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Non-Self-Adjoint Operators and Microlocal Analysis in the Complex Domain

A dissertation submitted in partial satisfaction
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by

Francis White

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ABSTRACT OF THE DISSERTATION

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Professor Michael Hitrik, Chair

In this thesis, we examine aspects of non-self-adjoint (NSA) operators using the theory of microlocal analysis in exponentially weighted spaces of holomorphic functions on $\mathbb{C}^n$. We present four main results. The first theorem, which is the result of joint work with L. Coburn, M. Hitrik, and J. Sjöstrand, establishes a boundedness criterion for a class of Toeplitz operators acting on Bargmann spaces with quadratic weights. The Toeplitz operators that this result applies to have symbols of the form $\exp Q(z)$, where $Q(z)$ is an inhomogenous quadratic polynomial on $\mathbb{C}^n$. The second and third results of this thesis establish properties of solutions of time-dependent Schrödinger equations on $\mathbb{R}^n$ with NSA quadratic Hamiltonians. More specifically, these results pertain to solutions $u = u(t,x)$ of the initial value problem $(\partial_t + q^w(x,D))u(t,x) = 0$, $x \in \mathbb{R}^n$, $t \geq 0$, on $\mathbb{R}^n$, where the initial data $u|_{t=0} = u_0$ is a tempered distribution on $\mathbb{R}^n$ and $q^w(x,D)$ is the Weyl quantization of a complex-valued quadratic form on the phase space $\mathbb{R}^{2n}$ with non-negative real part $\operatorname{Re} q \geq 0$. Our second result characterizes the propagation in time of global analytic singularities of initial data u_0 by this evolution, and our third result establishes L^p-bounds for the evolution semigroup $e^{-tq^w(x,D)}$ in the short $0 < t \ll 1$ and long $t \gg 1$ time regimes under the assumption that the so-called "singular space" of $q(x,\xi)$ is trivial. In the fourth result of this thesis, we establish optimal L^p-bounds for low-lying eigenfunctions of analytic NSA semiclassical pseudodifferential operators with double characteristics, improving upon prior results in the field.

The dissertation of Francis White is approved.

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To my family.

Contents

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Chapter 1

Introduction

1.1 Overview

The purpose of this dissertation is to address some problems related to the understanding of non-self-adjoint (NSA) operators arising in the fields of mathematical physics and partial differential equations. More specifically, this thesis is concerned with three major topics: (1) the boundedness of Toeplitz operators on Bargmann spaces with quadratic exponential weights, (2) the behavior of solutions of time-dependent Schrödinger equations on $\mathbb{R}^n$ with NSA quadratic Hamiltonians, and (3) bounds for L^p-norms for low-lying eigenfunctions of analytic NSA semiclassical pseudodifferential operators on $\mathbb{R}^n$ with double characteristics. In this dissertation, we present four main results. The text is correspondingly divided into four sections:

1. *Weyl Symbols and Boundedness of Toeplitz Operators*

2. *Propagation of Global Analytic Singularities for Schrödinger Equations with Quadratic Hamiltonians*

3. *L^p-Bounds for Semigroups Generated by Non-Elliptic Quadratic Differential Operators*

4. *L^p-Bounds for Eigenfunctions of Analytic Non-Self-Adjoint Operators with Double Characteristics*

In this introduction, we give background and motivation for each of these works and provide informal statements of our main results.

1.2 Boundedness of Toeplitz Operators and the Berger-Coburn Conjecture

Let Φ be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$, let $H_\Phi(\mathbb{C}^n) = L^2(\mathbb{C}^n, e^{-2\Phi(z)}\, L(dz)) \cap \mathrm{Hol}(\mathbb{C}^n)$, where $\mathrm{Hol}(\mathbb{C}^n)$ denotes the space of entire functions on $\mathbb{C}^n$, be the Bargmann space associated to Φ, and let Π_Φ be the orthogonal projection from $L^2(\mathbb{C}^n, e^{-2\Phi(z)}\, L(dz))$ to $H_\Phi(\mathbb{C}^n)$. Given a Lebesgue measurable function f on $\mathbb{C}^n$, we may define the **Toeplitz quantization** of f as the unbounded operator on $H_\Phi(\mathbb{C}^n)$ given by

$$\mathrm{Top}(f) = \Pi_\Phi \circ f \circ \Pi_\Phi \tag{1.2.1}$$

equipped with its maximal domain. Here $L(dz)$ denotes the Lebesgue measure on $\mathbb{C}^n$. It is reasonable to seek conditions on f that imply that the operator $\mathrm{Top}(f)$ is in fact bounded on $H_\Phi(\mathbb{C}^n)$. Under fairly general assumptions on f (see, for instance, Theorem 13.10 in [63]), one may write $\mathrm{Top}(f)$ as a complex Weyl pseudodifferential operator acting unboundedly on $H_\Phi(\mathbb{C}^n)$ whose symbol is obtained by applying an apropriate 'forwards heat flow' to f. For detailed background information regarding Toeplitz quantization and its relation to pseudodifferential quantization in the complex domain, see Chapter 13 of [63]. In the work [6], Berger and Coburn conjectured, essentially, that a Toeplitz operator $\mathrm{Top}(f)$ is bounded on $H_\Phi(\mathbb{C}^n)$ if and only if its associated Weyl symbol belongs to $L^\infty(\mathbb{C}^n)$. This conjecture has remained open since the publication of [6] in 1994. Namely, if proven true, the Berger-Coburn conjecture would provide a prime example of the classical — quantum correspondence for NSA operators since it relates a fundamental property of the operator $\mathrm{Top}(f)$ (boundedness) to a property of its associated classical Hamiltonian, f (boundedness of the corresponding Weyl symbol). Thus, even partial progress towards the resolution of the Berger-Coburn conjecture is of great interest. The following theorem establishes one direction of the conjecture for Toeplitz operators whose symbols are of the form $f(z) = e^{Q(z)}$, with $Q(z)$ a complex-valued quadratic polynomial on $\mathbb{C}^n$.

Theorem 1.2.1 (Coburn-Hitrik-Sjöstrand-White [13]). *Let Φ be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and let Q be a quadratic polynomial on $\mathbb{C}^n$ with the principal part q. Assume that*

$$\mathit{Re}\, q(z) < \Phi_{\mathit{Herm}}(z) := (1/2)(\Phi(z) + \Phi(iz)), \quad z \neq 0, \tag{1.2.2}$$

and

$$\det \partial_z \partial_{\overline{z}}(2\Phi - q) \neq 0. \tag{1.2.3}$$

If the Weyl symbol of the Toeplitz operator $\mathit{Top}(e^Q)$ is bounded, then the Toeplitz operator

$$\mathit{Top}(e^Q) : H_\Phi(\mathbb{C}^n) \to H_\Phi(\mathbb{C}^n) \tag{1.2.4}$$

is bounded.

The proof of this theorem is accomplished by showing that any such Toeplitz operator may be realized as a Fourier integral operator in the complex domain (see Appendix B of [9] and [12]) whose underlying complex canonical transformation enjoys certain positivity properties. For a complete proof, see Chapter 2 below.

1.3 Solutions of Schrödinger Equations with NSA Quadratic Hamiltonians

One major problem in the study of NSA operators is to understand, both qualitatively and quantitatively, the behavior of solutions of time-dependent Schrödinger equations with NSA Hamiltonians. The simplest example of such an evolution is the heat/Schrödinger equation,

$$\begin{cases} (\partial_t + q^w(x,D))u(t,x) = 0, \quad x \in \mathbb{R}^n, \quad t \geq 0, \\ u|_{t=0} = u_0 \in \mathcal{S}'(\mathbb{R}^n), \end{cases} \tag{1.3.1}$$

where $q^w(x,D)$ denotes the Weyl quantization of a complex-valued quadratic form q on $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ with non-negative real part, $\mathrm{Re}\, q \geq 0$, and the initial data u_0 is a tempered distribution on $\mathbb{R}^n$. The operator $q^w(x,D)$ is a differential operator with polynomial coefficients known as a **quadratic differential operator**. Obtaining information about solutions to (1.3.1) is tantamount to understanding properties of the contraction semigroup on $L^2(\mathbb{R}^n)$ generated by $q^w(x,D)$, which is a t-dependent family of Fourier integral operators on $\mathbb{R}^n$ associated to the complex Hamilton flow $\exp(-itH_q)$ on $\mathbb{C}^{2n}$ generated by q at the imaginary times $-it$, $t \geq 0$. In this thesis, we present two results concerning the propagation of global analytic singularities and the size of L^p-norms of solutions of the problem (1.3.1). The first of these concerns the propagation in time of the **global analytic** or $1/2$**-Gelfand-Shilov wavefront set** of $u(t,\cdot)$ in terms of that of the initial data. Roughly speaking, if $u_0 \in \mathcal{S}'(\mathbb{R}^n)$ is a tempered distribution on $\mathbb{R}^n$, then the global analytic wavefront set $\mathrm{WF}^{1/2}(u_0)$, is a closed conic subset of $\mathbb{R}^{2n}\backslash\{(0,0)\}$ that captures the microlocal failure of u_0 to behave like a Gaussian. For a precise definition, see either Section 6 of [30], the work [10], or Section 3 of Chapter 3 below. Our principal result concerning the propagation of global analytic singularities for the evolution equation (1.3.1) is the following theorem, which says that global analytic singularities of the initial data that lie within a distinguished linear subspace of the phase space of $\mathbb{R}^{2n}$, known as the **singular space** ([23]) of q are transported exactly by the Hamilton flow of the imaginary part of q, while all other global analytic singularities are instantaneously regularized. Dynamically, the singular space is important because it is precisely the set of points $X \in \mathbb{R}^{2n}$ such that $\exp(-itH_q)(X) \in \mathbb{R}^{2n}$ for all $t \in \mathbb{R}$. For a full discussion, see Chapter 3 below.

Theorem 1.3.1 (White [61]). *Let q be a complex-valued quadratic form on $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ with non-negative real part, $Re\, q \geq 0$, let $q^w(x,D)$ denote the Weyl quantization of q, let $\left(e^{-tq^w(x,D)}\right)_{t\geq 0}$, denote the 1-parameter contraction semigroup on $L^2(\mathbb{R}^n)$ generated by $q^w(x,D)$, and let S be the singular space of q. Then*

$$WF^{1/2}(e^{-tq^w(x,D)}u_0) = \exp(tH_{Im\, q})\left(WF^{1/2}(u_0) \cap S\right) \tag{1.3.2}$$

for all $u_0 \in \mathcal{S}'(\mathbb{R}^n)$ and $t > 0$. Here, $H_{Im\, q}$ denotes the Hamilton vector field on $\mathbb{R}^{2n}$ of the imaginary *part of q, $Im\, q$, and $\exp(tH_{Im\, q})$ denotes the Hamilton flow on $\mathbb{R}^{2n}$ of $Im\, q$.*

Regarding the size of L^p-norms of solutions of the initial value problem (1.3.1), we prove bounds for the propagator $e^{-tq^w(x,D)}$ in the long and short time regimes when the singular space of q is trivial, $S = \{0\}$. This case arises when, for instance, $\operatorname{Re} q$ is positive-definite. The following theorem is the main result of Chapter 4.

Theorem 1.3.2 (White [62]). *Let q, $q^w(x,D)$, $\left(e^{-tq^w(x,D)}\right)_{t\geq 0}$, and S be as in the statement of Theorem 1.3.1. If $S = \{0\}$, then*

1. *there exists $\gamma > 0$ such that for every $1 \leq p \leq q \leq \infty$ and every $\epsilon > 0$ there is $C, c > 0$ such that*

$$ce^{-\gamma t} \leq \|e^{-tq^w(x,D)}\|_{L^p \to L^q} \leq Ce^{-\gamma t}, \quad t \geq \epsilon, \tag{1.3.3}$$

2. *there exists an integer $k_0 \in \{0, 1, \ldots, 2n-1\}$ and a time $0 < t_0 \ll 1$ such that for any $1 \leq p \leq q \leq \infty$ there is $C, c > 0$ such that*

$$c \leq \|e^{-tq^w(x,D)}\|_{L^p \to L^q} \leq Ct^{-(2k_0+1)n}, \quad 0 < t \leq t_0. \tag{1.3.4}$$

Our strategy for proving both of these theorems is to analyze the conjugation of $e^{-tq^w(x,D)}$ by a suitable global metaplectic Fourier-Bros-Iagolnitzer (FBI) transform. As a byproduct of this approach, we obtain an alternative to Hörmander's generalized Mehler formula [31] for representing the semigroup $e^{-tq^w(x,D)}$ that may be of independent interest. In particular, we determine an integral formula for $e^{-tq^w(x,D)}$ that is valid for all $t \geq 0$.

1.4 L^p-Bounds for Eigenfunctions of Operators with Double Characteristics

The goal of Chapter 5 of this thesis is to obtain L^p-bounds for low-lying eigenfunctions of analytic NSA semiclassical operators on $\mathbb{R}^n$ whose principal symbols are complex-valued with non-negative real part and vanish to second order at the origin 0 of the classical phase space $\mathbb{R}^{2n}$. While the topic of eigenfunctions of self-adjoint operators is well explored and established within the literature (see, for instance, [33] and [35] and the references therein for the subject of L^p-bounds of self-adjoint operators), very little is known about eigenfunctions of NSA operators. Our main result in this direction concerns optimal L^p-bounds for L^2-normalized ground states $u(h)$ of NSA semiclassical pseudodifferential operators of the form $P(h) = \mathrm{Op}_h^w(p_0 + hp_1)$ when the quadratic approximation to the principal symbol p_0 at $0 \in \mathbb{R}^{2n}$ is elliptic along its singular space and p_0 and p_1 are real analytic. Here $0 < h \leq 1$ is a semiclassical parameter, and Op_h^w denotes the semiclassical Weyl quantization. This work is motivated by the fact that complete asymptotic expansions for the low-lying eigenvalues of $P(h)$ are known when p_1 admits a suitable asymptotic expansion (see [25]). Our result is in agreement with that of Krupchyk and Uhlmann [36], which established L^p-bounds for ground states $u(h)$ of $P(h)$ under the stronger assumption that $\mathrm{Re}\, q$ is positive-definite.

Theorem 1.4.1 (White [60]). *Let $P(h) = Op_h^w(p_0 + hp_1)$, where p_0 and p_1 are real analytic, $Re\, p_0 \geq 0$, $p_0(0) = \nabla p_0(0) = 0$, and suppose that the quadratic approximation to p_0 at $0 \in \mathbb{R}^{2n}$ is elliptic along its singular space. Assume also that the real part of p_0, $Re\ p_0$, is suitably ellipic at infinity. If $P(h)u(h) \equiv 0$, where $\|u(h)\|_{L^2} \equiv 1$, then there exists $0 < h_0 \leq 1$ such that*

$$\|u(h)\|_{L^p} \leq \mathcal{O}(1) h^{\frac{n}{2p} - \frac{n}{4}}, \quad 0 < h \leq h_0, \tag{1.4.1}$$

for every $1 \leq p \leq \infty$.

In contrast to the stationary approach taken in [**?**], the proof of Theorem 2.4 proceeds by using the complex Hamiltonian dynamics generated by the holomorphic extension of p_0 to establish microlocal estimates for FBI transforms of the ground states $u(h)$ that are sufficient to yield (1.4.1). For a full discussion, see Chapter 5 below.

We remark that the proof of the bounds (1.4.1) given in the work [36] requires only that the symbols p_0 and p_1 belong to C^∞. As a consequence, our Theorem 1.4.1 generalizes the main result of [36] in the case when p_0 and p_1 are real analytic. It is thus still an open question whether Theorem 1.4.1 holds under the weaker assumption that $p_0, p_1 \in C^\infty$. We are planning to explore this question in the future.

Chapter 2

Weyl Symbols and Boundedness of Toeplitz Operators

2.1 Introduction and Statement of Results

In the recent work [12], the authors have established some basic links between the theory of Toeplitz operators acting on exponentially weighted spaces of entire holomorphic functions and Fourier integral operators (FIOs) in the complex domain. The point of view of complex FIOs was used in [12] to show that the boundedness of a certain class of Toeplitz operators is implied by the boundedness of their Weyl symbols, in agreement with a general conjecture made in [7]. The purpose of this note is to obtain a slight, but perhaps natural, extension of this result, by taking a closer look at the arguments of [12]. In a special case, we show that the boundedness of the Weyl symbols is also a necessary condition for the boundedness of the corresponding Toeplitz operators. We shall now proceed to describe the assumptions and state the main results.

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and let us set

$$\Lambda_{\Phi_0} = \left\{ \left(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x) \right), \, x \in \mathbb{C}^n \right\} \subset \mathbb{C}^{2n}. \tag{2.1.1}$$

The real $2n$-dimensional linear subspace Λ_{Φ_0} is I-Lagrangian and R-symplectic, in the sense that the restriction of the complex symplectic form on $\mathbb{C}^{2n}$ to Λ_{Φ_0} is real and non-degenerate. In particular,

Λ_{Φ_0} is maximally totally real.

Let us introduce the Bargmann space

$$H_{\Phi_0}(\mathbb{C}^n) = L^2(\mathbb{C}^n, e^{-2\Phi_0} L(dx)) \cap \mathrm{Hol}(\mathbb{C}^n), \tag{2.1.2}$$

and the orthogonal projection

$$\Pi_{\Phi_0} : L^2(\mathbb{C}^n, e^{-2\Phi_0} L(dx)) \to H_{\Phi_0}(\mathbb{C}^n). \tag{2.1.3}$$

Here $L(dx)$ is the Lebesgue measure on $\mathbb{C}^n$. In this note we shall be concerned with the boundedness properties of Toeplitz operators of the form

$$\mathrm{Top}(e^Q) = \Pi_{\Phi_0} \circ e^Q \circ \Pi_{\Phi_0} : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n), \tag{2.1.4}$$

where Q is an inhomogeneous quadratic polynomial on $\mathbb{C}^n$ with complex coefficients. The following is the main result of this work.

Theorem 2.1.1. *Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and let Q be a quadratic polynomial on $\mathbb{C}^n$ with the principal part q. Assume that*

$$\mathrm{Re}\, q(x) < \Phi_{\mathrm{herm}}(x) := (1/2)\left(\Phi_0(x) + \Phi_0(ix)\right), \quad x \neq 0 \tag{2.1.5}$$

and

$$\det \partial_x \partial_{\overline{x}} \left(2\Phi_0 - q\right) \neq 0. \tag{2.1.6}$$

Let $a \in C^\infty(\Lambda_{\Phi_0})$ be the Weyl symbol of the operator $\mathrm{Top}(e^Q)$ and assume that $a \in L^\infty(\Lambda_{\Phi_0})$. Then the Toeplitz operator

$$\mathrm{Top}(e^Q) : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n)$$

is bounded.

Remark 2.1.2. In the homogeneous case, when Q is a quadratic form, Theorem 2.1.1 was established in [12]. In the general inhomogeneous case considered here, Theorem 2.1.1 provides further evidence for the conjecture of [7], [14], stating that a Toeplitz operator is bounded on $H_{\Phi_0}(\mathbb{C}^n)$ if and only if the corresponding Weyl symbol is bounded on Λ_{Φ_0}.

The plan of this note is as follows. In Section 2.2, we carry out the principal step in the proof of Theorem 2.1.1 by characterizing boundedness properties of operators given as Weyl quantizations of symbols of the form $e^{iP(x,\xi)}$, where P is a holomorphic inhomogeneous quadratic polynomial on $\mathbb{C}^{2n}$. The homogeneous case was discussed in [12], and the only additional idea required here consists of performing a factorization of a suitable complex affine canonical transformation associated to the Weyl quantization above. The proof of Theorem 2.1.1 is then completed in Section 2.3 by passing from the Toeplitz symbol to the Weyl one, along the lines of [12]. Section 2.4 is devoted to the discussion of an explicit family of metaplectic Toeplitz operators in a quadratic Bargmann space, where we also verify that the sufficient condition for the boundedness of the Toeplitz operator given in Theorem 2.1.1 is in fact necessary, in agreement with the conjecture of [7], [14].

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2.2 From bounded Weyl symbols to bounded Weyl quantizations

Let $F(x,\xi)$ be a holomorphic quadratic form on $\mathbb{C}^{2n}$, let $\ell(x,\xi)$ be a complex linear function on $\mathbb{C}^{2n}$, and let us consider formally the Weyl quantization of a symbol of the form

$$a(x,\xi) = \exp(i(F(x,\xi) + \ell(x,\xi))). \tag{2.2.1}$$

We have

$$Au(x) = a^w(x, D_x)u(x) = \frac{1}{(2\pi)^n} \iint e^{i((x-y)\cdot\theta + F((x+y)/2,\theta) + \ell((x+y)/2,\theta))} u(y)\,dy\,d\theta. \tag{2.2.2}$$

Following [12], we shall view A as a Fourier integral operator in the complex domain. The holomorphic quadratic polynomial

$$\Phi(x, y, \theta) = (x - y) \cdot \theta + F((x + y)/2, \theta) + \ell((x + y)/2, \theta) \tag{2.2.3}$$

is a non-degenerate phase function in the sense of Hörmander and defines a canonical relation

$$\kappa : (y, -\partial_y \Phi(x, y, \theta)) \mapsto (x, \partial_x \Phi(x, y, \theta)), \quad \partial_\theta \Phi(x, y, \theta) = 0. \tag{2.2.4}$$

Writing $\eta = -\partial_y \Phi(x, y, \theta)$ and $\xi = \partial_x \Phi(x, y, \theta)$ we see that κ is given by $(y, \eta) \mapsto (x, \xi)$, where

$$\begin{aligned} x &= \frac{x+y}{2} - \frac{1}{2} F'_\xi\left(\frac{x+y}{2}, \theta\right) - \frac{1}{2}\ell'_\xi, \\ y &= \frac{x+y}{2} + \frac{1}{2} F'_\xi\left(\frac{x+y}{2}, \theta\right) + \frac{1}{2}\ell'_\xi, \\ \xi &= \theta + \frac{1}{2} F'_x\left(\frac{x+y}{2}, \theta\right) + \frac{1}{2}\ell'_x, \\ \eta &= \theta - \frac{1}{2} F'_x\left(\frac{x+y}{2}, \theta\right) - \frac{1}{2}\ell'_x. \end{aligned} \tag{2.2.5}$$

Here ℓ'_x, $\ell'_\xi \in \mathbb{C}^n$ are constant. The graph of κ is parametrized by the midpoint coordinate

$$\rho = \left(\frac{x+y}{2}, \theta\right) \in \mathbb{C}^{2n},$$

and we may rewrite (2.2.5) in the form

$$\kappa : \rho + \frac{1}{2} H_{F+\ell}(\rho) \mapsto \rho - \frac{1}{2} H_{F+\ell}(\rho). \tag{2.2.6}$$

Here $H_{F+\ell}(\rho) = (F'_\xi(\rho) + \ell'_\xi, -F'_x(\rho) - \ell'_x)$ is the Hamilton vector field of the holomorphic function $F + \ell$ at ρ. Recalling as in [12] that the Hamilton vector field of F is given by $H_F(\rho) = \mathcal{F}\rho$, where

$$\mathcal{F} = \begin{pmatrix} F''_{\xi x} & F''_{\xi\xi} \\ -F''_{xx} & -F''_{x\xi} \end{pmatrix}$$

is the fundamental matrix of F, we see that (2.2.6) takes the form

$$\kappa : \left(1 + \frac{1}{2}\mathcal{F}\right)\rho + \frac{1}{2}H_\ell \mapsto \left(1 - \frac{1}{2}\mathcal{F}\right)\rho - \frac{1}{2}H_\ell. \tag{2.2.7}$$

In what follows we shall assume that $\pm 2 \notin \mathrm{Spec}(\mathcal{F})$, so that the canonical relation

$$\kappa_F : \left(1 + \frac{1}{2}\mathcal{F}\right)\rho \mapsto \left(1 - \frac{1}{2}\mathcal{F}\right)\rho \tag{2.2.8}$$

is a canonical transformation. We have

$$\kappa = \exp\left(-\frac{1}{2}H_\ell\right) \circ \kappa_F \circ \exp\left(-\frac{1}{2}H_\ell\right).$$

More explicitly, it follows from (2.2.7) that κ is a complex affine canonical transformation given by

$$\kappa : \rho \mapsto \kappa_F(\rho) - \frac{1}{2}\kappa_F(H_\ell) - \frac{1}{2}H_\ell. \tag{2.2.9}$$

In view of Jacobi's theorem, the right hand side of (2.2.9) is given by

$$\kappa_F(\rho) - \frac{1}{2}H_{\ell\circ\kappa_F^{-1}+\ell},$$

and we conclude that the map κ admits the following factorization

$$\kappa = \kappa_\ell \circ \kappa_F, \tag{2.2.10}$$

where κ_F is given in (2.2.8) and κ_ℓ is a complex phase space translation given by

$$\kappa_\ell(\rho) = \rho - \frac{1}{2}H_{\ell\circ\kappa_F^{-1}+\ell}. \tag{2.2.11}$$

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and let us recall the I-Lagrangian R-symplectic linear manifold Λ_{Φ_0} defined in (2.1.1). The following is the main result of this section.

Theorem 2.2.1. *Let F be a holomorphic quadratic form on $\mathbb{C}^{2n}$ such that the fundamental matrix of F does not have the eigenvalues ± 2, and let ℓ be a complex linear form on $\mathbb{C}^{2n}$. Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$. Let*

$$a(x,\xi) = \exp(i(F(x,\xi) + \ell(x,\xi)))$$

and assume that $a \in L^\infty(\Lambda_{\Phi_0})$. Then the operator

$$a^w(x, D_x) : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n)$$

is bounded.

When proving Theorem 2.2.1, we shall rely on some results of [12], and it will also be convenient to use the factorization (2.2.10). Our starting point is the following observation.

Lemma 2.2.2. *Let $m(x,\xi)$ be a complex linear form on $\mathbb{C}^{2n}$ and let us consider the complex canonical transformation $\exp(H_m)(\rho) = \rho + H_m$, $\rho \in \mathbb{C}^{2n}$, (a complex phase space translation). Let Φ be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$. Then we have*

$$\exp(H_m)\,(\Lambda_\Phi) = \Lambda_\Psi,$$

where Ψ is a strictly plurisubharmonic quadratic polynomial on $\mathbb{C}^n$ given by

$$\Psi(x) = \Phi(x) + \operatorname{Im}\left(m\left(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}(x)\right)\right), \quad x \in \mathbb{C}^n. \tag{2.2.12}$$

Proof. While (2.2.12) can be established by a straightforward computation, here we would like to indicate a more general approach, illustrating the point of view of evolution equations associated to the operator $m^w(x,D)$. See also [52], [26]. Let us consider the real Hamilton-Jacobi equation

$$\frac{\partial \Psi}{\partial t}(x,t) - \operatorname{Im} m\left(x, \frac{2}{i}\frac{\partial \Psi}{\partial x}(x,t)\right) = 0, \quad \Psi(x,0) = \Phi(x), \tag{2.2.13}$$

for $x \in \mathbb{C}^n$, $t \in \mathbb{R}$, $t \geq 0$. Associated to the function $\Psi(x,t)$ is the manifold

$$L_\Psi = \left\{ \left(t, \frac{\partial \Psi}{\partial t}, x, \frac{2}{i} \frac{\partial \Psi}{\partial x} \right) \right\} \subset \mathbb{R}^2_{t,\tau} \times \mathbb{C}^{2n}_{x,\xi},$$

which is Lagrangian with respect to the real symplectic form

$$d\tau \wedge dt - \operatorname{Im} \sigma, \tag{2.2.14}$$

where

$$\sigma = \sum_{j=1}^{n} d\xi_j \wedge dx_j$$

is the complex symplectic (2,0)–form on $\mathbb{C}^{2n}_{x,\xi}$. The equation (2.2.13) tells us that

$$(\tau - \operatorname{Im} m)\,|_{L_\Psi} = 0,$$

and therefore the Hamilton vector field of the function $\tau - \operatorname{Im} m$, computed with respect to the real symplectic form (2.2.14), is tangent to L_Ψ. Using the general relation

$$\widehat{H_m} = H^{-\operatorname{Im}\sigma}_{-\operatorname{Im} m},$$

valid for any $m(x,\xi)$ holomorphic, where $\widehat{H_m} = H_m + \overline{H_m}$ is the real vector field naturally associated to the holomorphic vector field H_m, see [51], we conclude that the vector field

$$\partial_t + H^{-\operatorname{Im}\sigma}_{-\operatorname{Im} m} = \partial_t + \widehat{H_m}$$

is tangent to L_Ψ. Identifying $\widehat{H_m}$ and H_m, we get

$$\Lambda_{\Psi(\cdot,t)} = \exp(tH_m)\,(\Lambda_\Phi)\,.$$

It is now easy to obtain (2.2.12) and to this end, we claim that the unique solution of the equation

(2.2.13) is given by

$$\Psi(x,t) = \Phi(x) + t\mathrm{Im}\left(m\left(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}(x)\right)\right) + C_t, \tag{2.2.15}$$

where C_t depends on t only. When verifying the claim, let us write $-\mathrm{Im}\, m = p$ and choose real linear coordinates on $\mathbb{C}^n$ so that $(x, \frac{2}{i}\partial_x \Psi(x,t))$ corresponds to $(x, \partial_x \Psi(x,t))$ in the usual $\mathbb{R}^{2n}$–sense. Then (2.2.13) becomes

$$\frac{\partial \Psi}{\partial t}(x,t) + p\left(x, \frac{\partial \Psi}{\partial x}(x,t)\right) = 0, \quad \Psi(x,0) = \Phi(x). \tag{2.2.16}$$

Here $p(x,\xi)$ is real linear on $\mathbb{R}^{2n}_x \times \mathbb{R}^{2n}_\xi$, $p(x,\xi) = p'_x \cdot x + p'_\xi \cdot \xi$ and $\Phi(x)$ is a real quadratic form on $\mathbb{R}^{2n}$,

$$\Phi(x) = \frac{1}{2} A_0 x \cdot x, \quad x \in \mathbb{R}^{2n},$$

with A_0 real symmetric. With

$$\Psi(x,t) = \frac{1}{2} A_t x \cdot x + B_t \cdot x + C_t,$$

the equation (2.2.16) becomes

$$\frac{1}{2}\partial_t A_t x \cdot x + \partial_t B_t \cdot x + \partial_t C_t + p'_x \cdot x + p'_\xi \cdot (A_t x + B_t) = 0,$$

and we immediately get the unique solution

$$\Psi(x,t) = \frac{1}{2} A_0 x \cdot x - t\left(p'_x \cdot x + p'_\xi \cdot A_0 x\right) + C_t = \Phi(x) - tp\left(x, \partial_x \Phi(x)\right) + C_t,$$

where

$$C_t = \frac{t^2}{2} p'_\xi \cdot \left(p'_x + A_0 p'_\xi\right)$$

depends on t only. This shows (2.2.15) and completes the proof. □

Remark. Associated to the canonical transformation $\exp(H_m)$ is the Fourier integral operator

$e^{-im^w(x,D)}$, and from [54] we may recall the explicit description

$$e^{-im^w(x,D)} = e^{-\frac{i}{2}m'_x\cdot x} \circ \tau_{m'_\xi} \circ e^{-\frac{i}{2}m'_x\cdot x},$$

where τ_s is the operator of translation by $s \in \mathbb{C}^n$, $(\tau_s u)(x) = u(x-s)$. We may then verify by an explicit computation that the operator $e^{-im^w(x,D)}$ is bounded,

$$e^{-im^w(x,D)} : H_\Phi(\mathbb{C}^n) \to H_\Psi(\mathbb{C}^n),$$

where Ψ is given by (2.2.12). Here the weighted spaces of holomorphic functions $H_\Phi(\mathbb{C}^n)$, $H_\Psi(\mathbb{C}^n)$ are defined analogously to (2.1.2).

Let a be of the form (2.2.1) and let us notice that $a \in L^\infty(\Lambda_{\Phi_0})$ precisely when

$$\operatorname{Im} F|_{\Lambda_{\Phi_0}} \geq 0 \tag{2.2.17}$$

and

$$\rho \in \Lambda_{\Phi_0},\ \operatorname{Im} F(\rho) = 0 \Longrightarrow \operatorname{Im} \ell(\rho) = 0. \tag{2.2.18}$$

It follows from (2.2.17) and Proposition B.1 in [12] that the canonical transformation κ_F in (2.2.8) is positive relative to Λ_{Φ_0}, and applying Theorem 1.1 of [12], we get

$$\kappa_F(\Lambda_{\Phi_0}) = \Lambda_\Phi, \tag{2.2.19}$$

where Φ is a strictly plurisubharmonic quadratic form such that $\Phi \leq \Phi_0$. We need to obtain an explicit description of the (clean) intersection $\Lambda_\Phi \cap \Lambda_{\Phi_0}$.

Proposition 2.2.3. *We have*

$$\Lambda_\Phi \cap \Lambda_{\Phi_0} = \left\{ \left(1 - \frac{1}{2}\mathcal{F}\right)\rho;\ \rho \in \Lambda_{\Phi_0},\ \operatorname{Im} F(\rho) = 0 \right\} \subset \mathbb{C}^{2n}. \tag{2.2.20}$$

Proof. It will be convenient to obtain a reduction to the case when Λ_{Φ_0} is replaced by the real phase space $\mathbb{R}^{2n}$. To this end, let $T : L^2(\mathbb{R}^n) \to H_{\Phi_0}(\mathbb{C}^n)$ be a unitary metaplectic Fourier integral operator with the associated complex linear canonical transformation κ_T such that

$$\kappa_T(\mathbb{R}^{2n}) = \Lambda_{\Phi_0}.$$

We have

$$\Lambda_\Phi \cap \Lambda_{\Phi_0} = \kappa_F(\Lambda_{\Phi_0}) \cap \Lambda_{\Phi_0} = \kappa_T\left(\kappa_G(\mathbb{R}^{2n}) \cap \mathbb{R}^{2n}\right),$$

where it follows from (2.2.8) that

$$\kappa_G : \left(1 + \frac{1}{2}\mathcal{G}\right)\rho \mapsto \left(1 - \frac{1}{2}\mathcal{G}\right)\rho,$$

and $\mathcal{G}$ is the fundamental matrix of the quadratic form $G = F \circ \kappa_T$. We have $\operatorname{Im} G|_{\mathbb{R}^{2n}} \geq 0$ and therefore

$$\left(1 \pm \frac{1}{2}\mathcal{G}\right)\rho \in \mathbb{R}^{2n}$$

precisely when $\rho \in \mathbb{R}^{2n}$, $\operatorname{Im} G(\rho) = 0$. It follows that

$$\kappa_G(\mathbb{R}^{2n}) \cap \mathbb{R}^{2n} = \left\{\left(1 - \frac{1}{2}\mathcal{G}\right)\rho;\ \rho \in \mathbb{R}^{2n},\ \operatorname{Im} G(\rho) = 0\right\},$$

and we obtain (2.2.20). $\square$

In what follows we shall use the notation

$$L = \left\{\left(1 - \frac{1}{2}\mathcal{F}\right)\rho;\ \rho \in \Lambda_{\Phi_0},\ \operatorname{Im} F(\rho) = 0\right\}. \tag{2.2.21}$$

Letting $\pi_x : \mathbb{C}^{2n} \ni (x, \xi) \mapsto x \in \mathbb{C}^n$ be the projection map, we notice that

$$\{x \in \mathbb{C}^n; \Phi_0(x) = \Phi(x)\} = \pi_x\left(\Lambda_\Phi \cap \Lambda_{\Phi_0}\right) = \pi_x L, \tag{2.2.22}$$

and the quadratic form $\Phi_0 - \Phi \geq 0$ satisfies

$$\Phi_0(x) - \Phi(x) \simeq \mathrm{dist}(x, \pi_x L)^2, \quad x \in \mathbb{C}^n. \tag{2.2.23}$$

We shall now consider the I-Lagrangian R-symplectic affine plane $\kappa(\Lambda_{\Phi_0})$, where κ is given by (2.2.7). It follows from (2.2.10), (2.2.11), Lemma 2.2.2, and (2.2.19) that

$$\kappa(\Lambda_{\Phi_0}) = \Lambda_\Psi, \tag{2.2.24}$$

where Ψ is a strictly plurisubharmonic quadratic polynomial on $\mathbb{C}^n$ given by

$$\Psi(x) = \Phi(x) + \mathrm{Im}\left(m\left(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}(x)\right)\right), \tag{2.2.25}$$

where

$$m = -\frac{1}{2}\left(\ell \circ \kappa_F^{-1} + \ell\right). \tag{2.2.26}$$

We claim that the quadratic polynomial $\Phi_0 - \Psi$ vanishes along the real linear subspace $\pi_x L \subset \mathbb{C}^n$ and to this end, it suffices to check that the linear form m is real along $L \subset \Lambda_\Phi$. It follows from (2.2.8), (2.2.18), and (2.2.26) that when $\rho \in \Lambda_{\Phi_0}$, $\mathrm{Im}\, F(\rho) = 0$, we have

$$m\left(\left(1 - \frac{1}{2}\mathcal{F}\right)\rho\right) = -\ell(\rho)$$

is real. We have therefore verified the claim and using also (2.2.23) and (2.2.25) we conclude that the inhomogeneous quadratic polynomial $\Phi_0 - \Psi$ is bounded below on $\mathbb{C}^n$. The general theory (see [51],[9]), together with (2.2.24), allows us to conclude that the operator

$$a^w(x, D_x) : H_{\Phi_0}(\mathbb{C}^n) \to H_\Psi(\mathbb{C}^n)$$

is bounded, and this completes the proof of Theorem 2.2.1.

2.3 Toeplitz operators and proof of Theorem 2.1.1

The purpose of this section is to apply the results of Section 2.2 to the study of boundedness properties of Toeplitz operators in the Bargmann space, establishing Theorem 2.1.1.

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and let Q be a quadratic polynomial with complex coefficients on $\mathbb{C}^n$, with the principal part q. Assume that the condition (2.1.5) holds. Arguing as in Section 4 of [12], we then see that when equipped with the natural domain

$$\mathcal{D}(\mathrm{Top}(e^Q)) = \left\{u \in H_{\Phi_0}(\mathbb{C}^n); e^Q u \in L^2(\mathbb{C}^n, e^{-2\Phi_0} L(dx))\right\}, \tag{2.3.1}$$

the Toeplitz operator

$$\mathrm{Top}(e^Q) = \Pi_{\Phi_0} \circ e^Q \circ \Pi_{\Phi_0} : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n) \tag{2.3.2}$$

is densely defined.

Recalling the integral representation for the orthogonal projection Π_{Φ_0} and following [12], we may write for $u \in \mathcal{D}(\mathrm{Top}(e^Q))$,

$$\mathrm{Top}(e^Q)u(x) = C \iint_\Gamma e^{2(\Psi_0(x,\theta)-\Psi_0(y,\theta))+Q(y,\theta)} u(y)\, dy\, d\theta, \tag{2.3.3}$$

where Ψ_0 is the polarization of Φ_0 and Γ is the contour in $\mathbb{C}^{2n}$, given by $\theta = \overline{y}$. Using the assumption (2.1.6), we conclude as in [12] that the operator $\mathrm{Top}(e^Q)$ can be viewed as a metaplectic Fourier integral operator associated to a complex affine canonical transformation: $\mathbb{C}^{2n} \to \mathbb{C}^{2n}$.

It is now easy to complete the proof of Theorem 2.1.1. Let us write, following [54], [12],

$$\mathrm{Top}(e^Q) = a^w(x, D_x), \tag{2.3.4}$$

where $a \in C^\infty(\Lambda_{\Phi_0})$ is the Weyl symbol of the Toeplitz operator $\mathrm{Top}(e^Q)$, given by

$$a\,(x,\xi) = \left(\exp\left(\frac{1}{4}\left(\Phi''_{0,x\overline{x}}\right)^{-1} \partial_x \cdot \partial_{\overline{x}}\right) e^Q\right)(x), \quad (x,\xi) \in \Lambda_{\Phi_0}. \tag{2.3.5}$$

In [12], we have seen that

$$a(x,\xi) = C_{\Phi_0} \int_{\mathbb{C}^n} \exp(-4\Phi_{\mathrm{herm}}(x-y)) e^{Q(y)} \, L(dy), \quad C_{\Phi_0} \neq 0, \tag{2.3.6}$$

where the integral converges thanks to (2.1.5). An application of the method of exact stationary phase allows us therefore to conclude that

$$a(x,\xi) = C \exp(i\,(F(x,\xi) + \ell(x,\xi))), \quad (x,\xi) \in \Lambda_{\Phi_0}, \tag{2.3.7}$$

for some $C \neq 0$, where F is a holomorphic quadratic form on $\mathbb{C}^{2n}$ and ℓ is a complex linear function on $\mathbb{C}^{2n}$. Theorem 2.1.1 follows therefore from Theorem 2.2.1.

2.4 Example: boundedness of a metaplectic Toeplitz operator

In the beginning of this section we shall illustrate Theorem 2.1.1 by applying it in the case when

$$\Phi_0(x) = \frac{|x|^2}{4}, \tag{2.4.1}$$

and

$$Q(x) = \lambda \, |x|^2 + \frac{1}{2} \overline{c} \cdot x - \frac{1}{2} d \cdot \overline{x}. \tag{2.4.2}$$

Here $c, d \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ satisfies $\operatorname{Re} \lambda < 1/4$, so that the conditions (2.1.5), (2.1.6) are satisfied. It follows from (2.3.5) that the Weyl symbol a of the operator $\mathrm{Top}(e^Q)$ is given by

$$a\left(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x)\right) = \left(\exp\left(\frac{1}{4}\Delta\right) e^Q\right)(x) = \frac{1}{\pi^n} \int_{\mathbb{C}^n} e^{-|x-y|^2} e^{Q(y)} \, L(dy). \tag{2.4.3}$$

Here Δ is the Laplacian on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. The Gaussian integral in (2.4.3) can be computed by the exact version of stationary phase and we get, after a straightforward computation,

$$a\left(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x)\right) = C\,\exp\left(\frac{1}{1-\lambda}\left(\lambda\,|x|^2 + \frac{1}{2}\overline{c}\cdot x - \frac{1}{2}d\cdot\overline{x}\right)\right). \tag{2.4.4}$$

Here $C \neq 0$ is a suitable constant depending on λ, c, d only.

Using (2.4.4), we may determine the explicit necessary and sufficient conditions for the boundedness of a along Λ_{Φ_0}. When doing so, it is convenient to introduce the parameter

$$\gamma = \frac{1}{1-2\lambda}, \tag{2.4.5}$$

and to observe that

$$\mathrm{Re}\left(\frac{\lambda}{1-\lambda}\right) = \frac{1}{4\,|1-\lambda|^2}\left(1 - \frac{1}{|\gamma|^2}\right). \tag{2.4.6}$$

It follows, in particular, that if $|\gamma| < 1$, then $a \in L^\infty(\Lambda_{\Phi_0})$ for all $c, d \in \mathbb{C}^n$, and if $|\gamma| > 1$, then a is unbounded for all $c, d \in \mathbb{C}^n$. In the "boundary" case when $|\gamma| = 1$, we have $a \in L^\infty(\Lambda_{\Phi_0})$ precisely when

$$\mathrm{Re}\left(\frac{\overline{c}}{1-\lambda}\cdot x - \frac{d}{1-\lambda}\cdot\overline{x}\right) = 0, \quad x \in \mathbb{C}^n. \tag{2.4.7}$$

Rewriting the condition (2.4.7) in the form

$$\left(\frac{\overline{c}}{1-\lambda} - \frac{\overline{d}}{\overline{(1-\lambda)}}\right)\cdot x + \left(\frac{c}{\overline{(1-\lambda)}} - \frac{d}{1-\lambda}\right)\cdot\overline{x} = 0, \quad x \in \mathbb{C}^n,$$

we conclude that $a \in L^\infty(\Lambda_{\Phi_0})$ precisely when

$$c = \frac{\overline{1-\lambda}}{1-\lambda}\,d \Longleftrightarrow c = \gamma d.$$

An application of Theorem 2.1.1 gives the following result.

Let $\Phi_0(x) = |x|^2/4$ and

$$Q(x) = \lambda |x|^2 + \frac{1}{2}\overline{c} \cdot x - \frac{1}{2} d \cdot \overline{x},$$

with $c, d \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$, $\mathrm{Re}\,\lambda < 1/4$. Let us define $\gamma \in \mathbb{C}$ as in (2.4.5). If $|\gamma| < 1$ then the operator

$$\mathrm{Top}(e^Q) : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n)$$

is bounded, for all $c, d \in \mathbb{C}^n$. The same conclusion holds if $|\gamma| = 1$ and $c = \gamma d$.

We shall finish this section by demonstrating that, in the special case at hand, the condition $a \in L^\infty(\Lambda_{\Phi_0})$ is in fact also necessary for the boundedness of the Toeplitz operator $\mathrm{Top}(e^Q)$. When doing so, we shall study the action of $\mathrm{Top}(e^Q)$ on the normalized reproducing kernels for the Bargmann space $H_{\Phi_0}(\mathbb{C}^n)$. To this end, let us first recall from [54] that the orthogonal projection $\Pi_{\Phi_0} : L^2(\mathbb{C}^n, e^{-2\Phi_0}\, L(dx)) \to H_{\Phi_0}(\mathbb{C}^n)$ is given by

$$\Pi_{\Phi_0} u(x) = a_{\Phi_0} \int e^{2\Psi_0(x,\overline{y})} u(y) e^{-2\Phi_0(y)}\, L(dy), \quad a_{\Phi_0} > 0. \tag{2.4.8}$$

Here

$$\Psi_0(x,y) = \frac{1}{4} x \cdot y, \quad x, y \in \mathbb{C}^n, \tag{2.4.9}$$

is the polarization of Φ_0. We have

$$2\mathrm{Re}\,\Psi_0(x,\overline{y}) - \Phi_0(x) - \Phi_0(y) = -\Phi''_{0,\overline{x}x}(x-y)\cdot(\overline{x-y}) = -\frac{1}{4}|x-y|^2. \tag{2.4.10}$$

Let us set

$$k_w(x) = (2\pi)^{-n/2} e^{2\Psi_0(x,\overline{w}) - \Phi_0(w)}, \quad w \in \mathbb{C}^n. \tag{2.4.11}$$

Using (2.4.10) we see that $k_w \in H_{\Phi_0}(\mathbb{C}^n)$ with

$$\|k_w\|^2_{H_{\Phi_0}(\mathbb{C}^n)} = \int |k_w(x)|^2 \, e^{-2\Phi_0(x)} \, L(dx) = 1, \quad w \in \mathbb{C}^n. \tag{2.4.12}$$

We shall now consider the operator $\mathrm{Top}(e^Q)$ acting on k_w. To this end, it will be convenient to start by making the following observations. First, letting $q(x) = \lambda \, |x|^2$ be the principal part of Q in (2.4.2), we obtain, in view of (2.4.8) and the exact stationary phase,

$$\left(\mathrm{Top}(e^q) e^{2\Psi_0(\cdot,\overline{w})}\right)(x) = C_\lambda e^{2\Psi_0(x,\gamma\overline{w})}, \quad w \in \mathbb{C}^n, \tag{2.4.13}$$

where C_λ is a constant depending on λ only, and the parameter γ has been defined in (2.4.5). Next, let h be entire holomorphic such that $\overline{h} e^{2\Psi_0(\cdot,\overline{w})} \in L^2(\mathbb{C}^n, e^{-2\Phi_0} L(dx))$ for all $w \in \mathbb{C}^n$. We then have

$$\left(\mathrm{Top}(\overline{h}) e^{2\Psi_0(\cdot,\overline{w})}\right)(x) = \overline{h(w)} e^{2\Psi_0(x,\overline{w})}. \tag{2.4.14}$$

Indeed, it suffices to observe that in view of (2.4.8), the left hand side of (2.4.14) is equal to

$$\overline{\left(\mathrm{Top}(h) e^{2\Psi_0(\cdot,\overline{x})}\right)(w)}.$$

Finally, let h be entire holomorphic such that

$$\overline{h} e^{2\Psi_0(\cdot,\overline{w})}, \overline{h} e^q e^{2\Psi_0(\cdot,\overline{w})} \in L^2(\mathbb{C}^n, e^{-2\Phi_0} L(dx)),$$

for all $w \in \mathbb{C}^n$. Directly from the definitions we then see that

$$\mathrm{Top}(\overline{h} e^q) = \mathrm{Top}(\overline{h}) \mathrm{Top}(e^q), \tag{2.4.15}$$

when acting on the linear span of $\{e^{2\Psi_0(\cdot,\overline{w})}, \, w \in \mathbb{C}^n\} \subset H_{\Phi_0}(\mathbb{C}^n)$.

Using (2.4.2), (2.4.11), (2.4.13), (2.4.14), and (2.4.15), we get

$$\left(\mathrm{Top}(e^{Q})k_w\right)(x) = C_\lambda \exp(2\Psi_0(x, \gamma(\overline{w}+\overline{c})) - 2\Psi_0(d, \gamma(\overline{w}+\overline{c})) - \Phi_0(w)). \tag{2.4.16}$$

Here, as above, $C_\lambda \neq 0$ is a constant which depends on λ only. Taking the norm in $H_{\Phi_0}(\mathbb{C}^n)$ and using (2.4.10), we obtain

$$\|\mathrm{Top}(e^{Q})k_w\|_{H_{\Phi_0}(\mathbb{C}^n)} = C\ \exp(\Phi_0(\overline{\gamma}(w+c)) - \Phi_0(w) - 2\mathrm{Re}\,\Psi_0(d, \gamma\overline{w})). \tag{2.4.17}$$

Here $C \neq 0$ is a constant depending on λ, c, d only. It follows from (2.4.17) that if $|\gamma| > 1$, the operator $\mathrm{Top}(e^{Q})$ is unbounded for all $c, d \in \mathbb{C}^n$. If $|\gamma| = 1$, we get with a new constant,

$$\|\mathrm{Top}(e^{Q})k_w\|_{H_{\Phi_0}(\mathbb{C}^n)} = C\ \exp(2\mathrm{Re}\,\Psi_0(w, \overline{c}) - 2\mathrm{Re}\,\Psi_0(w, \overline{\gamma}\overline{d})), \tag{2.4.18}$$

and it follows that if $c \neq \gamma d$, the operator $\mathrm{Top}(e^{Q})$ is unbounded.

The discussion above may be summarized in the following theorem.

Theorem 2.4.1. *Let $\Phi_0(x) = |x|^2/4$ and $Q(x) = \lambda|x|^2 + \frac{1}{2}\overline{c}\cdot x - \frac{1}{2}d\cdot\overline{x}$, with $c, d \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$, $\mathrm{Re}\,\lambda < 1/4$. The Toeplitz operator*

$$\mathrm{Top}(e^{Q}) : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n)$$

is bounded if and only if the Weyl symbol $a \in C^\infty(\Lambda_{\Phi_0})$ of $\mathrm{Top}(e^{Q})$ satisfies $a \in L^\infty(\Lambda_{\Phi_0})$.

Chapter 3

Propagation of Global Analytic Singularities for Schrödinger Equations with Quadratic Hamiltonians

3.1 Introduction and Statement of Results

In this chapter we consider the initial value problem for the Schrödinger equation

$$\begin{cases} \partial_t u + q^w(x,D)u = 0, \quad t \geq 0,\ x \in \mathbb{R}^n, \\ u|_{t=0} = u_0, \end{cases} \tag{3.1.1}$$

where the initial data u_0 is a tempered distribution on $\mathbb{R}^n$, $q = q(x,\xi)$ is a complex-valued quadratic form defined on the phase space $T^*\mathbb{R}^n \cong \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ with non-negative real part $\operatorname{Re} q \geq 0$, and $q^w(x,D)$ is the Weyl quantization of q given by

$$q^w(x,D)u(x) = \frac{1}{(2\pi)^n} \int\int e^{i(x-y)\cdot\xi} q\left(\frac{x+y}{2}, \xi\right) u(y)\, dy\, d\xi, \quad u \in \mathcal{S}'(\mathbb{R}^n),$$

in the sense of distributions. This class of equations comprises a number of well-known examples, including the free Schrödinger equation where $q(x,\xi) = i\,|\xi|^2$, the harmonic oscillator where $q(x,\xi) =$

$i(|x|^2 + |\xi|^2)$, the heat equation where $q(x, \xi) = |\xi|^2$, as well as the Kramers-Fokker-Planck equation with quadratic potential where $q(x, v, \xi, \eta) = \eta^2 + 1/4v^2 + i(v \cdot \xi - ax \cdot \eta)$ with $(x, v, \xi, \eta) \in \mathbb{R}^{4n} = \mathbb{R}^{2n}_{x,v} \times \mathbb{R}^{2n}_{\xi,\eta}$ and $a \in \mathbb{R}\backslash\{0\}$ a constant.

The problem has been studied by a number of authors, and the majority of works in this area ([44], [59], [43]) have focused on the propagation of Gabor singularities. The Gabor singularities of a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ are captured by the Gabor wavefront set of u, which may be defined as follows: given $u \in \mathcal{S}'(\mathbb{R}^n)$, the *Gabor wavefront of* u, denoted $\mathrm{WF}_G(u)$, is the complement in $\mathbb{R}^{2n}\backslash\{(0,0)\}$ of all points (x_0, ξ_0) for which there exists a symbol $a \in C^\infty(\mathbb{R}^{2n})$ satisfying

1. for all $\alpha, \beta \in \mathbb{N}^n$ there exists $C > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\right| \leq C\langle(x, \xi)\rangle^{-|\alpha|-|\beta|}, \quad (x, \xi) \in \mathbb{R}^{2n},$$

2. there exists an open conic neighborhood V of (x_0, ξ_0) in $\mathbb{R}^{2n}\backslash\{(0,0)\}$ and $c > 0$ so that $|a(x, \xi)| \geq c$ within V for all $|(x, \xi)|$ large, and

3. $a^w(x, D)u \in \mathcal{S}(\mathbb{R}^n)$.

In particular, a tempered distribution is a Schwartz function if and only if its Gabor wavefront set is empty. Thus, the Gabor wavefront set of a tempered distribution measures its deviation from Schwartz regularity in the sense of both smoothness and decay as $|x| \to \infty$. For more information, see [46] or the recent survey article [45].

The propagation of exponential phase space singularities by the evolution (3.1.1) has also been studied. In the work [10], the authors investigated the propagation of s-Gelfand-Shilov singularities for $s > 1/2$ with initial data belonging to classes of distributions larger than $\mathcal{S}'(\mathbb{R}^n)$. For $s > 1/2$, the s-Gelfand-Shilov wavefront of $u \in \mathcal{S}'(\mathbb{R}^n)$ is a closed conic subset of $\mathbb{R}^{2n}\backslash\{(0,0)\}$ that may be conveniently defined using metaplectic Fourier-Bros-Iagolnitzer (FBI) transforms. We recall (see [27] or Chapter 13 of [63]) that a metaplectic FBI transform on $\mathbb{R}^n$ is a Fourier integral operator

$\mathcal{T}_\varphi : \mathcal{S}'(\mathbb{R}^n) \to \mathrm{Hol}(\mathbb{C}^n)$ of the form

$$\mathcal{T}_\varphi u(z) = c_\varphi \int_{\mathbb{R}^n} e^{i\varphi(z,y)} u(y)\, dy, \quad u \in \mathcal{S}'(\mathbb{R}^n), \tag{3.1.2}$$

where $c_\varphi \neq 0$ is a normalizing constant and $\varphi(z,y)$ is a holomorphic quadratic form on $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_y$ such that $\det \partial^2_{zy}\varphi \neq 0$ and $\mathrm{Im}\ \partial^2_{yy}\varphi > 0$. The phase function φ generates a complex linear canonical transformation $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ given implicitly by

$$\kappa_\varphi : (y, -\partial_y\varphi(z,y)) \mapsto (z, \partial_z\varphi(z,y)), \quad (z,y) \in \mathbb{C}^{2n}, \tag{3.1.3}$$

and we have

$$\kappa_\varphi(\mathbb{R}^{2n}) = \left\{ \left(z, \frac{2}{i}\partial_z\Phi(z) \right) : z \in \mathbb{C}^n \right\}, \tag{3.1.4}$$

where

$$\Phi(z) = \sup_{y\in\mathbb{R}^n} (-\mathrm{Im}\ \varphi(z,y)) \tag{3.1.5}$$

is the strictly plurisubharmonic weight associated to φ. If φ is a metaplectic FBI phase function and $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is the complex linear canonical transformation generated by φ, let

$$\kappa^\flat_\varphi = \pi_1 \circ \kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^n, \tag{3.1.6}$$

where $\pi_1 : (z,\zeta) \mapsto z$ is the projection in $\mathbb{C}^{2n}$ onto the first factor. We say that a point $(x_0, \xi_0) \in \mathbb{R}^{2n}\backslash\{(0,0)\}$ does not belong to the *s-Gelfand-Shilov wavefront set of* u, denoted $\mathrm{WF}^s(u)$, if there exists a metaplectic FBI phase function $\varphi = \varphi(z,y)$, $(z,y) \in \mathbb{C}^n \times \mathbb{C}^n$, with associated plurisubharmonic weight Φ, and complex canonical transformation $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ such that

$$\exists C > 0 : \ |\mathcal{T}_\varphi u(z)| \leq C e^{\Phi(z) - c|z|^{1/s}}$$

for all z in some open conic neighborhood of $\kappa_\varphi^\flat(x_0, \xi_0)$ in $\mathbb{C}^n \backslash \{0\}$ and all $c > 0$. For a review of the basic properties of metaplectic FBI transforms on $\mathbb{R}^n$, see Section 2 below.

In this work, we shall be primarily interested in the propagation of global analytic singularities or 1/2-Gelfand-Shilov singularities. The notion of the 1/2-Gelfand-Shilov or global analytic wavefront set was originally introduced by L. Hörmander in the work [30] under the name 'analytic wavefront set.' The 1/2-Gelfand-Shilov wavefront set $\mathrm{WF}^{1/2}(u)$ of a tempered distribution u on $\mathbb{R}^n$ may be defined as follows: a point $(x_0, \xi_0) \in \mathbb{R}^{2n} \backslash \{(0,0)\}$ does not lie in $\mathrm{WF}^{1/2}(u)$ if there exists a metaplectic FBI phase function $\varphi = \varphi(z, y)$, $(z, y) \in \mathbb{C}^n \times \mathbb{C}^n$, with associated plurisubharmonic weight Φ, and complex canonical transformation $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ such that

$$\exists C, c > 0 : \ |\mathcal{T}_\varphi u(z)| \leq C e^{\Phi(z) - c|z|^2}$$

for all z in an open conic neighborhood of $\kappa_\varphi^\flat(x_0, \xi_0)$ in $\mathbb{C}^n \backslash \{0\}$. Roughly speaking, if $u \in \mathcal{S}'(\mathbb{R}^n)$, then $\mathrm{WF}^{1/2}(u)$ measures the failure of u to admit an extension to a holomorphic function U on $\mathbb{C}^n$ satisfying

$$|U(w)| \leq C e^{-c|x|^2 + C|y|^2}, \quad \forall w = x + iy \in \mathbb{C}^n, \tag{3.1.7}$$

for some $C, c > 0$. A function u on $\mathbb{R}^n$ admitting an extension $U \in \mathrm{Hol}(\mathbb{C}^n)$ obeying the estimate (3.1.7) is called a *Gelfand-Shilov test function*. Thus, $\mathrm{WF}^{1/2}(u)$ is a microlocal measure of how a distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ fails to be a Gelfand-Shilov test function. For additional information concerning general Gelfand-Shilov wavefront sets and classes of Gelfand-Shilov ultradistributions, see [10]. In the present work, our objective is to study the propagation in time of 1/2-Gelfand-Shilov singularities by the evolution (3.1.1). To the best of our knowledge, this case has not yet been explored.

We now proceed to state the main results of this chapter. Let $\mathbb{R}^{2n}$ be equipped with the standard symplectic form

$$\sigma((x, \xi), (y, \eta)) = \xi \cdot y - x \cdot \eta, \quad (x, \xi), \ (y, \eta) \in \mathbb{R}^{2n}. \tag{3.1.8}$$

Suppose $q : \mathbb{R}^{2n} \to \mathbb{C}$ is a complex-valued quadratic form and let $q(\cdot,\cdot)$ denote its symmetric $\mathbb{C}$-bilinear polarization. Because σ is non-degenerate, there is a unique $F \in M_{2n\times 2n}(\mathbb{C})$ such that

$$q((x,\xi),(y,\eta)) = \sigma((x,\xi),F(y,\eta))$$

for all $(x,\xi),(y,\eta) \in \mathbb{R}^{2n}$. This matrix F is called the *Hamilton map* or *Hamilton matrix of q* (see Section 21.5 of [32]). Explicitly, the Hamilton matrix of q is given by

$$F = JQ$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is the standard $2n \times 2n$ symplectic matrix and $Q \in M_{2n\times 2n}(\mathbb{C})$ is the unique complex symmetric matrix such that

$$q(X) = QX \cdot X, \quad X \in \mathbb{R}^{2n}.$$

Let

$$\text{Re } F = \frac{F + \overline{F}}{2} \text{ and } \text{Im } F = \frac{F - \overline{F}}{2i}$$

be the real and imaginary parts of F respectively. The *singular space S of q* is defined as the following finite intersection of kernels:

$$S = \left(\bigcap_{j=0}^{2n-1} \ker\left[(\text{Re } F)(\text{Im } F)^j\right] \right) \cap \mathbb{R}^{2n}. \tag{3.1.9}$$

The singular space was first introduced by M. Hitrik and K. Pravda-Starov in [23] where it arose naturally in the study of spectra and semi-group smoothing properties for non-self adjoint quadratic differential operators. The concept of the singular space has since been shown to play a key role in the understanding of hypoelliptic and spectral properties of non-elliptic quadratic differential operators. See for instance [24], [25], [41], [42], [57], and [58].

In the work [31], it was shown that when the quadratic form q has non-negative real part $\mathrm{Re}\, q \geq 0$, the maximal closed realization of the quadratic differential operator $q^w(x, D)$ on $L^2(\mathbb{R}^n)$ is maximally accretive and generates a contraction semigroup $(e^{-tq^w(x,D)})_{t\geq 0}$. In this same work it was also proven that for each $t \geq 0$ the operator $e^{-tq^w(x,D)}$ both restricts to a continuous linear transformation

$$\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \tag{3.1.10}$$

and admits a unique extension to a continuous linear transformation

$$\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n). \tag{3.1.11}$$

Consequently,

$$e^{-tq^w(x,D)}u_0$$

is a well-defined element of $\mathcal{S}'(\mathbb{R}^n)$ for any $u_0 \in \mathcal{S}'(\mathbb{R}^n)$ and $t \geq 0$. Our goal is to understand the relationship between $\mathrm{WF}^{1/2}(e^{-tq^w(x,D)}u_0)$ and $\mathrm{WF}^{1/2}(u_0)$ in terms of the Hamiltonian dynamics generated by q.

Let us point out that the topic of the partial Gelfand-Shilov regularizing properties for the semigroup $e^{-tq^w(x,D)}$ has received considerable attention over the last several years. The general aim of work in this direction has been to prove sharp microlocal Gelfand-Shilov smoothing estimates in directions transverse to the singular space S for various time regimes. We mention the works [26], [3], and [4]. In particular, it follows from Theorem 1.2 in [26] that

$$\mathrm{WF}^{1/2}\left(e^{-tq^w(x,D)}u_0\right) = \emptyset$$

for all $u_0 \in L^2(\mathbb{R}^n)$ and $0 < t \ll 1$ when $S = \{0\}$. In other words, if the singular space of the quadratic form q is trivial, then the propagator for the evolution (3.1.1) is 'instantaneously globally analytically regularizing.' Our main theorem extends this observation to the case of general S.

Theorem 3.1.1. *Let q be a complex-valued quadratic form on $\mathbb{R}^{2n}$ with Re $q \geq 0$ and S be the singular space of q. For every $u_0 \in \mathcal{S}'(\mathbb{R}^n)$ and $t > 0$, we have*

$$WF^{1/2}(e^{-tq^w(x,D)}u_0) = \exp{(tH_{\mathrm{Im}\ q})}(WF^{1/2}(u_0) \cap S). \tag{3.1.12}$$

Here $H_{Im\ q}$ denotes the Hamilton vector field of Im q on $\mathbb{R}^{2n}$ taken with respect to the symplectic form σ, and $\exp{(tH_{\mathrm{Im}\ q})}$ *is the Hamilton flow of Im q at time t.*

Remark 3.1.2. In [44] it was shown that

$$\mathrm{WF}_G(e^{-tq^w(x,D)}u_0) \subset \exp(tH_{\mathrm{Im}\ q})(\mathrm{WF}_G(u_0) \cap S), \quad \forall u_0 \in \mathcal{S}'(\mathbb{R}^n),\ \forall t > 0, \tag{3.1.13}$$

and that equality need not hold for any $t > 0$. In [11], it was established that for $s > 1/2$, there is the inclusion

$$\mathrm{WF}^s(e^{-tq^w(x,D)}u_0) \subset \exp{(tH_{\mathrm{Im}\ q})}(\mathrm{WF}^s(u_0) \cap S) \quad \forall u_0 \in \mathcal{S}'(\mathbb{R}^n),\ \forall t > 0. \tag{3.1.14}$$

In contrast to these results, Theorem 3.1.1 shows that the propagator $e^{-tq^w(x,D)}$ never regularizes any of the 1/2-Gelfand-Shilov singularities of u_0 that lie within S. Thus it is possible to recover $\mathrm{WF}^{1/2}(u_0) \cap S$ from knowledge of $\mathrm{WF}^{1/2}(e^{-tq^w(x,D)}u_0)$ for some $t > 0$. So far as we are aware, it is not known if the inclusion in (3.1.14) is in general strict for $s > 1/2$. It would be of interest to determine the range of $s \geq 1/2$ for which equality holds in (3.1.14) for all $u_0 \in \mathcal{S}'(\mathbb{R}^n)$ and $t > 0$.

Our approach to the proof of Theorem 3.1.1 is based entirely on FBI transform techniques. Let $\varphi(z, y)$, $(z, y) \in \mathbb{C}^{2n}$, be a metaplectic FBI phase function with associated plurisubharmonic weight $\Phi(z)$, $z \in \mathbb{C}^n$, and complex linear canonical transformation $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$. Our strategy is to study the conjugated propagator

$$\mathcal{T}_\varphi \circ e^{-tq^w(x,D)} \circ \mathcal{T}_\varphi^*, \quad t \geq 0, \tag{3.1.15}$$

acting on the extended Bargmann space $H_\Phi^{-\infty}(\mathbb{C}^n)$, which is the image of $\mathcal{S}'(\mathbb{R}^n)$ by $\mathcal{T}_\varphi$. The extended Bargmann space $H_\Phi^{-\infty}(\mathbb{C}^n)$ is the space of all $u \in \mathrm{Hol}(\mathbb{C}^n)$ such that

$$\int_{\mathbb{C}^n} \langle z \rangle^{2s} \, |u(z)|^2 \, e^{-2\Phi(z)} \, L(dz) < \infty,$$

for some $s \in \mathbb{R}$. See Section 2 below for a full discussion of the space $H_\Phi^{-\infty}(\mathbb{C}^n)$ and related exponentially weighted spaces of entire functions. By Egorov's theorem, the operator (3.1.15) is the evolution semigroup generated by the complex Weyl differential operator $\tilde{q}^w(z, D_z)$, where $\tilde{q} = q \circ \kappa_\varphi^{-1}$. We note that this approach differs significantly from that of the works [44], [59], [11], and [43], which rely heavily on representations of the Schwartz kernel of $e^{-tq^w(x,D)}$ as a Gaussian oscillatory integral in the sense of [31]. In the present work, we use an elementary geometrical optics construction to show that the semigroup $e^{-t\tilde{q}^w(z,D_z)}$ is a Fourier integral operator in the complex domain in the sense of Sjöstrand [51]. Motivated by recent results concerning Bergman representations of metaplectic Fourier integral operators in the work [12], we establish an integral representation for the semigroup $e^{-t\tilde{q}^w(z,D)}$ on the FBI transform side of the form

$$e^{-t\tilde{q}^w(z,D)} u(z) = \hat{a}(t) \int_{\mathbb{C}^n} e^{2\Psi_t(z,\overline{w})} u(w) e^{-2\Phi(w)} \, L(dw), \quad u \in H_\Phi(\mathbb{C}^n), \quad t \geq 0, \tag{3.1.16}$$

where $\hat{a}(t) \in \mathbb{C}$ depends analytically on t and $\Psi_t(\cdot, \cdot)$ is a holomorphic quadratic form on $\mathbb{C}^{2n}$ whose coefficients depend analytically on t. Here we denote the Lebesgue measure on $\mathbb{C}^n = \mathbb{C}^n_z$ by $L(dz)$ and $H_\Phi(\mathbb{C}^n)$ is the *Bargmann space*

$$H_\Phi(\mathbb{C}^n) = L^2(\mathbb{C}^n, e^{-2\Phi(z)} \, L(dz)) \cap \mathrm{Hol}(\mathbb{C}^n),$$

which is the unitary image of $L^2(\mathbb{R}^n)$ under the FBI transform $\mathcal{T}_\varphi$. Writing (3.1.16) as the contour integral

$$e^{-t\tilde{q}^w(z,D)}u(z) = a(t)\iint\limits_{\Gamma} e^{2\Psi_t(z,\theta)-2\Psi(w,\theta)}u(w)\,dw\wedge d\theta, \quad u\in H_\Phi(\mathbb{C}^n), \tag{3.1.17}$$

where

$$\Gamma = \{(w,\theta) : \theta = \overline{w}\} \tag{3.1.18}$$

is the anti-diagonal in $\mathbb{C}^{2n}$, $a(t) = (i/2)^n\hat{a}(t)$, and $\Psi(\cdot,\cdot)$ is the polarization of Φ, i.e. the unique holomorphic quadratic form on $\mathbb{C}^{2n}$ such that $\Psi(z,\overline{z}) = \Phi(z)$ for all $z\in\mathbb{C}^n$, we find that $e^{-t\tilde{q}^w(z,D)}$ is a metaplectic Fourier integral operator in the complex domain whose phase function generates the graph of the Hamilton flow $\tilde{\kappa}_t = \exp(tH_{-i\tilde{q}}) : \mathbb{C}^{2n}\to\mathbb{C}^{2n}$ of $-i\tilde{q}$ at time t for each $t\geq 0$. In particular, for every $t\geq 0$, the flow $\tilde{\kappa}_t$ is positive relative to the maximally totally real subspace

$$\Lambda_\Phi = \left\{\left(z, \frac{2}{i}\partial_z\Phi(z)\right) : z\in\mathbb{C}^n\right\} \tag{3.1.19}$$

of $\mathbb{C}^{2n}$ in the sense that

$$\frac{1}{i}\left(\sigma(\tilde{\kappa}_t(X), \iota_{\Lambda_\Phi}(\tilde{\kappa}_t(X))) - \sigma(X, \iota_{\Lambda_\Phi}X)\right)\geq 0, \quad X\in\mathbb{C}^{2n},$$

where ι_{Λ_Φ} is the unique antilinear involution of $\mathbb{C}^{2n}$ fixing Λ_Φ. By using some recent results from [12] concerning complex canonical transformations of $\mathbb{C}^{2n}$ that are positive relative to a pair of maximally totally real subspaces of the form (3.1.19) for some strictly plurisubharmonic quadratic form Φ on $\mathbb{C}^{2n}$, we show that it is possible to find $\hat{a}(t)$ and $\Psi_t(\cdot,\cdot)$ so that (3.1.16) is valid for all non-negative times $t\geq 0$. In particular, the Bergman representation (3.1.16) provides an alternative to Hörmander's generalized Mehler formula [31] for representing the semigroup $e^{-tq^w(x,D)}$. We feel that this result may be of independent interest, and we plan to explore additional applications in the future.

Having established the representation (3.1.16), we can invoke some generalities concerning how metaplectic Fourier integral operators associated to complex canonical transformations of $\mathbb{C}^{2n}$ propagate and/or regularize 1/2-Gelfand-Shilov singularities. Given $u \in H_\Phi^{-\infty}(\mathbb{C}^n)$, we define the *1/2-Gelfand-Shilov wavefront set of u relative to* Φ as the complement in $\mathbb{C}^n \backslash \{0\}$ of all points z_0 for which there exist $C, c > 0$ such that

$$|u(z)| \leq Ce^{\Phi(z) - c|z|^2}$$

for all z in some open conic neighborhood of z_0 in $\mathbb{C}^n \backslash \{0\}$. Thus, if $u = \mathcal{T}_\varphi v$ where $v \in \mathcal{S}'(\mathbb{R}^n)$, then

$$\mathrm{WF}_\Phi^{1/2}(u) = \kappa_\varphi^\flat(\mathrm{WF}^{1/2}(v)),$$

where $\kappa_\varphi^\flat$ is as in (3.1.6). Given a non-zero metaplectic Fourier integral operator G acting on $H_\Phi^{-\infty}(\mathbb{C}^n)$ whose underlying complex linear canonical transformation $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is positive relative to Λ_Φ and such that $\Lambda_\Phi \cap \kappa(\Lambda_\Phi)$ is invariant under κ, one has the following general relationship between $\mathrm{WF}_\Phi^{1/2}(Gu)$ and $\mathrm{WF}_\Phi^{1/2}(u)$:

$$\mathrm{WF}_\Phi^{1/2}(Gu) = \kappa^\flat(\mathrm{WF}_\Phi^{1/2}(u)) \cap \pi_1(\Lambda_\Phi \cap \kappa(\Lambda_\Phi)) \tag{3.1.20}$$

where

$$\kappa^\flat = \pi_1 \circ \kappa \circ (\pi_1|_{\Lambda_\Phi})^{-1}.$$

The proof of (3.1.20), which uses the result from [12] that every metaplectic Fourier integral operator possesses a unique Bergman form, is given in Section 5 below. Using (3.1.20) with $G = e^{-t\tilde{q}^w(z,D_z)}$, $\kappa = \tilde{\kappa}_t$, and the general geometric relationship

$$\Lambda_\Phi \cap \tilde{\kappa}_t(\Lambda_\Phi) = \kappa_\varphi(S), \quad t > 0, \tag{3.1.21}$$

which we establish in Section 6, will complete the proof of Theorem 3.1.1.

The plan for this chapter is as follows. In Section 2, we review background material on metaplectic FBI transforms and exponentially weighted spaces of entire functions. In Section 3, we recall the definition of the 1/2-Gelfand-Shilov wavefront set and state some basic properties. In Section 4, we discuss metaplectic Fourier integral operators acting on exponentially weighted spaces of entire functions and recall some results from [12] concerning Bergman representations of such operators. In Section 5, we prove some general results concerning how metaplectic Fourier integral operators in the complex domain move and/or regularize global analytic singularities. Finally, in Section 6, we establish the Bergman representation (3.1.16) for the evolution semigroup $e^{-tq^w(x,D)}$ on the FBI transform side and finish the proof of Theorem 3.1.1.

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Notation

- *Constants.* C and c stand for positive constants; C may increase from one inequality to the next, while c may decrease from one inequality to the next if necessary.
- *Dot Product.* If $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$,

$$z \cdot w = z_1 w_1 + \cdots + z_n \cdot w_n.$$

- $\langle z \rangle = (1 + |z|^2)^{1/2}$ denotes the Japanese bracket of $z \in \mathbb{C}^n$.
- $\pi_1 : \mathbb{C}^{2n} \to \mathbb{C}^n$ is the projection onto the first factor $(z, \zeta) \mapsto z$.
- $J \in M_{2n \times 2n}(\mathbb{R})$ is the standard symplectic matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

- *Conic Subsets*. If X is a real vector space with norm $\|\cdot\|$ and V, W are conic subsets of $X\backslash\{0\}$, we use the notation

$$V \subset\subset W$$

to mean that $V \cap \{x \in X : \|x\| = 1\}$ is compactly contained in $W \cap \{x \in X : \|x\| = 1\}$.

- *Radicals.* If P is a non-negative quadratic form on $\mathbb{R}^N$, then the radical of P, denoted $\mathrm{Rad}(P)$, is the zero set of P,

$$\mathrm{Rad}(P) = \left\{X \in \mathbb{R}^N : P(X) = 0\right\}.$$

- *Derivatives.* On $\mathbb{C}^n$ with coordinates $z = (z_1, \ldots, z_n)$,

$$\partial_{z_j} = \frac{1}{2}\left(\frac{\partial}{\partial \mathrm{Re}\ z_j} - i\frac{\partial}{\partial \mathrm{Im}\ z_j}\right), \quad j = 1, \ldots, n$$

and

$$\partial_{\overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial \mathrm{Re}\ z_j} + i\frac{\partial}{\partial \mathrm{Im}\ z_j}\right), \quad j = 1, \ldots, n.$$

For a sufficiently differentiable function $f : \mathbb{C}^n \to \mathbb{C}$,

$$\partial_z f = \begin{pmatrix} \frac{\partial f}{\partial z_1} \\ \vdots \\ \frac{\partial f}{\partial z_n} \end{pmatrix}, \quad \partial_{\overline{z}} f = \begin{pmatrix} \frac{\partial f}{\partial \overline{z}_1} \\ \vdots \\ \frac{\partial f}{\partial \overline{z}_n} \end{pmatrix}$$

and

$$\partial^2_{zz} f = \left(\frac{\partial^2 f}{\partial z_j \partial z_k}\right)_{1\le j,k\le n}, \quad \partial^2_{z\overline{z}} f = \left(\frac{\partial^2 f}{\partial z_j \partial \overline{z}_k}\right)_{1\le j,k\le n},$$
$$\partial^2_{\overline{z}z} f = \left(\frac{\partial^2 f}{\partial \overline{z}_j \partial z_k}\right)_{1\le j,k\le n,}, \quad \partial^2_{\overline{z}\overline{z}} f = \left(\frac{\partial^2 f}{\partial \overline{z}_j \partial \overline{z}_k}\right)_{1\le j,k\le n}.$$

- *Hamiltonian Dynamics in the Complex Domain.* Write $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_\zeta$ and let $\sigma = d\zeta \wedge dz$ be the standard complex symplectic form on $\mathbb{C}^{2n}$. Given $f \in \mathrm{Hol}(\mathbb{C}^{2n})$, the complex Hamilton vector field of f is denoted

$$H_f = \partial_\zeta f \cdot \partial_z - \partial_z f \cdot \partial_\zeta,$$

This is a complex vector field on $\mathbb{C}^{2n}$ of type $(1,0)$, which we identify with $(\partial_\zeta f, -\partial_z f) \in \mathbb{C}^{2n}$. We note that H_f is the unique complex vector field on $\mathbb{C}^{2n}$ of type $(1,0)$ such that $\sigma(t, H_f) = df(t)$ for all complex vector fields t on $\mathbb{C}^{2n}$ of type $(1,0)$. In this chapter we shall only ever be interested in the case where f is a holomorphic quadratic form. In this case the Hamilton flow of f is denoted by $\exp{(tH_f)}$, $t \in \mathbb{R}$, and defined as follows: for $(z_0, \zeta_0) \in \mathbb{C}^{2n}$ and $t \in \mathbb{R}$,

$$(z(t), \zeta(t)) = \exp{(tH_f)}(z_0, \zeta_0)$$

if and only if

$$\begin{cases} z'(t) = \partial_\zeta f(z(t), \zeta(t)) \\ \zeta'(t) = -\partial_z f(z(t), \zeta(t)), \\ z(0) = z_0, \;\; \zeta(0) = \zeta_0. \end{cases}$$

- *Schwartz Functions and Tempered Distributions.* $\mathcal{S}(\mathbb{R}^n)$ denotes class of Schwartz functions on $\mathbb{R}^n$ equipped with its usual locally convex Frechét topology; $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions on $\mathbb{R}^n$ endowed with the weak* topology. The distributional pairing of $u \in \mathcal{S}'(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$ is denoted $\langle u, f\rangle$.

3.2 Review of FBI Tools and Exponentially Weighted Spaces of Entire Functions

In this section we review basic definitions and facts concerning metaplectic Fourier-Bros-Iagolnitzer (FBI) transforms on $\mathbb{R}^n$. Standard references for this material include [27], Chapter 13 of [63], and [54].

Let $\varphi = \varphi(z, y)$ be a holomorphic quadratic form on $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_y$. Write

$$\varphi(z, y) = \frac{1}{2}Az \cdot z + Bz \cdot y + \frac{1}{2}Dy \cdot y, \quad (z, y) \in \mathbb{C}^n \times \mathbb{C}^n,$$

where $A, B, D \in M_{n\times n}(\mathbb{C})$ with $A = A^T$ and $D = D^T$. If $\det B \neq 0$ and Im D is positive-definite, we say that φ is an *FBI phase function* or *metaplectic FBI phase function*. If φ is an FBI phase function, the FBI transform associated to φ is the linear transformation $\mathcal{T}_\varphi : \mathcal{S}'(\mathbb{R}^n) \to \mathrm{Hol}(\mathbb{C}^n)$ defined by

$$\mathcal{T}_\varphi u(z) = c_\varphi \int_{\mathbb{R}^n} e^{i\varphi(z,y)} u(y)\, dy, \quad u \in \mathcal{S}'(\mathbb{R}^n), \tag{3.2.1}$$

where

$$c_\varphi = 2^{-n/2}\pi^{-3n/4}(\det \mathrm{Im}\ D)^{-1/4} \left|\det B\right|, \tag{3.2.2}$$

and the integral (3.2.1) is interpreted in the sense of distributions.

Let φ be an FBI phase function. To describe the range of the FBI transform $\mathcal{T}_\varphi$, we introduce the real-valued quadratic form

$$\Phi(z) = \sup_{y\in\mathbb{R}^n} (-\mathrm{Im}\ \varphi(z, y)), \quad z \in \mathbb{C}^n. \tag{3.2.3}$$

Since Im D is positive-definite, this supremum is really a maximum and we may write

$$\Phi(z) = -\mathrm{Im}\ \varphi(z, y(z))$$

where $y(z) \in \mathbb{R}^n$ is an $\mathbb{R}$-linear function of $z \in \mathbb{C}^n$. Because Φ is equal to the maximum of the family of pluriharmonic functions

$$\mathbb{C}^n \ni z \mapsto -\mathrm{Im}\, \varphi(z,y) \in \mathbb{R}, \quad y \in \mathbb{R}^n,$$

the form Φ is itself plurisubharmonic. In fact, Φ is strictly plurisubharmonic, i.e. the Levi matrix $\partial^2_{\bar{z}z}\Phi$ is Hermitian positive-definite. We refer to Proposition 1.3.2 of [27] for a proof. In the sequel, if φ is an FBI phase function, then we shall refer to Φ given by (3.2.3) as the strictly plurisubharmonic weight associated to φ.

For $s \in \mathbb{R}$, let

$$L^2_{\Phi,s}(\mathbb{C}^n) = L^2(\mathbb{C}^n, \langle z\rangle^{2s} e^{-2\Phi(z)}\, L(dz)), \tag{3.2.4}$$

equipped with the natural inner product,

$$(u_1,u_2)_s = \int_{\mathbb{C}^n} u_1(z)\overline{u_2(z)}\langle z\rangle^{2s} e^{-2\Phi(z)}\, L(dz), \quad u_1,u_2 \in L^2_{\Phi,s}(\mathbb{C}^n), \tag{3.2.5}$$

and associated norm

$$\|u\|_s^2 = \int_{\mathbb{C}^n} |u(z)|^2\, \langle z\rangle^{2s} e^{-2\Phi(z)}\, L(dz), \quad u \in L^2_{\Phi,s}(\mathbb{C}^n). \tag{3.2.6}$$

For $s \in \mathbb{R}$, let

$$H^s_\Phi(\mathbb{C}^n) = L^2_{\Phi,s}(\mathbb{C}^n) \cap \mathrm{Hol}(\mathbb{C}^n) \tag{3.2.7}$$

be the closed linear subspace of entire functions in $L^2_{\Phi,s}(\mathbb{C}^n)$. By convention, when $s=0$, we write $H_\Phi(\mathbb{C}^n)$ in place of $H^0_\Phi(\mathbb{C}^n)$, and we write $(\cdot,\cdot)$ and $\|\cdot\|$ in place of $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ respectively. In the literature, the space $H_\Phi(\mathbb{C}^n)$ is known as the *Bargmann* or *Bargmann-Fock* space of entire functions on $\mathbb{C}^n$. If $s_1 \le s_2$, then $H^{s_2}_\Phi(\mathbb{C}^n) \subset H^{s_1}_\Phi(\mathbb{C}^n)$, and the natural inclusion map $H^{s_2}_\Phi(\mathbb{C}^n) \hookrightarrow H^{s_1}_\Phi(\mathbb{C}^n)$

is bounded. Thus, if we let

$$H_\Phi^{-\infty}(\mathbb{C}^n) = \bigcup_{s\in\mathbb{R}} H_\Phi^s(\mathbb{C}^n), \tag{3.2.8}$$

we obtain an inductive system of Hilbert spaces $(H_\Phi^{-\infty}(\mathbb{C}^n), \{H_\Phi^s(\mathbb{C}^n)\}_{s\in\mathbb{R}})$ (see [15] Chapter IV Section 5). We equip $H_\Phi^{-\infty}(\mathbb{C}^n)$ with the corresponding inductive limit topology (see [15] Chapter IV Proposition 5.3 and Definition 5.4). We refer to $H_\Phi^{-\infty}(\mathbb{C}^n)$ as the *extended Bargmann space*. We also introduce the space

$$H_\Phi^{\infty}(\mathbb{C}^n) = \bigcap_{s\in\mathbb{R}} H_\Phi^s(\mathbb{C}^n), \tag{3.2.9}$$

equipped with the Frechét space topology induced by the family of norms $\{\|\cdot\|_s\}_{s\in\mathbb{R}}$. For every $s \in \mathbb{R}$, we have continuous inclusions

$$H_\Phi^{\infty}(\mathbb{C}^n) \hookrightarrow H_\Phi^s(\mathbb{C}^n) \hookrightarrow H_\Phi^{-\infty}(\mathbb{C}^n). \tag{3.2.10}$$

Thanks to the mean-value property of holomorphic functions, we have the following lemma characterizing functions in $H_\Phi^s(\mathbb{C}^n)$ as holomorphic functions on $\mathbb{C}^n$ obeying suitable weighted L^∞-estimates.

Lemma 3.2.1. *For any $s \in \mathbb{R}$ and $\epsilon > 0$, there is $C > 0$ such that*

$$\|u\|_{L^2_{\Phi,s}(\mathbb{C}^n)} \le C\|\langle z\rangle^{s+n+\epsilon} u(z) e^{-\Phi(z)}\|_{L^\infty(\mathbb{C}^n)} \tag{3.2.11}$$

for all measurable u on $\mathbb{C}^n$. For any $N \in \mathbb{R}$, there is $C > 0$ such that

$$\|\langle z\rangle^{n+N} u(z) e^{-\Phi(z)}\|_{L^\infty(\mathbb{C}^n)} \le C\|u\|_{L^2_{\Phi,N+2n}(\mathbb{C}^n)} \tag{3.2.12}$$

for all $u \in Hol(\mathbb{C}^n)$. Consequently, for $u \in Hol(\mathbb{C}^n)$,

$$u \in H_\Phi^{\infty}(\mathbb{C}^n) \iff \forall N \in \mathbb{R}\ \exists C > 0\ \forall z \in \mathbb{C}^n : |u(z)| \le C\langle z\rangle^N e^{\Phi(z)} \tag{3.2.13}$$

and

$$u \in H_\Phi^{-\infty}(\mathbb{C}^n) \iff \exists N \in \mathbb{R}\ \exists C > 0\ \forall z \in \mathbb{C}^n : |u(z)| \le C\langle z\rangle^N e^{\Phi(z)}. \tag{3.2.14}$$

Proof. Let $s \in \mathbb{R}$ and $\epsilon > 0$ be arbitrary. For any $N \in \mathbb{R}$ and measurable u on $\mathbb{C}^n$,

$$\|u\|_{L^2_{\Phi,s}(\mathbb{C}^n)} \le \left(\int_{\mathbb{C}^n} \langle z\rangle^{2(s-N)}\, L(dz)\right)^{1/2} \|\langle z\rangle^N u(z) e^{-\Phi(z)}\|_{L^\infty(\mathbb{C}^n)}. \tag{3.2.15}$$

Choosing $N = s + n + \epsilon$ gives

$$\|u\|_{L^2_{\Phi,s}(\mathbb{C}^n)} \le C\|\langle z\rangle^{s+n+\epsilon} u(z) e^{-\Phi(z)}\|_{L^\infty(\mathbb{C}^n)}, \tag{3.2.16}$$

where the constant $C > 0$ depends only on s, n, and ϵ. To prove the second claim, let $N \in \mathbb{R}$ and $u \in \mathrm{Hol}(\mathbb{C}^n)$ be arbitrary. By the mean-value theorem,

$$u(z) = \frac{1}{c_{2n}}\langle z\rangle^{2n} \int_{|z-w|\le\langle z\rangle^{-1}} u(w)\, L(dw), \quad z \in \mathbb{C}^n, \tag{3.2.17}$$

where c_{2n} is the volume of the unit ball in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Since Φ is a real quadratic form on $\mathbb{C}^n$, we get by Taylor expansion that

$$\Phi(w) - \Phi(z) = \partial_z\Phi(z)\cdot(w-z) + \partial_{\overline{z}}\Phi(z)\cdot\overline{(w-z)} + \Phi(w-z), \quad w, z \in \mathbb{C}^n. \tag{3.2.18}$$

Because Φ is quadratic, both $\partial_z\Phi$ and $\partial_{\overline{z}}\Phi$ are $\mathbb{R}$-linear forms on $\mathbb{C}^n$. Hence there is $C > 0$ such that

$$|\partial_z\Phi(z)| \le C\langle z\rangle \text{ and } |\partial_{\overline{z}}\Phi(z)| \le C\langle z\rangle, \quad z \in \mathbb{C}^n. \tag{3.2.19}$$

So when $|z-w| \leq \langle z\rangle^{-1}$, we may deduce from (3.2.18), (3.2.19), and Cauchy-Schwartz that

$$\begin{aligned} |\Phi(w)-\Phi(z)| &\leq C\langle z\rangle^{-1}\,|z-w| + C\langle z\rangle^{-1}\,|z-w| + |z-w|^2 \\ &\leq C + C + \langle z\rangle^{-2} \\ &\leq C \end{aligned} \tag{3.2.20}$$

for some large enough $C > 0$. Thus

$$\begin{aligned} \langle z\rangle^{N}\,|u(z)|\,e^{-\Phi(z)} &\leq C\int_{|z-w|\leq\langle z\rangle^{-1}} \langle z\rangle^{N+2n}\,|u(w)|\,e^{-\Phi(w)}\,L(dw) \\ &\leq C\int_{|z-w|\leq\langle z\rangle^{-1}} \langle z-w\rangle^{|N+2n|}\langle w\rangle^{N+2n}\,|u(w)|\,e^{-\Phi(w)}\,L(dw) \\ &\leq C\int_{|z-w|\leq\langle z\rangle^{-1}} \langle w\rangle^{N+2n}\,|u(w)|\,e^{-\Phi(w)}\,L(dw) \\ &\leq C\langle z\rangle^{-n}\|u\|_{H_{\Phi}^{N+2n}(\mathbb{C}^n)} \end{aligned} \tag{3.2.21}$$

for all $z \in \mathbb{C}^n$, where the constant $C > 0$ does not depend on u. The bound (3.2.12) follows. □

Next, we state a well-known proposition characterizing the range of a metaplectic FBI transform $\mathcal{T}_\varphi$ in terms of exponentially weighted spaces of entire functions on $\mathbb{C}^n$.

Proposition 3.2.2. *Let φ be an FBI phase function with associated FBI transform $\mathcal{T}_\varphi$ and strictly plurisubharmonic weight Φ. Then*

$$\mathcal{T}_\varphi : L^2(\mathbb{R}^n) \to H_\Phi(\mathbb{C}^n) \tag{3.2.22}$$

is a unitary transformation. Furthermore, $\mathcal{T}_\varphi$ is bijective

$$\mathcal{S}(\mathbb{R}^n) \to H_\Phi^\infty(\mathbb{C}^n) \quad \textit{and} \quad \mathcal{S}'(\mathbb{R}^n) \to H_\Phi^{-\infty}(\mathbb{C}^n). \tag{3.2.23}$$

Proof. There are many available proofs of the unitarity of $\mathcal{T}_\varphi : L^2(\mathbb{R}^n) \to H_\Phi(\mathbb{C}^n)$. The reader may consult, for instance, Theorem 13.7 of [63], Theorem 1.3.3 of [27], or Proposition 6.1 of [30]. The bijectivity of $\mathcal{T}_\varphi : \mathcal{S}(\mathbb{R}^n) \to H_\Phi^\infty(\mathbb{C}^n)$ and $\mathcal{T}_\varphi : \mathcal{S}'(\mathbb{R}^n) \to H_\Phi^{-\infty}(\mathbb{C}^n)$ is also well-known. It follows immediately from, for instance, Lemma 3.2.1 and Proposition 6.1 of [30]. See also Section 12.2 of [55]. ☐

We next discuss some functional analytic aspects of the spaces $H^s_\Phi(\mathbb{C}^n)$, $s \in \mathbb{R}$. Let Φ be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$. Let $\Psi(\cdot,\cdot)$ be the polarization of Φ, i.e. the unique holomorphic quadratic form on $\mathbb{C}^{2n}$ such that $\Psi(z,\overline{z}) = \Phi(z)$ for all $z \in \mathbb{C}^n$. We recall (see [27] or Chapter 13 of [63]) that the *Bergman projection* associated to Φ is the orthogonal projector $\Pi_\Phi : L^2_{\Phi,0}(\mathbb{C}^n) \to H_\Phi(\mathbb{C}^n)$. It is given explicitly by

$$\Pi_\Phi u(z) = C_\Phi \int_{\mathbb{C}^n} e^{2\Psi(z,\overline{w})} u(w) e^{-2\Phi(w)}\, L(dw), \quad u \in H_\Phi(\mathbb{C}^n), \tag{3.2.24}$$

where

$$C_\Phi = \left(\frac{2}{\pi}\right)^n \det \partial^2_{z\overline{z}} \Phi. \tag{3.2.25}$$

We recall that $\Psi(\cdot,\cdot)$ satisfies the 'fundamental estimate'

$$2\mathrm{Re}\ \Psi(z,\overline{w}) - \Phi(z) - \Phi(w) \asymp -\left|z-w\right|^2, \quad z,w \in \mathbb{C}^n. \tag{3.2.26}$$

For a proof, see page 492 of [27] or the proof of Theorem 13.6 in [63]. From (5.2.68) and Schur's lemma, it follows that the operator (5.2.65) is bounded $L^2_{\Phi,s}(\mathbb{C}^n) \to L^2_{\Phi,s}(\mathbb{C}^n)$ for every $s \in \mathbb{R}$. In fact, Π_Φ defined by (5.2.65) coincides with the orthogonal projection $L^2_{\Phi,s}(\mathbb{C}^n) \to H^s_\Phi(\mathbb{C}^n)$ (see [55] Section 12.2), and we have

$$\Pi_\Phi u = u, \quad \forall u \in H^{-\infty}_\Phi(\mathbb{C}^n). \tag{3.2.27}$$

Using (3.2.27), one can prove

Proposition 3.2.3 ([55] Section 12.2)**.** *For any strictly plurisubharmonic quadratic form Φ on $\mathbb{C}^n$, the space $H^\infty_\Phi(\mathbb{C}^n)$ is dense in $H^s_\Phi(\mathbb{C}^n)$ for every $s \in \mathbb{R}$. Consequently, $H^\infty_\Phi(\mathbb{C}^n)$ is dense in $H^{-\infty}_\Phi(\mathbb{C}^n)$.*

Proof. Let Φ be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$. Let $s \in \mathbb{R}$ and $u \in H^s_\Phi(\mathbb{C}^n)$ be arbitrary. Let $\chi \in C_0^\infty(\mathbb{C}^n)$ be such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in a neighborhood of $0 \in \mathbb{C}^n$. For $\epsilon > 0$, set

$$u_\epsilon = \Pi_\Phi(\chi(\epsilon z)u), \tag{3.2.28}$$

where Π_Φ is the Bergman projector (5.2.65). Using the fundamental estimate (5.2.68), we see that for any $\epsilon > 0$ and $N > 0$,

$$\begin{aligned}\langle z\rangle^N \left|u_\epsilon(z)\right| e^{-\Phi(z)} &\leq C \int_{\mathbb{C}^n} e^{-c|z-w|^2} \langle z\rangle^N \left|\chi(\epsilon w)u(w)\right| e^{-\Phi(w)}\, L(dw) \\ &\leq C \int_{\mathbb{C}^n} e^{-c|z-w|^2} \langle z-w\rangle^N \langle w\rangle^N \left|\chi(\epsilon w)u(w)\right| e^{-\Phi(w)}\, L(dw) \\ &\leq C_{\epsilon,N} \int_{\mathbb{C}^n} e^{-c|z-w|^2} \langle z-w\rangle^N \langle w\rangle^s \left|u(w)\right| e^{-\Phi(w)}\, L(dw),\end{aligned} \tag{3.2.29}$$

where $C_{\epsilon,N} > 0$ depends only on ϵ and N. Applying Schur's lemma, we find that $u_\epsilon \in H^N_\Phi(\mathbb{C}^n)$ for any $\epsilon > 0$ and $N > 0$. Thus $u_\epsilon \in H^\infty_\Phi(\mathbb{C}^n)$ for all $\epsilon > 0$. Now we claim that $u_\epsilon \to u$ in $H^s_\Phi(\mathbb{C}^n)$ as $\epsilon \to 0^+$. Indeed, from (3.2.27),

$$\begin{aligned}\langle z\rangle^s \left|u(z) - u_\epsilon(z)\right| e^{-\Phi(z)} &\leq C \int_{\mathbb{C}^n} e^{-c|z-w|^2} \langle z\rangle^s (1-\chi(\epsilon w)) \left|u(w)\right| e^{-\Phi(w)}\, L(dw) \\ &\leq C \int_{\mathbb{C}^n} K_s(z,w) \langle w\rangle^s (1-\chi(\epsilon w)) \left|u(w)\right| e^{-\Phi(w)}\, L(dw),\end{aligned} \tag{3.2.30}$$

where

$$K_s(z,w) = \langle z-w\rangle^{|s|} e^{-c|z-w|^2}, \quad z,w \in \mathbb{C}^n. \tag{3.2.31}$$

As

$$\sup_{z\in\mathbb{C}^n} \int_{\mathbb{C}^n} \left|K_s(z,w)\right| L(dw) < \infty \quad \text{and} \quad \sup_{w\in\mathbb{C}^n} \int_{\mathbb{C}^n} \left|K_s(z,w)\right| L(dz) < \infty, \tag{3.2.32}$$

we deduce from Schur's lemma that there is a $C > 0$ such that

$$\|u - u_\epsilon\|_{H^s_\Phi(\mathbb{C}^n)} \le C\|(1 - \chi(\epsilon z))u(z)\|_{L^2_{\Phi,s}(\mathbb{C}^n)} \tag{3.2.33}$$

for all $\epsilon > 0$. By dominated convergence, the righthand side of (3.2.33) converges to 0 as $\epsilon \to 0^+$. Therefore $\|u - u_\epsilon\|_{H^s_\Phi(\mathbb{C}^n)} \to 0$ as $\epsilon \to 0^+$. □

We conclude this section by establishing a proposition that identifies the dual space of $H^s_\Phi(\mathbb{C}^n)$ with $H^{-s}_\Phi(\mathbb{C}^n)$.

Proposition 3.2.4. *Let Φ be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and let $s \in \mathbb{R}$. For every $v \in H^{-s}_\Phi(\mathbb{C}^n)$, the functional*

$$\psi_v(u) = \int_{\mathbb{C}^n} u(z)\overline{v(z)}e^{-2\Phi(z)}\, L(dz), \quad u \in H^s_\Phi(\mathbb{C}^n), \tag{3.2.34}$$

defines an element of $(H^s_\Phi(\mathbb{C}^n))'$ with $\|\psi_v\| \le \|v\|_{-s}$. Moreover, the map $v \mapsto \psi_v$ is a bounded antilinear isomorphism $H^{-s}_\Phi(\mathbb{C}^n) \to (H^s_\Phi(\mathbb{C}^n))'$.

Proof. Let $s \in \mathbb{R}$ be fixed. For $v \in H^{-s}_\Phi(\mathbb{C}^n)$, let ψ_v be as in (3.2.34). An application of the Cauchy-Schwarz inequality gives

$$|\psi_v(u)| \le \|v\|_{-s}\|u\|_s \tag{3.2.35}$$

for all $u \in H^s_\Phi(\mathbb{C}^n)$, and so it is clear that the map $v \mapsto \psi_v$ is a bounded antilinear mapping $H^{-s}_\Phi(\mathbb{C}^n) \to (H^s_\Phi(\mathbb{C}^n))'$. To see that the map $v \mapsto \psi_v$ is injective, let $v \in H^{-s}_\Phi(\mathbb{C}^n)$ be such that $\psi_v \equiv 0$ on $H^s_\Phi(\mathbb{C}^n)$. Then, using (3.2.27) and the identity

$$\Psi(w, \overline{z}) = \overline{\Psi(z, \overline{w})}, \quad w, z \in \mathbb{C}^n, \tag{3.2.36}$$

(see Lemma 13.1 in [63]), we get, for any $u \in L^2_{\Phi,s}(\mathbb{C}^n)$,

$$\begin{aligned} &\int_{\mathbb{C}^n} u(z)\overline{v(z)}e^{-2\Phi(z)}\,L(dz) = \int_{\mathbb{C}^n} u(z)\overline{\Pi_\Phi v(z)}e^{-2\Phi(z)}\,L(dz) \\ &= \int_{\mathbb{C}^n} \Pi_\Phi u(z)\overline{v(z)}e^{-2\Phi(z)}\,L(dz) = 0. \end{aligned} \tag{3.2.37}$$

In particular,

$$\int_{\mathbb{C}^n} u(z)\overline{v(z)}e^{-2\Phi(z)}\,L(dz) = 0 \tag{3.2.38}$$

for all $u \in C_0^\infty(\mathbb{C}^n)$, and we deduce $v = 0$. To prove surjectivity, let $\psi \in (H^s_\Phi(\mathbb{C}^n))'$ be arbitrary. By the Riesz representation theorem, there exists a unique $v_1 \in H^s_\Phi(\mathbb{C}^n)$ such that

$$\psi(u) = \int_{\mathbb{C}^n} u(z)\overline{v_1(z)}\langle z\rangle^{2s}e^{-2\Phi(z)}\,L(dz), \quad u \in H^s_\Phi(\mathbb{C}^n). \tag{3.2.39}$$

Let

$$v = \Pi_\Phi(v_1(z)\langle z\rangle^{2s}). \tag{3.2.40}$$

Observing that

$$v_1(z)\langle z\rangle^{2s} \in L^2_{\Phi,-s}(\mathbb{C}^n), \tag{3.2.41}$$

we see that $v \in H^{-s}_\Phi(\mathbb{C}^n)$ and hence

$$\psi = \psi_v. \tag{3.2.42}$$

By the closed graph theorem, the map $v \mapsto \psi_v$ is a bounded antilinear isomorphism $H^{-s}_\Phi(\mathbb{C}^n) \to (H^s_\Phi(\mathbb{C}^n))'$. $\square$

3.3 The $1/2$-Gelfand-Shilov Wavefront Set

In this section we recall the basic definition and properties of the 1/2-Gelfand-Shilov wavefront set. For a full discussion, see [10] or [30].

Let $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_\zeta$ and let $\sigma = d\zeta \wedge dz$ be the standard complex symplectic form on $\mathbb{C}^{2n}$. When equipped with the form σ, the space $\mathbb{C}^{2n}$ becomes a complex symplectic vector space. A $\mathbb{C}$-linear map $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is said to be a *complex linear symplectomorphism* or *complex linear canonical transformation* if $\kappa^*\sigma = \sigma$. Note that if $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is a complex linear canonical transformation, then $\det \kappa = 1$, and hence κ is automatically bijective.

Associated to the complex symplectic form σ are the real 2-forms Re σ and Im σ. Suppose Σ is a real linear subspace of $\mathbb{C}^{2n}$. We say Σ is *R-symplectic* if the restriction of Re σ to Σ is non-degenerate, and we say that Σ is *I-Lagrangian* if Σ is a Lagrangian subspace of $\mathbb{C}^{2n}$ with respect to Im σ, i.e. $\dim_{\mathbb{R}}\Sigma = 2n$ and Im $\sigma|_\Sigma = 0$. We say that Σ is *totally real* if $\Sigma \cap i\Sigma = \{0\}$, and if, in addition $\dim_{\mathbb{R}}\Sigma = 2n$, we say Σ is *maximally totally real.* Any real linear subspace of $\mathbb{C}^{2n}$ that is I-Lagrangian and R-symplectic is automatically maximally totally real. For more background on complex symplectic linear algebra, see [27].

Let φ be an FBI phase function. By Proposition 1.3.2 of [27], φ generates a complex linear canonical transformation $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ given implicitly by

$$\mathbb{C}^{2n} \ni (y, -\partial_y\varphi(z,y)) \mapsto (z, \partial_z\varphi(z,y)) \in \mathbb{C}^{2n}, \quad (z,y) \in \mathbb{C}^{2n}. \tag{3.3.1}$$

It can be shown that the κ_φ maps $\mathbb{R}^{2n}$ bijectively onto the space

$$\Lambda_\Phi = \left\{ \left(z, \frac{2}{i}\partial_z\Phi(z) \right) \in \mathbb{C}^{2n} : z \in \mathbb{C}^n \right\}, \tag{3.3.2}$$

where Φ is the strictly plurisubharmonic weight associated to φ. See [63] Theorem 13.5 for a proof. Since κ_φ is a complex linear canonical transformation, the space Λ_Φ is I-Lagrangian and R-symplectic. Hence Λ_Φ is maximally totally real.

Suppose Φ is a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$. Let Λ_Φ be as in (3.3.2) and let

$\mathrm{pr}_\Phi = \pi_1|_{\Lambda_\Phi}$. Since Λ_Φ is the graph of the $\mathbb{R}$-linear map $\frac{2}{i}\partial_z\Phi : \mathbb{C}^n \to \mathbb{C}^n$, the projection pr_Φ is an $\mathbb{R}$-linear isomorphism $\Lambda_\Phi \to \mathbb{C}^n$. If Φ is the strictly plurisubharmonic weight associated to an FBI phase function φ and κ_φ is the complex canonical transformation (3.3.1) generated by φ, then the composition

$$\kappa_\varphi^\flat = \mathrm{pr}_\Phi \circ \kappa_\varphi \tag{3.3.3}$$

is an $\mathbb{R}$-linear isomorphism $\mathbb{R}^{2n} \to \mathbb{C}^n$.

Let X be a real vector space. A subset V of $X\backslash\{0\}$ is said to be *conic* if $tx \in V$ whenever $x \in V$ and $t > 0$. If X and Y are real vector spaces, $T : X \to Y$ is a linear map, and V is a conic subset of $X\backslash\{0\}$, then $T(V)$ is a conic subset of Y. In particular, if V is a conic subset of $\mathbb{R}^{2n}\backslash\{0\}$, φ is an FBI phase function, and $\kappa_\varphi^\flat$ is as in (3.3.3), then $\kappa_\varphi^\flat(V)$ is a conic subset of $\mathbb{C}^n\backslash\{0\}$.

Definition 3.3.1 ([10] Definition 3.1, [30] Definition 6.6)**.** Let $u \in \mathcal{S}'(\mathbb{R}^n)$. The *1/2-Gelfand-Shilov wavefront* of u, denoted $\mathrm{WF}^{1/2}(u)$, is the complement in $\mathbb{R}^{2n}\backslash\{(0,0)\}$ of the set of points (x_0,ξ_0) for which there exists an FBI phase function φ and constants $C, c > 0$ such that

$$|\mathcal{T}_\varphi u(z)| \le C e^{\Phi(z) - c|z|^2} \tag{3.3.4}$$

for all z within some open conic neighborhood V of $\kappa_\varphi^\flat(x_0,\xi_0)$ in $\mathbb{C}^n\backslash\{0\}$. Here Φ is the strictly plurisubharmonic weight associated to φ and $\kappa_\varphi^\flat$ is as in (3.3.3).

As shown in [30] Proposition 6.4, one may use any FBI phase to determine if a point $(x_0,\xi_0) \in \mathbb{R}^{2n}\backslash\{(0,0)\}$ lies in $\mathrm{WF}^{1/2}(u)$. Indeed, suppose $u \in \mathcal{S}'(\mathbb{R}^n)$, $(x_0,\xi_0) \in \mathbb{R}^{2n}\backslash\{(0,0)\}$, and that there is an FBI phase function φ, constants $C, c > 0$, and an open conic neighborhood V of $\kappa_\varphi^\flat(x_0,\xi_0)$ in $\mathbb{C}^n\backslash\{0\}$ such that the estimate (3.3.4) holds in V. If φ_1 is another FBI phase with associated strictly plurisubharmonic weight Φ_1 and associated canonical transformation κ_{φ_1}, then there exists an open conic neighborhood V_1 of $\kappa_{\varphi_1}^\flat(x_0,\xi_0)$ in $\mathbb{C}^n\backslash\{0\}$ and constants $C_1, c_1 > 0$ such that

$$|\mathcal{T}_{\varphi_1} u(z)| \le C_1 e^{\Phi_1(z) - c_1|z|^2}, \quad z \in V_1.$$

Thus we may reformulate the definition of $\mathrm{WF}^{1/2}(u)$ as follows: if $u \in \mathcal{S}'(\mathbb{R}^n)$ and $(x_0, \xi_0) \in \mathbb{R}^{2n}\backslash\{(0,0)\}$, then $(x_0, \xi_0) \in \mathrm{WF}^{1/2}(u)$ if and only if there is an FBI phase function φ such that for every open conic neighborhood V of $\kappa_\varphi^\flat(x_0, \xi_0)$ and every choice of constants $C, c > 0$ the estimate (3.3.4) fails to hold for every $z \in V$. In particular, $(x_0, \xi_0) \in \mathrm{WF}^{1/2}(u)$ if there is an FBI phase φ with associated weight Φ and canonical transformation κ_φ such that

$$\mathcal{T}_\varphi u(\lambda \kappa_\varphi^\flat(x_0, \xi_0)) e^{-\Phi(\lambda \kappa_\varphi^\flat(x_0, \xi_0))} \neq o(1) \text{ as } \lambda \to \infty.$$

For notational purposes, it is convenient to introduce the notion of the 1/2-Gelfand-Shilov wavefront set of elements of $H_\Phi^{-\infty}(\mathbb{C}^n)$ relative to the plurisubharmonic weight Φ. This definition appears to be original, but we feel its addition will help to make some of our results more transparent.

Definition 3.3.2. Let Φ be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$. For $u \in H_\Phi^{-\infty}(\mathbb{C}^n)$, we define 1/2-*Gelfand-Shilov wavefront set of* u *relative to* Φ, denoted $\mathrm{WF}_\Phi^{1/2}(u)$, as the complement in $\mathbb{C}^n\backslash\{0\}$ of all $z \in \mathbb{C}^n\backslash\{0\}$ for which there exists an open conic neighborhood V of z in $\mathbb{C}^n\backslash\{0\}$ and constants $C, c > 0$ such that

$$|u(z)| \leq Ce^{\Phi(z) - c|z|^2}, \quad z \in V.$$

If $u \in H_\Phi^{-\infty}(\mathbb{C}^n)$, then $\mathrm{WF}_\Phi^{1/2}(u)$ is a closed conic subset of $\mathbb{C}^n\backslash\{0\}$. Furthermore, using Definition 3.3.2, we may restate the definition of the 1/2-Gelfand-Shilov wavefront set of tempered distributions as follows: if $u \in \mathcal{S}'(\mathbb{R}^n)$, then a point $(x_0, \xi_0) \in \mathbb{R}^{2n}\backslash\{(0,0)\}$ belongs to $\mathrm{WF}^{1/2}(u)$ if and only if there is an FBI phase φ with associated weight Φ and canonical transformation κ_φ such that $\kappa_\varphi^\flat(x_0, \xi_0) \in \mathrm{WF}_\Phi^{1/2}(\mathcal{T}_\varphi u)$. If $u \in \mathcal{S}'(\mathbb{R}^n)$, $(x_0, \xi_0) \in \mathbb{R}^{2n}$, and φ is an FBI phase such that $\kappa_\varphi^\flat(x_0, \xi_0) \in \mathrm{WF}_\Phi^{1/2}(\mathcal{T}_\varphi u)$, then the same is true with φ replaced by any other FBI phase φ_1 and with Φ and κ_φ replaced by the strictly plurisubharmonic weight and complex canonical transformation associated to φ_1 respectively.

Proposition 3.3.3 ([30] Proposition 6.9)**.** *Suppose* $u \in \mathcal{S}'(\mathbb{R}^n)$. *Then* $WF^{1/2}(u) = \emptyset$ *if and only if*

u extends to a holomorphic function $U(z)$ on $\mathbb{C}^n$ satisfying

$$|U(z)| \le Ce^{C|y|^2 - c|x|^2}, \quad z = x + iy \in \mathbb{C}^n, \tag{3.3.5}$$

for some $C, c > 0$.

A function $u \in C^\infty(\mathbb{R}^n)$ that extends to an entire analytic function on $\mathbb{C}^n$ and whose extension satisfies the estimate (3.3.5) is known as a *Gelfand-Shilov test function.* For more information on the Gelfand-Shilov space of test functions and its topological dual, we refer the reader to [10].

Proposition 3.3.3 is significant because it shows that the 1/2-Gelfand-Shilov wavefront of a distribution u captures our intuition for what the set of global analytic singularities of a tempered distribution u should be. Namely, for $\mathrm{WF}^{1/2}(u)$ to be empty, u must not only be real analytic, but also satisfy a Gaussian type decay estimate as $|x| \to \infty$.

An elementary compactness argument also allows us to deduce a necessary and sufficient condition so that the 1/2-Gelfand-Shilov wavefront set of $u \in H_\Phi^{-\infty}(\mathbb{C}^n)$ relative to Φ is empty.

Proposition 3.3.4. *Let Φ be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$. Suppose $u \in H_\Phi^{-\infty}(\mathbb{C}^n)$. Then $WF_\Phi^{1/2}(u) = \emptyset$ if and only if there exist $C, c > 0$ such that*

$$|u(z)| \le Ce^{\Phi(z) - c|z|^2}, \quad z \in \mathbb{C}^n.$$

Combining Proposition 3.3.3 with Proposition 3.3.4 gives

Corollary 3.3.5. *Suppose $u \in \mathcal{S}'(\mathbb{R}^n)$. Then u is a Gelfand-Shilov test function if and only if there is an FBI phase φ with associated weight Φ and constants $C, c > 0$ such that*

$$|\mathcal{T}_\varphi u(z)| \le Ce^{\Phi(z) - c|z|^2}, \quad z \in \mathbb{C}^n. \tag{3.3.6}$$

If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{T}_\varphi u$ satisfies (3.3.6), then for every other choice of FBI phase φ_1 with associated weight Φ_1, the function $\mathcal{T}_{\varphi_1} u$ satisfies the estimate (3.3.6) for a potentially different choice of constants $C, c > 0$ and with Φ replaced by Φ_1.

3.4 Bergman Representations of Metaplectic Fourier Integral Operators

In this section we consider metaplectic Fourier integral operators and their action on exponentially weighted spaces of entire functions. Here, by 'metaplectic Fourier integral operator', we mean any type of Fourier integral operator whose phase function is a (real or complex-valued) quadratic form. There are several approaches available. For a treatment of semiclassical Fourier integral operators on $\mathbb{R}^n$ with real quadratic phase functions, the reader may consult the appendix to Chapter 7 of [19]. For the theory of Fourier integral operators on $\mathbb{R}^n$ whose phase functions are complex-valued quadratic forms with non-negative imaginary parts (the so-called 'Gaussian Fourier integral operators'), see Section 5 of [31]. Here, we shall follow (and slightly extend) the approach detailed in Appendix B of [9], which develops a calculus of Fourier integral operators with holomorphic quadratic phases acting on the FBI transform side. For the general theory of Fourier integral operators acting on exponentially weighted spaces of holomorphic functions, see the book [51].

We begin with a formal discussion. The metaplectic Fourier integral operators that we shall be interested in are operators of the form

$$Gu(z) = a \iint e^{i\phi(z,w,\theta)} u(w)\, dw \wedge d\theta, \quad z \in \mathbb{C}^n, \tag{3.4.1}$$

where $a \in \mathbb{C}$ is a constant, $\phi(z,w,\theta)$ is a holomorphic quadratic form on $\mathbb{C}^{2n+N} = \mathbb{C}^n_z \times \mathbb{C}^n_w \times \mathbb{C}^N_\theta$. We assume that ϕ is a non-degenerate phase function in the sense of Hörmander [29], i.e.

$$d\partial_{\theta_1}\phi, ..., d\partial_{\theta_N}\phi \text{ are linearly independent over } \mathbb{C}. \tag{3.4.2}$$

Let

$$C_\phi = \left\{(z,w,\theta) \in \mathbb{C}^{2n+N} : \partial_\theta \phi(z,w,\theta) = 0\right\}$$

be the *critical set of* ϕ. Since $\partial_\theta\phi$ is a $\mathbb{C}$-linear function of $(z,w,\theta) \in \mathbb{C}^{2n+N}$, the critical set of ϕ is a linear subspace of $\mathbb{C}^{2n+N}$, and the non-degeneracy of ϕ implies that $\dim_{\mathbb{C}} C_\phi = 2n$. We associate to G the complex canonical relation $\kappa \subset \mathbb{C}^{2n} \times \mathbb{C}^{2n}$ given implicitly by

$$\kappa : (w, -\partial_w \phi(z,w,\theta)) \mapsto (z, \partial_z \phi(z,w,\theta)), \quad (z,w,\theta) \in C_\phi. \tag{3.4.3}$$

In the sequel, we shall always assume that κ is the graph of a complex linear canonical transformation $\mathbb{C}^{2n} \to \mathbb{C}^{2n}$. In this situation, we say that G *quantizes* κ or that κ *is the underlying canonical transformation of* G.

We will now discuss how the formal Fourier integral operator (3.4.1) may be realized as a bounded linear operator between exponentially weighted spaces of entire functions by making an appropriate choice of the contour of integration. Let Φ_1 and Φ_2 be strictly plurisubharmonic quadratic forms on $\mathbb{C}^n$ and let $H_{\Phi_1}(\mathbb{C}^n)$ and $H_{\Phi_2}(\mathbb{C}^n)$ be their associated Bargmann spaces. Suppose that

$$\kappa(\Lambda_{\Phi_2}) = \Lambda_{\Phi_1}, \tag{3.4.4}$$

where Λ_{Φ_1} and Λ_{Φ_2} are as in (3.3.2) with Φ replaced by Φ_1 and Φ_2, respectively. Following [9] Appendix B, the plurisubharmonic quadratic form

$$\mathbb{C}^n \times \mathbb{C}^N \ni (w,\theta) \mapsto -\operatorname{Im} \phi(0,w,\theta) + \Phi_2(w)$$

is non-degenerate of signature $(n+N, n+N)$. Then, following either Proposition B.3 of [9] or the general theory of [51], we may conclude that there exists a real, smooth, $(n+N)$-dimensional contour $\Gamma(z)$ in $\mathbb{C}^{n+N}$, depending smoothly on $z \in \mathbb{C}^n$, such that Gu, when equipped with $\Gamma(z)$, is well-defined as an element of $H_{\Phi_1}(\mathbb{C}^n)$ for $u \in H_{\Phi_2}(\mathbb{C}^n)$ and that (3.4.1) defines a bounded linear transformation

$$G : H_{\Phi_2}(\mathbb{C}^n) \to H_{\Phi_1}(\mathbb{C}^n). \tag{3.4.5}$$

Next, we recount some recent results from [12] concerning the Bergman representation of a metaplectic Fourier integral operator G whose underlying complex canonical transformation κ satisfies (3.4.4). The following proposition summarizes the main results that we shall need. Recall that if

$\Phi(z)$ is a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$, then the polarization $\Psi(z,\theta)$ of $\Phi(z)$ is the unique holomorphic quadratic form on $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_\theta$ such that $\Psi(z,\overline{z}) = \Phi(z)$ for all $z \in \mathbb{C}^n$.

Proposition 3.4.1. *Let $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be a complex linear canonical transformation, let Φ_1 and Φ_2 be strictly plurisubharmonic quadratic forms on $\mathbb{C}^n$ such that*

$$\kappa(\Lambda_{\Phi_2}) = \Lambda_{\Phi_1}, \tag{3.4.6}$$

where

$$\Lambda_{\Phi_j} = \left\{ \left(z, \frac{2}{i}\partial_z \Phi_j(z) \right) : z \in \mathbb{C}^n \right\}, \quad j = 1, 2, \tag{3.4.7}$$

and let $G : H_{\Phi_2}(\mathbb{C}^n) \to H_{\Phi_1}(\mathbb{C}^n)$ be a metaplectic Fourier integral operator quantizing κ. Let

$$pr_{\Phi_j} = \pi_1|_{\Lambda_{\Phi_j}}, \quad j = 1, 2, \tag{3.4.8}$$

and let $\kappa^\flat$ be the $\mathbb{R}$-linear isomorphism

$$\kappa^\flat = pr_{\Phi_1} \circ \kappa \circ (pr_{\Phi_2})^{-1} : \mathbb{C}^n \to \mathbb{C}^n. \tag{3.4.9}$$

Then there exists a unique $\widehat{a} \in \mathbb{C}$ and a unique holomorphic quadratic form $\Psi(z,\theta)$ on $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_\theta$ such that

$$Gu(z) = \hat{a} \int_{\mathbb{C}^n} e^{2\Psi(z,\overline{w})} u(w) e^{-2\Phi_2(w)}\, L(dw), \quad u \in H_{\Phi_2}(\mathbb{C}^n). \tag{3.4.10}$$

The quadratic form $\Psi(z,\theta)$ has the following properties:

1. *if $\Psi_2(z,\theta)$ denotes the polarization of Φ_2, then*

$$\phi(z,w,\theta) = \frac{2}{i}\Psi(z,\theta) - \frac{2}{i}\Psi_2(w,\theta), \quad (z,w,\theta) \in \mathbb{C}^{3n}, \tag{3.4.11}$$

is a non-degenerate holomorphic phase function generating graph(κ*):*

$$\kappa : \left(w, \frac{2}{i}\partial_w \Psi_2(w,\theta)\right) \mapsto \left(z, \frac{2}{i}\partial_z \Psi(z,\theta)\right), \quad \partial_\theta \Psi(z,\theta) = \partial_\theta \Psi_2(w,\theta); \tag{3.4.12}$$

2. The real part of $\Psi(z,\theta)$ *satisfies*

$$2Re\ \Psi(z,\theta) = \Phi_1(z) + \Phi_2(\overline{\theta}) - R(z,\theta), \quad z,\theta \in \mathbb{C}^n, \tag{3.4.13}$$

where $R(z,\theta)$ *a non-negative quadratic form on* $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_\theta$ *such that*

$$c\left|z - \kappa^\flat(\overline{\theta})\right|^2 \leq R(z,\theta) \leq C\left|z - \kappa^\flat(\overline{\theta})\right|^2, \quad z,\theta \in \mathbb{C}^n, \tag{3.4.14}$$

for some $C, c > 0$*.*

If G is a metaplectic Fourier integral operator satisfying (3.4.5), then we refer to (3.4.10) as the *Bergman form of* G.

Example 1. Consider the formal Fourier integral operator

$$Gu(z) = \frac{1}{(2\pi)^n} \iint e^{i(z-w)\cdot\theta} u(w)\, dw \wedge d\theta, \quad z \in \mathbb{C}^n. \tag{3.4.15}$$

The phase function

$$\phi(z,w,\theta) = (z-w)\cdot\theta, \quad (z,w,\theta) \in \mathbb{C}^{3n}, \tag{3.4.16}$$

is easily seen to satisfy Hörmander's non-degeneracy condition (3.4.2). A direct computation shows that the complex linear canonical transformation $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ generated by ϕ is the identity

$$\kappa(z,\zeta) = (z,\zeta), \quad (z,\zeta) \in \mathbb{C}^{2n}. \tag{3.4.17}$$

If Φ is any strictly plurisubharmonic quadratic form on $\mathbb{C}^n$, the formal Fourier integral operator (3.4.15) may be realized as a bounded linear transformation $H_\Phi(\mathbb{C}^n) \to H_\Phi(\mathbb{C}^n)$ by integrating over the contour

$$\Gamma(z) : w \mapsto \theta = \frac{2}{i}\partial_z \Phi(z) + iC\overline{(z-w)}, \quad w \in \mathbb{C}^n, \tag{3.4.18}$$

where $C \gg 0$ is sufficiently large. By the 'complex Fourier inversion theorem' (see the proof of Proposition 1.3.4 in [27] or the proof of Theorem 13.6 in [63]),

$$Gu = u \tag{3.4.19}$$

for all $u \in H_\Phi(\mathbb{C}^n)$. After an appropriate contour deformation and $\mathbb{C}$-linear change of variables (again, see the proof of Proposition 1.3.4 in [27] or the proof of Theorem 13.6 in [63]), the operator (3.4.15) may be rewritten as

$$Gu(z) = C_\Phi \int_{\mathbb{C}^n} e^{2\Psi(z,\overline{w})} u(w) e^{-2\Phi(w)} \, L(dw), \quad u \in H_\Phi(\mathbb{C}^n), \tag{3.4.20}$$

where $\Psi(\cdot,\cdot)$ is the polarization of Φ and $C_\Phi = (2/\pi)^n \det \partial^2_{z\overline{z}} \Phi$. In other words, the Bergman form of G that is guaranteed to exist by Proposition 3.4.1 is precisely the Bergman projector (5.2.65) associated to the weight Φ. One may rewrite (3.4.20) in the form

$$Gu(z) = \widetilde{C}_\Phi \iint_\Gamma e^{2\Psi(z,\theta) - 2\Psi(w,\theta)} u(w) \, dw \wedge d\theta, \quad u \in H_\Phi(\mathbb{C}^n), \tag{3.4.21}$$

where $\widetilde{C}_\Phi = (i/2)^n C_\Phi$ and the contour of integration is the anti-diagonal

$$\Gamma = \left\{ (w,\theta) \in \mathbb{C}^{2n} : \theta = \overline{w} \right\}. \tag{3.4.22}$$

The strict plurisubharmonicity of Φ implies that the phase function

$$\frac{2}{i}\Psi(z,\theta) - \frac{2}{i}\Psi(w,\theta), \quad (z,w,\theta) \in \mathbb{C}^{3n}, \tag{3.4.23}$$

satisfies the non-degeneracy condition (3.4.2), and one may easily verify that (3.4.23) generates the identity map (3.4.17).

Using Proposition 3.4.1, we can give a simple proof that metaplectic Fourier integral operators extend uniquely to bounded linear transformations on $H^s_\Phi(\mathbb{C}^n)$ for every $s \in \mathbb{R}$.

Proposition 3.4.2. *Let Φ_1 and Φ_2 be strictly plurisubharmonic quadratic forms on $\mathbb{C}^n$ and let $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be a complex linear canonical transformation such that $\kappa(\Lambda_{\Phi_2}) = \Lambda_{\Phi_1}$. If $G : H_{\Phi_2}(\mathbb{C}^n) \to H_{\Phi_1}(\mathbb{C}^n)$ is a metaplectic Fourier integral operator quantizing κ, then G extends uniquely to a bounded linear transformation*

$$G : H^s_{\Phi_2}(\mathbb{C}^n) \to H^s_{\Phi_1}(\mathbb{C}^n) \tag{3.4.24}$$

for every $s \in \mathbb{R}$. Consequently, G restricts to a continuous linear transformation $H^\infty_{\Phi_2}(\mathbb{C}^n) \to H^\infty_{\Phi_1}(\mathbb{C}^n)$ and extends uniquely to a continuous linear transformation $H^{-\infty}_{\Phi_2}(\mathbb{C}^n) \to H^{-\infty}_{\Phi_1}(\mathbb{C}^n)$.

Proof. Let $\kappa^\flat : \mathbb{C}^n \to \mathbb{C}^n$ be as in (3.4.9). By Proposition 3.4.1, we may write G uniquely in Bergman form as

$$Gu(z) = \hat{a} \int_{\mathbb{C}^n} e^{2\Psi(z,\overline{w})} u(w) e^{-2\Phi_2(w)}\, L(dw), \quad u \in H_{\Phi_2}(\mathbb{C}^n), \tag{3.4.25}$$

where $\hat{a} \in \mathbb{C}$ and $\Psi(\cdot,\cdot)$ is a holomorphic quadratic form on $\mathbb{C}^{2n}$ such that

$$2\mathrm{Re}\, \Psi(z,\overline{w}) - \Phi_1(z) - \Phi_2(w) \le -c \left| z - \kappa^\flat(w) \right|^2, \quad z, w \in \mathbb{C}^n, \tag{3.4.26}$$

for some $c > 0$. Now (3.4.25) may be rewritten as

$$Gu(z) = \int_{\mathbb{C}^n} K(z,w) u(w)\, L(dw), \quad u \in H_{\Phi_2}(\mathbb{C}^n), \tag{3.4.27}$$

where

$$K(z,w) = \hat{a} e^{2\Psi(z,\overline{w}) - 2\Phi_2(w)}, \quad z, w \in \mathbb{C}^n. \tag{3.4.28}$$

Let $s \in \mathbb{R}$ be arbitrary. To see that

$$G = \mathcal{O}(1) : H^s_{\Phi_2}(\mathbb{C}^n) \to H^s_{\Phi_1}(\mathbb{C}^n), \tag{3.4.29}$$

we consider the reduced kernel

$$K_{\mathrm{red}}(z,w) = \langle z \rangle^s e^{-\Phi_1(z)} K(z,w) \langle w \rangle^{-s} e^{\Phi_2(w)}, \quad z, w \in \mathbb{C}^n. \tag{3.4.30}$$

Since $\kappa^\flat : \mathbb{C}^n \to \mathbb{C}^n$ is an invertible linear transformation, there are $C, c > 0$ such that

$$c\langle \kappa^\flat(w) \rangle \leq \langle w \rangle \leq C\langle \kappa^\flat(w) \rangle \tag{3.4.31}$$

for all $w \in \mathbb{C}^n$. From (3.4.26) and (3.4.31), we get that are $C, c > 0$ such that

$$|K_{\mathrm{red}}(z,w)| \leq C\langle z - \kappa^\flat(w) \rangle^{|s|} e^{-c\left|z - \kappa^\flat(w)\right|^2}, \quad z, w \in \mathbb{C}^n. \tag{3.4.32}$$

Because

$$\sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} |K_{\mathrm{red}}(z,w)| \, L(dw) < \infty \;\text{ and }\; \sup_{w \in \mathbb{C}^n} \int_{\mathbb{C}^n} |K_{\mathrm{red}}(z,w)| \, L(dz) < \infty, \tag{3.4.33}$$

Schur's lemma implies that the operator (3.4.25) is $\mathcal{O}(1) : H^s_{\Phi_2}(\mathbb{C}^n) \to H^s_{\Phi_1}(\mathbb{C}^n)$. As $H^\infty_{\Phi_2}(\mathbb{C}^n)$ is dense in $H^s_{\Phi_2}(\mathbb{C}^n)$ by Proposition 3.2.3, we conclude that G extends uniquely to a bounded linear transformation $H^s_{\Phi_2}(\mathbb{C}^n) \to H^s_{\Phi_1}(\mathbb{C}^n)$. It follows immediately that G restricts to a continuous linear transformation $H^\infty_{\Phi_2}(\mathbb{C}^n) \to H^\infty_{\Phi_1}(\mathbb{C}^n)$. To extend G to $H^{-\infty}_{\Phi_2}(\mathbb{C}^n)$, we define Gu for $u \in H^{-\infty}_{\Phi_2}(\mathbb{C}^n)$ using the formula (3.4.25). To prove the continuity of $G : H^{-\infty}_{\Phi_2}(\mathbb{C}^n) \to H^{-\infty}_{\Phi_1}(\mathbb{C}^n)$ it suffices to show that for every $s \in \mathbb{R}$ the restriction of G to $H^s_{\Phi_2}(\mathbb{C}^n)$ is continuous $H^s_{\Phi_2}(\mathbb{C}^n) \to H^{-\infty}_{\Phi_1}(\mathbb{C}^n)$ (see [15] Chapter IV Proposition 5.7). But this is immediate since we have already established the continuity of $G : H^s_{\Phi_2}(\mathbb{C}^n) \to H^s_{\Phi_1}(\mathbb{C}^n)$ and the inclusion $H^s_{\Phi_1}(\mathbb{C}^n) \hookrightarrow H^{-\infty}_{\Phi_1}(\mathbb{C}^n)$ is continuous by definition of the topology on $H^{-\infty}_{\Phi_1}(\mathbb{C}^n)$. Since $H^\infty_{\Phi_2}(\mathbb{C}^n)$ is dense in $H^{-\infty}_{\Phi_2}(\mathbb{C}^n)$ by Proposition 3.2.3, it follows that G extends uniquely to a continuous linear transformation $H^{-\infty}_{\Phi_2}(\mathbb{C}^n) \to H^{-\infty}_{\Phi_1}(\mathbb{C}^n)$. $\square$

3.5 Metaplectic Fourier Integral Operators and Propagation of 1/2-Gelfand-Shilov Singularities

Let $\mathbb{C}^{2n}$ be equipped with the standard complex symplectic form σ and let Σ be a maximally totally real subspace of $\mathbb{C}^{2n}$. Let ι_Σ be the unique antilinear involution of $\mathbb{C}^{2n}$ fixing Σ. Following the terminology of [27], we say that a $\mathbb{C}$-Lagrangian subspace Λ of $\mathbb{C}^{2n}$ is *positive relative to* Σ if

$$\frac{1}{i}\sigma(X, \iota_\Sigma X) \geq 0, \quad X \in \mathbb{C}^{2n}. \tag{3.5.1}$$

If equality holds in (3.5.1) only when $X = 0$, we say that Λ is *strictly positive relative to* Σ.

One may extend the notion of positivity to complex linear canonical transformations of $\mathbb{C}^{2n}$. If $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is a complex linear canonical transformation and $\Sigma_1, \Sigma_2 \subset \mathbb{C}^{2n}$ are maximally totally real subspaces of $\mathbb{C}^{2n}$ with associated antilinear involutions ι_{Σ_1} and ι_{Σ_2}, respectively, then we say that κ is *positive relative to* (Σ_1, Σ_2) if

$$\frac{1}{i}\left(\sigma(\kappa(X), \iota_{\Sigma_1}\kappa(X)) - \sigma(X, \iota_{\Sigma_2} X)\right) \geq 0, \quad X \in \mathbb{C}^{2n}. \tag{3.5.2}$$

If the inequality in (3.5.2) is strict for all $X \neq 0$, then κ is said to be *strictly positive relative to* (Σ_1, Σ_2). In the case when κ is positive, resp. strictly positive, relative to (Σ_1, Σ_2) and $\Sigma_1 = \Sigma_2 = \Sigma$, then we simply say that κ *is positive, resp. strictly positive, relative to* Σ.

Let Φ_1 and Φ_2 be strictly plurisubharmonic quadratic forms on $\mathbb{C}^n$. In [12], it was shown that a complex linear canonical transformation $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is positive relative to $(\Lambda_{\Phi_1}, \Lambda_{\Phi_2})$ if and only if

$$\kappa(\Lambda_{\Phi_2}) = \Lambda_\Phi \tag{3.5.3}$$

where Φ is a strictly plurisubharmonic quadratic form on $\mathbb{C}^{2n}$ such that $\Phi \leq \Phi_1$. In particular, if Φ

is a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and κ is positive relative to Λ_Φ, then

$$\kappa(\Lambda_\Phi) = \Lambda_{\tilde{\Phi}} \tag{3.5.4}$$

for some strictly plurisubharmonic quadratic form $\tilde{\Phi}$ on $\mathbb{C}^n$ such that

$$\tilde{\Phi} \leq \Phi. \tag{3.5.5}$$

In this case, there is a very useful characterization of the I-isotropic subspace $\Lambda_\Phi \cap \kappa(\Lambda_\Phi)$ in terms of Φ and $\tilde{\Phi}$. Namely, if Φ and $\tilde{\Phi}$ are strictly plurisubharmonic quadratic forms on $\mathbb{C}^n$ satisfying (3.5.4) and (3.5.5), then

$$\pi_1(\Lambda_\Phi \cap \kappa(\Lambda_\Phi)) = \mathrm{Rad}(\Phi - \tilde{\Phi}). \tag{3.5.6}$$

Indeed, since $\Phi - \tilde{\Phi}$ is a non-negative quadratic form, we have

$$\Phi(z) - \tilde{\Phi}(z) = 0 \iff \nabla_{\mathrm{Re}\ z, \mathrm{Im}\ z}(\Phi - \tilde{\Phi})(z) = 0 \iff \partial_z(\Phi - \tilde{\Phi})(z) = 0. \tag{3.5.7}$$

Hence

$$\left(z, \frac{2}{i}\partial_z \Phi(z)\right) = \left(z, \frac{2}{i}\partial_z \tilde{\Phi}(z)\right) \iff z \in \mathrm{Rad}(\Phi - \tilde{\Phi}). \tag{3.5.8}$$

Suppose $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is a complex linear canonical transformation that is positive relative to Λ_Φ and let $\tilde{\Phi}$ be as in (3.5.4). If G is a metaplectic Fourier integral operator quantizing κ, then G is a continuous linear transformation $H_\Phi^{-\infty}(\mathbb{C}^n) \to H_{\tilde{\Phi}}^{-\infty}(\mathbb{C}^n)$ by Proposition 3.4.2. Since also $H_{\tilde{\Phi}}^{-\infty}(\mathbb{C}^n) \hookrightarrow H_\Phi^{-\infty}(\mathbb{C}^n)$ continuously, we may regard G as a continuous linear transformation from $H_\Phi^{-\infty}(\mathbb{C}^n)$ to itself. Consequently, $\mathrm{WF}_\Phi^{1/2}(Gu)$ is well-defined for any $u \in H_\Phi^{-\infty}(\mathbb{C}^n)$.

We wish to explore the relationship between $\mathrm{WF}_\Phi^{1/2}(u)$ and $\mathrm{WF}_\Phi^{1/2}(Gu)$ when $u \in H_\Phi^{-\infty}(\mathbb{C}^n)$ and G is a metaplectic Fourier integral operator whose underlying canonical transformation κ is positive relative to Φ. The next theorem shows that G regularizes any 1/2-Gelfand-Shilov singularities of u

that are outside of $\mathrm{Rad}(\Phi - \tilde{\Phi})$ and transports those that lie within $\mathrm{Rad}(\Phi - \tilde{\Phi})$ by $\kappa^\flat$, where $\kappa^\flat$ is as in (3.4.9).

Theorem 3.5.1. *Let Φ be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$, let $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be a complex linear canonical transformation that is positive relative to Φ, let $\tilde{\Phi}$ be as in (3.5.4), and let $\kappa^\flat : \mathbb{C}^n \to \mathbb{C}^n$ be the $\mathbb{R}$-linear isomorphism*

$$\kappa^\flat = pr_{\tilde{\Phi}} \circ \kappa \circ pr_{\tilde{\Phi}}^{-1} : \mathbb{C}^n \to \mathbb{C}^n,$$

where pr_Φ and $pr_{\tilde{\Phi}}$ are the restrictions of π_1 to Λ_Φ and $\Lambda_{\tilde{\Phi}}$ respectively. If G is a metaplectic Fourier integral operator quantizing κ, realized as a continuous linear transformation from $H_\Phi^{-\infty}(\mathbb{C}^n)$ to itself, then, for any $u \in H_\Phi^{-\infty}(\mathbb{C}^n)$, we have

$$WF_\Phi^{1/2}(Gu) \subset \kappa^\flat(WF_\Phi^{1/2}(u)) \cap Rad(\Phi - \tilde{\Phi}). \tag{3.5.9}$$

If, in addition, G is non-zero and $\kappa(\Lambda_\Phi \cap \Lambda_{\tilde{\Phi}}) = \Lambda_\Phi \cap \Lambda_{\tilde{\Phi}}$, then

$$WF_\Phi^{1/2}(Gu) = \kappa^\flat(WF_\Phi^{1/2}(u)) \cap Rad(\Phi - \tilde{\Phi}) \tag{3.5.10}$$

for every $u \in H_\Phi^{-\infty}(\mathbb{C}^n)$.

We begin the proof of Theorem 3.5.1 by establishing the inclusion (3.5.9). It suffices to show

$$\mathbb{C}^n \backslash \mathrm{Rad}(\Phi - \tilde{\Phi}) \subset \mathbb{C}^n \backslash \mathrm{WF}_\Phi^{1/2}(Gu). \tag{3.5.11}$$

and

$$\kappa^\flat(\mathbb{C}^n \backslash \mathrm{WF}_\Phi^{1/2}(u)) \subset \mathbb{C}^n \backslash \mathrm{WF}_\Phi^{1/2}(Gu). \tag{3.5.12}$$

Let

$$Gu(z) = \hat{a} \int_{\mathbb{C}^n} e^{2\Psi(z,\overline{w})} u(w) e^{-2\Phi(w)} \, L(dw), \quad u \in H_{\Phi}^{-\infty}(\mathbb{C}^n),$$

where $\hat{a} \neq 0$, be the Bergman form of G given in Proposition 3.4.1. Suppose that $z_0 \in \mathbb{C}^n \backslash \mathrm{Rad}(\Phi - \tilde{\Phi})$. There is an open conic neighborhood V of z_0 in $\mathbb{C}^n \backslash \{0\}$ such that

$$\Phi(z) - \tilde{\Phi}(z) \geq c\,|z|^2$$

for all $z \in V$. In view of (3.4.13) and (3.4.14), for all $z \in V$, we have

$$\begin{aligned} |Gu(z)|\, e^{-\Phi(z)} &\leq C e^{-c|z|^2} \int_{\mathbb{C}^n} e^{-\Phi(w) - c\left|z - \kappa^\flat(w)\right|^2} \, |u(w)| \, L(dw) \\ &\leq C \|u\|_{H_\Phi^s(\mathbb{C}^n)} e^{-c|z|^2} \left(\int_{\mathbb{C}^n} \langle w \rangle^{-2s} e^{-c\left|z - \kappa^\flat(w)\right|^2} \, L(dw) \right)^{1/2}, \end{aligned}$$

where $s \in \mathbb{R}$ is such that $u \in H_\Phi^s(\mathbb{C}^n)$. Since $\kappa^\flat$ is a $\mathbb{R}$-linear isomorphism $\mathbb{C}^n \to \mathbb{C}^n$, we see that

$$\left(\int_{\mathbb{C}^n} \langle w \rangle^{-2s} e^{-c\left|z - \kappa^\flat(w)\right|^2} \, L(dw) \right)^{1/2} \leq C \langle z \rangle^s.$$

It follows

$$|Gu(z)|\, e^{-\Phi(z)} \leq C e^{-c|z|^2}, \quad z \in V.$$

Hence (3.5.11) holds.

Let $z_0 \in \kappa^\flat(\mathbb{C}^n \backslash \mathrm{WF}_\Phi^{1/2}(u))$. If $z_0 = 0$, then trivially $z_0 \in \mathbb{C}^n \backslash \mathrm{WF}_\Phi^{1/2}(Gu)$. If $z_0 \neq 0$, write $z_0 = \kappa^\flat(w_0)$ for some unique $w_0 \in \mathbb{C}^n \backslash \mathrm{WF}_\Phi^{1/2}(u)$ and let V be an open conic neighborhood of w_0 in $\mathbb{C}^n \backslash \{0\}$ such that

$$|u(w)| \leq C e^{\Phi(w) - c|w|^2}, \quad w \in V. \tag{3.5.13}$$

Let $\tilde{V}$ be an open conic neighborhood of z_0 in $\mathbb{C}^n \backslash \{0\}$ such that $\tilde{V} \subset\subset \kappa^\flat(V)$. From Proposition

3.4.1 and the fact that $\tilde{\Phi} \leq \Phi$, we get

$$\left|Gu(z)e^{-\Phi(z)}\right| \leq C\left(\int_V + \int_{\mathbb{C}^n\setminus V}\right) e^{-c\left|z-\kappa^\flat(w)\right|^2} |u(w)|\, e^{-\Phi(w)}\, L(dw) =: I(z) + II(z), \quad z \in \tilde{V}. \tag{3.5.14}$$

In view of (3.5.13),

$$I(z) \leq C\int_V e^{-c\left|(\kappa^\flat)^{-1}(z)-w\right|^2 - c|w|^2}\, L(dw) \leq Ce^{-c|z|^2}, \quad z \in \mathbb{C}^n.$$

To estimate $II(z)$, we observe that the quadratic form

$$(z,w) \mapsto \left|z - \kappa^\flat(w)\right|^2$$

is non-vanishing for $(z,w) \in \tilde{V} \times (\mathbb{C}^n\setminus\{0\})\setminus V$. By homogeneity, there is a constant $c > 0$ such that

$$\left|z - \kappa^\flat(w)\right|^2 \geq c(|z|^2 + |w|^2)$$

for all $z \in \tilde{V}$ and $w \in \mathbb{C}^n\setminus V$. As a result,

$$II(z) \leq C\|u\|_{H^s_\Phi(\mathbb{C}^n)} e^{-c|z|^2}\left(\int_{\mathbb{C}^n} \langle w\rangle^{-2s} e^{-c|w|^2}\, L(dw)\right)^{1/2} \leq Ce^{-c|z|^2}$$

for all $z \in \tilde{V}$. This establishes that

$$|Gu(z)|\, e^{-\Phi(z)} \leq Ce^{-c|z|^2}, \quad z \in \tilde{V}.$$

Therefore (3.5.12) holds. We conclude that the inclusion (3.5.9) is true.

To establish the equality (3.5.10) under the additional assumption that $\kappa(\Lambda_\Phi \cap \Lambda_{\tilde{\Phi}}) = \Lambda_\Phi \cap \Lambda_{\tilde{\Phi}}$, we first prove the following lemma.

Lemma 3.5.2. *Let Φ, $\tilde{\Phi}$, κ, and $\kappa^\flat$ be as in Theorem 3.5.1 and assume $\kappa(\Lambda_\Phi \cap \Lambda_{\tilde{\Phi}}) = \Lambda_\Phi \cap \Lambda_{\tilde{\Phi}}$. Suppose that $\widetilde{G}$ is a metaplectic Fourier integral operator quantizing κ^{-1}, realized as a continuous* *linear transformation* $H^{-\infty}_{\tilde{\Phi}}(\mathbb{C}^n) \to H^{-\infty}_{\Phi}(\mathbb{C}^n)$. *Then, for any* $v \in H^{-\infty}_{\tilde{\Phi}}(\mathbb{C}^n)$,

$$WF^{1/2}_{\Phi}(\widetilde{G}v) \cap Rad(\Phi - \tilde{\Phi}) \subset (\kappa^\flat)^{-1}(WF^{1/2}_{\Phi}(v)) \cap Rad(\Phi - \tilde{\Phi}). \tag{3.5.15}$$

Proof. Let $v \in H^{-\infty}_{\tilde{\Phi}}(\mathbb{C}^n)$ be arbitrary. It suffices to show that

$$\mathrm{Rad}(\Phi - \tilde{\Phi}) \backslash (\kappa^\flat)^{-1} \left(\mathrm{WF}^{1/2}_{\Phi}(v)\right) \subset \mathrm{Rad}(\Phi - \tilde{\Phi}) \backslash \mathrm{WF}^{1/2}_{\Phi}(\widetilde{G}v). \tag{3.5.16}$$

Because $(\kappa^\flat)^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ is invertible,

$$\begin{aligned} \mathrm{Rad}(\Phi - \tilde{\Phi}) \backslash (\kappa^\flat)^{-1} \left(\mathrm{WF}^{1/2}_{\Phi}(v)\right) &= \mathrm{Rad}(\Phi - \tilde{\Phi}) \cap \left[\mathbb{C}^n \backslash (\kappa^\flat)^{-1}(\mathrm{WF}^{1/2}_{\Phi}(v))\right] \\ &= \mathrm{Rad}(\Phi - \tilde{\Phi}) \cap (\kappa^\flat)^{-1} \left(\mathbb{C}^n \backslash \mathrm{WF}^{1/2}_{\Phi}(v)\right). \end{aligned} \tag{3.5.17}$$

Thus (3.5.16) is equivalent to

$$\mathrm{Rad}(\Phi - \tilde{\Phi}) \cap (\kappa^\flat)^{-1} \left(\mathbb{C}^n \backslash \mathrm{WF}^{1/2}_{\Phi}(v)\right) \subset \mathrm{Rad}(\Phi - \tilde{\Phi}) \cap \mathbb{C}^n \backslash \mathrm{WF}^{1/2}_{\Phi}(\widetilde{G}v). \tag{3.5.18}$$

Let $z_0 \in \mathrm{Rad}(\Phi - \tilde{\Phi}) \cap (\kappa^\flat)^{-1} \left(\mathbb{C}^n \backslash \mathrm{WF}^{1/2}_{\Phi}(v)\right)$. If $z_0 = 0$, then trivially $z_0 \in \mathrm{Rad}(\Phi - \tilde{\Phi}) \cap \mathbb{C}^n \backslash \mathrm{WF}^{1/2}_{\Phi}(\widetilde{G}v)$. If $z_0 \neq 0$, then we may write $z_0 = (\kappa^\flat)^{-1} w_0$ for some non-zero $w_0 \in \mathbb{C}^n \backslash \mathrm{WF}^{1/2}_{\Phi}(v)$. In view of (3.5.6) and our assumption that $\kappa(\Lambda_\Phi \cap \Lambda_{\tilde{\Phi}}) = \Lambda_\Phi \cap \Lambda_{\tilde{\Phi}}$, we have

$$\kappa^\flat(\mathrm{Rad}(\Phi - \tilde{\Phi})) = \pi_1(\kappa(\Lambda_\Phi \cap \Lambda_{\tilde{\Phi}})) = \pi_1(\Lambda_\Phi \cap \Lambda_{\tilde{\Phi}}) = \mathrm{Rad}(\Phi - \tilde{\Phi}). \tag{3.5.19}$$

It follows that $w_0 \in \mathrm{Rad}(\Phi - \tilde{\Phi}) \cap \mathbb{C}^n \backslash \mathrm{WF}^{1/2}_{\Phi}(v)$. Let $\tilde{V} \subset \mathbb{C}^n \backslash \{0\}$ be an open conic neighborhood of w_0 in $\mathbb{C}^n \backslash \{0\}$ such that

$$|v(w)| \, e^{-\Phi(w)} \leq C e^{-\delta |w|^2}, \quad w \in \tilde{V}, \tag{3.5.20}$$

for some $C, \delta > 0$. Since $\Phi(w_0) - \tilde{\Phi}(w_0) = 0$, we may ensure, by taking $\tilde{V}$ smaller if necessary, that

$$\Phi(w) - \tilde{\Phi}(w) \leq \frac{1}{2} \delta \, |w|^2 \tag{3.5.21}$$

holds for all $w \in \tilde{V}$. Let V be an open conic neighborhood of z_0 in $\mathbb{C}^n \backslash \{0\}$ such that

$$V \subset\subset (\kappa^\flat)^{-1}(\tilde{V}). \tag{3.5.22}$$

We claim that

$$\left| \widetilde{G}v(z) \right| e^{-\Phi(z)} \leq C e^{-c|z|^2} \tag{3.5.23}$$

for all $z \in V$. Let

$$\widetilde{G}u(z) = \tilde{a} \int_{\mathbb{C}^n} e^{2\tilde{\Psi}(z,\overline{w})} u(w) e^{-2\tilde{\Phi}(w)}\, L(dw), \quad u \in H_{\tilde{\Phi}}^{-\infty}(\mathbb{C}^n),$$

where $\tilde{a} \neq 0$, be the Bergman form of $\widetilde{G}$. Since there is $c > 0$ such that

$$2\text{Re }\tilde{\Psi}(z,\overline{w}) - \Phi(z) - \tilde{\Phi}(w) \leq -c \left| z - (\kappa^\flat)^{-1}(w) \right|^2, \quad z, w \in \mathbb{C}^n, \tag{3.5.24}$$

we get, for $z \in V$,

$$\left| \widetilde{G}v(z) e^{-\Phi(z)} \right| \leq C \left(\int_{\tilde{V}} + \int_{\mathbb{C}^n \backslash \tilde{V}} \right) e^{-c\left|z-(\kappa^\flat)^{-1}w\right|^2} |v(w)|\, e^{-\tilde{\Phi}(w)}\, L(dw) =: I(z) + II(z). \tag{3.5.25}$$

Because (3.5.20) and (3.5.21) hold within $\tilde{V}$,

$$\begin{aligned} I(z) &= C \int_{\tilde{V}} e^{-c\left|z-(\kappa^\flat)^{-1}w\right|^2} e^{\Phi(w)-\tilde{\Phi}(w)} e^{-\delta|w|^2}\, L(dw) \\ &\leq C \int_{\mathbb{C}^n} e^{-c\left|z-(\kappa^\flat)^{-1}w\right|^2} e^{-\frac{\delta}{2}|w|^2}\, L(dw) \\ &\leq C e^{-c|z|^2}, \quad z \in \mathbb{C}^n. \end{aligned}$$

To estimate $II(z)$, we notice that, thanks to (3.5.22), the quadratic form

$$(z, w) \mapsto \left| z - (\kappa^\flat)^{-1} w \right|^2$$

is non-vanishing for $(z,w) \in V \times (\mathbb{C}^n\backslash\{0\})\backslash\tilde{V}$. By homogeneity, there is a constant $\gamma > 0$ such that

$$\left|z - (\kappa^\flat)^{-1}w\right|^2 \geq \gamma(|z|^2 + |w|^2)$$

for all $(z,w) \in V \times \mathbb{C}^n\backslash\tilde{V}$. Letting $s \in \mathbb{R}$ be such that $v \in H^s_{\tilde{\Phi}}(\mathbb{C}^n)$, we obtain

$$II(z) \leq C\left(\int_{\mathbb{C}^n} e^{-2\gamma(|z|^2+|w|^2)}\langle w\rangle^{-2s}\,L(dw)\right)^{1/2} \leq Ce^{-\gamma|z|^2}, \quad z \in V.$$

This proves (3.5.23). Therefore $z_0 \in \mathrm{Rad}(\Phi - \tilde{\Phi}) \cap \mathbb{C}^n\backslash\mathrm{WF}^{1/2}_{\Phi}(\widetilde{G}v)$. The lemma is proved. ☐

Now we can complete the proof of Theorem 3.5.1. Assume that $\kappa(\Lambda_\Phi \cap \Lambda_{\tilde{\Phi}}) = \Lambda_\Phi \cap \Lambda_{\tilde{\Phi}}$. Let $\widetilde{G}$ be a non-zero metaplectic Fourier integral operator quantizing κ^{-1}, realized as a continuous linear transformation $H^{-\infty}_{\tilde{\Phi}}(\mathbb{C}^n) \to H^{-\infty}_{\Phi}(\mathbb{C}^n)$. By Proposition B.4 of [9], the operator $\widetilde{G}G : H^{-\infty}_{\Phi}(\mathbb{C}^n) \to H^{-\infty}_{\Phi}(\mathbb{C}^n)$ is a metaplectic Fourier integral operator quantizing the identity map on $\mathbb{C}^{2n}$. By Proposition 3.4.1 (see also Example 1), the Bergman form of $\widetilde{G}G$ must be

$$\widetilde{G}Gu = \hat{b}\int_{\mathbb{C}^n} e^{2\Psi(z,\overline{w})}u(w)e^{-2\Phi(w)}\,L(dw), \quad u \in H^{-\infty}_{\Phi}(\mathbb{C}^n), \tag{3.5.26}$$

where $\hat{b} \in \mathbb{C}$ and $\Psi(z,\theta)$ is the polarization of Φ. By multiplying $\widetilde{G}$ by a non-zero constant if necessary, we may ensure that $\hat{b} = 2^n\pi^{-n}\det\partial^2_{z\overline{z}}\Phi$. For this choice of $\hat{b}$, the righthand side of (3.5.26) is the Bergman projector (5.2.65). From (3.2.27), we have

$$\widetilde{G}G = I \;\text{ on }\; H^{-\infty}_{\Phi}(\mathbb{C}^n). \tag{3.5.27}$$

Let $u \in H^{-\infty}_{\Phi}(\mathbb{C}^n)$ be given. By Lemma 3.5.2 and (3.5.27), we have

$$\mathrm{WF}^{1/2}_{\Phi}(u) \cap \mathrm{Rad}(\Phi - \tilde{\Phi}) \subset (\kappa^\flat)^{-1}(\mathrm{WF}^{1/2}_{\Phi}(Gu)) \cap \mathrm{Rad}(\Phi - \tilde{\Phi}). \tag{3.5.28}$$

By (3.5.9) and (3.5.19),

$$(\kappa^\flat)^{-1}(\mathrm{WF}_\Phi^{1/2}(Gu)) \cap \mathrm{Rad}(\Phi - \tilde{\Phi}) \subset \mathrm{WF}_\Phi^{1/2}(u) \cap \mathrm{Rad}(\Phi - \tilde{\Phi}). \tag{3.5.29}$$

Combining (3.5.28) with (3.5.29) and using (3.5.19) gives

$$\mathrm{WF}_\Phi^{1/2}(Gu) \cap \mathrm{Rad}(\Phi - \tilde{\Phi}) = \kappa^\flat(\mathrm{WF}_\Phi^{1/2}(u)) \cap \mathrm{Rad}(\Phi - \tilde{\Phi}). \tag{3.5.30}$$

Since (3.5.9) also holds, we must have

$$\mathrm{WF}_\Phi^{1/2}(Gu) = \kappa^\flat(\mathrm{WF}_\Phi^{1/2}(u)) \cap \mathrm{Rad}(\Phi - \tilde{\Phi}). \tag{3.5.31}$$

The proof of Theorem 3.5.1 is complete.

3.6 The Bergman Representation of the Evolution Semigroup and the Proof of Theorem 3.1.1

Let $q = q(x, \xi)$ be a complex-valued quadratic form on $\mathbb{R}^{2n}$ with Re $q \geq 0$ and let $q^w(x, D)$ be the Weyl quantization of q. We consider the Schrödinger initial value problem

$$\begin{cases} \partial_t u(t,x) + q^w(x,D)u(t,x) = 0, \quad t \geq 0, \ x \in \mathbb{R}^n, \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}^n). \end{cases} \tag{3.6.1}$$

From the discussion on pages 425-426 of [31], we know that $q^w(x, D)$, regarded as an unbounded operator on $L^2(\mathbb{R}^n)$ equipped with its maximal domain

$$D_{\max} = \left\{ u \in L^2(\mathbb{R}^n) : q^w(x,D)u \in L^2(\mathbb{R}^n) \right\}, \tag{3.6.2}$$

generates a strongly continuous one-parameter semigroup $G(t) = e^{-tq^w(x,D)}$, $t \geq 0$, on $L^2(\mathbb{R}^n)$. We may regard $G(t)$ as the solution operator for the problem (3.6.1).

Let φ be an FBI phase function with associated FBI transform $\mathcal{T}_\varphi$ and strictly plurisubharmonic weight $\Phi(z)$. Let $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be the complex canonical transformation generated by φ, and let $\tilde{q} = q \circ \kappa_\varphi^{-1}$. Applying $\mathcal{T}_\varphi$ to (3.6.1) and using the complex Egorov theorem ([63] Theorem 13.9) gives

$$\begin{cases} \partial_t \mathcal{T}_\varphi u(t,z) + \tilde{q}^w(z,D)\mathcal{T}_\varphi u(t,z) = 0, \quad t \geq 0, \; z \in \mathbb{C}^n, \\ \mathcal{T}_\varphi u|_{t=0} = \mathcal{T}_\varphi u_0 \in H_\Phi(\mathbb{C}^n). \end{cases} \tag{3.6.3}$$

Here

$$\tilde{q}^w(z, D_z)u(z) = \frac{1}{(2\pi)^n} \iint\limits_{\Gamma_\Phi(z)} e^{i(z-w)\cdot\theta} \tilde{q}\left(\frac{z+w}{2}, \theta\right) u(w)\, dw \wedge d\theta, \tag{3.6.4}$$

where

$$\Gamma_\Phi(z) = \left\{ (w,\theta) \in \mathbb{C}^{2n} : \theta = \frac{2}{i}\frac{\partial \Phi}{\partial z}\left(\frac{z+w}{2}\right) \right\}, \tag{3.6.5}$$

is the complex Weyl quantization of the symbol $\tilde{q}$. For further information regarding complex Weyl quantization, we refer the reader to Section 1.4 of [27], Chapter 13 of [63], or Section 12.2 of [55]. In particular, since $\|\partial^\alpha \tilde{q}\|_{L^\infty(\mathbb{C}^{2n})} < \infty$ for all $|\alpha| \geq 2$, Proposition 12.6 of [55] implies that $\tilde{q}^w(z,D) = \mathcal{O}(1) : H_\Phi^s(\mathbb{C}^n) \to H_\Phi^{s-2}(\mathbb{C}^n)$ for every $s \in \mathbb{R}$. We also note that, since $\tilde{q}(z,\zeta)$ is a holomorphic quadratic form, the operator $\tilde{q}^w(z,D)$ acts as a quadratic differential operator on elements of $H_\Phi^{-\infty}(\mathbb{C}^n)$. Indeed, if

$$\tilde{q}(z,\zeta) = \frac{1}{2}A_1 z \cdot z + A_2 z \cdot \zeta + \frac{1}{2}A_3 \zeta \cdot \zeta, \quad (z,\zeta) \in \mathbb{C}^{2n}, \tag{3.6.6}$$

where $A_1, A_2, A_3 \in M_{n\times n}(\mathbb{C})$ with $A_1 = A_1^T$ and $A_3 = A_3^T$, then

$$\tilde{q}^w(z,D)u(z) = \left(\frac{1}{2}A_1 z \cdot z + A_2 z \cdot D_z + \frac{1}{2i}\mathrm{tr}\,(A_2) + \frac{1}{2}A_3 D_z \cdot D_z\right) u(z), \quad z \in \mathbb{C}^n, \tag{3.6.7}$$

where $D_z = \frac{1}{i}\partial_z$, for all $u \in H_\Phi^{-\infty}(\mathbb{C}^n)$.

We may view $\tilde{q}^w(z, D)$ as an unbounded operator on $H_\Phi(\mathbb{C}^n)$ equipped with the maximal domain

$$\widetilde{D}_{\max} = \{u \in H_\Phi(\mathbb{C}^n) : \tilde{q}^w(z, D)u \in H_\Phi(\mathbb{C}^n)\}. \tag{3.6.8}$$

As a consequence of the complex Egorov theorem, we have $\widetilde{D}_{\max} = \mathcal{T}_\varphi(D_{\max})$. Since also $\mathcal{T}_\varphi : L^2(\mathbb{R}^n) \to H_{\Phi(\mathbb{C}^n)}$ is unitary, it follows that $\tilde{q}^w(z, D)$ generates a strongly continuous one-parameter semigroup $\widetilde{G}(t) = e^{-t\tilde{q}^w(z,D)}$, $t \geq 0$, on $H_\Phi(\mathbb{C}^n)$. This semigroup is related to $G(t)$ by

$$\widetilde{G}(t) = \mathcal{T}_\varphi \circ G(t) \circ \mathcal{T}_\varphi^*, \quad t \geq 0. \tag{3.6.9}$$

Our goal is to prove that, for all $t \geq 0$, the semigroup $\widetilde{G}(t)$ is a metaplectic Fourier integral operator in the sense of Section 4 whose underlying complex canonical transformation is the Hamilton flow of $-i\tilde{q}$ at time t. To this end, we recall from [44] that $\kappa_t := \exp(tH_{-iq}) : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$, the Hamilton flow of $-iq$, is positive relative to $\mathbb{R}^{2n}$ for each $t \geq 0$. To see this, write

$$q(X) = QX \cdot X, \quad X \in \mathbb{C}^{2n},$$

where $Q \in M_{2n\times 2n}(\mathbb{C})$ is symmetric. Let

$$F = JQ \tag{3.6.10}$$

be the Hamilton matrix of q. We can express κ_t in terms of F as follows:

$$\kappa_t = e^{-2itF}, \quad t \in \mathbb{R}. \tag{3.6.11}$$

Because the unique antilinear involution of $\mathbb{C}^{2n}$ fixing $\mathbb{R}^{2n}$ is the usual map of complex conjugation $X \mapsto \overline{X}$, the complex canonical transformation κ_t is positive relative to $\mathbb{R}^{2n}$ for all $t \geq 0$ if and only if for every $X \in \mathbb{C}^{2n}$ the real-valued function

$$r(t) = \frac{1}{i}\left(\sigma(\kappa_t(X), \overline{\kappa_t(X)}) - \sigma(X, \overline{X})\right), \quad t \in \mathbb{R}, \tag{3.6.12}$$

is non-negative for all $t \geq 0$. Recalling that

$$\sigma(X,Y) = JX \cdot Y, \quad X, Y \in \mathbb{C}^n, \tag{3.6.13}$$

we see that (3.6.12) may be rewritten as

$$r(t) = \frac{1}{i}\left(Je^{-2itF}X \cdot e^{2it\overline{F}}\overline{X} - JX \cdot \overline{X}\right), \quad t \geq 0. \tag{3.6.14}$$

Differentiating (3.6.14) with respect to t gives

$$r'(t) = 2(\overline{F}^T J - JF)e^{-2itF}X \cdot \overline{e^{-2itF}X}. \tag{3.6.15}$$

In view of (3.6.10), we have

$$\overline{F}^T J - JF = 2\text{Re } Q \geq 0. \tag{3.6.16}$$

Integrating (3.6.15) from 0 to t and using (3.6.16), we find that (3.6.12) is non-negative for $t \geq 0$. Thus

$$\frac{1}{i}\left(\sigma(\kappa_t(X), \overline{\kappa_t(X)}) - \sigma(X, \overline{X})\right) \geq 0, \quad X \in \mathbb{C}^{2n}, \quad t \geq 0. \tag{3.6.17}$$

Let $\tilde{\kappa}_t := \exp(tH_{-i\tilde{q}})$, $t \in \mathbb{R}$, be the Hamilton flow of $-i\tilde{q}$. From Jacobi's theorem (see e.g. Theorem 2.10 in [63]) it follows that

$$\tilde{\kappa}_t = \kappa_\varphi \circ \kappa_t \circ \kappa_\varphi^{-1}, \quad t \in \mathbb{R}. \tag{3.6.18}$$

Let ι_{Λ_Φ} be the unique antilinear involution of $\mathbb{C}^{2n}$ fixing Λ_Φ. Since complex conjugation $Z \mapsto \overline{Z}$ is the unique antilinear involution of $\mathbb{C}^{2n}$ fixing $\mathbb{R}^{2n}$ and $\kappa_\varphi(\mathbb{R}^{2n}) = \Lambda_\Phi$, we have

$$\iota_{\Lambda_\Phi}(\kappa_\varphi(X)) = \kappa_\varphi(\overline{X}), \quad Z \in \mathbb{C}^{2n}. \tag{3.6.19}$$

From (3.6.17), (3.6.18), (3.6.19), and the invariance of σ under κ_φ, we may deduce that

$$\frac{1}{i}\left(\sigma(\tilde{\kappa}_t(X), \iota_{\Lambda_\Phi}(\tilde{\kappa}_t(X))) - \sigma(X, \iota_{\Lambda_\Phi}(X))\right) \geq 0, \quad X \in \mathbb{C}^{2n}, \quad t \geq 0. \tag{3.6.20}$$

Hence the flow $\tilde{\kappa}_t$ is positive relative to Λ_Φ for each $t \geq 0$.

From the results of [12] (see also the discussion at the beginning of Section 5 above) there is a one-parameter family Φ_t, $t \geq 0$, of strictly plurisubharmonic quadratic forms on $\mathbb{C}^n$ with $\Phi_t \leq \Phi$ for all $t \geq 0$ such that $\Phi_0 = \Phi$ and

$$\tilde{\kappa}_t(\Lambda_\Phi) = \Lambda_{\Phi_t}, \quad t \geq 0. \tag{3.6.21}$$

It turns out that Φ_t, $t \geq 0$, satisfies a natural eikonal equation associated to $\tilde{\kappa}_t$. To the function $\Phi(t, z) = \Phi_t(z)$, we may associate the submanifold

$$\left\{(t, \tau; z, \zeta) : t \geq 0, z \in \mathbb{C}^n, \tau = \frac{\partial \Phi}{\partial t},\ \zeta = \frac{2}{i}\frac{\partial \Phi}{\partial z}\right\}$$

of $\mathbb{R}^2_{t,\tau} \times \mathbb{C}^{2n}_{z,\zeta}$, which is Lagrangian with respect to the real symplectic form

$$d\tau \wedge dt - \mathrm{Im}\ \sigma. \tag{3.6.22}$$

For $g \in \mathrm{Hol}(\mathbb{C}^{2n})$, we denote by $\widehat{H_g}$ the real vector field on $\mathbb{C}^{2n}$ corresponding to the holomorphic vector field H_g:

$$\widehat{H_g} = H_g + \overline{H_g}.$$

From the discussion on pages 78-79 of [51], we know that

$$\widehat{H_{-i\tilde{q}}} = H^{-\mathrm{Im}\ \sigma}_{\mathrm{Re}\ \tilde{q}},$$

where $H^{-\mathrm{Im}\ \sigma}_{\mathrm{Re}\ \tilde{q}}$ denotes the Hamilton vector field of $\mathrm{Re}\ \tilde{q}$ on $\mathbb{C}^{2n}$ taken with respect to $-\mathrm{Im}\ \sigma$.

Applying the Hamilton-Jacobi theory (see Chapter 1 of [19]) with respect to the real symplectic form (3.6.22), we find that $\Phi(t,z)$ satisfies the eikonal equation

$$\begin{cases} \frac{\partial \Phi}{\partial t}(t,z) + \mathrm{Re}\, \tilde{q}\left(z, \frac{2}{i}\frac{\partial \Phi}{\partial z}(t,z)\right) = 0, \quad t \geq 0,\ z \in \mathbb{C}^n, \\ \Phi(0,\cdot) = \Phi \text{ on } \mathbb{C}^n. \end{cases} \tag{3.6.23}$$

Now we prove that $\widetilde{G}(t)$, $t \geq 0$, is a metaplectic Fourier integral operator in the complex domain whose underlying complex canonical transformation at time t is $\tilde{\kappa}_t$. For $0 \leq t \ll 1$, this may be accomplished by a standard geometrical optics construction (see for instance Section 3 of [23] or Section 2 of [26]). However, it is actually possible to construct $\widetilde{G}(t)$ as a metaplectic Fourier integral operator directly in the Bergman form (3.4.10) for all $t \geq 0$. To the best of our knowledge, the idea of representing evolution semigroups on the FBI transform side as Fourier integral operators in Bergman form was introduced by J. Sjöstrand in the work [56]. The technique we present below may be viewed as a linearized version of the construction given in [56], valid for all positive times thanks to the positivity of $\tilde{\kappa}_t$ relative to Λ_Φ.

We search for $\widetilde{G}(t)$ of the form

$$\widetilde{G}(t)u(z) = \hat{a}(t) \int_{\mathbb{C}^n} e^{2\Psi_t(z,\overline{w})} u(w) e^{-2\Phi(w)}\, L(dw), \quad u \in H_\Phi(\mathbb{C}^n), \tag{3.6.24}$$

where $\hat{a} \in C^\infty([0,\infty);\mathbb{C})$ is non-vanishing and $\Psi_t(\cdot,\cdot)$ is a holomorphic quadratic form on $\mathbb{C}^{2n}$ with coefficients depending smoothly on t for $t \geq 0$. Our objective is to choose $\hat{a}$ and $\Psi_t(\cdot,\cdot)$ so that $\widetilde{G}(t)$ solves the operator initial value problem

$$\begin{cases} \partial_t \widetilde{G}(t) + \tilde{q}^w(z,D)\widetilde{G}(t) = 0, \quad t \geq 0, \\ \widetilde{G}(0) = I \text{ on } H_\Phi(\mathbb{C}^n). \end{cases} \tag{3.6.25}$$

To this end, let us rewrite (3.6.24) in the form

$$\widetilde{G}(t)u(z) = a(t) \int_\Gamma e^{2\Psi_t(z,\theta) - 2\Psi(w,\theta)} u(w)\, dw \wedge d\theta, \quad u \in H_\Phi(\mathbb{C}^n), \tag{3.6.26}$$

where

$$\Gamma = \left\{(w,\theta) \in \mathbb{C}^{2n} : \theta = \overline{w}\right\} \tag{3.6.27}$$

is the anti-diagonal in $\mathbb{C}^{2n}$, $a(t) = (i/2)^n \hat{a}(t)$, $t \geq 0$, and $\Psi(\cdot,\cdot)$ is the polarization of Φ. Thanks to (3.6.7) and the well-known formula for the conjugation of a Weyl differential operator by a quadratic exponential (see, for instance, the proof of Theorem 10.6 in [63]), we know that

$$e^{-2\Psi_t(z,\theta)} \circ (\partial_t + \tilde{q}^w(z,D)) \circ e^{2\Psi_t(z,\theta)} = \partial_t + 2\partial_t \Psi_t(z,\theta) + \tilde{q}^w_{\frac{2}{i}\Psi_t(\cdot,\theta)}(z,D), \quad \theta \in \mathbb{C}^n, \tag{3.6.28}$$

where

$$\tilde{q}_{\frac{2}{i}\Psi_t(\cdot,\theta)}(z,\zeta) = \tilde{q}\left(z, \zeta + \frac{2}{i}\partial_z \Psi_t(z,\theta)\right), \quad (z,\zeta) \in \mathbb{C}^{2n}, \ \theta \in \mathbb{C}^n. \tag{3.6.29}$$

Assume that $\Psi_t(\cdot,\theta)$ satisfies the eikonal equation

$$2\partial_t \Psi_t(z,\theta) + \tilde{q}\left(z, \frac{2}{i}\partial_z \Psi_t(z,\theta)\right) = 0, \quad t \geq 0, \ z \in \mathbb{C}^n, \ \theta \in \mathbb{C}^n. \tag{3.6.30}$$

Using that $\tilde{q}$ is quadratic, we see that the conjugated operator (3.6.28) is equal to

$$\partial_t + \left(\partial_\zeta \tilde{q}\left(z, \frac{2}{i}\partial_z \Psi_t(z,\theta)\right) \cdot \zeta\right)^w + \frac{1}{2}\left(\partial^2_{\zeta\zeta}\tilde{q}\right) D_z \cdot D_z, \quad z \in \mathbb{C}^n, \ \theta \in \mathbb{C}^n, \ t \geq 0. \tag{3.6.31}$$

Introducing the holomorphic vector field

$$\nu(z,\partial_z) = \partial_\zeta \tilde{q}\left(z, \frac{2}{i}\partial_z \Psi_t(z,\theta)\right) \cdot \partial_z, \quad z \in \mathbb{C}^n, \tag{3.6.32}$$

we deduce

$$e^{-2\Psi_t(z,\theta)} \circ (\partial_t + \tilde{q}^w(z,D)) \circ e^{2\Psi_t(\cdot,\theta)} = \partial_t + \frac{1}{i}\nu(z,\partial_z) + \frac{1}{2i}\mathrm{div}(\nu) + \frac{1}{2}\left(\partial^2_{\zeta\zeta}\tilde{q}\right) D_z \cdot D_z \tag{3.6.33}$$

for $z \in \mathbb{C}^n$, $\theta \in \mathbb{C}^n$, and $t \geq 0$. Here $\mathrm{div}(\nu)$ denotes the holomorphic divergence of ν,

$$\mathrm{div}(\nu) = \sum_{j=1}^{n} \partial^2_{\zeta_j z_j} \tilde{q} + \sum_{j=1}^{n}\sum_{k=1}^{n} \left(\partial^2_{\zeta_j \zeta_k} \tilde{q}\right) \left(\frac{2}{i} \partial^2_{z_j z_k} \Psi_t\right) = \mathrm{tr}\left(\partial^2_{\zeta z} \tilde{q} + \partial^2_{\zeta\zeta} \tilde{q} \cdot \frac{2}{i} \partial^2_{zz} \Psi_t\right). \tag{3.6.34}$$

Thus, if we are to have

$$(\partial_t + \tilde{q}^w(z, D)) \left(e^{2\Psi_t(\cdot,\theta)} a(t)\right) \equiv 0 \text{ on } \mathbb{C}^n, \quad \theta \in \mathbb{C}^n, \quad t \geq 0, \tag{3.6.35}$$

it suffices to choose $a(t)$ so that

$$a'(t) + \frac{1}{2i} \beta(t) a(t) = 0, \quad t \geq 0, \tag{3.6.36}$$

where

$$\beta(t) = \mathrm{tr}\left(\partial^2_{\zeta z} \tilde{q} + \partial^2_{\zeta\zeta} \tilde{q} \cdot \frac{2}{i} \partial^2_{zz} \Psi_t\right), \quad t \geq 0. \tag{3.6.37}$$

Demanding also that $\Psi_t(z, \theta)|_{t=0} = \Psi(z, \theta)$ and $a(0) = (i/2)^n C_\Phi$, where C_Φ is as in (3.2.25), we may ensure that $\widetilde{G}(0)$ coincides with the Bergman projection (5.2.65) and hence that the initial condition $\widetilde{G}(0) = I$ on $H_\Phi(\mathbb{C}^n)$ is satisfied. We conclude that if we are to produce a solution $\widetilde{G}(t)$ of the operator initial value problem (3.6.25) of the form (3.6.24), we should choose $\Psi_t(\cdot, \cdot)$ so that

$$\begin{cases} 2\partial_t \Psi_t(z, \theta) + \tilde{q}\left(z, \frac{2}{i} \partial_z \Psi_t(z, \theta)\right) = 0, \quad z, \theta \in \mathbb{C}^n, \ t \geq 0, \\ \Psi_0(z, \theta) = \Psi(z, \theta), \quad z \in \mathbb{C}^n, \ \theta \in \mathbb{C}^n, \end{cases} \tag{3.6.38}$$

and choose $\hat{a}(t)$ so that

$$\begin{cases} \hat{a}'(t) + \frac{1}{2i} \beta(t) \hat{a}(t) = 0, \quad t \geq 0, \\ \hat{a}(0) = C_\Phi. \end{cases} \tag{3.6.39}$$

As the initial value problem (3.6.39) can be solved by elementary methods once $\Psi_t(\cdot, \cdot)$ is known, we will focus our attention on solving the problem (3.6.38). We note that, since (3.6.39) is a linear ordinary differential equation and $C_\Phi \neq 0$, the solution $\hat{a}(t)$ of (3.6.39) will be non-vanishing for all $t \geq 0$. Taking the real part of (3.6.38) gives

$$\begin{cases} \partial_t \left[2\mathrm{Re}\ \Psi_t(z,\theta)\right] + \mathrm{Re}\ \tilde{q}\left(z, \frac{2}{i}\partial_z \left[2\mathrm{Re}\ \Psi_t(z,\theta)\right]\right) = 0, \quad z,\theta \in \mathbb{C}^n,\ t \geq 0, \\ 2\mathrm{Re}\ \Psi_0(z,\theta) = 2\mathrm{Re}\ \Psi(z,\theta), \quad z \in \mathbb{C}^n,\ \theta \in \mathbb{C}^n. \end{cases} \tag{3.6.40}$$

Using Hamilton-Jacobi theory (see, for instance the proof of Lemma 2.2 in [13] or the proof of Proposition 5.3.1 below), we may deduce from (3.6.40) that

$$\Lambda_{2\mathrm{Re}\ \Psi_t(\cdot,\theta)} = \tilde{\kappa}_t\left(\Lambda_{2\mathrm{Re}\ \Psi(\cdot,\theta)}\right), \quad \theta \in \mathbb{C}^n,\ t \geq 0, \tag{3.6.41}$$

where $\Lambda_{2\mathrm{Re}\ \Psi_t(\cdot,\theta)}$ and $\Lambda_{2\mathrm{Re}\ \Psi(\cdot,\theta)}$ denote the $\mathbb{C}$-Lagrangian subspaces of $\mathbb{C}^{2n}$ given by

$$\Lambda_{2\mathrm{Re}\ \Psi_t(\cdot,\theta)} = \left\{\left(z, \frac{2}{i}\partial_z \left[2\mathrm{Re}\ \Psi_t(z,\theta)\right]\right) : z \in \mathbb{C}^n\right\}, \quad \theta \in \mathbb{C}^n, \quad t \geq 0, \tag{3.6.42}$$

and

$$\Lambda_{2\mathrm{Re}\ \Psi(\cdot,\theta)} = \left\{\left(z, \frac{2}{i}\partial_z \left[2\mathrm{Re}\ \Psi(z,\theta)\right]\right) : z \in \mathbb{C}^n\right\}, \quad \theta \in \mathbb{C}^n, \tag{3.6.43}$$

respectively. As a consequence of the fundamental estimate (5.2.68), there is a constant $c > 0$ such that

$$2\mathrm{Re}\ \Psi(z,0) \leq \Phi(z) - c\,|z|^2, \quad z \in \mathbb{C}^n. \tag{3.6.44}$$

From Theorem 2.1 of [12] it follows that the $\mathbb{C}$-Lagrangian subspace $\Lambda_{2\mathrm{Re}\ \Psi(\cdot,0)}$ of $\mathbb{C}^{2n}$ is positive relative to Λ_Φ. As $\tilde{\kappa}_t$ is positive relative to Λ_Φ for all $t \geq 0$, (3.6.41) implies that $\Lambda_{2\mathrm{Re}\ \Psi_t(\cdot,0)}$ is positive relative to Λ_Φ for all $t \geq 0$. This observation, combined with Theorem 2.1 of [12], implies that (3.6.40) may be solved for all $t \geq 0$, first in the case $\theta = 0$, and then for general $\theta \in \mathbb{C}^n$. Thus we obtain a holomorphic quadratic form $\Psi_t(\cdot,\cdot)$ depending analytically on t for $t \geq 0$ that solves the initial value problem (3.6.38). It follows that $\widetilde{G}(t)$ given by (3.6.24) satisfies (3.6.25).

Finally, let us check that $\widetilde{G}(t)$ is a metaplectic Fourier integral operator in the sense of Section 4. Writing $\widetilde{G}(t)$ as the contour integral (3.6.26), we see that $\widetilde{G}(t)$ is of the form (3.4.1) with phase function

$$\phi_t(z,w,\theta) = \frac{2}{i}\Psi_t(z,\theta) - \frac{2}{i}\Psi(w,\theta), \quad (z,w,\theta) \in \mathbb{C}^{3n}, \quad t \geq 0. \tag{3.6.45}$$

Since Φ is strictly plurisubharmonic, the phase ϕ_t is easily seen to satisfy Hörmander's non-degeneracy condition (3.4.2) for every $t \geq 0$. Moreover, the relation (3.6.41) implies that the phase ϕ_t generates graph$(\tilde{\kappa}_t)$ in the sense of (3.4.3) for all $t \geq 0$. As $\partial_\theta \phi_t(z,w,\theta) = 0$ for $(z,w,\theta) \in \mathbb{C}^{3n}$ and $t \geq 0$ if and only if $\partial_\theta \Psi_t(z,\theta) = \partial_\theta \Psi(w,\theta)$, it follows that $\tilde{\kappa}_t : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is given implicitly by

$$\tilde{\kappa}_t : \left(w, \frac{2}{i}\partial_w \Psi(w,\theta)\right) \mapsto \left(z, \frac{2}{i}\partial_z \Psi_t(z,\theta)\right), \ \partial_\theta \Psi_t(z,\theta) = \partial_\theta \Psi(w,\theta), \quad z,w,\theta \in \mathbb{C}^n, \ t \geq 0. \tag{3.6.46}$$

We conclude that for all $t \geq 0$ the operator $\widetilde{G}(t)$ is indeed a metaplectic Fourier integral operator in the complex domain with underlying canonical transformation $\tilde{\kappa}_t$. By Proposition 3.4.1, (3.6.24) is the Bergman form of $\widetilde{G}(t)$ for every $t \geq 0$.

Let us verify that

$$\forall u \in \mathcal{S}'(\mathbb{R}^n), \ \forall t \geq 0 : \quad \mathcal{T}_\varphi G(t) u = \widetilde{G}(t) \mathcal{T}_\varphi u. \tag{3.6.47}$$

From the work [31], we know that for every $t \geq 0$ the operator $G(t) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ extends uniquely to a sequentially continuous linear transformation $G(t) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. Thus, by Propositions 3.2.3, 3.2.4 and 3.4.2, it suffices to show that for every $u \in \mathcal{S}'(\mathbb{R}^n)$ and $t \geq 0$, we have

$$\int_{\mathbb{C}^n} \mathcal{T}_\varphi G(t) u(z) \overline{v(z)} e^{-2\Phi(z)} \, L(dz) = \int_{\mathbb{C}^n} \widetilde{G}(t) \mathcal{T}_\varphi u(z) \overline{v(z)} e^{-2\Phi(z)} \, L(dz) \tag{3.6.48}$$

for all $v \in H^\infty_\Phi(\mathbb{C}^n)$. For any $u \in \mathcal{S}'(\mathbb{R}^n)$, $v \in H^\infty_\Phi(\mathbb{C}^n)$, and $t \geq 0$, we have

$$\begin{aligned}\int_{\mathbb{C}^n} \mathcal{T}_\varphi G(t)u(z)\overline{v(z)}e^{-2\Phi(z)}\,L(dz) &= \int_{\mathbb{C}^n} \langle G(t)u, c_\varphi e^{i\varphi(z,\cdot)}\overline{v(z)}e^{-2\Phi(z)}\rangle\,L(dz)\\ &= \langle G(t)u, \overline{\mathcal{T}_\varphi^* v}\rangle \\ &= \langle u, \overline{G(t)^*\mathcal{T}_\varphi^* v}\rangle,\end{aligned} \tag{3.6.49}$$

where $\mathcal{T}_\varphi^*$ is the adjoint of $\mathcal{T}_\varphi : L^2(\mathbb{R}^n) \to H_\Phi(\mathbb{C}^n)$ and $G(t)^*$ is the adjoint of $G(t)$ taken in the sense of distributions. The identity (3.6.9) implies

$$G(t)^*\mathcal{T}_\varphi^* v = (\mathcal{T}_\varphi G(t))^* v = (\widetilde{G}(t)\mathcal{T}_\varphi)^* v = \mathcal{T}_\varphi^*\widetilde{G}(t)^* v. \tag{3.6.50}$$

Thus

$$\langle u, \overline{G(t)^*\mathcal{T}_\varphi^* v}\rangle = \langle u, \overline{\mathcal{T}_\varphi^*\widetilde{G}(t)^* v}\rangle = \int_{\mathbb{C}^n} \mathcal{T}_\varphi u(w)\overline{\widetilde{G}(t)^* v(w)}e^{-2\Phi(w)}\,L(dw). \tag{3.6.51}$$

As a consequence of (3.6.24),

$$\widetilde{G}(t)^* v(w) = \overline{\hat{a}(t)}\int_{\mathbb{C}^n} e^{2\overline{\Psi_t(z,\overline{w})}}v(z)e^{-2\Phi(z)}\,L(dz). \tag{3.6.52}$$

Putting (3.6.52) into (3.6.51), interchanging the order of integration, and using (3.6.24) gives (3.6.48).

Having established that $\widetilde{G}(t)$ is a metaplectic Fourier integral operator whose underlying complex canonical transformation at time t is $\tilde{\kappa}_t$, we can apply the results of Section 5 to study the propagation of 1/2-Gelfand-Shilov singularities by the semigroup $G(t)$, via the identity (3.6.47). We begin by giving a characterization of the singular space S of q in terms of the intersection $\Lambda_\Phi \cap \Lambda_{\Phi_t}$ for $t > 0$.

Proposition 3.6.1. *For all $t > 0$,*

$$S = \kappa_\varphi^{-1}(\Lambda_\Phi \cap \Lambda_{\Phi_t}). \tag{3.6.53}$$

Proof. By Theorem 1.1 of [12], the positivity of $\tilde{\kappa_t}$ relative to Λ_Φ implies that the quadratic form $\Phi - \Phi_t$ is non-negative for all $0 \le t < \infty$. Thus,

$$\forall z \in \mathbb{C}^n,\ \forall\, 0 \le t < \infty : \ \left(z, \frac{2}{i}\frac{\partial \Phi}{\partial z}(z)\right) \in \Lambda_{\Phi_t} \iff \frac{\partial \Phi}{\partial z}(z) = \frac{\partial \Phi_t}{\partial z}(z) \iff \Phi(z) - \Phi_t(z) = 0. \tag{3.6.54}$$

According to (3.6.23),

$$\frac{\partial \Phi_t}{\partial t}(z) = -\mathrm{Re}\ \tilde{q}\left(z, \frac{2}{i}\frac{\partial \Phi_t}{\partial z}(z)\right),\ z \in \mathbb{C}^n,\ 0 \le t < \infty. \tag{3.6.55}$$

For any fixed $z \in \mathbb{C}^n$ and $0 \le t < \infty$, the point $\left(z, \frac{2}{i}\frac{\partial \Phi_t}{\partial z}(z)\right)$ belongs to $\Lambda_{\Phi_t} = \tilde{\kappa_t}(\Lambda_\Phi)$. It follows that for any $z \in \mathbb{C}^n$ there is an $X \in \Lambda_\Phi$ such that

$$\left(z, \frac{2}{i}\frac{\partial \Phi_t}{\partial z}(z)\right) = \tilde{\kappa_t}(X).$$

Since $\tilde{q}$ is invariant under the flow $\tilde{\kappa_t}$,

$$\mathrm{Re}\ \tilde{q}\left(z, \frac{2}{i}\frac{\partial \Phi_t}{\partial z}(z)\right) = \mathrm{Re}\ \tilde{q}(X). \tag{3.6.56}$$

As $\mathrm{Re}\,\tilde{q}|_{\Lambda_\Phi} \ge 0$, (3.6.56) and (3.6.55) together imply that

$$\frac{\partial \Phi_t}{\partial t}(z) \le 0 \tag{3.6.57}$$

for every $z \in \mathbb{C}^n$ and $0 \le t < \infty$. Hence,

$$\forall z \in \mathbb{C}^n,\ \forall\, 0 \le t < \infty : \ \left(z, \frac{2}{i}\frac{\partial \Phi}{\partial z}(z)\right) \in \Lambda_{\Phi_t} \iff \Phi(z) - \Phi_s(z) = 0 \text{ for all } 0 \le s \le t.$$

Because the quadratic form $\Phi - \Phi_s$ is non-negative, $\Phi(z) - \Phi_s(z) = 0$ if and only if $\partial_z \Phi(z) = \partial_z \Phi_s(z)$, and we have

$$\forall z \in \mathbb{C}^n,\ \forall\ 0 \le t < \infty : \left(z, \frac{2}{i}\frac{\partial \Phi}{\partial z}(z)\right) \in \Lambda_{\Phi_t} \iff \left(z, \frac{2}{i}\frac{\partial \Phi}{\partial z}(z)\right) \in \Lambda_{\Phi_s} \text{ for all } 0 \le s \le t.$$

Therefore

$$\Lambda_\Phi \cap \Lambda_{\Phi_t} = \bigcap_{0 \le s \le t} \Lambda_\Phi \cap \Lambda_{\Phi_s} \tag{3.6.58}$$

for every $t > 0$. Applying κ_φ^{-1} to both sides of (3.6.58) and using Jacobi's theorem and (3.6.11), we find that

$$\kappa_\varphi^{-1}(\Lambda_\Phi \cap \Lambda_{\Phi_t}) = \left\{X \in \mathbb{R}^{2n} : e^{2isF} X \in \mathbb{R}^{2n} \text{ for all } 0 \le s \le t\right\}$$

for all $t > 0$. Thus, if $X \in \mathbb{C}^{2n}$, then

$$X \in \kappa_\varphi^{-1}(\Lambda_\Phi \cap \Lambda_{\Phi_t}) \iff \operatorname{Im}\left(e^{2isF} X\right) = 0 \tag{3.6.59}$$

for every $0 \le s \le t$. From the discussion on page 22 of [44] and the real analyticity of the mapping $t \mapsto \left(\operatorname{Im} e^{2itF}\right)(X)$ for any fixed $X \in \mathbb{R}^{2n}$, we know that

$$S = \bigcap_{0 \le s \le t} \ker\left(\operatorname{Im} e^{2isF}\right) \cap \mathbb{R}^{2n} = \bigcap_{s \in \mathbb{R}} \ker\left(\operatorname{Im} e^{2isF}\right) \cap \mathbb{R}^{2n} \tag{3.6.60}$$

for every $t > 0$. Therefore (3.6.53) must hold for every $t > 0$. □

We now conclude the proof of Theorem 3.1.1. First, we verify that S is invariant under κ_t for every $t \in \mathbb{R}$. From (3.6.11) and (5.1.23), we see that

$$\kappa_t(X) = e^{tH_{\operatorname{Im} q}} X = \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} (\operatorname{Im} F)^k X \tag{3.6.61}$$

for all $X \in S$ and $t \in \mathbb{R}$. Now, let $X \in S$, $Y \in \mathbb{C}^{2n}$, and $t \in \mathbb{R}$ be such that

$$\kappa_t(X) = Y. \tag{3.6.62}$$

As a consequence of (3.6.60), we have $Y \in \mathbb{R}^{2n}$. Also, (3.6.61) and (5.1.23) together imply

$$(\mathrm{Re}\ F)(\mathrm{Im}\ F)^j Y = \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} (\mathrm{Re}\ F)(\mathrm{Im}\ F)^{j+k} X = 0, \quad j \in \mathbb{N}.$$

Thus

$$\kappa_t(S) \subset S, \quad t \in \mathbb{R}. \tag{3.6.63}$$

Since (3.6.63) implies that

$$\kappa_{-t}(S) \subset S, \quad t \in \mathbb{R}, \tag{3.6.64}$$

holds as well, we deduce that

$$\kappa_t(S) = S, \quad t \in \mathbb{R}. \tag{3.6.65}$$

The invariance of S under κ_t for every $t \in \mathbb{R}$, Jacobi's theorem, and Proposition 3.6.1 give

$$\tilde{\kappa}_t(\Lambda_\Phi \cap \Lambda_{\Phi_t}) = \Lambda_\Phi \cap \Lambda_{\Phi_t} \tag{3.6.66}$$

for every $t \geq 0$. Thus, the metaplectic Fourier integral operator $\widetilde{G}(t)$ and its underlying canonical transformation $\tilde{\kappa}_t$ satisfy the hypotheses of Theorem 3.5.1 for every $t \geq 0$. Let $\mathrm{pr}_\Phi = \pi_1|_{\Lambda_\Phi}$, $\mathrm{pr}_{\Phi_t} = \pi_1|_{\Lambda_{\Phi_t}}$ for $t \geq 0$, and $\tilde{\kappa}_t^\flat = \mathrm{pr}_{\Phi_t} \circ \tilde{\kappa}_t \circ \mathrm{pr}_\Phi^{-1}$ for $t \geq 0$. By Theorem 3.5.1, we have

$$\mathrm{WF}_\Phi^{1/2}(\widetilde{G}(t)u) = \tilde{\kappa}_t^\flat \left(\mathrm{WF}_\Phi^{1/2}(u)\right) \cap \mathrm{Rad}(\Phi - \Phi_t) \tag{3.6.67}$$

for every $u \in H_\Phi^{-\infty}(\mathbb{C}^n)$ and every $t \geq 0$. Applying $(\mathrm{pr}_\Phi \circ \kappa_\varphi|_{\mathbb{R}^{2n}})^{-1}$ to both sides of (3.6.67) and using Jacobi's theorem, (3.6.47), (3.6.53), (3.6.65), and Proposition 3.2.2, we get

$$\mathrm{WF}^{1/2}(G(t)u_0) = \kappa_t(\mathrm{WF}^{1/2}(u_0) \cap S) \tag{3.6.68}$$

for every $u_0 \in \mathcal{S}'(\mathbb{R}^n)$ and every $t > 0$. From (3.6.68) and (3.6.61), we may therefore deduce

$$\mathrm{WF}^{1/2}(G(t)u_0) = \exp{(tH_{\mathrm{Im}\ q})}(\mathrm{WF}^{1/2}(u_0) \cap S) \tag{3.6.69}$$

for every $u_0 \in \mathcal{S}'(\mathbb{R}^n)$ and $t > 0$. The proof of Theorem 3.1.1 is complete.

Chapter 4

L^p-Bounds for Semigroups Generated by Non-Elliptic Quadratic Differential Operators

4.1 Introduction and Statement of Results

In this chapter, we prove L^p-bounds for the solution operator $e^{-tq^w(x,D)}$ of the Schrödinger initial value problem

$$\begin{cases} \partial_t u(t,x) + q^w(x,D)u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{4.1.1}$$

where $u_0 \in L^2(\mathbb{R}^n)$ is the initial data, $q = q(x,\xi)$ is a complex-valued quadratic form on the phase space $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ with non-negative real part $\operatorname{Re} q \geq 0$, and $q^w(x,D)$ is the Weyl quantization of $q(x,\xi)$, defined by

$$q^w(x,D)v(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} q\left(\frac{x+y}{2}, \xi\right) v(y)\, dy\, d\xi, \quad v \in \mathcal{S}'(\mathbb{R}^n), \tag{4.1.2}$$

in the sense of distributions. Operators of the form (4.1.2) are quadratic differential operators with a simple, explicit expression. This is because the Weyl quantization of a quadratic monomial of the form $x^\alpha \xi^\beta$, where $\alpha, \beta \in \mathbb{N}^n$, $|\alpha + \beta| = 2$, is

$$\frac{x^\alpha D^\beta + D^\beta x^\alpha}{2}, \quad D := \frac{1}{i}\partial. \tag{4.1.3}$$

The class of evolution equations of the form (4.1.1) contains a number of familiar examples, such as the free Schrödinger equation where $q(x,\xi) = i\,|\xi|^2$, $(x,\xi) \in \mathbb{R}^{2n}$, the quantum harmonic oscillator, where $q(x,\xi) = i(|x|^2 + |\xi|^2)$, $(x,\xi) \in \mathbb{R}^{2n}$, the heat equation, where $q(x,\xi) = |\xi|^2$, $(x,\xi) \in \mathbb{R}^{2n}$, and the Kramers-Fokker-Planck equation with a quadratic potential, where $q(x,v,\xi,\eta) = \eta^2 + \frac{1}{4}v^2 + i(v \cdot \xi - ax \cdot \eta)$, for $(x,v,\xi,\eta) \in \mathbb{R}^{4n} = \mathbb{R}^{2n}_{x,v} \times \mathbb{R}^{2n}_{\xi,\eta}$ and $a \in \mathbb{R}\backslash\{0\}$ a constant. From the work [31], it is known that the operator $q^w(x,D)$, regarded as an unbounded operator on $L^2(\mathbb{R}^n)$ equipped with the maximal domain

$$D_{\max} = \left\{u \in L^2(\mathbb{R}^n) : q^w(x,D)u \in L^2(\mathbb{R}^n)\right\}, \tag{4.1.4}$$

is maximally accretive and generates a strongly continuous contraction semigroup $G(t) := e^{-tq^w(x,D)}$, $t \geq 0$, on $L^2(\mathbb{R}^n)$. We may regard $G(t)$ as the solution operator for the problem (4.1.1). Given that a wide range of physical processes may be modeled by equations of the form (4.1.1), it is of interest to understand the $L^p \to L^q$ mapping properties of the evolution semigroup $G(t)$ and to obtain bounds for the operator norm $\|G(t)\|_{L^p \to L^q}$ at various time regimes. Let us mention that the study of L^p-bounds for semigroups generated by self-adjoint Schrödinger operators has a long and rich tradition in the field of mathematical physics. We refer to [17], [18], [47], and [48] for some fundamental results in this area. In particular, L^p-bounds for the propagator $G(t)$ were obtained in [34] in the case when (4.1.1) is the time evolution of the quantum harmonic oscillator. We also mention that the topic of L^p-bounds for operators with Gaussian kernels is a classical subject. In particular, it is known that the that the $L^p \to L^q$ norm of an operator on $\mathbb{R}^n$ with a Gaussian kernel must be realized by a Gaussian. For more information, see [37].

In this note, we shall be primarily interested in obtaining $L^p \to L^q$ bounds for $G(t)$ in the case when the quadratic form q is non-elliptic. In order to recount the known results in this direction, we pause to recall the notion of the singular space of a complex-valued quadratic form q on $\mathbb{R}^{2n}$ with non-negative real part $\operatorname{Re} q \geq 0$. Let $\mathbb{R}^{2n}$ be equipped with the standard symplectic form

$$\sigma((x,\xi),(y,\eta)) = \xi \cdot y - x \cdot \eta, \quad (x,\xi),\ (y,\eta) \in \mathbb{R}^{2n}. \tag{4.1.5}$$

Suppose $q : \mathbb{R}^{2n} \to \mathbb{C}$ is a complex-valued quadratic form with $\operatorname{Re} q \geq 0$ and let $q(\cdot,\cdot)$ denote its symmetric $\mathbb{C}$-bilinear polarization. Because σ is nondegenerate, there is a unique $F \in \operatorname{Mat}_{2n\times 2n}(\mathbb{C})$ such that

$$q((x,\xi),(y,\eta)) = \sigma((x,\xi),F(y,\eta)) \tag{4.1.6}$$

for all $(x,\xi),(y,\eta) \in \mathbb{R}^{2n}$. This matrix F is called the *Hamilton map* or *Hamilton matrix of* q (see Section 21.5 of [32]). Explicitly, the Hamilton matrix of q is given by

$$F = \frac{1}{2} H_q, \tag{4.1.7}$$

where $H_q = \left(q'_\xi, -q'_x\right)$ is the Hamilton vector field of q, viewed as a linear map $\mathbb{C}^{2n} \to \mathbb{C}^{2n}$. Let

$$\operatorname{Re} F = \frac{F + \overline{F}}{2}, \quad \operatorname{Im} F = \frac{F - \overline{F}}{2i}$$

be the real and imaginary parts of F respectively. The *singular space* S *of* q is defined as the following finite intersection of kernels:

$$S = \left(\bigcap_{j=0}^{2n-1} \ker \left[(\operatorname{Re} F)(\operatorname{Im} F)^j \right] \right) \cap \mathbb{R}^{2n}. \tag{4.1.8}$$

The singular space was first introduced by M. Hitrik and K. Pravda-Starov in [23] where it arose naturally in the study of spectra and semi-group smoothing properties for non-self adjoint quadratic differential operators. The concept of the singular space has since been shown to play a key role in the understanding of hypoelliptic and spectral properties of non-elliptic quadratic differential operators. See for instance [24], [25], [41], [42], [57], and [58]. Recent work has also shown that the singular space is vital for the description of the propagation of microlocal singularities for the evolution (4.1.1). We refer the reader to [11], [43], [44], [59], and [61], as well as [3] and [4].

Let q be a complex-valued quadratic form on $\mathbb{R}^{2n}$ with non-negative real part $\operatorname{Re} q \geq 0$. Let S be the singular space of q. The quadratic form q is said to be *elliptic* if

$$q(X) = 0, \quad X \in \mathbb{R}^{2n} \implies X = 0, \tag{4.1.9}$$

If (4.1.9) fails to hold, then we say that q is *non-elliptic*. To the best of our knowledge, there are currently only two general results regarding $L^p \to L^q$ bounds for the semigroup $G(t)$ in the case when q is non-elliptic. First, in Theorem 1.2.3 of [23], it was established that $\|G(t)\|_{L^2 \to L^2}$ decays exponentially as $t \to \infty$ whenever S is symplectic and distinct from the entire phase space $\mathbb{R}^{2n}$. In other words, if S is symplectic and $S \neq \mathbb{R}^{2n}$, then there are $C, c > 0$ such that

$$\|G(t)\|_{L^2 \to L^2} \leq Ce^{-ct}, \quad t \geq 0. \tag{4.1.10}$$

Thanks to the subsequent work [40], it is also known that if S is trivial, i.e. $S = \{0\}$, then the optimal rate of exponential decay of $\|G(t)\|_{L^2 \to L^2}$ is the quantity γ defined below in Theorem 4.1.1. The second general result concerning $L^p - L^q$ bounds for $G(t)$ is Theorem 1.2 of [26], which yields the following $L^2 - L^\infty$ estimate: if $S = \{0\}$, then, for every $s > n/2$, there is $C > 0$ such that

$$\|G(t)\|_{L^2 \to L^\infty} \leq Ct^{-\frac{1}{2}(2k_0+1)(2n+s)}, \quad 0 < t \ll 1, \tag{4.1.11}$$

where $k_0 \in \{0, 1, \ldots, 2n-1\}$ is the smallest non-negative integer such that

$$\bigcap_{j=0}^{k_0} \ker \left[(\mathrm{Re}\ F)(\mathrm{Im}\ F)^j\right] \cap \mathbb{R}^{2n} = \{0\}. \tag{4.1.12}$$

Our goal in the present work is to prove bounds for the operator norm $\|G(t)\|_{L^p \to L^q}$ with (p, q) more general than $(2, 2)$ and $(2, \infty)$. The main result of this note refines and extends the bounds

(4.1.10) and (4.1.11) under the assumption that $S = \{0\}$. We recall from Theorem 1.2.2 of [23] that when $S = \{0\}$ the spectrum of the quadratic differential operator $q^w(x, D)$ is only composed of eigenvalues of finite algebraic multiplicity with

$$\operatorname{Spec}(q^w(x,D)) = \left\{ \sum_{\substack{\lambda \in \operatorname{Spec}(F) \\ \operatorname{Im}(\lambda) > 0}} (r_\lambda + 2k_\lambda)(-i\lambda) : k_\lambda \in \mathbb{N} \right\}, \tag{4.1.13}$$

where r_λ is the dimension of the space of generalized eigenvectors of the Hamilton matrix F of q in $\mathbb{C}^{2n}$ corresponding to the eigenvalue $\lambda \in \mathbb{C}$. In particular, the eigenvalue of $q^w(x, D)$ obtained by setting $k_\lambda = 0$ for all $\lambda \in \operatorname{Spec}(F)$ in (4.1.13) is

$$\rho = \sum_{\substack{\lambda \in \operatorname{Spec}(F) \\ \operatorname{Im}(\lambda) > 0}} -ir_\lambda \lambda. \tag{4.1.14}$$

We may think of ρ as the 'lowest eigenvalue' or 'ground state energy' of the operator $q^w(x, D)$.

Theorem 4.1.1. *Let q, $q^w(x, D)$, $G(t)$, S, and F be as above. Assume that $S = \{0\}$.*

1. *Let $\gamma = Re(\rho) > 0$. For every $1 \le p \le q \le \infty$ and $\epsilon > 0$, there are constants $C = C_{\epsilon,p,q} > 0$ and $c = c_{p,q} > 0$, such that*

$$ce^{-\gamma t} \le \|G(t)\|_{L^p \to L^q} \le Ce^{-\gamma t}, \quad t \ge \epsilon. \tag{4.1.15}$$

2. *Let $k_0 \in \{0, 1, \ldots, 2n-1\}$ be the smallest non-negative integer such that (4.1.12) holds. There is a time $0 < t_0 \ll 1$ such that for any $1 \le p \le q \le \infty$ we have*

$$c \le \|G(t)\|_{L^p \to L^q} \le Ct^{-(2k_0+1)n}, \quad 0 < t \le t_0, \tag{4.1.16}$$

for some constants $C = C_{p,q} > 0$ and $c = c_{p,q} > 0$.

Remark 4.1.2. For any $1 \le p \le q \le \infty$, it is actually true that there is a constant $c = c_{p,q} > 0$

such that

$$ce^{-\gamma t} \leq \|G(t)\|_{L^p \to L^q}, \quad 0 \leq t < \infty. \tag{4.1.17}$$

In fact, we have $c \geq \|v\|_{L^q}$, where $v \in \mathcal{S}(\mathbb{R}^n)$ is the L^p-normalized 'ground state' for the operator $q^w(x, D)$. For a proof, see the derivation of (4.4.15) below.

Let us make some general comments regarding Theorem 4.1.1 First, the bounds (4.1.15) show that for any $1 \leq p \leq q \leq \infty$ the operator norm $\|G(t)\|_{L^p \to L^q}$ decays exponentially as $t \to \infty$, with γ being the precise rate of decay, independent of (p, q). To prove that γ is the exact rate of exponential decay, one may examine the action of the propagator $G(t)$ on the 'ground state' eigenfunction of $q^w(x, D)$ corresponding to the eigenvalue ρ (see Section 4 below). Regarding the short time $0 < t \ll 1$ bounds in Theorem 4.1.1, it is clear that (4.1.16) is not sharp for all $1 \leq p \leq q \leq \infty$. For instance, (4.1.16) fails to reproduce (4.1.10) when $p = q = 2$. However, one may interpolate (4.1.16) with the bound $G(t) = \mathcal{O}_{L^2 \to L^2}(1)$ as $t \to 0^+$ to obtain more precise estimates at short times. We also note that when $(p, q) = (2, \infty)$, the bound (4.1.16) gives $G(t) = \mathcal{O}_{L^2 \to L^\infty}(t^{-(2k_0+1)n})$ as $t \to 0^+$, which is an improvement over (4.1.11).

Finally, let us briefly touch on the main ideas involved in the proof of Theorem 4.1.1. In the recent work [61], we showed that if $\mathcal{T}_\varphi$ is a global metaplectic FBI transform on $\mathbb{R}^n$, in the sense of either Chapter 13 of [63] or the minicourse [27], then the conjugated propagator $\widetilde{G}(t) := \mathcal{T}_\varphi \circ G(t) \circ \mathcal{T}_\varphi^*$ is, for each $t \geq 0$, a metaplectic Fourier integral operator acting on the Bargmann space $H_{\Phi_0}(\mathbb{C}^n)$, which is the unitary image of $L^2(\mathbb{R}^n)$ under $\mathcal{T}_\varphi$. In particular, we showed that the 'Bergman form' ([12], [56]) of $\widetilde{G}(t)$ is given by

$$\widetilde{G}(t)u(z) = \hat{a}(t) \int_{\mathbb{C}^n} e^{2\Psi_t(z, \overline{w})} u(w) e^{-2\Phi_0(w)}\, L(dw), \quad z \in \mathbb{C}^n, \quad u \in H_{\Phi_0}(\mathbb{C}^n), \quad t \geq 0, \tag{4.1.18}$$

where $L(dw)$ is the Lebesgue measure on $\mathbb{C}^n$, $\Phi_0(w) := \sup_{y \in \mathbb{R}^n}(-\mathrm{Im}\ \varphi(w, y))$, $w \in \mathbb{C}^n$, is the strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ associated to φ, Ψ_t is a holomorphic quadratic form on $\mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n$ depending analytically on $t \geq 0$, and $\hat{a} \in C^\omega([0, \infty); \mathbb{C})$ is a non-vanishing amplitude. Moreover, we showed that Ψ_t and $\hat{a}$ are the solutions of an eikonal equation and a transport equation, respectively. In particular, we did not attempt to solve these equations explicitly for Ψ_t and $\hat{a}$. Now, thanks to the work [1], it is known that when the singular space is trivial $S = \{0\}$ it is possible to choose a metaplectic FBI transform $\mathcal{T}_\varphi$ so that conjugated semigroup has the simple form

$$\widetilde{G}(t)u(z) = e^{\frac{i}{2}\mathrm{tr}(M)t}u(e^{itM}z), \quad u \in H_{\Phi_0}(\mathbb{C}^n),\ t \geq 0, \tag{4.1.19}$$

where $M \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is a suitable matrix. In the present work, we show that this choice of $\mathcal{T}_\varphi$ leads to equations for Ψ_t and $\hat{a}$ that may be easily solved. One may then show that (4.1.18) coincides with (4.1.19), giving an alternative derivation of (4.1.19). Once the Bergman form of $\widetilde{G}(t)$ is known and a basic estimate for the real part of its phase function is established, the bounds (4.1.15) and (4.1.16) follow easily by writing down a formal expression for the Schwartz kernel of the composition $\mathcal{T}_\varphi^* \circ \widetilde{G}(t) \circ \mathcal{T}_\varphi$ using (4.1.18) and applying Young's integral inequality.

The plan for this chapter is as follows. In Section 2, we recall how to choose the FBI transform $\mathcal{T}_\varphi$ so that (4.1.19) holds. In Section 3, we determine the Bergman form (4.1.18) of $\widetilde{G}(t)$ for $t \geq 0$ and prove some basic estimates. In Section 4, we conclude the proof of Theorem 4.1.1, as outlined in this introduction.

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4.2 Reduction to a Normal Form on the FBI Transform Side

In this section, we follow the approach of [28] and [58] for reducing $q^w(x, D)$ to a normal form via a metaplectic FBI transform. We provide additional references where convenient.

Let q be a complex-valued quadratic form on $\mathbb{R}^{2n}$ with non-negative real part $\mathrm{Re}\, q \geq 0$ and trivial singular space $S = \{0\}$. Let $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_\zeta$ be equipped with the standard complex symplectic form $\sigma = d\zeta \wedge dz$. Let F be the Hamilton matrix of q introduced in (4.1.6). From the work [23], it is known that the matrix F has no real eigenvalues. Consequently,

$$\#\left\{\lambda \in \mathrm{Spec}(F) : \mathrm{Im}\,\lambda > 0\right\} = \#\left\{\lambda \in \mathrm{Spec}(F) : \mathrm{Im}\,\lambda < 0\right\}, \tag{4.2.1}$$

counting algebraic multiplicities. For $\lambda \in \mathrm{Spec}(F)$, let

$$V_\lambda = \ker\left((F-\lambda)^{2n}\right) \subset \mathbb{C}^{2n} \tag{4.2.2}$$

be the generalized eigenspace of F corresponding to λ. Let us also introduce the stable outgoing and stable incoming manifolds for the quadratic form $-iq$ given by

$$\Lambda^+ = \bigoplus_{\substack{\lambda\in\mathrm{Spec}(F)\\ \mathrm{Im}\ \lambda>0}} V_\lambda, \quad \Lambda^- = \bigoplus_{\substack{\lambda\in\mathrm{Spec}(F)\\ \mathrm{Im}\ \lambda<0}} V_\lambda, \tag{4.2.3}$$

respectively. By Proposition 2.1 of [58], Λ^+ is a strictly positive $\mathbb{C}$-Lagrangian subspace of $\mathbb{C}^{2n}$ in the sense that Λ^+ is Lagrangian with respect to the complex symplectic form σ and

$$\frac{1}{i}\sigma(Z,\overline{Z}) > 0, \quad Z \in \Lambda^+\backslash\{0\}, \tag{4.2.4}$$

and Λ^- is a strictly negative $\mathbb{C}$-Lagrangian subspace of $\mathbb{C}^{2n}$ in the sense that Λ^- is Lagrangian for the form σ and (4.2.4) holds for all $Z \in \Lambda^-\backslash\{0\}$ with '>' replaced by '<'. For background information regarding positive and negative $\mathbb{C}$-Lagrangian subspaces of $\mathbb{C}^{2n}$, we refer to either [27] or [12]. In particular, from the discussion on pages 488-489 of [27], we know that there exists a holomorphic quadratic form $\varphi = \varphi(z,y)$ on $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_y$ with

$$\det \varphi''_{zy} \neq 0, \quad \mathrm{Im}\ \varphi''_{yy} > 0, \tag{4.2.5}$$

such that the complex linear canonical transformation

$$\kappa_\varphi : \mathbb{C}^{2n} \ni (y, -\varphi'_y(z,y)) \mapsto (z, \varphi'_z(z,y)) \in \mathbb{C}^{2n}, \quad (z,y) \in \mathbb{C}^{2n}, \tag{4.2.6}$$

generated by φ satisfies

$$\kappa_{\varphi}(\Lambda^{+}) = \{(z,0) : z \in \mathbb{C}^n\}, \quad \kappa_{\varphi}(\Lambda^{-}) = \{(0,\zeta) : \zeta \in \mathbb{C}^n\}. \tag{4.2.7}$$

Let

$$\Phi_0(z) = \sup_{y\in\mathbb{R}^n} \left(-\mathrm{Im}\ \varphi(z,y)\right), \quad z \in \mathbb{C}^n, \tag{4.2.8}$$

be the strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ associated to the phase φ (see Chapter 13 of [63] or Section 1.3 of [27]), and let

$$\Lambda_{\Phi_0} = \left\{ \left(z, \frac{2}{i}\Phi'_{0,z}(z) \right) : z \in \mathbb{C}^n \right\}. \tag{4.2.9}$$

From either Theorem 13.5 of [63] or Proposition 1.3.2 of [27], we have

$$\kappa_{\varphi}\left(\mathbb{R}^{2n}\right) = \Lambda_{\Phi_0}, \tag{4.2.10}$$

and thus Λ_{Φ_0} is I-Lagrangian and R-symplectic for the complex symplectic form σ. Also, the strict positivity of Λ^{+} in conjunction with (4.2.7) gives that the base $\{(z,0) : z \in \mathbb{C}^n\}$ is strictly positive relative to Λ_{Φ_0} (see e.g. [12]). It then follows, as explained in Chapter 11 of [51], that the quadratic form Φ_0 is strictly convex.

Let

$$\tilde{q} = q \circ \kappa_{\varphi}^{-1}, \tag{4.2.11}$$

regarded as a holomorphic quadratic form on $\mathbb{C}^{2n}$. Since Λ^{+} and Λ^{-} are invariant under F and Lagrangian with respect to σ, we have

$$q(X) = \sigma(X, FX) = 0, \quad X \in \Lambda^{+} \cup \Lambda^{-}. \tag{4.2.12}$$

From (4.2.7) and (4.2.11), it follows that $\tilde{q}$ must be of the form

$$\tilde{q}(z,\zeta) = Mz \cdot \zeta, \quad (z,\zeta) \in \mathbb{C}^{2n}, \tag{4.2.13}$$

for some $M \in \mathrm{Mat}_{n\times n}(\mathbb{C}^n)$. In particular, the complex Hamilton vector field of $\tilde{q}$ with respect to σ is

$$H_{\tilde{q}} = \left(Mz, -M^T\zeta\right), \quad (z,\zeta) \in \mathbb{C}^{2n}. \tag{4.2.14}$$

The Hamilton map of $\tilde{q}$ is thus given by $\tilde{F} = \frac{1}{2}H_{\tilde{q}}$, and we have

$$\tilde{F} = \frac{1}{2}\begin{pmatrix} M & 0 \\ 0 & -M^T \end{pmatrix}. \tag{4.2.15}$$

As a consequence of (4.2.11), (4.1.6), and the invariance of σ under κ_φ, it is true that $\tilde{F} = \kappa_\varphi \circ F \circ \kappa_\varphi^{-1}$. Since also $\tilde{F}$ maps $(z,0) \in \kappa_\varphi(\Lambda^+)$ to $\frac{1}{2}(Mz,0) \in \kappa_\varphi(\Lambda^+)$, we have

$$\mathrm{Spec}(M) = \mathrm{Spec}(2F) \cap \{\mathrm{Im}\,\lambda > 0\}, \tag{4.2.16}$$

with agreement of algebraic multiplicities.

Let $\mathcal{T}_\varphi : \mathcal{S}'(\mathbb{R}^n) \to \mathrm{Hol}(\mathbb{C}^n)$ be the metaplectic FBI transform on $\mathbb{R}^n$ associated to φ, given in the sense of distributions by

$$\mathcal{T}_\varphi u(z) = c_\varphi \int_{\mathbb{R}^n} e^{i\varphi(z,y)} u(y)\, L(dy), \quad u \in \mathcal{S}'(\mathbb{R}^n), \tag{4.2.17}$$

where

$$c_\varphi = 2^{-n/2}\pi^{-3n/4}(\det \mathrm{Im}\, \varphi''_{yy})^{-1/4} \left|\det \varphi''_{zy}\right|. \tag{4.2.18}$$

By Theorem 13.7 of [63], $\mathcal{T}_\varphi$ is unitary $L^2(\mathbb{R}^n) \to H_{\Phi_0}(\mathbb{C}^n)$, where

$$H_{\Phi_0}(\mathbb{C}^n) := L^2(\mathbb{C}^n, e^{-2\Phi_0(z)}\, L(dz)) \cap \mathrm{Hol}(\mathbb{C}^n) \tag{4.2.19}$$

is the Bargmann space associated to the weight Φ_0, equipped with the natural Hilbert space structure inherited from $L^2(\mathbb{C}^n, e^{-2\Phi_0(z)}\, L(dz))$. Here $L(dz)$ denotes the Lebesgue measure on $\mathbb{C}^n$. Let $\tilde{q}^w(z, D)$ denote the complex Weyl quantization of the symbol $\tilde{q}$ with respect to the weight Φ_0. We recall that $\tilde{q}^w(z, D)$ is defined as an unbounded operator on $H_{\Phi_0}(\mathbb{C}^n)$ that acts on suitable $u \in H_{\Phi_0}(\mathbb{C}^n)$ by

$$\tilde{q}^w(z, D)u(z) = \frac{1}{(2\pi)^n} \iint_{\Gamma_{\Phi_0}(z)} e^{i(z-w)\cdot\zeta} \tilde{q}^w\left(\frac{z+w}{2}, \zeta\right) u(w)\, dw \wedge d\zeta, \quad z \in \mathbb{C}^n, \tag{4.2.20}$$

for the contour of integration

$$\Gamma_{\Phi_0}(z) : w \mapsto \zeta = \frac{2}{i}\Phi'_{0,z}\left(\frac{z+w}{2}\right), \quad w \in \mathbb{C}^n, \quad z \in \mathbb{C}^n. \tag{4.2.21}$$

For more information on Weyl quantization in the complex domain, see Chapter 13 of [63] or Section 1.4 of [27]. By Egorov's theorem (see Theorem 13.9 in [63] or Theorem 1.4.2 of [27]), we have

$$q^w(x, D) = \mathcal{T}_\varphi^* \circ \tilde{q}^w(z, D) \circ \mathcal{T}_\varphi \tag{4.2.22}$$

when both sides are viewed as operators acting on the maximal domain of $q^w(x, D)$,

$$D_{\max} = \left\{u \in L^2(\mathbb{R}^n) : q^w(x, D)u \in L^2(\mathbb{R}^n)\right\}. \tag{4.2.23}$$

Let

$$\widetilde{D}_{\max} = \{u \in H_{\Phi_0}(\mathbb{C}^n) : \tilde{q}^w(z, D)u \in H_{\Phi_0}(\mathbb{C}^n)\} \tag{4.2.24}$$

be the maximal domain of $\tilde{q}^w(z, D)$, and let us view $\tilde{q}^w(z, D)$ as an unbounded operator on $H_{\Phi_0}(\mathbb{C}^n)$

with the domain $\widetilde{D}_{\max}$. Thanks to (4.2.22), we have

$$\widetilde{D}_{\max} = \mathcal{T}_{\varphi}(D_{\max}). \tag{4.2.25}$$

Let $G(t) = e^{-tq^w(x,D)}$, $t \geq 0$, be the strongly continuous semigroup on $L^2(\mathbb{R}^n)$ generated by $q^w(x,D)$ (see [31]). From (4.2.22), (4.2.25), and the unitarity of $\mathcal{T}_{\varphi}$, it follows that $\tilde{q}^w(z,D)$ generates a strongly continuous semigroup $\widetilde{G}(t) = e^{-t\tilde{q}^w(z,D)}$, $t \geq 0$, on $H_{\Phi_0}(\mathbb{C}^n)$. The semigroups $G(t)$ and $\widetilde{G}(t)$ are related by

$$G(t) = \mathcal{T}_{\varphi}^* \circ \widetilde{G}(t) \circ \mathcal{T}_{\varphi} \tag{4.2.26}$$

for all $t \geq 0$.

We have established the following proposition, which summarizes the discussion in this section.

Proposition 4.2.1. *Let q be a complex-valued quadratic form on $\mathbb{R}^{2n}$ with non-negative real part $\mathrm{Re}\, q \geq 0$ and trivial singular space $S = \{0\}$. Let F be the Hamilton matrix of q, and let $q^w(x,D)$ be the Weyl quantization of q, viewed as an unbounded operator on $L^2(\mathbb{R}^n)$ equipped with its maximal domain D_{max} defined in (4.2.23). Let $G(t) = e^{-tq^w(x,D)}$, $t \geq 0$, be the strongly continuous semigroup on $L^2(\mathbb{R}^n)$ generated by $q^w(x,D)$.*

1. *There exists a holomorphic quadratic form φ on $\mathbb{C}^{2n}$ satisfying (4.2.5) such that the quadratic form Φ_0 defined by (4.2.8) is strictly convex and the complex linear canonical transformation $\kappa_{\varphi} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ defined implicitly by (4.2.6) has the property that*

$$\tilde{q}(z,\zeta) := \left(q \circ \kappa_{\varphi}^{-1}\right)(z,\zeta) = Mz \cdot \zeta, \quad (z,\zeta) \in \mathbb{C}^{2n}, \tag{4.2.27}$$

where $M \in Mat_{n\times n}(\mathbb{C})$ is such that $Spec(M) = Spec(2F) \cap \{Im\ \lambda > 0\}$ with agreement of algebraic multiplicities.

2. *Let $\tilde{q}^w(z,D)$ be the complex Weyl quantization (4.2.20) of $\tilde{q}$ with respect to the weight Φ_0, realized as an unbounded operator on the Bargmann space $H_{\Phi_0}(\mathbb{C}^n)$ introduced in (4.2.19)*

equipped with the maximal domain $\widetilde{D}_{max}$ defined in (4.2.24). The operator $\tilde{q}^w(z,D)$ generates a strongly continuous semigroup $\widetilde{G}(t) = e^{-t\tilde{q}^w(z,D)}$, $t \geq 0$, on $H_{\Phi_0}(\mathbb{C}^n)$ that is unitarily equivalent to $G(t)$ for each $t \geq 0$. This unitary equivalence is given by the FBI transform $\mathcal{T}_\varphi$ introduced in (4.2.17), i.e.

$$G(t) = \mathcal{T}_\varphi^* \circ \widetilde{G}(t) \circ \mathcal{T}_\varphi, \quad t \geq 0. \tag{4.2.28}$$

4.3 The Evolution Semigroup on the FBI Transform Side

We now study the semigroup $\widetilde{G}(t)$, $t \geq 0$. Let Ψ_0 be the polarization of Φ_0, i.e. Ψ_0 is the unique holomorphic quadratic form on $\mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n$ such that $\Psi_0(z,\overline{z}) = \Phi_0(z)$ for all $z \in \mathbb{C}^n$. Since

$$\Phi_0(z) = \frac{1}{2}\Phi''_{0,zz}z \cdot z + \Phi''_{0,\overline{z}z}z \cdot \overline{z} + \frac{1}{2}\Phi''_{0,\overline{z}\,\overline{z}}\overline{z} \cdot \overline{z}, \quad z \in \mathbb{C}^n, \tag{4.3.1}$$

we see that Ψ_0 is given explicitly by

$$\Psi_0(z,\theta) = \frac{1}{2}\Phi''_{0,zz}z \cdot z + \Phi''_{0,\overline{z}z}z \cdot \theta + \frac{1}{2}\Phi''_{0,\overline{z}\,\overline{z}}\theta \cdot \theta, \quad (z,\theta) \in \mathbb{C}^{2n}. \tag{4.3.2}$$

In the work [61], we showed that for every $t \geq 0$ the semigroup $\widetilde{G}(t)$ is a metaplectic Fourier integral operator in the complex domain whose underlying complex canonical transformation is the Hamilton flow $\tilde{\kappa}_t$ of the symbol $\tilde{q}$ at time t/i, i.e.

$$\tilde{\kappa}_t = \exp\left(\frac{t}{i}H_{\tilde{q}}\right), \quad t \geq 0. \tag{4.3.3}$$

In view of (4.2.14), we have

$$\tilde{\kappa}_t(z,\zeta) = \left(e^{-itM}z, e^{itM^T}\zeta\right), \quad (z,\zeta) \in \mathbb{C}^{2n}, \; t \geq 0. \tag{4.3.4}$$

For background information regarding metaplectic Fourier integral operators in the complex domain, see Appendix B of [9]. In particular, in the work [12], it was shown that every such metaplectic

Fourier integral operator in $\mathbb{C}^n$ possesses a unique 'Bergman form.' In Section 6 of [61], we proved that the Bergman form of $\tilde{G}(t)$ is given by

$$\widetilde{G}(t)u(z) = \hat{a}(t)\int_{\mathbb{C}^n} e^{2\Psi_t(z,\overline{w})}u(w)e^{-2\Phi_0(w)}\,L(dw), \quad z\in\mathbb{C}^n, \quad u\in H_{\Phi_0}(\mathbb{C}^n), \tag{4.3.5}$$

where Ψ_t is a holomorphic quadratic form on $\mathbb{C}^{2n}$, depending analytically on $t\geq 0$, and $\hat{a}\in C^\omega([0,\infty);\mathbb{C})$ is a non-vanishing amplitude. In addition, we showed that Ψ_t, $t\geq 0$, is the unique solution of the eikonal equation

$$\begin{cases} 2\partial_t\Psi_t(z,\theta) + \tilde{q}\left(z, \frac{2}{i}\Psi'_{t,z}(z,\theta)\right) = 0, \quad (z,\theta)\in\mathbb{C}^{2n}, \quad t\geq 0, \\ \Psi_t(z,\theta)|_{t=0} = \Psi_0(z,\theta), \quad (z,\theta)\in\mathbb{C}^{2n}, \end{cases} \tag{4.3.6}$$

and $\hat{a}$ is the unique solution of the transport equation

$$\begin{cases} \hat{a}'(t) + \frac{1}{2i}\beta(t)\hat{a}(t) = 0, \quad t\geq 0, \\ \hat{a}(0) = C_{\Phi_0}, \end{cases} \tag{4.3.7}$$

where

$$\beta(t) = \operatorname{tr}\left(\tilde{q}''_{\zeta z} + \tilde{q}''_{\zeta\zeta}\cdot\frac{2}{i}\Psi''_{t,zz}\right), \quad t\geq 0, \tag{4.3.8}$$

and

$$C_{\Phi_0} = 2^n\pi^{-n}\det\Phi''_{0,z\overline{z}}. \tag{4.3.9}$$

We note that the initial conditions in (4.3.6) and (4.3.7) are chosen so that when $t=0$ the righthand side of (4.3.5) coincides with the orthogonal projector $\Pi_{\Phi_0} : L^2(\mathbb{C}^n, e^{-2\Phi_0(z)}\,L(dz)) \to H_{\Phi_0}(\mathbb{C}^n)$, which has the explicit integral represenation

$$\Pi_{\Phi_0}u(z) = C_{\Phi_0}\int_{\mathbb{C}^n} e^{2\Psi_0(z,\overline{w})}u(w)e^{-2\Phi_0(w)}\,L(dw), \quad u\in L^2(\mathbb{C}^n, e^{-2\Phi(z)}\,L(dz)). \tag{4.3.10}$$

In the literature, the operator Π_{Φ_0} is known as the 'Bergman projector' associated to the weight Φ_0. For a proof of (4.3.10), see Theorem 13.6 of [63] or Proposition 1.3.4 of [27].

Since $\tilde{q}$ has the simple form (4.2.13), we may determine Ψ_t and $\hat{a}$ by solving (4.3.6) and (4.3.7) explicitly. We begin by studying the transport equation (4.3.7). Thanks to (4.2.13), we see that

$$\beta(t) = \operatorname{tr}(M), \quad t \geq 0. \tag{4.3.11}$$

The unique solution of (4.3.7) is

$$\hat{a}(t) = C_{\Phi_0} e^{\frac{i}{2}\operatorname{tr}(M)t}, \quad t \geq 0. \tag{4.3.12}$$

Next, we solve (4.3.6) for Ψ_t. We search for a solution to (4.3.6) of the form

$$\Psi_t(z,\theta) = \frac{1}{2}A_t z\cdot z + B_t z\cdot\theta + \frac{1}{2}D_t\theta\cdot\theta, \quad (z,\theta)\in\mathbb{C}^{2n}, \quad t\geq 0, \tag{4.3.13}$$

where $A_t, B_t, D_t \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ depend smoothly on t and $A_t = A_t^T$ and $D_t = D_t^T$ for all $t \geq 0$. Inserting (4.3.13) into (4.3.6) and using (4.2.13) and (4.3.2), we see that Ψ_t will be a solution of (4.3.6) provided A_t, B_t, and D_t satisfy

$$\begin{cases} \partial_t A_t z\cdot z + \frac{2}{i}A_t M z\cdot z = 0, \quad z\in\mathbb{C}^n,\ t\geq 0, \\ A_0 = \Phi''_{0,zz}, \end{cases} \tag{4.3.14}$$

$$\begin{cases} \partial_t B_t z\cdot\theta + \frac{1}{i}B_t M z\cdot\theta = 0, \quad z,\theta\in\mathbb{C}^n,\ t\geq 0, \\ B_0 = \Phi''_{0,\bar{z}z}, \end{cases} \tag{4.3.15}$$

and

$$
\begin{cases} \partial_t C_t \theta \cdot \theta = 0, \quad \theta \in \mathbb{C}^n,\ t \geq 0, \\ C_0 = \Phi''_{0,\overline{zz}}, \end{cases} \tag{4.3.16}
$$

respectively. The symmetry of A_t implies that

$$
2A_t M z \cdot z = \left(A_t M + M^T A_t\right) z \cdot z, \quad z \in \mathbb{C}^n,\ t \geq 0. \tag{4.3.17}
$$

Thus (4.3.14) holds if and only if

$$
\begin{cases} \partial_t A_t + \frac{1}{i} A_t M + \frac{1}{i} M^T A_t = 0, \quad t \geq 0, \\ A_0 = \Phi''_{0,zz}. \end{cases} \tag{4.3.18}
$$

The unique solution of (4.3.18) is

$$
A_t = e^{iM^T t} \Phi''_{0,zz} e^{iMt}, \quad t \geq 0. \tag{4.3.19}
$$

By inspection, the solutions of (4.3.15) and (4.3.16) are

$$
B_t = \Phi''_{0,\overline{z}z} e^{itM},\ C_t = \Phi''_{0,\overline{zz}}, \quad t \geq 0, \tag{4.3.20}
$$

respectively. Using (4.3.2), we get

$$
\Psi_t(z, \theta) = \Psi_0(e^{itM} z, \theta), \quad (z, \theta) \in \mathbb{C}^{2n},\ t \geq 0. \tag{4.3.21}
$$

From (4.3.5), (4.3.10), (4.3.12), and (4.3.21), we deduce that

$$
\widetilde{G}(t) u(z) = e^{\frac{i}{2}\operatorname{tr}(M) t} u\left(e^{itM} z\right), \quad u \in H_{\Phi_0}(\mathbb{C}^n),\ t \geq 0. \tag{4.3.22}
$$

The formula (4.3.22) for the semigroup $\widetilde{G}(t)$ was obtained by a different method in [1].

For $t \geq 0$, let us define

$$\Phi_t(z) = \Phi_0\left(e^{itM} z\right), \quad z \in \mathbb{C}^n,\ t \geq 0. \tag{4.3.23}$$

Since Φ_0 is strictly convex, Φ_t is a strictly convex quadratic form on $\mathbb{C}^n$ for all $t \geq 0$. In addition, we have $\Phi_t|_{t=0} = \Phi_0$. For $t \geq 0$, let

$$H_{\Phi_t}(\mathbb{C}^n) = L^2\left(\mathbb{C}^n, e^{-2\Phi_t(z)}\, L(dz)\right) \cap \mathrm{Hol}(\mathbb{C}^n) \tag{4.3.24}$$

be the Bargmann space associated to Φ_t, equipped with the natural Hilbert space structure induced from $L^2\left(\mathbb{C}^n, e^{-2\Phi_t(z)}\, L(dz)\right)$. From (4.3.22), it is clear that $\widetilde{G}(t)$ is bounded $H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_t}(\mathbb{C}^n)$ for every $t \geq 0$, and a direct computation using (4.3.22), (4.3.23), and (4.2.16) gives

$$\|\widetilde{G}(t)u\|_{H_{\Phi_t}(\mathbb{C}^n)} = e^{\gamma t}\|u\|_{H_{\Phi_0}(\mathbb{C}^n)}, \quad u \in H_{\Phi_0}(\mathbb{C}^n),\ t \geq 0, \tag{4.3.25}$$

where $\gamma > 0$ is as in the statement of Theorem 4.1.1.

The following proposition summarizes the discussion so far in this section and establishes some basic estimates that will be necessary for the proof of Theorem 4.1.1 in Section 4.

Proposition 4.3.1. *Let q, $\tilde{q}$, M, Φ_0, $H_{\Phi_0}(\mathbb{C}^n)$, and $\widetilde{G}(t)$ be as in Proposition 4.2.1.*

1. *For every $t \geq 0$, we have*

$$\widetilde{G}(t)u(z) = e^{\frac{i}{2} tr(M)t} u\left(e^{itM} z\right), \quad u \in H_{\Phi_0}(\mathbb{C}^n). \tag{4.3.26}$$

In addition,

$$\|\widetilde{G}(t)u\|_{H_{\Phi_t}(\mathbb{C}^n)} = e^{\gamma t}\|u\|_{H_{\Phi_0}(\mathbb{C}^n)}, \quad t \geq 0, \tag{4.3.27}$$

where

$$\Phi_t(z) = \Phi_0\left(e^{itM} z\right), \quad z \in \mathbb{C}^n,\ t \geq 0, \tag{4.3.28}$$

the norm $\|\cdot\|_{H_{\Phi_t}(\mathbb{C}^n)}$ *is the norm on the Bargmann space* $H_{\Phi_t}(\mathbb{C}^n)$ *introduced in (4.3.24), and* $\gamma > 0$ *is as in the statement of Theorem 1.1.*

2. *Let* $R_t = \Phi_0 - \Phi_t$, $t \geq 0$, *and let* $\alpha : [0, \infty) \to \mathbb{R}$ *be the continuous function defined by*

$$\alpha(t) = min_{|z|=1} R_t(z), \tag{4.3.29}$$

so that

$$R_t(z) \geq \alpha(t) \, |z|^2 , \quad z \in \mathbb{C}^n, \ t \geq 0. \tag{4.3.30}$$

The function α *has the following properties:*

   (a) $\alpha(0) = 0$ *and* $\alpha(t) > 0$ *for all* $t > 0$,

   (b) α *is non-decreasing,*

   (c) *there is* $0 < t_0 \ll 1$ *and* $c > 0$ *such that*

$$\alpha(t) \geq ct^{2k_0+1}, \quad 0 \leq t \leq t_0, \tag{4.3.31}$$

   where $k_0 \in \{0, 1, \ldots, 2n-1\}$ *is the smallest non-negative integer such that (4.1.12) holds, and*

   (d) $\alpha(t) \to \min_{|z|=1} \Phi_0(z) > 0$ *as* $t \to \infty$.

3. *Let* Ψ_0 *be the polarization of* Φ_0 *given by (4.3.2). For any* $t \geq 0$ *and* $u \in H_{\Phi_0}(\mathbb{C}^n)$, *we have*

$$\widetilde{G}(t)u(z) = C_{\Phi_0} e^{\frac{i}{2} tr(M)t} \int_{\mathbb{C}^n} e^{2\Psi_t(z,\overline{w})} u(w) e^{-2\Phi_0(w)} \, L(dw), \quad z \in \mathbb{C}^n, \tag{4.3.32}$$

where

$$\Psi_t(z, \theta) = \Psi_0 \left(e^{itM} z, \theta \right), \quad (z, \theta) \in \mathbb{C}^{2n}, \ t \geq 0. \tag{4.3.33}$$

Moreover, there are constants $C, c > 0$, independent of t, such that

$$-C\left|w - e^{itM}z\right|^2 \leq 2\,Re\,\Psi_t(z,\overline{w}) - \Phi_t(z) - \Phi_0(w) \leq -c\left|w - e^{itM}z\right|^2, \quad z, w \in \mathbb{C}^n,\ t \geq 0. \tag{4.3.34}$$

Proof. It remains to establish Point 2 and the estimate (5.2.68). To this end, let

$$R_t(z) = \Phi_0(z) - \Phi_t(z), \quad z \in \mathbb{C}^n,\ t \geq 0, \tag{4.3.35}$$

and let $\alpha : [0, \infty) \to \infty$ be as in (4.3.29). We will begin by showing that

$$R_t(z) \geq 0, \quad z \in \mathbb{C}^n,\ t \geq 0. \tag{4.3.36}$$

Let $\tilde{\kappa}_t$, $t \geq 0$, be as in (4.3.3). A straightforward computation using (4.2.9), (4.3.4), and (4.3.28) gives that

$$\tilde{\kappa}_t\left(\Lambda_{\Phi_0}\right) = \Lambda_{\Phi_t} := \left\{\left(z, \frac{2}{i}\Phi'_{t,z}(z)\right) : z \in \mathbb{C}^n\right\}, \quad t \geq 0. \tag{4.3.37}$$

From either the discussion in Section 6 of [61] or a direct computation, we know that the family $(\Phi_t)_{t\geq 0}$ satisfies the eikonal equation

$$\begin{cases} \partial_t \Phi_t(z) + \mathrm{Re}\,\tilde{q}\left(z, \frac{2}{i}\Phi'_{t,z}(z)\right) = 0, \quad z \in \mathbb{C}^n,\ t \geq 0, \\ \Phi_t|_{t=0} = \Phi_0 \text{ on } \mathbb{C}^n. \end{cases} \tag{4.3.38}$$

As a consequence of (4.3.37), for every $z \in \mathbb{C}^n$ and $t \geq 0$, there is a point $Z \in \Lambda_{\Phi_0}$ such that

$$\left(z, \frac{2}{i}\Phi'_{t,z}(z)\right) = \tilde{\kappa}_t(Z). \tag{4.3.39}$$

Since $\tilde{q}$ is invariant under the flow $\tilde{\kappa}_t$, for every $t \geq 0$ and $z \in \mathbb{C}^n$, there is $Z \in \Lambda_{\Phi_0}$ such that

$$\partial_t \Phi_t(z) = -\mathrm{Re}\,\tilde{q}(Z). \tag{4.3.40}$$

Because $\operatorname{Re} q \geq 0$, (4.2.10) and (4.2.11) imply that $\operatorname{Re} \tilde{q} \geq 0$ on Λ_{Φ_0}, and we have

$$\partial_t \Phi_t(z) \leq 0, \quad z \in \mathbb{C}^n, \ t \geq 0. \tag{4.3.41}$$

Thus, for any fixed $z \in \mathbb{C}^n$, the function

$$t \mapsto \Phi_0(z) - \Phi_t(z) \tag{4.3.42}$$

is non-decreasing. It follows that $R_t \geq 0$ for all $t \geq 0$ and that the function α is non-decreasing.

We next recall from Proposition 6.1 of [61] that

$$\Lambda_{\Phi_0} \cap \Lambda_{\Phi_t} = \pi_1 \left(\kappa_\varphi(S) \right), \quad t > 0, \tag{4.3.43}$$

where S is the singular space of q, $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is the complex linear canonical transformation defined by (4.2.6), and $\pi_1 : \mathbb{C}^{2n} \to \mathbb{C}^n$ is the projection $\pi_1 : (z, \zeta) \mapsto z$. Since we assume that $S = \{0\}$, we deduce from (4.3.43) that

$$\Lambda_{\Phi_0} \cap \Lambda_{\Phi_t} = \{0\}, \quad t > 0. \tag{4.3.44}$$

Thus, for every $t > 0$ and $z \in \mathbb{C}^n$,

$$\frac{2}{i} \Phi'_{0,z}(z) - \frac{2}{i} \Phi'_{t,z}(z) = 0 \iff z = 0. \tag{4.3.45}$$

Because R_t is a non-negative quadratic form for each $t \geq 0$, we have

$$R_t(z) = 0, \ z \in \mathbb{C}^n, \ t > 0 \iff \nabla_{\operatorname{Re} z, \operatorname{Im} z} R_t(z) = 0 \iff \frac{2}{i} \Phi'_{0,z}(z) - \frac{2}{i} \Phi'_{t,z}(z) = 0. \tag{4.3.46}$$

Hence, for any $z \in \mathbb{C}^n$ and $t > 0$,

$$R_t(z) = 0 \iff z = 0. \tag{4.3.47}$$

Thus $\alpha(t) > 0$ for all $t > 0$.

To establish (4.3.31), we recall the main result of Section 2 of [26], which states that if the singular space of q is trivial, $S = \{0\}$, then there is a small time $0 < t_0 \ll 1$ and a constant $c > 0$ such that

$$R_t(z) \geq ct^{2k_0+1} |z|^2, \quad z \in \mathbb{C}^n,\ 0 \leq t \leq t_0, \tag{4.3.48}$$

where $k_0 \in \{0, 1, \ldots, 2n-1\}$ is the smallest non-negative integer such that (4.1.12) holds. It is therefore true that

$$\alpha(t) \geq ct^{2k_0+1}, \quad 0 \leq t \leq t_0. \tag{4.3.49}$$

To prove the claim regarding the behavior of $\alpha(t)$ as $t \to \infty$, we note that (4.2.16) implies that $\mathrm{spec}(iM) \subset \{\mathrm{Re}\,\lambda < 0\}$. Thus there is $c > 0$ such that

$$R_t(z) = \Phi_0(z) + \mathcal{O}(e^{-ct} |z|^2) \text{ as } t \to \infty. \tag{4.3.50}$$

It follows that

$$\alpha(t) \to \min_{|z|=1} \Phi_0(z) \text{ as } t \to \infty. \tag{4.3.51}$$

The proof of Point 2 is complete.

Finally, we prove (5.2.68). Using (4.3.1), (4.3.2), (4.3.21), and (4.3.23), we obtain the following identity by elementary algebraic manipulations:

$$2\,\mathrm{Re}\,\Psi_t(z, \overline{w}) - \Phi_t(z) - \Phi_0(w) = -\Phi''_{0,\overline{z}z} \left(w - e^{iMt} z\right) \cdot \overline{(w - e^{iMt} z)}, \quad z, w \in \mathbb{C}^n,\ t \geq 0. \tag{4.3.52}$$

Because Φ_0 is a strictly plurisubharmonic quadratic form, the Levi matrix $\Phi''_{0,\overline{z}z}$ is Hermitian positive-definite. Consequently, there are constants $C, c > 0$, independent of t, such that

$$-C\left|w - e^{itM}z\right|^2 \leq 2\,\mathrm{Re}\,\Psi_t(z,\overline{w}) - \Phi_t(z) - \Phi_0(w) \leq -c\left|w - e^{itM}z\right|^2, \quad z, w \in \mathbb{C}^n,\ t \geq 0. \quad (4.3.53)$$

This proves (5.2.68). □

4.4 The Conclusion of the Proof of Theorem 4.1.1

In view of (4.2.26), (4.2.17), (4.3.5), and (4.3.12), the Schwartz kernel $K_t(x,y)$ of $G(t)$ is given, formally, by

$$K_t(x,y) = c_\varphi^2 C_{\Phi_0} e^{\frac{i}{2}\mathrm{tr}(M)t} \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} e^{P_t(x,y,z,w)}\, L(dw)\, L(dz), \quad (x,y) \in \mathbb{R}^{2n}, \quad t \geq 0, \qquad (4.4.1)$$

where

$$P_t(x,y,z,w) := -i\overline{\varphi(z,x)} - 2\Phi_0(z) + 2\Psi_t(z,\overline{w}) - 2\Phi_0(w) + i\varphi(w,y), \qquad (4.4.2)$$

for $x, y \in \mathbb{R}^n$, $z, w \in \mathbb{C}^n$, and $t \geq 0$. For $z \in \mathbb{C}^n$, let $r(z) \in \mathbb{R}^n$ be the unique point such that

$$\Phi_0(z) = -\mathrm{Im}\ \varphi(z, r(z)). \qquad (4.4.3)$$

We note that $r(z)$ is an $\mathbb{R}$-linear function of $z \in \mathbb{C}^n$. Since $\mathrm{Im}\ \varphi''_{yy} > 0$, there is $c > 0$ such that

$$-\mathrm{Im}\ \varphi(z,y) - \Phi_0(z) \leq -c\,|y - r(z)|^2, \quad z \in \mathbb{C}^n,\ y \in \mathbb{R}^n. \qquad (4.4.4)$$

Using (4.4.4) together with the estimate (5.2.68), we find that

$$\mathrm{Re}\ P_t(x,y,z,w) \leq -c\,|x - r(z)|^2 - R_t(z) - c\left|w - e^{itM}z\right|^2 - c\,|y - r(w)|^2, \qquad (4.4.5)$$

for all $x, y \in \mathbb{R}^n$, $z, w \in \mathbb{C}^n$, and $t \geq 0$, where $R_t(z)$ is as in Proposition 4.3.1. Let $\alpha : [0, \infty) \to \mathbb{R}$ be as in (4.3.29). Since (4.3.30) holds, there is $c > 0$ such that

$$\operatorname{Re} P_t(x, y, z, w) \leq -c\,|x - r(z)|^2 - \alpha(t)\,|z|^2 - c\,\left|w - e^{itM} z\right|^2 - c\,|y - r(w)|^2 \tag{4.4.6}$$

for all $x, y \in \mathbb{R}^n$, $z, w \in \mathbb{C}^n$, and $t \geq 0$.

Let γ be as in the statement of Theorem 4.1.1. Taking the absolute value of (4.4.1) and using (4.4.6) and (4.2.16), we find that there are constants $C, c > 0$ such that

$$|K_t(x, y)| \leq C e^{-\gamma t} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} e^{-c|x - r(z)|^2 - \alpha(t)|z|^2 - c|w - \exp(itM)z|^2 - c|y - r(w)|^2} \, L(dw)\, L(dz) \tag{4.4.7}$$

for every $x, y \in \mathbb{R}^n$ and $t \geq 0$. Let $1 \leq p \leq q \leq \infty$ be given, and let $1 \leq r \leq \infty$ be such that

$$1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}. \tag{4.4.8}$$

Using Minkowski's integral inequality and the fact that $\alpha(t) > 0$ for every $t > 0$, we get that

$$\begin{aligned} \|K_t(x, \cdot)\|_{L^r} &\leq C e^{-\gamma t} \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} e^{-c|x - r(z)|^2 - \alpha(t)|z|^2 - c|w - \exp(itM)z|^2} \|e^{-c|y - r(w)|^2}\|_{L^r_y} \, L(dw)\, L(dz) \\ &\leq C \alpha(t)^{-n} e^{-\gamma t}, \quad x \in \mathbb{R}^n, \ t > 0, \end{aligned} \tag{4.4.9}$$

where $C = C_{p,q} > 0$ depends only on p and q. By similar reasoning, there is $C = C_{p,q} > 0$ such that

$$\|K_t(\cdot, y)\|_{L^r} \leq C \alpha(t)^{-n} e^{-\gamma t}, \quad y \in \mathbb{R}^n, \ t > 0. \tag{4.4.10}$$

Applying Young's integral inequality with (4.4.9) and (4.4.10) gives

$$\|G(t)\|_{L^p \to L^q} \leq C \alpha(t)^{-n} e^{-\gamma t}, \quad t > 0, \tag{4.4.11}$$

for some $C = C_{p,q} > 0$.

Let $\epsilon > 0$ be arbitrary. From Proposition 4.3.1, we know that α is non-decreasing and $\alpha(t) > 0$

for all $t > 0$. Thus,

$$\alpha(t) \geq \alpha(\epsilon), \quad t \geq \epsilon. \tag{4.4.12}$$

In view of (4.4.11), we may deduce that there is $C = C_{\epsilon,p,q} > 0$ such that

$$\|G(t)\|_{L^p \to L^q} \leq C e^{-\gamma t}, \quad t \geq \epsilon. \tag{4.4.13}$$

To see that the bound (4.4.13) is sharp as $t \to \infty$, we recall from Theorem 2.1 of [40] that the lowest eigenvalue ρ of $q^w(x, D)$, introduced in (4.1.14), is simple and that the eigenspace of $q^w(x, D)$ corresponding to ρ is spanned by a 'ground state' of the form

$$u_0(x) = e^{-a(x)}, \quad x \in \mathbb{R}^n, \tag{4.4.14}$$

where a is a complex-valued quadratic form on $\mathbb{R}^n$ with positive-definite real part $\operatorname{Re} a > 0$. Let $v = \|u_0\|^{-1}_{L^p(\mathbb{R}^n)} u_0$. Since $q^w(x, D)v = \rho v$, is is clear that

$$\|e^{-tq^w(x,D)} v\|_{L^q} = e^{-t\gamma} \|v\|_{L^q}, \quad t \geq 0. \tag{4.4.15}$$

Hence there is a constant $c = c_{p,q} > 0$ such that

$$\|e^{-tq^w(x,D)}\|_{L^p \to L^q} \geq c e^{-\gamma t}, \quad t \geq 0. \tag{4.4.16}$$

We conclude that there are constants $C = C_{\epsilon,p,q} > 0$ and $c = c_{p,q} > 0$ such that (4.1.15) holds for all $t \geq \epsilon$.

Finally, we prove the bound (4.1.16). From (4.3.31), (4.4.11), and (4.4.16), we get that there are constants $C = C_{p,q} > 0$ and $c_{p,q} > 0$ such that

$$c \leq \|G(t)\|_{L^p \to L^q} \leq C t^{-(2k_0+1)n}, \quad 0 < t \leq t_0. \tag{4.4.17}$$

The proof of Theorem 4.1.1 is complete.

Chapter 5

L^p-Bounds for Eigenfunctions of Analytic Non-Self-Adjoint Operators with Double Characteristics

5.1 Introduction and Statement of Results

This work is inspired by the recent progress [36] in the understanding of L^p-norms of low-lying eigenfunctions of semiclassical pseudodifferential operators with double characteristics. More precisely, we are interested in eigenfunctions of semiclassical pseudodifferential operators on $\mathbb{R}^n$ of the form

$$P = \mathrm{Op}_h^w\left(p_0 + hp_1\right), \quad 0 < h \leq 1, \tag{5.1.1}$$

where $p_0, p_1 \in C^\infty(\mathbb{R}^{2n})$ belong to a symbol class that we shall specify shortly. Here $0 < h \leq 1$ is a semiclassical parameter, and $\mathrm{Op}_h^w(a)$ denotes the semiclassical Weyl quantization of a symbol a on $\mathbb{R}^{2n}$, which is defined formally by

$$\mathrm{Op}_h^w(a)\, u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi; h\right) u(y)\, dy\, d\xi, \quad x \in \mathbb{R}^n, \tag{5.1.2}$$

for $u \in \mathcal{S}'(\mathbb{R}^n)$. For background concerning semiclassical Weyl quantization on $\mathbb{R}^n$, we refer to the texts [63] and [19]. We make the following assumptions regarding the leading symbol p_0:

1. p_0 is h-independent,

2. p_0 has non-negative real part,

$$\operatorname{Re} p_0 \geq 0 \text{ on } \mathbb{R}^{2n}, \tag{5.1.3}$$

   with

$$(\operatorname{Re} p_0)^{-1}(0) = \{0\}, \tag{5.1.4}$$

   and

3. $\operatorname{Im} p_0$ vanishes to second order at $0 \in \mathbb{R}^{2n}$, i.e.

$$\operatorname{Im} p_0(0) = 0 \text{ and } \nabla(\operatorname{Im} p_0)(0) = 0. \tag{5.1.5}$$

Note that (5.1.3) and (5.1.4) imply that

$$\nabla(\operatorname{Re} p_0)(0) = 0, \tag{5.1.6}$$

and hence p_0 vanishes to second order at $0 \in \mathbb{R}^{2n}$,

$$p_0(0) = 0 \text{ and } \nabla p_0(0) = 0. \tag{5.1.7}$$

We summarize (5.1.7) by saying that p_0 is *doubly characteristic* at $0 \in \mathbb{R}^{2n}$. To state our assumptions concerning the growth of p_0 and p_1 at infinity, we first recall that an *order function* on $\mathbb{R}^{2n}$ is a Lebesgue measurable function $m : \mathbb{R}^{2n} \to (0, \infty)$ such that

$$\exists C > 0,\ \exists N > 0 :\ m(X) \leq C\langle X - Y\rangle^N m(Y), \quad X, Y \in \mathbb{R}^{2n}. \tag{5.1.8}$$

Here $\langle X\rangle = (1+|X|^2)^{1/2}$ denotes the Japanese bracket of $X \in \mathbb{R}^{2n}$. For any order function m on $\mathbb{R}^{2n}$, we have the symbol class $S(m)$ consisting of all $a : \mathbb{R}^{2n} \times (0,1] \to \mathbb{C}$ such that

$$a(\cdot;h) \in C^\infty(\mathbb{R}^{2n}), \quad 0 < h \leq 1, \tag{5.1.9}$$

and

$$\forall \alpha \in \mathbb{N}^{2n},\ \exists C > 0 : \ |\partial_X^\alpha a(X;h)| \leq C m(X), \quad X \in \mathbb{R}^{2n}, \quad 0 < h \leq 1. \tag{5.1.10}$$

Regarding the symbols p_0 and p_1, we assume that there is an order function m on $\mathbb{R}^{2n}$ with

$$m \in S(m) \text{ and } m \geq 1 \tag{5.1.11}$$

such that

$$p_0, p_1 \in S(m). \tag{5.1.12}$$

We also assume that $\operatorname{Re} p_0$ is elliptic at infinity in the sense that

$$\operatorname{Re} p_0(X) \geq \frac{1}{C} m(X), \quad |X| \geq C, \tag{5.1.13}$$

for some $C > 0$.

Operators of the form (5.1.1) with p_0 and p_1 satisfying the above assumptions include, for example, Schrödinger operators on $\mathbb{R}^n$ with complex potentials. Other examples of such semiclassical operators may be found in areas of mathematical physics such as fluid dynamics, superconductivity, and kinetic theory. See, for instance, [2], [16], and [21]. In many applications, the operator P arises as an unbounded operator $L^2(\mathbb{R}^n)$, and one is interested in determining the spectrum of P in the semiclassical limit as $h \to 0^+$. We may realize P as an unbounded operator on $L^2(\mathbb{R}^n)$ with the domain

$$H_h(m) := \mathrm{Op}_h^w(m)^{-1}\left(L^2(\mathbb{R}^n)\right), \tag{5.1.14}$$

for $h > 0$ sufficiently small. When equipped with the domain (5.1.14), the operator P becomes a closed and densely defined operator on $L^2(\mathbb{R}^n)$. Concerning the spectrum of P, the assumption (5.1.13) that $\mathrm{Re}\, p_0$ is elliptic at infinity implies that there is $0 < h_0 \leq 1$ and $C > 0$ such that

$$\begin{aligned} &\mathrm{Spec}(P) \cap D(0, C) \text{ is discrete and consists of eigenvalues} \\ &\text{of finite algebraic multiplicity for every } 0 < h \leq h_0. \end{aligned} \tag{5.1.15}$$

Here $D(0, r)$ denotes the open disc of radius r in r centered at the origin 0. For a proof of (5.1.15), see Section 3 of [20]. The eigenvalues $\lambda(h)$ of P that lie in a disc of the form $D(0, Ch)$ for some $C > 0$ are known as *low-lying eigenvalues* of P. Thanks to the works [49], [8], [22], and [25], complete asymptotic expansions for the low-lying eigenvalues of P are known when Weyl symbol of P admits an asymptotic expansion in the class $S(m)$, i.e. there exists a sequence of h-independent symbols $p_{1,j} \in S(m)$, $j \in \mathbb{N}$, such that

$$p_1 \sim \sum_{j=0}^{\infty} h^j p_{1,j} \text{ in } S(m), \tag{5.1.16}$$

and the quadratic approximation to p_0 at $0 \in \mathbb{R}^{2n}$,

$$q(X) = \frac{1}{2} p_0''(0) X \cdot X, \quad X \in \mathbb{R}^{2n}, \tag{5.1.17}$$

satisfies suitable partial ellipticity hypotheses. Note that the sign assumption (5.1.3) implies that the complex-valued quadratic form q defined by (5.1.17) has non-negative real part,

$$\mathrm{Re}\, q \geq 0. \tag{5.1.18}$$

In this chapter, we are primarily interested in the eigenfunctions that correspond to low-lying eigenvalues of P. A *low-lying eigenfunction of* P is a family $u = u(h) \in L^2(\mathbb{R}^n)$, $0 < h \leq 1$, such that

$$\begin{cases} Pu(h) = z(h)u(h), \\ \|u(h)\|_{L^2(\mathbb{R}^n)} = 1, \end{cases} \tag{5.1.19}$$

where $z(h) \in \mathbb{C}$ satisfies

$$z(h) = \mathcal{O}(h), \quad h \to 0^+. \tag{5.1.20}$$

When the operator P is non-self-adjoint, little is known about the low-lying eigenfunctions of P. In particular, relationship between the quadratic form q and properties of the low-lying eigenfunctions of P is not well-understood. In this work, our objective shall be to establish L^p-bounds for low-lying eigenfunctions of P under minimal partial ellipticity assumptions on q.

Remark 5.1.1. We remark the study of L^p-bounds of eigenfunctions of self-adjoint semiclassical pseudodifferential operators is well-established and has a long history. For more information, we refer to the works [35], [33], and the references therein.

To the best of our knowledge, the first and only work to undertake a study of L^p-bounds for low-lying eigenfunctions of non-self-adjoint semiclassical pseudodifferential operators of the above type has been the work [36], which showed, under the assumption that $\operatorname{Re} q$ is positive-definite, that any low-lying eigenfunction u of P satisfies the bound

$$\|u(h)\|_{L^p(\mathbb{R}^n)} \leq \mathcal{O}(1)h^{\frac{n}{2p}-\frac{n}{4}}, \quad h \to 0^+, \tag{5.1.21}$$

for p in the range $2 \leq p \leq \infty$. In particular, the bound (5.1.21) is saturated by the eigenfunctions of the semiclassical harmonic oscillator on $\mathbb{R}^n$, and thus the bounds (5.1.21) are sharp within the class of operators considered in [36].

In the present work, we will show that the bounds (5.1.21) also hold in cases where $\operatorname{Re} q$ need not be positive-definite, provided we assume in addition that p_0 and p_1 admit suitable holomorphic extensions to a tubular neighborhood of $\mathbb{R}^{2n}$ in $\mathbb{C}^{2n}$. This kind of result is of interest because in many physical applications the real quadratic form $\operatorname{Re} q$ does indeed fail to be positive-definite. Such a situation arises in, for example, the study of operators Kramers-Fokker-Planck type in kinetic theory. For more information, see e.g. [22]. Actually, our main result will be somewhat stronger than what we have just described. Namely, we will show that the low-lying eigenfunctions of semiclassical operators of the kind we consider satisfy (5.1.21) for p in the entire range $1 \leq p \leq \infty$. Thus our main result improves that of [36] in the case when $\operatorname{Re} q$ is positive-definite and p_0 and p_1 belong to a suitable holomorphic symbol class.

In the work [25], asymptotic expansions for the low-lying eigenvalues of P were established under the assumption that the quadratic approximation q to p_0 at $0 \in \mathbb{R}^{2n}$ is elliptic only along a certain, distinguished, subspace of $\mathbb{R}^{2n}$, known as the *singular space* of q. In the present work, we aim to establish L^p-bounds for low-lying eigenfunctions of P under this very same partial ellipticity assumption on q. In order to state our main result, we pause to recall the notion of the singular space of a complex-valued quadratic form q on $\mathbb{R}^{2n}$ with non-negative real part $\operatorname{Re} q \geq 0$.

Let $\mathbb{R}^{2n}$ be equipped with the standard symplectic form

$$\sigma((x,\xi),(y,\eta)) = \xi \cdot y - x \cdot \eta, \quad (x,\xi),\ (y,\eta) \in \mathbb{R}^{2n}. \tag{5.1.22}$$

Suppose $q : \mathbb{R}^{2n} \to \mathbb{C}$ is a complex-valued quadratic form with non-negative real part $\operatorname{Re} q \geq 0$. Let $q(\cdot,\cdot)$ denote the symmetric $\mathbb{C}$-bilinear polarization of q. Because σ is non-degenerate, there is a unique $F \in \operatorname{Mat}_{2n\times 2n}(\mathbb{C})$ such that

$$q((x,\xi),(y,\eta)) = \sigma((x,\xi), F(y,\eta)), \quad (x,\xi),(y,\eta) \in \mathbb{R}^{2n}.$$

This matrix F is called the *Hamilton map* or *Hamilton matrix of* q (see Section 21.5 of [32]). The *singular space* S *of* q is defined as the following finite intersection of kernels:

$$S = \left(\bigcap_{j=0}^{2n-1} \ker \left[(\operatorname{Re} F)(\operatorname{Im} F)^j \right] \right) \cap \mathbb{R}^{2n}. \tag{5.1.23}$$

The singular space was first introduced by M. Hitrik and K. Pravda-Starov in [23] where it arose naturally in the study of spectra and semi-group smoothing properties for non-self adjoint quadratic differential operators. The concept of the singular space has since been shown to play a key role in the understanding of hypoelliptic and spectral properties of non-elliptic quadratic differential operators. See for instance [24], [25], [41], [42], [57], and [58]. The notion of the singular space is also crucial for the description of the propagation of global microlocal singularities for Schrödinger equations on $\mathbb{R}^n$ with quadratic Hamiltonians. The interested reader may consult [44], [59], [43], [11], and [61], as well as [3], [4], and [62].

Next, we recall the definition of the holomorphic symbol class $S_{\mathrm{Hol}}(m)$. If m is an order function on $\mathbb{R}^{2n}$ and W is a bounded open neighborhood of $0 \in \mathbb{C}^{2n}$, we may extend m to a function $\tilde{m}$ on the tubular neighborhood $\mathbb{R}^{2n} + W$ of $\mathbb{R}^{2n}$ in $\mathbb{C}^{2n}$ by setting

$$\tilde{m}(Z) = m(\mathrm{Re}(Z)), \quad Z \in \mathbb{R}^{2n} + W. \tag{5.1.24}$$

We define $S_{\mathrm{Hol}}(m)$ as the set of all $a : \mathbb{R}^{2n} \times (0, 1] \to \mathbb{C}$ for which there is a bounded open neighborhood W of 0 in $\mathbb{C}^{2n}$ and a function $\tilde{a} : (\mathbb{R}^{2n} + W) \times (0, 1] \to \mathbb{C}$ extending a such that

$$\tilde{a}(\cdot; h) \in \mathrm{Hol}(\mathbb{R}^{2n} + W), \quad 0 < h \leq 1, \tag{5.1.25}$$

and

$$\exists C > 0 : \ |\tilde{a}(Z; h)| \leq C\tilde{m}(Z), \quad Z \in \mathbb{R}^{2n} + W, \ 0 < h \leq 1. \tag{5.1.26}$$

Note that by Cauchy's inequalities, the derivatives of symbols in $S_{\mathrm{Hol}}(m)$ are controlled by $\tilde{m}$ in any strictly smaller tubular neighborhood of $\mathbb{R}^{2n}$ as well: if $a \in S_{\mathrm{Hol}}(m)$, then for any open $\tilde{W} \subset\subset W$ we have

$$\forall \alpha \in \mathbb{N}^{2n}, \ \exists C = C(\alpha) > 0 : \ |\partial_Z^\alpha \tilde{a}(Z; h)| \leq C\tilde{m}(Z), \quad Z \in \mathbb{R}^{2n} + \tilde{W}, \quad 0 < h \leq 1, \tag{5.1.27}$$

where $\tilde{a}$ is the holomporphic extension of a and $\partial_Z = \frac{1}{2}\left(\partial_{\mathrm{Re}\, Z} - i\partial_{\mathrm{Im}\, Z}\right)$. For additional background concerning the holomorphic symbol class $S_{\rm Hol}(m)$, see Section 12.3 of [55].

The following theorem is the main result of this work, which establishes L^p bounds for low-lying eigenfunctions of the operator P defined in (5.1.1) under the assumption that $p_0, p_1 \in S_{\rm Hol}(m)$ and that the quadratic form q introduced in (5.1.17) is elliptic along its singular space S, i.e.

$$q(X) = 0,\ X \in S \implies X = 0. \tag{5.1.28}$$

In formulating our theorem, we shall be equivalently concerned with L^2-normalized solutions $u = u(h)$ of an equation of the form $Pu = 0$. In the sequel, we shall refer to a family $u = u(h) \in L^2(\mathbb{R}^n)$ satisfying

$$\begin{cases} Pu = 0 \text{ on } \mathbb{R}^n, \\ \|u\|_{L^2(\mathbb{R}^n)} = 1, \end{cases} \tag{5.1.29}$$

for all $0 < h \leq 1$, as a *ground state* for the operator P.

Theorem 5.1.2. *Suppose that $P = Op_h^w(p_0 + hp_1)$, where $p_0, p_1 \in S_{Hol}(m)$ for an order function m on $\mathbb{R}^{2n}$ satisfying (5.1.11). Assume that p_0 is h-independent and satisfies (5.1.3), (5.1.4), (5.1.5), and (5.1.13). Assume, in addition, that the quadratic approximation q to p_0 at $0 \in \mathbb{R}^{2n}$, defined in (5.1.17), is elliptic along its singular space S in the sense (5.1.28). If $u = u(h) \in L^2(\mathbb{R}^n)$ is such that*

$$\begin{cases} Pu = 0 \quad \textit{on} \quad \mathbb{R}^n, \quad n \geq 1, \\ \|u\|_{L^2(\mathbb{R}^n)} = 1, \end{cases} \tag{5.1.30}$$

for all $0 < h \leq 1$, then there exists $0 < h_0 \leq 1$ such that

$$\|u\|_{L^p(\mathbb{R}^n)} \leq \mathcal{O}(1) h^{\frac{n}{2p} - \frac{n}{4}}, \quad 1 \leq p \leq \infty, \tag{5.1.31}$$

for all $0 < h \leq h_0$.

Example 2. As noted in [36], the bounds (5.1.31) are saturated by the L^2-normalized eigenfunctions of the quantum harmonic oscillator on $\mathbb{R}^n$. Let

$$P = -h^2\Delta + |x|^2, \quad x \in \mathbb{R}^n, \quad n \geq 1. \tag{5.1.32}$$

The operator P, when viewed as an unbounded operator on $L^2(\mathbb{R}^n)$ with the domain

$$\mathcal{D}(P) = \left\{ u \in L^2(\mathbb{R}^n) : x^\alpha \partial_x^\beta u \in L^2(\mathbb{R}^n),\ |\alpha + \beta| \leq 2 \right\}, \tag{5.1.33}$$

is self-adjoint and has a discrete spectrum. Explicitly, the eigenvalues of P are given by

$$\lambda_\alpha(h) = (2|\alpha| + n)h, \quad \alpha \in \mathbb{N}^n, \tag{5.1.34}$$

and the corresponding L^2-normalized eigenfunctions have the form

$$u_\alpha(h)(x) = h^{-\frac{n}{4}} p_\alpha(h^{-\frac{1}{2}} x) e^{-\frac{|x|^2}{2h}}, \quad x \in \mathbb{R}^n, \tag{5.1.35}$$

where p_α is a Hermite polynomial of degree $|\alpha|$. For more information, see Section 6.1 of [63]. An explicit computation gives that

$$\|u_\alpha(h)\|_{L^p(\mathbb{R}^n)} = C(\alpha, p) h^{\frac{n}{2p} - \frac{n}{4}}, \quad 0 < h \leq 1, \tag{5.1.36}$$

where

$$C(\alpha, p) := \|p_\alpha(\cdot) e^{-\frac{|\cdot|^2}{2}}\|_{L^p(\mathbb{R}^n)}, \quad \alpha \in \mathbb{N}^n, \quad 1 \leq p \leq \infty. \tag{5.1.37}$$

Since

$$P = \mathrm{Op}_h^w(p_0), \tag{5.1.38}$$

for

$$p_0(x,\xi) = |x|^2 + |\xi|^2\,, \quad (x,\xi) \in \mathbb{R}^{2n}, \tag{5.1.39}$$

which is a non-negative elliptic quadratic form on $\mathbb{R}^{2n}$, we see that the bound (5.1.31) is sharp within the class of semiclassical pseudodifferential operators on $\mathbb{R}^n$ satisfying the hypotheses of Theorem 5.1.2.

Example 3. A more general class of semiclassical operators to which Theorem 5.1.2 applies are Schödinger operators on $\mathbb{R}^n$ with analytic complex-valued potentials. Let

$$P = -h^2\Delta + V(x) \;\text{ on }\; \mathbb{R}^n, \quad n \geq 1, \tag{5.1.40}$$

where $V \in C^\infty(\mathbb{R}^n; \mathbb{C})$ satisfies the following assumptions:

1. $\operatorname{Re} V \geq 0$,
2. $(\operatorname{Re} V)^{-1}(0) = \{0\}$,
3. $(\operatorname{Im} V)(0) = 0$ and $\nabla(\operatorname{Im} V)(0) = 0$,
4. $\det V''(0) \neq 0$,
5. there exists $s \geq 0$ such that

   $$\operatorname{Re} V(x) \geq \frac{1}{C}\,|x|^s\,, \quad |x| \geq C, \tag{5.1.41}$$

   for some $C > 0$, and
6. there exists $\epsilon > 0$ and a holomorphic extension $\tilde{V} \in \operatorname{Hol}(\mathbb{R}^n + i(-\epsilon,\epsilon)^n)$ of V satisfying

   $$\left|\tilde{V}(z)\right| \leq C\langle \operatorname{Re} z\rangle^s, \quad z \in \mathbb{R}^n + i(-\epsilon,\epsilon)^n, \tag{5.1.42}$$

   for some $C > 0$, where s is as in (5.1.41).

We may view the operator P as a closed, densely defined operator on $L^2(\mathbb{R}^n)$ with the the maximal domain

$$\mathcal{D}(P) = \left\{ u \in L^2(\mathbb{R}^n) : (-h^2\Delta + V(x))u \in L^2(\mathbb{R}^n) \right\}. \tag{5.1.43}$$

The spectrum of P in an open disc $D(0, Ch)$ of radius Ch, centered at $0 \in \mathbb{C}$, is discrete, and the low-lying eigenfunctions of P correspond to eigenvalues $z(h) \in D(0, Ch)$.

Let

$$p_0(x, \xi) = |\xi|^2 + V(x), \quad (x, \xi) \in \mathbb{R}^{2n}. \tag{5.1.44}$$

We observe that

$$P = \mathrm{Op}_h^w(p_0). \tag{5.1.45}$$

Thanks to our assumptions on V, the symbol p_0 admits the holomorphic extension

$$\tilde{p}_0(z, \zeta) = \zeta^2 + \tilde{V}(z), \quad (z, \zeta) \in \mathbb{R}^{2n} + i(-\epsilon, \epsilon)^{2n}, \tag{5.1.46}$$

for any $\epsilon > 0$. Clearly

$$\operatorname{Re} p_0 \geq 0 \quad \text{on} \quad \mathbb{R}^{2n}, \tag{5.1.47}$$

with

$$p_0(0) = 0, \quad \nabla p_0(0) = 0, \tag{5.1.48}$$

and we have

$$\tilde{p}_0 \in S_{\mathrm{Hol}}(m) \tag{5.1.49}$$

for the order function

$$m(x,\xi) = \langle \xi \rangle^2 + \langle x \rangle^s, \quad (x,\xi) \in \mathbb{R}^{2n}, \tag{5.1.50}$$

where s is as in (5.1.41). Also, from (5.1.41), we see that there is $C > 0$ such that

$$\operatorname{Re} p_0(x,\xi) \geq \frac{1}{C} m(x,\xi), \quad |(x,\xi)| \geq C. \tag{5.1.51}$$

The Hessian of p_0 at $0 \in \mathbb{R}^{2n}$ is

$$p_0''(0) = \begin{pmatrix} V''(0) & 0 \\ 0 & 2I_n \end{pmatrix}, \tag{5.1.52}$$

where I_n denotes the identity matrix of size $n \times n$. Let

$$q(Z) = \frac{1}{2} p_0''(0) Z \cdot Z, \quad Z \in \mathbb{C}^{2n}, \tag{5.1.53}$$

be the quadratic approximation to p_0 at $0 \in \mathbb{C}^{2n}$. A straightforward computation shows that the Hamilton matrix of q is

$$F = \begin{pmatrix} 0 & I \\ -\frac{1}{2} V''(0) & 0 \end{pmatrix}. \tag{5.1.54}$$

Thus

$$\operatorname{Re} F = \begin{pmatrix} 0 & I \\ -\frac{1}{2} (\operatorname{Re} V)''(0) & 0 \end{pmatrix}, \quad \operatorname{Im} F = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} (\operatorname{Im} V)''(0) & 0 \end{pmatrix}, \tag{5.1.55}$$

and hence

$$(\operatorname{Re} F)(\operatorname{Im} F) = \begin{pmatrix} -\frac{1}{2} (\operatorname{Im} V)''(0) & 0 \\ 0 & 0 \end{pmatrix}, \quad (\operatorname{Re} F)(\operatorname{Im} F)^j = 0 \text{ for all } j \geq 2. \tag{5.1.56}$$

It follows that

$$\ker(\operatorname{Re} F) = \ker\left[(\operatorname{Re} V)''(0)\right] \times \{0\}, \quad \ker\left[(\operatorname{Re} F)(\operatorname{Im} F)\right] = \ker\left[(\operatorname{Im} V)''(0)\right] \times \mathbb{C}^n, \tag{5.1.57}$$

and

$$\ker\left[(\operatorname{Re} F)(\operatorname{Im} F)^j\right] = \mathbb{C}^{2n} \text{ for all } j \geq 2. \tag{5.1.58}$$

We thus see that the singular space of q is

$$S = \left[\ker\left(V''(0)\right) \cap \mathbb{R}^n\right] \times \{0\}. \tag{5.1.59}$$

Because $\det V''(0) \neq 0$, we may deduce that the singular space of q is trivial,

$$S = \{0\}. \tag{5.1.60}$$

In particular, q is elliptic along S. By Theorem 5.1.2, any low-lying eigenfunction u of P must satisfy the bounds (5.1.31).

We point out that computation of the singular space S above is essentially a special case of Lemma 2.2 in [5]. We refer to this work for the related topic of magnetic Schrödinger operators with complex-valued potentials.

Remark 5.1.3. Theorem 5.1.2 improves the main result of [36] in several ways when it is assumed in addition that the symbols p_0 and p_1 belong to the holomorphic symbol class $S_{\mathrm{Hol}}(m)$. Most significantly, the bound (5.1.31) holds under the weaker assumption that the quadratic approximation q to p_0 at $0 \in \mathbb{R}^{2n}$ is elliptic only along its singular space S. It is also of note that the work [36] only establishes the bound (5.1.31) for p in the range $2 \leq p \leq \infty$. By contrast, the approach we give in this work yields the bound (5.1.31) for p in the entire range $1 \leq p \leq \infty$. That the bound (5.1.31) should also hold for p in the range $1 \leq p < 2$ is actually a very reasonable expectation in view of Example 2. Finally, in [36], it is assumed that the symbols p_0 and p_1 grow at most quadratically as

$|X| \to \infty$, i.e.

$$\|\partial^\alpha p_0\|_{L^\infty(\mathbb{R}^n)},\ \|\partial^\alpha p_1\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(1), \quad h \to 0^+, \quad |\alpha| \geq 2, \tag{5.1.61}$$

and that $\operatorname{Re} p_0$ grows at least quadratically as $|X| \to \infty$, i.e. there exists $C, c > 0$ such that

$$\operatorname{Re} p_0(X) \geq c\langle X\rangle^2, \quad |X| \geq C. \tag{5.1.62}$$

Our main result, on the other hand, applies to operators whose symbols may grow faster than $\langle X\rangle^2$ at infinity. For example, Theorem 5.1.2 applies to low-lying eigenfunctions of the operator

$$P = -h^2\Delta + |x|^2 + h^4\Delta^2 + |x|^4, \quad x \in \mathbb{R}^n, \quad n \geq 1, \tag{5.1.63}$$

whose Weyl symbol

$$p_0(x,\xi) = |x|^2 + |\xi|^2 + |x|^4 + |\xi|^4, \quad (x,\xi) \in \mathbb{R}^{2n}, \tag{5.1.64}$$

has size comparable to $\langle (x,\xi)\rangle^4$ when $|(x,\xi)| \to \infty$.

We conclude this introduction with an informal overview of the proof of Theorem 5.1.2. Since the symbol p_0 is elliptic away from 0, the ground states u of P are well microlocalized to any small but fixed neighborhood of 0 in the phase space $\mathbb{R}^{2n}$. This fact may be expressed conveniently in terms of any global metaplectic Fourier-Bros-Iagolnitzer (FBI) transform of u. Let φ be a holomorphic quadratic form on $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_y$ such that

$$\det \varphi''_{zy} \neq 0, \quad \operatorname{Im} \varphi''_{yy} > 0. \tag{5.1.65}$$

We refer to such a holomorphic quadratic form as an *FBI phase function on* $\mathbb{C}^{2n}$ ([63], [27], [54]). Let $\mathcal{T}_\varphi : \mathcal{S}'(\mathbb{R}^n) \to \operatorname{Hol}(\mathbb{C}^n)$, given by

$$\mathcal{T}_\varphi v(z) = c_\varphi h^{-3n/4} \int_{\mathbb{R}^n} e^{i\varphi(z,y)/h} v(y)\, dy, \quad v \in \mathcal{S}'(\mathbb{R}^n), \tag{5.1.66}$$

where

$$c_\varphi = 2^{-n/2}\pi^{-3n/4}(\det \mathrm{Im}\ \varphi''_{yy})^{-1/4}\left|\det \varphi''_{zy}\right|, \tag{5.1.67}$$

be the corresponding metaplectic FBI transform on $\mathbb{R}^n$. Let $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_\zeta$ be equipped with the standard complex symplectic form $\sigma = d\zeta \wedge dz \in \Lambda^{(2,0)}(\mathbb{C}^{2n})$. We recall that $\mathcal{T}_\varphi$ is a Fourier integral operator associated to the complex linear canonical transformation $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ given implicitly by

$$\kappa_\varphi : (w, -\partial_w \varphi(z,w)) \mapsto (z, \partial_z \varphi(z,w)), \quad (z,w) \in \mathbb{C}^{2n}, \tag{5.1.68}$$

and that $\mathcal{T}_\varphi$ maps $L^2(\mathbb{R}^n)$ unitarily onto the Bargmann space

$$H_{\Phi_0}(\mathbb{C}^n) = L^2(\mathbb{C}^n, e^{-2\Phi_0(z)/h}\, L(dz)) \cap \mathrm{Hol}(\mathbb{C}^n). \tag{5.1.69}$$

Here $L(dz)$ is the Lebesgue measure on $\mathbb{C}^n$, and $\Phi_0(z)$ is the strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ given by

$$\Phi_0(z) = \max_{y\in\mathbb{R}^n}\left(-\mathrm{Im}\ \varphi(z,y)\right), \quad z \in \mathbb{C}^n. \tag{5.1.70}$$

We also recall that κ_φ maps the real phase space $\mathbb{R}^{2n}$ isomorphically onto the I-Lagrangian, R-symplectic subspace

$$\Lambda_{\Phi_0} := \left\{\left(z, \frac{2}{i}\partial_z \Phi_0(z)\right) : z \in \mathbb{C}^n\right\}, \tag{5.1.71}$$

of $\mathbb{C}^{2n}$. Let $\pi_1 : \mathbb{C}^{2n} \to \mathbb{C}^n$ be the projection onto the first factor in $\mathbb{C}^{2n}$, $\pi_1 : (z,\zeta) \mapsto z$, and observe that π_1 restricts to an $\mathbb{R}$-linear isomorphism $\Lambda_\Phi \to \mathbb{C}^n$. Thus the map

$$\kappa^\flat_\varphi := \pi_1 \circ \kappa_\varphi|_{\mathbb{R}^{2n}} \tag{5.1.72}$$

is an $\mathbb{R}$-linear isomorphism $\mathbb{R}^{2n} \to \mathbb{C}^n$. Since u is microlocalized near 0 in $\mathbb{R}^{2n}$, it follows from standard arguments that the L^1-mass of the function $\mathcal{T}_\varphi u(z) e^{-\Phi_0(z)/h}$ with respect to $L(dz)$ is $\mathcal{O}(h^\infty)$ in the complement of any small but fixed neighborhood of 0 in $\mathbb{C}^n$ (see Proposition 5.2.2 below). By combining this observation with the identity $u = \mathcal{T}_\varphi^* \mathcal{T}_\varphi u$ and applying the Minkowski integral inequality, one can show that for any $N \in \mathbb{N}$, $\delta > 0$, and $1 \leq p \leq \infty$, there is $C = C(\delta, N, p) > 0$ and $0 < h_0 \leq 1$, independent of p, such that

$$\|u\|_{L^p(\mathbb{R}^n)} \leq C h^{\frac{n}{2p} - \frac{3n}{4}} \int_{|z|<\delta} |\mathcal{T}_\varphi u(z)| \, e^{-\Phi_0(z)/h} \, L(dz) + C h^N, \quad 0 < h \leq h_0. \tag{5.1.73}$$

For a proof, see Proposition 5.2.3 below.

Establishing the validity of (5.1.73) for any FBI phase function φ is the first main step in the proof of Theorem 5.1.2. The next step is to show that there exists an FBI phase function φ on $\mathbb{C}^{2n}$ and a real analytic strictly plurisubharmonic function $\Phi^* \in C^\omega(\mathrm{neigh}(0;\mathbb{C}^n);\mathbb{R})$ such that

$$\Phi_0(z) - \Phi^*(z) \geq c \, |z|^2, \quad z \in \mathrm{neigh}(0;\mathbb{C}^n), \tag{5.1.74}$$

for some $c > 0$, and

$$\|\mathcal{T}_\varphi u\|^2_{L^2_{\Phi^*}(\mathrm{neigh}(0;\mathbb{C}^n))} := \int_{\mathrm{neigh}(0;\mathbb{C}^n)} |\mathcal{T}_\varphi u(z)|^2 \, e^{-2\Phi^*(z)/h} \, L(dz) = \mathcal{O}(1), \quad h \to 0^+. \tag{5.1.75}$$

If such a weight Φ^* can be found, then, for $\delta > 0$ small enough, we have

$$\int_{|z|<\delta} |\mathcal{T}_\varphi u(z)| \, e^{-\Phi_0(z)/h} \, L(dz) = \int_{|z|<\delta} |\mathcal{T}_\varphi u(z)| \, e^{-\Phi^*(z)/h} e^{-c|z|^2/h} \, L(dz) \leq \mathcal{O}(1) h^{\frac{n}{2}}, \tag{5.1.76}$$

by the Cauchy-Schwart inequality. Combining (5.1.73) and (5.1.76) gives the desired bound (5.1.31).

The construction of the strictly plurisubharmonic weight Φ^* defined near 0 in $\mathbb{C}^n$ satisfying (5.1.74) and (5.1.75) is purely dynamical. We give a rough sketch of the procedure. Let

$$\mathfrak{p}_0 := p_0 \circ \kappa_\varphi^{-1} \in \mathrm{Hol}(\Lambda_{\Phi_0} + W), \tag{5.1.77}$$

where $W \subset\subset \mathbb{C}^{2n}$ is a sufficiently small open neighborhood of 0 in $\mathbb{C}^{2n}$. Let

$$H_{\mathfrak{p}_0} = \mathfrak{p}'_{0,\zeta} \cdot \partial_z - \mathfrak{p}'_{0,z} \cdot \partial_\zeta \in T^{(1,0)}(\Lambda_{\Phi_0} + W)$$

be the complex Hamilton vector field of $\mathfrak{p}_0$. For $t \in \mathbb{C}$ with $|t|$ sufficiently small, we may define the complex-time Hamilton flow $\kappa_t = \exp(tH_{\mathfrak{p}_0})$ of $\mathfrak{p}_0$ in a neighborhood of $0 \in \mathbb{C}^{2n}$ as follows:

$$(z(t), \zeta(t)) = \kappa_t(z_0, \zeta_0), \quad (z_0, \zeta_0) \in \mathrm{neigh}(0; \mathbb{C}^{2n}), \quad t \in \mathrm{neigh}(0; \mathbb{C}), \tag{5.1.78}$$

if and only if $z(t)$ and $\zeta(t)$ satisfy the complex Hamilton's equations

$$\begin{cases} \partial_t z(t) = \partial_\zeta \mathfrak{p}_0(z(t), \zeta(t)), \\ \partial_t \zeta(t) = -\partial_z \mathfrak{p}_0(z(t), \zeta(t)), \\ z(0) = z_0, \ \zeta(0) = \zeta_0, \end{cases} \tag{5.1.79}$$

where $\partial_t = \frac{1}{2}(\partial_{\mathrm{Re}\, t} - i\partial_{\mathrm{Im}\, t})$. From the work [52], it is known that there is $0 < T \ll 1$ such that

$$\kappa_t(\Lambda_{\Phi_0} \cap \mathrm{neigh}(0; \mathbb{C}^{2n})) \cap \mathrm{neigh}(0; \mathbb{C}^{2n}) = \Lambda_{\Phi_t}, \quad 0 \leq |t| < T, \tag{5.1.80}$$

where $(\Phi_t)_{|t|<T}$ is a family of real-valued, strictly plurisubharmonic functions defined in a neighborhood of $0 \in \mathbb{C}^n$ solving the complex-time eikonal equation

$$\begin{cases} 2\partial_t \Phi_t(z) + i\mathfrak{p}_0\left(z, \frac{2}{i}\partial_z \Phi_t(z)\right) = 0, \quad |t| < T, \quad z \in \mathrm{neigh}(0; \mathbb{C}^n), \\ \Phi_t|_{t=0}(z) = \Phi_0(z), \quad z \in \mathrm{neigh}(0; \mathbb{C}^n), \end{cases} \tag{5.1.81}$$

and $(\Lambda_{\Phi_t})_{|t|<T}$ is the family of I-Lagrangian, R-symplectic submanifolds of $\mathbb{C}^{2n}$ given by

$$\Lambda_{\Phi_t} := \left\{ \left(z, \frac{2}{i}\partial_z \Phi_t(z)\right) : z \in \mathrm{neigh}(0; \mathbb{C}^n) \right\}, \quad |t| < T. \tag{5.1.82}$$

For a proof, see the discussion in Section 3 below. In particular, since $\mathfrak{p}_0$ vanishes to second order at $0 \in \mathbb{C}^{2n}$ and $\Phi_0(0) = 0$, and we have that

$$\Phi_0 - \Phi_t \text{ vanishes to } 2^{\text{nd}} \text{ order at } z = 0 \text{ for all } 0 \le |t| < T.$$

For $|t| < T$, let Ξ_t be the quadratic approximation to Φ_t at 0, i.e. Ξ_t is the real quadratic form on $\mathbb{C}^n$ that begins the Taylor expansion of Φ_t at $0 \in \mathbb{C}^n$,

$$\Phi_t(z) = \Xi_t(z) + \mathcal{O}(|z|^3), \quad |z| \to 0^+, \quad 0 \le |t| < T. \tag{5.1.83}$$

Taylor expanding (5.1.81) to second order about $z = 0$ shows that Ξ_t must satisfy the quadratic complex-time eikonal equation

$$\begin{cases} 2\partial_t \Xi_t(z) + i\mathfrak{q}\left(z, \frac{2}{i}\partial_z \Xi_t(z)\right) = 0, \quad |t| < T, \quad z \in \mathbb{C}^n, \\ \Xi_t|_{t=0} = \Phi_0 \text{ on } \mathbb{C}^n, \end{cases} \tag{5.1.84}$$

where $\mathfrak{q} := q \circ \kappa_\varphi^{-1}$ is the quadratic approximation to $\mathfrak{p}_0$ at 0, i.e.

$$\mathfrak{p}_0(Z) = \mathfrak{q}(Z) + \mathcal{O}(|Z|^3), \quad Z \in \mathbb{C}^{2n}, \quad |Z| \to 0^+. \tag{5.1.85}$$

Using the assumption that q is elliptic along S, one can show that there exists an FBI phase function φ on $\mathbb{C}^{2n}$ with the property that for any $0 < T_0 < T$ there exists a non-zero complex time t_0 with $|t_0| < T_0$ such that

$$\Phi_0(z) - \Xi_{t_0}(z) > 0, \quad z \in \mathbb{C}^n \backslash \{0\}. \tag{5.1.86}$$

A proof of this claim is given in Proposition 5.4.2 below. Taking

$$\Phi^* := \Phi_{t_0}, \tag{5.1.87}$$

we obtain a strictly plurisubharmonic weight Φ^* defined in a neighborhood of 0 in $\mathbb{C}^n$ so that (5.1.74) holds. Now, by Egorov's theorem, $\mathcal{T}_\varphi u$ satisfies the FBI-side pseudodifferential equation

$$\mathrm{Op}^w_{\Phi_0,h}(\mathfrak{p}_0 + h\mathfrak{p}_1)\mathcal{T}_\varphi u = 0, \quad 0 < h \leq 1, \tag{5.1.88}$$

where $\mathfrak{p}_j := p_j \circ \kappa_\varphi^{-1} \in \mathrm{Hol}(\Lambda_{\Phi_0} + W)$, $j = 0, 1$, and $\mathrm{Op}^w_{\Phi_0,h}(\mathfrak{p}_0 + h\mathfrak{p}_1)$ denotes the complex Weyl quantization of $\mathfrak{p}_0 + h\mathfrak{p}_1$ with respect to the weight Φ_0 (see the discussion in Section 2 below). From (5.1.88) and (5.1.81), one may deduce that $\mathcal{T}_\varphi u$ satisfies the following dynamical bound in a sufficiently small neighborhood of 0 in $\mathbb{C}^n$:

$$\exists 0 < T_0 < T, \ \exists \delta > 0, \ \exists 0 < h_0 \leq 1 : \quad \sup_{\substack{0 \leq |t| < T_0 \\ 0 < h \leq h_0}} \|\mathcal{T}_\varphi u\|_{L^2_{\Phi_t}(\{|z|<\delta\})} < \infty. \tag{5.1.89}$$

The proof of (5.1.89), whose main ingredient is the quantization-multiplication theorem for pseudodifferential operators with holomorphic symbols ([27], [55], [53]), is given in Section 3 below. In particular, from (5.1.89), it follows that

$$\sup_{0 < h \leq h_0} \|\mathcal{T}_\varphi u\|_{L^2_{\Phi_{t_0}}(\{|z|<\delta\})} < \infty, \tag{5.1.90}$$

where $0 < |t_0| < T_0$ is such that (5.1.86) holds. Thus the weight Φ^* defined by (5.1.87) satisfies (5.1.74) and (5.1.75).

The plan for this chapter is as follows. In Section 2, we prove that the ground states u of P are well microlocalized to any fixed neighborhood of 0 in $\mathbb{R}^{2n}$ and we establish bounds for FBI transforms of u. In Section 3, we give a self-contained derivation of the complex-time eikonal equation (5.1.81), and we prove the dynamical bound (5.1.89). In Section 4, we conclude the proof of Theorem 5.1.2 as outlined in this introduction by proving the existence of an FBI phase function φ on $\mathbb{C}^{2n}$ and a small non-zero complex time t_0 such that the weight Φ^* defined by (5.1.87) satisfies (5.1.74).

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5.2 Microlocalization of the Ground States

In this section, we establish that any ground state $u = u(h) \in L^2(\mathbb{R}^n)$ of the operator P is well-microlocalized to any small, but fixed, neighborhood of $0 \in \mathbb{R}^{2n}$. We begin with an elementary parametrix construction, which we carry out using the calculus of semiclassical pseudodifferential operators on $\mathbb{R}^n$ with C^∞-symbols. For background on semiclassical pseudodifferential calculus on $\mathbb{R}^n$, including standard notation, we refer to Chapters 4 and 8 of [63].

Proposition 5.2.1. *Let $p_0, p_1 \in S(m)$, where m is an order function on $\mathbb{R}^{2n}$ satisfying (5.1.11). Assume that p_0 is h-independent and that p_0 satisfies (5.1.3), (5.1.4), (5.1.5), and (5.1.13). If $P = Op_h^w(p_0 + hp_1)$ and $u = u(h) \in L^2(\mathbb{R}^n)$ is such that*

$$\begin{cases} Pu = 0 \text{ on } \mathbb{R}^n, \ n \geq 1, \\ \|u\|_{L^2(\mathbb{R}^n)} = 1, \end{cases} \tag{5.2.1}$$

for all $0 < h \leq 1$, then, for any $\delta > 0$, there exists $\psi \in C_0^\infty(\mathbb{R}^{2n})$ with $\operatorname{supp} \psi \subset \left\{X \in \mathbb{R}^{2n} : |X| < \delta\right\}$ and there exists $R = \mathcal{O}_{\mathcal{S}' \to \mathcal{S}}(h^\infty)$ such that

$$u = Op_h^w(\psi)\, u + Ru, \quad 0 < h \leq h_0, \tag{5.2.2}$$

for some $0 < h_0 \leq 1$.

Proof. Let $\delta > 0$ be arbitrary and let $\chi \in C^\infty(\mathbb{R}^{2n}; [0,1])$ be such that

$$\chi(X) \equiv 1, \quad |X| \leq \frac{\delta}{2}, \tag{5.2.3}$$

and

$$\operatorname{supp} \chi \subset \left\{X \in \mathbb{R}^{2n} : |X| < \delta\right\}. \tag{5.2.4}$$

Set

$$\tilde{p} := p_0 + hp_1 + \chi, \quad 0 < h \le 1. \tag{5.2.5}$$

Note that $\tilde{p} \in S(m)$. Our objective is to construct a parametrix for the operator $\mathrm{Op}_h^w(\tilde{p})$. The assumption (5.1.13) that $\mathrm{Re}\, p_0$ is elliptic in the class $S(m)$ implies that there is $c > 0$ such that

$$\mathrm{Re}\, \tilde{p}(X; h) \ge cm(X), \quad X \in \mathbb{R}^{2n}, \tag{5.2.6}$$

for all $h > 0$ sufficiently small, depending on δ. Thus the symbol

$$e_0 := \frac{1}{\tilde{p}} \tag{5.2.7}$$

is well-defined and belongs to the class $S(\frac{1}{m})$. By the semiclassical composition calculus, there is a symbol $r_0 \in S(1)$ such that

$$\mathrm{Op}_h^w(e_0)\, \mathrm{Op}_h^w(\tilde{p}) = I + h\mathrm{Op}_h^w(r_0), \tag{5.2.8}$$

As a consequence of the semiclassical Calderon-Vaillancourt theorem (see Theorem 4.23 in [63]), we have $\mathrm{Op}_h^w(r_0) = \mathcal{O}_{L^2 \to L^2}(1)$ as $h \to 0^+$. Thus there is $0 < h_0 \le 1$ such that the operator $I + h\mathrm{Op}_h^w(r_0)$ is boundedly invertible on $L^2(\mathbb{R}^n)$ for all $0 < h \le h_0$. It follows that

$$(I + h\mathrm{Op}_h^w(r_0))^{-1} \mathrm{Op}_h^w(e_0)\, \mathrm{Op}_h^w(\tilde{p}) = I, \quad 0 < h \le h_0. \tag{5.2.9}$$

By Beals' Theorem (see the discussion on page 177 of [63]), there is $\tilde{r}_0 \in S(1)$ such that

$$\mathrm{Op}_h^w(\tilde{r}_0) = (I + h\mathrm{Op}_h^w(r_0))^{-1} \tag{5.2.10}$$

for all $h > 0$ sufficiently small. By the composition calculus, we have

$$\mathrm{Op}_h^w(\tilde{r}_0)\,\mathrm{Op}_h^w(e_0) = \mathrm{Op}_h^w(e_1)\,, \tag{5.2.11}$$

where $e_1 = \tilde{r}_0 \# e_0 \in S(\frac{1}{m})$ is the Moyal product of $\tilde{r}_0$ and e_0. From (5.2.9), (5.2.10), and (5.2.11), we deduce that

$$\mathrm{Op}_h^w(e_1)\,\mathrm{Op}_h^w(\tilde{p}) = I. \tag{5.2.12}$$

Since

$$\mathrm{Op}_h^w(\tilde{p}) = P + \mathrm{Op}_h^w(\chi)\,, \tag{5.2.13}$$

we have

$$u = \mathrm{Op}_h^w(e_1)\,\mathrm{Op}_h^w(\tilde{p})\,u = \mathrm{Op}_h^w(e_1)\,\mathrm{Op}_h^w(\chi)\,u \tag{5.2.14}$$

for all $h > 0$ sufficiently small. Let $\psi \in C_0^\infty(\mathbb{R}^{2n})$ be such that

$$\psi \equiv 1 \text{ in } \mathrm{neigh}(\mathrm{supp}\,\chi; \mathbb{R}^{2n}), \quad \mathrm{supp}\,\psi \subset \left\{X \in \mathbb{R}^{2n} : |X| < \delta\right\}. \tag{5.2.15}$$

Using (5.2.14), we get

$$\begin{aligned} u &= \mathrm{Op}_h^w(\psi)\,\mathrm{Op}_h^w(e_1)\,\mathrm{Op}_h^w(\chi)\,u + \mathrm{Op}_h^w(1-\psi)\,\mathrm{Op}_h^w(e_1)\,\mathrm{Op}_h^w(\chi)\,u \\ &= \mathrm{Op}_h^w(\psi)\,u + \mathrm{Op}_h^w(1-\psi)\,\mathrm{Op}_h^w(e_1)\,\mathrm{Op}_h^w(\chi)\,u. \end{aligned} \tag{5.2.16}$$

Because $\mathrm{supp}\,\chi$ is compact and $\mathrm{supp}\,(1-\psi) \cap \mathrm{supp}\,\chi = \emptyset$, it follows from the general theory that there is $r \in h^\infty \mathcal{S}(\mathbb{R}^n)$ such that

$$R := \mathrm{Op}_h^w((1-\psi))\,\mathrm{Op}_h^w(e_1)\,\mathrm{Op}_h^w(\chi) = \mathrm{Op}_h^w(r)\,. \tag{5.2.17}$$

Hence

$$R = \mathcal{O}_{\mathcal{S}' \to \mathcal{S}}(h^\infty). \tag{5.2.18}$$

From (5.2.16) and (5.2.17), we conclude that

$$u = \mathrm{Op}_h^w(\psi)u + Ru, \quad 0 < h \le h_0. \tag{5.2.19}$$

The proof of Proposition 5.2.1 is complete. □

We next explore the consequences of Proposition 5.2.1 for any FBI transform of a ground state u of P. Let $\mathbb{C}^{2n} = \mathbb{C}^n_\zeta \times \mathbb{C}^n_z$ be equipped with the standard complex symplectic form $\sigma = d\zeta \wedge dz \in \Lambda^{(2,0)}(\mathbb{C}^{2n})$. Let $\varphi = \varphi(z, y)$ be a holomorphic quadratic form on $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_y$ such that

$$\det \varphi''_{zy} \neq 0, \quad \mathrm{Im}\, \varphi''_{yy} > 0, \tag{5.2.20}$$

and let $\mathcal{T}_\varphi : \mathcal{S}'(\mathbb{R}^n) \to \mathrm{Hol}(\mathbb{C}^n)$ be its associated FBI transform introduced in (5.1.66). Let

$$\Phi_0(z) := \max_{y \in \mathbb{R}^n}(-\mathrm{Im}\, \varphi(z, y)), \quad z \in \mathbb{C}^n. \tag{5.2.21}$$

As the maximum of a family of pluriharmonic quadratic forms on $\mathbb{C}^n$, the function Φ_0 is a plurisubharmonic quadratic form on $\mathbb{C}^n$. In fact, the quadratic form Φ_0 is strictly plurisubharmonic, i.e. $\Phi''_{0,\bar{z}z} > 0$. See [63] or [27] for a proof. In the sequel, we shall refer to the quadratic form Φ_0 defined by (5.2.21) as the *strictly plurisubharmonic weight associated to the FBI phase function* φ. Let

$$\Lambda_{\Phi_0} := \left\{ \left(z, \frac{2}{i} \partial_z \Phi_0(z) \right) \in \mathbb{C}^{2n} : z \in \mathbb{C}^n \right\}, \tag{5.2.22}$$

and let

$$H_{\Phi_0}(\mathbb{C}^n) = L^2(\mathbb{C}^n, e^{-2\Phi_0(z)/h}\, L(dz)) \cap \mathrm{Hol}(\mathbb{C}^n) \tag{5.2.23}$$

be the Bargmann space of entire functions on $\mathbb{C}^n$ associated to the weight Φ_0. Here $L(dz)$ denotes the Lebesgue measure on $\mathbb{C}^n$. It is well known that $\mathcal{T}_\varphi$ is unitary $L^2(\mathbb{R}^n) \to H_{\Phi_0}(\mathbb{C}^n)$. For a proof, see Theorem 13.7 in [63], Theorem 1.3.3 in [27], or Section 12.2 of [55].

Following the discussion in Section 12.2 of [55], we recall that an order function on Λ_{Φ_0} is a Lebesgue measurable function $\mathfrak{m} : \Lambda_{\Phi_0} \to (0, \infty)$ such that

$$\exists C > 0,\ \exists N \in \mathbb{R} :\ \mathfrak{m}(Z) \leq C\langle Z - W\rangle^N \mathfrak{m}(W), \quad Z, W \in \Lambda_{\Phi_0}. \tag{5.2.24}$$

Given an order function $\mathfrak{m}$ on Λ_{Φ_0}, we may introduce the symbol class $S(\Lambda_{\Phi_0}, \mathfrak{m})$ consisting of all $a \in C^\infty(\Lambda_{\Phi_0})$ such that

$$\forall \alpha, \beta \in \mathbb{N}^n,\ \exists C > 0 :\ \left|\partial_z^\alpha \partial_{\bar{z}}^\beta \left(a\left(z, \frac{2}{i}\partial_z \Phi_0(z)\right)\right)\right| \leq C\mathfrak{m}\left(z, \frac{2}{i}\partial_z \Phi_0(z)\right), \quad z \in \mathbb{C}^n. \tag{5.2.25}$$

Also, if $\mathfrak{m}$ is an order function on Λ_{Φ_0}, then we may define the Sobolev space

$$H_{\Phi_0,\mathfrak{m}}(\mathbb{C}^n) = L^2\left(\mathbb{C}^n, \mathfrak{m}\left(z, \frac{2}{i}\partial_z \Phi_0(z)\right)^2 e^{-2\Phi_0(z)/h}\, L(dz)\right) \cap \mathrm{Hol}(\mathbb{C}^n), \tag{5.2.26}$$

which is a Hilbert space equipped with the norm

$$\|v\|^2_{L^2_{\Phi_0,\mathfrak{m}}(\mathbb{C}^n)} = \int_{\mathbb{C}^n} |v(z)|^2\, \mathfrak{m}\left(z, \frac{2}{i}\partial_z \Phi_0(z)\right)^2 e^{-2\Phi_0(z)/h}\, L(dz). \tag{5.2.27}$$

Note that $H_{\Phi_0,1}(\mathbb{C}^n) = H_{\Phi_0}(\mathbb{C}^n)$. Given a symbol $a \in S(\Lambda_{\Phi_0}, \mathfrak{m})$, where $\mathfrak{m}$ is an order function on Λ_{Φ_0}, we define the *complex Weyl quantization of* a formally by

$$\mathrm{Op}^w_{\Phi_0,h}(a)v(z) = \frac{1}{(2\pi h)^n} \iint\limits_{\Gamma_{\Phi_0}(z)} e^{\frac{i}{h}(z-w)\cdot\zeta} a\left(\frac{z+w}{2}, \zeta\right) v(w)\, dw \wedge d\zeta, \tag{5.2.28}$$

for the contour of integration

$$\Gamma_{\Phi_0}(z) : \mathbb{C}^n \ni w \mapsto \zeta = \frac{2}{i}\partial_z \Phi_0\left(\frac{z+w}{2}\right), \quad z \in \mathbb{C}^n. \tag{5.2.29}$$

Initially, $\mathrm{Op}^w_{\Phi_0,h}(a)v$ is defined for $v \in \mathrm{Hol}(\mathbb{C}^n)$ such that

$$\forall h \in (0,1],\ \forall N > 0,\ \exists C = C(N,h) > 0: \quad |v(z)| \leq C\langle z\rangle^{-N} e^{\Phi_0(z)/h}, \quad z \in \mathbb{C}^n. \tag{5.2.30}$$

One may then show that for any order function $\tilde{\mathfrak{m}}$ on Λ_{Φ_0} the operator $\mathrm{Op}^w_{\Phi_0,h}(a)$ defined by (5.2.28) extends by density to an operator $H_{\Phi_0,\tilde{\mathfrak{m}}}(\mathbb{C}^n) \to H_{\Phi_0,\frac{\tilde{\mathfrak{m}}}{\mathfrak{m}}}(\mathbb{C}^n)$ whose norm is uniformly bounded with respect to h, i.e.

$$\mathrm{Op}^w_{\Phi_0,h}(a) = \mathcal{O}(1) : H_{\Phi_0,\tilde{\mathfrak{m}}}(\mathbb{C}^n) \to H_{\Phi_0,\frac{\tilde{\mathfrak{m}}}{\mathfrak{m}}}(\mathbb{C}^n), \quad h \to 0^+. \tag{5.2.31}$$

For a proof, see Proposition 12.6 in [55].

We next review the relationship between the complex Weyl quantization and the ordinary Weyl quantization on $\mathbb{R}^n$. Let φ be an FBI phase function on $\mathbb{C}^{2n}$ and let Φ_0 be the strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ associated to φ. The FBI phase function φ generates a complex linear canonical transformation $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ implicitly by

$$\kappa_\varphi : (y, -\varphi'_y(z,y)) \mapsto (z, \varphi'_z(z,y)), \quad (z,y) \in \mathbb{C}^{2n}. \tag{5.2.32}$$

In the sequel, we refer to κ_φ as the *complex linear canonical transformation generated by* φ or as the *complex linear canonical transformation associated to* φ. From either Theorem 13.5 of [63] or Proposition 1.3.2 of [27], we know that

$$\kappa_\varphi(\mathbb{R}^{2n}) = \Lambda_{\Phi_0}, \tag{5.2.33}$$

and we have a version of Egorov's theorem. Namely, if m is any order function on $\mathbb{R}^{2n}$ and $a \in S(m)$, then

$$\mathfrak{a} := a \circ \kappa_\varphi^{-1} \in S(\Lambda_{\Phi_0}, \mathfrak{m}), \tag{5.2.34}$$

where $\mathfrak{m}$ is the order function on Λ_{Φ_0} given by

$$\mathfrak{m} = m \circ \kappa_\varphi^{-1}, \tag{5.2.35}$$

and we have

$$\mathrm{Op}_h^w(a) = \mathcal{T}_\varphi^* \circ \mathrm{Op}_{\Phi_0,h}^w(\mathfrak{a}) \circ \mathcal{T}_\varphi, \tag{5.2.36}$$

where both sides are viewed as operators on $\mathcal{S}'(\mathbb{R}^n)$. For a proof, see Section 12.2 of [55].

Let P and u be as in the statement of Proposition 5.2.1, and let φ be an FBI phase function on $\mathbb{C}^{2n}$ with associated FBI transform $\mathcal{T}_\varphi$ and strictly plurisubharmonic weight Φ_0. In view of Proposition 5.2.1, it is natural to expect that the mass of the entire function $\mathcal{T}_\varphi u$ will be concentrated near $0 \in \mathbb{C}^n$. Equivalently, we expect that the mass of $\mathcal{T}_\varphi u$ in the complement of any fixed neighborhood of 0 in $\mathbb{C}^n$ will be semiclassically negligible, i.e. $\mathcal{O}(h^\infty)$. The following proposition solidifies this intuition.

Proposition 5.2.2. *Let P and $u \in L^2(\mathbb{R}^n)$ be as in the statement of Proposition 5.2.1, and let φ be any FBI phase function on $\mathbb{C}^{2n}$ with associated FBI transform $\mathcal{T}_\varphi$ and associated strictly plurisubharmonic weight Φ_0. For any $\delta > 0$, there is $0 < h_0 \leq 1$ such that for any $1 \leq p \leq \infty$ we have*

$$\|\mathcal{T}_\varphi u(z) e^{-\Phi_0(z)/h}\|_{L^p(\{|z|\geq\delta\}, L(dz))} \leq \mathcal{O}_N(1) h^N, \quad 0 < h \leq h_0, \tag{5.2.37}$$

for any $N \in \mathbb{N}$.

Proof. By Hölder's inequality, it suffices to prove the proposition in the cases $p = 1$ and $p = \infty$. We begin by proving (5.2.37) in the case when $p = 1$. Let $\delta > 0$ be given, let $0 < \rho < \delta/2$, let Λ_{Φ_0} be as in (5.2.22), let $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be the complex linear canonical transformation generated by φ, let $\pi_1 : \mathbb{C}^{2n} \to \mathbb{C}^n$ be projection onto the first factor $\pi_1 : (z, \zeta) \mapsto z$, and let

$\kappa_\varphi^\flat := \pi_1 \circ \kappa_\varphi|_{\mathbb{R}^{2n}} : \mathbb{R}^{2n} \to \mathbb{C}^n$. By Proposition 5.2.1, there is $\psi \in C_0^\infty(\mathbb{R}^{2n})$ such that

$$\operatorname{supp}\left[\psi \circ \left(\kappa_\varphi^\flat\right)^{-1}\right] \subset \{z \in \mathbb{C}^n : |z| < \rho\}, \tag{5.2.38}$$

and $R = \mathcal{O}_{\mathcal{S}' \to \mathcal{S}}(h^\infty)$ such that

$$u = \operatorname{Op}_h^w(\psi)u + Ru, \quad 0 < h \leq h_0, \tag{5.2.39}$$

for some $0 < h_0 \leq 1$. By Egorov's theorem and (5.2.39),

$$\mathcal{T}_\varphi u = \operatorname{Op}_{\Phi_0,h}^w(\tilde{\psi})\mathcal{T}_\varphi u + \mathcal{T}_\varphi(Ru), \tag{5.2.40}$$

where

$$\tilde{\psi} := \psi \circ \kappa_\varphi^{-1} \in C_0^\infty(\Lambda_{\Phi_0}). \tag{5.2.41}$$

Thus

$$\int_{|z| \geq \delta} |\mathcal{T}_\varphi u(z)|\, e^{-\Phi_0(z)/h}\, L(dz) \leq \mathrm{I}(h) + \mathrm{II}(h), \tag{5.2.42}$$

where

$$\mathrm{I}(h) := \int_{|z| \geq \delta} \left|\operatorname{Op}_{\Phi_0,h}^w(\tilde{\psi})\mathcal{T}_\varphi u(z)\right| e^{-\Phi_0(z)/h}\, L(dz) \tag{5.2.43}$$

and

$$\mathrm{II}(h) := \int_{|z| \geq \delta} |\mathcal{T}_\varphi(Ru)(z)|\, e^{-\Phi_0(z)/h}\, L(dz). \tag{5.2.44}$$

We first consider $II(h)$. For $J \in \mathbb{N}$, let $\|\cdot\|_J$ denote the Schwartz seminorm

$$\|f\|_J := \sum_{|\alpha+\beta| \leq J} \|x^\alpha \partial^\beta f\|_{L^\infty(\mathbb{R}^n)}, \quad f \in \mathcal{S}(\mathbb{R}^n). \tag{5.2.45}$$

From Theorem 13.4 in [63], we know that for every $N > 0$ there exists $J \in \mathbb{N}$ and $C > 0$ such that

$$|\mathcal{T}_\varphi Ru(z)| \leq C\|Ru\|_J h^{-n/4} \langle z \rangle^{-N} e^{\Phi_0(z)/h}, \quad z \in \mathbb{C}^n, \quad 0 < h \leq 1. \tag{5.2.46}$$

Since $R = \mathcal{O}_{\mathcal{S}' \to \mathcal{S}}(h^\infty)$ and $\|u\|_{L^2(\mathbb{R}^n)} = 1$, we conclude that for every $M, N > 0$, there is $C = C(M,N) > 0$ such that

$$|\mathcal{T}_\varphi Ru(z)| \leq Ch^M \langle z \rangle^{-N} e^{\Phi_0(z)/h}, \quad z \in \mathbb{C}^n, \quad 0 < h \leq h_0. \tag{5.2.47}$$

In particular, for any $M > 0$ there is $C > 0$ such that

$$|\mathcal{T}_\varphi Ru(z)| \leq Ch^M \langle z \rangle^{-2n-1} e^{\Phi_0(z)/h}, \quad z \in \mathbb{C}^n, \quad 0 < h \leq h_0. \tag{5.2.48}$$

Bounding (5.2.44) using (5.2.48) gives that

$$\mathrm{II}(h) = \mathcal{O}(h^M) \tag{5.2.49}$$

for any $M > 0$. Therefore

$$\mathrm{II}(h) = \mathcal{O}(h^\infty). \tag{5.2.50}$$

It remains to show that $\mathrm{I}(h) = \mathcal{O}(h^\infty)$. From the definition (5.2.28) of the complex Weyl quantization, we have

$$\mathrm{Op}^w_{\Phi_0,h}(\tilde{\psi})\mathcal{T}_\varphi u(z) = \frac{1}{(2\pi h)^n} \iint_{\Gamma_{\Phi_0}(z)} e^{\frac{i}{h}(z-w)\cdot\zeta} \tilde{\psi}\left(\frac{z+w}{2}, \zeta\right) \mathcal{T}_\varphi u(w)\, dw \wedge d\zeta, \tag{5.2.51}$$

where the contour of integration $\Gamma_{\Phi_0}(z)$ is given by

$$\Gamma_{\Phi_0}(z) : \mathbb{C}^n \ni w \mapsto \zeta = \frac{2}{i}\partial_z \Phi_0\left(\frac{z+w}{2}\right), \quad z \in \mathbb{C}^n. \tag{5.2.52}$$

Let

$$L = \frac{1 + (\partial^2_{z\overline{z}}\Phi_0)^{-1}\overline{(z-w)} \cdot \partial_{\overline{w}}}{1 + |z-w|^2/h}, \quad z, w \in \mathbb{C}^n. \tag{5.2.53}$$

Observe that

$$Le^{\frac{i}{h}(z-w)\cdot\zeta} = e^{\frac{i}{h}(z-w)}, \quad z, w \in \mathbb{C}^n, \quad \zeta = \frac{2}{i}\partial_z \Phi_0\left(\frac{z+w}{2}\right). \tag{5.2.54}$$

Thus for any $z \in \mathbb{C}^n$ such that $|z| \geq \delta$ and any $N \in \mathbb{N}$ we have

$$\mathrm{Op}^w_{\Phi_0,h}(\tilde{\psi})\mathcal{T}_\varphi u(z) = \frac{1}{(2\pi h)^n} \iint\limits_{\Gamma_{\Phi_0}(z)} e^{\frac{i}{h}(z-w)\cdot\zeta}(L^T)^N \tilde{\psi}\left(\frac{z+w}{2}, \zeta\right) \mathcal{T}_\varphi u(w)\, dw \wedge d\zeta, \tag{5.2.55}$$

where L^T denotes the transpose of the differential operator L. Since $\tilde{\psi} \in C_0^\infty(\Lambda_{\Phi_0})$,

$$(L^T)^N \tilde{\psi}\left(\frac{z+w}{2}\right) = \mathcal{O}\left(\langle h^{-1/2}(z-w)\rangle^{-N}\right), \quad z, w \in \mathbb{C}^n, \quad 0 < h \leq 1, \tag{5.2.56}$$

for any $N \in N$. Now, it is true that

$$\mathrm{Re}\,[i(z-w)\cdot\zeta] = \Phi_0(z) - \Phi_0(w), \quad z, w \in \mathbb{C}^n,\ \zeta = \frac{2}{i}\partial_z \Phi_0\left(\frac{z+w}{2}\right). \tag{5.2.57}$$

For a proof, see Lemma 13.1 of [63]. From (5.2.43), (5.2.55), (5.2.56), and (5.2.57), it follows that for any $N \in \mathbb{N}$ there is $C > 0$ such that

$$\mathrm{I}(h) \leq Ch^{-n} \int_{|z|\geq\delta} \int_{\left|\frac{z+w}{2}\right|\leq\rho} \langle h^{-1/2}(z-w)\rangle^{-N} |\mathcal{T}_\varphi u(w)|\, e^{-\Phi_0(w)/h}\, L(dw)\, L(dz). \tag{5.2.58}$$

for all $0 < h \le h_0$. For any $z, w \in \mathbb{C}^n$ with $|z| \ge \delta$, we have

$$\left|\frac{z+w}{2}\right| \le \rho \implies |z| - \frac{|z-w|}{2} \le \rho \implies 2\,|z| - \delta \le |z-w| \implies |z| \le |z-w|. \tag{5.2.59}$$

Thus, from (5.2.58) we may deduce that for any $N \in \mathbb{N}$ there is $C > 0$ such that

$$\mathrm{I}(h) \le Ch^{-n+N/2} \int_{|z|\ge\delta} \int_{\left|\frac{z+w}{2}\right|\le\rho} |z|^{-N}\, |\mathcal{T}_\varphi u(w)|\, e^{-\Phi_0(w)/h}\, L(dw)\, L(dz) \tag{5.2.60}$$

for all $0 < h \le h_0$. Making the change of variables

$$\tilde{z} = z, \quad \tilde{w} = \frac{z+w}{2} \tag{5.2.61}$$

in (5.2.60) and applying Fubini's theorem and the Cauchy-Schwarz inequality yields

$$\begin{aligned} \mathrm{I}(h) &\le Ch^{-n+N/2} \int_{|\tilde{z}|\ge\delta} \int_{|\tilde{w}|\le\rho} |\tilde{z}|^{-N}\, |\mathcal{T}_\varphi u(2\tilde{w}-\tilde{z})|\, e^{-\frac{\Phi_0(2\tilde{w}-\tilde{z})}{h}}\, L(d\tilde{w})\, L(d\tilde{z}) \\ &\le C\|u\|_{H_{\Phi_0}(\mathbb{C}^n)} h^{-n+N/2} \int_{|\tilde{z}|\ge\delta} |\tilde{z}|^{-N}\, L(d\tilde{z}) \\ &\le Ch^{-n+N/2}, \quad 0 < h \le h_0, \end{aligned} \tag{5.2.62}$$

for any $N > 2n$. Thus

$$\mathrm{I}(h) = \mathcal{O}(h^\infty). \tag{5.2.63}$$

From (5.2.42), (5.2.63), and (5.2.50), we conclude

$$\int_{|z|\ge\delta} |\mathcal{T}_\varphi u(z)|\, e^{-\Phi_0(z)/h}\, L(dz) = \mathcal{O}(h^\infty). \tag{5.2.64}$$

Therefore the proposition is true when $p = 1$.

Next, we show that (5.2.37) holds when $p = \infty$. We recall that the orthogonal projection

$\Pi_{\Phi_0} : L^2_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n)$ is given by

$$\Pi_{\Phi_0} v(z) = C_{\Phi_0} h^{-n} \int_{\mathbb{C}^n} e^{\frac{2}{h}\Psi_0(z,\overline{w})} v(w) e^{-\frac{2}{h}\Phi_0(w)}\, L(dw), \quad v \in L^2_{\Phi_0}(\mathbb{C}^n), \tag{5.2.65}$$

where

$$C_{\Phi_0} = \left(\frac{2}{\pi}\right)^n \det \partial^2_{z\overline{z}} \Phi_0, \tag{5.2.66}$$

and $\Psi_0(\cdot,\cdot)$ is the polarization of Φ_0, i.e. $\Psi_0(\cdot,\cdot)$ is the unique holomorphic quadratic form on $\mathbb{C}^{2n}$ such that

$$\Psi_0(z,\overline{z}) = \Phi_0(z), \quad z \in \mathbb{C}^n, \tag{5.2.67}$$

In the literature, Π_{Φ_0} is known as the *Bergman projector* associated to the strictly plurisubharmonic weight Φ_0. For a proof that the Bergman projector associated to Φ_0 has the integral representation (5.2.65), we refer the reader to either Theorem 13.6 of [63], Proposition 1.3.4 of [27], or Section 12.2 of [55]. In particular, the real part of the polarization $\Psi_0(\cdot,\cdot)$ satisfies the 'fundamental estimate'

$$2\mathrm{Re}\,\Psi_0(z,\overline{w}) - \Phi_0(z) - \Phi_0(w) \asymp -\left|z-w\right|^2. \tag{5.2.68}$$

From (5.2.68) and the identity $\mathcal{T}_\varphi u = \Pi_{\Phi_0}\mathcal{T}_\varphi u$, it follows that

$$\left|\mathcal{T}_\varphi u(z) e^{-\Phi_0(z)/h}\right| \le C_{\Phi_0} h^{-n} \int_{\mathbb{C}^n} e^{-c|z-w|^2/h} \left|\mathcal{T}_\varphi u(w) e^{-\Phi_0(w)/h}\right| L(dw), \quad z \in \mathbb{C}^n, \quad 0 < h \le 1. \tag{5.2.69}$$

Let $\delta > 0$ be arbitary. For $|z| \ge \delta$, we have

$$\begin{aligned} \left|\mathcal{T}_\varphi u(z) e^{-\Phi_0(z)/h}\right| \le C_{\Phi_0} h^{-n} &\int_{|w|<\delta/2} e^{-c\delta^2/4h} \left|\mathcal{T}_\varphi u(w) e^{-\Phi_0(w)/h}\right| L(dw) \\ &+ \mathcal{O}(1) h^{-n} \|\mathcal{T}_\varphi u(z) e^{-\Phi_0(z)/h}\|_{L^1(\{|z|\ge\delta/2\}, L(dz))}. \end{aligned} \tag{5.2.70}$$

By the Cauchy-Schwarz inequality,

$$C_{\Phi_0} h^{-n} \int_{|w|<\delta/2} e^{-c\delta^2/4h} \left| \mathcal{T}_\varphi u(w) e^{-\Phi_0(w)/h} \right| L(dw) \leq \mathcal{O}(1) h^{-n} e^{-c/h} \leq \mathcal{O}(1) h^N, \quad 0 < h \leq 1, \tag{5.2.71}$$

for any $N \in \mathbb{N}$. Also, since the proposition has already been proven in the case $p = 1$, we know that there is $0 < h_0 \leq 1$ such that

$$\|\mathcal{T}_\varphi u(z) e^{-\Phi_0(z)/h}\|_{L^1(\{|z|\geq\delta/2\}, L(dz))} \leq \mathcal{O}(1) h^N, \quad 0 < h \leq h_0, \tag{5.2.72}$$

for any $N \in \mathbb{N}$. From (5.2.70), (5.2.71), and (5.2.72), we conclude that there is $0 < h_0 \leq 1$ such that

$$\|\mathcal{T}_\varphi u(z) e^{-\Phi_0(z)/h}\|_{L^\infty(\{|z|\geq\delta\}, L(dz))} \leq \mathcal{O}(1) h^N, \quad 0 < h \leq h_0, \tag{5.2.73}$$

for all $N \in \mathbb{N}$. Thus (5.2.37) holds when $p = \infty$. □

Using Proposition 5.2.2 and the unitarity of $\mathcal{T}_\varphi$, we can establish a simple upper bound for $\|u\|_{L^p(\mathbb{R}^n)}$ in terms of the L^1-norm of $\mathcal{T}_\varphi u(z) e^{-\Phi_0(z)/h}$ with respect to the Lebesgue measure $L(dz)$ over any bounded neighborhood of 0 in $\mathbb{C}^n$.

Proposition 5.2.3. *Let $u \in L^2(\mathbb{R}^n)$ be as in the statement of Proposition 5.2.1. Let φ be any FBI phase function on $\mathbb{C}^{2n}$ with associated FBI transform $\mathcal{T}_\varphi$ and associated strictly plurisubharmonic weight Φ_0. For any $\delta > 0$ and $N \in \mathbb{N}$, there is $0 < h_0 \leq 1$ such that for any $1 \leq p \leq \infty$ we have*

$$\|u\|_{L^p(\mathbb{R}^n)} \leq C h^{\frac{n}{2p} - \frac{3n}{4}} \int_{|z|<\delta} |\mathcal{T}_\varphi u(z)| \, e^{-\Phi_0(z)/h} \, L(dz) + C h^N, \quad 0 < h \leq h_0, \tag{5.2.74}$$

for some constant $C = C(\delta, N, p) > 0$.

Proof. Let φ be an FBI phase function on $\mathbb{C}^{2n}$ with associated FBI transform $\mathcal{T}_\varphi$ and strictly plurisubharmonic weight Φ_0. Let $H_{\Phi_0}(\mathbb{C}^n)$ be the Bargmann space introduced in (5.2.23). Let $0 < h_0 \leq 1$ be small enough so that the conclusions of Proposition 5.2.1 and Proposition 5.2.2 hold.

Since $\mathcal{T}_\varphi : L^2(\mathbb{R}^n) \to H_{\Phi_0}(\mathbb{C}^n)$ is unitary, we have

$$u(x) = \mathcal{T}_\varphi^* \mathcal{T}_\varphi u(x) = c_\varphi^2 h^{-3n/4} \int_{\mathbb{C}^n} e^{-\frac{i}{h}\overline{\varphi(z,x)}} \mathcal{T}_\varphi u(z) e^{-\frac{2}{h}\Phi_0(z)} \, L(dz), \quad x \in \mathbb{R}^n. \tag{5.2.75}$$

Because φ is quadratic and $\operatorname{Im} \varphi''_{yy} > 0$, for every $z \in \mathbb{C}^n$ there is a unique $x(z) \in \mathbb{R}^n$ such that

$$\Phi_0(z) = -\operatorname{Im} \varphi(z, x(z)). \tag{5.2.76}$$

Taking the absolute value of (5.2.75) and using (5.2.20), (5.2.21) and (5.2.76), we find that there are constants $C, c > 0$ and $0 < h_0 \leq 1$ such that

$$|u(x)| \leq C h^{-3n/4} \int_{\mathbb{C}^n} e^{-\frac{c}{h}|x - x(z)|^2} \, |\mathcal{T}_\varphi u(z)| \, e^{-\Phi_0(z)/h} \, L(dz), \quad x \in \mathbb{R}^n, \tag{5.2.77}$$

for all $0 < h \leq h_0$. Note that the righthand side of (5.2.77) is finite for every $x \in \mathbb{R}^n$ since Proposition 5.2.1 implies that $u \in \mathcal{S}(\mathbb{R}^n)$ for every $0 < h \leq h_0$ and hence $\mathcal{T}_\varphi u(z) e^{-\Phi_0(z)/h} = \mathcal{O}(h^{-n/4} \langle z \rangle^{-N})$ for any $N > 0$ by Theorem 13.4 of [63]. By direct calculation, we have

$$\| e^{-\frac{c}{h}|\cdot|^2} \|_{L^p(\mathbb{R}^n)} = \mathcal{O}(h^{\frac{n}{2p}}), \tag{5.2.78}$$

with the convention that $h^{\frac{n}{2p}} = 1$ when $p = \infty$. Taking the L^p-norm on both sides of (5.2.77), applying Minkowski's integral inequality, and using (5.2.78) gives

$$\|u\|_{L^p(\mathbb{R}^n)} \leq C h^{\frac{n}{2p} - \frac{3n}{4}} \int_{\mathbb{C}^n} |\mathcal{T}_\varphi u(z)| \, e^{-\Phi_0(z)/h} \, L(dz), \quad 0 < h \leq h_0, \tag{5.2.79}$$

for some $C > 0$. Let $\delta > 0$ be arbitrary. By Proposition 5.2.2, for every $N > 0$, there is $C > 0$ and $0 < h_0 \leq 1$ such that

$$\int_{\mathbb{C}^n} |\mathcal{T}_\varphi u(z)| \, e^{-\Phi_0(z)/h} \, L(dz) = \int_{|z| < \delta} |\mathcal{T}_\varphi u(z)| \, e^{-\Phi_0(z)/h} \, L(dz) + C h^N, \quad 0 < h \leq h_0. \tag{5.2.80}$$

Thus, for every $N \in \mathbb{N}$ and $1 \leq p \leq \infty$, there is $C = C(\delta, N, p) > 0$ such that

$$\|u\|_{L^p(\mathbb{R}^n)} \leq Ch^{\frac{n}{2p} - \frac{3n}{4}} \int_{|z|<\delta} |\mathcal{T}_\varphi u(z)|\, e^{-\Phi_0(z)/h}\, L(dz) + Ch^N, \quad 0 < h \leq h_0. \tag{5.2.81}$$

□

In the sequel, it will also be useful to have pointwise estimates available for $\mathcal{T}_\varphi u(z) e^{-\Phi_0(z)/h}$ in compact subsets of $\mathbb{C}^n$ that do not contain the origin 0. When we assume that $p_0, p_1 \in S_{\mathrm{Hol}}(m)$, it is actually true that $\mathcal{T}_\varphi u(z) e^{-\Phi_0(z)/h}$ is exponentially small in any fixed compact subset $K \subset \mathbb{C}^n$ that does not contain $0 \in \mathbb{C}^n$. This follows from standard analytic ellipticity arguments, which we review below.

We recall the notion of the semiclassical analytic wavefront set. Let φ be an FBI phase function on $\mathbb{C}^{2n}$ with associated FBI transform $\mathcal{T}_\varphi$ and strictly plurisubharmonic weight Φ_0. Let $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be the complex linear canonical transformation generated by φ, let $\pi_1 : \mathbb{C}^{2n} \to \mathbb{C}^n$ be projection onto the first factor $\pi_1 : (z, \zeta) \mapsto z$, and let $\kappa_\varphi^\flat := \pi_1 \circ \kappa_\varphi|_{\mathbb{R}^{2n}}$. For an h-dependent family $v = v(h) \in L^2(\mathbb{R}^n)$, $0 < h \leq 1$, such that $\|v\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(1)$ as $h \to 0^+$, we can define the *semiclassical analytic wavefront set* $\mathrm{WF}_{A,h}(v) \subset \mathbb{R}^{2n}$ of v as follows: a point $(x_0, \xi_0) \in \mathbb{R}^{2n}$ does not lie in $\mathrm{WF}_{A,h}(v)$ if there exist $C, c > 0$ and a bounded open neighborhood U_0 of $\kappa_\varphi^\flat(x_0, \xi_0)$ in $\mathbb{C}^n$ such that

$$|\mathcal{T}_\varphi u(z)| \leq Ce^{-c/h + \Phi_0(z)/h}, \quad z \in U_0, \quad 0 < h \leq 1. \tag{5.2.82}$$

It can be shown $\mathrm{WF}_{A,h}(v)$ is independent of the choice of FBI phase function φ. For more information regarding semiclassical analytic wavefront sets, we refer the reader to [51], [27], or [38]. The following corollary characterizes the semiclassical analytic wavefront set of any L^2-normalized ground state u of P.

Corollary 5.2.4. *If $u \in L^2(\mathbb{R}^n)$ is as in the statement of Theorem 5.1.2, then*

$$WF_{A,h}(u) = \{0\}. \tag{5.2.83}$$

Consequently, if φ is any FBI phase function on $\mathbb{C}^{2n}$ with associated strictly plurisubharmonic weight Φ_0 and K is any compact subset of $\mathbb{C}^n$ such that $0 \notin K$, then there are $C, c > 0$ and $0 < h_0 \leq 1$ such that

$$|\mathcal{T}_\varphi u(z)| \leq C e^{-c/h + \Phi_0(z)/h}, \quad z \in K, \quad 0 < h \leq h_0. \tag{5.2.84}$$

Proof. Let $P = \mathrm{Op}_h^w(p_0 + hp_1)$ be as in the statement of Theorem 5.1.2. Since $p_0, p_1 \in S_{\mathrm{Hol}}(m)$ and $p_0^{-1}(0) = \{0\}$, semiclassical analytic elliptic regularity (see, for example, Theorem 4.2.2 in [38]) implies that

$$\text{either} \quad WF_{A,h}(u) = \emptyset \quad \text{or} \quad WF_{A,h}(u) = \{0\}. \tag{5.2.85}$$

Let φ be an FBI phase function on $\mathbb{C}^{2n}$ with associated FBI transform $\mathcal{T}_\varphi$ and strictly plurisubharmonic weight Φ_0. Suppose towards contradiction that $\mathrm{WF}_{A,h}(u) = \emptyset$. Then there exist $C, c, \delta > 0$ and $0 < h_0 \leq 1$ such that

$$\sup_{|z|<\delta} \left| \mathcal{T}_\varphi u(z) e^{-\Phi_0(z)/h} \right| \leq C e^{-c/h}, \quad 0 < h \leq h_0. \tag{5.2.86}$$

Proposition 5.2.3 with $p = 2$ gives

$$\int_{|z|\geq\delta} |\mathcal{T}_\varphi u(z)|^2 \, e^{-2\Phi_0(z)/h} \, L(dz) = \mathcal{O}(h^\infty). \tag{5.2.87}$$

Combining (5.2.86) and (5.2.87) and using the unitarity of $\mathcal{T}_\varphi$ gives that

$$\|u\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h^\infty), \tag{5.2.88}$$

which is impossible since $\|u\|_{L^2(\mathbb{R}^n)} = 1$ for all $0 < h \leq 1$. Therefore, it must be the case that

$$\mathrm{WF}_{A,h}(u) = \{0\}. \tag{5.2.89}$$

If $K \subset \mathbb{C}^n$ is a compact subset such that $0 \notin K$, then we can find a finite collection of bounded open subsets U_j, $1 \leq j \leq k$, of $\mathbb{C}^n$, constants $C_j, c_j > 0$, $1 \leq j \leq k$, and $0 < h_0 \leq 1$ such that

1. $0 \notin U_j$, $1 \leq j \leq k$,
2. $K \subset \bigcup_{1\leq j\leq k} U_j$, and
3. $|\mathcal{T}_\varphi u(z)| \leq C_j e^{-c_j/h+\Phi_0(z)/h}, \quad z \in U_j, \quad 1 \leq j \leq k, \quad 0 < h \leq h_0.$

Let $U = \bigcup_{1\leq j\leq k} U_j$. We have

$$|\mathcal{T}_\varphi u(z)| \leq C e^{-c/h+\Phi_0(z)/h}, \quad z \in U, \quad 0 < h \leq h_0, \tag{5.2.90}$$

where $C = \max_{1\leq j\leq k} C_j > 0$ and $c = \min_{1\leq j\leq k} c_j > 0$. Therefore $\mathcal{T}_\varphi u$ satisfies (5.2.84). □

5.3 Dynamical Bounds on the FBI Transform Side

We begin this section a short review of Hamiltonian dynamics in the complex domain. For a textbook treatment of these concepts, see Chapter 11 of [51]. Here $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_\zeta$ is equipped with the standard holomorphic symplectic form

$$\sigma = d\zeta \wedge dz \in \Lambda^{(2,0)}(\mathbb{C}^{2n}). \tag{5.3.1}$$

Let $X \subset \mathbb{C}^{2n}$ be open and let $f \in \mathrm{Hol}(X)$. The *complex Hamilton vector field* of f is defined as the unique holomorphic vector field $H_f \in T^{1,0}(X)$ so that

$$H_f \lrcorner \sigma = -df \text{ in } X. \tag{5.3.2}$$

Explicitly,

$$H_f = \partial_\zeta f \cdot \partial_z - \partial_z f \cdot \partial_\zeta \text{ in } X. \tag{5.3.3}$$

From (5.3.2), it is clear that

$$\sigma(t, H_f) = df(t), \quad t \in T^{1,0}(X). \tag{5.3.4}$$

Since $df(t) = 0$ whenever $t \in T^{0,1}(X)$, the identity (5.3.4) also holds for every $t \in T(X) \otimes \mathbb{C}$. In particular, (5.3.4) holds whenever $t \in T(X)$. To H_f we associate the real vector field

$$\widehat{H_f} = H_f + \overline{H_f} \in T(X), \tag{5.3.5}$$

which is the unique real vector field on X such that $H_f - \widehat{H_f} \in T^{0,1}(X)$. Because σ is a holomorphic $(2,0)$-form,

$$\sigma(t, \widehat{H_f} - H_f) = 0, \quad t \in T(X) \otimes \mathbb{C}, \tag{5.3.6}$$

and hence

$$\widehat{H_f} \lrcorner \sigma = -df \text{ in } X. \tag{5.3.7}$$

When we separate the real and imaginary parts of (5.3.7), we obtain

$$\widehat{H_f} \lrcorner \operatorname{Re}\sigma = -d(\operatorname{Re} f), \quad \widehat{H_f} \lrcorner \operatorname{Im}\sigma = -d(\operatorname{Im} f). \tag{5.3.8}$$

Letting $H^{\operatorname{Re}\sigma}_{\operatorname{Re} f}$ and $H^{\operatorname{Im}\sigma}_{\operatorname{Im} f}$ denote Hamilton vector fields of $\operatorname{Re} f$ and $\operatorname{Im} f$ with respect to the real symplectic forms $\operatorname{Re}\sigma$ and $\operatorname{Im}\sigma$ on $\mathbb{C}^{2n}$, respectively, we conclude from (5.3.8) that

$$\widehat{H_f} = H^{\operatorname{Re}\sigma}_{\operatorname{Re} f} = H^{\operatorname{Im}\sigma}_{\operatorname{Im} f}. \tag{5.3.9}$$

Repeating this discussion with f replaced by if gives

$$\widehat{H_{if}} = H^{\operatorname{Im}\sigma}_{\operatorname{Re} f} = -H^{\operatorname{Re}\sigma}_{\operatorname{Im} f}, \tag{5.3.10}$$

where $H^{\operatorname{Im}\sigma}_{\operatorname{Re} f}$, resp. $H^{\operatorname{Re}\sigma}_{\operatorname{Im} f}$, is the Hamilton vector field of $\operatorname{Re} f$, resp. $\operatorname{Im} f$, with respect to $\operatorname{Im}\sigma$, resp. $\operatorname{Re}\sigma$.

Following [50], [39], and [51], we define the *complex-time Hamilton flow* κ_t, $t \in \mathbb{C}$, of f as the complex local 1-parameter family of holomorphic diffeomorphisms of X given by

$$\kappa_t = \exp(1\widehat{H_{tf}}), \quad t \in \mathbb{C}. \tag{5.3.11}$$

Here $\exp(1\widehat{H_{tf}})$ denotes the flow of the real vector field $\widehat{H_{tf}}$ in X at time 1. By abuse of notation, we shall often write $\exp(tH_f)$ in place of $\exp(1\widehat{H_{tf}})$ so that the complex-time Hamilton flow κ_t generated by H_f may be expressed as

$$\kappa_t = \exp(tH_f), \quad t \in \mathbb{C}. \tag{5.3.12}$$

Thanks to the Cauchy-Riemann equations, we have

$$[\widehat{H_f}, \widehat{H_{if}}] = 0 \tag{5.3.13}$$

for any $f \in \operatorname{Hol}(X)$. Also, if $f \in \operatorname{Hol}(X)$, it is true that

$$\widehat{H_{tf}} = (\operatorname{Re} t)\widehat{H_f} + (\operatorname{Im} t)\widehat{H_{if}} \tag{5.3.14}$$

for any $t \in \mathbb{C}$. This observation, in conjunction with (5.3.11) and (5.3.13), shows that the complex-time Hamilton flow κ_t on X generated by H_f is given by

$$\kappa_t = \exp\left[(\operatorname{Re} t)\widehat{H_f}\right] \exp\left[(\operatorname{Im} t)\widehat{H_{if}}\right], \quad t \in \mathbb{C}, \tag{5.3.15}$$

whenever the righthand side is defined. It follows that the flow κ_t is given explicitly by

$$Z(t) = \kappa_t(Z_0), \quad Z_0 \in X, \quad t \in \operatorname{neigh}(0; \mathbb{C}), \tag{5.3.16}$$

if and only if $Z(t)$ satisfies

$$\begin{cases} \partial_{\mathrm{Re}\,t} Z(t) = \widehat{H_f}(Z(t)) = (\partial_\zeta f(Z(t)), -\partial_z f(Z(t))), \\ \partial_{\mathrm{Im}\,t} Z(t) = \widehat{H_{if}}(Z(t)) = (i\partial_\zeta f(Z(t)), -i\partial_z f(Z(t))), \\ Z(0) = Z_0. \end{cases} \tag{5.3.17}$$

A straightforward verification shows that this description of the flow κ_t is equivalent to

$$(z(t), \zeta(t)) = \kappa_t(z_0, \zeta_0), \quad (z_0, \zeta_0) \in X, \quad t \in \mathrm{neigh}(0; \mathbb{C}), \tag{5.3.18}$$

if and only if $z(t)$ and $\zeta(t)$ satisfy the complex Hamilton's equations

$$\begin{cases} \partial_t z(t) = \partial_\zeta f(z(t), \zeta(t)), \\ \partial_t \zeta(t) = -\partial_z f(z(t), \zeta(t)), \\ z(0) = z_0, \ \zeta(0) = \zeta_0, \end{cases} \tag{5.3.19}$$

where $\partial_t = \frac{1}{2}(\partial_{\mathrm{Re}\,t} - i\partial_{\mathrm{Im}\,t})$. From (5.3.19), it is easy to check that κ_t preserves the complex symplectic form σ,

$$\kappa_t^* \sigma = \sigma, \quad t \in \mathbb{C}, \tag{5.3.20}$$

whenever the lefthand side is well-defined. For this reason, we say that κ_t is a *complex local* 1-*parameter family of complex canonical transformations* of X.

Let $p_0 \in S_{\mathrm{Hol}}(m)$ be as in the statement of Theorem 5.1.2, let φ be an FBI phase function on $\mathbb{C}^{2n}$ with associated FBI transform $\mathcal{T}_\varphi$, let Φ_0 be the strictly plurisubharmonic weight associated to φ, let $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be the complex linear canonical transformation generated by φ, and let

$$\mathfrak{p}_0 = p_0 \circ \kappa_\varphi^{-1} \in \mathrm{Hol}(\Lambda_{\Phi_0} + W), \tag{5.3.21}$$

where

$$\Lambda_{\Phi_0} = \left\{ \left(z, \frac{2}{i} \partial_z \Phi_0(z) \right) : z \in \mathbb{C}^n \right\}, \tag{5.3.22}$$

and W is a small bounded open neighborhood of $(0,0)$ in $\mathbb{C}^{2n}$. Let

$$H_{\mathfrak{p}_0} = \partial_\zeta \mathfrak{p}_0 \cdot \partial_z - \partial_z \mathfrak{p}_0 \cdot \partial_\zeta \in T^{1,0}(\Lambda_{\Phi_0} + W) \tag{5.3.23}$$

be the complex Hamilton vector field of $\mathfrak{p}_0$, and let

$$\kappa_t = \exp(tH_{\mathfrak{p}_0}), \quad t \in \mathbb{C}, \tag{5.3.24}$$

be the complex-time Hamilton flow of $\mathfrak{p}_0$, defined locally in $\Lambda_{\Phi_0} + W$ for sufficiently small complex times t. In this section, we shall be interested in the evolution of the subspace Λ_{Φ_0} by the flow κ_t in a small, but fixed, neighborhood of the origin $0 \in \mathbb{C}^{2n}$. Using Hamilton-Jacobi theory, it is possible to obtain a real analytic generating function for this evolution. We describe this in detail below, following essentially the discussion in [52].

Let $\mathbb{C}^{2+2n} = \mathbb{C}^2_{t,\tau} \times \mathbb{C}^{2n}_{z,\zeta}$ be equipped with the standard complex symplectic form

$$\Omega = d\tau \wedge dt + \sigma \in \Lambda^{(2,0)}(\mathbb{C}^{2+2n}), \tag{5.3.25}$$

where σ is as in (5.3.1). Let

$$G(t, \tau; z, \zeta) = \tau + \mathfrak{p}_0(z, \zeta), \quad (t, \tau; z, \zeta) \in \mathrm{neigh}((0,0;0,0); \mathbb{C}^2 \times \mathbb{C}^{2n}), \tag{5.3.26}$$

and let

$$H_G^\Omega = \frac{\partial}{\partial t} + H_{\mathfrak{p}_0} \tag{5.3.27}$$

be the complex Hamilton vector field of G with respect to Ω, defined in an open neighborhood of

$(0,0;0,0)$ in $\mathbb{C}^2 \times \mathbb{C}^{2n}$. Consider the following real $2n$-dimensional submanifold of $\mathbb{C}^{2+2n}$:

$$\mathcal{L}_0 = \left\{ \left(0, -\mathfrak{p}_0\left(z, \frac{2}{i}\partial_z \Phi_0(z)\right); z, \frac{2}{i}\partial_z \Phi_0(z)\right) : z \in \mathrm{neigh}(0; \mathbb{C}^n) \right\} \tag{5.3.28}$$

Observe that $\mathcal{L}_0$ is isotropic with respect to the real symplectic form

$$\mathrm{Im}(\Omega) = \mathrm{Im}(d\tau \wedge dt) + \mathrm{Im}\,\sigma \tag{5.3.29}$$

on $\mathbb{C}^2 \times \mathbb{C}^{2n}$ and that

$$\mathcal{L}_0 \subset G^{-1}(0). \tag{5.3.30}$$

Let

$$\exp\left(tH_G^{\Omega}\right), \quad t \in \mathrm{neigh}(0; \mathbb{C}), \tag{5.3.31}$$

be the complex-time Hamilton flow of G with respect to the complex symplectic form Ω, defined in a neighborhood of $(0,0;0,0) \in \mathbb{C}^2 \times \mathbb{C}^{2n}$ for sufficiently small complex times t. We observe that the complex-time flow generated by H_G^{Ω} is given explicitly by

$$\exp\left(tH_G^{\Omega}\right)(0,\tau;z,\zeta) = (t,\tau;\kappa_t(z,\zeta)), \quad (t,\tau;z,\zeta) \in \mathrm{neigh}((0,0;0,0); \mathbb{C}^2 \times \mathbb{C}^{2n}). \tag{5.3.32}$$

Let

$$\mathcal{L} := \bigcup_{t \in \mathrm{neigh}(0;\mathbb{C})} \exp\left(tH_G^{\Omega}\right)(\mathcal{L}_0) \tag{5.3.33}$$

be the complex time flowout of the manifold $\mathcal{L}_0$ by $\exp\left(tH_G^{\Omega}\right)$. In view of (5.3.32), we have

$$\mathcal{L} = \left\{ \left(t, -\mathfrak{p}_0\left(z, \frac{2}{i}\partial_z \Phi_0(z)\right); \kappa_t\left(z, \frac{2}{i}\partial_z \Phi_0(z)\right)\right) : t \in \mathrm{neigh}(0; \mathbb{C}),\ z \in \mathrm{neigh}(0; \mathbb{C}^n) \right\}, \tag{5.3.34}$$

when the flowout (5.3.33) is restricted to sufficiently small complex times. Also, from (5.3.11), (5.3.13), and the identities (5.3.9) and (5.3.10), we see that

$$\mathcal{L} = \bigcup_{t\in \text{neigh}(0;\mathbb{C})} \exp\left[(\operatorname{Re} t) H_{\operatorname{Im} G}^{\operatorname{Im}\Omega}\right] \exp\left[(\operatorname{Im} t) H_{\operatorname{Re} G}^{\operatorname{Im}\Omega}\right](\mathcal{L}_0), \tag{5.3.35}$$

where $\operatorname{Re} G$ and $\operatorname{Im} G$ are the real and imaginary parts of G, respectively. As is easily verified, the vector fields $H_{\operatorname{Im} G}^{\operatorname{Im}\Omega}$ and $H_{\operatorname{Re} G}^{\operatorname{Im}\Omega}$ are linearly independent and nowhere tangent to $\mathcal{L}_0$ in a neighborhood $(0,0;0,0) \in \mathbb{C}^2 \times \mathbb{C}^{2n}$. Since $\dim \mathcal{L}_0 = 2n$, it follows from Hamilton-Jacobi theory that the manifold $\mathcal{L}$ is Lagrangian for the real symplectic form $\operatorname{Im}(\Omega)$ on $\mathbb{C}^2 \times \mathbb{C}^{2n}$.

Let

$$\gamma = \operatorname{Im}(\tau\, dt) + \operatorname{Im}(\zeta\, dz) \in \Lambda^1(\mathbb{C}^2 \times \mathbb{C}^{2n}). \tag{5.3.36}$$

We observe that

$$d\gamma = \operatorname{Im}(\Omega). \tag{5.3.37}$$

Because $\mathcal{L}$ is Lagrangian with respect to $\operatorname{Im}(\Omega)$, we have

$$d\gamma|_{\mathcal{L}} \equiv 0. \tag{5.3.38}$$

Let $\pi : \mathcal{L} \to \mathbb{C} \times \mathbb{C}^n$ be the restriction of the projection $(t, \tau; z, \zeta) \mapsto (t, z)$ to $\mathcal{L}$. Since Λ_{Φ_0} is transverse to the fiber $\{0\} \times \mathbb{C}^n$ in $\mathbb{C}^{2n}$, the differential of π at $(0,0;0,0) \in \mathcal{L}$,

$$d_{(0,0;0,0)}\pi : T_{(0,0;0,0)}\mathcal{L} \to T_{(0;0)}(\mathbb{C} \times \mathbb{C}^n) \cong \mathbb{C} \times \mathbb{C}^n, \tag{5.3.39}$$

is an invertible linear transformation. Thus we may parametrize $\mathcal{L}$ by $(t, z) \in \mathbb{C} \times \mathbb{C}^n$ in a neighborhood of $(0,0;0,0)$ in $\mathcal{L}$. By (5.3.38) and Poincaré's lemma, there exists $\Phi \in C^\omega(\text{neigh}(0;\mathbb{C}) \times \text{neigh}(0;\mathbb{C}^n);\mathbb{R})$, unique up to an overall additive constant, such that

$$\gamma(t, z) = -d\Phi(t, z), \quad t \in \text{neigh}(0;\mathbb{C}), \quad z \in \text{neigh}(0;\mathbb{C}^n). \tag{5.3.40}$$

From (5.3.36), we see that

$$\gamma = \frac{1}{2i}(\tau\, dt - \overline{\tau}\, d\overline{t}) + \frac{1}{2i}\left(\zeta\, dz - \overline{\zeta}\, d\overline{z}\right), \quad (t, \tau; z, \zeta) \in \mathbb{C}^2 \times \mathbb{C}^{2n}. \tag{5.3.41}$$

We also have

$$\begin{aligned} -d\Phi(t,z) = -\partial_t \Phi(t,z)\, dt - \partial_{\overline{t}}\Phi(t,z)\, d\overline{t} \\ - \partial_z \Phi(t,z)\, dz - \partial_{\overline{z}}\Phi(t,z)\, d\overline{z}, \quad t \in \mathrm{neigh}(0;\mathbb{C}), \quad z \in \mathrm{neigh}(0;\mathbb{C}^n). \end{aligned} \tag{5.3.42}$$

From (5.3.40), (5.3.41), and (5.3.42), it follows that

$$(t, \tau; z, \zeta) \in \mathrm{neigh}((0,0;0,0);\mathcal{L}) \tag{5.3.43}$$

if and only if

$$t \in \mathrm{neigh}(0;\mathbb{C}), \quad z \in \mathrm{neigh}(0;\mathbb{C}^n), \text{ and } \begin{cases} \tau = \frac{2}{i}\partial_t \Phi(t,z), \\ \zeta = \frac{2}{i}\partial_z \Phi(t,z). \end{cases} \tag{5.3.44}$$

Thus there is $0 < T \ll 1$ and $U = \mathrm{neigh}(0;\mathbb{C}^n)$ such that

$$\kappa_t(\Lambda_{\Phi_0} \cap U \times U) \cap U \times U = \Lambda_{\Phi_t}, \quad t \in D(0,T), \tag{5.3.45}$$

where $D(0,T)$ denotes the open disc in $\mathbb{C}$ with radius T and center 0,

$$\Phi_t := \Phi(t, \cdot) \in C^\omega(U), \quad t \in D(0,T), \tag{5.3.46}$$

and

$$\Lambda_{\Phi_t} = \left\{ \left(z, \frac{2}{i}\partial_z \Phi_t(z)\right) : z \in U \right\}, \quad t \in D(0,T). \tag{5.3.47}$$

Adjusting Φ by a real constant if necessary, we may assume that

$$\Phi_t|_{t=0} = \Phi_0 \text{ in } U. \tag{5.3.48}$$

Also, it is true that Φ_t is the solution of a natural eikonal equation in $D(0,T) \times U$. Indeed, since $\mathfrak{p}_0$ is invariant under the flow κ_t, we have

$$\mathfrak{p}_0\left(z, \frac{2}{i}\partial_z \Phi_0(z)\right) = \mathfrak{p}_0\left(\kappa_t\left(z, \frac{2}{i}\partial_z\Phi_0(z)\right)\right), \quad t \in D(0,T), \quad z \in U. \tag{5.3.49}$$

From (5.3.34), (5.3.43), (5.3.44), (5.3.48), and (5.3.49), we deduce that Φ_t solves the initial value problem

$$\begin{cases} 2\partial_t \Phi_t(z) + i\mathfrak{p}_0\left(z, \frac{2}{i}\partial_z\Phi_t(z)\right) = 0, \quad t \in D(0,T), \quad z \in U, \\ \Phi_t|_{t=0} = \Phi_0 \text{ in } U, \end{cases} \tag{5.3.50}$$

where $\partial_t = \frac{1}{2}\left(\partial_{\operatorname{Re} t} - i\partial_{\operatorname{Im} t}\right)$. In the sequel, we shall refer to (5.3.50) as the *complex-time eikonal equation*.

Finally, we note that the function $\Phi_t \in C^\omega(U)$ is strictly plurisubharmonic in U for every $t \in D(0,T)$, i.e.

$$\Phi''_{t,\overline{z}z}(z) > 0, \quad t \in D(0,T), \quad z \in U. \tag{5.3.51}$$

To see why, we observe that since κ_t preserves the complex symplectic form σ, the manifold Λ_{Φ_t} is I-Lagrangian and R-symplectic for every $t \in D(0,T)$. Parametrizing Λ_{Φ_t} by $z \in U$, we find that

$$\sigma|_{\Lambda_{\Phi_t}} = \sum_{j=1}^n d\left(\frac{2}{i}\partial_{z_j}\Phi(z)\right) \wedge dz_j = \frac{2}{i}\sum_{k,j=1}^n \frac{\partial^2 \Phi_t}{\partial \overline{z}_k \partial z_j}\, d\overline{z}_k \wedge dz_j. \tag{5.3.52}$$

Because the two-form on the righthand side of (5.3.52) is real, we see that

$$\mathrm{Re}\,\sigma|_{\Lambda_{\Phi_t}} = \frac{2}{i}\sum_{k,j=1}^{n} \frac{\partial^2 \Phi_t}{\partial \overline{z}_k \partial z_j}\, d\overline{z}_k \wedge dz_j. \tag{5.3.53}$$

As $\mathrm{Re}\,\sigma|_{\Lambda_{\Phi_t}}$ is non-degenerate, we have

$$\det \Phi''_{t,\overline{z}z}(z) \neq 0, \quad t \in D(0,T), \quad z \in U. \tag{5.3.54}$$

Because also $\Phi_{0,\overline{z}z} > 0$ and $\Phi''_{t,\overline{z}z}$ depends continuously on t, it must therefore be the case that

$$\Phi''_{t,\overline{z}z}(z) > 0, \quad t \in D(0,T), \quad z \in U. \tag{5.3.55}$$

This discussion is summarized by the following proposition.

Proposition 5.3.1 ([52]). *Let $p_0 \in S_{Hol}(m)$ be as in the statement of Theorem 5.1.2, and let φ be an FBI phase function on $\mathbb{C}^{2n}$ with associated strictly plurisubharmonic weight Φ_0 and complex canonical transformation $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$. Let*

$$\Lambda_{\Phi_0} = \left\{ \left(z, \frac{2}{i}\partial_z \Phi_0(z) \right) : z \in \mathbb{C}^n \right\}, \tag{5.3.56}$$

let $\mathfrak{p}_0 := p_0 \circ \kappa_\varphi^{-1} \in Hol(\Lambda_{\Phi_0} + W)$, where W is a suitably small open neighborhood of 0 in $\mathbb{C}^{2n}$, and let $\kappa_t = \exp(tH_{\mathfrak{p}_0})$, $t \in \mathbb{C}$, be the complex-time Hamilton flow of $\mathfrak{p}_0$, defined in $\Lambda_{\Phi_0} + W$. Then there exists $0 < T \ll 1$, $U = neigh(0; \mathbb{C}^n)$, and a unique $\Phi \in C^\omega(D(0,T) \times U; \mathbb{R})$ such that

$$\kappa_t(\Lambda_{\Phi_0} \cap U \times U) \cap U \times U = \Lambda_{\Phi_t}, \quad t \in D(0,T), \tag{5.3.57}$$

and

$$\Phi_t|_{t=0} = \Phi_0 \text{ in } U, \tag{5.3.58}$$

where

$$\Phi_t := \Phi(t,\cdot) \in C^\omega(U;\mathbb{R}), \quad t \in D(0,T), \tag{5.3.59}$$

and

$$\Lambda_{\Phi_t} := \left\{ \left(z, \frac{2}{i}\partial_z \Phi_t(z) \right) : z \in U \right\}, \quad t \in D(0,T). \tag{5.3.60}$$

The function Φ_t is strictly plurisubharmonic in U for each $t \in D(0,T)$, and Φ is a solution of the complex-time eikonal equation

$$\begin{cases} 2\partial_t \Phi_t(z) + i\mathfrak{p}_0 \left(z, \tfrac{2}{i}\partial_z \Phi_t(z) \right) = 0, \quad (t,z) \in D(0,T) \times U, \\ \Phi_t|_{t=0} = \Phi_0 \ \textit{in}\ U, \end{cases} \tag{5.3.61}$$

where $\partial_t = \frac{1}{2}\left(\partial_{Re\,t} - i\partial_{Im\,t}\right)$.

Now we return our attention to eigenfunctions. Let $P = \mathrm{Op}_h^w(p_0 + hp_1)$, and $u \in L^2(\mathbb{R}^n)$ be as in the statement of Theorem 5.1.2. Let φ be any FBI phase function on $\mathbb{C}^{2n}$ with associated FBI transformation $\mathcal{T}_\varphi$ and strictly plurisubharmonic weight Φ_0, and let $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be the complex linear canonical transformation generated by φ. For $j = 0, 1$, let $\mathfrak{p}_j := p_j \circ \kappa_\varphi^{-1} \in \mathrm{Hol}(\Lambda_{\Phi_0} + W)$, where W is a suitably small bounded open neighborhood of 0 in $\mathbb{C}^{2n}$. Let $\kappa_t = \exp\left(tH_{\mathfrak{p}_0}\right)$, $t \in \mathbb{C}$ be the complex-time Hamilton flow of $\mathfrak{p}_0$, defined in $\Lambda_{\Phi_0} + W$. Suppose that $0 < T \ll 1$, $U = \mathrm{neigh}(0;\mathbb{C}^n)$, and $\Phi \in C^\omega(D(0,T) \times U;\mathbb{R})$ are as in the conclusion of Proposition 5.3.1.

For each $t \in D(0,T)$ and each open subset $U_0 \subset U$, let $H_{\Phi_t}(U_0)$ be the Hilbert space

$$H_{\Phi_t}(U_0) := L^2(U_0, e^{-2\Phi_t(z)/h}\, L(dz)) \cap \mathrm{Hol}(U_0), \tag{5.3.62}$$

equipped with the norm

$$\|v\|^2_{L^2_{\Phi_t}(U_0)} := \int_{U_0} |v(z)|^2\, e^{-2\Phi_t(z)/h}\, L(dz). \tag{5.3.63}$$

Here $\Phi_t = \Phi(t, \cdot) \in C^\omega(U)$, $t \in D(0,T)$. The main goal of this section is to show that there exists $\delta > 0$, $0 < T_0 < T$, $0 < C < \infty$, and $0 < h_0 \leq 1$, such that

$$\sup_{\substack{t \in D(0,T_0) \\ 0 < h \leq h_0}} \|\mathcal{T}_\varphi u\|_{L^2_{\Phi_t}(\{|z|<\delta\})} \leq C. \tag{5.3.64}$$

To begin the proof, let

$$\mathcal{U}(t,z;h) := \mathcal{T}_\varphi u(h)(z), \quad (t,z) \in \mathbb{C} \times \mathbb{C}^n, \quad 0 < h \leq 1. \tag{5.3.65}$$

Let

$$\tilde{P} = \mathrm{Op}^w_{\Phi_0,h}(\mathfrak{p}_0 + h\mathfrak{p}_1). \tag{5.3.66}$$

By Egorov's theorem,

$$\tilde{P}\mathcal{U}(t,z;h) = 0, \quad (t,z) \in \mathbb{C} \times \mathbb{C}^n, \quad 0 < h \leq 1. \tag{5.3.67}$$

Thus $\mathcal{U}$ is trivially a solution of the semiclassical Schrödinger initial value problem

$$\begin{cases} \left(hD_t + \tilde{P}\right)\mathcal{U}(t,z;h) = 0, \ (t,z) \in \mathbb{C} \times \mathbb{C}^n, \quad 0 < h \leq 1, \\ \mathcal{U}(0,z;h) = \mathcal{T}_\varphi u(h)(z), \quad z \in \mathbb{C}^n, \quad 0 < h \leq 1, \end{cases} \tag{5.3.68}$$

where

$$D_t := \frac{1}{i}\partial_t, \quad \partial_t := \frac{1}{2}\left(\partial_{\mathrm{Re}\, t} - i\partial_{\mathrm{Im}\, t}\right). \tag{5.3.69}$$

Let $\delta > 0$ be small enough so that

$$\{|z| < 5\delta\} \subset\subset U. \tag{5.3.70}$$

Let $\chi \in C_0^\infty(\mathbb{C}^n; [0,1])$ be such that

$$\chi(z) = 1, \quad |z| \leq 2\delta, \quad \text{and} \quad \chi(z) = 0, \quad |z| \geq 3\delta, \tag{5.3.71}$$

and let $\widetilde{\chi} \in C_0^\infty(\mathbb{C}^n; [0,1])$ be such that

$$\widetilde{\chi}(z) = 1, \quad |z| \leq 4\delta, \quad \text{and} \quad \widetilde{\chi}(z) = 0, \quad |z| \geq 5\delta. \tag{5.3.72}$$

Let

$$\widetilde{\Phi}_t := (1 - \widetilde{\chi})\Phi_0 + \widetilde{\chi}\Phi_t, \quad t \in D(0,T). \tag{5.3.73}$$

By construction, $\widetilde{\Phi}_t \in C^\infty(\mathbb{C}^n; \mathbb{R})$ for each $t \in D(0,T)$, and we have

$$\widetilde{\Phi}_t(z) = \Phi_t(z), \quad |z| \leq 2\delta, \quad t \in D(0,T), \tag{5.3.74}$$

and

$$\widetilde{\Phi}_t(z) = \Phi_0(z), \quad |z| \geq 5\delta, \quad t \in D(0,T). \tag{5.3.75}$$

To each $\widetilde{\Phi}_t$, we associate the Hilbert space

$$H_{\widetilde{\Phi}_t}(\mathbb{C}^n) = L^2(\mathbb{C}^n, e^{-2\widetilde{\Phi}_t(z)/h}\, L(dz)) \cap \mathrm{Hol}(\mathbb{C}^n), \tag{5.3.76}$$

which is equipped with the inner product

$$(v_1, v_2)_{L^2_{\widetilde{\Phi}_t}(\mathbb{C}^n)} := \int_{\mathbb{C}^n} v_1(z)\overline{v_2(z)} e^{-2\widetilde{\Phi}_t(z)/h}\, L(dz), \quad v_1, v_2 \in L^2_{\widetilde{\Phi}_t}(\mathbb{C}^n), \tag{5.3.77}$$

inherited from

$$L^2_{\widetilde{\Phi}_t}(\mathbb{C}^n) := L^2(\mathbb{C}^n, e^{-2\widetilde{\Phi}_t(z)/h}\, L(dz)). \tag{5.3.78}$$

Let

$$\|v\|^2_{L^2_{\widetilde{\Phi}_t}(\mathbb{C}^n)} = \int_{\mathbb{C}^n} |v(z)|^2 \, e^{-2\widetilde{\Phi}_t(z)/h} \, L(dz), \quad v \in L^2_{\widetilde{\Phi}_t}(\mathbb{C}^n), \tag{5.3.79}$$

denote the corresponding norm.

Consider the quantity

$$M_t := (\chi \mathcal{U}, \mathcal{U})_{L^2_{\widetilde{\Phi}_t}(\mathbb{C}^n)}, \quad t \in D(0,T). \tag{5.3.80}$$

In view of (5.3.68), we have

$$hD_t M_t = -\left(\chi \tilde{P} \mathcal{U}, \mathcal{U}\right)_{L^2_{\widetilde{\Phi}_t}(\mathbb{C}^n)} - \frac{2}{i}\left(\chi \partial_t \widetilde{\Phi}_t \mathcal{U}, \mathcal{U}\right)_{L^2_{\widetilde{\Phi}_t}(\mathbb{C}^n)} \tag{5.3.81}$$

Let $0 < T_0 < T$ be small enough so that

$$\Lambda_{\widetilde{\Phi}_t} := \left\{ \left(z, \frac{2}{i}\partial_z \widetilde{\Phi}_t(z)\right) : z \in \mathbb{C}^n \right\} \subset \Lambda_{\Phi_0} + W, \quad t \in D(0,T_0) \tag{5.3.82}$$

and

$$\widetilde{\Phi}''_{t,\bar{z}z}(z) > 0, \quad z \in \mathbb{C}^n, \quad t \in D(0,T_0). \tag{5.3.83}$$

By taking T_0 smaller, we may make the quantity

$$\max_{k=0,1,2} \sup_{t \in D(0,T_0)} \|\nabla^k \widetilde{\Phi}_t - \nabla^k \Phi_0\|_{L^\infty(\mathbb{C}^n)} \tag{5.3.84}$$

as small as we wish. Thus, if $0 < T_0 < T$ is sufficiently small, the Hilbert spaces $L^2_{\widetilde{\Phi}_t}(\mathbb{C}^n)$ and $L^2_{\Phi_0}(\mathbb{C}^n)$ agree for every $t \in D(0,T_0)$ and have equivalent norms. We note, however, that these norms are not uniformly equivalent as $h \to 0^+$. From Lemma 12.7 of [55], we know that

$$\chi \tilde{P} = \mathcal{O}(1) : H_{\widetilde{\Phi}_t}(\mathbb{C}^n) \to L^2_{\widetilde{\Phi}_t}(\mathbb{C}^n), \tag{5.3.85}$$

uniformly for $t \in D(0, T_0)$ and $0 < h \ll 1$. By the quantization-multiplication theorem (Proposition 12.10 in [55]), there is $0 < h_0 \leq 1$ such that

$$\begin{aligned} &\left(\chi \tilde{P} \mathcal{U}, \mathcal{U}\right)_{L^2_{\widetilde{\Phi}_t}(\mathbb{C}^n)} \\ &= \int_{\mathbb{C}^n} \chi(z) \left[\mathfrak{p}_0\left(z, \frac{2}{i}\partial_z \widetilde{\Phi}_t(z)\right) + h\mathfrak{p}_1\left(z, \frac{2}{i}\partial_z \widetilde{\Phi}_t(z); h\right)\right] |\mathcal{U}(t,z;h)|^2 \, e^{-2\widetilde{\Phi}_t(z)/h} \, L(dz) \\ &\quad + \mathcal{O}\left(h \|\mathcal{U}\|^2_{L^2_{\widetilde{\Phi}_t(\mathbb{C}^n)}}\right), \quad t \in D(0, T_0), \quad 0 < h \leq h_0. \end{aligned} \tag{5.3.86}$$

Thus

$$hD_t M_t = \mathrm{I}(t;h) + \mathrm{II}(t;h) + \mathrm{III}(t;h), \quad t \in D(0, T_0), \quad 0 < h \leq h_0, \tag{5.3.87}$$

where

$$\begin{aligned} \mathrm{I}(t;h) &:= i \int_{\mathbb{C}^n} \chi(z) \left[2\partial_t \widetilde{\Phi}_t(z) + i\mathfrak{p}_0\left(z, \frac{2}{i}\partial_z \widetilde{\Phi}_t(z)\right)\right] |\mathcal{U}(t,z;h)|^2 \, e^{-2\widetilde{\Phi}_t(z)/h} \, L(dz), \\ \mathrm{II}(t;h) &:= -h \int_{\mathbb{C}^n} \chi(z) \, \mathfrak{p}_1\left(z, \frac{2}{i}\partial_z \widetilde{\Phi}_t(z); h\right) |\mathcal{U}(t,z;h)|^2 \, e^{-2\widetilde{\Phi}_t(z)/h} \, L(dz), \end{aligned} \tag{5.3.88}$$

and $\mathrm{III}(t;h)$ is such that

$$\mathrm{III}(t;h) = \mathcal{O}\left(h \|\mathcal{U}\|^2_{L^2_{\widetilde{\Phi}_t(\mathbb{C}^n)}}\right), \quad t \in D(0, T_0), \quad 0 < h \leq h_0. \tag{5.3.89}$$

As a consequence of (5.3.72) and (5.3.73), we have

$$\widetilde{\Phi}_t(z) = \Phi_t(z), \quad \partial_z \widetilde{\Phi}_t(z) = \partial_z \Phi_t(z), \quad |z| \leq 3\delta, \quad t \in D(0, T_0). \tag{5.3.90}$$

From (5.3.90), (5.3.71), and (5.3.61), we deduce

$$\mathrm{I}(t;h) = i \int_{|z| \leq 3\delta} \chi(z) \left[2\partial_t \Phi_t(z) + i\mathfrak{p}_0\left(z, \frac{2}{i}\partial_z \Phi_t(z)\right)\right] |\mathcal{U}(t,z;h)|^2 \, e^{-2\Phi_t(z)/h} \, L(dz) = 0 \tag{5.3.91}$$

for all $t \in D(0, T_0)$ and $0 < h \leq h_0$. Since $p_1 \in S_{\mathrm{Hol}}(m)$ and $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is linear, there exists

$C > 0$ and $N \in \mathbb{R}$ such that

$$\left|\mathfrak{p}_1\left(z, \frac{2}{i}\partial_z \widetilde{\Phi}_t(z); h\right)\right| \leq C\langle z\rangle^N, \quad z \in \mathbb{C}^n, \quad t \in D(0, T_0), \quad 0 < h \leq h_0. \tag{5.3.92}$$

In particular, there is $C > 0$ such that

$$\left|\mathfrak{p}_1\left(z, \frac{2}{i}\partial_z \widetilde{\Phi}_t(z); h\right)\right| \leq C, \quad |z| \leq 3\delta, \quad t \in D(0, T_0), \quad 0 < h \leq h_0. \tag{5.3.93}$$

It follows that

$$\mathrm{II}(t; h) = \mathcal{O}\left(hM_t\right), \quad t \in D(0, T_0), \quad 0 < h \leq h_0. \tag{5.3.94}$$

Next, observe that

$$\|\mathcal{U}\|^2_{L^2_{\widetilde{\Phi}_t}(\mathbb{C}^n)} = M_t + \int_{\mathbb{C}^n} (1 - \chi(z)) \left|\mathcal{T}_\varphi u(z)\right|^2 e^{-2\widetilde{\Phi}_t(z)/h} \, L(dz), \quad t \in D(0, T_0), \quad 0 < h \leq h_0. \tag{5.3.95}$$

Since χ satisfies (5.3.71), we have

$$\int_{\mathbb{C}^n} (1 - \chi(z)) \left|\mathcal{T}_\varphi u(z)\right|^2 e^{-2\widetilde{\Phi}_t(z)/h} \, L(dz) \leq \int_{|z| \geq 2\delta} \left|\mathcal{T}_\varphi u(z)\right|^2 e^{-2\widetilde{\Phi}_t(z)/h} \, L(dz), \tag{5.3.96}$$

for all $t \in D(0, T_0)$ and $0 < h \leq h_0$. From (5.3.72) and (5.3.73), we see that

$$\widetilde{\Phi}_t(z) = \Phi_0(z), \quad |z| \geq 5\delta, \quad t \in D(0, T_0). \tag{5.3.97}$$

Hence

$$\int_{|z| \geq 2\delta} \left|\mathcal{T}_\varphi u(z)\right|^2 e^{-2\widetilde{\Phi}_t(z)/h} \, L(dz) = A(t; h) + B(t; h), \quad t \in D(0, T_0), \quad 0 < h \leq h_0, \tag{5.3.98}$$

where

$$A(t; h) := \int_{2\delta \leq |z| \leq 5\delta} \left|\mathcal{T}_\varphi u(z)\right|^2 e^{-2\widetilde{\Phi_t}(z)/h} \, L(dz), \quad t \in D(0, T_0), \quad 0 < h \leq h_0, \tag{5.3.99}$$

and

$$B(t;h) := \int_{|z|>5\delta} |\mathcal{T}_\varphi u(z)|^2 \, e^{-2\Phi_0(z)/h} \, L(dz), \quad t \in D(0,T_0), \quad 0 < h \leq h_0. \tag{5.3.100}$$

We note that $B(t;h)$ is independent of t. Taking h_0 smaller if necessary, we get from Corollary 5.2.4 that there are constants $C, c > 0$ such that

$$|\mathcal{T}_\varphi u(z)| \leq C e^{-c/h + \Phi_0(z)/h}, \quad 2\delta \leq |z| \leq 5\delta, \quad 0 < h \leq h_0. \tag{5.3.101}$$

Since $\widetilde{\Phi}_t$ depends continuously on t and $\widetilde{\Phi}_0 = \Phi_0$, we may assume, after taking T_0 smaller if necessary, that

$$\left|\Phi_0(z) - \widetilde{\Phi}_t(z)\right| \leq \frac{c}{2}, \quad |z| \leq 5\delta, \quad t \in D(0,T_0). \tag{5.3.102}$$

It follows that

$$|\mathcal{T}_\varphi u(z)|^2 \, e^{-2\widetilde{\Phi}_t(4z)/h} \leq |\mathcal{T}_\varphi u(z)|^2 \, e^{-2\Phi_0(z)/h} e^{c/h} = \mathcal{O}(e^{-c/h}), \quad 2\delta \leq |z| \leq 5\delta, \quad 0 < h \leq h_0. \tag{5.3.103}$$

Thus

$$A(t;h) = \mathcal{O}(e^{-c/h}), \quad t \in D(0,T_0), \quad 0 < h \leq h_0. \tag{5.3.104}$$

On the other hand, Proposition 5.2.2 gives

$$B(t;h) = \mathcal{O}(h^\infty), \quad t \in D(0,T_0), \quad 0 < h \leq h_0. \tag{5.3.105}$$

Putting (5.3.95), (5.3.104), and (5.3.105) together, we obtain

$$\|\mathcal{U}\|^2_{L^2_{\widetilde{\Phi}_t}(\mathbb{C}^n)} = M_t + \mathcal{O}(h^\infty), \quad t \in D(0,T_0), \quad 0 < h \leq h_0. \tag{5.3.106}$$

Then, from (5.3.89) and (5.3.106), we get

$$\mathrm{III}(t;h) = \mathcal{O}(hM_t) + \mathcal{O}(h^\infty), \quad t \in D(0,T_0), \quad 0 < h \le h_0. \tag{5.3.107}$$

Combining (5.3.87), (5.3.91), (5.3.95), and (5.3.107) therefore yields

$$hD_t M_t = \mathcal{O}(hM_t) + \mathcal{O}(h^\infty), \quad t \in D(0,T_0), \quad 0 < h \le h_0. \tag{5.3.108}$$

We conclude that

$$\partial_t M_t = \mathcal{O}(M_t) + \mathcal{O}(h^\infty), \quad t \in D(0,T_0), \quad 0 < h \le h_0. \tag{5.3.109}$$

Let $\alpha \in \{|z| = 1\}$ be arbitrary. The function

$$(-T_0, T_0) \ni s \mapsto M_{\alpha s} \tag{5.3.110}$$

is smooth and real-valued. By the chain rule,

$$\frac{d}{ds} M_{\alpha s} = \left.\partial_t M_t\right|_{t=\alpha s} \alpha + \left.\partial_{\bar{t}} M_t\right|_{t=\alpha s} \overline{\alpha} = 2\mathrm{Re}\,\left(\left.\partial_t M_t\right|_{t=\alpha s} \alpha\right), \quad s \in (-T_0, T_0). \tag{5.3.111}$$

Bounding the righthand side of (5.3.111) using (5.3.109), we find that for any $N > 0$, there is $C > 0$, independent of α, such that

$$\frac{d}{ds} M_{\alpha s} \le CM_{\alpha s} + Ch^N, \quad s \in (-T_0, T_0), \quad 0 < h \le h_0. \tag{5.3.112}$$

By Gronwall's inequality, for any $N > 0$, there is $C > 0$, independent of α, such that

$$M_{\alpha s} \le CM_0 + Ch^N, \quad s \in [0, T_0), \quad 0 < h \le h_0. \tag{5.3.113}$$

Since

$$M_0 \leq \|\mathcal{T}_\varphi u\|^2_{L^2_{\Phi_0}(\mathbb{C}^n)} = 1, \quad 0 < h \leq h_0, \tag{5.3.114}$$

and (5.3.113) holds for any $\alpha \in \{|z| = 1\}$, we conclude, after taking h_0 smaller if necessary, that

$$M_t \leq C, \quad t \in D(0, T_0), \quad 0 < h \leq h_0, \tag{5.3.115}$$

for some constant $0 < C < \infty$. In view of (5.3.71), (5.3.72), and (5.3.115), we have

$$\|\mathcal{T}_\varphi u\|_{L^2_{\Phi_t}(\{|z|<\delta\})} \leq M_t^{1/2} < C^{1/2}, \quad t \in D(0, T_0), \quad 0 < h \leq h_0. \tag{5.3.116}$$

Therefore it is true that (5.3.64) holds for some $\delta > 0$ and $0 < T_0 < T$. We have proved the following proposition.

Proposition 5.3.2. *Let $P = Op_h^w(p_0 + hp_1)$ and $u \in L^2(\mathbb{R}^n)$ be as in the statement of Theorem 5.1.2. Let φ be any FBI phase function on $\mathbb{C}^{2n}$ with associated FBI transformation $\mathcal{T}_\varphi$ and strictly plurisubharmonic weight Φ_0, and let $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be the complex linear canonical transformation generated by φ. Let $\mathfrak{p}_j := p_j \circ \kappa_\varphi^{-1} \in Hol(\Lambda_{\Phi_0} + W)$, $j = 0, 1$, where W is a suitably small bounded open neighborhood of 0 in $\mathbb{C}^{2n}$, and let $\kappa_t = \exp(tH_{\mathfrak{p}_0})$, $t \in \mathbb{C}$ be the complex-time Hamilton flow of $\mathfrak{p}_0$, defined in $\Lambda_{\Phi_0} + W$. Suppose that $0 < T \ll 1$, $U = neigh(0; \mathbb{C}^n)$, and $\Phi \in C^\omega(D(0,T) \times U; \mathbb{R})$ are such that*

$$\kappa_t(\Lambda_{\Phi_0} \cap U \times U) \cap U \times U = \Lambda_{\Phi_t}, \quad t \in D(0, T), \tag{5.3.117}$$

and

$$\Phi_t|_{t=0} = \Phi_0 \text{ in } U, \tag{5.3.118}$$

where

$$\Phi_t = \Phi(t,\cdot) \in C^\omega(U;\mathbb{R}), \quad t \in D(0,T), \tag{5.3.119}$$

and

$$\Lambda_{\Phi_t} = \left\{ \left(z, \frac{2}{i}\partial_z \Phi_t(z)\right) : z \in U \right\}, \quad t \in D(0,T). \tag{5.3.120}$$

Then there are constants $\delta > 0$, $0 < T_0 < T$, $0 < C < \infty$, *and* $0 < h_0 \leq 1$ *such that*

$$\sup_{\substack{t \in D(0,T_0) \\ 0<h\leq h_0}} \|\mathcal{T}_\varphi u\|_{L^2_{\Phi_t}(\{|z|<\delta\})} \leq C. \tag{5.3.121}$$

5.4 The Conclusion of the Proof of Theorem 5.1.2

Let p_0 be as in the statement of Theorem 5.1.2. We start this section by showing that there exists an FBI phase function φ on $\mathbb{C}^{2n}$ whose associated complex linear canonical transformation $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is such that the quadratic approximation $\mathfrak{q}$ to $\mathfrak{p}_0 := p \circ \kappa_\varphi^{-1}$ at the origin $0 \in \mathbb{C}^{2n}$ has the convenient form

$$\mathfrak{q}(z,\zeta) = Mz \cdot \zeta, \quad (z,\zeta) \in \mathbb{C}^{2n}. \tag{5.4.1}$$

where M is a suitable complex $n \times n$ matrix.

Let q be the quadratic approximation to p_0 at $0 \in \mathbb{C}^{2n}$. Since q is elliptic along its singular space S, it follows from Proposition 2.0.1 of [23] that there exists a symplectic splitting of the coordinates of $\mathbb{R}^{2n}$,

$$\mathbb{R}^{2n} = \mathbb{R}^{2n'} \times \mathbb{R}^{2n''}, \quad n = n' + n'', \quad 0 \leq n', n'' \leq n, \quad (x,\xi) = (x',\xi';x'',\xi''), \tag{5.4.2}$$

and an $\mathbb{R}$-linear canonical transformation $\kappa_{\Re} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that

$$(q \circ \kappa_{\Re}^{-1})(x, \xi) = q_1(x', \xi') + q_2(x'', \xi''), \quad (x, \xi) \in \mathbb{R}^{2n}, \tag{5.4.3}$$

where q_1 is a complex-valued quadratic form on $\mathbb{R}^{2n'}$ having non-negative real part $\operatorname{Re} q_1 \geq 0$ and trivial singular space, $S_1 = \{0\}$, and q_2 is a purely imaginary quadratic form on $\mathbb{R}^{2n''}$ of the form

$$q_2(x'', \xi'') = i\epsilon \sum_{j=1}^{n''} \lambda_j (|x''_j|^2 + |\xi''_j|^2), \quad (x'', \xi'') \in \mathbb{R}^{2n''}, \tag{5.4.4}$$

where $\epsilon \in \{\pm 1\}$ and $\lambda_j > 0$ for all $1 \leq j \leq n''$. We will show that there exist FBI phase functions φ_1 and φ_2 on $\mathbb{C}^{2n'}$ and $\mathbb{C}^{2n''}$, respectively, with associated complex linear canonical transformations $\kappa_{\varphi_1} : \mathbb{C}^{2n'} \to \mathbb{C}^{2n'}$ and $\kappa_{\varphi_2} : \mathbb{C}^{2n''} \to \mathbb{C}^{2n''}$ such that

$$\mathfrak{q}_1(z', \zeta') := \left(q_1 \circ \kappa_{\varphi_1}^{-1}\right)(z', \zeta') = M_1 z' \cdot \zeta', \quad (z', \zeta') \in \mathbb{C}^{2n'}, \tag{5.4.5}$$

and

$$\mathfrak{q}_2(z'', \zeta'') := \left(q_2 \circ \kappa_{\varphi_2}^{-1}\right)(z'', \zeta'') = M_2 z'' \cdot \zeta'', \quad (z'', \zeta'') \in \mathbb{C}^{2n''}, \tag{5.4.6}$$

for some $M_1 \in \mathrm{Mat}_{n' \times n'}(\mathbb{C})$ and $M_2 \in \mathrm{Mat}_{n'' \times n''}(\mathbb{C})$.

We begin by proving the existence of an FBI phase function φ_1 on $\mathbb{C}^{2n'}$ such that (5.4.5) holds. For this, we will closely follow the presentation of Section 2 of [62]. The method we employ originates from the works [28] and [58]. Let $\mathbb{C}^{2n'} = \mathbb{C}^{n'}_{z'} \times \mathbb{C}^{n'}_{\zeta'}$ be equipped with the standard complex symplectic form $\sigma^{(1)} = d\zeta' \wedge dz' \in \Lambda^{(2,0)}(\mathbb{C}^{2n'})$. Let F_1 be the Hamilton matrix of q_1. Since the singular space S_1 of q_1 is trivial, $S_1 = \{0\}$, we know from the work [23] that F_1 possesses no real eigenvalues. It follows that

$$\#\left\{\lambda \in \operatorname{Spec}(F_1) : \operatorname{Im} \lambda > 0\right\} = \#\left\{\lambda \in \operatorname{Spec}(F_1) : \operatorname{Im} \lambda < 0\right\}, \tag{5.4.7}$$

when counting algebraic multiplicities. For $\lambda \in \operatorname{Spec}(F_1)$, let us denote the generalized eigenspace of F_1 corresponding to $\lambda \in \mathrm{Spec}(F_1)$ by

$$V_\lambda = \ker((F_1 - \lambda I)^{2n}) \subset \mathbb{C}^{2n'}. \tag{5.4.8}$$

Next, let

$$\Lambda^+ := \bigoplus_{\substack{\lambda \in \mathrm{Spec}(F_1) \\ \mathrm{Im}\,\lambda > 0}} V_\lambda, \quad \Lambda^- := \bigoplus_{\substack{\lambda \in \mathrm{Spec}(F_1) \\ \mathrm{Im}\,\lambda < 0}} V_\lambda, \tag{5.4.9}$$

denote the stable outgoing and stable incoming manifolds for the quadratic form $-iq_1$, respectively. From Proposition 2.1 of the work [58], we know that Λ^+ is a strictly positive $\mathbb{C}$-Lagrangian subspace of $\mathbb{C}^{2n'}$ and that Λ^- is a strictly negative $\mathbb{C}$-Lagrangian subspace of $\mathbb{C}^{2n'}$. For background on positive and negative $\mathbb{C}$-Lagrangian subspaces of $\mathbb{C}^{2n}$, see [27] and [12]. From the discussion on pages 488-489 of [27], we may thus conclude that there is an FBI phase function φ_1 on $\mathbb{C}^{2n'}$ such that the complex linear canonical transformation $\kappa_{\varphi_1} : \mathbb{C}^{2n'} \to \mathbb{C}^{2n'}$ generated by φ_1 satisfies

$$\kappa_{\varphi_1}(\Lambda^+) = \left\{ (z', 0) : z' \in \mathbb{C}^{n'} \right\}, \quad \kappa_{\varphi_1}(\Lambda^-) = \left\{ (0, \zeta') : \zeta' \in \mathbb{C}^{n'} \right\}. \tag{5.4.10}$$

Let

$$\Phi_0^{(1)}(z') := \max_{y' \in \mathbb{R}^{n'}} \left(-\mathrm{Im}\, \varphi_1(z', y') \right), \quad z' \in \mathbb{C}^{n'}, \tag{5.4.11}$$

be the strictly plurisubharmonic weight on $\mathbb{C}^{n'}$ associated to φ_1, and let

$$\Lambda_{\Phi_0^{(1)}} = \left\{ \left(z, \frac{2}{i} \partial_{z'} \Phi_0^{(1)}(z') \right) : z' \in \mathbb{C}^{n'} \right\}. \tag{5.4.12}$$

The subspace $\Lambda_{\Phi_0^{(1)}}$ of $\mathbb{C}^{2n'}$ is I-Lagrangian and R-symplectic for the complex symplectic form $\sigma^{(1)}$, and we have

$$\kappa_{\varphi_1}(\mathbb{R}^{2n'}) = \Lambda_{\Phi_0^{(1)}}. \tag{5.4.13}$$

Since Λ^+ is a strictly positive $\mathbb{C}$-Lagrangian subspace of $\mathbb{C}^{2n'}$, the base $\left\{(z',0) : z' \in \mathbb{C}^{n'}\right\}$ is a $\mathbb{C}$-Lagrangian subspace of $\mathbb{C}^{2n'}$ that is strictly positive relative to $\Lambda_{\Phi_0^{(1)}}$. As explained in Chapter 11 of [51], we may therefore conclude that the strictly plurisubharmonic quadratic form $\Phi_0^{(1)}$ on $\mathbb{C}^{n'}$ is in fact strictly convex.

Let

$$\mathfrak{q}_1 := q \circ \kappa_{\varphi_1}^{-1} \in \mathrm{Hol}(\mathbb{C}^{2n'}). \tag{5.4.14}$$

Since V_λ is invariant under F_1 for every $\lambda \in \mathrm{Spec}(F_1)$, we know that Λ^+ and Λ^- are both invariant under F_1. Since also Λ^+ and Λ^- are Lagrangian with respect to $\sigma^{(1)}$, we have

$$q_1(Z') = \sigma^{(1)}(Z', FZ') = 0, \quad Z' \in \Lambda^+ \cup \Lambda^-. \tag{5.4.15}$$

From (5.4.10), (5.4.14), and (5.4.15), we deduce that

$$\mathfrak{q}_1(z',\zeta') = M_1 z' \cdot \zeta', \quad (z',\zeta') \in \mathbb{C}^{2n'}, \tag{5.4.16}$$

for some $M_1 \in \mathrm{Mat}_{n'\times n'}(\mathbb{C})$. Thus φ_1 is an FBI phase function on $\mathbb{C}^{2n'}$ satisfying (5.4.5).

Now we prove that there is an FBI phase function φ_2 on $\mathbb{C}^{2n''}$ satisfying (5.4.6). Let $\mathbb{C}^{2n''} = \mathbb{C}^{n'}_{z'} \times \mathbb{C}^{n'}_{\zeta'}$ be equipped with the complex symplectic form $\sigma^{(2)} = d\zeta'' \wedge dz'' \in \Lambda^{(2,0)}(\mathbb{C}^{2n''})$. Let

$$\varphi_2(z'',y'') = \frac{i}{2}(z'')^2 - i\sqrt{2}z''\cdot y'' + \frac{i}{2}y''\cdot y'', \quad (z'',y'') \in \mathbb{C}^{2n''}, \tag{5.4.17}$$

be the 'standard FBI phase' on $\mathbb{C}^{2n''}$ (see the discussion on pages 304-306 of [63]). A straightforward computation shows that the complex linear canonical transformation $\kappa_{\varphi_2} : \mathbb{C}^{2n''} \to \mathbb{C}^{2n''}$ generated by φ_1 is

$$\kappa_{\varphi_2}(x'',\xi'') = \frac{1}{\sqrt{2}}(x'' - i\xi'', \xi'' - ix''), \quad (x'',\xi'') \in \mathbb{C}^{2n''}. \tag{5.4.18}$$

The inverse of κ_{φ_2} is then easily seen to be

$$\kappa_{\varphi_2}^{-1}(z'', \zeta'') = \frac{1}{\sqrt{2}}(z'' + i\zeta'', \zeta'' + iz''), \quad (z'', \zeta'') \in \mathbb{C}^{2n''}. \tag{5.4.19}$$

A direct computation using (5.4.17) shows that the strictly plurisubharmonic weight associated to φ_2 is

$$\Phi_0^{(2)}(z'') := \frac{|z''|^2}{2}, \quad z'' \in \mathbb{C}^{n''}, \tag{5.4.20}$$

and we have

$$\kappa_{\varphi_2}(\mathbb{R}^{2n''}) = \Lambda_{\Phi_0^{(2)}} := \left\{ \left(z'', \frac{\overline{z''}}{i} \right) : z'' \in \mathbb{C}^{n''} \right\}. \tag{5.4.21}$$

Using (5.4.4) and (5.4.19), we see that

$$\begin{aligned} \mathfrak{q}_2(z'', \zeta'') := (q_2 \circ \kappa_{\varphi_2}^{-1})(z'', \zeta'') &= i\epsilon \sum_{j=1}^{n''} \lambda_j \left(\left(\frac{z_j'' + i\zeta_j''}{\sqrt{2}} \right)^2 + \left(\frac{\zeta_j'' + iz_j''}{\sqrt{2}} \right)^2 \right) \\ &= -2\epsilon \sum_{j=1}^{n''} \lambda_j z_j'' \zeta_j'', \quad (z'', \zeta'') \in \mathbb{C}^{2n''}. \end{aligned} \tag{5.4.22}$$

Hence

$$\mathfrak{q}_2(z'', \zeta'') = M_2 z'' \cdot \zeta'', \quad (z'', \zeta'') \in \mathbb{C}^{2n''}, \tag{5.4.23}$$

where $M_2 \in \mathrm{Mat}_{n'' \times n''}(\mathbb{C})$ is the diagonal matrix

$$M_2 = \mathrm{diag}(-2\epsilon\lambda_1, \ldots, -2\epsilon\lambda_{n''}). \tag{5.4.24}$$

Write

$$\mathbb{C}^{2n} = \mathbb{C}^{2n'} \times \mathbb{C}^{2n''}, \quad (z, \zeta) = (z', \zeta'; z'', \zeta''). \tag{5.4.25}$$

For $j = 0, 1$, let

$$\mathfrak{p}_j := p_j \circ (\kappa_{\mathfrak{R}}^{\mathbb{C}})^{-1} \circ (\kappa_{\varphi_1}^{-1} \times \kappa_{\varphi_2}^{-1}) \in \mathrm{Hol}\left(\Lambda_{\Phi_0^{(1)}} \times \Lambda_{\Phi_0^{(2)}} + W_1 \times W_2\right), \tag{5.4.26}$$

where W_1 and W_2 are sufficiently small bounded open neighborhoods of 0 in $\mathbb{C}^{2n'}$ and $\mathbb{C}^{2n''}$ respectively, and $\kappa_{\mathfrak{R}}^{\mathbb{C}} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is the complexification of $\kappa_{\mathfrak{R}} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. From (5.4.3), (5.4.16), (5.4.22), and (5.4.23), we deduce that the quadratic approximation $\mathfrak{q}$ to $\mathfrak{p}_0$ at $0 \in \mathbb{C}^{2n}$ is given by

$$\mathfrak{q}(z, \zeta) := Mz \cdot \zeta, \quad (z, \zeta) \in \mathbb{C}^{2n}, \tag{5.4.27}$$

for the matrix

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \in \mathrm{Mat}_{n\times n}(\mathbb{C}). \tag{5.4.28}$$

Let

$$\kappa = (\kappa_{\varphi_1} \times \kappa_{\varphi_2}) \circ \kappa_{\mathfrak{R}}^{\mathbb{C}} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}. \tag{5.4.29}$$

We would like to introduce a metaplectic FBI transform $\mathcal{T}_\varphi$ on $\mathbb{R}^n$ whose underlying complex linear canonical transformation is κ. The existence of such an FBI transform is guaranteed by the following proposition.

Proposition 5.4.1. *There exists a unique FBI phase function φ on $\mathbb{C}^{2n}$ whose associated complex linear canonical transformation κ_φ is precisely κ,*

$$\kappa_\varphi = \kappa. \tag{5.4.30}$$

Proof. Let

$$\Phi_0(z) := \Phi_0^{(1)}(z') + \Phi_0^{(2)}(z''), \quad z = (z', z'') \in \mathbb{C}^n = \mathbb{C}^{n'} \times \mathbb{C}^{n''}. \tag{5.4.31}$$

Since $\Phi_0^{(1)}$ and $\Phi_0^{(2)}$ are strictly plurisubharmonic quadratic forms on $\mathbb{C}^{n'}$ and $\mathbb{C}^{n''}$, respectively, the function Φ_0 is a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$. Let

$$\Lambda_{\Phi_0} := \left\{ \left(z, \frac{2}{i}\partial_z \Phi_0(z) \right) : z \in \mathbb{C}^n \right\}. \tag{5.4.32}$$

Thus

$$\Lambda_{\Phi_0} = \Lambda_{\Phi_0^{(1)}} \times \Lambda_{\Phi_0^{(2)}}. \tag{5.4.33}$$

From (5.4.13), (5.4.21), and (5.4.33), we deduce that

$$(\kappa_{\varphi_1} \times \kappa_{\varphi_2})(\mathbb{R}^{2n}) = \Lambda_{\Phi_0}. \tag{5.4.34}$$

Since we also have

$$\kappa_{\Re}^{\mathbb{C}}(\mathbb{R}^{2n}) = \mathbb{R}^{2n}, \tag{5.4.35}$$

it follows that

$$\kappa(\mathbb{R}^{2n}) = \Lambda_{\Phi_0}. \tag{5.4.36}$$

The existence and uniqueness of an FBI phase function φ on $\mathbb{C}^{2n}$ such that (5.4.30) holds now follows from well-known arguments. For the details, the reader may consult, for instance, the discussion on pages 393-394 of [52]. □

Having shown that κ is generated by an FBI phase function φ on $\mathbb{C}^{2n}$, we may now study the evolution of the subspace Λ_{Φ_0} introduced in (5.4.32) by the complex-time Hamilton flow generated by $\mathfrak{p}_0 := p_0 \circ \kappa^{-1}$.

Proposition 5.4.2. *Let p_0 be as in the statement of Theorem 5.1.2, let φ be as in the statement of Proposition 5.4.1, let Φ_0 be the strictly plurisubharmonic weight associated to φ, let $\kappa_\varphi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be the complex linear canonical transformation generated by φ, let $\mathfrak{p}_0 = p_0 \circ \kappa_\varphi^{-1} \in Hol(\Lambda_{\Phi_0} + W)$,* *where* W *is a sufficiently small open bounded neighborhood of* 0 *in* $\mathbb{C}^n$*, and let* $\kappa_t = \exp(tH_{\mathfrak{p}_0})$*,* $t \in \mathbb{C}$*, be the complex-time Hamilton flow of* $\mathfrak{p}_0$ *in* $\Lambda_{\Phi_0} + W$*. Suppose that* $0 < T \ll 1$*,* $U = neigh(0; \mathbb{C}^n)$*, and* $\Phi \in C^\omega(D(0,T) \times U; \mathbb{R})$ *are such that*

$$\kappa_t(\Lambda_{\Phi_0} \cap U \times U) \cap U \times U = \Lambda_{\Phi_t}, \quad t \in D(0,T), \tag{5.4.37}$$

and

$$\Phi_t|_{t=0} = \Phi_0 \text{ in } U, \tag{5.4.38}$$

where

$$\Phi_t = \Phi(t, \cdot) \in C^\omega(U; \mathbb{R}), \quad t \in D(0,T), \tag{5.4.39}$$

and

$$\Lambda_{\Phi_t} = \left\{ \left(z, \frac{2}{i} \partial_z \Phi_t(z) \right) : z \in U \right\}, \quad t \in D(0,T). \tag{5.4.40}$$

Then, for every $0 < T_0 < T$*, there exists* $t_0 \in D(0,T_0) \backslash \{0\}$*,* $\delta > 0$*, and* $c > 0$ *such that*

$$\Phi_0(z) - \Phi_{t_0}(z) \geq c\,|z|^2, \quad |z| < \delta. \tag{5.4.41}$$

Proof. By Proposition 5.3.1, the function Φ satisfies the complex-time eikonal equation

$$\begin{cases} 2\partial_t \Phi_t(z) + i\mathfrak{p}_0\left(z, \frac{2}{i}\partial_z \Phi_t(z)\right) = 0, & (t,z) \in D(0,T) \times U, \\ \Phi_t|_{t=0} = \Phi_0 \text{ in } U, \end{cases} \tag{5.4.42}$$

where $\partial_t = \frac{1}{2}(\partial_{\operatorname{Re} t} - i\partial_{\operatorname{Im} t})$. Since p_0 vanishes to second order at $0 \in \mathbb{C}^{2n}$ and $\kappa_\varphi(0) = 0$, the symbol $\mathfrak{p}_0$ must also vanish to second order at $0 \in \mathbb{C}^{2n}$. Thus the complex Hamilton vector field of

$H_{\mathfrak{p}_0}$ of $\mathfrak{p}_0$ vanishes at $0 \in \mathbb{C}^{2n}$,

$$H_{\mathfrak{p}_0}|_0 = 0. \tag{5.4.43}$$

Therefore

$$\kappa_t(0) = 0, \quad t \in D(0,T). \tag{5.4.44}$$

This observation, in conjunction with (5.4.37), implies that

$$\partial_z \Phi_t(0) = 0, \quad t \in D(0,T). \tag{5.4.45}$$

Hence, when we take $z = 0$ in (5.4.42), we obtain that

$$\partial_t \Phi_t(0) = 0, \quad t \in D(0,T). \tag{5.4.46}$$

Because Φ_0 is quadratic, we have $\Phi_0(0) = 0$, and so

$$\Phi_t(0) = 0, \quad t \in D(0,T). \tag{5.4.47}$$

We conclude that

$$\Phi_t \text{ vanishes to 2nd order at } z = 0 \text{ for all } t \in D(0,T). \tag{5.4.48}$$

Let $\Xi \in C^\omega(D(0,T) \times \mathbb{C}^n; \mathbb{R})$ be the unique analytic function on $D(0,T) \times \mathbb{C}^n$ such that $\Xi_t := \Xi(t, \cdot)$ is the quadratic approximation to Φ_t at $z = 0$ in $\mathbb{C}^n$ for each $t \in D(0,T)$, i.e. Ξ_t is the unique real quadratic form on $\mathbb{C}^n$ such that

$$\Phi_t(z) = \Xi_t(z) + \mathcal{O}(|z|^3), \quad |z| \to 0^+, \tag{5.4.49}$$

for each fixed $t \in D(0,T)$. Note that, by Proposition 5.3.1, Φ_t is strictly plurisubharmonic in U

for each $t \in D(0,T)$, and hence Ξ_t is a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ for each $t \in D(0,T)$, i.e.

$$\Xi''_{t,\bar{z}z} > 0, \quad t \in D(0,T). \tag{5.4.50}$$

Taylor expanding (5.4.42) to second order about the origin $z = 0$ shows that Ξ is the unique solution of the quadratic complex-time eikonal equation

$$\begin{cases} 2\partial_t \Xi_t(z) + i\mathfrak{q}\left(z, \frac{2}{i}\partial_z \Xi_t(z)\right) = 0, & (t,z) \in D(0,T) \times \mathbb{C}^n, \\ \Xi_t|_{t=0} = \Phi_0 \text{ on } \mathbb{C}^n, \end{cases} \tag{5.4.51}$$

where $\mathfrak{q}$ is the quadratic approximation to $\mathfrak{p}_0$ at $0 \in \mathbb{C}^{2n}$. Let us make the following splitting of coordinates in $\mathbb{C}^n$:

$$\mathbb{C}^n = \mathbb{C}^{n'} \times \mathbb{C}^{n''}, \quad z = (z', z''), \tag{5.4.52}$$

where $0 \leq n', n'' \leq n$ are as in (5.4.2). We search for a solution to (5.4.51) of the form

$$\Xi(t,z) = \Xi^{(1)}(t,z') + \Xi^{(2)}(t,z''), \quad (t,z) \in D(0,T) \times \mathbb{C}^n, \tag{5.4.53}$$

where $\Xi^{(1)} \in C^\omega(D(0,T) \times \mathbb{C}^{n'}; \mathbb{R})$, $\Xi^{(2)} \in C^\omega(D(0,T) \times \mathbb{C}^{n''}; \mathbb{R})$, and $\Xi^{(1)}_t := \Xi^{(1)}(t,\cdot)$ and $\Xi^{(2)}_t := \Xi^{(2)}(t,\cdot)$ are strictly plurisubharmonic quadratic forms on $\mathbb{C}^{n'}$ and $\mathbb{C}^{n''}$ for each $t \in D(0,T)$, respectively. Since $\mathfrak{q}$ is of the form (5.4.1), where the matrix M is given by (5.4.28), and since the strictly plurisubharmonic weight Φ_0 has the form (5.4.31), we see that (5.4.53) will be a solution of the problem (5.4.51) provided $\Xi^{(1)}_t$ and $\Xi^{(2)}_t$ solve the eikonal equations

$$\begin{cases} \partial_t \Xi^{(1)}_t(z') + M_1 z' \cdot \partial_{z'} \Xi^{(1)}_t(z') = 0, & (t,z') \in D(0,T) \times \mathbb{C}^{n'}, \\ \Xi^{(1)}_0 = \Phi^{(1)}_0 \text{ in } \mathbb{C}^{n'}, \end{cases} \tag{5.4.54}$$

and

$$\begin{cases} \partial_t \Xi_t^{(2)}(z'') + M_2 z'' \cdot \partial_{z''} \Xi_t^{(2)}(z'') = 0, & (t, z'') \in D(0,T) \times \mathbb{C}^{n''}, \\ \Xi_0^{(2)} = \Phi_0^{(2)} \text{ in } \mathbb{C}^{n''}, \end{cases} \tag{5.4.55}$$

respectively. The problems (5.4.54) and (5.4.55) are globally well-posed in time $t \in \mathbb{C}$, and by inspection we see that their solutions are

$$\Xi_t^{(1)}(z') = \Phi_0^{(1)}\left(e^{-tM_1} z'\right), \quad (t, z') \in \mathbb{C} \times \mathbb{C}^{n'} \tag{5.4.56}$$

and

$$\Xi_t^{(2)}(z'') = \Phi_0^{(2)}(e^{-tM_2} z''), \quad (t, z'') \in \mathbb{C} \times \mathbb{C}^{n''}, \tag{5.4.57}$$

respectively. Thus the unique solution to (5.4.51) is

$$\Xi_t(z) = \Phi_0^{(1)}\left(e^{-tM_1} z'\right) + \Phi_0^{(2)}(e^{-tM_2} z''), \quad (t, z) \in D(0,T) \times \mathbb{C}^n. \tag{5.4.58}$$

In view of (5.4.20) and (5.4.24), we have

$$\Xi_t(z) = \Phi_0^{(1)}\left(e^{-tM_1} z'\right) + \frac{1}{2} \sum_{j=1}^{n''} e^{4\mathrm{Re}(t)\epsilon\lambda_j} \left|z_j''\right|^2, \quad (t, z) \in D(0,T) \times \mathbb{C}^n. \tag{5.4.59}$$

Let $0 < T_0 < T$ be arbitary. We claim that there exists $t_0 \in D(0, T_0)\backslash\{0\}$ such that

$$\Phi_0(z) - \Xi_{t_0}(z) \geq c\,|z|^2, \quad z \in \mathbb{C}^n, \tag{5.4.60}$$

for some $c > 0$. Indeed, let us search for such a complex-time t_0 of the form

$$t_0 = -\epsilon\rho - is \tag{5.4.61}$$

where $0 < s, \rho \ll 1$. Taking $t = t_0$ in (5.4.59) gives

$$\Xi_{t_0}(z) = \Phi_0^{(1)}\left(e^{isM_1+\epsilon\rho M_1}z'\right) + \frac{1}{2}\sum_{j=1}^{n''} e^{-4\rho\lambda_j}\left|z_j''\right|^2, \quad z \in \mathbb{C}^n. \tag{5.4.62}$$

By Taylor expansion, we have

$$\Phi_0^{(1)}\left(e^{isM_1-\epsilon\rho M_1}z'\right) = \Xi_{-is}^{(1)}\left(e^{-\epsilon\rho M_1}z'\right) = \Xi_{-is}^{(1)}(z') + \mathcal{O}(\rho\left|z'\right|^2), \quad z' \in \mathbb{C}^{n'}, \quad 0 < s, \rho \ll 1. \tag{5.4.63}$$

From (5.4.54) and (5.4.16), we see that the s-dependent quadratic form Ξ_{-is} is the unique solution of the eikonal equation

$$\begin{cases} \partial_s \Xi_{-is}^{(1)}(z') + \operatorname{Re}\mathfrak{q}_1\left(z', \frac{2}{i}\partial_{z'}\Xi_{-is}(z')\right) = 0, \quad (s, z') \in [0,\infty)\times\mathbb{C}^{n'}, \\ \Xi_{-is}^{(1)}\Big|_{s=0} = \Phi_0^{(1)} \text{ on } \mathbb{C}^{n'}. \end{cases} \tag{5.4.64}$$

Since the quadratic form q_1 has trivial singular space, $S_1 = \{0\}$, we know from the results of Section 2 of [26] that there is $c > 0$ such that

$$\Phi_0^{(1)}(z') - \Xi_{-is}^{(1)}(z') \geq c s^{2k_0^{(1)}+1}\left|z'\right|^2, \quad z' \in \mathbb{C}^{n'}, \quad 0 \leq s \ll 1, \tag{5.4.65}$$

where $k_0^{(1)}$ is the smallest non-negative integer such that

$$\bigcap_{j=0}^{k_0^{(1)}} \ker\left[(\operatorname{Re} F_1)(\operatorname{Im} F_1)^j\right] \cap \mathbb{R}^{2n'} = \{0\}. \tag{5.4.66}$$

From (5.4.31), (5.4.20), (5.4.59), (5.4.62), (5.4.63), and (5.4.65), we deduce that there is $c > 0$ such that

$$\Phi_0(z) - \Xi_{t_0}(z) \geq c s^{2k_0^{(1)}+1}\left|z'\right|^2 + \mathcal{O}(\rho\left|z'\right|^2) + \frac{1}{2}\sum_{j=1}^{n''}\left(1 - e^{-4\rho\lambda_j}\right)\left|z_j''\right|^2, \quad z \in \mathbb{C}^n, \tag{5.4.67}$$

whenever $0 < s, \rho \ll 1$ are sufficiently small. Since $\lambda_j > 0$ for all $1 \leq j \leq n''$, for any $0 < s \ll 1$ sufficiently small, we can choose $0 < \rho \ll 1$ small enough so that the righthand side of (5.4.67) is bounded below by $c\,|z|^2$ for some $c > 0$. Thus, for any $0 < s \ll 1$, there is $0 < \rho \ll 1$ and $c > 0$ such that

$$\Phi_0(z) - \Xi_{t_0}(z) \geq c\,|z|^2\,, \quad z \in \mathbb{C}^n, \tag{5.4.68}$$

for the non-zero complex time t_0 given in (5.4.61). Taking s and ρ smaller if necessary, we may ensure that $|t_0| < T_0$. From (5.4.49) and (5.4.68), we conclude that there is $c > 0$ and $\delta > 0$ such that

$$\Phi_0(z) - \Phi_{t_0}(z) \geq c\,|z|^2\,, \quad |z| < \delta. \tag{5.4.69}$$

□

Remark 5.4.3. An alternative derivation of the quadratic complex-time eikonal equation (5.4.51) satisfied by Ξ_t may be obtained by considering the tangent spaces $T_0\Lambda_{\Phi_t}$ for $t \in D(0,T)$. Since $\kappa_t(0) = 0$ for all $t \in D(0,T)$, we have

$$T_0\Lambda_{\Phi_t} = d_0\kappa_t\left(T_0\Lambda_{\Phi_0}\right), \quad t \in D(0,T), \tag{5.4.70}$$

where $d_0\kappa_t$ denotes the differential of κ_t at $0 \in \mathbb{C}^{2n}$. Because Φ_t vanishes to second order at $0 \in \mathbb{C}^n$ for every $t \in D(0,T)$, we have a canonical identification

$$T_0\Lambda_{\Phi_t} \cong \Lambda_{\Xi_t}, \quad t \in D(0,T), \tag{5.4.71}$$

where Λ_{Ξ_t} is the I-Lagrangian, R-symplectic, subspace of $\mathbb{C}^{2n}$ given by

$$\Lambda_{\Xi_t} = \left\{\left(z, \frac{2}{i}\partial_z\Xi_t(z)\right) : z \in \mathbb{C}^n\right\}, \quad t \in D(0,T). \tag{5.4.72}$$

Moreover, since $\mathfrak{p}_\mathfrak{o}$ vanishes to second order at $0 \in \mathbb{C}^{2n}$, we have

$$d_0\kappa_t = \exp\left(tH_\mathfrak{q}\right), \quad t \in D(0,T), \tag{5.4.73}$$

when we view $d_0\kappa_t$ as a $\mathbb{C}$-linear transformation $\mathbb{C}^{2n} \to \mathbb{C}^{2n}$. From (5.4.70), (5.4.71), and (5.4.73), it follows that the family of real quadratic forms $(\Xi_t)_{t\in D(0,T)}$ satisfies

$$\begin{cases} \Lambda_{\Xi_t} = \exp(tH_{\mathfrak{q}})(\Lambda_{\Phi_0}), \quad t \in D(0,T), \\ \Xi_t|_{t=0} = \Phi_0. \end{cases} \tag{5.4.74}$$

Reasoning similarly to the proof of Proposition 5.3.1, we find that Ξ_t solves the quadratic complex-time eikonal equation (5.4.51).

We now conclude the proof of Theorem 5.1.2 following the argument sketched in the introduction to this chapter. Let $P = \mathrm{Op}_h^w(p_0 + hp_1)$ and $u = u(h) \in L^2(\mathbb{R}^n)$ be as in the statement of Theorem 5.1.2, and let φ, Φ_0, U, $0 < T \ll 1$, and Φ be as in the statement of Proposition 5.4.2. Let $\mathcal{T}_\varphi$ be the FBI transform on $\mathbb{R}^n$ associated to the FBI phase function φ. By Proposition 5.3.2, there are constants $\delta > 0$, $0 < T_0 < T$, $0 < C < \infty$, and $0 < h_0 \leq 1$ such that

$$\sup_{\substack{t\in D(0,T_0) \\ 0<h\leq h_0}} \|\mathcal{T}_\varphi u\|_{L^2_{\Phi_t}(\{|z|<\delta\})} \leq C, \tag{5.4.75}$$

where $\|\cdot\|_{L^2_{\Phi_t}(\{|z|<\delta\})}$ is the norm

$$\|v\|^2_{L^2_{\Phi_t}(\{|z|<\delta\})} = \int_{|z|<\delta} |v(z)|^2 \, e^{-2\Phi_t(z)/h} \, L(dz). \tag{5.4.76}$$

After taking $\delta > 0$ smaller if necessary, we get from Proposition 5.4.2 that there is $t_0 \in D(0,T_0)\backslash\{0\}$ and $c > 0$ such that

$$\Phi_0(z) - \Phi^*(z) \geq c\,|z|^2\,, \quad |z| < \delta, \tag{5.4.77}$$

where

$$\Phi^* := \Phi_{t_0}. \tag{5.4.78}$$

Let N be a positive integer that is strictly larger than $n/4$. After taking $0 < h_0 \leq 1$ smaller if necessary, we obtain from Proposition 5.2.3 that for any $1 \leq p \leq \infty$ there is $C > 0$ such that

$$\|u\|_{L^p(\mathbb{R}^n)} \leq C h^{\frac{n}{2p} - \frac{3n}{4}} \int_{|z|<\delta} |\mathcal{T}_\varphi u(z)|\, e^{-\Phi_0(z)/h}\, L(dz) + C h^N, \quad 0 < h \leq h_0. \tag{5.4.79}$$

Thanks to (5.4.75), we have

$$\sup_{0<h\leq h_0} \|\mathcal{T}_\varphi u\|_{L^2_{\Phi_{t_0}}(\{|z|<\delta\})} < C \tag{5.4.80}$$

for some $0 < C < \infty$. From (5.4.77) and (5.4.80), we may conclude that there is $C > 0$ such that

$$\int_{|z|<\delta} |\mathcal{T}_\varphi u(z)|\, e^{-\Phi_0(z)/h}\, L(dz) \leq \int_{|z|<\delta} |\mathcal{T}_\varphi u(z)|\, e^{-\Phi^*(z)/h} e^{-c|z|^2/h}\, L(dz) \leq C h^{\frac{n}{2}}, \tag{5.4.81}$$

where the second inequality follows from an application of Cauchy-Schwarz and the fact that

$$\left(\int_{|z|<\delta} e^{-c|z|^2/h}\, L(dz) \right)^{1/2} = \mathcal{O}(h^{\frac{n}{2}}), \tag{5.4.82}$$

which may be deduced from a direct calculation. Putting (5.4.79) and (5.4.81) together, we find that there is $0 < h_0 \leq 1$ such that for any $1 \leq p \leq \infty$ we have

$$\|u\|_{L^p(\mathbb{R}^n)} \leq \mathcal{O}(1) h^{\frac{n}{2p} - \frac{n}{4}} + \mathcal{O}(1) h^N = \mathcal{O}(1) h^{\frac{n}{2p} - \frac{n}{4}}, \quad 0 < h \leq h_0. \tag{5.4.83}$$

The proof of Theorem 5.1.2 is complete.

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