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**Essays on Disciplining Financial Frictions in Macroeconomic Models**

A dissertation submitted in partial satisfaction  
of the requirements for the degree of  
Doctor of Philosophy in Economics

by

**Robert J. Kurtzman**

2015

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ABSTRACT OF THE DISSERTATION

**Essays on Disciplining Financial Frictions in Macroeconomic Models**

by

Robert J. Kurtzman

Doctor of Philosophy in Economics

University of California, Los Angeles, 2015

Professor Andrew Atkeson, Co-Chair

Professor Pierre-Olivier Weill, Co-Chair

These essays contribute to the study of the role of financial frictions in the macroeconomy. **The Gains from Resolving Debt Overhang: Evidence from a Structural Estimation (Chapter 1):** What are the private and public gains from resolving the debt overhang problem, specifically for non-financial firms? To address this question, we develop a general equilibrium model of firm dynamics where firms make endogenous entry, leverage and bankruptcy decisions, are heterogeneous in their investment opportunities, and endogenously innovate but can suffer from debt overhang when making innovation decisions. We estimate key parameters in our model with data on U.S. public firms using a method of simulated moments procedure, relying on properties of firm growth, especially its significant relationship with risk-adjusted leverage. Under our estimates, the percent difference in firm value between two firms with the same investment opportunities and risk-adjusted leverage, but where one suffers from debt overhang and the other does not, are close to zero for firms with modest amounts of risk-adjusted leverage, but are nonlinear, potentially rising to close to 30% for firms with high risk-adjusted leverage. Ultimately, the gains from resolving this problem for the average entering firm are small compared to the literature, 0.5%, because

firms enter with modest amounts of risk-adjusted leverage. We also use our model to consider the welfare gains in general equilibrium from eliminating the agency problem of debt overhang for all firms, and estimate the welfare gains to be 0.05% of long-run consumption. The general equilibrium response of entry and the aggregate bankruptcy rate acts to dampen the gains from removing the debt overhang distortion.

**Resolving Debt Overhang over the Business Cycle (Chapter 2):** To what extent does alleviating the debt overhang problem change how the economy responds to aggregate shocks? We develop a dynamic, stochastic general equilibrium model where firms make endogenous leverage, bankruptcy, innovation, and entry decisions to help answer this question. The key parameter which governs the extent to which debt overhang affects firms as they have more risk-adjusted leverage, along with some others, is estimated in the *cross-section* in Kurtzman and Zeke (2015b). We perform a number of quantitative exercises with our estimated model. We analyze the extent to which aggregates and firm decisions respond to aggregate shocks to TFP, the level of idiosyncratic asset volatility, and the recovery rate on debt (a “financial shock”), when firms suffer from debt overhang, and when firms hit by a shock resolve the debt overhang problem. Under our estimates, resolving debt overhang after a shock increases productivity, entry, output, and consumption. A negative shock to TFP leads to relatively large movements (compared to the other shocks) in output and consumption with slow recoveries, and an especially slow recovery in productivity. A reasonably calibrated shock to volatility in our model leads to only modest losses in productivity. This finding sheds light on the maturity length and choice of debt as a key modeling assumption, as the literature that argues financial frictions can drastically amplify the effect of uncertainty shocks over the business cycle usually assumes that debt is only one period and finds much stronger results. Financial shocks behave similarly to volatility shocks, in that they lead to modest movements in aggregates.

**Accounting for Dispersion over the Business Cycle without Estimating Production Function Coefficients or Firm-Level TFP (Chapter 3):** The standard approach

to measuring the contribution of misallocation to aggregate productivity over time relies on the estimation of firm-level productivity and industry-level production function coefficients. We argue that when there is limited time series information, this is a task wrought with biases. In turn, we propose a novel decomposition of changes in aggregate productivity that does not require the identification of firm-level productivity or production function coefficients. We argue that changes in the within-sector dispersion component of our decomposition captures the effect of within-sector reallocation of inputs on aggregate productivity. To demonstrate this point, we perform a second-order Taylor expansion on an off-the-shelf model with heterogeneous firms and frictions on the input decisions of firms, and show that changes in the within-industry dispersion component of our decomposition are proportional to changes in within-sector allocative efficiency in the model. We perform our decomposition on U.S. public non-financial firms over the period of 1972-2012. We find our within-sector dispersion component does not contribute to the fall in productivity during recessions. If anything, it plays a stabilizing role in the recessions in our sample. We find similar results for the role of between-sector allocative efficiency. We extend our analysis to publicly listed firms in Japan. Public firms in Japan display similar time series behavior of the within-industry dispersion component to those in the U.S.

The dissertation of Robert J. Kurtzman is approved.

Andrea Eisfeldt

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2015

*This dissertation is dedicated to my great-grandfather, Karl Löwy, and his daughter, my grandmother, Ilse Nusbaum. Karl was a late-stage doctoral student at what is now the Vienna University of Economics and Business. He had passed the necessary examinations, and his dissertation was accepted in January, 1938. His degree was to have been conferred June, 1938. Sadly, the timing was unfortunate; he was denied the ability to defend his dissertation and was denied his doctorate. His dissertation still exists and is now stamped with a swastika, the symbol of the persecution that denied him his degree. He was lucky enough to escape with his life and much of our family to America. However, without his degree, he had trouble obtaining the work for which he was passionate. Ilse has fought for him to obtain a posthumous degree, but has been unable to do so due to university policy. To make amends, Vienna University of Economics and Business initiated a project that culminated in a monument on its campus with the names of Karl and the other students expelled in 1938 due to persecution, and an on-line Memory Book with the biographies of the expelled scholars. The monument was dedicated on May 8, 2014.*



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# Chapter 1: The Gains from Resolving Debt Overhang

## 1 Chapter 1: Introduction

Debt overhang can cause firms to make inefficient investment decisions. Myers (1977) first laid out the debt overhang problem; put simply, existing debt can lead equity holders to underinvest, since part of the expected cash flow generated by the investment goes to debt holders, while equity holders bear its costs.<sup>1</sup> In this paper, we are interested in quantitatively assessing how debt overhang affects firm growth and its welfare implications. To perform our analysis, we develop a general equilibrium model of firm dynamics where firms make endogenous entry, leverage and bankruptcy decisions, are heterogeneous in their investment opportunities, and endogenously innovate but can suffer from debt overhang when making innovation decisions.

To assess the extent to which debt overhang affects firm growth and to perform general equilibrium counterfactuals, we estimate key parameters in our model. Specifically, we perform a method of simulated moments procedure using data on U.S. public firms to estimate parameters governing the strength of debt overhang and extent of dispersion in investment opportunities in our model. We rely on properties of employment growth, especially its significant relationship with risk-adjusted leverage. We then perform counterfactuals to measure the welfare costs of debt overhang in partial and general equilibrium. The gains in partial equilibrium to a single firm were the agency problem of debt overhang resolved would be about 0.5% of firm value upon entry. This number is on the lower end of estimates in the literature, as our procedure finds firm heterogeneity in investment opportunities is the predominant driver of the negative relationship between firm risk-adjusted leverage and subsequent firm growth observed in the data.<sup>2</sup> However, the percent difference in firm value

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<sup>1</sup>Myers (1977) spurred a large literature to study the debt overhang problem; some important papers include: Lamont (1995), Hennessy (2004), Hennessy, Levy, and Whited (2007), Moyen (2007), Occhino and Pescatori (2012), Giroud, Mueller, Stomper, and Westerkamp (2012), Phillippon and Schnabl (2013), Diamond and He (2013), and Gomes, Jermann, and Schmid (2014). Chatterjee (2013) provides a nice overview on the topic, and its relevance for policy.

<sup>2</sup>Moyen (2007) provides an examination of previous estimates of debt overhang losses as a per-

between two firms with the same investment opportunities and risk-adjusted leverage, but where one suffers from debt overhang and the other does not are nonlinear. In our setup, firms with better investment opportunities and high risk-adjusted leverage can gain almost 30% of firm value by resolving this problem. Hence, the private benefits from resolving this problem are highly nonlinear, and can be large if the firm has high risk-adjusted leverage.

We also use our model to consider the welfare gains in general equilibrium from eliminating the agency problem of debt overhang for all firms, and estimate the gains to be an order of magnitude lower than the gains from the partial equilibrium counterfactual. Why do we recover an estimate that is an order of magnitude lower than the partial equilibrium estimate? Were all firms to resolve the debt overhang problem, there would be more demand for innovation by incumbent firms on average, since firms make better investment decisions when more highly levered. However, since investment requires labor, and since there is more demand for investment in the economy, wages rise. Innovation by incumbents and entry both require labor which is supplied inelastically, so both innovation and entry become more expensive. Firms also take out more leverage on average as they anticipate that they will make better investment decisions when more levered, so the bankruptcy rate rises. Bankruptcy is costly in terms of productivity, so aggregate productivity falls as the bankruptcy rate rises. Hence, the response of prices and the aggregate bankruptcy rate dampen the gains from resolving the debt overhang problem. We end up with slightly more productive firms in general equilibrium, but there are fewer of them, which leaves steady state output and aggregate productivity mostly unchanged.

In our model, an intermediate good firm is a monopoly producer of a differentiated product. The firm earns quasi-rents due to a constant markup of its price over marginal cost.

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cent of firm value, focusing particularly on Childs, Mauer, and Ott (2005), Mello and Parsons (1992), Titman and Tsyplakov (2007), and Parrino and Weisbach (1999), which have varying assumptions about the nature of the debt contract, the nature of the flexibility of the investment decision, and have different estimation/calibration techniques. Mello and Parsons (1992), Parrino and Weisbach (1999), and Childs et al. (2005), quantify the debt overhang losses should be no more than 1.54% of firm value. Moyen (2007), on the other hand, in a calibrated model with fully flexible investment finds losses of up to 4.7% of firm value. Hennessy, Levy, and Whited (2007) find an elasticity of leverage to investment of the median firm of -1 or -2 depending on their model specification.

Incumbent firms have an investment technology through which they can invest resources to lower their marginal cost of production and hence expand profits by expanding sales. We refer to this investment as *process innovation*. These incumbent firms differ in the productivity of this technology for investing to reduce marginal costs: for some firms it is cheap to invest to lower marginal cost and thus grow sales and profits, for others it is expensive to do so; hence, firms differ in their *investment opportunities*.

At any time, new firms can pay a fixed cost to enter with a new differentiated product; we refer to this mechanism as *product innovation*. After entering, an intermediate good firm realizes its investment opportunities and its initial level of productivity. The firm then makes an initial debt decision in the face of a classical trade-off to maximize the joint value of equity holders and new creditors. Firms take out debt because it has a tax advantage, but do not fully finance themselves with debt because it can lead to costly bankruptcy. Firms have rational expectations, so they will take out less debt if they know they will suffer more from debt overhang. The debt is perpetuity debt; it pays a fixed coupon every period, and there is no principal due. Conditional on its productivity, the firm makes production decisions using a constant returns to scale technology in labor. Each period, the firm enters with a productivity level, investment opportunities, and a coupon level. It has some probability of exiting exogenously. If it survives, equity holders *alone* decide whether or not to go bankrupt and then make the investment decision. Firm productivity follows a binomial process, and when firms invest, they invest in the drift of this process. The volatility of firm innovation represents firm business risk. If the firm does go bankrupt, creditors seize the firm (in effect, gaining full equity stake), the firm loses a fixed proportion of its productivity, and the firm makes a new leverage decision. Again, this decision is made to maximize the joint value of equity holders and creditors.

A competitive final good sector aggregates the differentiated products into a final good using a constant elasticity of substitution production function. Households derive utility from consumption of the final good, are risk-averse, and inelastically supply labor. All production

and innovation requires labor. Households also hold the equity and debt of intermediate good firms, and receive the dividend and coupon payments. We solve for a stationary competitive equilibrium of our model where all aggregates are in steady state.<sup>3</sup>

In order to assess the welfare effects of removing the debt overhang distortion, we need estimates of the extent to which debt overhang is affecting firms, and the extent to which the patterns in the data could be driven by firm heterogeneity in investment opportunities. We use a method of simulated moments procedure to locally identify the parameters that control these mechanisms, which requires moments from the data and simulated moments from our model. The data moments we use come from annual data on an unbalanced panel of U.S. public firms from 1982-2011. We rely heavily on two measures, the first of which is employment growth, which we measure as annual log differences in employment from year to year. The second measure we use is risk-adjusted leverage, or the log of the book value of a firm's debt relative to the value of its assets, scaled by its business risk. Risk-adjusted leverage is measured in units of the number of standard deviations of annual asset volatility by which the firm's assets must change to equal the firm's book value of debt. A value of -1 implies the firm is one standard deviation from its book value of debt exceeding its assets, -2 implies the firm is two standard deviations from its book value of debt exceeding its assets, and so on. In the data, we also detrend both of these measures by year, industry, size, age, and the Whited-Wu index, which is an index of firms' access to external finance. Two of our moments are the dispersion in growth rates and dispersion in mean growth rates. We find there is a substantial dispersion in growth rates, and that dispersion can partially be explained with dispersion in mean growth rates. The last set of moments we use come from the significant, nonlinear relationship between the detrended measures of risk-adjusted leverage and employment growth we find exists in the data. The strength of this relationship can exist in our model for two reasons. First, it can be driven by debt overhang:

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<sup>3</sup>Our model is in the spirit of the knowledge capital model of firm productivity of Griliches (1979); in particular, our model takes a one-country version of Atkeson and Burstein (2010), and embeds heterogeneity in investment opportunities and endogenous debt and default in such a general equilibrium model of firm dynamics with *process* and *product* innovation.

the endogenous response of firms changing their investment decisions. Second, it can be driven by firm heterogeneity in investment opportunities. Heterogeneity creates a selection effect where firms with worse investment opportunities have higher leverage relative to their business risk on average, whereas firms with better investment opportunities find themselves with less leverage relative to their business risk on average.

We argue the moments described above allow us to sharply locally identify the parameters which govern the mechanisms of interest in our model. We derive a closed-form expression for dispersion in firm growth rates in our model, which we show is to a first-order driven by the parameter which controls firm business risk. Were all firms unlevered, we can show that conditional on the business risk firms face, dispersion in investment opportunities will determine dispersion in mean growth rates. However, with leverage, firms also *choose* lower mean growth rates as they become more levered. Both mechanisms are important in determining this moment, but to different degrees. Lastly, we have a set of moments related to the shape and significance of the relationship between risk-adjusted leverage and employment growth that helps to further differentiate between the exogenous and endogenous factors driving growth rates in our model. We show our parameters are locally identified in a clear range. They imply that heterogeneity in firms' investment opportunities are the main driver of the relationship between risk-adjusted leverage and employment growth. Nonetheless, we also find that firms' investment decisions do seem to respond to how levered they are relative to their business risk. Next, we compute welfare counterfactuals using our estimates and the remaining calibration of our model.

The key welfare question we ask is: How does debt overhang affect firm value in partial and general equilibrium? We compare two steady states, one where equity holders make investment decisions and the firm suffers from debt overhang, and one where equity holders and creditors jointly make the investment decision so that the firm does not suffer from debt overhang. To perform our partial equilibrium exercise, we solve the firm's problem in the two steady states holding prices, the labor allocation, and the supply of firms constant. We

find losses of about 0.5% of firm value.

Our general equilibrium policy counterfactual asks the question, what if we could resolve the debt overhang problem for all firms? We find gains of about 0.05% of long-run consumption, driven mostly by a rise in aggregate productivity. Aggregate productivity is a function of process innovation, product innovation, the aggregate bankruptcy rate, and the exogenous exit rate. The exogenous exit rate does not differ between steady states. When we remove the debt overhang distortion, the aggregate bankruptcy rate rises because firms take out more leverage when we remove the debt overhang distortion. Product innovation falls because entry is then more expensive.

In our model, the gains to a single firm from removing the debt overhang distortion are higher than are the gains from removing the distortion for all firms. The response of prices, product innovation, and the aggregate bankruptcy rate all act to dampen gains in firm value upon entry and the long-run consumption gains from removing the debt overhang distortion.

The rest of the paper follows as such. Section 2 describes our general equilibrium model and characterizes its stationary competitive equilibrium. Section 3 describes the data we use and how we compute the moments in the data, and the details of our estimation procedure. Section 4 describes our estimation results and the remaining calibration of parameters. Section 5 describes our welfare results. Section 6 concludes.

## 2 The Model

In this section, we will describe the physical environment, innovation policies, define an equilibrium, and discuss how we construct our counterfactual objects. Our model takes a discrete-time, discretized Leland (1994) model of capital structure, and allows firms to make process innovation decisions. Because equity holders make innovation decisions, they can thus suffer from debt overhang when highly levered. Aside from the problem of the intermediate good firm, the demand and supply sides of our model are similar to a one-

country version of Atkeson and Burstein (2010). Such a framework allows us to quantify the losses from debt overhang in a general equilibrium model of firm dynamics.

## 2.1 Physical Environment

Time is discrete and indexed as  $t = 0, 1, 2, \dots$ . Households are endowed with  $L$  units of time which they supply inelastically. On the production side, there is a competitive final good sector and a monopolistically competitive intermediate good sector. The final good is produced from a continuum of differentiated intermediate goods that can be consumed by the household. Intermediate good firm productivities evolve endogenously through process innovation, and the measure of differentiated intermediate goods is determined endogenously through product innovation. All innovation and intermediate good firm production requires labor, which is paid wage  $w_t$ . Firms issue both equity and debt to finance their operations, which are held by consumers. Debt is infinitely-lived and pays a constant coupon. Firms take out debt, because it has a tax advantage,  $\tau^d$ , but they do not fully finance themselves with debt because it can lead to costly bankruptcy.

## 2.2 Production

The final good is produced from intermediate goods with constant elasticity of substitution (CES) production function:

$$Y_t = \left( \int y_t(z)^{1-\frac{1}{\rho}} dz \right)^{\frac{\rho}{\rho-1}}. \quad (2.1)$$

In our model, there is a standard inefficiency due to the monopoly markup in the production of intermediate goods. To undo this distortion, we allow for a per-unit subsidy,  $\tau^s$ , on the production of the consumption good.<sup>4</sup> In equilibrium, standard arguments show prices must satisfy:

$$(1 + \tau^s)P_t = \left[ \int_z (p_t(z))^{1-\rho} dJ_t(z) \right]^{\frac{1}{\rho-1}} \quad (2.2)$$

---

<sup>4</sup>If we left in this distortion, the tax advantage of debt can act as a welfare improving subsidy to entry.

where  $p_t(z)$  is the price set by firms with productivity index  $z$ , and  $P_t$  is the price set by final good firms. We choose the price of the final good to be numeraire. Thus, from profit maximization demand for intermediate goods is:

$$y_t(z) = (1 + \tau_s)^\rho p_t(z)^{-\rho} Y_t \quad (2.3)$$

given (2.2) and (2.3).

Intermediate good firms are indexed by three state variables: their productivity,  $z$ , their coupon level,  $d$ , and their investment opportunity,  $\theta$ . Firms have an investment technology through which they can lower the marginal cost of production. Investment requires labor. Upon entry, firms are endowed with  $\theta$ , which controls the inherent productivity of this technology. We assume  $\theta \in \{\theta^L, \theta^H\}$ .

A firm with state  $(z, d, \theta)$  produces output,  $y_t(z)$ , with labor,  $l_t(z)$ , using the following constant returns to scale production function:

$$y = e^{z \frac{1}{\rho-1}} l. \quad (2.4)$$

The productivity of an intermediate good firm is  $e^{z \frac{1}{\rho-1}}$ . We rescale productivity by  $\frac{1}{\rho-1}$ , so that each firm's variable profits and labor are proportional in  $e^z$ .

At every time  $t$ , the firm solves:

$$\pi_t(z) = \max_{y_t(z)} p(y_t(z)) y_t(z) - w_t \frac{y_t(z)}{e^{z \frac{1}{\rho-1}}} \quad (2.5)$$

subject to (2.4) and (2.5) to maximize profits.

Productivity at the firm level evolves conditional on the investments the firm has made in improving its productivity, and on idiosyncratic productivity shocks. At the start of each period,  $t$ , each incumbent firm has a probability,  $\delta$ , of exiting, and a probability,  $1 - \delta$ , of surviving to produce. If it survives, equity holders then choose whether to declare bankruptcy



or continue to operate. If it declares bankruptcy, the firm loses a fixed proportion,  $(1 - \alpha)$ , of its productivity, where  $\alpha \in (0,1]$ . The existing creditors then gain full equity control of the firm and take out new debt to maximize the joint value of equity holders and new creditors. If equity holders continue to operate, the firm with state  $(z, d, \theta)$  invests  $e^z \theta^{-b} e^{bq}$  units of labor to improve its productivity where  $b$  is the convexity of the cost function. With probability  $q$ , next period, the firm's productivity improves by  $\Delta_z$ , and with probability  $(1 - q)$ , the firm's productivity falls by  $\Delta_z$ . Note, the cost function is convex in  $q$ ; it is also proportional in  $e^z$ , so the cost of innovation is proportional in the size of the firm.

Equity holders of a firm with state variable  $(z, d, \theta)$  receive the following cash flows each period given (2.6):

$$(1 - \tau) \left( \pi_t(z) - w_t e^z \theta^{-b} e^{bq_t} \right) - d + \tau^d d.$$

They get after-tax profits including the cost of investment, pay the coupon, and receive the tax advantage of the coupon. Hence, the expected, discounted present value of profits for equity holders of a firm with state variable  $(z, d, \theta)$  satisfies the following Bellman equation:

$$\begin{aligned} V_{E,t}(z, d, \theta) = \max_{q_t} & \left\{ 0, (1 - \tau) \left( \pi_t(z) - w_t e^z \theta^{-b} e^{bq_t} \right) - d + \tau^d d + \right. & (2.6) \\ & \left. e^{-r_t} (1 - \delta) \left( q_t V_{E,t+1}(z + \Delta_z, d, \theta) + (1 - q_t) V_{E,t+1}(z - \Delta_z, d, \theta) \right) \right\}. \end{aligned}$$

Above and throughout, to save on notation, we write  $q_t$  rather than  $q_t(z, d, \theta)$ . Equity's bankruptcy decision can be rewritten as a function of productivity and the current debt level; we call this  $\bar{z}_t(d, \theta)$ .

From (2.6), the optimal choice of  $q_t$  is:

$$q_t^* = \frac{1}{b} \log \left( \frac{e^{-r_t} (1 - \delta) (V_{E,t+1}(z + \Delta_z, d, \theta) - V_{E,t+1}(z - \Delta_z, d, \theta))}{b(1 - \tau) w_t e^z} \right) + \log(\theta).$$

Marginal benefits from a unit of investment for equity holders vs. the firm as a whole

differ, and  $b$  governs how much the investment decision responds to this difference. The functional form of the cost function implies that as  $b \rightarrow \infty$ , the firm's choice of  $q$  still differs conditional on  $\theta$ . The fact that it is equity holders rather than the firm as a whole making this decision drives the debt overhang problem. The second we allow debt holders and equity holders to jointly make the investment decision, the debt overhang problem vanishes. The more levered the firm is relative to business risk, the lower the value of equity. Hence, as  $b$  is lower, so that the marginal benefits from a unit of investment for equity holders and the firm as a whole differ by more, firms will invest less as they are more levered relative to their business risk. In this sense,  $b$  controls the extent of the debt overhang distortion.

Creditors of a firm with state variable  $(z, d, \theta)$  receive the coupon payment as their cash flow, so the combined cash flow of creditors and equity holders of this firm is:

$$(1 - \tau) \left( \pi_t(z) - w_t e^z \theta^{-b} e^{bq_t} \right) + \tau^d d.$$

Since, the bankruptcy rule,  $\bar{z}_{d,\theta}$ , is taken as given by creditors, the expected, discounted present value of profits for equity holders and creditors combined of a firm with state  $(z, d, \theta)$  can be broken into two parts. If equity holders decide not to go bankrupt, the expected, discounted present value of profits for the joint value of equity holders and creditors of a firm with state variable  $(z, d, \theta)$  satisfies the following Bellman equation:

$$\begin{aligned} V_{A,t}(z, d, \theta) = & (1 - \tau) \left( \pi_t(z) - w_t e^z \theta^{-b} e^{bq_t} \right) + \tau^d d + \\ & e^{-r_t} (1 - \delta) \left( q_t V_{A,t+1}(z + \Delta_z, d, \theta) + (1 - q_t) V_{A,t+1}(z - \Delta_z, d, \theta) \right). \end{aligned} \quad (2.7)$$

If equity holders do decide to go bankrupt, creditors seize the firm, lose a fraction of their productivity, and make a new debt to decision to maximize the value of the creditors that seized the firm (which are now the equity holders) and the new creditors. Hence, the

Bellman equation in that case is:

$$V_{A,t}(z, d, \theta) = \max_{d'} V_{A,t}(z + \ln(\alpha), d', \theta). \quad (2.8)$$

The value of creditors,  $V_{B,t}$ , is defined as the difference between the value of the firm as a whole, (2.7) and (2.8), and the value of equity, (2.6); thus,

$$V_{B,t}(z, d, \theta) = V_{A,t}(z, d, \theta) - V_{E,t}(z, d, \theta).$$

New firms are created by purchasing  $n_e$  units of labor; a purchase in period  $t$  yields a new firm in period  $t + 1$  with initial state variables  $z$  and  $\theta$  drawn from a distribution  $G$ . After receiving  $z$  and  $\theta$ , the firm makes an initial debt decision to maximize the value of equity holders and new creditors. In any period with a positive mass of entering firms, we have:

$$w_t n_e = e^{-r_t} \sum_{\theta} \int \max_{d'} V_{A,t+1}(z, d', \theta) G(z, \theta) dz. \quad (2.9)$$

We define  $M_{e,t}$  as the measure of new firms entering the economy at period  $t$  that start producing in period  $t + 1$ .

Households have preferences over consumption of the form:  $\sum_{t=0}^{\infty} \beta^t \ln(C_t)$ , where  $C_t$  is aggregate consumption, and  $\beta \leq 1$  is their discount factor. Households face the following intertemporal budget constraint:

$$P_0 C_0 - w_0 L - T_0 + \sum_{t=1}^{\infty} \left( \prod_{j=1}^t e^{-r_j} \right) (P_t C_t - w_t L - T_t) \leq \bar{W}$$

where  $T_t$  are transfers from the government and  $\bar{W}$  is the time 0 stock of assets of the household.

In our simple setup, market clearing for the final good requires:

$$C_t = Y_t.$$

Market clearing for labor gives us:

$$\sum_{\theta} \int \int l_t(z) M_t(z, d, \theta) dz dd + L_{r,t} = L.$$

where  $\sum_{\theta} \int \int l_t(z) M_t(z) dz dd$  is total employment used to produce the intermediate good, whereas  $L_{r,t}$  denotes labor spent on process and product innovation.

We can write labor spent on research (process and product innovation) as:

$$M_{e,t} n_e + \sum_{\theta} \int \int e^z \theta^{-b} e^{bq_t} M_t(z, d, \theta) dz dd = L_{r,t}.$$

The distribution of operating firms,  $M_t$ , evolves over time as a function of the exogenous exit rate,  $\delta$ , the choices of  $q$  by incumbent firms, and the mass of entering firms each period,  $M_{e,t}$ . To simplify the definition of the mass of firms with state  $(z', d, \theta)$  in period  $t + 1$ ,  $M_{t+1}(z', d, \theta)$ , we break it into three pieces. First, there is a mass of continuing firms who did not go bankrupt who could enter period  $t + 1$  with state  $(z', d, \theta)$  which is a function of continuing firms with productivity  $z' - \Delta_z$  last period that drew positive productivity shocks and continuing firms with productivity  $z' + \Delta_z$  last period that drew negative productivity shocks:

$$\begin{aligned} M_{t+1}^C(z', d, \theta) &= (1 - \delta)(1 - q_t(z' + \Delta_z, d, \theta)) M_t(z' + \Delta_z, d, \theta) \\ &\quad + (1 - \delta) q_t(z' - \Delta_z, d, \theta) M_t(z' - \Delta_z, d, \theta). \end{aligned} \quad (2.10)$$

Second,  $M_{t+1}$  is also a function of the mass of entering firms who received productivity,  $z'$ , and investment opportunities,  $\theta$ , such that they chose debt  $d$ :

$$M_{t+1}^E(z', d, \theta) = M_{e,t} G(z', \theta). \quad (2.11)$$

Third,  $M_{t+1}$  is a function of the mass of firms who have productivity  $z' + \Delta_z - \log(\alpha)$  last

period, with type  $\theta$  and debt load  $d'$  that drew negative productivity shocks, went bankrupt, and chose debt  $d$ .<sup>5</sup>

$$M_{t+1}^B(z', d, \theta) = (1 - \delta) \int q_t(z' + \Delta_z - \ln(\alpha), d', \theta) M_t(z' + \Delta_z - \ln(\alpha), d', \theta) dd'. \quad (2.12)$$

Hence, we can define  $M_{t+1}(z', d, \theta)$ , as the sum of  $M_{t+1}^C(z', d, \theta)$ ,  $M_{t+1}^E(z', d, \theta)$ , and  $M_{t+1}^B(z', d, \theta)$  using (2.15), (2.16), and (2.17).

### 2.3 Equilibrium Definition

An equilibrium in this economy is a sequence of aggregate prices and wages,  $\{P_t, w_t\}$ , and prices of intermediate goods,  $\{p_t(s)\}$  where  $s = (z, d, \theta)$ ; a collection of sequences of aggregate quantities,  $\{C_t, Y_t, L_{rt}, T_t\}$ ; quantities of the intermediate goods,  $\{l_t(z), y_t(z)\}$ ; initial assets  $\bar{W}$ , and a collection of sequences of firm value functions, value functions for equity and debt holders, profit, bankruptcy decisions, capital structure decisions, and process innovation decisions  $\{V_{E,t}(s), V_{B,t}(s), V_{A,t}(s), \pi_t(s), \bar{z}_t(d), d_t(z, \theta), q_t(s)\}$ , together with distributions of operating firms and measures of entering firms  $\{M_t(s), M_{e,t}(s)\}$  such that households maximize their utility subject to their budget constraints, intermediate good firms maximize their within-period profits, innovation decisions, make optimal bankruptcy decisions, and choose optimal debt levels upon entry or restructuring, final good firms maximize profits, research good firms maximize profits, all market clearing conditions are satisfied, and the changes in the distribution of firms are consistent with firm decisions and entry.

A stationary competitive equilibrium is an equilibrium in which all aggregates, prices, and distributions are constant over time. In such an equilibrium, we say these aggregates are in steady state. We focus only on equilibria with positive entry.

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<sup>5</sup>It is also possible for firms to have had productivity  $z' - \Delta_z - \log(\alpha)$  last period, type  $\theta$ , and debt load  $d'$ , to go bankrupt and chose debt  $d$ . This does not occur in a steady state.

## 2.4 Planner's Problem and Other Counterfactuals

The ultimate goal of this paper is to evaluate the impact of the various distortions at play in our model. There are three distortions in the model: debt overhang, bankruptcy costs, and the tax advantage of debt. To achieve our goal, we develop a decomposition based on counterfactual objects. Each counterfactual consists of removing one distortion. First we remove debt overhang, then we remove the bankruptcy cost, then we remove the tax advantage of debt. These three objects add up to the difference between the baseline economy and the solution of the social planner's problem.

### 2.4.1 Planner's Problem

The social planner chooses consumption, product innovation, process innovation, and the labor allocation to maximize her discounted present value of utility, such that the final good market clears, the labor market clears, and the law of motion for productivity is satisfied. In our setup, the planner's problem is the equivalent of setting  $\tau^d = 0$  and  $\tau = 0$  with a per-unit subsidy,  $\tau^s$ , on production of the consumption good, to undo the distortion from the efficient allocation from the markup in our model. The subsidy takes value  $\tau^s = \frac{\rho}{\rho-1}$ . We discuss how to aggregate our model in a steady state in Appendix C. When aggregating our model, we include the subsidy in all counterfactuals and in our base case. We also set  $\tau = 0$  to focus on the distortion of interest, which is the social costs due to the tax advantage of debt, and the associated costs of debt overhang and bankruptcy.

### 2.4.2 Debt Overhang Counterfactual

The Bellman equations can also be solved if the firm as a whole rather than equity alone were to make the investment decision. It is always equity, however, who chooses the point at which the firm goes bankrupt. Were the firm as a whole to make the bankruptcy decision, it would never go bankrupt. We define the Bellman equation for equity without debt overhang,

$V_E^{ND}$ , in steady state below:

$$V_E^{ND}(z, d, \theta) = \max \left\{ 0, (1 - \tau) \left( \pi(z) - we^z \theta^{-b} e^{-bq} \right) - d + \tau^d d + e^{-r}(1 - \delta) \left( qV_E^{ND}(z + \Delta_z, d, \theta) + (1 - q)V_E^{ND}(z - \Delta_z, d, \theta) \right) \right\}.$$

We define the Bellman equation for equity holders and creditors combined,  $V_A^{ND}$ , below:

$$V_A^{ND}(z, d, \theta) = \max_q \begin{cases} \max_{d'} V_A^{ND}(z + \ln(\alpha), d', \theta) & V_E^{ND}(z, d, \theta) < 0 \\ (1 - \tau) \left( \pi(z) - we^z \theta^{-b} e^{-bq} \right) + \tau^d d & \text{else} \\ + e^{-r}(1 - \delta)qV_A^{ND}(z + \Delta_z, d, \theta) \\ + e^{-r}(1 - \delta)(1 - q)V_A^{ND}(z - \Delta_z, d, \theta). \end{cases} \quad (2.13)$$

We then use the first-order condition from (2.13) to find  $q$ :

$$q^* = \frac{1}{b} \log \left( \frac{e^{-r}(1 - \delta)(V_A(z + \Delta_z, d, \theta) - V_A(z - \Delta_z, d, \theta))}{b(1 - \tau)we^z} \right) + \log(\theta).$$

Notice, now, no matter the value of  $b$ , the firm does not suffer from debt overhang, as equity holders and creditors are jointly making the investment decision. Because they make the investment decision taking into account the possibility of bankruptcy, if  $b$  has any convexity, the firm will invest more as it is more levered relative to its business risk to avoid bankruptcy.

It is still the case, then, that the value of creditors,  $V_B^{ND}(z, d, \theta)$ , is defined as the difference between the value of the firm as a whole and the value of equity holders; thus,

$$V_B^{ND}(z, d, \theta) = V_A^{ND}(z, d, \theta) - V_E^{ND}(z, d, \theta).$$

### 2.4.3 Social Loss Decomposition

We define the planner’s problem in Subsection 2.4. Define consumption from the planner’s problem to be,  $C^{EFF}$ . We define  $LOSSES_{EFF}$  as the long-run differences in aggregate consumption between the planner’s problem and our base case with debt overhang:

$$LOSSES_{EFF} = \frac{C^{EFF} - C}{C}$$

where  $C$  is consumption from our baseline estimation. We call these losses “social losses,” and moving forward we describe welfare as differences in long-run consumption between steady states. To further decompose these social losses, we create two more consumption measures. To create our first additional consumption measure, we have the firm as a whole, rather than equity, make decisions as we describe in Subsection 2.4.2. We solve for a stationary competitive equilibrium given these decision rules, and recover a counterfactual object,  $C^{ND}$ . We define the losses from debt overhang as:

$$LOSSES_{DO} = \frac{C^{ND} - C}{C}.$$

We then create one more object to recover two other counterfactuals. We take the decision rules from when the firm as a whole makes the investment decision, but we aggregate our model slightly differently; we create the stationary distribution of firms not allowing any value to be destroyed in bankruptcy, so  $\alpha = 1$ . Notice,  $\alpha = 1$  only during the aggregation process, not when firms make decisions. Were  $\alpha = 1$  when the firm makes its initial debt decision, it would fully finance itself with debt. We then recover a new consumption measure:  $C^{NO\alpha}$ . This gives us the ability to create two counterfactuals, which along with  $LOSSES_{DO}$  should add up to  $LOSSES_{EFF}$ . The first counterfactual object represents the effect of bankruptcy on the total mass of productivity:

$$LOSSES_{NO\alpha} = \frac{C^{NO\alpha} - C^{ND}}{C}.$$



The second is the remaining loss, which can be interpreted as the degree to which  $\alpha$  and  $\tau^d$  distort firm decisions relative to the social planner's choice:

$$LOSSES_{REM} = \frac{C^{EFF} - C^{NO\alpha}}{C}.$$

## 2.5 Some Aggregate Relationships

We show how we aggregate our model in Appendix C, but here we want to outline a few key equations that will help us in clarifying the mechanisms driving our results. We detail these mechanisms in the context of the parameterization implied by our estimation procedure in Subsection 5.2. First, one can show real wages are a function of aggregate productivity,  $Z$ , in a steady state according to the formula below:

$$w = (1 + \tau^s) \frac{\rho - 1}{\rho} Z^{\frac{\rho-1}{\rho}}. \quad (2.14)$$

From (2.12) and (2.14), we can then rewrite the free entry condition as:

$$Z^{\frac{1}{\rho-1}} = \frac{e^{-r}}{n_e(1 + \tau^s)^{\frac{\rho-1}{\rho}}} \sum_{\theta} \int \max_{d'} V_A(z, d', \theta) G(z, \theta) dz. \quad (2.15)$$

Hence, the expected, discounted present value of the firm upon entry fully reflects aggregate productivity and vice-versa. We also note that aggregate consumption is a function of aggregate productivity and the labor allocation. It is defined as:

$$C = Z^{\frac{1}{\rho-1}} (L - L_r).$$

Because we are interested in comparing consumption across steady states, we can then see that changes in log consumption can be characterized in terms of changes in log productivity

and changes in log labor spent on production:

$$\Delta \log(C) = \frac{1}{\rho - 1} \Delta \log(Z) + \Delta \log(L - L_r). \quad (2.16)$$

But, from (2.15), we see that changes in log consumption are thus a function of the labor allocation and the real wage (or the value of the firm upon entry):

$$\Delta \log(C) = \Delta \log(w) + \Delta \log(L - L_r).$$

In turn, when we compare the value of the firm upon entry between two steady states in general equilibrium after a policy change, we capture how much aggregate productivity moves between steady states, which captures much of how aggregate consumption differs between the two steady states.

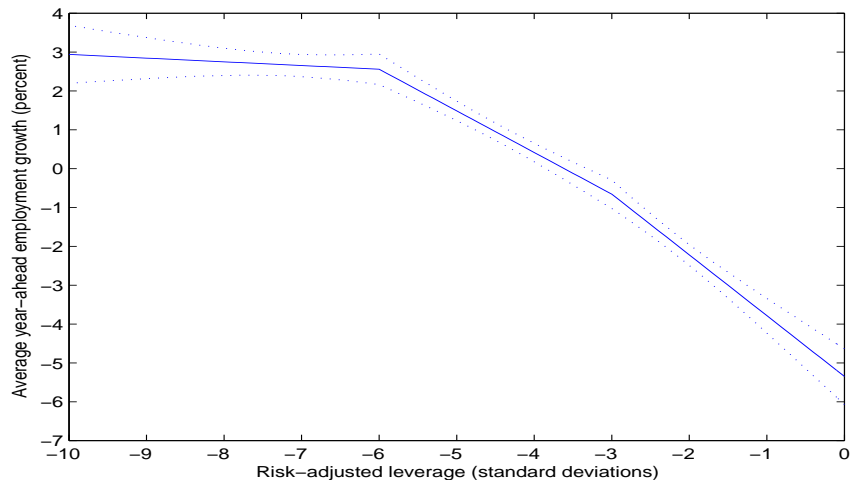
### 3 Details of Data Moments and Estimation Procedure

To perform the policy experiments described in Sections 1 and 2, we estimate key parameters of our model using a method of simulated moments procedure. In this section, we describe our data set, the facts we establish regarding the relationship between risk-adjusted leverage and growth, and our estimation procedure. In Section 4, we present estimation results and our calibration. In Section 5 we discuss our policy counterfactuals. We outline the numerical solution procedure in Appendix D. In Appendix E, we show closed-form approximations to the decision rules in our model which help to provide good guesses in our solution procedure, and should help to provide further intuition for the quantitative results.

#### 3.1 Data

For our analysis, we use CRSP/Compustat, a widely analyzed dataset on U.S. public firms. We discuss exact details of how our data is constructed in Appendix A and the methodology

Figure 3.1: Relationship between Risk-Adjusted Leverage and Employment Growth



\*Risk-adjusted leverage and employment growth are the residuals from a regression on industry dummies, year dummies, age effects, size effects, and the Whited-Wu index. Risk-adjusted leverage is measured in units of the number of standard deviations of annual asset volatility by which the firm's assets must change to equal the firm's book value of debt. A value of -1 implies the firm is 1 standard deviation from its book value of debt exceeding its assets, -2 implies the firm is 2 standard deviations from its book value of debt exceeding its assets, and so on. To construct the line in the figure, we run a pooled piecewise linear regression between risk-adjusted leverage and growth with breakpoints of -6 and -3. Dotted lines are 95% confidence intervals constructed with 15,000 bootstraps.

for constructing option-adjusted measures of asset value and volatility in Appendix B. In our estimation procedure, we will rely heavily on properties of employment growth, especially its relationship with risk-adjusted leverage, which we show in Figure 3.1. If we define  $V_B$  as the book value of debt,  $V_A$  as the value of assets, and  $\sigma_A$  as the value of the firm, we define risk-adjusted leverage as:

$$\frac{\ln\left(\frac{V_B}{V_A}\right)}{\sigma_A}$$

Risk-adjusted leverage (or leverage adjusted for business risk) is measured in units of the number of standard deviations of annual asset volatility by which the firm's assets must change to equal the firm's book value of its debt. We trim the measure to lie between -10 and 0, which leaves over 90% of firms. Our results only change slightly without trimming, and our estimate of the elasticity of investment to leverage due to debt overhang is quite close and slightly lower. Employment growth is measured as log differences in employment from year to year. Figure 3.1 shows the relationship between risk-adjusted leverage and average year-ahead employment growth. Having three slopes to the relationship is consistent with our theory, as we will show in the next subsection. We run the following pooled piecewise

Table 3.1: Data Estimates: Summary and Some Robustness

	<i>U.S. Data</i>	<i>No Business Cycles</i>	<i>Only Manufacturing</i>
Dispersion in growth rates	23.3 (0.12)	23.36 (0.13)	22.48 (0.16)
Dispersion in mean growth rates across firms	7.96 (0.047)	7.95 (0.053)	7.4 (0.059)
Relationship between $lv g^*$ & growth, $lv g^* < -6$	-0.1 (0.16)	-0.083 (0.18)	-0.16 (0.21)
Relationship between $lv g^*$ & growth, $lv g^* \in [-6, -3]$	-1.07 (0.14)	-0.93 (0.22)	-0.91 (0.26)
Relationship between $lv g^*$ & growth, $lv g^* > -3$	-1.56 (0.2)	-1.66 (0.15)	-1.32 (0.18)
Number of firms	5,241	5,194	2,682
Number of observations	45,785	36,891	25,009

\*  $lv g^*$  is risk-adjusted leverage. Risk-adjusted leverage is measured in units of the number of standard deviations of annual asset volatility by which the firm’s assets must change to equal the firm’s book value of debt. A value of -1 implies the firm is 1 standard deviation from its book value of debt exceeding its assets, -2 implies the firm is 2 standard deviations from its book value of debt exceeding its assets, and so on. Growth is employment growth. Risk-adjusted leverage in the data is the residuals from a regression on industry dummies, year dummies, age effects, size effects, and the Whited-Wu index. The “Only Manufacturing” column summarizes information on firms in our sample with SIC codes between 20 and 39. The “No Business Cycles” column summarizes information on firms where the start date or end date of the growth measure does not lie during NBER recession years.

regression using thresholds of -6 and -3 to assess this point:

$$y_t = \gamma + \beta_0 x_t + \beta_1 (x_t + 6) \mathbf{1}(x_t \leq -6) + \beta_2 (x_t + 3) \mathbf{1}(x_t > -3) + \epsilon_t \quad (3.1)$$

As long as the first threshold is between -6.5 and -5.5 and the second threshold is between -3.5 and -2.5, the coefficients will exist in the same range and give us the shape of the plot which is consistent with Figure 3.1. Our results are also similar if we perform our exercise on a Kernel-smoothing regression. We create standard errors using 15,000 bootstraps in standard fashion. We use detrended measures for risk-adjusted leverage and employment growth throughout our analysis, including in Figure 3.1. To be precise, risk-adjusted leverage and measures of growth are the residuals from a regression on industry dummies, year dummies, age dummies, log employment, and the Whited-Wu index, the last of which is an index of firms’ external financing constraints.<sup>6</sup> The independent variables are all measures that are

<sup>6</sup>See Whited and Wu (2006) for how to construct the Whited-Wu index.

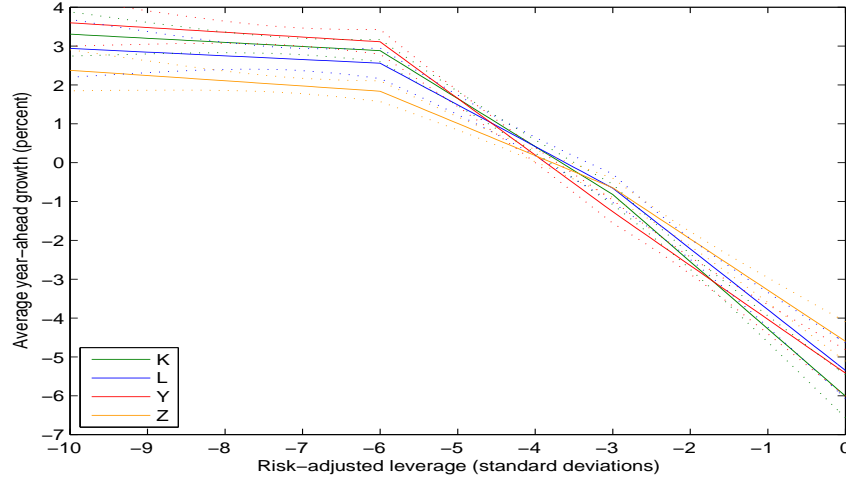
known to be strongly correlated with growth rates. We include polynomial terms if they are significant. We focus on employment growth when estimating our model, as it is a measure less prone to be distorted by measurement error or accounting manipulation. Yet, as one can see in Figure 3.2, the same striking pattern exists between detrended measures of risk-adjusted leverage and year-ahead growth in labor (employment), capital (PP&E), sales, and a stock measure of R&D created using the perpetual inventory method following the procedure in Falato, Kadyrzhanova, and Sim (2013). In our model, sales, employment, and the stock of R&D, all move together as firm leverage relative to business risk changes, as in the data. If one were to add capital (PP&E) to our model in a standard way as in Atkeson and Burstein (2014), this measure would move together with our other stock variables, as it does in the data.

In Table 3.1, we show data moments and standard errors of the regression coefficients plotted in Figure 3.1, along with some properties of firm growth and the sample. To allay concerns about whether the data is driven by firms in certain industries or years (even though it is detrended), we show the same moments for subsamples of the data. We break the data into only manufacturing firms. We also show moments without NBER recession years, i.e. we ignore data where the growth measure either started or ended during an NBER recession year. All moments are close to moments from the “full sample” and would leave us with very similar estimates of our parameters. After detrending, the properties of the sample generally hold for sample splits by age and size, and our measure of financial constraints as well.

The relationship we establish between growth rates and risk-adjusted leverage should not be a surprising one. We expect firms that are not growing are on average more likely to have higher leverage relative to their business risk, so, *ex ante*, we should expect risk-adjusted leverage and growth to have a monotonically decreasing relationship. However, the shape, economic significance, and robustness of the relationship are what appeal to us.

Were we to only include debt overhang and not heterogeneity in investment opportunities in our model and estimation procedure, we could not get the shape of the nonlinear relation-

Figure 3.2: Relationship between Risk-Adjusted Leverage and Growth in  $K$ ,  $L$ ,  $Y$ , &  $Z$

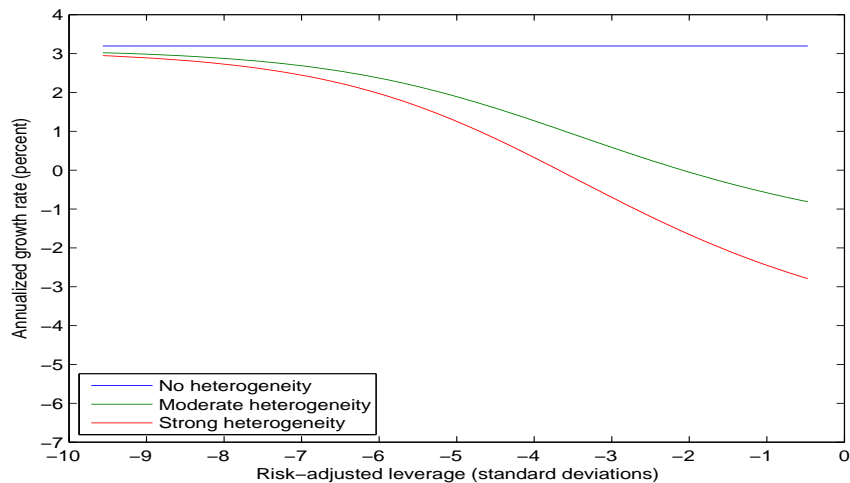


\* $K$  is capital (PP&E).  $L$  is employment.  $Y$  is output (sales).  $Z$  is a stock measure of  $R\&D$ . Risk-adjusted leverage and all stock measures are the residuals from a regression on industry dummies, year dummies, age effects, size effects, and the Whited-Wu index. Risk-adjusted leverage is measured in units of the number of standard deviations of annual asset volatility by which the firm's assets must change to equal the firm's book value of debt. A value of -1 implies the firm is 1 standard deviation from its book value of debt exceeding its assets, -2 implies the firm is 2 standard deviations from its book value of debt exceeding its assets, and so on.  $R\&D$  is a stock measure created using the perpetual inventory method. To construct the lines in the figure, we run a pooled piecewise linear regression between risk-adjusted leverage and growth with breakpoints of -6 and -3. Dotted lines are 95% confidence intervals constructed with 15,000 bootstraps.

ship between risk-adjusted leverage and growth similar to that in Figure 3.1. When there is no risk of bankruptcy, firms have the same investment decisions and mean growth rates, so we get the leveling off we see in the data. However, as we will show below, the *exact* pattern, where risk-adjusted leverage and growth have a decreasing relationship above -6 but less than -3 is difficult to recover with debt overhang alone, as firms with risk-adjusted leverage above -6 but less than -3 do not have much different investment decisions from those below -6, since the risk of bankruptcy for those firms is low. It is firms that are the most levered relative to their business risk which are affected by debt overhang most. Nonetheless, with debt overhang alone, we can get the significant correlation between risk-adjusted leverage and the leveling off we see in the data. And our estimate of the elasticity of investment to leverage due to debt overhang estimated without heterogeneity gives us costs of debt overhang for firms that are in line with findings in the corporate finance literature.

Were we to only include heterogeneity in investment opportunities in our model and estimation procedure, we could get the nonlinear relationship similar to that in Figure 3.1 because of a selection effect. Firms have different mean growth rates, and firms that grow

Figure 3.3: The Role of Heterogeneity in Investment Opportunities in Explaining the Relationship between Risk-Adjusted Leverage and Growth, Turning off Debt Overhang



Risk-adjusted leverage is measured in units of the number of standard deviations of annual asset volatility by which the firm's assets must change to equal the firm's book value of debt. A value of -1 implies the firm is 1 standard deviation from its book value of debt exceeding its assets, -2 implies the firm is 2 standard deviations from its book value of debt exceeding its assets, and so on.

faster on average find themselves with less risk-adjusted leverage. We can get leveling off, because only high type firms with the same average growth rate find themselves with such low leverage relative to their business risk. We will find the mixture of high types relative to low decreasing as firms are more levered relative to their business risk, which will change the composition of mean growth rates, leading to the relationship above. We provide a detailed description of the moments we choose and justification in the next section.

### 3.2 Estimation Procedure

In our method of simulated moments exercise, we allow four parameters to vary. First, we allow  $\theta$  to vary. Recall,  $\theta \in \{\theta^H, \theta^L\}$ . To minimize the number of parameters in our procedure, we pin down  $\theta^H$ , and we define  $m$  to be the difference between  $\theta^H$  and  $\theta^L$ .<sup>7</sup> Second, we allow  $b$  to vary. Recall,  $b$  is the convexity of the cost function and governs the extent to which debt overhang affects firms as they are more levered relative to their business

<sup>7</sup>We pin down  $\theta_H$  by picking  $q$  of unlevered high types to be 0.52, which implies a mean growth rate for high type firms that matches the average growth rate of firms in the data with a risk-adjusted leverage below -6. The details of how we derive this transformation can be found in Appendix D. When estimating  $\theta_H$  and using  $\gamma$  as a moment, we get similar results to those we get from our current estimation procedure.

risk. Third, we allow  $\Delta_z$  to vary, which represents the business risk firms face. Fourth, we allow the probability a firm enters as a high type or low type to vary.

What are the first-order effects of  $\theta$ ? Mechanically, increasing  $\theta$  increases the dispersion of mean growth rates, along with the dispersion in growth rates. However, allowing for heterogeneity also contributes to explaining the relationship between risk-adjusted leverage and growth. Because the size of the coupon is fixed until the firm goes bankrupt, the first-order reason risk-adjusted leverage moves is due to movements in the value of firm assets.<sup>8</sup> In equilibrium, slow growing firms end up with more leverage on average, and fast growing firms end up with less leverage on average. Fast growing firms still can end up highly levered, as they still face risk each period, but they are *more likely* to end up with less leverage. In our model, the proportion of high to low types decrease smoothly as firms have more risk-adjusted leverage. Conditional on the proportion of high to low types decreasing as firms are more highly levered, increasing the dispersion in types will create a stronger relationship between risk-adjusted leverage and growth. We show this in Figure 3.3, using estimates of the extent to which heterogeneity affects firms from Table 4.1, holding fixed the proportion of high to low types under our estimate. We see that increasing the dispersion in heterogeneity in investment opportunities increases the strength of the relationship between risk-adjusted leverage and growth.

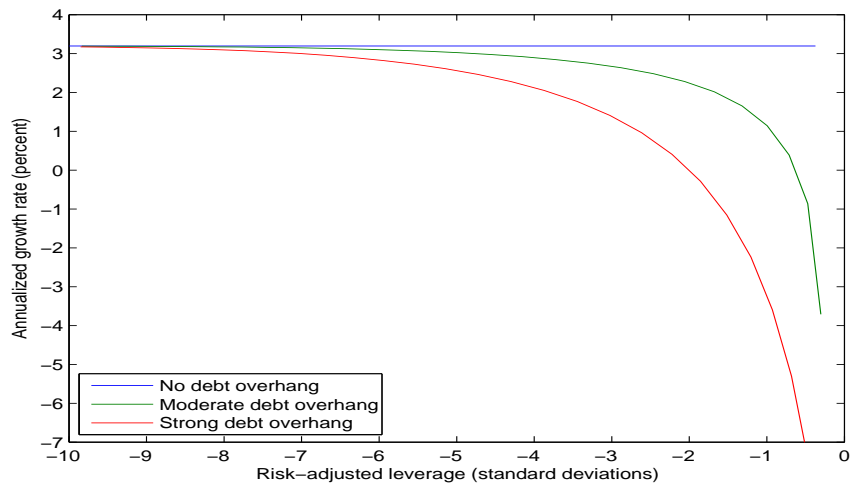
What is the first-order effect of  $b$ , i.e. the elasticity of investment to risk-adjusted leverage due to debt overhang? In Figure 3.4, we demonstrate that if firms suffer more from debt overhang as they are more levered, the relationship between risk-adjusted leverage and growth is stronger. We take estimates from Table 4.1 of the extent to which debt overhang affects firm growth. Unlike in Figure 3.3, debt overhang only acts for firms that have the highest risk-adjusted leverage. Yet, in the data, we see that there is a strong relationship between risk-adjusted leverage and growth for firms 4 or 5 standard deviations in units of annual asset volatility of their assets from their liabilities. This distinction will be a key

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<sup>8</sup>The second-order effect is there are movements in firm asset volatility conditional on firm's types.



Figure 3.4: The Role of Debt Overhang in Explaining the Relationship between Risk-Adjusted Leverage and Growth, Turning off Heterogeneity



Risk-adjusted leverage is measured in units of the number of standard deviations of annual asset volatility by which the firm's assets must change to equal the firm's book value of debt. A value of -1 implies the firm is 1 standard deviation from its book value of debt exceeding its assets, -2 implies the firm is 2 standard deviations from its book value of debt exceeding its assets, and so on.

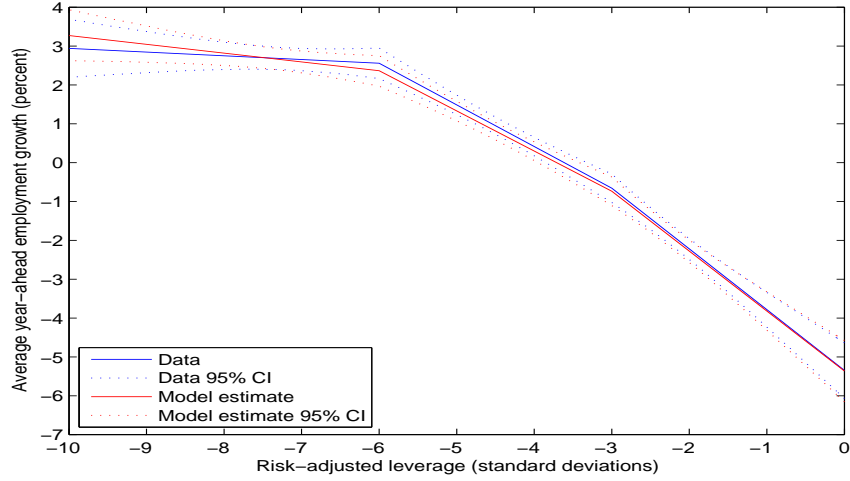
driver of local identification of our parameters.

What is the role of  $x$ , the probability of entering as a high type firm? As the probability of entering as a low type increases, the first-order effect is that low type firms are more likely to live with less leverage because there are more low type firms receiving positive shocks in the economy. In turn, the relationship between leverage and growth will grow further from zero. In practice, we need a probability of around 0.8 of entering as a high type,  $x$ , to get the leveling off behavior below a risk-adjusted leverage of -6 that we see in the data conditional on our other moments. Given we are using public firm data to estimate our model which is likely firms with good investment opportunities, our estimate seems reasonable.

In our exercise, we overidentify our problem by trying to match five moments from the data. We match the dispersion in growth rates, dispersion in mean growth rates, and  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$  from the piecewise linear regression we ran in (3.1). By pinning down  $\theta_H$ , we essentially pin down the constant,  $\gamma$ , from this regression. Notice, that to recover employment growth, year-over-year, we just look at log differences in productivities. We measure risk-adjusted leverage as in the data:

$$\frac{\ln\left(\frac{V_B}{V_A}\right)}{\sigma_A}$$

Figure 3.5: Our Model Estimate Against the Data



Risk-adjusted leverage in the data is the residuals from a regression on industry dummies, year dummies, age effects, size effects, and the Whited-Wu index. Risk-adjusted leverage is measured in units of the number of standard deviations of annual asset volatility by which the firm's assets must change to equal the firm's book value of debt. A value of -1 implies the firm is 1 standard deviation from its book value of debt exceeding its assets, -2 implies the firm is 2 standard deviations from its book value of debt exceeding its assets, and so on. To construct the lines in the figure, we run a pooled piecewise linear regression between risk-adjusted leverage and growth with breakpoints of -6 and -3. Dotted lines are 95% confidence intervals constructed with 15,000 bootstraps.

where  $V_A$  is the value of assets,  $V_B$  is the book value of debt, and  $\sigma_A$  is the firm's asset volatility. As is standard, in both the data and the model, we define  $V_B$  as the value of short-term +  $\frac{1}{2}$ \*(long-term debt). We assume short-term debt is the sum of all coupons due within a year, whereas long-term debt is the expected, discounted present value of all debt due after a year. We take  $V_A$  from our model, and we use Ito's Lemma to recover the following standard formula for  $\sigma_A$ :

$$\sigma_A = \sigma_E \frac{V_E}{V_A} \frac{\partial V_A}{\partial V_E} . \quad (3.2)$$

We derive  $\frac{\partial V_E}{\partial V_A}$  numerically along our grid, and we measure  $\sigma_E$  as the annualized volatility of monthly equity returns. As in the data, we trim risk-adjusted leverage to be between -10 and 0. We show in Figure 6.1 in Appendix G the relationship between risk-adjusted leverage in the data and the model before trimming. Clearly, we trim roughly equal percentages of firms, and we do a good job of hitting the distribution of risk-adjusted leverage even though we do not target any moments from this distribution.

We now describe in further detail the intuition behind which parameters drive which

moments in the full simulation. The dispersion in growth rates is affected by all four parameters, but is to a first-order driven by changes in  $\Delta_z$ . To better understand this, first denote the relative mass of firms at a given state  $(z, d, \theta)$  as:  $F(z, d, \theta) = \frac{M(z, d, \theta)}{\sum_{\theta} \int \int M(z, d, \theta)}$  where  $M(z, d, \theta)$  denotes the mass of firms for a given  $(z, d, \theta)$  in steady state. It is useful to first define the average choice of process innovation in the economy:

$$\bar{q} = \sum_{\theta} \int \int F(z, d, \theta) q(z, d, \theta) dz dd.$$

We can then find the average growth rate (in log differences) of the economy over one period:

$$2\Delta_z \bar{q} - \Delta_z.$$

In turn, with some algebra, which we show in the Appendix F, we can find the annualized dispersion in growth rates:

$$\frac{4}{\Delta} \Delta_z^2 \bar{q} (1 - \bar{q})$$

where  $\Delta$  denotes the number of periods in a year. In practice,  $\Delta_z$  drives this moment to a first-order, but  $b$ ,  $m$ , and the probability of entering as a high type will also affect this moment given they affect the average  $q$  in the economy.

The dispersion in mean growth rates is affected by all three parameters as well, but, conditional on the level of  $\Delta_z$ , is to a first-order driven by  $\theta$ . To demonstrate this, let's consider the case when there is no leverage in the economy. Let  $x$  denote the expected mass of entry into the high type (so  $(1 - x)$  is the expected mass into the low type), and  $q_1$  denote the unlevered high type's choice of  $q$ . With some algebra, one can show the variance in expected growth rates will be:

$$x(1 - x) \left( (2q_1 \Delta_z - \Delta_z + 1)^{\frac{1}{\Delta}} - (2q_1 \frac{1}{m^2} \Delta_z - \Delta_z + 1)^{\frac{1}{\Delta}} \right)^2.$$

Notice, given  $x$ ,  $q_1$ ,  $\Delta_z$ , and  $\Delta$ , it is  $m$ , which will drive this moment. Differences in

mean growth rates will be driven by the extent of dispersion in mean growth rates, as one would expect. That said, firm investment decisions are affected by firm leverage through debt overhang, and firm leverage will change over time, so  $b$  will also control the extent of dispersion in mean growth rates. Another important point here is that this moment will also be driven by the different tenures of firms which is why, when we estimate our model, we make sure it has the same properties in terms of number of firms, tenure of firms, and missing observations, as we explain in more detail below.

We implement our simulated method of moments procedure in the following standard way. Say our model moments are the  $1 \times n$  vector,  $\hat{M}(G)_t$ , where  $G$  represents the tuple  $(b, m, x, \Delta_z)$ , and our data moments are the  $1 \times n$  vector,  $\hat{D}_t$ . Define  $\hat{g} = \hat{M}(G)_t - \hat{D}_t$ . We want to minimize the following objective function over  $G$ :

$$\hat{g}W\hat{g}'.$$

We use the identity matrix as our weighting matrix  $W$ . Our estimates will, in turn, be consistent but not efficient.

The simulation procedure follows as such: for each guess of  $G$ , we draw 10,000 firms who each live for 32 years with starting values of  $z$  and  $\theta$  drawn from the stationary distribution. Each firm can exogenously or endogenously exit throughout its life. We then take our sample of firms and draw from them so as to match the sample of firms from the data. Without resampling, we create a dataset with the same number of observations and firms as in the Compustat data, making sure to correctly create gaps in our model sample for missing observations. If one wants to use as much data as possible, it is important to take this matched-sample approach, especially when looking at the dispersion in mean growth rates, as this moment is very much driven by the properties of the sample. So, this moment is not driven by firms who exist in the sample for short periods of time and so our procedure is more stable, we only compute it in the model and the data for firms that live at least seven

years.

We repeat this exercise 100 times and take the mean of the moments over the 100 samples to find  $\hat{M}(G)$ . We use simulated annealing in order to find a global minimum of the objective function. Given our estimates, we can then derive standard errors and the J-statistic in the usual way when using pre-specified weighting matrices. For these objects, we need the efficient weighting matrix, which we recover by creating a variance-covariance matrix of the the data using 15,000 bootstraps.

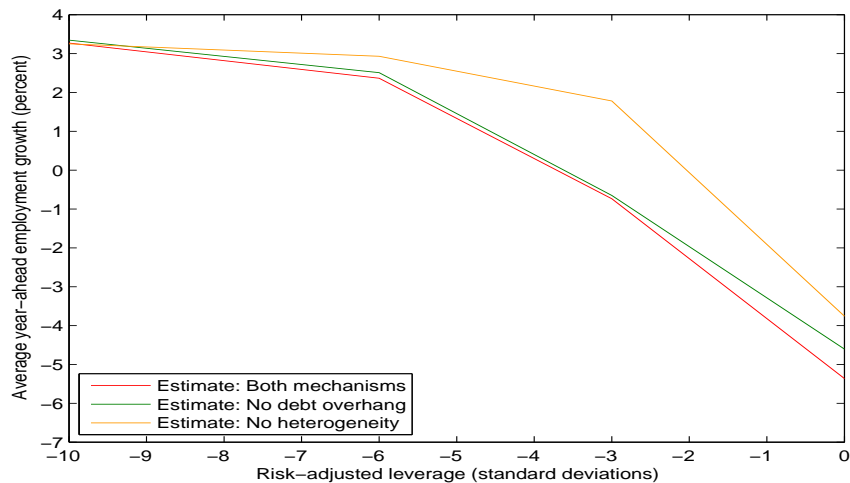
## 4 Estimation Results and Calibration

Table 4.1: Method of Simulated Moments Results

	<i>U.S.</i> <i>Data</i>	<i>Estimate</i> <i>(S.E.)</i>	<i>No</i> <i>Types</i>	<i>No Debt</i> <i>Overhang</i>
<i>Estimated Parameters</i>				
Convexity of cost function, $b$	–	35.81 (2.39)	12.34 (0.19)	$\infty$
$f(\theta)$ , Mean % diff in growth rts b/w types	–	5.29 (0.025)	–	8.85 (0.043)
$f(\Delta_z)$ , Avg. ann. standard deviation of high $lvq.^*$ firms if no debt overhang	–	22.73 (0.0041)	23 (0.0045)	22.82 (0.0041)
$P(\theta = \theta^H)$ , Probability of entry as high type	–	0.79 (0.023)		0.74 (0.0048)
<i>Targets</i>				
Dispersion in growth rates	23.3	23.32	23.31	23.31
Dispersion in mean growth rates across firms	7.96	7.92	6.72	7.92
Relationship between $lvq.^*$ & growth, $lvq.^* < -6$	-0.1	-0.23	-0.079	-0.21
Relationship between $lvq.^*$ & growth, $lvq.^* \in [-6, -3]$	-1.07	-1.03	-0.38	-1.05
Relationship between $lvq.^*$ & growth, $lvq.^* > -3$	-1.56	-1.54	-1.85	-1.32

\*  $lvq.^*$  is risk-adjusted leverage. Risk-adjusted leverage in the data is the residuals from a regression on industry dummies, year dummies, age effects, size effects, and the Whited-Wu index. Risk-adjusted leverage is measured in units of the number of standard deviations of annual asset volatility by which the firm's assets must change to equal the firm's book value of debt. A value of -1 implies the firm is 1 standard deviation from its book value of debt exceeding its assets, -2 implies the firm is 2 standard deviations from its book value of debt exceeding its assets, and so on. Growth is employment growth.

Figure 4.1: Model Estimates of the Relationship between Risk-Adjusted Leverage and Growth with and without Mechanisms



Risk-adjusted leverage is measured in units of the number of standard deviations of annual asset volatility by which the firm's assets must change to equal the firm's book value of debt. A value of -1 implies the firm is 1 standard deviation from its book value of debt exceeding its assets, -2 implies the firm is 2 standard deviations from its book value of debt exceeding its assets, and so on. To construct these lines, we run a pooled piecewise linear regression between risk-adjusted leverage and growth with breakpoints of -6 and -3.

## 4.1 Estimation Results

In the previous section, we walked the reader through the logic behind our estimation procedure. In this section, we present our estimation results along with some robustness checks. When we present our estimates of  $\theta$  and  $\Delta_z$ , we write them in terms of the intuitive moments we presented in Section 3. In Table 4.1, we show results when we estimate our model with debt overhang, without debt overhang, and without heterogeneity. In this table, we show that we can get quite close to our moments, and we show that we need both mechanisms to get close to our moments. We show without debt overhang, we end up estimating that types are stronger, and without heterogeneity, we estimate that debt overhang is stronger. Our standard errors are reasonable, and our results pass the J-test at the 5% level.

We plot the results from the piecewise-linear regressions for all columns of Table 4.1 in Figure 4.1. We find that the model without heterogeneity in investment opportunities is able to produce very close estimates to those with heterogeneity. This said, the relationship has trouble getting *as* strong a nonlinearity as the data projects. Hence, when we estimate our model with both mechanisms, it finds a role for debt overhang, especially since debt

overhang can also partially explain the dispersion in mean growth rates. Our procedure finds that debt overhang alone has a lot of trouble explaining the data. It overestimates the relationship between risk-adjusted leverage and growth for firms with the highest leverage, but can get little action for firms with risk-adjusted leverage between -6 and -3. It also has trouble hitting the dispersion in mean growth rates we see in the data.

We also show further robustness in Appendix G. In Table 6.1, we hold fixed our estimates of  $\Delta_z$ ,  $\theta$ , and  $x$ , and we vary  $b$ . We show that varying  $b$  gets us far from our moments with the clear directionality we would expect. In Table 6.2, we turn off debt overhang and hold fixed our estimates of  $\Delta_z$  and  $x$ . We then vary  $\theta$ . Again, our moments move with the clear directionality we expect.

## 4.2 Remaining Calibration

We only estimated the key parameters in our model that affect the mechanisms of interest, so as to make the moment conditions that lead to local identification clear. In Table 4.2, we show our remaining calibration. We set  $\tau^d$  to 0.2 to match the value chosen in Leland (1998). This number is in range of different estimates used in corporate finance. We have run our estimation procedure varying  $\tau^d$  between 0.15 and 0.35, a range of values covering most of those found in the literature, and found similar results. We set the corporate tax rate,  $\tau$ , to 0 in our procedure so that when we perform counterfactuals the tax advantage is a pure distortion. Had corporate taxes been positive, the tax advantage could act as a

Table 4.2: Remaining Calibration

Parameter	Value
Tax advantage of debt, $\tau_d$	0.2
Retained value of the firm after bankruptcy, $\alpha$	0.8
Period length, $\Delta$	$\frac{1}{12}$
Exogenous exit rate, $\delta$	0.053
Discount rate, $r$	$\log(1.001)$
Elasticity of substitution across intermediate goods, $\rho$	4
Per-period entry cost, $n_e \Delta$	1
Total labor supply, $L$	1

subsidy to entry, which is a second-best policy of less interest to this paper. The intermediate good firm’s problem scales in taxes, so only aggregates will be different (not decision rules) had they been positive. Hence, we get the same estimates of  $b$ ,  $\theta$ ,  $x$ , and  $\Delta_z$  no matter the level of  $\tau$ . We choose  $\alpha$  to be 0.8, which is the upper bound of bankruptcy costs found in Bris, Welch, and Zhu (2006). Our estimation procedure gives similar results were we to vary  $\alpha$  toward 1. The overall welfare losses are decreasing in  $\alpha$ , as more productivity is lost in bankruptcy. Generally, raising  $\alpha$  will exaggerate differences between the debt overhang and no debt overhang cases, as there is more bankruptcy when there is no debt overhang, because firms take out more leverage. We set our period length to  $\frac{1}{12}$ , so each period represents one month. We set the exogenous exit rate to 0.053 to match the employment-weighted exit rate of U.S. establishments in 2007. Given this model is not on a balanced growth path, we need to have a high exogenous exit rate in order to hit the mean growth rate of firms with risk-adjusted leverage below -6, which is greater than 2% annually. We set  $r$  to be  $\log(1.001)$ . We set  $r$  close to one so that consumption in the planner’s problem is close to the highest feasible allocation; this is relevant because transition dynamics are slow. We choose  $\rho$  to be 4 to match  $\rho$  in Atkeson and Burstein (2014). This parameter does not affect firm decisions, only aggregates. Assuming  $\rho > 1$ , holding all other parameters fixed, the welfare losses from debt overhang are decreasing in  $\rho$ . If  $\rho \rightarrow \infty$ , the CES production function becomes linear, and, in turn, the losses from debt overhang tend toward 0 in the limit.<sup>9</sup> We normalize the entry cost and total labor supply to 1.

## 5 Counterfactuals

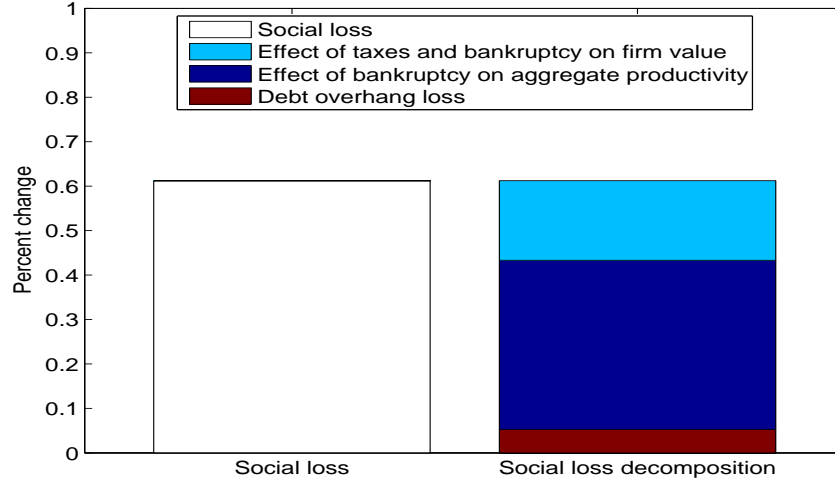
In Figure 5.1, we show the social losses from debt overhang are approximately 0.05%, and the gains from removing the tax advantage of debt are approximately 0.61%. We describe how we compute these numbers in Subsection 2.4.3. There are two other counterfactuals in this figure which we also explain how to compute in Subsection 2.4.3. The first other object is

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<sup>9</sup>See Acemoglu (2008) Chapter 2 for further discussion of the properties of the CES production function.



Figure 5.1: Social Losses and Losses from Debt Overhang



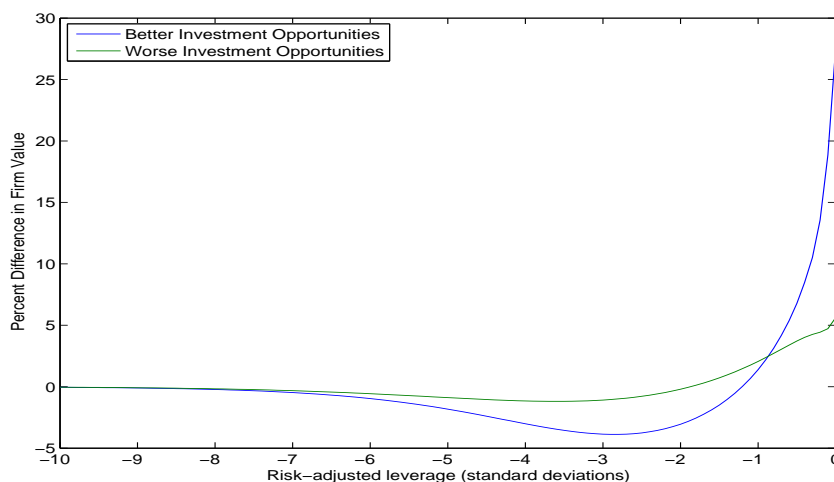
Social loss is the percentage difference between efficient consumption and baseline steady state consumption. Debt overhang loss is the percentage change between steady state consumption if there is no debt overhang against steady state consumption if there is debt overhang. Debt overhang loss is consumption in the steady state where debt overhang does not affect firm's investment decisions and steady state baseline consumption. The effect of bankruptcy on the total mass of productivity is the percentage change between steady state consumption if debt overhang does not affect firm investment and firms do not lose  $1 - \alpha$  of their productivity in bankruptcy and the steady state consumption if debt overhang does not affect firm investment decisions. The effect of taxes on bankruptcy is the percentage change between efficient consumption and steady state consumption if debt overhang does not affect firm investment and firms do not lose  $1 - \alpha$  of their productivity in bankruptcy.

the effect of bankruptcy on the total mass of productivity. Recall, the interpretation of  $\alpha$  in our model is that there is some mass of productivity which is being lost which will be costly to replace. We get rather large losses relative to the losses from debt overhang from this effect. There are a few points to make here given that there are other possible interpretations of  $\alpha$ . One could interpret the costs of bankruptcy as not destroying any productivity, but being costly in terms of labor. In this case, we have a smaller, but still significant blue bar. Another interpretation of  $\alpha$  is that the cost of bankruptcy is a direct financial transfer; this should make the dark blue bar zero. In these two cases, the white bar, the social loss, move close to proportionally with movements in the dark blue bar, and the light blue bar and red bars do not change by much. Even though our estimation procedure will not change with these different interpretations, the interpretation of bankruptcy is very important in translating the costs of bankruptcy into social losses. An interesting extension of this paper would be to allow  $\alpha$  to be endogenous as in Bernardo and Welch (2014) and reassess how firms' decisions change and what happens to social welfare under different interpretations of how bankruptcy is treated in aggregation.

The light blue bar is the effect of taxes and bankruptcy on firm value. This says that even without debt overhang, there are losses from firm decisions being distorted by the tax advantage and bankruptcy. Notice, the size of this effect is slightly larger than the losses from debt overhang.

## 5.1 Partial Equilibrium Firm Value Counterfactual

Figure 5.2: The Private Gains from Resolving the Debt Overhang Problem Conditional on Investment Opportunities and Risk-Adjusted Leverage



Percent difference in firm value compares firm value between two cases between which prices and the mass of firms remains constant: there is no debt overhang (the firm as a whole makes the investment decision) and equity holders make the investment decision, so there is debt overhang. The lines represent the percent differences in firm value between cases conditional on firm investment opportunities and risk-adjusted leverage.

Debt overhang in our model is a highly nonlinear problem. In Figure 5.2, we plot the percent difference in firm value between two firms, one that suffers from debt overhang (so equity holders make the investment decision), and one that does not (so the firm as a whole makes the investment decision), and compare their value functions assuming prices and the mass of firms do not change. The gains for firms with good investment opportunities reach nearly 30% under our estimate. Firms with worse investment opportunities have lower gains as they already invest so much less. However, the average entering firm, because it enters with a modest risk-adjusted leverage has relatively low gains.

We estimate resolving the debt overhang problem presents gains of about 0.5% for the

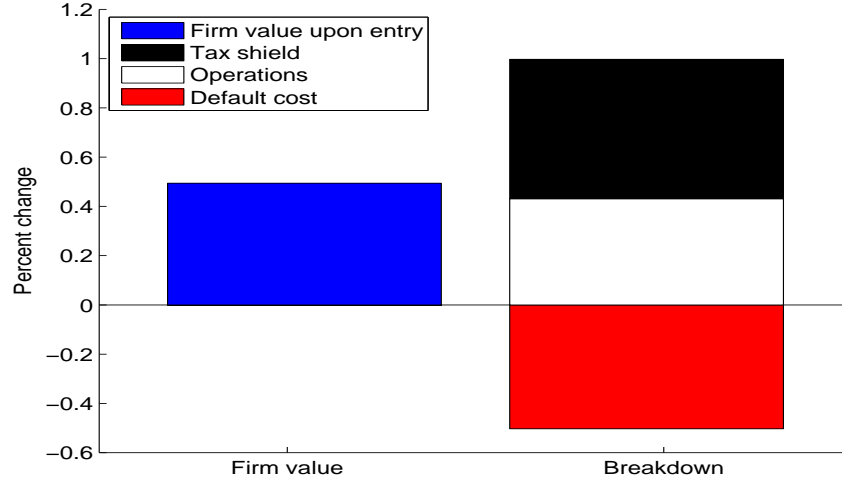
average entering firm (in terms of firm value) under our estimate. We are interested in what underlying general equilibrium effects are driving our results and how they compare with the literature that shows that debt overhang can destroy firm value. To answer these questions, we first perform a counterfactual experiment from Moyen (2007). To perform the same counterfactual as Moyen (2007), we hold fixed all general equilibrium effects that could affect firm value (prices, the labor allocation, and the supply of firms), and solve the model again, assuming there is no debt overhang. We then compare the difference between the value function upon entry with and without debt overhang. The blue bar on the left chart represents these gains as a percent of firm value upon entry, which are about 0.5%. Following Moyen (2007), we decompose these into the gains from operations, the gains from the tax advantage, and the losses from bankruptcy. The value from operations is the expected, discounted present value of the firm’s production and investment activities. The tax advantage of debt is the expected, discounted present value of all interest deductions. The default cost is the expected discounted, present value of the deadweight losses from bankruptcy. Most of the gains in partial equilibrium come from gains in terms of the value of operations, because the firm makes better investment decisions on average. The firm also anticipates that it will suffer less from debt overhang, so it takes on more debt. In turn, the average entering firm gains more from the tax shield, but also goes bankrupt more often, and these two effects offset.

## 5.2 Firm Value in General Equilibrium

In Subsection 2.5 in (2.15), we show that the value of the firm upon entry fully reflects movements in aggregate productivity, i.e. if we define the expected, discounted present value of the firm upon entry as  $V_A^E$ , we have that:

$$\Delta \log(w) = \Delta \log(V_A^E) = \Delta \frac{1}{\rho - 1} \log(Z). \quad (5.1)$$

Figure 5.3: How Firm Value Differs Between Steady States without General Equilibrium Effects



Firm Value upon entry is the percentage difference between firm value upon entry if a single firm does not suffer from debt overhang and firm value upon entry if it does suffer from debt overhang under our model estimates. The value from operations is the expected, discounted present value of the firm's production and investment activities. The tax shield is the expected, discounted present value of all interest deductions. The default cost is the expected discounted, present value of the deadweight losses from bankruptcy. The value of operations, the tax shield, and the default cost add up to firm value upon entry.

Aggregate productivity is a function of process innovation, product innovation, the aggregate bankruptcy rate, and the exogenous exit rate. The exogenous exit rate does not differ between steady states. When we remove the debt overhang distortion, the aggregate bankruptcy rate rises because firms take out more leverage as they anticipate they will suffer less from debt overhang when they are more highly levered, which lowers aggregate productivity. Product innovation falls when wages rise because entry is then more expensive. Process innovation rises because firms do not suffer from debt overhang, but the extent to which it rises is dampened because innovation is more expensive.

The gains in partial equilibrium to a single firm from removing the debt overhang distortion are an order of magnitude higher than the gains from removing the distortion for all firms in general equilibrium. In our framework, the response of prices, product innovation, and the aggregate bankruptcy rate all act to dampen the long-run consumption gains from removing the debt overhang distortion.

## 6 Conclusion

Through the lens of a traditional trade-off theory model of capital structure with endogenous investment in the face of a convex cost, we have shown how firm growth is related to firm risk-adjusted leverage, and assess the extent to which this relationship is driven by firm heterogeneity in investment opportunities rather than debt overhang. With our estimated model, we then infer the social costs of resolving the debt overhang problem. That this framework is able to match a number of key cross-sectional facts on firm growth, and that it is tractable, makes it amenable for extensions, especially for assessment of policies which mean to mitigate the impact of debt overhang over the business cycle.

## Appendix A

### Dataset Construction

Our empirical analysis relies on data from U.S. public firms.<sup>10</sup> We take daily stock returns from CRSP and merge them with the Industrial Annual Data from the CRSP/Compustat Merged database to get firm-level balance sheet measures.

We use stock returns and market capitalization data from CRSP. Market capitalization is defined as *closing price*  $\times$  *shares outstanding*, and is the data equivalent of the value of equity holders in the model. We define the value of debt in the data in three ways for sensitivity analysis, all of which give us similar results. In the tables in the paper, we define the value of debt as short-term debt  $+$   $\frac{1}{2}$  long-term debt, where short-term debt is defined as the max of debt in current liabilities (data item 34) and total current liabilities (data item

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<sup>10</sup>For a given date, we aggregate all data to the *permco* level, not the *gvkey* level or *permno* level, as we are interested in firms and not subentities or establishments. For each measure of interest, be it returns or a balance sheet measure, if there are multiple *gvkeys* or *permnos* for a given *permco* and *date* pair, we determine if it is a duplicate or from a separate part of the firm. For any two *gvkeys* or *permnos* for a given *permco* and *date* pair, if the observations are duplicates, we drop one of them; if not, then we create a weighted mean or a sum, depending the variable of interest, weighted by the respective market capitalization of the subentity to create a firm level measure. For further details, the SAS code is available upon request.

5), and long-term debt is data item 9. In the second case, we define the value of debt as total liabilities, or data item 181. Finally, we also define the value of debt as in Gomes and Schmid (2010), who define it as the book value of assets minus the book value of equity. Book assets is data item 6. Book equity is defined as the book value of stockholders' equity (data item 216), plus balance sheet deferred taxes and investment tax credit (data item 35), less the book value of preferred stock which is created using redemption, liquidation, or par value (data items 56, 10, and 130). Employment is defined as data item 29. Sales is defined as data item 12. We use year-ahead growth numbers for employment and sales growth. The other variable we include in our regression is age, which is a measure that we create from the CRSP/Compustat Merged database. To create our age measure, we download the entire time series for stock returns for each firm (*permco*) from CRSP. For each date, for each firm, dating back to 1926, we then state the age of a firm is 1 if it is the first date that shows up for the given firm. The age will then be two the next year and so on.

For our analysis, we only use data from 1982-2011. We keep only firms with December fiscal year-ends, but our results are generally robust to using firms with any fiscal year-end months. Following Hennessy and Whited (2007), we clean the data by trimming each series at the 2nd percentile, except for the data that are inherently bounded in nice ranges, e.g. market leverage or some log measures. Our results are generally robust to when we trim at the 5th, 2nd or 1st percentiles, or winsorize the data at these respective percentiles. We trim our measure of risk-adjusted leverage, which we explain how to construct in the body of the paper, to be between -10 and 0; our results are generally robust to being winsorized at -10 and 0, or looking at wider ranges. Following Hennessy and Whited (2007), we exclude firms with primary SIC codes between 4000 and 4999, between 6000 and 6999, or greater than 9000, as our model is not representative of regulated, financial, or public service firms.

# Appendix B

## Methodology for Computing $V_A$ and $\sigma_A$

We follow a procedure consistent with Bharath and Shumway (2008) and Gilchrist and Zakrajsek (2012) in measuring firm  $V_A$  and  $\sigma_A$ , whose procedures are all in the spirit of Merton (1974).  $V_A$  is the value of assets,  $V_B$  is the value of debt,  $\mu_A$  is the mean rate of asset growth, and  $\sigma_A$  is asset volatility. We provide a detailed description of how we measure and clean the data in Appendix A. We recover  $V_A$  and  $\sigma_A$  from the data closely following the procedure outlined by Gilchrist and Zakrajsek (2012). For each firm, we linearly interpolate our quarterly value of debt to a daily frequency. We use daily data on the market value of equity; call this  $V_E$ . We guess a value of asset volatility,  $\sigma_A = \sigma_E \frac{V_B}{V_E + V_B}$ , where the standard deviation of equity is calculated as the square root of the annualized 21-day moving average of squared returns for a firm. Here, we differ from Gilchrist and Zakrajsek (2012), in that they choose a 252-day horizon for the moving average. To recover the value of assets and the volatility of assets, we follow the procedure outlined in Merton (1974). For robustness, we also check if our results are robust to a Leland (1994) measure of  $V_A$  and  $\sigma_A$ .<sup>11</sup>

Given our guess of  $\sigma_A$ , we can then use equation

$$V_E(t) = V_A(t)\Phi(d_1) - e^{-r(T-t)} * V_B\Phi(d_2)$$

where  $d_1 = \frac{V_A + (r + \frac{1}{2}\sigma_A^2)T}{\sigma_A\sqrt{T}}$  and  $d_2 = d_1 - \sigma_A\sqrt{T}$  to recover the value of assets. We define  $r$  to be the one-year Treasury-constant maturity, which we take from the Federal Reserve's H.15 report. After converging on  $V_A$  for the given  $\sigma_A$ , we recompute  $\sigma_A$  from our implied  $V_A$  using the same methodology we use to compute  $\sigma_E$ . We ultimately converge on  $\sigma_A$  through

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<sup>11</sup>We follow Appendix A of Atkeson, Eisfeldt, and Weill (2013) to recover the Leland (1994) measures of  $V_A$  and  $\sigma_A$ . We guess  $\sigma_A$ , which implies  $\theta = \frac{r - \delta - \frac{\sigma_A^2}{2} + \sqrt{(r - \delta - \frac{\sigma_A^2}{2}) + 2\sigma_A^2 r}}{\sigma_A^2}$ . We set  $\delta = .01$ .  $\theta$  implies a  $V_{A^*} = V_B * \max\{.5, \frac{\theta}{1+\theta}\}$ . We can then recover  $V_A = V_E + V_B + (V_A - V_B)e^{-\theta \log(\frac{V_A}{V_{A^*}})}$ . As in our other procedure, for each  $\sigma_A$ , we iterate until convergence on  $V_A$ , and ultimately converge on  $\sigma_A$  through a slow-updating procedure.

a slow-updating procedure.<sup>12</sup>

## Appendix C

### Aggregation in a Steady State

Our aggregation is equivalent to a one country version of Atkeson and Burstein (2010) with a per-unit subsidy,  $\tau^s$ , on the production of the consumption good. We present our model aggregation below. We first note that:

$$\pi(z) = e^z Y (1 + \tau^s)^\rho w^{-\rho} \frac{1}{\rho^\rho (\rho - 1)^{1-\rho}}.$$

It is then useful to define:

$$\Pi = Y (1 + \tau^s)^\rho w^{-\rho} \frac{1}{\rho^\rho (\rho - 1)^{1-\rho}}.$$

The choice of labor by the intermediate good firm is:

$$l(z) = e^z Y (1 + \tau^s)^\rho \left( \frac{\rho - 1}{\rho} \right)^\rho w^{-\rho},$$

which is proportional in  $e^z$ .

Define the steady state scaled distribution of firms across states as  $\tilde{M}(z, d, \theta)$ . We then find scaled aggregate productivity as:

$$\tilde{Z} = \sum_{\theta} \int \int e^z \tilde{M}(z, d, \theta) dz dd.$$

Another useful aggregate to define is average expenditures on the research good per

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<sup>12</sup>We iterate on both  $\sigma_A$  and  $V_A$  until they converge to a tolerance of 1e-5. We choose updating parameters for the slow-updating procedure on  $V_A$  and  $\sigma_A$ , .25 and .15, respectively, such that 100% of firms converge.



entering firm, which we denote by  $\Upsilon$ :

$$\Upsilon = n_e + \sum_{\theta} \int \int e^z \theta^{-b} e^{bq} \tilde{M}(z, d, \theta) dz dd.$$

Given  $\Pi$ ,  $\tilde{Z}$ , and  $\Upsilon$ , we can recover the following equilibrium objects:

$$W = (1 + \tau^s) \frac{\rho - 1}{\rho} (M_e \tilde{Z})^{\frac{1}{\rho-1}}.$$

$$Y = (M_e \tilde{Z})^{\frac{1}{\rho-1}} (L - L_r).$$

$$L_r = \frac{1}{\rho \xi} L,$$

where  $\xi = \frac{\Pi Z}{\Upsilon}$  is the ratio of total variable profits to total expenditures on the research good. Total aggregate productivity is then:

$$Z = (M_e \tilde{Z})^{\frac{1}{\rho-1}}.$$

## Appendix D

### Overview of the Solution Procedure

#### Scaled Bellman Functions

We can write the Bellman equations of equity holders and creditors as scaled problems since our problem is homogenous of degree one in  $e^z$  and  $d$ . We then have only two state variables: the number of steps,  $i$ , to bankruptcy (spaced by  $\Delta_z$ ) and the type,  $\theta \in \{\theta_H, \theta_L\}$ . Defining  $i \equiv \frac{z - z^*(d, \theta)}{\Delta_z}$  where  $z^*(d, \theta)$  is the bankruptcy threshold of a given type, we redefine the the

expected, discounted present value of equity holders to be:

$$V_E(i, \theta) = \max_p \left\{ 0, (1 - \tau)(\Pi - \theta^{-b}e^{bq}) + \bar{a}e^{-i\Delta z} + \beta \left( \sum_{\theta'} P_{\theta, \theta'} \left( qe^{\Delta z} V_E(i + 1, \theta') + (1 - q)e^{-\Delta z} V_E(i - 1, \theta') \right) \right) \right\}, \quad (6.1)$$

where  $P_{\theta_{-1}, \theta}$  is the probability of transitioning from having investment opportunity  $\theta_{-1}$  to  $\theta$ ,  $\beta = e^{-r}(1 - \delta)$ , and  $\bar{a}$  is a constant defined such that  $z^*(d, \theta_1) = \log(\frac{(1-\tau^d)d}{\bar{a}})$ . Notice, in the paper, we assume you always stay the same type throughout your life, so  $P_{\theta_H, \theta_H} = 1$  and  $P_{\theta_L, \theta_L} = 1$ . Note, for  $\theta = \theta_H$ , the lowest  $i$  for which the firm does not exit is  $i = 0$ . For  $\theta \neq \theta_1$ , this lowest  $i$  is pinned down endogenously as the lowest  $i$  where 0 is not the chosen in the maximization above. This choice of a threshold  $\underline{i}(d, \theta)$  implies that the exit decision has  $z^*(d, \theta) = z^*(d, \theta_H) + \Delta_z \underline{i}(d, \theta)$ .

To pin down  $q$ , we use the first-order condition:

$$(1 - \tau)\theta^{-b}be^{bq} = \beta \sum_{\theta'} P_{\theta, \theta'} \left( e^{\Delta z} V_E(i + 1, \theta') - e^{-\Delta z} V_E(i - 1, \theta') \right). \quad (6.2)$$

We now define the scaled expected, discounted present value of the firm,  $V_F(i, \theta)$ :

$$V_F(i, \theta) = \begin{cases} \max_{i'} \alpha V_A(i', \theta) & z < z^*(d, \theta) \\ (1 - \tau)(\Pi - \theta^{-b}e^{bq}) + \frac{\tau^d}{1 - \tau^d} \bar{a}e^{-i\Delta z} \\ + \beta \sum_{\theta'} P_{\theta, \theta'} \left( qe^{\Delta z} V_A(i + 1, \theta') + (1 - q)e^{-\Delta z} V_A(i - 1, \theta') \right) & \textit{else.} \end{cases}$$

The unscaled problems can then be recovered as  $V_E(z, d, \theta) = e^z V_E(\frac{z\bar{a} - (1 - \tau^d)d}{\bar{a}\Delta z}, \theta)$  and  $V_A(z, d, \theta) = e^z V_A(\frac{z\bar{a} - (1 - \tau^d)d}{\bar{a}\Delta z}, \theta)$ .

Notice, if  $\theta$  were stochastic, because  $\theta$  is realized before the investment decision is made, but after the period starts, we need to account for the fact that some firms may go bankrupt

even though they move up a step in productivity when we solve for the stationary distribution.

The value of creditors is defined as the difference between the value of owners of the firm and the value of equity holders; thus,

$$V_B(z, d, \theta) = V_A(z, d, \theta) - V_E(z, d, \theta).$$

## Solution for $q_\infty$

If we allow  $i \rightarrow \infty$  in (6.2), we find:

$$(1 - \tau)\theta^{-b}be^{bq} = \beta \left( e^{\Delta z} - e^{-\Delta z} \right) * \sum_j \varphi_{Aj} V_{\infty,j}, \quad (6.3)$$

where  $\theta = A \in \{\theta_H, \theta_L\}$  and  $\varphi_{Aj}$  is the probability of going from type A to type  $j \in \{\theta_H, \theta_L\}$ .<sup>13</sup>

A little rearranging of (6.3)  $\implies$

$$\sum_j \varphi_{Aj} V_{\infty,j} = \frac{(1 - \tau)\theta_A^{-b}be^{bq}}{\beta(e^{\Delta z} - e^{-\Delta z})}. \quad (6.4)$$

We also have the following equation for  $V_E(\infty, \theta)$  from (6.1):

$$V_{\infty,A} = (1 - \tau) \left( \Pi - \theta_A^{-b}e^{bq} \right) + \beta \left( q_{\infty,A} \left( e^{\Delta z} - e^{-\Delta z} \right) + e^{-\Delta z} \right) * \sum_j \varphi_{Aj} V_{\infty,j}. \quad (6.5)$$

If we sum (6.5) over both types, we then recover:

$$\begin{aligned} \sum_A \varphi_{jA} V_{\infty,A} &= \sum_A \varphi_{jA} (1 - \tau) \left( \Pi - \theta_A^{-b}e^{bq} \right) \\ &\quad + \sum_A \varphi_{jA} \beta \left( q_{\infty,A} \left( e^{\Delta z} - e^{-\Delta z} \right) + e^{-\Delta z} \right) \sum_j \varphi_{Aj} V_{\infty,j}. \end{aligned} \quad (6.6)$$

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<sup>13</sup>Recall, in this paper, we set the probability of switching types to 0.

In turn, plugging (6.4) into (6.6), we get:

$$\begin{aligned} \frac{(1-\tau)\theta_j^{-b}be^{bq}}{\beta(e^{\Delta z} - e^{-\Delta z})} &= \sum_A \varphi_{jA}(1-\tau) \left( \Pi - \theta_A^{-b}be^{bq} \right) \\ &+ \sum_A \varphi_{jA} \beta \left( q_{\infty,A} \left( e^{\Delta z} - e^{-\Delta z} \right) + e^{-\Delta z} \right) \frac{(1-\tau)\theta_A^{-b}be^{bq}}{\beta(e^{\Delta z} - e^{-\Delta z})}. \end{aligned} \tag{6.7}$$

Now, if we assume symmetry, as we do in the paper, we can say  $\theta_H = m$  and  $\theta_L = \frac{1}{m}$ . Given  $q_{\infty,H}$ , we use (6.7) to recover  $q_{\infty,L}$ . In the paper, we target  $q_{\infty,H}$  to hit a moment in the data. When we allow  $q_{\infty,H}$  to vary in the counterfactuals, we can also just guess  $q_{\infty,H}$ , and iterate on it until free entry converges. Then, from (6.6), we can recover  $V_{\infty,H}$  and  $V_{\infty,L}$ .

## Further Details of the Procedure

We use the following algorithm to solve the intermediate-good firm's problem in a stationary equilibrium after recovering  $q_{\infty}$  and  $V_{\infty}$  of the two types.

We guess a value for  $\bar{a}$  and set  $\Pi = 1$ .  $V_E$  and  $V_A$  can then be scaled by  $\Pi$  such that  $\Pi$  can be recovered from the free entry condition. We also note the implied value of  $\bar{a}_{imp} = \bar{a} - \frac{V_E(z_d^*(\theta_H), \theta_H)}{\Delta}$ . So, we perform value function iteration on  $V_E$  while slowly updating on  $\bar{a}$  until the functions converge below a very low tolerance parameter.

With our value of  $\bar{a}$  and decision rules  $q$ , we then perform value function iteration on  $V_A$ . With the optimal leverage decision,  $n^*$ , we can then recover  $\Pi$  from the free entry condition:

$$\Pi = \frac{\frac{n_e}{\Delta}}{e^{-r} \sum_{\theta} \int V_{A,n_{\theta}^*} e^{\bar{z}} dz}.$$

We then recover  $\tilde{Z}$  and  $\Upsilon$ , and aggregate the model as described in Appendix C.

# Appendix E

## Closed Form Approximations

We show tight closed form approximations to the decision rules in our problem. The approximations rely on the fact that the value of equity holders when  $q$  is not endogenous is a close approximation to the value of equity holders when  $q$  is endogenous. Our approximations help to provide intuition as to how  $b$  drives the decision rules in our model, and may be useful in other iterations of trade-off models with endogenous investment. We will show results for a single type to make the math clearer; however, it can be easily extended to multiple types.

### Closed Form Approximation for Value of Equity Holders

Excusing the abuse of notation, the problem of equity holders,  $V^E$ , takes the form:

$$V_n^E = b + cd^n + eV_{n+1}^E + fV_{n-1}^E \text{ for } n > 0.$$

Also,

$$V_0^E = 0 = b + c + eV_1^E. \tag{6.8}$$

This problem has a closed form solution:

$$V_n^E = c_1 \left( \frac{1 - \sqrt{1 - 4ef}}{2e} \right)^n - \frac{b}{e + f - 1} - \frac{cd^{n+1}}{d(ed - 1) + f}.$$

Plugging this into (6.8), we find that:

$$c_1 = \frac{b}{e + f - 1} + \frac{cd}{d(ed - 1) + f}.$$

So,

$$V_n^E = \left( \frac{b}{e+f-1} + \frac{cd}{d(ed-1)+f} \right) \left( \frac{1-\sqrt{1-4ef}}{2e} \right)^n - \frac{b}{e+f-1} - \frac{cd^{n+1}}{d(ed-1)+f}.$$

We note that  $c$  is endogenous. We can recover it using (6.8) and noting:

$$eV_1^E = e \left( \left( \frac{b}{e+f-1} + \frac{cd}{d(ed-1)+f} \right) \left( \frac{1-\sqrt{1-4ef}}{2e} \right)^1 - \frac{b}{e+f-1} - \frac{cd^2}{d(ed-1)+f} \right).$$

We then find that

$$c = -b \frac{d(ed-1)+f}{e+f-1} \frac{(f-1+\frac{1}{2}(1-\sqrt{1-4ef}))}{(f-d+d\frac{1}{2}(1-\sqrt{1-4ef}))}.$$

Thus, we find that the value function takes the form:

$$V_n^E = \frac{b}{1-e-f} \left( 1 - d^{n+1} \frac{(f-1+\frac{1}{2}(1-\sqrt{1-4ef}))}{(f-d+d\frac{1}{2}(1-\sqrt{1-4ef}))} - \frac{(f-df)}{(f-d+d\frac{1}{2}(1-\sqrt{1-4ef}))} \left( \frac{1-\sqrt{1-4ef}}{2e} \right)^n \right). \quad (6.9)$$

We can translate  $b$ ,  $c$ ,  $d$ ,  $e$ , and  $f$  into parameters in our model. Given that we set  $\Pi = 1$ , this implies that in the above,  $b = \Delta(1-\tau)(1-\theta^{-b}e^{bq_\infty})$ ,  $d = e^{-\Delta z}$ ,  $e = \beta e^{\Delta z} q_\infty$ , and  $f = \beta e^{-\Delta z}(1-q_\infty)$ .

Plugging in the above parameters into (6.9), we find:

$$V_n^E = \frac{(1-\tau)\Delta(1-\theta^{-b}e^{bq_\infty})}{1-\beta(e^{\Delta z}q_\infty + e^{-\Delta z}(1-q_\infty))} \left( 1 - e^{-\Delta z n} \frac{(1-\beta e^{-\Delta z}(1-q_\infty) - \frac{1}{2}(1-\sqrt{1-4\beta^2 q_\infty(1-q_\infty)}))}{(1-\beta(1-q_\infty) - \frac{1}{2}(1-\sqrt{1-4\beta^2 q_\infty(1-q_\infty)}))} + \frac{\beta(1-q_\infty)(1-e^{-\Delta z})}{(1-\beta(1-q_\infty) - \frac{1}{2}(1-\sqrt{1-4\beta^2 q_\infty(1-q_\infty)}))} \left( \frac{1-\sqrt{1-4\beta^2 q_\infty(1-q_\infty)}}{2\beta e^{\Delta z} q_\infty} \right)^n \right). \quad (6.10)$$

We are interested in characterizing  $q$ . We will use the fact that (6.10) is a close approximation to the value of equity holders were the choice of  $q$  endogenous. To start, we characterize:

$$V_{n+1}^E e^{\Delta z} - V_{n-1}^E e^{-\Delta z}.$$

We note that  $\theta^{-b}$  is implied by a choice of  $q_\infty$  at calibration, i.e.:

$$\theta^{-b} = \frac{\beta(e^{\Delta z} - e^{-\Delta z})}{e^{bq_\infty \left( \beta(e^{\Delta z} - e^{-\Delta z}) + b \left( 1 - \beta(e^{\Delta z} q_\infty + e^{-\Delta z} (1 - q_\infty)) \right) \right)}}.$$

With some simplification, we find  $V_{n+1}^E e^{\Delta z} - V_{n-1}^E e^{-\Delta z}$  to be:

$$\begin{aligned} & \frac{(1 - \tau)\Delta b}{\beta(e^{\Delta z} - e^{-\Delta z}) + b(1 - \beta(e^{\Delta z} q_\infty + e^{-\Delta z} (1 - q_\infty)))} \left( (e^{\Delta z} - e^{-\Delta z}) + \right. \\ & \left. \left( \frac{1 - \sqrt{1 - 4\beta^2 q_\infty (1 - q_\infty)} - 2\beta^2 q_\infty}{q_\infty (1 - \sqrt{1 - 4\beta^2 q_\infty (1 - q_\infty)})} \right) \right. \\ & \left. * \left( \frac{(1 - q_\infty)(1 - e^{-\Delta z})}{(1 - \beta(1 - q_\infty) - \frac{1}{2}(1 - \sqrt{1 - 4\beta^2 q_\infty (1 - q_\infty)})} \right) \right. \\ & \left. * \left( \frac{(1 - \sqrt{1 - 4\beta^2 q_\infty (1 - q_\infty)})^n}{2\beta e^{\Delta z} q_\infty} \right)^n \right). \end{aligned} \quad (6.11)$$

## Closed Form Approximation for $q$

The optimal choice of  $q^*$  in the scaled problem (see Appendix D) can be written as:

$$q^* = \frac{1}{b} \log \left( \beta \frac{V_{n+1}^E e^{\Delta z} - V_{n-1}^E e^{-\Delta z}}{(1 - \tau)\Delta b} \right) + \log(\theta). \quad (6.12)$$

Plugging (6.11) into (6.12), we find:

$$q^* = q_\infty + \frac{1}{b} \log \left( 1 - \frac{(1 - \sqrt{1 - 4\beta^2 q_\infty (1 - q_\infty)}) - 2\beta^2 q_\infty (1 - e^{-\Delta z})}{(\beta(1 - \sqrt{1 - 4\beta^2 q_\infty (1 - q_\infty)}) - 2\beta^2 q_\infty)(e^{\Delta z} - e^{-\Delta z})q_\infty} \right) \\ * \left( \frac{1 - \sqrt{1 - 4\beta^2 q_\infty (1 - q_\infty)}}{2\beta e^{\Delta z} q_\infty} \right)^n.$$

Note that since  $\left( \frac{1 - \sqrt{1 - 4\beta^2 q_\infty (1 - q_\infty)}}{2\beta e^{\Delta z} q_\infty} \right) < 1$ , as  $n \rightarrow \infty$ ,  $q \rightarrow q_\infty$ . If we have multiple types, then around a baseline of  $\theta = 1$ , we have that:

$$q^* = q_{\infty,1} + \frac{1}{b} \log \left( 1 - \frac{\left( 1 - \sqrt{1 - 4\beta^2 q_\infty (1 - q_\infty)} - 2\beta^2 q_\infty \right) \left( 1 - e^{-\Delta z} \right)}{\left( \beta \left( 1 - \sqrt{1 - 4\beta^2 q_\infty (1 - q_\infty)} \right) - 2\beta^2 q_\infty \right) (e^{\Delta z} - e^{-\Delta z}) q_\infty} \right) \\ * \left( \frac{1 - \sqrt{1 - 4\beta^2 q_\infty (1 - q_\infty)}}{2\beta e^{\Delta z} q_\infty} \right)^n + \log(\theta),$$

where  $q_{\infty,1}$  is the value of  $q_\infty$  when  $\theta=1$ .

Now, that we have a closed form approximation for  $q$ , we want to analyze its properties.

$$\frac{\partial q}{\partial n} = \frac{1}{b} \frac{\log(c_4)}{c_5 c_4^n - 1},$$

where

$$c_4 = \frac{2\beta e^{\Delta z} q_\infty}{1 - \sqrt{1 - 4\beta^2 q_\infty (1 - q_\infty)}},$$

and

$$c_5 = \frac{\left( \beta \left( 1 - \sqrt{1 - 4\beta^2 q_\infty (1 - q_\infty)} \right) - 2\beta^2 q_\infty \right) (e^{\Delta z} - e^{-\Delta z}) q_\infty}{\left( 1 - \sqrt{1 - 4\beta^2 q_\infty (1 - q_\infty)} - 2\beta^2 q_\infty \right) (1 - e^{-\Delta z})}.$$

Note these constants depend only on parameters  $q_\infty, \beta$ , and  $\Delta_z$ .



Note  $c_4, c_5 > 1$ . Thus,  $\frac{\partial q}{\partial n} \geq 0$ .

$$\frac{\partial^2 q}{\partial n^2} = -\frac{1}{b} \log(c_1)^2 \frac{c_2 c_1^n}{(c_2 c_1^n - 1)^2},$$

where  $c_1$  and  $c_2$  are defined in the same manner as  $c_4$  and  $c_5$ . Since  $c_1, c_2 > 0$ ,  $\frac{\partial^2 q}{\partial n^2} < 0$ .

The choice of  $q$  is an upper bound under any parameterizations considered in the paper. This is because the  $V^E$  is a lower bound relative to when  $V^E$  depends on the choice of  $q$ , since in one case equity gets to make the optimal choice of  $q$ , whereas in the other case, it has to take  $q$  as given.

One can go much further with these approximations, showing approximate stationary distributions and the closed form solution for  $V^A$ , but at this point, we have demonstrated the mechanism and shown the approximations we use to improve the speed of our algorithm, so we stop here.

## Appendix F

### Variance of Growth Rates

Recall, productivity evolves according to a binomial process. With probability  $q$ , firms move up one step in productivity,  $\Delta_z$ , and with probability  $(1 - q)$ , firms move down one step in productivity,  $-\Delta_z$ . Also, firms choose  $q$  conditional on their productivity,  $z$ , coupon level,  $d$ , and type,  $\theta$ .

Denote the relative mass of firms at a given state  $(z, d, \theta)$  as:  $F(z, d, \theta) = \frac{M(z, d, \theta)}{\sum_{\theta} \int \int M(z, d, \theta)}$  where  $M(z, d, \theta)$  denotes the mass of firms for a given  $(z, d, \theta)$  in steady state. We can then write the average  $q$  of the economy,  $\bar{q}$ , as:

$$\bar{q} = \sum_{\theta} \int \int F(z, d, \theta) q(z, d, \theta) dz dd.$$

In turn, we can then write down the average one-period growth rate of the economy, measured

in log changes in productivity levels as:

$$\bar{q}\Delta_z - (1 - \bar{q})\Delta_z = 2\bar{q}\Delta_z - \Delta_z.$$

The per-period variance of growth rates is then:

$$\begin{aligned} &= \sum_{\theta} \int \int F(z, d, \theta) q(z, d, \theta) \left( \Delta_z - \left( 2\bar{q}\Delta_z - \Delta_z \right) \right)^2 dzdd + \\ &\sum_{\theta} \int \int F(z, d, \theta) (1 - q(z, d, \theta)) \left( -\Delta_z - \left( 2\bar{q}\Delta_z - \Delta_z \right) \right)^2 dzdd, \end{aligned}$$

from the formula  $V(x) = E[x - \bar{x}]^2$ .

$$\begin{aligned} &= \sum_{\theta} \int \int F(z, d, \theta) q(z, d, \theta) \left( \Delta_z - \left( 2\bar{q}\Delta_z - \Delta_z \right) \right)^2 dzdd \\ &+ \sum_{\theta} \int \int F(z, d, \theta) (1 - q(z, d, \theta)) \left( -\left( 2\bar{q}\Delta_z \right) \right)^2 dzdd. \\ &= 4\Delta_z^2 \left( 1 - \bar{q} \right)^2 \sum_{\theta} \int \int F(z, d, \theta) q(z, d, \theta) dzdd \\ &+ 4\Delta_z^2 \bar{q}^2 \sum_{\theta} \int \int F(z, d, \theta) (1 - q(z, d, \theta)) dzdd. \end{aligned}$$

Now notice that we can use the definition of  $\bar{q}$  to simplify it further:

$$\begin{aligned} &= 4\Delta_z^2 \left( 1 - \bar{q} \right)^2 \bar{q} + 4\Delta_z^2 \bar{q}^2 \left( 1 - \bar{q} \right). \\ &= 4\Delta_z^2 \bar{q} (1 - \bar{q}) \left( (1 - \bar{q}) + \bar{q} \right). \\ &= 4\Delta_z^2 \bar{q} (1 - \bar{q}). \end{aligned}$$

Thus, all we need to recover the per-period variance of growth rates is the step size,  $\Delta_z$ , and the average per-period growth rate of the economy, summarized by  $\bar{q}$ .

# Appendix G

## Robustness Tables and Figures

Table 6.1: Varying Convexity of Cost Function:  $b$

	<i>U.S. Data</i>	<i>Estimate (S.E.)</i>	<i>Strong Overhang</i>	<i>No Debt Overhang</i>
<i>Estimated Parameters</i>				
Convexity of cost function, $b$	–	35.81 (2.39)	10	$\infty$
$f(\theta)$ , Mean % diff in growth rts. b/w types	–	5.29 (0.025)	5.29	5.29
$f(\Delta_z)$ , Avg. ann. standard deviation of high $lv_g^*$ firm if no debt overhang	–	22.73 (0.0041)	22.73	22.73
$P(\theta = \theta^H)$ , Probability of entry as high type firm	–	0.79 (0.023)	0.79	0.79
<i>Targets</i>				
Dispersion in growth rates	23.3	23.32	24.38	22.86
Dispersion in mean growth rates across firms	7.96	7.92	10.37	6.75
Relationship between $lv_g^*$ & growth, $lv_g^* < -6$	-0.1	-0.23	-0.92	-0.25
Relationship between $lv_g^*$ & growth, $lv_g^* \in [-6, -3]$	-1.07	-1.03	-1.48	-0.4
Relationship between $lv_g^*$ & growth, $lv_g^* > -3$	-1.56	-1.54	-3.15	-0.026

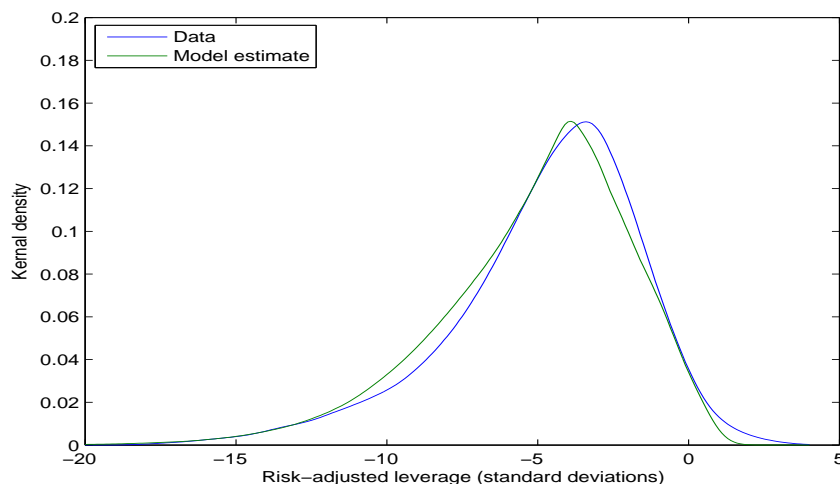
\*  $lv_g^*$  is risk-adjusted leverage. Risk-adjusted leverage in the data is the residuals from a regression on industry dummies, year dummies, age effects, size effects, and the Whited-Wu index. Risk-adjusted leverage is measured in units of the number of standard deviations of annual asset volatility by which the firm's assets must change to equal the firm's book value of debt. A value of -1 implies the firm is 1 standard deviation from its book value of debt exceeding its assets, -2 implies the firm is 2 standard deviations from its book value of debt exceeding its assets, and so on. Growth is employment growth.

Table 6.2: Turning off Types and Varying Convexity of Cost Function:  $b$

	<i>U.S.</i> <i>Data</i>	<i>Estimate</i> <i>(S.E.)</i>	<i>Estimate</i> –	<i>Strong</i> <i>Overhang</i>	<i>No Debt</i> <i>Overhang</i>
<i>Estimated Parameters</i>					
Convexity of cost function, $b$	–	35.81 (2.39)	35.81	10	$\infty$
$f(\theta)$ , Mean % diff in growth rts. b/w types	–	5.29 (0.025)	0	0	0
$f(\Delta_z)$ , Avg. ann. standard deviation of high $lv g^*$ firm if no debt overhang	–	22.73 (0.0041)	22.73	22.73	22.73
$P(\theta = \theta^H)$ , Probability of entry as high type	–	0.79 (0.023)	0.79	0.79	0.79
<i>Targets</i>					
Dispersion in growth rates	23.3	23.32	22.78	23.16	22.64
Dispersion in mean growth rates across firms	7.96	7.92	6.17	6.89	5.94
Relationship between $lv g^*$ & growth, $lv g^* < -6$	-0.1	-0.23	-0.041	-0.098	-0.015
Relationship between $lv g^*$ & growth, $lv g^* \in [-6, -3]$	-1.07	-1.03	-0.14	-0.39	0.038
Relationship between $lv g^*$ & growth, $lv g^* > -3$	-1.56	-1.54	-0.066	-2.99	0.56

\*  $lv g^*$  is risk-adjusted leverage. Risk-adjusted leverage in the data is the residuals from a regression on industry dummies, year dummies, age effects, size effects, and the Whited-Wu index. Risk-adjusted leverage is measured in units of the number of standard deviations of annual asset volatility by which the firm's assets must change to equal the firm's book value of debt. A value of -1 implies the firm is 1 standard deviation from its book value of debt exceeding its assets, -2 implies the firm is 2 standard deviations from its book value of debt exceeding its assets, and so on. Growth is employment growth.

Figure 6.1: Kernel Density of Risk-Adjusted Leverage: Model Estimates Against the Data



Risk-adjusted leverage in the data is the residuals from a regression on industry dummies, year dummies, age effects, size effects, and the Whited-Wu index. Risk-adjusted leverage is measured in units of the number of standard deviations of annual asset volatility by which the firm's assets must change to equal the firm's book value of debt. A value of -1 implies the firm is 1 standard deviation from its book value of debt exceeding its assets, -2 implies the firm is 2 standard deviations from its book value of debt exceeding its assets, and so on.

# Chapter 2: Resolving Debt Overhang over the Business Cycle

## 1 Chapter 2: Introduction

Debt overhang should always affect some firms, even when the economy is not in a recession, since there are always firms with high risk-adjusted leverage. In turn, recovering estimates from the *cross-section* should be useful in performing inference to how firms will respond to an aggregate shock. In this paper, we investigate the extent to which the costs of debt overhang are amplified by shocks to aggregate TFP, shocks to asset volatility, and financial shocks, in a model that has been estimated in the *cross-section*, i.e. Kurtzman and Zeke (2015b).

Our quantitative analysis leads to two sets of results. First, we find that resolving debt overhang dampens the negative effect of a negative TFP shock. *Ceteris paribus*, resolving debt overhang increases the average growth rate of firms. The average growth rate of firms is higher because firms invest more in process innovation, and, in general equilibrium, the increased demand for research increases the price of investing in research. In turn, the value of the firm at entry will increase, but the cost of entry will decrease, which makes the net effect on entry ambiguous. What follows is the net effect on aggregate productivity is ambiguous and may vary with the calibration. Under our estimate, however, the mass of entry rises as does productivity, output, and consumption, and the gains can be large.

Our second result relies on the modeling assumption that debt is long-lived. In our model, contrary to Arellano, Bai, and Kehoe (2012) and Gilchrist, Sim, and Zakrajsek (2014), to name a few papers, volatility shocks lead to only modest movements in aggregates. Volatility shocks decrease the value of creditors, but increase the option value of equity holders and alleviate the debt overhang problem. Given the firm only holds nominal long-lived debt which it cannot rechoose, firms will invest more and go bankrupt less. This finding sheds

light on the the maturity structure of debt as a key modeling assumption, as the literature that argues financial frictions can amplify the effect of uncertainty shocks over the business cycle usually assumes that debt is only one period and finds much stronger results.

In our model, there is a continuum of heterogeneous firms that produce differentiated products, and are financed with debt and equity. Firms differ in their productivity, since productivity has an idiosyncratic component that evolves randomly, according to a binomial process. The volatility of the binomial process is shared across firms and is time varying. Firm productivity has an aggregate component that is also time varying. Competitive final good firms aggregate the differentiated products of the intermediate good firms into a final good using a constant elasticity of substitution production function. New entrants into the economy, as well as a continuum of identical households, consume the final good. Households also inelastically supply labor each period, hold the equity and debt of the intermediate good firms, receive dividend and coupon payments, and receive lump-sum transfers from the government.

Conditional on its productivity, the intermediate good firm makes production decisions using a constant returns to scale technology in labor. Each period, the firm enters with a productivity level, investment opportunities, and a coupon level. It has some probability of exiting exogenously. If it survives, equity holders decide whether or not to go bankrupt and then make the investment decision. If the firm does go bankrupt, creditors seize the firm (in effect, gaining full equity stake), the firm loses a fixed proportion of its productivity, and the firm makes a new leverage decision. This decision is made to maximize the joint value of equity holders and creditors.

The firm has an investment technology through which it can lower the marginal cost of production, and equity holders invest in this technology in the face of a convex cost function. Firms also differ in the inherent productivity of this technology for investing: for some firms it is cheap to invest to lower marginal cost and thus grow sales and profits, for others it is expensive to do so; hence, firms differ in their *investment opportunities*.

In equilibrium, this implies that firms are investing to improve the drift of their binomial process. Firms with worse investment opportunities have lower drift and firms with better investment opportunities have higher drift. Firms can costlessly issue new equity, but we only allow firms to hold long-lived debt. Firm production only requires labor. On the other hand, investment requires a research good, which is produced by a competitive research good sector with Cobb-Douglas production function. The research good is a composite of the final good and labor.

At any time, new firms can pay a fixed cost to enter with a new differentiated product. This fixed cost is priced in the same units as investment. We call investment by incumbent firms *process innovation*, and investment by entering firms, *product innovation*. After entering, an intermediate good firm realizes its investment opportunities and its initial level of productivity. Firms then face a classical trade-off: firms can then take out debt because it has a tax advantage, but do not fully finance themselves with debt because it can lead to costly bankruptcy. Firms have rational expectations, so they will take out less debt if they know they will suffer more from debt overhang.

To assess the gains from resolving the debt overhang problem, we include an additional shock which controls the extent to which firm decisions are made by the firm as a whole rather than equity holders alone. When the economy is hit by an aggregate shock, to assess the gains from resolving the debt overhang problem, we simultaneously shock this parameter. This allows us to do a direct comparison between the response of an economy from a steady state where a shock hits the economy and policy makers are potentially able to resolve this problem, and the response when debt overhang continues to affect firms.

Aside from the results we detail above, we also analyze a shock to the retained value of the firm upon bankruptcy (a “financial shock.”), which has been less studied in the literature, yet has been argued as being relevant in the recent recession. Our shock processes are calibrated to those in Gilchrist et al. (2014), and how we obtain values for our other parameters is discussed in Kurtzman and Zeke (2015b).

In conjunction with demonstrating some new results, our paper displays the advantages of the solution method we use, which is described in Section 3. The solution procedure is commonly known as the “projection and perturbation” approach, wherein nonlinearities are taken into account at the firm level, but not at the aggregate level. This method is particularly useful for our analysis, because it allows one to easily incorporate additional aggregate shocks. In future work, we would like to extend this analysis to second-order perturbations, as nonlinearities may not only be crucial at the firm-level but also in the aggregate for the exercises of interest.

The rest of the paper follows as such. Section 2 describes our dynamic, stochastic, general equilibrium model and characterizes a recursive competitive equilibrium. Section 3 discusses our parameter choices and presents results from our numerical procedure. Section 4 concludes.

## 2 The Model

In this section, we will describe physical environment, innovation policies, the processes for aggregate shocks, define an equilibrium, and briefly discuss the solution method for solving our model which we describe in further detail in Appendix A. Our model takes a discrete-time, discretized Leland (1994) model of capital structure, and allows firms to make process innovation decisions. Because equity holders make innovation decisions, they can thus suffer from debt overhang when highly levered. We embed this partial equilibrium model in a general equilibrium model with free entry. Unlike in Kurtzman and Zeke (2015b), this model also incorporates aggregate shocks. Such a framework allows us to quantify the losses from debt overhang in a dynamic, stochastic, general equilibrium model of firm dynamics.



## 2.1 Physical Environment

Time is discrete and indexed as  $t = 0, 1, 2, \dots$ . Households are endowed with  $L$  units of time which they supply each period inelastically. On the production side, there is a competitive final good sector, a competitive research good sector, and a monopolistically competitive intermediate good sector. The final good is produced from a continuum of differentiated intermediate goods, and it can be consumed by the household or used in the production of the research good. Intermediate good firm productivities evolve endogenously through process innovation, and the measure of differentiated intermediate goods is determined endogenously through product innovation. Intermediate good firm production requires labor, which is paid wage  $w_t$ . All innovation requires the research good, which is a composite of labor and the final good made by a competitive research good sector. Intermediate good firms issue both equity and debt to finance their operations, which are held by consumers. Debt is infinitely-lived and pays a constant coupon. Firms take out debt, because it has a tax advantage,  $\tau^d$ , but they do not fully finance themselves with debt because it can lead to costly bankruptcy.

The timing of the model follows as such. All aggregate shocks and idiosyncratic shocks are realized. Equity holders then decide whether to go bankrupt or not. Afterwards, the following actions are simultaneously undertaken: equity holders make the innovation and dividend decision, the intermediate good firm sets its prices, sells its differentiated product to the final good firm, pays its workers, and dividends are distributed. Final good firms buy differentiated intermediate goods, aggregate them into the final good, and sell them to consumers and research good firms. Consumers inelastically supply labor to the firm, and, hence, the wage rate is determined so that intermediate goods are produced and the labor market clears. Research good firms purchase the final good and hire labor, aggregate them into research goods, and price and sell them to innovators. New entrants purchase the research good. Households consume, receive dividends and coupons, pay taxes, and price the available debt and equity.

## 2.2 Production

At time  $t$ , an intermediate good firm is indexed by six state variables: aggregate productivity,  $A_t$ , the aggregate “step size”,  $\Delta_{z,t}$ , the proportion of the firm’s productivity retained upon bankruptcy,  $\alpha_t$ , idiosyncratic productivity,  $z_t$ , its coupon level,  $d_t$ , and its investment opportunities,  $\theta_t$ . The natural logarithm of aggregate productivity follows an AR1 process:

$$\ln(A_t) = \rho_A \ln(A_{t-1}) + \epsilon_{A,t},$$

where  $\epsilon_{A,t} \sim (0, \omega_A^2)$ . The natural logarithm of the aggregate step size similarly follows an AR1 process:

$$\ln(\Delta_z) = (1 - \rho_{\Delta_z}) \ln(\bar{\Delta}_z) + \rho_{\Delta_z} \ln(\Delta_{z,t-1}) + \epsilon_{\Delta_z,t},$$

where  $\bar{\Delta}_z$  is some baseline level of the aggregate step size, and  $\epsilon_{\Delta_z,t} \sim (0, \omega_{\Delta_z}^2)$ .<sup>14</sup> The natural logarithm of a financial shock follows an AR1 process:

$$\ln(\alpha_t) = (1 - \rho_\alpha) \ln(\bar{\alpha}) + \rho_\alpha \ln(\alpha_{t-1}) + \epsilon_{\alpha,t},$$

where  $\bar{\alpha}$  is some baseline level of the aggregate step size, and  $\epsilon_{\alpha,t} \sim (0, \omega_\alpha^2)$ . We will also define  $\Gamma_t$  as the measure of firms indexed across  $(z_t, x_t)$  where  $x_t = (d_t, \theta_t)$  to simplify notation. We summarize the aggregate state the firm enters each period with as  $S_t = (\Gamma_t, A_t, \Delta_{z,t}, \alpha_t)$ . We will, in turn, summarize the transition function of the aggregate states as:

$$S_{t+1} = H(S_t) \tag{2.1}$$

The final good is produced from intermediate goods,  $y_t(z_t)$ , with constant elasticity of

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<sup>14</sup>In our numerical implementation, we log-linearize our model around a steady state. In turn,  $\bar{\Delta}_z$  is the steady state level of  $\Delta_z$ . Likewise,  $\bar{\alpha}$  is the steady state value of  $\alpha$ .

substitution (CES) production function:

$$Y_t = \left( \int y_t(z)^{1-\frac{1}{\rho}} dz \right)^{\frac{\rho}{\rho-1}}. \quad (2.2)$$

In our model, there is a standard inefficiency due to the monopoly markup in the production of intermediate goods. To undo this distortion, we allow for a per-unit subsidy,  $\tau^s$ , on the production of the consumption good.<sup>15</sup> In equilibrium, standard arguments show prices must satisfy:

$$P_t = \frac{1}{(1 + \tau^s)} \left[ \int_z (p_t(z)^{1-\rho} dJ_t(z)) \right]^{\frac{1}{\rho-1}}, \quad (2.3)$$

where  $p_t(z)$  is the price set at time  $t$  by firms with productivity index  $z$ , and  $P_t$  is the price set by final good firms at time  $t$ . Thus, from profit maximization demand for intermediate goods is:

$$y_t(z_t) = (1 + \tau^s)^\rho \left( \frac{p_t(z_t)}{P_t} \right)^{-\rho} Y_t, \quad (2.4)$$

given (2.2) and (2.3).

Firms have an investment technology through which they can lower the marginal cost of production. Investment requires the research good. Upon entry, firms are endowed with  $\theta$ , which controls the inherent productivity of this technology. We assume  $\theta \in \{\theta^L, \theta^H\}$ .

Research good firms are competitive and produce the research good,  $R_t$ , as a composite of final goods used for research,  $Y_{r,t}$ , and labor used in research,  $L_{r,t}$ , with Cobb-Douglas production:

$$R_t = L_{r,t}^\lambda Y_{r,t}^{(1-\lambda)},$$

where  $\lambda \in [0, 1]$ . We denote the price of the research good as  $P_r(S_t)$ . Notice if  $\lambda = 0$ , the research good is only produced with the final good, and if  $\lambda = 1$ , it is produced only with labor.

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<sup>15</sup> If we left in this distortion, the tax advantage of debt can act as a welfare improving subsidy to entry.

A firm with state  $(S_t, z_t, x_t)$  produces output,  $y_t(z_t)$ , with labor,  $l_t(z_t)$ , using the following constant returns to scale production function:

$$y_t(z_t) = (A_t e^{z_t})^{\frac{1}{\rho-1}} l_t(z_t). \quad (2.5)$$

The idiosyncratic productivity of an intermediate good firm is  $e^{z_t \frac{1}{\rho-1}}$ . We rescale productivity by  $\frac{1}{\rho-1}$ , so that each firm's variable profits and labor are proportional in  $A_t e^{z_t}$ . Notice, labor at time  $t$  comes directly from the production function, given  $y_t(z_t)$ .

At every time  $t$ , the firm solves:

$$\pi(S_t, z_t) = \max_{y_t(z_t)} p_t(z_t) y_t(z_t) - w(S_t) \frac{y_t(z_t)}{(A_t e^{z_t})^{\frac{1}{\rho-1}}} \quad (2.6)$$

subject to (2.4) and (2.5) to maximize profits.

Productivity at the firm level evolves conditional on the investments the firm has made in improving its productivity, idiosyncratic productivity shocks, and on shocks to the aggregate state. At the start of each period,  $t$ , each incumbent firm has a probability,  $\delta$ , of exiting, and a probability,  $1 - \delta$ , of surviving to produce. If it survives, equity holders then choose whether to declare bankruptcy or continue to operate. If it declares bankruptcy, the firm loses a proportion,  $(1 - \alpha_t)$ , of its productivity, where  $\alpha_t \in [0,1)$ . The existing creditors then gain full equity control of the firm and take out new debt to maximize the joint value of equity holders and new creditors. If equity holders continue to operate, the firm with state  $(S_t, z_t, x_t)$  invests  $\phi(\Delta_{z,t}, z_t, \theta_t, q_t)$  units of labor to improve its productivity. The cost function is similar to that in Kurtzman and Zeke (2015b), except because there is now an aggregate shock to asset volatility, we define the cost of investment, such that no matter how the level of  $\Delta_z$  changes compared to  $\bar{\Delta}_z$ , it costs the same amount to get the same expected

growth rate. To be precise,  $\phi(\Delta_{z,t}, z_t, \theta_t, q_t)$  is defined as:

$$\phi(\Delta_{z,t}, z_t, \theta_t, q_t) = e^{z_t} \theta^{-b} e^{\frac{e^{-\Delta_{z,t}} - e^{-\bar{\Delta}_z} + q_t \left( e^{\Delta_{z,t}} - e^{-\Delta_{z,t}} \right)}{(e^{\bar{\Delta}_z} - e^{-\bar{\Delta}_z})}}, \quad (2.7)$$

where  $b$  is the convexity of the cost function. With probability  $q_t$ , next period, the firm's productivity improves by  $\Delta_{z,t}$ , and with probability  $(1 - q_t)$ , the firm's productivity falls by  $\Delta_{z,t}$ . The cost function is convex in  $q_t$ . It is also proportional in  $e^{z_t}$ .

Given (2.6), equity holders of a firm with state variable  $(S_t, z_t, x_t)$  receive cash flows each period,  $CF_E$ , defined as:

$$CF_E(S_t, z_t, x_t) = (1 - \tau) \left( \pi(S_t, z_t) - P_r(S_t) \phi(\Delta_{z,t}, z_t, \theta_t, q_t) \right) - \frac{d_t}{P_t} + \tau^d \frac{d_t}{P_t}. \quad (2.8)$$

The expected, discounted present value of profits for equity holders of a firm with state variable  $(S_t, z_t, x_t)$  satisfies the following Bellman equation:

$$\begin{aligned} V_E(S_t, z_t, x_t) = \max_{q_t} \left\{ 0, CF_E(S_t, z_t, x_t) + \right. \\ \left. E_t[M_{t+1} \left( q_t V_E(S_{t+1}, z_t + \Delta_{z,t}, x_t) \right. \right. \\ \left. \left. + (1 - q_t) V_E(S_{t+1}, z_t - \Delta_{z,t}, x_t) \right) | S_t] \right\}. \end{aligned} \quad (2.9)$$

where  $M_{t+1} = e^{-r} (1 - \delta) \frac{P_{t+1}}{P_t}$  and  $CF_E(S_t, z_t, x_t)$  is defined in (2.8). Also, notice in our notation  $w(S_t)$  and  $P_r(S_t)$  are relative to  $P_t$ . Above and throughout, to save on notation, we write  $q_t$  rather than  $q(S_t, z_t, x_t)$ . Equity holder's bankruptcy decision can be rewritten as a function of the aggregate state, productivity, and the current debt level; we call this  $\bar{z}(S_t, x_t)$ . We define  $q_t^*$  to be the optimal choice of  $q_t$ .

Creditors of a firm with state variable  $(S_t, z_t, x_t)$  receive the coupon payment as their

cash flow, so the combined cash flow of creditors and equity holders of this firm is:

$$CF_F(S_t, z_t, x_t) = (1 - \tau) \left( \pi(S_t, z_t) - P_r(S_t) \phi(\Delta_{z,t}, z_t, \theta_t, q_t) + \tau^d \frac{d_t}{P_t} \right).$$

Since, the bankruptcy rule,  $\bar{z}(S_t, x_t)$ , is taken as given by creditors, the expected, discounted present value of profits for equity holders and creditors combined of a firm with state  $(S_t, z_t, x_t)$  can be broken into two parts. If equity holders decide not to go bankrupt, the expected, discounted present value of profits for the joint value of equity holders and creditors of a firm with state variable  $(S_t, z_t, x_t)$  satisfies the following Bellman equation:

$$\begin{aligned} V_F(S_t, z_t, x_t) &= CF_F(S_t, z_t, x_t) + \\ &E_t[M_{t+1} \left( q_t V_F(S_{t+1}, z_t + \Delta_{z,t}, x_t) + \right. \\ &\left. (1 - q_t) V_F(S_{t+1}, z_t - \Delta_{z,t}, x_t) \right) | S_t]. \end{aligned} \tag{2.10}$$

If equity holders do decide to go bankrupt, creditors seize the firm, lose a fraction of their productivity, and make a new debt decision to maximize the value of the creditors that seized the firm (which are now the equity holders) and the new creditors. Hence, the Bellman equation in that case is:

$$V_F(S_t, z_t, x_t) = \max_{d_{t+1}} V_F(S_t, z_t + \ln(\alpha_t), d_{t+1}, \theta_t). \tag{2.11}$$

Let  $d^*(S_t, z_t, \theta_t)$  be that optimal choice of  $d_{t+1}$  that satisfies (2.11). The value of debt holders,  $V_B(S_t, z_t, x_t)$ , is defined as the difference between the value of the firm as a whole, (2.10) and (2.11), and the value of equity, (2.9); thus,

$$V_B(S_t, z_t, x_t) = V_F(S_t, z_t, x_t) - V_E(S_t, z_t, x_t).$$

New firms are created by purchasing  $n_e$  units of labor; a purchase in period  $t$  yields a new firm in period  $t + 1$  with initial state variables  $z$  and  $\theta$  drawn from a distribution  $G$ . After receiving  $z$  and  $\theta$ , the firm makes an initial debt decision to maximize the value of equity and new creditors. In any period with a positive mass of entering firms, we have:

$$P_r(S_t)n_e = E_t[M_{t+1} \sum_{\theta} \int \max_{d_{t+1}} V_F(S_{t+1}, z, d_{t+1}, \theta) G(z, \theta) dz | S_t]. \quad (2.12)$$

We define  $M_{e,t}$  as the measure of new firms entering the economy at period  $t$  that start producing in period  $t + 1$ .

Households inelastically provide labor,  $L$ , to firms. After all (idiosyncratic and aggregate) shocks are realized, households make a consumption decision,  $C_t$ , get paid wages,  $w(S_t)$ , receive a lump sum transfer of dividends,  $D_t$ , and pay a lump-sum tax,  $T_t$ .

The recursive problem for households is the following:

$$V^H(S_t) = \max_{C_t} [\ln(C_t) + e^{-r} E_t[V^H(S_{t+1} | S_t)]] \quad (2.13)$$

subject to their budget constraint:

$$C_t = w(S_t)L + D_t - T_t, \quad (2.14)$$

and the aggregate law of motion for  $S_t$ , (2.1). Recall, the aggregate dividend is the sum of all-after tax profits from intermediate-good firms net of the entry costs of all newly entering firms.

The distribution of operating firms at time  $t$  with state  $(z, d, \theta)$ ,  $\Gamma_t(z, d, \theta)$ , evolves over time as a function of the exogenous exit rate,  $\delta$ , the choices of  $q_t$  by incumbent firms, and the mass of entering firms each period,  $M_{e,t}$ . To simplify the definition of the mass of firms with state  $(S_{t+1}, z_{t+1}, x_t)$  in period  $t + 1$ ,  $\Gamma_{t+1}(z_{t+1}, x_t)$ , we break it into four pieces. First, there is a mass of continuing firms who did not go bankrupt who could enter period  $t + 1$

with state  $(S_{t+1}, z_{t+1}, x_t)$  which is a function of continuing firms with productivity  $z_{t+1} - \Delta_{z,t}$  last period that drew positive productivity shocks and continuing firms with productivity  $z_{t+1} + \Delta_{z,t}$  last period that drew negative productivity shocks:

$$\begin{aligned} \Gamma_{t+1}^C(z_{t+1}, x_t) &= (1 - \delta)(1 - q(S_t, z_{t+1} + \Delta_{z,t}, x_t))\Gamma_t(z_{t+1} + \Delta_{z,t}, x_t) \\ &\quad + (1 - \delta)q(S_t, z_{t+1} - \Delta_{z,t}, x_t)\Gamma_t(z_{t+1} - \Delta_{z,t}, x_t). \end{aligned} \tag{2.15}$$

Second,  $\Gamma_{t+1}$  is also a function of the mass of entering firms who received productivity,  $z_{t+1}$ , and investment opportunities,  $\theta_t$ , such that they chose debt  $d_{t+1}$ :

$$\Gamma_{t+1}^E(z_{t+1}, d_{t+1}, \theta_t) = M_{e,t}G(z_{t+1}, \theta_t). \tag{2.16}$$

Third,  $\Gamma_{t+1}$  is a function of the mass of firms who have productivity  $z_{t+1} + \Delta_{z,t} - \log(\alpha_t)$  last period, with type  $\theta$  and debt load  $d_{t+1}$  that drew negative productivity shocks, went bankrupt, and chose debt  $d_{t+1}$ .

$$\begin{aligned} \Gamma_{t+1}^{B,1}(z_{t+1}, d_{t+1}, \theta_t) &= (1 - \delta) \int (1 - q(S_t, z_{t+1} + \Delta_{z,t} - \ln(\alpha_t), d', \theta_t)) * \\ &\quad \mathbf{1}_{(V_{t+1}^E(z_{t+1} - \ln(\alpha_t), d', \theta_t) < 0)} \Gamma_t(z_{t+1} + \Delta_{z,t} - \ln(\alpha_t), d', \theta_t) dd'. \end{aligned} \tag{2.17}$$

Fourth, it is also possible for firms to have had productivity  $z_{t+1} - \Delta_{z,t} - \log(\alpha_t)$  last period, type  $\theta_t$  and debt load  $d_{t+1}$ , to go bankrupt and chose debt  $d_{t+1}$ .



$$\begin{aligned} \Gamma_{t+1}^{B,2}(z_{t+1}, d_{t+1}, \theta_t) &= (1 - \delta) \int q(S_t, z_{t+1} - \Delta_{z,t} - \ln(\alpha_t), d', \theta_t) * \\ &\quad \mathbf{1}_{(V_{t+1}^E(z_{t+1} - \ln(\alpha_t), d', \theta_t) < 0)} \Gamma_t(z_{t+1} - \Delta_{z,t} - \ln(\alpha_t), d', \theta_t) dd'. \end{aligned} \tag{2.18}$$

Hence, we can define  $\Gamma_{t+1}(z_{t+1}, d_{t+1}, \theta_t)$ , as the sum of  $\Gamma_{t+1}^C(z_{t+1}, d_{t+1}, \theta_t)$ ,  $\Gamma_{t+1}^E(z_{t+1}, d_{t+1}, \theta_t)$ ,  $\Gamma_{t+1}^{B,1}(z_{t+1}, d_{t+1}, \theta_t)$ , and  $\Gamma_{t+1}^{B,2}(z_{t+1}, d_{t+1}, \theta_t)$ , using (2.15), (2.16), (2.17), and (2.18).

## 2.3 Equilibrium

Market clearing for the final good requires:

$$C(S_t) = Y(S_t) - Y_r(S_t).$$

Labor market clearing gives us:

$$\sum_{\theta} \int \int l(S_t, z) \Gamma_t(z, d, \theta) dz dd + L_r(S_t) = L.$$

where  $\sum_{\theta} \int \int l(S_t, z) \Gamma_t(z, d, \theta) dz dd$  is total employment used to produce the intermediate good, and  $L_{r,t}$  denotes labor spent on process and product innovation.

We can write total demand for the research good as:

$$M_e(S_t) n_e + \sum_{\theta} \int \int \phi(\Delta_{z,t}, z, d, \theta, q_t) \Gamma_t(z, d, \theta) dz dd = R(S_t).$$

Research good market clearing then comes directly from the production function for the research good.

$$R(S_t) = Y_r(S_t)^{1-\lambda} L_r(S_t)^{\lambda}.$$

Also, cost minimization for the research good implies:

$$(1 - \lambda)L_r(S_t)w(S_t) = \lambda Y_r(S_t).$$

A recursive competitive equilibrium in this economy is defined as follows. Given initial distribution,  $\Gamma_0$ , and initial aggregate shocks,  $A_0$ ,  $\Delta_{z,0}$ , and  $\alpha_0$ , a recursive equilibrium consists of policy and value functions of equity holders, creditors, and intermediate good firms,  $\{l(S_t, z_t), V_E(S_t, s_t), \bar{z}(S_t, x_t), p(S_t, s_t), V_B(S_t, s_t), V_F(S_t, s_t), d^*(S_t, z_t, \theta_t)\}$  where  $s_t = (z_t, x_t)$ , household policy functions for consumption,  $C(S_t)$ , aggregate prices,  $\{P(S_t), P_r(S_t), w(S_t)\}$ , the mass of new entrants,  $M_e(S_t)$ , and the aggregate states,  $S_t$ , which evolved according to transition function  $H(S_t)$  such that for all  $t$ : (i) the policy and value functions of intermediate good firms are consistent with the firm’s optimization problem (ii) the representative consumers policy function is consistent with its maximization problem (iii) debt and equity holders value functions and decision rules are priced such that they break even in expected value (iv) free entry holds (v) the labor, final good, and research good markets clear (vi) the measure of firms evolves in a manner consistent with the policy functions of firms, households, and shocks. We focus only on equilibria with positive entry.

## 2.4 Solution Method

It is well known that models with heterogeneous firms and aggregate shocks can be difficult to solve because the distribution of firms is part of the aggregate state vector. Following a recent literature that includes Costain and Nakov (2011), Haefke and Reiter (2011), McKay and Reis (2013), Terry (2014), and Winberry (2015), we implement the “projection and perturbation approach” of Reiter (2009) to overcome this problem.<sup>16</sup> The main advantage of this approach is that it is easier to introduce more aggregate shocks compared to the popular method used in Krussel and Smith (1998). To implement the method of Reiter

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<sup>16</sup>An approach in the same spirit was introduced by Campbell (1998).

(2009), we follow the algorithm in Kurtzman and Zeke (2015b) to find the stationary competitive equilibrium of our model with idiosyncratic uncertainty, but no aggregate shocks. We then linearize our model with respect to the aggregate shocks, and we solve for aggregate and firm-level responses to a shock as a perturbation around the steady state of our model. We provide richer detail on our solution method in Appendix A.

## 2.5 Debt Overhang Counterfactual

We also include an aggregate shock which controls the extent to which debt overhang affects firms. This will allow us to directly compare the effect of an aggregate shock with debt overhang affecting firms and with debt overhang resolved to some extent. We call this case a “convexification” of the investment decision between equity and debt, so that there is “partial debt overhang.” We take a reduced form approach to characterizing this case where debt overhang can be partially resolved and ignore the strategic problems inherent in such a convexification by solving the following system of equations. We define,  $\xi_t$ , which is the degree to which equity holder’s decision gets weighted relative to the creditor and equity holder’s combined decisions at time  $t$ . The natural logarithm of  $\xi_t$  follows an AR1 process:

$$\ln(\xi_t) = \rho_\xi \ln(\xi_{t-1}) + \epsilon_{\xi,t},$$

where  $\epsilon_{\xi,t} \sim (0, \omega_\xi^2)$ . We define  $q_t^E$  and  $q_t^F$  to demonstrate our “convexification” approach:

$$q_t^E = \max_{q_t} \left\{ 0, CF_E(S_t, z_t, x_t) + E_t[M_{t+1} \left( q_t V_E(S_{t+1}, z_t + \Delta_{z,t}, x_t) + (1 - q_t) V_E(S_{t+1}, z_t - \Delta_{z,t}, x_t) \right) | S_t] \right\}. \quad (2.19)$$

$$q_t^F = \max_{q_t} \left\{ 0, CF_A(S_t, z_t, x_t) + E_t[M_{t+1} \left( q_t V_F(S_{t+1}, z_t + \Delta_{z,t}, x_t) + (1 - q_t) V_F(S_{t+1}, z_t - \Delta_{z,t}, x_t) \right) | S_t] \right\}. \quad (2.20)$$

We can then define the Bellman's for equity holders, creditors, and the firm as a whole as before, noting that now incumbent firms also make a choice of  $q_t$ , which above we called  $q_t^F$ .

We then define  $q_t$  such that:

$$q_t = (1 - \xi_t + \bar{\xi}_t) q_t^E + (\xi_t - \bar{\xi}_t) q_t^F.$$

Notice, we now have to solve for a fixed point in both problems rather than first solving the problem of equity holders and then solving the problem of asset holders. When  $\xi_t = \bar{\xi}_t$ , equity holders completely makes the investment decision. As  $\xi_t$  rises, the firm as a whole's discounted present value receives more weight. Given our estimate of  $b$  implies some degree of debt overhang, in steady state, we always set  $\xi_t$  to  $\bar{\xi}$  (where  $\bar{\xi}$  is set to one) in steady state. In this way, the steady implies the same estimate of the extent to which debt overhang affects firms as in Kurtzman and Zeke (2015b), but an increase in  $\xi_t$  helps to resolve the debt overhang problem.

Table 2.1: Shock Calibration

Parameter	Steady State Value	Respective $\rho$	Respective $\omega$
$\ln(A)$	0	0.98	0.20
$\ln(\alpha)$	$\ln(0.8)$	0.99	0.005
$\ln(\Delta_z)$	$\ln(0.065)$	0.97	0.065

$\alpha$  is the retained value of the firm after bankruptcy.  $A$  is the level of TFP.  $\Delta_z$  is the aggregate “step size.” All values of  $\rho$  and  $\omega$  coincide with the parameter in the same row, are monthly, and are rounded to two decimals after any zeros. The choice of parameters is explained in the text of this section.

### 3 Quantitative Results

In this section, we present results from the numerical solution to the model developed in Section 2. We first discuss our parameter choices and some computational details.

#### 3.1 Calibration

As we discussed in Section 1, some model parameters are estimated in Kurtzman and Zeke (2015b). However, we need estimates of the processes for shocks. Where possible, we use the calibration of the shock processes of Gilchrist, Sim, and Zakrajsek (2014), and we refer the reader to their text for details of the calibration. A summary of these estimates is in Table 2.1. We depart from Kurtzman and Zeke (2015b) and set the production function coefficient,  $\lambda$ , to one half, so the assumption that labor is inelastically supplied does not too crucially affect how entry behaves. Nonetheless, we still want our results to be consistent with the fact that consumption and output are correlated over the business cycle, so we do not set  $\lambda$  to zero which would make consumption and output inversely correlated.<sup>17</sup> We also increase  $\rho$  to 7.15, so our value of  $Y_r$  relative to  $Y$  is more consistent with the share of intangible capital as a percent of GDP in the U.S., which is 0.15% according to Corrado, Hulten, and Sichel (2009). However, in our steady state, total spending on the research good,  $P_r R$ , as a percent

<sup>17</sup>In Kurtzman and Zeke (2015b), there is no research good, as innovation only requires labor. This is equivalent to setting  $\lambda$  to one.

Table 3.1: Remaining Parameters

Parameter	Value
Tax advantage of debt, $\tau_d$	0.2
Period length, $\Delta$	$\frac{1}{12}$
Exogenous exit rate, $\delta$	0.053
Discount rate, $r$	$\log(1.001)$
Convexity of Cost Function, $b$	35.81
Choice of $p$ in the limit of the high type	0.52
Probability of entry as a high type	0.79
Ratio of High to Low $\theta$	1.053
Elasticity of substitution across intermediate goods, $\rho$	7.15
Total labor supply, $L$	1
Fixed Cost of Entry, $n_e$	1
Research good production function coefficient, $\lambda$	0.5

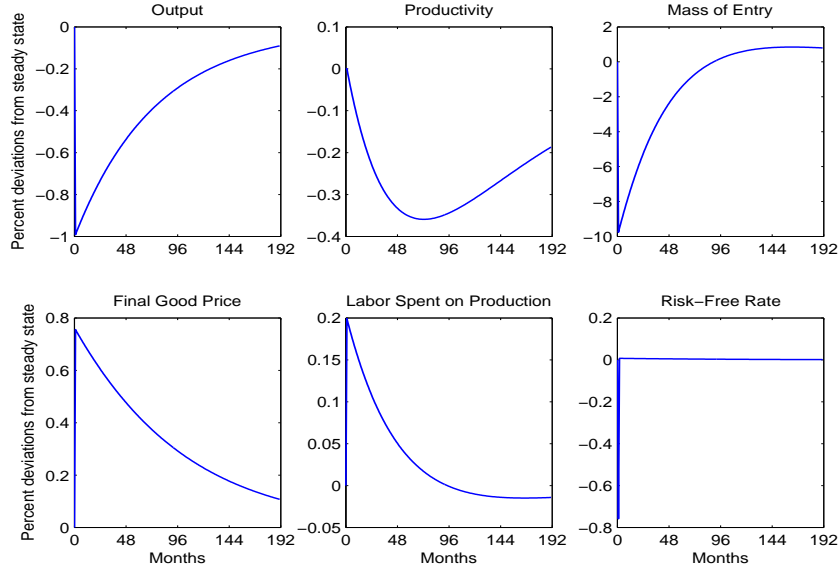
of total output,  $PY$ , is still much higher, at 8.73%. The remaining parameters in our model are summarized in Table 3.1.

## 3.2 Computational Notes

### 3.2.1 Numerical Accuracy

We check the accuracy of our solution method by computing Euler-equation errors in the same manner as McKay and Reis (2013). Euler equation errors arise because the compression of the state space at the firm level contains some numerical inaccuracy, because Jacobians calculated numerically can be inaccurate, and because we linearize our problem. At the steady state, the unit-free Euler equation errors are  $1e-9$ , which is highly accurate. As a comparison, McKay and Reis (2013) state their errors are on the order of 0.0002 in steady state. We simulate the economy with only TFP shocks and take the average of the maximum of the absolute value of the Euler equation errors in the aggregate state vector over 50 periods. We obtain a value of 0.0066. A similarly small number of 0.004 is found by McKay and Reis (2013), although they do not take the maximum but rather the average.

Figure 3.1: Impact of a Negative TFP Shock in Baseline Case



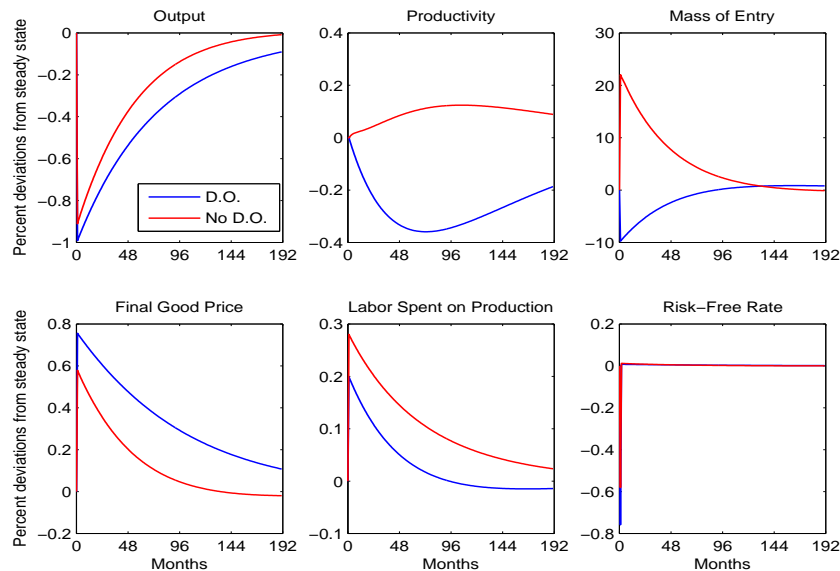
The blue line in each figure shows the response of the aggregate in the title of the subplot to a 1% standard deviation shock to TFP when firms can possibly suffer from debt overhang.

### 3.2.2 Existence

We describe the solution method we use in Subsection 2.4 and we describe the computational details in Appendix A, but we want to make one computational note that we see as important regarding this procedure and the literature to which our paper speaks. We believe this note should be the motivation for future computational work. As we describe in Appendix A, in the final step of this procedure, we use Chris Sims' program **gensys**. The existence and uniqueness variable that gensys provides consistently produces a value of  $[0;1]$ , respectively, which implies that this equilibrium does not exist, but is unique. We found that the only publicly available code we have found that uses the Reiter (2009) method, Terry (2014), finds the same output for Khan and Thomas (2008) and is able to replicate Khan and Thomas (2008) with some numerical error using the Reiter (2009) method. Does the equilibrium in work by Khan and Thomas (2008) or in our paper actually not exist, or is there a computational issue with this methodology?<sup>18</sup> Determining existence of equilibria

<sup>18</sup>We are skeptical that gensys is correct in stating this equilibrium does not exist, as we solved our same model in a special case without long-term debt or bankruptcy and without endogenous innovation decisions (essentially, a standard neoclassical model with free entry and aggregate shocks), and also obtained a value of  $[0;1]$  from the eu variable. Given the Jacobians can be calculated analytically, there is little room for

Figure 3.2: Impact of a Negative TFP Shock with and without Debt Overhang



The blue line in each figure shows the response of the aggregate in the title of the subplot to a 1 standard deviation shock to TFP when firms can possibly suffer from debt overhang. The red line is the same as the blue line except that an additional shock to  $\xi$ , which is the extent to which firm decisions are made to benefit the firm as a whole rather than equity holders alone, is fed in with the same persistence parameter.

in models with heterogeneous agents and business cycles is often intractable. This procedure would prove itself extremely useful if, besides its ability to tractably add aggregate shocks without much difficulty, it can also be used to help check existence. If our equilibrium truly does not exist, then we should look to better understand why and in what ways our model would need to change so that it does.

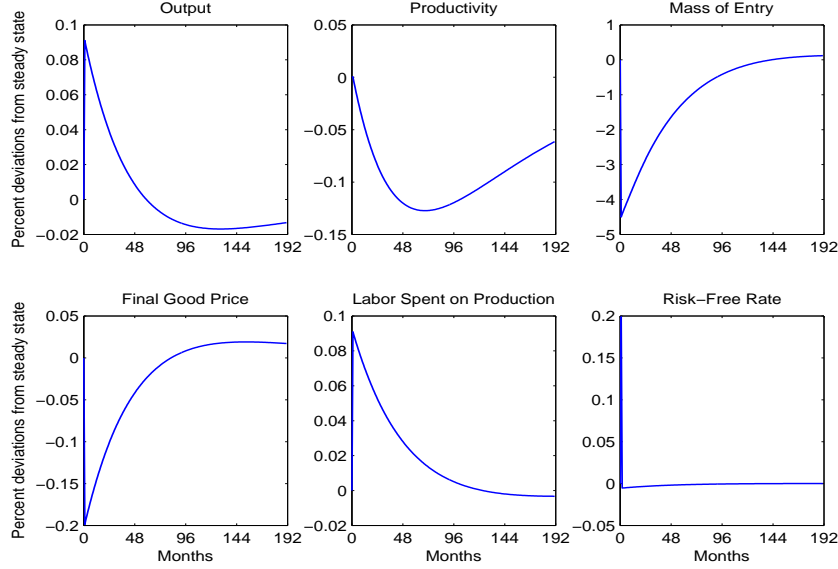
### 3.3 Impulse Responses

In our model, there are four aggregate shocks: a shock to TFP,  $A$ , a shock to the aggregate step size,  $\Delta_z$ , a shock to the retained value of the firm upon entry,  $\alpha$ , and a shock to  $\xi$ , which is the extent to which firm decisions are made to benefit the firm as a whole rather than equity holders alone.

A benefit of our approach is that it allows us to maintain the nonlinearities in the firm level that exist, while still easily adding and calculating the responses to a number of aggregate numerical error in creating such an output from the eu variable. We plan on formalizing this finding in future work.



Figure 3.3: Impact of a Positive Volatility Shock in Baseline Case



The blue line in each figure shows the response of the aggregate in the title of the subplot to a 1 standard deviation shock to asset volatility when firms can possibly suffer from debt overhang. The red line is the same as the blue line except that an additional shock to  $\xi$ , which is the extent to which firm decisions are made to benefit the firm as a whole rather than equity holders alone, is fed in with the same persistence and standard deviation.

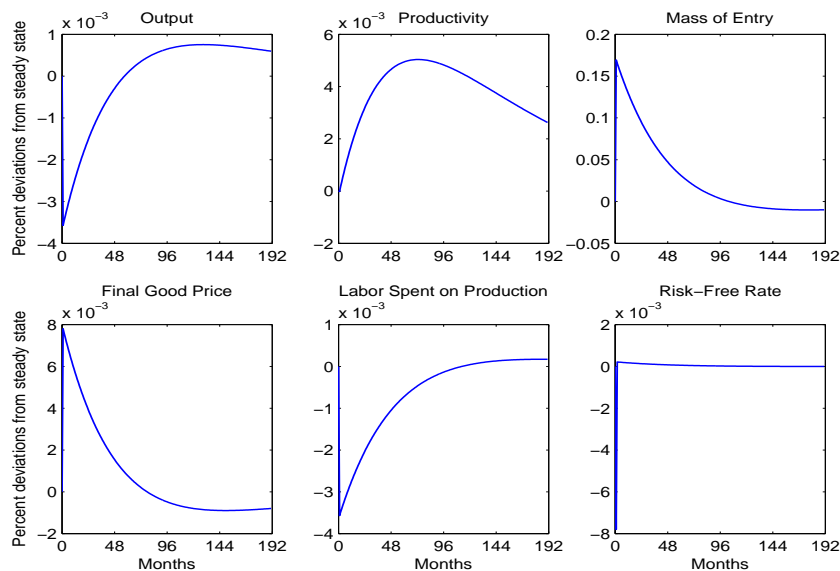
shocks. We spend most of our efforts in this section describing the response of the economy to shocks to TFP and shocks to asset volatility, as those shocks are generally the shocks most considered in the literature to which our paper speaks.

In Figure 3.1, we show the response of an economy to a negative shock to TFP. This figure shows that a negative TFP shock can generate relatively large movements in output, productivity, and consumption compared to the other aggregate shocks (see Figures 3.3 and 3.4). Output, consumption, and entry immediately respond. Productivity eventually falls, but it takes time to reach its trough, and the recovery is slow. This occurs because lower process innovation slowly detracts from the stock of firm productivity. That combined with more bankruptcy leads productivity to fall for longer than output and consumption, which are half way recovered by the time productivity is at its lowest.

In Figure 3.2, we also show the response of the economy if we simultaneously resolve debt overhang.<sup>19</sup> When debt overhang is resolved, firm value rises and the firm invests more.

<sup>19</sup>We calibrate the process for the shock to the extent to which the firm as a whole rather than equity holders make the investment decision ( $\xi$ ) to that of TFP, the idea being the policy response will be in proportion to the size of the shock, and there is some decay in its effectiveness.

Figure 3.4: Impact of a Positive “Financial Shock” in Baseline Case



The blue line in each figure shows the response of the aggregate in the title of the subplot to a 1% standard deviation shock to the retained value of the firm upon bankruptcy when firms can possibly suffer from debt overhang.

Productivity responds quickly because entry rises, but for the same reason that a negative shock to TFP leads to a slow recovery in aggregate productivity, productivity also peaks later in the business cycle.

In Figure 3.3, we plot the response of the economy to a positive one standard deviation shock to asset volatility. This shock has benefits to equity holders as the option value of equity increases. Because the firm only holds long-lived debt, this positive effect on equity can overwhelm the negative effect of an asset volatility shock on firm value depending on the calibration. Under our calibration, productivity and entry fall modestly. However, because labor is inelastically supplied, and demand for labor falls (so it gets cheaper), labor spent on production, output, and consumption rise in the short-run. In response to resolving debt overhang in conjunction with a volatility or financial shock, aggregates do not behave much differently than in response to resolving debt overhang in conjunction with a TFP shock.

In Figure 3.4, we show how the economy responds to a shock to the retained value of the firm upon bankruptcy (a “financial shock”). As one would expect, increasing the retained value of the firm upon bankruptcy increases the value of entry and the mass of

entry. However, there are general equilibrium effects associated with this shock that are similar to a shock to asset volatility. When there is more product innovation, there is more demand for the research good. Hence, labor is more expensive and shifts out of production, which lowers labor spent on production, output, and consumption. However, productivity rises and peaks later in the business cycle, as there is both less loss in bankruptcy, and firms invest more in process innovation. In both the case of a 1% standard deviation asset volatility shock and financial shock, the movements in aggregates are modest compared to those due to a TFP shock.

## 4 Conclusion

Our paper contributes to a growing literature that studies movements in aggregates with different forms of reasonably calibrated shocks in models with heterogeneous firms and financial frictions. Our paper raises a number of questions, a few of which we lay out below and plan to explore in future work. First, if in our models, we allow the average maturity of debt and other moments of the maturity structure of debt to be calibrated to match facts on U.S. credit markets, to what extent is the amplification mechanism dampened in the literature that argues uncertainty and financial frictions can combine to generate large, long-lasting business cycles? This is a question that has been unexplored to our knowledge, and our results suggest it is a salient one. Another key question is, when the average firm becomes close to default, what other externalities are present that may amplify the business cycle? Gourio (2014) puts forth a model where customers, suppliers and workers all suffer if a firm goes bankrupt, which generates a wedge that in the data would look like a labor wedge which he calls the *default wedge*. His model is similar to ours along a number of dimensions, and demonstrates the flexibility of our modeling approach and its usefulness in answering further questions. As we answer these questions, we will gain a better understanding of the nature of the business cycle, and the role financial frictions play in its propagation.

# Appendix A

## Solution Procedure

A standard way to apply the Reiter (2009) method is as follows: Solve for a steady state of the model with some compression of the state space. Then, set up a system of nonlinear equations that summarize the general equilibrium model with aggregate shocks, perturb the system around the original steady state, and recover the necessary Jacobians. One can then translate Jacobians into a linear system using standard methods. We detail this approach below. We describe how to solve for a steady state in Kurtzman and Zeke (2015b).

## Compression of State Space

Even though the Reiter (2009) method allows us to carry many states, it is extremely helpful to speed up the algorithm by compressing the state space. Importantly, it is still very important to be precise in our procedure, as the Jacobians are calculated numerically. The nonlinear system of equations for which we calculate our Jacobians is discussed in the next subsection. The first thing we do to compress the state space is to, as in Kurtzman and Zeke (2015b), reduce the number of state variables the firm and equity holders have to carry by rewriting their problems as a function of the number of steps the firm is from bankruptcy. Hence, the firm and equity holders must only carry two state variables: the number of steps a firm is from bankruptcy, and the firm's *investment opportunities*. We refer to the reader to Kurtzman and Zeke (2015b) for more detail.

We also compress the actual number of states we need to evaluate our Jacobians at for the Bellman functions of equity holders and the firm, along with the actual number of grid points for the size distribution in the following manner. We calculate these objects on fine grids. We then evaluate cubic splines at a finite number of grid points (40 for the Bellman functions and 100 for the size distribution). Notice, just counting these objects, we still

have 180 states. As we note in Section 3, we obtain small Euler equation errors with our methodology.

## System of Equations and other Details

We now define the system of equations that represents our general equilibrium model. We follow Terry (2014) in our notation.

$$\begin{aligned}
V_{t-1}^E(z, d, \theta) = & \max \left\{ 0, \max_q \left\{ (1 - \tau)(\pi(P_{t-1}, W_{t-1}, Y_{t-1}, A_{t-1}, z, d, \theta) - \right. \right. \\
& P_{r,t-1}\phi(\Delta_{z,t-1}, z, \theta, q') - (1 - \tau_d)d\frac{1}{P_{t-1}} + \\
& \left. \left. \beta M_{t+1}(q'V_t^E(z + \Delta_{z,t-1}, d, \theta) + (1 - q')V_t^E(z - \Delta_{z,t-1}, d, \theta)) \right\} + \right. \\
& \left. \eta_t^E(z, d, \theta) \right\},
\end{aligned} \tag{4.1}$$

where  $\beta = e^{-r}(1 - \delta)$ , and  $\eta_t^E$  are expectational errors in the equation for the value of equity holders. If the firm does not go bankrupt,

$$\begin{aligned}
V_{t-1}^F(z, d, \theta) = & (1 - \tau)(\pi(P_{t-1}, W_{t-1}, Y_{t-1}, A_{t-1}, z, d, \theta) - \\
& P_{r,t-1}\phi(\Delta_{z,t-1}, z, \theta, q^*) + \tau_d d\frac{1}{P_{t-1}} + \\
& \beta M_{t+1}(q^*V_t^F(z + \Delta_{z,t-1}, d, \theta) + (1 - q^*)V_t^F(z - \Delta_{z,t-1}, d, \theta)) + \\
& \eta_t^F(z, d, \theta),
\end{aligned} \tag{4.2}$$

where  $\eta_t^F$  are expectational errors in the equation for the value of the firm and  $q^*$  is the  $q$  that maximizes (4.1). Recall, the firm goes bankrupt if the value of equity holders falls below zero. If the firm does go bankrupt:

$$V_{t-1}^F(z, d, \theta) = \max_{d'} \{V_{t-1}^F(z + \ln(\alpha), d', \theta)\}. \tag{4.3}$$

We can define the mass of firms across states as follows:

$$\begin{aligned}
\Gamma_t(z, d, \theta) = & (1 - \delta)(\Gamma_{t-1}(z - \Delta_{z,t-1}, d, \theta)q'_{t-1}(z - \Delta_{z,t-1}, d, \theta) + \\
& (1 - \delta)(\Gamma_{t-1}(z + \Delta_{z,t-1}, d, \theta)(1 - q'_{t-1}(z + \Delta_{z,t-1}, d, \theta)) + \\
& \mathbf{1}_{(d=d_{t-1}^*(z, \theta))} \left( M_{e,t-1}G(z, \theta) + \mathbf{1}_{(V_t^E(z - \ln(\alpha_{t-1}), d', \theta) < 0)} * \right. \\
& \int \left( \Gamma_{t-1}(z - \Delta_{z,t-1} - \ln(\alpha_{t-1}), d', \theta)q'_{t-1}(z - \Delta_{z,t-1} - \ln(\alpha_{t-1}), d', \theta) + \right. \\
& \Gamma_{t-1}(z + \Delta_{z,t-1} - \ln(\alpha_{t-1}), d', \theta) * \\
& \left. \left. (1 - q'_{t-1}(z + \Delta_{z,t-1} - \ln(\alpha_{t-1}), d', \theta)) \right) dd' \right),
\end{aligned}$$

where  $d_{t-1}^*(z, \theta)$  is the choice of  $d'$  that maximizes (4.3) at time  $t - 1$ .

$$R_{t-1} = M_{e,t-1}e^{\bar{z}}n_e + \sum_{\theta} \sum_d \sum_z \Gamma_{t-1}(z, d, \theta)\phi(\Delta_{z,t-1}, z, \theta, q^*(z, d, \theta)).$$

$$\lambda Y_{r,t-1} = (1 - \lambda) * L_{r,t-1} * \frac{W_{t-1}}{P_{t-1}}.$$

$$\frac{P_{r,t-1}}{P_{t-1}} L_{r,t-1}^\lambda Y_{r,t-1}^{(1-\lambda)} = Y_{r,t-1} + \frac{W_{t-1}}{P_{t-1}} L_{r,t-1}$$

$$M_{e,t-1}P_{r,t-1}n_e = M_{e,t-1}e^{\bar{z}} \left( \beta M_{t+1} \sum_{\theta} \sum_d \sum_z g(\theta) V_t^F(z, d_{t-1}^*, \theta) + \eta_t^{M_e} \right),$$

where  $\eta_t^{M_e}$  is the expectational error for the free entry condition.

$$Z_{t-1} = \sum_{\theta} \sum_d \sum_z \Gamma_{t-1}(z, d, \theta).$$

$$P_{t-1} = \frac{1}{C_{t-1}},$$

with log utility.

$$\frac{W_{t-1}}{P_{t-1}} = (A_{t-1}Z_{t-1})^{\frac{1}{\rho-1}}.$$

$$R_{t-1} = L_{r,t-1}^\lambda Y_{r,t-1}^{(1-\lambda)}.$$

$$Y_{t-1} = (A_{t-1}Z_{t-1})^{\frac{1}{\rho-1}}(L - L_{r,t-1}).$$

$$C_{t-1} = Y_{t-1} - Y_{r,t-1}.$$

$$\ln(A_t) = \rho_A \ln(A_{t-1}) + \epsilon_{A,t}.$$

$$\ln(\Delta_z) = (1 - \rho_{\Delta_z}) \ln(\bar{\Delta}_z) + \rho_{\Delta_z} \ln(\Delta_{z,t-1}) + \epsilon_{\Delta_z,t},$$

$$\ln(\alpha_t) = (1 - \rho_\alpha) \ln(\bar{\alpha}) + \rho_\alpha \ln(\alpha_{t-1}) + \epsilon_{\alpha,t},$$

$$\ln(\xi_t) = (1 - \rho_\xi)\ln(\bar{\xi}) + \rho_\xi\ln(\xi_{t-1}) + \epsilon_{\xi,t}.$$

Hence, we can define matrices,  $X_{t-1}$  and  $X_t$ , such that

$$X_{t-1} = \begin{bmatrix} V_{t-1}^E \\ V_{t-1}^F \\ \Gamma_{t-1} \\ \ln(L_{r,t-1}) \\ \ln(M_{t-1}) \\ \ln(Z_{t-1}) \\ \ln(C_{t-1}) \\ \ln(P_{t-1}) \\ \ln\left(\frac{W_{t-1}}{P_{t-1}}\right) \\ \ln(Y_{t-1}) \\ \ln(A_{t-1}) \\ \ln(\Delta_{z,t-1}) \\ \ln(\alpha_{t-1}) \\ \ln(\xi_{t-1}) \end{bmatrix} \quad X_t = \begin{bmatrix} V_t^E \\ V_t^F \\ \Gamma_t \\ \ln(L_{r,t}) \\ \ln(M_t) \\ \ln(Z_t) \\ \ln(C_t) \\ \ln(P_t) \\ \ln\left(\frac{W_t}{P_t}\right) \\ \ln(Y_t) \\ \ln(A_t) \\ \ln(\Delta_{z,t}) \\ \ln(\alpha_t) \\ \ln(\xi_t) \end{bmatrix}.$$

Define  $X_{ss}$  as the steady state values of  $X$ . We want to transform the system into the canonical form:

$$X_t - X_{ss} = A(X_{t-1} - X_{ss}) + B\epsilon_t \quad (4.4)$$

To do so, we compute  $F_1$ ,  $F_2$ ,  $F_3$ , and  $F_4$  such that:

$$F_1(X_t - X_{ss}) + F_2(X_{t-1} - X_{ss}) + F_3\eta_t + F_4\epsilon_t = 0, \quad (4.5)$$



where  $\epsilon_t$  represents the aggregate shocks at time  $t$ , and where  $F_1 = \frac{\partial F}{\partial X_t}$ ,  $F_2 = \frac{\partial F}{\partial X_{t-1}}$ ,  $F_3 = \frac{\partial F}{\partial \eta_t}$ , and  $F_4 = \frac{\partial F}{\partial \epsilon_t}$ . With Chris Sims's **gensys** software, we can then recover  $A$  and  $B$  to get a system of the form in (4.4) from (4.5).

# Chapter 3: Accounting for Dispersion over the Business Cycle without Estimating Production Function Coefficients or Firm-Level TFP

## 1 Chapter 3: Introduction

Estimating firm-level total factor productivity (TFP) and production functions is a delicate problem, where issues of co-linearity, measurement error and misspecification can all severely bias results. Yet, in addressing the role of resource reallocation in productivity dynamics over the business cycle, the literature has relied on the estimation of these technological measures. This paper proposes measures of allocative efficiency which depend only on the measurement of firm-level value added, capital, and labor utilization, and do not require the estimation of firm-level production functions or productivity. We measure the role of the allocation of resources in productivity dynamics by decomposing changes in aggregate factor productivity ratios (labor to output and capital to output ratios) into components representing changes in the mean factor productivities of sectors, the dispersion of factor productivity within sectors, and the output share of sectors. Our decomposition is only relevant if it helps to measure the effect of the reallocation of resources; to address this point, in a general setting, we demonstrate that the within-industry component of our decomposition is reflective of within-sector allocative efficiency.

Our decomposition is useful in that it does not rely on estimating firm-level TFP and production function coefficients. In Figure 1.1, we demonstrate one possible issue with identification that can severely change the interpretation of the qualitative and quantitative importance of the role of reallocation over the business cycle. We show our results change substantially when we follow a standard procedure and only slightly vary the estimation procedure for production function coefficients.<sup>20</sup> Figure 1.1 shows the cumulative change in

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<sup>20</sup>Specifically, we implement the approach of Oberfeld (2013) to measure changes in allocative efficiency

the contribution of allocative efficiency to TFP over the recent recession for three different standard “versions” of estimating production function coefficients.<sup>21</sup> We estimate production function coefficients as the average of the ratio of capital expenditures to labor expenditures,  $\frac{rk}{wl}$ , across firms within a sector over time, where  $r$  is the rental rate,  $k$  is the capital stock in the firm,  $w$  is the wage, and  $l$  is labor utilization in the sector. In version 1, we drop all observations before 1974, we use capital and labor utilization from our firm-level data, and we estimate the rental rate and wage following Chari, Kehoe, and McGrattan (2007). We see that in version 1 of our estimation of production function coefficients, there seems to be a decrease in allocative efficiency from pre-recession levels to the trough in 2008-2009. In version 2, we take capital,  $k$ , and labor expenditures,  $wl$ , from the firm-level data, but still estimate  $r$  following Chari, Kehoe, and McGrattan (2007). Estimating labor expenditures using firm-level data changes the year-over-year behavior of the contribution of allocative efficiency to TFP. In version 3, we follow the same procedure as in version 1, but drop all observations before 1976. In this version, which is only different from version 1 in that we assume there is slightly less data available decades prior to the recession we are examining, we find an increase in allocative efficiency from 2006 to the trough of the recession. These results demonstrate just one of the potential problems with identification to which the standard model-based approaches are susceptible, a problem which can substantially change the qualitative and quantitative implications of the role of reallocation over the business cycle. Other issues with identification remain; another example is that we find that when we vary the elasticity of substitution across firms between 3 and 10 (standard values used in the literature as noted in Hsieh and Klenow (2009)), the qualitative and quantitative nature of our results change substantially.

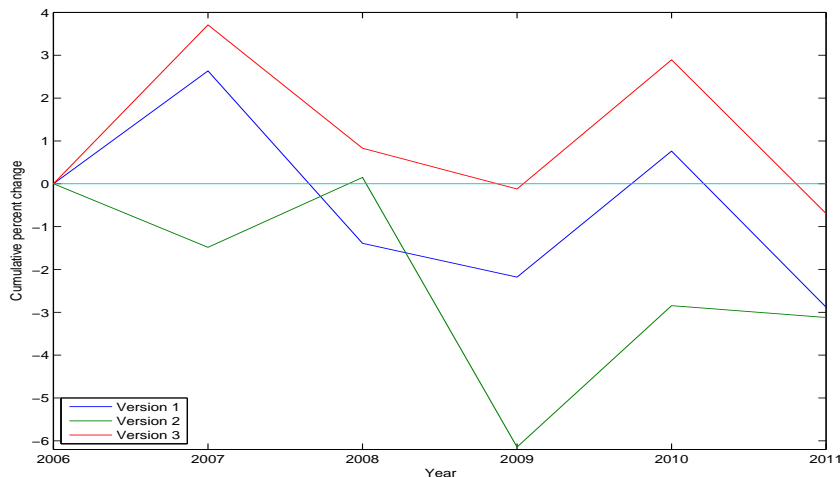
Nonetheless, the role of reallocation over the business cycle is an important question which

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in our sample of U.S. publicly listed firms. This model of production and aggregation is identical to that in Hsieh and Klenow (2009). Details of our dataset construction and measurement can be found in Section 3.1 and Appendix A. As in Oberfeld (2013) and Hsieh and Klenow (2009), we set the elasticity of substitution within sectors to 3.

<sup>21</sup>Positive changes indicate an increase in the extent of allocative efficiency.

Figure 1.1: Three Estimation Techniques for the Contribution of Allocative Efficiency to TFP



We can express TFP as a function of the hypothetical efficient productivity,  $TFP_t^{eff}$ , and the allocative efficiency of resources  $a_t$  such that  $TFP_t = a_t TFP_t^{eff}$ . The figure shows  $\ln(a_t) - \ln(a_{2006})$  for the range of sigma used in the literature. The lines can thus be interpreted as the cumulative percent change in TFP over 2006 levels due to changes in allocative efficiency. The estimation of the different “versions” only differs in the estimation of production function coefficients as described in paragraph two of Section 1.

the literature needs to address, and to the end of deriving measures of the role of reallocation over the business cycle, we differ from the literature in that we employ a decomposition of movements in TFP that does not rely on the estimation of firm-level TFP or production function coefficients. Our decomposition relies on two key facts. First, the natural logarithm of TFP can be expressed as a linear function of the natural logarithm of aggregate factor productivity ratios. Second, the natural logarithm of aggregate factor productivity ratios can be separated into mean and dispersion components for any grouping of firms. The dispersion component is always non-negative, and is similar to an entropy measure used widely in the finance literature. Using these two facts, we can decompose changes in the natural logarithm of factor productivity ratios into three components. The first component captures changes in the means of the factor productivity ratios of firms within sectors. The second component captures the effect of changes in the dispersion of factor productivity ratios within sectors. The third component captures the effect of changes in the output share of sectors. We express aggregate TFP as a function of these components for labor and capital factor productivity ratios. This decomposition, by construction, always sums to aggregate TFP. We are also

able to isolate the effect of entry and exit on TFP with such an approach.

We argue that our decomposition captures the role of allocative efficiency in driving time series properties of TFP. Specifically, we show within an off-the-shelf model with heterogeneous firms that the within-sector dispersion component of the decomposition measures changes in within-sector allocative efficiency. We apply a second-order Taylor approximation to this model, and we show that changes in the within-industry dispersion component of our decomposition are proportional to changes in within-sector allocative efficiency in the model. Separately, we also argue that changes in sectoral output shares reflect the effect of reallocation of resources between sectors in our sample. We note that changes in the relative productivity of sectors could also affect changes in sectoral output shares, but that our empirical results show that changes in sectoral productivity are not correlated with sectoral output shares. Lastly, we argue that changes in the mean component are indicative of changes in firm-level TFP or distortions that affect firms evenly.

We argue our measure has a clean interpretation under standard assumptions, such as the elasticity of demand within a sector is constant. It is possible that a firm could have its elasticity of demand be low relative to the average in the sector in one year but not be subject to input frictions, and then the next year have a higher elasticity of demand relative to the average but be subject to frictions. If this were the case, misallocation could worsen without dispersion increasing. This is a fair criticism to which - more generally - the literature we are aware of, to which our paper contributes, is subject. We see our measure as a first step in systematically avoiding issues in identification of production function coefficients and firm-level TFP while still understanding the role of reallocation over the business cycle, but we in no way see it as a last step, and argue more work needs to be done here.

We perform our decomposition on a sample of U.S. public firms from 1972-2012. Our analysis requires that we derive productivity ratios; the difficult task in constructing productivity ratios (labor to value added and capital to value added) is finding a good measure of value added. We use the income side of the balance sheet to get a measure of firm-

contribution to Gross Domestic Income (GDI) for U.S. public firms. This measure is the sum of net operating profits before depreciation and employee wages. This measure summed over every private and public sector establishment, adjusted for production and import taxes minus subsidies and the difference between accounting and economic treatment of certain variables, is equivalent to GDI.<sup>22</sup> We show the time series properties of a measure of aggregate TFP computed from public firm data behaves similarly to a measure of TFP computed from the National Income and Product Accounts (NIPA).

We find that neither changes in the dispersion in productivity ratios within industries nor changes in sectoral output shares amplify movements in aggregate productivity over the business cycle. The correlation of changes in these two measures with TFP is near zero. If anything, these two measures play a stabilizing role during the recessions in our sample. The *magnitude* of changes in the dispersion in productivity ratios within industries and changes in sectoral output shares are also second-order. It is changes in the mean productivity of sectors that drives the majority of year-over-year variation in aggregate productivity. We extend our decomposition to public firms in Japan and find very similar results.

We provide a short review of the highly related literature below. The rest of the paper follows as such. Section 2 describes our decomposition of aggregate productivity and its relationship with allocative efficiency. Section 3 details the implementation of our decomposition on public firm data in the U.S. and presents the results. Section 4 extends our analysis to public firms in Japan. Section 5 concludes.

**Highly Related Recent Literature** The role of allocative efficiency in driving business cycle movements has been previously explored in the literature. A collection of papers, for a variety of business cycle episodes in various countries, have analyzed the extent to which the allocation of resources between firms is significant. Here, we focus on a few recent papers. Osoimehin (2013) finds, for a large sample of French firms from the manufacturing

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<sup>22</sup>The Bureau of Economic Analysis computes adjustments for accounting versus economic treatment of depreciation and inventories, which affects operating profits. We do not have tax data to perform these adjustments.

and service sectors, that within-sector allocative efficiency tends to increase during recessions and dampen business cycle movements, while between-sector efficiency is acyclical and not important. When showing the within-industry component of our decomposition is reflective of within-sector allocative efficiency, we use some of the methods from her paper. Oberfeld (2013) finds, for a sample of manufacturing firms during the Chilean crisis of 1982, that within-sector allocative efficiency remained constant or improved, but between-sector allocative efficiency fell substantially and explained around a third of the TFP drop in that recession. Sandleris and Wright (2014) find, for a sample of manufacturing firms during the 2001 Argentinian crisis, that both within and between-sector allocative efficiency fell and explained more than half of the decline in aggregate productivity. As these papers rely on datasets from varying countries that cover different time periods, the disagreements in the role of allocative efficiency in driving the business cycle may reflect differences in the nature of recessions in these countries or particular time periods. Alternatively, all of these results are dependent on the identification of production function technologies and the technical efficiency of firms. Production function technologies and the technical efficiency of firms are required to compute how much higher productivity could be if resources were transferred from less productive firms or industries to more productive ones. Identification of production functions is a challenging task. Our paper, by contrast, uses a different approach to measure the changes in allocative efficiency that does not rely on identifying production coefficients or firm-level TFP.

## **2 TFP Decomposition**

Estimating firm-level TFP and production functions is a difficult problem, particularly for samples with a high degree of heterogeneity. The literature which analyzes the role of allocative efficiency relies on estimates of firm-level TFP and production function coefficients to measure changes in allocative efficiency. This section proposes a simpler approach for

decomposing aggregate productivity which avoids estimating firm-level TFP and production function coefficients.

## 2.1 Overview

We first note that aggregate productivity can be expressed as a function of aggregate factor productivity ratios and capital's share of income, which we denote with the variable,  $a$ :

$$\ln(TFP_t) = -a \ln\left(\frac{K_t}{Y_t}\right) - (1-a) \ln\left(\frac{L_t}{Y_t}\right), \quad (2.1)$$

where  $K$  is the aggregate capital stock,  $Y$  is real total output, and  $L$  is total labor supply, and all variables are indexed by time. We can express aggregate factor productivity ratios as a function of firm-factor productivity ratios and the relative distribution of nominal output:

$$\frac{K_t}{P_t Y_t} = \sum_i \frac{k_{it}}{p_{it} y_{it}} \frac{p_{it} y_{it}}{P_t Y_t}, \quad (2.2)$$

and

$$\frac{L_t}{P_t Y_t} = \sum_i \frac{l_{it}}{p_{it} y_{it}} \frac{p_{it} y_{it}}{P_t Y_t}, \quad (2.3)$$

where  $P_t Y_t$  is nominal output,  $p_{it} y_{it}$  is firm-level value added,  $k_{it}$  is firm-level capital stock, and  $l_{it}$  is firm-level labor utilization. This aggregation, (2.2) and (2.3), can also be applied for any grouping of firms, such as a sector. This same method can then be used to compute aggregate factor productivity ratios from sectoral factor productivity ratios and output weights. The aggregation of individual productivity ratios is useful because of the economic significance of productivity ratios. In any setting with Cobb-Douglas production functions, a factor productivity ratio times a constant reflects the marginal value of an input to the firm. The constant term depends on production function coefficients and the demand elasticity the firm faces. If these production function coefficients and demand elasticities are constant



for a group of firms, such as a specific sector of the economy, dispersion in productivity ratios within that sector is indicative of frictions preventing those firms from equalizing the marginal value of inputs to their marginal costs. Note that these frictions may either be socially inefficient, such as financial or regulatory frictions, or socially efficient, such as adjustment costs.<sup>23</sup> To measure the extent to which movements in sectoral factor productivity ratios are driven by within-sector allocative efficiency versus technological considerations or the total supply of input factors to the sector, we decompose each factor productivity ratio for a sector into a mean component and a dispersion component. Using this decomposition for each sector, we can calculate aggregate measures of these objects. We then decompose each aggregate factor productivity ratio into changes in the corresponding sector mean component, sectoral share of output, and the dispersion component.

We can easily express the aggregate factor productivities in real terms. For simplicity in notation, we normalize  $P_t = 1$ , so all variables denominated in dollars are in real terms.

## 2.2 Decomposing Sectoral TFP

Note that the decomposition of aggregate productivity ratios in (2.2) can be expressed as a function of a mean component and a dispersion component:

$$\ln\left(\frac{K_t}{Y_t}\right) = \underbrace{\int \ln\left(\frac{k_{it}}{p_{it}y_{it}}\right) \frac{p_{it}y_{it}}{Y_t} di}_{\text{mean}} + \underbrace{\left(\ln\left(\int \frac{k_{it}}{p_{it}y_{it}} \frac{p_{it}y_{it}}{Y_t} di\right) - \int \ln\left(\frac{k_{it}}{p_{it}y_{it}}\right) \frac{p_{it}y_{it}}{Y_t} di\right)}_{\text{dispersion component}}. \quad (2.4)$$

Note that the mean component is the output-weighted average of productivity ratios. The dispersion component is an entropy term; by Jensen's inequality it is always non-negative. If all firms have identical factor productivity ratios, the dispersion component is zero. If

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<sup>23</sup>Asker, De Loecker, and Collard-Wexler (2014) argue that much of the dispersion in measured capital misallocation across countries and industries can in fact be explained by the presence of adjustment costs and variation in the volatility of productivities. Thus, the allocation of resources can be dynamically efficient but significantly vary from the allocation that would maximize output in a given period.

one were to assume joint log-normality of output shares and factor productivity ratios, the dispersion component would be a function of the variance-covariance matrix. More generally, this component is a measure of how the non-constancy of individual factor productivity ratios affects the sectoral factor productivity ratio. Movements in the mean component are indicative of movements that affects all firms evenly, while movements in the dispersion component are indicative of changes in the dispersion of factor productivity ratios.

We can utilize this representation to get the impact of distribution of productivity ratios on TFP. To this end, we denote the mean and entropy components found in (2.4) as  $M_t^k$  and  $E_t^k$ , respectively. We define  $M_t^l$  and  $E_t^l$  analogously for the labor productivity ratio. We can then express TFP for a group of firms as:

$$\ln(TFP_t) = -a (M_t^k + E_t^k) - (1 - a) (M_t^l + E_t^l).$$

### 2.3 Aggregation of Multiple Sectors

Productivity ratios vary not only due to frictions, but also due to heterogeneity in production function coefficients or the demand elasticities faced by firms. We assume that there exist groupings (sectors) within which production function coefficients and demand elasticities are constant. This implies that the statically optimal productivity ratios should vary across sectors, but not within them. We then construct mean and entropy components, as above, for each productivity ratio in each sector. The advantage of our approach is that it is not necessary to estimate production function coefficients. For instance, the natural logarithm of the capital productivity ratio in sector  $j$  can be expressed as the sum of a mean component,  $M_{jt}^k$ , and a dispersion component,  $E_{jt}^k$ :

$$\ln\left(\frac{K_{jt}}{P_{jt}Y_{jt}}\right) = M_{jt}^k + E_{jt}^k,$$

where

$$M_{jt}^k = \int_{i \in j} \ln \left( \frac{k_{it}}{p_{it}y_{it}} \right) \frac{p_{it}y_{it}}{Y_t} di,$$

and

$$E_{jt}^k = \ln \left( \int_{i \in j} \frac{k_{it}}{p_{it}y_{it}} \frac{p_{it}y_{it}}{Y_t} di \right) - \int_{i \in j} \ln \left( \frac{k_{it}}{p_{it}y_{it}} \right) \frac{p_{it}y_{it}}{Y_t} di.$$

Using this notation, we can express the aggregate factor productivity ratios as a function of sectoral output shares and mean and dispersion components for each sector:

$$\frac{K_t}{Y_t} = \sum_j e^{M_{jt}^k + E_{jt}^k} \frac{P_{jt}Y_{jt}}{Y_t},$$

and

$$\frac{L_t}{Y_t} = \sum_j e^{M_{jt}^l + E_{jt}^l} \frac{P_{jt}Y_{jt}}{Y_t}.$$

We can also decompose the natural logarithm of each aggregate factor productivity ratio into a mean component and dispersion component:

$$\ln \left( \frac{K_t}{Y_t} \right) = \underbrace{\ln \left( \sum_j e^{M_{jt}^k} \frac{P_{jt}Y_{jt}}{Y_t} \right)}_{\text{mean component}} + \underbrace{\ln \left( \sum_j e^{M_{jt}^k + E_{jt}^k} \frac{P_{jt}Y_{jt}}{Y_t} \right) - \ln \left( \sum_j e^{M_{jt}^k} \frac{P_{jt}Y_{jt}}{Y_t} \right)}_{\text{dispersion component}},$$

and

$$\ln \left( \frac{L_t}{Y_t} \right) = \underbrace{\ln \left( \sum_j e^{M_{jt}^l} \frac{P_{jt}Y_{jt}}{Y_t} \right)}_{\text{mean component}} + \underbrace{\ln \left( \sum_j e^{M_{jt}^l + E_{jt}^l} \frac{P_{jt}Y_{jt}}{Y_t} \right) - \ln \left( \sum_j e^{M_{jt}^l} \frac{P_{jt}Y_{jt}}{Y_t} \right)}_{\text{dispersion component}}.$$

We denote the mean components of the natural logarithm of aggregate factor productivity

ratios found above for capital and labor as  $\overline{M}_t^k$  and  $\overline{M}_t^l$ , respectively. We similarly denote the dispersion components as  $\overline{E}_t^k$  and  $\overline{E}_t^l$ , respectively. These components summarize the impact of their respective sectoral-level counterparts of aggregate output. Similar to the sector-level measures, the dispersion component is always non-negative and is zero if there is no within-sector variation in factor productivity ratios. Using this notation, we express the natural logarithm of aggregate TFP as:

$$\ln(TFP_t) = -a \left( \overline{M}_t^k + \overline{E}_t^k \right) - (1 - a) \left( \overline{M}_t^l + \overline{E}_t^l \right). \quad (2.5)$$

We can therefore infer whether time series variation is driven primarily by changes in sectoral mean factor productivity ratios,  $\overline{M}_t^k$  and  $\overline{M}_t^l$ , or by changes in the dispersion in productivity ratios within sectors,  $\overline{E}_t^k$  and  $\overline{E}_t^l$ . We amend this decomposition in the following sections to account for changes in entry and exit, as well as in sectoral output shares.

## 2.4 Controlling for Entry and Exit

Entry and exit into our dataset could also account for some of the movement in TFP calculated from our sample. Entry and exit in our sample is not true entry and exit into the economy. Firms entering our sample most likely existed the previous year, but they were not publicly listed or their data is missing for that year. Similarly, many firms exiting our sample still continue to operate the next year, it is just they are either no longer publicly listed or data is missing for that year. To correct for entry and exit we note that (2.5) implies that we can express changes in the natural logarithm of TFP as:

$$\Delta \ln(\overline{TFP}_t) = -a \left( \Delta \overline{M}_t^k + \Delta \overline{E}_t^k \right) - (1 - a) \left( \Delta \overline{M}_t^l + \Delta \overline{E}_t^l \right).$$

For every adjacent pair of years, using only firms on which we have data in both years, we compute  $\overline{M}_t^k$ ,  $\overline{M}_t^l$ ,  $\overline{E}_t^k$ , and  $\overline{E}_t^l$ . This allows us to compute  $\Delta \ln(\overline{TFP}_t)$ ,  $\Delta \overline{M}_t^k$ ,  $\Delta \overline{M}_t^l$ ,  $\Delta \overline{E}_t^k$ , and  $\Delta \overline{E}_t^l$  without entry or exit into our dataset biasing results. We can construct adjusted

series for  $\overline{M}_t^k$ ,  $\overline{M}_t^l$ ,  $\overline{E}_t^k$ , and  $\overline{E}_t^l$  by taking their initial value and adding the values of  $\Delta\overline{M}_t^k$ ,  $\Delta\overline{M}_t^l$ ,  $\Delta\overline{E}_t^k$ , and  $\Delta\overline{E}_t^l$  for each progressive year. We can assess the impact of entry and exit on TFP and aggregate productivity ratios by computing  $\Delta\ln(TFP_t)$ ,  $\Delta\ln\left(\frac{L_t}{Y_t}\right)$ , and  $\Delta\ln\left(\frac{K_t}{Y_t}\right)$  both with all firms and only firms on whom we have data in both years. The difference between these measures computed with all firms and these measure computed only with firms for whom we have data in both years is the impact of entry and exit on the measured values.

## 2.5 The Role of Sectoral Composition

Thus far, we have decomposed aggregate factor productivity ratios into two components: a dispersion component indicative of within-sector allocative inefficiency and a mean component. However, inefficiencies can arise in resource allocation not only within sectors, but also between them.<sup>24</sup> Estimating between-sector allocative efficiency is difficult, as it requires identifying whether productivity differences between sectors are due to technology or frictions. To measure the effect of between-sector reallocation without estimating technology, we augment our decomposition to have an additional component capturing changes in the sectoral share of output. The logic behind this approach is that if frictions to between-sector allocative efficiency improve, then highly productive sectors should increase their output share (as their allocation of inputs increase), while unproductive sectors should see their shares decrease. However, sectoral shares can also be influenced by changes in the technical efficiency of sectors. Osotimehin (2013) describes such a bias in the decomposition performed by Foster, Haltiwanger, and Krizan (2001).<sup>25</sup> A large improvement in the TFP of firms within a sector will result, even for a fixed allocation of inputs between sectors, in an increase of that sector's output share. We control for this bias so that we measure the

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<sup>24</sup>Notably, Oberfeld (2013) attribute one third of the decline in TFP during the 1982 Chilean crisis to between-sector allocative efficiency.

<sup>25</sup>Our measure differs from Foster, Haltiwanger, and Krizan (2001) in several ways. First, our measure does not rely on the identification of firm-level TFP. Additionally, our breakdown assesses changes in aggregate TFP instead of average TFP. Our approach additionally only uses output share approach for measuring between-sector allocative efficiency, rather than all measures.

effect of changes in between-sector resource allocation, rather than changes in sector productivities, on TFP. If the relative allocation of inputs (capital and labor) between sectors remained constant, then the change in output shares could be expressed simply as a function of previous sectoral output shares, changes in sectoral factor productivity ratios, and changes in the total allocation of assets. We can thus express the output share of a given industry implied by the sectoral capital factor productivity ratios (assuming the relative distribution of capital remained constant) as:

$$s_{kjt} = \frac{P_{j,t-1}Y_{j,t-1}}{Y_{t-1}} \frac{\frac{P_{j,t}Y_{j,t}}{K_{j,t}}}{\frac{Y_t}{K_t}} \frac{\frac{Y_{t-1}}{K_{t-1}}}{\frac{P_{j,t-1}Y_{j,t-1}}{K_{j,t-1}}}.$$

Note that if  $\frac{K_{jt}}{K_t} = \frac{K_{j,t-1}}{K_{t-1}}$ , then  $s_{kjt} = \frac{P_{jt}Y_{jt}}{Y_t}$ . We analogously define  $s_{ljt}$  from the sectoral labor factor productivity ratios. We thus can understand the difference between the sectoral output shares,  $\frac{P_{jt}Y_{jt}}{Y_t}$ , and the hypothetical ones implied by sector factor productivity changes,  $s_{kjt}$  and  $s_{ljt}$ , as being due to changes in the allocation of inputs between sectors.

The new augmented decomposition of changes in aggregate factor ratios has three components: changes in the mean components of sector productivity ratios, changes in the sectoral output shares due to resource reallocation between sectors, and changes in the dispersion components of sectors. We can express the change in the natural logarithm of log capital factor productivity as:

$$\begin{aligned} \Delta \ln \left( \frac{K_t}{Y_t} \right) &= \underbrace{\ln \left( \frac{\sum_j e^{M_{jt}^k} s_{kjt}}{\sum_j e^{M_{jt-1}^k} \frac{P_{j,t-1}Y_{j,t-1}}{Y_{t-1}}} \right)}_{\Delta \text{ mean component}} + \underbrace{\ln \left( \frac{\sum_j e^{M_{jt}^k + E_{jt}^k} \frac{P_{jt}Y_{jt}}{Y_t}}{\sum_j e^{M_{jt}^k + E_{jt}^k} s_{kjt}} \right)}_{\Delta \text{ sectoral share component}} \\ &+ \underbrace{\ln \left( \frac{\sum_j e^{M_{jt}^k + E_{jt}^k} s_{kjt}}{\sum_j e^{M_{jt-1}^k + E_{jt-1}^k} \frac{P_{j,t-1}Y_{j,t-1}}{Y_{t-1}}} \right)}_{\Delta \text{ dispersion component}} - \ln \left( \frac{\sum_j e^{M_{jt}^k} s_{kjt}}{\sum_j e^{M_{jt-1}^k} \frac{P_{j,t-1}Y_{j,t-1}}{Y_{t-1}}} \right). \end{aligned} \quad (2.6)$$

We denote the above components as  $\Delta \tilde{M}_t^k$ ,  $\Delta \tilde{I}_t^k$ , and  $\Delta \tilde{E}_t^k$ , respectively, as well as analogous components of the labor productivity ratio  $\Delta \tilde{M}_t^l$ ,  $\Delta \tilde{I}_t^l$ , and  $\Delta \tilde{E}_t^l$ . We can then denote

changes in the natural logarithm of TFP as:

$$\Delta \ln(\overline{TFP}_t) = -a \left( \Delta \tilde{M}_t^k + \Delta \tilde{I}_t^k + \Delta \tilde{E}_t^k \right) - (1-a) \left( \Delta \tilde{M}_t^l + \Delta \tilde{I}_t^l + \Delta \tilde{E}_t^l \right). \quad (2.7)$$

We can then build adjusted series for mean, dispersion, and sectoral output share components by starting with the initial value and adding the changes year by year. For instance, we can construct a series for the contribution of within-sector dispersion in capital factor productivity ratios to TFP as:  $-a \left( \bar{E}_0^k + \sum_{t=1}^{\tau} \Delta \tilde{E}_t^k \right)$ .

Finally, we can combine corresponding labor and capital components in (2.7) to measure the total effect of within-sector allocative efficiency and between-sector allocative efficiency on TFP:

$$\begin{aligned} \Delta \ln(\overline{TFP}_t) = & \underbrace{- \left( a \Delta \tilde{M}_t^k + (1-a) \Delta \tilde{M}_t^l \right)}_{\Delta \text{ mean component}} - \underbrace{\left( a \Delta \tilde{E}_t^k + (1-a) \Delta \tilde{E}_t^l \right)}_{\Delta \text{ dispersion component}} \\ & - \underbrace{\left( a \Delta \tilde{I}_t^k + (1-a) \Delta \tilde{I}_t^l \right)}_{\Delta \text{ sectoral share component}}. \end{aligned} \quad (2.8)$$

Decompositions (2.7) and (2.8) are the decompositions of movements in the natural logarithm of TFP we use to understand the role of allocative efficiency.

## 2.6 Relationship with Models of Allocative Efficiency

This section demonstrates how the components of our decomposition capture changes in allocative efficiency in a simple, static model of heterogeneous firms with frictions on the input allocation. Consider a one sector economy in which firms produce differentiated goods with production function  $y_{it} = A_{it} k_{it}^\alpha l_{it}^\phi$ . Each firm's output is valued at a price  $p = Py^{-\eta}$ , indicating a constant elasticity of  $-\frac{1}{\eta}$ . There is an exogenously supplied stock of total capital and labor,  $K_t$  and  $L_t$ , and corresponding market-clearing wage and rental rates  $w_t$  and  $r_t$ .

There are also frictions on the input decisions of firms,  $\omega_{ilt}$  and  $\omega_{ikt}$ .<sup>26</sup> Firms solve the maximization problem:

$$\max_{k_{it}, l_{it}} P_t \left( A_{it} k_{it}^\alpha l_{it}^\phi \right)^{1-\eta} - k_{it} \frac{r_t}{\omega_{ikt}} - l_{it} \frac{w_t}{1 - \omega_{ilt}}.$$

Firms take wage and rental rate as given, which satisfy the clearing conditions  $K_t = \sum_i k_{it}$  and  $L_t = \sum_i l_{it}$ . The equilibrium decision rules and prices imply that the firm-level productivity ratios are:

$$\frac{l_{it}}{\frac{p_{it}}{P_t} y_{it}} = K_t^{-\alpha(1-\eta)} L_t^{1-\phi(1-\eta)} \frac{(\omega_{ilt}) \left( \sum_i \left( A_{it}^{1-\eta} (\omega_{ikt})^{1-\phi(1-\eta)} (\omega_{ilt})^{\phi(1-\eta)} \right)^{\frac{1}{1-(\alpha+\phi)(1-\eta)}} \right)^{\alpha(1-\eta)}}{\left( \sum_i \left( A_{it}^{1-\eta} (\omega_{ikt})^{\alpha(1-\eta)} (\omega_{ilt})^{1-\alpha(1-\eta)} \right)^{\frac{1}{1-(\alpha+\phi)(1-\eta)}} \right)^{1-\phi(1-\eta)}}, \quad (2.9)$$

and

$$\frac{k_{it}}{\frac{p_{it}}{P_t} y_{it}} = K_t^{1-\alpha(1-\eta)} L_t^{-\phi(1-\eta)} \frac{1}{P_t} \frac{(\omega_{ikt}) \left( \sum_i \left( A_{it}^{1-\eta} (\omega_{ikt})^{\alpha(1-\eta)} (\omega_{ilt})^{1-\alpha(1-\eta)} \right) \right)^{\phi(1-\eta)}}{\left( \sum_i \left( A_{it}^{1-\eta} (\omega_{ikt})^{1-\phi(1-\eta)} (\omega_{ilt})^{\phi(1-\eta)} \right)^{\frac{1}{1-(\alpha+\phi)(1-\eta)}} \right)^{1-\alpha(1-\eta)}}, \quad (2.10)$$

The output share of each firm is:

$$\frac{p_{it} y_{it}}{P_t Y_t} = \frac{\left( A_{it}^{1-\eta} (\omega_{ikt})^{\alpha(1-\eta)} (\omega_{ilt})^{\phi(1-\eta)} \right)^{\frac{1}{1-(\alpha+\phi)(1-\eta)}}}{\sum_i \left( A_{it}^{1-\eta} (\omega_{ikt})^{\alpha(1-\eta)} (\omega_{ilt})^{\phi(1-\eta)} \right)^{\frac{1}{1-(\alpha+\phi)(1-\eta)}}}. \quad (2.11)$$

Combining (2.9), (2.10), and (2.11) with (2.1), (2.2), and (2.3), we can characterize sectoral TFP as a function of firm-level frictions,  $\omega_{ikt}$  and  $\omega_{ilt}$ , and productivities,  $A_{it}$ . A second-order Taylor expansion of these expressions gives us closed-form approximations for the effect of

<sup>26</sup>In the spirit of Chari, Kehoe, and McGrattan (2007), the nature of these frictions is not specified.



TFP as a function of firm-level TFP and frictions.<sup>27</sup> In Appendix B, we provide the details of this computation. Hence, we can express the natural logarithm of TFP as:

$$\begin{aligned}
\ln(TFP) &\simeq \ln\left(\frac{\overline{A}_t^{(1-\eta)} n^{(1-(\alpha+\phi)(1-\eta))}}{K_t^{(a-\alpha(1-\eta))} L_t^{(1-a-\phi(1-\eta))}}\right) + (1-\eta) \left(\frac{-\eta + (\alpha + \phi)(1-\eta)}{1 - (\alpha + \phi)(1-\eta)}\right) \frac{1}{2} \frac{\sigma_{A_t}^2}{\overline{A}_t^2} \\
&- \frac{\alpha(1-\eta)(1-\phi(1-\eta))}{2(1-(\alpha+\phi)(1-\eta))} \frac{\sigma_{k_t}^2}{\overline{\omega_{k_t}}^2} - \frac{\phi(1-\eta)(1-\alpha(1-\eta))}{2(1-(\alpha+\phi)(1-\eta))} \frac{\sigma_{l_t}^2}{\overline{\omega_{l_t}}^2} \\
&- \frac{\phi\alpha(1-\eta)^2}{1-(\alpha+\phi)(1-\eta)} \frac{\sigma_{klt}}{\overline{\omega_{lt}\omega_{kt}}},
\end{aligned} \tag{2.12}$$

where  $A_{it}, \omega_{ilt}, \omega_{ikt}$  are distributed with means  $\overline{A}_t, \overline{\omega_{lt}}, \overline{\omega_{kt}}$ , variances  $\sigma_{A_t}^2, \sigma_{l_t}^2, \sigma_{k_t}^2$ , and covariances  $\sigma_{Alt}, \sigma_{Akt}, \sigma_{klt}$ .  $n$  denotes the number of firms. In order to isolate the effect of the distortions on the input allocation, we compute TFP in the socially efficient allocation (where  $\omega_{ilt} = \omega_{ikt} = 1$ ). This yields:

$$\ln(TFP_{eff}) \simeq \ln\left(\frac{\overline{A}_t^{(1-\eta)} n^{(1-(\alpha+\phi)(1-\eta))}}{K_t^{(a-\alpha(1-\eta))} L_t^{(1-a-\phi(1-\eta))}}\right) + \left(\frac{-\eta + (\alpha + \phi)(1-\eta)}{1 - (\alpha + \phi)(1-\eta)}\right) \frac{1-\eta}{2} \frac{\sigma_{A_t}^2}{\overline{A}_t^2} \tag{2.13}$$

The difference between (2.12) and (2.13) is the effect of distortions on aggregate productivity. This is a function of the variance of the distortions (relative to the mean factor distortion) and their covariance. The mean distortion does not enter by itself, because input prices adjust. We can use our decomposition in (2.7) to split sectoral TFP into mean and dispersion components. The contribution of the dispersion components to TFP,  $-a\tilde{E}_t^K$  and  $-(1-a)\tilde{E}_t^L$ , can be expressed as  $-a\frac{1}{2}\frac{\sigma_{k_t}^2}{\overline{\omega_{k_t}}^2}$  and  $-(1-a)\frac{1}{2}\frac{\sigma_{l_t}^2}{\overline{\omega_{l_t}}^2}$ . We can express the natural logarithm of TFP as:

$$\begin{aligned}
\ln(TFP) &\simeq \ln(TFP_{eff}) - \frac{\phi\alpha(1-\eta)^2}{1-(\alpha+\phi)(1-\eta)} \frac{\sigma_{klt}}{\overline{\omega_{lt}\omega_{kt}}} \\
&- \frac{\alpha(1-\eta)(1-\phi(1-\eta))}{(1-(\alpha+\phi)(1-\eta))} \tilde{E}_t^K - \frac{\phi(1-\eta)(1-\alpha(1-\eta))}{(1-(\alpha+\phi)(1-\eta))} \tilde{E}_t^L.
\end{aligned}$$

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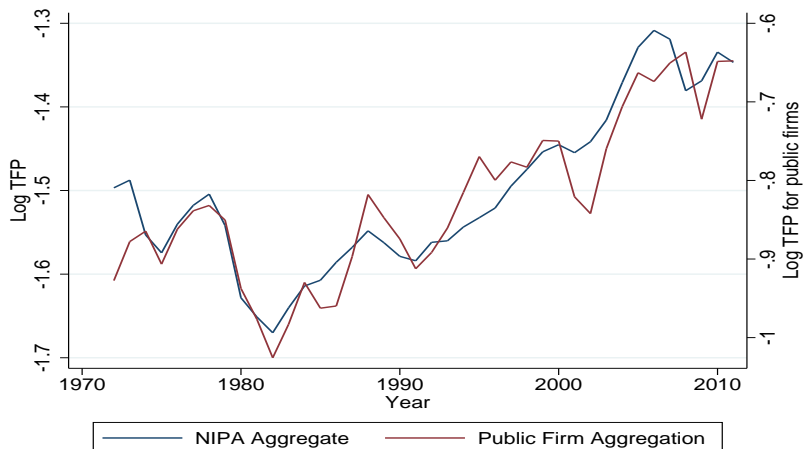
<sup>27</sup>We compute this Taylor expansion for TFP following Osotimehin (2013).

We can express the natural logarithm of TFP as a linear function of efficient TFP, the dispersion components of capital and labor productivity ratios, and a term capturing the interactions of the input frictions on capital and labor. If we assume that the interaction term is second-order (or at least changes in it are), then the impact of frictions on aggregate productivity can be approximated as a linear function of the dispersion components of factor productivity ratios. The constants on these components depend on the particular technology and elasticity of demand facing firms in the sector. Our decomposition (2.7), in comparison, has these dispersion components enter with coefficients equal to the negative of aggregate factor shares. This implies that the time series movements in the dispersion component of aggregate capital productivity is approximately proportional to movements in the degree of within-sector allocative inefficiency due to input frictions on capital decisions. An analogous approximation holds for input frictions on labor decisions. The mean components pick up the remainder of the terms in (2.12), including all of the productivity and input supply terms.

### 2.6.1 Inference Regarding the Magnitude of the Dispersion Component

While the proportionality of the relationship shown above is useful in determining the cyclical properties of distortions to input frictions, the magnitude of the constant reflects the importance of these frictions over time. The model implies that if  $\frac{\alpha(1-\eta)(1-\phi(1-\eta))}{(1-(\alpha+\phi)(1-\eta))} \geq a$ , the dispersion component of the capital productivity ratio is a lower bound for the magnitude of movements in within-sector allocative efficiency. Otherwise, it is an upper bound. Similarly, if  $\frac{\phi(1-\eta)(1-\alpha(1-\eta))}{(1-(\alpha+\phi)(1-\eta))} \geq (1-a)$ , the labor dispersion component of the labor productivity ratio is a lower bound for the magnitude of movements in within-sector allocative efficiency. Otherwise, it is an upper bound. We do not attempt to estimate  $\alpha$ ,  $\phi$ , or  $\eta$ ; our model-free decomposition is motivated by the difficulty in accurately identifying these parameters. However, we can characterize the threshold in a simplified setting. For instance, assume that firms produce with constant returns to scale technology ( $\alpha + \phi = 1$ ) and that  $\alpha = a$

Figure 2.1:  $\ln(TFP)$  from NIPA and Compustat



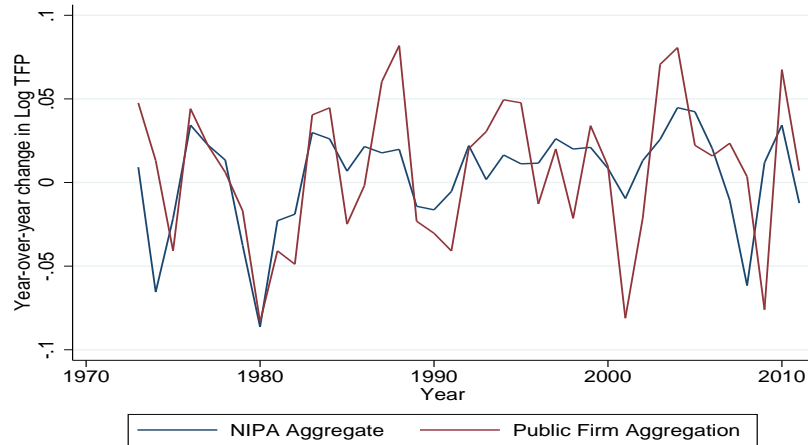
Note: Sample period: 1972-2011. The figure depicts the natural logarithm of aggregate productivity computed from two different samples. The red line and right axis correspond to a measure computed from the aggregation of public firms. The blue line and left axis correspond to a measure computed from the NIPA equivalent of our firm-level measures.

and  $\phi = 1 - a$ . In this case, we can characterize the threshold at which the magnitudes of our measures correspond to the magnitude of the effect of the corresponding input friction. For capital, this threshold is  $\eta = \frac{-\alpha + \sqrt{\alpha}}{(1-\alpha)}$ . For  $\eta$  greater than this threshold, our measure of the dispersion component of the aggregate capital productivity ratio is an upper bound for the magnitude of frictions. Alternatively, it is a lower bound if  $\eta$  is less than this threshold. Similarly, this threshold is  $\eta = \frac{-\phi + \sqrt{\phi}}{(1-\phi)}$  for the labor decomposition. These two thresholds are identical if  $\alpha = \phi = \frac{1}{2}$  and imply  $\eta = \sqrt{2} - 1 \approx .41$ , which can be interpreted as a 100% increase in output leads to a 50% increase in revenue. If we instead use the aggregate factor shares Chari, Kehoe, and McGrattan (2007), implying  $\alpha = .35$  and  $\phi = .65$ , these imply thresholds  $\eta = .37$  and  $\eta = .45$  for capital and labor respectively. These elasticities can be interpreted as a 100% increase in output leads to, respectively, a 55% or 45% increase in revenue.

### 3 Data and Results

In this section we implement our decomposition in (2.7) using data on U.S. publicly listed firms and aggregate data from NIPA, the Bureau of Labor Statistics, and the Penn World

Figure 3.1: Changes in  $\ln(TFP)$  from NIPA and Compustat



Note: Sample period: 1972-2011. The figure depicts changes in the natural logarithm of aggregate productivity computed from two different samples, and can thus be interpreted as the percent changes in aggregate productivity for two different samples. The red line correspond to a measure computed from the aggregation of public firms. The blue line correspond to a measure computed from the NIPA equivalent of our firm-level measures.

Tables.

### 3.1 Aggregation and Accounting

We create a measure of value added in public firms using income accounting. GDP has an income equivalent, GDI, which has similar time-series properties. The major components of this measure have equivalents to balance sheet measures that are required on 10-K forms for U.S. public firms. In order of magnitude, GDI is made up of the following components: compensation of employees, net operating surplus, consumption of fixed capital (depreciation), and taxes on production and imports less subsidies. While we do not observe the taxes or subsidies on production and imports firms pay in our dataset, we do observe measures of the other three components, all of which make up over 90% of GDI for all years in our sample. We observe labor compensation in Compustat annually. We also observe net operating profits before depreciation, which is the sum of a firm's net operating surplus and its capital consumption. We define a firm's contribution to output as the sum of labor compensation and operating profits before depreciation. In practice, the BEA does a similar, more detailed approach, where they use firm tax data to aggregate up the components of domestic income

Table 3.1: TFP Component Correlations

	Year-over-year changes	Deviations from trend
<i>Correlation of <math>\ln(TFP)</math> with:</i>		
Mean component	0.9683	0.9847
Dispersion component	-0.0580	-0.3139
Sectoral share component	-0.2430	-0.0132

Note: Sample period: 1972-2011. We compute the correlations of both year-over-year changes or deviations from the trend of the natural logarithm of TFP with the equivalent for its components as derived in (2.8). Our measure of TFP and the components are all calculated from data on U.S. public firms.

and make adjustments for differences between accounting and economic treatment of factors such as capital consumption and inventory valuation. We also use aggregate equivalents of our firm-level measures to compare the results of our aggregation. Our output measure, which at times we call GDI in the paper as it has similar time-series properties, is actually measured as the sum of compensation of employees excluding government employees, corporate profits before tax without inventory or capital consumption adjustments, and consumption of fixed capital. This measure is meant to mimic our firm-contribution to GDI measure at the aggregate.

We measure labor as the number of employees reported in Compustat.<sup>28</sup> We measure capital as the firm’s plant, property, and equipment, adjusted for accumulated depreciation (PPENT). To adjust for potential changes in the valuation of capital over time, we construct a perpetual inventory measure of the aggregate capital stock, and use the ratio of this measure to the value of the aggregate capital stock to deflate the firm-level measure of PPENT.<sup>29</sup>

We define sectors, within which sectoral dispersion and mean components are calculated, as 2 digit SIC codes. For the calculation of changes in TFP, we only include sectors that have 5 or more firms in the given years.

<sup>28</sup>For robustness we replicate our results using total labor compensation as the measure for labor; such an approach proxies for differences in human capital as in Hsieh and Klenow (2009). Our results are similar.

<sup>29</sup>To construct our measure of capital using the perpetual inventory method, we use a depreciation rate of 4.64% and growth rate of technology of 1.6%, following Chari, Kehoe, and McGrattan (2007). Our measure of the value of capital comes from the Penn World Tables. Our perpetual inventory measure comes from NIPA.

Table 3.2: TFP Component Correlations

	Year-over-year changes	Deviations from trend
<i>Correlation of <math>\ln(TFP)</math> with:</i>		
Labor mean component	0.8959	0.9493
Capital mean component	0.8924	0.9109
Labor dispersion component	-0.0286	-0.3403
Capital dispersion component	0.3558	0.2168
Labor sectoral composition component	0.0940	-0.0746
Capital sectoral composition component	-0.2638	-0.0364

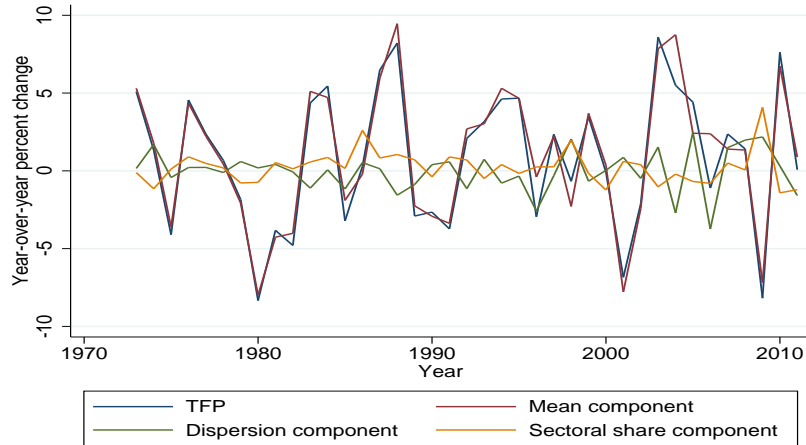
Note: Sample period: 1972-2011. We compute the correlations of both year-over-year changes or deviations from the trend of the natural logarithm of TFP with the equivalent for its components as derived in (2.7). Our measure of TFP and the components are all calculated from data on U.S. public firms.

We follow Chari, Kehoe, and McGrattan (2007), and set capital's share of income,  $a$  to be 0.35. Aggregate  $Y$  is our measure of GDI (explained above). Aggregate  $K$  is the stock of capital measured as above. Total labor is measured as total non-farm employees from the Bureau of Labor Statistics. All dollar measures are in 2005 chain-weighted dollars.

### 3.2 Aggregate TFP from Compustat Compared to TFP from NIPA

Our data only consists of publicly traded firms. Our sample represents a significant slice of the U.S. economy; in 2011, it accounted for over 15% of GDP and over 17 million employees. To understand the extent to which our sample reflects the mechanisms responsible for driving aggregate productivity changes, we compare the time-series behavior of TFP as aggregated from Compustat to that of TFP computed from NIPA. Figures 2.1 and 3.1 show that the time series properties of the two series are similar both in their cyclical dynamics and long term trends. This suggests that some of the key forces driving TFP over time are likely present in Compustat data. If there were significant factors driving TFP over the business cycle which existed only in small, private firms, we would expect systematic differences in the behavior of TFP and our measure computed from publicly listed firms over time. There are some differences in the measures. TFP from Compustat is more volatile. This is unsurprising given the documented greater volatility of corporate profits measured with generally accepted

Figure 3.2: Components of TFP



Note: Sample period: 1972-2011. We compute year-over-year changes in TFP and its components as derived in (2.8). Our measure of TFP and the components are all calculated from data on U.S. public firms.

accounting principles (GAAP) accounting than corporate profits as measured in NIPA.<sup>30</sup> There are also some slight timing differences, particularly in the timing of the trough (of TFP) of the 2007-2009 recession. This may be due to the reporting dates of firms in Compustat. However, the measure of TFP for the U.S. in the Penn World Tables (8.0) has the trough in 2009, so it may also be due to some technical adjustments made in the NIPA aggregation.

### 3.3 Role of Dispersion in Driving Aggregate TFP

Our decomposition, (2.7), gives us a time series of the contribution of changes in the mean, dispersion, and sectoral composition components. We examine the role of these components in driving year-over-year changes as well as individual business cycle episodes. Table 3.1 shows that changes in allocative efficiency within sectors (represented by the dispersion component) has a low correlation with changes in TFP, and over the business cycle it tends to have a slight dampening effect (HP-filtered deviations have a negative correlation). Changes in between-sector allocative efficiency, represented by the sectoral composition component, have a slight negative correlation with changes in TFP and a slight dampening effect over

<sup>30</sup>Hodge (2011) compares the properties of corporate profits computed from the GAAP accounting statements of firms in the S&P 500 index with the corresponding measure from NIPA, finding greater significantly greater volatility in the S&P measure.

Table 3.3: TFP Component Correlations for Japanese Public Firms

<i>Correlation of <math>\Delta \ln(TFP)</math> with <math>\Delta \ln :</math></i>	
Mean component	0.9891
Dispersion component	-0.1092
Sectoral composition component	-0.8433

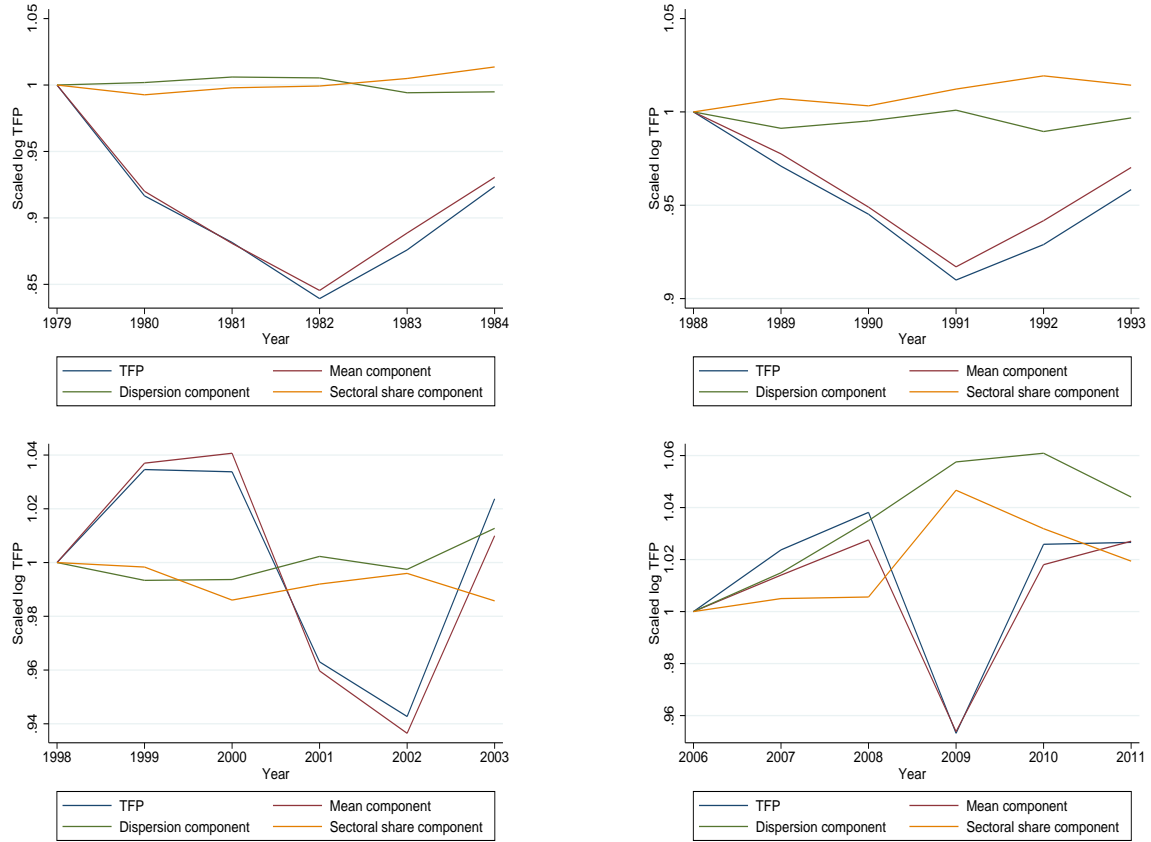
Note: Sample period: 2001-2012. We compute the correlations of both year-over-year changes or deviations from the trend of the natural logarithm of TFP with the equivalent for its components as derived in (2.8). Our measure of TFP and the components are all calculated from data on Japanese public firms in the Compustat global database.

the business cycle. Changes in the mean component (representing productivity changes or frictions affecting all firms evenly) correlate highly with changes in TFP. Table 3.2 describes the correlations of the components of factor productivity ratios with TFP. We see that the dampening effect of the dispersion component is driven by the dispersion in the labor productivity ratios, while the capital dispersion component tend to slightly co-vary with TFP.

Figure 3.2 shows year-over-year changes in the components of TFP. The extensive margin is omitted as variations in it are insignificant. As the components enter TFP multiplicatively (as their logs enter the natural logarithm of TFP additively), the relative magnitudes in the chart are indicative of the effect on TFP. We see that the mean component is by far the most important factor. Variations in the dispersion or sectoral share components are second-order in explaining year-over-year changes in TFP. Figure 3.3 shows the evolution of the components of TFP over the large recessions in our sample. We see that the dispersion component of TFP certainly does not seem to be a significant contributing force to the decline in TFP associated with these recessions; if it has any role at all, it is a minor stabilizing one. Similarly, the role of sectoral composition does not seem to be driving or correlated with the declines in productivity. These results indicate that allocative efficiency, rather than explaining the depth and persistence of recessions, if playing any role, seems to play a minor stabilizing role for U.S. public firms over the business cycle.



Figure 3.3: Contribution of dispersion over 4 recessions

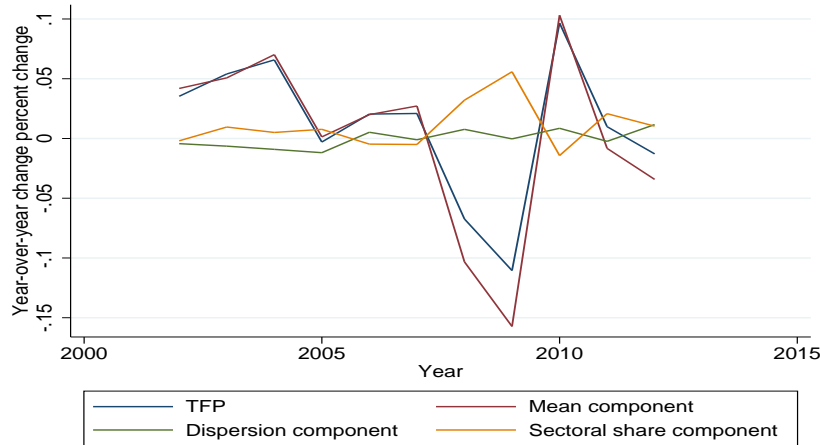


Note: We compare the behavior of  $\ln(TFP)$  and its components as derived in (2.8) over four different recessions in our sample. Values for all components and  $\ln(TFP)$  are scaled such that their pre-recession value is 1. We select pre-recession baseline years as 1979, 1988, 1998, and 2006 for the four recessions. The y axis can be interpreted as the cumulative change in  $\ln(TFP)$  or its components from their pre-recession values.

### 3.4 The Role of Entry and Exit

We find that entry and exit from our dataset has a negligible effect on changes in aggregate TFP. We caution that entry into our sample and exit from our sample do not correspond to a firm starting operations or shutting down. Public firm listings and delistings happen for a multitude of reasons. We can be satisfied, however, that entry and exit into our sample is not erroneously leading to our results from the application of our decomposition on U.S. public firms.

Figure 3.4: Components of TFP for Japanese Public Firms



Note: Sample period: 2001-2012. We compute year-over-year changes in TFP and its components as derived in (2.8). Our measure of TFP and the components are all calculated from data on Japanese public firms in the Compustat global database.

## 4 Decomposition on Public Firms in Japan

We also run our analysis on public firms in Japan using data from the Compustat global database. Our data on Japanese public firms spans from 2001 to 2013. We obtain price deflators from the Organization for Economic Co-operation and Development (OECD). We construct all of our measures identically to how we compute them in Section 3, as we have the same data fields. We find results consistent with those found for U.S. public firms. Table 3.3 shows the correlations of changes in the natural logarithm of TFP with its components. We see that both the dispersion component and sectoral output share component have moderate negative correlations with TFP. Figure 3.4 shows year-over-year changes in the logarithm of TFP and its components over the business cycle. We see that both the dispersion component and sectoral output share tend to moderate rather than amplify productivity movements over the recent recession. This stabilizing effect and negative correlation is driven primarily by the labor sub-component rather than the capital sub-component.

## 5 Conclusion

We present a novel decomposition of total factor productivity to measure the impact of changes in allocative efficiency that does not rely on the estimation of firm-level TFP or industry-level production function coefficients. We argue that the components of our decomposition reflect changes in the extent to which allocative inefficiency, both within and between sectors, affect aggregate productivity. We implement this decomposition using data on value added, the capital stock, and employment for U.S. public firms. Within our sample, we find that changes in allocative efficiency play a slight stabilizing role over the business cycle. We find similar results for Japanese public firms. Our results are inconsistent with mechanisms which explain large contractions in output and TFP by generating significant misallocation.

Although we show that changes in the within-industry dispersion component capture changes in allocative efficiency in a simple (albeit general) setting, it would be useful to further demonstrate this within models wherein allocative efficiency drives movements in aggregates. A natural next step would then be to apply our measure to larger datasets in different countries over different time periods.

## Appendix A

### Dataset Construction

Our empirical analysis relies on data from U.S. public firms. We use annual data with firms that exist in the CRSP/Compustat Merged database to get firm-level balance sheet measures.<sup>31</sup>

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<sup>31</sup>For a given date, we aggregate all data to the *permco* level, not the *gvkey* level or *permno* level, as we are interested in firms and not subentities or establishments. For each measure of interest, be it returns or a balance sheet measure, if there are multiple *gvkeys* or *permnos* for a given *permco* and *date* pair, we determine if it is a duplicate or from a separate part of the firm. For any two *gvkeys* or *permnos* for a given *permco* and *date* pair, if the observations are duplicates, we drop one of them; if not, then we create a weighted mean or a sum, depending the variable of interest, weighted by the respective market capitalization

For our analysis, we only use data from 1972-2011. We keep only firms with December fiscal year-ends, but our results are generally robust to using firms with any fiscal year-end months. We clean the data by winsorizing each series at the 1st percentile. Our results are generally robust to when we trim or winsorize at the 5th, 2nd or 1st percentiles. We exclude firms with primary SIC codes between 4000 and 4999, between 6000 and 6999, or greater than 9000, as our model is not representative of regulated, financial, or public service firms.

## Appendix B

### Model Derivations

In Section 2.6, we demonstrate how a model of heterogeneous firms and input allocation frictions map into our model. The derivations of the results stated there are derived here.

The maximization problem:

$$\max P_t \left( A_{it} k_{it}^{\alpha} l_{it}^{\phi} \right)^{1-\eta} - k_{it} \frac{r_t}{\omega_{ikt}} - l_{it} \frac{w_t}{1 - \omega_{ilt}}$$

has first-order conditions:

$$\frac{l_{it}}{\frac{p_{it}}{P_t} y_{it}} = \frac{\phi(1 - \eta)}{w_t},$$

and

$$\frac{k_{it}}{\frac{p_{it}}{P_t} y_{it}} = \frac{\alpha(1 - \eta)}{r_t}.$$

---

of the subentity to create a firm-level measure. For further details, the SAS code is available upon request.

These two conditions, combined with clearing conditions for inputs and production function in real terms  $\frac{p_{it}}{P_t} y_{it} = \left( A_{it} k_{it}^\alpha l_{it}^\phi \right)^{1-\eta}$  imply that firm-level productivity ratios are:

$$\frac{l_{it}}{\frac{p_{it}}{P_t} y_{it}} = K_t^{-\alpha(1-\eta)} L_t^{1-\phi(1-\eta)} \frac{(\omega_{ilt}) \left( \sum_i \left( A_{it}^{1-\eta} (\omega_{ikt})^{1-\phi(1-\eta)} (\omega_{ilt})^{\phi(1-\eta)} \right)^{\frac{1}{1-(\alpha+\phi)(1-\eta)}} \right)^{\alpha(1-\eta)}}{\left( \sum_i \left( A_{it}^{1-\eta} (\omega_{ikt})^{\alpha(1-\eta)} (\omega_{ilt})^{1-\alpha(1-\eta)} \right)^{\frac{1}{1-(\alpha+\phi)(1-\eta)}} \right)^{1-\phi(1-\eta)}},$$

and

$$\frac{k_{it}}{\frac{p_{it}}{P_t} y_{it}} = K_t^{1-\alpha(1-\eta)} L_t^{-\phi(1-\eta)} \frac{1}{P_t} \frac{(\omega_{ikt}) \left( \sum_i \left( A_{it}^{1-\eta} (\omega_{ikt})^{\alpha(1-\eta)} (\omega_{ilt})^{1-\alpha(1-\eta)} \right) \right)^{\phi(1-\eta)}}{\left( \sum_i \left( A_{it}^{1-\eta} (\omega_{ikt})^{1-\phi(1-\eta)} (\omega_{ilt})^{\phi(1-\eta)} \right)^{\frac{1}{1-(\alpha+\phi)(1-\eta)}} \right)^{1-\alpha(1-\eta)}}$$

and that the output share of each firm is the following:

$$\frac{p_{it} y_{it}}{P_t Y_t} = \frac{\left( A_{it}^{1-\eta} (\omega_{ikt})^{\alpha(1-\eta)} (\omega_{ilt})^{\phi(1-\eta)} \right)^{\frac{1}{1-(\alpha+\phi)(1-\eta)}}}{\sum_i \left( A_{it}^{1-\eta} (\omega_{ikt})^{\alpha(1-\eta)} (\omega_{ilt})^{\phi(1-\eta)} \right)^{\frac{1}{1-(\alpha+\phi)(1-\eta)}}}.$$

These equations allow us to use (2.2) and (2.3) to calculate aggregate productivity ratios and (2.1) to calculate aggregate TFP. Notably,  $\ln(TFP)$  is a linear combination of  $\ln\left(\frac{K_t}{Y_t}\right)$  and  $\ln\left(\frac{L_t}{Y_t}\right)$ . These are both functions of firm-level TFP,  $A_{it}$ , as well as input frictions,  $\omega_{ilt}$  and  $\omega_{ikt}$ , and parameters. Some simple algebra allows us to express the labor productivity ratio as:

$$\ln\left(\frac{L_t}{Y_t}\right) = \ln\left(\frac{(\sum_i f^1(A_{it}, \omega_{ilt}, \omega_{ikt})) (\sum_i f^2(A_{it}, \omega_{ilt}, \omega_{ikt}))^{\alpha(1-\eta)}}{(\sum_i f^3(A_{it}, \omega_{ilt}, \omega_{ikt})) (\sum_i f^4(A_{it}, \omega_{ilt}, \omega_{ikt}))^{1-\phi(1-\eta)}}\right) + \ln\left(K_t^{-\alpha(1-\eta)} L_t^{1-\phi(1-\eta)}\right),$$

where the  $f$  functions are given by:

$$f^1(A_{it}, \omega_{ilt}, \omega_{ikt}) = \omega_{ilt} \left( A_{it}^{1-\eta} (\omega_{ikt})^{\alpha(1-\eta)} (\omega_{ilt})^{\phi(1-\eta)} \right)^{\frac{1}{1-(\alpha+\phi)(1-\eta)}},$$

and

$$f^2(A_{it}, \omega_{ilt}, \omega_{ikt}) = \left( A_{it}^{1-\eta} (\omega_{ikt})^{1-\phi(1-\eta)} (\omega_{ilt})^{\phi(1-\eta)} \right)^{\frac{1}{1-(\alpha+\phi)(1-\eta)}},$$

and

$$f^3(A_{it}, \omega_{ilt}, \omega_{ikt}) = \left( A_{it}^{1-\eta} (\omega_{ikt})^{\alpha(1-\eta)} (\omega_{ilt})^{\phi(1-\eta)} \right)^{\frac{1}{1-(\alpha+\phi)(1-\eta)}},$$

and

$$f^4(A_{it}, \omega_{ilt}, \omega_{ikt}) = \left( A_{it}^{1-\eta} (\omega_{ikt})^{\alpha(1-\eta)} (\omega_{ilt})^{1-\alpha(1-\eta)} \right)^{\frac{1}{1-(\alpha+\phi)(1-\eta)}}.$$

To summarize the effect of distortions with a small number of moments, we use a Taylor approximation method following Osotimehin (2013). Assume that firm-level productivity and input distortions  $A_{it}, \omega_{ilt}, \omega_{ikt}$  are distributed with means  $\bar{A}_t, \bar{\omega}_{lt}, \bar{\omega}_{kt}$ , variances  $\sigma_{At}^2, \sigma_{lt}^2, \sigma_{kt}^2$ , and covariances  $\sigma_{Alt}, \sigma_{Akt}, \sigma_{klt}$ . A first-order Taylor approximation of  $\ln(\frac{L_t}{Y_t})$  yields:

$$\begin{aligned} \ln\left(\frac{L_t}{Y_t}\right) &\simeq \ln\left(\frac{\left(\sum_i \bar{f}^1\right) \left(\sum_i \bar{f}^2\right)^{\alpha(1-\eta)}}{\left(\sum_i \bar{f}^3\right) \left(\sum_i \bar{f}^4\right)^{1-\phi(1-\eta)}}\right) + \frac{\sum_i (f_i^1 - \bar{f}^1)}{\sum_i \bar{f}^1} + \alpha(1-\eta) \frac{\sum_i (f_i^2 - \bar{f}^2)}{\sum_i \bar{f}^2} \\ &- \frac{\sum_i (f_i^3 - \bar{f}^3)}{\sum_i \bar{f}^3} - \alpha(1-\eta) \frac{\sum_i (f_i^4 - (1-\phi(1-\eta))\bar{f}^4)}{\sum_i \bar{f}^4} + \\ &\ln\left(K_t^{-\alpha(1-\eta)} L_t^{1-\phi(1-\eta)}\right). \end{aligned}$$

Each  $f$  function can then be approximated by a second-order Taylor approximation as:

$$\begin{aligned} f_i^1 - \bar{f}^1 &= (A_{it} - \bar{A}_t) \frac{\partial \bar{f}^1}{\partial A_{it}} + (\omega_{ilt} - \bar{\omega}_{ilt}) \frac{\partial \bar{f}^1}{\partial \omega_{ilt}} + (\omega_{ikt} - \bar{\omega}_{ikt}) \frac{\partial \bar{f}^1}{\partial \omega_{ikt}} + (A_{it} - \bar{A}_t)^2 \frac{\partial^2 \bar{f}^1}{\partial A_{it}^2} \\ &+ (\omega_{ilt} - \bar{\omega}_{ilt})^2 \frac{\partial^2 \bar{f}^1}{\partial \omega_{ilt}^2} + (\omega_{ikt} - \bar{\omega}_{ikt})^2 \frac{\partial^2 \bar{f}^1}{\partial \omega_{ikt}^2} + (A_{it} - \bar{A}_t) (\omega_{ikt} - \bar{\omega}_{ikt}) \frac{\partial^2 \bar{f}^1}{\partial A_{it} \partial \omega_{ikt}} \\ &+ (A_{it} - \bar{A}_t) (\omega_{ilt} - \bar{\omega}_{ilt}) \frac{\partial^2 \bar{f}^1}{\partial A_{it} \partial \omega_{ilt}} + (\omega_{ilt} - \bar{\omega}_{ilt}) (\omega_{ikt} - \bar{\omega}_{ikt}) \frac{\partial^2 \bar{f}^1}{\partial \omega_{ilt} \partial \omega_{ikt}}. \end{aligned}$$

Taking derivatives and using the law of large numbers gives us an expression for  $\ln(\frac{L_t}{Y_t})$  as a function of moments of firm-level TFP and frictions, as well as aggregate input supply and parameters. Repeating this for capital and plugging the results into (2.1) yields:

$$\begin{aligned} \ln(TFP) &\simeq \ln\left(\frac{\bar{A}_t^{(1-\eta)} n^{(1-(\alpha+\phi)(1-\eta))}}{K_t^{(a-\alpha(1-\eta))} L_t^{(1-a-\phi(1-\eta))}}\right) + (1-\eta) \left(\frac{-\eta + (\alpha + \phi)(1-\eta)}{1 - (\alpha + \phi)(1-\eta)}\right) \frac{1}{2} \frac{\sigma_{A_t}^2}{\bar{A}_t^2} \\ &- \frac{\alpha(1-\eta)(1-\phi(1-\eta))}{2(1-(\alpha+\phi)(1-\eta))} \frac{\sigma_{kt}^2}{\bar{\omega}_{kt}^2} - \frac{\phi(1-\eta)(1-\alpha(1-\eta))}{2(1-(\alpha+\phi)(1-\eta))} \frac{\sigma_{lt}^2}{\bar{\omega}_{lt}^2} \\ &- \frac{\phi\alpha(1-\eta)^2}{1-(\alpha+\phi)(1-\eta)} \frac{\sigma_{klt}}{\bar{\omega}_{lt}\bar{\omega}_{kt}}. \end{aligned}$$

We use a similar methodology to compute Taylor approximations for the undistorted case, as well as for the dispersion components. We find that:

$$E_t^k = \frac{1}{2} \frac{\sigma_{kt}^2}{\bar{\omega}_{kt}^2},$$

and

$$E_t^l = \frac{1}{2} \frac{\sigma_{lt}^2}{\bar{\omega}_{lt}^2}.$$

Thus, the contributions of the dispersion components to TFP are, following (2.7), respectively,  $-a\frac{1}{2}\frac{\sigma_{kt}^2}{\bar{\omega}_{kt}^2}$  and  $-(1-a)\frac{1}{2}\frac{\sigma_{lt}^2}{\bar{\omega}_{lt}^2}$ . We note that we can express the natural logarithm of TFP as:

$$\begin{aligned} \ln(TFP) &\simeq \ln(TFP_{eff}) - \frac{\phi\alpha(1-\eta)^2}{1-(\alpha+\phi)(1-\eta)} \frac{\sigma_{klt}}{\bar{\omega}_{lt}\bar{\omega}_{kt}} \\ &- \frac{\alpha(1-\eta)(1-\phi(1-\eta))}{(1-(\alpha+\phi)(1-\eta))} \tilde{E}_t^K - \frac{\phi(1-\eta)(1-\alpha(1-\eta))}{(1-(\alpha+\phi)(1-\eta))} \tilde{E}_t^L. \end{aligned}$$

Thus, we see that the contribution of distortions to TFP can be approximated (assuming interaction between frictions is second-order) as:

$$\begin{aligned} \ln(TFP) &\simeq \ln(TFP_{eff}) - \frac{\alpha(1-\eta)(1-\phi(1-\eta))}{(1-(\alpha+\phi)(1-\eta))} \tilde{E}_t^K \\ &\quad - \frac{\phi(1-\eta)(1-\alpha(1-\eta))}{(1-(\alpha+\phi)(1-\eta))} \tilde{E}_t^L. \end{aligned}$$



## References

- ABEL, A. B. AND J. C. EBERLY (2011): “How Q and Cash Flow Affect Investment without Frictions: An Analytic Explanation,” *Review of Economic Studies*, 78, 1179–1200.
- ACEMOGLU, D. (2008): *Introduction to Modern Economic Growth*, Princeton University Press.
- ACEMOGULU, D., U. AKCIGIT, N. BLOOM, AND W. KERR (2013): “Innovation, Reallocation and Growth,” *Working Paper*.
- AKCIGIT, U. (2009): “Firm Size, Innovation Dynamics and Growth,” *Working Paper*.
- ARELLANO, C., Y. BAI, AND P. J. KEHOE (2012): “Financial Frictions and Fluctuations in Volatility,” *Working Paper*.
- ASKER, J., J. DE LOECKER, AND A. COLLARD-WEXLER (2014): “Dynamic Inputs and Resource (Mis) Allocation,” *The Journal of Political Economy*, 122, 1013 – 1063.
- ATKESON, A. AND A. BURSTEIN (2010): “Innovation, Firm Dynamics, and International Trade,” *Journal of Political Economy*, 118, 433–484.
- (2014): “Aggregate Implications of Innovation Policy,” *Working Paper*.
- ATKESON, A., A. EISFELDT, AND P.-O. WEILL (2013): “Measuring the Financial Soundness of U.S. Firms: 1926-2012,” *Working Paper*.
- BASU, S. AND J. FERNALD (2002): “Aggregate Productivity and Aggregate Technology,” *European Economic Review*, 46, 963–991.
- BASU, S., L. PASCALI, F. SCHIANTARELLI, AND L. SERVEN (2009): “Productivity, Welfare and Reallocation: Theory and Firm-Level Evidence,” *Working Paper*.
- BERNARDO, A. E. AND I. WELCH (2014): “Capital Structure with Endogenous Liquidation Value,” *Working Paper*.

- BHAMRA, H. S., L.-A. KUHN, AND I. A. STREBULAEV (2010): “The Levered Equity Risk Premium and Credit Spreads: A Unified Framework,” *Review of Financial Studies*, 23, 645–703.
- BHARATH, S. T. AND T. SHUMWAY (2008): “Forecasting Default with the Merton Distance to Default Model,” *Review of Financial Studies*, 21, 1339–1369.
- BINSBERGEN, J. H., J. R. GRAHAM, AND J. YANG (2010): “The Cost of Debt,” *Journal of Finance*, 65, 2089–2136.
- BRIS, A., I. WELCH, AND N. ZHU (2006): “The Costs of Bankruptcy: Chapter 7 Liquidation Versus Chapter 11 Reorganization,” *Journal of Finance*, 59, 1253–1303.
- BUERA, F. J., J. P. KABOSKI, AND Y. SHIN (2011): “Finance and Development: A Tale of Two Sectors,” *American Economic Review*, 101, 1964–2002.
- CAMPBELL, J. (1998): “Entry, Exit, Embodied Technology, and Business Cycles,” *Review of Economic Dynamics*, 1, 378–408.
- CHARI, V., P. J. KEHOE, AND E. R. MCGRATTAN (2007): “Business Cycle Accounting,” *Econometrica*, 75, 781–836.
- CHATTERJEE, S. (2013): “Debt Overhang: Why Recovery from a Financial Crisis Can be Slow,” *Business Review, Federal Reserve Bank of Philadelphia*.
- CHEN, H. (2010): “Macroeconomic Conditions and the Puzzles of Credit Spreads and Capital Structure,” *The Journal of Finance*, 65, 2171–2212.
- CHEN, H. AND G. MANSO (2010): “Macroeconomic Risk and Debt Overhang,” *Working Paper*.
- CHILDS, P. D., D. C. MAUER, AND S. H. OTT (2005): “Interactions of Corporate Financing and Investment Decisions,” *Journal of Financial Economics*, 76, 667–690.

- CHOI, J. AND M. RICHARDSON (2012): “The Volatility of the Firm’s Assets,” *Working Paper*.
- CLEMENTI, G. L. AND H. A. HOPENHAYN (2006): “A Theory of Financing Constraints and Firm Dynamics,” *The Quarterly Journal of Economics*, 229–265.
- CORRADO, C., C. HULTEN, AND D. SICHEL (2009): “Intangible Capital and U.S. Economic Growth,” *The Review of Income and Wealth*, 55, 661–685.
- COSTAIN, J. AND A. NAKOV (2011): “Distributional Dynamics under Smoothly State-Dependent Pricing,” *Journal of Monetary Economics*, 58, 646–665.
- COVAS, F. AND W. J. DEN HAAN (2011): “The Cyclical Behavior of Debt and Equity Finance,” *American Economic Review*, 101, 877–99.
- DAVID, J. M., H. HOPENHAYN, AND V. VENKATESWARAN (2013): “Information, Misallocation and Aggregate Productivity,” *Working Paper*.
- DIAMOND, D. AND Z. HE (2013): “A Theory of Debt Maturity: The Long and Short of Debt Overhang,” *Journal of Finance*, 69, 719–762.
- DUFFIE, D. (2011): “Measuring Corporate Default Risk,” *Clarendon Lectures in Finance*.
- EISFELDT, A. AND T. MUIR (2013): “Aggregate Issuance and Savings Waves,” *Working Paper*.
- EPSTEIN, L. G. AND S. E. ZIN (1991): “Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis,” *The Journal of Political Economy*, 99, 263–286.
- FALATO, A., D. KADYRZHANOVA, AND J. W. SIM (2013): “Rising Intangible Capital, Shrinking Debt Capacity, and the US Corporate Savings Glut,” *Working Paper*.

- FOSTER, L., J. HALTIWANGER, AND C. J. KRIZAN (2001): “Aggregate Productivity Growth. Lessons from Microeconomic Evidence,” in *New Developments in Productivity Analysis*, ed. by C. R. Hulten, E. R. Dean, and M. J. Harper, University of Chicago Press, 303 – 372.
- GALA, V. D. AND J. F. GOMES (2013): “Beyond Q: Investment without Asset Prices,” *Working Paper*.
- GILCHRIST, S., J. W. SIM, AND E. ZAKRAJSEK (2013): “Misallocation and Financial Market Frictions: Some Direct Evidence from the Dispersion in Borrowing Costs,” *Review of Economic Dynamics*, 16, 159–176.
- (2014): “Uncertainty, Financial Frictions, and Investment Dynamics,” *Working Paper*.
- GILCHRIST, S. AND E. ZAKRAJSEK (2012): “Credit Spreads and Business Cycle Fluctuations,” *American Economic Review*, 102, 1692–1720.
- GIROUD, X., H. M. MUELLER, A. STOMPER, AND A. WESTERKAMP (2012): “Snow and Leverage,” *The Review of Financial Studies*, 25, 681–710.
- GLOVER, B., J. GOMES, AND A. YARON (2011): “Corporate Taxes, Leverage, and Business Cycles,” *Working Paper*.
- GOLDSTEIN, R., N. JU, AND H. E. LELAND (2001): “An EBIT-Based Model of Dynamic Capital Structure,” *Journal of Business*, 74, 483–512.
- GOMES, J., U. JERMANN, AND L. SCHMID (2014): “Sticky Leverage,” *Working Paper*.
- GOMES, J. AND L. SCHMID (2010): “Levered Returns,” *The Journal of Finance*, 65, 467–494.
- (2012): “Equilibrium Credit Spreads and the Macroeconomy,” *Working Paper*.

- GOURIO, F. (2014): “Financial Distress and Endogenous Uncertainty,” *Working Paper*.
- GRAHAM, J. AND M. T. LEARY (2011): “A Review of Empirical Capital Structure Research and Directions for the Future,” *Annual Review of Financial Economics*, 3, 309–345.
- GRILICHES, Z. (1979): “Issues in Assessing the Contribution of Research and Development to Productivity Growth,” *Bell Journal of Economics*, 10, 92–116.
- HAEFKE, C. AND M. REITER (2011): “What do Participation Fluctuations Tell us About Labor Supply Elasticities,” *Working Paper*.
- HAYASHI, F. (1982): “Marginal Q and Average Q: A Neoclassical Interpretation,” *Econometrica*, 50, 213–224.
- HENNESSY, C. (2004): “Tobin’s Q, Debt Overhang, and Investment,” *The Journal of Finance*, 59, 1717–1742.
- HENNESSY, C. AND T. WHITED (2007): “How Costly is External Financing? Evidence from a Structural Estimation,” *The Journal of Finance*, 62, 1705–1745.
- HENNESSY, C. A., A. LEVY, AND T. M. WHITED (2007): “Testing Q Theory with Financing Frictions,” *Journal of Financial Economics*, 83, 691–717.
- HODGE, A. W. (2011): “Comparing NIPA Profits With S&P 500 Profits,” Tech. rep., Bureau of Economic Analysis.
- HOPENHAYN, H. (2011): “Firm Microstructure and Aggregate Productivity,” *Journal of Money, Credit and Banking*, 43, 111–145.
- HOPENHAYN, H. A. (1992): “Entry, Exit, and Firm Dynamics in Long Run Equilibrium,” *Econometrica*, 60, 1127–1150.
- HOPENHAYN, H. A. AND R. ROGERSON (1993): “Job Turnover and Policy Evaluation: A General Equilibrium Analysis,” *Journal of Political Economy*, 101, 915–938.

- HSIEH, C.-T. AND P. J. KLENOW (2009): “Misallocation and Manufacturing TFP in China and India,” *The Quarterly Journal of Economics*, 74, 1403–1448.
- KHAN, A. AND J. K. THOMAS (2008): “Idiosyncratic shocks and the role of nonconvexities in plant and aggregate investment dynamics,” *Econometrica*, 76, 395–436.
- (2013): “Credit Shocks and Aggregate Fluctuations in an Economy with Production Heterogeneity,” *Working Paper*.
- KORTEWEG, A. (2010): “The Net Benefits to Leverage,” *Journal of Finance*, 65, 2137–2170.
- KRUSSEL, P. AND A. A. SMITH (1998): “Income and Wealth Heterogeneity in the Macroeconomy,” *Journal of Political Economy*, 106, 867–896.
- KURTZMAN, R. AND D. ZEKE (2015a): “Accounting for Dispersion over the Business Cycle without Estimating Production Function Coefficients or Firm-Level TFP,” *Working Paper*.
- (2015b): “The Gains from Resolving Debt Overhang: Evidence from a Structural Estimation,” *Working Paper*.
- LAMONT, O. (1995): “Corporate-Debt Overhang and Macroeconomic Expectations,” *The American Economic Review*, 85, 1106–1117.
- LELAND, H. E. (1994): “Corporate Debt Value, Bond Covenants, and Optimal Capital Structure,” *The Journal of Finance*, 49, 1213–1252.
- (1998): “Agency Costs, Risk Management, and Capital Structure,” *The Journal of Finance*, LIII, 1213–1243.
- LEMMON, M. L., M. R. ROBERTS, AND J. F. ZENDER (2008): “Back to the Beginning: Persistence and the Cross-Section of Corporate Capital Structure,” *The Journal of Finance*, 63, 1575–1608.

- MCKAY, A. AND R. REIS (2013): “The Role of Automatic Stabilizers in the U.S. Business Cycle,” *Working Paper*.
- MELLO, A. S. AND J. E. PARSONS (1992): “Measuring the Agency Cost of Debt,” *The Journal of Finance*, 47, 1887–1904.
- MERTON, R. C. (1974): “On the Pricing of Corporate Debt: The Risk Structure of Interest Rates,” *The Journal of Finance*, 29, 449–470.
- MIDRIGAN, V. AND D. Y. XU (2014): “Finance and Misallocation: Evidence from Plant-Level Data,” *American Economic Review*, 104, 422–458.
- MILLER, M. H. (1977): “Debt and Taxes,” *The Journal of Finance*, 32, 261–275.
- MODIGLIANI, F. AND M. H. MILLER (1958): “The Cost of Capital, Corporation Finance, and the Theory of Investment,” *American Economic Review*, 48, 261–297.
- MOYEN, N. (2007): “How Big is the Debt Overhang Problem?” *Journal of Economic Dynamics and Control*, 31, 433–472.
- MYERS, S. (1977): “Determinants of Corporate Borrowing,” *Journal of Financial Economics*, 5, 147–175.
- OBERFELD, E. (2013): “Productivity and misallocation during a crisis: Evidence from the Chilean crisis of 1982,” *Review of Economic Dynamics*, 16, 100–119.
- OCCHINO, F. AND A. PESCATORI (2012): “Debt Overhang in a Business Cycle Model,” *Working Paper*.
- OSOTIMEHIN, S. (2013): “Aggregate Productivity and the Allocation of Resources over the Business Cycle,” *Working Paper*.
- PANIZZA, U. AND A. F. PRESBITERO (2014): “Public debt and economic growth: Is there a causal effect?” *Journal of Macroeconomics*, 41, 21–41.

- PARRINO, R. AND M. S. WEISBACH (1999): “Measuring Investment Distortions Arising from Stockholder-Bondholder Conflicts,” *Journal of Financial Economics*, 53, 3–42.
- PETRIN, A. AND J. LEVINSOHN (2012): “Measuring aggregate productivity growth using plant-level data,” *RAND Journal of Economics*, 43, 705–725.
- PHILLIPPON, T. AND P. SCHNABL (2013): “Efficient Recapitalization,” *The Journal of Finance*, 68, 1–42.
- REITER, M. (2009): “Solving Heterogeneous-Agent Models by Projection and Perturbation,” *Journal of Economic Dynamics and Control*, 33, 649–665.
- RESTUCCIA, D. AND R. ROGERSON (2008): “Policy Distortions and Aggregate Productivity with Heterogeneous Establishments,” *Review of Economic Dynamics*, 11, 707–720.
- ROMER, P. (1990): “Endogenous Technical Change,” *Journal of Political Economy*, 98, S71–102.
- SANDLERIS, G. AND M. L. J. WRIGHT (2014): “The Costs of Financial Crises: Resource Misallocation, Productivity and Welfare in the 2001 Argentine Crisis,” *The Scandinavian Journal of Economics*, 116, 87–127.
- SIMS, C. A. (2002): “Solving Linear Rational Expectations Models,” *Computational Economics*, 20, 1–20.
- TERRY, S. J. (2014): “Alternative Methods for Solving Heterogeneous Firm Models,” *Working Paper*.
- TITMAN, S. AND S. TSYPLAKOV (2007): “A Dynamic Model of Optimal Capital Structure,” *Review of Finance*, 11, 401–451.
- WELCH, I. (2004): “Capital Structure and Stock Returns,” *Journal of Political Economy*, 112, 106–131.



WHITED, T. AND G. WU (2006): “Financial Constraints Risk,” *Review of Financial Studies*, 19, 531–559.

WINBERRY, T. (2015): “Lumpy Investment, Business Cycles, and Stimulus Policy,” *Working Paper*.