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BEAM DYNAMICS OF THE LINEAR ACCELERATOR

Author
Panofsky, Wolfgang K.H.

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Subject: Corrections to UCRL-1095

It is requested that the following corrections be made to your copies of the document UCRL-1095, "Beam Dynamics of the Linear Accelerator", by Wolfgang K. H. Panofsky, February 15, 1951.

Pg. 4. Eq. (1): right hand side should be:

$$e\left[E^{o}_{z}(r,z)\cos(\omega t+\phi)+\frac{dr}{dt}B_{\phi}^{\ o}\sin(\omega t+\phi)\right]$$

Pg. 4. Eq. (2): put [after e,
and put] after sin $(\omega t+\phi)$.

Pg. 13. Two equations on top of page: multiply first member by $\frac{1}{2}$.

Pg. 15. Eq. (39): change sign in third term on R.H.S. of equation. Change γ to $\pi g_n/L_n$.

Eq. (40): put ; before $\gamma_n = 1/\sqrt{1-\beta_n^2}$

Pg. 16. Eq. (42): change R.H.S. to $-\dfrac{\beta_{n-1}-\beta_{n-1,s}}{\beta_{n-1,s}}$

Eq. (43): should be:

$$-\frac{\pi}{2}\left(1-\frac{2g_n}{L_n}\right)<\Phi_n<\frac{\pi}{2}\left(1+\frac{2g_n}{L_n}\right)$$

Line above Equation (45): write:
We obtain, neglecting coupling terms:

Eq. (45): delete [ ] term and add = 0.

Pg. 17. Eq. (48) (N.R.): change $n^{1/8}$ to $n^{1/4}$.

Corrections to UCRL-1095 - Cont.

Pg. 18. Eq. (53): put $_n$ on Φ

Label (54) equation below Eq. (53).

Pg. 19. Eq. (61) (E.R.): should read:

$$W_{\lambda}^{2} \frac{d}{dN}\left(N^3 \frac{d\phi_n}{dN}\right) - (2\pi \tan \phi_s)\, \phi_n = 0$$

Eq. (62) (E.R.): delete 2G in denominator.

delete exponent ^{2} over W

Eq. (63) (N.R.): multiply by 2.

Pg. 20. Eq. (64): Delete last bracket $\{\ \}$. Add on line below Eq. (64):

where couping terms have been neglected.

Pg. 25. Eq. (79):

$$\frac{d}{dt}\left\{m\, r^2\, \dot{\phi} + e\, r\, A_{\phi}\right\} = 0$$

footnote, second equation (six lines from bottom): put ^{2} on last term.

Pg. 22. Eq. (72): second member: change σ_n^{-2} to σ_n^{-1}.

Pg. 26. Eq. (87): change σ_n^{-2} to σ_n^{-1}.

Pg. 27. Eq. (88): denominator: delete σ_n^{-1}.

exponent: change σ_n^{-3} to σ_n^{-2}.

Pg. 28. Eq. (92): in bracket: delete σ_n^{-1}.

delete last member of equation.

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BEAM DYNAMICS OF THE LINEAR ACCELERATOR

Wolfgang K. H. Panofsky

February 15, 1951

Berkeley, California

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Beam Dynamics of the Linear Accelerator

Wolfgang K. H. Panofsky

Department of Physics, Radiation Laboratory
University of California, Berkeley, California

February 15, 1951

Introduction

This report summarizes some of the known facts concerning the dynamics of particle behavior in a heavy particle linear accelerator. This report is not an original contribution, but represents a compilation of relations derived by several members of this Laboratory, notably R. Serber, E. M. McMillan, L. Henrich and the author. The graphs (Figs. 11 and 12) concerning unstable operation of a linear accelerator are the results of numerical integrations performed under the direction of L. Henrich. With the exception of the short section on unstable operation, this report deals only with questions which can be handled by analytical methods. This implies that the important question of phase acceptance from a very low velocity injector, important in the MTA application, is not given here. This problem has been handled by numerical integration and on a differential analyzer by L. Henrich (UCRL-866) and A. Nordsieck. Also the effect of coupling between radial and phase motion has not been discussed in detail beyond giving the equations to be solved. This is justified here since the phase oscillations are very rapidly damped; also "resonances" which are of importance in the theory of the circular machines are of no significance here.

Beam Dynamics of the Linear Accelerator

Wolfgang K. H. Panofsky

Department of Physics, Radiation Laboratory
University of California, Berkeley, California

February 15, 1951

I. General Equations of Motion

A particle of charge e and rest mass M_o traveling in the linear accelerator will be acted on by both radial and longitudinal forces. The longitudinal forces are due to the longitudinal (axial) component of the electric field; the transverse components are due to a) the transverse component of the electric field, b) the effect of the radio-frequency magnetic field and c) any external focusing means.

In order to determine the motion precisely the two electric field components $E_z(r,z;t)$ and $E_r(r,z;t)$ have to be known and also the R.F. magnetic field $B_\phi(r,z;t)$. If the time variation is sinusoidal, the equations to be integrated are:

$$\frac{d}{dt}\left[\frac{M_o}{\sqrt{1-\beta^2}}\frac{dz}{dt}\right] = e\left[E^o_z(r,z)\cos(\omega t+\phi) + \frac{dr}{dt}B^o_\phi \sin(\omega t+\phi)\right] \tag{1}$$

$$\frac{d}{dt}\left[\frac{M_o}{\sqrt{1-\beta^2}}\frac{dr}{dt}\right] = e\left[E^o_r(r,z)\cos(\omega t+\phi) - B^o_\phi(r,z)\frac{dz}{dt}\sin(\omega t+\phi)\right] + F \tag{2}$$

where F is any external radial force. The superscript ^{o} denotes the amplitude of the respective field. Using empirical fields the equations can be integrated numerically; ~~this~~ is the only method feasible if exact results are desired. However, many general facts can be learned without a detailed integration.

In general the motion described by Eqs. (1) and (2) represent coupled motions between the r and Z behavior. In the case of large relative drift tube aperture the energy gain per gap depends on radial position and in general the radial motion depends on the phase of the particle relative to the R. F. crest. In case this coupling is not negligible, numerical integration is required. However, the dependence of energy gain/gap on radial position is usually small so that with negligible error the phase motion (i.e. the Z motion) can be integrated for constant r. Also owing to the rapid damping of the phase motions toward a constant phase, reasonable approximations to the radial motions can be made by assuming a constant phase. Under these simplifying assumptions analytical treatment is possible.

II. Synchronous Conditions

The basic geometry of the linear accelerator is shown in Fig. 1. At a given time the electric fields are everywhere in phase and the magnetic field in quadrature with the electric fields. Let L_n be the "repeat length" of the n^{th} gap and g_n be the gap length. Let us assume that the velocity increases from β_{n-1} to β_n when passing across the n^{th} gap (Fig. 2). Let us measure the coordinate z from the "electrical center" of each gap; we define the center by the condition that:

$$\int_{-\frac{g}{2}}^{+\frac{g}{2}} E_z(z) \sin\left(\frac{2\pi z}{L}\right) dz = 0 \tag{3}$$

If the gap is symmetrical this corresponds to the geometrical center. Let us define the "phase" ϕ_n of the particle to be the number of radians in time by which the particle crosses the electrical center of the n^{th} gap relative to the time at which the electric field reaches its crest value. Let

$$\phi_n > 0 \tag{4}$$

correspond to a particle which crosses after the crest of the field has been reached, and

$$\phi_n < 0 \tag{5}$$

to a particle which crosses before. These conventions are in agreement with Eqs. (1) and (2).

The machine is to be designed such that there shall be a particle called a synchronous particle, for which under a specific set of injection conditions ϕ_n shall be independent of n. This phase angle is called the synchronous phase angle ϕ_s and all quantities such as energy, velocity, etc., associated with the synchronous particle are designated by a subscript $_s$. In principle the machine cannot be designed to have a synchronous particle without knowing the motion; on the other hand, the motions cannot be integrated without knowledge of the fields in a given machine. In particular if the fractional changes in velocity expected per gap are large, the design can only be handled by successive approximations. On the other hand, if the velocity $c\beta_{n-1}$ before entering the gap and the velocity $c\beta_n$ differ only by a small amount, then

$$\frac{Ln}{\lambda} = (\beta_{n-1,s} + \beta_{n,s})/2 \tag{6}$$

is clearly the condition for a synchronous phase; here λ is the free space wavelength. The existence of a synchronous orbit also implies that the influence of radial position on phase motion is small.

The synchronous particle will increase its total relativistic energy from $W_{n,s}$ to $W_{n+1,s}$ when crossing the n^{th} gap; this gain is given by:

$$W_{n,s}-W_{n-1,s} = \int e\, E_z^o\,(z) \cos\left(\frac{\omega z}{V_s} + \phi_s\right) dz \tag{7}$$

where V_s is the synchronous velocity. In general, using the definition (3) of the electrical center, we can write Eq. (7) in the form:

$$W_{n,s}-W_{n-1,s} = e\,\lambda\, T\, E_o \left(\frac{\beta n_{,s} + \beta n\text{-}1_{,s}}{2}\right) \cos \phi_s = e\, T\, E_o\, L_n \cos \phi_s \tag{8}$$

where

$$E_o = \int E_z^o(z)dz \Big/ \int dz \tag{9}$$

is the mean effective field, and

$$T = \int E_z^o(z) \cos\left(\frac{2\pi z}{L_n}\right) dz \Big/ \int E_z^o(z)\, dz \qquad (10)$$

is the "transit time factor". It is useful to evaluate T in some simplified field shapes. The simplest case, often a good approximation, is to consider the field to be uniform in the gap, i.e.

$$E_z^o(z) = 0 \qquad -\frac{L_n}{2} < z < -\frac{g_n}{2} \; ; \; \frac{g_n}{2} < z < \frac{L_n}{2} \qquad (11)$$

$$E_z^o(z) = \frac{E_o L_n}{g_n} \qquad -\frac{g_n}{2} < z < \frac{g_n}{2} \qquad (12)$$

For this type field

$$T = \sin\left(\frac{\pi g_n}{L_n}\right) \Big/ \frac{\pi g}{L_n} \qquad (13)$$

If we consider the field solution corresponding to a drift tube of bore $2a_n$, where the field is given by Eq. (11) at $r = a_n$, then it can be shown that the transit time factor for a particle at an arbitrary radius r is given by:

$$T = \frac{I_o\left(\frac{2\pi r}{L_n}\right)}{I_o\left(\frac{2\pi a_n}{L_n}\right)} \cdot \frac{\sin\left(\frac{\pi g_n}{L_n}\right)}{\left(\frac{\pi g_n}{L_n}\right)} \qquad (14)$$

the extra factor* being due to the penetration of the field into the drift tubes. For most purposes Eq. (14) is of sufficient accuracy. The r-dependence of T is a factor which produces coupling between the radial and the phase motions.

Let the energy gain in $M_o c^2$ units per wavelength

$$W_\lambda = \frac{e\, E_o\, T\, \lambda}{M_o c^2} \cos \phi_s \qquad (15)$$

be a basic design parameter of the machine. In the non-relativistic range Eq. (8) can then be written as:

---

*Here $I_o(x) = J_o(ix)$ is the zeroth order Bessel function of the first kind with imaginary argument.

$$\beta_{n,s} - \beta_{n-1,s} = W_\lambda \tag{16(N.R.)}$$

In the relativistic range the fractional changes per gap are small; we can thus put $W_{n+1,s} - W_{n,s} = \Delta W_{n,s}$ and if $\Delta P_{n,s}$ is the momentum increase per gap we have $\Delta W_{n,s} = V_{n,s} \Delta P_{n,s}$. Hence relativistically, using Eq. (~~8~~ 15):

$$P_{n,s} - P_{n-1,s} = M_o c\, W_\lambda \tag{16}$$

Equation (16) evidently includes Eq. (16)(N.R.).

These equations give the simple relations between the momentum P_n, total energy W_n, kinetic energy K_n and velocity of the particle and the number of drift tubes. Evidently the momentum will vary linearly with the number of drift tubes. The synchronous relations are:

$$P_{n,s} \Big/ M_o c = (n + n_o)\, W_\lambda \tag{17}$$

$$W_{n,s} \Big/ M_o c^2 = \sqrt{1 + (n + n_o)^2\, W_\lambda^2} \tag{18}$$

$$\beta_{n,s} = (n + n_o)\, W_\lambda \Big/ \sqrt{1 + (n + n_o)^2\, W_\lambda^2} \tag{19}$$

$$K_{n,s} \Big/ M_o c^2 = \sqrt{1 + (n + n_o)^2\, W_\lambda^2} - 1 \tag{20}$$

Here n is taken to make n = 1 the first gap of the machine and n_o is the "effective number of gaps" corresponding to the injector. Non-relativistically,

$$\beta_{n,s} = (n + n_o)\, W_\lambda \tag{19(N.R.)}$$

$$K_{n,s} \Big/ M_o c^2 = (n + n_o)^2\, W_\lambda^2 / 2 \tag{20(N.R.)}$$

The repeat length is thus simply:

$$L_n = \lambda (n + n_o - \tfrac{1}{2})\, W_\lambda \Big/ \sqrt{1 + (n + n_o - \tfrac{1}{2})^2\, W_\lambda^2} \tag{21}$$

III. General Stability Considerations for a "Long" Accelerator.

a. Types of Stability

To obtain satisfactory operation for a "long" linear accelerator it is clearly necessary that the orbits be stable in phase and also stable radially. What length of such an accelerator would be considered "long" in this sense depends of course on the tolerances on injection conditions, voltage gradient, etc., which can be held. We shall show later that the periods of the various oscillations depend on the number $N = n + n_0$, i.e. the total effective number of drift tubes including the injector. A linear accelerator of this type is thus "long" in the sense of requiring stability if it increases the injection momentum by a large factor. A large injection voltage thus tends to make an accelerator effectively "short".

Phase stability is produced in a linear accelerator if a late particle receives a larger degree of acceleration. This in the case of a linear accelerator means that the particle whould traverse the center of each gap at a time when the field is increasing.* Specificially the condition for phase stability in a field $E_z(z, \omega t)$ is:

$$\frac{\partial}{\partial \Phi} \left\{ \int E_z\left(z, \frac{\omega z}{V} + \Phi\right) dz \right\} > 0 \tag{22}$$

The conditions for radial stability are more complicated. Focusing is obtained by the following mechanisms: 1) velocity focusing, sometimes called electrostatic or second order focusing. 2) phase focusing. 3) focusing produced by charges contained within the beam. 4) focusing produced by external means.

b. The "incompatibility" theorem.

If no charge is contained in the beam, a particle crossing a gap off the

*Note that this is the inverse of the condition pertaining to phase stability of a circular accelerator.

axis will cross as many lines of force directed toward the axis as away from the axis. A net radial momentum is thus produced if: a) the particle changes its velocity when crossing the gap and b) if the field varies in time. The former mechanism is the one responsible for the focusing action of electrostatic lenses. We shall show that its effect is only important in the first gaps of a linear accelerator. The second effect, due to the time variation of the field, rapidly becomes dominating further along the machine. It is clear that the condition for stable radial focusing is essentially that the field be decreasing during the time of passage through the gap. This condition appears incompatible with the phase stability condition expressed by Eq. (22). That this disagreement is a fundamental one and cannot be removed by any artifices of geometry has been shown by McMillan.[1] The theorem was interpreted by Ginzton, Hansen and Kennedy[2] as being simply a manifestation of Earnshaw's theorem as effective in a frame moving with the charged particle. We shall here reproduce McMillan's proof since it incorporates several useful relations to be used later. Using the two Maxwell relations $\vec{\nabla}\cdot\vec{E} = 0$ and* $\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$, valid in a charge and current free region, we can easily show that:

$$E_r = -\frac{r}{2}\frac{\partial E_z}{\partial z} \tag{23}$$

$$B_\phi = \frac{r}{2c^2}\frac{\partial E_z}{\partial t} \tag{24}$$

are the leading terms in a Taylor expansion of the radial electric field and azimuthal magnetic field in powers of the radial distance r. Accordingly for small r, the radial momentum of a particle crossing the gap is given by:

$$\Delta P_r = -\frac{e}{2}\int r\left[\frac{\partial E_z}{\partial z} + \frac{\beta}{c}\frac{\partial E_z}{\partial t}\right]\frac{dz}{\beta c} \tag{25}$$

---

*MKS units are used throughout.

Here $\frac{\partial E_z}{\partial z}$ and $\frac{\partial E_z}{\partial t}$ represent the values of the respective partial derivatives at the time of passage of the particle. If the particle moves with a velocity $v = \beta c$, then the rate $\frac{dE_z}{dz}$ at which the field varies with z for this particle is given by:

$$\frac{\partial E_z}{\partial z} = \frac{dE_z}{dz} - \frac{1}{V}\frac{\partial E_z}{\partial t} \qquad \beta = \text{const.} \tag{26}$$

If we are here concerned only with the phase focusing action, we may ignore the z dependence of β and r (which give rise to the electrostatic focusing!) and put

$$\triangle P_r = -\frac{e\,r}{2\beta c}\int\left(\frac{\partial E_z}{\partial z} + \frac{\beta}{c}\frac{\partial E_z}{\partial t}\right) dz \tag{27}$$

We can eliminate either the partial time or space derivative in Eq. (25) by the use of Eq. (26). Also since the phase ϕ and the quantity ωt are additive in all cases, we can put $\frac{\partial}{\partial t} = \omega\frac{\partial}{\partial \phi}$. Eliminating the partial space derivative, we have:

$$\triangle P_r = -\frac{er}{2V^2}\left\{V\int\frac{dE_z}{dz}\,dz - (1-\beta^2)\,\omega\frac{\partial}{\partial\phi}\int E_z\left(z, \frac{\omega z}{V} + \phi\right) dz\right\} \tag{28}$$

In the absence of foils or grids or other devices occluding charge within the beam the total derivative integral will vanish due to either the periodic end conditions or by extending the integral into zero field. The condition for radial phase-focusing is thus:

$$\frac{\partial}{\partial\phi}\int E_z\left(z, \frac{\omega z}{V} + \phi\right) dz < 0 \tag{29}$$

This is clearly incompatible with Eq. (22), the condition for phase stability.

c. Velocity focusing.

Note that in this discussion only phase focusing and not velocity focusing has been included. If the latter is included also, then there will actually be a small phase angle over which the orbits would both be radially stable and phase stable. We can estimate this range here by simply not assuming constancy of β as was done in Eq. (26). Thus re-writing Eq. (26) with the aid of Eq. (27) and omitting the total derivative we have:

$$\Delta p_r = -\frac{er}{2c}\int\left(\frac{1-\beta^2}{\beta}\right)\frac{\partial E^o_z}{\partial z}\cos(\omega t + \emptyset)\,dz \qquad (30)$$

Let us estimate this integral for the simple "square wave" field of Eqs.(11)(12). This can be written in the form:

$$\frac{\partial E}{\partial z} = \frac{E_o L_n}{g_n}\left\{\delta\left(z + \frac{g_n}{2}\right) - \delta\left(z - \frac{g_n}{2}\right)\right\} \qquad (31)$$

where δ is the ordinary δ-function, i.e. $\delta(z) = 0$ for $z \neq 0$ and $\int \delta(z)dz = 1$.

Hence:

$$\Delta p_r = -\frac{eE_o L_n r}{2g_n c}\left[\frac{1-\beta^2}{\beta}\cos(\omega t + \emptyset)\right]_{z = \frac{g_n}{2}}^{z = -\frac{g_n}{2}} \qquad (32)$$

It is only necessary to evaluate this term non-relativistically since velocity focusing is only of importance at low velocities. Let us neglect terms deriving from the fact that the velocity β_n is not the synchronous velocity given by Eq. (19). The β dependence effects the value of the radial momentum in two ways: 1) since the velocity is increased during passage, more time is spent during the focusing part of the field than the defocusing part. 2) Even if the particle crosses the center of the gap at the crest of the r.f. wave, the times of passing the entrance and exit of the gap are not symmetrical with respect to the time of passing the center; the particle takes a longer time to approach the center than to leave it. This causes the focusing field at the time of entry into the gap to be weaker than the defocusing field at the time of departure. This second effect, which increases with gap length, counteracts the first one; in fact there will be a critical gap length beyond which the second order focusing will be negative (defocusing).

In order to evaluate approximately Eq. (32) we have to determine the times

t_+ and t_- at which the particle crosses $z = \frac{+g_n}{2}$ and $z = \frac{-g_n}{2}$. Let us assume that the particle travels with the velocity:

$$\frac{\beta_{n-1} + \frac{\beta_n + \beta_{n-1}}{2}}{2} = \frac{3\beta_{n-1} + \beta_n}{4} \quad \text{in the range } \frac{-g_n}{2} < z < 0$$

and the velocity

$$\frac{\beta_n + \frac{\beta_n + \beta_{n-1}}{2}}{2} = \frac{3\beta_n + \beta_{n-1}}{4} \quad \text{in the range } 0 < z < \frac{g_n}{2} .$$

Since

$$\omega L_n = c\,\pi\,(\beta_n + \beta_{n-1})$$

we have

$$\omega t_- = -\frac{\pi g_n}{L_n}\left[1 + \frac{1}{4N-3}\right] \approx -\frac{\pi g_n}{L}\left[1 + \frac{1}{4N}\right]$$

and

$$\omega t_+ = \frac{\pi g_n}{L_n}\left[1 - \frac{1}{4N-1}\right] \approx \frac{\pi g_n}{L}\left[1 - \frac{1}{4N}\right]$$

Using Eqs. (15) and (19)*, the radial momentum becomes:

$$\frac{\Delta p_r}{M_o c} = -\frac{\pi r}{2\lambda \cos\phi_s}\,\frac{1}{\sin\frac{\pi g_n}{L_n}}\left\{\frac{1}{N-1}\cos\left[\frac{\pi g_n}{L_n}\left(1 + \frac{1}{4N}\right) - \Phi\right] - \frac{1}{N}\cos\left[\frac{\pi g_n}{L_n}\left(1 - \frac{1}{4N}\right) + \Phi\right]\right\} \qquad (33)$$

Expanded in orders of N^{-1}, this becomes, carrying terms to order N^{-2} only:

$$\frac{\Delta p_r}{M_o c} = -\frac{\pi r}{\lambda}\,\frac{1}{N}\,\frac{\sin\Phi}{\cos\phi_s} - \frac{\pi r}{2\lambda}\,\frac{1}{N^2}\left[\sin\Phi + \left(\cot\frac{\pi g_n}{L_n} - \frac{\pi g_n}{2L_n}\right)\cos\Phi\right]\Big/\cos\phi_s \qquad (34)$$

This is composed of a phase defocusing term:

$$\frac{\pi r}{\lambda}\left(\frac{-\sin\phi}{\cos\phi_s}\right)\frac{1}{N}\left(1 + \frac{1}{2N}\right)$$

and a second order focusing (or defocusing term)

$$-\frac{\pi r}{2\lambda N^2}\left(\cot\frac{\pi g}{L_n} - \frac{\pi g}{2L_n}\right)\frac{\cos\phi}{\cos\phi_s}$$

* We are assuming here that $\beta = \beta_s$, i.e. ignoring terms in $\frac{\partial\phi_n}{\partial n}$. These will later be included.

The specific form of this latter term (not the former!) will depend on the specific shape of the field.

Hence there will be a small phase angle of the order:

$$\delta \phi = \frac{1}{2}\left[\cot\left(\frac{\pi g_n}{L_n}\right) - \frac{\pi g_n}{2L_n}\right] \Big/ N \tag{35}$$

for which a particle is both phase stable and focusing. In the Berkeley 40-foot accelerator $g_n/L_n \approx 1/4$ and $n_o = 24$; therefore this angle is very small.

Note that at $g_n/L_n = .34$ the second order focusing vanishes; this is the point where the two effects mentioned above cancel. Particularly when large apertures are used, leading to a large "effective gap", the velocity focusing may easily become defocusing.

d. Radial oscillations of a grid or foil focused linear accelerator.

The theorem showing incompatibility of radial stability with phase stability is essentially based on the fact that a particle will cross no net radial electric lines of force if the field were static. Clearly if the entrance to the end drift tube were closed by a grid or foil (Fig. 4) a net inward momentum would result. Mathematically this corresponds to the total derivative integral in Eq. (28) not vanishing at its upper limit. Accordingly the first order focusing term becomes:

$$\triangle P_r = -\frac{er}{2V^2}\left\{VE_F - (1-\beta^2)\omega\frac{\partial}{\partial \Phi}\int E_z\left(z, \frac{z}{V} + \Phi\right) dz\right\} \tag{36}$$

This is the general formula for the radial momentum gained. Here E_F is the electric field at the foil at the time of passage of the particle through the foil (or grid).

We can evaluate Eq. (36) again only in terms of a specific model to give the field variations. Using again the "square wave" field of Eq.(11/12) we obtain:

$$\triangle P_r = -\frac{erL_nE_o}{2\,g_nc\beta}\left\{\cos\left(\frac{\pi g_n}{L_n}\frac{\beta_s}{\beta} + \phi\right) + 2(1-\beta^2)\left[\sin\left(\frac{\pi g_n}{L_n}\frac{\beta_s}{\beta}\right)\right]\sin\phi\right\} \tag{37}$$

We can simplify this expression by only carrying the first order in $\frac{\beta-\beta_s}{\beta_s}$;
we obtain:

$$\Delta P_r = -\frac{reE_o}{2\beta_s c}\frac{L_n}{g_n}\cdot\left\{\cos\left(\frac{\pi g_n}{L_n}-\Phi\right)-2\beta_s^2 \sin\left(\frac{\pi g_n}{L_n}\right)\sin\Phi+\left(\frac{\beta-\beta_s}{\beta_s}\right) F\right\} \tag{38}$$

where

$$F = \frac{\pi g_n}{L_n}\sin\left(\frac{\pi g_n}{L_n}-\Phi\right)-2\beta_s^2 \sin\left(\frac{\pi g_n}{L_n}\right)\sin\Phi-2\beta_s^2\frac{\pi g_n}{L_n}\cos\left(\frac{\pi g_n}{L_n}\right)\sin\Phi-\cos(\gamma-\Phi) \tag{39}$$

The correction term in F represents the influence on the radial momentum of the orbits not at synchronous velocity. These equations are relativistically correct.

We can convert Eq. (38) into the form of a difference equation; such an equation then forms a good basis for numerical solution. It shall be pointed out here again that this analysis treats each gap effectively as a "thin" lens, i.e., calculates the change in radial momentum in the lens but ignores radial motion in the lens itself. Again it will be evident by looking at the resulting motion that this approximation is justified if N is not too small.

We can put:

$$\frac{\Delta P_r}{M_o} = \gamma_n\frac{dr_n}{dt}-\gamma_{n-1}\frac{dr_{n-1}}{dt} \qquad \gamma_n = 1/\sqrt{1-\beta_n^2}$$

$$\cong \frac{\omega}{2\pi}\left[\gamma_n\frac{\beta_n}{\beta_{n,s}}(r_{n+1}-r_n)-\frac{\beta_{n-1}\gamma_{n-1}}{\beta_{n-1,s}}(r_n-r_{n-1})\right] \tag{40}$$

This relation in combination with Eqs. (38) and (39), gives a difference equation for r_{n+1} in terms of r_n and r_{n-1} if the phase motion has been obtained, i.e., if β_n and $\not{\Phi}_n$ are known as a function of n. We are going to write down the resultant equation only in the non-relativistic case; using Eqs. (13, 15 and 16) we obtain:

$$r_{n+1}-2r_n+r_{n-1} = -\frac{\pi}{2}\frac{r_n}{(n+n_o)}\frac{\cos\left[\left(\frac{\pi g_n}{L_n}\right)-\Phi_n\right]}{\sin\left(\frac{\pi g_n}{L_n}\right)\cos\phi_s} \tag{41}$$

We have also omitted the terms in $\beta_n-\beta_{n,s}$; these can be introduced using the relation:

$$\frac{\Phi_n - \Phi_{n-1}}{2\pi} = - \frac{\beta_n - \beta_{n,s}}{\beta_{n,s}}, \tag{42}$$

which expresses the rate of phase variation as a function of deviation from synchronous velocity. This coupling term is usually small.

Foil or grid focusing thus gives rise to stable radial oscillations in the range:

$$\left| -\Phi \right| < \frac{\pi}{2} \left(1 - \frac{2g_n}{L_n} \right) \tag{43}$$

Figure 5 shows the regions of stable focusing and of phase stability for $g_n/L_n = .25$; the region where both motions are stable is:

$$- \frac{\pi}{4} < \Phi < 0 \tag{44}$$

Note that if $g_n/L_n = 1/2$, foil or grid focusing becomes ineffective. (Fig. 5).

For large values of $N = n+n_o$, we can write these equations as differential equations. We obtain:

$$\frac{d}{dn}\left(\gamma_n \frac{dr_n}{dn}\right) + \frac{Kr_n}{n+n_o} \left[1 - \frac{1}{2\pi} \frac{d\phi_n}{dn} \left(\frac{F}{K} - 1\right)\right] \tag{45}$$

where

$$K = \frac{\pi}{2 \sin \frac{\pi g_n}{L_n}} \; \frac{\cos\left(\frac{\pi g_n}{L_n} - \bar{\phi}_n\right) - 2\beta_s^2 \sin \frac{\pi g_n}{L_n} \sin \bar{\phi}_n}{\cos \phi_s} \tag{46}$$

For a synchronous orbit this gives:

$$\frac{d}{dn}\left[\sqrt{1 + (n+n_o)^2 W_\lambda^2} \; \frac{dr_n}{dn}\right] + \frac{K_s}{(n+n_o)} r_n = 0 \tag{47}$$

Here $K_s = K$ for $\bar{\phi}_n = \phi_s$.

We thus find that we obtain oscillations which will be increasing in amplitude. Asymptotically the oscillations become (see Appendix I):

$$r_n \propto \beta_n^{1/4} \, e^{\pm i \int^n \frac{K^{1/2} \, dn}{(n+n_o)^{1/2} \left[1 + (n+n_o)^2 W_\lambda^2\right]^{1/4}}} \tag{48}$$

which is simply, non-relativistically:

$$r_n \propto n^{1/8} e^{\pm 2iK^{1/2}(n+n_o)^{1/2}} \qquad (48)(N.R.)$$

Eq. (47) can be integrated exactly non-relativistically giving:

$$r_n = \sqrt{N}\left\{A\, J_1\,(2\sqrt{K_sN}) + BY_1\,(2\sqrt{K_sN})\right\} \qquad (49)$$

This equation has been plotted for $n_o = 24$ and various values of ϕ_s in Fig. 6. Its asymptotic expansion is identical with Eq. (49)(N.R.).

In case foils of thickness N atoms/unit area and atomic number Z rather than grids are used, multiple Coulomb scattering will lose a certain fraction f of the beam up to the n^{th} drift tube. Serber[5] has shown that, approximately:

$$1 - f = 1.6\, e^{-.363\, d^2/a^2} \qquad (50)$$

where

$$\left(\frac{d}{\lambda}\right)^2 = \frac{32\pi\,(Nr_o^2)Z^2}{W_\lambda^2}\left(\frac{m}{M_o}\right)^2\left[\log\left(\frac{4\pi NZ^{4/3}\,\hbar^2}{mv^2}\right)\right]\left[\left(\frac{n}{n_o}\right)^{1/2} - 1\right]$$

here m is the electron mass and r_o is the classical electron radius.

If grids are used the transmission will depend on the opacity of the grids. The permissible transmission of the grids is limited by the field concentration on the grid wires.

5. Phase Oscillations in a Linear Accelerator.

Let us examine the oscillations of the phase Φ_n about the synchronous phase angle ϕ_s in the phase stable case, i.e. $\phi_s < 0$. Let us put

$$\begin{aligned} \beta_n &= \beta_{n,s} + \beta_n' \\ \Phi_n &= \phi_s + \phi_n \\ W_n &= W_{n,s} + W_n' \end{aligned} \qquad (51)$$

In general:

$$W_n - W_{n-1} = e\int_n E_z^o(Z)\cos\left(\frac{\omega z}{V} + \Phi_n\right)dz \qquad (52)$$

Using Eq. (10) we obtain:

$$\frac{W_n - W_{n-1}}{M_o c^2} = W_\lambda \frac{L_n}{\lambda} \frac{\cos \Phi}{\cos \phi_s} \left\{ 1 + G \frac{\beta_n'}{\beta_{n,s}} \right\} \tag{53}$$

where:

$$G = 1 - \frac{\pi g_n}{L_n} \cot \frac{\pi g_n}{L_n}$$

for a uniform field is a factor correlating the energy gain to the deviation from synchronous velocity. Eq. (53) can be re-written as:

$$\frac{W_n' - W_{n-1}'}{M_o c^2} = W_\lambda \frac{L_n}{\lambda} \left\{ \left(1 + G \frac{\beta_n'}{\beta_{n,s}} \right) \cos(\phi_n + \phi_s) - \cos \phi_s \right\} \Big/ \cos \phi_s \tag{55}$$

But in general, to the first order in β_n':

$$W_n' = M_o c^2 \gamma_{s,n}^3 \beta_{s,n} \beta_n' \tag{56}$$

Hence:

$$\gamma_{s,n}^3 \beta_{s,n} \beta_n' - \gamma_{s,n-1}^3 \beta_{s,n-1} \beta_{n-1}' = \frac{\beta_{n,s} + \beta_{n-1,s}}{2} \frac{W_\lambda}{\cos \phi_s} \left\{ \left(1 + \frac{G \beta_n'}{\beta_{n,s}} \right) \left[\cos(\phi_n + \phi_s) \right] - \cos \phi_s \right\} \tag{57}$$

This equation, in conjunction with the equation (cf. Eq. (42)):

$$\frac{\phi_n - \phi_{n-1}}{2\pi} = - \frac{\beta_n'}{\beta_{n,s}} \tag{58}$$

and the synchronous conditions Eq. (18) and (19), gives a second order difference equation for the phases, suitable for direct computation.

Written as a differential equation Eq. (57) becomes:

$$\frac{d}{dn}\left(\gamma_{s,n}^3 \beta_{s,n}^2 \frac{d\phi_n}{dn} \right) + 2\pi \beta_{n,s} W_\lambda \left\{ \left(1 - \frac{G}{2\pi} \frac{d\phi_n}{dn} \right) \frac{\cos(\phi_n + \phi_s)}{\cos \phi_s} - 1 \right\} = 0 \tag{59}$$

or in terms of $N = n + n_o$:

$$\frac{\sqrt{1 + W_\lambda^2 N^2}}{N} \frac{d}{dN} \left[\sqrt{1 + W_\lambda^2 N^2}\, N^2 \frac{d\phi_n}{dN} \right] + \frac{2\pi}{\cos \phi_s} \left\{ \left(1 - \frac{G}{2\pi} \frac{d\phi_n}{dN} \right) \cos(\phi_n + \phi_s) - \cos \phi_s \right\} = 0 \tag{60}$$

This is the general equation for the phase oscillations of a linear accelerator.

a. Small Oscillations.

If $\phi_n << 1$ we can write Eq. (60) as:

$$\frac{(1 + W_\lambda^2 N^2)^{(1-G)/2}}{N^{1-G}} \frac{d}{dN}\left[(1 + W_\lambda^2 N^2)^{(1+G)/2} N^{2-G} \frac{d\phi_n}{dN} \right] - (2\pi \tan \phi_s)\, \phi_n = 0 \qquad (61)$$

This can be written in the non-relativistic and extremely relativistic limits as:

$$\frac{1}{N^{1-G}} \frac{d}{dN}\left(N^{2-G} \frac{d\phi_n}{dN} \right) - (2\pi \tan \phi_s)\, \phi_n = 0 \qquad (61)(N.R.)$$

and:

$$\frac{W_\lambda^2}{N^{-G}} \frac{d}{dN}\left(N^{3-G} \frac{d\phi_n}{dN} \right) - (2\pi \tan \phi_s)\, \phi_n = 0 \qquad (61)(E.R.)$$

The resultant phase motions are then:

$$\phi_n \propto \frac{1}{N^{(3-2G)/4}} \cos\left[2 \sqrt{-2\pi \tan \phi_s}\, \sqrt{N} + \delta \right] \qquad (62)(N.R.)$$

$$\phi_n \propto \frac{1}{N^{(3-2G)/4}} \cos\left[2\left(\sqrt{-2\pi \tan \phi_s} \Big/ W_\lambda^2 \right) N^{-1/2} + \delta \right] \qquad (62)(E.R.)$$

In the non-relativistic case the period of phase oscillations is, measured in terms of number of drift tubes:

$$N_\phi = \frac{\pi \sqrt{N}}{\sqrt{-2\pi \tan \phi_s}} \qquad (63)(N.R.)$$

For g_n/L_n = .25 G = .213 and the amplitude damps as $N^{-.64}$.

In the extremely relativistic orbits, the phase motion becomes non-oscillatory as demanded by the asymptotic constancy of the velocity.

b. Large oscillations.

In the previous section we have derived the period of small oscillations about an equilibrium phase angle ϕ_s. The complete phase equation (60) is not integrable analytically, but we can estimate the limits of stability it represents. The principal interest in investigating the region of stability is the problem of calculating the range of input phase which will be stable. Since the phase oscillations are quite rapidly damped, we have to discuss the problem for low velocities only; any particles phase-stable early in the machine will remain well within the stable region; in fact it would be permissible to decrease $-\phi_s$ slightly toward higher energy. We are therefore justified to treat the problem non-relativistically only. In that case Eq. (60) can be written as:

$$\frac{d}{du}\left[u^{\left(1+\frac{1}{2-G}\right)}\frac{d\phi_n}{du}\right]+\frac{2\pi}{(2-G)^2}\left\{\frac{\cos(\phi_n+\phi_s)-\cos\phi_s}{\cos\phi_s}\right\}\left\{1-\frac{G}{2\pi}\frac{d\phi_n}{du}\,u^{\frac{1}{2}\left[1-G/(2-G)\right]}\right\} \tag{64}$$

where $u = N^{2-G}$.

This is identical to the differential equation of motion of a pendulum[6] of mass proportional to the $1 + 1/(2-G)$ power of the "time" u which is "biased" by a torque on its axis such that its equilibrium position is an angle $-\phi_s$ below the horizontal (Fig. 7). The motion can thus be described by a "potential" proportional to:

$$V(\phi_n) = -\phi_n + \sin\phi_n + \cos\phi_n \tan\phi_s + \text{constant} \tag{65}$$

which is plotted in Fig. 8. The last term in Eq. (64) which slightly affects the damping only, has been neglected. If particles are injected at the correct injection energy, the range of phase acceptance is approximately $3\,\phi_s$, i.e. for

$$-2\,\phi_s < \phi_n < +\,\phi_s \tag{66}$$

Since ϕ_s is given by $\cos\phi_s = V_{threshold}/V_{operating}$, this gives immediately the acceptance phase as a function of operating voltage near threshold.

The acceptance of a given particle for a given value of ϕ_s depends on the value of the starting phase and starting velocity. These are defined by the condition that the sum of the "potential energy" and "kinetic energy" of the equivalent pendulum shall not exceed the depth of the potential.

Since the effective "kinetic energy" is:

$$T_{eff} = \frac{1}{2} n_o^{2-G} \left(\frac{d\phi}{dn}\right)^2_{n=0} \tag{67}$$

and the effective potential energy, measured relative to its maximum $\Phi_n = +|\phi_s| = -\phi_s$ is:

$$V_{eff} = n_o^{1-G} \cdot 2\pi \left\{ \frac{\sin(\phi+\phi_s)+\sin\phi_s}{\cos\phi_s} - (2\phi_s + \phi_n) \right\} \tag{68}$$

Let ΔW be the deviation of the injection energy, W_o from its synchronous value. We can put:

$$\left.\frac{d\phi_n}{dn}\right|_{n=n_o} = -2\pi \frac{\beta_n'}{\beta_{n,s}} = -\pi\left(\frac{\Delta W}{W_o}\right) . \tag{69}$$

Hence the phase acceptance condition is:

$$\pi n_o\left(\frac{\Delta W}{W_o}\right)^2 + 4\left[\frac{\sin(\phi_n+\phi_s)+\sin\phi_s}{\cos\phi_s} - (2\phi_s + \phi_n)\right] < 0 \tag{70}$$

Fig. 9 shows a plot of ϕ_n vs $n_o^{1/2}\,\Delta W/W_o$ for various values of ϕ_s. This defines a set of closed curves such that any particle within these curves will be stably accelerated. For small oscillations Eq. (70) corresponds to the simple harmonic oscillator of total energy:

$$V_{eff} = \pi n_o\left(\frac{\Delta W}{W}\right)^2 + 4\left\{\frac{\phi_n^2}{2}(-\tan\phi_s) - 2\left[-\tan\phi_s + \phi_s\right]\right\} \tag{71}$$

Note that the per cent tolerance of the injection voltage becomes less critical as the design injection voltage decreases. Also note that the "depth" of the potential varies as ϕ_s^3 for small ϕ_s.

6. Unstable operation.

It is clear that if a linear accelerator is short enough it can be operated without grids. It will then be either phase unstable or radially unstable. The original r.f. linear accelerator of Sloan and Lawrence[3] was clearly operated in such a manner. Experiments with the 40-foot linear accelerator without grids have shown that one can obtain an "unstable" beam of essentially the same magnitude as is obtainable with grids. However, the criticality of adjustment is greatly increased.

Figure 10 shows some possible phase motions which might give rise to short time focused operation. Since during the time $\phi_n > 0$ one will obtain a stable radial restoring force, phase motions which will spend a sufficient time near the "hump" of the potential V_{eff} at $\phi_n = + \left| \phi_s \right|$ will give radially stable orbits. The question is thus one of the tolerances in injection energy required to achieve such operation.

Several orbits have been computed by first integrating Eq. (64) numerically in the form of a finite difference equation. Typical curves are shown in Fig. 11 for various initial values of $\Delta W/W_o$, and for $\phi_s = -10^\circ$ and a starting phase of -15°.

Ignoring the velocity dependent term in Eq. (34) we obtain the simple radial difference equation, analogous to Eq. (41):

$$r_{n+1} - 2\, r_n + r_{n-1} + \frac{\pi}{N}\, r_n \frac{\sin \Phi_n}{\cos \phi_s} = 0 \qquad (72)(\mathrm{N.R.})$$

Using the phase motions of Fig. 11, this equation has been integrated for both parallel and divergent injection. A typical stable orbit is shown in Fig. 12. With the parameters indicated the range of $\Delta W/W_o$ leading to orbits passing the 40-foot accelerator is only .05 percent. This is in fair agreement with the performance obtained.

For reference let us write Eq. (72), which describes the unstable radial motion, in completely relativistic, but differential, form. We obtain:

$$\frac{d}{dn}\left(\gamma_n \frac{dr_n}{dn}\right) + \frac{\pi \gamma_n^{-2}}{N}\, r_n \frac{\sin \Phi_n}{\cos \phi_s} = 0 \qquad (72)$$

if the $\frac{d\phi_n}{dn}$ coupling is ignored.

7. Space charge effects.

In the Berkeley 40-foot linear accelerator space charge effects are totally negligible owing to the limitation on the beam current by the injector. In the MTA application on the other hand, such effects may become important.

Exact treatment of the space charge effect is obviously impossible since one cannot treat the orbits independently. The only practical method seems to be to introduce a mean field corresponding to the expected motion as a whole and let each particle move in such a mean field.

If I is the beam current, the charge per bunch is $2\pi I/\omega$ and if the bunch is concentrated in a sphere of radius ρ, each particle will experience a radial field of order of magnitude:

$$E_r = \frac{I}{2\omega K_o} \frac{r}{\rho^3} = \frac{1}{4\pi} \sqrt{\frac{\mu_o}{K_o}} \, I \frac{\lambda r}{\rho^3} \qquad (73)$$

Hence a term is introduced into the radial equation which is of order:

$$\left.\frac{d^2r}{dn^2}\right|_{\text{space charge}} = \frac{r}{4\pi}\left(\frac{\lambda}{\rho}\right)^3 \cdot I_o \qquad (74)$$

where $I_o = e\sqrt{\frac{\mu_o}{K_o}} \, I/M_oc^2$ is the proton charge times the current times 376 Ω measured in rest energy units of the particle accelerated.

Clearly the value of λ/ρ will depend on the particular accelerator design, and even on the injection conditions. This fact, in addition to the crude initial assumptions, makes an evaluation of this term meaningless except as an estimate.

In order to keep the transit time factor T of Eq. (14) within reasonable limits, the drift tube radius a_n has to be a reasonably small fraction of the repeat length L_n. For a practical design:

$$a_n \backsim \rho \backsim L_n/2\pi \backsim (\lambda/2\pi)\, N\, W_\lambda \qquad (75)$$

hence:

$$\left.\frac{d^2r}{dn^2}\right|_{\text{space charge}} \backsim \frac{2r\pi^2}{N^3}\frac{I_o}{W_\lambda^3} \qquad (76)$$

Hence the space charge effect will become important only in the very first drift tubes. As a numerical example we might consider an energy gain of 350 Mev for deuterons in 300 meters and a beam current of 1 ampere and a wavelength of 25 meters. This gives $I_o = 2.0 \times 10^{-7}$; $W_\lambda = .0137$ and hence:

$$\left.\frac{d^2r}{dn^2}\right|_{\text{space charge}} \sim 1.5\ \frac{r}{N^3} \tag{77}$$

Although this effect is not serious here it must be included in a correct orbit calculation. Note that for a given energy gain/unit length of machine this expression scales as λ^{-3} and would become a serious current limitation at higher frequency.

8. Magnetic Lenses

Magnetic focusing has been successfully used for electron linear accelerators but its use with a heavy particle accelerator appeared impossible owing to the excessive magnet powers required. Early calculations in connection with the 40-foot Berkeley machine showed magnetic focusing to be totally impractical.. E. M. McMillan showed that at MTA frequencies magnetic focusing would again be practical.

The theory of magnetic lenses is thoroughly discussed by Zworykin, et al[4], and the reader is referred to them for a thorough discussion of the details of the motions.

Fig. 13 shows a diagram of a thin magnetic lens. Ions entering the fringing field at a radius r from the axis will experience a force in the azimuthal (ϕ) direction owing to the radial component of the magnetic field. This azimuthal motion in turn will interact with the longitudinal (z) component, resulting in a displacement toward the axis. As the particle crosses the central plane of the lens the azimuthal motion will be retarded, but never reversed; the radial force thus retains its direction toward the axis. Fig. 14 qualitatively shows the behavior of r, ϕ, $\frac{d\phi}{dz}$ and the radial force as a function of z. Note that a net rotation of the orbit is obtained; however, if the beam enters with zero angular momentum about the axis, it will leave with zero angular momentum. This is simply a consequence of the conservation of the canonical angular momentum:

$$\frac{d}{dt}\left\{r^2 \dot{\phi} - er A_\phi\right\} = 0 \tag{79}$$

Here A_ϕ is the magnetic vector potential. A beam with initial mechanical angular momentum before entering the lens or a beam generated in a magnetic field with zero angular momentum will never reach the point r = 0; injection into the linear accelerator with angular momentum (by injection mis-alignment) or the magnetic field in the ion source will generate a "hollow" beam.

Let us consider each drift tube to contain a magnet of the general shape of Fig. 13, and let B(z) be the magnetic field of induction along the axis. The relativistic differential equation of motion, if the canonical angular momentum of Eq. (79) is zero is given by (See Zworykin, et al[4], p. 656):*

$$\frac{d^2r}{dz^2} + \frac{1}{W_o\beta^2\gamma}\left(\frac{dW}{dz}\right)\left(\frac{dr}{dz}\right) + \frac{r}{4}\left(\frac{eB}{P}\right)^2 = 0 \tag{80}$$

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*Zworykin, et al[4], use in their equations the potential ϕ through which a particle would have to be accelerated to have a given energy. They also use the term $a = e/2M_oc^2$ (Gaussian). To convert their equations to our notations we use:

$$1 + 2a\phi = \gamma = 1/\sqrt{1-\beta^2} = W/W_o$$

$$a\phi(1 + a\phi) = (\beta\gamma/2)^2 = \frac{1}{4}(cP/W_o)$$

$$1 + a\phi = (\gamma + 1)/2 = (W + W_o)/2W_o$$

Equation 18.19 of Zworykin, et al:

$$r'' = -\frac{\phi'}{2\phi}\,\frac{1+2a\phi}{1+a\phi}\,r' - \left\{\frac{\phi''}{4\phi}\left(\frac{1+2a\phi}{1+a\phi}\right) + \frac{eH^2}{8M_oc^2(1+a\phi)\,\phi}\right\} r$$

becomes in our notation:

$$\frac{d}{dz}\left(\beta\gamma\,\frac{dr}{dz}\right) + \frac{r}{2}\left\{\left(\frac{eB}{W_o}\right)^2\frac{1}{2\beta\gamma} + \frac{1}{\beta}\,\frac{d^2\gamma}{dz^2}\right\} = 0$$

where W is the total relativistic energy and W_o is the rest energy. This can be written as, ignoring here velocity focusing:

$$\frac{1}{\beta\gamma}\frac{d}{dz}\left[\beta\gamma\left(\frac{dr}{dz}\right)\right]+\frac{r}{4}\left(\frac{eB}{P}\right)^2=0 \qquad (81)$$

Since the magnets are contained in the drift tube, let us call R_n the radial position in the drift tube center where the momentum is P_n. Let $\bar{B}_n$ be the root mean square magnetic field. Then:

$$\frac{(R_{n+1}-R_n)P_{n+1}}{L_{n+1}}-\frac{(R_n-R_{n-1})\,R_n}{L_n}=-R_nL_n\left(\frac{e\bar{B}_n}{2P_n}\right)^2P_n \qquad (82)$$

Hence, from Eq. (6), in the differential limit:

$$\frac{d}{dN}\left\{\sqrt{1+N^2W_\lambda^2}\left(\frac{P_n}{P_{n,s}}\right)\frac{dR_n}{dN}\right\}=-\frac{R_n}{\sqrt{1+N^2W_\lambda^2}}\left(\frac{e\bar{B}_n\lambda}{2M_oc}\right)^2\left(\frac{P_{n,s}}{P_n}\right) \qquad (83)$$

Let the dimensionless quantity B_λ be defined by:

$$B_\lambda=(e\,\bar{B}_n\,\lambda)\Big/(2M_oc) \qquad (84)$$

The radial equation is thus:

$$\sqrt{1+N^2W^2}\;\frac{d}{dN}\left\{\left(\frac{P_n}{P_{n,s}}\right)\left(\sqrt{1+N^2W_\lambda^2}\;\frac{dR_n}{dN}\right)\right\}+B_\lambda^2R_n\left(\frac{P_{n,s}}{P_n}\right)=0 \qquad (85)$$

The term $P_{n,s}/P_n$ is a coupling term between the derivative of the phase motion and the radial motion. For a synchronous orbit the radial motion under influence of the magnetic field only is thus:

$$R_n\propto\frac{1}{B_\lambda^{1/2}}\,e^{\pm i\int^n(B_\lambda\gamma_n^{-1})\,d_n} \qquad (86)$$

Let us now investigate the combined effect of phase defocusing and magnetic lens action. The radial equation, when combining Eq. (37) and Eq. (40), becomes, taking $r_n\approx R_n$ and using synchronous orbits:

$$\frac{d}{dN}\left\{\gamma_n\frac{dr_n}{dN}\right\}+\left[B_\lambda^2\,\gamma_n^{-1}-\frac{\pi}{N}\,\gamma_n^{-2}(-\tan\phi_s)\right]r_n=0 \qquad (87)$$

The motion is thus:

$$r_n \propto \frac{1}{\left[B_\lambda^2 - \frac{\pi}{N}\gamma_n^{-1}(-\tan\phi_s)\right]^{1/4}} \quad e^{\pm i\int^N \left[B_\lambda^2 \gamma_n^{-2} - \frac{\pi}{N}\gamma_n^{-3}(-\tan\phi_s)\right]^{1/2} dN} \tag{88}$$

To maintain constant amplitude we have to grade the magnetic field such that:

$$B_\lambda^2 = \frac{\pi W_o}{NW} (-\tan\phi_s) + \text{constant} \tag{89}$$

The value of the constant depends on the excess stability required at injection.*

A remark might be made here as to the order of the various kinds of focusing, expressed in terms of the dependence on N, the total number of drift tubes. This dependence is given in the following table.

	Order of radial term
Space Charge	N^{-3}
Velocity Focusing	N^{-2}
Phase Defocusing	N^{-1}
Magnetic Focusing for constant W_λ	N^0

Note also that other than for the magentic term, all non-relativistic equations depend only on the ordinal number N of drift tubes; the gradient and the nature of the particle are not contained explicitly in the equations describing the motions. The relativistic motions and the action of the magnetic lenses can be described in terms of the two dimensionless constants W_λ and B_λ.

9. Output Beam Shape

The details of the shape of the beam emerging from a linear accelerator depend of course on the injection conditions. However, if the machine is long, the radial oscillations will become essentially random in phase; in that case

*Detailed calculations based on this equation are given in UCRL-1073; Eng. Note 303-60, M3.

general things can be said without detailed integration.

Let A be the amplitude of a given radial oscillation. The probability of a particle emerging at a radius r is then:

$$P(r) = \frac{1}{\sqrt{A^2 - r^2}} \tag{90}$$

The output beam density $\sigma(r)$ is then given by:

$$\sigma(r) \sim \frac{1}{r} \int_r^{r_{max}} \frac{N(A)\ 2\pi A dA}{\sqrt{A^2 - r^2}} \tag{91}$$

Here N(A) is the initial probability distribution per unit area for a given amplitude. N(A) will of course depend on the injection conditions. Qualitatively the beam profile will be well represented by taking N(A) constant. The resultant beam shape is shown in Fig. 15. The r^{-1} singularity is due to the "crossing over" of radial oscillations of various azimuthal orientations. This singularity, which is independent of the location of any actual foci has been demonstrated to occur in the 40-foot linear accelerator both for grid and no grid operation.

The actual value of the central maximum is limited by: a) space charge, b) asymmetry of the accelerator, and c) angular momentum of the beam.

Space charge limiting of the singularity can easily be shown to be very small at reasonably high output energies. The effect of angular momentum has been studied* and in effect produces a "hollow beam".

The angular divergence of the outgoing beam depends on the wavelength of the radial oscillations. This wavelength, in the case of magnetic focusing, is given by: (See Eq. (88))

$$2\pi \lambda_R^{-1} = (\beta_n \lambda)^{-1} \gamma_n^{-1} \left[B_\lambda^2 - \frac{\pi}{N} \gamma_n^{-1} (-\tan \phi_s) \right]^{1/2} = \beta_n \lambda \gamma_n^{-1} \text{ x constant} \tag{92}$$

for a magnet system designed for constant amplitude of radial oscillation.

*Eng. Note UCRL-820, 200-10, M14, and Eng. Note UCRL-792, 205-10, M4.

If r_{max} is the maximum radius at the machine proper, the beam envelope as a function of the distance y from the machine will spread as:

$$\frac{r}{r_{max}} = \sqrt{1 + \left(\frac{2\pi y}{\lambda_R}\right)^2} \quad (93)$$

The beam shape will not deviate in its general behavior from that discussed previously as a function of the distance from the machine.

Appendix I.

The Adiabatic Theorem

The differential equations governing the oscillations in ϕ_n and r_n (if coupling is neglected) can always be put into the form:

$$\frac{d}{dn}\left(m \frac{dx}{dn}\right) + kx = 0 \qquad \text{(I-1)}$$

where m = m(n) and k=k(n). Let us assume that m(n) and k(n) are slowly varying; i.e., let

$$\frac{m'}{m} \ll \sqrt{\frac{k}{m}} \text{ and } \frac{k'}{k} \ll \sqrt{\frac{k}{m}} \qquad \text{(I-2)}$$

Let us try a solution of the form:

$$x = A(n)\, e^{i\int^n \omega(n)dn} \qquad \text{(I-3)}$$

Substituting into I-1:

$$(m'A' + m\,A'' - m\,A\omega^2 + k\,A) + i(2\omega m\,A' + \omega m'\,A + \omega' m\,A) = 0 \qquad \text{(I-4)}$$

ignoring higher order terms this gives:

$$-m\,A\omega^2 + k\,A = 0 \qquad \text{(I-5)}$$

$$2\omega m\,A' + \omega\,m'\,A + \omega'\,m\,A = 0 \qquad \text{(I-6)}$$

Hence:

$$\omega = \sqrt{\frac{k}{m}} \qquad \text{(I-7)}$$

and, from (I-6):

$$A = \frac{1}{\sqrt[4]{k\,m}} \qquad \text{(I-8)}$$

References

1. E. M. McMillan, Phys. Rev. 80, 493 (1950).

2. Ginzton, Hansen and Kennedy, Rev. Sci. Instr. 19, 89 (1948).

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4. V. K. Zworykin, et al, Electron Optics and the Electron Microscope, John Wiley and Sons, New York (1945).

5. R. Serber, Phys. Rev. 73, 535 (1948).

6. D. Bohm and L. Foldy, Phys. Rev. 72, 649 (1947).

Figure Captions

Fig. 1. Schematic diagram of drift tube geometry. The lines AB and CD denote division into unit cells.

Fig. 2. Diagram indicating notation used for the geometrical parameters.

Fig. 3. Lines of force between drift tubes.

Fig. 4. Grid or foil geometry.

Fig. 5. Diagram indicating range of foil (grid) focusing and phase stability for $g_n/L_n = .25$.

Fig. 6. Radial oscillations of foil focused linear accelerator for various synchronous phase angles.

Fig. 7. Pendulum model representing phase oscillations of linear accelerator. The sign conventions for $\bar{\phi}_n$ are indicated.

Fig. 8. Effective "potential" applying to pendulum model. Note the rapidity of potential depth increase with synchronous phase angle.

Fig. 9. Diagram showing tolerance of the injection conditions. All particles starting inside the indicated curves will be stably accelerated. ΔW = deviation in injection energy from synchronous value.

Fig. 10. Possible mode of phase unstable operation of a "short" linear accelerator.

Fig. 11. Phase motion of particles in a phase unstable accelerator for various injection energies.

Fig. 12. Radial motions for typical phase unstable motions.

Fig. 13. Magnetic lens.

Fig. 14. Behavior of radius (r); azimuth angle (ϕ), angular velocity $\left(\frac{d\phi}{dz}\right)$, radial force ($F_r$) as a function of axial distance.

Fig. 15. Output beam profile of a long accelerator. It is assumed that radial oscillations are random in phase.

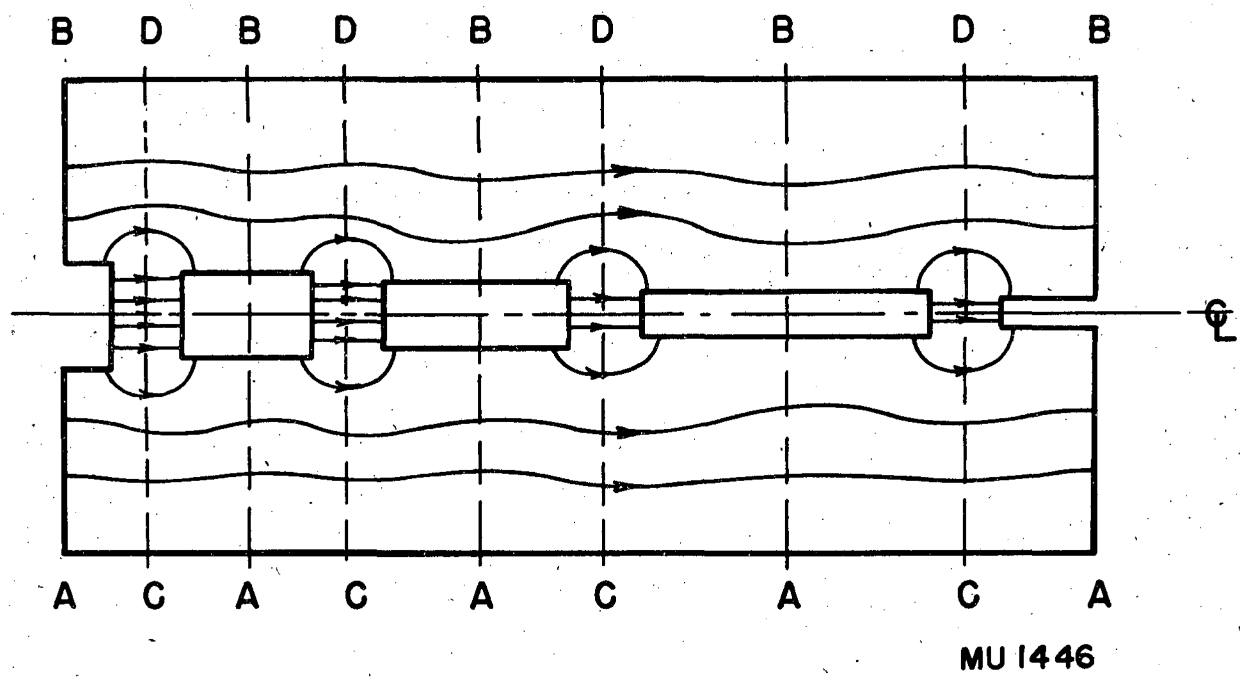


Fig. 1

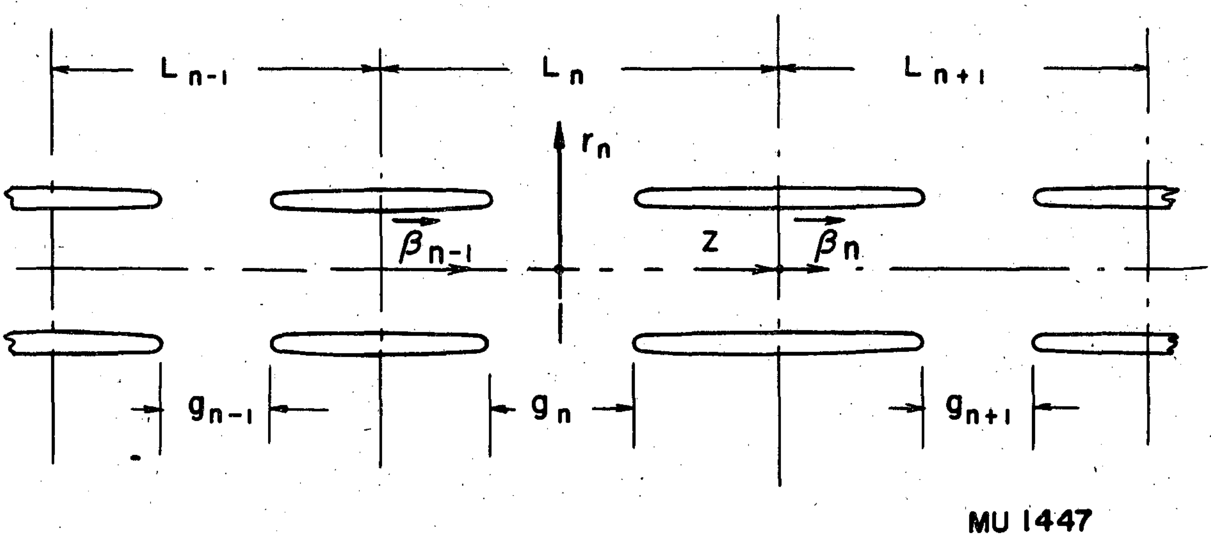
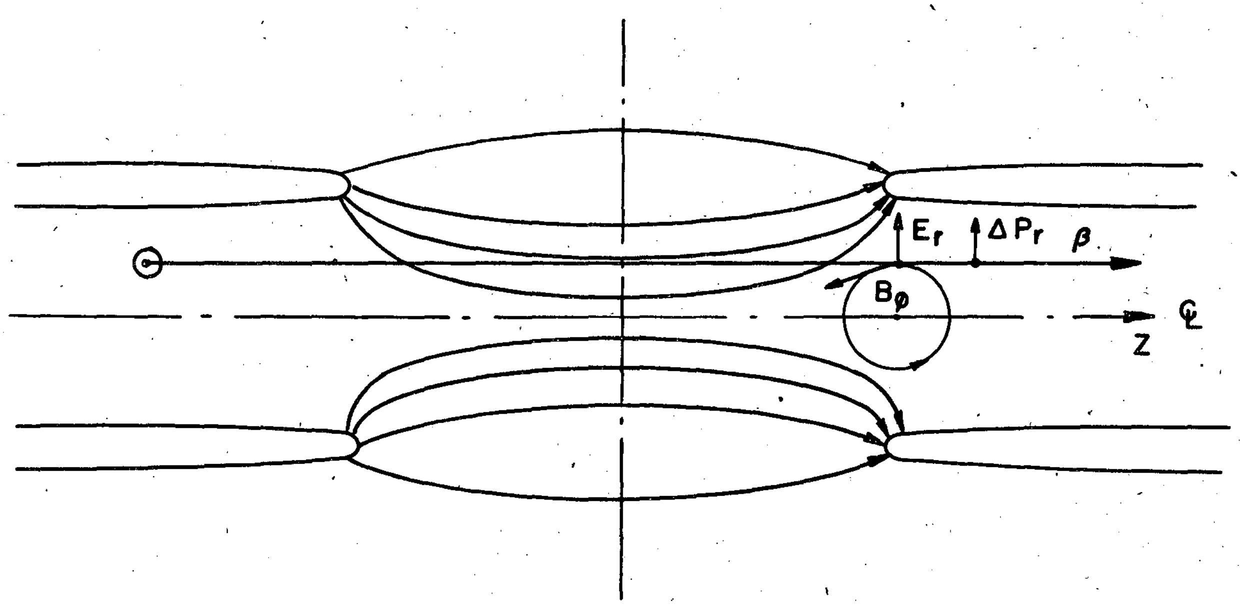


Fig. 2



MU 1448

Fig. 3

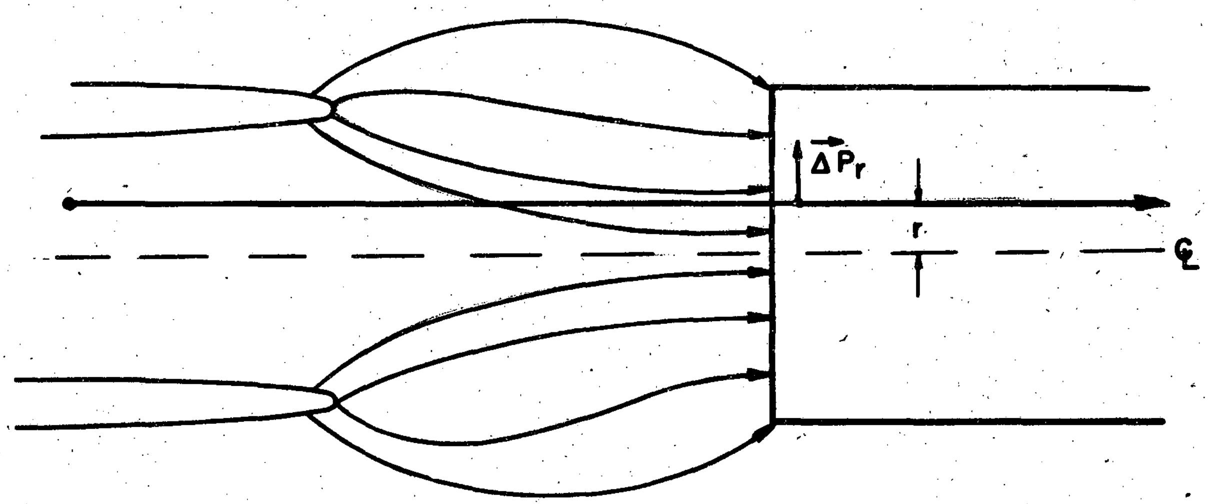


Fig. 4

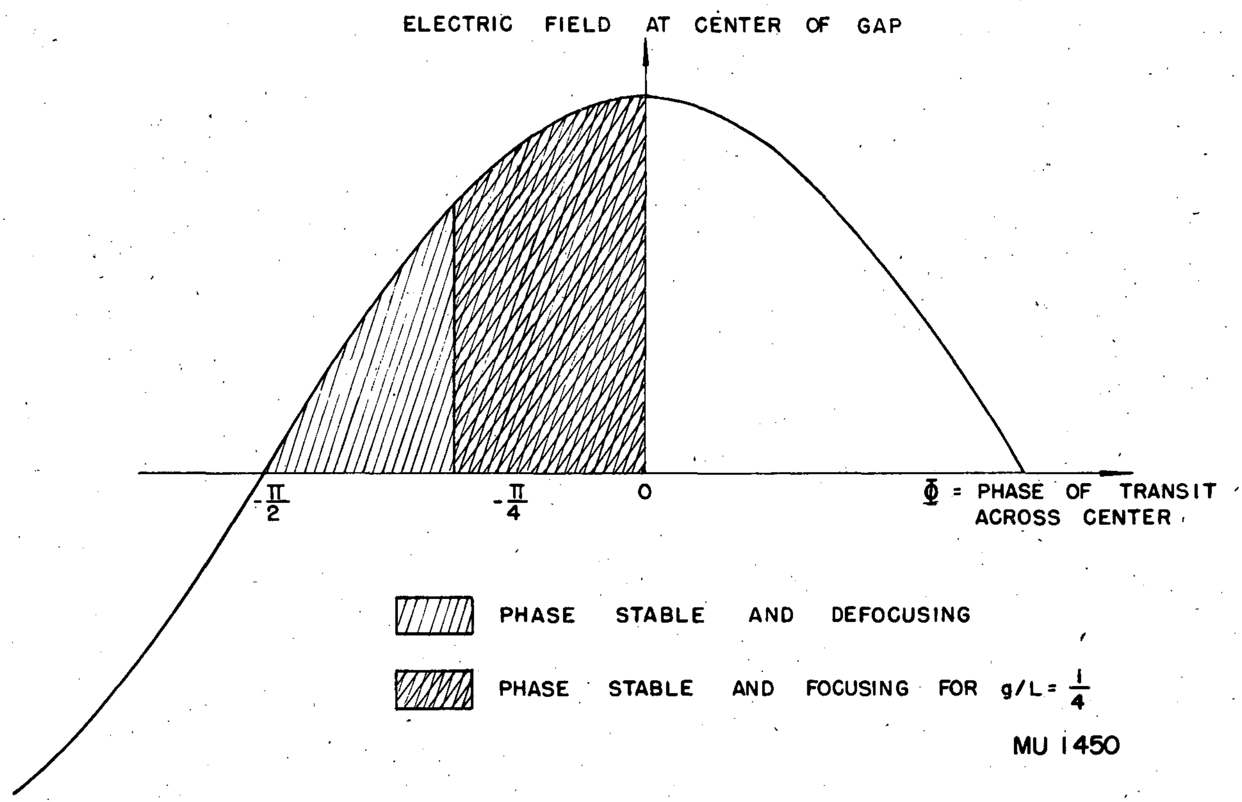

ELECTRIC FIELD AT CENTER OF GAP
$-\frac{\pi}{2}$
$-\frac{\pi}{4}$
0
Φ = PHASE OF TRANSIT ACROSS CENTER
PHASE STABLE AND DEFOCUSING
PHASE STABLE AND FOCUSING FOR $g/L = \frac{1}{4}$
MU 1450

Fig. 5

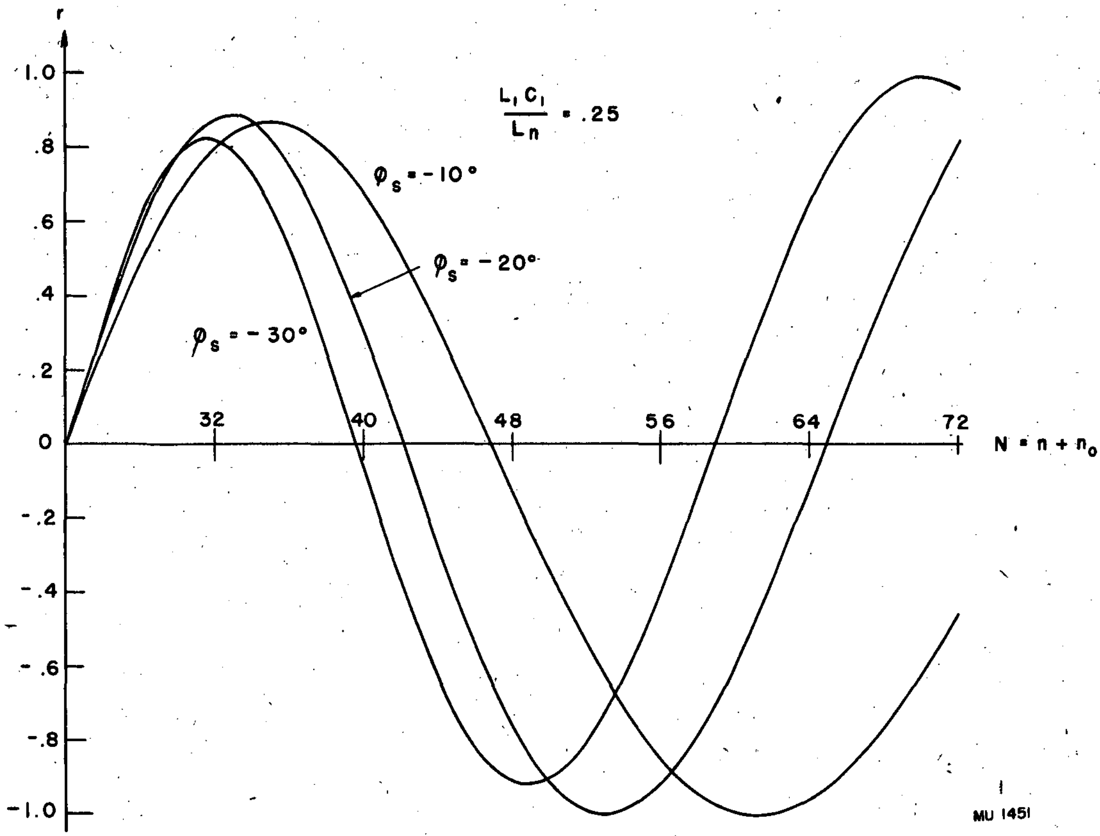
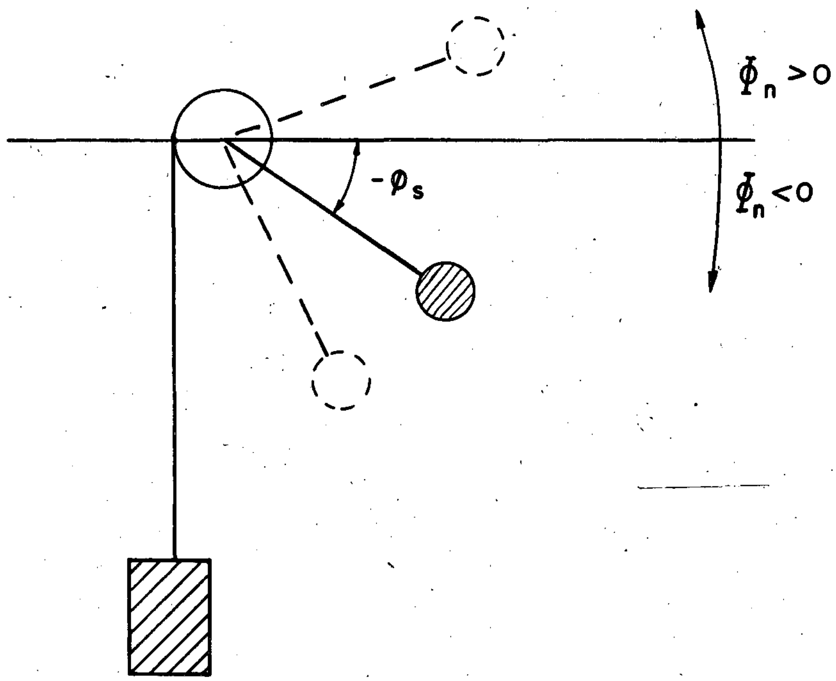

r
1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
32
40
48
56
64
72
N = n + n₀
L₁C₁ / Lₙ = .25
φs = -10°
φs = -20°
φs = -30°
MU 1451

Fig. 6



MU 1452

Fig. 7

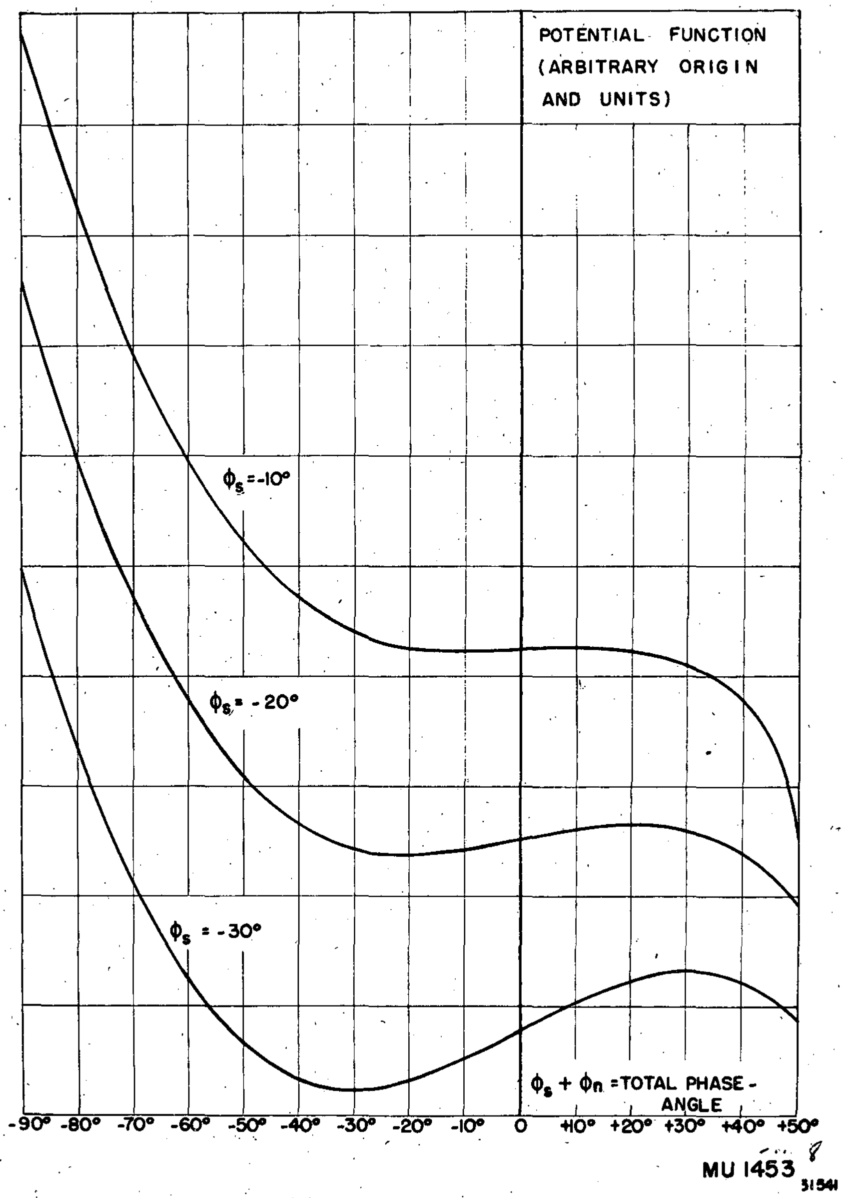
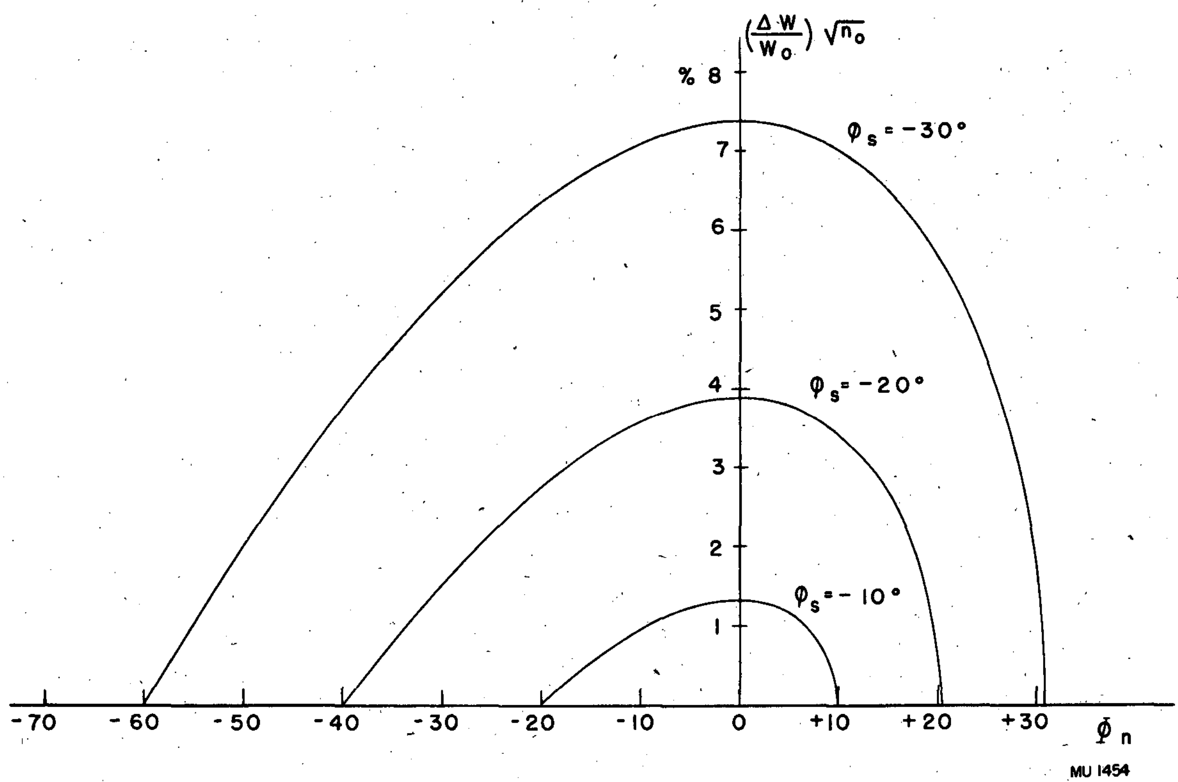

POTENTIAL FUNCTION (ARBITRARY ORIGIN AND UNITS)
ϕ_s = -10°
ϕ_s = -20°
ϕ_s = -30°
$\phi_s + \phi_n$ = TOTAL PHASE-ANGLE
-90° -80° -70° -60° -50° -40° -30° -20° -10° 0 +10° +20° +30° +40° +50°
MU 1453

Fig. 8



$\left(\frac{\Delta W}{W_0}\right)\sqrt{n_0}$
%
8
7
6
5
4
3
2
1
$\varphi_s = -30°$
$\varphi_s = -20°$
$\varphi_s = -10°$
-70
-60
-50
-40
-30
-20
-10
0
+10
+20
+30
Φ_n
MU 1454

Fig. 9

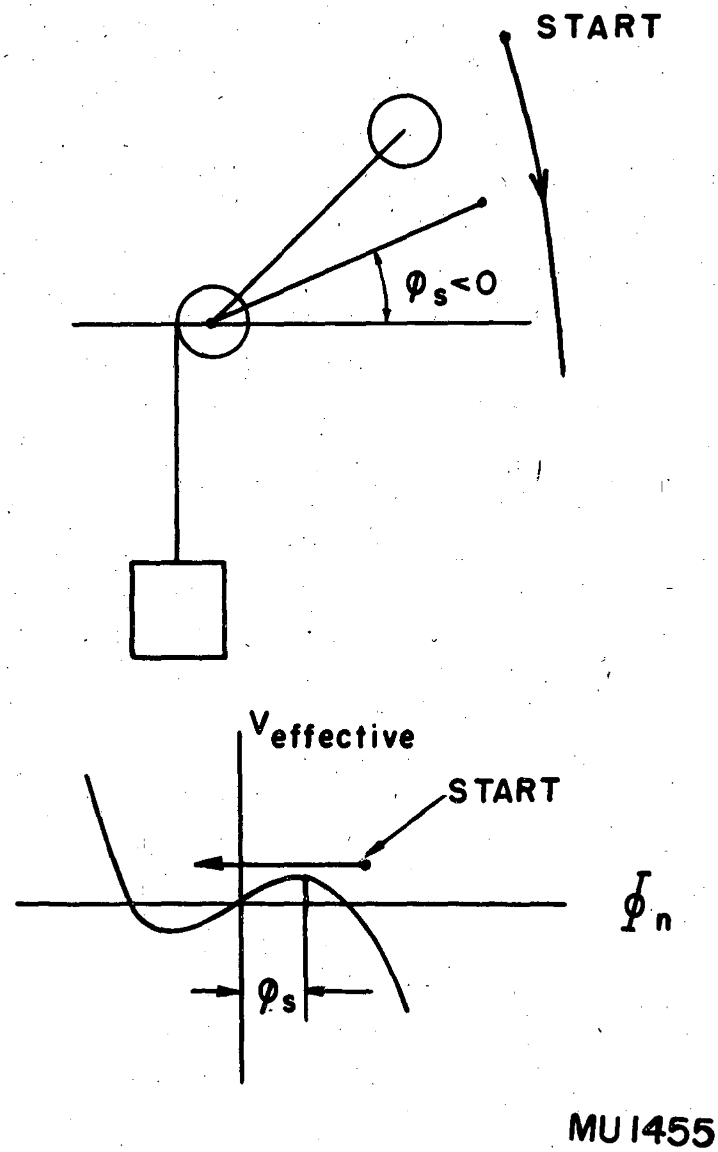
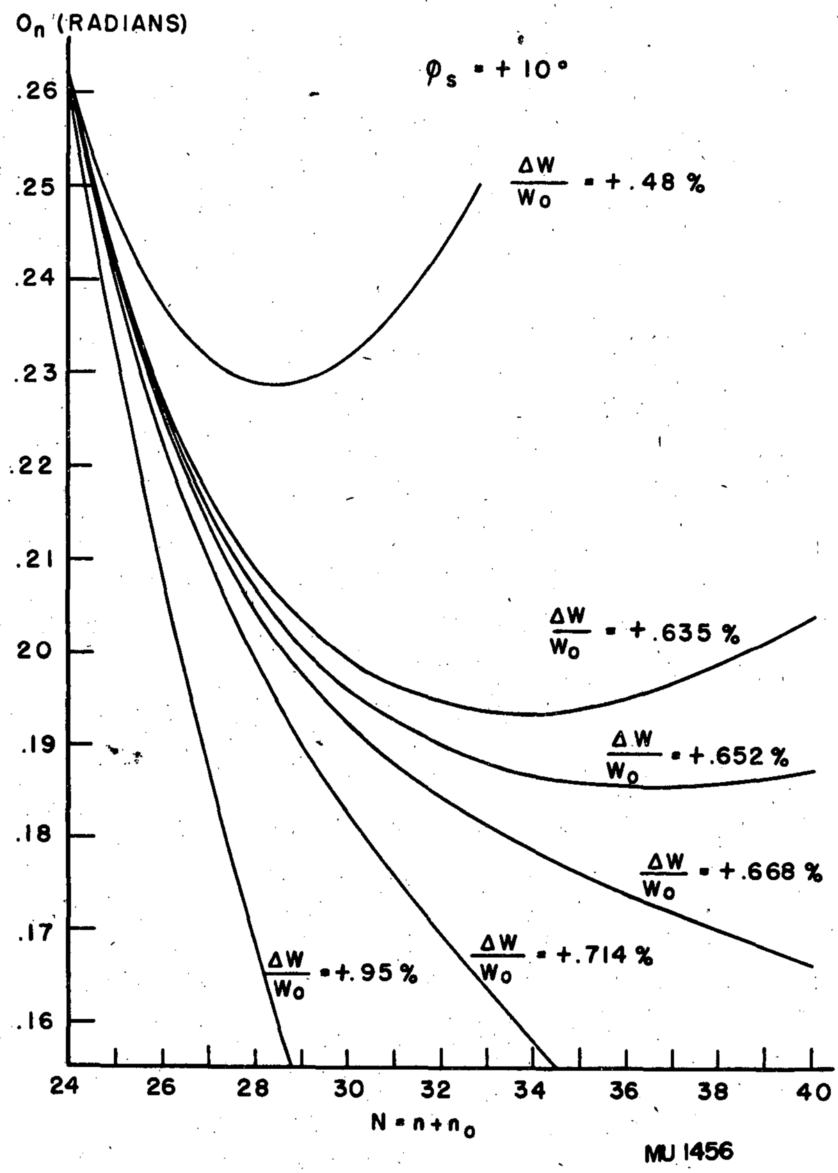


Fig. 10



On (RADIANS)
φs = +10°
ΔW/Wo = +.48 %
ΔW/Wo = +.635 %
ΔW/Wo = +.652 %
ΔW/Wo = +.668 %
ΔW/Wo = +.714 %
ΔW/Wo = +.95 %
.26
.25
.24
.23
.22
.21
20
.19
.18
.17
.16
24
26
28
30
32
34
36
38
40
N = n+no
MU 1456

Fig. 11

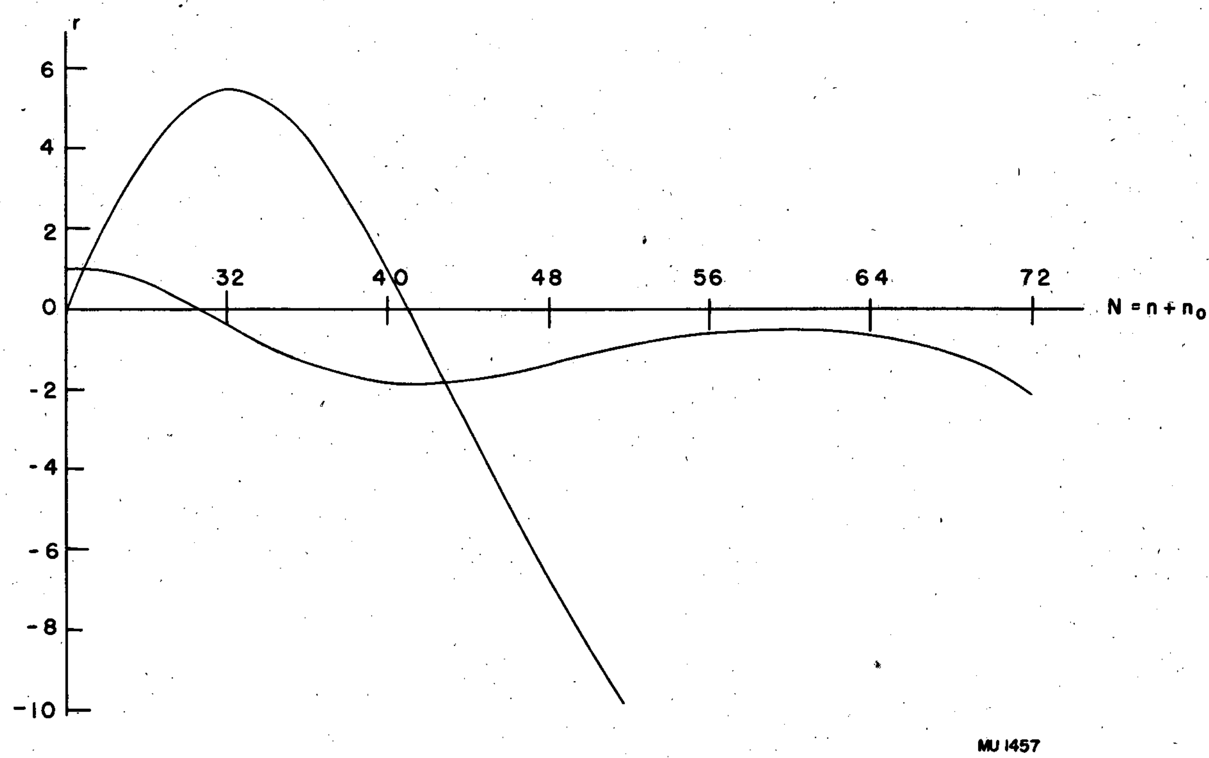


Fig. 12

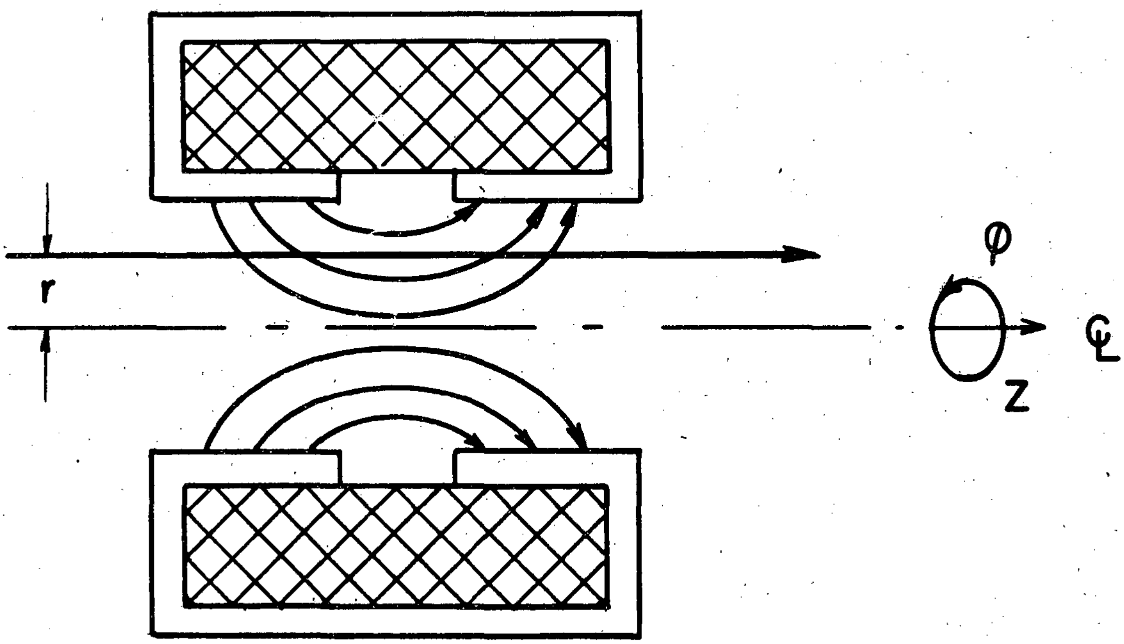


Fig. 13

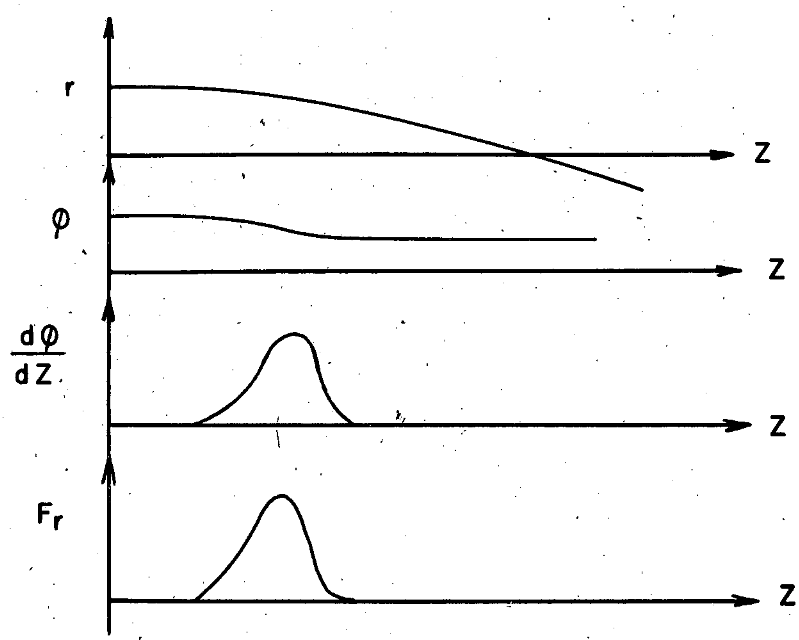


Fig. 14

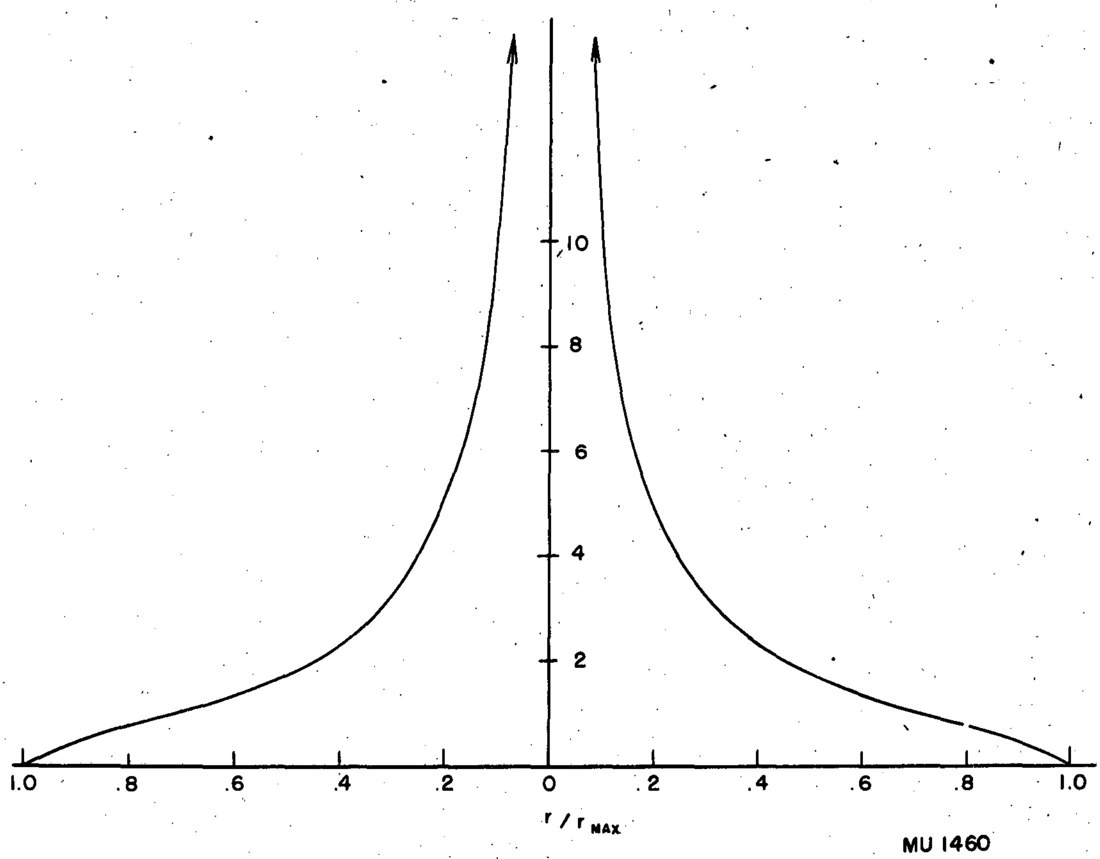


Fig. 15

DECLASSIFIED