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Demazure Flags of Local Weyl Modules

Author
Wand, Jeffrey Olan

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Demazure Flags for Local Weyl Modules

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by

Jeffrey Olan Wand

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Dissertation Committee:

Dr. Vyjayanthi Chari , Chairperson
Dr. Wee Liang Gan
Dr. Jacob Greenstein

The Dissertation of Jeffrey Olan Wand is approved:

Committee Chairperson

University of California, Riverside

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ABSTRACT OF THE DISSERTATION

Demazure Flags for Local Weyl Modules

by

Jeffrey Olan Wand

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, August 2015
Dr. Vyjayanthi Chari , Chairperson

In [3] it was shown that for $\mathfrak{sl}_2$, local Weyl modules have level 2 Demazure flags. In addition, a closed formula was given for the multiplicity of Demazure module in a level 2 Demazure flag. This thesis is a continuation of the work in [3]. More precisely, the objective of this thesis is to develop the framework for constructing level 2 Demazure flags for local Weyl modules for $\mathfrak{sl}_n$. We construct right exact sequences that will (provided they are exact) allow one to construct the level 2 Demazure flag and employ combinatorics to calculate multiplicities of Demazure modules in the flag. In the case of $\mathfrak{sl}_3$ we show that these right exact sequences are exact and give a closed formula for the multiplicities.

Contents

Chapter 1

Introduction

This thesis focuses on the representation theory of modules for the current algebra $\mathfrak{sl}_n[t]$, where $\mathfrak{sl}_n$ is the Lie algebra of $n \times n$ traceless complex matrices. In general, for any complex simple Lie algebra $\mathfrak{g}$, the current algebra is just the vector space $\mathfrak{g} \otimes \mathbb{C}[t]$, where $\mathbb{C}[t]$ is the polynomial ring over the complex numbers. Since the polynomial ring has the natural degree grading, $\mathfrak{g}[t]$ inherits a grading. We then say that V is a graded module over $\mathfrak{g}$ if V itself is graded, and the action of $\mathfrak{g}$ respects this grading.

We will be interested in two special finite dimensional cyclic modules for $\mathfrak{sl}_{\mathfrak{n}}[t]$ namely the local Weyl module, $W_{\text{loc}}(\lambda)$, and the level ℓ Demazure module, $D(\ell, \lambda)$, where λ is in the set of dominant integral weights of $\mathfrak{sl}_n$, denoted P^+. In addition, if we denote the generators of $W_{\text{loc}}(\lambda)$ and $D(\ell, \lambda)$ by w_λ and d_λ, these modules become graded once we declare w_λ, d_λ to have grade zero.

The local Weyl modules were first introduced and studied in [2]. Recently it has been shown that the local Weyl modules have a certain connection with Demazure modules. In particular it was shown in [5],[1],[7] that the local Weyl module is isomorphic to the level one Demazure module when $\mathfrak{g}$ is simply laced (i.e $\mathfrak{g}$ is of type A,D, or E). This is not true in the non-simply laced case (i.e $\mathfrak{g}$ =B,C,F,G). So a natural question is how are the local Weyl modules related to the level one Demazure modules when $\mathfrak{g}$ is not simply laced?

In 2011, it was proven in [11] that if $\mathfrak{g}$ is not simply laced then the local Weyl module has a **filtration** by level one Demazure modules. More precisely there exists a chain of modules:

$$0 \subset V_0 \subset V_1 \subset \cdots \subset V_k = W_{\text{loc}}(\lambda)$$

where $V_i/V_{i-1} \cong \tau_{q_i} D(1, \lambda_i)$ for some $\lambda_i \in P^+$ and τ_{q_i} is the uniform grade shift operator. To prove this, Naoi used the following result by Joseph: a level ℓ Demazure module for $\widehat{\mathfrak{sl}_{\mathfrak{n}}}$ has a level m Demazure filtration when $m \geq \ell$ (see [8] Thm 5.22). However, as Naoi notes in [11](see remark 4.15), Joseph's proof is only for the case when $\hat{\mathfrak{g}}$ is simply laced (i.e excluding $\widehat{\mathfrak{sl}_2}$). Naoi points out though, that the simply laced requirement was only necessary in Joseph's proof to use a positivity result of Lusztig in [9], but this result also works for $\widehat{\mathfrak{sl}_2}$ and hence Joseph's result holds for all $\widehat{\mathfrak{sl}_{\mathfrak{n}}}$.

To further discuss Naoi's results, we set up some notation. Given any $\mu, \lambda \in P^+$, we denote the multiplicity, $[W_{\text{loc}}(\lambda), D(1, \mu)]_q$, as the number of times $\tau_q D(1, \mu)$ occurs in a filtration of $W_{\text{loc}}(\lambda)$ (Note this is independent of the filtration). In [11] Naoi gave a way to find $[W_{\text{loc}}(\lambda), D(1, \mu)]_q$ by looking at a certain subalgebra of $\mathfrak{g}$. More precisely, if you let k be the number of simple short roots of $\mathfrak{g}$, Naoi constructs a Lie subalgebra of $\mathfrak{g}$ from these short roots, denoted $\mathfrak{g}^{\text{sh}}$, that is isomorphic to $\mathfrak{sl}_{k+1}$. His result is the following:

$$[W_{\text{loc}}(\lambda), D(1, \mu)]_q = [W_{\text{loc}}^{\text{sh}}(\lambda|_{\text{sh}}), D^{sh}(r, \mu|_{sh})]_q$$

where $\lambda|_{\text{sh}}, \mu|_{sh}$ are λ, μ restricted to a Cartan subalgebra of $\mathfrak{g}^{\text{sh}}$, $W_{\text{loc}}^{\text{sh}}(\lambda|_{\text{sh}}), D^{sh}(1, \mu|_{sh})$

are the corresponding local Weyl and Demazure modules associated to these weights, and r is 3 for $\mathfrak{g} = \mathrm{G}_2$ and 2 otherwise. Thus if we want to calculate multiplicities when $\mathfrak{g}$ is not simply laced, it is enough to calculate the multiplicity of $D(r, \mu)$ in $W_{\mathrm{loc}}(\lambda)$ for $\mathfrak{sl}_{k+1}$ and the appropriate r.

Giving a closed formula for the multiplicites of the Demazure filtrations was one of the driving motivations of [3]. More specifically, we wanted to calculate $[W_{\mathrm{loc}}(m), D(2, n)]_q$ in the case when $\mathfrak{g} = \mathfrak{sl}_2$ and arbitrary $m, n \in P^+ = \mathbb{Z}$. In addition we wanted to give a more direct proof of the fact that level ℓ Demazure modules for $\widehat{\mathfrak{sl}_2}$ have filtrations by level m Demazure modules for $m \geq \ell$. To accomplish these goals, we used the modules constructed in [5]. In the case of $\mathfrak{sl}_2$, Chari and Venkatesh defined the $V(\xi)$-modules (where ξ is a partition of the integer n), as the quotient of $W_{\mathrm{loc}}(n)$ by certain relations involving ξ (for a more general definition see [5] section 2). In addition, they developed short exact sequences involving the $V(\xi)$-modules. These sequences enabled us to explicitly construct the Demazure filtrations and find a closed formula for the multiplicity.

This thesis is a continuation of the work done in [3]. In particular, we make progress on showing that for arbitrary $\mathfrak{sl}_{k+1}$, $W_{\mathrm{loc}}(\lambda)$ has a Demazure filtration by level two Demazure modules. Also, we want to give a closed formula for the graded multiplicity $[W_{\mathrm{loc}}(\lambda), D(1, \mu)]_q$. This thesis will be organized as follows: Chapter 2 will be dedicated to setting up the notation, precisely defining the modules we will use, and stating the main results of this thesis. In particular, we will define the modules $D(\lambda, \mu)$ and state Theorem 14 which gives right exact sequences of these modules. Chapter 3 we will be dedicated to proving Theorem 14. In Chapter 4 we fix $\mathfrak{g} = \mathfrak{sl}_3$ and prove that the right exact sequences in Theorem 14 are short exact sequences. In addition, we will be able to prove that local Weyl modules have level 2 Demazure flags and give a closed formula for the multiplicity of a Demazure module in a flag of this type.

Chapter 2

Background and Notation

In this chapter we will develop the notation used throughout this thesis. In addition we will define the modules we are studying in terms of generators and relations and discuss some of their properties.

2.1 The Current Algebra and Graded Modules

Let $\mathbb{C}$ denote the complex numbers, $\mathbb{Z}$ denote the integers, $\mathbb{Z}_{\geq 0}$the nonnegative integers, and $\mathbb{N}$ denote the natural numbers. Given any two complex vector spaces V, W we denote their tensor product over $\mathbb{C}$ by $V \otimes W$. In addition, we say a vector space V is **$\mathbb{Z}$-graded** if we can write V as the direct sum $V = \bigoplus_{r \in \mathbb{Z}} V[r]$.

Let $\mathfrak{g}$ be an arbitrary complex Lie algebra, then we denote $U(\mathfrak{g})$ be the **universal enveloping algebra** of $\mathfrak{g}$. As mentioned in the introduction, the associated current algebra of $\mathfrak{g}$ is, as a vector space, $\mathfrak{g} \otimes \mathbb{C}[t]$ where $\mathbb{C}[t]$ is the set of polynomials with complex coefficients. We will denote the current algebra by $\mathfrak{g}[t]$. In addition, we can make $\mathfrak{g}[t]$ into a Lie algebra with the following action. Let $x \otimes f(t), y \otimes g(t) \in \mathfrak{g}[t]$, then we define the bracket action of $\mathfrak{g}[t]$ to be

$$[x \otimes f(t), y \otimes g(t)] = [x, y]_{\mathfrak{g}} \otimes f(t)g(t)$$

where $[\cdot, \cdot]_{\mathfrak{g}}$ is the bracket operation of $\mathfrak{g}$. Lastly the current algebra inherits a grading from the degree grading of $\mathbb{C}[t]$. In particular

$$\mathfrak{g}[t] = \bigoplus_{r \in \mathbb{Z}_{\geq 0}} (\mathfrak{g} \otimes \mathbb{C}[t])[r]$$

where $(\mathfrak{g} \otimes \mathbb{C}[t])[r] = \{x \otimes f(t) : x \in \mathfrak{g}, f(t) \in \mathbb{C}[t][r]\}$. This grading also induces a grading on $U(\mathfrak{g}[t])$. In particular, elements such as $(x_1 \otimes t^{r_1})(x_2 \otimes t^{r_2}) \cdots (x_m \otimes t^{r_m})$ has grade $r_1 + r_2 + \cdots + r_m$.

Since the current algebra is graded, we can also define the notion of a graded $\mathfrak{g}[t]$-module. We say that V is a **graded** $\mathfrak{g}[t]$-module if V itself is $\mathbb{Z}$-graded and the action of $\mathfrak{g}[t]$ respects this grading in the following way: let $f(t) \in \mathbb{C}[t][s]$ and $x \in \mathfrak{g}$

$$(x \otimes f(t))V[r] \subset V[r+s].$$

Notice that $V[r]$ is a $\mathfrak{g}$-module if V is a graded $\mathfrak{g}[t]$-module. Let V' and V be two graded $\mathfrak{g}[t]$-modules, then we say that $\phi : V \to V'$ is a map of graded $\mathfrak{g}[t]$modules if $\phi(V_1[r]) \subset V_2[r]$. In this thesis, it will be necessary to shift the grade of $V[r]$ by some number s (i.e send $V[r] \to V[r+s]$). For this we use the grade shift operator τ_s. In other words, $\tau_s V$ is the graded $\mathfrak{g}[t]$-module V where the graded pieces have been shifted uniformly by s, but the action of $\mathfrak{g}[t]$ is left unchanged.

We have a way of constructing $\mathfrak{g}[t]$-modules out of $\mathfrak{g}$-modules by doing the following. Let W be a $\mathfrak{g}$-module, $w \in W$, and $x \in \mathfrak{g}$ then we set

$$(x \otimes t^r)w = \delta_{r,0} x w.$$

We denote this induced $\mathfrak{g}[t]$-module by $\mathrm{ev}_0 W$. Note that $\mathrm{ev}_0 W$ is, in fact, a graded $\mathfrak{g}[t]$-module with $\mathrm{ev}_0 W[0] = W$.

2.2 Finite Dimensional Simple Lie Algebras

In this thesis we will be concerned with certain finite dimensional simple Lie Algebras. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra. We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ as $R \subset \mathfrak{h}^*$. The restriction of the killing form of $\mathfrak{g}$ to $\mathfrak{h}$ gives us an isomorphism from $\mathfrak{h}$ to $\mathfrak{h}^*$. From this isomorphism we can construct a symmetric, non-degenerate form $(\cdot,\cdot)$ on $\mathfrak{h}^*$. For simplicity, we assume that this form is normalized so that the square length of a long root is 2 and for $\alpha \in R$, let $d_\alpha = \frac{2}{(\alpha,\alpha)}$.

Let the dimension of $\mathfrak{h}$ be n, then we set $I = \{1, ..., n\}$ and a fix a set of simple roots $\{\alpha_i : i \in I\}$ for R and a set of fundamental weights $\{\omega_i : i \in I\} \subset \mathfrak{h}^*$. Let Q (respectively Q^+) denote the $\mathbb{Z}-$span ($\mathbb{Z}_{\geq 0}$-span) of the simple roots. Like wise we set P (respectively P^+) to be the $\mathbb{Z}-$span ($\mathbb{Z}_{\geq 0}$-span) of the fundamental weights. The positive roots, which we denote by R^+, is just the intersection $R \bigcap Q^+$. We can also define a partial order on P in the following way: let $\lambda, \mu \in P$ then we say $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q^+$. Lastly, let $\{x_\alpha^\pm, h_i : \alpha \in R^+, i \in I\}$ to be a Chevalley Basis of $\mathfrak{g}$ and for convenience we set $x_i^\pm = x_{\alpha_i}^\pm$.

Given $\lambda \in P^+$, we let $V(\lambda)$ be the finite dimensional irreducible $\mathfrak{g}-$module generated by an element v_λ with the following defining relations:

$$x_i^+ v_\lambda = 0, \quad h_i v_\lambda = \lambda(h_i) v_\lambda, \quad (x_i^-)^{\lambda(h_i)+1} v_\lambda = 0$$

where $i \in I$. These modules play an important role in a characterization of finite dimensional $\mathfrak{g}$ modules. In particular, given any finite dimensional $\mathfrak{g}$-module V, we can write V as a direct sum of irreducible modules. Weyl's dimension formula gives us the dimension of these modules:

Theorem 1 *(Weyl's dimension formula) Let* $\lambda \in P^+$ *then*

$$\dim V(\lambda) = \sum_{\alpha \in R^+} \frac{< \alpha + \rho, \alpha >}{< \alpha, \alpha >}$$

where $\rho = \sum_{i=1}^n \omega_i$.

We may turn $V(\lambda)$ into a $\mathfrak{g}[t]$-module by extending the following action of $\mathfrak{g}$ in the following way:

$$x \otimes f(t)v = f(0)xv$$

where $x \in \mathfrak{g}$, $f(t) \in \mathbb{C}[t]$, and v the generator of $V(\lambda)$. We denote this induced $\mathfrak{g}[t]$-module by $\mathrm{ev}_0 V(\lambda)$. From now on we will simply write $V(\lambda)$ for $\mathrm{ev}_0 V(\lambda)$.

We have the following two useful proposition which will be helpful in the Chapter 3 and can be found in [5].

Proposition 2 *([5] lemma 1.6) Let* V *be a* $\mathfrak{g}[t]-$*module and* $v \in V$ *such that*

$$(x_i^- \otimes t^{s_i})v = 0$$

for all $i \in I$ *and some* $s_i \in \mathbb{Z}_{\geq 0}$. *Set* $\lambda = \sum_{i \in I} d_i s_i \omega_i$. *For all* $\alpha \in R^+$, *we have*

$$(x_\alpha^- \otimes t^{s_\alpha})v = 0, \quad s_\alpha \in \mathbb{Z}_{\geq 0} \text{ such that } \lambda(h_\alpha) = d_\alpha s_\alpha.$$

Proposition 3 (Garland's Formula) *([5] lemma 2.3) Given* $s \in \mathbb{N}$, $r \in \mathbb{Z}_{\geq 0}$, *and* $\alpha \in R^+$ *then*

$$(x_\alpha^+ \otimes t)^{(s)}(x_\alpha^- \otimes 1)^{(s+r)} - (-1)^s x_\alpha^-(r,s) \in U(\mathfrak{g}[t])\mathfrak{n}^+ \bigoplus U(\mathfrak{n}^-[t] \oplus \mathfrak{h}[t]_+)\mathfrak{h}[t]_+$$

2.3 Characters and Graded Characters

Given a finite dimensional $\mathfrak{g}$- module, V, we may write

$$V = \bigoplus_{\nu \in P^+} V_\nu, \quad V_\nu = \{v \in v : h_i v = \nu(h_i)v \text{ for all } i \in I\}.$$

This is known as the **weight space decomposition** of V. In addition we define $\mathrm{wt}(V) = \{\nu \in \mathfrak{h}^* : V_\nu \neq 0\}$.

Let $\mathbb{Z}[P]$ be the group ring of P with integer coefficients and basis $e(\nu)$ with $\nu \in P$. Then we define the character of a finite-dimensional $\mathfrak{g}-$module V is the element of $\mathbb{Z}[P]$

$$ch_{\mathfrak{g}}(V) = \sum_{\nu \in P} \dim V_\nu e(\nu).$$

By using the partial order on P and the fact that $\mathrm{wt}(V(\nu)) \subset \nu - Q^+$, it is easy to show that the set $\{ch_{\mathfrak{g}} V(\nu) : \nu \in P^+\}$ is a linearly independent set of elements of $\mathbb{Z}[P]$. We also have a notion of the character for a graded $\mathfrak{g}[t]-$module. Let q be an indeterminate and V be a graded $\mathfrak{g}[t]-$module, then we let

$$ch_{gr}(V) = \sum_{r \geq 0} ch_{\mathfrak{g}} V[r] q^r.$$

Note this is an element of $\mathbb{Z}[P][q]$.

2.4 Fusion Products

In this section we associate a finite dimensional graded cyclic module to a the tensor product of graded $\mathfrak{g}[t]-$modules. This is know as the **fusion product**. To define this object, we first need to construct the associated graded space $\mathrm{gr}V$ of a $\mathfrak{g}[t]$-module V. Let V be a cyclic $\mathfrak{g}[t]$ module generated by an element $v \in V$. Let $r \in \mathbb{Z}_{\geq 0}$, we define the r-th filtration $F^r V$ of V as

$$F^r V = \bigoplus_{0 \leq s \leq r} U(\mathfrak{g}[t])[s] v.$$

Notice that each $F^r V$ is a $\mathfrak{g}$-module. Using this filtration, we construct the associated graded vector space:

$$\mathrm{gr}V = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} F^i V / F^{i-1} V,$$

where we set $F^{-1}V = 0$. gr$V$ becomes a graded cyclic $\mathfrak{g}[t]$-module with the following action:

$$(x \otimes t^m)(\bar{w}) = \overline{(x \otimes t^m)w}$$

where $w \in V$ and $\bar{w} \in F^iV/F^{i-1}V$. In addition, gr$V \cong V$ as $\mathfrak{g}$-modules and is generated by $\bar{v}$. The following lemma will be used many times in the following chapter.

Lemma 4 *Let V be a cyclic $\mathfrak{g}[t]$-module generated by $v \in V$. Then for all $u \in V$, $x \in \mathfrak{g}$, $r \in \mathbb{Z}_{\geq 0}$, $a_1, \cdots, a_r \in \mathbb{C}$, we have*

$$(x \otimes t^r)\bar{u} = \overline{(x \otimes (t-a_1)\cdots(t-a_r))u}$$

where $\bar{u}$ is in the image in gr$V$.

Proof. Let $u \in V$, $x \in \mathfrak{g}$, and $r \in \mathbb{Z}_{\geq 0}$. By definition of the action of $\mathfrak{g}[t]$ on gr$V$ we have that

$$(x \otimes t^r)\bar{u} = \overline{(x \otimes t^r)u}$$

On the other hand

$$(x \otimes (t-a_1)\cdots(t-a_r))u = (x \otimes t^r)u + (x \otimes f(t))u,$$

where $\deg(f(t)) = r - 1$, hence we have

$$\overline{(x \otimes (t-a_1)\cdots(t-a_r))u} = \overline{(x \otimes t^r)u + (x \otimes f(t))u} = \overline{(x \otimes t^r)u}.$$

■

Finally, we define the fusion product. Let $V_1, \cdots V_m$ be finite-dimensional graded cyclic modules generated by $v_1, \cdots, v_m$ respectively, and $z_1, \cdots, z_m \in \mathbb{C}$ be distinct. On each V_i we twist the action of $\mathfrak{g}[t]$ in the following way:

$$(x \otimes t^r)v_i = (x \otimes (t + z_i)^r)v_i$$

and call the new module $V_i^{z_i}$. Then we define the fusion product as the associated graded space of

$$V_1^{z_1} \otimes V_2^{z_2} \otimes \cdots \otimes V_m^{z_m}$$

and denote it by

$$V_1^{z_1} * V_2^{z_2} * \cdots * V_m^{z_m}.$$

We will often simplify the notation of the fusion product. In particular we will write $V_1 * V_2 * \cdots * V_m$ for $V_1^{z_1} * V_2^{z_2} * \cdots * V_m^{z_m}$. In addition, we note that $\dim V_1 * V_2 * \cdots * V_m = (\dim V_1)(\dim V_2) \cdots (\dim V_m)$. This enables us to use dimension arguments when using fusion products. However, a drawback is that we do not know defining relations for fusion products in general.

2.5 The Local Weyl Module and Demazure Module

In this section we define the main objects of this thesis, and state some of the known properties of them. Let $\lambda \in P^+$ then the local Weyl module, $W_{\mathrm{loc}}(\lambda)$, is the cyclic, $\mathfrak{g}[t]$-module generated by w_λ with the following defining relations:

$$(x_i^+ \otimes \mathbb{C}[t])w_\lambda = 0, \quad (h_i \otimes t^r)w_\lambda = \lambda(h_i)\delta_{0,r}w_\lambda, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}w_\lambda = 0 \qquad (2.5.1)$$

where $i \in I$ and $r \in \mathbb{Z}_{\geq 0}$. From the definition, it is easy to see that $\mathrm{wt}(W_{\mathrm{loc}}(\lambda)) \subset \lambda - Q^+$. As mentioned in the introduction, we also have that $W_{\mathrm{loc}}(\lambda)$ becomes a graded $\mathfrak{g}[t]$-module once we declare the grade of w_λ=0. In turn, we have that the 0th graded piece, $W_{\mathrm{loc}}(\lambda)[0]$ is just $V(\lambda)$. It is also known that the local Weyl module is finite dimensional, hence we can talk about its graded character. In particular we can write

$$\mathrm{ch}_{\mathrm{gr}} W_{\mathrm{loc}}(\lambda) = \mathrm{ch}_{\mathfrak{g}} V(\lambda) + \sum_{r>0} \mathrm{ch}_g W_{\mathrm{loc}}(\lambda)[r]q^r = \mathrm{ch}_{\mathfrak{g}} V(\lambda) + \sum_{\substack{r>0 \\ \mu<\lambda}} \dim \hom_{\mathfrak{g}}(V(\mu), W_{\mathrm{loc}}(\lambda))\mathrm{ch}_g V(\mu)q^r$$

For later use, we also have the following Lemma that relates quotients of local Weyl modules and fusion products.

Lemma 5 ([5]) *Let $1 \leq s \leq m$, $\lambda_s \in P^+$, and V_s be a quotient of $W_{loc}(\lambda_s)$. Then $V_1 * V_2 * \cdots * V_m$ is a quotient of $W_{loc}(\lambda)$ where $\lambda = \sum_s \lambda_s$.*

Proof. Let $z_1, \cdots, z_m$ be the parameters of the fusion product, $v_1, \cdots, v_m$ be the generators of $V_1, \cdots, V_m$, and w_λ be the generator of $W_{\rm loc}(\lambda)$. Our goal is to show that the map sending w_λ to $v_1 * \cdots * v_m$ is well-defined, and thus showing that $V_1 * \cdots * V_m$ is a quotient of $W_{\rm loc}(\lambda)$. To do this, we just need to show that $v_1 * \cdots * v_m$ satisfies the defining relations of w_λ. Since V_i are all quotients of $W_{\rm loc}(\lambda_i)$, $x_i^+ \otimes \mathbb{C}[t]v_i = 0$ and hence $(x_i^+ \otimes \mathbb{C}[t])v_1 * \cdots * v_m = 0$. Next, notice that

$$(x_i^- \otimes 1)^{\lambda(h_i)+1} v_1 \otimes \cdots \otimes v_m = \sum_{i=1}^{m} v_1 \otimes \cdots (x_i \otimes 1)^{\lambda(h_i)+1} v_i \otimes \cdots \otimes v_m = 0$$

where the last equality follows since $\lambda(h_i) \geq \lambda_s(h_i)$. Thus by lemma 4 we have that $(x_i^- \otimes 1)^{\lambda(h_i)+1}(v_1 * \cdots * v_m) = 0$. Finally let $r \in \mathbb{Z}_{\geq 0}$, then

$$(h_i \otimes t^r)(v_1 \otimes v_2 \cdots \otimes v_m) = \sum_{i=1}^{m} v_1 \otimes \cdots (h_i \otimes t^r) v_i \otimes \cdots \otimes v_m = \sum_{i=1}^{m} {z_i}^r \lambda(h_i)(v_1 \otimes v_2 \otimes \cdots \otimes v_m).$$

If $r = 0$ then by lemma 4 we have that $(h_i \otimes 1)(v_1 * v_2 * \cdots * v_m) = \lambda(h_i)(v_1 * v_2 * \cdots * v_m)$ as desired. If $r > 0$, then by lemma 4 $(h_i \otimes t^r)(v_1 * v_2 * \cdots * v_m)$ has both grade r and 0, and hence must be zero. ■

Given $\lambda \in P^+$ and $\ell \in \mathbb{N}$ we define the level ℓ Demazure module, $D(\ell, \lambda)$ as the quotient of the local Weyl module by the submodule generated by the following:

$$\{(x_\alpha^- \otimes t^p)^{r+1} w_\lambda : p \in \mathbb{Z}_{\geq 0}, \quad r \geq \max\{0, \lambda(h_\alpha) - d_\alpha \ell p\}, \quad \text{for } \alpha \in R^+\}.$$

We remark that in [5] [1],[7], [11], it was proved that $W_{\rm loc}(\lambda)$ is isomorphic to $D(1, \lambda)$ when $\mathfrak{g}$ is of type A, D, E. We will use this freely.

Like the local Weyl modules, $\mathrm{wt}(D(\ell,\lambda) \subset \lambda - Q^+$ and we can again write the graded character of the Demazure module as

$$\mathrm{ch}_{\mathrm{gr}} D(\ell,\lambda) = \mathrm{ch}_{\mathfrak{g}} V(\lambda) + \sum_{\substack{r>0 \\ \mu<\lambda}} \dim \hom_{\mathfrak{g}}(V(\mu), D(\ell,\lambda)) \mathrm{ch}_g V(\mu) q^r$$

We have the following simple lemma which we will use in the next section:

Lemma 6 *Let S be the $\mathbb{Z}[q]$ span of $\{ch_{\mathfrak{g}} V(\lambda) : \lambda \in P^+\}$ then both $\{ch_{gr} W_{loc}(\lambda) : \lambda \in P^+\}$ and $\{ch_{gr} D(\ell,\lambda) : \lambda \in P^+\}$ (for a fixed ℓ) form a $\mathbb{Z}[q]$ basis for S.*

If $\mathfrak{g}$ is simply laced it was proved in [5] that local Weyl module is isomorphic a level 1 Demazure module. In addition simpler relations were given for both the level 1 and 2 Demazure modules (See proposition 3.4 and Theorem 2). We now state a reformulation of these results.

Theorem 7 *Let $\lambda \in P^+$. If $\mathfrak{g}$ is simply laced then*

1. *$D(1,\lambda)$ is isomorphic to the quotient of $W_{loc}(\lambda)$ by the submodule generated by*

$$\{(x_\alpha^- \otimes t^{\lambda(h_\alpha)}) w_\lambda : \alpha \in R^+\} \tag{2.5.2}$$

2. *$D(2,\lambda)$ is isomorphic to the quotient of $W_{loc}(\lambda)$ by the submodule generated by*

$$\{(x_\alpha^- \otimes t^{\lceil \lambda(h_\alpha)/2 \rceil}) w_\lambda : \alpha \in R^+\} \tag{2.5.3}$$

In [4] we proved a factorization of Demazure modules into "smaller" Demazure modules. We state a restricted version of this result that will be used in later chapters.

Theorem 8 *Let $\mathfrak{g} = \mathfrak{sl}_n$, $k, \ell \in \mathbb{N}$, and $\lambda \in P^+$. Write $\lambda = \ell(\sum_{j=1}^k \lambda^j) + \lambda^0$ where $\lambda^j \in P^+$ for $0 \le j \le k$. Then we have*

$$D(\ell,\lambda) \cong D(\ell,\lambda^0) * D(\ell,\ell\lambda^1) * \cdots * D(\ell,\ell\lambda^k)$$

2.6 Demazure Flags

In this section we define the objects we will be constructing in this thesis.

Definition 9 *A level ℓ Demazure flag of a $\mathfrak{g}[t]$-module, W, is a filtration of W*

$$0 \subset W_1 \subset W_2 \subset \cdots \subset W_m = W$$

such that $W_i/W_{i-1} \cong \tau_{s_i} D(\ell, \lambda_i)$ for $1 \leq i \leq m$, $s_i \in \mathbb{Z}_{\geq 0}$, and $\lambda_i \in P^+$.

Let $[F(W) : \tau_s D(\ell, \lambda)]$ be the number of times $\tau_s D(\ell, \lambda)$ occurs in the Demazure flag, then we define

$$[F(W) : D(\ell, \lambda)]_q = \sum_{r \geq 0} [F(W) : \tau_s D(\ell, \lambda)] q^s$$

Since we know that $\{\mathrm{ch}_{\mathrm{gr}} D(\ell, \lambda) : \lambda \in P^+\}$ form a linearly independent set of the $\mathbb{Z}[q]$ span of the $\mathfrak{g}$ characters of the irreducible modules (lemma 6), we have that $[F(W) : D(\ell, \lambda)]_q$ is independent of the filtration. Thus we will remove the dependence of F and simply write $[W : D(\ell, \lambda)]_q$ for $[F(W) : D(\ell, \lambda)]_q$.

Lastly, we state a result that will enable us to calculate multiplicities using the short exact sequences we will construct.

Lemma 10 *Let V be a $\mathfrak{g}[t]$-module and let U be a submodule of V, then we have the following equality of Demazure multiplicities:*

$$[V : D(\ell, \lambda)]_q = [U : D(\ell, \lambda)]_q + [V/U : D(\ell, \lambda)]_q$$

Chapter 3

Main Results

In this chapter we will discuss the main results of this thesis. Again, our overall goal is to prove that local Weyl modules for $\mathfrak{g} = \mathfrak{sl}_n$ have level 2 Demazure flags. To do this we will need to develop a framework that will allow us to construct these filtrations. In particular, short exact sequences and Demazure filtrations are compatible, in the following sense, given a short exact sequence

$$0 \to A \to B \to C \to 0$$

we can construct a filtration of B if we have a filtration for A, C. Thus a way to construct the Level 2 Demazure flag of a local Weyl module is to develop short exact sequences involving a family of modules that include local Weyl modules and level 2 Demazure modules. This chapter is organized as follows: we will first develop some notation and define a family of modules, $D(\lambda, \mu)$ with certain $\lambda, \mu \in P^+$, which include the local Weyl module and level 2 Demazure modules. Next we state a theorem involving right exact sequences of this family. In the case of $\mathfrak{g} = \mathfrak{sl}_3$, we can prove that these right exact sequences are short exact sequences, and thus we state a result concerning the level 2 Demazure flags of the Local Weyl modules and give a closed formula for the multiplicity of a particular Demazure module in this flag.

3.1 Some Notation, Sets, and Maps

First we establish some notation. Let $\lambda \in P^+$, then we define the following integers:

$$j^o_\lambda = \max\{i \in I : \lambda(h_i) \in 2\mathbb{Z}+1\}, \quad j^e_\lambda = \max\{i \in I : \lambda(h_i) \geq 2\}$$

$$i_\lambda = \min\{i \in I : \lambda(h_i) > 0\}$$

where we set $j^o_\lambda = 0$ if $\{i \in I : \lambda(h_i) \in 2\mathbb{Z}+1\} = \emptyset$ and similarly for j^e_λ and i_λ. Let

$$P^+(1) = \{\lambda \in P^+ \backslash \{0\} : \lambda(h_i) \leq 1, \text{ for all } i \in I\}.$$

Next we define sets consisting of certain tuples of weights. These tuples will represent the the highest weights of the modules in our right exact sequences. Define $\mathbf{P}$ to be union of the following subsets of $P^+ \times P^+$:

$$\mathbf{P_1} = \{(\lambda, \mu) : j^o_\lambda = 0, \quad \mu \in P^+ \setminus \{\omega_i : 0 \leq i \leq n\}\},$$

$$\mathbf{P_2} = \{(\lambda, \mu) : \mu \in P^+(1), \quad 0 < j^o_\lambda < i_\mu\},$$

$$\mathbf{P_3} = \{(\lambda, \mu + \omega_i) : \mu \in P^+(1), \ \mu(h_i) = 1, \quad \mu(h_{i-1}) = 0, \quad 0 < j^0_\lambda < i_\mu - \delta_{i,i_\mu}\}.$$

where we establish the convention that $\omega_0 = 0$ and $h_0 = 0$. Also let

$$\mathbf{D} = \{(\lambda, \omega_i) : \lambda \in P^+, \quad j^o_\lambda = 0, \quad 0 \leq i \leq n\} \bigcup \{(\lambda, 0) : \lambda \in P^+\}.$$

Lastly we define two maps $^{\pm}: \mathbf{P} \to P^+ \times P^+$, $(\lambda,\mu) \to (\lambda^{\pm},\mu^{\pm})$ as follows:

$$(\lambda^+,\mu^+) = \begin{cases} (\lambda + 2\omega_{j^e_\mu}\ ,\ \mu - 2\omega_{j^e_\mu}),\ \mu \notin P^+(1), \\ (\lambda + \omega_{i_\mu}, \mu - \omega_{i_\mu}),\quad \mu \in P^+(1),\quad j^o_\lambda \neq 0 \\ (\lambda + \omega_{i_\mu} + \omega_{i_2}, \mu - \omega_{i_\mu} - \omega_{i_2}),\ \mu \in P^+(1),\quad j^o_\lambda = 0. \end{cases}$$

$$(\lambda^-,\mu^-) = \begin{cases} (\lambda,\quad \mu + \omega_{j^e_\mu - 1} - 2\omega_{j^e_\mu} + \omega_{j^e_\mu + 1}),\quad \mu \notin P^+(1), \\ (\lambda - \omega_{j^o_\lambda} + \omega_{j^o_\lambda - 1}\ ,\ \mu - \omega_{i_\mu} + \omega_{i_\mu + 1}),\quad \mu \in P^+(1),\quad j^o_\lambda \neq 0, \\ (\lambda, \mu + \omega_{i_\mu - 1} + \omega_{i_2 + 1} - \omega_{i_\mu} - \omega_{i_2}),\ \mu \in P^+(1),\quad j^o_\lambda = 0 \end{cases}$$

where we write $\mu = \omega_{i_\mu} + \omega_{i_2} + \cdots + \omega_{j^o_\mu}$. The following lemma shows us that the image of $\pm$ is well-behaved in the sense that they map into $\mathbf{P} \bigcup \mathbf{D}$.

Proposition 11 *The elements* $(\lambda^{\pm},\mu^{\pm})$ *are elements of* $\mathbf{P} \bigcup \mathbf{D}$.

Proof. First, suppose that $\mu \notin P^+(1)$, then $(\lambda,\mu) \in \mathbf{P_1} \cup \mathbf{P_3}$. In which case,

$$(\lambda,\mu) \in \mathbf{P_1} \Rightarrow (\lambda^{\pm},\mu^{\pm}) \in \mathbf{P_1} \cup \mathbf{D}$$

$$(\lambda,\mu) \in \mathbf{P_3} \Rightarrow (\lambda^{\pm},\mu^{\pm}) \in \mathbf{P_2} \cup \mathbf{D}$$

Next, suppose that $\mu \in P^+(1),\ \ j^o_\lambda = 0$ then $(\lambda,\mu) \in \mathbf{P_1}$ and $(\lambda^+,\mu^+) \in \mathbf{P_2} \cup \mathbf{D}$ while $(\lambda^-,\mu^-) \in \mathbf{P_1} \cup \mathbf{D}$.

Lastly, suppose that $\mu \in P^+(1),\ \ j^o_\lambda > 0$, then $(\lambda,\mu) \in \mathbf{P_2}$ and $(\lambda^+,\mu^+) \in \mathbf{P_2} \cup \mathbf{D}$. To see where (λ^-,μ^-) is contained, we have a couple of cases. First, if $\lambda = 2\nu + \omega_{j^o_\lambda - 1} + \omega_{j^o_\lambda}$, then $(\lambda^-,\mu^-) \in \mathbf{P_1} \cup \mathbf{D}$. Next if $\mu_{i_\mu + 1} = 1$ then $(\lambda^-,\mu^-) \in \mathbf{P_3} \cup \mathbf{D}$. In all other cases, $(\lambda^-,\mu^-) \in \mathbf{P_2} \cup \mathbf{D}$. ■

3.2 The Modules $D(\lambda, \mu)$

For convenience, we establish the following: Let $\lambda \in P^+$ and $1 \le i \le j \le n$, then we write $\lambda_{i,j} = \lambda(h_i + \cdots h_j)$ and $\lambda_i = \lambda(h_i)$. In addition we let

$$(\text{supp}(\lambda))^o = \{i \in I : \lambda(h_i) \in 2\mathbb{Z} + 1\}$$

Definition 12 *Let $\lambda, \mu \in P^+$. We define $D(\lambda, \mu)$ to be the quotient of $W_{loc}(\lambda + \mu)$ by the submodule generated by the elements:*

$$(x_i^- \otimes t^{\lceil \lambda_i/2 \rceil + \mu_i}) w_{\lambda+\mu}, \quad 1 \le i \le n \tag{3.2.1}$$

$$(x_{i,j}^- \otimes t^{\lceil \lambda_{i,j}/2 \rceil + \mu_{i,j}}) w_{\lambda+\mu}, \quad i, j \in (\text{supp}\,\lambda)^o, \quad (i,j) \bigcap (\text{supp}\,\lambda)^o = \emptyset \tag{3.2.2}$$

where $(i,j) = \{k : i < k < j\}$

Immediately from the relations we have the following proposition.

Proposition 13 *We have the following $\mathfrak{g}[t]$-module isomorphisms:*

$$D(\lambda, \mu) \cong \begin{cases} W_{loc}(\lambda + \mu), & \text{if } \lambda = \omega_i, \ 0 \le i \le n \\ D(2, \lambda + \mu), & \text{if } (\lambda, \mu) \in \mathbf{D} \end{cases}$$

In particular, the main theorem of this section is the following.

Theorem 14 *Let $(\lambda, \mu) \in \mathbf{P}$.*

(i) *There exists a surjection of $\mathfrak{g}[t]$-modules*

$$D(\lambda, \mu) \to D(2, \lambda) * W_{loc}(\mu) \to 0$$

(ii) *There exists the following right exact sequence*

$$\tau_{s_{\lambda,\mu}} D(\lambda^-, \mu^-) \xrightarrow{\varphi^-} D(\lambda, \mu) \xrightarrow{\varphi^+} D(\lambda^+, \mu^+) \to 0 \tag{3.2.3}$$

where $\varphi^-(v^-) = (x^-_{\alpha_{\lambda,\mu}} \otimes t^{s_{\lambda,\mu}})v,\ \varphi^+(v) = v^+,$ *and*

$$(s_{\lambda,\mu},\ \alpha_{\lambda,\mu}) = \begin{cases} (\lceil \lambda_i/2 \rceil + \mu_i - 1,\ \alpha_i), & \textit{for } \mu \notin P^+(1) \textit{ and } \mu(h_i) \geq 2 \\ (\lceil \lambda_{i_\mu + i_2}/2 \rceil + \mu_{i_{\mu+i_2}} - 1,\ \alpha_{i_\mu,i_2}), & \textit{for } \mu \in P^+(1),\quad j^o_\lambda = 0 \\ (\lceil \lambda_{j^o_\lambda,i_\mu}/2 \rceil + \mu_{j^o_\lambda,i_\mu} - 1,\ \alpha_{j^o_\lambda,i_\mu}), & \textit{for } \mu \in P^+(1),\quad j^o_\lambda > 0 \end{cases}$$

We conclude this section with a proof of the first part of the theorem. The proof of the second part will be carried out in the chapter 5.

Proof of Theorem 14(i). Let v be the generator of $D(\lambda,\mu)$ and $v_\lambda * w_\mu$ be the generator of $D(2,\lambda) * W_{\text{loc}}(\mu)$, and z_1, z_2 be the parameters of the fusion product. We define $\phi : D(\lambda,\mu) \to D(2,\lambda) * W_{\text{loc}}(\mu)$ to be the map sending v to $v_\lambda * w_\mu$. It remains to show that ϕ is well-defined. By lemma (5) we know $D(2,\lambda) * W_{\text{loc}}(\mu)$ is a quotient of $W_{\text{loc}}(\lambda+\mu)$. Thus we check the additional defining relations, (3.2.1) and (3.2.2), of $D(\lambda,\mu)$.

Notice that (3.2.1) follows from lemma 4 since

$$(x^-_i \otimes (t+z_1)^{\lceil \lambda_{i,j}/2 \rceil})(t+z_2)^{\mu_i})v_\lambda \otimes w_\mu = 0$$

where the equality is from Theorem 7. (3.2.2) follows in a similar fashion. By a combination of both Theorem (7) and a lemma (4) we have that

$$(x^-_{i,j} \otimes (t+z_1)^{\lceil \lambda_{i,j}/2 \rceil}(t+z_2)^{\mu_{i,j}})v_\lambda \otimes w_\mu = 0.$$

and we are done. ■

3.3 The $\mathfrak{sl}_3$ case

In the case of $\mathfrak{g} = \mathfrak{sl}_3$ we can use dimension arguments to show that the right exact sequences in Theorem (2) are short exact sequences.

Corollary 15 *Let $\mathfrak{g} = \mathfrak{sl}_3$ then we have the following short exact sequence*

$$0 \to \tau_{s_{\lambda,\mu}} D(\lambda^-, \mu^-) \overset{\varphi^-}{\to} D(\lambda, \mu) \overset{\varphi^+}{\to} D(\lambda^+, \mu^+) \to 0$$

In addition $W_{loc}(\lambda)$ has a level 2 Demazure flag.

An advantage of having a short exact sequences is that the Demazure module multiplicity distributes over a short exact sequence (see lemma 10). In the case of $\mathfrak{g} = \mathfrak{sl}_3$ we are able to give a closed formula for the multiplicity of a Demazure module in a local Weyl module. To state this result, we first develop some notation.

Let $\lambda, \mu \in P^+$ and write $\lambda = \sum_{i=1}^{2} r_i \omega_i$ and $\mu = \sum_{i=1}^{2} s_i \omega_i$. If $\lambda - \mu \in Q^+$ then we write $\lambda - \mu = \sum_{i=1}^{2} k_i \alpha_i$.

Theorem 16 *The multiplicity of $D(2, \mu)$ in a Demazure flag of $W_{loc}(\lambda)$ is given by*

$$[W_{loc}(\lambda) : D(2, \mu)]_q = \begin{cases} \prod_{i=1}^{2} q^{\frac{1}{2}(r_i + res_2(s_i))k_i} \begin{bmatrix} \lfloor \frac{s_i}{2} \rfloor + k_i \\ k_i \end{bmatrix}_q \\ 0, \ \text{otherwise} \end{cases} \quad (3.3.1)$$

where $res_2(s_i)$ is the residue of s_i modulo 2.

Chapter 4

Proof of Theorem 14 Part 2

In this chapter we will prove the second part of Theorem (14). Throughout this section we will fix the following notation. Let $(\lambda, \mu) \in \mathbf{P}$, $(\lambda^{\pm}, \mu^{\pm})$ the image of (λ, μ) under $\pm : \mathbf{P} \to \mathbf{P} \bigcup \mathbf{D}$, and let v, v^-, v^+ generate $D(\lambda, \mu), D(\lambda^-, \mu^-), D(\lambda^+, \mu^+)$ respectively. The section proceeds as follows in Section 4.1, we consider the right exact sequence when $\mu \notin P^+(1)$. In 4.2, we consider when $\mu \in P^+(1)$ and $j_\lambda^0 = 0$. Lastly, in Section 4.3, we consider when $\mu \in P^+(1)$ and $j_\lambda^0 > 0$.

4.1 $\mu \notin P^+(1)$

Note that since $\mu \notin P^+(1)$, then $(\lambda, \mu) \in \mathbf{P_1} \cup \mathbf{P}_3$.

Lemma 17 *Let $(\lambda, \mu) \in \mathbf{P}$ with $\mu \notin P^+(1)$. The map*

$$\varphi^+ : D(\lambda, \mu) \to D(\lambda + 2\omega_{j_\mu^e}, \mu - 2\omega_{j_\mu^e})$$

which sends v to v^+ is a well-defined surjection. In particular, the $\ker \phi^+$ is generated by the nonzero element $(x_{j_\mu^e}^- \otimes t^{\lceil \lambda_{j_\mu^e}/2 \rceil + \mu_{j_\mu^e} - 1})v$.

Proof. To show φ^+ is well-defined we show that v^+ satisfies the relations (3.2.1) and (3.2.2) of $D(\lambda,\mu)$. In other words, we show

$$(x_i^- \otimes t^{\lceil \lambda_i/2\rceil+\mu_i})v^+ = 0,\ \ 1 \le i \le n \tag{4.1.1}$$

$$(x_{i,j}^- \otimes t^{\lceil \lambda_{i,j}/2\rceil+\mu_{i,j}})v^+ = 0,\quad i,j \in (\mathrm{supp}(\lambda))^o,\quad (i,j)\bigcap(\mathrm{supp}(\lambda))^o = \emptyset \tag{4.1.2}$$

First, we show (4.1.1) holds. If $i \neq j_\mu^e$, then $\lceil \lambda_i^+/2\rceil + \mu_i^+ = \lceil \lambda_i/2\rceil + \mu_i$ and we are done by (3.2.1). On the other hand, if $i = j_\mu^e$, then $\lceil \lambda_{j_\mu^e}^+/2\rceil + \mu_{j_\mu^e}^+ = \lceil \lambda_{j_\mu^e}/2\rceil + \mu_{j_\mu^e} - 1$ and again we are done from (3.2.1). Next, we show (4.1.2) holds. In this case note that $(\mathrm{supp}\,\lambda)^o \neq \emptyset$ if and only if $(\lambda,\mu) \in \mathbf{P_3}$. In which case, $(\mathrm{supp}\,\lambda)^o = (\mathrm{supp}\,\lambda^+)^o$, and

$$\lceil \lambda_{i,j}/2\rceil + \mu_{i,j} \ge \lceil \lambda_{i,j}^+/2\rceil + \mu_{i,j}^+$$

which completes the proof that φ^+ is well-defined.

Next, we show that $\ker(\varphi^+)$ is generated by $(x_{j_\mu^e}^- \otimes t^{\lceil \lambda_{j_\mu^e}/2\rceil + \mu_{j_\mu^e} - 1})v$. It is clear from above that $(x_{j_\mu^e}^- \otimes t^{\lceil \lambda_{j_\mu^e}/2\rceil + \mu_{j_\mu^e} - 1})v \in \ker(\varphi^+)$. We also remark that the only different defining relations between $D(\lambda,\mu)$ and $D(\lambda^+,\mu^+)$ are

$$(x_{j_\mu^e}^- \otimes t^{\lceil \lambda_{j_\mu^e}/2\rceil + \mu_{j_\mu^e} - 1})v^+ = 0 \text{ whereas } (x_{j_\mu^e}^- \otimes t^{\lceil \lambda_{j_\mu^e}/2\rceil + \mu_{j_\mu^e}})v = 0.$$

In which case it is clear that $\ker(\varphi^+)$ is generated by $(x_{j_\mu^e}^- \otimes t^{\lceil \lambda_{j_\mu^e}/2\rceil + \mu_{j_\mu^e} - 1})v$.

■

Lemma 18 *Let $(\lambda,\mu) \in \mathbf{P}$ such that $\mu \notin P^+(1)$. Then the map*

$$\varphi^- : \tau_{\lceil \lambda_{j_\mu^e}/2\rceil + \mu_{j_\mu^e} - 1} D(\lambda, \mu + \omega_{j_\mu^e - 1} - 2\omega_{j_\mu^e} + \omega_{j_\mu^e + 1}) \to D(\lambda,\mu)$$

which sends v^- to the element $(x_{j_\mu^e}^- \otimes t^{\lceil \lambda_{j_\mu^e}/2\rceil + \mu_{j_\mu^e} - 1})v$ is a well-defined map of $\mathfrak{g}[t]$ modules.

Proof. We first show that $(x^-_{j^e_\mu} \otimes t^{\lceil \lambda_{j^e_\mu}/2 \rceil + \mu_{j^e_\mu} - 1})v$ is a highest weight vector. It is enough to show that

$$(x_i^+ \otimes 1)(x^-_{j^e_\mu} \otimes t^{\lceil \lambda_{j^e_\mu}/2 \rceil + \mu_{j^e_\mu} - 1})v = 0.$$

If $i \neq j^e_\mu$ then $[x_i, x_{j^e_\mu}] = 0$ and the result follows since v^+ is a highest weight vector.

If $i = j_{\mu^e}$ then

$$(x^+_{j^e_\mu} \otimes 1)(x^-_{j^e_\mu} \otimes t^{\lceil \lambda_{j^e_\mu}/2 \rceil + \mu_{j^e_\mu} - 1})v = c(h_{j^e_\mu} \otimes t^{\lceil \lambda_{j^e_\mu}/2 \rceil + \mu_{j^e_\mu} - 1})v = 0$$

where c is some non-zero scalar and we note that the last equality follows from the local Weyl relations since $\mu_{j^e_\mu} \geq 2$.

It remains to show that the relations (3.2.1) and (3.2.2) in $D(\lambda^-, \mu^-)$ are satisfied by the element $(x^-_{j^e_\mu} \otimes t^{\lceil \lambda_{j^e_\mu}/2 \rceil + \mu_{j^e_\mu} - 1})v$, i.e., we show

$$(x_i^- \otimes t^{\lceil \lambda_i^-/2 \rceil + \mu_i^-})(x^-_{j^e_\mu} \otimes t^{\lceil \lambda_{j^e_\mu}/2 \rceil + \mu_{j^e_\mu} - 1})v = 0, \quad 1 \leq i \leq n \tag{4.1.3}$$

$$(x^-_{i,j} \otimes t^{\lceil \lambda^-_{i,j}/2 \rceil + \mu^-_{i,j}})(x^-_{j^e_\mu} \otimes t^{\lceil \lambda_{j^e_\mu}/2 \rceil + \mu_{j^e_\mu} - 1})v = 0, \; i,j \in (\mathrm{supp}(\lambda^-))^o, \; (i,j) \bigcap (\mathrm{supp}(\lambda^-))^o = \emptyset \tag{4.1.4}$$

First, we show that (4.1.3) holds. Suppose $i \notin \{j^e_\mu - 1, j^e_\mu, j^e_\mu + 1\}$, then (4.1.3) follows by noting

$$\lceil \lambda_i^-/2 \rceil + \mu_i^- = \lceil \lambda_i/2 \rceil + \mu_i$$

$$[x_i^-, x^-_{j^e_\mu}] = 0.$$

On the other hand, suppose that $i = j^e_\mu - 1$ (similarly for $i = j^e_\mu + 1$), in which case we have that $\lceil \lambda_i^-/2 \rceil + \mu_i^- = \lceil \lambda_i/2 \rceil + \mu_i + 1$ and thus

$$(x^-_{j^e_\mu - 1} \otimes t^{\lceil \lambda^-_{j^e_\mu - 1}/2 \rceil + \mu^-_{j^e_\mu - 1}})(x^-_{j^e_\mu} \otimes t^{\lceil \lambda_{j^e_\mu}/2 \rceil + \mu_{j^e_\mu} - 1})v$$

$$= (x^-_{j^e_\mu - 1, j^e_\mu} \otimes t^{\lceil \lambda_{j^e_\mu - 1}/2 \rceil + \lceil \lambda_{j^e_\mu}/2 \rceil + \mu_{j^e_\lambda - 1, j^e_\mu}})v = 0$$

where the last equality is from lemma 2.

Lastly we consider the case when $i = j^e_\mu$ in which case $\lceil \lambda^-_i/2 \rceil + \mu^-_i = \lceil \lambda_i/2 \rceil + \mu_i - 2$. In which case (4.1.3) follows directly by Garland's lemma, i.e.,

$$\begin{aligned} 0 &= x^-_{j^e_\mu}(2, 2\lceil \lambda_{j^e_\mu}/2 \rceil + 2\mu_{j^e_\mu} - 3)v \\ &= \sum (x^-_{j^e_\mu} \otimes t^{n_1})(x^-_{j^e_\mu} \otimes t^{n_2})v \\ &= 2(x^-_{j^e_\mu} \otimes t^{\lceil \lambda_i/2 \rceil + \mu_i - 2})(x^-_{j^e_\mu} \otimes t^{\lceil \lambda_{j^e_\mu}/2 \rceil + \mu_{j^e_\mu} - 1})v \end{aligned}$$

where the above sum is over all $n_1,\ n_2$ satisfying $n_1 + n_2 = 2\lceil \lambda_{j^e_\mu}/2 \rceil + 2\mu_{j^e_\mu} - 3$.

Next, we show that (4.1.4) holds, in which case we need only consider the case when $(\operatorname{supp}(\lambda^-))^0 \neq \emptyset$, which happens if and only if $(\lambda, \mu) \in \mathbf{P}_3$. In this case, we make the following three remarks:

$$\mu = \nu + \omega_{j^e_\mu},\ \nu \in P^+(1),\ \nu_{j^e_\mu} = 1, \nu_{j^e_\mu - 1} = 0$$

$$\lambda = \lambda^- \Rightarrow (\operatorname{supp} \lambda^-)^o = (\operatorname{supp} \lambda)^o$$

$$j^o_\lambda < i_\mu - \delta_{j^e_\mu, i_\mu} \Rightarrow j^e_\mu, j^e_\mu - 1 \notin (\operatorname{supp} \lambda)^o$$

Then, it is clear that $[x^-_{j,k}, x^-_{j^e_\mu}] = 0$ for all $j, k \in (\operatorname{supp}(\lambda^-))^o$ with $(j,k) \bigcap (\operatorname{supp}(\lambda^-))^o = \emptyset$ and since $\lceil \lambda^-_{j,k}/2 \rceil + \mu^-_{j,k} = \lceil \lambda_{j,k}/2 \rceil + \mu_{j,k}$, we are done proving (4.1.4) and φ^- is well-defined. ■

4.2 $\mu \in P^+(1)$ and $j^o_\lambda = 0$

In this section, since we assume that $\mu \in P^+(1)$ and $j^o_\lambda = 0$, it follows that $\lceil \lambda^-_{i,j}/2 \rceil = \lceil \lambda_{i,j}/2 \rceil = \lambda_{i,j}/2$ for all $i, j \in I$.

Lemma 19 *Let $(\lambda, \mu) \in \mathbf{P}$ such that $\mu \in P^+(1)$ and $j^o_\lambda = 0$, the map*

$$\varphi^+ : D(\lambda, \mu) \to D(\lambda + \omega_{i_\mu} + \omega_{i_2}, \mu - \omega_{i_\mu} - \omega_{i_2})$$

which sends v to v^+ is well-defined, and the kernel is generated by the element $(x^-_{i_\mu,i_2} \otimes t^{\lceil \lambda_{i_\mu,i_2}/2\rceil + \mu_{i_\mu,i_2}-1})v$.

Proof. To show that φ^+ is well defined we show that the relations (3.2.1) and (3.2.2) of $D(\lambda,\mu)$ hold for the vector v^+. Since $j^o_\lambda = 0$, (3.2.2) is vacuously is true, threfore all we need to show is

$$(x^-_i \otimes t^{\lceil \lambda_i/2\rceil + \mu_i})v^+ = 0,\ 1 \le i \le n \qquad (4.2.1)$$

Notice that $\lceil \lambda^+_i/2\rceil + \mu^+_i = \lceil \lambda_i/2\rceil + \mu_i$ for all i. And so (4.2.1) follows immediately and φ^+ is well defined.

Next, we show that the kernel is in fact generated by $(x^-_{i_\mu,i_2} \otimes t^{\lceil \lambda_{i_\mu,i_2}/2\rceil + \mu_{i_\mu,i_2}-1})v$. First, since $\lceil \lambda^+_{i_\mu,i_2}/2\rceil + \mu^+_{i_\mu,i_2} = \lceil \lambda_{i_\mu,i_2}/2\rceil + \mu_{i_\mu,i_2} - 1$, then $(x^-_{i_\mu,i_2} \otimes t^{\lceil \lambda_{i_\mu,i_2}/2\rceil + \mu_{i_\mu,i_2}-1})v \in \ker(\varphi^+)$. By comparing the defining relations of $D(\lambda,\mu)$ and $D(\lambda^+,\mu^+)$, and observing that $(\mathrm{supp}(\lambda^+)^o) = \{i_\mu, i_2\}$ it is easy to see that $(x^-_{i_\mu,i_2} \otimes t^{\lceil \lambda_{i_\mu,i_2}/2\rceil + \mu_{i_\mu,i_2}-1})v$ must generate the kernel. ■

Lemma 20 *Let $(\lambda,\mu) \in \mathbf{P}$ such that $\mu \in P^+(1)$ and $j^0_\lambda = 0$. Then the map*

$$\varphi^- : \tau_{\lceil \lambda_{i_\mu+i_2}/2\rceil + \mu_{i_\mu+i_2}-1} D(\lambda, \mu - \omega_{i_\mu} - \omega_{i_2} + \omega_{i_\mu-1} + \omega_{i_2+1}) \to D(\lambda,\mu)$$

which sends v^- to the element $(x^-_{i_\mu,i_2} \otimes t^{\lceil \lambda_{i_\mu,i_2}/2\rceil + \mu_{i_\mu,i_2}-1})v$ is a well-defined map of $\mathfrak{g}[t]$ modules.

Proof. First we show that $(x^-_{i_\mu,i_2} \otimes t^{\lceil \lambda_{i_\mu,i_2}/2\rceil + \mu_{i_\mu,i_2}-1})v$ is a highest weight vector, i.e. we show that

$$(x^+_i \otimes 1)(x^-_{i_\mu,i_2} \otimes t^{\lceil \lambda_{i_\mu,i_2}/2\rceil + \mu_{i_\mu,i_2}-1})v = 0 \text{ for all } 1 \le i \le n.$$

If $i \notin \{i_\mu, i_2\}$ then $[x^+_i, x^-_{i_\mu,i_2}] = 0$ and we have the desired relation since v is a highest weight vector. If $i = i_\mu$ (and similarly for $i = i_2$) we have

$$(x^+_i \otimes 1)(x^-_{i_\mu,i_2} \otimes t^{\lceil \lambda_{i_\mu,i_2}/2\rceil + \mu_{i_\mu,i_2}-1})v = (x^-_{i_\mu+1,i_2} \otimes t^{\lceil \lambda_{i_\mu,i_2}/2\rceil + \mu_{i_\mu,i_2}-1})v = 0$$

by (3.2.1) and lemma 2 since

$$\lceil\lambda_{i_\mu,i_2}/2\rceil + \mu_{i_\mu,i_2} - 1 \geq \lceil\lambda_{i_\mu+1,i_2}/2\rceil + \mu_{i_\mu+1,i_2}$$

Lastly, we show that the relation (3.2.1) of $D(\lambda^-,\mu^-)$ holds, i.e. we show

$$(x_i^- \otimes t^{\lceil\lambda_i^-/2\rceil+\mu_i^-})(x_{i_\mu,i_2}^- \otimes t^{\lceil\lambda_{i_\mu,i_2}/2\rceil+\mu_{i_\mu,i_2}-1})v = 0 \tag{4.2.2}$$

For $i \notin \{i_\mu, i_2, i_\mu - 1, i_2+1\}$ we have that $[x_i^-, x_{i_\mu,i_2}^-] = 0$ and $\lceil\lambda_i^-/2\rceil + \mu_i^- = \lceil\lambda_i/2\rceil + \mu_i$ so (4.2.2) follows. Suppose $i = i_\mu$ (and similarly for $i = i_2$). We have

$$(x_{i_\mu}^- \otimes t^{\lceil\lambda_i/2\rceil+\mu_{i_\mu}-1})^2 v = 0$$

by Garlands lemma. Thus

$$(x_{i_\mu+1,i_2}^- \otimes t^{\lceil\lambda_{i_mu+1,i_2}/2\rceil+\mu_{i_\mu+1,i_2}})(x_{i_\mu}^- \otimes t^{\lceil\lambda_i/2\rceil+\mu_{i_\mu}-1})^2 v = 0.$$

However $(x_{i_\mu+1,i_2}^- \otimes t^{\lceil\lambda_{i_mu+1,i_2}/2\rceil+\mu_{i_\mu+1,i_2}})v = 0$ by (3.2.1) and lemma 2 and thus by commutation

$$2c(x_{i_\mu}^- \otimes t^{\lceil\lambda_{i_\mu}^-/2\rceil+\mu_{i_\mu}^-})(x_{i_\mu,i_2}^- \otimes t^{\lceil\lambda_{i_\mu,i_2}/2\rceil+\mu_{i_\mu,i_2}-1})v = 0$$

The case when $i = i_\mu - 1$ is clear from (3.2.1) and lemma 2. ■

4.3 $\mu \in P^+(1)$ and $j_\lambda^0 > 0$

In this section, we will write $\mu = \omega_{i_\mu} + \cdots + \omega_{j_\mu^o} \in P^+(1)$ and note that in this case we must have that $(\lambda,\mu) \in \mathbf{P}_2$ in which case we also have that $j_\lambda^o < i_\mu$.

Lemma 21 *Let $(\lambda,\mu) \in \mathbf{P}$ such that $\mu \in P^+(1)$ and $j_\lambda^o > 0$, the map*

$$\varphi^+ : D(\lambda,\mu) \to D(\lambda + \omega_{i_\mu}, \mu - \omega_{i_\mu})$$

which sends v to v^+ is well-defined, and the kernel is generated by the element $(x_{j_\lambda^o,i_\mu}^- \otimes t^{\lceil\lambda_{j_\lambda^o,i_\mu}/2\rceil+\mu_{j_\lambda^o,i_\mu}-1})v$.

Proof. First we show that φ^+ is well-defined. Here all we must show is

$$(x_i^- \otimes t^{\lceil \lambda_i/2 \rceil + \mu_i})v^+ = 0, \ 1 \le i \le n \tag{4.3.1}$$

$$(x_{i,j}^- \otimes t^{\lceil \lambda_{i,j}/2 \rceil + \mu_{i,j}})v^+ = 0, \ i, j \in (\mathrm{supp}(\lambda))^o, \ (i,j) \bigcap (\mathrm{supp}(\lambda))^o = \emptyset \tag{4.3.2}$$

We have that $\lceil \lambda_i/2 \rceil + \mu_i = \lceil \lambda_i^+/2 \rceil + \mu_i^+$ for all $1 \le i \le n$, so (4.3.1) is satisfied. Also, $(\mathrm{supp}(\lambda))^o \subset (\mathrm{supp}(\lambda^+))^o$, thus (4.3.2) is satisfied.

Next, notice that

$$(x^-_{j^o_\lambda, i_\mu} \otimes t^{\lceil \lambda^+_{j^o_\lambda, i_\mu}/2 \rceil + \mu^+_{j^o_\lambda, i_\mu}})v^+ = 0$$

and $\lceil \lambda^+_{j^o_\lambda, i_\mu}/2 \rceil + \mu^+_{j^o_\lambda, i_\mu} = \lceil \lambda_{j^o_\lambda, i_\mu}/2 \rceil + \mu_{j^o_\lambda, i_\mu} - 1$, thus

$$(x^-_{j^o_\lambda, i_\mu} \otimes t^{\lceil \lambda_{j^o_\lambda, i_\mu}/2 \rceil + \mu_{j^o_\lambda, i_\mu} - 1})v \in \ker(\varphi^+)$$

By comparing the relations of both $D(\lambda, \mu)$ and $D(\lambda^+, \mu^+)$ it is easy to see that $\ker(\varphi^+)$ is generated by $(x^-_{j^o_\lambda, i_\mu} \otimes t^{\lceil \lambda_{j^o_\lambda, i_\mu}/2 \rceil + \mu_{j^o_\lambda, i_\mu} - 1})v$. ■

Lemma 22 *Let $(\lambda, \mu) \in \mathbf{P}$ such that $\mu \in P^+(1)$ and $j^0_\lambda > 0$. Then the map*

$$\varphi^- : \tau_{\lceil \lambda_{j^o_\lambda, i_\mu}/2 \rceil + \mu_{j^o_\lambda, i_\mu} - 1} D(\lambda - \omega_{j^o_\lambda} + \omega_{j^o_\lambda - 1}, \mu - \omega_{i_\mu} + \omega_{i_\mu + 1}) \to D(\lambda, \mu)$$

which sends v^- to the element $(x^-_{j^o_\lambda, i_\mu} \otimes t^{\lceil \lambda_{j^o_\lambda, i_\mu}/2 \rceil + \mu_{j^o_\lambda, i_\mu} - 1})v$ is a well-defined map of $\mathfrak{g}[t]$ modules.

Proof. First, we show $(x^-_{j^o_\lambda, i_\mu} \otimes t^{\lceil \lambda_{j^o_\lambda, i_\mu}/2 \rceil + \mu_{j^o_\lambda, i_\mu} - 1})v$ is a highest weight vector, i.e., we show

$$(x_i^+)(x^-_{j^o_\lambda, i_\mu} \otimes t^{\lceil \lambda_{j^o_\lambda, i_\mu}/2 \rceil + \mu_{j^o_\lambda, i_\mu} - 1})v = 0$$

for all $i \in I$. For $i \notin \{j_{\lambda^o}, i_\mu\}$ then $[x_i^+, x^-_{j^o_\lambda, i_\mu}] = 0$ and the result follows since v is a highest weight vector. Suppose $i = i_\mu$ (similarly for $i = i_\mu$). We have that

$$(x^-_{j^o_\lambda + 1, i_\mu} \otimes t^{\lceil \lambda_{j^o_\lambda, i_\mu}/2 \rceil + \mu_{j^o_\lambda, i_\mu} - 1})v = 0$$

since $\lceil\lambda_{j^o_\lambda,i_\mu}/2\rceil + \mu_{j^o_\lambda,i_\mu} - 1 \geq \lceil\lambda_{j^o_\lambda+1,i_\mu}/2\rceil + \mu_{j^o_\lambda,i_\mu}$.

Next, we show that (3.2.1) and (3.2.2) of $D(\lambda^-,\mu^-)$ hold, i.e., we show

$$(x_i^- \otimes t^{\lceil\lambda_i^-/2\rceil+\mu_i^-})(x^-_{j^o_\lambda,i_\mu} \otimes t^{\lceil\lambda_{j^o_\lambda,i_\mu}/2\rceil+\mu_{j^o_\lambda,i_\mu}-1})w_{\lambda+\mu} = 0,\ i \in I \tag{4.3.3}$$

$$(x_{i,j}^- \otimes t^{\lceil\lambda_{i,j}^-/2\rceil+\mu_{i,j}^-})(x^-_{j^o_\lambda,i_\mu} \otimes t^{\lceil\lambda_{j^o_\lambda,i_\mu}/2\rceil+\mu_{j^o_\lambda,i_\mu}-1})w_{\lambda+\mu} = 0,\ i,j \in (supp(\lambda^-)^o),\ (i,j)\cap(supp(\lambda^-)^o) = \emptyset. \tag{4.3.4}$$

We show (4.3.3) first. If $i \neq j^o_\lambda, j^o_\lambda - 1, i_\mu, i_\mu + 1$ then the factors commute and $\lambda_i^- = \lambda_i$ and $\mu_i^- = \mu_i$ so (1) follows from (3.2.1).

Suppose $\mathbf{i = j^o_\lambda}$. In this case we have that $\lambda_i^- = \lambda_i - 1$, and we can write

$$\lceil\lambda_i^-/2\rceil = \frac{1}{2}(\lambda_i + 1) - 1 \text{ and } \lceil\lambda_{i,i_\mu}/2\rceil = \frac{1}{2}(\lambda_{i,i_\mu} + 1)$$

By Garland we know that

$$(x_i^- \otimes t^{\frac{1}{2}(\lambda_i+1)+\mu_i-1})^2 v = 0.$$

And by lemma (1.6) and the relations of $D(\lambda,\mu)$ we know that

$$(x^-_{i+1,i_\mu} \otimes t^{\frac{1}{2}(\lambda_{i+1,i_\mu})+\mu_{i+1,i_\mu}})v = 0$$

Therefore, putting the above two identities together it follows that

$$\begin{aligned}0 &= (x^-_{i+1,i_\mu} \otimes t^{\frac{1}{2}(\lambda_{i+1,i_\mu})+\mu_{i+1,i_\mu}})(x_i^- \otimes t^{\frac{1}{2}(\lambda_i+1)+\mu_i-1})^2 v \\ &= c(x_i^- \otimes t^{\frac{1}{2}(\lambda_i+1)+\mu_i-1})(x^-_{i,i_\mu} \otimes t^{\frac{1}{2}(\lambda_{i,i_\mu}+1)+\mu_{i,i_\mu}-1})v\end{aligned}$$

where c is a nonzero scalar.

Suppose $\mathbf{i = j^o_\lambda - 1}$. Then (4.3.3) is equivalent to showing

$$(x^-_{i,i_\mu} \otimes t^{\lceil(\lambda_i+1)/2\rceil+\lceil\lambda_{i+1,i_\mu}/2\rceil+\mu_{i,i_\mu}-1})v = 0$$

If $j^o_\lambda - 1 \notin \mathrm{supp}(\lambda)^o$ then $\lceil(\lambda_i+1)/2\rceil = \lceil\lambda_i/2\rceil + 1$, and we get the desired result by lemma 2 and relations of $D(\lambda,\mu)$. So, we suppose $j^o_\lambda - 1 \in \mathrm{supp}(\lambda)^o$. Then by the relations of $D(\lambda,\mu)$ that

$$(x^-_{i,i+1}\otimes t^{\lceil(\lambda_{i,i+1})/2\rceil+\mu_{i,i+1}})v=0$$

$$\Rightarrow (x^-_{i+2,i_\mu}\otimes t^{\lceil(\lambda_{i+2,i_\mu})/2\rceil+\mu_{i+2,i_\mu}})(x^-_{ji,j^o_\lambda}\otimes t^{\lceil(\lambda_{i,i+1})/2\rceil+\mu_{i,i+1}})v=0$$

$$\Rightarrow (x^-_{i,i_\mu}\otimes t^{\lceil(\lambda_{i+2,i_\mu})/2\rceil+\lceil(\lambda_{i,j^o_\lambda})/2\rceil+\mu_{i,i_\mu}})v=0$$

And since

$$\lceil(\lambda_{i+2,i_\mu})/2\rceil+\lceil(\lambda_{i,i+1})/2\rceil+\mu_{i,i_\mu}\le\lceil(\lambda_i+1)/2\rceil+\lceil(\lambda_{i+1,i_\mu})/2\rceil+\mu_{i,i_\mu}-1$$

(4.3.3) follows.

Suppose $\mathbf{i}=\mathbf{i}_\mu$. In this case we note that

$$\lceil(\lambda^-_{i_\mu})/2\rceil+\mu^-_{i_\mu}=\lceil(\lambda_{i_\mu})/2\rceil+\mu_{i_\mu}-1$$

By Garland's lemma, we have that $(x_{i_\mu}\otimes t^{\lceil(\lambda_{i_\mu})/2\rceil+\mu_{i_\mu}-1})^2 w_{\lambda+\mu}=0$ and hence

$$(x^-_{j^o_\lambda,i_\mu-1}\otimes t^{\lceil(\lambda_{j^o_\lambda,i_\mu-1})/2\rceil+\mu_{j^o_\lambda,i_\mu-1}})(x^-_{i_\mu}\otimes t^{\lceil(\lambda_{i_\mu})/2\rceil+\mu_{i_\mu}-1})^2 w_{\lambda+\mu}=0$$

Using commutation relations and defining relations of $D(\lambda,\mu)$, we have that

$$c(x^-_{i_\mu}\otimes t^{\lceil(\lambda_{i_\mu})/2\rceil+\mu_{i_\mu}-1})(x^-_{j^o_\lambda,i_\mu}\otimes t^{\lceil(\lambda_{j^o_\lambda,i_\mu-1})/2\rceil+\lceil(\lambda_{i_\mu})/2\rceil+\mu_{j^o_\lambda,i_\mu}-1})w_{\lambda+\mu}=0$$

where c is a nonzero scalar. Note that since $i_\mu\notin(\operatorname{supp}\lambda)^0$, then $\lceil(\lambda_{j^o_\lambda,i_\mu-1})/2\rceil+\lceil(\lambda_{i_\mu})/2\rceil=\lceil(\lambda_{j^o_\lambda,i_\mu})/2\rceil$, and so we have the desired result.

Suppose $\mathbf{i}=\mathbf{i}_\mu+\mathbf{1}$. In this case, (4.3.3) follows immediately by observing that

$$\lceil(\lambda^-_{i_\mu+1})/2\rceil+\mu^-_{i_\mu+1}=\lceil(\lambda_{i_\mu+1})/2\rceil+\mu_{i_\mu+1}+1$$

Lastly, we show that (4.3.4) holds. Suppose $j^0_\lambda-1\notin(\operatorname{supp}(\lambda))^o$. Then $(\operatorname{supp}(\lambda^-))^o\subset(\operatorname{supp}(\lambda))^o$ and hence the relation follows immediately. So we suppose that $j^0_\lambda - 1 \notin (\mathrm{supp}(\lambda))^o$ For convenience, we write

$$\lambda = 2\nu + \omega_{k_1} + \cdots + \omega_{k_m} + \omega_{j^o_\lambda}$$

$$\lambda^- = 2\nu + \omega_{k_1} + \cdots + \omega_{k_m} + \omega_{j^o_\lambda - 1}.$$

Consider $k_r, k_s \in (\mathrm{supp}(\lambda^-))^o$ with $1 \le s < r \le m$, then $[x^-_{k_r,k_s}, x^-_{j^o_\lambda, i_\mu}] = 0$ and the result follows since $\lceil \lambda^-_{k_r,k_s}/2 \rceil + \mu^-_{k_r,k_s} = \lceil \lambda_{k_r,k_s}/2 \rceil + \mu_{k_r,k_s}$.

It remains to check $i = k_m$ and $j = j^o_\lambda - 1$. First, we observe that

$$(x^-_{k_m, j^o_\lambda - 1} \otimes t^{\lceil \lambda^-_{km, j_\lambda o_{-1}}/2 \rceil + \mu^-_{k_m, j^o_\lambda - 1}})v = 0$$

follows from lemma 2 since

$$\lceil \lambda^-_{km, j_\lambda o_{-1}}/2 \rceil + \mu^-_{k_m, j^o_\lambda - 1} = \lceil (\lambda_{km, j_\lambda o_{-1}} + 1)/2 \rceil + \mu_{k_m, j^o_\lambda - 1}$$

$$= (\lambda_{k_m, j_\lambda o_{-1}} + 1)/2 + \mu_{k_m, j^o_\lambda - 1}$$

$$= (\lambda_{k_m} + 1)/2 + (\lambda_{k_m+1, j^o_\lambda - 1}/2) + \mu_{k_m, j^o_\lambda - 1}$$

$$= \lceil (\lambda_{k_m} + 1)/2 \rceil + \lceil (\lambda_{k_m+1, j^o_\lambda - 1}/2) \rceil + \mu_{k_m, j^o_\lambda - 1}.$$

Second, we observe that

$$(x^-_{k_m, i_\mu} \otimes t^{\lceil \lambda^-_{k_m, j_\lambda o_{-1}}/2 \rceil + \lceil \lambda_{j^o_\lambda, i_\mu}/2 \rceil \mu_{k_m, i_\mu} - 1})v = 0$$

since

$$\lceil \lambda^-_{k_m, j^o_\lambda - 1}/2 \rceil + \lceil \lambda_{j^o_\lambda, i_\mu}/2 \rceil + \mu_{k_m, i_\mu} - 1 =$$

$$(\lambda_{k_m, j^o_\lambda - 1} + 1)/2 + (\lambda_{j^o_\lambda, i_\mu} + 1)/2 + \mu_{k_m, i_\mu} - 1 =$$

$$\lceil \lambda_{k_m, j^o_\lambda}/2 \rceil + \mu_{k_m, j^o_\lambda} + \lceil \lambda_{j^o_\lambda + 1, i_\mu}/2 \rceil + \mu_{j^o_\lambda + 1, i_\mu}.$$

Thus (4.3.4) follows in this case, and φ^- is well-defined. ∎

Chapter 5

The $\mathfrak{sl}_3$ Case

As we saw in chapter 3, we were able to develop right exact sequences of the form:

$$\tau_{s_{\lambda,\mu}} D(\lambda^-, \mu^-) \to D(\lambda, \mu) \to D(\lambda^+, \mu^+) \to 0 \tag{5.0.1}$$

In addition, we showed that $D(\lambda, \mu)$ maps onto $D(2, \lambda) * W_{\text{loc}}(\mu)$. When we look at $\mathfrak{g}$ equal to either $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$ we can use dimension arguments to show that the right exact sequences are, in fact, short exact sequences and the surjection is an isomorphism. In addition, by using these short exact sequences we will be able to give a closed formula for $[W_{\text{loc}}(\lambda), D(2, \mu)]_q$ for arbitrary $\lambda, \mu \in P^+$. This chapter will be organized as follows. In the first section we will explicitly write out $\mathbf{P}$ for the case when $\mathfrak{g} = \mathfrak{sl}_3, \mathfrak{sl}_4$ and show that the right exact sequences in Theorem 2 are short exact. In section two and three we will give a closed formula for the graded multiplicities for $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$ respectively.

5.1 P and Short Exact Sequences for $\mathfrak{sl}_3$

Let $\mathfrak{g} = \mathfrak{sl}_3$, then it is easy to see that

$$\mathbf{P} = \{(2\nu, a_1\omega_1 + a_2\omega_2) : \nu \in P^+, \quad \text{with } a_i \geq 2 \text{ for some i}\} \bigcup \{(2\nu, \omega_1 + \omega_2) : \quad \nu \in P^+\}$$

$$\mathbf{D} = \{(2\nu, \omega_i) : \quad 0 \leq i \leq 2\} \bigcup \{(2\nu + \omega_1 + \omega_2, 0)\}$$

and 5.0.1 is one of the following three types of short exact sequences,

1. For $a_2 \geq 2$: $\tau_{a_2+b_2-1} D(2b_1\omega_1 + 2b_2\omega_2, (a_1+1)\omega_1 + (a_2-2)\omega_2) \to D(2b_1\omega_1 + 2b_2\omega_2, a_1\omega_1 + a_2\omega_2) \to D(2b_1\omega_1 + 2(b_2+1)\omega_2, a_1\omega_1 + (a_2-2)\omega_2) \to 0$

2. For $a_2 \leq 1$ and $a_1 \geq 2$: $\tau_{a_1+b_1-1} D(2b_1\omega_1 + 2b_2\omega_2, (a_1-2)\omega_1 + (a_2+1)\omega_2) \to D(2b_1\omega_1 + 2b_2\omega_2, a_1\omega_1 + a_2\omega_2) \to D(2(b_1+1)\omega_1 + 2b_2\omega_2, (a_1-2)\omega_1 + a_2\omega_2) \to 0$

3. For $a_1 = 1 = a_2$: $\tau_{b_1+b_2+1} D(2b_1\omega_1 + 2b_2\omega_2, 0) \to D(2b_1\omega_1 + 2b_2\omega_2, \omega_1 + \omega_2) \to D((2b_1+1)\omega_1 + (2b_2+1)\omega_2, 0) \to 0$

Theorem 23 *$D(\lambda, \mu) \cong D(2, \lambda) * W_{loc}(\mu)$ and hence (1), (2), (3) are short exact sequences.*

We immediately have the following corollary:

Corollary 24 *$D(\lambda, \mu)$ has a level 2 Demazure flag. In particular, the local Weyl module $W_{loc}(a_1\omega_1 + a_2\omega_2)$ has a level 2 Demazure Flag.*

Proof. Let $\mu = a_1\omega_1 + a_2\omega_2$. We induct on $s = a_1 + a_2$. If $s = 0$ or $s = 1$, then $D(\lambda, \mu) \cong D(2, \lambda + \mu)$ by Theorem (8) and we are done. If $s = 2$ then either $a_i = 2$ for some i or $a_1 = a_2 = 1$. If $a_2 = 2$ by the short exact sequence (1) we have

$$0 \to \tau_{a_2+b_2-1} D(2, \lambda + \omega_1) \to D(\lambda, 2\omega_2) \to D(2, \lambda + 2\omega_2) \to 0$$

and then a filtration for $D(\lambda, 2\omega_2)$ would be

$$0 \subset \tau_{a_2+b_2-1} D(2, \lambda + \omega_1) \subset \tau_{a_2+b_2-1} D(2, \lambda + \omega_1) \oplus D(2, \lambda + 2\omega_2) = D(\lambda, 2\omega_2).$$

A similar argument holds for the cases where $a_1 = 2$ or $a_1 = a_2 = 1$. Now suppose we that for $s \geq 2$ we have that $D(\lambda', \mu')$ has a level 2 Demazure flag and consider $D(\lambda, \mu)$

where $a_1+a_2=s+1$. Without loss of generality, let $a_2 \geq 2$. Then again by (1) we have

$$0 \to \tau_{a_2+b_2-1} D(\lambda, \mu+\omega_1-2\omega_2) \to D(\lambda,\mu) \to D(\lambda+2\omega_2, \mu-2\omega_2) \to 0.$$

By our inductive hypothesis, $D(\lambda,\mu+\omega_1-2\omega_2)$ and $D(\lambda+2\omega_2,\mu-2\omega_2)$ both have level 2 Demazure flags and thus we can construct a level 2 Demazure flag for $D(\lambda,\mu)$. ■

The rest of this section is devoted to proving Theorem 23.

Proof. We prove this by induction on $s=a_1+a_2$. If $s=0,1$ we are done by Theorem 8. Assume that $s=2$. In this case we have that either $a_i=2$ for some i or $a_1=a_2=1$. Let us first consider the former and without loss of generality assume that $a_2=2$. Then $D(\lambda,\mu)=D(\lambda,2\omega_2)$ and by the right exact sequence (1) we have that

$$\dim D(\lambda, 2\omega_2) \leq \dim D(\lambda+2\omega_2, 0) + \dim D(\lambda, \omega_1).$$

In addition, we know by proposition **??** that

$$\dim D(\lambda, 2\omega_2) \geq \dim D(2,\lambda) \dim W_{\mathrm{loc}}(2\omega_2) = \dim D(2,\lambda)(\dim V(\omega_2))^2.$$

By proposition 13, $D(\lambda+2\omega_2,0) \cong D(2,\lambda+2\omega_2) \cong D(2,\lambda) * V(2\omega_2)$ and $D(\lambda,\omega_1) \cong D(2,\lambda) * W_{\mathrm{loc}}(\omega_1) \cong D(2,\lambda) * V(\omega_1)$ where the third isomorphisms follow from Theorem 8. Thus

$$\dim D(\lambda, 2\omega_2) \leq \dim D(2,\lambda)(\dim V(2\omega_2)) + \dim V(\omega_1)).$$

Using Weyl's dimension formula, it follows that $\dim V(2\omega_2)+\dim V(\omega_1) = (\dim V(\omega_2))^2$ and we are done.

To complete the base case for the induction, we need to show that the result is true if $a_1=a_2=1$. Here $D(\lambda,\mu)=D(\lambda,\omega_1+\omega_2)$. Using the right exact sequence from Theorem 3 and Theorem 14 we have that

$$\dim D(\lambda, \omega_1+\omega_2) \leq \dim D(\lambda, 0) + \dim D(\lambda+\omega_1+\omega_2, 0)$$

and

$$\dim D(\lambda, \omega_1 + \omega_2) \geq \dim D(2, \lambda) \dim W_{\text{loc}}(\omega_1 + \omega_2) = \dim D(2, \lambda) \dim V(\omega_1) \dim V(\omega_2).$$

Using Theorem 13, we know that $D(\lambda, 0) \cong D(2, \lambda)$ and $D(\lambda + \omega_1 + \omega_2, 0) \cong D(2, \lambda + \omega_1 + \omega_2) \cong D(2, \lambda) * V(\omega_1 + \omega_2)$ where the third isomorphism comes from Theorem 8. Hence

$$\dim D(\lambda, \omega_1 + \omega_2) \leq \dim D(2, \lambda)(1 + \dim V(\omega_1 + \omega_2))$$

Appealing to Weyl's dimension formula again, we have that $(1 + \dim V(\omega_1 + \omega_2)) = \dim V(\omega_1) \dim V(\omega_2)$ and hence the desired result.

Assume now that that Theorem 23 holds for $D(\lambda', a_1'\omega_1 + a_2'\omega_2)$ with $a_1' + a_2' = s \geq 2$. Consider $D(\lambda, a_1\omega_1 + a_2\omega_2)$ with $a_1 + a_2 = s + 1$ and without loss of generality assume $a_2 \geq 2$. By the right exact sequence (1) and Theorem 14 we have that

$$\dim D(\lambda, a_1\omega_1 + a_2\omega_2) \geq \dim D(\lambda + 2\omega_2, a_1\omega_1 + (a_2 - 2)\omega_2) + \dim D(\lambda, (a_1 + 1)\omega_1 + (a_2 - 2)\omega_2).$$

and

$$D(\lambda, a_1\omega_1 + a_2\omega_2) \geq \dim(D(2, \lambda)) \dim W_{\text{loc}}(a_1\omega_1 + a_2\omega_2) = \dim(D(2, \lambda))(\dim V(\omega_1))^{a_1}(\dim V(\omega_2))^{a_2}$$

where the last equality is by Theorem 8. However, induction and Theorem 8, we have that

$$\dim D(\lambda + 2\omega_2, a_1\omega_1 + (a_2 - 2)\omega_2) + \dim D(\lambda, (a_1 + 1)\omega_1 + (a_2 - 2)\omega_2) =$$

$$[\dim(D(2, \lambda))(\dim V(\omega_1))^{a_1}(\dim V(\omega_2))^{a_2 - 2}](\dim V(2\omega_2) + \dim V(\omega_1)) =$$

$$\dim(D(2, \lambda))(\dim V(\omega_1))^{a_1}(\dim V(\omega_2))^{a_2}$$

Where the last equality is from Weyl's dimension formula. ■

To show that the right exact sequences of Theorem 14 are short exact sequences in the case of $\mathfrak{g} = \mathfrak{sl}_4$ is similar to the above argument. We refer the reader to the appendix for more details on the proof. However, we do make a couple of remarks about the proof. The amount of cases increases in this case due to the structure of $\mathfrak{sl}_4$. In addition, we make use of a $\mathfrak{g}-module$ decomposition of $D(2,\rho)$ proved in [13]. In particular, this decomposition gives us a way of computing the dimension of $D(2,\rho)$. In general, though, we do not know the dimension of the Demazure modules, thus we cannot generalize this method to the $\mathfrak{sl}_n$ case.

5.2 A Multiplicity Formula

In this section our goal is to prove Theorem 16. To do so, we the following two results.

Lemma 25 *Let* $(\lambda,\mu)\in\mathbf{P}$ *and* $\nu\in P^+$. *If* $\lambda(h_i)\geq 2$ *and* $[D(\lambda,\mu),D(2,\nu)]_q\neq 0$ *then we have that* $\nu(h_i)\geq 2$.

Proof. We will induct on $\mu(h_\theta)$. If $\mu(h_\theta)=0,\quad 1$ then we have that

$$[D(\lambda,\mu),D(2,\nu)]_q=[D(2\lambda',\omega_i,D(2,\nu)]_q$$

for some $\lambda'\in P^+$ and $i\in\{0,1,2\}$. Since $[D(\lambda,\mu),D(2,\nu)]_q\neq 0$ we must have that $\nu=\lambda+\omega_i$ and hence $\nu(h_i)\geq 2$. Next lets consider the case when $\mu(h_1)=\mu(h_2)=1$. In this case we have by corollary 15.

$$[D(\lambda,\mu),D(2,\nu)]_q=[D(2,2\lambda'),D(2,\nu)]_q+q^c[D(2,2\lambda'+\omega_1+\omega_2),D(2,\nu)]_q$$

for some $c\in\mathbb{Z}$ and $\lambda'\in P^+$. Again since $[D(\lambda,\mu),D(2,\nu)]_q\neq 0$ we must have that either $\nu=2\lambda'+\omega_1+\omega_2$ or $\nu=2\lambda'$.

Now assume the theorem holds when $2\leq\mu'(h_\theta)\leq n-1$ for some $n\in\mathbb{N}$. Let $\mu(h_\theta)=n$. Without loss of generality assume that $\mu(h_1)\geq 2$. Then by the short exact sequence in corollary 15 we have that

$$[D(\lambda,\mu), D(2,\nu)]_q = [D(\lambda+2\omega_1, \mu-2\omega_1), D(2,\nu)]_q + q^c[D(\lambda,\mu-\alpha_1), D(2,\nu)]_q.$$

and thus either $[D(\lambda+2\omega_1,\mu-2\omega_1), D(2,\nu)]_q \neq 0$ or $[D(\lambda,\mu-\alpha_1), D(2,\nu)]_q \neq 0$. Induction then gives that $\nu(h_i) \geq 2$ and we are finished. ■

This lemma tells us that if $\lambda(h_i) \geq 2$ for some $i \in I$ then we immediately know that $\nu(h_i) \geq 2$. This will be important for the proof of the Theorem 16 and the next lemma.

Lemma 26 *Let $(\lambda,\mu) \in \mathbf{P}$ where $\lambda(h_i) \geq 2$ for some $i \in I$ then we have the following equality of multiplicities*

$$[D(\lambda,\mu), D(2,\nu)]_q = q^{k_i}[D(\lambda-2\omega_i,\mu), D(2,\nu-2\omega_i)]_q$$

Proof. We will prove this by inducting on $\mu(h_\theta)$. If $\mu(h_\theta) = 0, 1$ then for some $\lambda' \in P^+$ and $i \in \{0,1,2\}$

$$\begin{aligned}[D(\lambda,\mu), D(2,\nu)]_q = [D(2\lambda',\omega_i), D(2,\nu)]_q &= [D(2, 2\lambda'+\omega_i) : D(2,\nu)]_q \\ &= \begin{cases} 1, \text{if } 2\lambda'+\omega_i = \nu \\ 0, \text{otherwise} \end{cases} \\ &= q^0[D(2\lambda'-2\omega_i,\mu), D(2,\nu-2\omega_i)]_q\end{aligned}$$

Note $k_i = 0$ in this case, so we are done. Now assume $\mu(h_1) = \mu(h_2) = 1$ and let $\lambda = 2\lambda'$ for some $\lambda' \in P^+$. By the short exact sequence in corollary 15 we have that

$$[D(\lambda,\mu), D(2,\nu)]_q = [D(2\lambda',\omega_1+\omega_2), D(2,\nu)]_q = \begin{cases} 1, \quad \nu = 2\lambda'+\omega_1+\omega_2 \\ q^{\lceil 2\lambda'_{1,2}/2\rceil+1}, \quad \nu = \lambda' \\ 0, \quad \text{otherwise} \end{cases}$$

$$[D(2\lambda' - 2\omega_i, \omega_1 + \omega_2), D(2, \nu - 2\omega_i)]_q = \begin{cases} 1, & \nu = 2\lambda' + \omega_1 + \omega_2 \\ q^{\lceil 2\lambda'_{1,2}/2 \rceil}, & \nu = \lambda' \\ 0, & \text{otherwise} \end{cases}$$

Since $k_i(2\lambda', 2\lambda') = 0$ and $k_i(2\lambda' + \omega_1 + \omega_2, 2\lambda') = 1$ we get the desired equality and induction begins. Assume $\mu(h_\theta) \geq 2$ and without loss of generality, assume $\mu(h_1) \geq 2$. Then by the short exact sequence in corollary 15 we have that

$$[D(\lambda, \mu)]_q \stackrel{*}{=} [D(\lambda + 2\omega_1, \mu - 2\omega_1]_q + q^{\lceil \lambda_1/2 \rceil + \mu_1 - 1}[D(\lambda, \mu - \alpha_1) : D(2, \nu)]_q$$

Notice that induction applies to both part of the summand, thus

$$\stackrel{*}{=} q^{k_i}[D(\lambda + 2\omega_1 - 2\omega_i, \mu - 2\omega_1) : D(2, \nu - 2\omega_i)]_q +$$

$$q^{\lceil \lambda_1/2 \rceil + \mu_1 - 1 + k_i - \delta_{i,1}}[D(\lambda - 2\omega_i, \mu - 2\omega_1 + \omega_2) : D(2, \nu - 2\omega_i)]_q$$

$$= q^{k_i}[D(\lambda - 2\omega_i, \mu) : D(2, \nu - 2\omega_i)]_q$$

and we are done. ■

We finally prove Theorem 16. Recall that we want to show

$$[W_{\text{loc}}(\lambda) : D(2, \mu)]_q = \begin{cases} \prod_{i=1}^{2} q^{\frac{1}{2}(r_i + \text{res}_2(s_i))k_i} \begin{bmatrix} \lfloor \frac{s_i}{2} \rfloor + k_i \\ k_i \end{bmatrix}_q \\ 0, \text{ otherwise} \end{cases}$$

where we write $\lambda = \sum_{i=1}^{2} r_i\omega_i$, $\mu = \sum_{i=1}^{2} s_i\omega_i$, and $\lambda - \mu = \sum_{i=1}^{2} k_i\alpha_i$. We fix this notation for the rest of the proof.

Proof. We start by defining

$$f(r_1, r_2, s_1, s_2) = \frac{1}{2}(r_1 + \text{res}_2(s_1))k_1 + \frac{1}{2}(r_2 + \text{res}_2(s_2))k_2$$

where $(r_1 - s_1)\omega_1 + (r_2 - s_2)\omega_2 = k_1\alpha_1 + k_2\alpha_2$. Next we establish the following two identities for $r_1 \geq 2$ (we have similar identities for $r_2 \geq 2$).

$$r_1 - 1 + f(r_1 - 2, r_2 + 1, s_1, s_2) = f(r_1, r_2, s_1, s_2) + \lceil \frac{s_1}{2} \rceil \tag{5.2.1}$$

$$k_1 + f(r_1 - 2, r_2, s_1 - 2, s_2) = f(r_1, r_2, s_1, s_2) \tag{5.2.2}$$

For (5.2.1) we have

$$r_1 - 1 + f(r_1 - 2, r_2 + 1, s_1, s_2) = r_1 - 1 + \frac{1}{2}(r_1 - 2 + \mathrm{res}_2(s_1))(k_1 - 1) + \frac{1}{2}(r_2 + 1 + \mathrm{res}_2(s_2))k_2 =$$
$$r_1 - 1 + \frac{1}{2}(r_1 - 2 + \mathrm{res}_2(s_1))k_1 - \frac{1}{2}(r_1 - 2 + \mathrm{res}_2(s_1)) + \frac{1}{2}(r_2 + \mathrm{res}_2(s_2))k_2 + \frac{k_2}{2} =$$
$$f(r_1, r_2, s_1, s_2) + \frac{1}{2}(2r_1 - 2 - r_1 + 2 - \mathrm{res}_2(s_1) - 2k_1 + k_2) =$$
$$f(r_1, r_2, s_1, s_2) + \lceil \frac{s_1}{2} \rceil.$$

Similarly it is easy to verify (5.2.2). Next assume that $\lambda(h_1) \geq 2$ (we have a similar argument for when $\lambda(h_2) \geq 2$). Then corollary 15 gives us

$$[W_{\mathrm{loc}}(\lambda) : D(2, \mu)]_q$$

$$\overset{*}{=} q^{r_1 - 1}[W_{\mathrm{loc}}(\lambda) : D(2, \mu)]_q + [D(2\omega_1, \lambda - 2\omega_1) : D(2, \mu)]_q$$

Now by applying the lemma 26 to the second summand we get

$$\overset{*}{=} q^{r_1 - 1}[W_{\mathrm{loc}}(\lambda) : D(2, \mu)]_q + q^{k_1}[W_{\mathrm{loc}}(\lambda - 2\omega_1) : D(2, \mu - 2\omega_1)]_q$$

Let

$$P^{r_1, r_2}_{s_1, s_2}(q) = q^{r_1 - 1} P^{r_1 - 2, r_2 + 1}_{s_1, s_2}(q) + q^{k_1} P^{r_1 - 2, r_2}_{s_1 - 2, s_2}(q), \quad r_1 \geq 2$$

with initial conditions $P^{0,0}_{0,0}(q) = P^{1,0}_{1,0}(q) = P^{0,1}_{0,1}(q) = P^{1,1}_{1,1}(q) = 1$ and

$$P^{1,1}_{0,0}(q) = q, P^{r_1, r_2}_{s_1, s_2}(q) = 0 \text{ if } 2r_1 - 2s_1 + r_2 - s_2 \neq 0 \mod 3.$$

Using (5.2.1) and (5.2.2) one can now see that

$$\prod_{i=1}^{2} q^{\frac{1}{2}(r_i + \mathrm{res}_2(s_i))k_i} \begin{bmatrix} \lfloor \frac{s_i}{2} \rfloor + k_i \\ k_i \end{bmatrix}_q$$

solves this recurrence relation and we are done.

■

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Appendix A

A.1 The $\mathfrak{sl}_4$ Case

We fix $\mathfrak{g}$ to be $\mathfrak{sl}_4$. In this section we will prove that the right exact sequences in Theorem 14 are short exact sequences in the case of $\mathfrak{g} = \mathfrak{sl}_4$. We will do this in a similar fashion as in Chapter 3. In addition, we will prove that local Weyl modules have level 2 Demazure flags. Lastly we will prove a multiplicity formula.

A.1.1 Short Exact Sequences

We first write, explicitly, what **P** and **D**.

P is the union of the following sets:

$$\{(2\nu, \mu) : \nu, \mu \in P^+, \text{ and } \mu(h_i) \geq 2 \text{ for some i}\}$$

$$\{(2\nu, \mu) : \nu \in P^+, \quad \mu \in P^+(1) : \quad \mu(h_\theta) \geq 2\}$$

$$\{(2\nu + \omega_1 + \omega_2, \omega_3) : \nu \in P^+\}$$

D is the union of the following sets:

$$\{(2\nu, 0) : \nu \in P^+\}$$

$$\{(2\nu + \lambda, 0) : \nu \in P^+, \quad \lambda \in P^+(1) \text{ with } \lambda(h_\theta) \geq 2\}$$

$$\{(2\nu, \omega_i) : \nu \in P^+, \quad i \in I\}$$

Theorem 27 *Given $D(\lambda, \mu) \in \mathbf{P}$ we have that*

$$D(\lambda, \mu) \cong D(2, \lambda) * W_{loc}(\mu)$$

and thus we have following short exact sequence of $\mathfrak{sl}_4[t]$-modules

$$0 \to \tau_{s_{\lambda,\mu}} D(\lambda^-, \mu^-) \stackrel{\varphi^-}{\to} D(\lambda, \mu) \stackrel{\varphi^+}{D} (\lambda, \mu) \to 0$$

To prove this, we will need the following theorem proved by Shereen in [13].

Theorem 28 *Let $\rho = \omega_1 + \omega_2 + \omega_3 \in P^+$, then*

$$D(2, \rho) \cong_{\mathfrak{g}} V(\theta) \oplus V(\rho - \theta)$$

We now prove Theorem 27.

Proof. Let $(\lambda, \mu) \in \mathbf{P} \bigcup \mathbf{D}$. We will induct on $\mu(h_\theta)$. The result is true if $\mu(h_\theta) = 0$ by Theorem 8. If $\mu(h_\theta) = 1$ then we have two possibilities.

1. $(\lambda, \mu) = (2\nu, \omega_i)$ for some $\nu \in P^+$ and $i \in I$.

   The result follows from Theorem 8.

2. $(\lambda, \mu) = (2\nu + \omega_1 + \omega_2, \omega_3)$ for some $\nu \in P^+$.

   By Theorem 2, we have the following inequalities of dimension:

$$\dim D(2\nu + \omega_1 + \omega_2, \omega_3) \leq \dim D(2, 2\nu + \rho) + \dim D(2, 2\nu + 2\omega_1)$$

$$\dim D(2\nu + \omega_1 + \omega_2, \omega_3) \geq \dim D(2, 2\nu) \dim V(\omega_1 + \omega_2) \dim V(\omega_3)$$

Now we have $\dim D(2, 2\nu + \rho) = \dim D(2, 2\nu)[\dim V(\rho) + \dim V(\omega_2)]$ by Theorem 8 and (28). In addition, by Weyl's dimension formula we have that

$$\dim V(\rho) + \dim V(\omega_2) + \dim V(2\omega_1) = \dim V(\omega_1 + \omega_2) \dim V(\omega_3)$$

Thus $\dim D(2\nu+\omega_1+\omega_2, \omega_3) = \dim D(2, 2\nu) \dim V(\omega_1+\omega_2) \dim V(\omega_3)$ and we are done. Assume that $\mu(h_\theta) = 2$, then we have two cases:

1. $(\lambda, \mu) = (2\nu, \omega_i + \omega_j)$ for $i < j$

   Theorem 14 gives us one of the following three right exact sequences depending on i and j

$$\tau_{\lceil \lambda_{i,j}/2 \rceil + \mu_{i,j} - 1} D(\lambda, \omega_{j+1}) \stackrel{\varphi^-}{\to} D(\lambda, \mu) \stackrel{\varphi^+}{\to} D(\lambda + \omega_i + \omega_j, \mu - \omega_i + \omega_j) \to 0$$

$$\tau_{\lceil \lambda_{i,j}/2 \rceil + \mu_{i,j} - 1} D(\lambda, \omega_{i-1}) \stackrel{\varphi^-}{\to} D(\lambda, \mu) \stackrel{\varphi^+}{\to} D(\lambda + \omega_i + \omega_j, \mu - \omega_i + \omega_j) \to 0$$

   or

$$\tau_{\lceil \lambda_{i,j}/2 \rceil + \mu_{i,j} - 1} D(\lambda, 0) \stackrel{\varphi^-}{\to} D(\lambda, \mu) \stackrel{\varphi^+}{\to} D(\lambda + \omega_i + \omega_j, \mu - \omega_i + \omega_j) \to 0.$$

   Assume we are in the first case (the other cases follow similarly) and let $\mu = \omega_1 + \omega_2$. By the first part of Theorem 14 we also know that $D(\lambda, \omega_1 + \omega_2)$ maps onto $D(2, \lambda) * W_{\text{loc}}(\omega_1 + \omega_2)$. By using the previous cases and Theorem 8 we get the following inequalities of dimension:

$$\dim D(\lambda, \omega_1 + \omega_2) \leq \dim D(2, \lambda) \dim V(\omega_3) + \dim D(2, \lambda) \dim D(2, \omega_1 + \omega_2)$$

$$\dim D(\lambda, \omega_1 + \omega_2) \geq \dim D(2, \lambda) \dim V(\omega_1) \dim V(\omega_2)$$

   Note that $D(2, \omega_1 + \omega_2) \cong V(\omega_1 + \omega_2)$ and so by Weyl's dimension formula we have

$$\dim D(2, \lambda) \dim V(\omega_3) + \dim D(2, \lambda) \dim D(2, \omega_1 + \omega_2) =$$

$$\dim D(2, \lambda)(\dim V(\omega_3) + \dim V(\omega_1 + \omega_2)) =$$

$$\dim D(2, \lambda)(\dim D(2, \lambda) \dim V(\omega_1) \dim V(\omega_2))$$

2. $(\lambda, \mu) = (2\nu, 2\omega_i)$ for some i

   In this case Theorem 14 gives us the right exact sequence:

$$\tau_{\lceil \lambda_i/2 \rceil + \mu_i - 1} D(2\nu, \mu - \alpha_i) \stackrel{\varphi^-}{\to} D(2\nu, \mu) \stackrel{\varphi^+}{\to} D(2\nu + 2\omega_i, 0) \to 0.$$

By the first part of Theorem 14 and the above right exact sequence we have the following inqualities of dimension:

$$\dim D(2\nu, \mu) \leq \dim D(2\nu, \mu - \alpha_i) + \dim D(2\nu + 2\omega_i, 0)$$

and

$$\dim D(2\nu, \mu) \geq \dim D(2, 2\nu) * \dim V(\omega_i)^2$$

we know that

$$\dim D(2\nu, \mu - \alpha_i) = \begin{cases} \dim D(2, 2\nu) \dim V(\omega_2), & \text{for } i \in \{1, 3\} \\ \dim D(2, 2\nu) \dim V(\omega_1) \dim V(\omega_3), & \text{for } i = 2 \end{cases}$$

and $\dim D(2\nu + 2\omega_i) = \dim D(2, 2\nu + 2\omega_i) = \dim D(2, 2\nu) \dim V(2\omega_i)$. Weyl's dimension formula then gives us that $\dim D(2\nu, \mu) = \dim D(2, 2\nu) * \dim V(\omega_i)^2$ and we are done.

The last base case we need to consider is when $\mu(h_\theta) = 3$. Again we have two cases.

1. $(\lambda, \mu) = (2\nu, \omega_1 + \omega_2 + \omega_3)$ In this case we have the following right exact sequence:

$$\tau_{\lceil \lambda_{1,2}/2 \rceil + \mu_{i,2} - 1} D(2\nu, 2\omega_3) \to D(2\nu, \omega_1 + \omega_2 + \omega_3) \to D(2\nu + \omega_1 + \omega_2, \omega_3) \to 0$$

Thus we have the following inequality

$$\dim D(2\nu, \omega_1 + \omega_2 + \omega_3) \leq \dim D(2\nu, 2\omega_3) + \dim D(2\nu + \omega_1 + \omega_2, \omega_3)$$

In addition we know that $\dim D(2\nu, \omega_1 + \omega_2 + \omega_3) \geq \dim D(2, 2\nu) \dim V(\omega_1) \dim V(\omega_2) \dim V(\omega_3)$ by Theorem 14. Now by the previous base cases, we know that $\dim D(2\nu, 2\omega_3) = \dim D(2, 2\nu)(\dim V(\omega_3))^2$ and

$$\dim D(2\nu + \omega_1 + \omega_2, \omega_3) = \dim D(2, 2\nu) \dim V(\omega_1 + \omega_2) \dim V(\omega_3).$$

By Weyl's dimension formula we have that

$$(\dim V(\omega_3))^2 + \dim V(\omega_1+\omega_2)\dim V(\omega_3) = \dim V(\omega_1)\dim V(\omega_2)\dim V(\omega_3),$$

and thus we have that $D(2\nu, \omega_1+\omega_2+\omega_3) \cong D(2,\lambda) * W_{\mathrm{loc}}(\omega_1+\omega_2+\omega_3)$.

2. $\mu(h_i) \geq 2$ for some $i \in I$.

   In this case we have that $(\lambda,\mu) = (2\nu, 2\omega_i+\omega_j)$ where $i,j \in I$. By Theorem 14 we have the following inequalities of dimension

$$\dim D(2\nu, 2\omega_i+\omega_j) \leq \dim D(2\nu, 2\omega_i+\omega_j-\alpha_i) + \dim D(2\nu+2\omega_i, \omega_j)$$

$$\dim D(2\nu, 2\omega_i+\omega_j) \geq \dim D(2,2\nu)\dim V(2\omega_i)\dim V(\omega_i)$$

   We have, by the previous base cases, that $\dim D(2\nu, 2\omega_i+\omega_j-\alpha_i) = \dim D(2,2\nu)\dim W_{\mathrm{loc}}(2\omega_i+\omega_j-\alpha_i)$ and Theorem 8 we have that $\dim D(2\nu+2\omega_i,\omega_j) = \dim D(2,2\nu)\dim V(2\omega_i)\dim V(\omega_j)$. Thus by Weyl's dimension formula we have that

$$\dim D(2\nu, 2\omega_i+\omega_j) = \dim D(2,2\nu)\dim W_{\mathrm{loc}}(2\omega_i+\omega_j)$$

   and we are done.

Now assume that $D(\lambda',\mu') \cong D(2,\lambda') * W_{\mathrm{loc}}(\mu')$ when $\mu'(h_\theta) = n-1 > 3$. We will show it is true for for $D(\lambda,\mu)$ with $\mu(h_\theta) = n$. In this case we must have that $\mu(h_i) \geq 2$ for some i. If $i \in \{1,3\}$ then a similar argument as in the base cases shows $D(\lambda,\mu) \cong D(2,\lambda) * W_{\mathrm{loc}}(\mu)$. Let us assume that $\lambda(h_2) \geq 2$ and $\lambda(h_1) \leq 1$.

- $\lambda(h_1) = 1$ Then by Theorem 14 we have the right exact sequence

$$\tau_{\lceil \lambda_{h_2}/2\rceil+\mu_2-1} D(\lambda, 2\omega_1+(\mu(h_2)-2)\omega_2+(\mu(h_3)+1)\omega_3) \to D(\lambda,\mu) \to D(\lambda+2\omega_2, \mu-2\omega_2) \to 0$$

Notice that induction applies to the quotient giving us $\dim D(\lambda + 2\omega_2, \mu - 2\omega_2) = \dim D(2, \lambda) \dim V(2\omega_2)(\dim V(\omega_1))(\dim V(\omega_2))^{\mu(h_2)-2}(\dim V(\omega_3))^{\mu(h_3)}$. For the kernel we, again, use Theorem 2 to obtain the following right exact sequence (with grade shifts omitted):

$$D(\lambda, (\mu(h_2) - 1)\omega_2 + (\mu(h_3) + 1)\omega_3) \to D(\lambda, 2\omega_1 + (\mu(h_2) - 2)\omega_2 + (\mu(h_3) + 1)\omega_3) \to$$

$$D(\lambda + 2\omega_1, (\mu(h_2) - 2)\omega_2 + (\mu(h_3) + 1)\omega_3) \to 0$$

Notice that induction now applies to both the kernel and quotient of this right exact sequence and by a familiar argument, we can see that

$$\dim D(\lambda, 2\omega_1 + (\mu(h_2) - 2)\omega_2 + (\mu(h_3) + 1)\omega_3) =$$

$$\dim D(\lambda, (\mu(h_2) - 1)\omega_2 + (\mu(h_3) + 1) + \dim D(\lambda + 2\omega_1, (\mu(h_2) - 2)\omega_2 + (\mu(h_3) + 1)\omega_3) =$$

$$\dim D(2, \lambda)(\dim V(\omega_1))^2(\dim V(\omega_2))^{\mu(h_2)-2}(\dim V(\omega_3))^{\mu(h_3)+1}$$

Thus we have that

$$\dim D(\lambda, \mu) \leq$$

$$\dim D(2, \lambda)(\dim V(\omega_1))^2(\dim V(\omega_2))^{\mu(h_2)-2}(\dim V(\omega_3))^{\mu(h_3)}(\dim V(2\omega_2) + \dim V(\omega_1) \dim V(\omega_3))$$

$$= \dim D(2, \lambda) \dim W_{\mathrm{loc}}(\mu)$$

Using Theorem 14 we are done.

- $\lambda(h_1) = 0$

This case is proved in a similar fashion as the previous case, so we omit the details. ■