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ABSTRACT

Series and rational function approximations provide a viable alternative to finite difference methods in the solution of partial differential initial value problems. Large scale symbolic algebra manipulation systems can be used to generate such approximations for a very general class of problems. The use of operator derivatives keeps the method from being limited to ordinary power series.

The procedure is applied to the non-linear Burgers' equation

$$u_t + u \cdot u_x = u_{xx}$$

with periodic initial condition

$$u(x,0) = \sin(x)$$

and the results are compared to an asymptotic expansion and a finite difference solution.

Introduction

If someone says he has "solved" a partial differential equation on a computer, we usually assume that he has found a numerical solution by means of a finite difference technique. However, a computer solution need not imply a finite difference or even a numerical solution, and in some cases it may be a considerable advantage to use a computer to obtain a quasi-analytic solution to a problem. One step towards using the computer to perform analytic tasks has been the development of computer systems to symbolically manipulate polynomials. We intend to show how "quasi-analytic" solutions to problems governed by parabolic and hyperbolic partial differential equations can be obtained, at least in principle, on a computer. We will implement our ideas with the ALTRAN language for symbolic algebra on a digital computer [1] although we wish to emphasize that this may not be the best tool for the job.

To illustrate the ideas involved, a quasilinear parabolic equation in two independent variables will be solved by our methods. Finally, we will discuss the practical limitations and potential usefulness of the method. To set the tone for our development, we will now discuss some failings of the common finite difference approach to partial differential equations.

Disadvantages of Finite Difference Methods

The most obvious drawback of finite difference methods is the specific nature of the results: the effects of parameter variations must be ascertained from specific solutions for many different values of the parameter. Hence it is clear that a computer method which permits the parameter to be varied after the computation would have considerable utility. The technique which we shall present is in a crude state and does not readily yield a "simple" analytic result; however, there is reason to believe that with sufficient effort useful results can be obtained.

Moreover, while stable finite difference schemes are generally available for linear problems, such is by no means the case for non-linear problems. No general guidelines are available for the generation of stable finite difference schemes for non-linear equations; each equation requires analysis oriented to its individual nature. Indeed, Richtmyer and Morton [2] have commented: for non-linear problems, stability depends not only on the structure of the finite difference system but also generally on the solution being obtained, and for a given solution, the system may be stable for some values of t and not for others.

Another problem with finite difference methods is that certain types of boundary and initial conditions present great difficulties, e.g. an **initial** condition such as $e^{-x^2/\epsilon}$, where $\epsilon \to 0$, requires an infinitely fine mesh, while with a quasi-analytic method we might be able to leave ϵ as a parameter and study the result of $\epsilon \to 0$ in the solution. This type of problem might arise if we wish to find the Greens function for a linear equation.

In many problems there are no physical boundaries; for such problems, the finite difference approach requires the introduction of spurious boundaries which may produce stability difficulties in the finite difference scheme. Finally, we note the exponential increase of required storage with the number of independent variables that occurs with finite difference schemes. This need not occur with a carefully chosen quasi-analytic scheme, at least not for linear problems. Of course, quasi-analytic schemes have many drawbacks of their own; however, for special cases and for obtaining insight such schemes may prove invaluable.

Series Solutions to Partial Differential Equations

Consider the following type of problem in two independent variables t and x.

$$(1) \qquad \begin{aligned} u_t &= f(x,u,u_x,u_{xx}) \\ u(x,0) &= \phi(x) \end{aligned}$$

where f is a polynomial function. An example is the heat equation.

This is a so-called "pure" initial-value problem for which the requirement that the solution be bounded is sufficient to guarantee uniqueness [3].

If $u(\vec{r},t)$ describes the state of a physical system, and we are given the initial state of the system, i.e., $u(\vec{r},0)$, then partial differential equations of parabolic and hyperbolic type, such as the heat and wave equation, provide a means of describing the time evolution of the system. Thus the solution of the equation takes the form $u(\vec{r},t) = E(t)\, u(\vec{r},0)$, where $E(t)$ is an operator which represents the action of the differential equation. In some sample cases like the one dimensional linear heat equation $u_t = u_{xx}$ with no boundary conditions and the initial condition $u(x,0) = u_o(x)$, $E(t)$ can be explicitly found. In the one dimensional heat equation case

$$E(t)u_o(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{4t}} u_o(z)dz.$$

Another approach to find the time evolution of a system is to generate a power series solution of the differential equation; thus in the case of the heat equation let $u(x,t) = \sum_{n=0}^{\infty} a_n(x)t^n$. The term $a_o(x) = u_o(x)$, and the general a_n is found by a recursion derived from $u_t = u_{xx}$. Thus $a_{n+1}(x) = (n+1)^{-1}a_n''(x)$. It is evident that this method requires $u_o(x)$ to be an analytic function.

The power series solution can also be obtained from the integral form of the solution. A simple transformation of variables shows that

$$u(x,t) = (\pi)^{-1/2} \int_{-\infty}^{\infty} \exp\{-y^2\}u_o(x-2\sqrt{t}\,y)dy.$$

A series expansion of $u_o(x - 2\sqrt{t}\,y)$ in the variable $2\sqrt{t}\,y$ then provides the series expansion. From this form it is clear that the singularities of $u_o(z)$, considered as a function of the complex variable "z", determine the radius of convergence of the power series solution. In general, we must expect a similar state of affairs to hold for series solutions of other hyperbolic and parabolic problems, i.e. the series will only converge in a limited region. To overcome this difficulty and to improve the convergence of the solution, we resort to techniques for analytic continuation and improvement of rate of series convergence.

It is well known that a meromorphic function can be represented at all non-singular points by a single rational fraction expansion. This expansion can be found by a recasting of the power series representation of the function as a rational fraction. The method has been used with success in fluid mechanics to solve both the blunt body problem and the shock on shock problem [4]. The rational fraction expansion can be obtained using a special nonlinear sequence to sequence transformation known as the Padé approximant method [5] which we will discuss in connection with the use of computer algebra manipulation to develop a solution to a quasilinear parabolic equation. Before doing this, we believe it will be profitable to discuss the general idea behind the approach to a solution of such problems.

For the simple case of a parabolic equation with no explicit dependence on x and t, we have a relation of the form $u_t = f(u,u_x,u_{xx})$, together with the initial condition $u(x,0) = u_o(x)$. We know that a power series of the form

$$u(x,t) = \sum_{n=0}^{\infty} a_n(x)t^n$$

can represent the solution in some region of the x,t-plane centered at $t = 0$ in the complex t-plane. The equation can then be expressed in terms of a recursion relation which yields the function $a_n(x)$ in terms of the given $u_o(x) = a_o(x)$. The recursion relation for a_N will require operations such as differentiation and multiplication of the a_n's for $n < N$, i.e., $a_N(x) = g(a_o, a_1, \ldots a_{N-1}, a_o', a_1' \ldots a_{N-1}')$. For many equations the function g will be simple enough to permit automation of the recursion process. As an example, suppose only multiplication and differentiation are involved in g, and the initial function $u_o(x)$ has the form $[1 + \exp(x/\delta)]^{-1}$, where δ is a parameter. If we differentiate $C(M,Q,S) = \delta^{-S} \exp(Mx/\delta)[1 + \exp(x/\delta)]^{-Q}$ we find:

$$\frac{d}{dx} C(M,Q,S) = M \cdot C(M,Q,S+1) - Q \cdot C(M+1,Q+1,S+1).$$

If two of these functions are multiplied together we find that $C(M,Q,S) \cdot C(M',Q',S') = C(M+M',Q+Q',S+S')$. Thus we can replace the operations of differentiation and multiplication by operations on the integers M,Q,S. The general term in the power series expansion would then have the form

$$a_N(x) = \sum_{M,Q,S} A(M,Q,S)C(M,Q,S).$$

A computer can be used to carry out the necessary arithmetic and to store the resulting symbolic representation of $a_N(x)$. The result is not a numerical solution but a power series in t with coefficients that are functions of x. In principle one can produce a_N for as large an N as desired; however, limitations on storage and the complexity of terms for large N will in most cases limit the number of terms that can be found.

As an example of the above idea, we will "solve" a nonlinear parabolic equation with a sinusoidal initial condition. Before proceeding to this problem, we will first examine how ALTRAN can be used for our purpose and how the convergence of a series can be improved by recasting as a rational fraction.

Operator Derivatives and Transcendental Functions

ALTRAN is based on the ALPAK system for non-numerical algebra on the digital computer [6]. ALPAK and similar systems such as PM [7] can deal only with polynomials (and, in the case of ALPAK, with rational functions considered as two polynomials, the numerator and denominator); they cannot explicitly handle the elementary transcendental functions sine, cosine, and exp. However, by the simple method of introducing additional independent variables and making use of operator derivatives obtained by the chain rule, one can easily generate series coefficients in terms of these transcendental functions.

Sine and cosine are handled by introducing the variables $S = \sin(x)$ and $C = \cos(x)$ and then taking derivatives using a subroutine which defines an operator L acting on polynomials P by

$$L = DIFF(P,S)*C - DIFF(P,C)*S$$

where DIFF is the system differentiation function, i.e. $DIFF(P,X) = \frac{\partial P}{\partial X}$. In this way we have $L(S) = C$ and $L(C) = -S$.

Rational Fraction Approximations

As was noted above, we should not expect the power series to converge for all t; and even if the function does converge for all x in the complex t plane, it is unlikely that convergence will be rapid. Also, in our method the complexity of higher order a_n rapidly increase, and it is desirable that we obtain a reasonable approximation of the solution with as few a_n as possible. To accomplish this purpose, we shall make use of the so called Padé approximants to a power series. The (N,M) Padé approximant of a function is a rational fraction of the form:

$$\frac{\alpha_{0,M} + \alpha_{1,M}t + \alpha_{2,M}t^2 + \ldots + \alpha_{M,M}t^M}{1 + \beta_{1,N}t + \beta_{2,N}t^2 + \ldots + \beta_{N,N}t^N} .$$

If (N,M) is expanded in a power series, it must have the property that the coefficients of the terms up to t^{M+N} match the coefficients of the power series expansion of the function being approximated. It can be shown [5] that this uniquely determines (N,M).

The current theory of Padé approximants does not appear to do justice to their power as an approximation tool, however those theorems that do exist give some indication of this power. For example, suppose f(z) has a finite number of poles, i.e., is a meromorphic function, then the limit as $N \to \infty$ of the (N,N) Padé approximant of f(z) converges uniformly to f(z) except in the neighborhood of the poles. Thus, unlike power series, the Padé approximants' region of convergence is not limited to circular regions in the complex plane. Some theorems for functions with branch points also show the power of the Padé approximant. We do not wish to discuss Padé approximants in detail here, and we refer the reader to [5] and [8] for a complete discussion. However as a simple example consider

$$f(z) = (1+z)^{-1} = 1 - z + z^2 - z^3 + z^4 - \dots .$$

To compute (1,1) consider

$$\frac{\alpha_{0,1} + \alpha_{1,1}t}{1 + \beta_{1,1}t} = (\alpha_{0,1} + \alpha_{1,1}t)\left(1-\beta_{1,1}t + \beta_{1,1}^2 t^2 - \dots\right)$$

$$= \alpha_{0,1} + (\alpha_{1,1} - \alpha_{0,1}\beta_{1,1})t$$

$$+ \left(\alpha_{0,1}\beta_{1,1}^2 - \alpha_{1,1}\beta_{1,1}\right)t^2 + O(t^3).$$

If we match coefficients we find

$$\alpha_{0,1} = 1 ,$$

$$\alpha_{1,1} - \alpha_{0,1}\beta_{1,1} = -1 ,$$

and $$\alpha_{0,1}\beta^2_{1,1} - \alpha_{1,1}\beta_{1,1} = +1 .$$

Hence we find:

$$(1,1) \text{ of } f(z) = \frac{1}{1+z} \text{ is just } \frac{1}{1+z} .$$

In this case (N,N) is also $(1+z)^{-1}$. Of course the Padé method works so well on this function because it is a rational function to begin with. But equally remarkable results exist for functions such as $\exp\left\{-\frac{z}{1+z}\right\}$, a function which possesses an essential singularity at $z = -1$.

Example: The Burgers' equation

Burgers' equation [9] provides us with an interesting non-trivial equation which contains many of the features indicative of problems in non-linear acoustics, and fluid dynamics. A good deal is known about the behavior of Burgers' equation, which will permit us to ascertain at least qualitatively, the behavior of the true solution. Thus we solve the following initial value problem

$$(2) \qquad \begin{aligned} u_t + u \cdot u_x &= u_{xx} , \\ u(x,0) &= \sin x . \end{aligned}$$

If we consider a series solution of the form

$$u = \Sigma a_i(x)t^i$$

we can easily obtain the following recursion formula for the a_i

$$a_{i+1}(x) = \frac{1}{i+1}\left\{ a_i''(x) - \sum_{j+k=i} a_j(x)\cdot a_k'(x) \right\}$$

$$a_o(x) = \sin(x)$$

Introducing the dummy variables

$$S = \sin(x) \qquad C = \cos(x)$$

an ALTRAN program can be written to obtain the a_i as polynomials in the variables S and C .

The first ten coefficients a_i are listed in Appendix A. Only the first three could be obtained by hand without undue effort. Because of the explosive growth with increasing i of the integer coefficients of the terms of a_i , this seemed an ideal situation to apply the Padé approximant method.

(2,2) as computed by ALTRAN, contains 30 terms in the numerator and 34 terms in the denominator; an attempt to reduce it to lowest form failed because of coefficient overflow during a greatest common denominator execution. The higher order Padé approximants (3,3) and (4,4) could not be obtained symbolically using ALTRAN, but they were obtained numerically. In view of the complicated symbolic structure of (2,2), no significant advantage is apparent in a symbolic over a numerical solution. A selective sampling of the results, in comparison with two other solution methods to be described below, is shown in Figures 1-5 .

Finite Difference Solution

Problem (2) was formulated as an initial-boundary problem for finite difference solution by imposing the additional boundary values

$$u(0,t) = u(2\pi,t) = 0$$

As might be expected, an explicit difference scheme proved unstable for all but very small values of t . However, a predictor-corrector modification of an implicit Crank-Nicholson scheme as suggested by Douglas & Jones [10] provided a stable finite difference solution to the problem.

An Asymptotic Solution for Small t

As discussed in [9], Burgers' equation can be transformed into the linear heat equation by use of the so called Hopf-Cole transformation.

Thus letting $u = -2 \frac{\partial}{\partial x} \ell n \psi$ we see that Burgers' equation implies that $\psi_t = \psi_{xx}$. Using the Green's function form of the solution to the appropriate transformed initial value problem for the heat equation, one finds that

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \left(\frac{x-y}{t}\right) \exp\left\{\frac{1}{2}\cos y - (x-y)^2/4t\right\} dy}{\int_{-\infty}^{\infty} \exp\left\{\frac{1}{2}\cos y - (x-y)^2/4t\right\} dy} .$$

We have not been able to obtain a closed form evaluation of the above; however, standard techniques (Laplace's method) have led us to the following asymptotic result for small t:

$$u \sim \sin x - \frac{\sin x(\cos x+1)t}{1 + t \cos x} - \frac{1}{2} \frac{\sin^3 x}{(1+t \cos x)^2} t^2 + O(t^3) .$$

Fourier Series Form of Solution

As may be seen in Appendix A, the coefficients a_i of the series solution to Burgers equation are in the form of polynomials in sine and cosine. This is necessarily the form of output obtained using ALTRAN. For numerical purposes of evaluation of the solution, this form is entirely satisfactory. However, for purposes of insight into the nature of the solution and the possible location of singular points, the equivalent finite Fourier series may be more appropriate. This equivalent form for the series coefficients has been obtained using identities in [11] which were programmed for this purpose. The coefficients are shown in Fourier series form in Appendix B.

This result was also recast as a (2,2) Padé approximant which can be found in Appendix B. Inspection shows that the resulting fraction is much simpler, and more compact than the previous form.

Evaluation of Results

Figures 1-5 plot the various solutions u for five different values of time. The solid line is the finite difference solution. By $t = 0.6$ the asymptotic expansion values are no longer resonable. At this point the Padé approximants seem to be the only analytic representation behaving as the true solution (except in the neighborhood of singular points, as will be discussed below). By $t = 2.0$ the Padé approximants begin to diverge from the finite difference solution.

Poles of Pade Approximants

The appearance of singular behavior in the Padé approximants (2,2), (3,3), and (4,4) in Figures 2-5 displays **a** disadvantage of Padé approximants which must be allowed for in their use as a calculation tool. An (n,n) Padé solution to a differential equation in 2 independent variables of the form

$$(n,n) = \frac{P_n}{Q_n} = \frac{P_n(x,t)}{q_o(x) + q_1(x)t + \dots + q_n(x)t^n}$$

will have at most **n (complex)** poles for each value of x within the range of the problem. As $t \to r$ where $Q(x,r) = 0$ the Padé approximant becomes infinite. Since we are solving an initial value problem, we need only consider positive real roots of $Q_n(x,t)$. By obtaining a profile of the positive real roots, it is possible to practically determine regions in which the Padé approximant will not be an accurate representation of the solution to the equation. Root profiles for the (2,2) (3,3) and (4,4) Padé approximants to Burgers' equation are shown in Figures 6-8.

There is a large body of literature devoted to convergence proofs for sequences of Padé approximants [8] . Recently Walsh [12] has obtained new results in this area. However, little work has been done in the area of error estimation in the neighborhood of poles of a given approximant. Thus root profiles remain the practical tool for determination of the region of validity of Padé approximants as a solution to a partial differential equation.

Three Dimensional Problems

The application of finite difference methods to equations of three independent variables squares the amount of space and alloted time required to solve problems of magnitude equivalent to two-dimensional ones. This is to say nothing of additional difficulties introduced in the theory by such transition to higher-dimensional problems. Furthermore, in these problems, the task of fitting extraneous boundary conditions is more complicated. The introduction of an additional spatial variable will not, however, significantly complicate or overwhelm the series and rational function generation method which we have described. Taking, for example the three-dimensional wave equation

$$u_{xx} + u_{yy} = \frac{1}{c(x,y)} u_{tt}$$

$$u(x,y,0) = f_1(x,y)$$

$$u_t(x,y,0) = f_2(x,y)$$

it would be easily possible to automate and generate a series of the form

$$u = a_o(x,y) + a_1(x,y)t + \ldots + a_n(x,y)t^{n+1} + \ldots$$

to which Padé approximants could be applied. The storage space requirements which are the bane of finite difference methods do not bother the symbolic series method, since it deals with functions rather than values of functions at specific points of a finite difference mesh.

Summary

This paper has presented a solution method for partial differential initial value problems which rests upon three basic ideas. First, a solution is formulated as a power series in the relevant variable, time, whose coefficients are functions of the other independent variables. Second, computer symbolic algebra manipulation systems are used to generate these coefficients from recursion relationships derived from the differential equation. This may be accomplished only if the initial function is a polynomial over a finite set of functions which are closed under the operations found in the recursion relationships (usually addition, subtraction, multiplication and differentiation). Finally, since the series thus obtained will, in general, be a poor representation of the actual solution, the Padé approximant method is applied to obtain a better analytic approximation to the actual solution.

Our results demonstrate that it is feasible, using existing systems, to generate accurate analytic approximations to the solution of partial differential initial value problems. The primary importance of the method lies in that it provides a general solution method even for those non-linear equations for which stable finite difference schemes may be unavailable.

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REFERENCES

1. M. D. McIlroy and W. S. Brown, The ALTRAN Language for Symbolic Algebra on the Digital Computer, in manuscript, summary in Comm. of the ACM, V. 9, No. 6, Aug. 1966, p. 548.

2. R. D. Richtmyer and K. L. Morton, Difference Methods for Initial Value Problems, (2nd edition) New York: Interscience, 1967.

3. L. Bers, F. John, and M. Schecter, Partial Differential Equations, New York: Interscience, 1964.

4. J. Moran, The Blunt Body Problem, PhD Thesis, Cornell University, September 1965.

5. H. S. Wall, Analytic Theory of Continued Fractions, New York: D. Van Nostrand Co., 1948.

6. W. S. Brown, B. A. Tague, and J. P. Hyde, The ALPAK System for Non-numerical Algebra on the Digital Computer, BSTJ, V. 42, pp. 2081-2120, Sept. 1963 (Part I), V. 43, pp. 785-804 (Part II), V. 43, pp. 1547-1562, July, part 2, 1964 (Part III).

7. G. E. Collins, PM, A System for Polynomial Manipulation, Communications of the ACM, V. 9, No. 6, August 1966, pp. 578-589.

8. George A. Baker, Jr., The Theory and Application of the Padé Approximant Method, Advances in Theoretical Physics, V. 1, K. A. Brueckner, ed., New York: Academic Press, 1965, pp. 1-58.

9. M. J. Lighthill, Viscosity in Waves of Finite Amplitude, in Surveys in Mechanics Ed. by G. K. Batchelor and R. M. Davies, London-Cambridge University Press 1956.

10. J. Douglas, Jr., and B. F. Jones, Jr., On Predictor-Corrector Methods for Nonlinear Parabolic Differential Equations, SIAM Journal of Applied Math. V. 11, No. 1, March 1963, pp. 195-204.

11. L. B. W. Jolley, The Summation of Series, New York: Dover 1961.

12. J. L. Walsh, On the Convergence of Sequences of Rational Functions, SIAM Journal on Numerical Analysis, V. 4, No. 2, June 1967, pp. 211-221.

APPENDIX A

Listing of Series Coefficients and Rational Approximations for Burgers Equation

Appendix A - 2

$s = \text{sine}(x)$

$c = \text{cosine}(x)$

$a_o(x) = s$

$a_1(x) = -s(1+c)$

$a_2(x) = \frac{s}{2}(1+6c+2c^2-s^2)$

$a_3(x) = -\frac{s}{6}(1+28c+36c^2+6c^3-15s^2-10s^2c)$

$$a_4(x) = \frac{s}{24}(1+120c+440c^2+240c^3+24c^4-166s^2$$
$$-328s^2c-88s^2c^2+13s^4)$$

$$a_5(x) = -\frac{s}{120}(1+496c+4620c^2+6000c^3+1800c^4$$
$$+120c^5-1650s^2-7280s^2c-5340s^2c^2$$
$$-800s^2c^3+705s^4+376s^4c)$$

Appendix A - 3

$$a_6(x) = \frac{5}{720}(1+2016c+45122c^2+122640c^3+79800c^4$$

$$+15120c^5+720c^6-15631s^2-138320s^2c$$

$$-207140s^2c^2-80640s^2c^3-7800s^2c^4$$

$$+25235s^4+33696s^4c+7746s^4c^2-541s^6)$$

$$a_7(x) = \frac{s}{5040}[-1-8128c-424536c^2-2259936c^3$$

$$-2787120c^4-141120c^5-5040c^6$$

$$-s^2(144585+2436000c+6632640c^2$$

$$+4990720c^3+1211280c^4+82320c^5$$

$$-762615s^2-1912736s^2c-1064616s^2c^2$$

$$-143472s^2c^3+68551s^4+31312s^4c)]$$

$$a_8(x) = \frac{s}{40320} [1+32640c+3915200c^2+39312000c^3$$

$$+85536864c^4+59270400c^5+15240960c^6$$

$$+1451520c^7+40320c^8+s^2(-1320796$$

$$-41196288c-192204768c^2-247914240c^3$$

$$-112069440c^4-18627840c^5-940800c^6$$

$$+21200718s^2+89112576s^2c$$

$$+89889408s^2c^2+28619136s^2c^3$$

$$+2582496s^2c^4-5426988s^4-5743872s^4c$$

$$-1171808s^4c^2+47545s^6)]$$

Appendix A - 5

$$a_9(x) = \frac{s}{362880}\{-1-35715240c-35715240c^2-621401600c^3$$

$$-2433624480c^4-2793934080c^5$$

$$-1231977600c^6-224985600c^7-16329600c^8$$

$$-362880c^9+s^2[11984820+681018880c$$

$$+5263466040c^2+10925859840c^3$$

$$+8140305600c^4+2453552640c^5$$

$$+297198720c^6+11612160c^7$$

$$+s^2(-563638950-3744314112c$$

$$-6091860600c^2-3425184000c^3$$

$$-716597280c^4-46738944c^5+349425300s^2$$

$$+642724608s^2c+298338984s^2c^2$$

$$+36679680s^2c^3-11357793s^4-4606336s^4c)]\}$$

Appendix A - 6

$$(2,2) = \frac{\begin{aligned}[(&-12s+264sc+96sc^2+144sc^3-288s^3\\ &-168s^3c-96s^3c^2-36s^5)+(6s+66sc\\ &+528sc^2+504sc^3+144sc^4-516s^3\\ &-972s^3c-720s^3c^2-120s^3c^3-66s^5\\ &-6s^5c)t+(-s-62sc-316sc^2\\ &+240sc^3+285s^3+260s^3c-400s^3c^2\\ &-135s^5+138s^5c-4s^5c^2-21s^7)t^2]\end{aligned}}{\begin{aligned}[&-12+264c+96c^2+144c^3-288s^2-168s^2c\\ &-96s^2c^2-36s^4+(-6+318c+888c^2\\ &+744c^3+288c^4-804s^2-1428s^2c\\ &-984s^2c^2-216s^2c^3-102s^4-42s^4c)t\\ &+(-1+154c+62c^2+1248c^3+504c^4\\ &+144c^5-381s^2-892s^2c-1924s^2c^2\\ &-672s^2c^3-120s^2c^4-363s^4+18s^4c\\ &-58s^4c^2-39s^6)t^2]\end{aligned}}$$

APPENDIX B

Fourier Series Form of Coefficients

Appendix B - 2

$$a_o(x) = \sin(x)$$

$$a_1(x) = -\sin(x) - \frac{1}{2}\sin(2x)$$

$$a_2(x) = \frac{3}{8}\sin(x) + \frac{3}{2}\sin(2x) + \frac{3}{8}\sin(3x)$$

$$a_3(x) = \frac{5}{24}\sin(x) - \frac{13}{6}\sin(2x) - \frac{17}{8}\sin(3x) - \frac{1}{3}\sin(4x)$$

$$a_4(x) = -\frac{107}{192}\sin(x) + \frac{19}{12}\sin(2x) + \frac{781}{128}\sin(3x)$$

$$+ \frac{71}{24}\sin(4x) + \frac{125}{384}\sin(5x)$$

$$a_5(x) = \frac{667}{960}\sin(x) + \frac{139}{240}\sin(2x) - \frac{1441}{128}\sin(3x)$$

$$- \frac{407}{30}\sin(4x) - \frac{523}{128}\sin(5x) - \frac{27}{80}\sin(6x)$$

$$a_6(x) = -\frac{37597}{46080}\sin(x) - \frac{5689}{1440}\sin(2x) + \frac{39905}{3072}\sin(3x)$$

$$+ \frac{15149}{360}\sin(4x) + \frac{82205}{3072}\sin(5x) + \frac{899}{160}\sin(6x)$$

$$+ \frac{16807}{46080}\sin(7x)$$

Appendix B - 3

$$a_7(x) = \frac{82289}{64512}\sin(x) + \frac{1109}{140}\sin(2x) - \frac{43283}{15360}\sin(3x)$$

$$- \frac{1919}{20}\sin(4x) - \frac{1113505}{9216}\sin(5x) - \frac{6871}{140}\sin(6x)$$

$$- \frac{355081}{46080}\sin(7x) - \frac{128}{315}\sin(8x)$$

$$a_8(x) = -\frac{2628205}{1032192}\sin(x) - \frac{85501}{6720}\sin(2x)$$

$$- \frac{25937987}{860160}\sin(3x) + \frac{50651}{320}\sin(4x)$$

$$+ \frac{215355425}{516096}\sin(5x) + \frac{133633}{448}\sin(6x)$$

$$+ \frac{126391769}{1474560}\sin(7x) + \frac{47259}{4480}\sin(8x)$$

$$+ \frac{531441}{1146880}\sin(9x)$$

Appendix B - 4

$$a_9(x) = \frac{257555521}{46448640}\sin(x) + \frac{1536391}{72576}\sin(2x)$$

$$+ \frac{755093851}{7741440}\sin(3x) - \frac{3253091}{22680}\sin(4x)$$

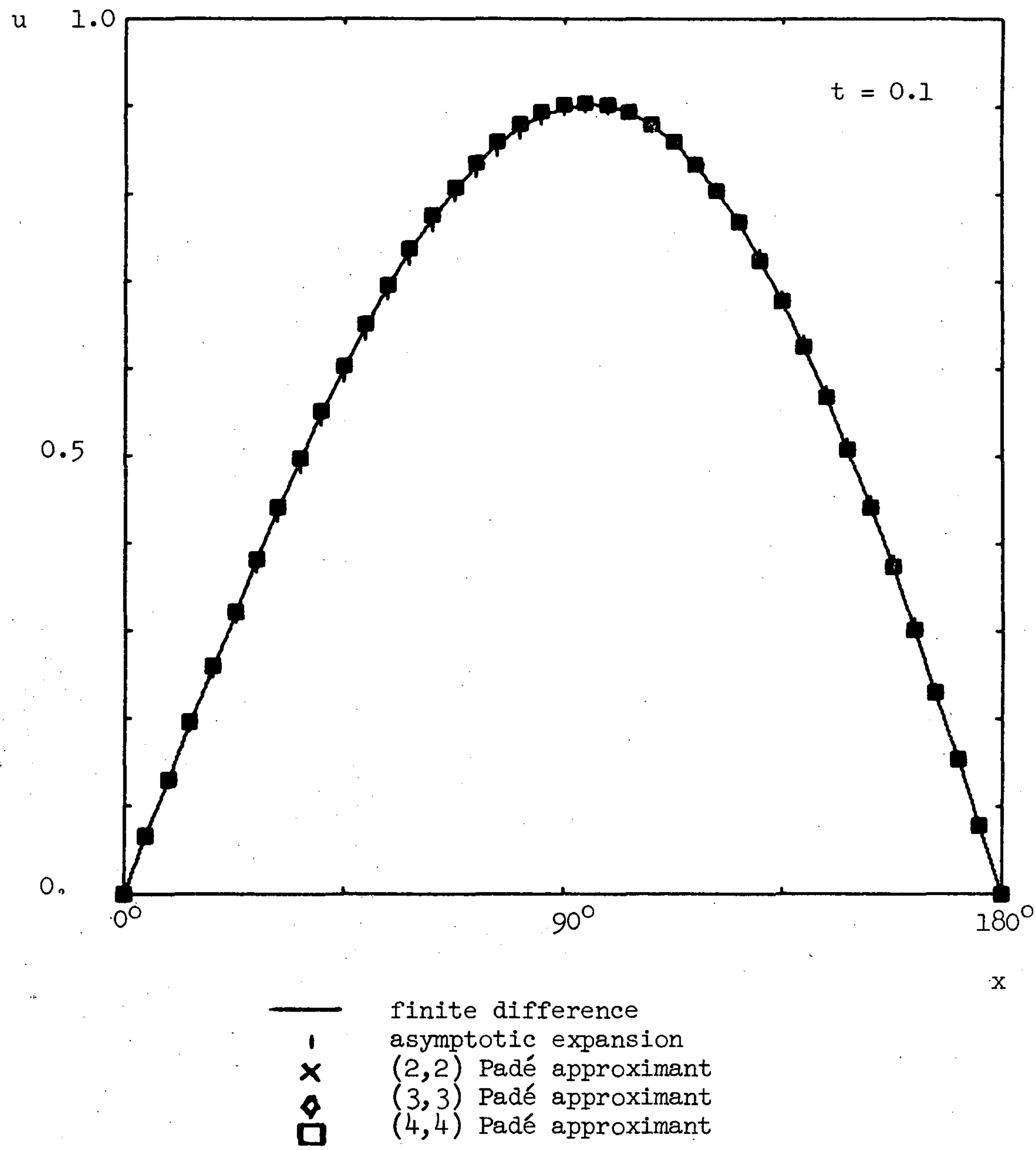
$$- \frac{1779033427}{1548288}\sin(5x) - \frac{12619227}{8960}\sin(6x)$$

$$+ \frac{13709849227}{30965760}\sin(7x) - \frac{6555559}{45360}\sin(8x)$$

$$- \frac{16541017}{1146880}\sin(9x) - \frac{78125}{145152}\sin(10x)$$

Appendix B - 5

$$(2,2) = \frac{\left[\left(-\frac{477}{2}\sin(x) + 126\sin(2x) + \frac{405}{4}\sin(3x) + 39\sin(4x) + \frac{15}{4}\sin(5x)\right) + \left(-\frac{1449}{4}\sin(x) - \frac{1539}{16}\sin(2x) + \frac{2101}{9}\sin(3x) + \frac{741}{4}\sin(4x) + \frac{399}{8}\sin(5x) + \frac{57}{16}\sin(6x)\right)t + \left(-\frac{795}{64}\sin(x) + \frac{1849}{16}\sin(2x) - \frac{8079}{64}\sin(3x) - \frac{79}{4}\sin(4x) + \frac{929}{64}\sin(5x) + \frac{69}{16}\sin(6x) + \frac{17}{64}\sin(7x)\right)t^2\right]}{\left[\left(-\frac{267}{2} + 330\cos(x) + 210\cos(2x) + 78\cos(3x) + \frac{15}{2}\cos(4x)\right) + \left(-\frac{69}{4} + \frac{1947}{4}\cos(x) + 1041\cos(2x) + \frac{4515}{8}\cos(3x) + \frac{585}{4}\cos(4x) + \frac{87}{8}\cos(5x)\right)t + \left(-\frac{5493}{16} + \frac{3501}{4}\cos(x) + \frac{21483}{32}\cos(2x) + \frac{4949}{8}\cos(3x) + \frac{4191}{16}\cos(4x) + \frac{417}{8}\cos(5x) + \frac{101}{32}\cos(6x)\right)t^2\right]}$$



XBL 696-624

Fig. 1. Solutions to Burgers' equation.

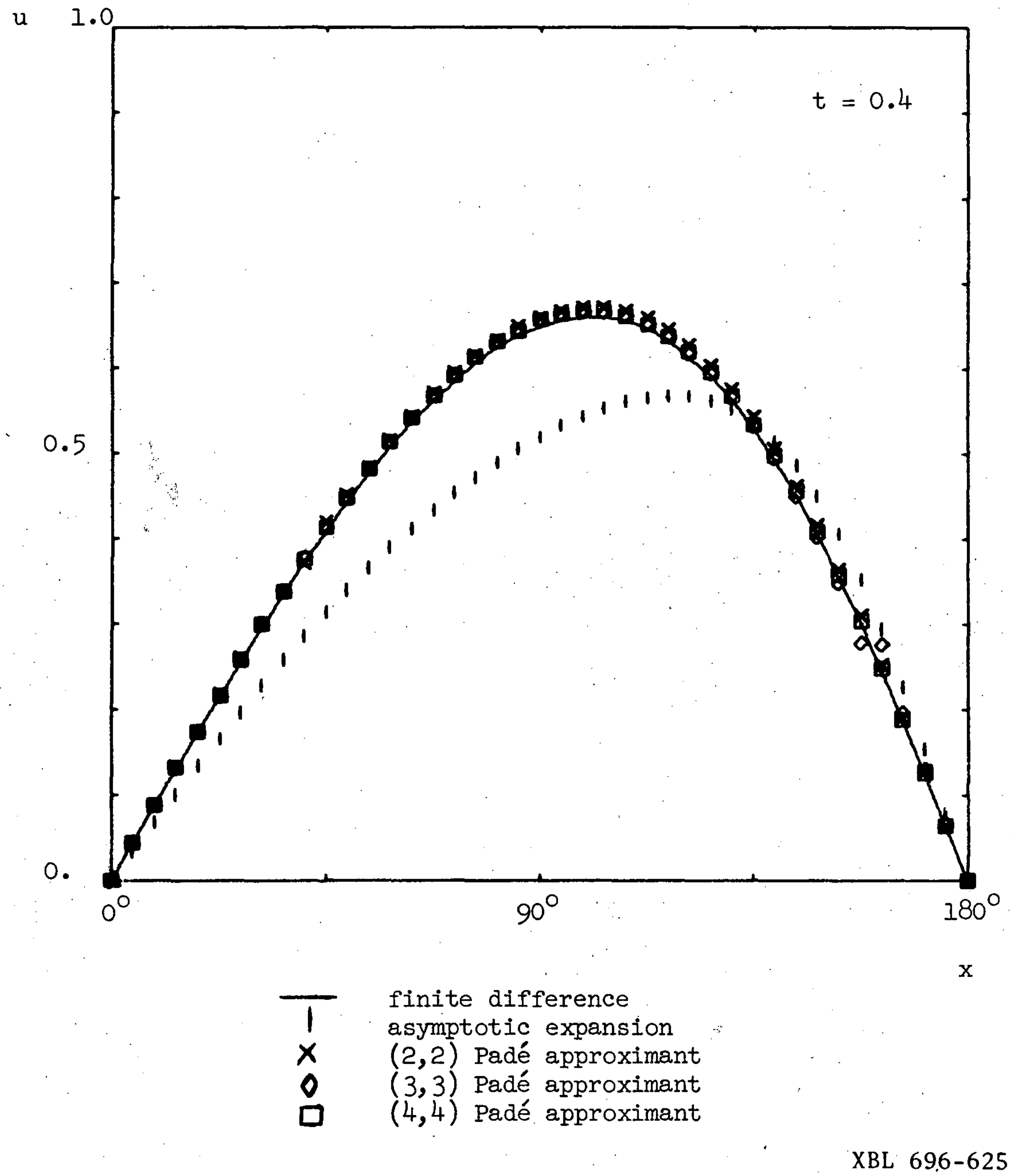
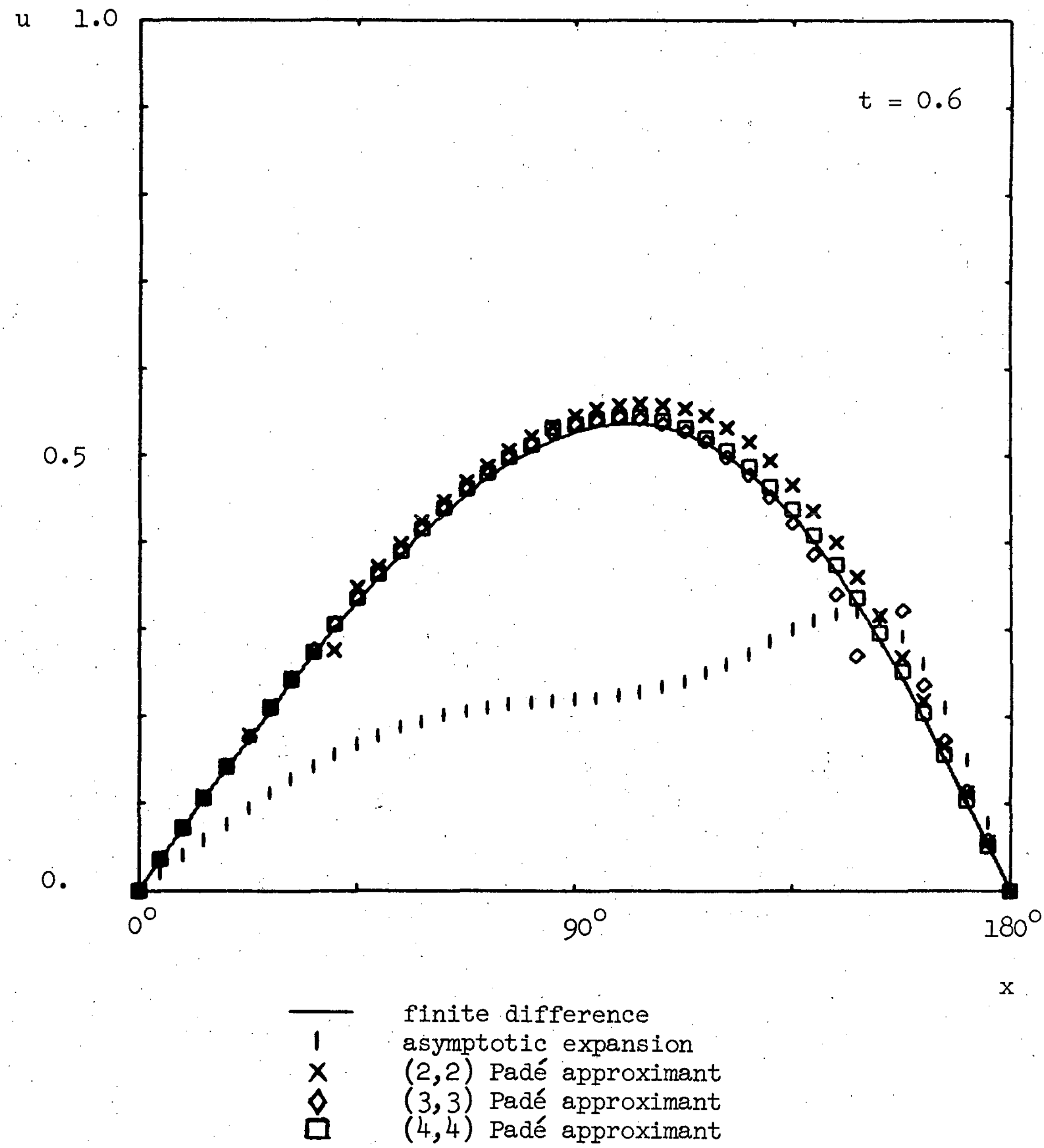
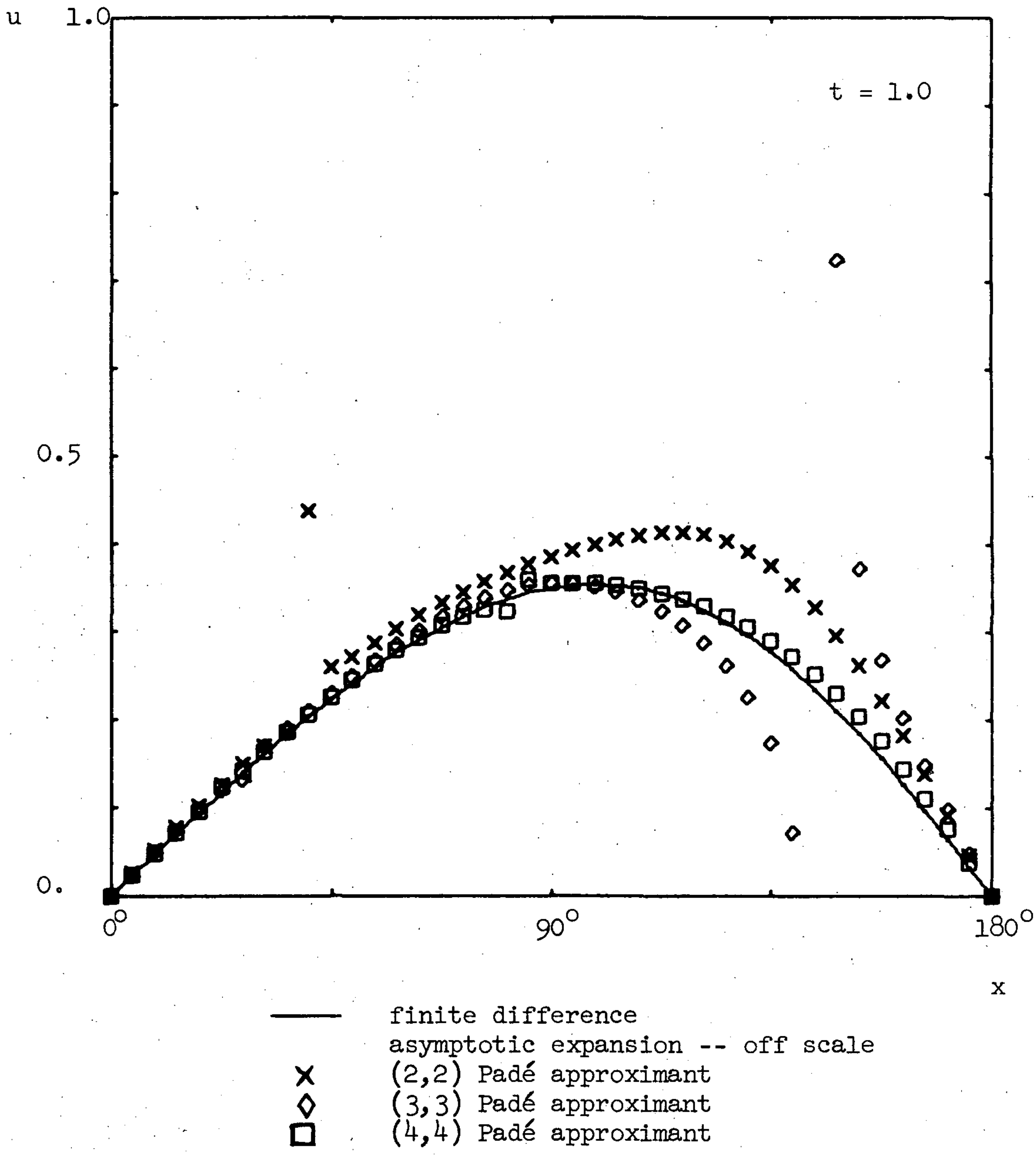


Fig. 2. Solutions to Burgers' equation.



XBL 696-626

Fig. 3. Solutions to Burgers' equation.



XBL 696-627

Fig. 4. Solutions to Burgers' equation.

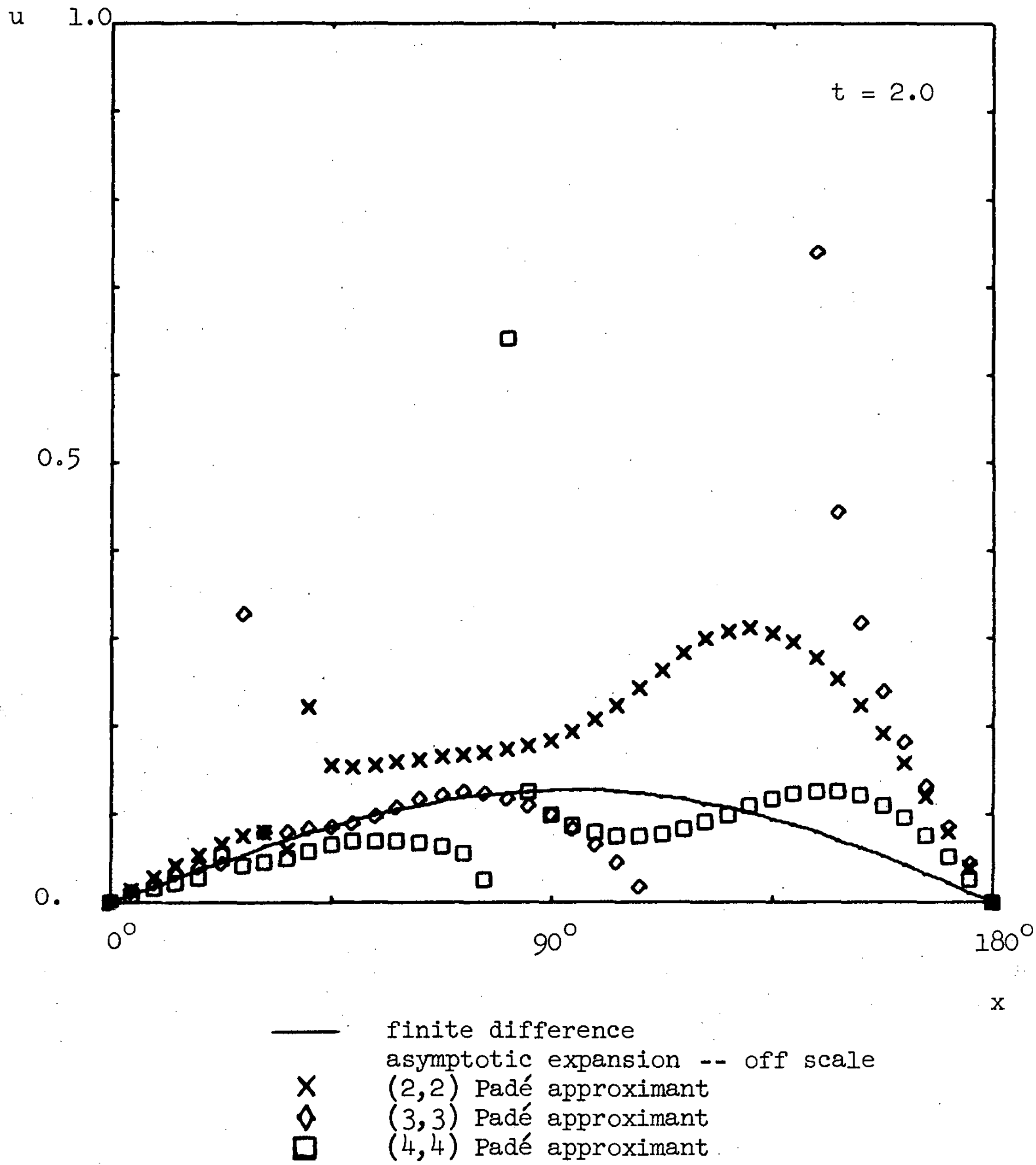


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Fig. 5. Solutions to Burgers' equation.

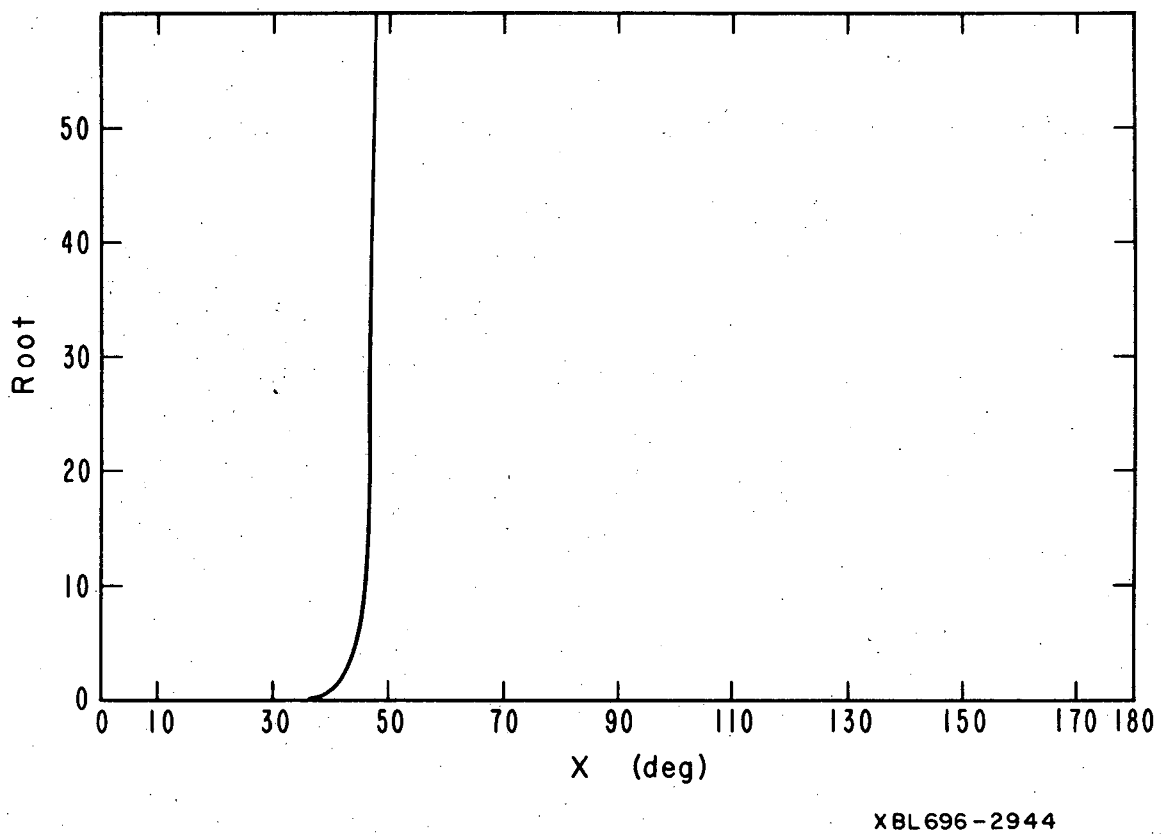

Fig. 6. Root profile, Denominator of (2, 2) Padé approximant.

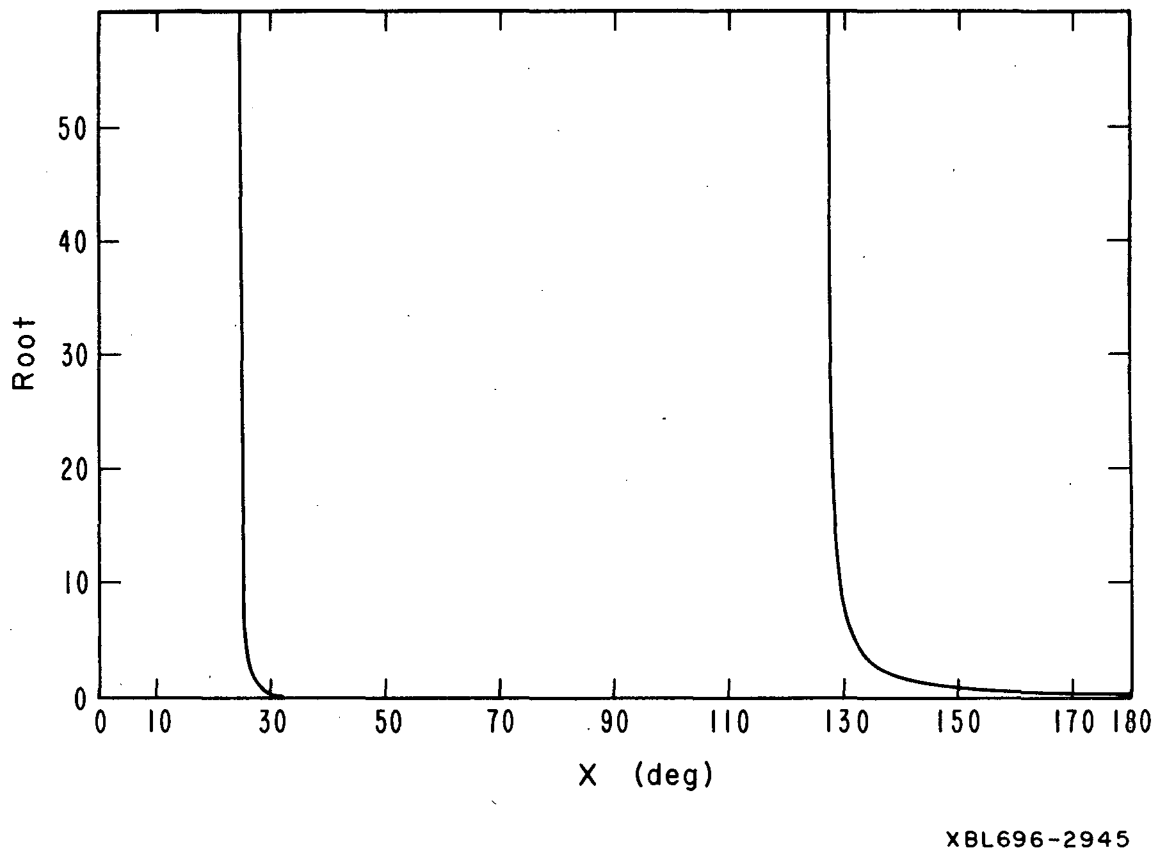

Fig. 7. Root profile. Denominator of (3, 3) Padé approximant.

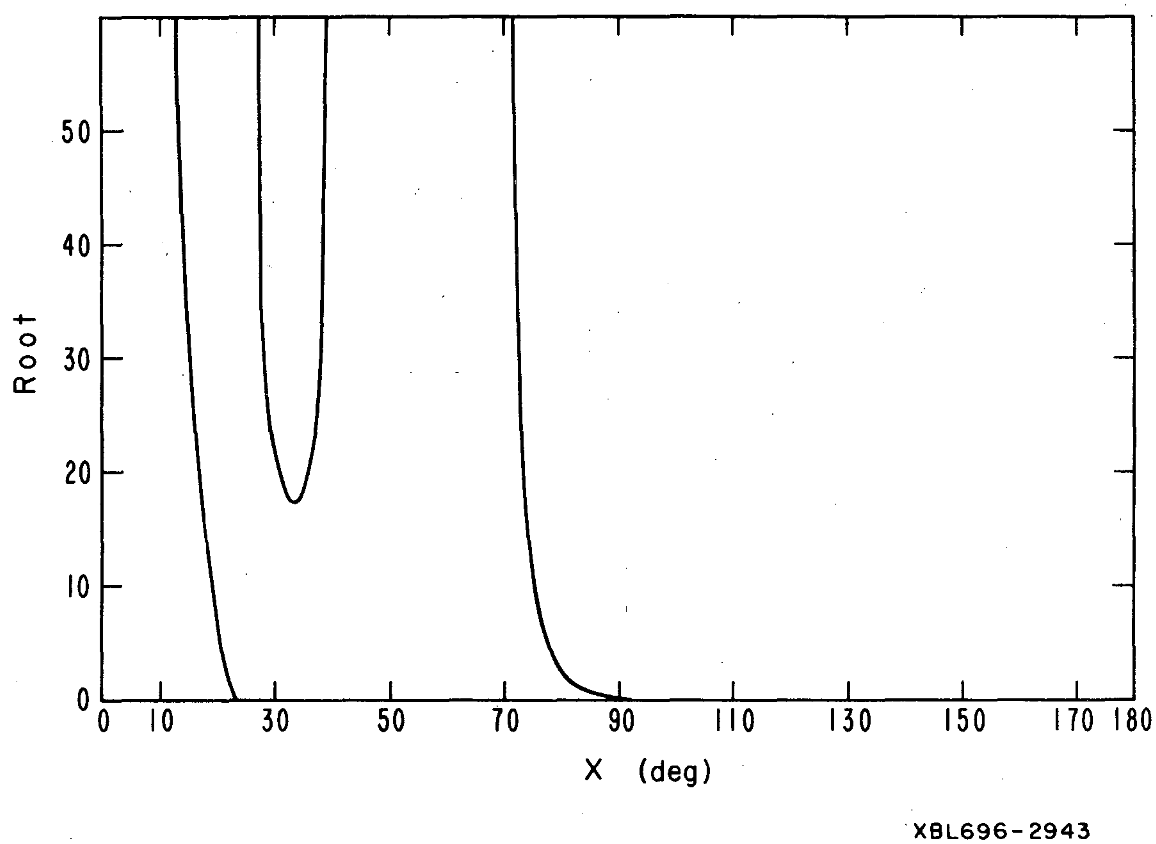

Fig. 8. Root profile. Denominator of (4, 4) Padé approximant.

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