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Author
Matic, Ivan

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Homogenizations and Large Deviations

by

Ivan Matic

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Committee in charge:

Professor Fraydoun Rezakhanlou, Chair
Professor Lawrence C. Evans
Professor Venkat Anantharam

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Abstract

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Ivan Matic

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Professor Fraydoun Rezakhanlou, Chair

The object of study is homogenization and large deviations for various stochastic models. We start by presenting large deviation bounds for certain Hamilton-Jacobi equations. The discrete analogue of the control curves from the variational formula brings us to the study of the deterministic walks in random environments. The discretization of time in variational formula of Hamilton-Jacobi equation is related to the Frenkel-Kontorova model, for which we do the homogenization.

To my mom and dad.

Contents

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Chapter 1

Introduction

The limiting behavior of stochastic models is used to describe many processes in nature: formation of crystals, spread of infections, traffic jams, and a variety of interface growths. The study of these models encourages and uses the mutual interactions of various math disciplines. For example, in the case of Hamilton-Jacobi (HJ) partial differential equations, homogenization is related to the study of invariant measures for Hamiltonian systems. This brings us to the celebrated weak KAM theory. The homogenization for Hamilton-Jacobi-Bellman (HJB) equations gives the (quenched) large deviations for diffusions with random drifts and obstacles. When the time is discrete, from the PDE world we get to the Frenkel-Kontorova (FK) and Gibbs measures (GM) models. Homogenization for FK is related to finding invariant measures for symplectic maps. When the space becomes discrete we arrive at the study of random walks in random environments (RWRE) and deterministic walks in random environment (DWRE). Totally asymmetric exclusion processes (TASEP) and k-exclusions are famous interacting particle systems used to model traffic flows. These two models are related to the FK model for special case of Lagrangians.

HJ and HJB are very popular in the PDE community. The FK model is of great interest to the researchers in dynamical systems, while GM is studied in equilibrium statistical mechanics. RWRE and percolation are some of the most important topics in probability theory, and they have been very popular among probabilists in recent years. Many of those studying hydrodynamic limits showed an interest in TASEP. Understanding the connection between these models would be appreciated by many mathematicians.

The results from this thesis help in understanding the FK model, deterministic walks in random environments, and solutions to some Hamilton-Jacobi equations.

1.1 Large Deviations

Large deviation theory is the study of the exponential rate of decay of probabilities of some rare events. The simplest example is the sum S_n of independent random variables $X_1, \ldots, X_n$ with mean m. The events $\{|S_n| \geq n(m+\delta)\}$ are rare for all $\delta > 0$, meaning that $\mathbb{P}(|S_n| \geq m(1+\delta)) \to 0$

as $n \to \infty$. However, it is possible to get a more precise result, namely there exists a function $I : \mathbb{R}_+ \to \overline{\mathbb{R}}$ such that $\mathbb{P}(|S_n| \geq m(1+\delta)) \approx e^{-nI(\delta)}$. The study of the function I and the large deviation probabilities is closely tied to the study of the limits of moment generating functions, i.e. $\Lambda(\lambda) = \lim \frac{1}{n} \log \mathbb{E}\left(e^{\lambda \cdot X_n}\right)$. For more details, see [8].

1.1.1 Stochastic Hamilton-Jacobi and Hamilton-Jacobi-Bellman equations

We are interested in the large deviations for the viscosity solutions of the following PDEs in the cases $\delta = 0$ (Hamilton-Jacobi) and $\delta = 1$ (Hamilton-Jacobi-Bellman):

$$\begin{aligned} u_t^\varepsilon(x,t,\omega) + H(\nabla u^\varepsilon(x,t,\omega), \tau_{x/\varepsilon}\omega)) &= \delta \frac{\varepsilon}{2} \Delta u^\varepsilon(x,t,\omega) \\ u^\varepsilon(x,0,\omega) &= g^\varepsilon(x). \end{aligned} \tag{1.1.1}$$

One of the central problems in this area is:

Problem 1.1.1. *Do there exist large deviations for the viscosity solutions of the Hamilton-Jacobi and Hamilton-Jacobi-Bellman equations? Identify and prove the regularity properties for the rate function.*

In the case of a convex Hamiltonian, viscosity solutions can be characterized by variational formulas. In the work [16] of Kosygina, Rezakhanlou, and Varadhan, various controls have been sampled from the variational formulas to obtain estimates. These estimates were good approximations for solutions to establish homogenization. We hope that some similar idea can be applied to the study of large deviations. The goal is to find a set of controls that are close enough to the solutions to cause sub-exponential error in probabilities of large deviations. At the same time, those controls should be sufficiently simple to establish the large deviations principle. That method would hopefully be robust enough to implement in some other models.

In the special case of quadratic Hamiltonian $H(p,x,\omega) = \frac{1}{2}|p|^2 + V(x,\omega)$, the Hopf-Cole transform makes connections between HJB equations and Brownian motions killed with rate V. When the environment is Poissonian, Sznitman in [28] managed to establish the shape theorem for the annealed Lyapunov exponents. His quenched Lyapunov exponents correspond to homogenization of Hamilton-Jacobi equations, while the annealed case corresponds to some extent to the large deviations. However, the connection is still not fully there, because his shape theorem covers the convergence of the moment generating function for the value $\lambda = 1$. This could be easily extended to positive integers but it is not yet clear how to go beyond this. The importance of establishing those connections is that the results of the two areas would immediately enhance each other.

In one-dimensional case, we will be able to say something more about large deviations when the Hamiltonian has the form $H(p,x,\omega) = K(p) - V(\tau_x\omega)$ for certain convex functions K .

1.1.2 Large deviations for deterministic walks in random environments

Motivation and relation to other models

An important class of controls featured in the variational formulas for HJ are those that solve an ODE of the form $dX/dt = b(X,\omega)$. It would be beneficial to understand the large deviations of the controls themselves. If the ODE is replaced by a recursive equation for sequences, we get an analogous model which is called a deterministic walk in a random environment. Another interest in this model comes from the fact that it can be viewed as a very degenerate case of a random walk in a random environment. An averaged (annealed) walk does not posses the Markov property and as such it has presented a big challenge in recent years. Varadhan in [30] and Rassoul-Agha in [20] used the point of view of the particle to understand this walk as a Markov chain on a much larger probability space and derived the large deviations from the Donsker-Varadhan theory applied to such a Markov chain. The problem they had to face is that the state space was extremely large and establishing compactness that is needed for the large deviations was a very difficult task. Following this approach, several refinements of the results were possible, and these all assumed the ellipticity for the environment. The most degenerate case when this ellipticity is not present is our model of the deterministic walk in a random environment.

Definition of the model

A deterministic walk in a random environment can be understood as a general finite-range dependent random walk that starts repeating the loop once it reaches a site it has visited before. For a more precise definition let us assume that $b : \mathbb{Z}^d \times \Omega \to \mathbb{Z}^d$ is a random field on the probability space $(\Omega, \mathbb{P})$. We now define the walk X_n recursively as $X_0 = 0$, $X_{n+1} = X_n + b(X_n,\omega)$. Without stating all the assumptions now, we will able to prove the following theorem (see also [17]):

Theorem 1.1.2. *There exists a convex function $\Lambda : \mathbb{R}^d \to \overline{\mathbb{R}}$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[e^{\lambda\cdot X_n}\right] = \Lambda(\lambda).$$

When time and space are continuous we get a generalization stated in the next problem. Solving this problem would directly benefit the understanding of large deviations for HJ equations.

Problem 1.1.3. *For reasonable random fields $b : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$, do we have large deviations for the solution X_t of the system of differential equations*

$$\frac{dX_t}{dt} = b(X_t,\omega)?$$

Before attempting to solve this problem, it might be useful to consider the following simplification in which the time is discrete but the space is continuous.

Problem 1.1.4. *Do we have large deviations for the sequence X_n on $\mathbb{R}^d$ defined as $X_{n+1} = X_n + b(X_n,\omega)$, for suitable random field $b : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$?*

1.2 Homogenization

Homogenization deals with studying the limiting behavior of solutions to partial differential equations with rapidly oscillating coefficients. Relatively well understood examples are ones of the Hamilton-Jacobi and Hamilton-Jacobi-Bellman equations. The homogenizations for the periodic models are done by Evans and Gomes in [13] and [14] while the results for stochastic models are due to Lions, Kosygina, Rezakhanlou, Souganidis, Tarver, and Varadhan in [16], [19], [24]. The limiting functions turned out to be the solution of a deterministic Hamilton-Jacobi PDE. The Hamiltonian of such a PDE (so called the effective Hamiltonian) is given by the cell problem.

1.2.1 Homogenization for the Frenkel-Kontorova model

The Frenkel-Kontorova model can be understood as a discrete analogue of a solution to the stochastic Hamilton-Jacobi equation. Let $(\Omega,\mathbb{P})$ be a probability space such that $\mathbb{P}$ is invariant under the family $(\tau_z)_{z\in\mathbb{R}^d}$ of transformations on Ω (a good way to think about τ_z is as translations). We also assume that τ_z is ergodic. In other words, we are assuming $\mathbb{P}(\tau_z A) = \mathbb{P}(A)$ (for all z and A); and only trivial sets (i.e. those of measure 0 or 1) can satisfy $\mathbb{P}(A\Delta\tau_z A) = 0$ for all $z\in\mathbb{R}^d$. Here Δ denotes the symmetric difference.

Let $L:\mathbb{R}^d\times\Omega$ be a convex function. Define

$$S_n(p,q,\omega) = \inf\left\{\sum_{i=0}^{n-1} L(q_{i+1}-q_i,\tau_{q_i}\omega)\right\},$$

where the infimum is taken over all sequences $q_0,q_1,\dots,q_n$ that satisfy $q_0 = p$ and $q_n = q$. Assume that $h:\mathbb{R}^d\to\mathbb{R}$ is a Lipschitz-continuous function and let us define

$$u_n(q,\omega) = \sup_Q\left\{nh\left(\frac{Q}{n}\right) - S_n(q,Q,\omega)\right\}.$$

We will prove (see also [18]) the following theorem under some coercivity assumptions on the function L:

Theorem 1.2.1. *There exists a convex and coercive function $\bar{L}:\mathbb{R}^d\to\mathbb{R}$ such that for every $\eta>0$, all $q\in\mathbb{R}^d$, and almost all $\omega\in\Omega$, the following holds:*

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{q'\in B(q,\eta)}\left|\frac{1}{n}u_n(nq',\omega) - u(q')\right|\right] = 0,$$

where

$$u(q) = \sup_Q\left\{h(Q) - \bar{L}(Q-q)\right\}.$$

The existence of $\bar{L}$ is established using the subadditive ergodic theorem. However, once the homogenization is obtained for all functions h plugging in some particular perturbations of linear initial data one can obtain a variational formula for $\bar{L}$. Denote by $\mathscr{K}$ the class of functions $g : \mathbb{R}^d \times \Omega \to \mathbb{R}$ that satisfy the following three conditions:

(i) For each $a \in \mathbb{R}^d$ the function $g(a,\omega)$ belongs to $L^{d+\alpha}(\Omega)$ for some $\alpha > 0$;

(ii) For each $a \in \mathbb{R}^d$ $\mathbb{E}(g(a,\omega)) = 0$, and

(iii) For each sequence $q_0, q_1, q_2, \dots, q_n \in \mathbb{R}^d$ such that $q_n = q_0$ the following equality holds:

$$\sum_{k=0}^{n-1} g(q_{k+1} - q_k, \tau_{q_k}\omega) = 0.$$

Theorem 1.2.2. *$\bar{L}$ is the convex conjugate of the function $\bar{H}$ defined as:*

$$\bar{H}(p) = \inf_{g \in \mathscr{K}} \sup_a \operatorname{esssup}_{\mathbb{P}} \{g(a,\omega) + p \cdot a - L(a,\omega)\}.$$

A discrete analogue to the class $\mathscr{K}$ turned out to be of great use to Rosenbluth and Yilmaz in studying RWRE. Some of its important properties are established in [25].

Let us conclude by mentioning a particularly interesting problem that arises from this study.

Problem 1.2.3. *Do we have large deviations for the Frenkel-Kontorova model under certain assumptions on L and h?*

Since the time here is discrete while the space is not, this would be a step towards understanding large deviations for Hamilton-Jacobi equations.

Connections to last passage percolation and k-exclusions

Methods used in the study of the homogenizations for the Frenkel-Kontorova model can be slightly modified to work in the discrete space. Interesting problems would be establishing the relations of this model to the last passage percolations and k-exclusions.

Homogenization results for percolation are obtained by Grimmett and Kesten in [15] and large deviations are studied by Chow and Zheng in [3], and Cranston, Gauthier, and Mountford in [6]. The proof of Theorem 1.2.2 utilized the limits obtained by the subadditive ergodic theorem. This gives a hope that similar arguments would allow us to get variational formulas for the limits derived by Grimmett and Kesten and prove some regularity of such limits.

Consider the following modification of the Frenkel-Kontorova model in which the space is also discrete and $L(a, \tau_z\omega)$ represents the time the particle needs to travel over the edge between z and $z+a$. We assume here that $L(a,\omega) = +\infty$ for all a such that $\|a\| > 1$. $S_n(q,Q,\omega)$ now has an interpretation as the time that a particle needs if it wants to travel between q and Q using exactly

n steps. The passage time between the sites q and Q is defined as $S(q,Q,\omega) = \inf_n S_n(q,Q,\omega)$. If we denote by $l(q,Q,\omega)$ the value for n that minimizes $S_n(q,Q,\omega)$ we see that l and S satisfy the subadditivity property and as such we have the limits of $\frac{1}{n}S(0,na,\omega)$ and $\frac{1}{n}l(0,na,\omega)$. We can obtain similar results to the ones for the FK model and analogous versions of $\bar{L}$ and $\bar{H}$ and get a candidate for the variational formula for $\bar{L}$. In this case $\bar{L}$ and $\bar{l}$ are norms, and $\bar{H}$ (that can be defined similarly to the definition from theorem 1.2.2) is not a conjugate of $\bar{L}$ for every probability distribution $\mathbb{P}$.

Another model is k-exclusions studied by Seppalainen in [26] and Rezakhanlou in [22] and [23]. The model is the following: The initial interface is a graph of a function $h:\mathbb{Z}^d \to \mathbb{Z}$. Each site has its Poisson clock. When the clock rings at the point a, our interface wants to change from h to h^a, where $h^a(b) = h(b)$ for $b \neq a$, and $h^a(a) = h(a)+1$. The change would happen if the function h^a belonged to the class of so called admissible functions. The set of admissible functions is usually chosen to consist of the functions with moderate jumps, i.e. of the size at most k. Totally asymmetric simple exclusion processes are special cases when $d = 1$ and $k = 1$. The problem we want to study is the regularity and convexity of the Hamiltonians that are obtained as the hydrodynamic limits of those processes.

Problem 1.2.4. *Is it possible to obtain regularity and convexity properties for the Hamiltonians in percolations and k exclusions?*

Chapter 2

Large Deviations for Hamilton-Jacobi Equations

2.1 Introduction

We are interested in the large deviations for the viscosity solutions of the following PDEs in the cases $\delta = 0$ (Hamilton-Jacobi) and $\delta = 1$ (Hamilton-Jacobi-Bellman):

$$\begin{aligned} u_t^\varepsilon(x,t,\omega) + H(\nabla u^\varepsilon(x,t,\omega), \tau_{x/\varepsilon}\omega)) &= \delta \frac{\varepsilon}{2} \Delta u^\varepsilon(x,t,\omega) \\ u^\varepsilon(x,0,\omega) &= g^\varepsilon(x). \end{aligned} \tag{2.1.1}$$

We will study certain Hamiltonians, and certain environments for which there are large deviations for solutions of (2.1.1). It is useful to consider the function $v^\varepsilon(x,t,\omega) = \frac{1}{\varepsilon} u^\varepsilon(\varepsilon x, \varepsilon t, \omega)$. Then the derivatives of v^ε satisfy:

$$\begin{aligned} v_t^\varepsilon(x,t,\omega) &= u_t^\varepsilon(\varepsilon x, \varepsilon t, \omega), \\ \nabla v^\varepsilon(x,t,\omega) &= \nabla u_t^\varepsilon(\varepsilon x, \varepsilon t, \omega), \text{ and} \\ \Delta v^\varepsilon(x,t,\omega) &= \varepsilon \Delta u^\varepsilon(\varepsilon x, \varepsilon t, \omega). \end{aligned}$$

The initial value problem (2.1.1) can be rephrased in terms of v^ε in the following way:

$$\begin{aligned} v_t^\varepsilon(x,t,\omega) + H(\nabla v^\varepsilon(x,t,\omega), \tau_x\omega) &= \delta \frac{1}{2} \Delta v^\varepsilon(x,t,\omega) \\ v^\varepsilon(x,0,\omega) &= \frac{1}{\varepsilon} g^\varepsilon(\varepsilon x). \end{aligned} \tag{2.1.2}$$

The advantage of the above equation is that each member of the family v^ε satisfies the same equation with different initial values.

We will focus on the case $\delta = 0$. Denote $f^\varepsilon(x) = \frac{1}{\varepsilon} g(\varepsilon x)$. Denote by L the convex conjugate of H. We have the following variational formula for representing the solution

$$\begin{aligned} v^\varepsilon(x,t,\omega) &= \inf_y \{f^\varepsilon(y) + S(x,y,t,\omega)\} \text{ where} \\ S(x,y,t,\omega) &= \inf\left\{ \int_0^t L(\dot{\xi}(s), \tau_{\xi(s)}\omega)\, ds : \xi(0) = y, \xi(t) = x \right\}. \end{aligned}$$

The function we are after is $u^\varepsilon(x,t,\omega) = \varepsilon v^\varepsilon\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \omega\right)$, and in the case $g^\varepsilon = g$ does not depend on ε we have:

$$u^\varepsilon(x,t,\omega) = \inf_y \left\{ g(y) + \varepsilon S\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{t}{\varepsilon}, \omega\right) \right\}. \tag{2.1.3}$$

Hopf-Cole Transform for the Hamilton-Jacobi-Bellman Equation

Let us briefly mention one approach that can yield to some results in the case when $\delta = 1$ is chosen in (2.1.1). If the Hamiltonian has the following special form

$$H(p,x) = \frac{1}{2}|p|^2 + b(\tau_x\omega) \cdot p + V(\tau_x\omega),$$

then we can use the Hopf-Cole transform to simplify the problem to the one in which the PDE is linear. The idea is to start from the solution v^ε of (2.1.2) and consider the function $w^\varepsilon(x,t,\omega) = e^{-v^\varepsilon(x,t,\omega)}$. Then $w_t^\varepsilon = -v_t^\varepsilon e^{-v^\varepsilon}$, $\nabla w^\varepsilon = -\nabla v^\varepsilon e^{-v^\varepsilon}$, and $\Delta w^\varepsilon = -\Delta v^\varepsilon e^{-v^\varepsilon} + |\nabla v^\varepsilon|^2 e^{-v^\varepsilon}$. The equation now becomes

$$\begin{aligned} w_t^\varepsilon(x,t,\omega) + b(\tau_x\omega) \cdot \nabla w^\varepsilon(x,t,\omega) - V(\tau_x\omega) w^\varepsilon(x,t,\omega) &= \frac{1}{2}\Delta w^\varepsilon(x,t,\omega) \\ w^\varepsilon(x,t,\omega) &= e^{-\frac{1}{\varepsilon} g^\varepsilon(\varepsilon x)}. \end{aligned}$$

Now we can apply the Feynman-Kac formula to obtain

$$w^\varepsilon(x,t,\omega) = E_x^\omega\left[e^{-\frac{1}{\varepsilon} g^\varepsilon(\varepsilon Z_t) + \int_0^t V(Z_s)\, ds} \right]$$

where Z_t is the diffusion with the generator $\frac{1}{2}\Delta - b \cdot \nabla$. Such Z_s is a Brownian motion with drift b. Consequently $v^\varepsilon(x,t,\omega) = -\log E_x\left[e^{-\frac{1}{\varepsilon} g^\varepsilon(\varepsilon Z_t) + \int_0^t V(Z_s)\, ds} \right]$ and

$$u^\varepsilon(x,t,\omega) = -\varepsilon \log E_{x/\varepsilon}\left[e^{-\frac{1}{\varepsilon} g^\varepsilon(\varepsilon Z_{t/\varepsilon}) + \int_0^{t/\varepsilon} V(Z_s)\, ds} \right].$$

Therefore doing quenched large deviations for Z_s is related to doing the homogenization for the corresponding HJ. On the other hand:

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon\log\mathbb{P}\left[u^{\varepsilon}(x,t,\omega)\leq\alpha\right] &= \lim_{\varepsilon\to 0}\varepsilon\log\mathbb{P}\left[\varepsilon v^{\varepsilon}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon},\omega\right)\leq\alpha\right]\\ &= \lim_{\varepsilon\to 0}\varepsilon\log\mathbb{P}\left[e^{-v^{\varepsilon}(x,t,\omega)}\geq e^{-\alpha/\varepsilon}\right]\\ &\leq -\alpha+\lim_{\varepsilon\to 0}\varepsilon\log\mathbb{E}\left[w^{\varepsilon}(x,y,\omega)\right]\\ &= \alpha+\lim_{\varepsilon\to 0}\varepsilon\log\mathbb{E}\left[E_x\left[e^{-\frac{1}{\varepsilon}g^{\varepsilon}(\varepsilon Z_t)+\int_0^t V(Z_s)\,ds}\right]\right].\end{aligned}$$

The problem of annealed large deviations for the drifted Brownian motion killed with rate V is tied to a bound for the large deviations for the Hamilton-Jacobi-Bellman equaiton.

2.2 Large Deviations for the Environment

As an illustration of the large deviations we will consider the Hamilton-Jacobi equation in one-dimensional space. For a Hamiltonian of a special form $H(p,x,\omega)=K(p)-V(\tau_x\omega)$ there is a convenient way to represent solutions to (2.1.1) and use the large deviation properties of the environment to study the solutions.

In this section we will see that Poisson fields posses a large deviation property that would be sufficient for our purposes. The section 2.3 will assume the existence of such large deviations for the environment and use that to establish the result for the solutions of (2.1.1).

First we need to recall some standard results about the entropy function. Then we will define the entropy function for Poisson fields. This argument is analogous to the one for Markov processes presented in the famous paper [11] of Donsker and Varadhan.

2.2.1 Entropy

If λ and μ are two probability measures on $(X,\mathscr{F})$ we define the entropy of μ with respect to λ as:

$$h(\mu|\lambda) = \sup_{\Phi\in\mathscr{B}(X)}\left\{\mathbb{E}^{\mu}(\Phi)-\log\mathbb{E}^{\lambda}\left(e^{\Phi}\right)\right\}. \tag{2.2.1}$$

Then we have the following result:

$$h(\mu|\lambda) = \begin{cases}\mathbb{E}^{\mu}\left(\log\frac{d\mu}{d\lambda}\right) & \text{if } \mu<<\lambda,\\ +\infty, & \text{otherwise.}\end{cases} \tag{2.2.2}$$

Indeed, if μ is not absolutely continuous with respect to λ then taking $\Phi_n=n\cdot 1_A$ for some set A that satisfies $0=\lambda(A)<\mu(A)$ we have $\mathbb{E}^{\mu}(\Phi_n)-\log\mathbb{E}^{\mu}(\Phi_n)=n\mu(A)\to+\infty$ as $n\to\infty$. On the other hand, if $\mu << \lambda$ we have by Jensen's inequality: $\mathbb{E}^{\mu}\left[\log\left(e^{\phi}/\frac{d\mu}{d\lambda}\right)\right] \leq \log \mathbb{E}^{\mu}\left[e^{\phi}/\frac{d\mu}{d\lambda}\right] = \log \mathbb{E}^{\lambda}(e^{\Phi})$. This implies that $h(\mu|\lambda) \leq \sup_{\Phi\in\mathscr{B}(X)}\left\{\mathbb{E}^{\mu}(\Phi) - \log \mathbb{E}^{\lambda}(e^{\Phi})\right\}$. Taking $\Phi = \log\frac{d\mu}{d\lambda}$ we get $\mathbb{E}^{\mu}(\Phi) - \log\mathbb{E}^{\lambda}(e^{\Phi}) = \mathbb{E}^{\mu}\left(\frac{d\mu}{d\lambda}\right)$ giving the other inequality. Thus (2.2.2) is established.

There is another way to express the entropy. If we denote by $\mathscr{B}_1(X) = \{f \in \mathscr{B}(X) : \mathbb{E}^{\lambda}\left(e^f\right) \leq 1\}$ then

$$h(\mu|\lambda) = \sup_{f\in\mathscr{B}_1(X)} \mathbb{E}^{\mu}(f). \tag{2.2.3}$$

This is easy to verify. If μ is not absolutely continuous with respect to λ then taking A be the set such that $\lambda(A) = 0 < \mu(A)$ and $f_n = n1_A$ we have $\mathbb{E}^{\lambda}(e^f) = 1$ and $\mathbb{E}^{\mu}(f) = n\mu(A) \to +\infty$ as $n \to \infty$. Assume now that $\mu << \lambda$. Let $f \in \mathscr{B}_1(X)$. Then $f \in \mathscr{B}(X)$ hence from (2.2.1) we get $h(\mu|\lambda) \geq \mathbb{E}^{\mu}(f) - \log\mathbb{E}^{\lambda}(e^f) \geq \mathbb{E}^{\mu}(f)$. Taking supremum over $f \in \mathscr{B}_1(X)$ we obtain $h(\mu|\lambda) \geq \sup_{f\in\mathscr{B}_1(X)}\mathbb{E}^{\mu}(f)$. If we now choose $f = \log\frac{d\mu}{d\lambda}$ we get $f \in \mathscr{B}_1(X)$ and $\mathbb{E}^{\mu}(f) = h(\mu|\lambda)$ which implies (2.2.3).

If $\mathscr{G} \subseteq \mathscr{F}$ is a sigma-field we can define the entropy of μ with respect to λ in the sigma-field $\mathscr{G}$ as

$$h_{\mathscr{G}}(\mu|\lambda) = \sup_{\Phi\in\mathscr{B}(X)}\left\{\mathbb{E}^{\mu}(\Phi) - \log\mathbb{E}^{\lambda}\left(e^{\Phi}\right)\right\}.$$

We have the following theorem relating the entropies of the smaller σ-fields. The proof is presented in [11] (lemma 2.3).

Theorem 2.2.1. *Let $\mathscr{G} \subseteq \mathscr{F}$ be a σ-field, and let $\hat{\mu}$ and $\hat{\lambda}$ denote the regular conditional probability distributions of μ and λ with respect to $\mathscr{G}$. Then the following equality holds:*

$$h(\mu|\lambda) = h_{\mathscr{G}}(\mu|\lambda) + \mathbb{E}^{\mu}\left(h(\hat{\mu}|\hat{\lambda})\right).$$

Let $v \in \mathbb{R}^d$ be a vector and $a \in \mathbb{R}^d$ a point. Define the strip starting at a in the direction v as

$$\Sigma_{v;a} = \{x \in \mathbb{R}^d : 0 \leq (x-a)\cdot v \leq |v|^2\}.$$

Each point $a \in \mathbb{R}^d$ and each vector $v \in \mathbb{R}^d$ determine the half-space $\Sigma^-_{v;a} = \{x \in \mathbb{R}^d : (x-a)\cdot v \leq 0\}$. Denote by $\mathscr{F}_{\Sigma} = \mathscr{F}_{\Sigma_{v;a}}$ the σ-field generated by those functions f that depend only on the points in Σ. Denote by $\hat{Q}$ and $\hat{\mathbb{P}}$ the r.c.p.d. of Q and $\mathbb{P}$ with respect to the sigma field $\mathscr{F}_{\Sigma^-_{v;a}}$. We define the entropy with respect to the strip $\Sigma_{v;a}$ as

$$H(v;a,Q) = \mathbb{E}^{Q}\left[h_{\mathscr{F}_{\Sigma}}(\hat{Q}|\hat{\mathbb{P}})\right].$$

The following theorem holds:

Theorem 2.2.2. *Let $v, a \in \mathbb{R}^d$ be fixed (v is assumed to be a vector, while a is a point). Either $H(tv; a, Q) = \infty$ for every $t > 0$ or*

$$H(tv; a, Q) = tH(v; a, Q).$$

Proof. It suffices to prove that for any two collinear vectors v_1 and v_1 we have $H(v_1 + v_2; a, Q) = H(v_1; a, Q) + H(v_2; a, Q)$. We will use $\hat{Q}$ and $\hat{\mathbb{P}}$ to denote the r.c.p.d. of Q and $\mathbb{P}$ with respect to $\mathscr{F}_{\Sigma^-_{v_1;a}}$. Let us denote by $\hat{Q}'$ and $\hat{\mathbb{P}}'$ the r.c.p.d. of $\hat{Q}$ and $\hat{\mathbb{P}}$ with respect to $\mathscr{F}_{\Sigma_{v_1;a}}$. Using the theorem 2.2.1 we get

$$\begin{aligned} H(v_1 + v_2; a, Q) &= \mathbb{E}^Q \left[h_{\mathscr{F}_{\Sigma_{v_1+v_2;a}}}(\hat{Q}|\hat{\mathbb{P}}) \right] \\ &= \mathbb{E}^Q \left[h_{\mathscr{F}_{\Sigma_{v_1;a}}}(\hat{Q}|\hat{\mathbb{P}}) + \mathbb{E}^{\hat{Q}} \left[h_{\mathscr{F}_{\Sigma_{v_1+v_2;a}}}(\hat{Q}'|\hat{\mathbb{P}}') \right] \right] \\ &= H(v_1; a, Q) + \mathbb{E}^Q \left[\mathbb{E}^{\hat{Q}} \left[h_{\mathscr{F}_{\Sigma_{v_1+v_2;a}}}(\hat{Q}'|\hat{\mathbb{P}}') \right] \right]. \end{aligned}$$

Let us denote by $\bar{Q}$ and $\bar{\mathbb{P}}$ the r.c.p.d. of Q and $\mathbb{P}$ with respect to $\mathscr{F}_{\Sigma^-_{v_2;a+v_1}}$. The last term is equal to $\mathbb{E}^Q \left[h_{\mathscr{F}_{\Sigma_{v_2;a+v_1}}}(\bar{Q}|\bar{\mathbb{P}}) \right]$. Because of the stationarity we have that $\mathbb{E}^Q \left[h_{\mathscr{F}_{\Sigma_{v_2;a+v_1}}}(\bar{Q}|\bar{\mathbb{P}}) \right] = \mathbb{E}^Q \left[h_{\mathscr{F}_{\Sigma_{v_2;a}}}(\hat{Q}|\hat{\mathbb{P}}) \right] = H(v_2; a, Q)$. □

Assume that $v, a \in \mathbb{R}^d$ are fixed. For $\bar{\omega} \in \Omega$, let us define the measure $\delta_\omega \otimes_{v;a} Q$ in the following way: For every $k, l \in \mathbb{N}$ and any sets $A_1, \dots, A_k \in \mathscr{F}_{\Sigma^-_{v;a}}$, $B_1, \dots, B_l \in \mathscr{F}_{\left(\Sigma^-_{v;a}\right)^c}$, and any natural numbers $a_1, \dots, a_k, b_1, \dots, b_l$

$$\begin{aligned} & \delta_{\bar{\omega}} \otimes_{v;a} Q(\omega(A_1) = a_1, \dots, \omega(A_k) = a_k, \omega(B_1) = b_1, \dots, \omega(B_k) = b_k) \\ = \; & 1_{\bar{\omega}(A_1)=a_1} \cdots 1_{\bar{\omega}(A_k)=a_k} \cdot Q(\omega(B_1) = b_1, \omega(B_l) = b_l). \end{aligned}$$

Theorem 2.2.3. *Let $\mathbb{P}_\omega = \delta_\omega \oplus_{v;a} \hat{\mathbb{P}}$. Then*

$$H(v; a, Q) = \sup_{\Phi \in \mathscr{F}_{\Sigma^-_{v;a+v}}} \left[\mathbb{E}^Q(\Phi) - \mathbb{E}^Q \left(\log \mathbb{E}^{\mathbb{P}_\omega} \left(e^\Phi \right) \right) \right].$$

Proof. Take any $\Phi \in \mathscr{F}_{\Sigma^-_{v;a+v}}$. Then

$$\begin{aligned} \mathbb{E}^Q(\Phi) &= \mathbb{E}^Q \left[\mathbb{E}^{\hat{Q}}(\Phi) \right] \le \mathbb{E}^Q \left[\log \mathbb{E}^{\mathbb{P}_\omega} \left(e^\Phi \right) + h_{\mathscr{F}_{\Sigma^-_{v;a+v}}}(\hat{Q}|\mathbb{P}_\omega) \right] \\ &= \mathbb{E}^Q \left[h_{\mathscr{F}_{\Sigma_{v;a+v}}}(\hat{Q}|\hat{\mathbb{P}}) \right] + \mathbb{E}^Q \left[\log \mathbb{E}^{\mathbb{P}_\omega} \left(e^\Phi \right) \right]. \end{aligned}$$

This implies that $H(v; a, Q) \ge \sup_{\Phi \in \mathscr{F}_{\Sigma^-_{v;a+v}}} \left[\mathbb{E}^Q(\Phi) - \mathbb{E}^Q \left(\log \mathbb{E}^{\mathbb{P}_\omega} \left(e^\Phi \right) \right) \right]$.

To prove the other inequality assume that the supremum on the right-hand side is equal to l. Let $\bar{Q}$ denote the measure $\int \mathbb{P}_\omega \, dQ(\omega)$. By Jensen's inequality we have

$$\sup_{\Phi \in \mathscr{F}_{\Sigma^-_{v;a+v}}} \left[\mathbb{E}^Q(\Phi) - \log \mathbb{E}^{\bar{Q}} \left(e^{\Phi} \right) \right] \leq l,$$

which implies that $h_{\mathscr{F}_{\Sigma^-_{v;a+v}}}(Q|\bar{Q}) \leq l$. According to the theorem 2.2.1 we have

$$h_{\mathscr{F}_{\Sigma^-_{v;a+v}}}(Q|\bar{Q}) = h_{\mathscr{F}_{\Sigma^-_{v;a}}}(Q|\bar{Q}) + \mathbb{E}^Q \left[h_{\mathscr{F}_{\Sigma^-_{v;a+v}}}(\hat{Q}|\hat{\bar{Q}}) \right] \leq l.$$

Since $Q = \hat{Q}$ on $\mathscr{F}_{\Sigma^-_{v;a}}$ we have that $h_{\mathscr{F}_{\Sigma^-_{v;a}}}(Q|\bar{Q}) = 0$. We also have $\hat{\bar{Q}} = \mathbb{P}_\omega$. We can now conclude that $\mathbb{E}^Q \left[h_{\mathscr{F}_{\Sigma^-_{v;a+v}}}(\hat{Q}|\hat{\bar{Q}}) \right] = \mathbb{E}^Q \left[h_{\mathscr{F}_{\Sigma_{v;a}}}(\hat{Q}|\hat{\mathbb{P}}) \right] = H(v;a,Q)$. This completes the proof. □

Define the following quantity:

$$\bar{H}(v;a,Q) \quad = \quad \sup_{\Phi \in \mathscr{F}_{\Sigma_{v;a}}} \left[\mathbb{E}^Q[\Phi] - \mathbb{E}^Q \left[\log E^{\mathbb{P}_\omega} \left(e^{\Phi} \right) \right] \right].$$

Theorem 2.2.4. *The following relation holds:*

$$\lim_{t \to +\infty} \frac{\bar{H}(tv;a,Q)}{t} = H(v;a,Q).$$

Proof. The proof is the same as the proof of the theorem 3.6 from [11]. □

Theorem 2.2.5. *Let* $\mathscr{F}'_{v;a} = \left\{ \Phi \in \mathscr{B}\left(\mathscr{F}_{\Sigma_{v;a}} \right) : \mathbb{E}^{\mathbb{P}} \left(e^{\Phi} \right) \leq 1 \right\}$. *Then*

$$\bar{H}(v;a,Q) = \sup_{\Phi \in \mathscr{F}'_{v;a}} \mathbb{E}^Q(\Phi).$$

Proof. This follows immediately from (2.2.3). □

2.2.2 Large Deviations for Poisson Field

Let Ω be the set of all point measures on $\mathbb{R}^d$. Fix $a, v \in \mathbb{R}^d$. For some subset $K \in \mathbb{R}^d$, let $\omega_K = \operatorname{supp} \omega \cap K$ and let $\mathscr{F}_K$ be the set of functions that depend on ω_K only. Let $\mathscr{F}_t = \mathscr{F}_{\Sigma_{tv;0}}$. Let $\mathscr{F}'_t = \{\Phi \in \mathscr{F}_t : \mathbb{E}(\exp(\Phi)) \leq 1\}$.

For every $\omega \in \Omega$ let $\omega^{v;a}$ be the process that is periodization of ω from the strip $\Sigma_{v;a}$ to the entire $\mathbb{R}^d$. Define $R_{v;a,\omega}(A) = \frac{1}{\|v\|} \int_0^{\|v\|} \delta_{\tau^{-1}_{a-v+z\frac{v}{\|v\|}} \omega}(A) \, dz$. The measure $R_{v;a,\omega}$ is not stationary on

Ω with respect to τ_{tv}^{-1}, so we define the modified measure $\hat{R}_{v;a,\omega}(A) = \frac{1}{\|v\|}\int_0^{\|v\|} \delta_{\tau^{-1}_{a-v+z\frac{v}{\|v\|}}\omega^{v;a}}(A)\,dz$. The measure $\hat{R}_{v;a,\omega}$ is stationary since

$$\begin{aligned}\hat{R}_{v;a,\omega}(\tau_{tv}A) &= \frac{1}{\|v\|}\int_0^{\|v\|} \delta_{\tau^{-1}_{a-v+z\frac{v}{\|v\|}}\omega^{v;a}}(\tau_{tv}A)\,dz \\ &= \frac{1}{\|v\|}\int_0^{\|v\|} \delta_{\tau^{-1}_{a-v+(z+t\|v\|)\frac{v}{\|v\|}}\omega^{v;a}}(A)\,dz = \hat{R}_{v;a,\omega}(A).\end{aligned}$$

Let $\mathscr{M}(\Omega)$ be the set of all probability measures on Ω and let $\mathscr{M}_S(\Omega)$ be the space of stationary measures on Ω. Then we have $\hat{R}_{v;a,\omega} \in \mathscr{M}_S(\Omega)$ for every ω and $v, a \in \mathbb{R}^d$.

If a function Φ is continuous and bounded then we have the following relations:

$$\begin{aligned}\int \Phi\, dR_{v;a,\omega} &= \frac{1}{\|v\|}\int_0^{\|v\|} \Phi(\tau^{-1}_{a-v+z\frac{v}{\|v\|}}\omega)\,dz, \text{ and} \\ \int \Phi\, d\hat{R}_{v;a,\omega} &= \frac{1}{\|v\|}\int_0^{\|v\|} \Phi(\tau^{-1}_{a-v+z\frac{v}{\|v\|}}\omega^{v;a})\,dz.\end{aligned}$$

The sets $\tau_{a-v+z\frac{v}{\|v\|}}(\omega)\cap\Sigma_{tv;0}$ and $\tau_{a-v+z\frac{v}{\|v\|}}(\omega^{v;a})\cap\Sigma_{tv,0}$ are the same for each $z \in (0,\|v\|)\setminus(\|v\|-t,v)$. Therefore $\left|\int \Phi\, dR_{v;a,\omega} - \int \Phi\, d\hat{R}_{v;a,\omega}\right| \le 2\frac{t}{l}\|\Phi\|_\infty$. Hence for every $\varepsilon > 0$ we have

$$\begin{aligned}&\limsup_{l\to\infty}\frac{1}{\|lv\|}\log\mathbb{P}\left(\int \Phi\, dR_{lv;a,\omega} < \tau\right) \\ \le\ &\limsup_{l\to\infty}\frac{1}{\|lv\|}\log\mathbb{P}\left(\int \Phi\, d\hat{R}_{lv;a,\omega} < \tau+\varepsilon\right).\end{aligned}$$

Theorem 2.2.6. *If $G \subseteq \mathscr{M}_S(\Omega)$ is a compact set then*

$$\limsup_{l\to\infty}\frac{1}{l}\log\mathbb{P}\left(\hat{R}_{lv;a,\omega}\in G\right) \le -\inf_{Q\in G}\left\{\sup_t \sup_{\Phi\in\mathscr{F}'_t}\int \frac{1}{t}\Phi\, dQ\right\}.$$

Proof. Take arbitrary $\Phi \in \mathscr{F}'_t$, and assume that $\|v\| = 1$. Then

$$\begin{aligned}\mathbb{P}(\hat{R}_{lv;a,\omega}\in G) &\le \mathbb{P}\left(\frac{l}{t}\int \Phi d\hat{R}_{lv;a,\omega} \ge \frac{l}{t}\inf_{Q\in G}\int \Phi\, dQ\right) \\ &\le e^{-\frac{l}{t}\inf_{Q\in G}\int \Phi\, dQ} \\ &\quad \cdot\mathbb{E}\left(e^{\frac{l}{t}\int \Phi\, dR_{lv;a,\omega}}\cdot e^{\frac{l}{t}\int \Phi\, d(\hat{R}_{lv;a,\omega}-R_{lv;a,\omega})}\right) \\ &\le e^{-\frac{l}{t}\inf_{Q\in G}\int \Phi\, dQ}\cdot\mathbb{E}\left(e^{\frac{l}{t}\int \Phi\, dR_{lv;a,\omega}}\right)\cdot e^{c\|\Phi\|}.\end{aligned}$$

Now we have

$$\limsup \frac{1}{l} \log \mathbb{P}(\hat{R}_{lv;a,\omega} \in G) \leq -\frac{1}{t} \inf_{Q \in G} \int \Phi \, dQ + \limsup \frac{1}{l} \log \mathbb{E}\left(e^{\frac{l}{t} \int \Phi \, dR_{lv;a,\omega}} \right).$$

Assume that $l = nt + \sigma$, for some $0 \leq \sigma < t$ and $n \in \mathbb{N}$. We can rewrite

$$\begin{aligned} \frac{l}{t} \int \Phi \, dR_{lv;a,\omega} &= \frac{1}{t} \int_0^l \Phi(\tau^{-1}_{a-v+zv}\omega) \, dz \\ &= \frac{1}{t} \sum_{k \in \mathbb{Z}_n} \int_{kt}^{(k+1)t} \Phi(\tau^{-1}_{a-v+zv}\omega) \, dz + \frac{1}{t} \int_B \Phi(\tau^{-1}_{a-v+zv}\omega) \, dz \\ &= \frac{1}{t} \int_0^t \sum_{k=1}^{n} \Phi(\tau^{-1}_{a-v+ktv+zv}\omega) \, dz + \frac{1}{t} \int_B \Phi(\tau^{-1}_{a-v+zv}\omega) \, dz, \end{aligned}$$

where B is defined as $B = \{z : nt \leq z < l\}$. Now we will use the Jensen's inequality and the bound $\exp(\frac{1}{t} \int_B \Phi(\tau^{-1}_{a-v+zv}\omega \, dz)) \leq c$ to get

$$\begin{aligned} \mathbb{E}\left(e^{\frac{l}{t} \int \Phi \, dR_{lv;a,\omega}} \right) &\leq c \mathbb{E}\left(\frac{1}{t} \int_0^t e^{\sum_{k=1}^{n} \Phi(\tau^{-1}_{a-v+ktv+zv}\omega)} \, dz \right) \\ &= c \frac{1}{t} \int_0^t \mathbb{E}\left(e^{\sum_{k=1}^{n} \Phi(\tau^{-1}_{a-v+ktv+zv}\omega)} \right) dz. \end{aligned}$$

The independence implies

$$\mathbb{E}\left(e^{\frac{l}{t} \int \Phi \, dR_{lv;a,\omega}} \right) = c \cdot \frac{1}{t} \int_0^t \left[\mathbb{E}\left(e^{\Phi(\tau^{-1}_{a-v+ktv+zv}\omega)} \right) \right]^n dz \leq c.$$

The last inequality follows from $\Phi \in \mathscr{F}'_t$. Hence, this term will dissapear after taking the logarithm and dividing by l. Therefore

$$\limsup \frac{1}{l} \log \mathbb{P}(\hat{R}_{lv;a,\omega} \in G) \leq -\sup_t \sup_{\Phi \in \mathscr{F}'_t} \inf_{Q \in G} \int \frac{1}{t} \Phi \, dQ.$$

To change the order of sup and inf, let us assume for the beginning that G is compact.

For every $\varepsilon > 0$ let $\bigcup_{j=1}^{J} U_j$ be the finite open cover of G with balls $U_j = B(Q_j, \varepsilon)$.

Then we have

$$\begin{aligned}\limsup \frac{1}{l}\log \mathbb{P}\left(\hat{R}_{lv;a,\omega}\in G\right) &= \limsup \frac{1}{l}\log \mathbb{P}\left(\hat{R}_{lv;a,\omega}\in \bigcup_j U_j\right)\\ &= \limsup \frac{1}{l}\log \mathbb{P}\left(\bigcup_j (\hat{R}_{l,\omega}\in U_j)\right)\\ &\leq \limsup \frac{1}{l}\log \sum_j \mathbb{P}\left(\hat{R}_{lv;a,\omega}\in U_j\right)\\ &\leq \limsup \frac{1}{l}\log \left[J\cdot \max_{j\leq J}\mathbb{P}\left(\hat{R}_{lv;a,\omega}\in U_j\right)\right]\\ &= \max_{j\leq J}\limsup \frac{1}{l}\log \mathbb{P}\left(\hat{R}_{lv;a,\omega}\in U_j\right)\\ &\leq -\min_{j\leq J}\sup_t \sup_{\Phi\in\mathscr{F}_t'}\inf_{Q\in U_j}\int \frac{1}{t}\Phi\, dQ.\end{aligned}$$

Because of the continuity of the linear functional $Q\mapsto \int \Phi\, dQ$ we have that

$$\limsup \frac{1}{l}\log \mathbb{P}\left(\hat{R}_{lv;a,\omega}\in G\right)\leq -\inf_{Q\in G}\left\{\sup_t \frac{1}{t}\sup_{\Phi\in\mathscr{F}_t'}\int \Phi\, dQ\right\}.$$

This completes the proof. □

In order to establish the upper large deviations for the environment we need to prove the previous theorem for the case of closed sets.

Topology on $\mathscr{M}(\Omega)$ and $\mathscr{M}_S(\Omega)$

The sequence ω^n converges to ω in vague topology ($\omega^n,\omega\in\Omega$) if for every compact set $K\subseteq\mathbb{R}^d$ we have $\omega^n(K)\to\omega(K)$. A set $A\subseteq\Omega$ is precompact if there exist a sequence l_n of real numbers such that $\sup_{\omega\in A}\omega([-n,n]^d)\leq l_n$.

In general, if $L_k\subseteq\mathbb{R}^d$ are compact sets whose union is Ω and $l_k\uparrow\infty$ the set of real numbers then the set $M_{(L_k),(l_k)}=\{\omega\in\Omega:\forall k\ \omega(L_k)\leq l_k\}$ is compact subset of Ω. For a sequence $m_n\uparrow\infty$ of real numbers and a sequence (L_k^n,l_k^n) of pairs of compact sets and real numbers such that as $k\to\infty$ $l_k^n\to\infty$ for each n, define

$$\mathscr{L}=\left\{Q\in\mathscr{M}(\Omega):\forall n\ Q\left(M^C_{(L_k^n),(l_k^n)}\right)\leq\frac{1}{m_n}\right\}.$$

According to the Prohorov's theorem, the set $\mathscr{L}$ is precompact in $\mathscr{M}(\Omega)$. This means that the set $\mathscr{L}\cap\mathscr{M}_S(\Omega)$ is precompact in $\mathscr{M}_S(\Omega)$.

We will frequently use that for every integer α

$$\mathbb{P}(\omega(L) \geq \alpha) \leq \sum_{k=\alpha}^{\infty} e^{-|L|} \frac{|L|^k}{k!} \leq \frac{|L|^\alpha}{\alpha!}.$$

Let us recall the Stirling's formula:

$$\lim_{n\to\infty} \frac{e^n n!}{n^n \sqrt{n}} = \sqrt{2\pi}.$$

If $\alpha > m|L|$ for some constant m then the Stirling formula implies that

$$\mathbb{P}(\omega(L) \geq \alpha) \quad \leq \quad \frac{(e|L|)^\alpha}{\alpha^\alpha} \leq \left(\frac{e}{m}\right)^\alpha. \tag{2.2.4}$$

Theorem 2.2.7. *If $G \subseteq \mathscr{M}_S(\Omega)$ is a csed set then*

$$\limsup_{l\to\infty} \frac{1}{l} \log \mathbb{P}\left(\hat{R}_{lv;a,\omega} \in G\right) \leq -\inf_{Q\in G} \left\{ \sup_t \sup_{\Phi\in\mathscr{F}_t'} \int \frac{1}{t} \Phi \, dQ \right\}.$$

Proof. Let us find some upper bounds on $\mathbb{P}(\hat{R}_{lv;a,\omega} \in \mathscr{L}^C)$. We have that

$$\mathbb{P}(\hat{R}_{lv;a,\omega} \in \mathscr{L}^C) \leq \sum_{n=1}^{\infty} \mathbb{P}\left(\hat{R}_{lv;a,\omega}\left(M^C_{(L^n_k),(l^n_k)}\right) > \frac{1}{m_n}\right).$$

Therefore

$$\begin{aligned} & \hat{R}_{lv;a,\omega}\left(M^C_{(L^n_k),(l^n_k)}\right) \\ = \quad & \frac{1}{l}\int_0^l \delta_{\tau^{-1}_{a-lv+zv}\omega^{lv;a}}(M^C_{(L^n_k),(l^n_k)})\, dz \\ = \quad & \frac{1}{l}\int_0^l 1\left\{\bigcup_{k=1}^{\infty}\left\{\tau^{-1}_{a-lv+zv}\omega^{lv;a}(L^n_k) > l^n_k\right\}\right\} dz. \end{aligned}$$

Let $k^n(l) = \min\{k : l^n_k \geq 4l\}$. Let d_k be any sequence such that $\sum_{k=1}^{\infty} d_k = 1$. Now we can choose $(L^n_k), (l^n_k)$ in such a way that $m_n(1/\sqrt{l^n_k})^{l^n_k} < \exp(-4m_n l^n_k)$ and for each $k > k^n(l)$ we have

$$\{\omega : \omega^{lv;a}(L^n_k) > l^n_k\} \quad \supseteq \quad \{\omega : \omega^{lv;a}(L^n_{k+1}) > l^n_{k+1}\}. \tag{2.2.5}$$

To show how this can be done let us first fix n and construct the sequence $(L^n_k, l^n_k)_{k=1}^{\infty}$ inductively as follows: If (L^n_k, l^n_k) is given, choose (L^n_{k+1}, l^n_{k+1}) such that (2.2.5) is satisfied for all $l \leq (k+1)^{k+1}$. Increasing l^n_{k+1} further won't ruin the previous relation so we can assume that $l^n_{k+1} > |L^n_{k+1}|^{|L^n_{k+1}|}$

as well as $l_k^n > e^{4k}$. We could also choose l_k^n a bit bigger to satisfy: $\frac{d_k l_k^n}{eCm_n|L_k^n|} \geq \sqrt{l_k^n}$ for any constant C. The last inequality implies that $l_{\log l}^n \geq 4l$ which gives that $k^n(l) \leq \log l$.

Then we have

$$\begin{aligned}
\mathbb{P}\left(\hat{R}_{lv;a,\omega}\left(M^C_{(L_k^n),(l_k^n)}\right) > \frac{1}{m_n}\right) &\leq \mathbb{P}\left(\int_0^l \sum_{k=1}^{k^n(l)} 1_{\omega^{lv;a}(\tau_{a-lv+zv}L_k^n) > l_k^n}\, dz > \frac{l}{m_n}\right) \\
&\leq \mathbb{P}\left(\bigcup_{k=1}^{k^n(l)} \left\{\int_0^l 1_{\omega^{lv;a}(\tau_{a-lv+zv}L_k^n) > l_k^n}\, dz > \frac{d_k l}{m_n}\right\}\right) \\
&\leq \sum_{k=1}^{k^n(l)} \mathbb{P}\left(\int_0^l 1_{\omega^{lv;a}(\tau_{a-lv+zv}L_k^n) > l_k^n}\, dz > \frac{d_k l}{m_n}\right).
\end{aligned}$$

We will consider two cases:

1° If $k < k^n(l)$. Then $l_k^n < 4l$. This also means that $|L_k^n|$ is much smaller than l. We can divide the strip $\Sigma_{lv;a}$ into many smaller strips of size congruent to the diameter of L_k^n. Let $\Sigma_{|L_k^n|v;z_i}$ be these small strips. Then according to the pigeon-hole principle we have:

$$\begin{aligned}
&\int_0^l 1_{\omega^{lv;a}(\tau_{a-lv+zv}L_k^n) > l_k^n}\, dz > \frac{d_k l}{m_n} \\
\Rightarrow\ & |L_k^n| \sum_i 1_{\omega^{lv;a}(\tau_{z_i}(2L_k^n)) > l_k^n} > \frac{d_k l}{m_n}.
\end{aligned}$$

However it is easy to see that the last inequality implies that $\omega([0,l] \times [0,|L_k^n|^{d-1}]) > l_k^n \frac{d_k l}{Cm_n} \cdot \frac{1}{|L_k^n|}$. Here C is some constant depending only on the dimension d. It relates to how many times the double cube is bigger than the original one and it also has the number of overlapping when the big $[0,l]$ is covered with the smaller L_k^n. Now from (2.2.4) we have that

$$\begin{aligned}
\mathbb{P}\left(\int_0^l 1_{\omega^{lv;a}(\tau_{a-lv+zv}L_k^n) > l_k^n}\, dz > \frac{d_k l}{m_n}\right) &\leq \left(\frac{eCm_n|L_k^n|}{d_k l_k^n}\right)^{\frac{l_k^n d_k l}{Cm_n|L_k^n|}} \leq \left(\frac{1}{\sqrt{l_k^n}}\right)^{\frac{l_k^n d_k l}{Cm_n|L_k^n|}} \\
&\leq \left(\frac{1}{\sqrt{l_k^n}}\right)^{\sqrt{l_k^n}\, l}
\end{aligned}$$

2° If $k = k^n(l)$ then $l_k^n \geq 4l$. Using Markov's inequality we get

$$\begin{aligned}
\mathbb{P}\left(\int_0^l 1_{\omega^{lv;a}(\tau_{a-lv+zv}L_k^n)>l_k^n}\,dz > \frac{d_k l}{m_n}\right) &\leq \frac{m_n}{d_k l}\mathbb{E}\left(\int_0^l 1_{\omega^{lv;a}(\tau_{a-lv+zv}L_k^n)>l_k^n}\,dz\right) \\
&= \frac{m_n}{d_k l}\int_0^l \mathbb{P}(\omega^{lv;a}(L_k^n) > l_k^n)\,dz \\
&\leq m_n \mathbb{P}(\omega^{lv;a}(L_k^n) > l_k^n) m_n \left(\frac{e|L_k^n|}{l_k^n}\right)^{l_k^n} \\
&\leq m_n \left(\frac{1}{\sqrt{l_k^n}}\right)^{l_k^n} \leq e^{-4m_n l},
\end{aligned}$$

because we have chosen l_k^n to be bigger than m_n.

Therefore we have

$$\begin{aligned}
\mathbb{P}\left(\hat{R}_{lv;a,\omega}\left(M^C_{(L_k^n),(l_k^n)}\right) > \frac{1}{m_n}\right) &\leq e^{-4m_n l} + \sum_{k=1}^{k^n(l)}\left(\frac{1}{\sqrt{l_k^n}}\right)^{\sqrt{l_k^n}\,l} \\
&\leq e^{-4m_n l} + \sum_{k=1}^{\infty}\left(\frac{1}{l_k^n}\right)^{\sqrt{l_k^n}\,l} \\
&\leq e^{-2m_n l}.
\end{aligned}$$

We finally get

$$\mathbb{P}\left(\hat{R}_{lv;a,\omega} \in \mathscr{L}^C\right) \leq \sum_{n=1}^{\infty} e^{-2m_n l} \leq e^{-lC_{\mathscr{L}}},$$

where $C_{\mathscr{L}}$ can be any pre-assigned constant after which we could choose m_n to be big enough. Therefore

$$\frac{1}{l}\log\mathbb{P}(\hat{R}_{lv;a,\omega} \in \mathscr{L}^C) \leq -C_{\mathscr{L}}.$$

We can make sure this thing goes to $-\infty$ and then we would have the large deviations for every closed set. □

Theorem 2.2.8. *If $N \subseteq \mathscr{M}_S(\Omega)$ is an open set then*

$$\liminf_{l\to\infty}\frac{1}{l}\log\mathbb{P}\left(\hat{R}_{lv;a,\omega} \in N\right) \geq -\inf_{Q\in N}\left\{\sup_t \sup_{\Phi\in\mathscr{F}_t'}\int \frac{1}{t}\Phi\, dQ\right\}.$$

Proof. This theorem can be now established in the same way as in [11]. It suffices to prove that for every $Q \in \mathscr{M}_S(\Omega)$ and every neighborhood N of Q we have

$$\liminf \frac{1}{l} \log \mathbb{P}(\hat{R}_{lv;a,\omega} \in N) \geq -H(v;a,Q).$$

Using the Jensen's inequality we get

$$\begin{aligned}
\log \mathbb{P}(\hat{R}_{lv;a,\omega} \in N) &= \log E^Q \left(1_{\hat{R}_{lv;a,\omega} \in N} \cdot \frac{d\mathbb{P}}{dQ}\bigg|_{\mathscr{F}_l} \right) \\
&= \log Q(\hat{R}_{lv;a,\omega} \in N) + \log E^Q \left(\frac{d\mathbb{P}}{dQ}\bigg|_{\mathscr{F}_l} \cdot 1_{\hat{R}_{lv;a,\omega} \in N} \cdot \frac{1}{Q(\hat{R}_{lv;a,\omega} \in N)} \right) \\
&\geq \log Q(\hat{R}_{lv;a,\omega} \in N) + E^Q \left(\log \frac{d\mathbb{P}}{dQ}\bigg|_{\mathscr{F}_l} \cdot 1_{\hat{R}_{lv;a,\omega} \in N} \cdot \frac{1}{Q(\hat{R}_{lv;a,\omega} \in N)} \right).
\end{aligned}$$

The first quantity on the right is finite, so after dividing by l that will go converge 0. The second quantity satisfies:

$$\begin{aligned}
& \liminf \frac{1}{l} E^Q \left(\log \frac{d\mathbb{P}}{dQ}\bigg|_{\mathscr{F}_l} \cdot 1_{\hat{R}_{lv;a,\omega} \in N} \cdot \frac{1}{Q(\hat{R}_{lv;a,\omega} \in N)} \right) \\
= & \liminf \cdot \frac{1}{Q(\hat{R}_{lv;a,\omega} \in N)} \cdot E^Q \left(\frac{1}{l} \log \frac{d\mathbb{P}}{dQ}\bigg|_{\mathscr{F}_l} \cdot 1_{\hat{R}_{lv;a,\omega} \in N} \right) \\
= & \lim_{l \to \infty} \frac{1}{Q(\hat{R}_{lv;a,\omega} \in N)} \cdot \liminf E^Q \left(\frac{1}{l} \log \frac{d\mathbb{P}}{dQ}\bigg|_{\mathscr{F}_l} \cdot 1_{\hat{R}_{lv;a,\omega} \in N} \right) \\
= & -H(v;a,Q).
\end{aligned}$$

□

2.3 Large Deviation Estimates

Assuming that the environment has the level 3 large deviations discussed in the section 2.2 we will establish the large deviations for the solutions of (2.1.1) in the case when the Hamiltonian is of a special form $H(p,x,\omega) = K(p) - V(\tau_x \omega)$.

We will start by listing the assumptions on the functions K and Ω. Then we will use a result of Rezakhanlou from [21] to get a representation of the solution as a continuous functional of $R_{\frac{x-y}{\varepsilon},\omega}$.

2.3.1 Introduction

Let $(\Omega, \mathbb{P})$ be a probability space on which we have a group $(\tau_z)_{z \in \mathbb{R}}$ of transformations. Assume that $\mathbb{P}$ is invariant under τ_z, and that the environment is ergodic and satisfies the level-3 large deviations formulated in the section 2.2.

Assume that $H(p,\tau_x\omega) = K(p) - V(\tau_x\omega)$, where $K:\mathbb{R}\to\mathbb{R}$ is strictly convex, non-negative, and satisfies $K(0)=0$. Assume that $A^+:[0,+\infty)\to[0,+\infty)$ and $A^-:[0,+\infty)\to(-\infty,0]$ are the inverses of K. Assume further that $V:\Omega\to\mathbb{R}$ is bounded, stationary, integrable, and satisfies:

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\frac{1}{n}\left|\int_0^n A^{\pm\prime}(V(\tau_z\omega))\,dz\right|\leq M\right) = -\infty, \text{ for all } M>0. \tag{2.3.1}$$

Theorem 2.3.1. *Assume that u^ε is a solution to the initial problem (2.1.1) with $\delta=0$ and $g^\varepsilon = g$ does not depend on ε. Then there exists a continuous function $\Lambda_{x,t,\delta}$ on $\mathscr{M}_S(\Omega)$ such that*

$$\begin{aligned}
\lim_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(u^\varepsilon(x,t,\omega)\geq\alpha) &\leq -\lim_{\delta\to 0}\inf\{H(Q):Q\in\mathscr{M}_S(\Omega),\Lambda_{x,t,\delta}(Q)\geq\alpha\}, \quad \textit{and}\\
\lim_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(u^\varepsilon(x,t,\omega)\leq\alpha) &\geq -\lim_{\delta\to 0}\inf\{H(Q):Q\in\mathscr{M}_S(\Omega),\Lambda_{x,t,\delta}(Q)\leq\alpha\}
\end{aligned}$$

We will use a theorem of Rezakhanlou (5.3 from [21]) to express the solution to (2.1.1) in a form convenient enough to allow us to use the contraction principle.

Let us first go over the general idea. Consider the following *cell problem*

$$\begin{aligned}
K(\nabla w(p,x,\omega)) - V(\tau_x\omega) &= \bar{H}(p),\\
w(p,x,\omega) &= p\cdot x + o(|x|), \quad \text{as } |x|\to+\infty.
\end{aligned}\tag{2.3.2}$$

In order for this to be solvable we need $\nabla w(p,x,\omega) = A^\pm(\bar{H}(p)+V(\tau_x\omega))$. For this quantity to be defined we must have $\bar{H}(p)+V(\tau_x\omega)\geq 0$, and we will have this under suitable conditions. Assume for the moment that $p>0$. In order to have $w(p,x,\omega) = p\cdot x + o(|x|)$ we need

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x A^+(\bar{H}(p)+V(\tau_z\omega))\,dz = p.$$

However, from the ergodic theorem we have that the previous limit is equal to $\mathbb{E}(A^+(\bar{H}(p)+V(\cdot)))$. Let us denote $\varphi^\pm(\lambda) = E(A^\pm(\lambda+V(\cdot)))$. The function φ is fully determined for $\lambda\notin[\lambda^-,\lambda^+]$, where λ^- and λ^+ are two real numbers. Denote by $\psi^\pm$ the inverses of φ. The solution w is now given by $w(p,x,\omega) = \int_0^x A^+(\psi(p)+V(\tau_z\omega))\,dz$. The idea is that the existence of such a solution for sufficiently large class of p's would allow us to make the following solutions $z(p,x,t,\omega) = w(p,x,\omega) - t\bar{H}(p)$ to the initial value problem $z_t(p,x,t,\omega)+H(z_x(p,x,t,\omega),\tau_x\omega) = 0$, $z(p,x,0,\omega) = w(p,x,\omega)$. Then, "inverting" the variational formula enables us to retrieve valuable information on S.

2.3.2 Large Deviations

Now we present the rigorous argument. We start by defining:

$$\bar{H}(p) = \begin{cases} \psi^-(p), & \text{if } p\leq\varphi^-(0)\\ \psi^+(p), & \text{if } p\geq\varphi^+(0)\\ 0, & \text{if } \phi^-(0)\leq p\leq\varphi^+(0).\end{cases}$$

Denote $p^{\pm} = \varphi^{\pm}(0)$, and let $\bar{L}$ denote the convex conjugate of $\bar{H}$. Furthermore, let us define

$$\begin{aligned} v^{\pm}(x,\lambda,\omega) &= \int_0^x A^{\pm}(\lambda + V(y))\,dy,\ \lambda \geq 0 \\ w(p,x,\omega) &= \begin{cases} v^{+}(x,\bar{H}(p),\omega), & \text{if } p \geq p^{+}, \\ v^{-}(x,\bar{H}(p),\omega), & \text{if } p \leq p^{-}. \end{cases} \end{aligned} \tag{2.3.3}$$

Also set

$$\Gamma^{\pm}(z,\lambda,\omega) = \frac{1}{K'\left(A^{\pm}(\lambda + V(\tau_z\omega))\right)}.$$

The following three lemmas are proved in [21] (lemmas 5.1-5.3):

Lemma 2.3.1. *If there exists some real λ such that $H(x,w_x(x)) = \lambda$ has a solution w, then $S(x,y,t) = \int_0^t L(x,\dot{x})\,ds = w(x) - w(y) - t\lambda$ where $x(\cdot)$ solves $\dot{x}(s) = H_p(x,w_x(x))$.*

Lemma 2.3.2. *If for some $\lambda, t > 0$ we have $\int_y^x \Gamma^{+}(z,\lambda) = t$, then*

$$S(x,y,t,\omega) = \int_y^x A^{+}(\lambda + V(\tau_z\omega))\,dz - \lambda t.$$

The analogous relation holds when Γ^{+}, A^{+} are replaced by Γ^{-}, A^{-}.

Lemma 2.3.3. *Define $z^{\pm}(x,t)$ implicitly by*

$$\int_{z^{+}(x,t)}^{x} \Gamma^{+}(z,0,\omega)\,dz = -\int_x^{z^{-}(x,t)} \Gamma^{-}(z,0,\omega)\,dz = t.$$

Then for $y \notin [z^{+}(x,t), z^{-}(x,t)]$ we have

$$S(x,y,t,\omega) = \sup_{p \notin [p^{-},p^{+}]} \left[w(p,x) - w(p,y) - t\bar{H}(p) \right]. \tag{2.3.4}$$

For each measure $Q \in \mathscr{M}(\Omega)$ and each $p \notin [p^{-}, p^{+}]$ let us denote

$$\Lambda_{p,x,y,t}(Q) = g(y) - t\bar{H}(p) + |x-y| \cdot \int A^{p}(\bar{H}(p) + V(\tilde{\omega}))\,dQ(\tilde{\omega}),$$

where $A^p = A^{+}$ for $p > p+$, and $A^p = A^{-}$ for $p < p^{-}$. Let us denote $C_\varepsilon = \left[z^{+}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right), z^{-}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right)\right]$. Assume that B is a measurable subset of $\mathbb{R}$. Our goal is to relate the probability $\mathbb{P}(u^{\varepsilon}(x,t,\omega) \in B)$ to the probability $\mathbb{P}(\Phi(\hat{R}_{\frac{x-y}{\varepsilon},\omega}))$ for some continuous functional Φ and then use the contraction principle to establish the large deviations. Let $C_\varepsilon = \left[z^{+}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right), z^{-}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon}\right)\right]$. Restricting ourselves to the sets of the form $B=[\alpha,+\infty)$ and using the equality (2.1.3) we get that for each $\delta>0$:

$$\begin{aligned}
&\mathbb{P}\left(\varepsilon u^{\varepsilon}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon},\omega\right)\geq\alpha\right)\\
=\;&\mathbb{P}\left(\inf_{y}\left\{g(y)+\varepsilon S\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{t}{\varepsilon},\omega\right)\right\}\geq\alpha\right)\\
\leq\;&\mathbb{P}\left(\inf_{y\notin B_{\delta}(x)}\left\{g(y)+\varepsilon S\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{t}{\varepsilon},\omega\right)\right\}\geq\alpha\right)\\
=\;&\mathbb{P}\left(\inf_{y\notin B_{\delta}(x)}\left\{g(y)+\varepsilon S\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{t}{\varepsilon},\omega\right)\right\}\geq\alpha,\varepsilon C_{\varepsilon}\subseteq B_{\delta}(x)\right)\\
&+\mathbb{P}\left(\inf_{y\notin B_{\delta}(x)}\left\{g(y)+\varepsilon S\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{t}{\varepsilon},\omega\right)\right\}\geq\alpha,\varepsilon C_{\varepsilon}\not\subseteq B_{\delta}(x)\right).
\end{aligned}\tag{2.3.5}$$

Here $B_{\delta}(x)=(x-\delta,x+\delta)$. We will prove that the probability $\mathbb{P}(\varepsilon C_{\varepsilon}\not\subseteq B_{\delta}(x))$ is negligible in comparison to the probability of the first summand in (2.3.5). In order to do this we will use the assumption (2.3.1). If $\varepsilon C_{\varepsilon}\not\subseteq B_{\delta}(x)$ then $x-\delta>\varepsilon z^{+}(\frac{x}{\varepsilon},\frac{t}{\varepsilon})$ or $x+\delta<\varepsilon z^{-}(\frac{x}{\varepsilon},\frac{t}{\varepsilon})$. Assume that the first of these two inequalities is taking place. From the definition of z^{+} we obtain the inequality $\int_{(x-\delta)/\varepsilon}^{x/\varepsilon}\Gamma^{+}(z,0,\omega)\,dz\leq t/\varepsilon$. Therefore

$$\frac{\varepsilon}{\delta}\int_{(x-\delta)/\varepsilon}^{x/\varepsilon}\frac{dz}{K'(A^{+}(V(\tau_z\omega)))}\leq t.$$

We know that $\frac{1}{K'\circ A^{+}}=A^{+\prime}$, hence if we assume (2.3.1) we derive the following inequality from (2.3.5):

$$\begin{aligned}
&\lim_{\varepsilon\to0}\varepsilon\log\mathbb{P}\left(\inf_{y\notin B_{\delta}(x)}\left\{g(y)+\varepsilon S\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{t}{\varepsilon},\omega\right)\right\}\geq\alpha,\varepsilon C_{\varepsilon}\subseteq B_{\delta}(x)\right)\\
\leq\;&\lim_{\varepsilon\to0}\varepsilon\log\mathbb{P}\left(\inf_{y\notin B_{\delta}(x)}\left\{g(y)+\varepsilon S\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{t}{\varepsilon},\omega\right)\right\}\geq\alpha\right)\\
\leq\;&\lim_{\varepsilon\to0}\varepsilon\log\left[\mathbb{P}\left(\inf_{y\notin B_{\delta}(x)}\left\{g(y)+\varepsilon S\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{t}{\varepsilon},\omega\right)\right\}\geq\alpha,\varepsilon C_{\varepsilon}\subseteq B_{\delta}(x)\right)\right.\\
&\left.+\mathbb{P}(\varepsilon C_{\varepsilon}\not\subseteq B_{\delta}(x)\right]\\
\leq\;&\lim_{\varepsilon\to0}\varepsilon\log\left[2\mathbb{P}\left(\inf_{y\notin B_{\delta}(x)}\left\{g(y)+\varepsilon S\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{t}{\varepsilon},\omega\right)\right\}\geq\alpha,\varepsilon C_{\varepsilon}\subseteq B_{\delta}(x)\right)\right].
\end{aligned}$$

The last inequality will follow once we prove that the given limit is not $-\infty$. Hence

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon\log\mathbb{P}\left(\inf_{y\notin B_\delta(x)}\left\{g(y)+\varepsilon S\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{t}{\varepsilon},\omega\right)\right\}\geq\alpha\right)\\
=\;&\lim_{\varepsilon\to 0}\varepsilon\log\mathbb{P}\left(\inf_{y\notin B_\delta(x)}\left\{g(y)+\varepsilon S\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{t}{\varepsilon},\omega\right)\right\}\geq\alpha,\varepsilon C_\varepsilon\subseteq B_\delta(x)\right)\\
=\;&\lim_{\varepsilon\to 0}\varepsilon\log\mathbb{P}\left(\inf_{y\notin B_\delta(x)}\sup_{p\notin[p^-,p^+]}\Lambda_{p,x,y,t}\left(R_{\frac{|x-y|}{\varepsilon},\omega}\right)\geq\alpha,\varepsilon C_\varepsilon\subseteq B_\delta(x)\right).
\end{aligned}$$

In the same way as before we use the fact that the probability $\mathbb{P}(\varepsilon C_\varepsilon\not\subseteq B_\delta(x))$ is of the smaller order of magnitude than $e^{-1/\varepsilon}$ to prove that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon\log\mathbb{P}\left(\inf_{y\notin B_\delta(x)}\sup_{p\notin[p^-,p^+]}\Lambda_{p,x,y,t}\left(R_{\frac{|x-y|}{\varepsilon},\omega}\right)\geq\alpha,\varepsilon C_\varepsilon\subseteq B_\delta(x)\right)\\
=\;&\lim_{\varepsilon\to 0}\varepsilon\log\mathbb{P}\left(\inf_{y\notin B_\delta(x)}\sup_{p\notin[p^-,p^+]}\Lambda_{p,x,y,t}\left(R_{\frac{|x-y|}{\varepsilon},\omega}\right)\geq\alpha\right).
\end{aligned}$$

For $Q\in\mathscr{M}(\Omega)$, consider the following functional:

$$\Lambda_{x,t,\delta}(Q)=\inf_{y\notin B_\delta(x)}\sup_{p\in[p^-,p^+]}\Lambda_{p,x,y,t}(Q).$$

Lemma 2.3.4. *The functional $\Lambda_{x,t,\delta}(Q)$ is continuous in Q, where $\mathscr{M}(\Omega)$ is equipped with the weak topology.*

Proof. Recall that a sequence of measures Q_n converges weakly to a measure Q if for all bounded continuous f we have $\int f\,dQ_n\to\int f\,dQ$. We first prove that $\Lambda_{x,y,t}(Q)=\sup_{p\notin[p^-,p^+]}\Lambda_{p,x,y,t}(Q)$ is continuous. Assume that Q_n weakly converges to Q, and let us prove that $\lim_{n\to\infty}\Lambda_{x,y,t}(Q_n)=\Lambda_{x,y,t}(Q)$. We immediately get that $\Lambda_{p,x,y,t}(Q_n)\to\Lambda_{p,x,y,t}(Q)$ since $\bar{H}$ and $A^{\pm}$ are continuous. For each $\varepsilon>0$ there exists $p^\varepsilon\notin[p^-,p^+]$ such that $\Lambda_{p^\varepsilon,x,y,t}(Q)>\Lambda_{x,y,t}(Q)-\varepsilon$. Now for each $n\in\mathbb{N}$ we have

$$\sup_{p\notin[p^-,p^+]}\Lambda_{p,x,y,t}(Q_n)\geq\Lambda_{p^\varepsilon,x,y,t}(Q_n)\to\Lambda_{p^\varepsilon,x,y,t}(Q)\geq\Lambda_{x,y,t}(Q)-\varepsilon$$

which gives that

$$\liminf_{n\to\infty}\Lambda_{x,y,t}(Q_n)\geq\Lambda_{x,y,t}(Q).$$

Assume now that there exists $\theta>0$, a subsequence $n_k\in\mathbb{N}$, a sequence $p_{n_k}\notin[p^-,p^+]$ such that $\Lambda_{p_{n_k},x,y,t}(Q_{n_k})\to\Lambda_{x,y,t}(Q)+\theta$ as $k\to\infty$. Let us now prove that $\Lambda_{p,x,y,t}(Q)\to-\infty$ as $p\to-\infty$ uniformly in Q. Since V is bounded we have that $\Lambda_{p,x,y,t}(Q)\leq g(y)-t\bar{H}(p)+|x-y|A^+(\bar{H}(p)+\|V\|_\infty)$. Since $\bar{H}(p)\to\infty$ as $p\to\infty$ it suffices to prove that $\lim_{h\to\infty}\frac{A^+(h+\gamma)}{h}=0$ for each $\gamma>0$.

The last limit can be established using the L'Hospital's theorem. Indeed, from $A^{+\prime} = \frac{1}{K' \circ A^+}$ we conclude that $\lim_{h\to\infty} \frac{1}{K'(A^+(h+\gamma))} = 0$ because K' and A^+ both go to $+\infty$ as $h \to \infty$. Similarly we prove that $\Lambda_{p,x,y,t}(Q) \to -\infty$ as $p \to -\infty$ uniformly in Q. Therefore we may assume that p_{n_k} belong to a bounded interval, and there is a convergent subsequence of the sequence (p_{n_k}). For simplicity let us denote that sequence by p_{n_k} as well, and let us assume that its limit is p^*. $\Lambda_{p,x,y,t}$ is continuous in p uniformly in Q. Indeed $A^p(\bar{H}(p)+V(\omega))$ is Lipschitz in p when p is bounded. Therefore

$$\Lambda_{p_{n_k},x,y,t}(Q_{n_k}) = (\Lambda_{p_{n_k},x,y,t}(Q_{n_k}) - \Lambda_{p^*,x,y,t}(Q_{n_k})) + (\Lambda_{p^*,x,y,t}(Q_{n_k}) - \Lambda_{p^*,x,y,t}(Q)) + \Lambda_{p^*,x,y,t}(Q).$$

We now take $k \to \infty$. We deduce that $\Lambda_{x,y,t}(Q) + \theta = \Lambda_{p^*,x,y,t}(Q) \leq \Lambda_{x,y,t}(Q)$ which contradicts our assumption $\theta > 0$.

In order to prove that $\Lambda_{x,t,\delta}$ is continuous in Q, assume again that Q_n weakly converges to Q. Notice that as $y \to \pm\infty$ we have for arbitrary $p_0 \notin [p^-, p^+]$:

$$\begin{aligned}\sup_p \{g(y) - t\bar{H}(p) + |x-y| \int A^p(\bar{H}(p)+V(\omega))\, dQ(\omega)\} &\geq g(y) - t\bar{H}(p_0) + |x-y| A^p(H(p_0)) \\ &\to +\infty.\end{aligned}$$

Hence $\lim_{y\to\pm\infty} \Lambda_{x,y,t}(Q) = +\infty$ uniformly in Q which yields that there is $N > 0$ such that

$$\inf_{y\notin B_\delta(x)} \Lambda_{x,y,t}(Q) = \inf_{y\notin (B_\delta(x)\cup[-N,N]^C} \Lambda_{x,y,t}(Q).$$

For each $\varepsilon > 0$ there exists y_0 such that $\Lambda_{x,y_0,t}(Q) < \Lambda_{x,t,\delta}(Q) + \varepsilon$. From the continuity of $\Lambda_{x,y_0,t}(Q)$ we conclude that $\Lambda_{x,y_0,t}(Q_n) \to \Lambda_{x,y_0,t}(Q)$ hence

$$\limsup_{n\to\infty} \Lambda_{x,t,\delta}(Q_n) \leq \Lambda_{x,t,\delta}(Q).$$

Assume now that there exists $\theta > 0$, a sequence $n_k \to \infty$ and a sequence $y_{n_k} \in [-N,N] \setminus B_\delta(x)$ such that $\Lambda_{x,y_{n_k},t}(Q_{n_k}) \to \Lambda_{x,t,\delta}(Q) - \theta$. There is a subsequence of y_{n_k} (that we will continue calling y_{n_k}) with a limit y^*. We have proved above that he function $\Lambda_{x,y,t}(Q)$ is continuous both in Q. Moreover, it is uniformly (in Q) continuous in y. The last follows from the fact that the function is Lipschitz because the supremum over p is attained in a bounded interval. This now implies that:

$$\Lambda_{x,y_{n_k},t}(Q_{n_k}) = (\Lambda_{x,y_{n_k},t}(Q_{n_k}) - \Lambda_{x,y^*,t}(Q_{n_k})) + (\Lambda_{x,y^*,t}(Q_{n_k}) - \Lambda_{x,y^*,t}(Q)) + \Lambda_{x,y^*,t}(Q).$$

The previous equality now implies that $\Lambda_{x,t,\delta}(Q) - \theta = \Lambda_{x,y^*,t}(Q) \geq \Lambda_{x,t,\delta}(Q)$ which is a contradiction. This completes the proof of the continuity of $\Lambda_{x,t,\delta}$. □

Proof of the Theorem 2.3.1. Applying the contraction principle yields the following large deviations bound:

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(u^\varepsilon(x,t,\omega) \geq \alpha) \leq -\lim_{\delta\to 0} \inf\{H(Q) : Q \in \mathscr{M}_S(\Omega), \Lambda_{x,t,\delta}(Q) \geq \alpha\}.$$

The other inequality is analogous. □

Chapter 3

Deterministic Walks in Random Environments

3.1 Introduction

Let $(\Omega,\mathbb{P})$ be a probability space. We study a random sequence X_n defined recursively as $X_{n+1}-X_n=b(X_n,\omega)$, for suitable function $b:\mathbb{Z}^d\times\Omega\to\mathbb{Z}^d$. Our aim is to establish some large deviation estimates for X_n.

Theorem 3.1.1. *Let $(\eta_z)_{z\in\mathbb{Z}^d}$ be a stationary $\mathbb{Z}^d$-valued random field that satisfy the assumptions (i)–(iii) (we refer to the assumptions from the section 2). The random variable X_n is defined as $X_0=0$, $X_{n+1}=X_n+\eta_{X_n}$. Then there exists a convex function $\Lambda:\mathbb{R}^d\to\overline{\mathbb{R}}$ such that*

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\left[e^{\lambda\cdot X_n}\right]=\Lambda(\lambda).$$

Assume that $b(x,\omega)=\eta_x$, where η is a random field on $\mathbb{Z}^d$. η itself could be understood as the random environment in which the walk occurs. When the particular realization of the environment is fixed, the walk becomes deterministic. One of the main characteristics of this walk is that once the loop occurs, the walk will start repeating the loop forever.

Let us make some connections of this model with the one of the random walk in a random environment. The random walk in a random environment could be defined as $X_{n+1}-X_n=b(X_n,\omega,\pi)$. Here $b:\mathbb{Z}^d\times\Omega\times\Pi\to\mathbb{Z}^d$ is a random variable, and $(\Omega,\mathbb{P})$, (Π,P) are probability spaces. In this model, $(\Omega,\mathbb{P})$ is the environment, and for each fixed $\omega\in\Omega$, the walk X_n could be understood as a random walk on probability space (Π,P). Recent works [30] and [20] have established the quenched and annealed large deviations for the random walk in a random environment under certain non-degeneracy assumptions on (Π,P). The articles [25] and [32] found variational formulas for the rate functions and some connections between the two rate functions. The model we are studying is related to the annealed (averaged) case stdudied in the mentioned papers. Here, the probability space (Π,P) is very degenerate.

The motivation for studying deterministic walk in random environment comes from the importance of understanding the behavior of solutions of certain ODEs that appear in [16] and [24].

Many of the arguments here involve the consideration of the position of the particle at the time of the first self-intersection. Other interesting results regarding self-intersection times of random walks can be found in [1] and [2].

We are assuming that the environment has finite-range dependence. The special case is the iid environment where many of the presented arguments can be simplified. We are not assuming that the walk is nearest–neighbor. We do assume however, certain *ellipticity* conditions on the environment: There is a *nice* set of vectors such that using only these vectors the walk can move in any direction. Our ellipticity condition is the assumption that the probability that the walk at any position of the environment takes any particular vector from the *nice* set is uniformly positive.

The organization of the article is as follows. In the next section we state the assumptions we impose to the model and state the theorem 3.2.1 that will be the main focus of our study. In order to prove it we will need some deterministic lemmas that establish the uniform bounds on the probabilities that the walk can go virtually in any direction. These statements are stated and proved in the third section.

The main idea for the proof is the adaptation of the subadditive argument. However, we need some work to find a quantity that happens to be subadditive. This quantity is found in the section 4 where we prove that it has the same exponential rate of decay as our large deviation probabilities. After that we are ready for the proofs of the main theorems which are presented in the section 5. Section 6 contains an easy consequence, i.e. the law of large numbers. The law of large numbers with limiting velocity 0 for the deterministic walk in random environment is not a surprising fact. It is intuitive that the loop will occur, the expected time of the occurrence of the loop is finite, hence the walk can't go to far.

3.2 Definitions and Assumptions

Definition 3.2.1. *The set of vectors* $\{u_1,\dots,u_m\}\subseteq\mathbb{Z}^d$ *is called* nice *if for every* $l\in\mathbb{R}^d$ *there exists* $i\in\{1,2,\dots,m\}$ *such that* $l\cdot u_i>0$.

Let $(\eta_z)_{z\in\mathbb{Z}^d}$ be a stationary $\mathbb{Z}^d$-valued random field that satisfies the following conditions:

(i) There exists a positive real number L such that $|\eta_z|\leq L$ for all $z\in\mathbb{Z}^d$.

(ii) There exists a real number M such that (η_{z_1},η_{z_2}) are independent whenever $|z_1-z_2|>M$.

(iii) There exist a nice set of vectors $\{u_1,\dots,u_m\}\in\mathbb{Z}^d$ and a constant $c>0$ such that $\mathbb{P}(\eta_z=u_i|\mathscr{F}_z)>c$ for all $i\in\{1,2,\dots,m\}$, where $\mathscr{F}_z$ is a sigma-algebra generated by all η_w for $w\in\mathbb{Z}^d$ such that $0<|w-z|\leq M$.

The last assumption implies the existence of a loop in any half-space with a positive probability (see theorem 3.3.1). It also implies that there exists a constant $c>0$ such that $\mathbb{P}(\eta_z\cdot l>0|\mathscr{F}_z)>c$ for every z.

A special case is the iid environment when in condition (ii) we require $M < 1$. The condition (iii) is then replaced by $\mathbb{P}(\eta_z = u_i) > c$ for all i. Many of the arguments would become simpler and/or less technical if we assumed that the environment is iid.

The random variable X_n is defined recursively as $X_0 = 0$ and:

$$X_{n+1} = X_n + \eta_{X_n}.$$

We will use the following equivalent interpretation of X_n. X_n behaves like a random walk until the first self intersection. The increments of the random walk are sampled at every step according to the law of the random field η_z. After the self intersection occurs, the walk becomes deterministic and repeats the loop.

Here is the precise definition of the walk X_n that we will use. Let ξ_n be a $\mathbb{Z}^d$-valued random sequence defined recursively: $\xi_0 = 0$, and $\xi_n = \eta_{\xi_0+\cdots+\xi_{n-1}}$. Let us define $Y_n = \sum_{k=1}^n \xi_k$ and

$$\tau = \inf\{n : Y_n \in \{Y_1, \dots, Y_{n-1}\}\}.$$

Let $\theta \leq \tau$ be the smallest integer such that $Y_\tau = Y_\theta$. We define the walk X_n using the formula:

$$X_n = \begin{cases} Y_n, & n \leq \tau, \\ Y_{\theta+v}, & n - \tau = u(\tau - \theta) + v, 0 \leq v < \theta - \tau. \end{cases}$$

Let $l \in \mathbb{R}^d$ be a unit vector. Define $T_m^l = \inf\{n : X_n \cdot l \geq m\}$. Denote by Z_x^l the hyperplane through $x \in \mathbb{Z}^d$ orthogonal to the vector l. Let us further denote by H_x^l the half-space through x determined by the vector l as

$$H_x^l = \mathbb{Z}^d \cap \left\{x + v : v \in \mathbb{R}^d, l \cdot v \geq 0\right\}.$$

Our goal is to prove the large deviations for X_n (see the theorem 3.5.2). We will be able to use the Ellis-Gartner theorem to get some further bounds once we establish the following result:

Theorem 3.2.1. *Let X_n be the random walk defined as above. Assume that the random environment satisfies the conditions (i)–(iii). For each unit vector $l \in \mathbb{R}^d$ there exists a concave function $\phi^l : \mathbb{R}_+ \to \bar{\mathbb{R}}$ such that for all $k \in \mathbb{R}_+$:*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq nk) = \phi^l(k). \tag{3.2.1}$$

Remark. Notice that $\phi^l(k) = \phi^{tl}(tk)$ for all $t \in \mathbb{R}_+$. Therefore $\phi^l(k) = \Phi\left(\frac{1}{k} l\right)$ for a suitable function $\Phi : \mathbb{R}^d \to \bar{\mathbb{R}}$.

3.3 Existence of a Loop

In this section we prove that the previously defined random walk will have a loop in each half-space with a positive probability. This fact will be a consequence of the following elementary lemma. The lemma states that there exists a loop consisting entirely of vectors from a nice set.

Lemma 3.3.1. *Let $\{u_1, \dots, u_m\}$ be a nice set of non-zero vectors. There exist non-negative integers $q_1, q_2, \dots, q_m$ not all equal to 0 such that $q_1u_1 + \cdots + q_mu_m = 0$.*

Proof. We will prove the statement using the induction on the dimension d. The statement is easy to prove for $d = 1$ and $d = 2$. We may assume that $\{u_1, \dots, u_m\}$ is a minimal nice set, i.e. there is no proper nice subset of $\{u_1, \dots, u_m\}$. If not, take the proper nice subset and repeat the argument. Let us fix the vector u_m, and let $v_i = u_i - \frac{u_i \cdot u_m}{|u_m|^2} u_m$ for $i = 1, \dots, m-1$. All vectors $v_1, \dots, v_{m-1}$ have rational coordinates. Let r be the common denominator of those fractions and consider the lattice D of size $1/r$ in the vector space determined by the span W of $v_1, \dots, v_{m-1}$. Let us prove that the set $\{v_1, \dots, v_{m-1}\}$ is nice in W. Let $\tilde{l} \in W$ be a vector with real coordinates. There exists $i \in \{1, 2, \dots, m\}$ such that $u_i \cdot \tilde{l} > 0$. Since $\tilde{l} \in W$ we immediately have that $u_m \cdot \tilde{l} = 0$ and $u_i \cdot \tilde{l} = v_i \cdot \tilde{l}$ hence $v_i \cdot \tilde{l} > 0$. This implies that $\{v_1, \dots, v_{m-1}\}$ is a nice set of vectors in W. According to the inductional hypothesis there are non-negative integers $q'_1, \dots, q'_{m-1}$ such that $q'_1 v_1 + \cdots + q'_{m-1} v_{m-1} = 0$. We now have

$$|u_m|^2 (q'_1 u_1 + \cdots + q'_{m-1} u_{m-1}) = \left(q'_1 u_1 \cdot u_m + \cdots + q'_{m-1} u_{m-1} \cdot u_m\right) u_m.$$

Let us now prove that $q'_1 u_1 \cdot u_m + \cdots + q'_{m-1} u_{m-1} \cdot u_m \leq 0$. Assume the contrary, that this number were greater than 0. Since $\{u_1, \dots, u_{m-1}\}$ is not a nice set (due to our minimality assumption for $\{u_1, \dots, u_m\}$) there exists a vector $l \in \mathbb{R}^d$ such that $l \cdot u_m > 0$ but $l \cdot u_k \leq 0$ for each $k \in \{1, 2, \dots, m-1\}$. This gives that

$$0 \quad \geq \quad l \cdot (q'_1 u_1 + \cdots + q'_{m-1} u_{m-1}) |u_m|^2 = \left(q'_1 u_1 \cdot u_m + \cdots + q'_{m-1} u_{m-1} \cdot u_m\right) u_m \cdot l > 0,$$

a contradiction. Therefore $q'_1 u_1 \cdot u_m + \cdots + q'_{m-1} u_{m-1} \cdot u_m \leq 0$. We can now choose $q_i = |u_m|^2 q'_i$, $i = 1, 2, \dots, m-1$, and $q_m = -(q'_1 u_1 \cdot u_m + \cdots + q'_{m-1} u_{m-1} \cdot u_m)$ to obtain $q_1 u_1 + \cdots + q_m u_m = 0$. This completes the proof of the lemma. □

The following theorem says that in each half-space H_x^l a walk X_n starting from x can have a loop in H_x^l with a probability that is strictly greater than 0. Because of the stationarity it suffices to prove this for half-spaces through the origin.

Theorem 3.3.1. *There exist constants $m \in \mathbb{N}$, $c_1 \in \mathbb{R}_+$ such that: For each unit vector $l \in \mathbb{R}^d$ there exist an integer $s \leq m$ and a sequence $x_0 = 0, x_1, x_2, \dots, x_s \in \mathbb{Z}^d$ such that:*

(i) $\sum_{i=0}^{p} x_i \cdot l \geq 0$ *for all* $0 \leq p \leq s$,

(ii) $\sum_{i=0}^{p} x_i \neq \sum_{i=0}^{q} x_i$ *for all* $0 \leq p < q \leq s-1$,

(iii) $\sum_{i=0}^{s} x_i = 0$*, and*

(iv) For $y_i = x_0 + \cdots + x_{i-1}$*, let us denote by* $\mathscr{F}_{y_1,\dots,y_s}$ *the* σ*-algebra generated by all random variables* η_z *for* $z \in \mathbb{Z}^d \setminus \{y_1,\dots,y_s\}$ *such that* $\min_{1\leq i\leq s} |z-y_i| \leq M$*. Then:*

$$\mathbb{P}(\xi_1 = x_1, \xi_2 = x_2, \dots, \xi_s = x_s | \mathscr{F}_{y_1,\dots,y_s}) \geq c_1.$$

Proof. Let us prove that there exist vectors $x_1,\dots,x_s \in \{u_1,\dots,u_m\}$ for which (i)–(iii) are satisfied. Then (iv) will be satisfied as well. To see that let us denote by $\mathscr{G}_{y_s}^{y_1,\dots,y_{s-1}}$ the sigma algebra generated by all random variables η_z for $z \in \mathbb{Z}^d \setminus \{y_s\}$ such that $\min_{1\leq i\leq s} |z-y_i| \leq M$.

$$\begin{aligned} & \mathbb{P}(\xi_1 = x_1, \xi_2 = x_2, \dots, \xi_s = x_s | \mathscr{F}_{y_1,\dots,y_s}) \\ = \; & \mathbb{E}\left(\mathbb{E}\left(1(\xi_1 = x_1)\cdots 1(\xi_s = x_s) | \mathscr{G}_{y_s}^{y_1,\dots,y_{s-1}}\right)\Big| \mathscr{F}_{y_1,\dots,y_s}\right) \\ = \; & \mathbb{E}\left(1(\xi_1 = x_1)\cdots 1(\xi_{s-1} = x_{s-1}) \cdot \mathbb{E}\left(1(\xi_s = x_s) | \mathscr{G}_{y_s}^{y_1,\dots,y_{s-1}}\right)\Big| \mathscr{F}_{y_1,\dots,y_s}\right) \\ > \; & c \cdot \mathbb{P}(\xi_1 = x_1, \dots, \xi_{s-1} = x_{s-1} | \mathscr{F}_{y_1,\dots,y_s}) \\ = \; & c \cdot \mathbb{P}(\mathbb{P}(\xi_1 = x_1, \dots, \xi_{s-1} = x_{s-1} | \mathscr{F}_{y_1,\dots,y_{s-1}}) | \mathscr{F}_{y_1,\dots,y_s}). \end{aligned}$$

Now we can continue by induction to obtain that

$$\mathbb{P}(\xi_1 = x_1, \dots, \xi_s = x_s | \mathscr{F}_{y_1,\dots,y_s}) > c^s.$$

Using the lemma 3.3.1 we have that there exists a sequence $z_1,\dots,z_s \in \{u_1,\dots,u_m\}$ such that $z_1 + \cdots + z_s = 0$. Assume that this sequence is the sequence of minimal length. Let us choose the index π such that that $(z_1 + \cdots + z_\pi) \cdot l$ is minimal (of all $j = 1,2,\dots,s$). Then $x_1 = z_{\pi+1}$, $z_2 = y_{\pi+2}$, $\dots$, $z_s = y_{\pi+s}$ (indeces are modulo s) satisfy $z_1 + z_2 + \cdots + z_s = 0$, and

$$(x_1 + x_2 + \cdots + x_j) \cdot l \geq 0 \quad \text{for all } j = 1,2,\dots,s.$$

Let us prove the last inequality. Assume the contrary, that $(x_1 + \cdots + x_j) \cdot l < 0$. Assume first that $\pi + j \leq s$. Then $(z_1 + \cdots + z_{\pi+j}) \cdot l = (z_1 + \cdots + z_\pi) \cdot l + (x_1 + \cdots + x_j) \cdot l < (z_1 + \cdots + z_\pi) \cdot l$ which contradicts the choice of π. If $\pi + j > s$ and $\pi + j \equiv t \pmod{s}$ then $0 = l \cdot (z_1 + \cdots + z_\pi + x_1 + \cdots + x_{s-\pi}) < l \cdot (z_1 + \cdots + z_t + x_1 + \cdots + x_{s-\pi}) = l \cdot (x_1 + \cdots + x_j) < 0$, a contradiction. This proves (i). The condition (ii) follows from the requirement that the sequence $z_1,\dots,z_n$ is the shortest. □

Using the previous two lemmas we can establish the equality analogous to the one from the theorem 3.2.1 in which $k = 0$.

Theorem 3.3.2. *For each vector* $l \in \mathbb{R}^d$ *and* X_n *defined as before the following equality holds:*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq 0) = 0.$$

Proof. The inequality $\mathbb{P}(X_n \cdot l \geq 0) \leq 1$ implies that $\limsup \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq 0) \leq 0$. For the other inequality we use the theorem 3.3.1. Let $x_1, \dots, x_s$ be the sequence whose existence is claimed by that theorem. Let $y_i = x_0 + \cdots + x_{i-1}$ as before. Notice that

$$\begin{aligned} \mathbb{P}(X_n \cdot l \geq 0) &\geq \mathbb{P}(\xi_1 = x_1, \xi_2 = x_2, \dots, \xi_s = x_s) \\ &= \mathbb{P}\left(\mathbb{P}\left(\xi_1 = x_1, \xi_2 = x_2, \dots, \xi_s = x_s \middle| \mathscr{F}_{y_1,\dots,y_s}\right)\right) \\ &\geq c_1. \end{aligned}$$

Therefore $\liminf \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq 0) \geq \liminf \frac{1}{n} \log c_1 = 0$. □

We will also need the following deterministic lemma.

Lemma 3.3.2. *Assume that $\{u_1, \dots, u_m\} \subseteq \mathbb{Z}^d$ is a nice set of vectors. Let $\rho : \mathbb{R}^d \to \mathbb{R}_+ \cup \{0\}$ be the function defined as $\rho(l) = \max_i \{u_i \cdot l\}$. Then*

$$\inf_{l:|l|=1} \rho(l) > 0.$$

Proof. First notice that $\rho(l) > 0$ for each $l \in \mathbb{R}^d \setminus \{0\}$. Otherwise the set $\{u_1, \dots, u_m\}$ would not be nice. Notice also that ρ is a continuous function (because it is a maximum of m continuous functions) and the unit sphere is a compact set. Thus the infimum of ρ over the unit sphere must be attained at some point, and we have just proved that value of ρ at any single point is not 0. □

3.4 Hitting Times of Hyperplanes

The main idea for the proof of the theorem 3.2.1 is to establish the asymptotic equivalence of $\frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq nk)$ and a sequence to which we can apply the deterministic superadditive lemmas. First we will prove that the previous sequence behaves like $\frac{1}{n} \log \mathbb{P}(T^l_{nk} \leq n)$. Then we will see that the asymptotic behavior of the latter sequence is the same as the behavior of $\frac{1}{n} \log \mathbb{P}(T^l_{nk} \leq n, T^l_{nk} \leq D^l_1)$ where D^l_1 is the first time the walk returns over the hyperplane Z^l_0. This probability captures those walks that don't backtrack over the hyperplane Z^l_0.

We will be able to prove the existence of the limit of the last sequence using a modification of the standard subadditive lemma that states that $\lim \frac{a_n}{n} = \inf \frac{a_n}{n}$ if $a_{n+m} \leq a_n + a_m$ for all $m, n \in \mathbb{N}$.

From now on let us fix the vector $l \in \mathbb{R}^d$ and let us omit the superscript l in the variables. Also, some of the new variables that will be defined would need to have a superscript l but we will omit it as well.

Our first result in carrying out the formerly described plan is the following lemma:

Lemma 3.4.1. *The following inequality holds:*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq nk) \quad \leq \quad \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(T_{nk} \leq n)$$

In addition, for each $\varepsilon > 0$ we have:

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(T_{n(k+\varepsilon)} \leq n) \quad \leq \quad \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq nk).$$

Proof. Clearly, $\{X_n \cdot l \geq kn\} \subseteq \{T_{kn} \leq n\}$. Therefore $\mathbb{P}(X_n \cdot l \geq kn) \leq \mathbb{P}(T_{kn} \leq n)$. This establishes the first inequality. Let $x_0, \dots, x_s$ be the sequence whose existence follows from the theorem 3.3.1. We now have have

$$\begin{aligned} \mathbb{P}(X_n \cdot l \geq kn) &= \mathbb{P}(X_n \cdot l \geq kn, T_{kn} \leq n) \\ &\geq \mathbb{P}\left(X_n \cdot l \geq kn, T_{kn} \leq n-s, X_{T_{kn}+1} - X_{T_{kn}} = x_1, \dots, \right. \\ &\quad \left. X_{T_{kn}+s} - X_{T_{kn}+s-1} = x_s\right) \\ &= \mathbb{P}\left(T_{kn} \leq n-s, X_{T_{kn}+1} - X_{T_{kn}} = x_1, \dots, \right. \\ &\quad \left. X_{T_{kn}+s} - X_{T_{kn}+s-1} = x_s\right) \\ &= \mathbb{E}\left[1(T_{nk} \leq n-s)\mathbb{E}\left[1(\xi_{T_{kn}} = x_1) \cdots \right.\right. \\ &\quad \left.\left. \cdot 1(\xi_{T_{kn}+s-1} = x_s) \middle| \mathscr{F}_{T_{kn}}\right]\right] \\ &\geq c_1 \cdot \mathbb{P}\left(T_{kn} \leq n-s\right). \end{aligned} \tag{3.4.1}$$

Here $\mathscr{F}_{T_{kn}}$ denotes the σ-algebra defined by η_z for $z \in \mathbb{Z}^d$ such that $|z - X_i| \leq M$ for $i = 1, 2, \dots, T_{kn}$. The equality in (2) holds because if $T_{kn} \leq n-s$ and $X_{T_{kn}+1} - X_{T_{kn}} = x_1, \dots, X_{T_{kn}+s} - X_{T_{kn}+s-1} = x_s$, then the walk will enter in a loop. This loop will be in the half-space H_{kn} which would guarantee that $X_n \cdot l > kn$.

For each $\varepsilon > 0$, if $n > s + sk/\varepsilon$ we have $\{T_{(k+\varepsilon)(n-s)} \leq n-s\} \subseteq \{T_{kn} \leq n-s\}$. This completes the proof of the lemma. □

Remark. In the same way we could obtain the analogous inequalities with the walk X_n replaced by $X_{n\wedge\tau}$.

For each integer $i \geq 1$ denote by D_i the time of the ith jump over the hyperplane Z_0 in the direction of the vector l. Define $D_0 = 0$ and we allow for D_i to be ∞.

Lemma 3.4.2. *Let k and k' be two real numbers such that $0 < k' < k$. Then the following two inequalities hold:*

$$\begin{aligned} \limsup \frac{1}{n} \log \mathbb{P}(T_{nk} \leq n) &\leq \limsup \frac{1}{n} \log \mathbb{P}(T_{nk'} \leq n, T_{nk'} \leq D_1) \\ \liminf \frac{1}{n} \log \mathbb{P}\left(T_{nk} \leq n, T_{nk} \leq D_1\right) &\leq \liminf \frac{1}{n} \log \mathbb{P}\left(T_{nk} \leq n\right). \end{aligned}$$

Proof. We have:

$$\mathbb{P}\left(T_u \leq n\right) \quad = \quad \sum_{i=1}^{n} \mathbb{P}\left(T_u \leq n, D_i \leq T_u \leq D_{i+1}\right).$$

We will prove that each term from the right-hand side of the previous equality is bounded above by $L^d n^{d-1} \cdot \mathbb{P}(T_{u-L} \le n, T_{u-L} \le D_1)$. Let Z_0' be the set of all points $z \in \mathbb{Z}^d$ such that $z \cdot l > 0$ and the distance between z and Z_0 is at most L. We have:

$$\begin{aligned} & \mathbb{P}(T_u \le n, D_i \le T_u \le D_{i+1}) \\ = & \sum_{z \in Z_0', |z| \le nL} \mathbb{E}\left[1(T_u \le n) \cdot 1(X_{D_i} = z) \cdot 1(T_u \ge D_i) \right. \\ & \left. \cdot 1(T_u \le D_{i+1})\right] \\ = & \sum_{z \in Z_0', |z| \le nL} \mathbb{E}\left[\mathbb{E}\left[1(T_u \le n) \cdot 1(X_{D_i} = z) \cdot 1(T_u \ge D_i) \right.\right. \\ & \left.\left. \cdot 1(T_u \le D_{i+1}) \middle| \mathscr{F}_{D_i, T_u}\right]\right]. \end{aligned}$$

Here $\mathscr{F}_{D_i,T_u}$ denotes the σ-algebra generated by the random environment that is contained in the M-neighborhood of the walk from D_i to T_u. When conditioning on this σ-field we essentially understand our environment in the following way: It consists of two walks: One deterministic that goes from z to Z_u without crossing the hyperplane Z_0, and another walk that starts at 0 and ends in z by making exactly i crossings over Z_0, not intersecting the other deterministic walk, and not crossing over Z_u. Therefore:

$$\begin{aligned} & \sum_{z \in Z_0', |z| \le nL} \mathbb{E}\left[\mathbb{E}\left[1(T_u \le n) \cdot 1(X_{D_i} = z) \cdot 1(T_u \ge D_i) \right.\right. \\ & \left.\left. \cdot 1(T_u \le D_{i+1}) \middle| \mathscr{F}_{D_i, T_u}\right]\right] \\ \le & \sum_{z \in Z_0', |z| \le nL} \mathbb{E}\left[1(\tilde{T}_{u-L} \le n - D_i) \cdot 1(\tilde{T}_{u-L} \le \tilde{D}_1) \right. \\ & \left. \cdot \mathbb{E}\left[1(X_{D_i} = z) \cdot 1(T_u \ge D_i) \middle| \mathscr{F}_{D_i, T_u}\right]\right] \\ \le & \sum_{z \in Z_0, |z| \le nL} \mathbb{E}\left[1(\tilde{T}_{u-L} \le n) \cdot 1(\tilde{T}_{u-L} \le \tilde{D}_1)\right], \end{aligned}$$

where $\tilde{T}_u$ is defined in analogous way as T_u to correspond to the new walk $\tilde{X}_j = X_{D_i+j}$. In the last equation $\tilde{D}_1$ is defined as the first time of crossing over the hyperplane Γ parallel to Z_0 that is shifted by the vector $-L\frac{l}{|l|}$. Let us now prove that $\mathbb{P}(T_{u-L} \le n, T_{u-L} \le \tilde{D}_1) \le L^d n^{d-1} \cdot \mathbb{P}(T_{u-2L} \le n, T_{u-L} \le D_1)$. Denote by J the closest time the walk comes to the hyperplane Γ. The number of possible positions of the walk is at most $L^d n^{d-1}$ and similarly as above, conditioning on the σ-filed between J and T_{u-L} we get $\mathbb{P}(T_{u-L} \le n, T_{u-L} \le \tilde{D}_1) \le L^d n^{d-1} \cdot \mathbb{P}(T_{u-2L} \le n, T_{u-L} \le D_1)$. We now have

$$\mathbb{P}(T_u \le n, D_i \le T_u \le D_{i+1}) \le L^{2d} n^{2d-2} \mathbb{P}(T_{u-2L} \le n, T_{u-2L} \le D_1).$$

This implies that $\mathbb{P}(T_u \le n) \le L^{2d} n^{2d-1} \mathbb{P}(T_{u-2L} \le n, T_{u-2L} \le D_1)$. Therefore

$$\limsup \frac{1}{n} \log \mathbb{P}(T_{nk} \le n) \le \limsup \frac{1}{n} \log \mathbb{P}(T_{nk-2L} \le n, T_{nk-2L} \le D_1).$$

For each $k' < k$ we have

$$\limsup \frac{1}{n} \log \mathbb{P}(T_{nk} \leq n) \leq \limsup \frac{1}{n} \log \mathbb{P}(T_{nk'} \leq n, T_{nk'} \leq D_1).$$

On the other hand, it is obvious that

$$\liminf \frac{1}{n} \log \mathbb{P}\left(T_{nk} \leq n, T_{nk} \leq D_1\right) \leq \liminf \frac{1}{n} \log \mathbb{P}\left(T_{nk} \leq n\right),$$

because $\{T_{nk} \leq n\} \supseteq \{T_{nk} \leq n, T_{nk} \leq D_1\}$. □

3.5 Large Deviations Estimates

Now we are ready to prove the main theorem:

Proof of the Theorem 3.2.1. We will prove that for each unit vector l there exists a concave function $\psi : \mathbb{R}_+ \to \bar{\mathbb{R}}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(T_{nk} \leq n) \quad = \quad \psi(k). \tag{3.5.1}$$

Because of the Lemma 3.4.2 it suffices to prove that there exists a concave function $\gamma : \mathbb{R}_+ \to \bar{\mathbb{R}}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(T_{nk} \leq n, T_{nk} \leq D_1\right) \quad = \quad \gamma(k). \tag{3.5.2}$$

Let $w \in \mathbb{Z}^d$ be a vector such that $w \cdot l > 0$ and $\mathbb{P}(\eta_z = w|\mathscr{F}_z) \geq c > 0$ for some constant c. Assume that r is an integer such that the distance between the hyperplanes Z_{rw} and Z_0 is at least M. If u, v, p, q are any four positive real numbers such that $q > r$, then the following inequality holds:

$$\begin{aligned} \mathbb{P}\left(T_{u+v} \leq p+q, T_{u+v} \leq D_1\right) \quad \geq \quad & c \cdot \mathbb{P}(T_u \leq p, T_u \leq D_1) \cdot \\ & \mathbb{P}(T_v \leq q-r, T_v \leq D_1). \end{aligned} \tag{3.5.3}$$

If the environment were iid this could have been done by conditioning on $\mathscr{F}_u$. In our situation the idea is the same, we just need some more work to compensate for the lack of independence.

$$\begin{aligned} & \mathbb{P}\left(T_{u+v} \leq p+q, T_{u+v} \leq D_1\right) \\ \geq\ & \mathbb{P}\left(T_{u+v} \leq p+q, T_{u+v} \leq D_1, T_u \leq p\right) \\ \geq\ & \mathbb{E}\left[1(T_u \leq p) \cdot 1(T_u \leq D_1) \cdot 1(\xi_{T_u} = w) \cdot \cdots \right. \\ & \left. \cdot 1(\xi_{T_u+r-1} = w) \cdot 1(T_{u+v} \leq p+q) \cdot 1(T_{u+v} \leq D_1)\right] \\ =\ & \mathbb{E}\left[\mathbb{E}\left[1(T_u \leq p) \cdot 1(T_u \leq D_1) \cdot 1(\xi_{T_u} = w) \cdot \cdots \cdot 1(\xi_{T_u+r-1} = w)\right.\right. \\ & \left.\left. \cdot 1(T_{u+v} \leq p+q) \cdot 1(T_{u+v} \leq D_1) | \mathscr{F}_{[X_{T_u}, X_{T_{u+rw}}]}\right]\right], \end{aligned}$$

where $\mathscr{F}_{[a,b]}$ is a σ-algebra determined by the environment outside of the strip $[Z_a, Z_b]$. Let us introduce the following notation: $\hat{X}_i = X_{T_u+r-1+i}$, $\hat{D}_1$ the first time $\hat{X}_i$ jumps over $Z_{X_{T_u}+rw}$, and $\hat{T}_v = \inf\{i : \hat{X}_i \cdot l \geq v\}$. We now have that

$$\begin{aligned}
& \mathbb{P}(T_{u+v} \leq p+q, T_{u+v} \leq D_1) \\
\geq \; & \mathbb{E}\left[\mathbb{E}\left[1(T_u \leq p) \cdot 1(T_u \leq D_1) \cdot 1(\xi_{T_u} = w) \cdot \dots \cdot 1(\xi_{T_u+r-1} = w)\right.\right. \\
& \left.\left. \cdot 1(\hat{T}_v \leq q-r) \cdot 1(\hat{T}_v \leq \hat{D}_1) \middle| \mathscr{F}_{[X_{T_u}, X_{T_{u+rw}}]}\right]\right] \\
= \; & \mathbb{E}\left[1(T_u \leq p) \cdot 1(T_u \leq D_1) \cdot 1(\hat{T}_v \leq q-r) \cdot 1(\hat{T}_v \leq \hat{D}_1) \cdot \right. \\
& \left. \mathbb{E}\left[1(\xi_{T_u} = w) \cdot \dots \cdot 1(\xi_{T_u+r-1} = w) \middle| \mathscr{F}_{[X_{T_u}, X_{T_{u+rw}}]}\right]\right] \\
\geq \; & c \cdot \mathbb{P}(T_u \leq p, T_u \leq D_1) \cdot \mathbb{P}(T_v \leq q-r, T_v \leq D_1).
\end{aligned}$$

This establishes (3.5.3). Let $\delta(m,x) = \log \mathbb{P}(T_{mx} \leq m, T_{mx} \leq D_1)$. Applying the inequality (3.5.3) to the numbers $u = nk$, $v = mk$, $p = n$, $q = m$ yields to:

$$\begin{aligned}
\delta(m+n,k) & \geq \log c + \delta(n,k) + \log \mathbb{P}(T_{mk} \leq m-r, T_{mk} \leq D_1) \\
& \geq \log c + \delta(n,k) + \delta(m-r,k'),
\end{aligned}$$

where k' is any real number greater than k for which $(m-r)k' \geq mk$. For now on we will write c instead of $\log c$. In other words, for each k, and each $k' > k$ we have

$$\delta(m+n,k) \geq c + \delta(m,k) + \delta(n-r,k'), \tag{3.5.4}$$

for all m, n such that $n \geq \frac{rk'}{k'-k}$. Let

$$\underline{\delta}(k) = \liminf_{n\to\infty} \frac{\delta(n,k)}{n}, \quad \overline{\delta}(k) = \limsup_{n\to\infty} \frac{\delta(n,k)}{n}.$$

If $k < k'$ then for each $\alpha < \overline{\delta}(k')$ there exists a sequence n_t that goes to infinity such that $\frac{\delta(n_t,k')}{n_t} \geq \alpha$. For each fixed n_t and each $n \geq n_t$ there exist integers $a \geq 0$ and $b \in \{0,1,2,\dots,n_t+r-1\}$ such that $n = a(n_t+r)+b$. Therefore

$$\begin{aligned}
\frac{\delta(n,k)}{n} & = \frac{\delta(a(n_t+r)+b,n)}{a(n_t+r)+b} \geq \frac{(a-1)\delta(n_t,k') + \delta(n_t+r+b,k) + (a-1)c}{a(n_t+r)+b} \\
& \geq \frac{\delta(n_t,k')}{n_t} \cdot \frac{(a-1)\cdot n_t}{a(n_t+r)+b} + \frac{\delta(n_t+r+b,k)}{a(n_t+r)+b} + \frac{(a-1)c}{a(n_t+r)+b} \\
& \geq \alpha \cdot \frac{(a-1)\cdot n_t}{a(n_t+r)+b} + \frac{\delta(n_t+r+b,k)}{a(n_t+r)+b} + \frac{(a-1)c}{a(n_t+r)+b}.
\end{aligned}$$

For each $\mu > 0$ there exists t_0 such that for all $t > t_0$ we have

$$\left|\frac{(a-1)c}{a(n_t+r)+b}\right| < \mu$$

uniformly in a and b. There exists n_0 such that for all $n > n_0$, the number $a = \left[\frac{n}{n_t+r}\right]$ would be large enough to guarantee

$$\left|\frac{\delta(n_t+r+b,k)}{a(n_t+r)+b}\right| < \mu \text{ and } \left|\frac{(a-1)n_t}{a(n_t+r)+b} - 1\right| \leq \frac{\mu}{|\alpha|+1}.$$

This implies that $\underline{\delta}(k) \geq \alpha - 3\mu$ for all $\mu > 0$, therefore $\underline{\delta}(k) > \alpha$ for all $\alpha < \overline{\delta}(k')$. This immediately yields to $\underline{\delta}(k) \geq \overline{\delta}(k')$ for all $k < k'$. Obviously $\overline{\delta}(k) \geq \underline{\delta}(k)$ hence we have

$$\underline{\delta}(k') \leq \overline{\delta}(k') \leq \underline{\delta}(k) \leq \overline{\delta}(k), \text{ for all } k < k'.$$

A consequence of the previous inequality is the monotonicity of the functions $\overline{\delta}$ and $\underline{\delta}$. They are both non-increasing. Let α and β be two positive rational numbers such that $\alpha + \beta = 1$. Let k_1 and k_2 be any two positive real numbers. According to (3.5.3) we know that for each $n \in \mathbb{N}$ we have:

$$\begin{aligned}
\delta(n, \alpha k_1 + \beta k_2) &= \log \mathbb{P}(T_{n(\alpha k_1+\beta k_2)} \leq \alpha n + \beta n, T_{n(\alpha k_1+\beta k_2)} \leq D_1) \\
&\geq \log \mathbb{P}(T_{n\alpha k_1} \leq \alpha n, T_{n\alpha k_1} \leq D_1) + \\
&\quad\ \log \mathbb{P}(T_{n\beta k_2} \leq \beta n - r, T_{n\beta k_2} \leq D_1) + c \\
&\geq \delta(\alpha n, k_1) + \delta(\beta n, k_2') + c,
\end{aligned}$$

for sufficiently large n, where k_2' is any number larger than k_2. This implies that

$$\begin{aligned}
\overline{\delta}(\alpha k_1 + \beta k_2) &\geq \limsup \frac{1}{n}\left(\delta(\alpha n, k_1) + \delta(\beta n, k_2')\right) \\
&\geq \alpha \underline{\delta}(k_1) + \beta \underline{\delta}(k_2').
\end{aligned} \tag{3.5.5}$$

Let us justify the second inequality. The previous lim sup is definitely larger than the lim inf over the sequence of those integers n that are divisible by the denominators of both α and β.

Consider now two positive real numbers $k < k'$ for which $\underline{\delta}(k')$ is a real number (i.e. not $-\infty$). Let α_n and β_n be two sequences of positive rational numbers such that $\alpha_n \to 1$, $\beta_n \to 0$, $\alpha_n + \beta_n = 1$. Let k'' be a real number such that $k < k'' < k'$. Then the inequality (3.5.5) implies:

$$\overline{\delta}(\alpha_n k + \beta_n k'') \geq \alpha_n \underline{\delta}(k) + \beta_n \underline{\delta}(k').$$

Taking the limit of both sides as $n \to \infty$ and using the monotonicity of $\overline{\delta}$ we get:

$$\lim_{\varepsilon > 0, \varepsilon \to 0} \overline{\delta}(k+\varepsilon) \geq \underline{\delta}(k).$$

This inequality together with $\underline{\delta}(k+\varepsilon/2) > \overline{\delta}(k+\varepsilon)$ implies that $\underline{\delta}$ is right-continuous.

Let us now choose the sequences α_n, β_n, k_n^1, k_n^2, k_n^3 that satisfy:

$$\begin{aligned}
&\alpha_n, \beta_n \in \mathbb{Q}_+, \ k_n^1, k_n^2, k_n^3 \in \mathbb{R}_+, \\
&\alpha_n + \beta_n = 1, \ \alpha_n \to 0, \ \beta_n \to 1, \\
&k_n^2 < k_n^3 < k < k_n^1, \ k_n^2, k_n^3 \nearrow k, \ k_n^1 \searrow k, \\
&\alpha_n k_n^1 + \beta_n k_n^2 = k.
\end{aligned}$$

Then the inequality (3.5.5) implies

$$\overline{\delta}(k) \geq \alpha_n \underline{\delta}(k_n^1) + \beta_n \underline{\delta}(k_n^3).$$

Letting $n \to \infty$ and using the monotonicity of $\underline{\delta}$ gives us:

$$\overline{\delta}(k) \geq \lim_{\varepsilon>0,\varepsilon\to 0} \underline{\delta}(k-\varepsilon).$$

Using the fact $\underline{\delta}(k-\varepsilon) \geq \overline{\delta}(k-\varepsilon/2)$ gives that $\overline{\delta}(k) \geq \overline{\delta}(k-0)$. Here $\overline{\delta}(k-0)$ is defined as

$$\overline{\delta}(k-0) = \lim_{\varepsilon\to 0+} \overline{\delta}(k-\varepsilon).$$

In other words, $\overline{\delta}$ is left-continuous.

Let us now choose the sequences:

$$\begin{aligned}
&\alpha_n, \beta_n \in \mathbb{Q}_+,\ \ k_n^1, k_n^2, k_n^3 \in \mathbb{R}_+,\\
&\alpha_n + \beta_n = 1,\ \ \alpha_n \to 1,\ \ \beta_n \to 0,\\
&k_n^3 > k_n^2 > k > k_n^1,\ \ k_n^2, k_n^3 \searrow k,\ \ k_n^1 \nearrow k,\\
&\alpha_n k_n^1 + \beta_n k_n^2 > k.
\end{aligned}$$

Placing these sequences in the inequality (3.5.5) gives us the following relation:

$$\underline{\delta}(k) \geq \overline{\delta}(\alpha_n k_n^1 + \beta_n k_n^2) \geq \alpha_n \underline{\delta}(k_n^1) + \beta_n \underline{\delta}(k_n^3) \geq \alpha_n \overline{\delta}(k) + \beta_n \underline{\delta}(k_n^3).$$

Letting $n \to \infty$ implies $\underline{\delta}(k) \geq \overline{\delta}(k)$. Summing up all the facts we got for $\underline{\delta}$ and $\overline{\delta}$ we have that $\underline{\delta}(k) = \overline{\delta}(k) = \gamma(k)$ for some function γ. Moreover, γ is continuous and $\gamma(\alpha k_1 + \beta k_2) \geq \alpha\gamma(k_1) + \beta\gamma(k_2)$ for $\alpha, \beta \in \mathbb{Q}_+$ such that $\alpha + \beta = 1$. Because of the continuity, the last inequality holds for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha + \beta = 1$. This means that γ is concave and the equality (3.5.2) is established. Now we have the relation (3.5.1) and the concavity of the function ψ. Namely, the lemma 3.4.2 implies that $\psi = \gamma$.

Using the lemma 3.4.1 we get that

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq nk) &\leq \psi(k), \text{ and}\\
\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq nk) &\geq \psi(k+\varepsilon),
\end{aligned}$$

for all $\varepsilon > 0$. If k belongs to the interior of $\psi^{-1}(\mathbb{R})$, we can take $\varepsilon \to 0$ in the previous inequality and use the continuity of ψ to obtain

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq nk) \geq \psi(k).$$

This in turn implies (3.2.1) and the concavity of ϕ. □

The Gartner-Ellis theorem will enable us to get some more information on lower and upper bound large deviations for general sets.

Definition 3.5.1. *Assume that* Λ^* *is a convex conjugate of the function* Λ. $y \in \mathbb{R}^d$ *is an* exposed point *of* Λ^* *if for some* $\lambda \in \mathbb{R}^d$ *and all* $x \neq y$,

$$\lambda \cdot y - \Lambda^*(y) > \lambda \cdot x - \Lambda^*(x). \tag{3.5.6}$$

λ *in (3.5.6) is called an* exposing hyperplane*.*

We are now ready to prove the theorem stated in the introduction.

Theorem 3.5.1. *There exists a convex function* $\Lambda : \mathbb{R}^d \to \bar{\mathbb{R}}$ *such that* $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[e^{X_n\cdot\lambda}\right] = \Lambda(\lambda)$. *The origin belongs to the interior of the set* $\{\lambda \in \mathbb{R}^d : \Lambda(\lambda) < +\infty\}$.

Proof. As noted in the remark after the theorem 3.2.1 there exists a function $\Phi : \mathbb{R}^d \to \bar{\mathbb{R}}$ such that for all $l \in \mathbb{R}^d$ and $k \in \mathbb{R}_+$:

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \cdot l \geq kn) = \Phi\left(\frac{1}{k} l\right).$$

For each $\lambda \in \mathbb{R}^d$ and each $k > 0$ we have that

$$\begin{aligned} \liminf \frac{1}{n} \log \mathbb{E}\left(e^{X_n\cdot\lambda}\right) &\geq \liminf \frac{1}{n} \log \mathbb{E}\left(e^{X_n\cdot\lambda} \cdot 1(X_n \cdot \lambda > kn)\right) \\ &\geq \liminf \frac{1}{n} \log \mathbb{E}\left(e^{kn} \cdot 1(X_n \cdot \lambda > kn)\right) = k + \Phi\left(\frac{1}{k}\lambda\right). \end{aligned}$$

Moreover, from the theorem 3.3.2 we get

$$\begin{aligned} \liminf \frac{1}{n} \log \mathbb{E}\left(e^{X_n\cdot\lambda}\right) &\geq \liminf \frac{1}{n} \log \mathbb{E}\left(e^{X_n\cdot\lambda}\right) \\ &\geq \liminf \frac{1}{n} \log \mathbb{E}\left(e^{0} \cdot 1(X_n \cdot \lambda \geq 0)\right) = 0. \end{aligned}$$

Therefore

$$\liminf \frac{1}{n} \log \mathbb{E}\left(e^{X_n\cdot\lambda}\right) \geq \max\left\{0, \sup_{k>0}\left\{k + \Phi\left(\frac{\lambda}{k}\right)\right\}\right\}.$$

From the boundedness of jumps of the random walk X_n we have that $|X_n \cdot \lambda| < L|\lambda|$. Let $r \in \mathbb{N}$ and

$0 = k_0 < k_1 < k_2 < \cdots < k_r = L|\lambda|$. Then

$$\begin{aligned}
\overline{\lim}\frac{1}{n}\log\mathbb{E}\left(e^{X_n\cdot\lambda}\right) &= \overline{\lim}\frac{1}{n}\log\left(\mathbb{E}\left((e^{X_n\cdot\lambda}\cdot 1(X_n\cdot\lambda\le 0)\right)+\right.\\
&\left.\sum_{i=0}^{r-1}\mathbb{E}\left(e^{X_n\cdot\lambda}\cdot 1(nk_i < X_n\cdot\lambda \le nk_{i+1})\right)\right)\\
&\le \overline{\lim}\frac{1}{n}\log\left(\mathbb{P}(X_n\cdot\lambda\le 0)+\sum_{i=0}^{r-1}e^{nk_{i+1}}\cdot\mathbb{P}(X_n\cdot\lambda > nk_i)\right)\\
&\le \overline{\lim}\frac{1}{n}\log\left(\mathbb{P}(X_n\cdot\lambda\le 0)+\right.\\
&\left. r\max_{0\le i\le r-1}e^{nk_{i+1}}\cdot\mathbb{P}(X_n\cdot\lambda > nk_i)\right)\\
&= \max\left\{0,\max_{0\le i\le r-1}\left\{k_{i+1}+\Phi\left(\frac{\lambda}{k_i}\right)\right\}\right\}.
\end{aligned}$$

The last equality is true because $r \in \mathbb{N}$ is a fixed number as $n \to \infty$ and the lemma 3.3.2 implies that $\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(X_n\cdot(-\lambda)\ge 0) = 0$. The theorem 3.2.1 implies that the function $\Phi\left(\frac{1}{k}\lambda\right)$ is continuous in k hence taking $r \to \infty$ and $k_{i+1}-k_i$ constant we get:

$$\limsup\frac{1}{n}\log\mathbb{E}\left(e^{X_n\cdot\lambda}\right) \le \max\left\{0,\sup_{k>0}\left\{k+\Phi\left(\frac{\lambda}{k}\right)\right\}\right\}.$$

This proves the existence of the limit in from the statement of the theorem with

$$\Lambda(\lambda) = \max\left\{0,\sup_{k>0}\left\{k+\Phi\left(\frac{\lambda}{k}\right)\right\}\right\}.$$

We will not use this representation for Λ to prove its convexity. Notice that all functions $\Lambda_n(\lambda) = \log\mathbb{E}\left(e^{X_n\cdot\lambda}\right)$ are convex when n is fixed. Indeed, for all $\alpha,\beta \in \mathbb{R}_+$ with $\alpha+\beta = 1$ and all $\lambda,\mu \in \mathbb{R}^d$ according to the Holder's inequality we have:

$$\begin{aligned}
e^{\Lambda_n(\alpha\lambda+\beta\mu)} &= \mathbb{E}\left[\left(e^{X_n\cdot\lambda}\right)^{\alpha}\cdot\left(e^{X_n\cdot\mu}\right)^{\beta}\right]\\
&\le \left(\mathbb{E}\left[e^{X_n\cdot\lambda}\right]\right)^{\alpha}\cdot\left(\mathbb{E}\left[e^{X_n\cdot\mu}\right]\right)^{\beta} = e^{\alpha\Lambda_n(\lambda)+\beta\Lambda_n(\mu)}.
\end{aligned}$$

Since the limit of convex functions is convex, as well as the maximum of two convex functions, we are able to conclude that Λ is convex. Obviously, the origin belongs to the interior of the set $\{\lambda \in \mathbb{R}^d : \Lambda(\lambda) < +\infty\}$ because Λ is bounded. □

Theorem 3.5.2. *Let X_n be the previously defined deterministic walk in a random environment that satisfies the conditions (i)–(iii). Let Λ be the function from the theorem 3.5.1 and let Λ^* be its* *convex conjugate. Let $\mathscr{F}$ be the set of exposed points of Λ^* whose exposing hyperplane belongs to the interior of the set $\{\lambda \in \mathbb{R}^d : \Lambda(\lambda) < +\infty\}$. For any closed set $F \subseteq \mathbb{R}^d$,*

$$\limsup \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n} X_n \in F\right) \leq -\inf_{x \in F} \Lambda^*(x),$$

and for any open set $G \subseteq \mathbb{R}^d$

$$\liminf \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n} X_n \in G\right) \geq -\inf_{x \in G \cap \mathscr{F}} \Lambda^*(x).$$

Moreover, there exists $\delta > 0$ such that $\Lambda^*(x) < +\infty$ for $|x| < \delta$.

Proof. The conditions for Gartner-Ellis theorem are now satisfied because of the theorem 3.5.1. Direct application of that result proves the first part of the statement.

For the second part, we will use the lemma 3.3.2. There exists $\kappa > 0$ such that for each $\lambda \in \mathbb{R}^d$ there exists $i \in \{1, 2, \dots, m\}$ such that $\lambda \cdot u_i > \kappa|\lambda|$. Then we have $\Lambda(\lambda) \geq \kappa|\lambda| + \Phi\left(\frac{\lambda}{\kappa|\lambda|}\right)$ and:

$$\Phi\left(\frac{\lambda}{\kappa|\lambda|}\right) \geq \liminf \frac{1}{n} \log \mathbb{P}(\xi_1 = \xi_2 = \cdots = \xi_n = u_i) \geq c,$$

for some constant c. Therefore $\Lambda(\lambda) \geq \kappa|\lambda| + c$, and

$$\Lambda^*(x) \leq \sup_{\lambda} \{\lambda \cdot x - \kappa|\lambda| - c \leq -c + \sup_{\lambda} \{|\lambda| \cdot |x| - \kappa|\lambda|\}.$$

Hence if $|x| \leq \kappa$ then $\Lambda^*(x) \leq -c < +\infty$. □

3.6 Law of Large Numbers

Let us end with a note about the law of large numbers for this deterministic walk in a random environment. It is not surprising that the walk will have 0 limiting velocity because it is expected that the walk will eventually end in a loop.

Theorem 3.6.1. *If X_n is defined as before then*

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E}(X_n) = 0.$$

Proof. It suffices to prove that $\lim_{n \to \infty} \frac{1}{n} \mathbb{E}(X_n \cdot l) = 0$ for each $l \in \mathbb{R}^d$, because the zero vector is the only one orthogonal to the entire $\mathbb{R}^d$. Furthermore, the problem can be reduced to proving that $\frac{1}{n} \mathbb{E}[X_n \cdot l]^+$ converges to 0 because $X_n \cdot l = (X_n \cdot l)^+ + (X_n \cdot (-l))^+$. By the Fubini's theorem we have

$$\mathbb{E}\left[\left(\frac{1}{n} X_n \cdot l\right)^+\right] = \int_0^{+\infty} \mathbb{P}(X_n \cdot l > nt)\, dt.$$

Since $\{X_n \cdot l > nt\} = \emptyset$ for $t > L$ the previous integration could be performed on the interval $(0,L)$ only. Let $x_1, \dots, x_s$ be a sequence from theorem 3.3.1, and let $y_k = \sum_{i=1}^{k} x_i$. Define the random walk Y_i as $Y_i = X_{s+i}$. The probability that the walk will reach the half-space H^l_{nt} before time n is smaller than the probability of the following event: The walk does not make a loop in first s steps, and after that it reaches the half-space H^l_{nt-sL}. Therefore we deduce that for each $t \in (0,L)$ the following inequality holds:

$$\begin{aligned}
\mathbb{P}(X_n \cdot l > nt) &\leq \mathbb{E}\left[1((X_1,\dots,X_s) \neq (y_1,\dots,y_s)) \cdot 1(Y_{n-s} \cdot l \geq nt - sL)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[1((X_1,\dots,X_s) \neq (y_1,\dots,y_s)) \cdot \right.\right. \\
&\qquad \left.\left. 1(Y_{n-s} \cdot l \geq nt - sL) | \mathscr{F}_{Y_1,\dots,Y_{n-s}}\right]\right],
\end{aligned}$$

where $\mathscr{F}_{Y_1,\dots,Y_s}$ is a σ-field determined by η_z for all $z \in \mathbb{Z}^d \setminus \{X_1,\dots,X_s\}$ such that $\min_{i=1}^{n} |z - X_i| \leq M$. The previous inequality now implies that

$$\begin{aligned}
\mathbb{P}(X_n \cdot l \geq nt) &\leq \mathbb{E}\left[1(Y_{n-s} \geq nt - sL) \cdot \right. \\
&\qquad \left. \mathbb{E}\left[1((X_1,\dots,X_s) \neq (y_1,\dots,y_s)) | \mathscr{F}_{Y_1,\dots,Y_{n-s}}\right]\right].
\end{aligned}$$

From theorem 3.3.1 we have that $\mathbb{E}\left[1((X_1,\dots,X_s) \neq (y_1,\dots,y_s)) | \mathscr{F}_{Y_1,\dots,Y_{n-s}}\right] \leq 1 - c$ for some constant $c > 0$. Let us denote $g = 1 - c$. We know that $g \in (0,1)$. Using mathematical induction, we can repeat the previous sequence of inequalities $[nt/sL]$ times to obtain that $\mathbb{P}(X_n \cdot l \geq nt) \leq g^{[nt/sL]}$. Now we have that for all $t_0 > 0$ the following inequality holds:

$$\begin{aligned}
\frac{1}{n}\mathbb{E}(X_n \cdot l)^+ &= \int_0^L \mathbb{P}(X_n \cdot l > nt)\, dt \\
&= \int_0^{t_0} \mathbb{P}(X_n \cdot l > nt)\, dt + \int_{t_0}^{L} \mathbb{P}(X_n \cdot l > nt)\, dt \\
&\leq t_0 + (L - t_0) \cdot g^{[nt_0/sL]} \\
&\leq t_0 + L \cdot g^{[nt_0/sL]}.
\end{aligned}$$

If we keep t_0 fixed and let $n \to \infty$ it is easy to see that the last quantity converges to 0. Therefore $\limsup \frac{1}{n}\mathbb{E}(X_n \cdot l) + \leq t_0$. However, this holds for every $t_0 > 0$ hence $\limsup \frac{1}{n}\mathbb{E}(X_n \cdot l)^+ \leq 0$. This finishes the proof of the theorem. □

Chapter 4

Homogenization for Stochastic Frenkel-Kontorova Models

4.1 Introduction

Let $(\Omega, \mathbb{P})$ be a probability space such that $\mathbb{P}$ is invariant under the family $(\tau_z)_{z\in\mathbb{R}^d}$ of transformations on Ω. We also assume that τ_z are ergodic. In other words, we are assuming $\mathbb{P}(\tau_z A) = \mathbb{P}(A)$ for all z and A; and only trivial sets (i.e. those of measure 0 or 1) can satisfy $\mathbb{P}(A\Delta\tau_z A) = 0$ for all $z \in \mathbb{R}^d$. Here Δ denotes the symmetric difference of sets: $A\Delta B = (A \setminus B) \cup (B \setminus A)$.

Let $L : \mathbb{R}^d \times \Omega \to \overline{\mathbb{R}}$ be a convex function that is finite in an open neighborhood of 0. Define

$$S_n(p,q,\omega) = \inf\left\{\sum_{i=0}^{n-1} L(q_{i+1} - q_i, \tau_{q_i}\omega)\right\},$$

where the infimum is taken over all sequences $q_0, q_1, \ldots, q_n$ that satisfy $q_0 = p$ and $q_n = q$. Assume that $h : \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz-continuous function and let us define

$$u_n(q,\omega) = \sup_Q \left\{nh\left(\frac{Q}{n}\right) - S_n(q,Q,\omega)\right\}.$$

We will assume that L is coercive. In other words, we assume that there are functions $C_0 \in L^r(\Omega)$ (for some real $r > 1$) and $\Phi : \mathbb{R}^d \to \mathbb{R}$ such that Φ is convex, $L(p,\omega) \geq \Phi(p) - C_0(\omega)$ and

$$\lim_{|p|\to+\infty} \frac{\Phi(p)}{|p|} = +\infty.$$

Assume that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{q,|p|\leq\delta,|z|\leq\delta} |L(q+p,\omega) - L(q,\tau_z\omega)| \leq C|q| \cdot |p| + \varepsilon.$$

Under the above assumptions on L we will prove the following result.

Theorem 4.1.1. *There exists convex and coercive function $\bar{L} : \mathbb{R}^d \to \mathbb{R}$ such that for every $\eta > 0$, all $q \in \mathbb{R}^d$, and almost all $\omega \in \Omega$ the following holds:*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{q'\in B(q,\eta)} \left|\frac{1}{n}u_n(nq',\omega) - u(q')\right|\right] \quad = \quad 0,$$

where

$$u(q) = \sup_Q \left\{h(Q) - \bar{L}(Q-q)\right\}.$$

First we will use methods similar to ones developed by Rezakhanlou and Tarver in [24] for Hamilton-Jacobi equations to establish the previous theorem. Then we will establish a variational formula for $\bar{L}$. In order to formulate the latter result we need to define a class of functions.

Denote by $\mathscr{K}$ the set of all functions $g : \mathbb{R}^d \times \Omega \to \mathbb{R}$ such that

(i) For each $a \in \mathbb{R}^d$ the function $g(a,\omega)$ belongs to $L^{d+\alpha}(\Omega)$ for some $\alpha > 0$,

(ii) For almost all ω the function $g(a,\omega)$ is Lipschitz continuous,

(iii) For each $a \in \mathbb{R}^d$ we have $\mathbb{E}(g(a,\omega)) = 0$, and

(iv) For each sequence $q_0, q_1, q_2, \ldots, q_n \in \mathbb{R}^d$ such that $q_n = q_0$ the following equality holds:

$$\sum_{k=0}^{n-1} g(q_{k+1} - q_k, \tau_{q_k}\omega) = 0.$$

Theorem 4.1.2. *The function $\bar{L}$ introduced by the theorem 4.1.1 satisfies $\bar{L}(Q) = \sup_p\{Q\cdot p - \bar{H}(p)\}$, where*

$$\bar{H}(p) = \inf_{g\in\mathscr{K}} \sup_{a\in\mathbb{R}^d} \operatorname{esssup}\left\{p\cdot a + g(a,\omega) - L(a,\omega)\right\}.$$

In this theorem we need to prove that $\bar{L}^* \leq \bar{H}$ and $\bar{L}^* \geq \bar{H}$. The first inequality is done by using the existence of the limit and using special initial data to take advantage of the theorem 4.1.1. For the second inequality one can go by using minimax principle similar to [16]. However, it is possible to avoid such approach by constructing a subsolution to the cell problem. Similar tricks have been used by Concordel in [4] and [5].

Functions from $\mathscr{K}$ are sublinear at infinity (theorem 4.3.1) so modifying the initial data by such functions doesn't change the macroscopic behavior.

With minor modifications, this entire program can be repeated for the case when $\mathbb{R}^d$ is replaced by $\mathbb{Z}^d$. The goal is to find appropriate L that corresponds to k-exclusion and v-exclusion processes defined and studied in [26], [22], and [23]. Also, one hope is to try to apply these methods to the last passage time in last passage percolation under the condition that the length of a trip has to be exactly n. In order to apply the methods to percolation, still some work has to be done to modify the quantity S_n.

4.2 Homogenization

We are studying the convergence of:

$$\begin{aligned}\frac{1}{n}u_n(nq,\omega) &= \frac{1}{n}\sup_Q\{nh(Q/n)-S_n(nq,Q,\omega)\}\\ &= \sup_Q\left\{h(Q)-\frac{1}{n}S_n(nq,nQ,\omega)\right\}.\end{aligned}$$

Let us first prove the following lemma under the above assumptions on L:

Lemma 4.2.1. *For each compact set $K\in\mathbb{R}^d$ there exists $\mu_0>0$ such that for each $\mu>\mu_0$ there exists a real number M such that for all $\lambda>0$ the functions $\frac{1}{n}S_n(nq,nQ,\omega)$ of $q,Q\in K$ are equicontinuous as $n\in\mathbb{N}$ and $\omega\in\Omega(\lambda,\mu)$ where*

$$\Omega(\lambda,\mu)=\{\omega\in\Omega:\sup_z\sup_{|p|\le M}L(p,\tau_z\omega)\le\lambda, C_0(\omega)\le\mu, C_1(\omega)\le\mu\}.$$

Moreover, $\mathbb{P}\left(\bigcup_{\mu>0}\bigcup_{\lambda>0}\Omega(\lambda,\mu)\right)=1.$

Proof. Fix $q,Q\in\mathbb{R}^d$. Let us choose M to be any real number greater than $2(|Q|+|q|+1)$ such that $\Phi(q)>2A|q|$ for $|q|>\frac{M}{2}$, where $A=8(5\mu+|\inf\Phi|)$. Let us prove that for each $\lambda>0$ and $\varepsilon>0$ there exists $\delta>0$ such that for all $v,V\in\mathbb{B}_0(\delta)$ the following inequality holds for all $n\in\mathbb{N}$ and all $\omega\in\Omega(\lambda,\mu)$: $\left|\frac{1}{n}S_n(n(q+v),n(Q+V),\omega)-\frac{1}{n}S_n(nq,nQ,\omega)\right|<\varepsilon$. From the inequality

$$\begin{aligned}&\left|\frac{1}{n}S_n(n(q+v),n(Q+V),\omega)-\frac{1}{n}S_n(nq,nQ,\omega)\right|\\ \le&\left|\frac{1}{n}S_n(n(q+v),n(Q+V),\omega)-\frac{1}{n}S_n(n(q+v),nQ,\omega)\right|+\\ &\left|\frac{1}{n}S_n(n(q+v),nQ,\omega)-\frac{1}{n}S_n(nq,nQ,\omega)\right|\end{aligned}$$

we deduce that it is sufficient to prove that for each $\varepsilon>0$ there exists $\delta>0$ such that the inequality $\left|\frac{1}{n}S_n(nq,n(Q+V),\omega)-\frac{1}{n}S_n(nq,nQ,\omega)\right|\le\varepsilon$ holds for every $V\in\mathbb{B}_0(\delta)$, every n, and every $\omega\in\Omega(\lambda,\mu)$. Let us take $\delta=\frac{\varepsilon}{2^{10}M\lambda}$ and let n_0 be some integer such that $\lceil n\delta\rceil<\left\lfloor\frac{n}{4}\right\rfloor$ and $\lceil n\delta\rceil<2n\delta$ for all $n\ge n_0$. We now have that

$$\begin{aligned}&S_n(nq,n(Q+V),\omega)-S_n(nq,nQ,\omega)\\ =\;&S_n(nq,n(Q+V),\omega)-S_{\lceil n(1+\delta)\rceil}(nq,n(Q+V),\omega)\\ &+S_{\lceil n(1+\delta)\rceil}(nq,n(Q+V),\omega)-S_n(nq,nQ,\omega).\end{aligned}\tag{4.2.1}$$

If we take any sequence $q_0=nq$, q_1, …, $q_n=nQ$ we can form a new sequence q'_0, q'_1, …, q'_n, q'_{n+1}, …, $q_{\lceil n(1+\delta)\rceil}$ such that $q'_i=q_i$ for $i\in\{0,1,2,\dots,n\}$, and $q'_j=nQ+(j-n)\cdot\frac{nV}{\lceil n(1+\delta)\rceil-n}$ for $j\in\{n+$

$1, \ldots, \lceil n(1+\delta)\rceil\}$ hence $S_{\lceil n(1+\delta)\rceil}(nq, n(Q+V), \omega) - S_n(nq, nQ, \omega) \leq \lceil n\delta\rceil \cdot \sup_z |L(V/\delta, \tau_z\omega)| \leq 2n\delta\lambda$. This gives an upper bound for the second term of the right-hand side of (4.2.1).

To bound the first term of the right-hand side of (4.2.1) let us start by taking an arbitrary sequence $q_0, q_1, \ldots, q_{\lceil n(1+\delta)\rceil}$ with $q_0 = nq$ and $q_{\lceil n(1+\delta)\rceil} = n(Q+V)$ for which

$$\sum_{i=0}^{\lceil n(1+\delta)\rceil - 1} L(q_{i+1} - q_i, \tau_{q_i}\omega) \leq \frac{\varepsilon}{2^{10}} + S_{\lceil n(1+\delta)\rceil}(nq, n(Q+V), \omega).$$

Observe that there are at most $n/8$ elements q_i of the previous sequence for which $|q_{i+1} - q_i| > M/2$. Otherwise we would have $\sum L(q_{i+1} - q_i, \tau_{q_i}\omega) \geq \sum \Phi(q_{i+1} - q_i) - \sum C_0(\tau_{q_i}\omega) \geq 2nA/8 - n|\inf \Phi| - n\mu > 4\mu n + 1$. However by choosing μ_0 such that $\|L(Q+V-q, \omega)\|_{Q,v\in K} \leq \mu_0$ we can guarantee that $4\mu n + 1 > S_{\lceil n(1+\delta)\rceil}(nq, n(Q+V), \omega) < 4\mu n$ which is impossible.

Therefore there exist $\lceil n(1+\delta)\rceil - n$ indeces $i \in \{1, 2, \ldots, \lceil n(1+\delta)\rceil\}$ such that $|q_{i-1} - q_i| < M/2$, and $|q_i - q_{i+1}| < M/2$. Let us color these indeces in green and the others in red. We may further assume that none of the green vertices are adjacent. Let $r_0, r_1, \ldots, r_n$ be the red subsequence of the sequence $q_0, \ldots, q_n$. We clearly have $r_0 = nq$, $r_n = n(Q+V)$ and $|r_i - r_{i+1}| \leq M$ for all i. We are now able to control the difference $\sum_{i=1}^{n-1} L(r_{i+1} - r_i, \tau_{r_i}\omega) - \sum_{i=0}^{\lceil n(1+\delta)\rceil - 1} L(q_{i+1} - q_i, \tau_{q_i}\omega)$ because these two sums have many terms in common. Whenever r_i, and r_{i+1} are two consecutive terms in the sequence $\{q_j\}$ the term $L(r_{i+1} - r_i, \tau_{r_i}\omega)$ will cancel with the corresponding term of the other sum. The only problem are those pairs (r_i, r_{i+1}) that correspond to the pair (q_j, q_{j+2}) such that q_{j+1} is a green vertex. However we then have $|r_i - r_{i+1}| = |q_j - q_{j+2}| \leq |q_j - q_{j+1}| + |q_{j+1} - q_{j+2}| \leq M$ giving us $|L(r_{i+1} - r_i, \tau_{r_i}\omega)| \leq \lambda$. We also have $|L(q_{i+1} - q_i, \tau_{q_i}\omega)| \leq \lambda$ as well as $|L(q_{i+2} - q_{i+1}, \tau_{q_{i+1}}\omega)| \leq \lambda$. There are at most $(\lceil n(1+\delta) - n)$ such pairs (r_i, r_{i+1}) hence the difference between two sums can be bounded as follows:

$$\begin{aligned} \sum_{i=0}^{n-1} L(r_{i+1} - r_i, \tau_{r_i}\omega) - \sum_{i=0}^{\lceil n(1+\delta)\rceil - 1} L(q_{i+1} - q_i, \tau_{q_i}\omega) &\leq \frac{\varepsilon}{2^{10}} + 3(\lceil n(1+\delta)\rceil - n)\lambda \\ &\leq \frac{\varepsilon}{2^{10}} + 6n\delta\lambda. \end{aligned}$$

Now (4.2.1) implies that

$$\left| \frac{1}{n} S_n(nq, n(Q+V), \omega) - \frac{1}{n} S_n(nq, nQ, \omega) \right| \leq 2\delta\lambda + 6\delta\lambda + \frac{\varepsilon}{2^{10} n} < \varepsilon.$$

The previous inequalities hold for $n \geq n_0$. However, it is obvious that each S_n is equicontinuous for $\omega \in \Omega(\lambda, \mu)$ and there are only finitely many of them, so the first part of the lemma is proved.

For each $\varepsilon > 0$ there exists $\mu > 0$ such that $\mathbb{P}(C_0(\omega) \leq \mu, C_1(\omega) \leq \mu) \geq 1 - \varepsilon$. Once such μ is fixed we clearly have $\bigcup_{\lambda > 0} \{\omega \in \Omega : \sup_z \sup_{|p| \leq M} L(p, \tau_z\omega) \leq \lambda\} = \Omega$ almost surely and this proves the second part of the lemma. □

Lemma 4.2.2. *Under the above conditions on L there exists a function $\bar{L}:\mathbb{R}^d\to\overline{\mathbb{R}}$ that is convex, coercive, finite in an open set around* 0*, and for which*

$$\lim_{n\to\infty}\frac{1}{n}S_n(0,nq,\omega)=\bar{L}(q)$$

almost surely and in L^1. Such L further satisfies

$$\lim_{n\to\infty}\mathbb{E}\left|\frac{1}{n}S_n(nq,nQ,\omega)-\bar{L}(Q-q)\right|=0$$

for all $q,Q\in\mathbb{R}^d$.

Proof. For integers $m<n$ and any $q\in\mathbb{R}^d$ let us denote:

$$A_{m,n}(q,\omega)=\inf\left\{\sum_{k=0}^{n-m-1}L(q_{k+1}-q_k,\tau_{q_k}\omega):q_0=mq,q_1,\dots,q_{n-m}=nq\right\}.$$

Obviously, we have $A_{m,n}(q,\omega)\leq A_{m,k}(q,\omega)+A_{k,n}(q,\omega)$ for all $m<k<n$. The distribution of $A_{m,n}(q,\cdot)$ is the same as $A_{m+1,n+1}(q,\cdot)$. We also have that $\mathbb{E}(|A_{0,n}(q,\omega)|)<+\infty$ hence by Kingman's Subadditive Ergodic Theorem we have that there exists a limit

$$\lim_{n\to\infty}\frac{1}{n}A_{0,n}(q,\omega)=\bar{L}(q,\omega)$$

for almost all $\omega\in\Omega$. Let us now prove that $\bar{L}(q,\omega)$ does not depend on ω. Because of the ergodicity of $(\tau_z)_{z\in\mathbb{R}^d}$ it suffices to show that $\bar{L}(q,\omega)=\bar{L}(q,\tau_z\omega)$ for almost all ω. We will use $S_n(0,nq,\tau_z\omega)=S_n(z,nq+z,\omega)$ and the lemma 4.2.1. For each $\lambda>0$, $\mu>0$, and each $\varepsilon>0$ there exists $\delta>0$ such that $\left|\frac{1}{n}S_n(nu,nq+nu,\omega)-\frac{1}{n}S_n(0,nq,\omega)\right|<\varepsilon$ whenever $|u|\leq\delta$ and $\omega\in\Omega(\lambda,\mu)$. For such δ there exists n_0 such that $\frac{|z|}{n}<\delta$ whenever $n\geq n_0$. This establishes that there exists $\bar{L}(q)$ such that $\bar{L}(q)=\bar{L}(q,\omega)$ for almost all $\omega\in\Omega(\lambda,\mu)$. This holds for all λ and μ therefore $\bar{L}(q)=\bar{L}(q,\omega)$ for almost all ω.

Since $\frac{1}{n}S_n(nq,nQ,\omega)=\frac{1}{n}S_n(0,n(Q-q),\tau_{nq}\omega)$ and the last quantity has the same distribution as $\frac{1}{n}S_n(0,n(Q-q),\omega)$ we conclude that

$$\lim_{n\to\infty}\mathbb{E}\left|\frac{1}{n}S_n(nq,nQ,\omega)-\bar{L}(Q-q)\right|=0.$$

Let us prove that $\bar{L}$ is convex function. Let $\alpha,\beta\in\mathbb{Q}\cap(0,1)$ satisfy $\alpha+\beta=1$. We will now prove that for any two $Q_1,Q_2\in\mathbb{R}^d$ $\bar{L}(\alpha Q_1+\beta Q_2)\leq\alpha\bar{L}(Q_1)+\beta\bar{L}(Q_2)$. Assume that n is integer such that αn and βn are both integers. Let q_0, q_1, $\dots$, q_n be elements of $\mathbb{R}^d$ for which $q_0=0$, $q_{\alpha n}=\alpha nQ_1$, $q_n=\alpha nQ_1+\beta nQ_2$. We have

$$\begin{aligned}\frac{1}{n}S_n(0,n(\alpha Q_1+\beta Q_2),\omega) \quad&\leq\quad \alpha\cdot\frac{1}{\alpha n}\sum_{k=0}^{\alpha n-1}L(q_{k+1}-q_k,\omega)+\\ &\qquad \beta\cdot\frac{1}{\beta n}\sum_{k=\alpha n}^{n-1}L(q_{k+1}-q_k,\omega).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{n}S_n(0,n(\alpha Q_1+\beta Q_2),\omega) &\leq \alpha\cdot\frac{1}{\alpha n}S_{\alpha n}(0,\alpha nQ_1,\omega)+\\ &\quad \beta\cdot\frac{1}{\beta n}S_{\beta n}(\alpha nQ_1,n(\alpha Q_1+\beta Q_2),\omega).\end{aligned}$$

We know that the left-hand side and the first term on the right-hand side converge almost surely to $\bar{L}(\alpha Q_1+\beta Q_2)$ and $\alpha\bar{L}(Q_1)$. We also know that the second term on the right-hand side converge to $\beta\bar{L}(Q_2)$ in L^1. According to the Riesz-Weyl theorem there exists a subsequence n_k for which this term converge almost surely to $\beta\bar{L}(Q_2)$ which finally gives us

$$\bar{L}(\alpha Q_1+\beta Q_2)\leq\alpha\bar{L}(Q_1)+\beta\bar{L}(Q_2).$$

Now we will prove that $\bar{L}(Q)$ is continuous. For each $\varepsilon>0$ there exists a sequence $q_0=0,\dots,q_n=nQ$ such that $\bar{L}(Q)\geq-\varepsilon+\frac{1}{n}\sum_{k=0}^{n-1}L(q_{k+1}-q_k,\tau_{q_k}\omega)$ and $|q_n-q_{n-1}|\leq 2|Q|$. We now have that for each $z\in\mathbb{B}_0(\delta/n)\cap\mathbb{B}_0(\varepsilon/n)$ the following inequalities hold:

$$\begin{aligned}\bar{L}(Q+z)-\bar{L}(Q) &\leq \frac{1}{n}\big(L(nQ+nz-q_{n-1},\tau_{q_{n-1}}\omega)-\\ &\quad L(nQ-q_{n-1},\tau_{q_{n-1}}\omega)+\varepsilon\big)\\ &\leq \frac{Cn|Q|\cdot|z|}{n}+2\varepsilon\leq C\cdot|Q|\cdot|z|+2\varepsilon.\end{aligned}$$

This implies the lower semi-continuity of $\bar{L}$, i.e. $\bar{L}(Q)\geq\limsup_{z\to 0}\bar{L}(Q+z)$. Let us show that $\bar{L}$ is convex. Let $\alpha,\beta\in\mathbb{R}\cap(0,1)$ satisfy $\alpha+\beta=1$, and let $Q_1,Q_2\in\mathbb{R}^d$. Let α_n, β_n be two sequences of rational numbers from $(0,1)$ such that $\alpha_n+\beta_n=1$, $\alpha_n\to\alpha$, $\beta_n\to\beta$, and let Q_1^n, Q_2^n bet two sequences from $\mathbb{R}^d$ such that $Q_1^n\to Q_1$, $Q_2^n\to Q_2$, and $\alpha_nQ_1^n+\beta_nQ_2^n=\alpha Q_1+\beta Q_2$. Now we get

$$\bar{L}(\alpha Q_1+\beta Q_2) = \bar{L}(\alpha_nQ_1^n+\beta_nQ_2^n)\leq\alpha_n\bar{L}(Q_1^n)+\beta_n\bar{L}(Q_2^n).$$

Taking the lim sup on the right-hand side we get $\bar{L}(\alpha Q_1+\beta Q_2)\leq\alpha\bar{L}(Q_1)+\beta\bar{L}(Q_2)$ which means that $\bar{L}$ is convex.

The coercivity of $\bar{L}$ follows from the coercivity of L. For every choice of $q_1,\dots,q_n=nQ$ we have

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1}L(q_{i+1}-q_i,\tau_{q_i}\omega) &\geq \frac{1}{n}\sum_{i=0}^{n-1}\Phi(q_{i+1}-q_i)-\frac{1}{n}\sum_{i=0}^{n-1}C_0(\omega)\\ &\geq \Phi\left(\frac{1}{n}\sum_{i=0}^{n-1}q_{i+1}-q_i\right)-C_0(\omega)\\ &= \Phi(Q)-C_0(\omega).\end{aligned}$$

It is obvious to see that $\bar{L}$ is finite for all q's for which L is finite. □

Proof of the theorem 4.1.1. There exists a compact set $\Gamma \subseteq \mathbb{R}^d$ such that for all $n \in \mathbb{N}$

$$\sup_Q \left\{ h(Q) - \frac{1}{n} S_n(nq, nQ, \omega) \right\} = \sup_{Q \in \Gamma} \left\{ h(Q) - \frac{1}{n} S_n(nq, nQ, \omega) \right\}.$$

In order to prove the previous equality let us start by observing that the quantity on the left-hand side is bounded below by $h(q) - L(0, \tau_{nq}\omega) \geq |h(q)| - \|L(0,\omega)\|_\infty$. This bound can be obtained by choosing $Q = q$, and $q_i = nq$ for all i. Using the coercivity of L we get that $h(Q) - \frac{1}{n} S_n(nq, nQ, \omega) \geq h(Q) - \Phi(Q-q) + C_0(\omega)$ and this will be definitely smaller than $h(q) - \|L(0,\omega)\|_\infty$ if $|Q|$ is large enough.

In a similar way we can prove that there exists a compact set Γ for which

$$\sup_Q \{h(Q) - \bar{L}(Q-q)\} = \sup_{Q \in \Gamma} \{h(Q) - \bar{L}(Q-q)\}.$$

Notice that if $u, v : B \to \mathbb{R}$ are any two real valued functions then $\sup_b u(b) - \sup_b v(b) \leq \sup_b |u(b) - v(b)|$. This means that for each $\eta > 0$ we have

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}\left[\sup_{q' \in B(q,\eta)} \left| \frac{1}{n} u_n(nq', \omega) - u(q') \right| \right] \\ \leq \; &\lim_{n\to\infty} \mathbb{E}\left[\sup_{q' \in B(q,\eta)} \sup_{Q \in \Gamma} \left| \frac{1}{n} S_n(nq', nQ, \omega) - \bar{L}(Q-q') \right| \right]. \end{aligned}$$

Assume that $\Gamma \subseteq \mathbb{B}_0(M)$. Let $\lambda, \mu > 0$ and $\Omega(\lambda,\mu)$ be as in lemma 4.2.1. It is easy to prove that for each $\lambda, \mu > 0$:

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{q' \in B(q,\eta)} \sup_{Q \in \Gamma} \left| \frac{1}{n} S_n(nq', nQ, \omega) - \bar{L}(Q-q') \right| 1(\Omega(\lambda,\mu)) \right] = 0.$$

The last limit can be made arbitrarily small by choosing first a subset of Ω on which S_n are equicontinuous and a finite grid q_i', Q_i on which S_n and $\bar{L}$ don't differ by much from the values outside of the grid. Then the given supremums can be replaced by sums over the grid. Since these sums are finite and each summand has zero limit we have established the convergence to zero on $\Omega(\lambda,\mu)$.

Outside of $\Omega(\lambda,\mu)$ things are nice because the integrand is bounded uniformly in $|q| + |\eta| + |Q|$, and $\mathbb{P}(\Omega(\lambda,\mu)) \to 0$ by monotone convergence theorem. $\square$

4.3 Sub-linearity of Functions from $\mathscr{K}$

It is immediate that if $g \in \mathscr{K}$ then $g(q_2 - q_1, \tau_{q_1}\omega) = -g(q_1 - q_2, \tau_{q_2}\omega)$. Therefore $g(q_n - q_0, \tau_{q_0}\omega) = \sum_{i=0}^{n-1} g(q_{i+1} - q_i, \tau_{q_i}\omega)$. We also have $g(q_2 - q_1, \tau_{q_1}\omega) = g(q_2,\omega) - g(q_1,\omega)$.

Lemma 4.3.1. *For $f \in \mathscr{K}$ we have*

$$\lim_{n\to\infty} \sup_{|z|\leq n} \frac{f(z)}{n} = 0$$

for almost all ω.

This lemma is a continuous analog to the theorem 3 from [25] and the proof is quite similar. We will use the following theorem of Garsia, Rodemich, and Rumsey.

Theorem 4.3.1. *Let $h : \mathbb{R}^d \to \mathbb{R}$ be a continuous function on B_1. Assume that $\gamma > 2d$. If*

$$\int_{B_1}\int_{B_1} \frac{|h(x)-h(y)|^{d+\alpha}}{|x-y|^{\gamma}}\, dxdy \leq c_0$$

for some $c_0 \in \mathbb{R}$ then for $x, y \in \mathbb{B}_1$ we have

$$|h(x)-h(y)| \leq c_1 |x-y|^{\frac{\gamma-2d}{d+\alpha}}$$

where c_1 depends on c_0 and on the dimension d.

We will first prove the following lemma:

Lemma 4.3.2. *The functions $f_n(x,\omega) = \frac{1}{n} f(nx,\omega)$ are equicontinuous.*

Proof. We will use the theorem 4.3.1 to establish this fact. Notice that

$$\int_{B_d}\int_{B_d} \frac{|f_n(x,\omega)-f_n(y,\omega)|^{d+\beta}}{|x-y|^{\gamma}}\, dxdy = \frac{1}{n^{3d+\beta-\gamma}} \int_{B_n}\int_{B_n} \frac{|f(x,\omega)-f(y,\omega)|^{d+\beta}}{|x-y|^{\gamma}}\, dxdy.$$

We break the last expression as the sum $A_1 + A_2$ where

$$\begin{aligned} A_1 &= \frac{1}{n^{3d+\beta-\gamma}} \int_{B_n}\int_{B_n\cap B_d(y)} \frac{|f(x,\omega)-f(y,\omega)|^{d+\beta}}{|x-y|^{\gamma}}\, dxdy, \\ A_2 &= \frac{1}{n^{3d+\beta-\gamma}} \int_{B_n}\int_{B_n\cap B_d(y)^C} \frac{|f(x,\omega)-f(y,\omega)|^{d+\beta}}{|x-y|^{\gamma}}\, dxdy. \end{aligned}$$

In order to bound the quantity A_1 we use the Lipschitz property of f: $|f(x,\omega)-f(y,\omega)| \leq K|x-y|$. Therefore $A_1 \leq \frac{1}{n^{3d+\beta-\gamma}} \int_{B_n} K^{d+\beta} \cdot \int_{B_d} |z|^{d+\beta-\gamma} dzdy$. In order for this integral to be finite we need $2d+\beta-\gamma > 0$.

We now want to bound the term A_2. For each $x \in \mathbb{R}^d$, let us denote by $[x]$ the unique vector $v \in \mathbb{Z}^d$ for which $x - v \in [0,1]^d \setminus \{(1,1,\ldots,1)\}$. Since f is Lipschitz we have that $|f(x,\omega) -$

$f(y,\omega)| \leq 2K + |f([x],\omega) - f([y],\omega)|$. For (x,y) in our domain of integration we have $[x] \neq [y]$ hence $|x-y| \leq 2 + |[x]-[y]| \leq 3|[x]-[y]|$. This implies that

$$A_2 \leq C\frac{1}{n^{3d+\beta-\gamma}} \sum_{i,j\in[-n,n]^d, i\neq j} \frac{|f(i,\omega)-f(j,\omega)|^{d+\beta}}{|i-j|^{\gamma}}.$$

Notice that in the previous sum we have $|f(i,\omega)-f(j,\omega)|^{d+\beta} = \left|\sum_k f(q_{k+1}-q_k, \tau_{q_k}\omega)\right|^{d+\beta}$, where the last sum is taken over the sequence $q_0,\dots,q_{|j-i|} \in \mathbb{Z}^d$ of length $|j-i|$ that satisfies

(i) $q_0 = i$, $q_{|j-i|} = j$, $|q_{k+1}-q_k| = 1$.

(ii) Given the standard basis $e_1,\dots,e_d$ of $\mathbb{Z}^d$, we assume that if q_{k+1} and q_k differ only in the coordinate e_i then for $l > k$, q_{l+1} and q_l differ in the coordinate e_j for $j \geq i$.

Such a sequence q is obviously unique. Using the Hölder's inequality we get

$$|f(i,\omega)-f(j,\omega)|^{d+\beta} \leq |j-i|^{d+\beta-1} \cdot \sum_k |f(q_{k+1}-q_k, \tau_{q_k}\omega)|^{d+\beta}, \text{ hence}$$

$$A_2 \leq C\frac{1}{n^{3d+\beta-\gamma}} \sum_{i,j\in[-n,n]^d, i\neq j} \sum_k \frac{|f(q_{k+1}-q_k, \tau_{q_k}\omega)|^{d+\beta}}{|i-j|^{\gamma+1-d-\beta}}.$$

Let $v \in [-n,n]^d \cap \mathbb{Z}^d$ be fixed. Let us fix a unit vector from $\mathbb{Z}^d$. That vector must be of the form $\pm e_i$, but for the clarity let us assume that it is e_1. We want to see the $D_{e_1,v}$ coefficient next to $|f(e_1,\tau_v\omega)|$. The elements v and $v+e_1$ could be adjacent terms in some of the sums if the first element i in the sequence has the same projection to the $e_1^{\perp}$ hyperplane as v; and the last element j has the same projection to e_1 as v. Therefore

$$D_{e_1,v} \leq C\sum_{p=1}^{n}\sum_{q=1}^{n} \frac{d_q}{p^{\gamma+1-d-\beta}+q^{\gamma+1-d-\beta}}.$$

The coefficient d_q from the last sum represents the number of length q sequences of points in the integer lattice that satisfy (i) and (ii). Such number is equal to the number of non-negative integer solutions to $x_1+\cdots+x_d = q$. This number is equal to $\binom{q+d-1}{d-1} \leq Cq^{d-1}$ hence $D_{e_1,v} \leq C\sum_{p=1}^{n}\sum_{q=1}^{n} \frac{q^{d-1}}{p^{\gamma+1-d-\beta}+q^{\gamma+1-d-\beta}}$. Consider the function $\phi(a) = \sum_{i=1}^{\infty} \frac{i^{\mu}}{a+i^{\nu}}$ where μ,ν are positive real numbers satisfying $\mu < \nu - 1$. Then

$$\begin{aligned}
\phi(a) &\leq C\int_0^{+\infty} \frac{x^{\mu}}{a+x^{\nu}}\,dx = C\int_0^{a^{1/\nu}} \frac{x^{\mu}}{a+x^{\nu}}\,dx + C\int_{a^{1/\nu}}^{+\infty} \frac{x^{\mu}}{a+x^{\nu}}\,dx \\
&\leq C\int_0^{a^{1/\nu}} \frac{x^{\mu}}{a}\,dx + C\int_{a^{1/\nu}}^{+\infty} \frac{x^{\mu}}{x^{\nu}}\,dx \\
&= Ca^{\frac{\mu+1}{\nu}-1} + Ca^{\frac{\mu-\nu+1}{\nu}} \leq Ca^{\frac{\mu-\nu+1}{\nu}}.
\end{aligned}$$

From there we can conclude that $D_{e_1,v} \leq C\sum_{p=1}^{n} p^{2d+\beta-\gamma-1} < Cn$, if $2d+\beta-\gamma-1<0$, or $\gamma > 2d-1+\beta$. We now have

$$\begin{aligned} A_2 &\leq C\frac{1}{n^{3d+\beta-\gamma-1}} \cdot \sum_{v\in\mathbb{Z}^d\cap[-n,n]^d} \sum_{e\in\mathbb{Z}^d,\|e\|=1} |f(e,\tau_v\omega)|^{d+\beta} \\ &\leq \frac{C}{n^{2d+\beta-\gamma-1}} \cdot \frac{1}{n^d} \sum_{v\in\mathbb{Z}^d\cap[-n,n]^d} \sum_{e\in\mathbb{Z}^d,\|e\|=1} |f(e,\tau_v\omega)|^{d+\beta} \\ &\leq \frac{C}{n^{2d+\beta-\gamma-1}} \cdot \sum_{e\in\mathbb{Z}^d,\|e\|=1} \frac{1}{n^d}\int_{B_n} |f(e,\tau_v\omega)|^{d+\beta}\,dv. \end{aligned}$$

We can now use the ergodic theorem to conclude that

$$\lim_{n\to\infty}\frac{1}{n^d}\int_{B_n}|f(e,\tau_v\omega)|^{d+\beta}\,dv = \mathbb{E}[|f(e,\omega)|^{d+\beta}].$$

Therefore the quantity A_2 as a function of n converges to 0, hence it is bounded.

We can now use the theorem of Garsia, Rodemich, and Rumsey to conclude that the sequence of functions $f_n(x,\omega)$ is equicontinuous. □

Proof of the lemma 4.3.1. The family $\{f_n(x,\omega)\}$ of functions is equicontinuous as shown in the lemma 4.3.2. It is also uniformly bounded because the functions are Lipschitz. According to the Arzela-Ascoli theorem we get that there is a subsequence f_{n_k} that converges to some function g. We will prove that g has to be zero. This will furthermore imply that even the original sequence f_n has to converge to 0 (because 0 is the only accumulation point of the sequence).

In order to prove that $g=0$ we will first prove that g is a constant function. It suffices to prove that $g(x,\omega)=g(y,\omega)$ for any two $x,y\in\mathbb{R}^d$ that differ in the first coordinate only. We will prove that for any $d-1$ dimensional box B we have

$$\int_B g((x_1,z),\omega)\,dz = \int_B g((y_1,z),\omega)\,dz.$$

It is enough to prove this result only for boxes B of the type $[0,b_2]\times[0,b_3]\times\cdots\times[0,b_d]$, and $y_1=0$. Because of the uniform convergence it is enough to prove that for sufficiently large n we have that the following quantity is small:

$$\frac{1}{n}\int_B f(n(x_1,z),\omega)-f(n(0,z),\omega)\,dz \;=\; \frac{1}{n^d}\int_{nB} f((nx_1,z),\omega)-f((0,z),\omega)\,dz.$$

We have

$$
\begin{aligned}
f((nx_1,z),\omega)-f((0,z),\omega) &= \sum_{i=1}^{n} f\left((x_1,0), \tau_{((i-1)x_1,z)}\omega\right) \\
&= \frac{1}{|x_1|}\int_0^{nx_1} f((x_1,0),\tau_{(x,z)}\omega)\,dx \\
&\quad +\sum_{i=1}^{n}\frac{1}{|x_1|}\int_{(i-1)x_1}^{ix_1} f((x_1,0),\tau_{((i-1)x_1,z)}\omega)-f((x_1,0),\tau_{(x,z)}\omega)\,dx.
\end{aligned}
$$

The Lipschitzness of f gives that the absolute value of the last term is smaller than or equal to $\frac{nC}{|x_1|}$. Once divided by n^d this will tend to 0 so it suffices to prove that

$$
\lim_{n\to\infty}\frac{1}{n^d}\int_{[0,nx_1]\times nB} f((x_1,0),\tau_{(x,z)}\omega)\,dxdz = 0.
$$

However, according to the ergodic theorem, this quantity converges to $\mathbb{E}(f((x_1,0),\omega)) = 0$.

Now we have proved that g is constant. The equality $g(0,\omega) = g(0,\omega)+g(0,\omega)$ implies that this constant must be 0. This completes the proof. □

4.4 Variational Formula for $\bar{L}$

In this section we prove the variational formula $\bar{L}$ that is stated in the theorem 4.1.2. One direction of the proof involves a method analogous to constructing a subsolution to the cell problem of the Hamilton-Jacobi equation. Sublinearity of the correctors is necessary in order for them to form a small perturbation, and in order to guarantee this property we use the lemma 4.3.1.

Proof of the theorem 4.1.2. The lemma 4.3.1 implies that for every $g \in \mathscr{K}$ the following relation holds:

$$
\lim_{n\to\infty}\sup_{|z|\leq n}\frac{1}{n}g(z,\omega) = 0.
$$

Assume that $h(q) = p\cdot q$ for some $p\in\mathbb{R}^d$. In the theorem 4.1.1 we already proved the existence of the limit for any initial data. Using this particular initial data we conclude that

$$
\lim_{n\to\infty}\mathbb{E}\left[\frac{1}{n}u_n(nq,\omega)\right] = \sup_{Q}\{p\cdot Q-\bar{L}(Q-q)\} = p\cdot q+\bar{L}^*(p).
$$

In the last equality we used that $\sup_Q\{p\cdot Q-\bar{L}(Q-q)\} = p\cdot q+\sup_Q\{p\cdot(Q-q)-\bar{L}(Q-q)\} = p\cdot q+\sup_a\{p\cdot a-L(a)\} = p\cdot q+\bar{L}^*(p)$.

Let us denote $\bar{H}^g(p) = \sup_{a\in\mathbb{R}^d}\text{esssup}\{p\cdot a+g(a,\omega)-L(a,\omega)\}$. Our first goal is to prove the following inequality for each $g\in\mathscr{K}$:

$$
\lim_{n\to\infty}\mathbb{E}\left[\frac{1}{n}u_n(nq,\omega)\right] \leq p\cdot q+\bar{H}^g(p).
$$

Once the function g is fixed, let us introduce

$$v_n^g(q,\omega) = \sup_{q_1,\dots,q_n} \left\{ p\cdot q_n + \sum_{k=0}^{n-1} g(q_{k+1}-q_k, \tau_{q_k}\omega) - \sum_{k=0}^{n-1} L(q_{k+1}-q_k,\tau_{q_k}\omega) \right\}.$$

It is easy to see that $\lim_{n\to\infty}\mathbb{E}\left[\frac{1}{n}v_n^g(nq,\omega)\right] = \lim_{n\to\infty}\mathbb{E}\left[\frac{1}{n}u_n(nq,\omega)\right]$. For almost all ω we have:

$$\begin{aligned}
v_n^g(q,\omega) &= \sup_{q_1,\dots,q_{n-1}} \left\{ p\cdot q_{n-1} + g(q_{n-1}-q_0,\tau_{q_0}\omega) - \sum_{k=0}^{n-2} L(q_{k+1}-q_k,\tau_{q_k}\omega) + \right. \\
&\quad \left. \sup_{q_n}\left\{ p\cdot(q_n-q_{n-1}) + g(q_n-q_{n-1},\tau_{q_{n-1}}\omega) - L(q_n-q_{n-1},\tau_{q_{n-1}}\omega)\right\} \right\} \\
&= \sup_{q_1,\dots,q_{n-1}} \left\{ p\cdot q_{n-1} + g(q_{n-1}-q_0,\tau_{q_0}\omega) - \sum_{k=0}^{n-2} L(q_{k+1}-q_k,\tau_{q_k}\omega) + \right. \\
&\quad \left. \sup_{a}\left\{ p\cdot a + g(a,\tau_{q_{n-1}}\omega) - L(a,\tau_{q_{n-1}}\omega)\right\} \right\} \\
&\leq v_{n-1}^g(q,\omega) + \bar{H}^g(p) \leq \cdots \\
&\leq p\cdot q + n\bar{H}^g(p).
\end{aligned}$$

This implies that $\frac{1}{n}v_n^g(nq,\omega) \leq q + \bar{H}^g(p)$, hence $\bar{L}^*(p) \leq \bar{H}(p)$. To prove the reverse inequality we need to show that for each $\varepsilon > 0$ there exists a function g such that for almost all ω:

$$\sup_a \{g(a,\omega) + p\cdot a - L(a,\omega)\} \leq \varepsilon + \bar{L}^*(p).$$

Denote $L_p(a,\omega) = L(a,\omega) - p\cdot a$. For each $\lambda \in (0,1)$ let us consider the function

$$f^\lambda(\omega) = -\inf \sum_{n=0}^{\infty} L_p(q_{n+1}-q_n,\tau_{q_n}\omega)\lambda^n$$

where the last infimum is taken over all infinite sequences $q_0, q_1, \dots$ that satisfy $q_0 = 0$. Then $\lambda f^\lambda(\tau_a\omega) = -\inf\sum_{n=0}^{\infty} L_p(q_{n+1}-q_n,\tau_{q_n}\tau_a\omega)\lambda^{n+1}$. As a consequence, we derive the following equality: $\sup_a\left\{\lambda f^\lambda(\tau_a\omega) - L_p(a,\omega)\right\} = f^\lambda(\omega)$ which further gives us

$$\sup_a\left\{\lambda f^\lambda(\tau_a\omega) - \lambda f^\lambda(\omega) - L_p(a,\omega)\right\} = (1-\lambda)f^\lambda(\omega).$$

Our next goal is to prove that $\limsup(1-\lambda)f^\lambda(\omega) \leq \bar{L}^*(p)$ as $\lambda \to 1$. The idea is to change the order of summations from the definition of f^λ, like in Abel-Dirichlet's theorem. However, since we have infimum here in the definition of f^λ things don't follow the standard argument. Fortunately, we are happy with the inequality only, and infimums go in correct directions so we can get rid of them without much trouble. To carry this out, let us start our proof by making the following observation: For each $N \in \mathbb{N}$ and each $\lambda \in (0,1)$ we have:

$$\begin{aligned}
& \sum_{n=0}^{N} L_p(q_{n+1}-q_n, \tau_{q_n}\omega)\lambda^n \\
= & \sum_{n=0}^{N} \left(\sum_{k=0}^{n} L_p(q_{k+1}-q_k, \tau_{q_k}\omega) \right) \cdot \left(\lambda^n - \lambda^{n+1}\right) + \sum_{k=0}^{N} L_p(q_{k+1}-q_k, \tau_{q_k}\omega)\lambda^{N+1} \\
= & (1-\lambda) \sum_{n=0}^{N} \left(\sum_{k=0}^{n} L_p(q_{k+1}-q_k, \tau_{q_k}\omega) \right) \cdot \lambda^n + \sum_{k=0}^{N} L_p(q_{k+1}-q_k, \tau_{q_k}\omega)\lambda^{N+1}.
\end{aligned}$$

The previous equalty now implies that for each $N \in \mathbb{N}$ and each $\lambda \in (0,1)$ we have the following upper bound:

$$\begin{aligned}
(1-\lambda)f^{\lambda}(\omega) \leq & -(1-\lambda)^2 \sum_{n=0}^{N} \inf \left\{ \sum_{k=0}^{n} L_p(q_{k+1}-q_k, \tau_{q_k}\omega) \right\} \lambda^n \\
& -(1-\lambda) \cdot \inf \sum_{k=0}^{N} L_p(q_{k+1}-q_k, \tau_{q_k}\omega)\lambda^{N+1} \\
& -(1-\lambda) \inf \sum_{n=N+1}^{\infty} L_p(q_{n+1}-q_n, \tau_{q_n}\omega)\lambda^n.
\end{aligned} \quad (4.4.1)$$

Each of the previous infimums was taken over the sequences $q_0, q_1, \dots$ of elements from $\mathbb{R}^d$ such that $q_0 = 0$.

We will show that the last two terms of (4.4.1) are small, while the first converges to $\bar{L}^*(p)$. Let us first restrict ourselves to $\omega \in \Omega(\Lambda)$ for arbitrary fixed $\Lambda > 0$. Assume that $\varepsilon > 0$ is a fixed real number. We have that:

$$-\frac{1}{n} \inf \sum_{k=0}^{n} L_p(q_{k+1}-q_k, \tau_{q_k}\omega) = \sup_a \left\{ p \cdot a - \frac{1}{n} S_n(0, na, \omega) \right\}.$$

From the theorem 4.1.1 we know that $\frac{1}{n}S_n(0,na,\omega) \to \bar{L}(a)$ almost surely and in L^1. According to Egorov's theorem this means that the convergence is almost uniform. Therefore for every $\delta > 0$ there exists a set Ω_δ such that $\mathbb{P}(\Omega_\delta) > 1-\delta$ and the previous convergence is uniform on Ω_δ. This implies that for each $\varepsilon > 0$ there exists N_0 such that for $N \geq N_0$ we have

$$\frac{1}{n} S_n(0, na, \omega) \in (\bar{L}(a) - \varepsilon, \bar{L}(a) + \varepsilon)$$

for all $\omega \in \Omega_\delta$. This further allows us to deduce that for all $\omega \in \Omega_\delta$ and all $n > N_0$ the following inequality holds:

$$-\frac{1}{n} \inf \sum_{k=0}^{n} L_p(q_{k+1}-q_k, \tau_{q_k}\omega) \leq \bar{L}^*(p) + \varepsilon.$$

Therefore we have the following bound on the first term of the right-hand side of (4.4.1) for each $\omega \in \Omega_\delta \cap \Omega(\Lambda)$ and $\lambda \in (0,1)$:

$$\begin{aligned}
& -(1-\lambda)^2 \sum_{n=0}^{N} \inf\left\{ \sum_{k=0}^{n} L_p(q_{k+1}-q_k, \tau_{q_k}\omega) \right\} \lambda^n \\
= \;& -(1-\lambda)^2 \sum_{n=0}^{N_0} \inf\left\{ \sum_{k=0}^{n} L_p(q_{k+1}-q_k, \tau_{q_k}\omega) \right\} \lambda^n \\
& -(1-\lambda)^2 \sum_{n=N_0+1}^{N} \inf\left\{ \sum_{k=0}^{n} L_p(q_{k+1}-q_k, \tau_{q_k}\omega) \right\} \lambda^n \\
\leq \;& (1-\lambda)^2 \sum_{n=0}^{N_0} N_0(p\cdot a - \Phi(a) + \Lambda) + (1-\lambda)^2 \sum_{n=0}^{N} (\bar{L}^*(p)+\varepsilon) n\lambda^n + \varepsilon \\
\leq \;& (1-\lambda)^2 N_0^2(|\Phi^*(p)| + \Lambda) + \bar{L}^*(p) + \varepsilon.
\end{aligned}$$

So far N_0 is chosen. We now choose λ_0 such that for $\lambda > \lambda_0$ we have $(1-\lambda)^2 N_0^2(|\Phi^*(p)+\Lambda) \leq \varepsilon$. After this we choose N so that the last two terms on the right-hand side of (4.4.1) are small. Take N to be an integer such that $(|\Phi^*(p)|+\Lambda)N\lambda^N < \varepsilon$. We notice that for all q_n, q_{n+1} we have

$$\begin{aligned}
L_p(q_{n+1}-q_n, \tau_{q_n}\omega) &= -p\cdot(q_{n+1}-q_n) + L(q_{n+1}-q_n, \tau_{q_n}\omega) \\
&\geq -p\cdot(q_{n+1}-q_n) + \Phi(q_{n+1}-q_n) - \Lambda \\
&\geq -|\Phi^*(p)| - \Lambda
\end{aligned}$$

hence we see that the second term of the right-hand side of (4.4.1) is bounded by

$$N\lambda^{N+1}(1-\lambda)(|\Phi^*(p)|+\Lambda) < \varepsilon.$$

The third term is bounded by $\lambda^{N+1}\cdot(|\Phi^*(p)|+\Lambda)$ which is also smaller than ε.

Since this holds for every $\varepsilon > 0$ we have $\limsup_{\lambda\to 1}(1-\lambda)f^\lambda(\omega) \leq \bar{L}^*(p)$ for all ω that belongs to the set $\bigcup_{\delta>0}\bigcup_{\Lambda>0}\Omega_\delta \cap \Omega(\Lambda)$. Hence $\limsup(1-\lambda)f^\lambda(\omega) \leq \bar{L}^*(p)$ for almost all ω as $\lambda \to 1$.

Now we will prove that $f^\lambda(\tau_a\omega) - f^\lambda(\omega)$ is Lipschitz continuous in a with a Lipschitz constant uniform in λ.

For each $\varepsilon > 0$ there exists a sequence $q_0 = 0, q_1, q_2, \ldots$ such that

$$f^\lambda(\tau_a\omega) - \varepsilon \leq -\sum_{n=0}^{\infty} L_p(q_{n+1}-q_n, \tau_{q_n+a}\omega)\lambda^n.$$

Then $-f^\lambda(\omega) \leq L_p(q_1+a,\omega) + \sum_{n=1}^{\infty} L_p(q_{n+1}-q_n, \tau_{q_n+a}\omega)\lambda^n$ hence

$$\begin{aligned}
f^\lambda(\tau_a\omega) - f^\lambda(\omega) &\leq \varepsilon + L_p(q_1+a,\omega) - L_p(q_1,\tau_a\omega) \\
&\leq \varepsilon + |L_p(q_1+a,\omega) - L_p(q_1,\omega)| + |L_p(q_1,\omega) - L_p(q_1,\tau_a\omega)|.
\end{aligned}$$

The functions $L(q_1+\cdot,\omega)$ and $L(q_1,\tau.\omega)$ are Lipschitz for bounded q_1. We may assume that $q_1\in\Gamma$ because of the coercivity of L_p and this finishes the proof that $f^\lambda(\tau_a\omega)-f^\lambda(\omega)$ is Lipschitz with a constant independent on λ.

Let us recall the inequality

$$f^\lambda(\tau_a\omega)-f^\lambda(\omega)\leq\frac{1}{\lambda}\Big(L_p(a,\omega)+(1-\lambda)f^\lambda(\omega)\Big).$$

For almost all $\omega\in\Omega$ we have $\limsup(1-\lambda)f^\lambda(\omega)\leq\bar{L}^*(p)$. Let us use the following modification of the Egorov' theorem whose proof is presented in the end:

Lemma 4.4.1. *Assume that* $\limsup f_n(\omega)\leq K$ *for almost all* ω*. Then for every* $\delta>0$ *there exists a set* $\Omega_\delta\subseteq\Omega$ *with* $\mathbb{P}(\Omega_\delta)\geq 1-\delta$ *such that for all* $\varepsilon>0$ *there exists* n_0 *for which* $f_n(\omega)\leq K+\varepsilon$ *for every* $\omega\in\Omega_\delta$ *and every* $n\geq n_0$*.*

Using the previous lemma we see that for each $\delta>0$ there exists a set $\Omega_\delta\subseteq\Omega$ whose complement is of measure at most δ and $\lambda_0\in(1/2,1)$ such that $(1-\lambda)f^\lambda(\omega)\leq\bar{L}^*(p)+1$ for all $\lambda\in(\lambda_0,1)$ and all $\omega\in\Omega_\delta$. For all $\lambda>\lambda_0$ we have

$$f^\lambda(\tau_a\omega)-f^\lambda(\omega)\leq 2(L_p(a,\omega)+1+\bar{L}^*(p)).$$

We also have $f^\lambda(\omega)-f^\lambda(\tau_a\omega)\leq 2(L_p(-a,\omega)+1+\bar{L}^*(p))$. For each fixed a we have that there exists a sequence λ_i such that $f^{\lambda_i}(\tau_a\omega)-f^{\lambda_i}(\omega)$ converges to some function $g(a,\omega)$ integrable on Ω_δ. Since the function is integrable on each Ω_δ with a finite integral, it is integrable on Ω as well. It is easy to see that the integral of such a function is equal to 0, by the dominated convergence theorem and the invariance of the measure $\mathbb{P}$ under τ_a.

Using a diagonal argument we can find a sequence λ_i $(i\in\mathbb{N})$ such that $f^{\lambda_i}(\tau_a\omega)-f^{\lambda_i}(\omega)\to g(a,\omega)$ for each $a\in\mathbb{Q}^d$. Let us prove that g defined by previous relation can be extended to entire $\mathbb{R}^d$ to a function that belongs to $\mathscr{K}$. For $a\in\mathbb{R}^d$ and each $\varepsilon>0$ there exists $b\in\mathbb{Q}^d$ such that $|a-b|<\varepsilon$. For all $\lambda>0$ we have $f^\lambda(\tau_a\omega)-f^\lambda(\tau_b\omega)=f^\lambda(\tau_{a-b}\tau_b\omega)-f^\lambda(\tau_b\omega)$. We have proved that the previous quantity is Lipschitz uniformly in ω and λ hence we can continuously extend g from $\mathbb{Q}^d$ to $\mathbb{R}^d$. We saw that $\mathbb{E}(g(a,\omega))=0$ for all $a\in\mathbb{Q}^d$. Using the Lipschitzness we see that the previous relation will hold for all $a\in\mathbb{R}^d$. Also for any sequence $r_0,\dots,r_n$ such that $r_n=r_0$ from $\mathbb{R}^d$ we can find a sequence $q_0,\dots,q_n,q_0$ sufficiently close to $r_0,\dots,r_n,r_0$ such that $r_i\in\mathbb{Q}^d$ and $g(q_{i+1}-q_i,\tau_{q_i}\omega)-g(r_{i+1}-r_i,\tau_{r_i}\omega)\in(-\varepsilon,\varepsilon)$. Using the dominated convergence theorem we conclude that $\sum_{i=0}^{n-1}g(q_{i+1}-q_i,\tau_{q_i}\omega)=0$. This further implies that

$$\sum_{i=0}^{n-1}g(r_{i+1}-r_i,\tau_{r_i}\omega)\in(-\varepsilon,\varepsilon)$$

for all $\varepsilon>0$ meaning that this sum must be 0. This proves that $g\in\mathscr{K}$. □

For the sake of completeness, we will present the proof of the lemma 4.4.1. It is very similar in spirit to the Egorov's theorem, and is probably known in the literature.

Proof of the lemma 4.4.1. For each pair (p,q) of positive integers let us denote

$$F_{p,q} = \{\omega \in \Omega : f_n(\omega) \leq K + \frac{1}{p} \text{ for all } n \geq q\}.$$

We clearly have $\bigcup_{q=1}^{\infty} F_{p,q} = \{\omega \in \Omega : \limsup f_n(\omega) \leq K\}$. Since the last set is of measure 1 for each p there exists $q_p \in \mathbb{N}$ such that $\mathbb{P}(F_{p,q_p}) \geq 1 - \frac{\delta}{2^p}$. We claim that

$$\Omega_\delta = \bigcap_{p=1}^{\infty} F_{p,q_p}$$

satisfies the required properties. Indeed, the measure of Ω_δ^C is clearly smaller than δ, and we have that for each $\varepsilon > 0$ there exists p such that $\varepsilon > \frac{1}{p}$. For each $n > q_p$ we have that $f_n(\omega) < K + 1/p < K + \varepsilon$ for all $\omega \in F_{p,q_p}$ hence for all $\omega \in \Omega_\delta$. □

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