritchken

[18] Ang, Bekaert, and Wei (2008) show that inflation compensation explains about 80 percent of the variation in nominal rates for both short and long maturities.

[19] A partial listing of theoretical interest rate models includes those by Merton (1973), Brennan and Schwartz (1977, 1980), Vasicek (1977), Dothan (1978), Cox, Ingersoll, and Ross (1980, 1985), Constantinides and Ingersoll (1984), Schaefer and Schwartz (1984), Sundaresan (1984), Feldman (1989), Longstaff (1989), and Longstaff and Schwartz (1992).

(2012) develop and estimate a model of nominal and real bond yield curves. They show that time-varying volatility is particularly apparent in short-term real rates and expected inflation.

It is typical to follow the standard stochastic discount factor approach and assume that the real interest rate is a function only of fundamentals or of a vector of state variables (see, for example, Ang, Bekaert, and Wei (2008), Chernov and Mueller (2012), and Haubrich, Pennacchi, and Ritchken (2012)). In these models, the variance of the real interest rate does not depend on the level of interest rate, but instead is assumed to be constant or to have a GARCH structure. This approach allows estimating risk premia, inflation expectations, and various parameters of models, though it suffers from overly simplistic assumptions about the dynamics of the real interest rate.

A number of theoretical models of the short-term interest rate have been built. Canonical term structure models imply dynamics for the short-term riskless rate that can be nested in a single-factor stochastic differential equation of the form: $dr = \kappa(\mu - r)dt + \sigma r^{\gamma} dz$, where r is the interest rate and dz is the Brownian motion. An important volatility structure parameter that distinguishes models from each other is the elasticity of volatility with respect to the level of interest rates, γ. While other parameters are parts of the linear structure of the interest rate model, the elasticity of volatility of the interest rate adds a non-linearity component.

Studies of the nominal interest rate dynamics show a relatively high level of elasticity of interest rate volatility in the U.S. In the class of single-factor term structure models, a famous result is that of Chan, Karolyi, Longstaff, and Sanders (CKLS, 1992), who compare a series of models for the short-term 1-month Treasury-Bill nominal interest rate over the period 1964 through 1989 for the U.S. They found that an elasticity of volatility with respect to the interest rate level, γ, of 1.5 is required to properly model the nominal interest rate dynamics. Bliss and

Smith (1998) provide a re-examination of the CKLS (1992) results and find the elasticity of interest rate volatility to be around 1 if the structural changes in monetary policy in the 1980s are properly taken into account. Empirical estimates of the elasticity of volatility vary among countries. Nowman (1997) shows that the volatility of the short-term interest rate is highly sensitive to its level in the U.S. (the elasticity is about 1.5), while it is not in the U.K. (the elasticity is about 0.28). More advanced estimation methods found lower levels of elasticity of volatility of the nominal interest rate in the U.S. (Episcopos (2000) and Andersen and Lund (1997)). Evidence for other countries is mixed (Episcopos (2000), Hiraki and Takezawa (1997)).

Much less has been done in the analysis of the real interest rate dynamics. The major problem here is that real interest rates are not directly observed. In the U.S., Treasury Inflation-Protection Securities (TIPS), “real” bonds, are issued in terms of 5, 10, and 30 years and, therefore, do not allow extracting short-tem inflation expectations. Furthermore, TIPS did not start trading until 1997 and had considerable liquidity problems during the first few years, making a consistent analysis of real interest rates for the entire interest rate history of the U.S. almost impossible.

In theory, the Fisher equation tells us that the nominal interest rate is simply the sum of the real interest rate and expected inflation. When inflation is stochastic, the Fisher equation is extended by inflation risk premia and other “higher-order” components, related to nonlinearities, when calculating inflation expectations (see the discussion in Sarte (1998)).

The problem is more complex with longer-term real interest rates and different econometric methods have been applied to estimate real interest rates and their term structure. Older papers simply used projected ex-post real interest rates on instrumental variables (Mishkin (1981) and Huizinga and Mishkin (1986)). Hamilton (1985), Fama and Gibbons (1982), and

Burmeister, Wall, and Hamilton (1986) use ARIMA models and identify expected inflation and real interest rates under the assumption of rational expectations using the Kalman filter. Ang, Bekaert, and Wei (2008) were the first to establish a comprehensive set of stylized facts regarding the term structure of real interest rates. They found that the term structure of real interest rates has a fairly flat shape and that the real short-term interest rate is negatively correlated with both expected and unexpected inflation.

Another problem for calculating the real interest rate is expected inflation. There are a variety of methods for forecasting inflation and evaluating inflation expectations. The most popular are: time-series ARIMA models; regressions based on the Phillips curve; term structure models that include linear, non-linear, and arbitrage-free specifications; and survey-based measures. Ang, Bekaert, and Wei (2007) examine the forecasting power of these four methods and show that surveys outperform other methods for the U.S. To calculate real interest rates, this paper uses two major expected inflation surveys in the U.S.—the Michigan Survey of Consumer Attitudes and Behavior (MICH), which surveys a cross-section of the population on their inflation expectations, and the Livingston Survey, which surveys economists from industry, government, banking, and academia.

To summarize, a lot has been done in the field of nominal interest rate modeling, while the dynamics of the real interest rate are rarely studied. Ang, Bekaert, and Wei (2007) recently documented some stylized facts about the real interest rate dynamics, though some basic questions about the dynamics of the real interest rate are still to be answered. This paper proposes an answer to one of them: What should be the parameters of a model of the short-term real interest rate? This paper estimates parameters of the real interest rate model in the broad class of single-factor continuous interest rate diffusion processes. The empirical estimates show that the key parameters of the nominal and real interest rate models differ substantially. The major difference comes from the volatility structure of these models, mainly related to the elasticity of interest rate volatility, which is estimated to be much lower for the real interest rate model. The empirical estimates of this paper document the fact that the square root process, as in the Cox, Ingersoll, and Ross (1985) model, provides a good characterization of the short-term real interest rate process.

The remainder of paper is organized as follows. Section 2 discusses different theoretical single-factor short-term interest rate models. Section 3 provides the estimation methodology, data description, and empirical results. In Section 4, potential implications of the results of this paper are discussed. Section 5 concludes.

2. Models of the Short-Term Interest Rate

In this section, I briefly discuss canonical models that can be nested in the broad class of single-factor continuous interest rate diffusion processes. To model the interest rate dynamics, it is common to consider a continuous-time diffusion process defined by:

(1) $$dr = \kappa(\mu - r)dt + \sigma r^{\gamma} dz ,$$

where r is the continuous (real) interest rate and dz is the Brownian motion.

This continuous-time model can be represented as the following discrete-time analog:

(2) $$r_{t+1} - r_t = \alpha + \beta r_t + \varepsilon_{r,t+1},$$

(3) $$E_t[\varepsilon_{r,t+1}^2] = \sigma^2 r_t^{2\gamma} ,$$

where r_t is the (real) interest rate and $\varepsilon_{r,t+1}$ is the *iid* shock with the variance $\sigma^2 r_t^{2\gamma}$. In this model, α represents a drift, β is the parameter of mean-reversal, σ is the variance level, and γ

is a measure of the dependence of volatility on the interest rate level, or the elasticity of the interest rate volatility. This general version of the model comprises nine special cases that impose restrictions on the values of α, β, σ, and γ (Table 1).

Models	Model name	Parameters				Degrees of freedom
		α	β	σ	γ	
Model 1	Unrestricted	-	-	-	-	0
Model 2	CEV	0	-	-	-	1
Model 3	$\alpha = 0, \beta = 0$	0	0	-	-	2
Model 4	Merton	-	0	-	0	2
Model 5	Vasicek	-	-	-	0	1
Model 6	GBM	0	-	-	1	2
Model 7	CIR-SR	-	-	-	0.5	1
Model 8	Dothan	0	0	-	1.0	3
Model 9	Brennan-Schwartz	-	-	-	1	1
Model 10	CIR-VR	0	0	-	1.5	3

Table 1. Parameter restrictions (degrees of freedom) imposed by alternative models of the short-term interest rate.

Model 1 is an "unrestricted" version of the single-factor interest rate diffusion processes in discrete time, estimated by CKLS (1992). Model 2 is the constant elasticity of variance (CEV) process introduced by Cox (1975) and by Cox and Ross (1976). Model 3 is a version of the constant elasticity of variance (CEV) process with $\alpha = 0$ and $\beta = 0$. Model 4 is used in Merton (1973) to derive a model of discount bond prices. Model 5 is the Ornstein-Uhlenbeck process used by Vasicek (1977) in deriving an equilibrium model of discount bond prices. Model 6 is the geometric Brownian motion (GBM) process. Model 7 is the square root (SR) process, which appears in Cox, Ingersoll, and Ross (CIR, 1985). Model 8 is used by Dothan (1978) in valuing discount bonds. Model 9 is used by Brennan and Schwartz (1980) in deriving a numerical model for convertible bond prices. Model 10 is introduced by Cox, Ingersoll, and Ross (1980) in their study of variable-rate (VR) securities.

3. Empirical Estimates

3.1. The Real Interest Rate.

Many theoretical models use a certain interest rate process as an assumption. From a theoretical standpoint, many canonical models mentioned above do not require interest rates to be positive, implying a better fit with real interest rates. A theoretical calculation of the real interest rate is usually based on the stochastic discount factor approach. To satisfy the no-arbitrage condition, the real price of an arbitrary financial instrument must adhere to the law of one price:

$$P_t = E_t(M_{t+1}P_{t+1}), \tag{4}$$

where M_{t+1} is a real stochastic discount factor, P_t is the price level, and E_t is the conditional expectation operator at time *t*.

As the nominal and real stochastic discount factors are connected through inflation, under standard assumptions of log-normality, the one-period nominal interest rate can be expressed as:

$$R_t = r_t + E_t(\pi_{t+1}) + Cov_t(m_{t+1}, \pi_{t+1}) - \frac{1}{2}Var_t(\pi_{t+1}), \tag{5}$$

where R_t is the nominal interest rate, r_t is the real interest rate, π_{t+1} is inflation in period *t+1*, and m_{t+1} is a log of the real stochastic discount factor. This equation is different from the standard Fisher equation through the third and four terms, which account for the inflation premium and Jensen's inequality "higher-order" term, respectively.

For short horizons, it is typical to assume that the interest rate is risk-free and the inflation risk premium is negligible (see, for example, Ang, Bekaert, and Wei (2007)). Also, if interest rates are small, second-order components that come from Jensen's inequality are insignificant. Therefore, the canonical Fisher equation would hold for short horizons and the calculation of the real interest rate boils down to subtracting the expected inflation from the nominal interest rate:

$$r_t = R_t - E_t(\pi_{t+1}) \,. \qquad (6)$$

In this paper, I study short term (3 months) interest rates and assume that there is only a negligible inflation risk in it. I intentionally do not attempt to decompose the nominal interest rate into other components, as they are very small for the short-term interest rate and any procedure for estimating the risk premia would demand prior *ad hoc* assumptions about the structure of the real interest rate model. Instead, I focus on estimating the model of the short-term real interest rate using only data on 3-month Treasury-Bill interest rates and expected inflation.

3.2. Data

The real interest rate is calculated using the standard Fisher equation (6). For the short-term nominal interest rate, I use the 3-month Treasury-Bill interest rate included in the Federal Reserve's weekly H.15 release (monthly data is available from January 1934 to December 2012).

While there are many models of inflation expectations, the necessity of extended historical data on inflation expectations limits choice options. A typical approach of using TIPS for measuring expected inflation would not work either, as TIPS are issued in terms of 5, 10, and 30 years and, therefore, do not allow extracting short-tem inflation expectations. Ang, Bekaert, and Wei (2007) show that surveys outperform other forecasting methods. Therefore, two inflation expectation surveys are used in this paper to measure expected inflation: (1) monthly data from University of Michigan Inflation Expectation survey (MICH) available from the St. Louis Fed database and (2) the Livingston Survey from the Philadelphia Fed database. MICH data is available from January 1978 to December 2012 on a monthly basis. As the Livingston Survey data is available from 1954 to 2012 only on a semiannual basis, a linear interpolation is used to transform data into monthly series.

Dynamics of expected inflation, nominal interest rates, and real interest rates are presented in Figures 1 and 2. Both surveys provide similar dynamics of real interest rates. Since 1947, the dynamics of real interest rates was usually between -5 percent and 5 percent (Figure 1) and the dynamics of real interest rates looks more like a random process without clear trends, although expected inflation and nominal interest rates have historical trends and were influenced by economic policies. In the early 80s, inflation was high and Paul Volker, the chairman of the Federal Reserve, implemented the policy of high interest rates, pushing real interest rates up. Since the beginning of the 2008 crisis, nominal interest rates fell almost to zero, while inflation expectations were rather volatile, leading to substantial volatility in real interest rates.

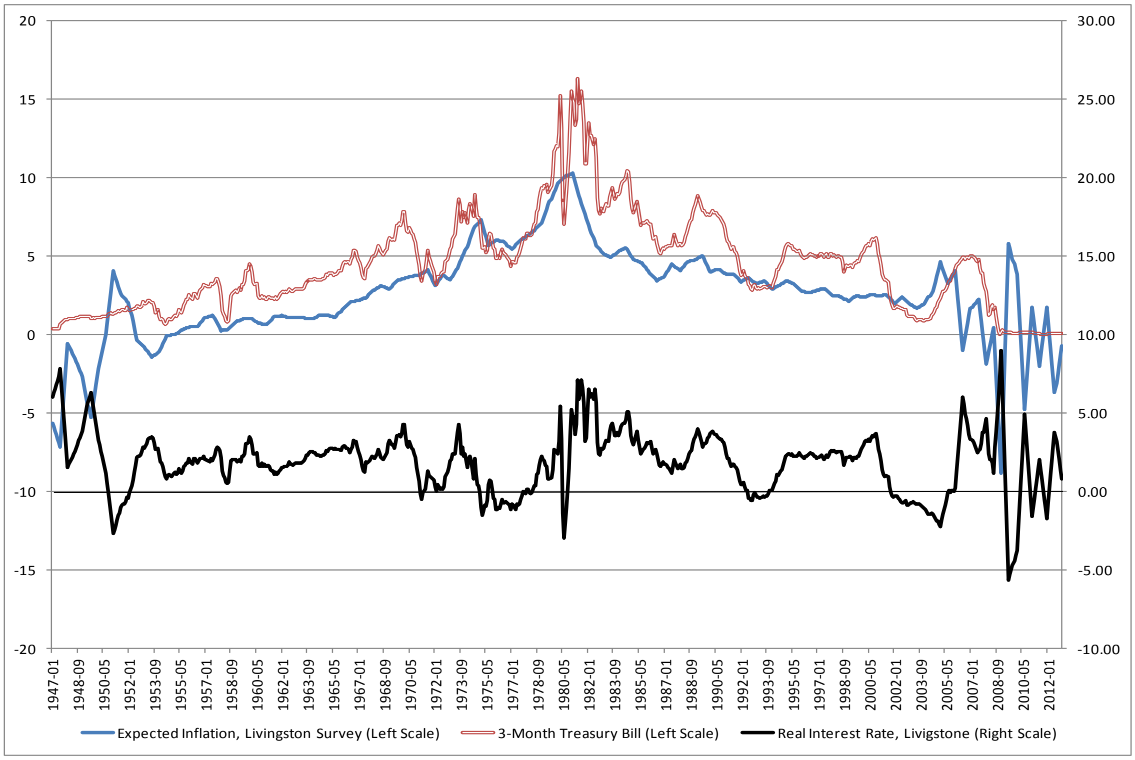


Figure 1. Expected inflation (Livingston Survey), 3-Month Treasury-Bill rate, and real interest rate, Jan 1947-Dec 2012.

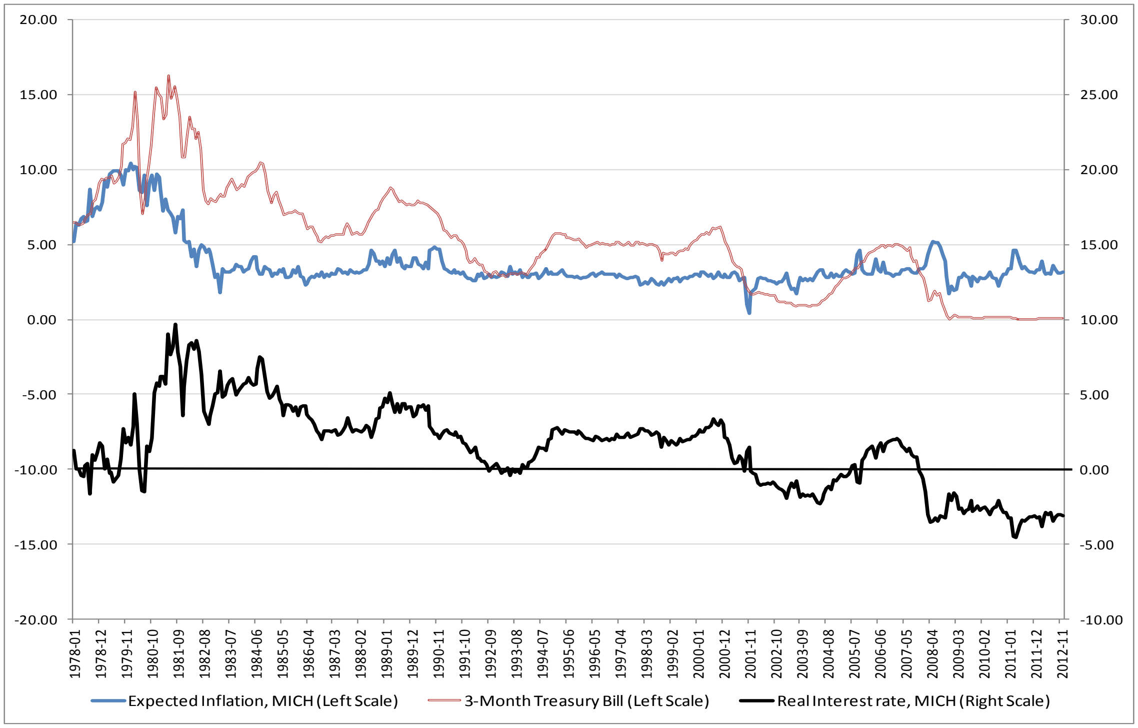


Figure 2. Expected inflation (MICH Survey), 3-Month Treasury-Bill rate, and real interest rate, Jan 1978-Dec 2012.

3.3. Empirical Estimates

I begin by estimating a single-equation model for the short-term interest rate of the form:

(7) $$r_{t+1} - r_t = \alpha + \beta r_t + \varepsilon_{r,t+1},$$

(8) $$E[\varepsilon_{r,t+1}^2] = \sigma^2 r_t^{2\gamma},$$

where r_t is the real interest rate and $\varepsilon_{r,t+1}$ is a shock.

I follow CKLS (1992) and use the GMM to estimate the model, a logical choice for the estimation of the single-factor interest rate processes. GMM estimators are consistent even if the errors are conditionally heteroskedastic, which is important in our case, as the variance of the interest rate process, $E[\varepsilon_{r,t+1}^2]$, depends on the level of interest rates. Also, the justification for the GMM procedure only requires that the distribution of interest rate changes be stationary and ergodic and that the relevant expectations exist.

For comparison, I estimate both the real and nominal interest rate models.[20] As expected inflation data from two surveys are available for different periods, empirical estimates are provided for two samples: from January 1978 to December 2012 and from January 1947 to December 2012. As only the MICH survey has monthly data, estimates for the January 1978-December 2012 period should be considered to be the most robust.

First, for comparison purposes, the process for the nominal interest rate is estimated (Table 1). The estimates of α and β are not statistically different from zero, which is consistent with no-arbitrage assumptions. The estimate for σ is very small. The estimate of the elasticity of the volatility of the nominal interest rate, γ, for the 1978-2012 period is about 1.8, which is

[20] The nominal interest rate model is the same as the real interest rate model, expect for the use of the nominal interest rate instead of the real.

consistent with the CKLS (1992) finding of about 1.5 for the 1964-1989 period. Such a high level of γ explains that the nominal interest rate becomes much more volatile when the level of interest rates is high.

Second, the process for the real interest rate, with the expected inflation taken from the MICH and Livingston surveys, is estimated. Results are similar for both real interest rate data series. The estimates of α are not statistically different from zero, which explains the absence of drift in the real interest rate dynamics. The estimate of the mean-reversal parameter, β, is very small and negative, which is consistent with standard properties of interest rate processes. The estimates for σ are very close to zero, meaning that the variance level is relatively small, which is consistent with the observation of Ang, Bekaert, and Wei (2008).

The estimates show that the real interest rate process has a much lower value for the elasticity of the interest rate volatility, γ, than the nominal interest rate process. The estimated elasticities of volatility of the real interest rate are about 0.55 and 0.47 with standard errors of about 0.2 for both data series. These results are striking, as they are much smaller than 1.8 for nominal interest rates and a canonical value of about 1.5.

As the Livingston survey of inflation expectations has data available starting from 1947, I estimate the nominal and real interest rate models from January 1947 to December 2012 separately. The estimates confirm the finding that the nominal interest rate process has a very high γ but the real interest rate process has a much lower one. Estimates on a full data set for the nominal interest rates from 1934 to 2012 confirm the high levels of γ of about 1.58 for nominal interest rates.

	Parameters			
	α	β	σ	γ
Period: 1978/01-2012/12				
Nominal interest rate model	0.025	-0.008	0.013***	1.820***
s.e.	*0.046*	*0.012*	*0.007*	*0.252*
t-stat	*0.549*	*-0.678*	*1.829*	*7.229*
Real interest rate model (MICH survey of inflation expectations)	0.000	-0.025***	0.047**	0.545***
s.e.	*0.000*	*0.011*	*0.034*	*0.219*
t-stat	*0.846*	*-2.233*	*1.376*	*2.487*
Real interest rate model (Livingston survey of inflation expectations)	0.000	-0.008	0.032**	0.468***
s.e.	*0.000*	*0.019*	*0.023*	*0.234*
t-stat	*-0.335*	*-0.425*	*1.378*	*1.996*
Period: 1947/01-2012/12				
Nominal interest rate model	0.040	-0.009	0.022***	1.595***
s.e.	*0.031*	*0.009*	*0.009*	*0.219*
t-stat	*1.287*	*-1.061*	*2.359*	*7.281*
Real interest rate model (Livingston survey of inflation expectations)	0.000	-0.010	0.045**	0.606***
s.e.	*0.000*	*0.017*	*0.028*	*0.193*
t-stat	*-0.192*	*-0.579*	*1.624*	*3.134*
Period: 1934-2012				
Nominal interest rate model	0.025	-0.007	0.022***	1.588***
s.e.	*0.020*	*0.007*	*0.009*	*0.218*
t-stat	*1.226*	*-0.951*	*2.362*	*7.285*

Table 2. GMM estimation results of the single-equation real and nominal interest rate models.

*Note: *** indicates coefficients significant at the 5% level, ** indicates coefficients significant at the 10% level, ** indicates coefficients significant at the 15% level. Coefficients σ and γ are assumed to be non-negative.*

3.4. Model Comparison

In this section, I compare the "unrestricted" model for the real interest rate with the nine other standard nested models discussed before. Table 3 reports parameter estimates, their standard errors, asymptotic t-statistics, and GMM minimized criterion (χ^2) values for each of the nine nested models. Each model imposes restriction(s) on the parameters of the interest rate model, influencing estimates of other parameters. The χ^2 goodness-of-fit test shows the "validity" of each model and the restrictions it imposes. The model comparison shows that the major parameter that influences the goodness of fit of the model is the parameter of the elasticity of volatility of the interest rate. The χ^2-test suggests that the CIR-VR, Brennan-Schwartz, and Merton are misspecified and can be rejected at the 90% confidence level. These are followed by the Vasicek, GBM, "$\alpha = 0, \beta = 0$", and Dothan models, all of which have lower χ^2 values.

These estimates show that the CIR-SR model provides a good characterization of the short-term real interest rate process. The estimates of this model show that, if γ is pinned down to be 0.5, the estimate of α is not statistically different from zero (which explains the absence of a drift in the real interest rate dynamics) and a mean-reversal coefficient, β, is slightly negative (explaining the mean-reversal dynamics of the real interest rate). These facts are consistent with the stylized facts about real interest rates, established by Ang, Bekaert, and Wei (2008).

		α	β	σ	γ	d.f.	χ^2	*P-value*
Model 1	Unrestricted	0.000	-0.025***	0.047**	0.545***	0	-	-
	s.e.	*0.000*	*0.011*	*0.034*	*0.219*			
	t-stat	*0.846*	*-2.233*	*1.376*	*2.487*			
Model 2	CEV	**0.000**	-0.021***	0.055**	0.603***	1	0.941	0.332
	s.e.		*0.010*	*0.041*	*0.230*			
	t-stat		*-2.066*	*1.321*	*2.624*			
Model 3	$\alpha = 0, \beta = 0$	**0.000**	**0.000**	0.055*	0.603***	2	4.271	0.118
	s.e.			*0.039*	*0.277*			
	t-stat			*1.081*	*1.991*			
Model 4	Merton	0.000	**0.000**	0.005***	**0.000**	2	4.821	0.090
	s.e.	*0.000*		*0.001*				
	t-stat	*0.130*		*6.768*				
Model 5	Vasicek	0.000	-0.023***	0.006***	**0.000**	1	2.611	0.106
	s.e.	*0.000*	*0.011*	*0.001*				
	t-stat	*0.732*	*-2.072*	*7.434*				
Model 6	GBM	**0.000**	-0.018**	0.173***	**1.000**	2	3.494	0.174
	s.e.		*0.010*	*0.020*				
	t-stat		*-1.766*	*8.860*				
Model 7	CIR-SR	0.000	-0.025***	0.040***	**0.500**	1	0.048	0.827
	s.e.	*0.000*	*0.011*	*0.005*				
	t-stat	*1.020*	*-2.309*	*8.653*				
Model 8	Dothan	**0.000**	**0.000**	0.165***	**1.000**	3	6.096	0.107
	s.e.			*0.020*				
	t-stat			*8.061*				
Model 9	Brennan-Schwartz	0.000	-0.018**	0.174***	**1.000**	1	3.401	0.065
	s.e.	*0.000*	*0.010*	*0.020*				
	t-stat	*0.334*	*-1.766*	*8.827*				
Model 10	CIR-VR	**0.000**	**0.000**	0.610***	**1.500**	3	9.280	0.026
	s.e.			*0.079*				
	t-stat			*7.719*				

Table 3. GMM estimates of alternative models for the short-term real interest rates, Jan 1978-Dec 2012.

Note 1. The MICH survey of inflation expectations and 3-month Treasury-Bill interest rates are used to compute real interest rates. Number of degrees of freedom (d.f.) are equal to the number of restrictions in the nested model.
*Note 2. *** indicates coefficients significant at the 5% level, ** indicates coefficients significant at the 10% level, ** indicates coefficients significant at the 15% level. Coefficients σ and γ are assumed to be non-negative.*
Note 3. Restrictions imposed by each model are in bold.

3.5. Structural Breaks

Many empirical studies conclude that a change in the Federal Reserve's monetary policy during the Volker period led to changes or structural breaks in interest rate processes. To test this hypothesis, I introduce a dummy variable, D_t, that equals unity for monthly observations from October 1979 through September 1982 (as in Bliss and Smith (1988)) and zero otherwise. The model takes the form:

$$r_{t+1} - r_t = (\alpha + \delta_1 D_t) + (\beta + \delta_2 D_t) r_t + \varepsilon_{r,t+1}, \tag{9}$$

$$E[\varepsilon^2_{r,t+1}] = (\sigma + \delta_3 D_t)^2 r_t^{2(\gamma + \delta_4 D_t)}, \tag{10}$$

where parameters δ_1, δ_2, δ_3, and δ_4 represent marginal changes during the 1979-1982 period of α, β, σ, and γ, respectively. As four additional parameters are introduced into the model, for GMM estimation purposes the orthogonal vector of instruments is extended by the corresponding series of the dummy variables and their products with other variables. The model is estimated on the real interest rates data series from January 1947 to December 2012, based on the Livingston survey of inflation expectations.[21]

[21] The MICH survey started only in 1978 and, therefore, does not have enough data points to consistently evaluate the existence of the structural break in the data.

	α	β	σ	γ	δ_1	δ_2	δ_3	δ_4
	0.000	-0.008	0.076***	0.849***	0.014**	-0.291***	0.001***	-0.171
s.e.	*0.000*	*0.017*	*0.032*	*0.118*	*0.007*	*0.119*	*41.739*	*1.388*
t-stat	*-0.220*	*-0.495*	*2.367*	*7.168*	*1.909*	*-2.444*	*0.000*	*-0.123*

Table 4. Test for structural breaks in the models of the short-term real interest rate, Jan 1947-Dec 2012.

Note 1. The Livingston survey of inflation expectations is used.
*Note 2. *** indicates coefficients significant at the 5% level, ** indicates coefficients significant at the 10% level, ** indicates coefficients significant at the 15% level. Coefficients σ and γ are assumed to be non-negative.*

The empirical results are striking (Table 4). The estimates of α and β are not statistically different from zero. At the same time, estimates show that there was a statistically significant change in the value of these parameters between October 1979 and September 1982. The drift parameter α increased slightly ($\delta_1 = 0.014$). Parameter β became substantially smaller $(\delta_2 = -0.291)$, reflecting more active mean-reversing dynamics of the real interest rate, which can be explained by an aggressive policy of the Federal Reserve during this period. While there was a statistically significant positive change in σ, it was very small $(\delta_3 = 0.0001)$. It is important to notice that there was no statistically significant change in the volatility structure of the real interest rates during this period, which is consistent with the CLKS (1992) estimates for the nominal interest rate model.

4. Potential Implications

The empirical results of this paper are important as building blocks for more sophisticated interest rate models. Modeling dynamics of the real interest rate simultaneous with dynamics of inflation would give a better perspective on the volatility of the nominal interest rate dynamics. The key findings of this paper are the estimates of the parameters of the volatility structure of the real interest rate model. The results of this paper can be extended and applied to different multi-factor models of interest rates with implications on bond and option pricing.

One of the potential applications of the results of this paper is the improvement of TIPS pricing. The estimated square root process for the real interest rate can be incorporated into a model of the term structure of real interest rates, expected inflation, and inflation risk premia, similar to Haubrich, Pennacchi, and Ritchken (2012) and Grishchenko and Huang (2012). Haubrich, Pennacchi, and Ritchken (2012) construct the model with an *ad hoc* assumption that the real interest rate process has a volatility structure that does not depend on the level of the interest rate. Somewhat similar assumptions are used in Grishchenko and Huang (2012). Both papers have important empirical implications for pricing TIPS. Using the estimated process for the short-term real interest rate of this paper, one might better understand the inflation risk premium for longer maturities and pricing of inflation-protected securities.

Real interest rates might play an important role in understanding the connection between yields on Treasury-Bill and the Federal Funds rate. Piazzesi (2005) shows that nominal bond yields respond to policy decisions of the Federal Reserve and vice versa and, therefore, suggests that models of the yield curve should take into account monetary policy actions of the Federal Reserve. As the Federal Reserve changes its nominal interest rate in response to changes in inflation and other macroeconomic variables, incorporating dynamics of the real interest rate from this paper in Piazzesi’s framework might provide a better understanding of the connection between different short-term interest rates.

All of these applications are left for future research.

5. Conclusion

While parameters of nominal interest rate models are well studied, not much is done in the field of real interest rates. This paper demonstrates that a canonical level of the parameter of the elasticity of nominal interest rate volatility of about 1.5 cannot be applied to the real interest rate model. Instead, the empirical estimates of this paper on U.S. data show that the short-term real interest rate has a much lower level of elasticity of interest rate volatility in the class of single-factor diffusion processes.

Using the 3-month Treasury-Bill interest rate and inflation expectations data, time series for real interest rates are constructed. The empirical estimates of this paper found the elasticity of the real interest rate volatility to be about 0.5, consistent with the square root single-factor diffusion process. The model comparison confirms that the Cox, Ingersoll, and Ross (1985) model provides a good characterization of the short-term real interest rate process. The analysis of structural changes during the Volcker disinflation period did not confirm the existence of a structural break in the volatility structure of the real interest rate model.

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