Visibility Laboratory
Scripps Institution of Oceanography
University of California
San Diego 52, California

RADIATIVE TRANSFER ON DISCRETE SPACES

Rudolph W. Preisendorfer

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Radiative Transfer on Discrete Spaces

Rudolph W. Preisendorfer

Scripps Institution of Oceanography, University of California
La Jolla, California

INTRODUCTION

We shall be concerned here with a study of radiative transfer processes on discrete rather than continuous optical media. That is, we shall explore the consequences of replacing the usual geometric setting for radiative transfer processes -- namely some continuum of points in three dimensional euclidean space E_3 -- by a spatially bounded set of points, finite in number, each of which is located in E_3 in accordance with some explicit rule of selection. Each point of the set is then assigned certain scattering and absorbing properties with respect to impinging radiant energy, and each is generally allowed to interact radiometrically in a specified way with a certain preselected subset of the given collection. The main object of the present study is to formulate and solve, within this context and on a phenomenological level, the problem of the steady state radiance distribution at each point of the set, under prescribed boundary conditions.

Several papers are planned for the exploration of radiative transfer theory on discrete spaces. The present note is concerned with the

formulation of the general theoretical foundation of the discrete theory. In particular we shall motivate and formulate the principle of local interaction, deduce from it the general equations governing the discrete radiance distributions about each point of the general discrete space and, finally, solve these equations in complete theoretical detail. Subsequent papers will exhibit further consequences of the principle of local interaction and will range in sophistication from the deduction of the principle of invariant imbedding and the appropriate forms of the principles of invariance on general discrete spaces down to numerical tabulations of radiance distributions for certain special discrete spaces.

The motivations for the present series of studies are many; the four most immediate and objective motivations are discussed below. Briefly, these are: (i) the need for the formulation of novel radiative transfer settings which possess a high utility potential on the numerical analysis level, but which also retain a high fidelity potential; (ii) the need for a simple formalism which can, by means of an hypothetical microstructure, explain such basic concepts as the volume scattering and volume attenuation functions which occur in the continuous theory; (iii) the need for an analytical approach which holds promise for the solution of the problem of the abiogenetic character of the principles of invariance, the fundamental tool in the continuous theory, which ranks second in importance to only the equation of transfer; (iv) the need for a revision of certain parts of the mathematical substructure of radiative transfer theory in

accordance with modern standards of mathematical and epistemological rigor. These motivations will be explained now in more detail.

Fidelity versus Utility in Physical Theories

One may liken physical theories to life masks of their associated natural phenomena. The hills and hollows of the face of nature in many places can be reproduced by mathematical casts with great detail. The fidelity of the cast can be high even in the reproduction of the more dynamical aspects of natural processes. But if the theorist is too enthusiastic in his quest for detail he may find, like the artist, that he has inextricably sealed the features of the mold from view; he will have fitted his mold so well that either it comes off as inscrutably detailed as its inscrutable original or, more frustratingly, that the mold can come free from the face of nature only by being critically distorted with an attendant critical impairment of fidelity.

As far as radiative transfer theory is concerned, the theorist has at hand initially, in the form of the equation of transfer, a model of reality which, within the experimental framework it represents, indeed fills every hollow and follows every hill. It would therefore be singularly fortunate if this theory were endowed also with an equal measure of practical utility. For then no compromises between fidelity and utility would be necessary. As it happens the equation of transfer actually has been used in its full generality

many times to deduce in minute qualitative detail the salient features of the light field in real optical media (see, e.g., references 1 and 2). In a far greater number of instances, however, it was required of the theory to produce detailed quantitative information about the light field; and the intractability of the general theory under such a requirement soon became all too evident.

As a consequence of its inherently intractable form, the equation of transfer was eventually subject to many fidelity-reducing procedures in order to gain access to much needed quantitative information about the structure of natural light fields. In the course of such quests the equation of transfer (or the radiance function it governs) was subject to every immediately obvious simplification device in order to make it more amenable to numerical representation or simple symbolic analysis. Specific examples are easily cited: Generally inhomogeneous media were replaced by homogeneous models (which was effected by considering the scattering and absorbing functions in the equations as constant functions of position); highly anisotropic scattering functions usually found in nature were replaced by isotropic scattering functions, the effect of the replacement not being known nor easily estimable; the intricate and subtle location-dependent angular structure of the radiance distributions was smoothed to a relatively innocuous spherical or step-function shape in order to reduce the integrodifferential equation to sets of differential equations; scattering orders higher than the first or second order were dismissed as negligible; and so on.

There is a single and important feature common to all these procedures by means of which they may be broadly classified. Without exception, each concession to complexity modified the analytical structure of the equation of transfer itself, or the function it governs. To see the significance of this type of theoretical activity and place it in a perspective by means of which we can anticipate and evaluate related and possibly novel types of theoretical activity we now briefly recall the basic mathematical structures underlying radiative transfer theory.

Radiative transfer theory is a phenomenological theory which mimics in minute detail results of real (or imagined) probings of natural light fields. These probings in principle can be carried out by means of a single basic instrument, the radiance meter. The mathematical theory that follows on the heels of the amassed experimental evidence should then, at its core, consist of not more than three basic ingredients: (a) a mathematical representation of the space in which the experimental probings take place; (b) a mathematical representation of the quantity which the physical probe detects and records; and (c) a mathematical representation of the observed behavior of the quantities detected and recorded by the probe. These three notions, which we summarize briefly by the symbols X, N, and T, respectively, are sufficient to form a mathematical foundation from which the entire existing superstructure of present day radiative transfer theory can be deduced.[3] The mathematical roles of these concepts are quite simple: the equation of transfer T is an operator which within X acts on the radiance function N.

The radiance function N in turn is defined on X . In this way the fundamental role of the space X in the theory becomes unmistakably clear: it sets the stage for the other two concepts. It is the substance from which the mathematical mold of physical reality is fashioned.

Returning now to the main line of the discussion we see, in the light of the preceding observations, that most of the theoretical activity in the domain of radiative transfer has been centered on modifications of the basic structure of the equation T , or of the function N . Thus the classical modification procedures may be classed as process-modifying procedures; the third basic concept namely that of the space X , was virtually untouched in these procedures. Of course, the space X has taken many superficially distinct forms such as half-spaces, slabs, cylinders, spheres, etc.; however, an important underlying topological structure is common to all; each is still a connected sub-set of (more than one) points in E_3 , with a non-empty interior, or a union of such sets.

It appears possible then to explore at least one further modification procedure of a general kind, namely that associated with the modification of the basic space X in the fundamental triple (X, N, T). This space-modifying procedure affects only the character of X , and leaves free the choice of the character of the radiative transfer process governing N and T on the space. Thus the various possibilities such as inhomogeneity

of optical structure, anisotropic scattering, polarization processes, etc., are all possible in their full generality for each new choice of χ . This space-modifying procedure of course gives rise to possible fidelity-reducing consequences in the resulting theory just as do the process-modifying procedures. But whatever the consequences, they still demand exploration for possible novel numerical procedures and theoretical methods;this then is the first motivation for the present study.

On the Use of Hypothetical Microstructures in Physical Theory

The second of the four motivations for a study of radiative transfer on discrete spaces arises from the need for a suitable extension of the theory under which macroscopically defined (i.e., observable) concepts of the theory can be explained and profitably studied in terms of hypothesized microscopic (normally non-observable) entities which, however, still obey as far as possible the basic laws extant in the macroscopic domain. In this way the complex radiometric behavior of apparently continuous bulk media could possibly be explained in relatively simple terms by means of the behavior of aggregates of irreducible (molecule-like) components without the necessity of introducing any new laws or novel principles.

As an example of a problem of this kind, consider the case of the volume scattering and volume attenuation functions in radiative

transfer theory. Even though these concepts take the status of point functions in the continuous formulations, they are nevertheless evaluated by experimental determinations of the radiometric response of small but finite volumes of material comprising an optical medium. By the introduction of a molecular level, however naive, into the theory, these properties of macroscopic aggregates can then be related in a known and perhaps informative way to the properties of the individual molecular components comprising the aggregate.

The classical instance of this type of extension of a phenomenological theory occurred in the general thermodynamical theory of bulk matter which describes the macroscopic behavior of gases, liquids, and solids in terms of the directly observable phenomena of pressure, volume, and temperature. Right from the outset, i.e., from initial introduction of a rather primitive form of the hypothesis of the molecular structure of bulk matter, namely that bulk matter was considered to consist of aggregates of small hard balls or of point masses which nevertheless were still subject to the same macroscopic (e.g., Newtonian and Coulombian) laws, the old familiar thermodynamic laws governing pressure, volume, temperature, heat, and work took on a new and lastingly deeper significance and descriptive power.

As is well known radiative transfer theory was also offered the opportunity to adopt a molecular substratum by means of the

maxwellian theory of light and later still in the form of the quantum theory of matter and radiation. But these models of light in the microcosm involved concepts foreign to the relatively uncluttered array of concepts used in the classical phenomenological theory. For all practical purposes of the theory, radiant energy (light) was still usefully viewed as an apparently continuous, non-interfering, non-diffracting type of phenomenon. The only wavelike phenomena of radiant energy that were detectable by the radiance-measuring apparatus were those associated with polarization, and those detectable by the attachment of various colored filters (or spectrum analyzers) to the radiance meters. Hence the classical theory was obliged to describe and predict those and only those features of the light field in natural media which were observable by means of such radiance-measuring apparatus, and to accomplish these descriptions and predictions by adopting only macroscopically defined concepts.

These observations show that radiative transfer theory still awaits its own extension to a microstructure theory. In such an extended theory it would be desirable to retain as far as possible the usual concepts and laws presently used within the theory. The extension would be made by means of these concepts and laws now applied to the hypothesized microstructure. Thus the shift of emphasis in the use of the fundamental laws and concepts would be from a continuous to the discrete space setting. In this way the radiometric behavior of the apparently continuous bulk media would possibly be explained in terms of the behavior of aggregates of irreducible (molecule-like) components without the introduction of any new concepts and radically different laws or principles.

Abiogenetic Principles in Physical Theory

We come now to the discussion of the third motivation of the study of discrete-space formulations of radiative transfer theory. This motivation is derived from the apparent promise shown by these formulations in resolving an outstanding problem in the foundations of the theory. This is the problem of the apparent mathematical abiogenesis of the principles of invariance which form an important cornerstone in the modern theory.

Now an abiogenetic principle in a given physical theory is (a) a principle, (i.e., a rule of action, together with a set of associated constructs, which may be followed in the formulation of a wide class of problems and laws in the theory) (b) whose statement has (as yet) no demonstrable theoretical or empirical basis within the main body of established principles and constructs of the theory.

According to this definition virtually all of the presently established principles were at one time in the limbo of abiogenetic principles. Usually, with increased empirical knowledge and an attendant growth of comprehensiveness of physical theory, more and increasingly rigorous theoretical and empirically-based connections are established between the principle and the main body of the theory so that eventually the abiogenetic character of the principle is correspondingly decreased. Thus at one time the far-reaching principle of least action[4] was certainly abiogenetic in character. Now it is solidly established, at least in man's immediate neighborhood of space-time. On the other hand, Mach's principle[5] which asserts

that the value at a point in space of the metric tensor in general relativity theory (hence the inertial mass of a given object) is determinable by means of a well-defined procedure from knowledge of the total space-time distribution of matter and energy — is an outstanding example of an abiogenetic principle.

The principles of invariance in radiative transfer theory are, first of all, principles in the sense of part (a) in the definition above. The constructs involved in their statement are the diffuse reflectance R and the diffuse transmittance T functions associated with homogeneous slabs of light-scattering materials. In the statements of the principles, the slabs may have finite or infinite optical depth but always infinite lateral extent. The statements of the principles give rise to explicit formulae which relate, by means of the R and T functions, the radiance distribution at some interior point of a slab to the incident boundary radiance distributions on each face of the slab.

The basic idea behind the modern form of the principles made its first appearance in a paper[6] by Ambarzumian published in 1943. In that paper the optical medium was an infinitely deep homogeneous isotropically-scattering slab. Only one of the two functions (the R function) was used, and only one of the four statements of the principles were given. This unheralded appearance of the R function and its first invariance statement without any substantiating ground-work and without any subsequent justification, either theoretical or empirical, immediately set the abiogenetic character of the

principles of invariance. Specifically, there was no immediate connection between the principle and the established equation of transfer for radiance which lay at the foundation of the theory. If any connection at all could be drawn between the principle and the existing body of radiative transfer it could perhaps be made, though tenuously, with some early peripheral studies first by Stokes[7] and later by Rayleigh[8] dealing with the reflection of light from piles of glass plates or regularly stratified media.

Subsequently, the Ambarzumian principle of invariance was rounded out by Chandrasekhar[9] to four comprehensive statements involving two functions, the R and T functions, which pertained to homogeneous plane-parallel slabs with arbitrary volume scattering (phase) function. However their abiogenetic character remained unchanged. Still later, the four principles were generalized to be applicable to a wider class of spaces, namely non-homogeneous, curvilinear, non steady state spaces; and the number of functions was increased from two to four (two R and two T).[10,11,12] Again, the principles remained as abiogenetic as ever.

In a subsequent note we will show that--in the context of arbitrary discrete spaces--the existence of the general R and T functions (and even more general counterparts) can be rigorously established and, furthermore, the principles of invariance can be derived from an intuitively simpler (and empirically meaningful) principle, the so-called principle of local interaction (developed below) on which, incidentally, the whole of discrete-space radiative

transfer theory can be based, including the appropriate forms of the equation of transfer. Further, it appears that on the basis of the local interaction principle, and by means of a sequential transition from discrete to arbitrary proximations of continuous spaces, the continuous-space, arbitrary-geometry counterparts of tho invariance principles may thus be proved with arbitrarily great precision.

Other approaches to the solution of the problem of the abiogenetic principles of invariance possibly can be made by capitalizing on the apparent Green-function nature of the $\mathcal{R}$ and $\mathcal{T}$ functions with respect to the equation of transfer. Such an approach has been considered elsewhere.[11,13] But such approaches involve mathematical techniques of possibly questionable rigor for reasons which will be touched on in the following and final discussion.

Finitary Formulations of Physical Theories

The fourth and final motivation to be discussed here is concerned with the possibility of a finitary formulation of radiative transfer theory, that is, a mathematical formulation of the theory solely in terms of intuitively based mathematical constructs which are manipulated in accordance with intuitively sound rules of logic. The term "intuitive" used here is in the strictly defined sense used by the intuitionistic school of modern mathematics as established by Brouwer (see, e.g., references 14).

The intuitionistic philosophy in the domain of pure mathematics is analogous in some ways to the operationism philosophy in the domain of physics: each requires its concepts to be formulated in accordance with performable activities, noumenal in mathematics, phenomenal in physics. The existence of an object (or its accepted representation) must be established on the authority of only those principles which yield explicit performable rules of action culminating in that object (or its accepted representation).

The similarity between physical operationism and mathematical intuitionism clearly gains in depth when one observes that the notion of the completed infinite is excluded from the intuitionistic mathematical philosophy. Only the potentially infinite sets are allowed, that is, arbitrarily large sets which are constructable element by element according to known performable rules (algorithms). The analogy is deepened even further when one observes that the rules of logic in either discipline do not make unqualified use of the law of the excluded middle which states that: with respect of a given proposition P, precisely one of the two and only two possibilities, " P true" or " P not true" must always hold. The proposition may be a mathematical statement about the property of a set of numbers, or it may be — in the physical case -- a statement about the phase space coordinate of an elementary particle.

Our purpose here is not to expound on the tenets of operationism and intuitionism, but to bring to the reader's attention a hint of the logically unsatisfactory state of affairs extant in physical

theories which are associated with non-operationalistic principles and which still couch their concepts in and operate with non-intuitive mathematics. The obvious remedy for this state of affairs is to combine the best elements of each school of thought. The preliminary finitary (in fact, strictly finite) formulation given below (and in subsequent papers) is restricted to one small segment of physical theory, namely radiative transfer theory; but it serves to point up the fact that, despite the adoption of the severly attenuated mathematical methodology associated with the finitary formulation, many of the so-called classical results are recoverable in this new and intuitively defensible formulation. For one of the few presently existing applications of finitary methods to other physical theories, in particular to measurement processes and to quantum mechanics, respectively, see references 15 and 16.

The reader may argue that a finitary formulation is apparently clumsy and relatively intractable when compared with the infinitary formulations using the ideas of uncountably infinite sets along with integrals and derivatives of functions over them. However, this criticism can be directed only at the calculus level. It will be admitted that the calculus of finitary formulations is presently not as well developed as the infinitary formulations. However, it would be just as foolish to ignore finitary formulations on such grounds as it would be to reject the electrification of a new city because it is apparently hopelessly isolated from the customary power sources: the problem of electrification should not be solved by returning to the oil lamp, but rather by finding a new way of generating the necessary supply of electrical power.

A different objection to the use of finitary formulations in physics may be made on pragmatic grounds: the infinitary formulations using the concepts of the completed infinite, the unqualified use of the law of the excluded middle, and such admittedly disguised chimeras as the axiom of choice, are harmless ploys that quickly and painlessly allow one to arrive at the desired results eventually attainable by the more conservative finitary tactics. So why not take the simplest and most practical route? This is a compelling argument, especially in this age of accelerating progress made possible by the hack 'n' tack approaches to the solution of everyday physical problems. The argument, however, is patently false. Elementary counterexamples to the argument can be drawn from the ranks of number theory, and other branches of mathematics, or even mathematical physics. Such an illustration would, however, be intrinsically difficult and uninformative for the general reader. Thus, the following simile, though crude, would perhaps serve better to bring home the fallacy along with a hint of its potentially catastrophic consequences.

The infinitary programs in pure mathematics and mathematical physics can be likened to the economies of separate cities each of which has evolved from the clumsy but workable barter level up through the more elegant but still practical treasury-backed money certificate level and, finally, reaches the ethereal levels in which only the abstract concept of credit in all its symbolic glory remains. The commodities and services available in each city are supposed evenly matched <u>within</u> each city. Thus, for example, inhabitants of each

city who remain in their city experience no difficulties in building, buying and selling houses solely by the internal exchange of credit; each city could actually thrive materially (in the literal sense) on the intramural flow of financial accreditation. Someday somewhere in either city, however, it is reasonable to expect that at least one creditor may for some reason be compelled to move to the other city. The creditor, preparatory to moving, is disturbed to find that the accreditation structures of the two cities are incompatible, even though each has at its roots a hard-cash equivalent of services or commodities. He therefore demands of his debtors in his own city a hard-cash liquidation of all his debts so that he may start afresh in the new city but without any loss of accumulated progress or wealth status. His dismay deepens and his hopes for a lossless move into the new city crumble when he finds that not only can he not equate his services or commodities to its counterparts but, more horribly, he finds that there is in his city no hard-cash equivalent to his own accumulated credits for services or commodities. He panics, so do <u>his</u> creditors; and the house of economic cards begins its kaleidoscopic collapse.

Someday, somewhere in the intricate lacework of infinitary mathematical and physical theories enough demands for "hard-cash" liquidation of debts may be brought to bear on them so that a serious halt in material progress may occur while a forced re-evaluation of the accreditation system takes place. The debt may be in the form of an hitherto mutually agreed upon (but never quite verified) "existing point" somewhere in an uncountably infinite set; the

creditor (nature) for some reason might demand of the debtor (the physicist or mathematician) that he produce the "point" immediately and palpably. Or the debt may be in the form of an action conveniently based on an "either-or" dichotomy postulated by some lazy physicist or mathematician who ruled out the actual possibility of "neither-nor" so that his inability to literally produce one or the other of the possibilities at a crucial moment may abruptly annihilate at this juncture a vital bridge to the remaining subnetwork of consequences of his theory.

HEURISTIC INTRODUCTION TO THE PRINCIPLE OF LOCAL INTERACTION

We now present an intuitively simple experiment one may perform with two particles which are allowed to interact radiometrically. From results of this experiment we proceed to trace a chain of deductions to a primitive form of the local interaction principle. From this we make an inductive leap to the general form of the principle presently envisioned. We may observe here that an alternative approach to the principle is possible by suitably dissecting the equation of transfer for radiance. However, such a biopsy could be performed only after several layers of infinitary fat have been laboriously stripped away. For the reasons cited at length above, we have chosen relatively more naive finitary approach to the principle. The question of the interrelation of the principle of local interaction and the equation of transfer will be touched on again in subsequent papers wherein some connections between the two will be made. We now turn directly to the heuristic introduction of the principle.

The Classical Interreflection Problem

Suppose it has been found by direct experimentation that a small volume or "point" of material exhibits the following radiometric response to a finite set of incoming pencils of radiation: for an incoming set of beams of radiation of total radiance N^o and arbitrary directions of incidence, the resultant radiance

distribution is angularly uniform of magnitude $N^{\circ}\Sigma$ where $\Sigma < 1$ is some number obtained from the experiment. Suppose now there are two such "points" or "particles" P_1 and P_2 in an otherwise empty region of space. For our immediate purpose P_1 and P_2 may be thought of as representing small disjoint spheres of scattering material separated a distance which is large compared to their radii. One of the points, say P_1, is now irradiated by a single pencil of radiation of radiance magnitude N°. Since the region between P_1 and P_2 is hypothesized to be empty of any other scattering material, it follows that the initial or immediate resultant radiance of P_1 as seen at P_2 is given by $N^{\circ}\Sigma$, that is, P_2 at this stage of the discussion is irradiated by primary scattered radiant flux from the direction of P_1, the magnitude of the primary radiance being $N^{\circ}\Sigma$.

The particle P_2 now scatters an amount $(N^{\circ}\Sigma)\Sigma = N^{\circ}\Sigma^2$ in the direction of P_1 (the quantity $N^{\circ}\Sigma$ acting as the initial irradiating pencil of flux on P_2). For the same reason as before P_1 sees precisely this amount of radiance arriving from P_2 which is composed of secondary scattered flux with respect to the source flux comprising N°. The particle P_1 in turn scatters the fractional amount $(N^{\circ}\Sigma^2)\Sigma$ of tertiary scattered flux back to P_2. This amount is in addition to the amount $N^{\circ}\Sigma$ of primary scattered flux originally sent to P_2. This process is continued indefinitely. Thus P_2 at the next stage, scatters back an amount $N^{\circ}\Sigma^4$ to P_1 in addition to the secondary scattered flux $N^{\circ}\Sigma^2$ determined during the immediately preceding interchange.

The total radiance received by P_2 as a result of the first n scatterings by P_1 in the direction of P_2 is represented by

$$N_{12}^{(n)} = N^{\circ}\Sigma + N^{\circ}\Sigma^{3} + \cdots + N^{\circ}\Sigma^{2n-1},$$

whereas the total radiance received by P_1 as a result of the first n flux scattering interchanges by P_2 in the direction of P_1 is represented by

$$N_{21}^{(n)} = N^{\circ}\Sigma^{2} + N^{\circ}\Sigma^{4} + \cdots + N^{\circ}\Sigma^{2n}.$$

To summarize the first n interchanges between P_1 and P_2 we may write $N_{12}^{(n)}$ and $N_{21}^{(n)}$ as:

$$N_{12}^{(n)} = \frac{N^{\circ}\Sigma(1-\Sigma^{2n})}{1-\Sigma^{2}}, \qquad (1)$$

$$N_{21}^{(n)} = \frac{N^{\circ}\Sigma^{2}(1-\Sigma^{2n})}{1-\Sigma^{2}}, \qquad (2)$$

which simply represent the sums of the corresponding finite geometric series with common ratio Σ^{2}.

Passage to the Steady State Limit

Now, because of the finite speed of propagation of scattered flux and because Σ applies to any irradiation, however small or large, there will be at any finite time after the initial irradiation of P_1, an infinite number of remaining interchanges still possible between P_1 and P_2. However, because $\Sigma < 1$ for sufficiently large values of n, the numerical difference between $N_{12}^{(n)}$ and N_{12} defined by

$$N_{12} \equiv \frac{N^0 \Sigma}{1 - \Sigma^2} \quad , \tag{3}$$

can be made arbitrarily small. That is, for any positive number ϵ, an integer n_0 can be found (explicitly) such that if n is an integer greater than n_0, we have

$$|N_{12} - N_{12}^{(n)}| = \frac{\Sigma^{2n+1}}{1 - \Sigma^2} < \epsilon$$

A similar statement holds for $N_{21}^{(n)}$ and the corresponding quantity N_{21} defined as:

$$N_{21} \equiv \frac{N^0 \Sigma^2}{1 - \Sigma^2} \quad . \tag{4}$$

We define the quantities N_{12} and N_{21} exhibited above as the <u>steady state radiances</u> between P_1 and P_2.

Preliminary Formulation of the Local Interaction Principle

We now come to a key observation on the mutual relations between $N_{12}^{(n)}$, $N_{21}^{(n)}$ and between N_{12} , N_{21} . An examination of (1) and (2) shows that if $N_{12}^{(n)}$ is multiplied by Σ , we obtain $N_{21}^{(n)}$, i.e., for each given integer n we see that the relation

$$N_{21}^{(n)} = N_{12}^{(n)} \Sigma \tag{5}$$

holds. Furthermore, a more detailed examination of (1) and (2) could show that if $N_{21}^{(n-1)}$ is multiplied by Σ and if to this is added $N^0\Sigma$ we would obtain $N_{12}^{(n)}$ i.e., in symbols:

$$N_{12}^{(n)} = N^0\Sigma + N_{21}^{(n-1)}\Sigma \tag{6}$$

which holds for each given integer n .

A similar set of relations may be shown to hold between the steady state radiances N_{12} and N_{21} . Thus, if N_{12} in (3) is multiplied by Σ , we obtain the expression for N_{21} in (4):

$$N_{21} = N_{12} \Sigma , \tag{7}$$

which is analogous to (5) above. Similarly we have

$$N_{12} = N^{\circ}\Sigma + N_{21}\Sigma \ , \qquad (8)$$

which is analogous to (6).

It is immediately verifiable that the pair of relations (5), (6) is equivalent to the pair (1), (2) in the sense that either pair is derivable from the other by a finite number of algebraic operations. More importantly, a similar observation may be made for the pair (3), (4) and the pair (7), (8), i.e., each is derivable from the other by a finite number of algebraic operations. The pair (7), (8) constitutes the germ of the principle of local interaction developed and exploited below for the case of an arbitrary finite number of radiometrically interacting points.

Symmetric Formulation of the Local Interaction Principle

A more symmetric formulation of the principle, as summarized in (7) and (8), can be given by allowing P_2 also to be irradiated by an outside source. Thus if N°_{01} replaces N°, and N°_{02} represents the initial irradiation of P_1 by some exterior source, it is easy to see that, by following once again the steps leading to (7) and (8), we have for the more general case:

$$N_{21} = N_{12}\Sigma + N^{\circ}_{02}\Sigma \ , \qquad (9)$$

$$N_{12} = N_{21}\Sigma + N^{\circ}_{01}\Sigma \qquad (10)$$

Various Levels of Interpretation of the Principle

The special principle of local interaction as summarized by the pair (9), (10), despite its apparently restrictive derivation, is of far-reaching generality and applicability. Its generality may be discerned by a suitably general reinterpretation of the notion of "point" or "particle" as summarized by P_1 and P_2 in the derivation. By interpreting P_1 and P_2 not as simple "geometric points" but as parallel planes which are (sets of points) separated by a vacuum, and by considering Σ as the reflectance (or transmittance) of the planes, N_{12} and N_{21} are then interpreted as the steady state radiances set up between P_1 and P_2. A broader interpretation would consider the "points" P_1 and P_2 as extended three dimensional sets of points, i.e., bodies of scattering material which are in radiometric interaction (such as two clouds, or the atmosphere and the earth, or the atmosphere and the sea). In such an interpretation N°, N_{12} and N_{21} would not be numbers but functions and Σ would then be considered as functional operator on N_{12} and N_{21} rather than just a single number. The juxtaposition of two symbols such as $N_{21}\Sigma$ ($N_{12}\Sigma$, or $N^{\circ}\Sigma$) would now be interpreted as a generalized linear operation on the function N_{21} (N_{12} , or N°)

by the operator $\sum$, rather than just a simple multiplication of numbers. These ideas lend great depth to the discrete-space formulations. They will be amplified in a subsequent paper.

BASIC DEFINITIONS

The discrete location space X_n is a bounded finite subset of distinct points of euclidean 3-space E_3 , i.e., $X_n = \{x_1, \ldots, x_n\}$, $x_i \in E_3$ where $i = 1, \ldots, n$, and n is a finite integer and there exists a positive integer $a < \infty$ such that each i, $1 \leq i \leq n, \Rightarrow |x_i| < a$. In any specific application of the subsequent results, a specific rule of construction must be given for X_n . For example, X_n may consist of all integral lattice points x_i of E_3 in the sphere C_a defined by the inequalities $|x_i| \leq a$, $i = 1, \ldots, n$ where a is some given integer. In what follows the spatial disposition of the elements x_i of X_n is arbitrary but fixed.

The Local Direction Space Ξ_i , $i = 1, \ldots, n$, is a set of unit vectors in E_3 defined as $\Xi_i = \{\xi_{i1}, \ldots, \xi_{i\Delta_i}\}$ where $\xi_{ij} = (x_j - x_i)/|x_j - x_i|$, and $x_i \neq x_j$. In other words Ξ_i is generally the set of unit vectors at x_i each of which points to some specific subset of $\Delta_i \leq n-1$ other elements of X_n; e.g., ξ_{ij} points from x_i to x_j . For each i , the set Ξ_i is clearly a subset of Ξ , , the unit sphere in E_3.

The local scattering function $\Sigma(x_i; \cdot; \cdot)$, $x_i \in X_n$ is a function defined in $\Xi \times \Xi$ to the set non negative real numbers.

The local absorption function $A(x_i, \cdot)$; $x_i \in X_n$ is a function defined in Ξ to the set of non negative real numbers.

The local conservation property of the local scattering and local absorption functions at x_i, $i = 1, \ldots, n$, is defined by:

$$A(x_i, \xi) + \sum_{\xi' \in \Xi_i} \Sigma(x_i; \xi; \xi') = 1, \qquad (11)$$

where ξ is an element of a given subset Ξ' of Ξ which must be defined at each x_i for each discrete space X_n. Setting $\sum_{\xi' \in \Xi_i} \Sigma(x_i; \xi; \xi') = S(x_i, \xi)$, (11) becomes:

$$A(x_i, \xi) + S(x_i, \xi) = 1. \qquad (11')$$

For our present purposes, we will say that X_n is conservative if $A(x_i, \cdot) \equiv 0$ on Ξ for every $x_i \in X_n$; and non-conservative if $1 > A(x_i, \xi) > 0$ for some $\xi \in \Xi$ at every $x_i \in X_n$. There are intermediate possibilities,

but they are of no interest here. Henceforth, unless otherwise specified, we will assume that X_n is non-conservative.* The specific radiance distribution $N(x_i, \cdot)$ at $x_i \in X_n$ is a function on Ξ to the set of non negative real numbers.

Two points x_i, x_k of X_n are said to be eclipsed if there is a point $x_j \in X_n$ such that $x_i \neq x_j$, $x_k \neq x_j$; and $\xi_{ij} = \xi_{jk}$.

Eclipse convention for radiance distributions: If x_i and x_k are eclipsed, then we set, by definition, $N(x_i, \xi_{ik}) = N(x_k, \xi_{ki}) = 0$.

Restriction convention for radiance distributions: In certain applications of the results developed below, it will be necessary to restrict the domain of the radiance distribution at $x_i \in X_n$ to certain subsets of Ξ. (i) the most common restriction is to Ξ_i. For convenience we will write $N(x_i, \xi_{ij})$ as N_{ij}, and set $N(x_i, \xi) = 0$ for all $\xi \notin \Xi_i$. In particular, to simplify notation, we set $N_{ii} = 0$, $i = 1, \ldots, n$. (ii) In general, if $N(x_i, \cdot)$ is restricted to a given fixed subset $\Xi_i' \subset \Xi_i \subset \Xi$, then we set $N(x_i, \xi) = 0$ for $\xi \neq \Xi_i'$. For the remainder of the present discussion, restriction convention (ii) will in force, unless explicitly noted otherwise.

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* This assumption's primary job is to insure the validity of equation (29) below. Actually this assumption may be relaxed somewhat. We shall not, however, go into such details here.

Source Convention. To each element $x_i \in X_n$ is assigned an incident radiance distribution $N^o(x_i, \cdot)$ defined on some finite subset Ξ_i^o of Ξ ; i.e., $\Xi_i^o = \{\xi_{1i}^o, \ldots, \xi_{\Delta_i i}^o\}$. For brevity, we will write $N^o(x_i, \xi_{ji}^o) = N_{ji}^o$. In general Ξ_i^o contains directions which are not in $-(\Xi_i)$; in this way we provide the system X_n with the possibility of exterior sources.

LOCAL INTERACTION PRINCIPLE

The statement of the local interaction principle for X_n is an immediate generalization of the special two-point case developed in the Introduction. Specifically, let $x_i \in X_n$. Then for every $\xi \in \Xi_i$,

$$N(x_i, \xi) = \sum_{k=1}^{n} N(x_k, \xi_{ki}) \Sigma(x_i ; \xi_{ki} ; \xi) + \sum_{k=1}^{\Delta_i} N_{ki}^o \Sigma(x_i ; \xi_{ki}^o ; \xi) . \tag{12}$$

Relation (12) is the principle of local interaction for X_n ; it takes the following special form for $\xi = \xi_{ij} \in \Xi_i$:

$$N_{ij} = \sum_{k=1}^{n} N_{ki} \Sigma(x_i ; \xi_{ki} ; \xi_{ij}) + \sum_{k=1}^{\Delta_i} N_{ki}^o \Sigma(x_i ; \xi_{ki}^o ; \xi_{ij}) , \quad i, j = 1, \ldots, n . \tag{13}$$

VECTOR FORMULATION OF THE PRINCIPLE

Relation (13) for the principle of local interaction is formally a set of n^2 equations in the n^2 unknowns N_{ij}, $i,j = 1,\ldots,n$; however, with the eclipse and restriction conventions on $N(x_i,\cdot)$ the actual number of unknowns may be less than n^2. Nevertheless, in order to emphasize the inherent symmetry of the following formulations, we can and shall explicitly carry through the discussions employing all n^2 symbols N_{ij}. The inherent symmetry of the local interaction principle is most clearly seen when vector notation is employed. We now consider the necessary steps leading to such a formulation.

Specific Radiance and Field Radiance Vectors

We shall designate by N_+ the $1 \times n^2$ array of N_{ij} values:

$$(N_{11}, N_{12}, \ldots, N_{1n}, N_{21}, N_{22}, \ldots, N_{2n}, \ldots, N_{n1}, N_{n2}, \ldots, N_{nn}).$$

This vector is called the <u>specific radiance vector</u>; the word "specific" serves to recall the important fact that N_+ is manufactured, in the above manner, from the <u>specific</u> radiance distributions $N(x_i,\cdot)$. There is another radiance vector which can be obtained from $N(x_i,\cdot)$ by a certain permutation of the entries of N_+. This new arrangement is of the form:

$$(N_{11}, N_{21}, \ldots, N_{n1}, N_{12}, N_{22}, \ldots, N_{n2}, \ldots, N_{1n}, N_{2n}, \ldots, N_{nn}).$$

We shall be designated this vector by the symbol N_- and call it the field radiance vector. The word "field" is used to point up the similarity of N_- and the well known concept of field radiance used in the continuous theory of radiative transfer. To see this, one may divide N_- into n blocks each of n entries, starting with N_{11}. Then the first block consists of radiance values of the form N_{i1} with i running from 1 to n. The second block consists of radiance values of the form N_{i2}, with i running from 1 to n. The ith block therefore contains the radiance values for all the incoming directions to $x_i \in X_n$ (These incoming directions are simply the negatives of the elements of Ξ_i, hence the minus sign subscript in N_-). It will be convenient to have an explicit notation for these subsets of N_-, and for corresponding subsets of N_+. Thus we set

$N_+(x_i) = (N_{i1}, \ldots, N_{in})$, $i = 1, \ldots, n$, so that $N_+ = (N_+(x_1), \ldots, N_+(x_n))$. Furthermore, we define $N_-(x_i) = \ldots$ $= (N_{1i}, \ldots, N_{ni})$ so that $N_- = (N_-(x_1), \ldots, N_-(x_n))$. $N_+(x_i)$ is the specific radiance vector at x_i; $N_-(x_i)$, the field radiance vector at x_i. Furthermore, to put the source or incident radiance distributions in vector form, we write $N^\circ(x_i) = (N^\circ_{1i}, \ldots, N^\circ_{n,i})$, $i = 1, \ldots, n$. $N^\circ(x_i)$ is the incident radiance vector at x_i. Finally, set $N^\circ = (N^\circ(x_1), \ldots, N^\circ(x_n))$.

Scattering Matrices

With the preceding definitions of the various radiance vectors at x_i we see that the form of the local interaction principle given in (13) motivates the definition of the following set of matrices. For each i , $i = 1, \dots, n$, set:

$$\Sigma(x_i) = \begin{pmatrix} \Sigma(x_i; \xi_{1i}; \xi_{i1}) & \Sigma(x_i; \xi_{1i}; \xi_{i2}) & \cdots & \Sigma(x_i; \xi_{1i}; \xi_{in}) \\ \Sigma(x_i; \xi_{2i}; \xi_{i1}) & \Sigma(x_i; \xi_{2i}; \xi_{i2}) & \cdots & \Sigma(x_i; \xi_{2i}; \xi_{in}) \\ \vdots & \vdots & \cdots & \vdots \\ \Sigma(x_i; \xi_{ni}; \xi_{i1}) & \Sigma(x_i; \xi_{ni}; \xi_{i2}) & \cdots & \Sigma(x_i; \xi_{ni}; \xi_{in}) \end{pmatrix} \quad (14)$$

In short $\Sigma(x_i)$ for each $i = 1, \dots, n$, is an $n \times n$ matrix whose entry in the α th row and β th column is $\Sigma(x_i; \xi_{\alpha i}; \xi_{i\beta})$. $\Sigma(x_i)$ is the <u>local scattering matrix.</u>

Furthermore, we define the $n^2 \times n^2$ matrix Σ as:

$$\Sigma = \begin{pmatrix} \Sigma(x_1) & & & \\ & \Sigma(x_2) & & \\ & & \ddots & \\ & & & \Sigma(x_n) \end{pmatrix}. \quad (15)$$

Thus, Σ is an $n^2 \times n^2$ matrix with n blocks along its main diagonal consisting of the $n \times n$ matrices $\Sigma(x_i)$; the zeros indicate that all other $n^4 - n^3$ entries of Σ are set equal to zero. Σ is the <u>scattering matrix</u> for X_n.

In order to place the incident radiance terms of (13) into vector form, we define a matrix $\Sigma^\circ(x_i)$, $i = 1, \ldots, n$, which has a similar structure to $\Sigma(x_i)$:

$$\Sigma^\circ(x_i) = \begin{pmatrix} \Sigma(x_i; \xi^\circ_{1i}; \xi_{i1}) & \Sigma(x_i; \xi^\circ_{1i}; \xi_{i2}) & \cdots & \Sigma(x_i; \xi^\circ_{1i}; \xi_{in}) \\ \Sigma(x_i; \xi^\circ_{2i}; \xi_{i1}) & \Sigma(x_i; \xi^\circ_{2i}; \xi_{i2}) & \cdots & \Sigma(x_i; \xi^\circ_{2i}; \xi_{in}) \\ \vdots & \vdots & \cdots & \vdots \\ \Sigma(x_i; \xi^\circ_{\Delta_i i}; \xi_{i1}) & \Sigma(x_i; \xi^\circ_{\Delta_i i}; \xi_{i2}) & \cdots & \Sigma(x_i; \xi^\circ_{\Delta_i i}; \xi_{in}) \end{pmatrix} \quad (16)$$

Thus $\Sigma^\circ(x_i)$ is, for each $i = 1, \ldots, n$ an $\Delta_i \times n$ matrix whose entry in the ath row and bth column is $\Sigma(x_i; \xi^\circ_{ai}; \xi_{ib})$.

Furthermore, analogously to Σ, we define the $\left(\sum_{i=1}^{n} \Delta_i\right) \times n^2$ matrix Σ° as:

$$\Sigma^\circ = \begin{pmatrix} \Sigma^\circ(x_1) & & & \bigcirc \\ & \Sigma^\circ(x_2) & & \\ & & \ddots & \\ \bigcirc & & & \Sigma^\circ(x_n) \end{pmatrix}. \quad (17)$$

Thus Σ° is an $\left(\sum_{i=1}^{n} \Delta_i\right) \times n^2$ matrix with n blocks along its main diagonal such that the ith block (counting from the top left) has the order $\Delta_i \times n$. The zeros indicate that all the other $\left(\sum_{i=1}^{n} \Delta_i\right) n(n-1)$ entries are zero.

Vector Synthesis of the Principle

Using the notation just defined, the principle of local interaction (13) can be put into the following vector form:

$$N_+(x_i) = N_-(x_i)\,\Sigma(x_i) + N^{\circ}(x_i)\,\Sigma^{\circ}(x_i), \qquad i = 1, \ldots, n \tag{18}$$

Relation (18) is made even more compact by using the remaining notation introduced above:

$$N_+ = N_-\Sigma + N^{\circ}\Sigma^{\circ}, \tag{19}$$

which is the required vector formulation of the local interaction principle.

It appears from (19) that we have completely rounded the notaticnal circle and have returned to the original speical forms (9), (10) of the principle deduced in the Introduction. However, now (19) represents in matricial form a system of n^2 linear algebraic equations in the unknowns N_{ij} with $\sum_{i=1}^{n} \Delta_i$ prescribed

source conditions. In order to place the system (19) in a form which immediately suggests its solution it will be necessary to first introduce the notion of the permutation matrix M .

FUNCTIONAL RELATIONS FOR THE RADIANCE VECTORS

The Permutation Matrix M

Equation (19) is the most compact form of the principle of local interaction on an arbitrary discrete location space X_n . It is not, however, in the form most suitable for obtaining solutions for the quantities N_{ij} . What we require is the so-called symmetric form which involves only N_+ or N_- . In order to cast (19) into its symmetric form and thereby ready it for solution, we recall that the vectors N_+ and N_- are composed of identical sets of numbers N_{ij} , $i,j = 1, \ldots, n$; their salient difference arises from the order of appearance of the numbers N_{ij} in each. Accordingly, there should exist an $n^2 \times n^2$ matrix M with the property that

$$N_+ = N_- M \; ; \tag{20}$$

in fact the required M is the form:

$$M = \begin{pmatrix} E_{11} & E_{21} & \cdots & E_{n1} \\ E_{12} & E_{22} & & E_{n2} \\ \vdots & & \ddots & \vdots \\ E_{1n} & E_{2n} & \cdots & E_{nn} \end{pmatrix} , \tag{21}$$

where E_{ij} is an $n \times n$ matrix of the form $E_{ij} = (e_{ij})$ and where $e_{kl} = 0$ unless $k = i$ and $l = j$, in which case $e_{ij} = 1$. In other words E_{ij} has all entries zero except for a unit entry in the ith row and jth column.

It follows immediately that

$$M^2 = I , \tag{22}$$

so that

$$M = M^{-1} , \tag{23}$$

and

$$|M| = 1 , \tag{24}$$

i.e., M represents a norm-preserving transformation in an n^2-dimensional vector space V_{n^2}. Therefore, in addition to (20), we have

$$N_- = N_+ M . \tag{25}$$

The Functional Relationships for N+ and N-

From (19),

$$N_+ = N_- \Sigma + N^\circ \Sigma^\circ ,$$

and making use of M and property (22):

$$N_+ = (N_- M)(M\Sigma) + N^\circ \Sigma^\circ$$

$$= N_+ M\Sigma + N^\circ \Sigma^\circ$$

Hence

$$\boxed{N_+ (I - M\Sigma) = N^\circ \Sigma^\circ .} \tag{26}$$

Furthermore, from (19) and (25):

$$N_- = N_- \Sigma M + N^\circ \Sigma M ,$$

so that

$$\boxed{N_- (I - \Sigma M) = N^\circ \Sigma^\circ M .} \tag{27}$$

Equations (26) and (27) are the required functional relations governing N_+ and N_-. We now turn to the question of the solvability of (26) and (27) for N_+ and N_- respectively. The answer to this question devolves on the invertibility of the operation $(I - \Sigma M)$ or, equivalently, on that of the operation $(I - M\Sigma)$.

SOLUTIONS OF THE FUNCTIONAL RELATIONS

The Norm Contracting Property of the Scattering Matrix

Let $V = (v_1, \ldots, v_{n^2})$ be an element of V_{n^2}, the n^2- dimensional (radiance) vector space associated with X_n. The norm $|V|$ of V is defined in the usual way as $|V| = \left[\sum_1^{n^2} v_i^2\right]^{1/2}$. Let S be a linear transformation on V_{n^2} into itself. S is said to be norm-contracting if there is a real number γ, $0 < \gamma < 1$, such that for every $V \in V_{n^2}$,

$$|VS| < \gamma |V| . \tag{28}$$

It is easy to show that the linear transformation Σ on V_{n^2} into itself is norm-contracting. To this end we recall that the location space X_n is hypothesized to be non-conservative which by (11) implies that every element $\Sigma(x_i; s_{ai}; s_{ib})$ of Σ is less than unity and not all such elements are zero. Set $\gamma = \max\{\Sigma(x_i; s_{ai}; s_{ib}) : a, b, i = 1, \ldots, n\}$. Clearly $0 < \gamma < 1$, and for every $V \in V_{n^2}$, we have

$$|V\Sigma| < \gamma |V| \tag{29}$$

It follows immediately from (24) that ΣM and $M\Sigma$ are also norm-contracting linear transformations on V_{n^2} into itself; furthermore, the associated γ in each case is precisely that for Σ.

Solutions

We now make use of the following well-known property of norm-contracting linear transformations (see, e.g., ref. (17)) : Let S be a norm-contracting linear transformation with a given contraction factor γ* on a finite dimensional vector space V_m. Then the transformation $(I-S)$ has an inverse $(I-S)^{-1}$ where I is the identity transformation, and

$$(I-S)^{-1} = I + S + S^2 + \cdots \quad . \tag{30}$$

Since ΣM and $M\Sigma$ are norm-contracting it follows immediately that (26) and (27) possess unique non trivial solution vectors N_+ and N_- whenever N° is not the zero vector. Thus

$$N_+ = N^\circ \Sigma \circ (I - M\Sigma)^{-1},$$

* V_{n^2} may be infinite dimensional; however, we shall not need this stronger condition. Furthermore, a strictly finitary demonstration of the existence of $(I-S)^{-1}$ may be given.

or

$$N_+ = \frac{N^\circ \Sigma^\circ}{I - M\Sigma} , \tag{31}$$

and

$$N_- = \frac{N^\circ \Sigma^\circ M}{I - \Sigma M} . \tag{32}$$

Equations (31) and (32) constitute the required solutions for the specific and field radiance vectors, respectively, under a given incident source condition N° and for the discrete location space X_n .

SCATTERING-ORDER DECOMPOSITION OF THE SOLUTIONS

In the continuous theory of radiative transfer it is quite often helpful in a physical or mathematical analysis of a multiple scattering problem to decompose the radiance distribution $N(x, \cdot)$ at a point x in the optical medium into an infinite series, the nth term of which consists of a radiance distribution associated with radiant flux scattered precisely n times; thus:

$$N(x, \cdot) = \sum_{n=0}^{\infty} N^n(x, \cdot) , \tag{33}$$

where $N^0(x,\cdot)$ is the reduced radiance distribution with respect to which the scattering orders are usually indexed. More often than not, owing to the complexity of the multiple scattering process, the theorist is unable to obtain explicit closed analytical forms for the $n-ary$ radiance distributions in terms of the reduced radiance distribution (which is generally the given radiometric datum). Usually only estimates and bounds on $N^n(x,\cdot)$ of a very crude (but occasionally helpful) nature can be wrung from the general transport equation.[18,19]

Not the least of the many analytically pleasant features of the discrete space formulations of radiative transfer theory is the possibility of obtaining simple closed forms for the $n-ary$ radiance distributions about each point of the location space X_n. These forms are obtained directly from the infinite series representation of $I - M\Sigma$. We now briefly outline the details of obtaining the desired decomposition.

To begin with, we set

$$N_+^0 = N^0\Sigma^0. \tag{34}$$

N_+^0 is the reduced radiance vector with respect to which the following $n-ary$ scattering orders are indexed. N_+^0 may be interpreted as the radiant flux of zero scattering order which is fed into the system X_n. We continue by inductively defining N_+^n as:

$$N_+^n = N_+^{n-1}(M\Sigma),$$

so that

$$N_+^n = N_+^0 (M\Sigma)^n. \tag{35}$$

N_+^n may be interpreted as the portion of N_+ representing the n-ary radiance vector, i.e., a radiance distribution consisting of radiant flux which, with respect to that comprising N_+^0, has been scattered at most n times. It follows that, by means of (30) and (31), the specific radiance vector N_+ may be represented by the following formal infinite series:

$$\boxed{N_+ = \sum_{n=0}^{\infty} N_+^n ,} \tag{36}$$

which is analogous to its continuous counterpart (33).

It is of interest to observe that (35) (and hence (36)) is especially amenable to numerical evaluation on large-scale automatic computors.

The Truncated Decomposition

In addition to its useful conceptual features and its basic numerical tractability, the scattering order decomposition of the radiance vector as given in (36) yields a relatively straightforward and practical truncation formula with a well-defined error bound.

That is, if the infinite series (36) is truncated at the Kth term, $K \geq 0$, we can estimate the norm of the difference between N_+ and the finite sum of the first K terms. Specifically, write

$$N_+ = \sum_{n=0}^{K} N_+^n + \sum_{n=k+1}^{\infty} N_+^n ,$$

so that

$$\left| N_+ - \sum_{n=0}^{K} N_+^n \right| = \left| \sum_{n=K+1}^{\infty} N_+^n \right| .$$

But then, according to (35) we may write:

$$\left| \sum_{n=K+1}^{\infty} N_+^n \right| = \left| \sum_{n=K+1}^{\infty} N_+^0 (M\Sigma)^n \right| ,$$

which, by virtue of the triangle inequality which holds in V_{n^2} (the present vector space), may be replaced by the inequality

$$\left| \sum_{n=K+1}^{\infty} N_+^n \right| \leq \sum_{n=K+1}^{\infty} \left| N_+^0 (M\Sigma)^n \right| .$$

This inequality may be strengthened further by using the norm-contracting property of the operator $M\Sigma$. That is

$$|N_+^0 (M\Sigma)^n| = |[N_+^0 (M\Sigma)^{n-1}](M\Sigma)|$$

$$< \gamma |N_+^0 (M\Sigma)^{n-1}| = \gamma |[N_+^0 (M\Sigma)^{n-2}](M\Sigma)|$$

$$< \gamma^2 |N_+^0 (M\Sigma)^{n-2}| .$$

Continuing to apply the norm-contracting properties in this way, we eventually arrive at the inequality:

$$|N_+^\circ (M\Sigma)^n| < \gamma^n |N_+^\circ| ,$$

so that the error estimate becomes

$$|N_+ - \sum_{n=0}^{K} N_+^n| < \sum_{n=K+1}^{\infty} \gamma^n |N_+^\circ| ,$$

which reduces to

$$\boxed{|N_+ - \sum_{n=0}^{K} N_+^n| < \frac{|N_+^\circ| \gamma^{K+1}}{1-\gamma}} \qquad (37)$$

For non-trivial scattering problems we have $|N_+| \geq |N_+^\circ|$; therefore, the relative error associated with the K-term truncation certainly cannot exceed $\gamma^{K+1}/(1-\gamma)$, i.e.,

$$\boxed{\frac{\left|N_+ - \sum_{n=0}^{K} N_+^n\right|}{N_+} < \frac{\gamma^{K+1}}{1-\gamma}} \qquad (38)$$

As an example of the use of (38), suppose that $\gamma = 1/2$. Then a truncation of the infinite series at $K = 5$, has associated with it a relative error less than 4%.

SUMMARY AND PROSPECTUS

Starting with an arbitrary discrete space X_n and the principle of local interaction (12) on X_n, it has been possible to formulate the complete multiple scattering problem on X_n (equation (19)) and to solve it explicitly (equation (31)) along with estimates on the accuracy of an iteration-type approximate solution (equation (38)).

These results only begin to show the analytic advantages of a discrete-space approach to radiative transfer problems. In subsequent papers we will show how the principles of invariance on X_n may be derived from the local interaction principle. The results of such a derivation may be applied to such practical problems as the numerical computation of the light field in natural aerosols and hydrosols. This also will be shown in subsequent works.

The local interaction principle will also be used to formulate the basic transfer equation on discrete spaces and, by limiting processes, also those on continua. Thus the principle is capable of tying together all the fundamental concepts and laws of radiative transfer theory.

For all its power and comprehensiveness, the principle has nevertheless a commonplace origin in everyday knowledge of radiative transfer: Its underlying precept was implicitly known and instinctively perceived by every investigator who ever engaged in an interreflection study, or who made use of the concept of the volume scattering (phase) function. In this light the following observation by Mach forms a particularly apt conclusion to this work:

"Let it be remarked in conclusion that ... every general principle, brings with it, by the insight which it furnishes, dis-illusionment as well as elucidation. It brings with it disillusionment to the extent that we recognize it in facts which were long before known and even instinctively perceived, although our present recognition is more distinct and more definite; and elucidation, in that it enables us to see everywhere throughout the most complicated relations the same simple facts."

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